ABSTRACT

LI, ZIYUE. Interpolation from Instant Form Dynamics to Light-Front Dynamics and Update on Light-Front Quark Model Analysis in Meson Phenomenology. (Under the direction of Dr. Chueng Ji.)

In this dissertation, we present the interpolation of the quantum electrodynamics (QED) theory between the instant form and the front form of relativistic Hamiltonian dynamics. A unified general physical gauge that interpolates between the Coulomb gauge in the instant form dynamics (IFD) and the light-front gauge in the light-front dynamics (LFD) is obtained. Photon polarization vectors and corresponding photon propagator for any interpolation angle are derived. Using these results, we calculate time-order scattering amplitudes for an arbitrary interpolation angle. The emerged universal J-shaped correlation between the total momentum of the system and the interpolation angle, coined as the “J-curve”, characterizes how the IFD changes to the LFD. We discuss the singular behavior of this correlation in conjunction with the zero-mode issue in LFD. We also derive the generalized helicity spinor in the \((0, J) \oplus (J, 0)\) chiral representation that naturally links the instant form helicity spinors and the light-front spinor as well as the generalized Melosh transformation that relates these generalized helicity spinors to the usual Dirac spinors. Our analysis of the spin orientation angle reveals that its behavior as a function of the momentum direction is bifurcated at a critical interpolation angle and the IFD and the LFD separately belong to the two different branches. In conjunction with the bifurcation of branches, two boundaries appear in the interpolating helicity amplitudes and the same J-curve persists within these two boundaries. These interpolating amplitudes provide examples of broken reflection symmetry under the longitudinal boost, \(P^z \leftrightarrow -P^z\), and clarify the confusion in the prevailing notion of the equivalence between the infinite momentum frame (IMF) and the LFD.

To conclude this interpolation study, we give the full Feynman rules for interpolating time-ordered diagrams in QED, including the interpolation of time-ordered fermion propagators. The canonical field theory approach is explored and the interpolating QED Hamiltonian is obtained.

We end this dissertation with an updated analysis of our light-front quark model (LFQM) for ground state pseudoscalar and vector mesons, using the variational principle to compute mass spectra and decay constants with the QCD-motivated effective Hamiltonian including a smeared out hyperfine interaction. We also consider the flavor mixing effect in our analysis and determine the mixing angles from the mass spectra of \((\omega, \phi)\) and \((\eta, \eta')\).
Interpolation from Instant Form Dynamics to Light-Front Dynamics and Update on
Light-Front Quark Model Analysis in Meson Phenomenology

by
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DEDICATION

To my mom.
BIOGRAPHY

The author was born and raised in Jishou, a small town of western Hunan province in southwest China. After high school, he went to University of Science and Technology Beijing to major in applied physics. During this time, his family moved from Jishou to Changsha, the capital city of Hunan province. In 2008, he came to the United States to study theoretical particle physics at North Carolina State University.
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Among the three different forms of relativistic dynamics proposed by Dirac [Dir49] in 1949, the instant form \( (x^0 = 0) \) and the front form \( (x^+ \equiv (x^0 + x^3)/\sqrt{2} = 0) \) have been the most popular choices in studying hadron physics. While the quantization at the equal time \( t = x^0 \) (in \( c = 1 \) unit) produces the instant form dynamics (IFD) of quantum field theory, the quantization at equal light-front time \( \tau = x^+ = (t + z)/\sqrt{2} \) yields the front form dynamics, now known as the light-front dynamics (LFD).

The LFD is especially useful in hadron physics, because of its unique properties. As first demonstrated by Dirac [Dir49], in LFD, we have the maximum number (seven) of kinematic (i.e. interaction independent) operators out of the ten Poincaré generators and they leave the state at \( \tau = 0 \) unchanged. In particular, the longitudinal boost operator joins the stability group of kinematic operators in LFD, thus the description of a hadron at fixed \( x^+ \) is independent of the observer’s frame and this gives a boost invariant wave function. Another remarkable feature of the light-front quantization is that the vacuum state of the full theory coincides with the free theory. This simpler vacuum property together with the built-in boost invariance makes the LFD a natural candidate for addressing dynamical processes in quantum chromodynamics (QCD), which are very difficult to deal with in the IFD. In fact it provides a possible first principle method for linking the constituent quark model, or the quark parton model, to the fundamental QCD. And given the frame-independent light-front wave function, one can calculate a large range of hadronic variables such as form factors, structure functions, generalized parton
distribution functions, etc.

The light-front quantization [Dir49; Ste80] has been applied successfully in the context of current algebra [Wei66; JS68] and the parton model [Bjo69; Dre69] in the past. With further advances in the Hamiltonian renormalization program [PB85; BP91; Per90; PH91; Mus91], the LFD appears to be even more promising for the relativistic treatment of hadrons. In the work of Brodsky et al. [Bro98b; Hil98], it is demonstrated how to solve the problem of renormalizing light-front Hamiltonian theories while maintaining Lorentz symmetry and other symmetries. The genesis of the work presented in Ref. [Bro98b; Hil98] may be found in Ref. [RM92; MR92] and additional examples including the use of LFD methods to solve the bound-state problems in field theory can be found in the review of QCD and other field theories on the light front [Bro98a]. A possible realization of chiral symmetry breaking in the light-front vacuum has also been discussed in the literature [SB94; WR94].

On the other hand, the light-front formulation has its own difficulties. For one, the transverse rotation whose direction is perpendicular to the direction of the quantization axis \( z \) at equal \( \tau \) becomes a dynamical problem in the LFD because the quantization surface \( \tau \) is not invariant under the transverse rotation and the transverse angular momentum operator involves the interaction that changes the particle number [JS92]. For another, processes for which the light-front momentum \( P^+ = 0 \) can have non-vanishing contributions in LFD, giving rise to the so-called "zero-mode" issue.

In light of the current 12 GeV energy upgrade at Jefferson lab, where the hadron structure and exotic bound states of partons will be studied in detail, it is more important than ever to have a better grasp of LFD, which is an essential tool for studying hadron properties and one of the few candidates for making a connection between the fundamental QCD theory and the constituent quark model. So as an effort to further our understanding of the unique properties of LFD, we link it to the familiar IFD, and interpolate between the two forms of dynamics by introducing an interpolation angle \( \delta \) that changes the ordinary time \( t \) to the light front time \( \tau \). The same method of interpolating hypersurfaces has been used by Hornbostel [Hor92] to analyze various aspects of field theories including the issue of nontrivial vacuum. The same vein of application to study the axial anomaly in the Schwinger model has also been presented [JR96], and other related works [Che71; EG76; Fri77; Saw91] can also be found in the literature. But a detailed interpolation study between the IFD and the LFD has not been pursued anywhere else as far as we know.
Our end game, of course, is to interpolate the QCD theory. But in this thesis we form the foundation and pave the way by studying the interpolation of a simpler theory, the quantum electrodynamics (QED). Our interpolation study between the IFD and the LFD provides the whole landscape picture between the two, brings valuable insights into the light-front formulation itself and clarifies the issue, if any, in linking them to each other. It not only reveals how the unique properties of LFD come about, but also clarifies some prevailing confusions such as the notion of equivalence between the LFD and the infinite momentum frame (IMF).

This thesis is organized as follows:

We will start out by reviewing the Hamiltonian formulation of dynamics in relativistic quantum mechanics, introducing some of the basics of the IFD and the LFD in Chapter 2. We then explain our interpolation method in Chapter 3, where we list the interpolating Poincaré algebra and formulate a $T$ transformation that interpolates between the helicity transformation in the IFD and the light-front boost in the LFD.

With these preliminary preparations, we can start interpolating the QED theory. This is divide up into three parts: Chapter 4, Chapter 5 and Chapter 6. In Chapter 4, we focus on interpolating the photon polarization vectors and the photon propagator, using the simpler scalar quantum electrodynamics (sQED) where the fermion spinor fields are replaced by scalar fields. The covariant photon propagator is decomposed into interpolating “time”-ordered propagators, and corresponding interpolating “time”-ordered scattering and annihilation amplitudes are calculated. When these “time”-ordered amplitudes are plotted in terms of both the interpolation angle and the total momentum of the system to reflect both the interpolation angle dependence and the frame dependence, a universal J-shaped curve emerges, characterizing how the instant form changes to the light-front form. A detailed discussion of the zero-mode contribution resulted from this J-curve is given. In Chapter 5, the interpolating fermion spinors are derived and analyzed in detail. Two distinct branches of behavior for the spin orientation angle appear, which are related to the sudden spin flip across a critical interpolation angle $\delta$ when the particle is moving in $\pm z$ direction. This then results in two boundaries in plots of the helicity scattering and annihilation probabilities. The same J-curve that’s in the interpolating “time”-ordered diagrams in Chapter 4 also persists here in the helicity probabilities. In Chapter 6, we derive the final piece of the interpolating QED theory, i.e. the interpolating fermion propagator, and give a formal derivation of “time”-ordered diagrams from which
the corresponding Feynman rules for any interpolation angle are obtained. The canonical field theory approach is also studied to some extent, and the interpolated Hamiltonian form is given. As the application of interpolation to QED was successful, we may envision forthcoming successful application to QCD, although the nonabelian gauge theory involves new sets of problems such as gauge copies, Gribov ambiguity, etc.

These interpolation studies lend great support to the light-front theory with valuable insights, but our ultimate goal lies in the light-front theory itself. Using LFD, we want to find effective degrees of freedom in hadron phenomenology that can hopefully one day be connected to the fundamental QCD theory. Therefore, to demonstrate how the light-front formulation can be useful, in Chapter 7 we present our updated analysis of the ground-state pseudoscalar and vector mesons in the light-front quark model (LFQM), where we fitted the mass spectrum using variational principle and calculated decay constants and compared the results with experimental values and other people’s calculations. Although the connection between QCD and the LFQM is still not fully clear, we believe our interpolation studies together with our phenomenological model are valuable contributions to this effort.

Our conclusions are summarized in Chapter 8.

The detailed structures of each chapter is explained in the beginning of that chapter. Because of the complexity, for Chapters 4 - 7, we also provide a summary at the end of each chapter. The appendices are grouped for their corresponding chapters for convenience and clarity.

Finally, we want to mention that Chapter 4, Chapter 5 and Chapter 7 are each adapted from our published work [Ji15; Li15; Cho15], while Chapter 6 about the formal derivation of the interpolating QED theory is based on our current ongoing research, of which a more detailed and complete version will be published in the future.
For the purpose of hadron physics, it is highly desirable to combine the principle of relativity with the Hamiltonian formulation of dynamics. In this chapter we give a quick review of these two subjects.

The principle of relativity, discovered in the early 20th century by Einstein, imposes conditions which all physical laws should satisfy. Although a perfect marriage between general relativity and quantum mechanics is still elusive, in hadron physics the departure from flat space-time due to gravitational effects is so excessively small we can simply consider special relativity. In Sec. 2.1 we review some of the basic results from combining special relativity with quantum mechanics.

To understand a hadron as a bound state of partons, a Hamiltonian formulation is not only desirable but also convenient. By choosing a quantization surface and the evolutionary direction, one defines a certain formulation of the Hamiltonian dynamics. The three forms of dynamics proposed by Dirac [Dir49], namely the instant form, the front form and the point form are briefly reviewed in Sec. 2.2.

Finally, in Sec. 2.3, we give a short introduction to some of the advantages of the light-front formulation and why it is an area of great interest.
2.1 Combining Special Relativity with Quantum Mechanics

2.1.1 Poincaré Transformation

According to Einstein’s theory of special relativity, the space-time coordinates $x^\mu$ and $x'^\mu$ in two different inertial frames should satisfy the following relation:

$$g_{\mu\nu}dx'^\mu dx'^\nu = g_{\rho\sigma}dx^\rho dx^\sigma,$$  \hspace{1cm} (2.1)

or equivalently

$$g_{\mu\nu} \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x'^\nu}{\partial x^\sigma} = g_{\rho\sigma}.  \hspace{1cm} (2.2)$$

When written in the ordinary space-time coordinates the flat space-time metric is

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \hspace{1cm} (2.3)$$

Although we’ve written this metric on a particularly simple coordinate basis so that it’s diagonal, the rest of the derivations in this section only depends on the fact that we’ve chosen a time-positive metric signature, and do not depend on the specific coordinate basis one may choose.

The transformation that satisfies Eq. (2.2) is linear

$$x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu, \hspace{1cm} (2.4)$$

with $a^\mu$ an arbitrary constant, and $\Lambda^\mu_\nu$ a constant matrix satisfying

$$g_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma = g_{\rho\sigma}. \hspace{1cm} (2.5)$$
The inverse of $\Lambda^\mu{}_{\nu}$ can be calculated from Eq. (2.5) to be

$$(\Lambda^{-1})^\alpha{}_{\beta} = \Lambda^\alpha{}_{\beta} \equiv g_{\beta\mu}g^{\alpha\nu}\Lambda^\mu{}_{\nu}. \quad (2.6)$$

The transformation in Eq. (2.4) may contain time and/or spacial reflections, but here we only consider transformations that preserve both orientation and direction of time. These transformations can be obtained by continuous built up of infinitesimal transformations from the identity.

We now use $T(\Lambda, a)$ to denote the said transformation on the space-time coordinates. The whole group $T(\Lambda, a)$ is known as the inhomogeneous Lorentz group, or Poincaré group. When $a = 0$, $T(\Lambda, 0)$ forms a subgroup called the homogeneous Lorentz group, or Lorentz group in short. Using Eq. (2.4), one can easily verify that

$$T(\tilde{\Lambda}, \tilde{a})T(\Lambda, a) = T(\tilde{\Lambda}\Lambda, \tilde{a} + \Lambda a), \quad (2.7)$$

where the bar is used to distinguish one Poincaré transformation from another.

The restricted principle of relativity says that physical laws should be invariant under Poincaré transformation. In the next subsection, we review how a symmetry transformation is incorporated in quantum mechanics.

### 2.1.2 Special Unitary Transformation

In quantum mechanics, a physical state is represented by a state vector $|\Psi\rangle$ in Hilbert space and a transformation on the state is represented as an operation on the state vector. For example, we may define an operator $U$ on the Hilbert space, so that $U|\Psi\rangle$ is the transformed state. To determine the explicit form of $U$ for a symmetry transformation, we take a look at all the properties it should have.

1. First, a certain probability $\langle \phi | \Psi \rangle^2$ shouldn’t change under a symmetry transformations, i.e.

$$\langle \phi | U^\dagger U | \Psi \rangle^2 = \langle \phi | \Psi \rangle^2. \quad (2.8)$$

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1Here the positions of the indices are important. The up and down positions indicate the contravariant and covariant components while the left and right positions indicate the row and column the matrix. Therefore, $\Lambda^\mu{}_{\nu}$ is not the same as $\Lambda^\mu{}_{\nu}$. 

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To satisfy this condition, we take $U$ to be a special unitary matrix, i.e.

$$U^\dagger = U^{-1}, \text{ with } \text{Det}(U) = 1. \quad (2.9)$$

2. For a continuous transformation that's a function of a continuous parameter, we should have\(^2\)

$$U(\theta_1)U(\theta_2) = U(\theta_1 + \theta_2). \quad (2.10)$$

For example, if $U$ represents a rotation around a certain axis, then two subsequent rotations of $\theta_1$ and $\theta_2$ should be the same as a single rotation of $\theta_1 + \theta_2$. For any transformation that corresponds to $\theta$, we can write it as an accumulation of many small transformations

$$U(\theta) = \left[U\left(\frac{\theta}{n}\right)\right]^n. \quad (2.11)$$

On the other hand, assuming $U$ to be an analytical function of $\theta$, we can expand $U(\theta/n)$ into a Taylor series. Coupled with the fact that when $\theta = 0$, $U$ should reduce to the identity operator, we have

$$U\left(\frac{\theta}{n}\right) = 1 + c_1 \frac{\theta}{n} + \frac{1}{2} c_2 \left(\frac{\theta}{n}\right)^2 + \cdots, \quad (2.12)$$

where $c_m \ (m = 1, 2, \ldots)$ are expansion coefficients. Since Eq. (2.11) should hold for arbitrarily large $n$, we can plug Eq. (2.12) into Eq. (2.11) and take the limit of $n \to \infty$ to get

$$U(\theta) = [1 + c_1 \frac{\theta}{n} + \cdots]^n \lim_{n \to \infty} \frac{1}{n} e^{c_1 \theta}. \quad (2.13)$$

From Eq. (2.9) and Eq. (2.13), we conclude that $U$ assumes the form

$$U(\theta) = e^{-iC\theta}, \quad (2.14)$$

where the $"-i"$ is pulled out of $c_1$ in Eq. (2.13) so that $C$ now is a Hermitian operator, i.e. $C^\dagger = C$. The hermitian operator $C$ is known as the generator for the symmetry. The negative sign in front of $"i"$ is our convention.

\(^2\)Strictly speaking, there could be a phase factor in this relation, i.e. $U(\theta_1)U(\theta_2) = e^{i\phi}U(\theta_1 + \theta_2)$. But since we take $U$ to be special unitary, the phase $\phi$ reduces to 0. For a detailed discussion of the phase factor, the readers are referred to Weinberg's book [Wei95].
2.1.3 Poincaré Algebra

In order for the special unitary transformations $U(\Lambda, a)$ on the Hilbert space to represent the Poincaré transformation $T(\Lambda, a)$ which is in the coordinate space, there are certain relations the corresponding symmetry generators should satisfy.

From Eq. (2.4), we see that the identity for Poincaré transformation is obtained when $\Lambda^\mu_\nu = \delta^\mu_\nu, \ a^\mu = 0$. So for an infinitesimal transformation, we can write

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu, \ a^\mu = \epsilon^\mu,$$

with both $\omega^\mu_\nu$ and $\epsilon^\mu$ infinitesimally small. Then Lorentz condition given by Eq. (2.5) now becomes

$$g_\rho\sigma = g^\mu_\nu (\delta^\mu_\rho + \omega^\mu_\rho)(\delta^\sigma_\nu + \omega^\sigma_\nu) = g_\rho\sigma + \omega_\rho\sigma + \omega_\sigma\rho + O(\omega^2).$$

(2.16)

Keeping only the first order terms in $\omega$, we see that the Lorentz condition reduces to the antisymmetry of $\omega^\mu_\nu$

$$\omega^\mu_\nu = -\omega^\nu_\mu.$$  

(2.17)

An infinitesimal Poincaré transformation in the Hilbert space can therefore be written as

$$U(1 + \omega, \epsilon) = 1 - \frac{1}{2}i\omega^\mu_\nu M^\mu_\nu + i\epsilon^\mu P^\mu + \cdots.$$  

(2.18)

Because of the antisymmetry of $\omega^\mu_\nu$, we can take $M^\mu_\nu$ to be antisymmetric also. This introduces two copies of the same terms, so we have the $\frac{1}{2}$ factor in front. This antisymmetry also tells us that there are only $(4 \times 3)/2 = 6$ independent components in $M^\mu_\nu$. So including the four components of $P^\mu$, a Poincaré transformation has $6 + 4 = 10$ generators. These are the ten fundamental quantities for a dynamical system.

We can now calculate the relations between these 10 generators by considering the

---

Note that the signs in front of the first order terms are opposite of that used in Weinberg’s book [Wei95]. This is because he used the space-positive metric signature $g^\mu_\nu = \text{diag}(-1, 1, 1, 1)$, while I use the time-positive metric signature $g^\mu_\nu = \text{diag}(1, -1, -1, -1)$. This results in, for example, $\omega^{12} = \omega^{12} = \omega^{12}$ in Weinberg’s convention and $\omega^{12} = \omega^{12} = -\omega^{12}$ in my convention.
product

\[ U(\Lambda, a)U(1 + \omega, \epsilon)U^{-1}(\Lambda, a), \quad (2.19) \]

where \( \Lambda^\mu_\nu \) and \( a^\mu \) are parameters of a new transformation unrelated to \( \omega \) and \( \epsilon \). First of all, Eq. (2.7) correspond to the following relation in the Hilbert space:

\[ U(\tilde{\Lambda}, \tilde{a})U(\Lambda, a) = U(\tilde{\Lambda}\Lambda, \tilde{a}a + \tilde{a}). \quad (2.20) \]

Setting \( \tilde{\Lambda}\Lambda = 1 \) and \( \tilde{a}a + \tilde{a} = 0 \), we find that the inverse of \( U(\Lambda, a) \) is \( U(\Lambda^{-1}, -\Lambda^{-1}a) \). One can then use Eq. (2.20) twice to obtain

\[ U(\Lambda, a)U(1 + \omega, a)U^{-1}(\Lambda, a) = U(\Lambda(1 + \omega)\Lambda^{-1}, \Lambda\epsilon - \Lambda\omega\Lambda^{-1}a). \quad (2.21) \]

Keeping both sides to the first order in \( \omega \) and \( \epsilon \), we have

\[
U(\Lambda, a) \left[ -\frac{1}{2} \omega^{\rho\sigma} M_{\rho\sigma} + \epsilon^\mu P_\mu \right] U^{-1}(\Lambda, a) = -\frac{1}{2}(\Lambda\omega\Lambda^{-1})^{\mu\nu} M_{\mu\nu} + (\Lambda\epsilon - \Lambda\omega\Lambda^{-1}a)^\mu P_\mu \\
= -\frac{1}{2}(\Lambda^\mu_\nu \omega^\rho_\sigma (\Lambda^{-1})^\sigma_\gamma) g^{\gamma\nu} M_{\mu\nu} - (\Lambda^\mu_\rho \omega^\rho_\sigma (\Lambda^{-1})^\sigma_\nu a^\nu) P_\mu + (\Lambda^\mu_\nu \epsilon^\nu) P_\mu \\
= -\frac{1}{2} \Lambda^\rho_\mu \omega^{\rho\sigma} \Lambda^\nu_\sigma M_{\mu\nu} - \Lambda^\mu_\rho \omega^{\rho\sigma} \Lambda^\nu_\sigma a_\nu P_\mu + \Lambda^\mu_\nu \epsilon^\nu P_\mu, \quad (2.22) \]

where Eq. (2.6) is used in the last step. Equating coefficients of \( \omega^{\rho\sigma} \) and \( \epsilon^\nu \) on both sides and using the antisymmetry of \( \omega^{\rho\sigma} \), we find

\[
U(\Lambda, a)M_{\rho\sigma}U^{-1}(\Lambda, a) = \Lambda^\mu_\rho \Lambda^\nu_\sigma (M_{\mu\nu} - a_\mu P_\nu + a_\nu P_\mu), \quad (2.23) \\
U(\Lambda, a)P_\rho U^{-1}(\Lambda, a) = \Lambda^\mu_\rho P_\mu. \quad (2.24) \]

Next, we treat \( U(\Lambda, a) \) and \( U^{-1}(\Lambda, a) \) also as infinitesimal transformations with parameters \( \omega^{\mu\nu} \) and \( \epsilon^\mu \) unrelated to the previous \( \omega \) and \( \epsilon \). Keeping everything to the first order, we get

\[
 i \left[ -\frac{1}{2} \omega^{\mu\nu} M_{\mu\nu} + \epsilon^\mu P_\mu, M_{\rho\sigma} \right] = \omega^\mu_\rho M_{\mu\rho} + \omega^\nu_\sigma M_{\rho\nu} - \epsilon_\rho P_\sigma + \epsilon_\sigma P_\rho \quad (2.25) \\
 i \left[ -\frac{1}{2} \omega^{\mu\nu} M_{\mu\nu} + \epsilon^\mu P_\mu, P_\rho \right] = \omega^\mu_\rho P_\mu \quad (2.26) \]
Figure 2.1 Three forms of relativistic dynamics proposed by Dirac: (a) the instant form, (b) the light-front form, and (c) the point form. They are defined by different quantization surfaces, shown as the thick black lines. The black arrows indicate the direction of evolution.

Finally, equating coefficients of $\omega$ and $\epsilon$ on both sides, we find the commutation rules:

\[
\begin{align*}
    i[M_{\mu\nu}, M_{\rho\sigma}] &= g_{\mu\rho}M_{\nu\sigma} - g_{\nu\rho}M_{\mu\sigma} - g_{\mu\sigma}M_{\nu\rho} + g_{\nu\sigma}M_{\mu\rho}, \\
    i[P_{\mu}, M_{\rho\sigma}] &= g_{\mu\sigma}P_{\rho} - g_{\mu\rho}P_{\sigma}, \\
    [P_\mu, P_\rho] &= 0.
\end{align*}
\] (2.27) (2.28) (2.29)

This is the Poincaré Algebra.

Notice that although the derived form of Poincaré algebra is for time-positive metric signature $(+,−,−,−)$, we haven’t invoked any specific choice of space-time coordinate basis in our derivation. The choice of the quantization surface, evolution direction and the associated system of coordinates makes Eqs. (2.27) - (2.29) assume specific form for the relevant dynamics, and is the subject of the next section.

2.2 Hamiltonian Dynamics

To obtain a Hamiltonian formulation of dynamics, we need to choose a surface on which quantization conditions (commutation relations) are defined. The fundamental quantities associated with infinitesimal transformations that leave this surface invariant are specially simple. They are called the kinematic operators. The rest of the ten fundamental quantities will be complicated in general and are dynamical.
In 1949, Dirac proposed three forms of Hamiltonian dynamics [Dir49], the instant form, the front-form and the point form, with three different choices of quantization surfaces, as demonstrated in Fig. 2.1. We will give a brief review of them in this section.

2.2.1 Instant Form

In conventional quantization, time is selected as the direction of propagation, and the quantization conditions are imposed on surfaces of instant time. The simplest instant surface is given by \( t = 0 \), and we take \( t > 0 \) to be the direction of evolution. This is called the instant form. The natural coordinate system for instant form is the ordinary space-time coordinates \((x^0, x^2, x^3, x^4)\) or \((t, x, y, z)\).

In the instant form, \( H = P^0 \) is the Hamiltonian, and operators that commute with it, namely the momentum 3-vector

\[
P = (P^1, P^2, P^3)
\]  

and the angular momentum 3-vector

\[
J = (M_{23}, M_{31}, M_{12}),
\]

are kinematic and they generate transformations that don’t leave the \( t = \) constant surface. The remaining generators form the so-called “boost” 3-vector

\[
K = (M_{10}, M_{20}, M_{30}).
\]

With our metric convention \((g_{00} = 1, g_{11} = g_{22} = g_{33} = -1)\), the Poincaré algebra in Eqs. (2.27) - (2.29) can now be written in the instant form as

\[
\begin{align*}
[J^i, J^j] &= i\epsilon_{ijk} J^k, \\
[J^i, K^j] &= i\epsilon_{ijk} K^k, \\
[J^i, P^j] &= i\epsilon_{ijk} P^k, \\
[J^i, H] &= 0, \\
[K^i, K^j] &= -i\epsilon_{ijk} J^k, \\
[K^i, P^j] &= -i\delta_{ij} H, \\
[P^i, P^j] &= 0, \\
[P^i, H] &= 0,
\end{align*}
\]

\[
i, j, k = 1, 2, 3.
\]
2.2.2 Light-front Form

The light-front form, or front form in short, is formulated on a plane wave front advancing with the speed of light. This is most conveniently expressed in the light-front coordinates \( x^\mu = (x^+, x^-, x^-) \), where \( x^+ = (x^0 + x^3/c)/\sqrt{2} \), \( x^- = (x^0 - x^3/c)/\sqrt{2} \), \( x^+ = (x^1, x^2) \), where the speed of light \( c \) will be taken to be one unit in this work. The simplest example of a front is given by \( \tau \equiv x^+ = 0 \), with the evolutionary direction given by \( x^- > 0 \).

Writing this as a transformation relation, we have

\[
\begin{pmatrix}
  x^+ \\
  x^1 \\
  x^2 \\
  x^-
\end{pmatrix} = \begin{pmatrix}
  1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  1/\sqrt{2} & 0 & 0 & -1/\sqrt{2}
\end{pmatrix}
\begin{pmatrix}
  x^0 \\
  x^1 \\
  x^2 \\
  x^3
\end{pmatrix}.
\]

(2.34)

Traditionally, \( x^- \) is chosen to be the evolutionary direction, and \( x^+ = 0 \) the quantization surface. The metric tensor in this new basis then becomes

\[
g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix}
  0 & 0 & 0 & 1 \\
  0 & -1 & 0 & 0 \\
  0 & 0 & -1 & 0 \\
  1 & 0 & 0 & 0
\end{pmatrix}.
\]

(2.35)

Thus \( x^+ = x^- \), \( x^- = x^+ \).

The 4-momentum transforms the same way as the coordinates, i.e. \( P_+ = P^- = (P^0 - P^3)/\sqrt{2} \), \( P_- = P^+ = (P^0 + P^3)/\sqrt{2} \), \( P_\perp = (P_1, P_2) \), where \( P_+ = P^- \) is the light-front energy. And the Poincaré matrix in the light-front basis becomes

\[
M_{\mu\nu} = \begin{pmatrix}
  1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  1/\sqrt{2} & 0 & 0 & -1/\sqrt{2}
\end{pmatrix}
\begin{pmatrix}
  0 & -K^1 & -K^2 & -K^3 \\
  K^1 & 0 & J^3 & -J^2 \\
  K^2 & -J^3 & 0 & J^1 \\
  K^3 & J^2 & -J^1 & 0
\end{pmatrix}
\begin{pmatrix}
  1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  1/\sqrt{2} & 0 & 0 & -1/\sqrt{2}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
  0 & -F^1 & -F^2 & K^3 \\
  F^1 & 0 & J^3 & E^1 \\
  F^2 & -J^3 & 0 & E^2 \\
  -K^3 & -E^1 & -E^2 & 0
\end{pmatrix}.
\]

(2.36)
where

\[
\begin{align*}
E^1 &= (K^1 + J^2)/\sqrt{2}, \\
E^2 &= (K^2 - J^1)/\sqrt{2}, \\
F^1 &= (K^1 - J^2)/\sqrt{2}, \\
F^2 &= (K^2 + J^1)/\sqrt{2}.
\end{align*}
\] (2.37)

Table 2.1 Poincare Algebra on the Light Front. The commutator \([\text{element in the first column, element in the first row}] = \text{element at the intersection of the row and column}].

<table>
<thead>
<tr>
<th></th>
<th>(P^+)</th>
<th>(P^1)</th>
<th>(P^2)</th>
<th>(K^3)</th>
<th>(E^1)</th>
<th>(E^2)</th>
<th>(J^3)</th>
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<th>(P^-)</th>
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<tbody>
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</tbody>
</table>

The fundamental quantities \(P_1, P_2, P_-, M_{1-} = E^1, M_{2-} = E^2, M_{12} = J^3, M_{+-} = K^3\) are associated with transformations of coordinates that leave the front \(\tau = 0\) invariant, and will be specially simple. While the remaining ones \(P_+, M_{1+}, M_{2+}\) are complicated and dynamical. Using Eq. (2.35) in Eqs. (2.27) - (2.29), one can obtain the corresponding Poincaré algebra on the light-front, which is listed in Table 2.1.
2.2.3 Other Forms of Dynamics

One can build up a dynamic theory in terms of physical conditions on some other three dimensional surfaces. Such surface must satisfy the condition that the world line of every particle must meet it, and preferably only once for the sake of uniqueness. Dirac [Dir49] proposed to use a branch of hyperboloid

\[ x^\mu x_\mu = k^2, \quad x^0 > 0, \]  

(2.38)

with \( k \) a constant, as another possible choice for quantization surface. This is called the “point form”.

As one can easily see, now the rotation and boost operations leave this surface invariant, while translation and time evolutions goes outside of the surface. Therefore, in the point form, all six \( M_{\mu\nu} \) will be kinematic operators, whereas all four translations will be dynamical. On the one hand, this makes the point form completely Lorentz invariant, but on the other hand, it also means that we will have explicit contribution from the interaction Lagrangian to both energy and momentum. And this can cause real difficulty in constructing a theory of relativistic dynamic system.

2.3 Advantages of the Light-front Formulation

The change from one form of Hamiltonian dynamics to another is not just a simple change of coordinate basis. It involves a whole new way of constructing the dynamics starting from the quantization surface. The instant form dynamics (IFD) is the most familiar form. But the light-front dynamics (LFD) exhibit some very appealing properties that can be taken advantage of.

One of the reasons why the LFD is useful may be attributed to the energy-momentum dispersion relation. For a particle of mass \( m \) that has 4-momentum \( k = (k^0, k^1, k^2, k^3) \), its energy-momentum dispersion relation at equal-\( t \) (instant form) is given by

\[ k^0 = \sqrt{k^2 + m^2}, \]  

(2.39)

where the energy \( k^0 \) is conjugate to \( t \) and the 3-momentum vector \( \mathbf{k} \) is given by \( \mathbf{k} = (k^1, k^2, k^3) \). On the other hand, the corresponding energy-momentum relation at equal-\( r \)
(light-front form) is given by

\[ k^- = \frac{k_1^2 + m^2}{k^+}, \quad (2.40) \]

where the light-front energy \( k^- = (k_0 - k^3)/\sqrt{2} \) is conjugate to \( \tau \), and the light-front momenta \( k^+ = (k_0 + k^3)/\sqrt{2} \) and \( k_\perp = (k^1, k^2) \) are orthogonal to \( k^- \). In contrast to the irrational dispersion relation Eq. (2.39) in the IFD, this rational energy-momentum relation Eq. (2.40) in the LFD not only makes the relation simpler, but also correlates the sign of \( k^- \) and \( k^+ \). When the system is evolving to the future direction (i.e. positive \( \tau \)), in order for \( k^- \) to be positive, \( k^+ \) also has to be positive. This feature prevents certain processes from happening in the LFD, for example, the spontaneous pair production from vacuum is forbidden unless \( k^+ = 0 \) for both particles due to the momentum conservation. Some dynamic processes are therefore eliminated in the LFD and correspondingly the required computation may be simplified, although the zero-mode issue involving the case of \( k^+ = 0 \) should also be taken care of. In the IFD, however, this type of sign correlation does not exist, and the vacuum structure appears much more complicated than in the case of LFD due to the quantum fluctuations. This difference in the energy-momentum dispersion relation makes the LFD quite distinct from other forms of the relativistic Hamiltonian dynamics.

The Poincaré algebra is also drastically changed in the LFD compared to the IFD. The light-front form has the maximum number (7 out of 10) of kinematic generators of the Poincaré group, whereas the instant form has 6 kinematic generators. This may make the problem of solving a dynamic system substantially easier in the LFD. In particular, the longitudinal boost operator joins the stability group of kinematic operators in LFD. This built-in boost invariance together with the simpler vacuum property makes the LFD quite appealing and may save substantial computational efforts to get the QCD solutions that reflect the full Poincaré symmetries.

Furthermore, because \( k^+ \) is non-negative and \( \sum k^+ \) is conserved in LFD, one can define the boost invariant momentum fraction \( x_i = k_i^+ / \sum k_i^+ \) which can be associated with a “parton” inside the hadron. This provides a possible way to link the successful parton model or the constituent quark model picture to the fundamental QCD theory. The discretized light cone quantization (DLCQ) method is a great effort in this direction. But we will not discuss this in detail here, since it’s beyond the scope of the current
dissertation. Readers who are interested in DLCQ are referred to Ref. [Bro98a] and the relevant references therein.

One downside of the LFD is that the transverse rotations become dynamical because the quantization surface \( \tau = 0 \) is not invariant under the transverse rotation and the transverse angular momentum operator involves the interaction that changes the particle number [JS92]. However, compared to the non-compact parameter space for boost \( K^1 \) and \( K^2 \), at least the parameter space for rotation is compact.
3.1 Interpolating Poincaré Algebra

The interpolating space-time coordinates may be defined as a transformation from the ordinary space-time coordinates, $x^\hat{\mu} = \mathcal{R}^\hat{\mu}_{\nu} x^\nu$, i.e.

$$
\begin{pmatrix}
  x^+ \\
  x^1 \\
  x^2 \\
  x^- 
\end{pmatrix} =
\begin{pmatrix}
  \cos \delta & 0 & 0 & \sin \delta \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  \sin \delta & 0 & 0 & -\cos \delta 
\end{pmatrix}
\begin{pmatrix}
  x^0 \\
  x^1 \\
  x^2 \\
  x^3 
\end{pmatrix},
$$

(3.1)

where $0 \leq \delta \leq \pi/4$ is the interpolation angle. Following [JS12], we use the hat notation “^” on the indices to denote the interpolating variables with the parameter $\delta$. In the limits $\delta \to 0$ and $\delta \to \pi/4$, we recover the corresponding variables in the instant form and the front form, respectively. For example, the interpolating coordinates $x^\pm$ in the limit $\delta \to \pi/4$ become the light-front coordinates $x^\pm = (x^0 \pm x^3)/\sqrt{2}$ without “^”.
In this interpolating basis, the metric becomes

\[ g_{\hat{\mu}\hat{\nu}} = g^{\hat{\mu}\hat{\nu}} = \mathcal{R}_\alpha^\hat{\mu} g^{\alpha\beta} \mathcal{R}_\beta^\hat{\nu} = \begin{pmatrix} C & 0 & 0 & S \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ S & 0 & 0 & -C \end{pmatrix}, \]  

(3.2)

where \( S = \sin 2\delta \) and \( C = \cos 2\delta \). The covariant interpolating space-time coordinates are then easily obtained as

\[ x_{\hat{\mu}} = g_{\hat{\mu}\hat{\nu}} x^{\hat{\nu}} = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} \cos \delta & 0 & 0 & -\sin \delta \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}. \]  

(3.3)

The same transformation also applies to the momentum:

\[ P^\hat{\mu} = P^0 \cos \delta + P^3 \sin \delta, \]  

(3.4)

\[ P^\hat{\nu} = P^0 \sin \delta - P^3 \cos \delta, \]  

(3.5)

\[ P_{\hat{\nu}} = P^0 \cos \delta - P^3 \sin \delta, \]  

(3.6)

\[ P_{\hat{\mu}} = P^0 \sin \delta + P^3 \cos \delta. \]  

(3.7)

Since the perpendicular components remain the same \((a^\hat{j} = a^j, a_{\hat{j}} = a_j, j = 1, 2)\), we will omit the “\(^\hat{}\)” notation unless necessary from now on for the perpendicular indices \(j = 1, 2\) in a 4-vector.

Using \( g^{\hat{\mu}\hat{\nu}} \) and \( g_{\hat{\mu}\hat{\nu}} \), we see that the covariant and contravariant components are related by

\[ a_{\hat{\mu}} = Ca^\hat{\mu} + Sa^\hat{\mu}; \quad a^\hat{\mu} = Ca_{\hat{\mu}} + Sa_{\hat{\mu}} \]

\[ a_{\hat{\nu}} = Sa^\hat{\nu} - Ca^\hat{\nu}; \quad a^\hat{\nu} = Sa_{\hat{\nu}} - Ca_{\hat{\nu}} \]

\[ a_j = a_{\hat{j}} = -a^j = -a_{\hat{j}}, \quad (j = 1, 2). \]  

(3.8)

The inner product of two 4-vectors must be interpolation angle independent as one
can verify
\[ a_\hat{\mu} b_\hat{\nu} = (a_\hat{\tau} b_\hat{\tau} - a_\hat{\tau} b_\hat{\tau}) C + (a_\hat{\tau} b_\hat{\tau} + a_\hat{\tau} b_\hat{\tau}) S - a_1 b_1 - a_2 b_2 = a_\mu b_\mu. \] (3.9)

In particular, we have the energy-momentum dispersion relation given by
\[ P_\hat{\mu} P_\hat{\nu} = P_\hat{\tau} C - P_\tau C + 2P_\tau S - P_\perp. \] (3.10)

Another useful relation
\[ P_\hat{\mu} P_\hat{\nu} C = P_\hat{\tau}^2 - P_\tau^2 - P_\perp C \] (3.11)
can also be verified, the derivation of which is very similar to the one we will give in Eq. (4.11). For particles of mass \( M \), \( P_\mu P_\mu \) on the mass shell equals \( M^2 \) of course.

Accordingly, the Poincaré matrix
\[
M^{\mu\nu} = \begin{pmatrix}
0 & K_1 & K_2 & K_3 \\
-K_1 & 0 & J_3 & -J_2 \\
-K_2 & -J_3 & 0 & J_1 \\
-K_3 & J_2 & -J_1 & 0
\end{pmatrix}
\] (3.12)
transforms as well, so that
\[
M^{\hat{\mu}\hat{\nu}} = R_\alpha^{\hat{\mu}} M^{\alpha\beta} R_\beta^{\hat{\nu}} = \begin{pmatrix}
0 & E_\hat{1} & E_\hat{2} & -K_3 \\
-E_\hat{1} & 0 & J_3 & -F_\hat{1} \\
-E_\hat{2} & -J_3 & 0 & -F_\hat{2} \\
K_3 & F_\hat{1} & F_\hat{2} & 0
\end{pmatrix}
\] (3.13)
and
\[
M_{\hat{\mu}\hat{\nu}} = g_{\hat{\mu}\hat{\alpha}} M^{\hat{\alpha}\hat{\beta}} g_{\hat{\beta}\hat{\nu}} = \begin{pmatrix}
0 & D_\hat{1} & D_\hat{2} & K_3 \\
-D_\hat{1} & 0 & J_3 & -K_\hat{1} \\
-D_\hat{2} & -J_3 & 0 & -K_\hat{2} \\
-K_3 & K_\hat{1} & K_\hat{2} & 0
\end{pmatrix},
\] (3.14)
where
\[
E_\hat{1} = J^2 \sin \delta + K_1 \cos \delta, \quad E_\hat{2} = K^2 \cos \delta - J^1 \sin \delta,
\]
\[
K_\hat{1} = -K_1 \sin \delta - J^2 \cos \delta, \quad K_\hat{2} = J^1 \cos \delta - K^2 \sin \delta.
\]
\[ F^\hat{1} = K^1 \sin \delta - J^2 \cos \delta, \quad D^\hat{1} = -K^1 \cos \delta + J^2 \sin \delta, \]
\[ F^\hat{2} = K^2 \sin \delta + J^1 \cos \delta, \quad D^\hat{2} = -J^1 \sin \delta - K^2 \cos \delta. \] (3.15)

The interpolating \( E^\hat{1} \) and \( F^\hat{2} \) will coincide with the usual \( E^\hat{1} \) and \( F^\hat{2} \) of LFD in the limit \( \delta = \pi/4 \). Note here that the “\( \wedge \)” notation is reinstated for 1, 2 to emphasize the angle \( \delta \) dependence and that the position of the indices on \( K, J, E, F, D, K \) won’t matter as they are not the 4-vectors: i.e. \( E_1 = E^\hat{1} \), \( F_1 = F^\hat{1} \) and etc. Of course, \( M^{\mu \nu} \) and \( M_{\mu \nu} \) should be distinguished in any case.

**Table 3.1** Poincaré Algebra for any interpolation angle. The commutation relation reads [element in the first column, element in the first row] = element at the intersection of the corresponding row and column.

<table>
<thead>
<tr>
<th>( P_\perp )</th>
<th>( D^1 )</th>
<th>( D^2 )</th>
<th>( K^3 )</th>
<th>( K^1 )</th>
<th>( K^2 )</th>
<th>( J^3 )</th>
<th>( P_1 )</th>
<th>( P_2 )</th>
<th>( P_\perp )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_\perp )</td>
<td>0</td>
<td>( iCP_1 )</td>
<td>( iCP_2 )</td>
<td>( -iP^\perp )</td>
<td>( iSP_1 )</td>
<td>( iSP_2 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( D^1 )</td>
<td>( -iCP_1 )</td>
<td>0</td>
<td>( -iJ^3 C )</td>
<td>( iF^1 )</td>
<td>( iK^3 )</td>
<td>( -iJ^3 S )</td>
<td>( -iD^2 )</td>
<td>( -iP_\perp )</td>
<td>0</td>
</tr>
<tr>
<td>( D^2 )</td>
<td>( -iCP_2 )</td>
<td>( iJ^3 C )</td>
<td>0</td>
<td>( iF^2 )</td>
<td>( iJ^3 S )</td>
<td>( iK^3 )</td>
<td>( iD^1 )</td>
<td>0</td>
<td>( -iP_\perp )</td>
</tr>
<tr>
<td>( K^3 )</td>
<td>( iP^\perp )</td>
<td>( -iF^1 )</td>
<td>( -iF^2 )</td>
<td>0</td>
<td>( iE^1 )</td>
<td>( iE^2 )</td>
<td>0</td>
<td>0</td>
<td>( -iP_\perp )</td>
</tr>
<tr>
<td>( K^1 )</td>
<td>( -iSP_1 )</td>
<td>( -iK^3 )</td>
<td>( -iJ^3 S )</td>
<td>( -iE^1 )</td>
<td>0</td>
<td>( iJ^3 C )</td>
<td>( -iK^2 )</td>
<td>( -iP_\perp )</td>
<td>0</td>
</tr>
<tr>
<td>( K^2 )</td>
<td>( -iSP_2 )</td>
<td>( iJ^3 S )</td>
<td>( -iK^3 )</td>
<td>( -iE^2 )</td>
<td>( -iJ^3 C )</td>
<td>0</td>
<td>( iK^1 )</td>
<td>0</td>
<td>( -iP_\perp )</td>
</tr>
<tr>
<td>( J^3 )</td>
<td>0</td>
<td>( iD^2 )</td>
<td>( -iD^1 )</td>
<td>0</td>
<td>( iK^2 )</td>
<td>( -iK^1 )</td>
<td>0</td>
<td>( iP_2 )</td>
<td>( -iP_1 )</td>
</tr>
<tr>
<td>( P_1 )</td>
<td>0</td>
<td>( iP^\perp )</td>
<td>0</td>
<td>0</td>
<td>( iP_\perp )</td>
<td>0</td>
<td>( -iP_2 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( P_2 )</td>
<td>0</td>
<td>0</td>
<td>( iP^\perp )</td>
<td>0</td>
<td>0</td>
<td>( iP_\perp )</td>
<td>( iP_1 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( P_\perp )</td>
<td>0</td>
<td>( iSP_1 )</td>
<td>( iSP_2 )</td>
<td>( iP^\perp )</td>
<td>( -iCP_1 )</td>
<td>( -iCP_2 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The commutation relations between these operators are again given by Eqs. (2.27) - (2.29) with interpolating \( g_{\mu \nu} \), \( P_\mu \) and \( M_{\mu \nu} \). They constitute the generalized Poincaré algebra, which is listed in Table 3.1. Among the ten Poincaré generators, the six generators \((K^1, K^2, J^3, P_1, P_2, P_\perp)\) are always kinematic in the sense that the \( x^\perp = 0 \) plane is intact under the transformations generated by them. As discussed in Refs. [JM01; JS12], the operator \( K^3 = M_{\perp \perp} \) is dynamical in the region where \( 0 \leq \delta < \pi/4 \) but becomes kinematic in the light-front limit \((\delta = \pi/4)\). The set of kinematic and dynamic generators depending on the interpolation angle are summarized in Table. 3.2. Since the kinematic
transformations don’t alter $x^+$, the individual time-ordered amplitude must be invariant 
under the kinematic transformations. This can be seen explicitly in the example of 
scattering process discussed in Sec. 4.3.

Table 3.2 Kinematic and dynamic generators for different interpolation angles.

<table>
<thead>
<tr>
<th></th>
<th>Kinematic</th>
<th>Dynamic</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta = 0$</td>
<td>$K^1 = -J^2, K^2 = J^1, J^3, P^1, P^2, P^3$</td>
<td>$D^1 = -K^1, D^2 = -K^2, K^3, P^0$</td>
</tr>
<tr>
<td>$0 \leq \delta &lt; \pi/4$</td>
<td>$K^1, K^2, J^3, P^1, P^2, P_\perp$</td>
<td>$D^1, D^2, K^3, P_\perp$</td>
</tr>
<tr>
<td>$\delta = \pi/4$</td>
<td>$K^1 = -E^1, K^2 = -E^2, J^3, K^3, P^1, P^2, P^+$</td>
<td>$D^1 = -F^1, D^2 = -F^2, P^-$</td>
</tr>
</tbody>
</table>

3.2 Interpolating Between the Helicity Boost and the Light-front Boost

3.2.1 Three Types of Relativistic Boosts

In the instant form, two different types of boosts are commonly used. The first one, defined
on the Hilbert space as

$$T_c = e^{-i\beta \cdot K},$$

is the rotationless boost. This is the most familiar one, composed of a pure boost in the
$\beta$ direction. For this reason, it’s also referred to as the canonical boost [KP91], hence the
subscript “c”. The second type is the helicity boost, defined by Jacob and Wick [JW59] as

$$T_h = e^{-i\theta \cdot J^1} e^{-i\beta^3 K_3}.$$

It’s a canonical boost in the $z$ direction to a momentum of the desired magnitude, followed
by a rotation from the $z$ axis to the axis that defines the direction of momentum. This
transformation keeps the spin aligned (anti-aligned) with the momentum direction when
applied to a rest frame state whose spin is pointing in the $+z$ ($-z$) direction.
In the light-front formulation, people often use the *front-form* boost [Sop71]

\[ T_f = e^{-i(\beta_1 E_1 + \beta_2 E_2)} e^{-i\beta_3 K^3}. \]  

(3.18)

Using the light-front Poincaré algebra listed in Table 2.1 and the Baker-Hausdorff theorem, one obtains the following momentum transformation relations

\[ T_f^{-1} P^- T_f = e^{-\beta_3}(P^- + 2e^{\beta_3} \mathbf{P}_\perp \cdot \mathbf{P}_\perp + \beta_\perp^2 e^{2\beta_3} P^+), \]  

(3.19)

\[ T_f^{-1} P^+ T_f = e^{\beta_3} P^+, \]  

(3.20)

\[ T_f^{-1} \mathbf{P}_\perp T_f = \mathbf{P}_\perp + \beta_\perp e^{\beta_3} P^+. \]  

(3.21)

Particularly, when it’s boosted from the rest frame,

\[ P^- = e^{-\beta_3} m + m e^{\beta_3} \beta_\perp^2, \]  

(3.22)

\[ P^+ = m e^{\beta_3}, \]  

(3.23)

\[ \mathbf{P}_\perp = m e^{\beta_3} \beta_\perp. \]  

(3.24)

If we define \( Q^\mu \equiv P^\mu/m \), then

\[ Q^- = e^{-\eta_3} + \eta_\perp^2, \]  

(3.25)

\[ Q^+ = e^{\eta_3}, \]  

(3.26)

\[ Q_\perp = e^{\eta_3} \eta_\perp. \]  

(3.27)

Using \( Q^\mu \), we can rewrite Eqs. (3.19) - (3.21) as

\[ T_f^{-1} P^- T_f = \frac{1}{Q^-}(P^- + 2Q_\perp \cdot \mathbf{P}_\perp + Q_\perp^2 P^+) \]  

(3.28)

\[ T_f^{-1} P^+ T_f = Q^+ P^+ \]  

(3.29)

\[ T_f^{-1} \mathbf{P}_\perp T_f = \mathbf{P}_\perp + Q_\perp P^+. \]  

(3.30)

The front-form boost is special in that \( E_1, E_2 \) and \( K^3 \) are all kinematic operators, so the boost leaves the quantization surface \( x^+ = 0 \) invariant. Furthermore, because the operators \( E_1, E_2 \) and \( K^3 \) form a subalgebra, as can be seen from the light-front Poincaré algebra in Table 2.1, the front-form boosts form a subgroup. In other words, the
composition of two front-form boosts yields a third front-form boost:

\[ T_f(Q^+, Q_\perp)T_f(q^+, q_\perp) = T_f(Q^+ q^+, q_\perp + Q_\perp q^+), \tag{3.31} \]

where I use \( T_f(Q^+, Q_\perp) \) to denote a front-form boost that transforms a rest 4-velocity \((1, 0, 0, 0)\) to the 4-velocity \(Q\). This is in stark contrast to the canonical boost, which induces a \( \gamma \) angle Wigner rotation \( R(\gamma) \):

\[ T_c(\theta_2, \beta)T_c(\theta_1, \eta) = T_c(\theta_3, \xi)R(\gamma), \tag{3.32} \]

where I use the notation \( T_c(\theta, \beta) \) to denote a canonical boost that makes an angle \( \theta \) with the +z axis and has rapidity \( \beta \). As derived and written in Eq. (A10) in Osborn’s paper [Osb68], the Wigner rotation angle can be put in the following form

\[ \frac{\gamma}{2} = \arctan \left( \frac{e^\beta (E + P \cdot \hat{v}) + M}{|P \times \hat{v}|} \right) - \arctan \left( \frac{(E + P \cdot \hat{v}) + M}{|P \times \hat{v}|} \right), \tag{3.33} \]

where \( M \) is the mass of the particle, \((E, P)\) is the 4-momentum after the first boost from the rest frame, and \( \beta, \hat{v} \) indicate the rapidity and direction of the second boost. The rotation is around the axis \( \hat{n} = P \times \hat{v}/|P \times \hat{v}| \).

### 3.2.2 Interpolating Helicity Boost

Notice that the front-form boost given by Eq. (3.18) and the helicity boost given by Eq. (3.17) are very similar in form. So in order to get to the front-form boost in the light-front limit, we want to interpolate between these two types of boosts. Following the procedure presented by Jacob and Wick [JW59] to define the helicity in the IFD, we can extend the definition of helicity to any arbitrary interpolation angle \( \delta \). For this purpose, we introduce the transformation \( T \) given by

\[ T = T_{12}T_3 = e^{i\beta_1 K^1 + i\beta_2 K^2} e^{-i\beta_3 K^3}, \tag{3.34} \]

where the sign convention of the exponents \( i\beta_1 K^1 \) and \( i\beta_2 K^2 \) of \( T_{12} \) is chosen to make the resulted momentum positive for infinitesimal positive values of \( \beta_1 \) and \( \beta_2 \). Our sign convention turns out to be consistent with Soper’s notation [Sop71] in the LFD.

Using the transformation given by Eq. (3.34), we obtain the following 4-momentum
components \((P_{\hat{\gamma}}, P'^1, P'^2\) and \(P'_{\perp})\) from the initial 4-momentum components \((P_{\hat{\gamma}}, P^1, P^2\) and \(P_{\perp})\) [JM01; JS12]:

\[
P'_{\hat{\gamma}} = P_{\hat{\gamma}} \cosh \beta_3 + P_{\perp} \sinh \beta_3, \tag{3.35}
\]
\[
P'^1 = P^1 + \beta_1 \frac{\sin \alpha}{\alpha} \left( P_{\perp} \cosh \beta_3 + P_{\hat{\gamma}} \sinh \beta_3 \right) + \frac{\cos \alpha - 1}{\alpha^2} \mathcal{C} \beta_1 \left( \beta_1 P^1 + \beta_2 P^2 \right), \tag{3.36}
\]
\[
P'^2 = P^2 + \beta_2 \frac{\sin \alpha}{\alpha} \left( P_{\perp} \cosh \beta_3 + P_{\hat{\gamma}} \sinh \beta_3 \right) + \frac{\cos \alpha - 1}{\alpha^2} \mathcal{C} \beta_2 \left( \beta_1 P^1 + \beta_2 P^2 \right), \tag{3.37}
\]
\[
P'_{\perp} = \left( P_{\perp} \cosh \beta_3 + P_{\hat{\gamma}} \sinh \beta_3 \right) \cos \alpha + \frac{\sin \alpha}{\alpha} \mathcal{C} \beta_1 \left( \beta_1 P^1 + \beta_2 P^2 \right), \tag{3.38}
\]

where \(\alpha = \sqrt{\mathcal{C} (\beta_1^2 + \beta_2^2)}\). These four momentum components \((P'_{\hat{\gamma}}, P'^1, P'^2\) and \(P'_{\perp})\) given by Eqs. (3.35) - (3.38) turn out to be most convenient in carrying out our calculations, but other choice of components such as \(P'_{\hat{\gamma}}\) and \(P_{\perp}'\) can be easily obtained using Eq. (3.8).

As an example of using Eqs. (3.35) - (3.38), we may find that the particle of mass \(M\) at rest gains the following momentum components under the transformation \(T\):

\[
P_{\hat{\gamma}} = (\cos \delta \cosh \beta_3 + \sin \delta \sinh \beta_3) M, \tag{3.39}
\]
\[
P^1 = \beta_1 \frac{\sin \alpha}{\alpha} (\sin \delta \cosh \beta_3 + \cos \delta \sinh \beta_3) M, \tag{3.40}
\]
\[
P^2 = \beta_2 \frac{\sin \alpha}{\alpha} (\sin \delta \cosh \beta_3 + \cos \delta \sinh \beta_3) M, \tag{3.41}
\]
\[
P_{\perp} = \cos \alpha (\sin \delta \cosh \beta_3 + \cos \delta \sinh \beta_3) M, \tag{3.42}
\]

where the factor \((\sin \delta \cosh \beta_3 + \cos \delta \sinh \beta_3)\) in the 3-momentum \((P^1, P^2, P_{\perp})\) is due to the first boost \(T_3 = e^{-i\beta_3 K^3}\). Our convention taking this factor to be positive, i.e. \((\sin \delta \cosh \beta_3 + \cos \delta \sinh \beta_3) > 0\), is consistent with the convention taken by Jacob and Wick [JW59] in their procedure to define the helicity in the IFD. We also note that \(P_{\perp} = M \sin \delta, P_{\hat{\gamma}} = M \cos \delta,\) and \(P^1 = P^2 = 0\) in the particle rest frame.

From Eq. (3.11), a useful relation for \(P_{\hat{\gamma}}, P_{\perp}, P_{\perp}'\) can be found:

\[
(P_{\hat{\gamma}})^2 - M^2 \mathcal{C} = P_{\perp}^2 + P_{\perp}'^2 \mathcal{C}. \tag{3.43}
\]

Since this quantity in Eq. (3.43) appears so often in our calculations, we now give it a
special symbol to simplify our notation

\[ P \equiv \sqrt{(P^\perp)^2 - M^2 C} = \sqrt{P_\perp^2 + P_\perp^2 C}, \]  

(3.44)

which corresponds to the magnitude of the particle’s 3-momentum \(|P|\) in the IFD when \(\delta \to 0\) (or \(C \to 1\)), and becomes identical to \(P^+\) in the LFD when \(\delta \to \pi/4\) (or \(C \to 0\)).

Solving Eqs. (3.39) - (3.42), one gets the following useful relations between parameters \(\beta_1, \beta_2, \beta_3, \alpha\) and the momentum components:

\[
\sin \delta \cosh \beta_3 + \cos \delta \sinh \beta_3 = \frac{P}{M}, \]  

(3.45)

\[
\cos \delta \cosh \beta_3 + \sin \delta \sinh \beta_3 = \frac{P^\perp}{M}, \]  

(3.46)

and

\[
\cos \alpha = \frac{P_\perp}{P}, \]  

(3.47)

\[
\sin \alpha = \frac{\sqrt{P^2 + P_\perp^2 C}}{P}, \]  

(3.48)

\[
e^{\beta_3} = \frac{P^\perp + P}{M(\sin \delta + \cos \delta)}, \]  

(3.49)

\[
e^{-\beta_3} = \frac{P^\perp - P}{M(\cos \delta - \sin \delta)}, \]  

(3.50)

\[
\frac{\beta_j}{\alpha} = \frac{P_j}{\sqrt{P^2 + P_\perp^2 C}}, \quad (j = 1, 2). \]  

(3.51)
As the first step to interpolate the quantum electrodynamics (QED), we present in this chapter the electromagnetic gauge field interpolation between the instant form and the front form of the relativistic Hamiltonian dynamics. In order to simplify our discussion, and concentrate on how the interpolated gauge propagator behaves, we consider the case of the electromagnetic gauge field theory with the scalar fermion fields known as the scalar quantum electrodynamics (sQED) theory. We will leave the discussion of interpolating fermion spinors to the next chapter.

In Sec. 4.1, we derive the photon polarization vector for any interpolation angle. Using this derivation, we present the general gauge that links the light-front gauge to the Coulomb gauge, and construct the corresponding photon propagator for an arbitrary interpolation angle. In Sec. 4.2, we decompose this interpolating gauge propagator according to the time ordering and apply it to the lowest scattering process such as an analogue of the well-known QED process $e\mu \rightarrow e\mu$ in sQED without involving the fermion spins. We also take a close look at the limiting cases of $\mathcal{C} \rightarrow 0$ and compare the results with the exact $\mathcal{C} = 0$ (LFD) results in this section. In Sec. 4.3, we plot the time-ordered amplitudes in terms of the total momentum of the system and the interpolation angle to reveal both the frame dependence and interpolation angle dependence of these amplitudes. We then give a detailed discussion of the universal J-shaped correlation curve that emerges from these time-ordered diagrams and how it gives rise to the zero-mode contributions at $P^z = -\infty$. A summary follows in Sec. 4.4.
In Appendix A.1, we list the explicit matrix representation of the boost $K$ and rotation $J$ generators and provide the explicit description of the steps involved in deriving the photon polarization vectors for an arbitrary interpolation angle. In Appendix A.2, we derive the numerator of the photon propagator from the photon polarization vectors for an arbitrary interpolation angle. In Appendix A.3, we decompose the photon propagator on the light-front in terms of the transverse and longitudinal components. For the completeness and the comparison with our analysis of sQED scattering process (analogous to $e\mu \rightarrow e\mu$ in QED) presented in Secs. 4.2 and 4.3, we present in Appendices A.4 and A.5 our calculations for the same process $e\mu \rightarrow e\mu$ in the scalar field theory and the process related by the crossing symmetry $e^+e^- \rightarrow \mu^+\mu^-$ in sQED. Together with the scalar field theory discussed in Ref. [JS12], this completes our study of the lowest order scattering processes related by the crossing symmetry in the scalar field theory as well as in sQED.

4.1 The Link between the Coulomb Gauge and the Light-Front Gauge

To discuss the gauge field in an arbitrary interpolation angle, rather than fixing the gauge first, we start from the explicit physical polarization 4-vectors of the spin-1 particle and then identify the corresponding gauge that these explicit representations of the gauge field polarization satisfy. This procedure is possible because we can follow the Jacob-Wick procedure discussed in Chapter 3 and apply the corresponding $T$ transformation [Eq. (3.34)] to the rest frame spin-1 particle polarization vectors which may be naturally given by the spherical harmonics. Although this procedure applies to the physical spin-1 particle with a non-zero mass $M$ such as the $\rho$-meson, we may take advantage of the Lorentz invariance of the 4-momentum squared $P^\mu P_\mu = M^2$ and replace $M^2$ by $P^\mu P_\mu$ to extend the obtained polarization 4-vectors to the virtual gauge particle. For the real photon, of course $M = 0$ and the longitudinal polarization vector should be discarded. From this procedure, we find that the identified gauge interpolates between the Coulomb gauge in the instant form and the light-front gauge in the front form.
4.1.1 Spin-1 Polarization Vector for Any Interpolation Angle

We use the 4-vector representation of Lorentz group for the spin-1 particle. The polarization vector in a specific frame is obtained by boosting the 4-vectors to that frame. In the rest frame, where the 4-momentum is \((M, 0, 0, 0)\), the polarization vectors are taken to be

\[
\epsilon(\pm) = \mp \frac{1}{\sqrt{2}} (0, 1, \pm i, 0), \quad \epsilon(0) = (0, 0, 0, 1),
\]

since the spherical harmonics \(Y_{1}^{\pm 1}\) and \(Y_{1}^{0}\) naturally correspond to the transverse and longitudinal polarization vectors, respectively.

To find the polarization vectors of the spin-1 particle with an arbitrary momentum \(P^\mu\), we apply the \(T\) transformation given by Eq. (3.34) with the explicit 4-vector representation of the operators \(K\) and \(J\). The details of the calculation are summarized in Appendix A.1. The result of the polarization vectors written in the form of \(\epsilon_\hat{\mu}\) is given by

\[
\epsilon_\hat{\mu}(P, +) = -\frac{1}{\sqrt{2}|P|} \left( S|P_\perp|, \frac{P_1 P_\perp - iP_2 P^\perp}{|P_\perp|}, \frac{P_2 P_\perp + iP_1 P^\perp}{|P_\perp|}, -C|P_\perp| \right),
\]

\[
\epsilon_\hat{\mu}(P, -) = \frac{1}{\sqrt{2}|P|} \left( S|P_\perp|, \frac{P_1 P_\perp + iP_2 P^\perp}{|P_\perp|}, \frac{P_2 P_\perp - iP_1 P^\perp}{|P_\perp|}, -C|P_\perp| \right),
\]

\[
\epsilon_\hat{\mu}(P, 0) = \frac{P^\perp}{M|P|} \left( P_\perp - \frac{M^2}{P^\perp}, P_1, P_2, P_\perp \right).
\]

They satisfy the transversality and orthogonality constraints

\[
\epsilon_\hat{\mu}(P, \lambda) P^\mu = 0, \quad \epsilon^*(P, \lambda) \cdot \epsilon(P, \lambda') = -\delta_{\lambda\lambda'}.
\]

It is also obvious that \(\epsilon(P, 0)\) is “parallel” to the three momentum \(P\), since \((\epsilon_1, \epsilon_2, \epsilon_\perp) \sim (P_1, P_2, P_\perp)\). Noticing that \(S \to 0\), \(C \to 1\), \(P \to |P|\) when \(\delta \to 0\), and \(S \to 1\), \(C \to 0\), \(P \to P^+\) when \(\delta \to \pi/4\), one can easily check that these polarization vectors have correct limits of the instant form and the light-front form at \(\delta = 0\) and \(\delta = \pi/4\), respectively.

4.1.2 Interpolating Transverse Gauge

Having obtained the explicit polarization 4-vectors of the spin-1 particle with mass \(M\), we notice that the transverse polarizations given by Eqs. (4.2) and (4.3) are independent of the particle mass \(M\). Thus, they can be used also as the transverse polarization 4-vectors.
of the gauge field such as the photon.

We observe that the transverse polarization vectors ($\lambda = \pm$) in Eqs. (4.2) and (4.3) satisfy the following conditions:

$$
\epsilon^\dagger(\lambda) = C\epsilon^\dagger_\lambda + S\epsilon_\lambda = 0,
$$

$$
\epsilon_\lambda P_\lambda + \epsilon_\perp(\lambda)P_\perp C = 0,
$$

where we used $\epsilon_\lambda(\lambda) = \epsilon_\lambda(P, \lambda)$ for convenience. So, we can write the gauge condition for transverse photons as

$$
A^\dagger = 0,
$$

$$
A_\perp P_\perp + A_\perp P_\perp C = 0,
$$

where the second condition can also be written as

$$
\partial_\perp A_\perp + \partial_\perp A_\perp C = 0.
$$

We now demonstrate that these two conditions are closely related to each other. First of all, the Lorentz condition $\partial_\dagger A^\dagger = 0$ is always satisfied, as already verified in Eq. (4.5). But the Lorentz condition can be rewritten in the following way

$$
0 = \partial_\dagger A^\dagger
$$

$$
= \partial_\dagger A^\dagger C - \partial_\perp A_\perp C + \partial_\parallel A_\parallel S + \partial_\perp A_\perp S - \partial_\perp A_\perp
$$

$$
= \partial_\dagger A^\dagger - \partial_\perp A_\perp C + \partial_\parallel A_\parallel S - \partial_\perp A_\perp
$$

$$
= \partial_\dagger A^\dagger - \left(1 - \frac{S^2}{C}\right)\partial_\perp A_\perp + \partial_\parallel A_\parallel S - \partial_\perp A_\perp
$$

$$
= \partial_\dagger A^\dagger + \frac{S}{C} (S\partial_\perp A_\perp + C\partial_\parallel A_\parallel) - \left(\frac{\partial_\perp A_\perp}{C} + \partial_\perp A_\perp\right)
$$

$$
= \frac{C}{\partial_\dagger} A^\dagger + \frac{S}{C} \left(\partial_\perp A_\perp\right) - \left(\frac{\partial_\perp A_\perp}{C} + \partial_\perp A_\perp\right)
$$

$$
= \frac{1}{\partial_\dagger} \partial_\perp A^\dagger - \left(\frac{\partial_\perp A_\perp}{C} + \partial_\perp A_\perp\right)
$$

$$
= \frac{1}{\partial_\dagger} \left[\partial_\perp A^\dagger - \left(\partial_\perp A_\perp + \partial_\perp A_\perp C\right)\right],
$$

(4.11)
where Eq. (3.9) is used in the second line and the relations in Eq. (3.8) are used to go between the superscript components and the subscript components. Eq. (4.11) indicates that if Eq. (4.8) holds, then we have Eq. (4.10). On the other hand, if Eq. (4.10) holds, then we have

\[ \partial^\perp A^\perp = 0. \]  

(4.12)

However, this is a differential equation, and we have the freedom to specify the boundary condition. We can choose our boundary condition to make \( A^\perp = 0 \). This same trick was used in the instant form [Ryd96] where the Coulomb gauge \( \nabla \cdot A = 0 \) gives \( \partial_0 A^0 = 0 \), but we can choose our initial condition, or gauge, so that \( A^0 = 0 \). Therefore, for a free gauge field, these two gauge conditions are effectively equivalent.

However, in order to be consistent with conventions used in the instant form, where the radiation gauge condition is specified as “\( \nabla \cdot A = 0 \)”, here we say that the radiation gauge condition for any interpolating angle is Eq. (4.10). In the instant form limit (\( \delta = 0 \)), \( A^- \to A^3, \mathbb{C} \to 1 \), and Eq. (4.10) becomes \( \nabla \cdot A = 0 \), which is the familiar Coulomb gauge. In the light-front limit (\( \delta = \pi/4 \)), \( A^- \to A^+, \mathbb{C} \to 0 \), and the gauge condition reduces to “\( \partial^+ A^+ = 0 \)”, which we will show in Sec. 6.2.1 gives \( A^+ = 0 \) under our choice of boundary condition.

### 4.1.3 Propagator for Transverse Photons

Choosing the transverse gauge fields as the dynamical degrees of freedom, we get the photon propagator in the interpolating transverse gauge given by

\[
\langle 0 | T(A^\mu_\perp(y)A^\nu_\perp(x)) | 0 \rangle = i \int \frac{d^4q}{(2\pi)^4} \frac{e^{-iq(y-x)}}{q^2 + i\epsilon} \mathcal{T}_{\mu\nu},
\]

(4.13)

where \( \mathcal{T}_{\mu\nu} \equiv \sum_{\lambda = \pm} \epsilon^\dagger_\mu(\lambda)\epsilon^\nu(\lambda) \) and \( \epsilon_\mu(\lambda) = \epsilon_\mu(q, \lambda) \) taking the photon momentum \( P = q \).

Here, we use the obvious familiar notation \( q^2 = q\bar{q} \). Although we can compute \( \mathcal{T}_{\mu\nu} \) directly using Eqs. (4.2) and (4.3) as shown in Appendix A.2, we demonstrate here the method of vierbein as a cross-check to our result.

To construct a vierbein, we just need a temporal basis 4-vector and a longitudinal basis 4-vector which we denote as \( \hat{n}_\mu \) and \( \hat{q}_\mu \), respectively, since the transverse basis 4-vectors are already given by \( \epsilon_\mu(\pm) \). The temporal basis 4-vector can be taken as a unit timelike 4-vector given by \( \hat{n}_\mu = \frac{1}{\sqrt{\mathbb{C}}} n_\mu = \frac{1}{\sqrt{\mathbb{C}}}(1, 0, 0, 0) \) whose dual vector is \( \hat{n}^\dagger = (\sqrt{\mathbb{C}}, 0, 0, S/\sqrt{\mathbb{C}}) \).
The $\frac{1}{\sqrt{C}}$ in $n_{\hat{\mu}}$ is a normalization factor to get $\hat{n}_{\hat{\mu}} \hat{n}_{\hat{\nu}} = 1$. The longitudinal basis 4-vector $\hat{q}_{\hat{\mu}}$ can also be rather easily found from the gauge condition given by Eq. (4.7) which can be written as $\epsilon_{\hat{\mu}}(\lambda) \hat{q}^\mu = 0$ with $\hat{q}^\mu = N(0, -q_1 C, -q_2 C, -q_\perp)$. The normalization factor $N$ is determined by $\hat{q}_{\hat{\mu}} \hat{q}^\mu = \hat{q}_{\hat{\mu}} \hat{q}_{\hat{\nu}} \hat{q}^\nu = -1$ with $g_{\hat{\mu}\hat{\nu}}$ given by Eq. (3.2), so that we have

$$\hat{q}^\mu = \frac{1}{\sqrt{C(q_1^2 C + q_\perp^2)}} (0, -q_1 C, -q_2 C, -q_\perp)$$

(4.14)

and

$$\hat{q}_{\hat{\mu}} = \frac{1}{\sqrt{C(q_1^2 C + q_\perp^2)}} (-q_\perp C, q_1 C, q_2 C, q_\perp C).$$

(4.15)

These four basis 4-vectors are of course mutually orthogonal: $\hat{q}_{\hat{\mu}} \hat{n}_{\hat{\nu}} = 0$, $\epsilon_{\hat{\mu}}(\lambda) \hat{n}_{\hat{\nu}} = 0$ and $\epsilon_{\hat{\nu}}(\lambda) \hat{q}_{\hat{\mu}} = 0$, where $\lambda = \pm$.

Since $\epsilon_{\hat{\mu}}(\pm), \hat{n}_{\hat{\mu}}, \hat{q}_{\hat{\mu}}$ form a vierbein, we may start from

$$g_{\hat{\mu}\hat{\nu}} = \hat{n}_{\hat{\mu}} \hat{n}_{\hat{\nu}} - \sum_{\lambda=\pm} \epsilon^*_\lambda(\lambda) \epsilon_\lambda(\lambda) - \hat{q}_{\hat{\mu}} \hat{q}_{\hat{\nu}},$$

(4.16)

and obtain

$$\mathcal{T}_{\hat{\mu}\hat{\nu}} \equiv \sum_{\lambda=\pm} \epsilon^*_\lambda(\lambda) \epsilon_\lambda(\lambda)$$

$$= -g_{\hat{\mu}\hat{\nu}} + \hat{n}_{\hat{\mu}} \hat{n}_{\hat{\nu}} - \hat{q}_{\hat{\mu}} \hat{q}_{\hat{\nu}}$$

$$= -g_{\hat{\mu}\hat{\nu}} + \frac{(q \cdot n)(q_{\hat{\mu}} n_{\hat{\nu}} + q_{\hat{\nu}} n_{\hat{\mu}})}{q_1^2 C + q_\perp^2} - \frac{Cq_{\hat{\mu}}q_{\hat{\nu}}}{q_1^2 C + q_\perp^2} - \frac{q^2 n_{\hat{\mu}} n_{\hat{\nu}}}{q_1^2 C + q_\perp^2},$$

(4.17)

where we used $\hat{n}_{\hat{\mu}} = \frac{1}{\sqrt{C}} n_{\hat{\mu}}$ and rewrote $\hat{q}_{\hat{\mu}}$ in terms of $q_{\hat{\mu}}$ and $n_{\hat{\mu}}$ as

$$\hat{q}_{\hat{\mu}} = \frac{C q_{\hat{\mu}} - (q \cdot n) n_{\hat{\mu}}}{\sqrt{C(q_1^2 C + q_\perp^2)}}$$

(4.18)

in the last step to remove any artifact of divergence in $\hat{q}_{\hat{\mu}}$ and $\hat{n}_{\hat{\mu}}$ in the light-front limit. The photon propagator constructed out of $n_{\hat{\mu}}$ and $q_{\hat{\mu}}$ interpolates smoothly and correctly to the one corresponding to the light-front gauge.

In the instant form limit, $C \to 1$, $q_1^2 C + q_\perp^2 \to q^2 = (q \cdot n)^2 - q^2$, and this reduces to
the well-known photon propagator in Coulomb gauge ($\nabla \cdot A = 0$) [Ryd96]:

$$T_{\mu\nu} = -\eta_{\mu\nu} + \frac{(q \cdot n)(q_\mu n_\nu + q_\nu n_\mu)}{(q \cdot n)^2 - q^2} - \frac{q_\mu q_\nu}{(q \cdot n)^2 - q^2} - \frac{q^2 n_\mu n_\nu}{(q \cdot n)^2 - q^2},$$ (4.19)

where $\eta_{\mu\nu} = \text{diag}(1,-1,-1,-1)$, and $\mu$ and $\nu$ run for $0,1,2,3$.

In the light-front limit, $C \to 0$, $q^2 \hat{\to} q^+ = (q \cdot n)^2$, and this becomes precisely the photon propagator under light-front gauge ($A^+ = 0$), and we have

$$T_{\mu\nu} = -g_{\mu\nu} + \frac{(q \cdot n)(q_\mu n_\nu + q_\nu n_\mu)}{(q \cdot n)^2} - \frac{q^2 n_\mu n_\nu}{(q \cdot n)^2},$$ (4.20)

where $g_{\mu\nu}$ is given by Eq. (3.2) with $\delta = \pi/4$, and $\mu$ and $\nu$ run for $+,1,2,3$. From this derivation, we see that the appropriate photon propagator for the interpolating gauge given by Eqs. (4.8) and (4.9) (or Eq. (4.10)) has three terms, consistent with [Lei84; SB01; SS03; SS04b; SS04a; MW05]. As we will see in the next section, the last term in Eq. (4.20) is canceled by the instantaneous interaction. Therefore, the two term gauge propagator [Mus91] can be used effectively without involving the instantaneous interaction.

### 4.1.4 Longitudinal Photons

Having used the transverse gauge fields as the dynamical degrees of the freedom, we are left with the longitudinal degree of freedom which also deserves a physical interpretation. This leftover longitudinal degree of freedom is necessary to describe the virtual photon. First, to find the longitudinal polarization ($\lambda = 0$) of the virtual photon, we need to generalize Eq. (4.4) replacing $M^2$ by $P^\mu P_\mu = q^2$ which can be either positive (timelike) or negative (spacelike). From the direct computation shown in Appendix A.2 with this replacement, we find

$$L_{\hat{\mu}\hat{\nu}} = c_\mu^*(0)c_\nu(0)$$

$$= -\frac{(q \cdot n)(q_\mu n_\nu + q_\nu n_\mu)}{q_+^2 C + q_-^2} + \frac{(q^\pm)^2 q_\mu q_\nu}{q_+^2 (q^2 C + q_-^2)} + \frac{q^2 n_\mu n_\nu}{q_+^2 C + q_-^2}.$$ (4.21)

Together with $T_{\hat{\mu}\hat{\nu}}$ given by Eq. (4.17), it verifies the well-known completeness relation [Ryd96]

$$T_{\hat{\mu}\hat{\nu}} + L_{\hat{\mu}\hat{\nu}} = -g_{\hat{\mu}\hat{\nu}} + \frac{q_{\hat{\mu}} q_{\hat{\nu}}}{q^2}.$$ (4.22)
We note that the last term in Eq. (4.21) without the dependence on $q_{\hat{\mu}}$ or $q_{\hat{\nu}}$ provides the instantaneous contribution. As we show in the next section, the instantaneous interaction appearing in the transverse gauge corresponds to the contribution from the longitudinal polarization.

### 4.2 Time-ordered Photon Exchange

As Kogut and Soper [KS70] regarded the theory of quantum electrodynamics as being defined by the usual perturbation expansion of the S-matrix in Feynman diagrams, we rewrite the sQED theory here by systematically decomposing each covariant Feynman diagram into a sum of interpolating $x^{+}$-ordered diagrams. Since we consider the Feynman expansion as a formal expansion as Kogut and Soper did, we also shall not be concerned in this paper with the convergence of the perturbation series, or convergence and regularization of the integrals.

For clarity, we split this section into three subsections. In the first subsection, we decompose the covariant photon propagator in an arbitrary interpolation angle as a sum of $x^{+}$-ordered terms. In the second part, we use the obtained propagator to derive the $x^{+}$-ordered diagrams and their amplitudes for the lowest order photon exchange process. In the final subsection, we verify the invariance of the corresponding total amplitude and discuss about the instant form and light-front limits of the $x^{+}$-ordered photon propagators.

#### 4.2.1 Photon Propagator Decomposition

The completeness relation given by Eq. (4.22) corresponds to the numerator of the covariant photon propagator in Landau gauge. We start from the covariant photon propagator in position space

$$D_F(x)_{\hat{\mu}\hat{\nu}} = \int \frac{d^4q}{(2\pi)^4} \frac{-ig_{\hat{\mu}\hat{\nu}} + i\frac{q_{\hat{\nu}}q_{\hat{\mu}}}{q^2} + i\epsilon e^{-iq_{\hat{\mu}}x_{\hat{\mu}}}}{q^2 q_{\hat{\mu}} + i\epsilon} e^{-iq_{\hat{\mu}}x_{\hat{\mu}}}.$$  

(4.23)

Due to the current conservation, however, the term involving $q_{\hat{\mu}}$ doesn’t contribute to any physical process and our starting point is equivalent to the Feynman photon propagator that Kogut and Soper used for their starting point [KS70]. For the same reason, the first and second terms that involve $q_{\hat{\mu}}$ can be dropped in Eq. (4.21) and the covariant photon
propagator can be written as

\[ D_F(x)_{\mu\nu} = i \int \frac{d^4 q}{(2\pi)^4} q^\mu q^\nu e^{-iq_\mu x^\mu} + \int \frac{d^4 q}{(2\pi)^4} n_\mu n_\nu e^{-iq_\mu x^\mu} \]

\[ = \int \frac{d^2 q_\perp dq^- dq_+}{(2\pi)^4} e^{-i(q_+ x^+ + q_- x^- + q_\perp x^\perp)} \times \left[ \frac{i \mathcal{T}_{\mu\nu}}{C q_+^2 + 2 S q_0 q_+ - C q_-^2 - q_\perp^2 + i\epsilon} + \frac{i n_\mu n_\nu}{q_0^2 C + q_-^2} \right], \tag{4.24} \]

where we used Eq. (3.9) for \( q_0^2 q_\perp^2 \) in the denominator.

To get the \( x^+ \)-ordered contributions, we now evaluate the \( q_+ \) integral in Eq. (4.24). We note here that the case of \( C = 0 \) should be distinguished from the case of \( C \neq 0 \) because the pole structures in the \( \mathcal{T}_{\mu\nu} \) term of Eq. (4.24) are different between the two cases, i.e. a single pole for \( C = 0 \) vs. two poles for \( C \neq 0 \).

For \( C \neq 0 \), the \( \mathcal{T}_{\mu\nu} \) term of Eq. (4.24) has two poles at

\[ A - i\epsilon' = \left(-S q_- + \sqrt{q_0^2 + C q_\perp^2} \right) / C - i\epsilon', \tag{4.25} \]

\[ B + i\epsilon' = \left(-S q_- - \sqrt{q_0^2 + C q_\perp^2} \right) / C + i\epsilon', \tag{4.26} \]

where \( \epsilon' > 0 \). In calculating the contour integration, we close the contour in the lower (upper) half plane for \( x^+ > 0 \) (\( x^+ < 0 \)). This produces a term proportional to the step function \( \Theta(x^+) \) and the other term proportional to \( \Theta(-x^+) \). We then make changes of the variables \( q_\perp \to -q_\perp \) and \( q_- \to -q_- \) which lead to \( B \to -A \) for the \( \Theta(-x^+) \) term and simplify the result expressing \( q_+ \) in terms of \( A \). The last term in Eq. (4.24) immediately gives a delta function of \( x^+ \) after the \( q_+ \) integration and provides the instantaneous contribution. We note that the instantaneous contribution stems from the longitudinal polarization of the virtual photon. Putting all terms together, we obtain the following result for the case of \( C \neq 0 \):

\[ D_F(x)_{\mu\nu} = \int \frac{d^2 q_\perp}{(2\pi)^3} \int_{-\infty}^{\infty} dq^- \quad \mathcal{T}_{\mu\nu} \left[ \Theta(x^+) e^{-iq_\perp x^\perp} + \Theta(-x^+) e^{iq_\perp x^\perp} \right] \]

\[ + i\delta(x^+) \int \frac{d^2 q_\perp}{(2\pi)^3} \int_{-\infty}^{\infty} dq^- \frac{n_\mu n_\nu}{q_0^2 C + q_-^2} e^{-i(q_+ x^+ + q_\perp x^\perp)}, \tag{4.27} \]
where $q_\perp^2$ in the exponent of the first two terms should be taken as $A$ given by Eq. (4.25).

Now let’s look at the light-front case. Because $C = 0$, there’s only one pole at $q_\perp^2/2q_- - i\epsilon/2q_- = q_\perp^2/2q^+ - i\epsilon/2q^+$. It depends on the sign of $q^+$ whether this pole is in the upper half plane or the lower half plane. As the integration over $q^-$ needs to be done in $q^+ > 0$ and $q^+ < 0$ regions separately, each region will get a step function after closing the contour in the plane where the arc contribution is absent. We again make changes of the variables $q^+ \rightarrow -q^-$ and $q_\perp \rightarrow -q_\perp$ in the term proportional to $\Theta(-x^+)$. We then obtain the result at the light-front [KS70]:

$$D_F(x)_{\mu\nu} = \int \frac{d^2 q_\perp}{(2\pi)^3} \int_0^\infty dq^+ \frac{\mathcal{T}_{\mu\nu}}{2q^+} \left[ \Theta(x^+) e^{-iq_\mu x^\mu} + \Theta(-x^+) e^{iq_\mu x^\mu} \right]$$

$$+ i\delta(x^+) \int \frac{d^2 q_\perp}{(2\pi)^3} \int_{-\infty}^\infty dq^+ \frac{n_\mu n_\nu}{(q^+)^2} e^{-i(q^+ x^- - q^+ x^+)}$$

(4.28)

where the indices $\mu$ and $\nu$ run for $+, 1, 2, -$ and $q^-$ in the exponent of the first two terms should be taken as $q_\perp^2/2q^+$. More details of the derivation for this equation can be found in Appendix A.3.

We note that the result for $C \neq 0$ given by Eq. (4.27) doesn’t coincide with the result for $C = 0$ given by Eq. (4.28) as we take the limit $C \rightarrow 0$, because the integration range $(-\infty, \infty)$ in $q_\perp$ is different from the integration range $(0, \infty)$ in $q^+$ due to the difference in the pole structure between the two cases, $C \neq 0$ and $C = 0$, as we mentioned earlier. Nevertheless, it can be written in a unified form by introducing an interpolating step function $\hat{\Theta}(q_-)$ given by

$$\hat{\Theta}(q_-) = \Theta(q_-) + (1 - \delta_{C0})\Theta(-q_-)$$

$$= \begin{cases} 1 & (C \neq 0) \\ \Theta(q^+) & (C = 0) \end{cases}$$

(4.29)

and realizing that $A$ given by Eq. (4.25) coincides with $q^- = q_\perp^2/2q^+$ in the limit $C \rightarrow 0$, i.e.

$$D_F(x)_{\hat{\mu}\hat{\nu}} = \int \frac{d^2 q_\perp}{(2\pi)^3} \int_{-\infty}^\infty dq_- \hat{\Theta}(q_-) \frac{\mathcal{T}_{\hat{\mu}\hat{\nu}}}{2\sqrt{q_-^2 + Cq_\perp^2}} \left[ \Theta(x^+) e^{-iq_\mu x^\mu} + \Theta(-x^+) e^{iq_\mu x^\mu} \right]$$
\[ + i \delta (x^+) \int \frac{d^2 q_\perp}{(2\pi)^3} \int_{-\infty}^{\infty} dq_- \frac{n_{\hat{q}} n_{\hat{p}}}{q_\perp^2 C + q_-^2} e^{-i(q_- x^+ + q_\perp x^)}, \quad (4.30) \]

where again \( q_\perp \) in the exponent of the first two terms should be taken as \( A \) given by Eq. (4.25). From now on, we will use Eq. (4.30) for all interpolation angles including \( C = 0 \).

4.2.2 Time-ordered Diagrams

The Lagrangian for sQED can be written as

\[
\mathcal{L} = D_\mu \phi D^\mu \phi^* - m^2 \phi^* \phi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \\
= \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi^* \phi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - e J^\mu A_\mu + e^2 A_\mu A^\mu \phi^* \phi, \quad (4.31)
\]

where

\[
D_\mu = \partial_\mu + ie A_\mu, \quad (4.32) \\
J^\mu = -i(\phi \partial^\mu \phi^* - \phi^* \partial^\mu \phi). \quad (4.33)
\]

To examine the contribution of each term in Eq. (4.30), we compute the lowest order tree level scattering amplitude starting from the usual Feynman amplitude in coordinate space. For the lowest tree level scattering diagram, \( e^2 A_\mu A^\mu \phi^* \phi \) doesn’t contribute, and the amplitude can be written as

\[
i \mathcal{M} = (-ie)^2 \int d^4 x d^4 y [J^\mu (y) D_F (y - x) \bar{J}^\rho (x)]. \quad (4.34)
\]

The scalar wave functions used here are the plane waves

\[
\phi(x) = e^{-ip_\mu x^\mu}. \quad (4.35)
\]

For a specific scattering process shown in Fig. 4.1, the currents from \( p_1 \) to \( p_3 \) and from \( p_2 \) to \( p_4 \) are respectively given by

\[
J^\nu = -i(\phi_1 \partial^\nu \phi_3^* - \phi_3^* \partial^\nu \phi_1) = (\not{p}_1 + \not{p}_3) e^{i(p_3 - p_1) x}, \quad (4.36) \\
J^\mu = -i(\phi_2 \partial^\mu \phi_4^* - \phi_4^* \partial^\mu \phi_2) = (\not{p}_2 + \not{p}_4) e^{i(p_4 - p_2) y}. \quad (4.37)
\]
With the change of variables

\[ x \rightarrow x, \quad y \rightarrow T = y - x, \]  

(4.38)

Eq. (4.34) becomes

\[ i\mathcal{M} = (-ie)^2 \int d^4x d^4T(p_2^\mu + p_4^\mu)D_{\mu\nu}(T)(p_1^\mu + p_3^\mu)e^{i(p_4 - p_2)T}e^{i(p_4 + p_1 - p_2 - p_1)x}. \]  

(4.39)

The \( x \) integration resulting in \((2\pi)^4\delta^4(p_4 + p_3 - p_2 - p_1)\) provides the total energy and momentum conservation. For the \( T \) integration, we use \( D_{\mu\nu} \) given by Eq. (4.30) as well as the following relations:

\[ \int_{-\infty}^{\infty} dT^\uparrow \Theta(T^\uparrow) e^{iP^\mu T^\uparrow} = \frac{i}{P^\uparrow}; \]  

(4.40)

\[ \int_{-\infty}^{\infty} dT^\uparrow \Theta(-T^\uparrow) e^{iP^\mu T^\uparrow} = -\frac{i}{P^\uparrow}; \]  

(4.41)

\[ \int_{-\infty}^{\infty} dT^\uparrow e^{iP^\mu T^\uparrow} = 2\pi \delta(P^\uparrow). \]  

(4.42)

After the \( x \) and \( T \) integration, we get

\[ i\mathcal{M} = (-ie)^2(p_2^\mu + p_4^\mu)\Pi_{\hat{\mu}\hat{\nu}}(p_3^\mu + p_1^\mu)(2\pi)^4\delta^4(p_4 + p_3 - p_2 - p_1), \]  

(4.43)

where

\[ \Pi_{\hat{\mu}\hat{\nu}} = \int \frac{d^2q_1 d^2q_2}{2\sqrt{q_2^2 + Cq_1^2}} \left[ \frac{i\mathcal{F}_{\hat{\mu}\hat{\nu}}}{p_4^\mu - p_2^\mu + A} \delta(p_4^\mu - p_2^\mu - q_2^\mu)\delta^2(p_4 - p_2 + q_2)\delta^2(p_4 + p_2 - q_2 - q_2) \right] \]

\[ - \frac{i\mathcal{F}_{\hat{\mu}\hat{\nu}}}{p_4^\mu - p_2^\mu + A} \delta(p_4^\mu - p_2^\mu + q_2^\mu)\delta^2(p_4 + p_2 + q_2)\delta^2(p_4 + p_2 - q_2 + q_2) \]

\[ + \int d^2q_1 d^2q_2 \frac{n_{q_1} n_{q_2}}{q_1^2 C + q_2^2} \delta(p_4^\mu - p_2^\mu - q_2^\mu)\delta^2(p_4 + p_2 - q_2 - q_2). \]  

(4.44)

The three terms in \( \Pi_{\hat{\mu}\hat{\nu}} \) corresponds to three different “time” orderings \( y^\uparrow > x^\uparrow \), \( y^\uparrow < x^\uparrow \) and \( y^\uparrow = x^\uparrow \) respectively\(^1\). The associated delta functions provide the momentum

\(^1\)In this paper, the “time” means the generalized interpolation time \( x^\uparrow \) unless specified otherwise.
conservation at each vertex as well as the conservation of total energy and momentum between the initial and final particles. In Fig. 4.1, the three “time” ordered diagrams are depicted with the momentum conservation at each vertex.

The corresponding photon propagators for $T^\hat{+} > 0$ (or $y^\hat{+} > x^\hat{+}$) and $T^\hat{+} < 0$ (or $y^\hat{+} < x^\hat{+}$) are respectively given by

$$\Pi^{(a)}_{\hat{\mu}\hat{\nu}} = \frac{1}{2Q^+(a)} \frac{i\,\hat{T}_{\hat{\mu}\hat{\nu}}\Theta(p_4^- - p_2^-)}{p_4^\hat{+} - p_2^\hat{+} - Q^{(a)}_{\hat{+}}}$$

$$= \frac{1}{2Q^+(a)} \frac{i\,\hat{T}_{\hat{\mu}\hat{\nu}}\Theta(p_1^- - p_3^-)}{p_1^\hat{+} - p_3^\hat{+} - Q^{(a)}_{\hat{+}}}, \quad (4.45)$$

and

$$\Pi^{(b)}_{\hat{\mu}\hat{\nu}} = -\frac{1}{2Q^+(b)} \frac{i\,\hat{T}_{\hat{\mu}\hat{\nu}}\Theta(p_2^- - p_4^-)}{p_4^\hat{+} - p_2^\hat{+} + Q^{(b)}_{\hat{+}}}$$

$$= \frac{1}{2Q^+(b)} \frac{i\,\hat{T}_{\hat{\mu}\hat{\nu}}\Theta(p_2^- - p_4^-)}{p_2^\hat{+} - p_4^\hat{+} - Q^{(b)}_{\hat{+}}}$$

$$= \frac{1}{2Q^+(b)} \frac{i\,\hat{T}_{\hat{\mu}\hat{\nu}}\Theta(p_3^- - p_1^-)}{p_3^\hat{+} - p_1^\hat{+} - Q^{(b)}_{\hat{+}}}, \quad (4.46)$$

where

$$Q^{\hat{z}(i)} = \sqrt{[q_{z}^{(i)}]^2 + C[q_{\perp}^{(i)}]^2}, \quad (i = a, b) \quad (4.47)$$
\[ Q_{\pm}^{(i)} = \frac{-S q_{\pm}^{(i)} + \sqrt{[q_{\pm}^{(i)}]^2 + C[q_{\perp}^{(i)}]^2}}{C}, \quad (i = a, b) \] (4.48)

and

\[ q_{\pm}^{(a)} = -q_{\pm}^{(b)} = p_1 - p_3, \quad (4.49) \]
\[ q_{\perp}^{(a)} = -q_{\perp}^{(b)} = p_1 - p_3 \perp. \quad (4.50) \]

The total energy-momentum conservation in Eq. (4.43) as well as the momentum conservation at each vertex were used here. \( Q_{\pm}^{(i)} \) and \( \hat{Q}_{\pm}^{(i)} \) satisfy the on-mass-shell condition of the propagating photon with momentum \( (q_{\pm}^{(i)}, q_{\perp}^{(i)}) \). To see this, one can use Eq. (3.11) for a massless photon to derive the \( Q_{\pm}^{(i)} \) in terms of \( q_{\pm}^{(i)} \) and \( q_{\perp}^{(i)} \) noting that \( Q_{\pm}^{(i)} \) is positive definite for an on-mass-shell particle due to Eq. (3.4). The formula for \( Q_{\pm}^{(i)} \) can also be obtained in terms of \( Q_{\pm}^{(i)} \) and \( \hat{Q}_{\pm}^{(i)} \) by using Eq. (3.8). Eqs. (4.45) and (4.46) can then be written in a unified form:

\[ \Pi_{\mu\nu}^{(i)} = \frac{1}{2Q_{\pm}^{(i)}} i \mathcal{F}_{\mu\nu} \Theta(q_{\pm}^{(i)}) P_{ini\pm} - P_{inter\pm}, \quad (i = a, b) \] (4.51)

where \( P_{ini\pm} \) \( [P_{inter\pm}] \) is the sum of the energy of the initial [intermediate] particles. All the initial and intermediate particles are now on their mass shells. This agrees with the familiar time-ordered perturbation theory in the IFD, although we now see it as a generalization to any interpolation angle. We note here that the propagating photon as the dynamical degree of freedom has only the transverse polarization. This is consistent with the interpretation that the intermediate particles are now “on-mass-shell” or “physical”.

Besides the two propagating terms, there is a third term in Eq. (4.44) that represents the instantaneous contribution. For the sake of consistency in our notation, we denote this part of \( \Pi_{\mu\nu} \) as

\[ \Pi_{\mu\nu}^{(c)} = \frac{in_{\mu}n_{\nu}}{q_{\perp}^2 C + q_{\pm}^2}, \quad (4.52) \]

where \( q_{\pm} \) and \( q_{\perp} \) are given by Eqs. (4.49) and (4.50), respectively. As noted earlier, the instantaneous contribution stems from the longitudinal polarization of the virtual photon.

We thus have three time-ordered diagrams as shown in Fig. 4.1, the first two of

\[ \text{Fig. 4.1: Time-ordered diagrams for the instantaneous contribution.} \]

\[ \text{Note that the subscripts '}' and ']' denote the energy and the longitudinal momentum for a given interpolating momentum, respectively.} \]
which represent time-ordered exchanges of propagating photon and the third of which represents the instantaneous interaction. The invariant amplitude is then the sum of all three time-ordered amplitudes:

\[ iM = \sum_{j=a,b,c} iM^{(j)} = (-i\epsilon)^2 \sum_{j=a,b,c} (p_4^\mu + p_2^\mu) \Pi^{(j)}_{\mu\nu}(p_3 + p_1^\nu), \]  

(4.53)

where the total energy-momentum conservation factor \((2\pi)^4\delta^4(p_4 + p_3 - p_2 - p_1)\) is implied.

### 4.2.3 LFD vs. the Limit to LFD

Since the time-ordered gauge propagators have the interpolating step function given by Eq. (4.29), we take a close look at the limiting cases of \(C \to 0\) and compare the results with the exact \(C = 0\) (LFD) results in this subsection. For convenience and simplicity of our discussion, we drop the \((i)\) superscript in Eq. (4.47) and have

\[ Q^+(a) = Q^+(b) = \sqrt{(p_1^\perp - p_3^\perp)^2 + (p_1\perp - p_3\perp)^2} \equiv Q^+. \]  

(4.54)

Because the terms proportional to \(q_{\mu\nu}\) or \(q_{\perp}\) in \(T_{\mu\nu}\) can be discarded due to the current conservation in the physical process, we may also replace \(T_{\mu\nu}\) by

\[ T_{\mu\nu} \equiv -g_{\mu\nu} - \frac{Q^+n_\mu n_\nu}{Q^+ C + Q^2}. \]  

(4.55)

Then, Eq. (4.45), Eq. (4.46) and Eq. (4.52) can be rewritten as

\[ \Pi_{\mu\nu}^{(a)} = \frac{1}{2Q^+} i\hat{T}_{\mu\nu} \hat{\Theta}(p_1^\perp - p_3^\perp), \]  

(4.56)

\[ \Pi_{\mu\nu}^{(b)} = -\frac{1}{2Q^+} i\hat{T}_{\mu\nu} \hat{\Theta}(p_3^\perp - p_1^\perp), \]  

(4.57)

\[ \Pi_{\mu\nu}^{(c)} = \frac{in_\mu n_\nu}{(Q^+)^2}, \]  

(4.58)

where \(Q^+_a\) and \(Q^+_b\) are given by Eq. (4.48).
For $C \neq 0$, assigning the 4-momentum transfer $q$ as

$$q_\perp \equiv p_{1\perp} - p_{3\perp} = q_\perp^{(a)} = -q_\perp^{(b)}, \quad (4.59)$$

$$q_\| \equiv p_{1\|} - p_{3\|} = q_\|^{(a)} = -q_\|^{(b)}, \quad (4.60)$$

$$q_\perp \equiv p_{1\perp} - p_{3\perp} = -(p_{2\perp}^* - p_{4\perp}^*), \quad (4.61)$$

we have

$$Q^{(a)}_\perp = \frac{-S q_\perp + \sqrt{q_\perp^2 + C q_\perp^2}}{C} = \frac{-S q_\perp + Q_\perp^\perp}{C}, \quad (4.62)$$

$$Q^{(b)}_\perp = \frac{S q_\perp + \sqrt{q_\perp^2 + C q_\perp^2}}{C} = \frac{S q_\perp + Q_\perp^\perp}{C}. \quad (4.63)$$

Since both $\hat{\Theta}(q_\perp)$ and $\hat{\Theta}(-q_\perp)$ are unity for $C \neq 0$, $\Pi^{(a)}_{\mu\nu}$ and $\Pi^{(b)}_{\mu\nu}$ can be rewritten as

$$\Pi^{(a)}_{\mu\nu} = \frac{1}{2Q^+ C q_\perp + S q_\perp - Q^+} \frac{i \tilde{F}_{\mu\nu} C}{(q_\perp^\perp)^2 - (Q_\perp^\perp)^2} = \frac{1}{2Q^+ q^\perp - Q^+}, \quad (4.64)$$

$$\Pi^{(b)}_{\mu\nu} = \frac{1}{2Q^+ C q_\perp + S q_\perp + Q^+} \frac{i \tilde{F}_{\mu\nu} C}{q^\perp + Q_\perp^\perp} = \frac{i \tilde{F}_{\mu\nu} C}{q^\perp + Q_\perp^\perp}. \quad (4.65)$$

Summing all contributions, we use Eqs. (3.11), (4.54) and (4.55) to verify

$$\Pi^{(a)}_{\mu\nu} + \Pi^{(b)}_{\mu\nu} + \Pi^{(c)}_{\mu\nu} = \frac{i \tilde{F}_{\mu\nu} C}{(q_\perp^\perp)^2 - (Q_\perp^\perp)^2} + \frac{in_\mu n_\nu}{q_\perp^2 C + q_\perp^2}$$

$$= \frac{i \tilde{F}_{\mu\nu} C}{q^2 C + q_\perp^2} = \frac{-i g_{\mu\nu}}{q^2} \quad (4.66)$$

It is clear from this derivation that the contribution from the instantaneous interaction is cancelled by the corresponding $n_\mu n_\nu$ term in the transverse photon propagator and the sum of all the contributions is totally Lorentz invariant.

For $C = 0$, $\hat{\Theta}(q_\perp)$ and $\hat{\Theta}(-q_\perp)$ are neither unity nor same to each other. Thus, we need to look into the details of each contribution in Eq. (4.56), Eq. (4.57) and Eq. (4.58) very carefully to understand the Lorentz invariance of the total amplitude. Taking $C = 0$
in Eqs. (4.56) - (4.58), we get:

\[
\Pi^{(a)}_{\mu\nu} = \frac{1}{2q^+} \frac{i(-g_{\mu\nu} - \frac{q^2 n_\mu n_\nu}{(q^+)^2})\Theta(q^+)}{p_1^- - p_3^- - \frac{q_1^2}{2q^+}},
\]

\[
\Pi^{(b)}_{\mu\nu} = -\frac{1}{2|q^+|} \frac{i(-g_{\mu\nu} - \frac{q^2 n_\mu n_\nu}{(q^+)^2})\Theta(-q^+)}{p_1^- - p_3^- + \frac{q_1^2}{2|q^+|}},
\]

\[
\Pi^{(c)}_{\mu\nu} = \frac{in_\mu n_\nu}{q^{+2}},
\]

where \(q^+ = p_1^+ - p_3^+\) and of course the conservation of total LF energy in the initial and final states is imposed such as \(p_1^- - p_3^- = p_4^- - p_2^-\). In this case, the contribution from the LF time-ordered process (a) or (b) in Fig. 4.1 depends on the values of external momenta \(p_1^+\) and \(p_3^+\), i.e. whether \(p_1^+ > p_3^+\) or \(p_1^+ < p_3^+\). As indicated by \(\Theta(q^+\) and \(\Theta(-q^+)\) in Eqs. (4.67) and (4.68), respectively, only one of the two LF time-ordering processes, (a) or (b), not both contributes once the external kinematic situation is given. For example, if \(p_1^+ > p_3^+\), then only (a) contributes. Similarly, if \(p_1^+ < p_3^+\), then only (b) contributes. This is dramatically different from the \(C \neq 0\) case where both (a) and (b) contributes regardless of \(p_1^- > p_3^-\) or \(p_1^- < p_3^-\). The limiting case \(C \to 0\) will be separately discussed later after this discussion of LFD, i.e. \(C = 0\). Although either (a) or (b) (not both) contributes to the total amplitude, the Lorentz invariance of the total amplitude is assured in LFD. For example, if \(p_1^+ > p_3^+\), then the sum of all contributions is given by

\[
\Pi^{(a)}_{\mu\nu} + \Pi^{(c)}_{\mu\nu} = \frac{1}{2q^+} \frac{i(-g_{\mu\nu} - \frac{q^2 n_\mu n_\nu}{(q^+)^2})}{p_1^- - p_3^- - \frac{q_1^2}{2q^+}} + \frac{in_\mu n_\nu}{q^{+2}}
\]

\[
= \frac{i(-g_{\mu\nu} - \frac{q^2 n_\mu n_\nu}{(q^+)^2})}{2q^+q^- - q_\perp^2} + \frac{in_\mu n_\nu}{q^{+2}}
\]

\[
= \frac{-ig_{\mu\nu}}{q^2},
\]

where \(p_1^- - p_3^- = q^-\) since the 4-momentum transfer \(q\) is assigned as \(p_1 - p_3\). As in the \(C \neq 0\) case, the contribution from the instantaneous interaction is cancelled by the
corresponding $n_\mu n_\nu$ term in the transverse photon propagator and the sum of all the contributions is totally Lorentz invariant. Similarly, if $p_1^+ < p_3^+$, then $\Pi^{(a)}_{\mu\nu}$ is replaced by $\Pi^{(b)}_{\mu\nu}$ with $|q^+| = -q^+$ and the sum $\Pi^{(b)}_{\mu\nu} + \Pi^{(c)}_{\mu\nu} = -ig_{\mu\nu} / q^2$ is totally Lorentz invariant.

Now, let’s consider the limit $C \to 0$ (i.e., $\delta \to \pi/4$ or $S \to 1$) from the $C \neq 0$ case given by Eqs. (4.64) and (4.65), where we take $q_+ \to q^- = q^-$, $q_- \to q^+ = q^+$, $Q^+ \to Q^+ = q^+$. To obtain the correct limits, we need to make a careful expansion. For $q^+ = p_1^+ - p_3^+ > 0$, we get

$$
\mathcal{C} q_+ + S q_- - Q^+ = \mathcal{C} q_+ + S q_- - \sqrt{q_+^2 + \mathcal{C} q_-^2} \\
\to \mathcal{C} q^- + q^+ - \sqrt{q^+ + 2 + \mathcal{C} q_-^2} \\
\to \mathcal{C} q^- - C \frac{q_-^2}{2q^+} + \mathcal{O}(C^2),
$$

(4.71)

$$
\mathcal{C} q_+ + S q_- + Q^+ \to 2q^+ + \mathcal{C} q_+ + \mathcal{C} \frac{q_-^2}{2q^+} + \mathcal{O}(C^2),
$$

(4.72)

$$
\mathcal{F}_{\mu\nu} \to -g_{\mu\nu} - 2q^- \frac{q_-^2}{2q^+} n_\mu n_\nu.
$$

(4.73)

Thus, in the limit $C \to 0$ for $p_1^+ > p_3^+$, Eqs. (4.64) and (4.65) as well as Eq. (4.58) become

$$
\Pi^{(a)}_{\mu\nu} = \frac{i \mathcal{F}_{\mu\nu}}{2q^+} \frac{1}{q^- - \frac{q_+^2}{2q^+}} = -i g_{\mu\nu} \frac{1}{2q^+} \frac{1}{q^- - \frac{q_+^2}{2q^+}} - i \frac{n_\mu n_\nu}{(q^+)^2},
$$

(4.74)

$$
\Pi^{(b)}_{\mu\nu} = \lim_{C \to 0} \frac{-i \mathcal{F}_{\mu\nu}}{2q^+} \frac{\mathcal{C}}{2q^+ + \mathcal{C} q_+ + \mathcal{C} \frac{q_-^2}{q^+}} \to 0,
$$

(4.75)

$$
\Pi^{(c)}_{\mu\nu} = \frac{i n_\mu n_\nu}{(q^+)^2}.
$$

(4.76)

This shows the agreement between the $C = 0$ result and the $C \to 0$ result for the kinematic situation $p_1^+ > p_3^+$. Note here that $\Pi^{(b)}_{\mu\nu} = 0$ was given in LFD ($C = 0$) via the factor $\Theta(-q^+)$ in Eq. (4.68) while $\Pi^{(b)}_{\mu\nu} \to 0$ as $C \to 0$ in Eq. (4.75) is obtained without the factor $\Theta(-q^+)$. Similarly, the agreement between the $C = 0$ result and the $C \to 0$ result is also found for the kinematic situation $p_1^+ < p_3^+$ following the procedure described above.

Having shown the agreement between the $C = 0$ result and the $C \to 0$ result for the kinematic situation $p_1^+ \neq p_3^+$, let’s now consider the special kinematic situation $p_1^+ = p_3^+$. If this kinematic situation is provided with the non-zero values of $p_1^+$ and $p_3^+$,
i.e. \( p_1^+ = p_3^+ \neq 0 \), then the LF time-ordered processes (a) and (b) are indistinguishable and the results of \( \Pi_{\mu\nu}^{(a)} \) and \( \Pi_{\mu\nu}^{(b)} \) given by Eqs. (4.67) and (4.68), respectively, are identical with the factor \( \Theta(q^+ = -q^+ = 0) = 1/2 \). We also find that the limit \( C \to 0 \) of Eqs. (4.64) and (4.65) agree with these results, precisely yielding the 1/2 factor both for (a) and (b) contributions as in the \( C = 0 \) result. Thus, both in \( C = 0 \) and the limit \( C \to 0 \), the total amplitude is given by

\[
\Pi_{\mu\nu}^{(a)} + \Pi_{\mu\nu}^{(b)} + \Pi_{\mu\nu}^{(c)} = \frac{-ig_{\mu\nu}}{q^2},
\]

(4.77)

where \( q^2 = -q_1^2 \) and the divergent instantaneous interaction is cancelled by the corresponding divergent \( n_\mu n_\nu \) term in the transverse photon propagator.

Although the LFD results at exact \( C = 0 \) are attainable from the \( C \to 0 \) results for all the kinematic regions that we discussed in this subsection so far such as \( p_1^+ > p_3^+ \), \( p_1^+ < p_3^+ \) and \( p_1^+ = p_3^+ \neq 0 \), the agreement between the two, i.e. LFD vs. the limit to LFD, should be looked into more carefully for the case \( p_1^+ = p_3^+ = 0 \) which should be distinguished from the case \( p_1^+ = p_3^+ \neq 0 \) that we have discussed in this subsection. This special kinematic situation \( p_1^+ = p_3^+ = 0 \) involves the discussion of the infinite momentum frame (IMF) with \( P^z \to -\infty \), since all the plus momenta go to zero in this frame. Presenting the numerical results with the frame dependence of the time-ordered amplitudes in the next section, we will discuss a particular case of correlating the two limits, \( C \to 0 \) and \( q^+ \to 0 \), in conjunction with the J-shaped correlation coined in Ref. [JS12]. As we show in the next section, the results in the limit \( C \to 0 \) following the J-curve are different from those in the LFD or at the exact \( C = 0 \).

### 4.3 Frame Dependence and Interpolation Angle Dependence of Time-ordered Amplitudes

In this section, we numerically compute the scattering amplitudes shown in Fig. 4.1 and discuss both the frame dependence and the interpolation angle dependence of each and every \( x^\pm \)-ordered amplitudes. Putting Eqs. (4.56) - (4.58) back into Eq. (4.53), we have the time-ordered amplitudes in the following form:

\[
\mathcal{M}^{(a)} = -(-ie)^2 \left( p_{24} \cdot p_{13} + p_{24}^\wedge p_{13}^\wedge g^2 /(Q^\wedge)^2 \right) \frac{\Theta(p_{1\wedge} - p_{3\wedge})}{2Q^+ (q^+ - Q^+)}.
\]

(4.78)
\[
\mathcal{M}^{(b)} = (-ie)^2 \frac{[p_{24} \cdot p_{13} + p_{24} \hat{p}_{13} q^2 / (Q^+)] \mathcal{C} \hat{\Theta}(p_{32} - p_{10})}{2Q^+(q^+ + Q^+)},
\]
\[
\mathcal{M}^{(c)} = (-ie)^2 \frac{\hat{p}_{24} \hat{p}_{13}}{(Q^+)^2},
\]

where

\[
p_{24} = p_2 + p_4,
\]
\[
p_{13} = p_1 + p_3.
\]

As we have shown in the last section, the sum of these three \(x^+\)-ordered amplitudes agree with the manifestly invariant total amplitude regardless of whether \(C \neq 0\) or \(C = 0\):

\[
\mathcal{M} = \Sigma_{j=a,b,c} \mathcal{M}^{(j)} = -(-ie)^2 \frac{p_{24}p_{13}}{q^2}.
\]

To investigate the frame dependence of these amplitudes we first look at how they change under different transformations. From Eq. (3.35), one can see that when \(\beta_3 = 0\), \(T_{12}^\dagger P^+ T_{12} = P^+\). So under the kinematic transformation \(T_{12}\), all the "\(\hat{+}\)" components, namely \(q^+, Q^+, p_{13}^+\) and \(p_{24}^+\) remain the same. We also note that the factors of \(\hat{\Theta}(p_{12} - p_{32})\) and \(\hat{\Theta}(p_{32} - p_{12})\) are invariant under \(T_{12}\) because these factors are unity for \(C \neq 0\) and become \(\Theta(p_{32} - p_{12})\) and \(\Theta(p_{12} - p_{32})\), respectively, for \(C = 0\). Thus, all three \(x^+\)-ordered amplitudes are invariant under \(T_{12}\) regardless of \(C\) values as they should be.

Now applying the longitudinal boost \(T_3\) to the three time-ordered amplitudes, we note that the operator \(K^3 = M_+\) changes its characteristic from dynamic for \(C \neq 0\) to kinematic in the light-front \((C = 0)\) as shown in Table 3.2 which summarized the set of kinematic and dynamic generators depending on the interpolation angle. We mentioned this point in Section 3.1. In applying \(T_3 = e^{-iK^3\beta_3}\), we thus distinguish the values of \(C\) between \(C \neq 0\) and \(C = 0\). We first consider the \(C \neq 0\) case and later compare the results in this case with the LFD \((C = 0)\) results.

For the kinematics of our two-body scattering process analogous to \(e\mu \rightarrow e\mu\), we choose the scattering plane as the \(x-z\) plane and the directions of the initial two particles as the parallel/antiparallel to the \(z\)-axis such that \(p_{1\perp} = p_{2\perp} = 0\) as shown in Fig. 4.1. In the center of momentum frame (CMF), we have \(p_1^0 = -p_2^0 \equiv p, \epsilon_1 \equiv p_1^0 = p_0^0 = \sqrt{p^2 + m_1^2}, \epsilon_2 \equiv p_2^0 = p_0^0 = \sqrt{p^2 + m_2^2}\) and \(M \equiv \epsilon_1 + \epsilon_2\), where we denote the total energy in CMF.
as “M” for the convenience in the later discussion. We should note that the invariant Mandelstam variable \( s = (p_1 + p_2)^2 \) is given by \( M^2 \). The final 4-momenta \( p_3 \) and \( p_4 \) of the particles that are going back to back at angle \( \theta \) in CMF are depicted in Fig. 4.1. Correspondingly, the 4-momenta of the initial and final particles in CMF are given by

\[
\begin{align*}
p_1 &= (\epsilon_1, 0, 0, p), \\
p_2 &= (\epsilon_2, 0, 0, -p), \\
p_3 &= (\epsilon_1, p \sin \theta, 0, p \cos \theta), \\
p_4 &= (\epsilon_2, -p \sin \theta, 0, -p \cos \theta).
\end{align*}
\]

To discuss the longitudinal boost \( (T_3) \) effect on time-ordered scattering amplitudes, let’s now boost the whole system to the total momentum \( P_z \). From the Lorentz transformation for a composite free particle system, we know that the total energy in the new frame is \( E = \sqrt{(P_z)^2 + M^2} \). The Lorentz transformation for each \( p_i \) \((i = 1, 2, 3, 4)\) can then be described in terms of \( \gamma = E/M \) and \( \gamma \beta = P_z/M \):

\[
\begin{align*}
p_i^0' &= \gamma p_i^0 + \gamma \beta p_i^z, \\
p_i^z' &= \gamma p_i^z + \gamma \beta p_i^0, \\
P_i^\perp &= P_i^\perp
\end{align*}
\]

The boosted four momentum transfer can therefore be written as

\[
\begin{align*}
q_{13}^x &= p_{13}^x - p_{3}^x = -p \sin \theta, \\
q_{13}^y &= p_{13}^y - p_{3}^y = 0, \\
q_{13}^z &= p_{13}^z - p_{3}^z = \frac{E}{M} p(1 - \cos \theta), \\
q_0 &= p_1^0 - p_3^0 = \frac{P_z}{M} p(1 - \cos \theta),
\end{align*}
\]

where \( M \) and \( E \) are functions of \( m_1, m_2, p \) and \( P_z \). We can use these 4-momentum components given by Eqs. (4.91) - (4.94) and apply Eqs. (3.4) - (3.7) to form \( q_\mu' \) and \( q_\bar{\mu}' \) for any interpolation angle. Eqs. (4.88) - (4.90) can also be used together with Eqs. (3.4) - (3.7) and Eqs. (4.81) and (4.82) to express all the components of the boosted 4-momenta. Plugging \( q_\bar{\mu}' \) into Eq. (4.54), we can get \( Q^\dagger \). Computing \( p_{13}' \cdot p_{13}' \) and plugging it with other
factors such as $p_{24}^\hat{+}, p_{13}^\hat{+}, q^\hat{+}, q^2$ and $Q^\hat{+}$ into Eq. (4.78) - Eq. (4.80), we get the interpolating time-ordered amplitudes $\mathcal{M}^{(j)}$ in a boosted frame. In CMF, all of these amplitudes are then given by functions of each particle's initial momentum $p(-p)$, individual particle's rest mass $m_1$ and $m_2$, scattering angle $\theta$, the total momentum $P^z$ and the interpolation angle $\delta$. For given values of $m_1, m_2, p$ and $\theta$, the $x^\hat{+}$-ordered amplitudes are dependent on the frame ($P^z$) and the interpolation angle ($\delta$ or $\mathcal{C}$).

To exhibit this feature quantitatively, we choose $m_1 = 1$, $m_2 = 2$, $p = 3$, in the same energy unit (i.e., $m_2$ and $p$ scaled by $m_1$), and $\theta = \pi/3$. For simplicity, we also take the charge $e$ to be 1 unit in our calculation. In Fig. 4.2, we plot $\mathcal{M}^{(a)}$, $\mathcal{M}^{(b)}$, $\mathcal{M}^{(c)}$ as well as the sum of all three amplitudes $\mathcal{M}^{(tot)}$ as functions of both the interpolation angle $\delta$ and the total momentum $P^z$ to reflect not only the interpolation angle dependence but
also the frame dependence. The total amplitude $\mathcal{M}^{(\text{tot})}$ shown in Fig. 4.2 is both frame independent and interpolation angle independent as it should be.

The detailed structures of these three time-ordered diagrams are very interesting. Just like in the $\phi^3$ toy model theory studied in Ref. [JS12], the $P^z > 0$ region is smooth for all three amplitudes, while a J-shaped correlation curve exists in the $P^z < 0$ region, which we plotted as the red solid line in Fig. 4.2 (color online). This is the curve that starts out in the center of mass frame ($P^z = 0$) in the $\delta = 0$ limit, but maintains the same value of amplitude throughout the whole range of interpolation angle. The values maintained by these curves can be found for arbitrary $m_1, m_2, p$ and $\theta$, and given by

$$
\mathcal{M}^{(a)} = \mathcal{M}^{(b)} = -\frac{\cot^2 \frac{\theta}{2}}{2},
$$

$$
\mathcal{M}^{(c)} = -\frac{\epsilon_1 \epsilon_2}{p^2 \sin^2 \frac{\theta}{2}}.
$$

Again, $\epsilon_i = \sqrt{q_i^2 + m_i^2}$, $(i = 1, 2)$ and the charge “$e$” in Eqs. (4.78) - (4.80) is taken as 1 unit.

In Fig. 4.3, we plot the profiles of $\mathcal{M}^{(a)}$, $\mathcal{M}^{(b)}$ and $\mathcal{M}^{(c)}$ in IFD ($C = 1$ or $\delta = 0$) for the given values of $m_1, m_2, p$ and $\theta$ and depict the starting point of the J-curve corresponding to the values of $\mathcal{M}^{(a)}$, $\mathcal{M}^{(b)}$ and $\mathcal{M}^{(c)}$ at $P^z = 0$ given by Eqs. (4.95) and (4.96). The maxima for $\mathcal{M}^{(a)}$ and $\mathcal{M}^{(b)}$ stay in the $P^z > 0$ and $P^z < 0$ regions respectively, but as the scattering angle $\theta$ approaches 0, the maximum for $\mathcal{M}^{(a)}$ and $\mathcal{M}^{(b)}$ will move towards $P^z = 0$. As the J-curve does not track the maximum or the minimum in $\mathcal{M}^{(a)}$ and $\mathcal{M}^{(b)}$, we note that the J-curve does not track the maximum of the surface $\mathcal{M}^{(c)}$ either.

It is interesting to note that the J-curve itself does not change with the scattering angle $\theta$. In fact, we find that it follows exactly the same formula as the one found in the $\phi^3$ theory [JS12]:

$$
P^z = -\sqrt{\frac{M^2(1 - C)}{2C}},
$$

where $M^2$ is identical to the invariant Mandelstam variable $s = (p_1 + p_2)^2$, i.e. $M = \sqrt{s}$. We note that the J-curve is “universal” in the sense that it doesn’t depend on the specific kinematics like the scattering angle or particle masses, but scales with $\sqrt{s}$.

If we follow this J-curve plotted as the red solid line in Fig. 4.2, we see that as $C \to 0$ or $\delta \to \pi/4$, it is pushed to $P^z \to -\infty$ and the constant values given by Eqs. (4.95) and
Figure 4.3 (color online) Time-ordered amplitudes $M^{(a)}$, $M^{(b)}$ and $M^{(c)}$ as a function of the total momentum $P^z$ at the instant form limit ($\delta = 0$). The red dots on the $P^z = 0$ axis denotes the starting position of the J-curves on each surface.

(4.96) are maintained throughout the curve. On the other hand, the $x^+$-ordered light-front amplitudes are invariant under the longitudinal boost $T_3 = e^{-iK_3\beta^3}$ and thus each of $M^{(a)}$, $M^{(b)}$ and $M^{(c)}$ show up individually as a constant independent of $P^z$ along the $\delta = \pi/4$ or $C = 0$ line in Fig. 4.2. Thus, the values of $M^{(a)}$, $M^{(b)}$ and $M^{(c)}$ in the limit $C \to 0$ following the J-curve are different from those in the LFD, i.e. at the exact $C = 0$. This brings up our discussion in the last subsection, Sec. 4.2.3, about the issue whether the LFD results with the exact $C = 0$ can be reproduced by taking the limit $C \to 0$. Because all the plus components of the particle momenta vanish in the limit $P^z \to -\infty$, we now should look closely at the kinematic situation $p_1^+ = p_3^+ = q^+ = 0$ which we have postponed its discussion in Sec. 4.2.3.

Since the details of each contribution depend on the values of $C$ and $P^z$, the results are in general dependent on the order of taking the limits to $C \to 0$ and $P^z \to -\infty$. In particular, if $C = 0$ is taken first, then the results must be independent of the $P^z$ values due to the longitudinal boost invariance in LFD as discussed before and the result at $P^z = -\infty$ is identical to that for any other $P^z$ value. However, if we take $P^z \to -\infty$ first and consider $C \to 0$, then we should first examine the $C \neq 0$ results in the $P^z \to -\infty$ limit with a great care. We begin with the discussion on the $C = 1$ (i.e. IFD) results and
vary the interpolation angle from $\delta = 0$ to $\delta \rightarrow \pi/4$.

As shown in Figs. 4.2 and 4.3, each of the time-ordered amplitudes $\mathcal{M}^{(a)}$, $\mathcal{M}^{(b)}$ and $\mathcal{M}^{(c)}$ at $C = 1(\delta = 0)$, are not symmetric under the reflection of $P^z \rightarrow -P^z$. Although the IFD results coincide with the LFD results as $P^z \rightarrow \infty$, they do not agree in the limit $P^z \rightarrow -\infty$ as shown in Fig. 4.2. Thus, the prevailing notion of the equivalence between IFD and LFD in the infinite momentum frame (IMF) should be taken with a great caution since it works for the limit of $P_z \rightarrow \infty$ but not in the limit $P_z \rightarrow -\infty$. We have already discussed the treachery in taking the limit $P_z \rightarrow -\infty$ even for the amplitudes with the reflection symmetry under $P^z \leftrightarrow -P^z$ in IFD such as the scalar annihilation process analogous to QED $e^+e^- \rightarrow \mu^+\mu^-$ in Ref. [JS12]. The present analysis of the sQED scattering process analogous to QED $e\mu \rightarrow e\mu$ provides a clear example of breaking the reflection symmetry under $P^z \leftrightarrow -P^z$ in IFD and fortify the clarification of the confusion in the folklore of the equivalence between the IMF and the LFD. As we vary the interpolation angle from $\delta = 0$ and approach to $\delta = \pi/4$, this broken reflection symmetry under $P^z \leftrightarrow -P^z$ persists while the LFD results are symmetric under $P^z \leftrightarrow -P^z$ due to the longitudinal boost invariance discussed previously. The J-curve correlation between $P^z$ and $C$ given by Eq. (4.97) provides a simultaneous limit of $P^z \sim -1/C^{1/2} \rightarrow -\infty$ as $C \rightarrow 0$ and the corresponding results of $\mathcal{M}^{(a)}$, $\mathcal{M}^{(b)}$ and $\mathcal{M}^{(c)}$ are uniquely given by Eq. (4.95) and Eq. (4.96). As mentioned earlier, the numerical values along the J-curve plotted as the red solid line in Fig. 4.2 (color online) are identical all the way to the $P^z \rightarrow -\infty$ limit and thus the clear distinction is manifest between the results in the $C \rightarrow 0$ limit with the J-curve correlation and the LFD results with the exact $C = 0$. Since the J-curve exists only for $P^z < 0$, it is also self-evident that the broken reflection symmetry under $P^z \leftrightarrow -P^z$ still persists for the correlated limit of $P^z \sim -1/C^{1/2} \rightarrow -\infty$ and the results obtained in this limit must be different from the LFD results. Therefore, the limit $P^z \rightarrow -\infty$ is treacherous and requires a great caution in taking the light-front limit from $C \neq 0$. As it must be, however, the sum of all time-ordered amplitudes $\mathcal{M}^{(tot)} = \mathcal{M}^{(a)} + \mathcal{M}^{(b)} + \mathcal{M}^{(c)}$ is completely independent of $C$ and $P^z$. The total amplitude $\mathcal{M}^{(tot)}$ is of course identical regardless of whether $C$ is exactly zero or not, however the limit $P^z \rightarrow -\infty$ is taken.

Finally, we also calculated the time-ordered scattering amplitudes discussed in this section for the scalar $\phi^3$ theory that was used in Ref. [JS12] and summarized the results in Appendix A.4 for a comparison with the sQED results. For completeness in comparing the results between the $\phi^3$ theory and sQED, the time-ordered annihilation amplitudes

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for sQED are also summarized in Appendix A.5.

4.4 Summary

In this chapter, we the sQED theory to discuss the electromagnetic gauge degree of freedom interpolated between the IFD and the LFD. We developed the electromagnetic gauge field propagator interpolated between the IFD and the LFD and found that the light-front gauge $A^+ = 0$ in the LFD is naturally linked to the Coulomb gauge $\nabla \cdot A = 0$ in IFD. We identified the dynamical degrees of freedom for the electromagnetic gauge fields as the transverse photon fields and clarified the equivalence between the contribution of the instantaneous interaction and the contribution from the longitudinal polarization of the virtual photon.

Our results for the gauge propagator and time-ordered diagrams clarified whether one should choose the two-term form [Mus91] or the three-term form [Lei84; SB01; SS03; SS04b; SS04a; MW05] for the gauge propagator in LFD. Our transverse photon propagator in LFD assumes the three-term form, but the third term cancels the instantaneous interaction contribution. Thus, one can use the two-term form of the gauge propagator for effective calculation of amplitudes if one also omits the instantaneous interaction from the Hamiltonian. But if one wants to show equivalence to the covariant theory, all three terms should be kept because the instantaneous interaction is a natural result of the decomposition of Feynman diagrams, and the third term in the propagator is necessary for the total amplitudes to be covariant. We also see that the photon propagator was derived according to the generalized gauge that links the Coulomb gauge to light-front gauge and thus the three-term form appears appropriate in order to be consistent with the appropriate gauge.

Using the interpolating photon propagator, we computed the lowest-order scattering process such as an analogue of the well-known QED process $e\mu \rightarrow e\mu$ in sQED and analyzed the three corresponding $x^\pm$-ordered diagrams, two of which are associated with the transverse propagating photon and one of which is associated with the instantaneous interaction. We analyzed both the frame and interpolation angle dependence of each $x^\pm$-ordered diagram including the instantaneous interaction, varying the total momentum $P^z$ of the system and the interpolation angle parameter $C = \cos 2\delta$. Our analysis provided a clear example of breaking the reflection symmetry under $P^z \leftrightarrow -P^z$ in IFD and clarified
the confusion in the folklore of the equivalence between the IMF and the LFD. As we vary
the interpolation angle from \( \delta = 0 \) (\( C = 1 \)) and approach to \( \delta = \pi/4 \) (\( C = 0 \)), this broken
reflection symmetry under \( P^z \leftrightarrow -P^z \) persists while the LFD results are symmetric under
\( P^z \leftrightarrow -P^z \) due to the longitudinal boost invariance. Moreover, the universal correlation
between \( P^z \) and \( C \) given by Eq. (4.97) which was coined as the J-curve in Ref. [JS12] is
intact in each of these \( x^\pm \)-ordered diagrams as plotted by the red solid line in Fig. 4.2.
The J-curve starts out from the center of mass frame in the instant form and goes to
\( P^z = -\infty \) as it approaches to the light-front limit. This correlation is independent of the
specific kinematics of the scattering process and manifests the difference in the results
between the light-front limit and the exact light-front (LFD). Since the J-curve exists only
for \( P^z < 0 \), it is also self-evident that the broken reflection symmetry under \( P^z \leftrightarrow -P^z \)
still persists for the correlated limit of \( P^z \sim -1/C^{1/2} \rightarrow -\infty \) and the results obtained
in this limit must be different from the LFD results. The J-curve not only provides a
representative characterization of how the \( x^\pm \)-ordered amplitudes change from IFD to
LFD, but also gives rise to the issue of zero-modes at \( P^z = -\infty \) since all the plus momenta
of particles vanish in this limit. The limit \( P^z \rightarrow -\infty \) is thus treacherous and requires
a great caution in taking the light-front limit from \( C \neq 0 \). The sum of all \( x^\pm \)-ordered
amplitudes is however completely independent of \( C \) and \( P^z \) and its full manifest Poincaré
invariance provides a useful guidance in handling the treacherous zero-mode issue.
CHAPTER 5

INTERPOLATING HELICITY SPINORS

To promote the interpolation from the sQED theory to the QED theory, we now start discussing the interpolation of the fermion spinors. Due to a few different representations available for the spinors, we need to examine the relations among the available spinor representations with the interpolation angle parameter $0 \leq \delta \leq \pi/4$ in this work. As the first step, we limit our discussion here only for the on-mass-shell spinors and analyze the interpolating helicity amplitudes for processes involving fermions as the external particles. We discuss a fermion and another fermion scattering process as well as a fermion and anti-fermion pair annihilation and creation process analogous to "$e\mu \rightarrow e\mu$" and "$e^+ e^- \rightarrow \mu^+ \mu^-$", respectively. In this chapter, we focus on the effects from the initial and final fermion degrees of freedom rather than from the intermediate gauge boson which we have studied in the previous chapter.

This chapter is organized as follows. In Sec. 5.1, we give a review of different types of spinors and different representations people often use, and clarify their relation with each other and our choice of conventions. In Sec. 5.2, we derive the generalized helicity operator for any interpolation angle and the corresponding $(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)$ helicity spinors. The spin orientations of these spinors are discussed in Sec. 5.3, where we consider the two subsequent operations $B(\eta)D(\hat{m}, \theta_s)$ on a spin-up spinor in the rest frame with the first operation $\mathcal{D}(\hat{m}, \theta_s) = e^{-i\hat{m} \cdot \mathbf{J} \theta_s}$ that rotates the spin around axis $\hat{m} = (-\sin \phi_s, \cos \phi_s, 0)$ by angle $\theta_s$ and the second operation $B(\eta) = e^{-i\eta \mathbf{K}}$ that boosts the spinor to momentum $\mathbf{P}$. In Sec. 5.3, we also derive the generalized Melosh transformation that relates these
generalized helicity spinors to the Dirac spinors. In Sec. 5.4, we calculate the generalized helicity probabilities for the fermion-fermion scattering process analogous to “$e\mu \rightarrow e\mu$” and plot their results in terms of both the total momentum $P^z$ of the system and the interpolation angle $\delta$ to reveal the entire landscape of the frame dependence and interpolation angle dependence of these probabilities. The summary follows in Sec. 5.5.

For the completeness and clarity, we added several Appendices. We exhibit the explicit matrix form of operators $T$ and $B(\eta)D(\tilde{m},\theta_s)$ in Appendix B.1, the $(0, \frac{1}{2}) \oplus (\frac{1}{2},0)$ helicity spinors for anti-particles in Appendix B.2, and the Dirac spinors in Appendix B.3. In Appendices B.4 and B.5, we list the $(0, J) \oplus (J, 0)$ helicity spinors and corresponding generalized Melosh transformations for higher spins up to $J = 2$. In Appendix B.6, the “apparent” spin orientation angle $\theta_a$ plotted in Fig. 5.5 is derived for the discussion presented in Sec. 5.3. The plots of the scattering amplitudes corresponding to the generalized helicity probabilities presented in Sec. 5.4 are shown in Appendix B.7. We also calculate and plot the generalized helicity amplitudes and probabilities for the fermion and anti-fermion pair annihilation and creation process analogous to “$e^+e^- \rightarrow \mu^+\mu^-$” in Appendix B.8.

5.1 Construction of Spinors

The Lorentz group is defined by the six generators of rotation ($J$) and boost ($K$) satisfying the commutation relations given in Eq. (2.33). These commutation rules can be decoupled by defining a pair of non-hermitian operators:

\[
A = \frac{1}{2}(J + iK),
\]

\[
B = \frac{1}{2}(J - iK).
\]

The commutation relations then become

\[
[A_i, A_j] = i\epsilon_{ijk}A_k,
\]

\[
[B_i, B_j] = i\epsilon_{ijk}B_k,
\]

\[
[A_i, B_j] = 0, (i, j, k = 1, 2, 3).
\]

$A$ and $B$ each generates a separate $SU(2)$ group and the Lorentz group is then essentially $SU(2) \otimes SU(2)$. The irreducible representation may then be labeled by two angular
momenta \((j, j')\), where \(j\) and \(j'\) denote the quantum numbers corresponding to each individual \(SU(2)\) subgroup consisted of \(A\) and \(B\) generators, respectively. In the case that one of the two angular momenta is spin zero, \((j, j')\) becomes

\[
(0, j) \rightarrow J = -iK \quad (A = 0),
\]

\[
(j, 0) \rightarrow J = iK \quad (B = 0),
\]

corresponding to the right-handed and left-handed components respectively. To represent the fermion state, we use the \((0, J) \oplus (J, 0)\) spinor.

The decoupling of the right-handed and left-handed components is most easily seen in the chiral representation, where

\[
J_C = \frac{1}{2} \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}, \quad K_C = \frac{i}{2} \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix},
\]

and \(\sigma\) denotes the Pauli matrices. It is clear that the combinations of rotation and boost given by \(J_C + iK_C\) and \(J_C - iK_C\) operate only on the corresponding block-diagonal \(2 \times 2\) matrices as their explicit representations are

\[
J_C + iK_C = \begin{pmatrix} 0 & 0 \\ 0 & \sigma \end{pmatrix},
\]

\[
J_C - iK_C = \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}.
\]

Due to such a transparent decoupling, we will write all of our spinors in this dissertation in the \((0, J) \oplus (J, 0)\) chiral representation of the Lorentz group.

For an overview of the commonly used spinor representations, we provide a schematic illustration as shown in Fig. 5.1 and discuss the relations among those representations. In Fig. 5.1, we denote the so-called “chiral representation” and “standard representation” as “C” and “S”. The standard representations can be found straightforwardly from the chiral representation using the transformation matrix \(S\):

\[
S = S^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ I & -I \end{pmatrix}.
\]
where $I$ is the $2 \times 2$ identity matrix. The standard presentation $(0, J) \oplus (J, 0)$ spinors can be obtained by

$$u_S = S u_C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \phi_R \\ \phi_L \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_R + \phi_L \\ \phi_R - \phi_L \end{pmatrix},$$

(5.11)

where $\phi_R$ and $\phi_L$ are the right-handed and left-handed components in the chiral representation. The Dirac matrices and the Lorentz group generators of rotation ($J$) and boost ($K$) in standard and chiral representations are related by $\gamma_\mu^S = S\gamma_\mu^CS^{-1}$, $J_S = SJS^{-1}$, $K_S = SKCS^{-1}$. Since one can easily get the corresponding standard representation using these

\[\text{Figure 5.1 (color online) An illustration of the relations between different conventions and names we use in this paper. The red letters C and S stand for chiral and standard representations respectively. The blue letters H and D stand for helicity spinor and Dirac spinor respectively. The solid black arrow points from the instant form to the light-front form, and the interpolation angle $\delta$ goes from 0 to $\pi/4$. Both the helicity and Dirac spinors can be generated for arbitrary interpolation angle. The dashed purple arrow indicates the Melosh transformation, which relates the instant Dirac spinor to the light-front helicity spinor.}\]
relations, we won’t list them explicitly in this dissertation and use exclusively the chiral representation from now on.

In Fig. 5.1, we also denote the so-called “helicity spinors” and “Dirac spinors” as “H” and “D”. They represent the spinors obtained by two different procedures.

As we briefly mentioned in Sec. 3.2, to define the helicity spinor in IFD, Jacob and Wick [JW59] started with a state at rest having a spin projection along the z direction equal to the desired helicity, then boosted it in the z direction to get the desired magnitude of momentum $|\vec{P}|$, and then rotated it subsequently to get the momentum and spin projection in the desired direction. This is given by the $T_h$ transformation in Eq. (3.17). We follow the same procedure in an arbitrary interpolation angle $\delta$, replacing the kinematic generators $J^1$ and $J^2$ in IFD by the corresponding kinematic generators $K^1$ and $K^2$ as we have done in Eq. (3.34). The “helicity spinors” denoted by “H” are thus obtained by applying this generalized transformation $T$ given by Eq. (3.34) to the initial state at rest that has a spin projection along the z direction.

On the other hand, the Dirac spinors in IFD are obtained through the canonical boost $T_c$ given by Eq. (3.16), applied to the initial state at rest that has a spin projection along the z direction. Because all the generators are dynamical, the spin direction will not be aligned with the momentum direction in general. Although one may still consider such interpolating “D” spinors for any interpolation $\delta$ by changing the dynamical generators $K^1$ and $K^2$ in IFD to the interpolating dynamical generators $D^1$ and $D^2$, the operation to get those interpolating “D” spinors is too dynamical to lend any useful discussion for the present work, and it doesn’t connect to the light-front spinors people use in the LFD. Thus, we will refer the “D” spinors in this dissertation to the familiar Dirac spinors in IFD only. We exhibit them for spin $J = \frac{1}{2}$ in Appendix B.3 using the $(0, J) \oplus (J, 0)$ chiral representation.

### 5.2 Helicity Operator and Helicity Spinor for Any Interpolation Angle

As introduced in Sec. 5.1, the helicity spinors denoted by “H” are obtained by applying the transformation $T$ given by Eq. (3.34) to the initial state at rest that has a spin projection along the z direction. We may denote a generalized helicity spinor in a given interpolation angle $\delta$ as $|p; j, m\rangle_\delta$ for a particle of spin $j$ moving with momentum $p$ and helicity $m$. This
state $|p; j, m\rangle_\delta$ is obtained by the transformation $T$ from the spin eigenstate $|0; j, m\rangle$ at rest, which has a spin projection along the $z$ direction satisfying $J_3|0; j, m\rangle = m|0; j, m\rangle$. Thus, we may specify $|p; j, m\rangle_\delta = T|0; j, m\rangle$.

Following the procedure of Leutwyler and Stern [LS78], we may then define a new spin operator $J_i$ for a moving particle as $J_i = TJ_iT^{-1}$ to get

$$J_3|p; j, m\rangle_\delta = TJ_3^{-1}T|0; j, m\rangle = m|p; j, m\rangle_\delta,$$

(5.12)

where $m$ is now not only the eigenvalue of the ordinary spin operator $J_3$ for the initial state at rest $|0; j, m\rangle$ but also the eigenvalue of the operator $J_3$ for the generalized helicity spinor state $|p; j, m\rangle_\delta$. It is straightforward to verify that $J_i$ satisfies the $SU(2)$ algebra as $J_i$ does:

$$[J_i, J_j] = TJ_iJ_jT^{-1} - TJ_jJ_iT^{-1} = i\epsilon_{ijk}TJ_kT^{-1} = i\epsilon_{ijk}J_k.$$

(5.13)

As shown in Ref. [JM01], the new spin operator $J_i$ commutes with the mass operator $M$ defined by $M^2 = \hat{P}^\mu\hat{P}_\mu = P_+^2C - P_-^2C + 2P_+P_-S - \hat{P}_\perp^2$ for any generalized helicity spinor state, i.e. $[J_i, M]|p; j, m\rangle_\delta = 0$. The operator $J_3$ intermediates between the usual Jacob-Wick helicity operator in IFD and the light-front helicity operator in LFD and thus offers the role of general helicity operator in-between for any interpolation angle $\delta$ as we discuss below.

Using the Poincaré algebra for any arbitrary interpolation angle, we find that the new spin operator written in terms of the parameters $\beta_1$, $\beta_2$ and $\beta_3$ remains unchanged with or without including $T_3$ [JM01] as $[J^3, K^3] = 0$ and is given by

$$J_3 = J_3 \cos \alpha + (\beta_1\mathcal{K}^2 - \beta_2\mathcal{K}^1)\frac{\sin \alpha}{\alpha}.\quad\quad (5.14)$$

We can then use the relations in Eqs. (3.47) - (3.51) to rewrite it in terms of the particle’s momentum, and get

$$J_3 = \frac{1}{\hat{P}}(P_-J_3 + P_+\mathcal{K}^2 - P^2\mathcal{K}^1),\quad\quad (5.15)$$

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where $\mathbb{P} \equiv \sqrt{(P^\perp)^2 - M^2C} = \sqrt{P^2 + P_\perp^2C}$. It is interesting to note that this operator $J_3$ can also be written in terms of the Pauli-Lubanski operator $[LS78]$ $W^\mu = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} P_\nu M_{\alpha\beta}$ simply as $J_3 = W^\perp / \mathbb{P}$. In the instant form limit ($\delta \to 0$), $K^1 \to -\mathbb{J}^2$, $K^2 \to J^1$, $P_- \to P^3$ and $\mathbb{P} \to \sqrt{(P^0)^2 - M^2} = |P|$, and thus the operator $J_3$ coincides with the familiar IFD helicity operator $\mathbb{P} \cdot J / |\mathbb{P}|$. In the light-front limit ($\delta \to \pi/4$), $K^1 \to -E_1$, $K^2 \to -E_2$, $P_- \to P^+$ and $\mathbb{P} \to \sqrt{(P^+)^2} = P^+$, and thus the operator $J_3$ coincides with the light-front helicity operator $J_3 + \frac{1}{P^+} (P^2 E_1 - P^1 E_2)$ as discussed in Ref. [CJ03]. Thus, $J_3$ intermediates between the usual Jacob-Wick helicity operator in IFD and the light-front helicity operator in LFD and it is reasonable to identify the operator $J_3$ as the general helicity operator for any interpolation angle $\delta$. The helicity eigenvalue for the state $|p; j, m\rangle_\delta$ is $m$ as previously given in Eq. (5.12). One should note, however, that the generalized helicity defined by the operator $J_3$ agrees with the ordinary notion of helicity defined usually by the spin parallel or anti-parallel to the particle momentum direction only in the IFD. For different interpolation angles, in general, there’s a relative angle between the spin orientation and the momentum direction according to the generalized helicity designated by $J_3$. In LFD, the transverse light-front boost operators $E_1$ and $E_2$ involve the rotations and they generate the angle between the spin orientation and the momentum direction.

If the particle is moving in $+z$ or $-z$ direction, so that $P^1 = P^2 = 0$, then the generalized helicity operator given by Eq. (5.15) becomes

$$J_3 = \frac{P_- J_3}{\mathbb{P}} = \frac{P_-}{|P_-|} J_3.$$  (5.16)

Thus, for an arbitrary interpolation angle, the helicity sign of a particle moving in $\pm z$ direction depends on the sign of $P_-$. In the light-front limit, $P_- \to P^+$ which is always positive, and thus the light-front helicity of the particle is positive once the spin is parallel to the $+z$ direction regardless whether the particle is moving in the $+z$ direction or the $-z$ direction. This is dramatically different from the ordinary helicity defined in the IFD where $P_- \to P^3$. For a particle moving in the $-z$ direction, the light-front helicity and the ordinary Jacob-Wick helicity is therefore opposite to each other. The swap of helicity amplitudes caused by such dramatic difference between the light-front helicity and the ordinary Jacob-Wick helicity has been noticed previously in deeply virtual Compton scattering process [JB13]. We will discuss the sign flip effect of $P_-$ factor in Eq. (5.16) for the more general directions of the particle spin and momentum in the next section.
As mentioned in Sec. 5.1, we write all of our spinors in the \((0, J) \oplus (J, 0)\) chiral representation of the Lorentz group due to a clear decoupling between the right-handed and left-handed components in the chiral representation. In this representation, the transformation \(T\) given by Eq. (3.34) is a block diagonal matrix given by

\[
T = \begin{pmatrix} T_R & 0 \\ 0 & T_L \end{pmatrix}.
\]

(5.17)

For spin 1/2 fermions in the chiral representation \((0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)\) of the Lorentz group, the right-handed part \(T_R\) corresponds to the \((0, \frac{1}{2})\) representation \((A = 0)\): \(K = iJ = i\sigma/2\). Plugging \(K\) and \(J\) into Eq. (3.34), we get

\[
T_R = e^{b_\perp \cdot \sigma_\perp/2} e^{\beta_3\sigma_3/2},
\]

(5.18)

where \(b_\perp = (\beta_1 \sin \delta + i \beta_2 \cos \delta, -i \beta_1 \cos \delta + \beta_2 \sin \delta)\). This transforms the right-handed component of the spinor. Using the relation \((b_\perp \cdot \sigma_\perp)^2 = -\alpha^2 \cdot I\) and \(\sigma_3^2 = I\) where we recall \(\alpha = \sqrt{C (\beta_1^2 + \beta_2^2)}\), we obtain

\[
T_{12R} = e^{b_\perp \cdot \sigma_\perp/2} = \cos \left(\frac{\alpha}{2}\right) \cdot I + \frac{b_\perp \cdot \sigma_\perp}{\alpha} \sin \left(\frac{\alpha}{2}\right),
\]

(5.19)

\[
T_{3R} = e^{\beta_3\sigma_3/2} = \begin{pmatrix} e^{\beta_3/2} & 0 \\ 0 & e^{-\beta_3/2} \end{pmatrix}.
\]

(5.20)

Similarly, the left-handed part \(T_L\) corresponds to the \((\frac{1}{2}, 0)\) representation \((B = 0)\): \(K = -iJ = -i\sigma/2\). Thus, we get

\[
T_L = e^{-b_\perp \cdot \sigma_\perp/2} e^{-\beta_3\sigma_3/2},
\]

(5.21)

where \(b_\perp^* = (\beta_1 \sin \delta - i \beta_2 \cos \delta, i \beta_1 \cos \delta + \beta_2 \sin \delta)\). With this, we obtain

\[
T_{12L} = e^{-b_\perp^* \cdot \sigma_\perp/2} = \cos \left(\frac{\alpha}{2}\right) \cdot I - \frac{b_\perp^* \cdot \sigma_\perp}{\alpha} \sin \left(\frac{\alpha}{2}\right),
\]

(5.22)

\[
T_{3L} = e^{-\beta_3\sigma_3/2} = \begin{pmatrix} e^{-\beta_3/2} & 0 \\ 0 & e^{\beta_3/2} \end{pmatrix}.
\]

(5.23)
The explicit matrix form of $T$ can be found in Appendix B.1.

To obtain the helicity spinor with arbitrary momentum, we apply this $T$ transformation on a spin eigenstate in its rest frame. The definition for helicity operator $J_3$ in Eq. (5.12) assures that if $|0; j, m\rangle$ is an eigenstate of $J_3$, then $T|0; j, m\rangle$ is an eigenstate of $J_3$. Because here we use the chiral representation, the spinors in the rest frame are given by

$$u^{(1/2)}(0) = \begin{pmatrix} \sqrt{M} \\ 0 \\ \sqrt{M} \\ 0 \end{pmatrix}, \quad u^{(-1/2)}(0) = \begin{pmatrix} 0 \\ \sqrt{M} \\ 0 \\ \sqrt{M} \end{pmatrix},$$

(5.24)

with the normalization $\bar{u}^{(\lambda)} u^{(\lambda)} = 2M$. After the $T$ transformation given by Eq. (5.17), we obtain

$$u^{(1/2)}_H(\beta) = \sqrt{M} \begin{pmatrix} \cos \frac{\alpha}{2} e^{\beta_3/2} \\ \frac{\beta_R (\sin \delta + \cos \delta)}{\alpha} \sin \frac{\alpha}{2} e^{\beta_3/2} \\ \cos \frac{\alpha}{2} e^{-\beta_3/2} \\ \frac{\beta_R (\cos \delta - \sin \delta)}{\alpha} \sin \frac{\alpha}{2} e^{-\beta_3/2} \end{pmatrix},$$

(5.25)

$$u^{(-1/2)}_H(\beta) = \sqrt{M} \begin{pmatrix} -\frac{\beta_L (\cos \delta - \sin \delta)}{\alpha} \sin \frac{\alpha}{2} e^{-\beta_3/2} \\ \cos \frac{\alpha}{2} e^{-\beta_3/2} \\ -\frac{\beta_L (\sin \delta + \cos \delta)}{\alpha} \sin \frac{\alpha}{2} e^{\beta_3/2} \\ \cos \frac{\alpha}{2} e^{\beta_3/2} \end{pmatrix},$$

(5.26)

where $\beta_R = \beta_1 + i\beta_2$ and $\beta_L = \beta_1 - i\beta_2$. The subscript “H” is used to denote that these are the generalized helicity spinors as mentioned in Sec. 5.1 to distinguish them from the Dirac spinors that we will discuss in the next section, Sec. 5.3. With the relations listed in
Eqs. (3.47) - (3.51), these spinors can be rewritten in terms of the particle’s momentum:

\[
\begin{align*}
  u^{(1/2)}_H(P) &= \left( \begin{array}{c}
    \sqrt{\frac{P_+ + P}{2P}} \sqrt{\frac{P^+ + P}{(\sin \delta + \cos \delta)}} \\
    PR \frac{\sin \delta + \cos \delta}{2P(P + P_\perp)} \sqrt{P^+ + P} \\
    \sqrt{\frac{P_\perp + P}{2P}} \sqrt{\frac{P^+ - P}{(\cos \delta - \sin \delta)}} \\
    PR \frac{\cos \delta - \sin \delta}{2P(P + P_\perp)} \sqrt{P^+ - P}
  \end{array} \right) \\
  u^{(-1/2)}_H(P) &= \left( \begin{array}{c}
    -P^L \frac{\cos \delta - \sin \delta}{2P(P + P_\perp)} \sqrt{P^+ - P} \\
    \sqrt{\frac{P_\perp + P}{2P}} \sqrt{\frac{P^+ - P}{(\cos \delta - \sin \delta)}} \\
    -P^L \frac{\sin \delta + \cos \delta}{2P(P + P_\perp)} \sqrt{P^+ + P} \\
    \sqrt{\frac{P_\perp + P}{2P}} \sqrt{\frac{P^+ + P}{(\sin \delta + \cos \delta)}}
  \end{array} \right),
\end{align*}
\]

where \( PR = P^1 + iP^2 \) and \( PL = P^1 - iP^2 \). From \( P^2 = (P^\perp)^2 - M^2C = P_\perp^2 + P_\perp^2C \), we get

\[
\sqrt{\frac{(P - P_\perp)}{(P_\perp^2C)}} = 1/\sqrt{P + P_\perp}
\]

and use this equation, i.e. Eq. (5.29), to obtain the above results given by Eqs. (5.27) and (5.28). One should note that there are two equivalent ways of writing the same factor as given by Eq. (5.29), i.e. the left hand side or the right hand side of Eq. (5.29). The form in the right hand side is convenient for obtaining the light-front limit of these spinors since both \( P \to P^+ \) and \( P_\perp \to P^+ \) are positive and they cannot cancel each other. On the other hand, \( P_\perp \) can be negative for other interpolation angle \( \delta \neq \pi/4 \) and may cancel
\( \mathbb{P} \) to end up with a treacherous singularity in the form of the right hand side. In such case, it would be more convenient to use the form in the left hand side as \( C \neq 0 \) and instead take care of the limit \( \mathbf{P}_\perp \to 0 \) noting that the factor \( 1/|\mathbf{P}_\perp| \) comes with \( P_R \) or \( P_L \) in Eqs. (5.27) and (5.28) and the factor \( P_R/|\mathbf{P}_\perp| \) or \( P_L/|\mathbf{P}_\perp| \) may be taken to be the unity by choosing an appropriate coordinate system.

The antiparticle spinors are generated through charge conjugation as \( \nu = C\bar{u}^T = i\gamma^0\gamma^2\gamma^0 u^* \). Here, the gamma matrices are written on the chiral basis as (see e.g. Ref. [Dzi88]):

\[
\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.
\]

(5.30)

The results of antiparticle spinors are summarized in Appendix (B.2).

In the instant form limit \( (\delta \to 0) \), \( C \to 1 \), \( \mathbf{P}^\perp \to \mathbf{P}^0 = E \), \( P_{\perp} \to P^3 \) and \( \mathbb{P} \to |\mathbf{P}| \).

Using the relation

\[
\sqrt{(E \pm |\mathbf{P}|)} = (E + M \pm |\mathbf{P}|)/\sqrt{2(E + M)},
\]

(5.31)

one can show that spinors in Eqs. (5.27) and (5.28) reduce to their instant form in chiral representation as follows:

\[
u_H^{(1/2)} = \sqrt{\frac{E + M}{2}} \left( \begin{array}{c}
\frac{E + M + |\mathbf{P}|}{E + M} \cos \frac{\theta}{2} \\
\frac{E + M + |\mathbf{P}|}{E + M} \sin \frac{\theta}{2} e^{i\phi} \\
\frac{E + M - |\mathbf{P}|}{E + M} \cos \frac{\theta}{2} \\
\frac{E + M - |\mathbf{P}|}{E + M} \sin \frac{\theta}{2} e^{i\phi}
\end{array} \right),
\]

(5.32)

\[
u_H^{(-1/2)} = \sqrt{\frac{E + M}{2}} \left( \begin{array}{c}
-\frac{E + M - |\mathbf{P}|}{E + M} \sin \frac{\theta}{2} e^{-i\phi} \\
\frac{E + M - |\mathbf{P}|}{E + M} \frac{\theta}{2} \\
\frac{E + M + |\mathbf{P}|}{E + M} \cos \frac{\theta}{2} \\
-\frac{E + M + |\mathbf{P}|}{E + M} \sin \frac{\theta}{2} e^{-i\phi}
\end{array} \right),
\]

(5.33)
where \( \theta \) and \( \phi \) specify the direction of the momentum \( \vec{P} \):

\[
\sin \theta \cos \phi = \frac{P^1}{|\vec{P}|}, \quad \sin \theta \sin \phi = \frac{P^2}{|\vec{P}|}, \quad \cos \theta = \frac{P^3}{|\vec{P}|},
\]

i.e.

\[
\cos(\theta/2) = \sqrt{(|\vec{P}| + P^3)/(2|\vec{P}|)}, \quad (5.34)
\]
\[
\sin(\theta/2) = \sqrt{(|\vec{P}| - P^3)/(2|\vec{P}|)}, \quad (5.35)
\]
\[
e^{i\phi} = \frac{P_R}{|\vec{P}_\perp|}, \quad (5.36)
\]
\[
e^{-i\phi} = \frac{P_L}{|\vec{P}_\perp|}. \quad (5.37)
\]

In the light-front limit (\( \delta \to \pi/4 \)), \( C \to 0 \), \( P^+ \to P^+ \), \( P^- \to P^+ \), \( P^0 \to P^+ \), and \( (P^+ - \vec{P})/C \xrightarrow{\delta \to 0} M^2/(2P^+) \). One can show that the spinors in Eqs. (5.27) and (5.28) become the light-front spinors in the chiral representation [Dzi88], or identically the Kogut-Soper spinors [Bro98b]:

\[
u^{(1/2)}_H = \frac{1}{\sqrt{2P^+}} \begin{pmatrix} \sqrt{2P^+} \\ P_R \\ M \\ 0 \end{pmatrix}, \quad \nu^{(-1/2)}_H = \frac{1}{\sqrt{2P^+}} \begin{pmatrix} 0 \\ M \\ -P_L \\ \sqrt{2P^+} \end{pmatrix}. \quad (5.38)
\]

The \( \sqrt{2} \) factors in front of \( P^+ \) in Eq. (5.38) are due to our definition of \( P^+ \) with the factor \( 1/\sqrt{2} \), i.e. \( P^+ = (P^0 + P^3)/\sqrt{2} \).

It is straightforward to extend our discussion for spin 1/2 fermions and derive the generalized helicity spinors for any spin \( J \). The only thing one needs to do is change the \( K \) and \( J \) matrices as well as the spinors in Eq. (5.24) from spin 1/2 representation to spin \( J \) representation. The results of the generalized helicity spinors for spin up to \( J = 2 \) are summarized in Appendix B.4.

### 5.3 Spin Orientation and Generalized Melosh Transformation

#### 5.3.1 Spin Orientation for Generalized Helicity Spinors

A spinor for a particle carries two pieces of information: the momentum of the particle and its spin orientation. We may denote the momentum as \( \vec{P} \), and the direction of momentum
$P$ as $(\theta, \phi)$, where $\theta$ is the angle between the momentum direction and the $z$ axis, and $\phi$ is the azimuthal angle. Likewise, we may denote spin as $S$, and the direction of spin $S$ as $(\theta_s, \phi_s)$. Since the direction of $S$ depends on the reference frame in general, we take $(\theta_s, \phi_s)$ in the rest frame for the discussion of this section.

We consider the following transformation on a spin-up spinor in the rest frame:

$$B(\eta)\mathcal{D}(\hat{m},\theta_s) = e^{-i\eta \cdot \vec{K}} e^{-i\hat{m} \cdot J \theta_s},$$

where the first operation $\mathcal{D}(\hat{m},\theta_s) = e^{-i\hat{m} \cdot J \theta_s}$ rotates the spin around the axis given by a unit vector $\hat{m} = (-\sin \phi_s, \cos \phi_s, 0)$ by angle $\theta_s$ and then the second operation $B(\eta) = e^{-i\eta \cdot \vec{K}}$ boosts the spinor to momentum $P$. By combining these two operations, all possible spinors can be reached. Therefore, we should be able to rewrite the $T$ transformation that we defined in Eq. (3.34) as a combination of a rotation and a boost with a suitable set of $(\theta_s, \phi_s)$:

$$T = B(\eta)\mathcal{D}(\hat{m},\theta_s).$$

In the $(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)$ chiral representation, we have

$$\mathcal{D}(\hat{m},\theta_s) = e^{-i \hat{m} \cdot J \theta_s} = \begin{pmatrix} e^{-i \hat{m} \cdot \sigma \theta_s/2} & 0 \\ 0 & e^{-i \hat{m} \cdot \sigma \theta_s/2} \end{pmatrix},$$

$$B(\eta) = e^{-i \eta \cdot \vec{K}} = \begin{pmatrix} e^{\eta \cdot \sigma/2} & 0 \\ 0 & e^{-\eta \cdot \sigma/2} \end{pmatrix},$$

with

$$e^{-i \hat{m} \cdot \sigma \theta_s/2} = \mathbf{I} \cos \frac{\theta_s}{2} - i \hat{\mathbf{m}} \cdot \sigma \sin \frac{\theta_s}{2},$$

$$e^{\pm \sigma \cdot \eta/2} = \mathbf{I} \cosh \frac{\eta}{2} \pm \hat{\mathbf{n}} \cdot \sigma \sinh \frac{\eta}{2}.$$
holds, provided that the angles $(\theta_s, \phi_s)$ are given by

$$
\cos \frac{\theta_s}{2} = \sqrt{\frac{2M}{E + M}} \cos \frac{\alpha}{2} \cosh \frac{\beta_3}{2} = \frac{1}{2} \sqrt{\frac{2M}{E + M}} \left( \sqrt{\frac{P^+ + \mathbb{P}}{2\mathbb{P}}} + \sqrt{\frac{M(\sin \delta + \cos \delta)}{P^+ + \mathbb{P}}} \right), \quad (5.45)
$$

where we used $\sinh(\eta/2) = |\mathbb{P}|/\sqrt{2M(E + M)}$ and $\cosh(\eta/2) = (E + M)/\sqrt{2M(E + M)}$.

Because our generalized helicity spinors are obtained by applying the $T$ transformation on a rest spinor, and now we know that the $T$ transformation is the same as first a rotation around the axis $\hat{m}$ and then a boost, we conclude that for the positive helicity spinors the spin points in the $(\theta_s, \phi_s)$ direction, and for the negative helicity spinors, the spin will point in the exact opposite direction, namely $(\pi - \theta_s, \pi + \phi_s)$. Since Eq. (5.40) is an operator equation, which doesn’t depend on the total spin of the system, this expression should also hold for higher spins although we obtained it using the $J = 1/2$ representation.

We notice that the spin $S$, momentum $P$ and $z$ axis are all in the same plane due to the fact that $\phi_s$ is the same as $\phi$ as shown in Eq. (5.46). Therefore, for the following discussions, we may take $\phi = \phi_s = 0$ without loss of any generality and focus on the relation between $\theta$ and $\theta_s$.

First of all, in the instant form limit ($\delta \to 0$), $P^\pm \to P^0 = E$, $P_\pm \to P^3$ and $\mathbb{P} \to |\mathbb{P}|$, and we get

$$
\cos(\theta_s/2) = \sqrt{(|\mathbb{P}| + P^3)/(2|\mathbb{P}|)} = \cos(\theta/2).
$$

So $\theta = \theta_s$ when $\delta = 0$, and the spin and the momentum directions are aligned, as expected from the usual Jacob and Wick helicity [JW59] defined in the IFD. However, as $\delta$ gets different from zero, the two angles $\theta$ and $\theta_s$ differ from each other. To give an illustration of the angles, we plot in Fig. 5.2 the spin orientation (red solid vector) and the momentum direction (black dashed vector) of the positive helicity spinor $u_H^{(1/2)}$ for a particle with mass $M = 1$ GeV and total momentum $|\mathbb{P}| = 1$ GeV when $\theta = \pi/3$ and $\delta = \pi/6$. It’s clear from Eq. (5.45) that the spin orientation depends on both the particle’s momentum and the interpolation angle. In particular, using Eq. (5.45), one can analyze how the angle difference between the momentum direction and the spin direction, i.e. $\theta - \theta_s$, changes as
the momentum direction $\theta$ changes for each different interpolation angle $\delta$.

To show how the spin orientation changes with the interpolation angle and the

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure_5.2}
\caption{(color online) At interpolation angle $\delta = \pi/6$, a particle’s momentum direction at $\theta = \pi/3$ (dashed black arrow) and the corresponding spin orientation (solid red arrow) of the positive helicity spinor. We take as an example a particle with mass $M = 1$ GeV and total momentum $|\mathbf{P}| = 1$ GeV.}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure_5.3}
\caption{(color online) The dependence of $\theta_s$ of a positive helicity spinor on the $\theta$ angle and the interpolation angle $\delta$. Again we take a particle with mass $M = 1$ GeV and total momentum $|\mathbf{P}| = 1$ GeV as our illustrative example.}
\end{figure}
momentum direction, we plot $\theta_s$ of a positive helicity spinor $u_H^{(1/2)}$ given by Eq. (5.45) in terms of both $\delta$ and $\theta$ in Fig. 5.3. Here again, we take the particle mass $M = 1$ GeV and total momentum $|P| = 1$ GeV for illustration. For $\delta = 0$, the equality $\theta = \theta_s$ from Eq. (5.47) yields the straight line relation at $\delta = 0$ as shown in Fig. 5.3. As we vary the interpolation angle between $\delta = 0$ and $\delta = \pi/4$, we see that for a fixed $\theta$, i.e. fixed momentum direction, $\theta_s$ decreases as $\delta$ increases in general, and correspondingly the angle between the spin and the momentum will increase with $\delta$. In fact, the angle between the momentum and the spin directions, i.e. $\theta - \theta_s$, increases with the interpolation angle $\delta$ and becomes the largest at the light front. This feature is shown in Fig. 5.4 as what’s shown in Fig. 5.3 is now plotted in terms of the angle difference $\theta - \theta_s$ between the momentum and the spin directions for various interpolation angles listed in the legend. Note here that the increment of the angle difference $\theta - \theta_s$ with the increment of the interpolation angle $\delta$ bifurcates at a critical interpolation angle $\delta_c$. This bifurcation of two branches shown in Fig. 5.4 is also indicated by the transparent vertical plane in Fig. 5.3. At a fixed

![Figure 5.4](image_url)

Figure 5.4 (color online) The angle between the momentum direction and the spin direction, i.e. $\theta - \theta_s$, as a function of $\theta$ with the variation of the interpolation angle $\delta$. The bifurcation point appears at the critical interpolation angle $\delta_c = \arctan \left( \frac{|P|}{E} \right) \approx 0.61548$ as discussed in the text.
interpolation angle $\delta < \delta_c$, the spin orientation $\theta_s$ increases with $\theta$, and when $\theta = \pi$, we also have $\theta_s = \pi$. On the other hand, at a fixed interpolation angle $\delta > \delta_c$, $\theta_s$ doesn’t follow $\theta$ all the way to $\pi$, but instead starts to decrease beyond a certain point and goes back to 0 when $\theta = \pi$. This phenomenon is due to the change of sign for $P_\perp$ in Eq. (5.16) as we discussed in Sec. 5.2, and the critical interpolation angle $\delta_c$ is thus given by $P_{\perp} = 0$ when $\theta = \pi$:

$$\delta_c = \arctan \left( \frac{|P|}{E} \right).$$

The critical value for the case $M = |P| = 1$ GeV shown in Figs. 5.3 and 5.4 is $\delta_c \approx 0.61548$. If we change the particle’s mass and total momentum, the position of this critical plane in Fig. 5.3 will change accordingly. The swap of helicity amplitudes [JB13] mentioned in the previous section, Sec. 5.2, is directly linked to the fact that the IFD and the LFD separately belong to the two different branches bifurcated and divided out at the critical interpolation angle $\delta_c$. This bifurcation indicates the necessity of the distinction in the spin orientation between the IFD and the LFD and clarifies further the characteristic of helicity amplitudes in the two distinguished forms of the relativistic dynamics. In particular, the discussion of the spin orientation and the helicity amplitudes clarifies any conceivable confusion in the prevailing notion of the equivalence between the IMF formulated in IFD and the LFD. We thus discuss below the distinguished features of the light-front helicity.

As we get to the light-front limit ($\delta \to \pi/4$), $P^\perp \to P^+$, $P_\perp \to P^+$, $P^\perp \to P^+$, Eq. (5.45) becomes

$$\cos \frac{\theta_s}{2} = \frac{\sqrt{2} P^+ + M}{\sqrt{2} \sqrt{2} P^+ (E + M)}.$$  

For a particle that is moving in the $\pm z$ direction, the numerator becomes $E \pm |P| + M$ and the denominator becomes $\sqrt{2(E \pm |P|)(E + M)}$. Using Eq. (5.31), we get $\cos(\theta_s/2) \to 1$. Therefore in the light-front limit, a positive helicity spinor, regardless of whether the particle is moving in the $+z$ or $-z$ direction, always has its spin pointed in the $+z$ direction. This produces the feature of $\theta_s = 0$ at both $\theta = 0$ and $\theta = \pi$ as shown in Fig. 5.3 as well as in Fig. 5.4. By the same token, a negative helicity spinor moving in $\pm z$ direction always has its spin pointed to the $-z$ direction in LFD, even when the particle is moving in the $-z$ direction. This already reveals a distinct feature of the light-front helicity and provides the rationale for the swap of helicity amplitudes discussed in deeply virtual Compton scattering process [JB13]. Because the light-front helicity is obtained
in general by the two kinds of boost operations, i.e. the light-front transverse boost $E_\perp$ and the longitudinal boost $K^3$, as one can get $T = e^{-i\beta_\perp \cdot E_\perp} e^{-i\beta_3 K^3}$ from Eq. (3.34) in the limit $\delta \to \pi/4$, we now discuss the salient features of the spin orientations due to each of these two kinds of light-front kinematic operations separately.

5.3.1.1 Light-front Transverse Boost $E_\perp$ Operation

Without loss of any generality, we may take the coordinate system where everything lies in the $x - z$ plane and consider just the transverse boost $E^1 = (K^1 + J^2)/\sqrt{2}$ to discuss the effect of the light-front boost operation on the spin orientation. Suppose that the particle at rest with the spin up in the positive $z$ direction is boosted by $E_1$ to gain the particle energy $E = \sqrt{P^2 + M^2}$. As $E^1$ consists of not only the boost in the $x$ direction, $K^1$, but also the rotation around the $y$ axis, $J^2$, the particle gains the momentum in the $-z$ direction as well as in the $x$ direction. Interestingly, the momentum gained in the $-z$ direction takes the non-relativistic form $P^z = -\vec{P}_\perp^2 2M$ (here of course $\vec{P}_\perp = P^x \hat{x}$) in such a way that the energy of the particle becomes $E = P^0 = M + \vec{P}_\perp^2 2M$ while the light-front plus momentum $P^+ = (P^0 + P^z)/\sqrt{2}$ gets invariant [JS13]. Of course, the relativistic energy-momentum dispersion relation still holds as one may expect:

$$ (P^0)^2 - \vec{P}^2 = \left( M + \frac{\vec{P}_\perp^2}{2M} \right)^2 - \vec{P}_\perp^2 - \left( \frac{-\vec{P}_\perp^2}{2M} \right)^2 = M^2. \quad (5.50) $$

In terms of the energy $E$ and the particle mass $M$, one may express the particle momentum components as $P^x = \sqrt{2M(E - M)}$ and $P^z = -\frac{(P^x)^2}{2M} = M - E$ as well as $\sqrt{2}P^+ = M$. Thus, the resulted momentum direction is given by $\cos \theta_0 = \frac{P^z}{|P|} = \frac{M - E}{\sqrt{E^2 - M^2}} = -\sqrt{\frac{E-M}{E+M}}$. Since the 4-vector of the particle spin $S^\mu$ is transverse to the 4-momentum $P^\mu$, i.e. $S \cdot P = 0$, while its normalization is fixed by $S^2 = -1$, one can find the specific direction of the spin that the particle takes under the $E^1$ operation. Thus, we consider here effectively the $E^1$ operation that results in the final state with the four-momentum $P^\mu$ and the spin 4-vector $S^\mu$ starting from the initial state at rest with the spin up in the $\hat{z}$ direction. As the $E^1$ operation is the only source of the spin rotation here, the rotated spin angle gets maximum only when the given energy $E = \sqrt{\vec{P}^2 + M^2}$ is attained entirely by the $E^1$ operation. If the particle is tempered with the $K^3$ operation prior to $E^1$ operation, then the amount of the $E^1$ operation required to achieve the same energy $E = \sqrt{\vec{P}^2 + M^2}$
would be reduced. We may discuss the “apparent” orientation of the spin in the moving frame by analyzing $S^\mu$ directly as we describe in Appendix B.6, where we define the “apparent” spin orientation angle $\theta_a$ and derive the relationship between $\theta_a$ and $\theta$. Using Eq. (5.39), however, one may also take a look at the spin orientation in the particle’s rest frame, i.e. the rotated $\theta_s$ angle prior to the boost operation $B(\eta) = e^{-i\eta \cdot K}$ with which the final 4-momentum $P^\mu$ and the spin 4-vector $S^\mu$ are obtained. The maximum “apparent” spin orientation in the moving frame, i.e. the final state, must correspond to the maximum initial spin orientation $(\theta_s)_{\text{max}}$ since the boost $B(\eta) = e^{-i\eta \cdot K}$ tilts the initial spin orientation toward the direction of the rapidity $\eta$ or the final momentum $P$. Therefore, for a certain total momentum $|P|$ or energy $E$, $(\theta_s)_{\text{max}}$ should also be obtained at $\theta = \theta_0$ that was attained entirely by the $E1$ operation.

![Figure 5.5](color online) The light-front profiles of $\theta_s$ vs $\theta$ for a particle with mass $M = 1$ GeV and the magnitude of the momentum $|P| = 1, 1.2, 2, 4, 100$ GeV. The thick gray dashed line indicates the position of $\theta_0$ which provides the maximum $\theta_s$. As a comparison, the magenta dot-dashed line indicates how the “apparent” spin orientation angle $\theta_a$ changes with the momentum direction. The gray circle is marked for the intersecting point between the line of $|P| = 1$ GeV (blue solid line) and the thick gray dashed line. The blue circle corresponds to the case when the spinor doesn’t have any momentum in $z$ direction but moves perpendicular to the $z$ direction, e.g. $x$ direction.
This initial spin orientation $\theta_s$ given by Eq. (5.49) can be written as

$$\frac{\theta_s}{2} = \frac{\pi}{2} - \arctan \left( \frac{\sqrt{2P^+ + M}}{P^x} \right),$$

(5.51)

since $\cot \frac{\theta_s}{2} = \left( \frac{\sqrt{2P^+ + M}}{P^x} \right)$ or $\tan \frac{\theta_s}{2} = \left( \frac{P^x}{\sqrt{2P^+ + M}} \right)$. One may obtain $(\theta_s)_{\text{max}}$ from Eq. (5.51) by taking $P^x = |P| \sin \theta$ and $\sqrt{2P^+} = \sqrt{|P|^2 + M^2 + |P| \cos \theta}$ and differentiating $\theta_s$ with respect to $\theta$ to get zero. As expected, we find the angle $\theta_0$ precisely given by $\cos \theta_0 = \left( \frac{P^x}{E + M} \right) = -\sqrt{\frac{E-M}{E+M}}$ with the corresponding $(\theta_s)_{\text{max}} = \pi - 2 \arctan \left( \sqrt{\frac{2M}{E-M}} \right)$ or equivalently $(\theta_s)_{\text{max}} = 2 \arctan \sqrt{\frac{E-M}{2M}}$. The negative sign here indicates that the $z$ component of the particle momentum must be negative. As $|P|$ (or $E$) gets larger, $\theta_0$ gets closer to $\pi$. This behavior is shown in Fig. 5.5. We choose $M = 1$ GeV for an illustration and first plot $\theta_s$ given by Eq. (5.51) as a function of $\theta$ for fixed values of $|P|$, e.g. $|P| = 1, 1.2, 2, 4, 100$ GeV, as indicated in the legend of Fig. 5.5. The values of $(\theta_s)_{\text{max}}$ is traced out with the thick gray dashed line along the values of $\theta_0$ as $|P|$ varies in Fig. 5.5. This thick gray dashed line is obtained entirely by the $E^1$ operation on the initial rest particle and thus the value of $\sqrt{2P^+}$ is identical to the particle mass $M$ throughout this line. As expected, this line approaches to $\theta = \pi/2$ when $|P|$ gets close to zero. We also plot the “apparent” spin orientation angle $\theta_a$ defined in Appendix B.6 as the magenta dot-dashed line, overlaying with the thick gray dashed line of $(\theta_s)_{\text{max}}$.

As an example of the $(\theta_s)_{\text{max}}$ state lying on the thick gray dashed line, a gray circle is marked for the intersecting point between the line of $|P| = 1$ GeV (blue solid line) and the thick gray dashed line in Fig. 5.5. Since $|P| = 1$ GeV for this state, the value of $\theta_0$ is given by $\arccos \left( -\frac{1}{1+\sqrt{2}} \right) \approx 1.99787$ and the corresponding value of $(\theta_s)_{\text{max}}$ is given by $2 \arctan \left( \frac{1}{\sqrt{2(1+\sqrt{2})}} \right) \approx 0.854157$. As a comparison, we displayed the state with the final momentum entirely given by $P^x$ without any $P^z$ component, e.g. $P^x = |P| = 1$ GeV, or $\theta = \pi/2$, marked by the blue circle in Fig. 5.5. To get this final momentum, the initial momentum prior to the $E^1$ operation should have had the positive $P^z$ momentum to compensate the negative $P^z$ component, i.e. $P^z = -\frac{P^2}{2M}$, gained by the $E^1$ operation. The initial positive $P^z$ momentum should have been attained by the $K^3$ operation prior to the $E^1$ operation and consequently the amount of $E^1$ operation required to achieve the state with $|P| = 1$ GeV gets reduced. Thus, the corresponding $\theta_s$ for the
blue circle is smaller than \((\theta_s)_{\text{max}}\) marked by gray circle. Numerically, the corresponding \(\theta_s\) value \(\frac{2\arctan\left(\frac{1}{\sqrt{2}+1}\right)}{\sqrt{2}} \approx 0.785398\) for \(\theta = \pi/2\) (blue circle) may be compared with \((\theta_s)_{\text{max}} = \frac{2\arctan\left(\frac{1}{\sqrt{2}+1}\right)}{\sqrt{2}} \approx 0.854157\) for \(\theta_0 = \arccos\left(-\frac{1}{1+\sqrt{2}}\right) \approx 1.99787\) (gray circle) on the line of \(|P| = 1\) GeV. When \(E \rightarrow \infty\), both \(\theta_0 \rightarrow \pi\) and \((\theta_s)_{\text{max}} \rightarrow \pi\) as shown in Fig. 5.5.

5.3.1.2 Longitudinal Boost \(K^3\) Operation

As we have completed our discussion on the \(E_\perp\) operation, we now take a look at how the spin direction changes under the longitudinal boost \(K^3\) when the particle is not moving in the \(\pm z\) direction. The change in \(|P|\) will result in a change in the \(\theta_s\) vs. \(\theta\) profile as we have already discussed and the changed momentum direction \(\theta\) pinpoints the resulted spin direction \(\theta_s\) on the corresponding profile. For an illustration, we start with the case

![Figure 5.6](color online) The light-front profiles of \(\theta_s\) vs \(\theta\) for a particle with mass \(M = 1\) GeV and the magnitude of the momentum \(|P| = 1, 1.2, 2, 4, 100\) GeV. The red solid dots indicates how the momentum direction \(\theta\) and the spin direction \(\theta_s\) changes as the spinor is boosted in \(\pm z\) direction. The blue circle corresponds to the case when the spinor doesn’t have any momentum in \(z\) direction but moves perpendicular to the \(z\) direction, e.g. \(x\) direction.
when the 3-momentum is perpendicular to the \( z \) direction, e.g. \( |\mathbf{P}| = P^x = 1 \text{ GeV} \), which we again denote by the blue circle now in Fig. 5.6. As discussed earlier, the corresponding spin direction is given by Eq. (5.51), i.e. \( \theta_s = 2 \arctan \left( \frac{1}{\sqrt{2}+1} \right) \approx 0.785398 \). As the particle is boosted in the \( +z \) direction while \( P^x \) is fixed as 1 GeV, the momentum angle \( \theta \) is decreased but the profile is lifted higher as the magnitude of momentum \( |\mathbf{P}| \) is increased.

The corresponding change of \((\theta, \theta_s)\) values follows the red solid dots in the region where \( \theta < \pi/2 \) as shown in Fig. 5.6. It is interesting to note that the angle \( \theta \) decreases fast enough to get \( \theta_s \) decreased despite the fact that the profiles are lifted. When boosted in the \(-z\) direction, the momentum angle \( \theta \) is increased and in fact it quickly goes over to the \( \theta > \theta_0 \) side, where \( \cos \theta_0 = -\sqrt{\frac{E-M}{E+M}} \) as previously discussed. As the whole profile moves upward, the resulted \( \theta_s \) is increased following the red solid dots in the region where \( \theta > \pi/2 \). As can be seen from the trend of the red dots as \( \theta \to \pi \) in Fig. 5.6, the \( \theta_s \) approaches to a value that’s not \( \pi \). In other words, even under the maximum boost in the \(-z\) direction, the spin does not align in parallel with the momentum direction. While \( P^x \) is fixed, the maximum \( \theta_s \) value that can be achieved in Eq. (5.51) is \( \pi - 2 \arctan(M/P^x) \) as \( P^x \to -\infty \). For \( P^x = 1 \text{ GeV} \), \( \pi - 2 \arctan(M/P^x) \approx 1.5708 \). As \( P^x \) is fixed in this case, \( P^+ \) can approach to zero. This is in constrast to the maximum on the \( \theta_s \) vs. \( \theta \) profile (i.e. \( (\theta_s)_{\text{max}} \)) discussed before, for which \( P^+ \) is fixed as \( M/\sqrt{2} \) and \( P^x = \sqrt{2M(E-M)} \), so that \( (\theta_s)_{\text{max}} \to \pi - 2 \arctan(2M/P^x) \approx \pi \) as the \( |\mathbf{P}| \) or \( E \) goes to \( \infty \). Effectively, in the fixed \( P^x = 1 \text{ GeV} \) case, the red solid dots follows a monotonic relation between the \( \theta_s \) and \( \theta \) under the boost in \( z \) direction going through the blue circle in Fig. 5.6.

This may also be understood from Eq. (5.51) by noting that the rotated angle of the spin after the boost in \( \pm z \) direction satisfies the following equation:

\[
-\frac{\Delta \theta_s}{2} = \arctan \left( \frac{\sqrt{2}P^+ + M}{P^x} \right) - \arctan \left( \frac{\sqrt{2}P^+ + M}{P^x} \right), \quad (5.52)
\]

where ‘\( \cdot \)' denotes the boosted quantities. For boosting the particle in the \( +z \) (\( -z \)) direction without changing its momentum in the perpendicular direction, i.e. \( x \) direction, the right-hand-side is positive (negative), because \( P^{x+} > P^+ \) (\( P^{x-} < P^+ \)) and \( P^{x+} = P^x \). The negative sign in front of \( \Delta \theta_s \) indicates that \( \theta_s \) is decreased (increased) as the particle is boosted in the \( +z \) (\( -z \)) direction. This illustrates that the rotation of the spin occurs around the \( \mathbf{P} \times (\pm \mathbf{\hat{z}}) \) axis as the particle is boosted in the \( \pm z \) direction, respectively. This is of course the well-known Wigner rotation and \( \Delta \theta_s \) is exactly the corresponding
Wigner rotation angle given by Eq. (3.33) when it is written using our coordinate system and notations. The fact that the Wigner rotation is not needed when boosting the light-front helicity spinors has been known for a long time, but our discussion using the spin orientation angle provides a clear understanding on this interesting feature of the light-front helicity. This is also related with the reason why the amplitudes computed with the light-front helicity, i.e. the light-front helicity amplitudes, are independent of the reference frame as we shall demonstrate in the scattering processes discussed in the next section, Sec. 5.4. Since both $E_\perp$ and $K^3$ in $T = e^{-i\beta_\perp E_\perp} e^{-i\beta_3 K^3}$ are the kinematic operators in LFD, the light-front helicity formulation takes care of the Wigner rotation which is a rather complicate dynamic effect, and offers an effective computation of spin observables in hadron physics.

5.3.2 Generalized Melosh Transformation for Any Interpolation Angle

On the light-front, people usually use the Melosh transformation [Mel74] to connect between the Dirac spinors and the light-front spinors. Now that we have our generalized helicity spinors, we can derive a generalized Melosh transformation that connects between the Dirac spinors and our generalized helicity spinors.

Dirac spinors $u_D^{\pm 1/2}(P)$ can be obtained by applying the boost operator $e^{-i\eta \cdot K}$ directly on the rest frame spinors $u^{\pm 1/2}(0)$ (see Appendix B.3 for explicit expressions). Using Eq. (5.40), this means that the Dirac spinors represent particles moving with momentum $P$ while their spins are parallel or anti-parallel to the $z$ axis in their rest frame. Pictorially, when the red solid arrow in Fig. 5.2 is horizontal while the dashed black arrow remains in its original position, it becomes a Dirac spinor defined for that momentum. Therefore, it is clear that the helicity spinors are related to the Dirac spinors by a pure rotation of the spin.

One can also prove this mathematically. Suppose that the Dirac spinors and the helicity spinors are related by the generalized Melosh transformation in the following way [Dzi88; AS93]:

$$u_D^{(\lambda)} = \Omega_{\lambda\rho}^{[u_D u_H]} u_H^{(\rho)} + \Omega_{\lambda\rho}^{[u_D v_H]} v_H^{(\rho)},$$  \hspace{1cm} (5.53)

$$v_D^{(\lambda)} = \Omega_{\lambda\rho}^{[v_D u_H]} u_H^{(\rho)} + \Omega_{\lambda\rho}^{[v_D v_H]} v_H^{(\rho)},$$  \hspace{1cm} (5.54)
where $H$ and $D$ subscripts denote the generalized helicity spinors and the Dirac spinors respectively, and $\rho, \lambda = \frac{1}{2}, -\frac{1}{2}$ denote positive or negative helicity in the case of our generalized helicity spinors and spin up or spin down in the case of Dirac spinors. Using our orthonormal convention

$$\bar{u}(\rho) H \hspace{1pt} \gamma^\lambda \hspace{1pt} u(\lambda) \hspace{1pt} = \hspace{1pt} 2M \delta_{\rho \lambda},$$

we then have

$$\Omega_{\lambda \rho}^{[uDvH]} \hspace{1pt} = \hspace{1pt} \frac{1}{2M} \bar{u}(\rho) (P) u_D(\lambda) (P) \hspace{1pt} = \hspace{1pt} \frac{1}{2M} u(\rho)^\dagger (0) e^{i\hat{m} \cdot J^s} e^{-i\eta \cdot K} \gamma^0 e^{-i\eta \cdot K} u(\lambda) (0) \hspace{1pt} = \hspace{1pt} \frac{1}{2M} u(\rho)^\dagger (0) e^{i\hat{m} \cdot J^s} \lambda^0 u(\lambda) (0) \hspace{1pt} = \hspace{1pt} \xi(\rho)^\dagger e^{i\hat{m} \cdot J^s} \xi(\lambda) \hspace{1pt} = \hspace{1pt} \mathcal{D}(\hat{\mathbf{m}}, -\theta_s)_{\rho \lambda}, \quad (5.55)$$

where $(e^{-i\eta \cdot K})^\dagger = e^{-i\eta \cdot K}$ because the boost operator $K$ is antihermitian, and we’ve rewritten Eq. (5.24) as

$$u(i) = \sqrt{M} \begin{pmatrix} \xi(i) \\ \xi(i) \end{pmatrix} \hspace{1pt} \text{with} \hspace{1pt} \xi(1/2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \hspace{1pt} \xi(-1/2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (5.56)$$

Since our antiparticle spinors are defined as $\bar{v} = C\bar{u} = i\gamma^0 \gamma^2 \gamma^0 u^*$, we have

$$\Omega_{\lambda \rho}^{[vDvH]} \hspace{1pt} = \hspace{1pt} [u_H(\rho)^\dagger (P)]^* [\gamma^0 \gamma^2 \gamma^0]^\dagger [i\gamma^0 \gamma^2 \gamma^0] [u_D(\lambda) (P)]^* \hspace{1pt} = \hspace{1pt} [u_H(\rho)^\dagger (P)]^* \gamma^0 [u_D(\lambda) (P)]^* \hspace{1pt} = \hspace{1pt} [\bar{u}(\rho) (P) u_D(\lambda) (P)]^* \hspace{1pt} = \hspace{1pt} \mathcal{D}(\hat{\mathbf{m}}, -\theta_s)^*_{\rho \lambda}, \quad (5.57)$$

the complex conjugate of $\mathcal{D}(\hat{\mathbf{m}}, -\theta_s)_{\rho \lambda}$. Using the same method and the orthogonal condition $\bar{u}(\rho) u(\lambda) = \bar{v}(\rho) u(\lambda) = 0$, one can easily check that

$$\Omega_{\lambda \rho}^{[uDvH]} = \Omega_{\lambda \rho}^{[vDvH]} = 0. \quad (5.58)$$

From Eqs. (5.55) and (5.57), we see that the generalized Melosh transformation is nothing but the transpose of the Wigner rotation matrix $\mathcal{D}(\hat{\mathbf{m}}, -\theta_s)$. Plugging $\hat{\mathbf{m}} = (-\sin \phi_s, \cos \phi_s, 0)$ and $-\theta_s$ into Eq. (5.43), we find the generalized Melosh transformation.
for spin $1/2$ as

$$\Omega\left(\frac{1}{2}\right) = \begin{pmatrix}
\omega\left(\frac{1}{2}\right) & 0 \\
0 & \omega\left(\frac{1}{2}\right)^{*}
\end{pmatrix},$$

(5.59)

where

$$\omega\left(\frac{1}{2}\right) = \begin{pmatrix}
\cos\frac{\theta_s}{2} & -e^{i\phi_s} \sin\frac{\theta_s}{2} \\
e^{-i\phi_s} \sin\frac{\theta_s}{2} & \cos\frac{\theta_s}{2}
\end{pmatrix},$$

(5.60)

with $\theta_s, \phi_s$ given by Eqs. (5.45) and (5.46) in terms of the particle’s momentum and “1/2” denotes the fact that this is for spin $J = 1/2$.

In the light-front limit ($\delta \to \pi/4$), we have Eqs. (5.46) and (5.49), so $\Omega(1/2)$ reduces to the well-known Melosh transformation that relates the light-front spinors in Eq. (5.38) to the Dirac spinors in Eq. (B.9), i.e.

$$\omega\left(\frac{1}{2}\right) = \frac{1}{\sqrt{2\sqrt{2}P^+(E + M)}} \begin{pmatrix}
\sqrt{2}P^+ + M & -P^R \\
P^L & \sqrt{2}P^+ + M
\end{pmatrix}. $$

(5.61)

The $\sqrt{2}$ in front of $P^+$ is due to our convention. This expression agrees with Eq. (A8) in Ref. [Dzi88] and Eq. (C1), Eq. (C2) in Ref. [AS93].

As Eqs. (5.55), (5.57) and (5.58) are independent of the spin $J$, our derivation is also applicable to higher spins. We can therefore quickly write down the generalized Melosh transformations for any spin $J$, as the transpose of the corresponding Wigner rotation matrix $D(\hat{m}, -\theta_s)$. The generalized Melosh transformation matrices for spins up to $J = 2$ are listed in Appendix B.5 for reference.

## 5.4 Interpolating Helicity Scattering Probabilities

Using the generalized helicity spinors we can calculate the helicity-dependent amplitudes, for example, for a scattering process depicted in Fig. 5.7. At the lowest order, the helicity
amplitude $M(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is given by

$$\bar{u}^{(\lambda_3)}(p_3)\gamma_{\mu}u^{(\lambda_1)}(p_1)\bar{u}^{(\lambda_4)}(p_4)\gamma_{\mu}u^{(\lambda_2)}(p_2),$$

(5.62)

where $\lambda$ denotes the helicity of the particle. We dropped the coupling constant factor $(-ie)^2$ and the Lorentz invariant part of the propagator $-1/q^2$, since they are irrelevant to our discussion. In this section, we investigate the scattering probabilities which are the square of Eq. (5.62).

It needs to be emphasized that the “helicity” we discuss here is the “generalized helicity” for an arbitrary interpolation angle, as defined in Sec. 5.2. For a certain helicity, the spin orientation depends on both the interpolation angle $\delta$ and the momentum of the particle, as given by Eq. (5.45). Because a boost in the $z$ direction changes the 4-momentum, it will also change the spin orientation in general. As a result, for a certain helicity configuration, the scattering probability will have both the interpolation angle dependence and the frame dependence. To demonstrate both dependences, we plot the helicity probabilities in terms of both the interpolation angle $\delta$ and the total momentum of the system $P_z$, which reflects the boost effect in the $z$ direction.

As illustrated in Fig. 5.7, we set our coordinate system in such a way that the initial particles are lying on the $z$ axis, and $\theta$ is the scattering angle between the momentum $p_1$ and momentum $p_3$. In the center of mass frame, the 4-momentum of these four particles
are given by

\[ p_1 = (\epsilon_1, 0, 0, p_{\text{initial}}), \quad (5.63) \]
\[ p_2 = (\epsilon_2, 0, 0, -p_{\text{initial}}), \quad (5.64) \]
\[ p_3 = (\epsilon_3, p_{\text{final}} \sin \theta, 0, p_{\text{final}} \cos \theta), \quad (5.65) \]
\[ p_4 = (\epsilon_4, -p_{\text{final}} \sin \theta, 0, -p_{\text{final}} \cos \theta), \quad (5.66) \]

where \( p_{\text{initial}} = p_{\text{final}} \equiv p \), \( \epsilon_1 = \epsilon_3 = \sqrt{m_1^2 + p^2} \) and \( \epsilon_2 = \epsilon_4 = \sqrt{m_2^2 + p^2} \), from the conservation of energy and momentum. To incorporate the frame dependence for the probabilities, we boost the whole system to the total momentum \( P_z \), and get

\[ p_i^0 = \gamma p_i^0 + \gamma \beta p_i^z = \frac{E}{M} p_i^0 + \frac{P^z}{M} p_i^z, \quad (5.67) \]
\[ p_i^z = \gamma p_i^z + \gamma \beta p_i^0 = \frac{E}{M} p_i^z + \frac{P^z}{M} p_i^0, \quad (5.68) \]
\[ P_i^\perp = P_i^\perp, \quad (i = 1, 2, 3, 4) \quad (5.69) \]

where \( M = \epsilon_1 + \epsilon_2 \) is the center of mass energy and \( E = \sqrt{(P^z)^2 + M^2} \) is the total energy in the boosted frame. We use these frame-dependent 4-momentum in Eqs. (5.27) and (5.28) to get frame dependent helicity spinors for an arbitrary interpolation angle \( \delta \). The probabilities as the square of Eq. (5.62) are then calculated using these generalized helicity spinors.

In our calculation, we choose \( \theta = \pi/3 \) in the center of mass frame, and the center of mass momentum for each particle as \( p = 2 \) GeV. The masses of the colliding particles are chosen to be \( m_1 = m_3 = 1 \) GeV and \( m_2 = m_4 = 1.5 \) GeV as an illustrative example. The scattering probabilities for all 16 different helicity configurations are plotted in terms of both the interpolation angle \( \delta \) and the total momentum \( P_z \) in the first 4 rows of Fig. 5.8, where we denote the positive and negative helicities with “+” and “−”. For example, “+− → +−” represents a process where particle 1 with positive helicity and particle 2 with negative helicity scatter to particle 3 with positive helicity and particle 4 with negative helicity.

The first striking feature that these plots exhibit is the two boundaries indicated by the blue dashed lines in Fig. 5.8, across which the probabilities suddenly change their values. The appearance of these boundaries is due to the bifurcation of the branches that
Figure 5.8 (Color online) Scattering probabilities (with the factor $e^4/q^4$ dropped) for 16 different helicity configurations, where the masses of the two colliding particles are $m_1 = 1$ GeV and $m_2 = 1.5$ GeV, the center of mass momentum for each particle is $p = 2$ GeV and the scattering angle $\theta = \pi/3$. 
are distinguished by the critical interpolation angle $\delta_c$ as we have discussed in Sec. 5.3.1. Since $\delta_c$ is obtained from the condition $P_\perp = 0$, the initial two particles that move only in the ±z direction yield the corresponding two boundaries upon the boost of the reference frame in the ±z direction. The sudden change of the helicity amplitude-square values is thus caused by the spin flip of the incoming scattering particles 1 and 2 for the given helicity assignment in the amplitude. When this spin flip happens, the probabilities for the same helicity configuration represent different spin configurations. In other words, we are not calculating the same physical process any more but three different initial spin configurations for the corresponding three different branches divided by the two boundaries although the given helicity configuration is identical regardless of the boundaries. Thus, the helicity probability suddenly shifts to a different value crossing a boundary. The only helicity amplitude immune from such bizarre behaviors is the light-front helicity amplitude without any border at $\delta = \pi/4$.

To see why the spin flips for $\delta \neq \pi/4$, we can approach it in two different ways. First, we can fix a certain value for the total $P_\perp$, i.e. we fix the inertial frame, and look at what happens when we change the interpolation angle $\delta$. From the $\theta = \pi$ edge of Fig. 5.3, it’s clear that for a particle moving in the $-z$ direction, as we increase the $\delta$, at some critical angle $\delta_c$, the spin of “positive helicity” will flip from $\theta_s = \pi$ to $\theta_s = 0$. And because the spin direction of “negative helicity” is always exactly the opposite of that of the “positive helicity”, the spin of the “negative helicity” will flip at this $\delta_c$ from $\theta_s = 0$ to $\theta_s = \pi$. Alternatively, and perhaps more conveniently, this spin flip can be understood by fixing a certain value for the interpolation angle $\delta$, and see what happens when we boost the

**Figure 5.9** Each plot is a summation of 4 plots in Fig. 5.8 with the same initial helicity configurations, i.e. each plot represents the probability of a certain helicity configuration going into all possible final helicity states.
system in the $z$ direction. As we discussed in Sec. 5.2, using Eq. (5.16), the “helicity” for particles moving in $\pm z$ direction is defined as whether the spin has the same sign as $P_\mp$.

So for the same helicity configuration, as we boost the system in $\pm z$ direction, the spin of particle 1 or 2 will flip if the sign of its $P_\mp$ changes. Therefore, we can find these boundary lines by setting $P_\mp = 0$ for particles 1 and 2, with their momenta given by Eqs. (5.67) and (5.69) and the form of $P_\mp$ given by Eq. (3.7). The equations that describe these two boundaries are respectively given by

$$\tan \delta = -\frac{\epsilon_1 P^z + pE}{\epsilon_1 E + pP^z}, \quad (5.70)$$

$$\tan \delta = -\frac{\epsilon_2 P^z - pE}{\epsilon_2 E - pP^z}. \quad (5.71)$$

To separate out the effects of the initial helicity states, we also plot in Fig. 5.9 probabilities of a certain helicity configuration going into all possible final helicity configurations. Each of the 4 plots is therefore a summation of 4 plots in Fig. 5.8 with the same initial helicity configurations. By summing over all final states, we see clearly that the varying landscapes due to the helicity configurations of the scattered particles 3 and 4 disappears and the two boundaries due to the spin flips of the incoming particles 1 and 2 remain. They divide the whole landscape into three regions. To show the positions of these boundaries more clearly, we plot them in the $\delta$ vs $P^z$ plane as the two blue dashed lines in Fig. 5.10.

Another interesting feature exhibited by all 16 helicity probability plots is the J-curve indicated by the red solid line in both Fig. 5.8 and Fig. 5.10. This is the same curve that appeared in the time-ordered amplitudes in the $\phi^3$ theory [JS13] and also the sQED theory we studied in Chapter 4. It starts out in the center of mass frame in the $\delta = 0$ limit and maintains the same probability value throughout the whole range of interpolation angle. It follows exactly the same formula as the one we found in the time-ordered amplitudes in the last chapter given by Eq. (4.97):

$$P^z = -\sqrt{\frac{M^2(1 - C)}{2C}}, \quad (5.72)$$

where $M^2$ is the same as the invariant Mandelstam variable “$s$” used in Ref. [JS13]. Therefore, this J-curve is truly universal in that it appears in not only the time-ordered amplitudes [JS13; Ji15], but also now in the helicity probabilities and amplitudes (see
Appendix B.7). Again, this curve is independent of the specific kinematics of the scattering process, e.g. the \( \theta \) angle, the masses of these four particles and so on. It’s only scaled by the center of mass energy \( M \), and has a universal shape. In our example, we chose \( \theta = \pi/3 \) in the center of mass frame. As the \( \theta \) angle changes, the landscape will continuously change, but the main features remain the same, and this J-curve is always present. Incidentally, this J-curve as given by Eq. (4.97) is exactly \( P_z = 0 \) for the system, which can be verified by setting \( P^0 = \sqrt{M^2 + (P_z)^2} \) and \( P^3 = P^z \) in Eq. (3.7). As \( \delta \to \pi/4 \), \( P^z \to -\infty \), and therefore if we take the limit to the light front along this J-curve, we will have the same probability value as in the center of mass frame in the IFD, which doesn’t agree with the invariant light-front result in general, as can be seen from Fig. 5.8. This raises the issue of non-commuting orders in taking the two independent limits, \( \delta \to \pi/4 \) and \( P_z \to -\infty \). Thus, it requires a great caution in taking the limit to \( \delta = \pi/4 \). This is reminiscent of the zero-mode issue in LFD as we have previously discussed in great detail in Sec. 4.3.

In LFD (\( \delta = \pi/4 \)), all 16 helicity probabilities are frame independent as we have discussed. We may relate this LFD invariance with our previous examination on the spin orientations of the light-front spinors. First of all, as we discussed in Sec. 5.3.1, for
light-front spinors moving in the $\pm z$ directions, positive (negative) helicity always means the spin is orientated in the $+z$ ($-z$) direction. Thus, for a certain helicity configuration, the spin orientations of particles 1 and 2 are fixed and there are no boundaries generated by spin flips. If $\theta = 0$ or $\pi$, then particles 3 and 4 are also moving in $\pm z$ directions, and their spin orientations will also be fixed. In this case, the probabilities are clearly frame independent, since the helicity configuration in different inertial frames represent the same physical spin configuration. What about the case where $0 < \theta < \pi$ (the case plotted in Fig. 5.8)? As we discussed in Sec. 5.3.1 using Eq. (5.52), the light-front spinors naturally generates the required Wigner rotations when boosted in $\pm z$ directions. This means even for the scattered particles 3 and 4 not moving in the $\pm z$ direction, the light-front spinors in different inertial frames still represent the same physical spin states. Since we are calculating the same physical process in different frames, we are guaranteed to get the same probability.

Figure 5.11 The total scattering probability (with the factor $(e^2/q^2)^2$ dropped) as a sum of all 16 spin contributions for the lowest tree diagram shown in Fig. 5.8.

Finally, the helicity probability plots in Fig. 5.8 show a clear asymmetry between the $+P^z$ direction and the $-P^z$ direction unless $\delta = \pi/4$. The corresponding helicity
amplitudes that provide the probabilities shown in Fig. 5.8 are plotted in Appendix B.7. The LFD result is smoothly connected only to the $P^z = +\infty$ result but not to the $P^z = -\infty$ result. This disparity is a good example of showing the difference between the IMF in IFD and the LFD. Our plots obviously clear up the prevailing confusion between the two. Regardless of all the discussions on individual helicity probabilities, however, the sum of all 16 helicity probabilities plotted in Fig. 5.11 is both frame independent and interpolation angle independent as it must be.

For completeness, the helicity amplitudes and probabilities for annihilation to the lowest order are calculated and plotted in Appendix B.8.

5.5 Summary

As a continuation of our effort to interpolate between the IFD and the LFD, in this chapter we generalized the helicity operator to any interpolation angle $\delta$ and derived the generalized helicity spinor that links the instant form helicity spinor to the light-front helicity spinor.

For a given generalized helicity spinor, the spin direction does not coincide with the momentum direction in general. Thus, we studied how the spin orientation angle changes in terms of both $\delta$ and the angle $\theta$ that defines the momentum direction of the particle. Applying the transformation matrix $T$ given by Eq. (3.34) to an initial spin state at rest, we obtained a generalized helicity spinor as we discussed in Sec. 5.2. We then used the operator relation given by Eq. (5.40) to analyze the spin orientation angles $(\theta_s, \phi_s)$, where $\phi_s$ can be taken to be zero without loss of any generality because the spin $S$, momentum $P$ and $z$ axis are all in the same plane. As shown in Figs. 5.3 and 5.4, the angle between the momentum and the spin directions, i.e. $\theta - \theta_s$, increases with the interpolation angle $\delta$ and becomes the largest at the light front. In particular, the increment of the angle difference $\theta - \theta_s$ with the increment of the interpolation angle $\delta$ bifurcates at a critical interpolation angle $\delta_c$ as shown in Fig. 5.4. We found $\delta_c$ as given by Eq. (5.48) and noted that the IFD and the LFD separately belong to the two different branches bifurcated and divided out at the critical interpolation angle $\delta_c$. This bifurcation indicates the necessity of the distinction in the spin orientation between the IFD and the LFD and clarifies any conceivable confusion in the prevailing notion of the equivalence between the IMF formulated in IFD and the LFD.
As the light-front helicity is obtained in general by the two kinds of boost operations, i.e. the light-front transverse boost $E_\perp$ and the longitudinal boost $K^3$, we discussed the salient features of the spin orientations due to each of these two kinds of light-front kinematic operations separately in Sec. 5.3.1. It is interesting to note that the $E_\perp$ operation on a rest particle of mass $M$ provides the non-relativistic form of energy gain $E = M + \frac{\vec{P}_\perp^2}{2M}$, yet satisfying the relativistic energy-momentum dispersion relation given by Eq. (5.50).

We have analyzed the change of spin orientation under the $E_\perp$ operation and obtained the “apparent” spin orientation angle $\theta_a$ given by Eq. (B.21). In Fig. 5.5, we plotted $\theta_a$ in comparison with the corresponding initial spin orientation $(\theta_s)_{\text{max}}$ discussed in Sec. 5.3.1 along with various $\theta_s$ profiles depending on the particle momentum $|\vec{P}|$ or the energy $E$. The change of the spin direction under the $K^3$ operation was also very interesting and our result given by Eq. (5.52) agrees with the well-known result of the Wigner rotation angle given by Eq. (A10) in Ref. [Osb68]. Thus, the light-front helicity formulation takes care of the Wigner rotation which is a rather complicate dynamic effect and offers an effective computation of spin observables in hadron physics. With the spin orientation analysis presented in Sec. 5.3.1, we have also derived the generalized Melosh transformation that connects between the Dirac spinors and our generalized helicity spinors as shown in Sec. 5.3.2.

Using the generalized helicity spinors, we computed all 16 helicity amplitudes and their squares (or probabilities) for the scattering process analogous to the QED process $\text{“}\epsilon\mu \rightarrow \epsilon\mu\text{”}$ in Sec. 5.4.¹ This computation provided an explicit demonstration of the whole landscape picture for the helicity amplitudes and amplitude-squares that depend on the reference frame as well as the interpolation angle. From this picture, one can see clearly that the helicity probabilities in LFD is independent of the reference frames while all other interpolating helicity probabilities for $0 \leq \delta < \pi/4$ do depend on the reference frames. The striking feature of the interpolating helicity probabilities as well as amplitudes is the appearance of the two boundaries due to the the bifurcation of the branches that are distinguished by the critical interpolation angle $\delta_c$. As $\delta_c$ is obtained from the condition $P_\perp = 0$, the initial two particles that move only in the $\pm z$ direction yield the corresponding two boundaries upon the boost of the reference frame in the $\pm z$ direction. For the given initial helicity states, the two boundaries indicate the spin flip of

¹For completeness of our discussion, we have also plotted the helicity amplitudes and probabilities for the fermion and anti-fermion pair annihilation and creation process analogous to the QED process $\text{“}e^+e^- \rightarrow \mu^+\mu^-\text{”}$ in in Appendix B.8.
each of the two incoming scattering particles as we have shown explicitly in Fig. 5.8 by the blue dashed lines.

Another interesting feature exhibited by all 16 helicity probability plots in Fig. 5.8 is the J-shaped curved indicated by the red solid line which is the same curve that appeared in the $\phi^3$ theory [JS13] and also the sQED theory we studied in Chapter 4. In Fig. 5.10, we showed that the J-curve lies between the two boundaries discussed above. The J-curve has a universal shape independent of the kinematics in the scattering process no matter what the underlying theory is. The scattering amplitudes maintain the same value along the J-curve in the whole range of interpolation angle $0 \leq \delta \leq \pi/4$. However, in taking the limit to $\delta = \pi/4$, it requires a great caution due to the non-commutativity in the order of taking the two independent limits, $\delta \to \pi/4$ and $P_z \to -\infty$. This non-commutativity brings up the zero-mode issue in LFD as we have discussed previously in great detail in Sec. 4.3. The disparity between $P_z \to \infty$ and $P_z \to -\infty$ also further clears up the confusion between the IMF in IFD and the LFD.
In Chapter 4 and Chapter 5, we studied in great detail the interpolation of the photon polarization vectors, the gauge propagator and the on-mass-shell helicity spinors. In this chapter, we complete the interpolation of the QED theory by providing the final piece of the puzzle: the interpolating fermion propagator. The form of this interpolating fermion propagator is derived, but we leave the detailed discussion of its interpolation properties to future researches. Instead, this chapter focuses on the formal derivation of the interpolating QED theory. In Sec. 6.1, we decompose the covariant Feynman diagrams into $x^\tau$-ordered diagrams, from which a general set of Feynman rules for any $x^\tau$-ordered scattering theory is obtained. In Sec. 6.2 the canonical field theory approach is studied, and the corresponding Hamiltonian for the old fashioned perturbation theory is derived.

### 6.1 Scattering Theory

Following what Kogut and Soper did in their light-front QED paper [KS70], we regard the perturbative expansion of the S matrix in Feynman diagrams as the foundation of quantum electrodynamics. In this section, we decompose the covariant Feynman diagram into a sum of $x^\tau$-ordered diagrams. We shall not be concerned with the convergence of the perturbation series, or convergence and regularization of the integrals.
6.1.1 Propagator Decomposition

The electron propagator can be written as

\[ S_F = (i\partial_\mu \gamma^\mu + m)\Delta_F(x), \quad (6.1) \]

where \( \Delta_F \) is the Klein-Gordon propagator:

\[ \Delta_F(x) \equiv \int \frac{d^4 q}{(2\pi)^4} e^{i(q_\mu x^\mu)} \frac{i}{q^2 - m^2 + i\epsilon}, \quad (6.2) \]

\[ = \int \frac{d^2 q_\perp dq_+ dq_-}{(2\pi)^4} e^{i(q_\perp x^\perp + q_+ x^+ + q_- x^-)} \]

\[ \times \left[ \frac{i}{Cq_+^2 + 2Sq_- q_+ - Cq_-^2 - q_\perp^2 - m^2 + i\epsilon} \right]. \quad (6.3) \]

If \( C \neq 0 \), we have two poles for \( q_\perp^2 \) at

\[ A - i\epsilon' = \left( -S q_+ + \sqrt{q_+^2 + C(q_\perp^2 + m^2)} \right) / C - i\epsilon', \quad (6.4) \]

\[ B + i\epsilon' = \left( -S q_- - \sqrt{q_-^2 + C(q_\perp^2 + m^2)} \right) / C + i\epsilon', \quad (6.5) \]

where \( \epsilon' > 0 \). We evaluate the \( q_\perp^2 \) integral by contour integration, closing the contour in the lower (upper) half plane if \( x^+ > 0 \) (\( x^- < 0 \)). This produces the desired decomposition for \( \Delta_F \):

\[ \Delta_F(x) = \int d^2 q_\perp \int_{-\infty}^{\infty} dq_- \frac{1}{2Q^\perp} \left[ \Theta(x^+) e^{-iq_\perp x^\perp} + \Theta(-x^+) e^{iq_\perp x^\perp} \right], \quad (6.6) \]

where

\[ Q^\perp \equiv \sqrt{q_-^2 + C(q_\perp^2 + m^2)} \quad (6.7) \]

and \( q_\perp = A \) with \( A \) given by Eq. (6.4) are both on-mass-shell components of the momentum. Notice that here

\[ \int \frac{d^2 q_\perp dq_-}{(2\pi)^3 2Q^\perp} = \int \frac{d^4 q}{(2\pi)^4} 2\pi \delta(q^2 - m^2) \quad (6.8) \]

is the invariant differential surface element on the mass shell.

We can now use Eqs. (6.1) and (6.6) to derive a decomposition for the electron propagator. The differentiation of \( \Theta(x^+) \) and \( \Theta(-x^+) \) in Eq. (6.6) with respect to \( x^+ \) gives
us two terms: $\delta(x^+)e^{-iq_{\perp}x^\parallel}$ and $-\delta(x^+)e^{iq_{\perp}x^\parallel}$. These two will cancel each other exactly when an integration with respect to $x^+$ is performed in the next subsection, so they don’t contribute to the diagrams. We can therefore drop them from this electron propagator decomposition. We finally arrive at

$$ S_F(x) = \int \frac{d^2q_\perp}{(2\pi)^3} \int_{-\infty}^{\infty} dq^+ \frac{1}{2q^\pm} \left[ \Theta(x^+)(\hat{q} + m)e^{-iq_{\perp}x^\parallel} + \Theta(-x^+)(-\hat{q} + m)e^{iq_{\perp}x^\parallel} \right]. \quad (6.9) $$

Notice that the situation is very different when we are at the exact light front where $C = 0$. Then there is only one pole for Eq. (6.3) at $q^+ = q^- = (q_\perp^2 + m^2)/2q^+-i\epsilon/2q^+$. This pole’s position depends on the sign of $q^+$. After completing the $q^+$ integral by contour integration and appropriate change of variables, the $q^+$ integral only goes from $0$ to $\infty$ instead of from $-\infty$ to $\infty$. And the $x^+$ integral to be performed in the next subsection will not cancel the two $\delta(x^+)$ terms resulted from the differentiation of $\Theta(x^+)$ and $\Theta(-x^+)$. The decomposition of electron propagator at the light front ($C = 0$) was derived by Kogut and Soper [KS70] to be

$$ S_F(x)_{LF} = \int \frac{d^2q_\perp}{(2\pi)^3} \int_{0}^{\infty} dq^+ \frac{1}{2q^+} \left[ \Theta(x^+)(\hat{q} + m)e^{-iq_{\perp}x^\parallel} + \Theta(-x^+)(-\hat{q} + m)e^{iq_{\perp}x^\parallel} \right] $$

$$ + i\delta(x^+)\gamma^+ \int \frac{d^2q_\perp}{(2\pi)^3} \int_{-\infty}^{\infty} dq^+ \frac{1}{2q^+} e^{-iq^+x^- + iq_{\perp}x^\parallel}. \quad (6.10) $$

From the above discussion, we see that the third term in Eq. (6.10) which results in an extra instantaneous contribution is unique to the exact light-front ($C = 0$).

Next, we also need a decomposition for the photon propagator. This was done in Sec. 4.2, and we got

$$ D_F(x)_{\mu\nu} = \int \frac{d^2q_\perp}{(2\pi)^3} \int_{-\infty}^{\infty} dq^- \Theta(q^-) \frac{\mathcal{Z}_{\mu\nu}^-}{2q^+} \left[ \Theta(x^+)e^{-iq_{\perp}x^\parallel} + \Theta(-x^+)e^{iq_{\perp}x^\parallel} \right] $$

$$ + i\delta(x^+) \int \frac{d^2q_\perp}{(2\pi)^3} \int_{-\infty}^{\infty} dq^- \frac{n_\mu n_\nu}{Q^+} e^{-i(q^-z + q_{\perp}x^\parallel)}, \quad (6.11) $$

where $q^- = A$ with $A$ given by Eq. (6.4), and $Q^+$ given by Eq. (6.7) with $m = 0$ are both on-mass-shell components of the momentum. The explicit form of $\mathcal{Z}_{\mu\nu}^\pm \equiv \sum_{\lambda=\pm} \epsilon_\mu^*(\lambda)\epsilon_\nu(\lambda)$
was given by Eq. (4.17). The interpolating step function

\[
\hat{\Theta}(q \mp) = \Theta(q \mp) + (1 - \delta_{C0})\Theta(-q \mp)
\]

\[
= \begin{cases} 
1 & (C \neq 0) \\
\Theta(q^+) & (C = 0)
\end{cases}
\]  

(6.12)

was introduced to combine the results of $C \neq 0$ and $C = 0$. This is possible because the third term in Eq. (6.11) which gives instantaneous contribution is present in both cases ($C \neq 0$ and $C = 0$), unlike the instantaneous term in the fermion propagator.

### 6.1.2 Feynman Rules for $x^\mp$-ordered Diagrams

To find the rules for $x^\mp$-ordered diagrams, we start with the Feynman diagrams in coordinate space. The amplitude for diagram shown in Fig. 6.1a can be written as

\[
i \mathcal{M} = (-ie)^2 \int d^4x d^4y \; \epsilon_\mu^*(y) \bar{\psi}_2(y) \gamma^\mu \\
\times S_F(y - x) \gamma^\nu \psi_1(x) \epsilon_\nu^*(x).
\]

(6.13)

Here we use plane waves as the electron wave functions, and the charge conjugate wave functions for positrons:

\[
\psi_1(x) = e^{-ip_\mu x^\mu} u(p, s),
\]

(6.14)

\[
\psi_2(y) = e^{ip'_\mu y^\mu} v(p', s'),
\]

(6.15)

where $p$ and $s$ are the momentum and spin of the particle. The photon wave function is

\[
\epsilon_\mu(x) = e^{-ikx} \epsilon_\mu(k, \lambda),
\]

(6.16)

where $\epsilon_\mu(k, \lambda)$ is the polarization vector with momentum $k$ and helicity $\lambda$, the explicit form of which was given in Eqs. (4.2) - (4.4).

With the change of variables

\[
x \to x, \quad y \to T = y - x,
\]

(6.17)
Eq. (6.13) becomes

\[ i\mathcal{M} = (-i\epsilon)^2 \int d^4x d^4T \, e^{i(k' - p')T} \epsilon^*_\mu(k', \lambda') \bar{\psi}(p', s') \gamma^\mu S_F(T) \gamma^\nu u(p, s) \epsilon^*_\nu(k, \lambda) e^{ix(k + k' - p - p')} . \]  

(6.18)

The x integration immediately gives the total energy-momentum conservation condition. After we plug in the decomposed \( S_F \) given by Eq. (6.9) when \( C \neq 0 \) and Eq. (6.10) when \( C = 0 \), we finish the \( T^\pm \) integration using the following relations

\[ \int_{-\infty}^{\infty} dT^\pm \Theta(T^\pm) e^{iP^\pm T^\pm} = \frac{i}{P^\pm} , \]  

(6.19)
Figure 6.2 Vertices that appear in the $x^\perp$-ordered diagrams. When $\mathbb{C} \neq 0$, only two kinds of vertices (a) and (b) exist. When $\mathbb{C} = 0$, all three vertices (a), (b), (c) are present.

\[
\int_{-\infty}^{\infty} dT^\perp \Theta(-T^\perp) e^{iP_+ T^\perp} = -\frac{i}{P_+}. \tag{6.20}
\]

Thus we get a factor of $i(P_{\text{ini}+} - P_{\text{inter}+})^{-1}$ for each intermediate state. Here $P_{\text{ini}+} = p_+ + p'_+$ is the total “energy” of the initial particles, and $P_{\text{inter}+}$ gives the total “energy” of the intermediate particles, which is $k_+ + q_+ + p'_+$ when $y^\perp > x^\perp$ and $p_+ + q_+ + k'_+$ when $y^\perp < x^\perp$. On the other hand, the $dT^\perp d^2T^\perp$ integration gives straightforwardly $(2\pi)^3\delta(P_{\text{ini}}^+ - P_{\text{out}}^+)\delta^2(P_{\perp}^\text{ini} - P_{\perp}^\text{out})$ at each vertex. And lastly, the $\delta(T^\perp)$ term in Eq. (6.10) gives an extra instantaneous contribution at the light-front ($\mathbb{C} = 0$) and is easy to calculate.

Similar analysis can be done for another diagram shown in Fig. 6.1c, with the decomposition equation of the photon propagator given by Eq. (6.11). In fact, the photon propagator decomposition has already been done in the sQED theory in Chapter 4, so we won’t repeat it here. After the above analysis, with a little thought, one can summarize and write down the rules for $x^\perp$-ordered diagrams when $\mathbb{C} \neq 0$ as the following:

1. $u(p, s), \bar{u}(p, s), v(p, s), \bar{v}(p, s), \epsilon_\mu(p, \lambda), \text{ and } \epsilon^*_\mu(p, \lambda)$ for each external line;

2. $(\not{\mathbf{p}} + m) = \Sigma_s u(p, s) \bar{u}(p, s)$ for electron propagators; $(-\not{\mathbf{p}} + m) = -\Sigma_s v(p, s) \bar{v}(p, s)$ for positron propagators; $\mathcal{F}_{\mu\nu} \equiv \sum_{\lambda = \pm} \epsilon^*_\mu(\lambda)\epsilon_\nu(\lambda)$ for photon propagators;
3. \(-ie\gamma^\hat{\mu}(2\pi)^3\delta(P^\text{in} - P^\text{out})\delta^2(P^\text{in} - P^\text{out})\) for each vertex as shown in Fig. 6.2a;

\[-e^2 \frac{in^\mu n^\nu}{q^2 + \mathcal{C}q^2_\perp}\delta(P^\text{in} - P^\text{out})\delta^2(P^\text{in} - P^\text{out})\]

\[\times \ldots \gamma^\hat{\mu} \ldots \gamma^\hat{\nu} \ldots\]

for each vertex as shown in Fig. 6.2b, where \(q^\perp\), \(q^\parallel\) are the total momentum transferred;

4. \(i(P^\text{ini} - P^\text{inter})^{-1}\) for each internal line, where \(P^\text{ini}\) and \(P^\text{inter}\) are the sums of energies for the initial and intermediate particles;

5. an over-all factor of \((2\pi)\delta(P^\text{in} - P^\text{out})\) for energy conservation.

6. an integration

\[\int \frac{dq^\parallel}{(2\pi)^3} \int_{-\infty}^{\infty} dq^\perp \ldots\]

for every internal propagating line, with \(m\) in Eq. (6.7) being the mass of the exchanged particle.

The rules for \(x^+\)-ordered diagrams on the light front (\(\mathcal{C} = 0\)), first derived by Kogut and Soper [KS70], are a little different from the above rules for \(\mathcal{C} \neq 0\). Besides the superficial differences in conventions used, the rules themselves only differ fundamentally in two places. The first is in rule 3. Because of the instantaneous contribution from the third term in Eq. (6.10), we have one more instantaneous vertex on the light-front as shown in Fig. 6.2c, with which we associate the following factor

\[-ie^2\gamma^\mu \gamma^\perp \gamma^\nu \frac{1}{2q^+} (2\pi)^3\delta(P^\text{ini} - P^\text{inter})\delta^2(P^\text{ini} - P^\text{inter})\]

where \(q^+ = k^\perp - p^\perp\). The second is that the integration limits for \(q^\perp\) in rule 6 change to \((0, \infty)\) on the light front, and we have

\[\int \frac{dq^\parallel}{(2\pi)^3} \int_0^{\infty} dq^+\]

for every internal line. All the other rules agree when we take the \(\mathcal{C} \to 0\) limit.

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In the next section, we develop the canonical field theory of quantum electrodynamics in any interpolating angle. And we will see that it reproduces the Feynman rules we obtained here.

6.2 Canonical Field Theory

6.2.1 Equations of Motion

The Lagrangian for QED is

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} (i\gamma^\mu D_\mu - m) \psi, \] (6.21)

where \( D_\mu = \partial_\mu + ieA_\mu \), and \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \). The equations of motions are therefore

\[ \partial_\mu F^{\mu\nu} = eJ^\mu = e\bar{\psi}\gamma^\mu \psi. \] (6.22)

\[ (i\gamma^\mu \partial_\mu - eA_\mu - m)\psi = 0, \] (6.23)

The "\( \hat{+} \)" component of Eq. (6.22) does not contain any time derivative \( \partial_\hat{+} \), and by converting all the upper index components into lower index components using Eq. (3.8) it can be written as

\[ (\partial_\bot^2 C + \partial_\bot^2) A_\hat{+} = (C\partial_\hat{+} + S\partial_-)\partial_\bot A_\bot + (\partial_\bot \partial_- - S\partial_\bot^2) A_- - eJ_\hat{+}. \] (6.24)

Next we apply the generalized transverse gauge condition we obtained in Sec. 4.1.2:

\[ \partial_- A_- + \partial_\bot A_\bot C = 0, \] (6.25)

and Eq. (6.24) simplifies to

\[ (\partial_\bot^2 C + \partial_-)(CA_\hat{+} + SA_-) = (\partial_\bot^2 C + \partial_-^2) A_\hat{+} = -eJ_\hat{+}C. \] (6.26)

From Eqs. (6.25) and (6.26), we see that we can regard \( A_1 \) and \( A_2 \) as the two independent free components, while at any given "time" \( x_\hat{+} \), \( A_- \) can be determined by \( A_1, A_2, \) and \( A_\hat{+} \) determined by \( A_1, A_2 \) and \( \psi \). For reasons of symmetry, we choose \( A_- (x_\hat{+}, x^1, x^2, +\infty) = -A_- (x_\hat{+}, x^1, x^2, -\infty) \), and the solution to Eq. (6.25) under this
boundary condition is

\[ A_\pm = -\frac{1}{2} \int d\xi \epsilon(x^\pm - \xi) \partial_\perp A_\perp(x^\pm, x^1, x^2, \xi) C, \tag{6.27} \]

where

\[ \epsilon(x) = \begin{cases} 
1, & x > 1, \\
-1, & x < 1.
\end{cases} \]

By a simple change of variables \( X^i \equiv x^i / \sqrt{C}, \) \( X^- \equiv x^-, \) Eq. (6.26) becomes

\[ \nabla^2 A_\pm \equiv \left( \frac{\partial^2}{\partial (X^i)^2} + \frac{\partial^2}{\partial (X^-)^2} \right) A_\pm = -eJ_\pm C, \quad (i = 1, 2). \tag{6.28} \]

Eq. (6.28) has the solution

\[ A_\pm = e \int d^2X^\perp dX^- \frac{J_\pm(X^-) C}{4\pi|X - X'|} \]
\[ = e \int d^2x^\perp dx^- \frac{J_\pm(x^-) \sqrt{C}}{4\pi \sqrt{(x^-)^2 + (x^- - x^-')^2} C}. \tag{6.29} \]

With \( A^- \) and \( A^\pm \) solved, other components can be obtained by using Eq. (3.8).

Notice that at the instant limit (\( C \to 1 \)), \( A_- \to A^3, A^\pm \to A^0, J^\pm \to J^0 \) and the above solutions agree with the instant form results. At the light-front limit (\( C \to 0 \)), according to both Eqs. (6.27) and (6.29) \( A^+ \to 0 \), which is consistent with the light-front gauge \( A^+ = 0 \). However, for \( A_\pm = A^\pm / C - SA_- / C \) with \( A_- \) and \( A^\pm \) given by Eqs. (6.27) and (6.29), the first term does not seem to have a clear limit at \( C = 0 \). A closer inspection reveals the problem. The \( A_\perp \) component satisfies the following equation:

\[ \nabla^2 (A_\perp - \frac{SA_-}{C}) = (\partial_\perp^2 C + \partial_\perp^2) (A_\perp - \frac{SA_-}{C}) = -eJ^\perp, \tag{6.30} \]

where the three dimensional Laplace operator reduces to a one dimensional operator when \( C = 0 \), and the solution in Eq. (6.29) is no longer valid. Instead, the solution is [KS70]

\[ A^- = -\frac{1}{2} \int d\xi |x^- - \xi| [\partial_\perp A_\perp(x^+, x^1, x^2, \xi) + J^+(x^+, x^1, x^2, \xi)], \tag{6.31} \]
as derived by Kogut and Soper.

To simplify the notations and make the derivations easier to follow, from now on we will write

\[ A^\perp = \frac{\partial^\perp A^\perp}{\partial^\perp} C, \]

\[ A^{\perp} = -\frac{eJ^{\perp} C}{\partial_\perp^2 C + \partial^2_\perp}, \]

instead of the explicit integral forms shown in Eqs. (6.27) and (6.29). We also write \( A^{\perp} \) as

\[ A^{\perp} = \frac{\partial^\perp A^\perp}{\partial^\perp} S - \frac{eJ^{\perp}}{\partial^2_\perp C + \partial^2_\perp}, \]

which represents \( A^{\perp} = A^{\perp}/C - S A^\perp/C \) with \( A^\perp \) and \( A^{\perp} \) given by Eqs. (6.27) and (6.29) when \( C \neq 0 \), and represents the light-front solution given by Eq. (6.31) when \( C = 0 \). Written in this way, Eqs. (6.32) - (6.34) also show very clearly, that only the \( A_1 \) and \( A_2 \) components of \( A^\perp \) are dynamical variables.

On the other hand, all four components of \( \psi \) field are independent when \( C \neq 0 \). Therefore, we don’t need to rewrite the fermion fields. Situation changes when \( C = 0 \) on the exact light-front. As discussed by Kogut and Soper [KS70], only two components of the \( \psi \) field are independent on the light-front. The other two are totally determined at any light-front time \( x^+ \), by the independent components.

From the previous discussion in Sec. 6.1 that the instantaneous fermion interaction only exist at the exact light-front \( (C = 0) \), and the above discussion that two components of the \( \psi \) field become non-dynamical only at the exact light front \( (C = 0) \), we see that although the interpolated theory connects smoothly to the instant form \( (C = 1) \), it seems that the derivation can not be easily combined with the exact light-front case when the limit \( C \to 0 \) is taken. In the following sections, we will develop the canonical field theory for any interpolating angle with \( C \neq 0 \), in comparison with the exact light-front results obtained by Kogut and Soper [KS70].
6.2.2 Free Fields

The Fourier expansion of the fermion field $\psi(x)$ takes the form

$$\psi(x^+, x^-, x^\perp) = \int \frac{d^2p_\perp dp^-}{(2\pi)^3 2p^+} \sum_{s=\pm 1/2} \left[ u^{(s)} e^{-ix^-p^- - i x^\perp p^\perp} b(p_\perp, p^-; s; x^+) \\
+ v^{(s)} e^{ix^-p^- + i x^\perp p^\perp} d\dagger(p_\perp, p^-; s; x^+) \right],$$  \hspace{1cm} (6.35)

where the positive and negative energy spinors $u$ and $v$ satisfy the Dirac equation:

$$\begin{align*}
(\gamma^\mu p^\mu - m) u &= 0, \hspace{1cm} (6.36) \\
(-\gamma^\mu p^\mu - m) v &= 0. \hspace{1cm} (6.37)
\end{align*}$$

Here we take $u$ and $v$ to be the generalized helicity spinors $u_H$ and $v_H$ whose explicit form in the chiral basis have been given by Eqs. (5.25) and (5.26) and Eqs. (B.7) and (B.8) respectively. For simplicity, we will omit the subscript “H” throughout this chapter.

Without any interaction, the fermion field should satisfy the Dirac equation:

$$(i\gamma^\mu \partial^\mu - m) \psi(x) = 0.$$  \hspace{1cm} (6.38)

Using this with the Fourier expansion of $\psi$ given by Eq. (6.35) and the relations in Eqs. (6.36) and (6.37), we find $b(p_\perp, p^-; s; x^+)$ and $d\dagger(p_\perp, p^-; s; x^+)$ satisfy the following differential equations:

$$\begin{align*}
[i\gamma^\mu \partial^\mu - \gamma^\mu p^\mu] b(p_\perp, p^-; s; x^+) &= 0, \hspace{1cm} (6.39) \\
[i\gamma^\mu \partial^\mu + \gamma^\mu p^\mu] d\dagger(p_\perp, p^-; s; x^+) &= 0. \hspace{1cm} (6.40)
\end{align*}$$

Solving these equations gives

$$\begin{align*}
b(p_\perp, p^-; s; x^+) &= e^{-ix^\perp p^\perp} b(p_\perp, p^-; s; 0), \hspace{1cm} (6.41) \\
d\dagger(p_\perp, p^-; s; x^+) &= e^{ix^\perp p^\perp} d\dagger(p_\perp, p^-; s; 0). \hspace{1cm} (6.42)
\end{align*}$$

Since the time dependence decouples from the rest of the operator, we will from now on...
drop the time indices and define

\[ b(p_\perp, p_\parallel; s) \equiv b(p_\perp, p_\parallel; s; 0), \]
\[ d'(p_\perp, p_\parallel; s) \equiv d'(p_\perp, p_\parallel; s; 0). \]

(6.43)
(6.44)

Therefore, the fermion field can be written as

\[ \psi(x) = \int \frac{d^2 p_\perp dp_\parallel}{(2\pi)^3 2p^+} \sum_{s = \pm 1/2} \left[ u^{(s)} e^{-i x^\mu p^\mu} b(p_\perp, p_\parallel; s) + v^{(s)} e^{i x^\mu p^\mu} d'(p_\perp, p_\parallel; s) \right]. \]

(6.45)

Following a similar procedure, we can find the free photon field to be

\[ A^\mu(x) = \int \frac{d^2 p_\perp dp_\parallel}{(2\pi)^3 2p^+} \sum_{\lambda = \pm} e^\lambda(p, \lambda) \left[ e^{-i x^\mu p^\mu} a(p_\perp, p_\parallel; s) + e^{i x^\mu p^\mu} a^\dagger(p_\perp, p_\parallel; s) \right], \]

(6.46)

where the polarization vectors \( e^\lambda(p, \pm) \) are given by Eqs. (4.2) and (4.3).

6.2.3 Momentum and Angular Momentum

Using Noether’s theorem, the conserved energy-momentum tensor and angular momentum tensor can be written as

\[ T^\lambda_\mu = i \bar{\psi} \gamma^\lambda \partial^\mu \psi + (\partial_\mu A_\lambda) F^\lambda_\mu - g^\lambda_\mu \mathcal{L}, \]
\[ J^\lambda_\mu = x^\mu T^\lambda_\mu - x_\mu T^\lambda_\mu + S^\lambda_\mu, \]

(6.47)
(6.48)

where

\[ S^\lambda_\mu = i \frac{1}{4} \bar{\psi} \gamma^\lambda [\gamma_\mu, \gamma_\nu] \psi + F^\lambda_\mu A_\nu - F^\lambda_\nu A_\mu. \]

(6.49)

In particular, the total momentum and total angular momentum

\[ P_\mu = \int d^2 x^\perp dx^\parallel \tilde{T}^\perp_\mu, \]
\[ M_\mu^\parallel = \int d^2 x^\perp dx^\parallel \tilde{J}^\parallel_\mu, \]

(6.50)
(6.51)
are constants of motion. The “kinematic” operators $P_1, P_2, P_3, M_{12}, M_{23}, M_{13}$ are given by

\[
T_\alpha^\hat{=} = i\bar{\psi}\gamma^\hat{=} \partial_\alpha \psi - \partial_\alpha A_i (\partial^\hat{=} A^i - \partial^i A^\hat{=}), \quad (\alpha = 1, 2, \hat{=}) \tag{6.52}
\]

\[
J_{12}^\hat{=} = x_1 T_2^\hat{=} - x_2 T_1^\hat{=} + \frac{1}{2}i\bar{\psi}\gamma^\hat{=} \gamma_1 \gamma_2 \psi
\]

\[+ A^2 \partial^\hat{=} A^1 - A^1 \partial^\hat{=} A^2 + A^1 \partial^2 A^\hat{=} - A^2 \partial^1 A^\hat{=}, \tag{6.53}
\]

\[
J_{1\hat{=}}^\hat{=} = x_1 T_\hat{=}^\hat{=} - x_\hat{=} T_1^\hat{=} + \frac{1}{2}i\bar{\psi}\gamma^\hat{=} \gamma_1 \gamma_\hat{=} \psi
\]

\[+ A_\hat{=} \partial^\hat{=} A_1 - A_1 \partial^\hat{=} A_\hat{=} + A_1 \partial_\hat{=} A^\hat{=} - A_\hat{=} \partial_1 A^\hat{=}, \tag{6.54}
\]

\[
J_{2\hat{=}}^\hat{=} = x_2 T_\hat{=}^\hat{=} - x_\hat{=} T_2^\hat{=} + \frac{2}{2}i\bar{\psi}\gamma^\hat{=} \gamma_2 \gamma_\hat{=} \psi
\]

\[+ A_\hat{=} \partial^\hat{=} A_2 - A_2 \partial^\hat{=} A_\hat{=} + A_2 \partial_\hat{=} A^\hat{=} - A_\hat{=} \partial_2 A^\hat{=}, \tag{6.55}
\]

where $A_\hat{=}$ and $A^\hat{=}$ are given by Eq. (6.32) and Eq. (6.33), and thus these operators only involve independent fields $\psi$ and $A_i$, and are independent of the coupling constant $e$.

And finally, the most important operator of the theory of course is the Hamiltonian:

\[
P_\hat{=} = T_\hat{=}^\hat{=} = -\bar{\psi} \left[ \sum_{j=1}^2 i\gamma^j \partial_j + i\gamma^\hat{=} \partial_\hat{=} - m \right] \psi + e\bar{A}_\mu \gamma^\mu \psi + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \partial_\hat{=} A_j F^\hat{=} j. \tag{6.56}
\]

**6.2.4 Hamiltonian and the Old-fashioned Perturbation Theory**

With Eqs. (6.32) - (6.34), we can rewrite $T_\hat{=}^\hat{=}$ in terms of the independent degrees of freedom $A^1, A^2, \psi$ and separate out the interaction part of the Hamiltonian from the free part. Eq. (6.56) becomes

\[
H \equiv T_\hat{=}^\hat{=} = H_0 + V \tag{6.57}
\]

with

\[
H_0 = \frac{1}{2} \partial^k \tilde{A}^h \partial_k \tilde{A}_h - \frac{1}{2} \partial^\hat{=} \tilde{A}^k \partial_\hat{=} \tilde{A}_k + \bar{\psi}(-i\gamma^\hat{=} \partial_\hat{=}) - i\gamma^\hat{=} \partial_\hat{=} \psi,
\]

\[
V = e\bar{A}_\mu J^\mu - \frac{1}{2} e^2 J^\hat{=} \frac{1}{\partial^2 - \partial_\hat{=}^2} J^\hat{=}. \tag{6.59}
\]

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where we’ve defined $\tilde{A}_\mu$ in terms of $A_1$ and $A_2$ as

$$\left( \tilde{A}_\perp, \tilde{A}_1, \tilde{A}_2, \tilde{A}_\perp \right) \equiv \left( S \frac{\partial_\perp A_1}{\partial_\perp}, A_1, A_2, -\frac{C \partial_\perp A_1}{\partial_\perp} \right). \quad (6.60)$$

We can then calculate the scattering matrix with the “old-fashioned” perturbation theory expansion

$$S_{fi} = 1 - i2\pi\delta(H_i - H_f)[V + V(H - H_0 + i\epsilon)^{-1}V + \cdots]. \quad (6.61)$$

This leads to the same rules for $x^{\hat{}}$-ordered diagrams which we obtained in Sec. 6.1.2 by directly decomposing the covariant Feynman diagrams. This can be seen by calculating a few matrix elements of the interaction Hamiltonian $V$. The first term in Eq. (6.59) gives the interactions

$$-iV_1 = -i \int d^2x^\perp dx^\perp e\tilde{A}_\mu \bar{\psi}^\gamma \gamma^\mu \psi \quad (6.62)$$

which is the “ordinary” vertex interaction as demonstrated in Fig. 6.2a. The second term in Eq. (6.59) written out in full is

$$-iV_2 = -\frac{1}{2}ie^2 \int d^2x^\perp dx^\perp d^2y^\perp dy^\perp e^2\bar{\psi}(0, x^\perp, x^\perp)\gamma^\hat{\mu} \psi(0, x^\perp, x^\perp)$$

$$\times \frac{1}{q_\perp^2 + q_\perp^2} \bar{\psi}(0, y^\perp, y^\perp)\gamma^\hat{\mu} \psi(0, y^\perp, y^\perp), \quad (6.63)$$

This gives the instantaneous interaction vertex shown in Fig. 6.2b. Thus, when we calculate the scattering matrix formally, we get the same rules as when we decompose the covariant Feynman diagrams directly.

### 6.3 Summary

In this chapter, the interpolation of the QED theory is presented. We outline two different derivations of the Feynman rules for $x^{\hat{}}$-ordered diagrams formulated at any interpolation angle. The first approach directly decomposes the covariant Feynman diagram, and the second one utilizes the canonical field theory and the old-fashioned perturbation theory.

The interpolating fermion propagators which provides the last piece to the interpolating
QED theory, gives an instantaneous contribution only at the exact light-front ($\delta = \pi/4$). This also makes the Feynman rules differ in form for $\delta \neq \pi/4$ and $\delta = \pi/4$. A more detailed study of the fermion propagator in calculations of Compton scattering as well as the two-photon production in lepton and antilepton pair annihilation may bring more insights into how this sudden appearance of the instantaneous interaction at the light-front can be understood. These are our ongoing and forthcoming researches, and we will leave the discussion for the future.

The canonical field theory is studied for interpolation angle $\delta < \pi/4$. Equations of motion, free fields, gauge condition, momentum and angular momentum tensor were examined, and the Hamiltonian at constant $x^\hat{r}$ is found. Although this by no means exhausts all the things one can study in the interpolating canonical field theory, it paves the ground for detailed researches in this direction.
The previous chapters dealt with the interpolation of QED theory, which lends great support to the light-front theory, and reveals the advantages of LFD. It is then natural to build a light-front quark model (LFQM) based on LFD for hadron phenomenology. In this chapter, we take a look at how the light-front theory can be applied in a constituent quark model for ground state mesons. This work is mainly an update on the analysis of a previously used LFQM [CJ99]. Using the variational principle, we compute mass spectra and decay constants of ground state pseudoscalar and vector mesons in the light-front quark model (LFQM) with the QCD-motivated effective Hamiltonian including the hyperfine interaction. By smearing out the Dirac delta function in the hyperfine interaction, we avoid the issue of negative infinity in applying the variational principle to the computation of meson mass spectra and provide analytic expressions for the meson mass spectra. Our analysis with the smeared hyperfine interaction indicates that the interaction for the heavy meson sector including the bottom and charm quarks gets more point-like. We also consider the flavor mixing effect in our analysis and determine the mixing angles from the mass spectra of \((\omega, \phi)\) and \((\eta, \eta')\). Our variational analysis with
the trial wave function including the two lowest order harmonic oscillator basis functions appears to improve the agreement with the data of meson decay constants and the heavy meson mass spectra over the previous computation handling the hyperfine interaction as perturbation.

This chapter is organized as follows: In Sec. 7.2, we describe our QCD-motivated effective Hamiltonian with the smeared-out hyperfine interaction. Using the mixture of the two lowest order HO states as our trial wave function of the variational principle, we find the analytic formula of the mass eigenvalues for the ground state pseudoscalar and vector mesons. The optimum values of model parameters are also presented in this section. In Sec. 7.3, we present our numerical results of the mass spectra obtained by taking a larger HO basis in the trial wave function and compare them with the experimental data as well as our previous calculations [CJ99; CJ09]. To test our trial wave function with the parameters obtained from the variational principle, we also calculate the meson decay constants and compare them with the experimental data as well as other available theoretical predictions. A summary follows in Sec. 7.4. The detailed procedure of fixing our parameters through variational principle is presented in C.1.

7.1 Introduction

Effective degrees of freedom to describe a strongly interacting system of hadrons have been one of the key issues in understanding the non-perturbative nature of QCD in the low energy regime. Within an impressive array of effective theories available nowadays, the constituent quark model has been quite useful in providing a good physical picture of hadrons just like the atomic model for the system of atoms. Absorbing the complicated effect of quark, antiquark and gluon interactions into the effective constituent degrees of freedom, one may make the problem more tractable yet still keep some key features of the underlying QCD to provide useful predictions [Bag97]. The effective potentials used in constituent quark models are typically described by the flux tube configurations generated by the gluon fields as well as the effective “one-gluon-exchange” calculation in QCD [Luc91; De 75]. In the QCD-motivated effective Hamiltonian, a proper way of dealing with the relativistic effects in the hadron system is quite essential due to the nature of strong interactions. In particular, proper care and handling of relativistic effects has been emphasized in describing the hadrons made of $u$, $d$, and $s$ quarks and antiquarks.
As a proper way of handling relativistic effects, the light-front quark model (LFQM) [KT01; Che97; Hwa10; Chu88; Car95] appears to be one of the most efficient and effective tools in hadron physics as it takes advantage of the distinguished features of the light-front dynamics (LFD) [Dir49; Bro98a]. In particular, the LFD carries the maximum number (seven) of the kinetic (or interaction independent) generators and thus the less effort in dynamics is necessary in order to get the QCD solutions that reflect the full Poincaré symmetries. Moreover, the rational energy-momentum dispersion relation of LFD, namely \( p^- = (p^2_\perp + m^2)/p^+ \), yields the sign correlation between the light-front (LF) energy \( p^- = (p^0 - p^3) \) and the LF longitudinal momentum \( p^+ = (p^0 + p^3) \) and leads to the suppression of quantum fluctuations of the vacuum, sweeping the complicated vacuum fluctuations into the zero-modes in the limit of \( p^+ \to 0 \) [BS99; Mel98; CJ98]. This simplification is a remarkable advantage in LFD and facilitates the partonic interpretation of the amplitudes. Based on the advantages of the LFD, the LFQM has been developed [CJ99] and subsequently applied for various meson phenomenologies such as the mass spectra of both heavy and light mesons [CJ09], the decay constants, distribution amplitudes, form factors and generalized parton distributions [Bro98a; CJ99; CJ09; CJ07; Jau99; Jau03; Che04; CJ11; MF12; Jau90; Cho07].

Despite these successes in reproducing the general features of the data, however, it has proved very difficult to obtain direct connection between the LFQM and QCD. Typically, rigorous derivations of the connection between the effective constituent degrees of freedom and the fundamental QCD quark, antiquark and gluon degrees of freedom have been explored by solving momentum-dependent mass gap equations as discussed in many-body Hamiltonian approach [AD84; Le 85; LE04], Dyson-Schwinger approach [RW94], etc. Although one has not yet explored solving the momentum-dependent mass gap equation in LFD, there has been some attempt to derive an effective LF Hamiltonian starting from QCD using the discrete light-cone quantization (DLCQ) and solve the corresponding equation of motion approximately for the quark and antiquark boundstates to provide semianalytical expressions for the masses of pseudoscalar and vector mesons [PM97]. The attempt to link between QCD and LFQM is also supported by our recent analyses of quark-antiquark distribution amplitudes for pseudoscalar and vector mesons in LFQM [CJ14], where we presented a self-consistent covariant description of twist 2 and twist 3 quark-antiquark distribution amplitudes for pseudoscalar and vector mesons in LFQM to discuss the link between the chiral symmetry of QCD and the LFQM.
Our results for the pseudoscalar and vector mesons [CJ14] effectively indicated that the constituent quark and antiquark in the LFQM could be considered as the dressed constituents including the zero-mode quantum fluctuations from the QCD vacuum. These developments motivate our present work for the more-in-depth analysis of the mass spectra and decay constants for the ground state pseudoscalar and vector mesons in LFQM.

In LFQM, the LF wave function is independent of all reference frames related by the front-form boosts because the longitudinal boost operator as well as the LF transverse boost operators are all kinematical. This is clearly an advantageous feature unique to LFQM, which makes the calculation of observables such as mass spectra, decay constants, form factors, etc. much more effective. Computing the meson mass spectra, however, we have previously [CJ99; CJ09] treated the hyperfine interaction as a perturbation rather than including it in the variation procedure to avoid the negative infinity from the Dirac delta function contained in the hyperfine interaction. In the present work, we smear out the Dirac delta function by a Gaussian distribution and resolve the infinity problem when variational principle is applied to the hyperfine interaction. We obtain optimal model parameters in our variational analysis including the hyperfine interaction and examine if it improves phenomenologically our numerical results compared to the ones obtained by the perturbative treatment of the hyperfine interaction. For our trial wave function, we also take a larger harmonic oscillator (HO) basis to see if it provides any phenomenological improvement in our predictions of mass spectra and decay constants for ground state pseudoscalar and vector mesons.

7.2 Model Description

As mentioned in the introduction, there has been an attempt to derive an effective LF Hamiltonian starting from QCD using DLCQ [PM97]. Transforming the LFD variables to the ordinary variables in the instant form dynamics (IFD), one may see the equivalence between the resulting effective LF Hamiltonian for the quark and antiquark bound-states and the usual relativistic constituent quark model Hamiltonian for mesons typically given in the rest frame of the meson, i.e. the center of mass (C.M.) frame for the constituent quark and antiquark system. It may be more intuitive to express the effective LF Hamiltonian describing the relativistic constituent quark model system for mesons in terms of the ordinary IFD variables. Effectively, the meson system at rest is then described as an
interacting bound system of effectively dressed valence quark and antiquark typically
given by the following QCD-motivated effective Hamiltonian in the quark and antiquark
C.M. frame [CJ99; CJ09]:

\[ H_{\text{C.M.}} = \sqrt{m_q^2 + \vec{k}^2} + \sqrt{m_{\bar{q}}^2 + \vec{k}^2} + V, \]  

(7.1)

where \( \vec{k} = (k_\perp, k_z) \) is the relativistic three-momentum of the constituent quarks and \( V \) 
is the effective potential between quark and antiquark in the rest frame of the meson. The effective potential \( V \) is typically given by the linear confining potential \( V_{\text{conf}} \) plus the effective one-gluon-exchange potential \( V_{\text{oge}} \). For \( S \)-wave pseudoscalar and vector mesons, the effective one-gluon-exchange potential reduces to the coulomb potential \( V_{\text{coul}} \) plus the hyperfine interaction \( V_{\text{hyp}} \). Thus, one may summarize \( V \) as

\[ V = V_{\text{conf}} + V_{\text{oge}} \]

\[ = a + b r - \frac{4\alpha_s}{3r} + \frac{2}{3m_q m_{\bar{q}}} \left( \frac{S_q \cdot S_{\bar{q}}}{m_q m_{\bar{q}}} \nabla^2 V_{\text{coul}} \right), \]  

(7.2)

where \( \alpha_s \) is the strong interaction coupling constant\(^1\), \( \langle S_q \cdot S_{\bar{q}} \rangle = 1/4 \) \((-3/4)\) for the vector (pseudoscalar) meson and \( \nabla^2 V_{\text{coul}} = (16\pi\alpha_s/3)\delta^3(\mathbf{r}) \). Reduction of the LF Hamiltonian in QCD to a similar form of the effective Hamiltonian in the C.M. frame of the quark and antiquark system given by Eqs. (7.1) and (7.2) was discussed in Ref. [PM97]. For the hyperfine interaction \( V_{\text{hyp}} \), one may consider the relativization such as \( V_{\text{hyp}} \rightarrow \sqrt{m_q m_{\bar{q}}/E_q E_{\bar{q}}} V_{\text{hyp}} \sqrt{m_q m_{\bar{q}}/E_q E_{\bar{q}}} \) [CI86; GI85]. Such relativization may be important for the \( \delta^3(\mathbf{r}) \)-type potential without any smearing in computing particularly the light meson sector. Since we apply the variational principle even for the hyperfine interaction in this work smearing out the Dirac delta function to resolve the infinity problem, we naturally introduce a smearing parameter which may effectively compensate the factor due to the relativization. With this treatment, we are able to provide explicit analytic expressions for the meson mass spectra (see Eq. (7.9)).

While the effective bound-state mass square \( M_{qq}^2 \) is given by \( M_{qq}^2 = (P_{\text{C.M.}}^0)^2 \) in the C.M. frame of the constituent quark and antiquark system, the energy-momentum dispersion

\(^1\)Although one may consider a running coupling constant, we take \( \alpha_s \) as one of the variation parameters in this work.
relation in LFD is given by $M_{qq}^2 = P^+ P^- - P_\perp^2$, where the four-momentum of the bound system is denoted by $P^\mu = (P^+, P^-, P_\perp) = (P^0 + P^3, P^0 - P^3, P_\perp)$. From this, one may consider the LFQM mass square operator $\hat{P}^+ \hat{P}^- - \hat{P}_\perp^2$ (that provides the eigenvalues $P^+ P^- - P_\perp^2$) as the square of the effective Hamiltonian given by Eq. (7.1), i.e. $H_{\text{C.M.}}^2$. Since the eigenvalues and the expectation values are same for the eigenstates, we compute the expectation value $\langle H_{\text{C.M.}} \rangle$ using the variation principle. Alternatively, one may consider computing the expectation value $\langle H_{\text{C.M.}}^2 \rangle$ in view of the LFQM mass square operator $\hat{P}^+ \hat{P}^- - \hat{P}_\perp^2$ being $\langle H_{\text{C.M.}}^2 \rangle$. Although $\langle (\Delta H_{\text{C.M.}})^2 \rangle = 0$ in principle for the eigenstates, it may be interesting to examine numerically how small the corresponding deviation $\langle (\Delta H_{\text{C.M.}})^2 \rangle = \langle H_{\text{C.M.}}^2 \rangle - \langle H_{\text{C.M.}} \rangle^2$ is. More future works complementary to our present computation of $\langle H_{\text{C.M.}} \rangle$ can be suggested in variational analysis. In this work, we examine the $\chi^2$ values of our computational results in comparison with experimental data to get optimal parameter values in the $\chi$ computation. This will provide useful ground information for any alternative and/or further works beyond the present analysis.

As discussed earlier, the longitudinal boost operator as well as the LF transverse boost operators are all kinematical and thus the LF wave function does not depend on the external momentum, i.e. $P^+$ and $P_\perp$. In effect, the determination of the LF wave function in the meson rest frame such as $P^+ = M_{qq}$ and $P_\perp = 0$ won’t hinder its use for any other values of $P^+$ and $P_\perp$. This provides the applicability of LFQM for the computation of observables beyond the meson mass spectra.

The wave function is thus represented by the Lorentz invariant internal variables $x_i = \frac{p_i^+}{P^+}$, $k_{\perp i} = P_{\perp i} - x_i P_\perp$ and helicity $\lambda_i$, where $p_i^\mu$ is the momenta of constituent quarks. Explicitly, the LF wave function of the ground state mesons is given by

$$\Psi_{100}^{J_{z}}(x_i, k_{\perp i}, \lambda_i) = \mathcal{R}_{\lambda_i \lambda_q}^{J_{z} q} (x_i, k_{\perp i}) \Phi(x_i, k_{\perp i}), \quad (7.3)$$

where $\Phi$ is the radial wave function and $\mathcal{R}_{\lambda_q \lambda_q}^{J_{z} q}$ is the interaction-independent spin-orbit wave function. The spin-orbit wave functions for pseudoscalar and vector mesons are given by [CJ99; Jau91]

$$\mathcal{R}_{\lambda_q \lambda_q}^{00} = \frac{-\bar{u}_{\lambda_q} (p_q) \gamma^5 \nu_{\lambda_q} (p_q)}{\sqrt{2} \sqrt{M_0^2 - (m_q - m_q)^2}}, \quad (7.4)$$

$$\mathcal{R}_{\lambda_q \lambda_q}^{1J_{z}} = \frac{-\bar{u}_{\lambda_q} (p_q) \left[ f(J_z) - \frac{c(J_z)}{M_0 + m_q + m_q} \right] \nu_{\lambda_q} (p_q)}{\sqrt{2} \sqrt{M_0^2 - (m_q - m_q)^2}}.$$
where $\epsilon^\mu(J_z)$ is the polarization vector of the vector meson and the boost invariant meson mass squared $M_0^2$ obtained from the free energies of the constituents is given by

$$M_0^2 = \frac{k^2 + m_q^2}{x} + \frac{k_\perp^2 + m_{\bar{q}}^2}{1-x}. \quad (7.5)$$

The spin-orbit wave functions satisfy the relation $\sum_\lambda \lambda_q \hat{R}^{J_z}_\lambda R^{J_z}_\lambda = 1$ for both pseudoscalar and vector mesons.

To use a variational principle, we take our trial wave function as an expansion of the true wave function in the HO basis. We use the same trial wave function expanded with the two lowest order HO wave functions

$$\Phi = \sum_{n=1}^2 c_n \phi_{nS} \text{ for both pseudoscalar and vector mesons, where}$$

$$\phi_{1S}(x_i, k_{\perp i}) = \frac{4\pi^{3/4}}{\beta^{3/2}} \sqrt{\frac{\partial k_z}{\partial x}} e^{-k^2/2\beta^2}, \quad (7.6)$$

$$\phi_{2S}(x_i, k_{\perp i}) = \frac{4\pi^{3/4}}{\sqrt{6\beta^{7/2}}} \left(2k^2 - 3\beta^2\right) \sqrt{\frac{\partial k_z}{\partial x}} e^{-k^2/2\beta^2}. \quad (7.7)$$

and $\beta$ is the variational parameter. We should note that our LF wave functions $\phi_{nS}$ depends on $M_0^2$ since $k^2 = k_{\perp}^2 + k_z^2$ and $k_z = (x - 1/2)M_0 + (m_q^2 - m_{\bar{q}}^2)/2M_0$. For instance, $e^{-k^2/2\beta^2} = e^{-m^2/2\beta^2} e^{-M_0^2/8\beta^2}$ for the equal quark and antiquark mass $m_q = m_{\bar{q}} = m$. The Jacobian of the variable transformation $(x, k_{\perp}) \to (k_{\perp}, k_z)$ is given by $\partial k_z/\partial x = M_0[1 - (m_q^2 - m_{\bar{q}}^2)^2/M_0^4]/4x(1-x)$ and thus the wave function $\phi_{nS}$ satisfies the following normalization:

$$\int_0^1 dx \int \frac{d^2k_{\perp}}{16\pi^3} |\phi_{nS}(x_i, k_{\perp i})|^2 = 1. \quad (7.8)$$

With $\Phi = \sum_{n=1}^2 c_n \phi_{nS}$, we evaluate the expectation value of the Hamiltonian in Eq. (7.1), i.e. $\langle \Phi | H_{C.M.} | \Phi \rangle$ which depends on the variational parameter $\beta$. According to the variational principle, we can set the upper limit of the ground state’s energy by calculating the expectation value of the system’s Hamiltonian with a trial wave function. In our previous calculations [CJ99; CJ09], which we call “CJ model", we first evaluate the expectation value of the central Hamiltonian $T + V_{\text{conf}} + V_{\text{coul}}$ with the trial function $\phi_{1S}$, where $T$ is the kinetic energy part of the Hamiltonian. Once the model parameters are fixed by minimizing the expectation value $\langle \phi_{1S} | (T + V_{\text{conf}} + V_{\text{coul}}) | \phi_{1S} \rangle$, then the mass eigenvalue of each meson is obtained as $M_{qq} = \langle \phi_{1S} | H_{C.M.} | \phi_{1S} \rangle$. The hyperfine interaction $V_{\text{hyp}}$ in CJ model, which contains a Dirac delta function, was treated as perturbation to...
the Hamiltonian and was left out in the variational process that optimizes the model parameters. The main reason for doing this was to avoid the negative infinity generated by the delta function as was pointed out in [CI86]. Specifically, \( \langle \phi_{1S} | V_{\text{hyp}} | \phi_{1S} \rangle \) for pseudoscalar mesons decreases faster than other terms that increase as \( \beta \) increases and the expectation value of the Hamiltonian is unbounded from below.

To avoid the negative infinity, we use a Gaussian smearing function to weaken the singularity of \( \delta^3(\mathbf{r}) \) in hyperfine interaction, viz. [CI86; GI85], \( \delta^3(\mathbf{r}) \rightarrow (\sigma^3/\pi^{3/2})e^{-\sigma^2\mathbf{r}^2} \). Once the delta function is smeared out like this, a true minimum for the mass occurs at a finite value of \( \beta \). The analytic formulae of mass eigenvalues for our modified Hamiltonian with the smeared-out hyperfine interaction, i.e. \( M_{q\bar{q}} = \langle \Phi | H_{C.M.} | \Phi \rangle \), are found as follows:

\[
M_{q\bar{q}} = a + \frac{b}{\beta \sqrt{\pi}} \left( 3 - c_1^2 - 2 \sqrt{\frac{2}{3}} c_1 c_2 \right) + \frac{\beta}{\sqrt{\pi}} \sum_{i=q,\bar{q}} \left\{ \sqrt{\pi} \left( \sqrt{6} c_1 c_2 - 3 c_2^2 \right) U \left( -\frac{1}{2}, -2, z_i \right) \right. \\
+ \frac{1}{3} c_2^2 z_i^2 e^{z_i^2} (3 - z_i) K_2 \left( \frac{z_i}{2} \right) + \frac{1}{6} z_i c_2 z_i \left( 2 c_2^2 z_i^2 - 3 c_1^2 - 6 \sqrt{6} c_1 c_2 + 9 \right) K_1 \left( \frac{z_i}{2} \right) \left\} \\
- \frac{4\alpha_s \beta}{9\sqrt{\pi}} \left\{ 5 + c_1^2 + 6 \sqrt{2/3} c_1 c_2 - \frac{4\beta^2 \sigma^3 (S_q \cdot S_{\bar{q}})}{\left( \beta^2 + \sigma^2 \right)^{7/2} m_q m_{\bar{q}}} \left[ \left( 2 \sqrt{6} c_1 c_2 + 3 - c_1^2 \right) \sigma^4 \right. \\
\left. + 2\beta^2 \left( 2 c_1^2 + \sqrt{6} c_1 c_2 \right) \sigma^2 + 2 \beta^4 \right] \right\}, \quad (7.9)
\]

where \( z_i = m_i^2/\beta^2 \) and \( K_1 \) is the modified Bessel function of the second kind and \( U(a, b, z) \) is Tricomi’s (confluent hypergeometric) function. We should note that the mass formula for the delta-function hyperfine interaction corresponds to Eq. (7.9) in the limit of \( \sigma \rightarrow \infty \). We then apply the variational principle, i.e. \( \partial M_{q\bar{q}} / \partial \beta = 0 \), to find the optimal model parameters in order to get a best fit for the mass spectra of ground state pseudoscalar and vector mesons (a more detailed description of this procedure can be found in Appendix C.1).

Our optimized potential parameters are obtained as \( \{ a = -0.6699 \text{ GeV}, b = 0.18 \text{ GeV}^2, \alpha_s = 0.4829 \} \). For the best fit of the ground state mass spectra, we obtain \( c_1 = +\sqrt{0.7} \) and \( c_2 = +\sqrt{0.3} \). We should note that our potential parameters are quite comparable with the ones suggested by Scora and Isgur [SI95], where they obtained \( a = -0.81 \text{ GeV}, b = 0.18 \text{ GeV}^2 \), and \( \alpha_s = 0.3 \sim 0.6 \). For a comparison, the coupling constant we found in our previous CJ model [CJ99; CJ09] was \( \alpha_s = 0.31 \).

\[2\]Although the true minimum occurs with the smeared-out hyperfine interaction even for \( \phi_{1S} \) case, we found that the phenomenological results do not show any improvement compared to CJ model.
Table 7.1 Constituent quark masses [GeV] and the smearing parameter $\sigma$ [GeV] obtained by the variational principle for the Hamiltonian with a smeared-out hyperfine interaction. Here $q = u$ and $d$.

<table>
<thead>
<tr>
<th>$m_q$</th>
<th>$m_s$</th>
<th>$m_c$</th>
<th>$m_b$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.205</td>
<td>0.380</td>
<td>1.75</td>
<td>5.15</td>
<td>0.423</td>
</tr>
</tbody>
</table>

While we use the common potential parameters $(a, b, \alpha_s)$ for all the mesons, it was shown in [GI85; Bad11] that if a smearing procedure for the $\delta^3(r)$ function is used, then a large Gaussian parameter $\sigma$ is obtained for the heavy quark sector. In our updated potential model using the smeared hyperfine interaction $(\sigma^2/\pi^{3/2})e^{-\sigma^2r^2}$, we also confirm the same observation as in [GI85] for the heavy meson sector including $(b, c)$ quarks. Thus, we differentiate the smearing parameter $\sigma$ for the heavy $(b, c)$ sectors such as $(\bar{c}c, \bar{b}c, \bar{b}\bar{b})$ from the other $(q\bar{q})$ sectors by introducing multiplicative factor in front of $\sigma$, i.e. $\sigma \to \lambda \sigma$ with $\lambda > 1$, while other potential parameters $(a, b, \alpha_s)$ remain the same for all $(q\bar{q})$ sectors. This differentiation is to accommodate the hyperfine splittings for the heavy $(b, c)$ quark sectors as we will show in the next section. Our new updated results with $\lambda$ differentiation using the common potential parameters $(a, b, \alpha_s, \sigma)$ for all meson sectors show definite improvement in the $\chi^2$ fit of the experimental data for meson masses.

Our optimal constituent quark masses and the smearing parameters $\sigma$ are listed in Table 7.1. Since we included the hyperfine interaction with smearing function entirely in our variational process, we now obtain the two different sets of $\beta$ values, one for pseudoscalar and the other for vector mesons, respectively. The optimal Gaussian parameters $\beta_{q\bar{q}}$ for pseudoscalar and vector mesons are also listed in Table 7.2. We should note that the values of the multiplicative factor $\lambda$ to get the best fits for the mass eigenvalues are obtained as $\lambda = (2, 2.3, 3)$ for $(\bar{c}c, \bar{b}c, \bar{b}\bar{b})$ sectors. As a sensitivity check, however, we present the numerical results with the following theoretical error bars $\lambda = (2^{+1}_{-1}, 2.3^{+1}_{-1}, 3^{+2}_{-2})$ for $(\bar{c}c, \bar{b}c, \bar{b}\bar{b})$ sectors, respectively. Although one may fine-tune more to improve the hyperfine splittings for the heavy-light sectors by using different set of $\lambda$ parameters, we set $\lambda = 1$ for any other $q\bar{q}$ sectors except $(\bar{c}c, \bar{b}c, \bar{b}\bar{b})$ sectors in this work for simplicity.

We also determine the mixing angles from the mass spectra of $(\omega, \phi)$ and $(\eta, \eta')$. Identifying $(\mathcal{F}, \mathcal{F}') = (\phi, \omega)$ and $(\eta, \eta')$ for vector and pseudoscalar nonets, the flavor
Table 7.2 The Gaussian parameter $\beta$ [GeV] for ground state pseudoscalar ($J^{PC} = 0^{-+}$) and vector ($1^{-+}$) mesons obtained by the variational principle. $q = u$ and $d$. We should note that $\lambda = (2_{-1}^{+1}, 2.3_{-1}^{+1}, 3_{-2}^{+2})$ are used to get $(\beta_{cc}, \beta_{bc}, \beta_{bb})$ values and $\lambda = 1$ is used to get the rest of $\beta_{qq}$ values.

<table>
<thead>
<tr>
<th>$J^{PC}$</th>
<th>$\beta_{qq}$</th>
<th>$\beta_{qs}$</th>
<th>$\beta_{ss}$</th>
<th>$\beta_{qs}$</th>
<th>$\beta_{bc}$</th>
<th>$\beta_{bb}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0^{-+}</td>
<td>0.4465</td>
<td>0.3759</td>
<td>0.3445</td>
<td>0.3861</td>
<td>0.3859</td>
<td>0.4226</td>
</tr>
<tr>
<td>1^{-+}</td>
<td>0.2346</td>
<td>0.2598</td>
<td>0.2820</td>
<td>0.3445</td>
<td>0.3667</td>
<td>0.4914+0.0062-0.0058</td>
</tr>
</tbody>
</table>

Assignment of $\mathcal{F}$ and $\mathcal{F}'$ mesons in the quark-flavor basis $n\bar{n} = (u\bar{u} + d\bar{d})/\sqrt{2}$ and $s\bar{s}$ is given by [Fel98; Fel00; Leu98]

$\begin{pmatrix} \mathcal{F} \\ \mathcal{F}' \end{pmatrix} = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} n\bar{n} \\ s\bar{s} \end{pmatrix} = U(\alpha) \begin{pmatrix} n\bar{n} \\ s\bar{s} \end{pmatrix},$ (7.10)

where $\alpha$ is the mixing angle in the quark-flavor basis. For the $\eta - \eta'$ mixing, the SU(3) mixing angle $\theta$ in the the flavor SU(3) octet-singlet basis ($\eta_8, \eta_1$) can also be used and the relation between the mixing angles is given by $\theta = \alpha - \arctan{\sqrt{2}} \simeq 54.7^\circ$[OG14].

Taking into account SU(3) symmetry breaking and using the parametrization for the (mass)$^2$ matrix suggested by Scadron [Sca84], we obtain [CJ99]

$\tan^2 \alpha = \frac{(M_{\phi}^2 - M_{\eta}^2)(M_{\otimes}^2 - M_{\eta}^2)}{(M_{\otimes}^2 - M_{\eta}^2)(M_{\eta}^2 - M_{\phi}^2)},$ (7.11)

which is the model-independent equation for any $q\bar{q}$ meson nonets. The details of obtaining meson mixing angles using quark-annihilation diagrams are summarized in [CJ99], where the mixing angle $\delta = \alpha - 90^\circ$ is used in the quark-flavor basis. In order to predict the $\omega - \phi$ and $\eta - \eta'$ mixing angles, we use the experimental values of $M_{\mathcal{F}} = (M_\phi, M_\eta)$ and $M_{\mathcal{F}'} = (M_\omega, M_{\eta'})$ as well as the masses of $M_V^{\eta} [M_{s\bar{s}}] = 780$ (901) MeV and $M_P^{\eta} [M_{s\bar{s}]} = 140$ (726) MeV obtained from $\langle \Phi |H_{s\bar{s}}| \Phi \rangle$ for both vector ($V$) and pseudoscalar ($P$) mesons, respectively.

Our prediction for $\omega - \phi$ mixing angle is $\alpha_{\omega - \phi} = 84.8^\circ$, which is about $5.2^\circ$ deviated from the ideal mixing $\alpha_{\omega - \phi} = 90^\circ$. Our prediction for $\eta - \eta'$ mixing angle is $\alpha_{\eta - \eta'} = 36.3^\circ$, which is in agreement with the range $34.7^\circ$ to $44.7^\circ$ of phenomenological values [Fel98; OG14].

Our updated model with the smeared hyperfine interaction appears to improve the result of mass spectrum, which is presented in the next section. This may suggest that
when using constituent quark models, the contact interactions has to be smeared out in general. In fact, we think this smeared interaction is more consistent with the physical picture for a system of the effective constituent quarks which are not point-like.

For practical application of our model, we also compute the decay constants for the ground state pseudoscalar and vector mesons. The decay constants are typically defined by

\[
\langle 0 | \bar{q} \gamma^\mu \gamma_5 q | P \rangle = i f_P P^\mu, \\
\langle 0 | \bar{q} \gamma^\mu q | V(P, h) \rangle = f_V M_V e^\mu(h),
\]

for pseudoscalar and vector mesons, respectively. For the \( \eta \) and \( \eta' \) case, one may also define decay constants through matrix elements of octet and singlet axial-vector currents. However, as discussed in \[Fel98; Fel00\], they cannot be expressed as \( U(\theta) \text{diag}[f_8, f_1] \) due to the \( U(1)_A \) anomaly. Thus, the following two mixing angle parametrization is adopted \[Fel98; Fel00\]

\[
f_8^{\eta} = f_8 \cos \theta_8, \quad f_1^{\eta} = -f_1 \sin \theta_1, \\
f_8^{\eta'} = f_8 \sin \theta_8, \quad f_1^{\eta'} = f_1 \cos \theta_1.
\]

The parameters appearing in Eq. (7.13) are related to the basis parameters \( \alpha, f_q \equiv f_{n\bar{n}} \) and \( f_s \equiv f_{s\bar{s}} \), characterizing the quark-flavor mixing scheme as follows \[Fel98\]:

\[
f_8^2 = \frac{f_q^2 + 2f_s^2}{3}, \quad \theta_8 = \alpha - \arctan \left( \frac{\sqrt{2}f_s}{f_q} \right), \\
f_1^2 = \frac{2f_q^2 + f_s^2}{3}, \quad \theta_1 = \alpha - \arctan \left( \frac{\sqrt{2}f_q}{f_s} \right).
\]

Using the plus component \( (\mu = +) \) of the currents, one can calculate the decay constants. The explicit formulae of pseudoscalar and vector meson decay constants in quark-flavor basis are given by \[CJ99; Jau91\]

\[
f_P = \sqrt{6} \int_0^1 dx \int \frac{d^2 k_\perp}{8\pi^3} \frac{\Phi(x, k_\perp)}{\sqrt{A^2 + k_\perp^2}} A, \\
f_V = \sqrt{6} \int_0^1 dx \int \frac{d^2 k_\perp}{8\pi^3} \frac{\Phi(x, k_\perp)}{\sqrt{A^2 + k_\perp^2}} \left[ A + \frac{2k_\perp^2}{D_{LF}} \right],
\]

(7.15)
where $A = (1 - x)m_q + x\bar{m}_q$ and $D_{LF} = M_0 + m_q + m_{\bar{q}}$.

### 7.3 Results and Discussion

In Fig. 7.1, we show the masses (and hyperfine splittings) and the corresponding decay constants of heavy quarkonia depending on the variation of the multiplicative factor $\lambda$. For $(c\bar{c})$ [Fig. 7.1(a) and Fig. 7.1(b)] and $(b\bar{b})$ [Fig. 7.1(c) and Fig. 7.1(d)], we plot the curves corresponding three different $\lambda$ values, i.e. $\lambda\sigma = (1, 2, 3)\sigma$ and $\lambda\sigma = (1, 3, 5)\sigma$, respectively. The results indicate that the interaction between heavier quarks gets more point-like as the larger $\lambda$ values are favored in comparison with data. In general, as one can see from Fig. 7.1, the hyperfine splittings for both charmonium [Fig. 7.1(a)] and bottomonium [Fig. 7.1(c)] states increases as $\lambda$ increases while other potential parameters remain the same. On the other hand, the decay constants of vector mesons decrease while the corresponding decay constants of pseudoscalar mesons increase as $\lambda$ increases [see Fig. 7.1(b) and Fig. 7.1(d)]. One may increase $\lambda$ value even further to have better hyperfine splittings compared to the data. However, one may not increase $\lambda$ value arbitrarily to accommodate the empirical constraint $f_V \geq f_P$. Similarly, we could improve the hyperfine splitting for $(b\bar{c})$ sector by using $\lambda = 2.3$.

We show in Fig. 7.2 our prediction of the meson mass spectra obtained from the variational principle to the effective Hamiltonian with the smeared-out hyperfine interaction using the trial function $\Phi = \sum c_i \phi_{iS}$ and compare them with the experimental data [OG14] as well as the results obtained from the CJ model with the linear confining potential [CJ99]. We should note that the $(\pi, \rho)$ masses are used as inputs. The $(\eta, \eta', \omega, \phi)$ masses are also used as inputs to find the $(\eta - \eta')$ and $(\omega - \phi)$ mixing angles. The theoretical error bars for $(c\bar{c}, b\bar{c}, b\bar{b})$ sectors are due to the usage of $\lambda = (2^{+1}_{-1}, 2.3^{+1}_{-1}, 3^{+2}_{-2})$ values, respectively. As one can see, our trial wave function $\Phi$ including more HO basis generates overall better results than our CJ model. This can be seen from our $\chi^2 = 0.008$ compared to $\chi^2 = 0.012$ obtained from the CJ model [CJ09]. Except the mass of $K$, our predictions for the masses of $1S$-state pseudoscalar and vector mesons are within 4% error. Especially, our effective Hamiltonian with the smeared hyperfine interaction using $\Phi$ clearly improves the predictions of heavy-light and heavy quarkonia systems such as $(\eta_c, J/\psi, B_c, \eta_b, \Upsilon)$ compared to the CJ model adopting the contact hyperfine interaction.
Figure 7.1 (color online). The masses (and hyperfine splittings) and the corresponding decay constants of heavy quarkonia depending on the variation of the multiplicative factor $\lambda$, i.e. 
(a) [Fig. 7.1(a) and Fig. 7.1(b)] with two different $\lambda \sigma = (1, 2) \sigma$ and (b) [Fig. 7.1(c) and Fig. 7.1(d)] with two different $\lambda \sigma = (1, 3) \sigma$ values, respectively.
Figure 7.2 (color online). Fit of the ground state meson masses [MeV] with the parameters given in Table 7.2 and 7.1 compared with the fit from our previous calculations using CJ model [CJ09] as well as the experimental values. The $(\pi, \rho)$ masses are our input data. The $(\eta, \eta', \omega, \phi)$ masses are also used as input to find the $(\eta - \eta')$ and $(\omega - \phi)$ mixing angles. The theoretical error bars for $(c\bar{c}, b\bar{c}, bb)$ sectors are due to the usage of $\lambda = (2^+_1, 2.3^+_1, 3^+_2)$ values, respectively.
Table 7.3 Decay Constants for light mesons (in unit of MeV) obtained from our updated LFQM.

<table>
<thead>
<tr>
<th>Model</th>
<th>$f_\pi$</th>
<th>$f_\rho$</th>
<th>$f_K$</th>
<th>$f_{K^*}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>This work</td>
<td>130</td>
<td>205</td>
<td>161</td>
<td>224</td>
</tr>
<tr>
<td>CJ [CJ07]</td>
<td>130</td>
<td>246</td>
<td>161</td>
<td>256</td>
</tr>
<tr>
<td>Exp. [OG14]</td>
<td>130.4(2)</td>
<td>208$^{(a)}$, 216(5)$^{(b)}$</td>
<td>156.1(8)</td>
<td>217(7)</td>
</tr>
</tbody>
</table>

$^{(a)}$ Exp. value for $\Gamma(\tau \rightarrow \rho \nu_{\tau})$.
$^{(b)}$ Exp. value for $\rho^0 \rightarrow e^+e^-$. 

Table 7.4 Decay constants in the singlet-octet basis and the mixing angle in the quark-flavor basis.

<table>
<thead>
<tr>
<th>Reference</th>
<th>$f_8/f_{f_\pi}$</th>
<th>$\theta_8$</th>
<th>$f_1/f_{f_\pi}$</th>
<th>$\theta_1$</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>This work</td>
<td>1.30</td>
<td>-27.3$^\circ$</td>
<td>1.16</td>
<td>-8.6$^\circ$</td>
<td>36.3$^\circ$</td>
</tr>
<tr>
<td>[Fel98]</td>
<td>1.26</td>
<td>-21.2$^\circ$</td>
<td>1.17</td>
<td>-9.2$^\circ$</td>
<td>39.3$^\circ$</td>
</tr>
<tr>
<td>[Leu98]</td>
<td>1.28</td>
<td>-20.5$^\circ$</td>
<td>1.25</td>
<td>-4$^\circ$</td>
<td>-</td>
</tr>
<tr>
<td>[EF05]</td>
<td>1.51</td>
<td>-23.8$^\circ$</td>
<td>1.29</td>
<td>-2.4$^\circ$</td>
<td>40.7$^\circ$</td>
</tr>
<tr>
<td>[Sch93]</td>
<td>1.27</td>
<td>-19.5$^\circ$</td>
<td>1.17</td>
<td>-5.5$^\circ$</td>
<td>42.1$^\circ$</td>
</tr>
</tbody>
</table>

Although the experimental data for $B_c^*$ is not yet available, our predictions of $B_c^*$, i.e. $6330^{+5}_{-3}$ MeV, are quite comparable with the lattice prediction 6331(9) MeV [Dow12] as well as other quark model predictions such as 6340 MeV [GI85] and 6345.8 MeV [Fre02].

In Table 7.3, we list our predictions for the decay constants of light mesons ($\pi, K, \rho, K^*$) obtained by using the mixed wave function $\Phi$ of $1S$ and $2S$ HO states and compare them with the results from the CJ model [CJ07] and the experimental data [OG14]. As one can see, our updated model calculation including the hyperfine interaction in the variation procedure clearly improves the results over the CJ model.

For the decay constant of $\phi$ meson, our prediction for the ideal mixing angle ($\alpha_{\text{ideal}} = 90^\circ$) is given by $f_\phi = f_{ss}^V = 245.1$ MeV. However, we obtain $f_\phi = f_{ss}^V = 226$ MeV using our predicted mixing angle $\alpha_{\omega-\phi} = 84.8^\circ$. Comparing to the experimental value $f_\phi^e = 233$ MeV [OG14] (extracted from the partial width of $\phi \rightarrow e^+e^-$ decay), our prediction for $f_\phi$ prefers a rather small $\omega - \phi$ mixing angle such as $\alpha_{\omega-\phi} \simeq 87.5^\circ$ than the ideal mixing.

For the decay constants of $\eta$ and $\eta'$, our predictions of the decay constants $f_\eta$ and
\(f_s\) are given by \(f_q = 130\) MeV and \(f_s = 184.8\) MeV so that \(f_q/f_\pi = 1\) and \(f_s/f_\pi = 1.42\), where the SU(3) breaking effect is manifest in the ratio \(f_q/f_s \neq 1\). Using Eq. (7.14), we obtain \(f_8/f_\pi = 1.30\) and \(f_1/f_\pi = 1.16\) with \(\theta_s = -27.3^\circ\) and \(\theta_1 = -8.6^\circ\), respectively.

In Table 7.4, we compare our results for the decay constants in the singlet-octet basis and the mixing angle in the quark-flavor basis with other theoretical predictions [Fel98; Leu98; EF05; Sch93]. As one can see, our results are consistent with other theoretical model results. Since the experimental values are very well known for light mesons, this improvement is very encouraging.

In Table 7.5, we list our predictions for the charmed meson decay constants \((f_D, f_{D^*}, f_{D_s}, f_{D_s^*}, f_{D_c}, f_{J/\Psi})\) together with CJ model [Cho07], lattice QCD [Beč99; Aok14; Beč14; Beč12], QCD sum rules [Gel13], relativistic Bethe-Salpeter (BS) model [Cve04], relativized quark model [CG90], and other relativistic quark model (RQM) [Ebe06] predictions as well as the available experimental data [OG14; EC00]. We extract the experimental value \((f_{J/\Psi})_{\text{exp}} = (407 \pm 5)\) MeV from the data \(\Gamma_{\text{exp}}(J/\Psi \rightarrow e^+e^-) = (5.55 \pm 0.14)\) keV [OG14] and the formula

\[
\Gamma(V \rightarrow e^+e^-) = \frac{4\pi}{3}\alpha_{\text{QED}}^2 e_Q^2 f_V^2 M_V^3, \tag{7.16}
\]

where \(e_Q\) is the electric charge of the heavy quark in units of \(e\) (2/3 for \(c\) and \(-1/3\) for \(b\)). We should note that our results of the ratios \(f_{D_s}/f_D = 1.11\) and \(f_{D_c}/f_{J/\Psi} = 0.98^{+0.08}_{-0.07}\) are quite comparable with the available experimental data, \(f_{D_s}/f_D = 1.25 \pm 0.06\) [OG14] and \(f_{D_c}/f_{J/\Psi} = 0.81 \pm 0.19\) [OG14; EC00], respectively. Our result of the ratios \(f_{D_s^*}/f_{D^*} = 1.13\) is also in good agreement with other theoretical model calculations such as \(1.16 \pm 0.02 \pm 0.06\) from the lattice QCD [Beč12] and \(1.10 \pm 0.06\) from the BS model [Cve04].

We list our results for the bottomed mesons \((f_B, f_{B^*}, f_{B_s}, f_{B_s^*}, f_{b}, f_{\Upsilon})\) in Table 7.6, and compare with CJ model [Cho07], lattice QCD [Beč99; Aok14; Col14], QCD sum rules [Gel13], BS model [Cve04], relativized quark model [CG90], and RQM [Ebe06] predictions as well as the available experimental data [OG14; Ika06]. Note that we extract the experimental value \((f_{\Upsilon})_{\text{exp}} = (689 \pm 5)\) MeV from the data \(\Gamma_{\text{exp}}(\Upsilon \rightarrow e^+e^-) = 1.340 \pm 0.018\) keV [OG14] and Eq. (7.16) with \(e_Q^2 = 1/9\) for \(V = \Upsilon\). Our results for the ratios \(f_{B_s}/f_B = 1.13\) and \(f_{B_s^*}/f_{B_s^*} = 1.15\) are in good agreement with the QCD sum rules [Gel13] predictions: \(f_{B_s}/f_B = 1.17^{+0.04}_{-0.03}\), and \(f_{B_s^*}/f_{B_s^*} = 1.20 \pm 0.04\). Ours are also in good agreement with the lattice results, \(f_{B_s}/f_B = 1.206(24)\) [Aok14] and \(f_{B_s^*}/f_{B_s^*} = 1.17(4)^{+1}_{-3}\) [Beč99]. Our result for the ratio \(f_b/f_{\Upsilon} = 0.99^{+0.07}_{-0.04}\) is consistent with
Table 7.5 Charmed meson decay constants (in unit of MeV) obtained from our updated LFQM. The theoretical error bars for \( f_{\eta_c(J/\psi)} \) come from the variation of the smearing parameters \( \sigma \), i.e. \( f_{\eta_c(J/\psi)}(2\sigma^{+\sigma} - 2\sigma^{-\sigma}) \).

<table>
<thead>
<tr>
<th>Model</th>
<th>( f_D )</th>
<th>( f_{D^*} )</th>
<th>( f_{D_s} )</th>
<th>( f_{D_s^*} )</th>
<th>( f_{\eta_c} )</th>
<th>( f_{J/\psi} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>This work</td>
<td>208</td>
<td>230</td>
<td>231</td>
<td>260</td>
<td>353^{+22}_{-17}</td>
<td>361^{+6}_{-7}</td>
</tr>
<tr>
<td>CJ [Cho07]</td>
<td>197</td>
<td>239</td>
<td>232</td>
<td>273</td>
<td>326</td>
<td>360</td>
</tr>
<tr>
<td>Lattice [Beč99]</td>
<td>211±3±17</td>
<td>245±20^{+3}_{-2}</td>
<td>231±12^{+8}_{-17}</td>
<td>272±16^{+3}_{-20}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sum-rules [Gel13]</td>
<td>201^{+13}_{-15}</td>
<td>242^{+20}_{-12}</td>
<td>238^{+13}_{-23}</td>
<td>293^{+19}_{-14}</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>QM [CG90]</td>
<td>240±20</td>
<td>-</td>
<td>290±20</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>RQM [Ebe06]</td>
<td>234</td>
<td>310</td>
<td>268</td>
<td>315</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Exp.</td>
<td>206.7±8.9 [OG14]</td>
<td>-</td>
<td>257.5±6.1 [OG14]</td>
<td>-</td>
<td>335±75 [EC00]</td>
<td>407±5 [OG14]</td>
</tr>
</tbody>
</table>

Table 7.6 Bottomed meson decay constants (in unit of MeV) obtained from our updated LFQM. The theoretical error bars for \( f_{\eta_b(\Upsilon)} \) come from the variation of the smearing parameters \( \sigma \), i.e. \( f_{\eta_b(\Upsilon)}(3\sigma^{+2\sigma} - 2\sigma^{-2\sigma}) \).

<table>
<thead>
<tr>
<th>Model</th>
<th>( f_B )</th>
<th>( f_{B^*} )</th>
<th>( f_{B_s} )</th>
<th>( f_{B_s^*} )</th>
<th>( f_{\eta_b} )</th>
<th>( f_\Upsilon )</th>
</tr>
</thead>
<tbody>
<tr>
<td>This work</td>
<td>181</td>
<td>188</td>
<td>205</td>
<td>216</td>
<td>605^{+12}_{-17}</td>
<td>611^{+11}_{-14}</td>
</tr>
<tr>
<td>CJ [Cho07]</td>
<td>171</td>
<td>185</td>
<td>205</td>
<td>220</td>
<td>507</td>
<td>529</td>
</tr>
<tr>
<td>Lattice [Beč99]</td>
<td>179±18^{+34}_{-39}</td>
<td>196±24^{+39}_{-2}</td>
<td>204±16^{+41}_{-9}</td>
<td>229±20^{+43}_{-16}</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>QCD [Aok14; Col14]</td>
<td>189±8 [Aok14]</td>
<td>-</td>
<td>228±8 [Aok14]</td>
<td>-</td>
<td>649±31 [Col14]</td>
<td></td>
</tr>
<tr>
<td>Sum-rules [Gel13]</td>
<td>207^{+17}_{-9}</td>
<td>210^{+10}_{-12}</td>
<td>242^{+17}_{-12}</td>
<td>251^{+14}_{-16}</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>BS [Cve04]</td>
<td>196±29</td>
<td>238±18</td>
<td>216±32</td>
<td>272±20</td>
<td>498±20</td>
<td></td>
</tr>
<tr>
<td>QM [CG90]</td>
<td>155±15</td>
<td>-</td>
<td>210±20</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>RQM [Ebe06]</td>
<td>189</td>
<td>219</td>
<td>218</td>
<td>251</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Exp.</td>
<td>229^{+36}<em>{-31}^{+34}</em>{-37} [Ika00]</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>689±5 [OG14]</td>
</tr>
</tbody>
</table>

the heavy quark symmetry \( f_{\eta_b}/f_\Upsilon = 1 \) [LP07]. One can also see that for heavy charmed and bottomed mesons, the trial wave function \( \phi_B \) produces better results when compared with the experimental data as well as the lattice results.

In Table 7.7, we present our model predictions for the decay constants of \( f_{B_c} \) and \( f_{B_c^*} \), and compare them with other model calculations [CJ09; Iva01; Ebe03; EQ94; Ger95; Ful98; CG90]. Our results are comparable with other model calculations.
Table 7.7: Bottom-charmed meson decay constants (in unit of MeV) obtained from our updated LFQM. The theoretical error bars for $f_{B_c(B_c^*)}$ come from the variation of the smearing parameters $\sigma$, i.e. $f_{B_c(B_c^*)}(2.3\sigma^+ - \sigma^-)$.

<table>
<thead>
<tr>
<th>Model</th>
<th>This work</th>
<th>CJ [CJ09]</th>
<th>[Iva01]</th>
<th>[Ebe03]</th>
<th>[EQ94]</th>
<th>[Ger95]</th>
<th>[Ful98]</th>
<th>[CG90]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_{B_c}$</td>
<td>$389^{+16}_{-36}$</td>
<td>349</td>
<td>360</td>
<td>433</td>
<td>500</td>
<td>$460 \pm 60$</td>
<td>517</td>
<td>410 $\pm$ 40</td>
</tr>
<tr>
<td>$f_{B_c^*}$</td>
<td>$391^{+53}_{-44}$</td>
<td>369</td>
<td>–</td>
<td>503</td>
<td>500</td>
<td>$460 \pm 60$</td>
<td>517</td>
<td>–</td>
</tr>
</tbody>
</table>

7.4 Summary

In this chapter, we updated our LFQM by smearing out the Dirac delta function in the hyperfine interaction to avoid the issue of negative infinity in applying the variational principle to the computation of meson mass spectra, while our previous model (CJ model) used the perturbation method to handle the delta function in the contact hyperfine interaction. Using the mixed wave function $\Phi$ of $1S$ and $2S$ HO states as the trial wave function, we calculated both the mass spectra of the ground state pseudoscalar and vector mesons and the decay constants of the corresponding mesons. The flavor mixing effect has also been implemented for the meson systems of $(\omega,\phi)$ and $(\eta,\eta')$.

The variational analysis with $\Phi$ seems to improve the agreement with the data of meson decay constants over the results of the CJ model. It also appears to provide better agreement with data in the heavy meson mass spectra. Accommodating the empirical constraint $f_V \geq f_P$, we have shown that the mass spectra and the hyperfine splittings for heavy $(b,c)$ quark sector get improved by introducing the multiplicative factor $\lambda$ in front of the smearing parameter $\sigma$, i.e. $\sigma \rightarrow \lambda \sigma$, in the smeared hyperfine interaction $(\sigma^3/\pi^{3/2})e^{-\sigma^2 r^2}$. Our results indicate that the interaction between heavier quarks gets more point-like as the larger $\lambda$ values are favored in comparison with data. The distinction between the heavy meson sector and the light meson sector is rather natural in our LFQM analysis. To get more definite conclusion in this respect, further analysis of other wave function related observables such as various meson elastic and transition form factors may be useful. It may also be interesting to analyze radially excited meson states using the larger HO basis.
In this thesis, we investigated the interpolation between the instant form and the light-front form of the QED theory. This not only provides a rich landscape between the IFD and the LFD to study, but also gives valuable insights into the LFD itself.

In LFD, the light-front gauge \((A^+ = (A^0 + A^3)/\sqrt{2} = 0)\) is commonly used, since the transverse polarizations of the gauge field can be immediately identified as the dynamical degrees of freedom, and ghost fields can be ignored in the quantum action of non-Abelian gauge theory [LB80; Bas87; Bas98; Lei87; Lei88; LW00]. This makes it especially attractive in various QCD applications. In Chapter 4, we found that the light-front gauge in the LFD is naturally linked to the Coulomb gauge in the IFD through the interpolation angle, at least in the QED theory we are studying here. The corresponding gauge propagator that interpolates between the IFD and the LFD also sheds light on the debate about whether the gauge propagator should be the two-term form [Mus91] or the three-term form [Lei84; SB01; SS03; SS04b; SS04a; MW05]. For example, by analyzing the lowest-order sQED Feynman amplitude, one may typically get the corresponding three time-ordered amplitudes, one of which corresponds to the contribution from the instantaneous interaction of the gauge field. This contribution from the instantaneous interaction is, however, precisely canceled by one of the terms in the three-term gauge propagator and thus the two-term gauge propagator may also be used effectively for the calculation of the same Feynman amplitude without involving the instantaneous interaction of the gauge field. Otherwise, to maintain the equivalence
to the covariant formulation, the three-term propagator should be used including the instantaneous interaction of the gauge field.

Our study of the interpolating helicity spinor in Chapter 5 suggests that the light-front spinor is naturally connected to the instant form helicity spinor. Therefore, instead of trying to compare the light-front spinor to the Dirac spinor as in the traditional Melosh transformation, the light-front spinor may be more appropriately thought of as the “generalized helicity spinor”. We also found that the behavior of the spin orientation angle $\theta_s$ as a function of the momentum direction $\theta$ bifurcates into two branches. The IFD and LFD each belongs to a separate branch. This not only demonstrates in a larger context the main difference between the instant form helicity spinor and the light-front spinor, but also demands a clear distinction between the IMF and the LFD, as the helicity spinors in the $-\infty$ momentum frame does not coincide with the light-front spinors. The analysis of this spin orientation angle at the light-front reveals that the change of this angle when boosted in the $\pm z$ direction is exactly the Wigner rotation angle required, and thus explains physically why using the light-front spinors guarantees the boost invariance of the helicity amplitudes and no additional Wigner rotation is needed.

When plotting the $x^\pm$-ordered amplitudes in Chapter 4 and the helicity amplitudes in Chapter 5 in terms of both the interpolation angle $\delta$ and the total momentum of the system $P^z$, we discovered a universal J-curve which is independent of the specific kinematics and is only scaled by the center of mass energy. This J-curve appears in all the amplitudes and probabilities we calculated, and gives rise to the singular behavior at $P^z = -\infty$ or $P^+ = 0$. The careful analysis of this J-curve provides a deeper understanding of the treacherous zero-mode issue in the LFD. Depending on how the total momentum and the interpolation angle is correlated when going to the light-front limit, it may or may not agree with the exact light-front result at that treacherous point ($P^+ = 0$). This fact that the J-curve and this singularity exist only in the $P^z < 0$ half of the plane clearly demonstrates the asymmetry between the positive IMF and the negative IMF. Our plots show that the LFD results are only equivalent to the positive IMF results and disconnected from the negative IMF. Furthermore, the LFD is invariant in all frames and does not require an infinite momentum frame. This further clears up the confusion in the prevailing notion of equivalence between the LFD and the IMF.

In Chapter 6, we presented the interpolating $x^\pm$-ordered fermion propagators, which unlike the photon propagators, do not produce the instantaneous interaction term unless
it’s exactly on the light-front ($\delta = \pi/4$). This singular behavior makes it seemingly impossible to combine the Feynman rules for $x^\pm$-ordered diagrams when $\delta < \pi/4$ and when $\delta = \pi/4$, although a more in-depth analysis of the interpolating fermion propagators and a careful study of their limits to the light-front case is needed. Canonical field theory approach is also briefly covered, but a more complete and detailed study is still desirable. Although our current work on the abelian QED theory has already brought a lot of great insights into the connection between the instant form and the light-front form, it would be also interesting to see what happens when one extends this interpolation study to the non-abelian QCD theory. We will leave these further investigations to future researches.

Finally, the updated analysis in our simple LFQM for ground-state pseudoscalar and vector mesons in Chapter 7 shows the advantage of the light-front formulation. In the LFD, because the vacuum is simple, the wave function has a simple interpretation, and because the wave function is boost invariant, once we fix our model parameters by fitting the mass spectrum, we can immediately use our obtained light-front wave function to calculate other quantities, in this case, the decay constants. Although our phenomenological model Hamiltonian is relatively simple and a clear connection to the fundamental QCD theory is still lacking, it demonstrates the principle of light-front quark model calculations. With the smeared out hyperfine interaction, we avoided the negative infinity in applying the variational principle to the full Hamiltonian, and obtained an analytical form for the mass spectrum, and improved decay constants results.
BIBLIOGRAPHY


[Col14] Colquhoun, B. et al. “\(\Upsilon\) and \(\Upsilon'\) leptonic widths, \(a_b^\mu\) and \(m_b\) from full lattice QCD”. *Phys. Rev. D* **91.7** (2014), p. 074514.


APPENDICES
A.1 Derivation of photon polarization vectors

We use the following explicit 4-vector representation of $K$ and $J$ operators given by

\[
\begin{align*}
K_1 &= \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & K_2 &= \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & K_3 &= \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \\
J_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, & J_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, & J_3 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{align*}
\] (A.1)

Upon the application of $T$ transformation given by Eq. (3.34) with the above 4-vector
representation for operators $K$ and $J$, we find the polarization vectors:

$$
\epsilon_{\hat{\mu}}(P,+) = -\sqrt{2} \begin{pmatrix}
\sin \delta \left( \frac{\beta_1 \sin \alpha}{\alpha} + i \beta_2 \sin \alpha \right) \\
\frac{C}{\sqrt{2}} \left( \frac{\beta_2^2 + \beta_1^2 \cos \alpha}{\alpha^2} + i \beta_1 \beta_2 (-1 + \cos \alpha) \right) \\
-\cos \delta \left( \frac{\beta_1 \sin \alpha}{\alpha} + i \beta_2 \sin \alpha \right)
\end{pmatrix},
$$

(A.2a)

$$
\epsilon_{\hat{\mu}}(P,-) = \sqrt{2} \begin{pmatrix}
\sin \delta \left( \frac{\beta_1 \sin \alpha}{\alpha} - i \beta_2 \sin \alpha \right) \\
\frac{C}{\sqrt{2}} \left( \frac{\beta_2^2 + \beta_1^2 \cos \alpha}{\alpha^2} - i \beta_1 \beta_2 (-1 + \cos \alpha) \right) \\
-\cos \delta \left( \frac{\beta_1 \sin \alpha}{\alpha} - i \beta_2 \sin \alpha \right)
\end{pmatrix},
$$

(A.2b)

$$
\epsilon_{\hat{\mu}}(P,0) = \sqrt{2} \begin{pmatrix}
\cos \delta (\sin \delta \cosh \beta_3 + \cos \delta \sinh \beta_3) - \sin \delta \cos \alpha (\cos \delta \cosh \beta_3 + \sin \delta \sinh \beta_3) \\
\frac{C}{\alpha} \beta_1 \sin \alpha (\sin \delta \cosh \beta_3 + \cos \delta \sinh \beta_3) \\
\frac{\alpha}{\beta_2 \sin \alpha (\sin \delta \cosh \beta_3 + \cos \delta \sinh \beta_3)} \\
\cos \delta \cos \alpha (\cos \delta \cosh \beta_3 + \sin \delta \sinh \beta_3) - \sin \delta (\sin \delta \cosh \beta_3 + \cos \delta \sinh \beta_3)
\end{pmatrix}
$$

(A.2c)

Using the relations listed in Eqs. (3.45) - (3.51), and multiplying the transverse polarization vectors $\epsilon_{\hat{\mu}}(P,\pm)$ with the phase factor $P_{\perp}^{1-2iP_{\perp}}$ to simplify, we arrive at the following polarization vectors for the 4-momentum $P_{\mu}$:

$$
\epsilon_{\hat{\mu}}(P,+) = -\frac{1}{\sqrt{2}P_{\perp}} \left( \sin \delta |P_{\perp}|, \frac{P_{\perp} P_{\perp}^{1-2iP_{\perp}}}{|P_{\perp}|}, \frac{P_{\perp}^2 P_{\perp}^{1-2iP_{\perp}}}{|P_{\perp}|}, -\cos \delta |P_{\perp}| \right),
$$

(A.3)
$$\epsilon_{\hat{\mu}}(P, -) = \frac{1}{\sqrt{2P}} \left( \sin \delta |P_\perp|, \frac{P_1 P_\perp + iP_2 P_\perp}{|P_\perp|}, -\cos \delta |P_\perp| \right), \quad (A.4)$$

$$\epsilon_{\hat{\mu}}(P, 0) = \frac{1}{M_P} \left( \cos \delta - \frac{P_2 P_\perp \sin \delta}{C}, P_1 P_\perp, \frac{P_2 P_\perp - iP_1 P_\perp}{C}, P_1 - \cos \delta \frac{P_2 P_\perp}{C} \right). \quad (A.5)$$

The polarization vectors listed above are written in the form $\epsilon_{\hat{\mu}}(P, \lambda) = (\epsilon^0, \epsilon^1, \epsilon^2, \epsilon^3)$. We then change the basis to $(\epsilon_{\hat{\mu}}, \epsilon_1, \epsilon_2, \epsilon_3)$ using the relations listed in Eq. (3.8). Finally, we obtain the polarization vectors given by Eqs. (4.2) - (4.4):

$$\epsilon_{\hat{\mu}}(P, +) = -\frac{1}{\sqrt{2P}} \left( S|P_\perp|, \frac{P_1 P_\perp - iP_2 P_\perp}{|P_\perp|}, \frac{P_2 P_\perp + iP_1 P_\perp}{|P_\perp|}, -C|P_\perp| \right), \quad (A.6)$$

$$\epsilon_{\hat{\mu}}(P, -) = \frac{1}{\sqrt{2P}} \left( S|P_\perp|, \frac{P_1 P_\perp + iP_2 P_\perp}{|P_\perp|}, \frac{P_2 P_\perp - iP_1 P_\perp}{|P_\perp|}, -C|P_\perp| \right), \quad (A.7)$$

$$\epsilon_{\hat{\mu}}(P, 0) = \frac{P^\perp}{M_P} \left( P^\perp - \frac{M^2}{P^\perp}, P_1, P_2, P_\perp \right), \quad (A.8)$$

where the relation $P^2 = (P^\perp)^2 - M^2C$ is used to derive the first component of $\epsilon_{\hat{\mu}}(P, 0)$.

### A.2 $\mathcal{F}_{\hat{\mu}\hat{\nu}}$ and $\mathcal{L}_{\hat{\mu}\hat{\nu}}$ derived from the interpolating polarization vectors

In this Appendix, we derive the numerator of the photon propagator directly from the polarization vectors listed in Eqs. (4.2) - (4.4).

We evaluate the relevant matrix elements. With the 4-momentum $P = q$ for virtual photon, these are, for $\lambda = +1$:

$$\epsilon^*_\pm(q, \lambda = +)\epsilon_\pm(q, \lambda = +) = \frac{S^2|q_\perp|^2}{2Q^2}, \quad (A.9)$$

$$\epsilon^*_\pm(q, \lambda = +)\epsilon_1(q, \lambda = +) = \frac{S(q_2 - iq_1 Q)}{2Q^2}, \quad (A.10)$$

$$\epsilon^*_\pm(q, \lambda = +)\epsilon_2(q, \lambda = +) = \frac{S(q_2 + iq_1 Q)}{2Q^2}, \quad (A.11)$$

$$\epsilon^*_\pm(q, \lambda = +)\epsilon_\mp(q, \lambda = +) = -\frac{CS|q_\perp|^2}{2Q^2}, \quad (A.12)$$

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\[ \epsilon^*_\pm (q, \lambda = +)\epsilon^\pm (q, \lambda = +) = -\frac{CS|q_\perp|^2}{2Q^2}, \quad (A.13) \]

\[ \epsilon^*_\pm (q, \lambda = +)\epsilon^1 (q, \lambda = +) = -\frac{C (q_\perp q_1 - i q_2 Q)}{2Q^2}, \quad (A.14) \]

\[ \epsilon^*_\pm (q, \lambda = +)\epsilon^2 (q, \lambda = +) = -\frac{C (q_\perp q_2 + i q_1 Q)}{2Q^2}, \quad (A.15) \]

\[ \epsilon^*_\pm (q, \lambda = +)\epsilon^\pm (q, \lambda = +) = \frac{C^2|q_\perp|^2}{2Q^2}, \quad (A.16) \]

\[ \epsilon^*_1 (q, \lambda = +)\epsilon^\pm (q, \lambda = +) = \frac{S (q_\perp q_1 + i q_2 Q)}{2Q^2}, \quad (A.17) \]

\[ \epsilon^*_1 (q, \lambda = +)\epsilon^1 (q, \lambda = +) = \frac{q^2 q_1 q_1^2 + q_2^2 Q^2}{2|q_\perp|^2 Q^2}, \quad (A.18) \]

\[ \epsilon^*_1 (q, \lambda = +)\epsilon^2 (q, \lambda = +) = -\frac{C q_1 q_2 + i q_\perp Q}{2Q^2}, \quad (A.19) \]

\[ \epsilon^*_1 (q, \lambda = +)\epsilon^\pm (q, \lambda = +) = -\frac{C (q_\perp q_1 + i q_2 Q)}{2Q^2}, \quad (A.20) \]

\[ \epsilon^*_2 (q, \lambda = +)\epsilon^\pm (q, \lambda = +) = \frac{S (q_\perp q_2 - i q_1 Q)}{2Q^2}, \quad (A.21) \]

\[ \epsilon^*_2 (q, \lambda = +)\epsilon^1 (q, \lambda = +) = \frac{-C q_1 q_2 - i q_\perp Q}{2Q^2}, \quad (A.22) \]

\[ \epsilon^*_2 (q, \lambda = +)\epsilon^2 (q, \lambda = +) = \frac{q^2 q_2^2 + q_1^2 Q^2}{2|q_\perp|^2 Q^2}, \quad (A.23) \]

\[ \epsilon^*_2 (q, \lambda = +)\epsilon^\pm (q, \lambda = +) = \frac{-C (q_\perp q_2 - i q_1 Q)}{2Q^2}, \quad (A.24) \]

and for \( \lambda = -1 \):

\[ \epsilon^*_\pm (q, \lambda = -)\epsilon^\pm (q, \lambda = -) = \frac{S^2|q_\perp|^2}{2Q^2}, \quad (A.25) \]

\[ \epsilon^*_\pm (q, \lambda = -)\epsilon^1 (q, \lambda = -) = \frac{S (q_\perp q_1 + i q_2 Q)}{2Q^2}, \quad (A.26) \]

\[ \epsilon^*_\pm (q, \lambda = -)\epsilon^2 (q, \lambda = -) = \frac{S (q_\perp q_2 - i q_1 Q)}{2Q^2}, \quad (A.27) \]

\[ \epsilon^*_\pm (q, \lambda = -)\epsilon^\pm (q, \lambda = -) = -\frac{CS|q_\perp|^2}{2Q^2}, \quad (A.28) \]
\[ 
\begin{align*}
\epsilon^{-}(q, \lambda = -)\epsilon^{+}(q, \lambda = -) &= -\frac{\mathcal{C}S|q_\perp|^2}{2Q^2}, \\
\epsilon^{-}(q, \lambda = -)\epsilon(1)(q, \lambda = -) &= -\frac{\mathcal{C}(q_2q_2 - iq_1Q)}{2Q^2}, \\
\epsilon^{-}(q, \lambda = -)\epsilon_{2}(q, \lambda = -) &= -\frac{\mathcal{C}(q_2q_2 - iq_1Q)}{2Q^2}, \\
\epsilon^{-}(q, \lambda = -)\epsilon_{-}(q, \lambda = -) &= \frac{\mathcal{C}^2|q_\perp|^2}{2Q^2}, \\
\epsilon^{+}(q, \lambda = -)\epsilon_{-}(q, \lambda = -) &= \frac{\mathcal{C}(q_2q_2 - iq_1Q)}{2Q^2}, \\
\epsilon^{+}(q, \lambda = -)\epsilon_{1}(q, \lambda = -) &= \frac{\mathcal{C}(q_2q_2 + iq_1Q)}{2Q^2}, \\
\epsilon^{+}(q, \lambda = -)\epsilon_{-}(q, \lambda = -) &= \frac{\mathcal{C}(q_2q_2 + iq_1Q)}{2Q^2}, \\
\epsilon^{+}(q, \lambda = -)\epsilon_{-}(q, \lambda = -) &= \frac{\mathcal{C}^2|q_\perp|^2}{2Q^2}, \quad (A.30) \\
\epsilon^{-}(q, \lambda = -)\epsilon_{-}(q, \lambda = -) &= \frac{\mathcal{C}^2|q_\perp|^2}{2Q^2}, \quad (A.35) \\
\epsilon^{-}(q, \lambda = -)\epsilon_{-}(q, \lambda = -) &= \frac{\mathcal{C}^2|q_\perp|^2}{2Q^2}, \quad (A.40)
\end{align*} 
\]

where \( Q \equiv \sqrt{q_\perp^2 + q_\perp^2} \).

By defining \( \mathcal{F}_{\mu\nu} = \sum_{\lambda = +, -} \epsilon^{+}_{\mu}(q, \lambda)\epsilon_{\nu}(q, \lambda) \), we obtain the following matrix form:

\[
\mathcal{F}_{\mu\nu} = \frac{1}{Q^2} \begin{pmatrix}
S^2|q_\perp|^2 & Sq_\perp q_1 & Sq_\perp q_2 & -\mathcal{C}S|q_\perp|^2 \\
Sq_\perp q_1 & q_\perp^2 + Cq_\perp^2 & -Cq_1q_2 & -Cq_\perp q_1 \\
Sq_\perp q_2 & -Cq_1q_2 & q_\perp^2 + Cq_\perp^2 & -Cq_\perp q_2 \\
-\mathcal{C}S|q_\perp|^2 & -Cq_\perp q_1 & -Cq_\perp q_2 & C^2|q_\perp|^2 \\
\end{pmatrix}.
\]
This can be written in general as

$$\mathcal{T}_{\mu\nu} = -g_{\mu\nu} + X(q_\mu n_\nu + n_\mu q_\nu) + Y n_\mu n_\nu + Z q_\mu q_\nu,$$

(A.42)

where the coefficients $X$, $Y$, $Z$ can be fixed by comparing this equation with the explicit matrix form given by Eq. (A.41). We find

$$X = \frac{q_+^2}{Q^2} = \frac{(q \cdot n)}{Q^2},$$

(A.43)

$$Y = -\frac{q^2}{Q^2},$$

(A.44)

$$Z = -\frac{C}{Q^2}.$$  

(A.45)

Thus, our result agrees with Eq. (4.17) derived in the main text.

Similar manipulation can be done for the longitudinal part, $\mathcal{L}_{\mu\nu} = \epsilon_\mu^\ast(q,0)\epsilon^\nu(q,0)$, yielding

$$\mathcal{L}_{\mu\nu} = \frac{(q_+^2)^2}{(q^2)Q^2} \begin{pmatrix} L^2 & Lq_1 & Lq_2 & Lq_\perp \\ Lq_1 & q_1^2 & q_1q_2 & q_1q_1 \\ Lq_2 & q_1q_2 & q_2^2 & q_2q_2 \\ Lq_\perp & q_1q_1 & q_1q_2 & q_2^2 \end{pmatrix},$$

(A.46)

where we introduced for convenience, $L = q_+ - (q^2)/q_+^2$. Notice that because we are dealing with virtual photons, we have replaced $M^2$ by $P_{\mu}P^\mu = q^2$ in the longitudinal polarization vector given by Eq. (4.4). Written in a general form, this gives Eq. (4.21).

### A.3 Photon propagator decomposition on the light front

Because $C = 0$, there’s only one pole at $q_1^2/2q^+ - i\varepsilon/2q^+$ for the first term in Eq. (4.24). When $q^+ > 0$, the pole is in the lower half plane, and when $q^+ < 0$, the pole is in the upper half plane. So, for the first term in Eq. (4.24), we separate the $q^+$ integral into two
\[
\int \frac{d^2 \mathbf{q}_\perp}{(2\pi)^4} \left[ \int_0^\infty dq^+ e^{-i(q^+ x^- - q^\perp \cdot x^\perp)} I + \int_{-\infty}^0 dq^+ e^{-i(q^+ x^- - q^\perp \cdot x^\perp)} I \right], \tag{A.47}
\]

where

\[
I = \int_{-\infty}^\infty dq^- e^{-iq^- x^+} \frac{\mathcal{T}_{\mu\nu}}{2q^- q^+ - q^\perp_\perp^2 + i\epsilon}. \tag{A.48}
\]

To evaluate \(I\), we close the contour in the plane where the arc contribution is zero. This means we close the contour in the upper (lower) half plane when \(x^+ < 0\) (\(x^+ > 0\)). For the factor \(I\) in the first term of Eq. (A.47), the only non-zero contribution comes from the pole that’s in the lower half plane, which is picked up when we close the contour from below (\(x^+ > 0\)). Similarly, for the factor \(I\) in the second term of Eq. (A.47), the only non-zero contribution comes from the pole that’s in the upper half plane, which is picked up when we close the contour from above (\(x^+ < 0\)). Thus, Eq. (A.47) becomes

\[
\int \frac{d^2 \mathbf{q}_\perp}{(2\pi)^4} \mathcal{T}_{\mu\nu} \left[ \int_0^\infty dq^+ e^{-i(q^+ x^- - q^\perp \cdot x^\perp)(-2\pi i)} \Theta(x^+) \frac{e^{-iHx^+}}{2q^+} \right. \\
\left. + \int_{-\infty}^0 dq^+ e^{-i(q^+ x^- - q^\perp \cdot x^\perp)} (2\pi i) \Theta(-x^+) \frac{e^{iHx^+}}{2q^+} \right], \tag{A.49}
\]

where \(H = |q^\perp_\perp^2/2q^+|\).

We now make changes of the variables \(q^+ \to -q^+\) and \(q^\perp_\perp \to -q^\perp_\perp\) in the second term that’s proportional to \(\Theta(-x^+)\) in order to have the limits on the \(q^+\) integral become the same as the first term. We can then combine these two terms. This gives us

\[
i \int \frac{d^2 \mathbf{q}_\perp}{(2\pi)^3} \int_0^\infty dq^+ \mathcal{T}_{\mu\nu} \left[ \Theta(x^+) e^{-iHx^+} e^{-i(q^+ x^- - q^\perp \cdot x^\perp)} + \Theta(-x^+) e^{iHx^+} e^{i(q^+ x^- - q^\perp \cdot x^\perp)} \right] \\
= i \int \frac{d^2 \mathbf{q}_\perp}{(2\pi)^3} \int_0^\infty dq^+ \mathcal{T}_{\mu\nu} \left[ \Theta(x^+) e^{-iq^+ x^\mu} + \Theta(-x^+) e^{iq^+ x^\mu} \right]. \tag{A.50}
\]

Since we’ve made the change of variables and the \(q^+\) can not be negative anymore, we now have \(q^- = H = q^\perp_\perp^2/2q^+\) in \(q_\mu\).

The second term in Eq. (4.24) gives straightforwardly the same result as the last term in Eq. (4.27), where \(C = 0\) (LFD). Putting everything together, we can write Eq. (4.24)
in LFD as
\[
\iint d^2q_\perp \frac{d^2q_\perp}{(2\pi)^3} \frac{dx^+}{2q^+} \mathcal{M}(\mu \nu) \left[ \Theta(x^+) e^{-iq_\perp x^\mu} + \Theta(-x^+) e^{iq_\perp x^\mu} \right]
\]
\[+ i\delta(x^+) \int d^2q_\perp \frac{d^2q_\perp}{(2\pi)^3} \frac{dx^+}{2q^+} e^{-i(q^+ x^+ + q_\perp x^\perp)}. \tag{A.51} \]

### A.4 Time-ordered scattering amplitudes for $\phi^3$ Theory

In $\phi^3$ theory, the scattering shown in Fig. 4.1 involves the exchange of scalar particle. Ignoring the inessential factors including the square of the coupling constant, we can write the time-ordered scattering amplitudes as those used in Ref. [JS12]:

\[
2\mathcal{M}^{(a)} = \frac{1}{2Q^+ q^+ - Q^+}, \tag{A.52}
\]
\[
\mathcal{M}^{(b)} = -\frac{1}{2Q^+ q^+ + Q^+}; \tag{A.53}
\]

where $Q^+$ is defined in Eq. (4.54).

We use the same kinematics used for the sQED case, i.e. $q_\perp = 0$. Because the kinematics is the same, the boosted $q'$ satisfies the same equations as listed in Eqs. (4.91) - (4.94). Using Eqs. (3.4) - (3.7), we can also get the boosted $q'_\mu$ and $q'^\mu$, which can then be used to calculate the scattering amplitudes in the boosted frame.

The amplitudes are given by the functions of $m_1$, $m_2$, $p$ and $\theta$. For comparison, we use the same parameter values that we used for plotting Fig. 4.2 in sQED, i.e. $m_1 = 1$, $m_2 = 2$, $p = 3$, and $\theta = \pi/3$. $\mathcal{M}^{(a)}$ and $\mathcal{M}^{(b)}$ with the J-curve depicted by the solid red line are plotted in Fig. A.1. Comparing it to Fig. 4.2, we see that the photon propagator modifies the profiles of the time-ordered amplitudes. Nevertheless, the J-curve remains the same and is still given by Eq. (4.97).

### A.5 Time-ordered annihilation amplitudes in sQED

For the sQED annihilation process analogous to $e^+e^- \rightarrow \mu^+\mu^-$ in QED, we can still use Fig. 4.1 for the kinematics. Three time-ordered diagrams corresponding to the lowest
order amplitudes are shown in Fig. A.2. Here, the amplitude of Eq. (4.53) changes to

\[ i \mathcal{M} = \sum_{j=a,b,c} i \mathcal{M}^{(j)} \]

\[ = (-ie)^2 \sum_{j=a,b,c} (\vec{p}_3 - \vec{p}_4) \Sigma^{(j)}_{\mu\nu}(\vec{p}_1 - \vec{p}_2). \] (A.54)

The time-ordered amplitudes are given by

\[ x^+ \]

\[ \begin{array}{cccc}
  & \text{initial} & \text{intermediate} & \text{initial} \\
  p_1 & x & p_3 & p_1 \\
  p_2 & y & p_4 & p_2 \\
\end{array} \]

Figure A.1 (Color Online) Time-ordered scattering diagrams and corresponding amplitudes for the $\phi^3$ theory.
Figure A.2 Time-ordered annihilation diagrams and their amplitudes for sQED
\[ M^{(a)} = -(ie)^2 \left[ \frac{p_{34} \cdot p_{12} + p_{34} \hat{p}_{12} q^2 / (Q^\hat{+})^2}{2Q^\hat{+}(q^+ - q^\hat{+})} \right] C, \] (A.55)
\[ M^{(b)} = -(ie)^2 \left[ \frac{p_{34} \cdot p_{12} + p_{34} \hat{p}_{12} q^2 / (Q^\hat{+})^2}{2Q^\hat{+}(q^+ + Q^\hat{+})} \right] C, \] (A.56)
\[ M^{(c)} = -(ie)^2 \frac{p_{34} \hat{p}_{12}}{(Q^\hat{+})^2}, \] (A.57)

where

\[ p_{34} = p_3 - p_4, \] (A.58)
\[ p_{12} = p_1 - p_2. \] (A.59)

and \( q = p_1 + p_2 \). We use the same kinematics (CMF) as before so that \( p_i \)'s are given by

\[ p_1 = (\epsilon, 0, 0, p), \] (A.60)
\[ p_2 = (\epsilon, 0, 0, -p), \] (A.61)
\[ p_3 = (\epsilon, p_f \sin \theta, 0, p_f \cos \theta), \] (A.62)
\[ p_4 = (\epsilon, -p_f \sin \theta, 0, -p_f \cos \theta). \] (A.63)

Since we are considering the annihilation process, the initial particles 1 and 2 should have the same mass, and the final particles 3 and 4 should have the same mass. So, we set \( m_1 = m_2 = m \) and \( m_3 = m_4 = m_f \). Due to the energy conservation, all four particles should have the same energy in CMF, and \( p \) and \( p_f \) are therefore related by \( p_f = \sqrt{q^2 + m^2 - m_f^2} \). The boosted \( p_i \)'s still follow Eqs. (4.88) - (4.90). Thus, \( q' \) in the annihilation process is rather simple:

\[ q'^x = p_1'^x + p_2'^x = 0, \] (A.64)
\[ q'^y = p_1'^y + p_2'^y = 0, \] (A.65)
\[ q'^z = p_1'^z + p_2'^z = P^z, \] (A.66)
\[ q'^0 = p_1'^0 + p_2'^0 = E. \] (A.67)

Applying Eqs. (4.88) - (4.90) to Eqs. (A.58) and (A.59) with \( p_i \) given by Eqs. (A.60) -
(A.63), we find that

\[
p'_{34} \cdot p'_{12} = -4p \sqrt{m^2 - m_f^2 + p^2 \cos \theta},
\]

(A.68)

\[
q'^2 \frac{p'_{34} \cdot p'_{12}}{(Q'^+)^2} = 4p \sqrt{m^2 - m_f^2 + p^2 \cos \theta}.
\]

(A.69)

So the numerators in Eqs. (A.55) - (A.56) are 0, regardless of the frame or the interpolation angle \(\delta\). Therefore, \(\mathcal{M}^{(a)} = \mathcal{M}^{(b)} = 0\), and the only non-zero contribution comes from the instantaneous interaction. One can verify that indeed the covariant total amplitude is the same as \(\mathcal{M}^{(c)}\):

\[
\mathcal{M}^{(c)} = \mathcal{M} = -\frac{p_{34} \cdot p_{12}}{q^2} = -\frac{p \sqrt{m^2 - m_f^2 + p^2 \cos \theta}}{m_f^2 + p^2},
\]

(A.70)

where the coupling constant \(e\) is taken as 1. We can see that this result is also independent of the frame and interpolation angle, just as expected.

In Fig. A.2, we again use the same parameter values as before: \(m_1 = 1\), \(m_2 = 2\), \(p = 3\), and \(\theta = \pi/3\). The corresponding time-ordered amplitudes shown here are all flat planes.
APPENDIX B

APPENDIX FOR CHAPTER 5

B.1 \((0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)\) chiral representation of \(T\) and \(B(\eta)\mathcal{D}(\hat{m}, \theta_s)\)

Written in the \((0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)\) chiral representation, \(T\) and \(B(\eta)\mathcal{D}(\hat{m}, \theta_s)\) are

\[
T = \begin{pmatrix} T_R & 0 \\ 0 & T_L \end{pmatrix}, \tag{B.1}
\]

\[
B(\eta)\mathcal{D}(\hat{m}, \theta_s) = \begin{pmatrix} (BD)_R & 0 \\ 0 & (BD)_L \end{pmatrix}, \tag{B.2}
\]

where

\[
T_R = \begin{pmatrix} \cos \frac{\alpha}{2} e^{\frac{\beta_3}{2}} & -\frac{\beta_R (\cos \delta - \sin \delta)}{\alpha} \sin \frac{\alpha}{2} e^{-\frac{\beta_3}{2}} \\ \frac{\beta_R (\sin \delta + \cos \delta)}{\alpha} \sin \frac{\alpha}{2} e^{\frac{\beta_3}{2}} & \cos \frac{\alpha}{2} e^{\frac{\beta_3}{2}} \end{pmatrix}, \tag{B.3}
\]

\[
T_L = \begin{pmatrix} \cos \frac{\alpha}{2} e^{-\frac{\beta_3}{2}} & -\frac{\beta_L (\sin \delta + \cos \delta)}{\alpha} \sin \frac{\alpha}{2} e^{\frac{\beta_3}{2}} \\ \frac{\beta_R (\cos \delta - \sin \delta)}{\alpha} \sin \frac{\alpha}{2} e^{-\frac{\beta_3}{2}} & \cos \frac{\alpha}{2} e^{-\frac{\beta_3}{2}} \end{pmatrix}, \tag{B.4}
\]
and

$$(BD)_R = \begin{pmatrix} F \cos \frac{\theta_s}{2} + e^{i\phi_s} n_L \sinh \frac{\eta}{2} \sin \frac{\theta_s}{2} & -e^{-i\phi_s} F \sin \frac{\theta_s}{2} + n_L \sinh \frac{\eta}{2} \cos \frac{\theta_s}{2} \\ n_R \sinh \frac{\eta}{2} \cos \frac{\theta_s}{2} + e^{i\phi_s} G \sin \frac{\theta_s}{2} & -e^{-i\phi_s} n_R \sinh \frac{\eta}{2} \sin \frac{\theta_s}{2} + G \cos \frac{\theta_s}{2} \end{pmatrix}, \quad (B.5)$$

$$(BD)_L = \begin{pmatrix} G \cos \frac{\theta_s}{2} - e^{i\phi_s} n_L \sinh \frac{\eta}{2} \sin \frac{\theta_s}{2} & -e^{-i\phi_s} G \sin \frac{\theta_s}{2} - n_L \sinh \frac{\eta}{2} \cos \frac{\theta_s}{2} \\ -n_R \sinh \frac{\eta}{2} \cos \frac{\theta_s}{2} + e^{i\phi_s} F \sin \frac{\theta_s}{2} & e^{-i\phi_s} n_R \sinh \frac{\eta}{2} \sin \frac{\theta_s}{2} + F \cos \frac{\theta_s}{2} \end{pmatrix}, \quad (B.6)$$

with $n_L = n_1 - in_2$, $n_R = n_1 + in_2$, $(n_1, n_2, n_3) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$, and $F \equiv \cosh \frac{\eta}{2} + n_3 \sinh \frac{\eta}{2}$, $G \equiv \cosh \frac{\eta}{2} - n_3 \sinh \frac{\eta}{2}$.

B.2 $(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)$ helicity spinors for anti-particles

$$
\hat{v}^{(1/2)}_H(P) = \begin{pmatrix} P_L \sqrt{\frac{\cos \delta - \sin \delta}{2P^2 + P \cdot P \gamma_0}} \sqrt{P^+ - P} \\ -P_L \sqrt{\frac{\cos \delta + \sin \delta}{2P^2 + P \cdot P \gamma_0}} \sqrt{P^+ - P} \\ -P_L \sqrt{\frac{\sin \delta + \cos \delta}{2P^2 + P \cdot P \gamma_0}} \sqrt{P^+ + P} \sqrt{\frac{\cos \delta - \sin \delta}{2P^2 + P \cdot P \gamma_0}} \sqrt{P^+ + P} \end{pmatrix},
$$

$$
\hat{v}^{(-1/2)}_H(P) = \begin{pmatrix} P_R \sqrt{\frac{\sin \delta + \cos \delta}{2P^2 + P \cdot P \gamma_0}} \sqrt{P^+ + P} \\ -P_R \sqrt{\frac{\cos \delta - \sin \delta}{2P^2 + P \cdot P \gamma_0}} \sqrt{P^+ + P} \sqrt{\frac{\cos \delta + \sin \delta}{2P^2 + P \cdot P \gamma_0}} \sqrt{P^+ + P} \end{pmatrix}.
$$
B.3 Dirac spinors in chiral representation

The Dirac spinors \( u^{(1/2)} \) and \( u^{(-1/2)} \) in chiral representation are given by

\[
\begin{align*}
    u^{(1/2)}(P) &= \frac{1}{\sqrt{2(E + M)}} \begin{pmatrix} E + P^3 + M \\ P^R \\ -P^L \\ E - P^3 + M \end{pmatrix}, \\
    u^{(-1/2)}(P) &= \frac{1}{\sqrt{2(E + M)}} \begin{pmatrix} P^L \\ E - P^3 + M \\ -P^L \\ E + P^3 + M \end{pmatrix}.
\end{align*}
\]

(B.9)

B.4 Generalized \((0, J) \oplus (J, 0)\) helicity spinors for arbitrary interpolation angle up to spin 2

Following the notations in Ref. [JM01], we will use \( A = \cos \delta, B = -\sin \delta \) in the spinors listed below. We also define \( X \equiv \frac{P^\perp - \|P\|}{C} = \frac{P^\perp - \sqrt{(P^\perp)^2 - M^2 C}}{C} \) for convenience.

Spin-1 helicity spinors for any interpolation angle in chiral representation:

\[
\begin{align*}
    u^{(+1)}_H &= \frac{1}{2\sqrt{M^2}} \begin{pmatrix} (P_+ + P)(P^\perp + P) \\ (A - B) \sqrt{2P^R(P^\perp + P)} \\ (A - B)(P_+ + P)X \\ \sqrt{2P^R(P^\perp - P)} \\ (A + B)(P^R)^2(P^\perp + P) \end{pmatrix}, \\
    u^{(-1)}_H &= \frac{1}{2\sqrt{M^2}} \begin{pmatrix} (A + B)(P^L)^2(P^\perp - P) \\ (P_+ + P) \\ -\sqrt{2P^L(P^\perp - P)} \\ (A - B)(P_+ + P)X \\ (A - B)(P^L)^2(P^\perp + P) \\ \sqrt{2P^L(P^\perp + P)} \end{pmatrix}, \\
    u^{(0)}_H &= \sqrt{\frac{M}{2P^2}} \begin{pmatrix} -(A + B)P^L \\ A - B \sqrt{P^L} \\ -A + B \sqrt{P^L} \end{pmatrix}.
\end{align*}
\]

(B.10)

Spin-\(\frac{3}{2}\) helicity spinors for any interpolation angle in chiral representation:
where \( N = \frac{1}{2\sqrt{2P^3}} \).
Spin-2 helicity spinors for any interpolation angle in chiral representation:

\[ u^{(2)}_H = \frac{1}{4M^4 \sqrt{p^2}} \begin{pmatrix} (P_- + \not{p})(P_\uparrow + \not{p})^2 \\ (A - B)^2 \\ 2P_L(P_- + \not{p})(P_\uparrow + \not{p})^2 \\ A - B \\ \sqrt{6}(P_R)^2(P_\uparrow + \not{p})^2 \\ 2(A - B)(P_R)^3(P_\uparrow + \not{p})^2 \\ P_- + \not{p} \\ (A - B)^2(P_- + \not{p})^2 \end{pmatrix} \]

\[ u^{(-2)}_H = \frac{1}{4M^4 \sqrt{p^2}} \begin{pmatrix} (P_- + \not{p})(P_\uparrow + \not{p})^2 \\ (A - B)^2 \\ 2P_L(P_- + \not{p})(P_\uparrow + \not{p})^2 \\ A - B \\ \sqrt{6}(P_L)^2(P_\uparrow + \not{p})^2 \\ 2(P_- + \not{p})(P_\uparrow + \not{p})^2 \\ (A - B)^2(P_- + \not{p})^2 \end{pmatrix} \]
\[
\begin{align*}
\mu^{(1)}_H &= \frac{1}{2\pi^2\sqrt{M}} \\
&\quad \begin{pmatrix}
-\frac{(A + B)P_L(P^+ + \mathbb{P})(P^- + \mathbb{P})}{A - B} \\
\frac{(2P_- - \mathbb{P})(P^+ + \mathbb{P})(P^- + \mathbb{P})}{A - B}
\end{pmatrix} \\
&\quad \begin{pmatrix}
\sqrt{6}P_R P_-(P^+ + \mathbb{P}) \\
(A - B)(P^- + \mathbb{P})
\end{pmatrix} \\
&\quad \begin{pmatrix}
\sqrt{6}P_R P_-(P^+ - \mathbb{P}) \\
\mathbb{P}
\end{pmatrix} \\
&\quad \begin{pmatrix}
\mathbb{P} \\
A - B
\end{pmatrix}
\end{align*}
\]

\[
\mu^{(-1)}_H = \frac{1}{2\pi^2\sqrt{M}} \\
\begin{pmatrix}
\frac{-C^2(P_L)^3\mathbb{X}}{(A - B)^2(\mathbb{P} + P_\perp)} \\
\frac{-C^2(2P_\perp + P)(P_R)^2\mathbb{X}}{(A - B)(P^- + \mathbb{P})}
\end{pmatrix} \\
\begin{pmatrix}
-\sqrt{6}(A + B)P_L P_\perp \\
3P_\perp^2 - \mathbb{P}^2
\end{pmatrix} \\
\begin{pmatrix}
\sqrt{6}(A - B)P_R P_\perp \\
\sqrt{3/2}(A - B)^2(P_R)^2
\end{pmatrix} \\
\begin{pmatrix}
\sqrt{3/2}(A - B)^2(P_L)^2 \\
-\sqrt{6}(A - B)P_L P_\perp
\end{pmatrix} \\
\begin{pmatrix}
3P_\perp^2 - \mathbb{P}^2 \\
\sqrt{6}(A + B)P_R P_\perp
\end{pmatrix} \\
\begin{pmatrix}
\sqrt{3/2}(A + B)^2(P_R)^2
\end{pmatrix}
\]

In the light-front limit ($\delta \to \pi/4$), $A - B \to \sqrt{2}$, $A + B \to 0$, $\mathbb{X} \to \frac{M^2}{2P^+}$, and
\( P_- = P^\hat{+} = \mathbb{P} \rightarrow P^\hat{+} \), and one can verify that our spinors agree with the light-front spinors listed in Ref. [AS93]. To compare our results to those in that paper, one should note that our spinors are written in the chiral representation while those in Ref. [AS93] are in the standard representation. The standard representation (SR) and the chiral representation (CR) used in this paper are related by:

\[
    u_{SR} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 1 & -1 \end{pmatrix} u = \begin{pmatrix} \chi_R \\ \chi_L \end{pmatrix}, \tag{B.13}
\]

where \( \chi_R \) and \( \chi_L \) are each 3, 4 and 5 components long for spin 1, 3/2, and 2. In Ref. [AS93], the notations of \( P_R = P^R \) and \( P_L = P^L \) were used. One should also note that in our paper \( P^+ = (P^0 + P^3)/\sqrt{2} \), and the spinors are all normalized so that \( \bar{u} u = 2M \) while the spinors in Ref. [AS93] are normalized to \( \bar{u} u = M^{2j} \) where \( j = 1/2, 1, 3/2, 2 \) is the total spin.

### B.5 Generalized Melosh transformation

For convenience, we write everything in terms of the angles \( \theta_s \) and \( \phi_s \) which are given by Eqs. (5.45) and (5.46). For spin \( i \), the generalized Melosh transformation matrix \( \Omega(i) \) which connect the Dirac spinors with our generalized helicity spinors is

\[
    \Omega(i) = \begin{pmatrix} \omega(i) & 0 \\ 0 & \omega(i)^* \end{pmatrix}. \tag{B.14}
\]

For spin 1,

\[
    \omega(1) = \begin{pmatrix} 
        \cos^2 \frac{\theta_s}{2} & -\frac{e^{i\phi_s} \sin \theta_s}{\sqrt{2}} & \frac{e^{2i\phi_s} \sin^2 \theta_s}{2} \\
        \frac{e^{-i\phi_s} \sin \theta_s}{\sqrt{2}} & \cos \theta_s & -\frac{e^{i\phi_s} \sin \theta_s}{\sqrt{2}} \\
        \frac{e^{-2i\phi_s} \sin^2 \theta_s}{2} & \frac{e^{-i\phi_s} \sin \theta_s}{\sqrt{2}} & \cos^2 \frac{\theta_s}{2} \end{pmatrix}. \tag{B.15}
\]
For spin $\frac{3}{2}$, $\omega(\frac{3}{2})$ is defined via the four columns

$$\omega \left( \frac{3}{2} \right)_{\alpha,1} = \begin{pmatrix} \cos^3 \theta_s/2 \\ \frac{1}{4} \sqrt{3} e^{-i \phi_s} \csc \frac{\theta_s}{2} \sin^2 \theta_s \\ \frac{1}{2} \sqrt{3} e^{-2i \phi_s} \sin \frac{\theta_s}{2} \sin \theta_s \\ e^{-3i \phi_s} \sin^3 \frac{\theta_s}{2} \end{pmatrix}, \quad \omega \left( \frac{3}{2} \right)_{\alpha,2} = \begin{pmatrix} -\frac{1}{4} \sqrt{3} e^{i \phi_s} \csc \frac{\theta_s}{2} \sin^2 \theta_s \\ \frac{1}{4} \left( \cos \frac{\theta_s}{2} + 3 \cos \frac{3 \theta_s}{2} \right) \\ -\frac{1}{4} e^{-i \phi_s} \left( \sin \frac{\theta_s}{2} - 3 \sin \frac{3 \theta_s}{2} \right) \\ \frac{1}{2} \sqrt{3} e^{-2i \phi_s} \sin \frac{\theta_s}{2} \sin \theta_s \end{pmatrix},$$

$$\omega \left( \frac{3}{2} \right)_{\alpha,3} = \begin{pmatrix} \frac{1}{2} \sqrt{3} e^{2i \phi_s} \sin \frac{\theta_s}{2} \sin \theta_s \\ \frac{1}{4} e^{i \phi_s} \left( \sin \frac{\theta_s}{2} - 3 \sin \frac{3 \theta_s}{2} \right) \sin \frac{\theta_s}{2} \\ \frac{1}{4} \left( \cos \frac{\theta_s}{2} + 3 \cos \frac{3 \theta_s}{2} \right) \\ \frac{1}{4} \sqrt{3} e^{-2i \phi_s} \csc \frac{\theta_s}{2} \sin^2 \theta_s \end{pmatrix}, \quad \omega \left( \frac{3}{2} \right)_{\alpha,4} = \begin{pmatrix} -e^{3i \phi_s} \sin^3 \frac{\theta_s}{2} \\ \frac{1}{2} \sqrt{3} e^{2i \phi_s} \sin \frac{\theta_s}{2} \sin \theta_s \\ -\frac{1}{4} \sqrt{3} e^{i \phi_s} \csc \frac{\theta_s}{2} \sin^2 \theta_s \\ \cos^3 \frac{\theta_s}{2} \end{pmatrix}. \tag{B.16}$$

For spin 2, $\omega(2)$ is defined via the five columns

$$\omega(2)_{\alpha,1} = \begin{pmatrix} \cos^4 \frac{\theta_s}{2} \\ 2 e^{-i \phi_s} \sin \frac{\theta_s}{2} \cos^3 \frac{\theta_s}{2} \\ \frac{1}{2} \sqrt{3} e^{-2i \phi_s} \sin^2 \theta_s \\ e^{-3i \phi_s} \sin^3 \frac{\theta_s}{2} \sin \theta_s \\ e^{-4i \phi_s} \sin^4 \frac{\theta_s}{2} \end{pmatrix}, \quad \omega(2)_{\alpha,2} = \begin{pmatrix} -2 e^{i \phi_s} \sin \frac{\theta_s}{2} \cos^3 \frac{\theta_s}{2} \\ \cos^2 \frac{\theta_s}{2} \left( 2 \cos \theta_s - 1 \right) \\ \sqrt{3} e^{-i \phi_s} \sin \theta_s \cos \theta_s \\ e^{-2i \phi_s} \sin^2 \frac{\theta_s}{2} \left( 2 \cos \theta_s + 1 \right) \\ e^{-3i \phi_s} \sin^3 \frac{\theta_s}{2} \sin \theta_s \end{pmatrix}.$$
B.6 Apparent Spin Orientation

As explained in Sec. 5.3.1, we apply the light-front boost in the $x$ direction to a spin 4-vector and figure out the resulted “apparent” spin direction. With the boost and rotation generators written in the following 4-vector representation:

\[
\begin{align*}
K_1 &= \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & K_2 &= \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & K_3 &= \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \\
J_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, & J_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, & J_3 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{align*}
\]
and $\beta_3 = 0, \beta_1 = 0$ in Eq. (3.34), the $T_{12}$ transformation can be applied to the momentum 4-vector $P_0^\mu = (M, 0, 0, 0)$ and the spin 4-vector $S_0^\mu = (0, 0, 0, 1)$ defined in the rest frame. The resulted momentum 4-vector is $P^\mu = (M + M\beta_1^2/2, M\beta_1, 0, -M\beta_1^2/2)$, where $\beta_1 = P^x/M$ and the relation $P^z = -(P^x)^2/2M$ is satisfied [JS13]. The resulted spin 4-vector is $S^\mu = (\beta_1^2/2, \beta_1, 0, 1 - \beta_1^2/2)$, or written in terms of the final momentum

$$S^\mu = \left( \frac{(P^x)^2}{2M^2}, \frac{P^x}{M}, 0, \frac{2M^2 - (P^x)^2}{2M^2} \right).$$

Therefore, after this transverse light-front boost, the momentum direction and the “apparent” spin direction are given by

$$\theta = -\arctan \left( \frac{2M}{P^x} \right), \quad \theta_a = \arccos \left( \frac{2M^2 - (P^x)^2}{\sqrt{4M^4 + (P^x)^4}} \right).$$

Finally, the “apparent” spin orientation angle $\theta_a$ as a function of the momentum direction $\theta$ is

$$\theta_a = \arccos \left( \frac{\tan^2 \theta - 2}{\sqrt{\tan^4 \theta + 4}} \right),$$

which is plotted as the magenta dot-dashed line in Fig. 5.5. From Eq. (B.21), we note that $\theta_a \approx 0$ as $\theta \approx \pi/2$ and $\theta_a \approx \pi$ as $\theta \approx \pi$. This behavior can be seen in Fig. 5.5.

### B.7 Interpolated helicity scattering amplitudes

In this Appendix, we plot the helicity amplitudes of two particle scattering, as given by Eq. (5.62), in terms of both the interpolation angle $\delta$ and the total momentum $P^z$. The same parameters in Sec. 5.4 are used. All the 16 helicity amplitudes are shown in Fig. B.1.
Figure B.1 Scattering Amplitudes (with the factor $-\frac{e^2}{q^2}$ dropped) for 16 different helicity configurations, where the masses of the two colliding particles are $m_1 = 1$ GeV and $m_2 = 1.5$ GeV, the center of mass momentum for each particle is $p = 2$ GeV and the scattering angle $\theta = \pi/3$. 
B.8 Interpolated helicity annihilation amplitudes and probabilities

For a annihilation process depicted in Fig. B.2, the helicity-dependent amplitude $\mathcal{M}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is given by

$$\bar{v}^{(\lambda_2)}(p_2)\gamma^\mu u^{(\lambda_1)}(p_1)\bar{u}^{(\lambda_3)}(p_3)\gamma_\mu v^{(\lambda_4)}(p_4), \quad (B.22)$$

where $\lambda$ denotes the helicity of the particle. Again, we dropped the coupling constant factor $(-ie)^2$ and the Lorentz invariant part of the propagator $-1/q^2$, since they are irrelevant to our discussion.

The 4-momentum of these four particles in the center of mass frame are given by Eqs. (5.63) - (5.66), with $m_1 = m_2 \equiv m_{\text{ini}}, m_3 = m_4 = m_{\text{final}}$ and $\epsilon_1 = \epsilon_2 = \sqrt{m_{\text{ini}}^2 + p_{\text{ini}}^2} = \sqrt{m_{\text{final}}^2 + p_{\text{final}}^2} = \epsilon_3 = \epsilon_4 \equiv \epsilon$. The boosted 4-momentum for each particle is given by Eqs. (5.67) - (5.69), with the center of mass energy $M = 2\epsilon$ and the total energy in the boosted frame $E = \sqrt{(P_z)^2 + M^2}$. Just as in the scattering case, we use these frame dependent helicity spinors to calculate the helicity amplitudes, as well as the probabilities which are given by the square of Eq. (B.22). In our calculation, we choose $m_{\text{ini}} = 1 \text{ GeV}$ and $m_{\text{final}} = 1.5 \text{ GeV}$, $\theta = \pi/3$ in the center of mass frame, and $M = 2\epsilon = 4 \text{ GeV}$.

The annihilation amplitudes and probabilities for all 16 different spin configurations are
plotted in terms of both the interpolation angle \( \delta \) and the total momentum \( P^z \) in Figs. B.3 and B.4. In Fig. B.5, we also plot the probabilities of a certain helicity configuration going into all possible helicity configurations.

The boundaries lines and the J-curve are still described by Eqs. (5.70), (5.71) and (5.72), except here \( \epsilon_1 = \epsilon_2 = \epsilon = 2 \text{ GeV}, \ p = \sqrt{\epsilon^2 - m_{\text{ini}}^2}, \ M = 2\epsilon, \) and \( E = \sqrt{(P^z)^2 + 4\epsilon^2}. \)
Figure B.3  (Color online) Fermion annihilation amplitudes (with the factor $-e^2/q^2$ dropped) for 16 different spin configurations with the center of mass energy $M = 4$ GeV and the annihilation angle $\theta = \pi/3$. The masses of the initial and final particles are $m_{\text{ini}} = 1$ GeV and $m_{\text{final}} = 1.5$ GeV.
Figure B.4 (Color online) fermion annihilation probabilities (with the factor $e^4/q^4$ dropped) for 16 different spin configurations with the center of mass energy $M = 4$ GeV and the annihilation angle $\theta = \pi/3$. The masses of the initial and final particles are $m_{\text{ini}} = 1$ GeV and $m_{\text{final}} = 1.5$ GeV.
Figure B.5 Each plot is a summation of 4 plots in Fig. B.4 with the same initial helicity configurations, i.e. each plot represents the probability of a certain helicity configuration going into all possible final helicity states.
APPENDIX C

APPENDIX FOR CHAPTER 7

C.1 Fixation of the model parameters using variational principle

In our model, we assumed \( SU(2) \) flavor symmetry and have the following parameters that need to be fixed: constituent quark masses \((m_u(d), m_s, m_c, m_b)\), potential parameters \((a, b, \alpha_s)\), gaussian parameter \(\beta\), and the smearing parameter \(\sigma\). For our trial wave function \(\Phi = \sum_{n=1}^{2} c_n \phi_n S\), we also have the mixing factor \(c_n(n = 1, 2)\) that we have to adjust. Notice that the \(\beta\) values here are not only different for different quark combinations, but also different for pseudoscalar and vector mesons of the same quark combination. The reason for this is that the hyperfine interaction we included in our parameterization process gives different contributions to the masses of pseudoscalar and vector mesons and thus induces different parameterizations under variational principle.

We now illustrate our procedure for fixing these parameters. The variational principle gives us one constraint:

\[
\frac{\partial \langle \Phi | H | \Phi \rangle}{\partial \beta} = \frac{\partial M_{\bar{q}q}}{\partial \beta} = 0. \tag{C.1}
\]

We can use this equation to rewrite the coupling constant \(\alpha_s\) in terms of other parameters and plug it back into Eq. (7.9) and thus eliminate \(\alpha_s\). The string tension \(b\) is fixed to be 0.18 GeV, a well known value from other quark model analysis [GI85; SI95; Isg89].
We will leave the quark masses and smearing parameter $\sigma$ and the mixing factor $c_1$ as externally adjustable variables. We picked a set of values for $(m_{u(d)}, m_s, m_c, m_b, \sigma, c_1)$ and proceed with the following procedure to solve for the rest of parameters.

We are left with 3 more parameters $(a, \beta^P_{q\bar{q}}, \beta^V_{q\bar{q}})$ for mesons of a certain quark combination $(q\bar{q})$, where $\beta^P_{q\bar{q}}$, $\beta^V_{q\bar{q}}$ are the gaussian parameters for pseudoscalar and vector mesons, respectively. Using the masses of $\pi$ and $\rho$ as our input values for $M_{q\bar{q}}$ in Eq. (7.9), and the condition that our coupling constants $\alpha_s$ are the same for all these ground state pseudoscalar and vector mesons, we can fix the three model parameters $(a, \beta^P_{q\bar{q}}, \beta^V_{q\bar{q}})$ for $q = u$ or $d$ from the following three equations:

\[
\begin{align*}
M_\pi(\beta^P_{q\bar{q}}, a) &= 0.140, \\
M_\rho(\beta^V_{q\bar{q}}, a) &= 0.780, \\
\alpha_s(\beta^P_{q\bar{q}}, a) &= \alpha_s(\beta^V_{q\bar{q}}, a).
\end{align*}
\]

Solving these equations not only gives us the remaining parameters $a, \beta^P_{q\bar{q}}$, and $\beta^V_{q\bar{q}}$, but also the coupling constant $\alpha_s$ which we assumed to be the same for all the mesons we consider here. We can then solve for the $\beta$ values of all the other mesons using the known $\alpha_s$ value, by equating the $\alpha_s$ expressions for different mesons that we got from Eq. (C.1). We thus fixed all parameters for the ground state pseudoscalar and vector mesons we consider here.

We then assign a different set of values to the externally adjustable variables, i.e. $(m_{u(d)}, m_s, m_c, m_b, \sigma, c_1)$, and repeat the above procedure until we find a set of values that give best fit for the meson mass spectra.

Through our trial and error type of analysis, we found $m_q = 0.205$ GeV, $m_s = 0.38$ GeV, $m_c = 1.75$ GeV, $m_b = 5.15$ GeV, $\sigma = 0.423$ GeV, $c_1 = \sqrt{0.7}$ gives best fit. We then determine the mixing angles from the mass spectra of $(\omega, \phi)$ and $(\eta, \eta')$ using Eqs. (7.10) and (7.11) as we have described in Sec. 7.2. In addition, the multiplicative factor $\lambda$ in front of the smearing parameter $\sigma$ for the $(c\bar{c}, b\bar{c}, b\bar{b})$ systems were adjusted utilizing the hyperfine splittings of $(b, c)$ quark sectors as we have discussed in Sec. 7.3. The updated $\beta$ values with this $\lambda$ adjustment are listed in Table 7.2.