

## **Abstract**

VARGA, KATHERINE YVONNE. Portfolio Optimization with Stochastic Dividends and Stochastic Volatility. (Under the direction of Tao Pang.)

We consider an optimal investment-consumption portfolio optimization model in which an investor receives stochastic dividends. As a first problem, we allow the drift of stock price to be a bounded function. Next, we consider a stochastic volatility model. In each problem, we use the dynamic programming method to derive the Hamilton-Jacobi-Bellman equation, which we proceed to prove existence of a classical solution. Our value function is chosen to maximize the expected total discounted HARA utility of consumption. In the Verification Theorem, we prove that the solution to the HJB equation is equal to the value function.

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Portfolio Optimization with Stochastic Dividends  
and Stochastic Volatility

by  
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## **Dedication**

To my parents, Michael and Yvonne, who have instilled in me the values of education and hard work; and to my fiancé, Brandon, who gives me the drive to keep on when the road gets rough. I love you all.

## Acknowledgements

I would like to thank my advisor, Dr. Tao Pang, for his help in this research project from beginning to end. I have learned a great deal from him, and appreciate all of the time and effort he has put in along the way.

I want to thank my fiancé, Brandon, for his constant support and understanding. He has given me the motivation I needed to finish and accomplish my goals. Thank you, honey.

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# Table of Contents

<b>Chapter 1 Investment with Dividends and Debt . . . . .</b>	<b>1</b>
1.1 Introduction . . . . .	1
1.2 Problem Formulation . . . . .	3
<b>Chapter 2 Drift as a Bounded Function of Price . . . . .</b>	<b>7</b>
2.1 Derivation of the HJB Equation . . . . .	8
2.2 Subsolution and Supersolution . . . . .	19
2.3 Existence of Solution . . . . .	26
2.4 Verification Theorem . . . . .	28
<b>Chapter 3 Stochastic Volatility . . . . .</b>	<b>43</b>
3.1 Background and Motivation . . . . .	43
3.2 Derivation of the HJB Equation . . . . .	45
3.3 Subsolution and Supersolution . . . . .	51
3.4 Existence of Solution . . . . .	56
3.4.1 Sufficient Conditions for Existence of Solution to the Boundary Value Problem . . . . .	57
3.4.2 Using Theorem 3.4.1 . . . . .	61
3.4.3 A Uniform Bound for $\sup_{B_R}  DZ ^2$ . . . . .	86
3.4.4 Existence of Solution to the HJB Equation . . . . .	99
3.5 Verification Theorem . . . . .	101
<b>Bibliography . . . . .</b>	<b>111</b>

# Chapter 1

## Investment with Dividends and Debt

### 1.1 Introduction

In the classical Merton portfolio optimization problem, an investor distributes his wealth between a risky asset (e.g., stock) and a riskless asset. The price  $P_t$  of the risky asset is governed by geometric Brownian motion:

$$\frac{dP_t}{P_t} = \mu dt + \sigma dw_t,$$

where  $\mu$  and  $\sigma$  are the positive constant drift and volatility, respectively, and  $w_t$  is a one-dimensional standard Brownian motion. The controls are investment and consumption; that is, at any time  $t$  the investor chooses how much of his wealth to invest and how much to consume. The goal is to maximize his expected discounted utility of consumption.

Our model is an extension of Merton's model. We consider an economic unit with productive capital and liabilities in the form of debt. Investment and consumption controls

are chosen to maximize the expected discounted utility of consumption. In particular, we consider the hyperbolic absolute risk aversion (HARA) utility function

$$U(C) = \frac{1}{\gamma} C^\gamma, \tag{1.1}$$

where  $\gamma$  is a constant parameter such that  $\gamma < 1$  and  $\gamma \neq 0$ .

As an example in the Merton problem context, the economic unit may be interpreted as an investor who receives dividends on his investments. The worth of the investor depends on particular sources of uncertainty. Dividends and stock prices fluctuate over time, so we allow them to be stochastic.

In this thesis, we diverge from the Merton model which assumes stock price is governed by geometric Brownian motion. As a first problem, we consider a constant volatility and allow the drift to be a bounded function of price. Several financial models, such as the popular Black-Scholes-Merton model for option-pricing, assume for simplicity that volatility is constant. However, historical data shows that volatility fluctuates over time, and appears to have a mean-reverting quality [FPS00]. For this reason, in our second problem we choose an Ornstein-Uhlenbeck (OU) model for a factor process driving the stock price volatility.

Our work is motivated by similar models for optimal investment and consumption. Fleming and Pang, in [FP04] and [FP05], assume stock price is governed by geometric Brownian motion. In [FP04] they allow interest rates to be stochastic, and in [FP05] they consider stochastic productivity. Hata and Sheu ([HS12a] and [HS12b]) consider investment in  $m$  risky assets and one riskless asset. They allow both the drift and volatility of price to be stochastic. In [KS06], Kaise and Sheu prove existence and uniqueness of



ergodic type Bellman equations.

There are several extensions to the Merton portfolio optimization problem which follow the dynamic programming approach. Here we note some other work in the field. In [BP99] and [FS00], mean returns of securities are explicitly affected by underlying economic factors. In addition to [FP04], [Pa02] and [Pa04] also allow interest rates to be stochastic. Some stochastic volatility models are given in [FH03], [FPS00], and [Za01]. In [FH05], [Na15], and [NK11], stochastic interest rates and stochastic volatility are both incorporated.

## 1.2 Problem Formulation

Consider an investor who, at time  $t$ , owns  $N_t$  shares of stock at price per share  $P_t$ . The total worth of investments is given by  $K_t = N_t P_t$ . Then we can write

$$\begin{aligned} dK_t &= N_t dP_t + P_t dN_t \\ &= K_t \frac{dP_t}{P_t} + I_t dt, \end{aligned} \tag{1.2}$$

where  $I_t$  is the investment rate at time  $t$ . For example, if  $dt$  represents one month and he decides at time  $t$  to invest an additional \$1000 per month at a \$10 price per share, then the number of shares purchased will be

$$dN_t = \frac{I_t dt}{P_t} = 100.$$

The investor's debt,  $L_t$ , increases with interest payments, investment, and consumption,

## 1.2. PROBLEM FORMULATION

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and decreases with income. Thus our equation for the change in debt is given by

$$dL_t = (rL_t + I_t + C_t - Y_t)dt, \quad (1.3)$$

where  $r \geq 0$  is a constant interest rate,  $C_t$  is the consumption rate, and  $Y_t$  is the rate of income from production. In other words,  $Y_t$  is the total earned in dividends per unit time at time  $t$ . The total in dividends is proportional to the productivity of capital, or dividend rate,  $b_t$ :

$$Y_t dt = b_t N_t dt. \quad (1.4)$$

The investor's net worth is given by

$$X_t = K_t - L_t. \quad (1.5)$$

We require  $X_t > 0$ .

It is worth noting that the model applies to any economic unit with productive capital and liabilities. For another example, consider a farm on which produce is grown and then sold for profit.  $N_t$  may represent the number of acres of the farm, with  $P_t$  being the property value per acre. Debt increases with property taxes, the purchase of new land, and consumption, and decreases with income from selling the produce. From here on, we continue our explanations with the example of the investor.

We consider the dividend rate and stock price to be stochastic. In particular, we allow the dividend rate to be governed by the following equation:

$$b_t dt = b dt + \sigma dw_t, \quad (1.6)$$

## 1.2. PROBLEM FORMULATION

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where  $b, \sigma > 0$  are constants. The general equation for stock price is

$$\frac{dP_t}{P_t} = \mu_t dt + \tilde{\sigma}_t d\tilde{w}_t. \quad (1.7)$$

Here  $w_t$  and  $\tilde{w}_t$  are one-dimensional standard Brownian motions, which we allow to be correlated with correlation constant  $\rho \in (-1, 1]$ . That is,

$$\mathbb{E}[d\tilde{w}_t \cdot dw_t] = \rho dt.$$

We restrict  $\rho \neq -1$  for technical reasons, which will be apparent later. In addition, it is reasonable to assume that the dividend process and the stock price process are not perfectly negatively correlated. In Chapters 2 and 3 we place specific conditions on the drift  $\mu_t$  and volatility  $\tilde{\sigma}_t$  of stock price.

The change in net worth of the investor is given by

$$dX_t = K_t \frac{dP_t}{P_t} + I_t dt - (rL_t + I_t + C_t - Y_t) dt. \quad (1.8)$$

The controls  $k_t = \frac{K_t}{X_t}$  and  $c_t = \frac{C_t}{X_t}$  are the proportion of wealth invested and the proportion of wealth consumed, respectively, with initial values  $k_0 = k$  and  $c_0 = c$ . Substituting  $L_t = K_t - X_t$ , (1.4), (1.7), and our controls into (1.8), we simplify to obtain

$$dX_t = X_t \left[ \left( \frac{b}{P_t} + \mu_t - r \right) k_t + (r - c_t) \right] dt + \frac{\sigma k_t X_t}{P_t} dw_t + \tilde{\sigma}_t k_t X_t d\tilde{w}_t. \quad (1.9)$$

In this thesis we specify certain conditions on the the stock price drift and volatility. In Chapter 2 we allow the drift to be a bounded function of price, and take the volatility

## 1.2. PROBLEM FORMULATION

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to be constant. In Chapter 3, we consider a constant  $\mu$  and a stochastic volatility. In each case we use the dynamic programming principle to derive the Hamilton-Jacobi-Bellman (HJB) equation, from which we proceed to prove existence of a classical solution using a subsolution/supersolution method, and then verify that the solution obtained is equal to our value function, the maximum expected discounted utility of consumption.

## Chapter 2

# Drift as a Bounded Function of Price

In this chapter we suppose the unit price of capital,  $P_t$ , satisfies the stochastic differential equation

$$\frac{dP_t}{P_t} = \mu(P_t)dt + \tilde{\sigma}d\tilde{w}_t, \tag{2.1}$$

where the volatility  $\tilde{\sigma} > 0$  is constant, and the drift parameter  $\mu$  is a bounded function price. That is, we suppose there exists a constant  $M_1 > 0$  such that

$$|\mu(P_t)| \leq M_1.$$

Using this particular equation for  $P_t$ , we will derive the Hamilton-Jacobi-Bellman equation in Section 2.1. In Section 2.3, we will prove existence of solution using the particular subsolution and supersolution obtained in Section 2.2, and in Section 2.4, we will verify that our solution is equal to the value function.

## 2.1 Derivation of the HJB Equation

In Chapter 1 we derived a general equation for  $dX_t$ . We wish to write this equation with our current specifications. Before doing so, we first introduce a new variable. Let

$$\lambda_t = \log P_t.$$

For notational purposes, we will now write  $\mu(\lambda_t)$  rather than  $\mu(P_t)$ . Using Ito's rule with (2.1), we have

$$\begin{aligned} d\lambda_t &= \frac{1}{P_t} dP_t - \frac{1}{2} \frac{1}{P_t^2} (dP_t)^2 \\ &= \mu(\lambda_t) dt + \tilde{\sigma} d\tilde{w}_t - \frac{1}{2} \tilde{\sigma}^2 dt \\ &= \left( \mu(\lambda_t) - \frac{1}{2} \tilde{\sigma}^2 \right) dt + \tilde{\sigma} d\tilde{w}_t, \end{aligned}$$

and so

$$d\lambda_t = \tilde{\mu}(\lambda_t) dt + \tilde{\sigma} d\tilde{w}_t, \tag{2.2}$$

where  $\tilde{\mu}(\lambda_t) = \mu(\lambda_t) - \frac{1}{2} \tilde{\sigma}^2$ . Note that  $\tilde{\mu}$  is also bounded:

$$|\tilde{\mu}(\lambda)| \leq |\mu(\lambda)| + \frac{1}{2} \tilde{\sigma}^2 \leq M_1 + \frac{1}{2} \tilde{\sigma}^2 \equiv M_2. \tag{2.3}$$

Now we can rewrite (1.9) as

$$dX_t = X_t \left[ (be^{-\lambda_t} + \mu(\lambda_t) - r) k_t + (r - c_t) \right] dt + X_t \left[ \sigma k_t e^{-\lambda_t} dw_t + \tilde{\sigma} k_t d\tilde{w}_t \right]. \tag{2.4}$$

## 2.1. DERIVATION OF THE HJB EQUATION

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Note that we have two state variables,  $X_t$  and  $\lambda_t$ . Let their initial values be given by  $X_0 = x$  and  $\lambda_0 = \lambda$ . Recall that our controls  $k_t = K_t/X_t$  and  $c_t = C_t/X_t$  are the proportions of wealth invested and consumed, respectively. We require that they belong to the admissible control space,  $\Pi$ , as defined below.

### Definition 2.1.1. (Admissible Control Space)

The pair  $(k_t, c_t)$  is said to be in the **admissible control space**  $\Pi$  if  $(k_t, c_t)$  is an  $\mathbb{R}^2$ -process which is progressively measurable with respect to a  $(w_t, \tilde{w}_t)$ -adapted family of  $\sigma$ -algebras  $(\mathcal{F}_t, t \geq 0)$ . Moreover, we require that  $k_t, c_t \geq 0$ , and

$$Pr \left( \int_0^T k_t^2 dt < \infty \right) = 1, \quad Pr \left( \int_0^T c_t dt < \infty \right) = 1 \quad \text{for all } T > 0.$$

Our goal is to maximize the expected total discounted HARA utility of consumption. The objective function is given by

$$J(x, \lambda, k., c.) = \mathbb{E}_{x, \lambda} \left[ \int_0^\infty e^{-\delta t} \frac{1}{\gamma} (c_t X_t)^\gamma dt \right], \quad (2.5)$$

which we wish to maximize subject to the constraints  $(k_t, c_t) \in \Pi$  and  $x > 0$ , while  $\lambda \in \mathbb{R}$ . The parameter  $\delta$  is a positive discount factor. The corresponding value function is

$$V(x, \lambda) = \sup_{(k_t, c_t) \in \Pi} \mathbb{E}_{x, \lambda} \left[ \int_0^\infty e^{-\delta t} \frac{1}{\gamma} (c_t X_t)^\gamma dt \right]. \quad (2.6)$$

**Remark 2.1.1.** The HARA utility function, as defined in Chapter 1, allows for any values of  $\gamma$  less than 1, excluding zero. The case of  $\gamma = 0$  corresponds to the log utility function. In this thesis, we consider only positive values of the parameter. That is, we

## 2.1. DERIVATION OF THE HJB EQUATION

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restrict  $0 < \gamma < 1$ . A natural extension for future work would be to consider negative values for the parameter.

To get a partial differential equation for  $V(x, \lambda)$ , we use the dynamic programming method. This involves introducing a dummy variable,  $u$ , and splitting our integral into the sum of an integral from 0 to  $t$ , and an integral from  $t$  to infinity:

$$\int_0^\infty e^{-\delta t} \frac{1}{\gamma} (c_t X_t)^\gamma du = \int_0^t e^{-\delta u} \frac{1}{\gamma} (c_u X_u)^\gamma du + \int_t^\infty e^{-\delta u} \frac{1}{\gamma} (c_u X_u)^\gamma du. \quad (2.7)$$

We apply a change of variables so that the second integral is rewritten as an integral from 0 to infinity. Then we can write our objective function as follows:

$$\begin{aligned} J(x, \lambda, k., c.) &= \mathbb{E}_{x, \lambda} \left[ \int_0^t e^{-\delta u} \frac{1}{\gamma} (c_u X_u)^\gamma du + \int_0^\infty e^{-\delta(v+t)} \frac{1}{\gamma} (c_v^t X_v^t)^\gamma dv \right] \\ &= \mathbb{E}_{x, \lambda} \left[ \int_0^t e^{-\delta u} \frac{1}{\gamma} (c_u X_u)^\gamma du + e^{-\delta t} \mathbb{E}_{X_t, \lambda_t} \int_0^\infty e^{-\delta v} \frac{1}{\gamma} (c_v^t X_v^t)^\gamma dv \right]. \end{aligned} \quad (2.8)$$

The second line is true because of the tower property of conditional expectations (see [Et02], Section 2.3), which says that if  $\mathcal{F}_i$  and  $\mathcal{F}_j$  are  $\sigma$ -algebras such that  $\mathcal{F}_i \subseteq \mathcal{F}_j$ , then

$$\mathbb{E}[\mathbb{E}[X|\mathcal{F}_j]|\mathcal{F}_i] = \mathbb{E}[X|\mathcal{F}_i]. \quad (2.9)$$

In the second term of (2.8),  $v$  is treated as a fixed value, and  $X_v^t$  is a process with initial value  $X_t$ . Therefore we have

$$J(x, \lambda, k., c.) = \mathbb{E}_{x, \lambda} \left[ \int_0^t e^{-\delta u} \frac{1}{\gamma} (c_u X_u)^\gamma du + e^{-\delta t} J(X_t, \lambda_t, k., c.) \right], \quad (2.10)$$



## 2.1. DERIVATION OF THE HJB EQUATION

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and so

$$\begin{aligned} V(x, \lambda) &= \sup_{(k., c.) \in \Pi} J(x, \lambda, k., c.) \\ &= \sup_{(k., c.) \in \Pi} \left\{ \mathbb{E}_{x, \lambda} \left[ \int_0^t e^{-\delta u} \frac{1}{\gamma} (c_u X_u)^\gamma du \right] + \mathbb{E}_{x, \lambda} \left[ e^{-\delta t} \sup_{(k., c.) \in \Pi} J(X_t, \lambda_t, k., c.) \right] \right\}. \end{aligned}$$

Note that

$$\sup_{(k., c.) \in \Pi} J(X_t, \lambda_t, k., c.) = V(X_t, \lambda_t),$$

so that

$$V(x, \lambda) = \sup_{(k., c.) \in \Pi} \left\{ \mathbb{E}_{x, \lambda} \left[ \int_0^t e^{-\delta u} \frac{1}{\gamma} (c_u X_u)^\gamma du \right] + \mathbb{E}_{x, \lambda} [e^{-\delta t} V(X_t, \lambda_t)] \right\}. \quad (2.11)$$

Then

$$\sup_{(k., c.) \in \Pi} \left\{ \mathbb{E}_{x, \lambda} \left[ \int_0^t e^{-\delta u} \frac{1}{\gamma} (c_u X_u)^\gamma du \right] + \mathbb{E}_{x, \lambda} [e^{-\delta t} V(X_t, \lambda_t)] - V(x, \lambda) \right\} = 0. \quad (2.12)$$

Working with the expression inside the braces, we divide by  $t$  and take the limit as  $t$  approaches zero. By the mean value theorem,

$$\lim_{t \downarrow 0} \frac{\mathbb{E}_{x, \lambda} \left[ \int_0^t e^{-\delta u} \frac{1}{\gamma} (c_u X_u)^\gamma du \right]}{t} = \frac{1}{\gamma} (cx)^\gamma. \quad (2.13)$$

We also have that

$$\lim_{t \downarrow 0} \frac{\mathbb{E}_{x, \lambda} [V(X_t, \lambda_t)] - V(x, \lambda)}{t} = \mathcal{A}V(x, \lambda), \quad (2.14)$$

where  $\mathcal{A}$  is called the *infinitesimal generator* of the process  $(X_t, \lambda_t)$ . (See [Ok07], p. 121).

## 2.1. DERIVATION OF THE HJB EQUATION

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We manipulate the second term of (2.12) and apply (2.14) to get

$$\begin{aligned}
& \lim_{t \downarrow 0} \frac{\mathbb{E}_{x,\lambda} [e^{-\delta t} V(X_t, \lambda_t)] - V(x, \lambda)}{t} \\
&= \lim_{t \downarrow 0} \left\{ \frac{e^{-\delta t} [\mathbb{E}_{x,\lambda} V(X_t, \lambda_t) - V(x, \lambda)]}{t} + \frac{e^{-\delta t} V(x, \lambda) - V(x, \lambda)}{t} \right\} \\
&= \left( \lim_{t \downarrow 0} e^{-\delta t} \right) \left( \lim_{t \downarrow 0} \frac{\mathbb{E}_{x,\lambda} V(X_t, \lambda_t) - V(x, \lambda)}{t} \right) + V(x, \lambda) \left( \lim_{t \downarrow 0} \frac{e^{-\delta t} - 1}{t} \right) \\
&= \mathcal{A}V(x, \lambda) - \delta V(x, \lambda). \tag{2.15}
\end{aligned}$$

To derive the generator,  $\mathcal{A}$ , we apply Ito's formula to  $V(X_t, \lambda_t)$ :

$$\begin{aligned}
dV(X_t, \lambda_t) &= V_x X_t \left\{ \left[ (be^{-\lambda_t} + \mu(\lambda_t) - r)k_t + (r - c_t) \right] dt + \sigma k_t e^{-\lambda_t} dw_t + \tilde{\sigma} k_t d\tilde{w}_t \right\} \\
&\quad + V_\lambda \left[ \tilde{\mu}(\lambda_t) dt + \tilde{\sigma} d\tilde{w}_t \right] + \frac{1}{2} V_{xx} X_t^2 \left[ \sigma^2 k_t^2 e^{-2\lambda_t} + 2\rho\sigma\tilde{\sigma} k_t^2 e^{-\lambda_t} + \tilde{\sigma}^2 k_t^2 \right] dt \\
&\quad + \frac{1}{2} V_{\lambda\lambda} \tilde{\sigma}^2 dt + V_{x\lambda} \left[ \tilde{\sigma}^2 k_t X_t + \rho\sigma\tilde{\sigma} k_t X_t e^{-\lambda_t} \right] dt.
\end{aligned}$$

Rearranging and integrating from 0 to  $t$ , we get

$$\begin{aligned}
V(X_t, \lambda_t) - V(x, \lambda) &= \int_0^t \left[ X_u \left( be^{-\lambda_u} + \mu(\lambda_u) - r \right) k_u + (r - c_u) \right] V_x + \tilde{\mu}(\lambda_u) V_\lambda \\
&\quad + \frac{1}{2} k_u^2 X_u^2 \left( \sigma^2 e^{-2\lambda_u} + 2\rho\sigma\tilde{\sigma} e^{-\lambda_u} + \tilde{\sigma}^2 \right) V_{xx} \\
&\quad + \frac{1}{2} \tilde{\sigma} V_{\lambda\lambda} + k_u X_u \left( \tilde{\sigma}^2 + \rho\sigma\tilde{\sigma} e^{-\lambda_u} \right) V_{x\lambda} \right] du \\
&\quad + \int_0^t [\sigma k_u X_u e^{-\lambda_u} V_x] dw_u + \int_0^t [\tilde{\sigma} k_u X_u V_x + \tilde{\sigma} V_\lambda] d\tilde{w}_u. \tag{2.16}
\end{aligned}$$

We assume that  $\mathbb{E}[\lambda_t^m] < \infty$  and  $\mathbb{E}[X_t^m] < \infty$  for all  $m > 0$ , and also that there exists a constant  $A > 0$  such that  $|Z_\lambda(\lambda)| \leq A$ , where  $Z(\lambda)$  is defined as a function such that

## 2.1. DERIVATION OF THE HJB EQUATION

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$V(x, \lambda) = \frac{1}{\gamma} x^\gamma e^{Z(\lambda)}$ . This will ensure that the last two integrals in (2.16) are martingales. In Section 2.4, we will prove that these conditions hold for our optimal control policy  $(k_t^*, c_t^*)$ , and solution  $\tilde{Z}(\lambda, \theta)$ .

Taking the expectation in (2.16) causes the last two terms to vanish because they are martingales. We then divide by  $t$ , and take the limit as  $t$  approaches zero. This give us the following equality:

$$\lim_{t \downarrow 0} \frac{\mathbb{E}_{x, \lambda} [V(X_t, \lambda_t)] - V(x, \lambda)}{t} = \lim_{t \downarrow 0} \frac{\mathbb{E}_{x, \lambda} \left[ \int_0^t f(X_u, \lambda_u) du \right]}{t}, \quad (2.17)$$

where

$$\begin{aligned} f(X_t, \lambda_t) &= X_t \left( b e^{-\lambda_t} + \mu(\lambda_t) - r \right) k_t + (r - c_t) V_x + \tilde{\mu}(\lambda_t) V_\lambda \\ &\quad + \frac{1}{2} k_t^2 X_t^2 \left( \sigma^2 e^{-2\lambda_t} + 2\rho\sigma\tilde{\sigma}e^{-\lambda_t} + \tilde{\sigma}^2 \right) V_{xx} \\ &\quad + \frac{1}{2} \tilde{\sigma} V_{\lambda\lambda} + k_t X_t \left( \tilde{\sigma}^2 + \rho\sigma\tilde{\sigma}e^{-\lambda_t} \right) V_{x\lambda}. \end{aligned}$$

Let

$$q(\lambda_t) = \sigma^2 e^{-2\lambda_t} + 2\rho\sigma\tilde{\sigma}e^{-\lambda_t} + \tilde{\sigma}^2. \quad (2.18)$$

The left-hand side of (2.17) is equal to  $\mathcal{A}V(x, \lambda)$ . By the mean value theorem, the right-hand side is equal to  $f(x, \lambda)$ . Therefore, we have that

$$\begin{aligned} \mathcal{A}V(x, \lambda) &= x \left( b e^{-\lambda} + \mu(\lambda) - r \right) k + (r - c) V_x + \tilde{\mu}(\lambda) V_\lambda + \frac{1}{2} k^2 x^2 q(\lambda) V_{xx} \\ &\quad + \frac{1}{2} \tilde{\sigma} V_{\lambda\lambda} + k x \left( \tilde{\sigma}^2 + \rho\sigma\tilde{\sigma}e^{-\lambda} \right) V_{x\lambda}. \end{aligned} \quad (2.19)$$

## 2.1. DERIVATION OF THE HJB EQUATION

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We now combine (2.12), (2.13), and (2.15) to get

$$\sup_{(k,c) \in \Pi} \left\{ \frac{1}{\gamma} (cx)^\gamma + \mathcal{A}V(x, \lambda) - \delta V(x, \lambda) \right\} = 0. \quad (2.20)$$

Substituting (2.19) into (2.20) and simplifying, we get the following dynamic programming equation for  $V(x, \lambda)$  :

$$\begin{aligned} \delta V = & \frac{\tilde{\sigma}^2}{2} V_{\lambda\lambda} + \tilde{\mu}(\lambda) V_\lambda + rx V_x + \max_{c \geq 0} \left[ -cx V_x + \frac{1}{\gamma} (cx)^\gamma \right] \\ & + \max_{k \geq 0} \left[ (be^{-\lambda} + \mu(\lambda) - r) kx V_x + (\tilde{\sigma}^2 + \rho \sigma \tilde{\sigma} e^{-\lambda}) kx V_{x\lambda} + \frac{k^2 x^2 q(\lambda)}{2} V_{xx} \right]. \end{aligned} \quad (2.21)$$

Equation (2.21) is the *Hamilton-Jacobi-Bellman (HJB) equation* for  $V(x, \lambda)$  as defined in (2.6).

We assume  $\rho \neq -1$ . To be more precise, we define a constant  $-1 < \rho_0 < 0$  (that may be arbitrarily close to  $-1$ ), and require  $\rho \in [\rho_0, 1]$ . Then  $q(\lambda)$  is bounded below by a positive constant. This is easily seen by looking at two cases, and using the fact that  $\sigma, \tilde{\sigma}$ , and  $e^{-\lambda}$  are all positive. For  $\rho \in [0, 1]$ ,

$$q(\lambda) \geq \sigma^2 e^{-2\lambda} + \tilde{\sigma}^2 > \tilde{\sigma}^2 > 0.$$

For  $\rho \in [\rho_0, 0)$ ,

$$q'(\lambda) = -2\sigma^2 e^{-2\lambda} - 2\rho\sigma\tilde{\sigma}e^{-\lambda} = -2\sigma e^{-\lambda}(\sigma e^{-\lambda} + \rho\tilde{\sigma}).$$

## 2.1. DERIVATION OF THE HJB EQUATION

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Now  $q'(\lambda) = 0$  when  $e^{-\lambda} = -\rho \frac{\tilde{\sigma}}{\sigma}$ . Substituting this into the equation for  $q$ , we have

$$q_{\min} = \tilde{\sigma}^2(1 - \rho^2) \geq \tilde{\sigma}^2(1 - \rho_0^2) \geq 0.$$

We see that  $\tilde{\sigma}^2 \geq \tilde{\sigma}^2(1 - \rho_0^2)$ , and set  $q_0 \equiv \tilde{\sigma}^2(1 - \rho_0^2)$ . Thus  $q(\lambda) \geq q_0$  for all  $\rho \in [\rho_0, 1]$ . Notice that if we allowed  $\rho$  to equal  $-1$ , then  $q$  would obtain the value zero. We need a positive lower bound on  $q$  for later proofs. It is for this reason that we restrict  $\rho \in [\rho_0, 1]$ .

For the non-log HARA utility function, we can show that  $V(x, \lambda)$  is homogeneous in  $x$  with order  $\lambda$ . Using equation (2.4) for  $dX_t$ , and for any constant  $\xi > 0$ , we have

$$d(\xi X_t) = \xi \cdot dX_t, \tag{2.22}$$

$$\xi x_0 = \xi x, \tag{2.23}$$

and

$$\begin{aligned} J(\xi x, \lambda, k., c.) &= \mathbb{E}_{x, \lambda} \left[ \int_0^\infty e^{-\delta t} \frac{1}{\gamma} (c_t \xi X_t)^\gamma dt \right] \\ &= \xi^\gamma \mathbb{E}_{x, \lambda} \left[ \int_0^\infty e^{-\delta t} \frac{1}{\gamma} (c_t X_t)^\gamma dt \right] \\ &= \xi^\gamma J(x, \lambda, k., c.). \end{aligned}$$

## 2.1. DERIVATION OF THE HJB EQUATION

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Then

$$\begin{aligned}
 V(x, \lambda) &= \sup_{(k, c.) \in \Pi} J(x, \lambda, k., c.) \\
 &= \sup_{(k, c.) \in \Pi} x^\gamma J(1, \lambda, k., c.) \\
 &= x^\gamma V(1, \lambda).
 \end{aligned}$$

We can then write  $V$  in the form

$$V(x, \lambda) = \frac{1}{x^\gamma} W(\lambda). \quad (2.24)$$

Using equation (2.24), we get a reduced DPE for  $W$ :

$$\begin{aligned}
 \delta W &= \frac{\tilde{\sigma}^2}{2} W_{\lambda\lambda} + \tilde{\mu}(\lambda) W_\lambda + \gamma r W + \max_{c \geq 0} [-\gamma c W + c^\gamma] \\
 &\quad + \gamma \max_{k \geq 0} \left[ (be^{-\lambda} + \mu(\lambda) - r) kW + (\tilde{\sigma}^2 + \rho \sigma \tilde{\sigma} e^{-\lambda}) kW_\lambda - \frac{k^2}{2} (1 - \gamma) q(\lambda) W \right].
 \end{aligned} \quad (2.25)$$

Let  $Z(\lambda) = \log W(\lambda)$ . Then

$$\frac{W_\lambda}{W} = Z_\lambda \quad \text{and} \quad \frac{W_{\lambda\lambda}}{W} = Z_{\lambda\lambda} + Z_\lambda^2. \quad (2.26)$$

So the equation for  $Z$  is

$$\begin{aligned}
 \delta &= \frac{\tilde{\sigma}^2}{2} (Z_{\lambda\lambda} + Z_\lambda^2) + \tilde{\mu}(\lambda) Z_\lambda + \gamma r + \max_{c \geq 0} [-\gamma ce^{Z_\lambda} + c^\gamma] \\
 &\quad + \gamma \max_{k \geq 0} \left\{ \left[ be^{-\lambda} + \mu(\lambda) - r + (\tilde{\sigma}^2 + \rho \sigma \tilde{\sigma} e^{-\lambda}) Z_\lambda \right] k - \frac{k^2}{2} (1 - \gamma) q(\lambda) \right\}.
 \end{aligned} \quad (2.27)$$

## 2.1. DERIVATION OF THE HJB EQUATION

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The maximum over all  $c \geq 0$  occurs at

$$c^* = e^{\frac{Z(\lambda)}{\gamma-1}}, \quad (2.28)$$

which gives us

$$\max_{c \geq 0} [-\gamma c e^Z + c^\gamma] = (1 - \gamma) e^{\frac{Z}{\gamma-1}}.$$

Using this expression, we simplify (2.27) to get

$$\begin{aligned} \delta = & \frac{\tilde{\sigma}^2}{2} (Z_{\lambda\lambda} + Z_\lambda^2) + \tilde{\mu}(\lambda) Z_\lambda + \gamma r + (1 - \gamma) e^{\frac{Z}{\gamma-1}} \\ & + \gamma \max_{k \geq 0} \left\{ \left[ b e^{-\lambda} + \mu(\lambda) - r + (\tilde{\sigma}^2 + \rho \sigma \tilde{\sigma} e^{-\lambda}) Z_\lambda \right] k - \frac{k^2}{2} (1 - \gamma) q(\lambda) \right\}. \end{aligned} \quad (2.29)$$

The maximum over all  $k \geq 0$  occurs at

$$k^* = \left[ \frac{b e^{-\lambda} + \mu(\lambda) - r + (\tilde{\sigma}^2 + \rho \sigma \tilde{\sigma} e^{-\lambda}) Z_\lambda}{(1 - \gamma) q(\lambda)} \right]^+, \quad (2.30)$$

where  $a^+ \equiv \max(a, 0)$ . Note that  $(k^*, c^*)$  is our candidate for admissible control. In the Verification Theorem (Section 2.4), we will prove that  $(k^*, c^*) \in \Pi$ .

Define

$$G(\lambda, p) \equiv \max_{k \geq 0} \left\{ \left[ b e^{-\lambda} + \mu(\lambda) - r + (\tilde{\sigma}^2 + \rho \sigma \tilde{\sigma} e^{-\lambda}) p \right] k - \frac{k^2}{2} (1 - \gamma) q(\lambda) \right\}, \quad (2.31)$$

and

$$\Psi(\lambda) \equiv \begin{cases} \frac{(b e^{-\lambda} + \mu(\lambda) - r)^2}{2 q(\lambda) (1 - \gamma)} & \text{if } b e^{-\lambda} + \mu(\lambda) - r > 0, \\ 0 & \text{otherwise.} \end{cases} \quad (2.32)$$

## 2.1. DERIVATION OF THE HJB EQUATION

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Then

$$G(\lambda, p) \geq 0, \quad G(\lambda, 0) = \Psi(\lambda), \quad (2.33)$$

and Equation (2.29) reduces to

$$\delta = \frac{\tilde{\sigma}^2}{2}(Z_{\lambda\lambda} + Z_\lambda^2) + \tilde{\mu}(\lambda)Z_\lambda + \gamma r + (1 - \gamma)e^{\frac{Z}{\gamma-1}} + \gamma G(\lambda, Z_\lambda). \quad (2.34)$$

To prove (2.33), note that

$$k^*(\lambda, p) = \max \left( \frac{be^{-\lambda} + \mu(\lambda) - r + (\tilde{\sigma}^2 + \rho\sigma\tilde{\sigma}e^{-\lambda})p}{(1 - \gamma)q(\lambda)}, 0 \right) \geq 0.$$

We can write  $G$  as a quadratic in  $k^*$ :

$$\begin{aligned} G(\lambda, p) &= \left[ be^{-\lambda} + \mu(\lambda) - r + (\tilde{\sigma}^2 + \rho\sigma\tilde{\sigma}e^{-\lambda})p \right] k^* - \frac{(k^*)^2}{2}(1 - \gamma)q(\lambda) \\ &= a(k^*)^2 + bk^*. \end{aligned}$$

If  $k^* = 0$ , then  $G = 0$ .

If  $k^* = \frac{be^{-\lambda} + \mu(\lambda) - r + (\tilde{\sigma}^2 + \rho\sigma\tilde{\sigma}e^{-\lambda})p}{(1 - \gamma)q(\lambda)} \geq 0$ , then

$$\begin{aligned} G(\lambda, p) &= \frac{-b^2}{4a} \\ &= \frac{\left[ be^{-\lambda} + \mu(\lambda) - r + (\tilde{\sigma}^2 + \rho\sigma\tilde{\sigma}e^{-\lambda})p \right]^2}{2(1 - \gamma)q(\lambda)} \geq 0, \end{aligned}$$



## 2.2. SUBSOLUTION AND SUPERSOLUTION

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since  $\gamma < 1$  and  $q(\lambda) > 0$ . So  $G(\lambda, p) \geq 0$ . Substituting in  $p = 0$  above, we see that

$$G(\lambda, 0) = \Psi(\lambda).$$

Define

$$H(\lambda, z, p) \equiv -\frac{\tilde{\sigma}^2}{2}p^2 - \tilde{\mu}(\lambda)p - \gamma G(\lambda, p) - \gamma r + \delta - (1 - \gamma)e^{\frac{z}{\gamma-1}}. \quad (2.35)$$

Note that  $H$  is a continuous function in each variable. Now we can rewrite Equation (2.34) as

$$\frac{\tilde{\sigma}^2}{2}Z_{\lambda\lambda} = H(\lambda, Z, Z_\lambda). \quad (2.36)$$

Equation (2.36) is the Hamilton-Jacobi-Bellman equation for  $Z(\lambda)$  corresponding to the value function (2.6), where  $V(x, \lambda) = \frac{1}{\gamma}x^\gamma e^{Z(\lambda)}$ .

## 2.2 Subsolution and Supersolution

To prove existence of solution to (2.36), we use the method of subsolution/supersolution. Information on the use of this method can be found in [BSW68], [Pa92], and [Wa98].

**Definition 2.2.1.**  $Z(\lambda)$  is a **subsolution (supersolution) of (2.36)** provided that

$$\frac{\tilde{\sigma}^2}{2}Z_{\lambda\lambda} \geq (\leq) H(\lambda, Z, Z_\lambda). \quad (2.37)$$

In addition, if  $\hat{Z}$  is a subsolution,  $\bar{Z}$  is a supersolution, and  $\hat{Z} \leq \bar{Z}$ , then we say that  $\langle \hat{Z}, \bar{Z} \rangle$  is an **ordered pair of subsolution/supersolution**.

We wish to identify a particular subsolution and supersolution. In doing so, we will

## 2.2. SUBSOLUTION AND SUPERSOLUTION

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make use of the following lemma.

**Lemma 2.2.1.**  $\Psi(\lambda)$ , as defined in (2.32), is bounded.

*Proof.* We first consider the case  $0 \leq \rho \leq 1$ . Using the identity

$$(a + b)^2 \leq 2a^2 + 2b^2, \quad (2.38)$$

and the fact that  $\sigma, \tilde{\sigma}, e^{-\lambda} > 0$ , we have

$$\begin{aligned} \Psi(\lambda) &= \frac{(be^{-\lambda} + \mu(\lambda) - r)^2}{2(1 - \gamma)(\sigma^2 e^{-2\lambda} + 2\rho\sigma\tilde{\sigma}e^{-\lambda} + \tilde{\sigma}^2)} \\ &\leq \frac{(be^{-\lambda} + \mu(\lambda) - r)^2}{2(1 - \gamma)(\sigma^2 e^{-2\lambda} + \tilde{\sigma}^2)} \\ &\leq \frac{2b^2 e^{-2\lambda} + 2(\mu(\lambda) - r)^2}{2(1 - \gamma)(\sigma^2 e^{-2\lambda} + \tilde{\sigma}^2)}. \end{aligned}$$

By performing some simplifications and applying our upper bound on  $\mu(\lambda)$ , we get

$$\begin{aligned} \frac{2b^2 e^{-2\lambda} + 2(\mu(\lambda) - r)^2}{2(1 - \gamma)(\sigma^2 e^{-2\lambda} + \tilde{\sigma}^2)} &= \frac{b^2 e^{-2\lambda} \sigma^2 + \sigma^2 (\mu(\lambda)^2 - r)^2}{\sigma^2 (1 - \gamma)(\sigma^2 e^{-2\lambda} + \tilde{\sigma}^2)} \\ &= \frac{b^2 (\sigma^2 e^{-2\lambda} + \tilde{\sigma}^2) + \sigma^2 (\mu(\lambda)^2 - r)^2 - b^2 \tilde{\sigma}^2}{\sigma^2 (1 - \gamma)(\sigma^2 e^{-2\lambda} + \tilde{\sigma}^2)} \\ &= \frac{b^2}{\sigma^2 (1 - \gamma)} + \frac{(\mu(\lambda) - r)^2 - \frac{b^2 \tilde{\sigma}^2}{\sigma^2}}{(1 - \gamma)(\sigma^2 e^{-2\lambda} + \tilde{\sigma}^2)} \\ &\leq \frac{b^2}{\sigma^2 (1 - \gamma)} + \frac{(M_1 + r)^2 - \frac{b^2 \tilde{\sigma}^2}{\sigma^2}}{(1 - \gamma)(\sigma^2 e^{-2\lambda} + \tilde{\sigma}^2)}. \end{aligned}$$

The last step is true because

$$|\mu(\lambda) - r| \leq |\mu(\lambda)| + r \leq M_1 + r, \quad r \geq 0.$$

## 2.2. SUBSOLUTION AND SUPERSOLUTION

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If  $(M_1 + r)^2 - b^2\tilde{\sigma}^2/\sigma^2 \leq 0$ , then

$$\Psi(\lambda) \leq \frac{b^2}{\sigma^2(1-\gamma)}. \quad (2.39)$$

Otherwise, we have

$$\begin{aligned} \Psi(\lambda) &\leq \frac{b^2}{\sigma^2(1-\gamma)} + \frac{(M_1 + r)^2 - \frac{b^2\tilde{\sigma}^2}{\sigma^2}}{(1-\gamma)\tilde{\sigma}^2} \\ &= \frac{(M_1 + r)^2}{\tilde{\sigma}^2(1-\gamma)}. \end{aligned}$$

Thus, for  $0 \leq \rho \leq 1$ ,

$$\Psi(\lambda) \leq \max \left\{ \frac{b^2}{\sigma^2(1-\gamma)}, \frac{(M_1 + r)^2}{\tilde{\sigma}^2(1-\gamma)} \right\}.$$

We can get a similar bound if  $\rho$  is negative. If  $\rho_0 < \rho < 0$ , then

$$\rho(\sigma e^{-\lambda} - \tilde{\sigma})^2 \leq 0,$$

which implies

$$2\rho\sigma\tilde{\sigma}e^{-\lambda} \geq \rho\sigma^2e^{-2\lambda} + \rho\tilde{\sigma}^2.$$

## 2.2. SUBSOLUTION AND SUPERSOLUTION

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Then we have

$$\begin{aligned}
\Psi(\lambda) &= \frac{(be^{-\lambda} + \mu(\lambda) - r)^2}{2(1 - \gamma)(\sigma^2 e^{-2\lambda} + 2\rho\sigma\tilde{\sigma}e^{-\lambda} + \tilde{\sigma}^2)} \\
&\leq \frac{(be^{-\lambda} + \mu(\lambda) - r)^2}{2(1 - \gamma)(\sigma^2 e^{-2\lambda} + \rho\sigma^2 e^{-2\lambda} + \rho\tilde{\sigma}^2 + \tilde{\sigma}^2)} \\
&= \frac{(be^{-\lambda} + \mu(\lambda) - r)^2}{2(1 - \gamma)(1 + \rho)(\sigma^2 e^{-2\lambda} + \tilde{\sigma}^2)} \\
&\leq \frac{2b^2 e^{-2\lambda} + 2(\mu(\lambda) - r)^2}{2(1 - \gamma)(1 + \rho)(\sigma^2 e^{-2\lambda} + \tilde{\sigma}^2)} \\
&\leq \frac{1}{1 + \rho_0} \max \left\{ \frac{b^2}{\sigma^2(1 - \gamma)}, \frac{(M_1 + r)^2}{\tilde{\sigma}^2(1 - \gamma)} \right\}.
\end{aligned}$$

Let

$$\tilde{\Psi}(\lambda) \equiv \max \left\{ \frac{b^2}{\sigma^2(1 - \gamma)}, \frac{(M_1 + r)^2}{\tilde{\sigma}^2(1 - \gamma)} \right\} \quad (2.40)$$

and

$$\bar{\Psi} \equiv \max \left\{ \tilde{\Psi}, \frac{\tilde{\Psi}}{1 + \rho_0} \right\}. \quad (2.41)$$

Then

$$0 \leq \Psi(\lambda) \leq \bar{\Psi} < \infty, \quad (2.42)$$

and so we see that  $\Psi(\lambda)$  is bounded.  $\square$

We now identify an ordered pair of subsolution/supersolution to (2.36).

**Lemma 2.2.2.** *Suppose  $0 < \gamma < 1$  and*

$$\delta > \gamma r. \quad (2.43)$$

## 2.2. SUBSOLUTION AND SUPERSOLUTION

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In addition, define

$$K_1 \equiv (\gamma - 1) \log \left[ \frac{\delta - \gamma r}{1 - \gamma} \right]. \quad (2.44)$$

Then any constant  $K \leq K_1$  is a subsolution of (2.36).

*Proof.* We want to show

$$\frac{\tilde{\sigma}^2}{2} K_{\lambda\lambda} \geq H(\lambda, K, K_\lambda).$$

Since  $K$  is a constant,  $K_\lambda = 0$  and  $K_{\lambda\lambda} = 0$ . So we want to show that

$$0 \geq -\frac{\tilde{\sigma}^2}{2} (0)^2 - \tilde{\mu}(\lambda)(0) - \gamma G(\lambda, 0) - \gamma r + \delta - (1 - \gamma)e^{\frac{K}{\gamma-1}},$$

or

$$0 \geq -\gamma \Psi(\lambda) - \gamma r + \delta - (1 - \gamma)e^{\frac{K}{\gamma-1}}.$$

Since  $K \leq K_1 = (\gamma - 1) \log \left( \frac{\delta - \gamma r}{1 - \gamma} \right)$ , and  $(\gamma - 1) < 0$ , we have

$$\begin{aligned} e^{\frac{K}{\gamma-1}} &\geq e^{\frac{K_1}{\gamma-1}} \\ &= \exp \left[ \log \left( \frac{\delta - \gamma r}{1 - \gamma} \right) \right] \\ &= \frac{\delta - \gamma r}{1 - \gamma}, \end{aligned}$$

and so

$$\begin{aligned} -(1 - \gamma)e^{\frac{K}{\gamma-1}} &\leq -(1 - \gamma) \frac{\delta - \gamma r}{1 - \gamma} \\ &= \gamma r - \delta. \end{aligned}$$

## 2.2. SUBSOLUTION AND SUPERSOLUTION

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Therefore

$$\begin{aligned}
-\gamma\Psi(\lambda) - \gamma r + \delta - (1 - \gamma)e^{\frac{K}{\gamma-1}} &\leq -\gamma\Psi(\lambda) - \gamma r + \delta + \gamma r - \delta \\
&= -\gamma\Psi(\lambda) \\
&\leq 0,
\end{aligned}$$

since  $\Psi(\lambda) \geq 0$ . □

**Lemma 2.2.3.** *Suppose  $0 < \gamma < 1$  and*

$$\delta > \gamma(r + \bar{\Psi}), \tag{2.45}$$

where  $\bar{\Psi}$  is defined by (3.27). In addition, define

$$K_2 \equiv (\gamma - 1) \log \left[ \frac{\delta - \gamma(r + \bar{\Psi})}{1 - \gamma} \right]. \tag{2.46}$$

Then any constant  $K \geq K_2$  is a supersolution of (2.36).

*Proof.* The proof is similar to the proof of Lemma (2.2.2), but instead we want to show that

$$-\gamma\Psi(\lambda) - \gamma r + \delta - (1 - \gamma)e^{\frac{K}{\gamma-1}} \geq 0.$$

Since  $K \geq K_2 = (\gamma - 1) \log \left( \frac{\delta - \gamma(r + \bar{\Psi})}{1 - \gamma} \right)$  and  $(\gamma - 1) < 0$ , we have

$$\begin{aligned}
e^{\frac{K}{\gamma-1}} &\leq \exp \left[ \log \left( \frac{\delta - \gamma(r + \bar{\Psi})}{1 - \gamma} \right) \right] \\
&= \frac{\delta - \gamma(r + \bar{\Psi})}{1 - \gamma},
\end{aligned}$$

## 2.2. SUBSOLUTION AND SUPERSOLUTION

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and so

$$-(1 - \gamma)e^{\frac{K}{\gamma-1}} \geq -\delta + \gamma(r + \bar{\Psi}).$$

Therefore

$$\begin{aligned} -\gamma\Psi(\lambda) - \gamma r + \delta - (1 - \gamma)e^{\frac{K}{\gamma-1}} &\geq -\gamma\Psi(\lambda) - \gamma r + \delta - \delta + \gamma(r + \bar{\Psi}) \\ &= \gamma(\bar{\Psi} - \Psi(\lambda)) \\ &\geq 0, \end{aligned}$$

since  $\Psi(\lambda) \leq \bar{\Psi}$ . □

**Lemma 2.2.4.** *Suppose  $K_1$  and  $K_2$  are defined by (2.44) and (2.46) respectively, (2.45) holds, and  $0 < \gamma < 1$ . Then  $\langle K_1, K_2 \rangle$  is an ordered subsolution/supersolution pair of (2.36).*

*Proof.* Since  $\bar{\Psi} \geq 0$ ,  $\delta > \gamma(r + \bar{\Psi})$  implies  $\delta > \gamma r$ , and so

$$\begin{aligned} \delta - \gamma(r + \bar{\Psi}) &\leq \delta - \gamma r \\ \log \left[ \frac{\delta - \gamma(r + \bar{\Psi})}{1 - \gamma} \right] &\leq \log \left[ \frac{\delta - \gamma r}{1 - \gamma} \right] \\ (\gamma - 1) \log \left[ \frac{\delta - \gamma(r + \bar{\Psi})}{1 - \gamma} \right] &\geq (\gamma - 1) \log \left[ \frac{\delta - \gamma r}{1 - \gamma} \right] \\ K_2 &\geq K_1. \end{aligned}$$

We have that  $K_1$  is a subsolution,  $K_2$  is a supersolution, and  $K_1 \leq K_2$ . Thus  $\langle K_1, K_2 \rangle$  is an ordered subsolution/supersolution pair of (2.36). □

## 2.3 Existence of Solution

The following theorem gives existence of a classical solution to (2.36). The result follows from Lemma 3.8 of [FP04].

**Theorem 2.3.1.** *Suppose  $K_1$  and  $K_2$  are defined by (2.44) and (2.46) respectively, (2.45) holds, and  $0 < \gamma < 1$ . Then (2.36) has a bounded classical solution  $\tilde{Z}(\lambda)$  which satisfies*

$$K_1 \leq \tilde{Z}(\lambda) \leq K_2. \quad (2.47)$$

*Proof.* By Lemma 2.2.4, we have that  $\langle K_1, K_2 \rangle$  is an ordered subsolution/supersolution pair of (2.36). Note that  $H$  is strictly increasing with respect to  $Z$ . In order to use Lemma 3.8 from [FP04], we need a bound on  $H$  in the following form:

$$|H(\lambda, z, p)| \leq C_1(p^2 + C_2), \quad (2.48)$$

for some constants  $C_1 > 0$  and  $C_2 > 0$ . We start by obtaining such a bound on  $G(\lambda, p)$ :

$$\begin{aligned} 0 &\leq G(\lambda, p) \\ &= \max_{k \geq 0} \left\{ \left[ be^{-\lambda} + \mu(\lambda) - r + (\tilde{\sigma}^2 + \rho\sigma\tilde{\sigma}e^{-\lambda})p \right] k - \frac{k^2}{2}(1 - \gamma)q(\lambda) \right\} \\ &\leq \frac{\left[ be^{-\lambda} + \mu(\lambda) - r + (\tilde{\sigma}^2 + \rho\sigma\tilde{\sigma}e^{-\lambda})p \right]^2}{2(1 - \gamma)q(\lambda)} \\ &\leq \frac{2(be^{-\lambda} + \mu(\lambda) - r)^2 + 2(\tilde{\sigma}^2 + \rho\sigma\tilde{\sigma}e^{-\lambda})^2 p^2}{2(1 - \gamma)q(\lambda)} \\ &= 2\Psi(\lambda) + \frac{2(\tilde{\sigma}^2 + \rho\sigma\tilde{\sigma}e^{-\lambda})^2}{(1 - \gamma)q(\lambda)} p^2. \end{aligned} \quad (2.49)$$



### 2.3. EXISTENCE OF SOLUTION

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Consider the coefficient of  $p^2$  in the second term of (2.51), and note that it is positive. If we let  $y = e^{-\lambda}$ , we see that this coefficient is a rational expression involving a quadratic in  $y$  divided by another quadratic in  $y$ , where  $y > 0$ . The limit as  $y$  tends to infinity ( $\lambda \rightarrow -\infty$ ) of such an expression is a constant. Since  $q(\lambda) \geq q_0 > 0$ , the denominator does not approach zero as  $y$  tends to 0. Therefore this coefficient is bounded above by some positive constant  $\tilde{C}_1$ :

$$\frac{2(\tilde{\sigma}^2 + \rho\sigma\tilde{\sigma}e^{-\lambda})^2}{(1-\gamma)q(\lambda)} \leq \tilde{C}_1. \quad (2.50)$$

Using (2.50) along with (2.42), we have that

$$0 \leq G(\lambda, p) \leq \tilde{C}_1 p^2 + 2\bar{\Psi}. \quad (2.51)$$

Now using (2.35), (2.3), (2.51), and (2.47), we get the following bound for  $K_1 \leq z \leq K_2$ :

$$\begin{aligned} |H(\lambda, z, p)| &\leq \frac{\tilde{\sigma}^2}{2} p^2 + |\tilde{\mu}(\lambda)p| + \gamma G(\lambda, p) + \gamma r + \delta + (1-\gamma)e^{\frac{z}{\gamma-1}} \\ &\leq \frac{\tilde{\sigma}^2}{2} p^2 + M_2|p| + \gamma(\tilde{C}_1 p^2 + 2\bar{\Psi}) + \gamma r + \delta + (1-\gamma)e^{\frac{K_1}{\gamma-1}}. \end{aligned}$$

Each term above is a constant or a constant multiple of  $p^2$ , other than the term  $M_2|p|$ .

But we can bound this term as well. We use the fact that

$$(M_2 + p)^2 \geq 0 \quad \text{and} \quad (M_2 - p)^2 \geq 0$$

imply

$$-\frac{1}{2}(M_2^2 + p^2) \leq M_2 p \quad \text{and} \quad M_2 p \leq \frac{1}{2}(M_2^2 + p^2),$$

respectively. This gives us the inequality

$$|M_2 p| = M_2 |p| \leq \frac{1}{2}(M_2^2 + p^2).$$

So we have that

$$|H(\lambda, z, p)| \leq C_1(p^2 + C_2), \quad z \in [K_1, K_2], \quad (2.52)$$

where  $C_1$  and  $C_2$  are positive constants. Therefore by [FP04], since  $H$  is strictly increasing with respect to  $Z$  and we can bound  $H$  as in (2.52), (2.36) has a classical solution  $\tilde{Z}(\lambda)$  such that

$$K_1 \leq \tilde{Z}(\lambda) \leq K_2.$$

□

## 2.4 Verification Theorem

Now that we have existence of a classical solution  $\tilde{Z}(\lambda)$  to the HJB Equation (2.36), we want to show that

$$\tilde{V}(x, \lambda) = \frac{1}{\gamma} x^\gamma e^{\tilde{Z}(\lambda)} \quad (2.53)$$

is equal to the value function defined by (2.6). In effect, we will have maximized the expected total discounted utility of consumption of an investor whose net worth is given by (2.4). This so-called “Verification Theorem” is given in Theorem 2.4.4.

Before stating the Verification Theorem, we will need a few results.

**Lemma 2.4.1.** *Suppose  $0 < \gamma < 1$  and (2.45) holds. If  $\tilde{Z}(\lambda)$  is a classical solution of*

## 2.4. VERIFICATION THEOREM

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(2.36) which satisfies (2.47), then  $\tilde{Z}_\lambda(\lambda)$  is bounded. That is, there exists a constant  $A > 0$  such that

$$|\tilde{Z}_\lambda(\lambda)| \leq A. \quad (2.54)$$

*Proof.* We know  $\tilde{Z}(\lambda)$  is bounded by constants  $K_1$  and  $K_2$ . Since  $\tilde{Z}$  is a classical solution of (2.36),  $\tilde{Z} \in C^2$ . So its first and second derivatives are defined everywhere. For this reason, we do not need to worry about  $\tilde{Z}_\lambda$  being undefined anywhere, but we must verify that  $\tilde{Z}_\lambda$  is bounded at its local maxima and minima. Such an example where this may not be true is if  $\tilde{Z}$  is oscillatory, with slopes becoming more steep with each oscillation. Thus to prove that  $\tilde{Z}_\lambda$  is bounded, it is sufficient to prove that it is bounded at its local extrema.

Suppose  $\tilde{Z}_\lambda$  has a local max or min at  $\lambda_0$ . Then  $\tilde{Z}_{\lambda\lambda}(\lambda_0) = 0$ . Therefore we have

$$0 = \frac{\tilde{\sigma}^2}{2} \tilde{Z}_\lambda^2(\lambda_0) + \tilde{\mu}(\lambda_0) \tilde{Z}_\lambda(\lambda_0) + \gamma r - \delta + (1 - \gamma) e^{\frac{\tilde{Z}(\lambda_0)}{\gamma-1}} + \gamma G(\lambda_0, \tilde{Z}_\lambda(\lambda_0)).$$

Since  $G \geq 0$  and  $(1 - \gamma) e^{\frac{\tilde{Z}}{\gamma-1}} \geq (1 - \gamma) e^{\frac{K_2}{\gamma-1}} = \delta - \gamma r - \gamma \bar{\Psi}$ , we have

$$\frac{\tilde{\sigma}^2}{2} \tilde{Z}_\lambda^2(\lambda_0) + \tilde{\mu}(\lambda_0) \tilde{Z}_\lambda(\lambda_0) + \gamma r - \delta + \delta - \gamma r - \gamma \bar{\Psi} \leq 0,$$

or

$$\frac{\tilde{\sigma}^2}{2} \tilde{Z}_\lambda^2(\lambda_0) + \tilde{\mu}(\lambda_0) \tilde{Z}_\lambda(\lambda_0) - \gamma \bar{\Psi} \leq 0. \quad (2.55)$$

The left-hand side of (2.55) is a quadratic expression in  $\tilde{Z}_\lambda(\lambda_0)$ . Since the quadratic is concave up, the only  $\tilde{Z}_\lambda$ -values that satisfy (2.55) are those bounded between the two

## 2.4. VERIFICATION THEOREM

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roots of the quadratic. The roots are given by

$$r_1 \equiv \frac{-\tilde{\mu}(\lambda_0) - \sqrt{\tilde{\mu}^2(\lambda_0) - 2\tilde{\sigma}^2\gamma\bar{\Psi}}}{\tilde{\sigma}^2} \quad \text{and} \quad r_2 \equiv \frac{-\tilde{\mu}(\lambda_0) + \sqrt{\tilde{\mu}^2(\lambda_0) - 2\tilde{\sigma}^2\gamma\bar{\Psi}}}{\tilde{\sigma}^2}, \quad (2.56)$$

where  $r_1 \leq r_2$ . We need bounds for  $\tilde{Z}_\lambda$  that are independent of  $\lambda_0$ . Using (2.3), we have

$$r_1 \geq \frac{-M_2 - \sqrt{M_2^2 - 2\tilde{\sigma}^2\gamma\bar{\Psi}}}{\tilde{\sigma}^2} \equiv \tilde{r}_1 \quad \text{and} \quad r_2 \leq \frac{M_2 + \sqrt{M_2^2 - 2\tilde{\sigma}^2\gamma\bar{\Psi}}}{\tilde{\sigma}^2} \equiv \tilde{r}_2,$$

so that

$$\tilde{r}_1 \leq \tilde{Z}_\lambda(\lambda_0) \leq \tilde{r}_2. \quad (2.57)$$

Thus  $\tilde{Z}_\lambda$  is bounded at  $\lambda_0$ . Since  $\lambda_0$  was an arbitrary maximum or minimum point of  $\tilde{Z}_\lambda$ ,  $\tilde{Z}_\lambda$  is bounded at its maxima and minima. Therefore,  $\tilde{Z}_\lambda$  is bounded.  $\square$

**Lemma 2.4.2.** *Suppose  $0 < \gamma < 1$  and (2.45) holds. Let  $\tilde{Z}(\lambda)$  denote a classical solution of (2.36) which satisfies (2.47). Define  $k^*(\lambda, p)$  as in (2.30). Then there exist positive constants  $\Lambda_1$  and  $\Lambda_2$  such that*

$$k^*(\lambda_t, \tilde{Z}_\lambda) \leq \Lambda_1, \quad (2.58)$$

and

$$e^{-\lambda_t} k^*(\lambda_t, \tilde{Z}_\lambda) \leq \Lambda_2. \quad (2.59)$$

*Proof.* Note that  $k_t^* \geq 0$ , so we do not need to bound it in absolute value. We bound

## 2.4. VERIFICATION THEOREM

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$k^*(\lambda_t, \tilde{Z}_\lambda)$  and  $e^{-\lambda_t} k^*(\lambda_t, \tilde{Z}_\lambda)$  as follows:

$$\begin{aligned}
k^*(\lambda_t, \tilde{Z}_\lambda) &= \left[ \frac{be^{-\lambda_t} + \mu(\lambda_t) - r + (\tilde{\sigma}^2 + \rho\sigma\tilde{\sigma}e^{-\lambda_t})\tilde{Z}_\lambda(\lambda_t)}{(1-\gamma)q(\lambda_t)} \right]^+ \\
&\leq \frac{|be^{-\lambda_t} + \mu(\lambda_t) - r + (\tilde{\sigma}^2 + \rho\sigma\tilde{\sigma}e^{-\lambda_t})\tilde{Z}_\lambda(\lambda_t)|}{(1-\gamma)q(\lambda_t)} \\
&\leq \frac{be^{-\lambda_t} + M_1 + r + (\tilde{\sigma}^2 + |\rho|\sigma\tilde{\sigma}e^{-\lambda_t})A}{(1-\gamma)q(\lambda_t)} \\
&\leq \Lambda_1,
\end{aligned} \tag{2.60}$$

and

$$\begin{aligned}
e^{-\lambda_t} k^*(\lambda_t, \tilde{Z}_\lambda) &\leq \frac{e^{-\lambda_t} [be^{-\lambda_t} + M_1 + r + (\tilde{\sigma}^2 + |\rho|\sigma\tilde{\sigma}e^{-\lambda_t})A]}{(1-\gamma)q(\lambda_t)} \\
&\leq \Lambda_2.
\end{aligned} \tag{2.61}$$

To justify (2.60) and (2.61), we apply the same reasoning we used for the bound (2.50) in the proof of Theorem 2.3.1. Let  $y_t = e^{-\lambda_t}$ . Note that (2.60) is the quotient of a linear expression in  $y_t$  and a quadratic expression in  $y_t$ , where  $y_t > 0$ . Since  $q(\lambda_t)$  is bounded below by a positive constant, the limits of (2.60) as  $\lambda_t$  approach positive and negative infinity both exist. Therefore the quotient is bounded.

The expression in (2.61) is the quotient of a quadratic in  $y_t$  by another quadratic in  $y_t$ , and its limits as  $\lambda_t$  approaches both positive and negative infinity exist as well. Therefore (2.61) holds.  $\square$

**Lemma 2.4.3.** *Suppose  $0 < \gamma < 1$  and (2.45) holds. Let  $\tilde{Z}(\lambda)$  denote a classical solution of (2.36) which satisfies (2.47). For the process  $X_t$  defined in (2.4), and  $k^*(\lambda, p)$  and*

## 2.4. VERIFICATION THEOREM

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$c^*(\lambda, p)$  defined by (2.30) and (2.28), respectively, we have

$$\mathbb{E}[X_t^m] < \infty \quad (2.62)$$

for any fixed  $m > 0$ .

For a proof, refer to [Pa02] Chapter 1, Lemma 1.1.

We now state our main result.

**Theorem 2.4.4. (Verification Theorem)** *Suppose  $0 < \gamma < 1$  and (2.45) holds. Let  $\tilde{Z}(\lambda)$  denote a classical solution of (2.36) which satisfies (2.47). Denote*

$$\tilde{V}(x, \lambda) \equiv \frac{1}{\gamma} x^\gamma e^{\tilde{Z}(\lambda)}. \quad (2.63)$$

Then we have

$$\tilde{V}(x, \lambda) \equiv V(x, \lambda), \quad (2.64)$$

where  $V(x, \lambda)$  is the value function defined by (2.6). Moreover, the optimal control policy is

$$k^*(\lambda, \tilde{Z}_\lambda) = \left[ \frac{be^{-\lambda} + \mu(\lambda) - r + (\tilde{\sigma}^2 + \rho\sigma\tilde{\sigma}e^{-\lambda})\tilde{Z}_\lambda(\lambda)}{(1-\gamma)q(\lambda)} \right]^+, \quad c^*(\lambda, \tilde{Z}) = e^{\frac{\tilde{Z}(\lambda)}{\gamma-1}}. \quad (2.65)$$

*Proof.* Since  $\tilde{Z}$  is a classical solution of (2.36), we have

$$\delta = \frac{\tilde{\sigma}^2}{2}(\tilde{Z}_{\lambda\lambda} + \tilde{Z}_\lambda^2) + \tilde{\mu}(\lambda)\tilde{Z}_\lambda + \gamma G(\lambda, \tilde{Z}_\lambda) + \gamma r + (1-\gamma)e^{\frac{\tilde{Z}}{\gamma-1}}. \quad (2.66)$$

## 2.4. VERIFICATION THEOREM

---

We can show that  $\tilde{V}$  is a classical solution of (2.21). From (2.63) we get

$$\tilde{V}_\lambda = \tilde{Z}_\lambda \tilde{V}, \quad \tilde{V}_{\lambda\lambda} = (\tilde{Z}_\lambda^2 + \tilde{Z}_{\lambda\lambda}) \tilde{V},$$

$$\tilde{V}_x = \frac{\gamma}{x} \tilde{V}, \quad \tilde{V}_{xx} = \frac{\gamma(\gamma-1)}{x^2} \tilde{V}, \quad \text{and} \quad \tilde{V}_{x\lambda} = \frac{\gamma}{x} \tilde{Z}_\lambda \tilde{V}.$$

Substituting these values into the right-hand-side of (2.21) and simplifying, we get

$$\begin{aligned} & \frac{\tilde{\sigma}^2}{2} \tilde{V}_{\lambda\lambda} + \tilde{\mu}(\lambda) \tilde{V}_\lambda + rx \tilde{V}_x + \max_{c \geq 0} \left[ -cx \tilde{V}_x + \frac{1}{\gamma} (cx)^\gamma \right] \\ & \quad + \max_{k \geq 0} \left[ (be^{-\lambda} + \mu(\lambda) - r) kx \tilde{V}_x + (\tilde{\sigma}^2 + \rho\sigma\tilde{\sigma}e^{-\lambda}) kx \tilde{V}_{x\lambda} + \frac{k^2 x^2 q(\lambda)}{2} \tilde{V}_{xx} \right] \\ & = \frac{\tilde{\sigma}^2}{2} (\tilde{Z}_\lambda^2 + \tilde{Z}_{\lambda\lambda}) \tilde{V} + \tilde{\mu}(\lambda) \tilde{Z}_\lambda \tilde{V} + r\gamma \tilde{V} + \left( \frac{1}{\gamma} - 1 \right) \left( \frac{\gamma}{x} \tilde{V} \right)^{\frac{\gamma}{\gamma-1}} \\ & \quad + \max_{k \geq 0} \left[ (be^{-\lambda} + \mu(\lambda) - r) k\gamma \tilde{V} + (\tilde{\sigma}^2 + \rho\sigma\tilde{\sigma}e^{-\lambda}) k\gamma \tilde{V} \tilde{Z}_\lambda + \frac{k^2 q(\lambda)}{2} \gamma(\gamma-1) \tilde{V} \right] \\ & = \tilde{V} \left\{ \frac{\tilde{\sigma}^2}{2} (\tilde{Z}_\lambda^2 + \tilde{Z}_{\lambda\lambda}) + \tilde{\mu}(\lambda) \tilde{Z}_\lambda + r\gamma + \left( \frac{1}{\gamma} - 1 \right) \left( \frac{\gamma}{x} \right)^{\frac{\gamma}{\gamma-1}} \tilde{V}^{\frac{1}{\gamma-1}} \right. \\ & \quad \left. + \gamma \max_{k \geq 0} \left[ (be^{-\lambda} + \mu(\lambda) - r + (\tilde{\sigma}^2 + \rho\sigma\tilde{\sigma}e^{-\lambda}) \tilde{Z}_\lambda) k - \frac{k^2}{2} (1-\gamma) q(\lambda) \right] \right\} \\ & = \tilde{V} \left\{ \frac{\tilde{\sigma}^2}{2} (\tilde{Z}_\lambda^2 + \tilde{Z}_{\lambda\lambda}) + \tilde{\mu}(\lambda) \tilde{Z}_\lambda + r\gamma + (1-\gamma) e^{\frac{\tilde{Z}}{\gamma-1}} + \gamma G(\lambda, \tilde{Z}_\lambda) \right\} \\ & = \tilde{V} \delta. \end{aligned} \tag{2.67}$$

Therefore  $\tilde{V}$  is a classical solution of (2.21).

## 2.4. VERIFICATION THEOREM

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For any admissible control  $(k_t, c_t) \in \Pi$ , using Ito's rule for  $f(t, \tilde{V}) = e^{-\delta t} \tilde{V}$ , we get

$$d[e^{-\delta t} \tilde{V}(X_t, \lambda_t)] = -\delta e^{-\delta t} \tilde{V}(X_t, \lambda_t) dt + e^{-\delta t} d\tilde{V}(X_t, \lambda_t). \quad (2.68)$$

Applying Ito's rule to  $\tilde{V}$ , we have

$$\begin{aligned} d\tilde{V}(X_t, \lambda_t) &= \tilde{V}_x dX_t + \tilde{V}_\lambda d\lambda_t + \frac{1}{2} \tilde{V}_{xx} (dX_t)^2 + \tilde{V}_{x\lambda} (dX_t)(d\lambda_t) + \frac{1}{2} \tilde{V}_{\lambda\lambda} (d\lambda_t)^2 \\ &= \tilde{V}_x X_t \left[ (\mu(\lambda_t) + be^{-\lambda_t} - r) k_t dt + (r - c_t) dt + k_t (\tilde{\sigma} d\tilde{w}_t + \sigma e^{-\lambda_t} dw_t) \right] \\ &\quad + \tilde{V}_\lambda (\tilde{\mu}(\lambda_t) dt + \tilde{\sigma} d\tilde{w}_t) + \frac{1}{2} \tilde{V}_{xx} \left[ k_t^2 X_t^2 (\tilde{\sigma}^2 + 2\rho\tilde{\sigma}\sigma e^{-\lambda_t} + \sigma^2 e^{-2\lambda_t}) dt \right] \\ &\quad + \tilde{V}_{x\lambda} k_t X_t (\tilde{\sigma}^2 + \rho\tilde{\sigma}\sigma e^{-\lambda_t}) dt + \frac{1}{2} \tilde{V}_{\lambda\lambda} \tilde{\sigma}^2 dt \\ &= \left[ \frac{\tilde{\sigma}^2}{2} \tilde{V}_{\lambda\lambda} + \tilde{\mu}(\lambda_t) \tilde{V}_\lambda + r X_t \tilde{V}_x - c X_t \tilde{V}_x + (be^{-\lambda_t} + \mu(\lambda_t) - r) k_t X_t \tilde{V}_x \right. \\ &\quad \left. + k_t X_t (\tilde{\sigma}^2 + \rho\tilde{\sigma}\sigma e^{-\lambda_t}) \tilde{V}_{x\lambda} + \frac{k_t^2 X_t^2 q(\lambda_t)}{2} \tilde{V}_{xx} \right] dt \\ &\quad + \sigma e^{-\lambda_t} k_t X_t \tilde{V}_x dw_t + (\tilde{\sigma} k_t X_t \tilde{V}_x + \tilde{\sigma} \tilde{V}_\lambda) d\tilde{w}_t. \end{aligned}$$

The integral form of (2.68) is

$$\begin{aligned} e^{-\delta T} \tilde{V}(X_T, \lambda_T) - \tilde{V}(x, \lambda) &= \int_0^T e^{-\delta t} d\tilde{V}(X_t, \lambda_t) - \int_0^T \delta e^{-\delta t} \tilde{V}(X_t, \lambda_t) dt \\ &= \int_0^T e^{-\delta t} \left[ \frac{\tilde{\sigma}^2}{2} \tilde{V}_{\lambda\lambda} + \tilde{\mu}(\lambda) \tilde{V}_\lambda + r X_t \tilde{V}_x - c X_t \tilde{V}_x + (be^{-\lambda} + \mu(\lambda) - r) k X_t \tilde{V}_x \right. \\ &\quad \left. + k X_t (\tilde{\sigma}^2 + \rho\tilde{\sigma}\sigma e^{-\lambda}) \tilde{V}_{x\lambda} + \frac{k^2 X_t^2 q(\lambda)}{2} \tilde{V}_{xx} \right] dt - \int_0^T \delta e^{-\delta t} \tilde{V}(X_t, \lambda_t) dt \\ &\quad + \int_0^T \sigma e^{-\lambda} k X_t \tilde{V}_x dw_t + \int_0^T (\tilde{\sigma} k X_t \tilde{V}_x + \tilde{\sigma} \tilde{V}_\lambda) d\tilde{w}_t. \end{aligned} \quad (2.69)$$



## 2.4. VERIFICATION THEOREM

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In (2.67) we have equality when the maximum over all  $k, c \geq 0$  is obtained. For arbitrary  $k, c \geq 0$ , we have the following inequality:

$$\begin{aligned} \delta \tilde{V}(x, \lambda) &\geq \frac{\tilde{\sigma}^2}{2} \tilde{V}_{\lambda\lambda} + \tilde{\mu}(\lambda) \tilde{V}_\lambda + rx \tilde{V}_x - cx \tilde{V}_x + \frac{1}{\gamma} (cx)^\gamma + (be^{-\lambda} + \mu(\lambda) - r) kx \tilde{V}_x \\ &\quad + (\tilde{\sigma}^2 + \rho\sigma\tilde{\sigma}e^{-\lambda}) kx \tilde{V}_{x\lambda} + \frac{k^2 x^2 q(\lambda)}{2} \tilde{V}_{xx}. \end{aligned}$$

We replace  $x, \lambda, k$ , and  $c$  with  $X_t, \lambda_t, k_t$ , and  $c_t$ , respectively, and rearrange to get

$$\begin{aligned} \frac{\tilde{\sigma}^2}{2} \tilde{V}_{\lambda\lambda} + \tilde{\mu}(\lambda_t) \tilde{V}_\lambda + rX_t \tilde{V}_x - c_t X_t \tilde{V}_x + (be^{-\lambda_t} + \mu(\lambda_t) - r) k_t X_t \tilde{V}_x \\ + (\tilde{\sigma}^2 + \rho\sigma\tilde{\sigma}e^{-\lambda_t}) k_t X_t \tilde{V}_{x\lambda} + \frac{k_t^2 X_t^2 q(\lambda_t)}{2} \tilde{V}_{xx} \leq \delta \tilde{V}(x, \lambda) - \frac{1}{\gamma} (cX_t)^\gamma. \end{aligned} \tag{2.70}$$

Combining (2.69) and (2.70), we have

$$\begin{aligned} e^{-\delta T} \tilde{V}(X_T, \lambda_T) - \tilde{V}(x, \lambda) \\ \leq \int_0^T e^{-\delta t} \left[ \delta \tilde{V}(X_t, \lambda_t) - \frac{1}{\gamma} (c_t X_t)^\gamma \right] dt - \int_0^T \delta e^{-\delta t} \tilde{V}(X_t, \lambda_t) dt \\ \quad + \int_0^T \sigma e^{-\lambda_t} k_t X_t \tilde{V}_x dw_t + \int_0^T \left( \tilde{\sigma} k_t X_t \tilde{V}_x + \tilde{\sigma} \tilde{V}_\lambda \right) d\tilde{w}_t \\ = - \int_0^T e^{-\delta t} \frac{1}{\gamma} (c_t X_t)^\gamma dt + m_T + \tilde{m}_T, \end{aligned} \tag{2.71}$$

where

$$m_T = \int_0^T \sigma e^{-\lambda_t} k_t X_t \tilde{V}_x dw_t \quad \text{and} \quad \tilde{m}_T = \int_0^T \left( \tilde{\sigma} k_t X_t \tilde{V}_x + \tilde{\sigma} \tilde{V}_\lambda \right) d\tilde{w}_t$$

are local martingales. They are local martingales because they are martingales on the

## 2.4. VERIFICATION THEOREM

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closed ball  $\{x^2 + \lambda^2 \leq R^2\}$ .

Let us start by proving that  $m_T$  is a local martingale. It is sufficient to show that

$$Pr \left( \int_0^{T \wedge \tau_R} (\sigma e^{-\lambda_t} k_t X_t \tilde{V}_x)^2 dt < \infty \right) = 1,$$

where

$$\tau_R = \inf\{t > 0; X_t^2 + \lambda_t^2 = R^2\}.$$

(See [KS91], p. 146). Since we are restricted to the region  $\{x^2 + \lambda^2 \leq R^2\}$  on which  $X_t$  and  $\lambda_t$  are bounded, we know that  $\tilde{V}_x = X_t^{\gamma-1} e^{\tilde{Z}(\lambda_t)}$  is also bounded. So there exists a constant  $\Lambda$  for which

$$\int_0^{T \wedge \tau_R} (\sigma e^{-\lambda_t} k_t X_t \tilde{V}_x)^2 dt \leq \Lambda \int_0^{T \wedge \tau_R} k_t^2 dt.$$

Recall that  $(k_t, c_t) \in \Pi$ , so that  $Pr \left( \int_0^T k_t^2 dt < \infty \right) = 1$ . Then  $Pr \left( \Lambda \int_0^{T \wedge \tau_R} k_t^2 dt < \infty \right) = 1$ , and so

$$Pr \left( \int_0^{T \wedge \tau_R} (\sigma e^{-\lambda_t} k_t X_t \tilde{V}_x)^2 dt < \infty \right) = 1.$$

Therefore  $m_T = \int_0^T \sigma e^{-\lambda_t} k_t X_t \tilde{V}_x dw_t$  is a local martingale.

Similarly, we can show that  $\tilde{m}_T$  is a local martingale. By Lemma 2.4.1,  $\tilde{Z}_\lambda$  is bounded. Then  $\tilde{V}_\lambda = \tilde{V} \tilde{Z}_\lambda(\lambda)$  is bounded on  $\{x^2 + \lambda^2 = R^2\}$ . So there exists a constant  $\Lambda$  such that

$$\begin{aligned} \int_0^{T \wedge \tau_R} \left( \tilde{\sigma} k_t X_t \tilde{V}_x + \tilde{\sigma} \tilde{V}_\lambda \right)^2 dt &\leq \Lambda \int_0^{T \wedge \tau_R} (k_t^2 + 1)^2 dt \\ &\leq 2\Lambda \int_0^{T \wedge \tau_R} k_t^2 dt + 2\Lambda(T \wedge \tau_R). \end{aligned}$$

## 2.4. VERIFICATION THEOREM

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Again, since  $(k_t, c_t) \in \Pi$ ,

$$Pr \left( \int_0^{T \wedge \tau_R} \left( \tilde{\sigma} k_t X_t \tilde{V}_x + \tilde{\sigma} \tilde{V}_\lambda \right)^2 dt < \infty \right) = 1.$$

So  $\tilde{m}_T$  is a local martingale.

Rearranging equation (2.71) to obtain a bound on  $\tilde{V}$ , we have

$$\tilde{V}(x, \lambda) \geq \int_0^T e^{-\delta t} \frac{1}{\gamma} (c_t X_t)^\gamma dt + e^{-\delta T} \tilde{V}(X_T, \lambda_T) - m_T - \tilde{m}_T. \quad (2.72)$$

Next we replace  $T$  with  $T \wedge \tau_R$ ,

$$\tilde{V}(x, \lambda) \geq \int_0^{T \wedge \tau_R} e^{-\delta t} \frac{1}{\gamma} (c_t X_t)^\gamma dt + e^{-\delta T \wedge \tau_R} \tilde{V}(X_{T \wedge \tau_R}, \lambda_{T \wedge \tau_R}) - m_{T \wedge \tau_R} - \tilde{m}_{T \wedge \tau_R}. \quad (2.73)$$

Since  $m_T$  and  $\tilde{m}_T$  are local martingales,  $\mathbb{E}[m_{T \wedge \tau_R}] = 0$  and  $\mathbb{E}[\tilde{m}_{T \wedge \tau_R}] = 0$ . So taking the expectation of both sides gives us

$$\tilde{V}(x, \lambda) \geq \mathbb{E}_{x, \lambda} \int_0^{T \wedge \tau_R} e^{-\delta t} \frac{1}{\gamma} (c_t X_t)^\gamma dt + \mathbb{E}_{x, \lambda} e^{-\delta T \wedge \tau_R} \tilde{V}(X_{T \wedge \tau_R}, \lambda_{T \wedge \tau_R}) \quad (2.74)$$

$$\geq \mathbb{E}_{x, \lambda} \int_0^{T \wedge \tau_R} e^{-\delta t} \frac{1}{\gamma} (c_t X_t)^\gamma dt. \quad (2.75)$$

The second inequality is true because  $\tilde{V} > 0$ . Now we take the limit as  $R$  tends to infinity and use Fatou's lemma, which states

$$\mathbb{E} \left[ \liminf_{n \rightarrow \infty} X_n \right] \leq \liminf_{n \rightarrow \infty} \mathbb{E}[X_n].$$

## 2.4. VERIFICATION THEOREM

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So we have

$$\begin{aligned}\tilde{V}(x, \lambda) &\geq \lim_{R \rightarrow \infty} \mathbb{E}_{x, \lambda} \left[ \int_0^{T \wedge \tau_R} e^{-\delta t} \frac{1}{\gamma} (c_t X_t)^\gamma dt \right] \\ &\geq \mathbb{E}_{x, \lambda} \left[ \int_0^T e^{-\delta t} \frac{1}{\gamma} (c_t X_t)^\gamma dt \right].\end{aligned}$$

Now, letting  $T$  tend to infinity and using Fatou's lemma again,

$$\begin{aligned}\tilde{V}(x, \lambda) &\geq \lim_{T \rightarrow \infty} \mathbb{E}_{x, \lambda} \int_0^T e^{-\delta t} \frac{1}{\gamma} (c_t X_t)^\gamma dt \\ &\geq \mathbb{E}_{x, \lambda} \int_0^\infty e^{-\delta t} \frac{1}{\gamma} (c_t X_t)^\gamma dt.\end{aligned}\tag{2.76}$$

Since (2.76) holds for any arbitrary values of  $c_t \geq 0$  and  $k_t \geq 0$ , we have

$$\begin{aligned}\tilde{V}(x, \lambda) &\geq \sup_{(k_t, c_t) \in \Pi} \mathbb{E}_{x, \lambda} \int_0^\infty e^{-\delta t} \frac{1}{\gamma} (c_t X_t)^\gamma dt \\ &= V(x, \lambda).\end{aligned}\tag{2.77}$$

Next we must show the reverse inequality,  $\tilde{V}(x, \lambda) \leq V(x, \lambda)$ . We start by showing that  $(k_t^*, c_t^*) \in \Pi$ . First, we note that

$$k^*(\lambda_t, \tilde{Z}_\lambda) = \left[ \frac{be^{-\lambda_t} + \mu(\lambda_t) - r + (\tilde{\sigma}^2 + \rho\sigma\tilde{\sigma}e^{-\lambda_t})\tilde{Z}_\lambda(\lambda_t)}{(1 - \gamma)q(\lambda_t)} \right]^+ \quad \text{and} \quad c^*(\lambda_t, \tilde{Z}) = e^{\frac{\tilde{Z}(\lambda_t)}{\gamma-1}}$$

are progressively measurable because at every time  $t$  their values are known. Second, we must show that

$$Pr \left( \int_0^T (k_t^*)^2 dt < \infty \right) = 1\tag{2.78}$$

and

$$Pr \left( \int_0^T c_t^* dt < \infty \right) = 1. \quad (2.79)$$

By Lemma 2.4.2, we have that  $k_t^* \leq \Lambda_1$ , and so (2.78) follows. (2.79) follows from the fact that  $K_1 \leq \tilde{Z}(\lambda) \leq K_2$ . Therefore  $(k_t^*, c_t^*) \in \Pi$ .

Using  $k_t^*$  and  $c_t^*$  instead of arbitrary  $k_t, c_t > 0$ , we have equality in (2.72):

$$\tilde{V}(x, \lambda) = \int_0^T e^{-\delta t} \frac{1}{\gamma} (c_t^* X_t)^\gamma dt + e^{-\delta T} \tilde{V}(X_T, \lambda_T) - m_T^* - \tilde{m}_T^*, \quad (2.80)$$

where  $m_T^* = \int_0^T \sigma e^{-\lambda t} k_t^* X_t \tilde{V}_x dw_t$  and  $\tilde{m}_T^* = \int_0^T \left( \tilde{\sigma} k_t^* X_t \tilde{V}_x + \tilde{\sigma} \tilde{V}_\lambda \right) d\tilde{w}_t$ .

We wish to show that  $m_T^*$  and  $\tilde{m}_T^*$  are martingales, so that these terms vanish when taking the expectation of both sides in (2.80). We will do so by using the fact that

$$\mathbb{E}_{x,\lambda} \left[ \int_0^T (f(X_t))^2 dt \right] < \infty \quad (2.81)$$

implies that  $\int_0^T f(X_t) dw_t$  is a martingale. Using Lemmas 2.4.1, 2.4.2, and 2.4.3, we have

$$\begin{aligned} \mathbb{E}_{x,\lambda} \left[ \int_0^T (\sigma e^{-\lambda t} k_t^* X_t \tilde{V}_x)^2 dt \right] &= \mathbb{E}_{x,\lambda} \left[ \int_0^T \left( \sigma e^{-\lambda t} k_t^* X_t \cdot \frac{1}{\gamma} X_t^\gamma e^{\tilde{Z}} \right)^2 dt \right] \\ &\leq \frac{\sigma^2}{\gamma^2} \Lambda_2^2 e^{2K_2} \int_0^T \mathbb{E}_{x,\lambda} \left[ X_t^{2(1+\gamma)} \right] dt \\ &< \infty, \end{aligned}$$

and

$$\begin{aligned}
 \mathbb{E}_{x,\lambda} \left[ \int_0^T (\tilde{\sigma} k_t^* X_t \tilde{V}_x + \tilde{\sigma} \tilde{V}_\lambda)^2 dt \right] \\
 &= \mathbb{E}_{x,\lambda} \left[ \int_0^T \left( \tilde{\sigma} k_t^* X_t \cdot \frac{1}{\gamma} X_t^\gamma e^{\tilde{Z}} + \tilde{\sigma} \cdot \frac{1}{\gamma} X_t^\gamma e^{\tilde{Z}} \tilde{Z}_\lambda \right)^2 dt \right] \\
 &\leq \mathbb{E}_{x,\lambda} \left[ \int_0^T 2 \frac{\tilde{\sigma}^2}{\gamma^2} (k_t^*)^2 X_t^{2(1+\gamma)} e^{2\tilde{Z}} dt \right] + \mathbb{E}_{x,\lambda} \left[ \int_0^T 2 \frac{\tilde{\sigma}^2}{\gamma^2} X_t^{2\gamma} e^{2\tilde{Z}} \tilde{Z}_\lambda^2 dt \right] \\
 &\leq 2 \frac{\tilde{\sigma}^2}{\gamma^2} \Lambda_1^2 e^{2K_2} \int_0^T \mathbb{E}_{x,\lambda} [X_t^{2(1+\gamma)}] dt + 2 \frac{\tilde{\sigma}^2}{\gamma^2} e^{2K_2} A^2 \int_0^T \mathbb{E}_{x,\lambda} [X_t^{2\gamma}] dt \\
 &< \infty.
 \end{aligned}$$

Therefore  $m_T^*$  and  $\tilde{m}_T^*$  are martingales. Taking the expectation in (2.80), we obtain

$$\tilde{V}(x, \lambda) = \mathbb{E}_{x,\lambda} \left[ \int_0^T e^{-\delta t} \frac{1}{\gamma} (c_t^* X_t)^\gamma dt \right] + e^{-\delta T} \mathbb{E}_{x,\lambda} [\tilde{V}(X_T, \lambda_T)]. \quad (2.82)$$

Since  $V(x, \lambda) \leq \tilde{V}(x, \lambda)$ , as shown above, we see that

$$\mathbb{E}_{x,\lambda} \left[ \int_0^\infty e^{-\delta t} \frac{1}{\gamma} (c_t^* X_t)^\gamma dt \right] \leq \tilde{V}(x, \lambda) < \infty. \quad (2.83)$$

This implies that

$$\liminf_{T \rightarrow \infty} \mathbb{E}_{x,\lambda} \left[ e^{-\delta T} \frac{1}{\gamma} (c_T^* X_T)^\gamma \right] = 0, \quad (2.84)$$

which is easily seen with a proof by contradiction. Note that  $c_t^*$  is bounded below by a positive constant:

$$c_t^* \geq e^{\frac{K_2}{\gamma-1}} \equiv \underline{c} > 0.$$

## 2.4. VERIFICATION THEOREM

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Then we have

$$0 = \liminf_{T \rightarrow \infty} \mathbb{E}_{x,\lambda} \left[ e^{-\delta T} \frac{1}{\gamma} (c_T^* X_T)^\gamma \right] \geq \underline{c}^\gamma \liminf_{T \rightarrow \infty} \mathbb{E}_{x,\lambda} \left[ e^{-\delta T} \frac{1}{\gamma} X_T^\gamma \right] \geq 0,$$

which implies

$$\liminf_{T \rightarrow \infty} \mathbb{E}_{x,\lambda} \left[ e^{-\delta T} \frac{1}{\gamma} X_T^\gamma \right] = 0. \quad (2.85)$$

Using (2.85) and the fact that  $e^{\tilde{Z}} \leq e^{K_2}$ , we have

$$\liminf_{T \rightarrow \infty} \mathbb{E}_{x,\lambda} \left[ e^{-\delta T} \tilde{V}(X_T, \lambda_T) \right] = 0. \quad (2.86)$$

Now taking the  $\liminf$  of (2.82) as  $T$  approaches infinity, we get

$$\begin{aligned} \tilde{V}(x, \lambda) &= \liminf_{T \rightarrow \infty} \mathbb{E}_{x,\lambda} \left[ \int_0^T e^{-\delta t} \frac{1}{\gamma} (c_t^* X_t)^\gamma dt \right] \\ &= \mathbb{E}_{x,\lambda} \left[ \int_0^\infty e^{-\delta t} \frac{1}{\gamma} (c_t^* X_t)^\gamma dt \right], \end{aligned} \quad (2.87)$$

by the monotone convergence theorem. Finally, by (2.87) and the definition of  $V$ , we have

$$\tilde{V}(x, \lambda) \leq V(x, \lambda). \quad (2.88)$$

Combining (2.77) and (2.88), we get

$$\tilde{V}(x, \lambda) = V(x, \lambda).$$

□

#### 2.4. VERIFICATION THEOREM

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In conclusion, we have proved that our solution  $\tilde{V}(x, \lambda)$  to the HJB equation for  $V$  given by (2.21) is equal to the value function. In effect, we have maximized the expected discounted HARA utility of consumption.



# Chapter 3

## Stochastic Volatility

### 3.1 Background and Motivation

It is widely accepted that volatilities exhibit random characteristics, which presents limitations in models that assume a constant volatility for stock prices. One commonly noted example is the discrepancy between observed option prices and the prices predicted by the Black-Scholes formula [FPS00]. If observed prices were equal to the Black-Scholes prices, then the implied volatility would equal the historical volatility. In other words, the implied volatility would be the same for all derivative contracts. In reality, this is not the case. Implied volatilities are known to vary with strike price (for a fixed maturity), creating a *volatility smile* or *smirk*. Stochastic volatility models capture this smile/smirk effect.

Fouque, Papanicolaou, & Sircar discuss further benefits of stochastic volatility models, including the generation of more realistic return distributions with fatter tails. They also discuss more difficulties presented by such models, and ways to address the challenges.

### 3.1. BACKGROUND AND MOTIVATION

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See Chapter 2 of [FPS00] for a detailed discussion.

In this chapter, we return to our stock price equation and now consider a constant drift and stochastic volatility. That is,

$$\frac{dP_t}{P_t} = \mu dt + \tilde{\sigma}(\theta_t) d\tilde{w}_t, \quad (3.1)$$

where  $\mu$  is constant.

**Remark 3.1.1.** *We assume that  $\mu$  is constant here because our focus is on the stochastic volatility condition. A natural extension would be to allow  $\mu$  to be a bounded function of  $\lambda$ , (or  $\theta$ ), as in Chapter 2. Note that  $\tilde{\mu}$ , defined in the next section, is a function of  $\theta$ , whether or not  $\mu$  is constant.*

We require that the function  $\tilde{\sigma}(\theta) \in C^1(\mathbb{R})$ , and that it is bounded above and below by positive constants,

$$0 < L \leq \tilde{\sigma}(\theta_t) \leq U. \quad (3.2)$$

We also require its derivative to be bounded:

$$\left| \frac{d\tilde{\sigma}(\theta)}{d\theta} \right| = |\tilde{\sigma}'(\theta)| \leq \tilde{U}. \quad (3.3)$$

These conditions result in the Lipschitz continuity of  $\tilde{\sigma}(\theta)$  and  $\tilde{\sigma}^2(\theta)$ .

In addition, we suppose that the volatility is driven by a mean-reverting OU-process. This captures the tendency of the random volatility to revert back to its invariant, or

long-run, distribution. The driving process is given by

$$d\theta_t = a(\theta_L - \theta_t)dt + \pi d\hat{w}_t, \quad (3.4)$$

where  $a, \pi > 0$ , and  $\theta_L$  are constants, and  $\hat{w}_t$  is a one-dimensional standard Brownian motion.

We proceed as in Chapter 2, using our new equation for  $P_t$ . In Section 3.2, we derive the HJB equation, to which we identify a subsolution/supersolution pair in Section 3.3. In Section 3.4, we prove existence of a classical solution, and in Section 3.5, we state and prove the Verification Theorem.

## 3.2 Derivation of the HJB Equation

Using (3.1), Equation (1.9) can be written as follows:

$$dX_t = X_t \left[ \left( \frac{b}{P_t} + \mu - r \right) k_t + (r - c_t) \right] dt + \frac{\sigma k_t X_t}{P_t} dw_t + \tilde{\sigma}(\theta_t) k_t X_t d\tilde{w}_t. \quad (3.5)$$

Let  $\lambda_t \equiv \log P_t$ . Then

$$d\lambda_t = \tilde{\mu}(\theta_t)dt + \tilde{\sigma}(\theta_t)d\tilde{w}_t, \quad (3.6)$$

where

$$\tilde{\mu}(\theta_t) = \mu - \frac{1}{2}\tilde{\sigma}^2(\theta_t). \quad (3.7)$$

### 3.2. DERIVATION OF THE HJB EQUATION

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Note that  $\tilde{\mu}(\theta)$  is bounded:

$$|\tilde{\mu}(\theta_t)| = \left| \mu - \frac{1}{2}\tilde{\sigma}^2(\theta_t) \right| \leq |\mu| + \frac{1}{2}U^2 \equiv \tilde{M}. \quad (3.8)$$

Our equation for  $dX_t$  becomes

$$dX_t = X_t \left[ (be^{-\lambda_t} + \mu - r)k_t + (r - c_t) \right] dt + X_t \left[ \sigma k_t e^{-\lambda_t} dw_t + \tilde{\sigma}(\theta_t) k_t d\tilde{w}_t \right]. \quad (3.9)$$

With the introduction of a new stochastic process driving the volatility, we now have three one-dimensional standard Brownian motions,  $w_t$ ,  $\tilde{w}_t$ , and  $\hat{w}_t$ . We again allow  $w_t$  and  $\tilde{w}_t$  to be correlated with correlation constant  $\rho \in [\rho_0, 1]$  for some constant  $\rho_0 \in (-1, 0)$ , and we suppose  $\hat{w}_t$  is uncorrelated with  $w_t$  and  $\tilde{w}_t$ . That is,

$$\mathbb{E}[w_t \cdot \tilde{w}_t] = \rho dt, \quad \mathbb{E}[w_t \cdot \hat{w}_t] = \mathbb{E}[\tilde{w}_t \cdot \hat{w}_t] = 0.$$

We wish to maximize the expected total discounted HARA utility of consumption subject to the constraints  $(k_t, c_t) \in \Pi$  and  $X_t > 0$ , while  $(\lambda_t, \theta_t) \in \mathbb{R}^2$ . The objective function is

$$J(x, \lambda, \theta, k., c.) = \mathbb{E}_{x, \lambda, \theta} \left[ \int_0^\infty e^{-\delta t} \frac{1}{\gamma} (c_t X_t)^\gamma dt \right], \quad (3.10)$$

and the corresponding value function is given by

$$V(x, \lambda, \theta) = \sup_{(c_t, k_t) \in \Pi} \mathbb{E}_{x, \lambda, \theta} \left[ \int_0^\infty e^{-\delta t} \frac{1}{\gamma} (c_t X_t)^\gamma dt \right]. \quad (3.11)$$

In the above equations,  $x$ ,  $\lambda$ , and  $\theta$  are the initial values of  $X_t$ ,  $\lambda_t$ , and  $\theta_t$ , respectively.

### 3.2. DERIVATION OF THE HJB EQUATION

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The discount factor  $\delta > 0$  is constant, and  $\gamma$  is the HARA utility parameter. We restrict  $\gamma$  to the interval  $(0,1)$ . The admissible control space,  $\Pi$ , is defined in a similar manner as in Chapter 2, now with the inclusion of our third Brownian motion,  $\hat{w}_t$ .

**Definition 3.2.1.** *The pair  $(k_t, c_t)$  is said to be in the **admissible control space**  $\Pi$  if  $(k_t, c_t)$  is an  $\mathbb{R}^2$ -process which is progressively measurable with respect to a  $(w_t, \tilde{w}_t, \hat{w}_t)$ -adapted family of  $\sigma$ -algebras  $(\mathcal{F}_t, t \geq 0)$ . Moreover, we require that  $k_t, c_t \geq 0$ , and*

$$Pr \left( \int_0^T k_t^2 dt < \infty \right) = 1, \quad Pr \left( \int_0^T c_t dt < \infty \right) = 1 \quad \text{for all } T > 0.$$

Our 3 state variables are  $X_t$ ,  $\lambda_t$ , and  $\theta_t$ , given by equations (3.9), (3.6), and (3.4), respectively.

Using the dynamic programming method, we get the following HJB equation for  $V(x, \lambda, \theta)$  :

$$\begin{aligned} \delta V = & \frac{\tilde{\sigma}^2(\theta)}{2} V_{\lambda\lambda} + \frac{\pi^2}{2} V_{\theta\theta} + rxV_x + \tilde{\mu}(\theta)V_\lambda + a(\theta_L - \theta)V_\theta + \max_{c \geq 0} \left[ \frac{1}{\gamma} (cx)^\gamma - cxV_x \right] \\ & + \max_{k \geq 0} \left\{ (by + \mu - r)kxV_x + \frac{k^2 x^2}{2} q(\lambda, \theta)V_{xx} + kx(\tilde{\sigma}^2(\theta) + \rho\sigma\tilde{\sigma}(\theta)y)V_{x\lambda} \right\}, \end{aligned} \quad (3.12)$$

where  $q(\lambda, \theta) = \sigma^2 e^{-2\lambda} + 2\rho\sigma\tilde{\sigma}(\theta)e^{-\lambda} + \tilde{\sigma}^2(\theta)$ . We leave out the details in the derivation of Equation (3.12) because they are similar to the steps for deriving Equation (2.21) in Section 2.1.

Restricting  $\rho \in [\rho_0, 1]$ , we can show that  $q(\lambda, \theta)$  is bounded below by a positive constant. That is,

$$q(\lambda, \theta) \geq q_0 > 0. \quad (3.13)$$

### 3.2. DERIVATION OF THE HJB EQUATION

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Consider  $\rho \in [0, 1]$ . We note that  $\sigma^2 e^{-2\lambda} > 0$  and  $2\rho\sigma\tilde{\sigma}(\theta)e^{-\lambda} \geq 0$ . Then

$$q(\lambda, \theta) > \tilde{\sigma}^2(\theta) \geq L^2 > 0.$$

Now consider  $\rho \in [\rho_0, 0)$ . The minimum of  $q$  in the  $\lambda$ -direction is given by the function  $\tilde{\sigma}^2(\theta)(1 - \rho^2)$ , which we can then bound below in  $\theta$ . So we have

$$q(\lambda, \theta) \geq \tilde{\sigma}^2(\theta)(1 - \rho^2) \geq L^2(1 - \rho_0^2) > 0.$$

Since  $L^2 \geq L^2(1 - \rho_0^2)$ , we set  $q_0 \equiv L^2(1 - \rho_0)$ . Therefore (3.13) holds for all  $\rho \in [\rho_0, 1]$ .

The value function  $V(x, \lambda, \theta)$  is homogeneous in  $x$  with order  $\gamma$ , which allows us to write it in the following form:

$$V(x, \lambda, \theta) = \frac{1}{\gamma} x^\gamma W(\lambda, \theta). \quad (3.14)$$

Substituting this form into the DPE for  $V$  gives us an equation for  $W$ :

$$\begin{aligned} \delta W = & \frac{\tilde{\sigma}^2(\theta)}{2} W_{\lambda\lambda} + \frac{\pi^2}{2} W_{\theta\theta} + \gamma r W + \tilde{\mu}(\theta) W_\lambda + a(\theta_L - \theta) W_\theta + \max_{c \geq 0} [c^\gamma - \gamma c W] \\ & + \gamma \max_{k \geq 0} \left\{ (be^{-\lambda} + \mu - r) kW + (\tilde{\sigma}^2(\theta) + \rho\sigma\tilde{\sigma}(\theta)e^{-\lambda}) kW_\lambda - \frac{k^2}{2} (1 - \gamma) q(\lambda, \theta) \right\}. \end{aligned} \quad (3.15)$$

Let  $Z = \log W$ . Then

$$\frac{W_\lambda}{W} = Z_\lambda, \quad \frac{W_{\lambda\lambda}}{W} = Z_{\lambda\lambda} + Z_\lambda^2,$$

### 3.2. DERIVATION OF THE HJB EQUATION

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$$\frac{W_\theta}{W} = Z_\theta, \quad \text{and} \quad \frac{W_{\theta\theta}}{W} = Z_{\theta\theta} + Z_\theta^2.$$

Substituting these values into (3.15) gives us the following equation for  $Z$ :

$$\begin{aligned} \delta = & \frac{\tilde{\sigma}^2(\theta)}{2}(Z_\lambda^2 + Z_{\lambda\lambda}) + \frac{\pi^2}{2}(Z_\theta^2 + Z_{\theta\theta}) + \gamma r + \tilde{\mu}(\theta)Z_\lambda + a(\theta_L - \theta)Z_\theta + \max_{c \geq 0} [c^\gamma - \gamma c e^Z] \\ & + \gamma \max_{k \geq 0} \left\{ \left[ b e^{-\lambda} + \mu - r + (\tilde{\sigma}^2(\theta) + \rho \sigma \tilde{\sigma}(\theta) e^{-\lambda}) Z_\lambda \right] k - \frac{k^2}{2} (1 - \gamma) q(\lambda, \theta) \right\}. \end{aligned} \quad (3.16)$$

The maximum of the expression involving  $c$  is obtained at

$$c^*(\lambda, \theta, Z) = e^{\frac{Z(\lambda, \theta)}{\gamma - 1}}, \quad (3.17)$$

and so

$$\max_{c \geq 0} [c^\gamma - \gamma c e^Z] = (1 - \gamma) e^{\frac{Z}{\gamma - 1}}.$$

Let

$$G(\lambda, \theta, p) = \max_{k \geq 0} \left\{ \left[ b e^{-\lambda} + \mu - r + (\tilde{\sigma}^2(\theta) + \rho \sigma \tilde{\sigma}(\theta) e^{-\lambda}) p \right] k - \frac{k^2}{2} (1 - \gamma) q(\lambda, \theta) \right\}. \quad (3.18)$$

The maximum occurs at

$$k^*(\lambda, \theta, Z_\lambda) = \left[ \frac{b e^{-\lambda} + \mu - r + (\tilde{\sigma}^2(\theta) + \rho \sigma \tilde{\sigma}(\theta) e^{-\lambda}) Z_\lambda}{(1 - \gamma) q(\lambda, \theta)} \right]^+. \quad (3.19)$$

The pair  $(k^*, c^*)$  is our candidate for optimal control. Note that if  $k^* = 0$ , then  $G = 0$ . If

### 3.2. DERIVATION OF THE HJB EQUATION

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$k^* > 0$ , then

$$G = \frac{[be^{-\lambda} + \mu - r + (\tilde{\sigma}^2(\theta) + \rho\sigma\tilde{\sigma}(\theta)e^{-\lambda})p]^2}{2(1-\gamma)q(\lambda, \theta)} \geq 0, \quad (3.20)$$

and so  $G(\lambda, \theta, p) \geq 0$ . We define

$$\Psi(\lambda, \theta) \equiv \begin{cases} \frac{(be^{-\lambda} + \mu - r)^2}{2(1-\gamma)q(\lambda, \theta)} & \text{if } be^{-\lambda} + \mu - r > 0, \\ 0 & \text{otherwise.} \end{cases} \quad (3.21)$$

Then

$$G(\lambda, \theta, 0) = \Psi(\lambda, \theta).$$

Our simplified equation for  $Z$  is

$$\begin{aligned} \delta = & \frac{\tilde{\sigma}^2(\theta)}{2}(Z_\lambda^2 + Z_{\lambda\lambda}) + \frac{\pi^2}{2}(Z_\theta^2 + Z_{\theta\theta}) + \gamma r + \tilde{\mu}(\theta)Z_\lambda + a(\theta_L - \theta)Z_\theta \\ & + \gamma G(\lambda, \theta, Z_\lambda) + (1-\gamma)e^{\frac{Z}{\gamma-1}}. \end{aligned} \quad (3.22)$$

Let

$$H(\lambda, \theta, z, p, s) \equiv -\frac{\tilde{\sigma}^2(\theta)}{2}p^2 - \frac{\pi^2}{2}s^2 - \tilde{\mu}(\theta)p - a(\theta_L - \theta)s - \gamma G(\lambda, \theta, p) + \delta - \gamma r - (1-\gamma)e^{\frac{z}{\gamma-1}}. \quad (3.23)$$

Then we have

$$\frac{\tilde{\sigma}^2(\theta)}{2}Z_{\lambda\lambda} + \frac{\pi^2}{2}Z_{\theta\theta} = H(\lambda, \theta, Z, Z_\lambda, Z_\theta), \quad (3.24)$$

which is our reduced HJB equation for  $Z(\lambda, \theta)$  corresponding to the value function (3.11),

where  $V(x, \lambda, \theta) = \frac{1}{\gamma}x^\gamma e^{Z(\lambda, \theta)}$ .



### 3.3 Subsolution and Supersolution

In order to prove existence of solution to (3.24), we will use the method of subsolution/supersolution.

**Definition 3.3.1.**  $Z(\lambda, \theta)$  is a **subsolution (supersolution)** of (3.24) if

$$\frac{\tilde{\sigma}^2(\theta)}{2}Z_{\lambda\lambda} + \frac{\pi^2}{2}Z_{\theta\theta} \geq (\leq) H(\lambda, \theta, Z, Z_\lambda, Z_\theta). \quad (3.25)$$

In addition, if  $\hat{Z}$  is a subsolution,  $\bar{Z}$  is a supersolution, and  $\hat{Z} \leq \bar{Z}$ , then  $\langle \hat{Z}, \bar{Z} \rangle$  is an **ordered pair of subsolution/supersolution**.

To identify a subsolution and supersolution to (3.24), we will use the fact that  $\Phi(\lambda, \theta)$  is bounded. To prove this, we consider two cases. First, let  $0 \leq \rho \leq 1$ . Then

$$\begin{aligned} \Psi(\lambda, \theta) &= \frac{(be^{-\lambda} + \mu - r)^2}{2(1 - \gamma)(\sigma^2 e^{-2\lambda} + 2\rho\sigma\tilde{\sigma}(\theta)e^{-\lambda} + \tilde{\sigma}^2(\theta))} \\ &\leq \frac{(be^{-\lambda} + \mu - r)^2}{2(1 - \gamma)(\sigma^2 e^{-2\lambda} + L^2)} \\ &\leq \frac{2b^2 e^{-2\lambda} + 2(\mu - r)^2}{2(1 - \gamma)(\sigma^2 e^{-2\lambda} + L^2)} \\ &= \frac{b^2 e^{-2\lambda} \sigma^2 + \sigma^2(\mu^2 - r)^2}{\sigma^2(1 - \gamma)(\sigma^2 e^{-2\lambda} + L^2)} \\ &= \frac{b^2(\sigma^2 e^{-2\lambda} + L^2) + \sigma^2(\mu^2 - r)^2 - b^2 L^2}{\sigma^2(1 - \gamma)(\sigma^2 e^{-2\lambda} + L^2)} \\ &= \frac{b^2}{\sigma^2(1 - \gamma)} + \frac{(\mu - r)^2 - \frac{b^2 L^2}{\sigma^2}}{(1 - \gamma)(\sigma^2 e^{-2\lambda} + L^2)}. \end{aligned}$$

### 3.3. SUBSOLUTION AND SUPERSOLUTION

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If  $(\mu - r)^2 - b^2 L^2 / \sigma^2 \leq 0$ , then

$$\Psi(\lambda, \theta) \leq \frac{b^2}{\sigma^2(1 - \gamma)}. \quad (3.26)$$

Otherwise, we have

$$\begin{aligned} \Psi(\lambda, \theta) &\leq \frac{b^2}{\sigma^2(1 - \gamma)} + \frac{(\mu - r)^2 - \frac{b^2 L^2}{\sigma^2}}{(1 - \gamma)L^2} \\ &= \frac{(\mu - r)^2}{L^2(1 - \gamma)}. \end{aligned}$$

Thus, if  $0 \leq \rho \leq 1$ , then

$$\Psi(\lambda, \theta) \leq \max \left\{ \frac{b^2}{\sigma^2(1 - \gamma)}, \frac{(\mu - r)^2}{L^2(1 - \gamma)} \right\} \equiv \tilde{\Psi}.$$

We can get a similar bound if  $\rho$  is negative. If  $\rho_0 < \rho < 0$ , then  $\rho(\sigma e^{-\lambda} - \tilde{\sigma}(\theta))^2 \leq 0$  implies

$$2\rho\sigma\tilde{\sigma}(\theta)e^{-\lambda} \geq \rho\sigma^2e^{-2\lambda} + \rho\tilde{\sigma}^2(\theta),$$

### 3.3. SUBSOLUTION AND SUPERSOLUTION

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and so

$$\begin{aligned}
\Psi(\lambda, \theta) &= \frac{(be^{-\lambda} + \mu - r)^2}{2(1 - \gamma)(\sigma^2 e^{-2\lambda} + 2\rho\sigma\tilde{\sigma}(\theta)e^{-\lambda} + \tilde{\sigma}^2(\theta))} \\
&\leq \frac{(be^{-\lambda} + \mu - r)^2}{2(1 - \gamma)(\sigma^2 e^{-2\lambda} + \rho\sigma^2 e^{-2\lambda} + \rho\tilde{\sigma}^2(\theta) + \tilde{\sigma}^2(\theta))} \\
&= \frac{(be^{-\lambda} + \mu - r)^2}{2(1 - \gamma)(1 + \rho)(\sigma^2 e^{-2\lambda} + \tilde{\sigma}^2(\theta))} \\
&\leq \frac{2b^2 e^{-2\lambda} + 2(\mu - r)^2}{2(1 - \gamma)(1 + \rho)(\sigma^2 e^{-2\lambda} + L^2)} \\
&\leq \frac{1}{1 + \rho} \max \left\{ \frac{b^2}{\sigma^2(1 - \gamma)}, \frac{(\mu - r)^2}{L^2(1 - \gamma)} \right\} \\
&\leq \frac{1}{1 + \rho_0} \tilde{\Psi}.
\end{aligned}$$

Let

$$\bar{\Psi} \equiv \max \left\{ \tilde{\Psi}, \frac{\tilde{\Psi}}{1 + \rho_0} \right\}. \quad (3.27)$$

Then

$$0 \leq \Psi(\lambda, \theta) \leq \bar{\Psi} < \infty. \quad (3.28)$$

Thus,  $\Psi(\lambda, \theta)$  is bounded.

**Lemma 3.3.1.** *Suppose  $0 < \gamma < 1$  and*

$$\delta > \gamma r. \quad (3.29)$$

*In addition, define*

$$K_1 \equiv (\gamma - 1) \log \left[ \frac{\delta - \gamma r}{1 - \gamma} \right]. \quad (3.30)$$

*Then any constant  $K \leq K_1$  is a subsolution of (3.24).*

### 3.3. SUBSOLUTION AND SUPERSOLUTION

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*Proof.* We want to show

$$\frac{\tilde{\sigma}^2}{2}K_{\lambda\lambda} + \frac{\pi^2}{2}K_{\theta\theta} \geq H(\lambda, \theta, K, K_\lambda, K_\theta)$$

$K$  is a constant, so all of its derivatives equal 0. This reduces the inequality we are trying to prove to the following:

$$0 \geq -\gamma\Psi(\lambda, \theta) - \gamma r + \delta - (1 - \gamma)e^{\frac{K}{\gamma-1}}.$$

Since  $K \leq K_1 = (\gamma - 1) \log \left[ \frac{\delta - \gamma r}{1 - \gamma} \right]$ , and  $(\gamma - 1) < 0$ , we have

$$\begin{aligned} e^{\frac{K}{\gamma-1}} &\geq e^{\frac{K_1}{\gamma-1}} \\ &= \exp \left[ \log \left( \frac{\delta - \gamma r}{1 - \gamma} \right) \right] \\ &= \frac{\delta - \gamma r}{1 - \gamma}. \end{aligned}$$

Then

$$\begin{aligned} -(1 - \gamma)e^{\frac{K}{\gamma-1}} &\leq -(1 - \gamma)\frac{\delta - \gamma r}{1 - \gamma} \\ &= \gamma r - \delta, \end{aligned}$$

### 3.3. SUBSOLUTION AND SUPERSOLUTION

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and

$$\begin{aligned}
-\gamma\Psi(\lambda, \theta) - \gamma r + \delta - (1 - \gamma)e^{\frac{K}{\gamma-1}} &\leq -\gamma\Psi(\lambda, \theta) - \gamma r + \delta + \gamma r - \delta \\
&= -\gamma\Psi(\lambda, \theta) \\
&\leq 0,
\end{aligned}$$

since  $\Psi(\lambda) \geq 0$ . □

**Lemma 3.3.2.** *Suppose  $0 < \gamma < 1$  and*

$$\delta > \gamma(r + \bar{\Psi}), \tag{3.31}$$

where  $\bar{\Psi}$  is defined by (3.27). In addition, define

$$K_2 \equiv (\gamma - 1) \log \left[ \frac{\delta - \gamma(r + \bar{\Psi})}{1 - \gamma} \right]. \tag{3.32}$$

Then any constant  $K \geq K_2$  is a supersolution of (3.24).

*Proof.* The proof is similar to the proof of Lemma (3.3.1), but instead we want to show that

$$-\gamma\Psi(\lambda, \theta) - \gamma r + \delta - (1 - \gamma)e^{\frac{K}{\gamma-1}} \geq 0.$$

Since  $K \geq K_2 = (\gamma - 1) \log \left[ \frac{\delta - \gamma(r + \bar{\Psi})}{1 - \gamma} \right]$  and  $(\gamma - 1) < 0$ , we have

$$\begin{aligned} e^{\frac{K}{\gamma-1}} &\leq \exp \left[ \log \left( \frac{\delta - \gamma(r + \bar{\Psi})}{1 - \gamma} \right) \right] \\ &= \frac{\delta - \gamma(r + \bar{\Psi})}{1 - \gamma} \\ -(1 - \gamma)e^{\frac{K}{\gamma-1}} &\geq -\delta + \gamma(r + \bar{\Psi}). \end{aligned}$$

Therefore

$$\begin{aligned} -\gamma\Psi(\lambda, \theta) - \gamma r + \delta - (1 - \gamma)e^{\frac{K}{\gamma-1}} &\geq -\gamma\Psi(\lambda, \theta) - \gamma r + \delta - \delta + \gamma(r + \bar{\Psi}) \\ &= \gamma(\bar{\Psi} - \Psi(\lambda, \theta)) \\ &\geq 0, \end{aligned}$$

since  $\Psi(\lambda, \theta) \leq \bar{\Psi}$ . □

**Lemma 3.3.3.** *Suppose  $0 < \gamma < 1$  and (3.31) holds. Then  $\langle K_1, K_2 \rangle$  is an ordered pair of subsolution/supersolution to (3.24), where  $K_1$  and  $K_2$  are defined by (3.30) and (3.32), respectively.*

The proof follows from Lemmas 3.3.1 and 3.3.2.

## 3.4 Existence of Solution

In this section we prove the existence of a classical solution to (3.24):

$$\frac{\tilde{\sigma}^2(\theta)}{2} Z_{\lambda\lambda} + \frac{\pi^2}{2} Z_{\theta\theta} = H(\lambda, \theta, Z, Z_\lambda, Z_\theta).$$

### 3.4. EXISTENCE OF SOLUTION

---

Our method follows the approach used in Section 3 of [HS12a]. We first prove that there exists a classical solution to the following boundary value problem on the closed ball  $\bar{B}_R \equiv \{(\lambda, \theta) \in \mathbb{R}^2 : \lambda^2 + \theta^2 \leq R^2\}$ ,

$$\begin{cases} \frac{\tilde{\sigma}^2(\theta)}{2} Z_{\lambda\lambda} + \frac{\pi^2}{2} Z_{\theta\theta} - H(\lambda, \theta, Z, Z_\lambda, Z_\theta) = 0 & \text{on } B_R, \\ Z = \psi & \text{on } \partial B_R, \end{cases} \quad (3.33)$$

for a particular choice of  $\psi$ . Once we have a solution to (3.33) on  $\bar{B}_R$  for each  $R$ , we take the limit as  $R \rightarrow \infty$  to show existence of solution to (3.24). The details are provided in the proof of Theorem 3.4.9.

#### 3.4.1 Sufficient Conditions for Existence of Solution to the Boundary Value Problem

We wish to use to prove existence of a classical solution to (3.33), and we use the approach taken by Hata and Sheu in [HS12a]. We start by introducing the parameter  $\tau \in [0, 1]$  into our equation:

$$\frac{\tilde{\sigma}^2(\theta)}{2} Z_{\lambda\lambda}^\tau + \frac{\pi^2}{2} Z_{\theta\theta}^\tau - H(\lambda, \theta, Z^\tau, Z_\lambda^\tau, Z_\theta^\tau, \tau) = 0. \quad (3.34)$$

Here,  $H(\lambda, \theta, z, p, s, \tau)$  is equal to  $H(\lambda, \theta, z, p, s)$  with  $\gamma$  replaced by  $\tau\gamma$ :

$$\begin{aligned} H(\lambda, \theta, z, p, s, \tau) \equiv & -\frac{\tilde{\sigma}^2(\theta)}{2} p^2 - \frac{\pi^2}{2} s^2 - \tilde{\mu}(\theta)p - a(\theta_L - \theta)s - \tau\gamma G^\tau(\lambda, \theta, p) \\ & + \delta - \tau\gamma r - (1 - \tau\gamma) e^{\frac{z}{\tau\gamma}} - 1, \end{aligned} \quad (3.35)$$

### 3.4. EXISTENCE OF SOLUTION

---

where

$$G^\tau(\lambda, \theta, p) = \max_{k \geq 0} \left\{ \left[ be^{-\lambda} + \mu - r + (\tilde{\sigma}^2(\theta) + \rho\sigma\tilde{\sigma}(\theta)e^{-\lambda})p \right] k - \frac{k^2}{2}(1 - \tau\gamma)q(\lambda, \theta) \right\},$$

and  $G^\tau(\lambda, \theta, 0) = \Psi^\tau(\lambda, \theta)$ , where

$$\Psi^\tau(\lambda, \theta) \equiv \begin{cases} \frac{(be^{-\lambda} + \mu - r)^2}{2(1 - \tau\gamma)q(\lambda, \theta)} & \text{if } be^{-\lambda} + \mu - r > 0, \\ 0 & \text{otherwise.} \end{cases} \quad (3.36)$$

For  $\tau \in (0, 1]$ , (3.34) is the HJB equation corresponding to the value function

$$V^\tau(x, \lambda, \theta) = \sup_{(k_t, c_t) \in \Pi} \mathbb{E}_{x, \lambda, \theta} \left[ \int_0^\infty e^{-\delta t} \frac{1}{\tau\gamma} (c_t X_t)^{\tau\gamma} dt \right]. \quad (3.37)$$

We can consider  $\tau = 0$  to be a limiting case of  $\tau \in (0, 1]$ . This corresponds to the consumption problem for log utility, for which (3.34) has a unique solution given by (3.111).

The corresponding boundary value problem is

$$\begin{cases} \frac{\tilde{\sigma}^2(\theta)}{2} Z_{\lambda\lambda}^\tau + \frac{\pi^2}{2} Z_{\theta\theta}^\tau - H(\lambda, \theta, Z^\tau, Z_\lambda^\tau, Z_\theta^\tau, \tau) = 0 & \text{on } B_R, \\ Z^\tau = \tau\psi & \text{on } \partial B_R. \end{cases} \quad (3.38)$$

Theorem 3.4.1 states the sufficient conditions for existence of solution to (3.33).

**Theorem 3.4.1.** *Let  $\alpha \in (0, 1)$  and  $R > 0$  be fixed. We assume the following conditions:*

(a)  $H(\lambda, \theta, z, p, s, 1) = H(\lambda, \theta, z, p, s)$ .



### 3.4. EXISTENCE OF SOLUTION

---

(b)  $\tilde{\sigma}^2(\theta) \in C^{1,\alpha}(\bar{B}_R)$ ;  $H(\cdot, \cdot, \cdot, \cdot, \cdot, \tau) \in C^\alpha(\bar{B}_R \times \mathbb{R} \times \mathbb{R}^2)$  for  $\tau \in [0, 1]$ , and is considered as a mapping from  $[0, 1]$  into  $C^\alpha(\bar{B}_R \times \mathbb{R} \times \mathbb{R}^2)$ . The functions  $\tilde{\sigma}^2(\theta)$  and  $H(\lambda, \theta, z, p, s, \tau)$  are continuous.

(c)  $\psi \in C^{2,\alpha}(\bar{B}_R)$ .

(d) There exists a constant  $M$  such that every  $C^{2,\alpha}(\bar{B}_R)$ -classical solution  $Z^\tau$  of (3.38) satisfies

$$|Z^\tau(\lambda, \theta)| < M, \quad (\lambda, \theta) \in \bar{B}_R,$$

where  $M$  is independent of  $Z^\tau$  and  $\tau$ .

(e) There are  $\bar{k} > 0, \underline{c}, \bar{c}$ , such that the following inequalities hold for  $(\lambda, \theta) \in \bar{B}_R, |z| \leq M, \eta \in \mathbb{R}^2, \tau \in [0, 1]$ , and arbitrary  $(p, s)$ :

$$\begin{cases} \underline{c} \sum_{i=1}^2 \eta_i^2 \leq \tilde{\sigma}^2(\theta) \eta_1^2 + \pi^2 \eta_2^2 \leq \bar{c} \sum_{i=1}^n \eta_i^2, \\ |H(\lambda, \theta, z, p, s, \tau)| + \left| \frac{\partial \tilde{\sigma}^2(\theta)}{\partial \theta} \right| \leq \bar{c} (1 + p^2 + s^2)^{\bar{k}/2}. \end{cases} \quad (3.39)$$

(f) There is an  $M_1 > 0$  such that

$$|Z_\tau^0(\lambda, \theta)| < M_1, \quad (\lambda, \theta) \in \bar{B}_R,$$

where  $Z_\tau^0(\cdot)$  is an arbitrary solution of

$$\begin{cases} \frac{\tilde{\sigma}^2(\theta)}{2} Z_{\lambda\lambda} + \frac{\pi^2}{2} Z_{\theta\theta} - \tau H(\lambda, \theta, Z, Z_\lambda, Z_\theta, 0) = 0 & \text{on } B_R, \\ Z = 0 & \text{on } \partial B_R. \end{cases}$$

### 3.4. EXISTENCE OF SOLUTION

---

where  $\tau \in [0, 1]$ .

Then the boundary value problem (3.33) is solvable in  $C^{2,\alpha}(\bar{B}_R)$ .

In order to prove Theorem 3.4.1, we need the following fixed-point theorem from Hata and Sheu in [HS12a].

**Theorem 3.4.2.** *Let  $\mathcal{B}$  be a Banach space with norm  $\|\cdot\|_{\mathcal{B}}$ .  $T$  is a continuous compact mapping of  $\mathcal{B} \times [0, 1]$  into  $\mathcal{B}$ . Assume there exists an  $M > 0$  such that*

$$\|\xi\|_{\mathcal{B}} < M$$

*holds if  $\xi$  satisfies*

$$\xi = T(\xi, \tau) \quad \text{or} \quad \xi = \tau T(\xi, 0)$$

*for some  $\tau \in [0, 1]$ . Then there exists a  $\xi \in \mathcal{B}$  such that  $T(\xi, 1) = \xi$ .*

For a proof of Theorem 3.4.2, see Appendix A of [HS12a].

We now prove Theorem 3.4.1. The proof resembles that of Theorem 3.4 in [HS12a].

*Proof of Theorem 3.4.1.* Consider the following boundary value problem for  $u$ :

$$\begin{cases} \frac{\tilde{\sigma}^2(\theta)}{2} u_{\lambda\lambda} + \frac{\pi^2}{2} u_{\theta\theta} - H(\lambda, \theta, v, v_\lambda, v_\theta, \tau) = 0 & \text{on } B_R, \\ u = \tau\psi & \text{on } \partial B_R, \end{cases} \quad (3.40)$$

where  $v$  is any  $C^{1,\beta}(\bar{B}_R)$  function, and  $\tau \in [0, 1]$ . It follows from assumptions (b) and (c) that (3.40) has a unique solution  $\tilde{u} \in C^{2,\alpha\beta}(\bar{B}_R)$ , by Theorem 6.14 of [GT77].

For all  $v \in C^{1,\beta}$ ,  $\tau \in [0, 1]$ , the operator  $T$  is defined by  $T(v, \tau) \equiv \tilde{u}$ . It follows from assumption (b) that  $T$  is continuous and compact.

### 3.4. EXISTENCE OF SOLUTION

---

If  $Z^\tau = T(Z^\tau, \tau)$ , then substituting into (3.40) gives us

$$\begin{cases} \frac{\tilde{\sigma}^2(\theta)}{2} Z_{\lambda\lambda}^\tau + \frac{\pi^2}{2} Z_{\theta\theta}^\tau - H(\lambda, \theta, Z^\tau, Z_\lambda^\tau, Z_\theta^\tau, \tau) = 0 & \text{on } B_R, \\ Z^\tau = \tau\psi & \text{on } \partial B_R, \end{cases}$$

and so (3.38) is satisfied. By assumption (d), there exists a constant  $M$  such that  $|Z^\tau(\lambda, \theta)| < M$ . Then by Theorem 3.4.2, there exists a  $Z \in C^{1,\beta}(\bar{B}_R)$  such that

$$T(Z, 1) = Z.$$

Substituting this into (3.40) gives us (3.33). Hence there exists a solution  $Z \in C^{2,\beta}(\bar{B}_R)$  to (3.33).  $\square$

#### 3.4.2 Using Theorem 3.4.1

The next step is to prove that the conditions in Theorem 3.4.1 hold for our particular problem. This will ensure existence of solution to the boundary value problem (3.33). Several results are needed before this can be done. We begin with the following definition.

**Definition 3.4.1.** Let  $\hat{Z}, \bar{Z}$  be continuous second order differentiable functions defined on  $\bar{B}_R$ .  $\hat{Z}$  is called a **subsolution of (3.33)** if

$$\begin{cases} \frac{\tilde{\sigma}^2(\theta)}{2} \hat{Z}_{\lambda\lambda} + \frac{\pi^2}{2} \hat{Z}_{\theta\theta} - H(\lambda, \theta, \hat{Z}, \hat{Z}_\lambda, \hat{Z}_\theta) \geq 0 & \text{on } B_R, \\ \hat{Z} \leq \psi & \text{on } \partial B_R. \end{cases}$$

### 3.4. EXISTENCE OF SOLUTION

---

$\bar{Z}$  is called a **supersolution of (3.33)** if

$$\begin{cases} \frac{\tilde{\sigma}^2(\theta)}{2} \bar{Z}_{\lambda\lambda} + \frac{\pi^2}{2} \bar{Z}_{\theta\theta} - H(\lambda, \theta, \bar{Z}, \bar{Z}_\lambda, \bar{Z}_\theta) \leq 0 & \text{on } B_R, \\ \bar{Z} \geq \psi & \text{on } \partial B_R. \end{cases}$$

In addition,  $\langle \hat{Z}, \bar{Z} \rangle$  is called an **ordered pair of subsolution/supersolution** if they also satisfy

$$\hat{Z}(\lambda, \theta) \leq \bar{Z}(\lambda, \theta), \quad (\lambda, \theta) \in \bar{B}_R.$$

We can define an ordered pair of subsolution/supersolution to (3.38) in a similar manner.

**Lemma 3.4.3.** *Let  $\tau \in (0, 1]$ . Assume  $\hat{Z}, \bar{Z}$  are second order continuous differentiable functions on  $\bar{B}_R$  and satisfy*

$$\begin{cases} \frac{\tilde{\sigma}^2(\theta)}{2} \hat{Z}_{\lambda\lambda} + \frac{\pi^2}{2} \hat{Z}_{\theta\theta} - H(\lambda, \theta, \hat{Z}, \hat{Z}_\lambda, \hat{Z}_\theta, \tau) \geq 0 & \text{on } B_R, \\ \frac{\tilde{\sigma}^2(\theta)}{2} \bar{Z}_{\lambda\lambda} + \frac{\pi^2}{2} \bar{Z}_{\theta\theta} - H(\lambda, \theta, \bar{Z}, \bar{Z}_\lambda, \bar{Z}_\theta, \tau) \leq 0 & \text{on } B_R, \\ \hat{Z} \leq \bar{Z} & \text{on } \partial B_R. \end{cases} \quad (3.41)$$

Then  $\hat{Z} \leq \bar{Z}$  holds in  $\bar{B}_R$ .

*Proof.* Suppose  $\sup_{(\lambda, \theta) \in \bar{B}_R} (\hat{Z} - \bar{Z})(\lambda, \theta) > 0$ . Since  $\hat{Z} \leq \bar{Z}$  on  $\partial B_R$ , this implies that  $\hat{Z} - \bar{Z}$  obtains its maximum at  $(\lambda_0, \theta_0) \in B_R$ . So we have that

$$(\hat{Z} - \bar{Z})_{\lambda\lambda}(\lambda_0, \theta_0) < 0, \quad (\hat{Z} - \bar{Z})_{\theta\theta}(\lambda_0, \theta_0) < 0, \quad (3.42)$$

$$\hat{Z}_\lambda(\lambda_0, \theta_0) = \bar{Z}_\lambda(\lambda_0, \theta_0), \quad \text{and} \quad \hat{Z}_\theta(\lambda_0, \theta_0) = \bar{Z}_\theta(\lambda_0, \theta_0). \quad (3.43)$$

### 3.4. EXISTENCE OF SOLUTION

---

By (3.41) we have that, for  $(\lambda, \theta) \in B_R$ ,

$$\frac{\tilde{\sigma}^2(\theta)}{2}(\hat{Z}_{\lambda\lambda} - \bar{Z}_{\lambda\lambda}) + \frac{\pi^2}{2}(\hat{Z}_{\theta\theta} - \bar{Z}_{\theta\theta}) - H(\lambda, \theta, \hat{Z}, \hat{Z}_\lambda, \hat{Z}_\theta, \tau) + H(\lambda, \theta, \bar{Z}, \bar{Z}_\lambda, \bar{Z}_\theta, \tau) \geq 0.$$

Evaluating this inequality at  $(\lambda_0, \theta_0)$  and applying (3.42), we have

$$H(\lambda_0, \theta_0, \bar{Z}(\lambda_0, \theta_0), \bar{Z}_\lambda(\lambda_0, \theta_0), \bar{Z}_\theta(\lambda_0, \theta_0), \tau) > H(\lambda_0, \theta_0, \hat{Z}(\lambda_0, \theta_0), \hat{Z}_\lambda(\lambda_0, \theta_0), \hat{Z}_\theta(\lambda_0, \theta_0), \tau).$$

By applying (3.43), this reduces to

$$-(1 - \tau\gamma)e^{\frac{\bar{Z}(\lambda_0, \theta_0)}{\tau\gamma - 1}} > -(1 - \tau\gamma)e^{\frac{\hat{Z}(\lambda_0, \theta_0)}{\tau\gamma - 1}},$$

or  $\bar{Z}(\lambda_0, \theta_0) > \hat{Z}(\lambda_0, \theta_0)$ , which is a contradiction. Therefore  $\hat{Z} \leq \bar{Z}$  on  $\bar{B}_R$ .  $\square$

**Corollary 3.4.3.1.** *For  $\tau \in (0, 1]$ , the solution  $Z^\tau \in C^2(\bar{B}_R)$  of (3.38) is unique.*

*Proof.* Suppose  $Z^1, Z^2 \in C^2(\bar{B}_R)$  are both solutions of (3.38). Then  $Z^1 = Z^2$  on  $\partial B_R$ , and

$$\frac{\tilde{\sigma}^2(\theta)}{2}(Z_{\lambda\lambda}^2 - Z_{\lambda\lambda}^1) + \frac{\pi^2}{2}(Z_{\theta\theta}^2 - Z_{\theta\theta}^1) - H(\lambda, \theta, Z^2, Z_\lambda^2, Z_\theta^2, \tau) + H(\lambda, \theta, Z^1, Z_\lambda^1, Z_\theta^1, \tau) = 0 \quad (3.44)$$

in  $B_R$ . Suppose  $\sup_{(\lambda, \theta) \in B_R} (Z^2 - Z^1) > 0$ . Then  $Z^2 - Z^1$  attains its maximum at some  $(\lambda_0, \theta_0) \in B_R$ . Then

$$(Z_{\lambda\lambda}^2 - Z_{\lambda\lambda}^1)(\lambda_0, \theta_0) < 0, \quad (Z_{\theta\theta}^2 - Z_{\theta\theta}^1)(\lambda_0, \theta_0) < 0, \quad (3.45)$$

### 3.4. EXISTENCE OF SOLUTION

---

$$Z_\lambda^1(\lambda_0, \theta_0) = Z_\lambda^2(\lambda_0, \theta_0), \quad \text{and} \quad Z_\theta^1(\lambda_0, \theta_0) = Z_\theta^2(\lambda_0, \theta_0). \quad (3.46)$$

Evaluating (3.44) at  $(\lambda_0, \theta_0)$  and applying (3.46) and (3.45), we arrive at

$$(1 - \tau\gamma)e^{\frac{Z^2(\lambda_0, \theta_0)}{\tau\gamma-1}} > (1 - \tau\gamma)e^{\frac{Z^1(\lambda_0, \theta_0)}{\tau\gamma-1}},$$

or  $Z^2(\lambda_0, \theta_0) < Z^1(\lambda_0, \theta_0)$ , which is a contradiction.

Note that if we assume  $\sup_{(\lambda, \theta) \in B_R} (Z^1 - Z^2) > 0$ , we would arrive at another contradiction. Therefore  $Z^1 \equiv Z^2$  on  $\bar{B}_R$ .  $\square$

We now wish to discuss some properties of the coefficients of the HJB equation (3.24). Recall from (3.20) that for  $k^* > 0$ ,  $G$  is defined by

$$G(\lambda, \theta, Z_\lambda) = \frac{[be^{-\lambda} + \mu - r + (\tilde{\sigma}^2(\theta) + \rho\sigma\tilde{\sigma}(\theta)e^{-\lambda})Z_\lambda]^2}{2(1 - \gamma)q(\lambda, \theta)} \geq 0, \quad (3.47)$$

or in the trivial case,  $G \equiv 0$  otherwise. Consider the expression for  $G$  given above. We can expand  $G$  to quadratic form in  $Z_\lambda$ :

$$G(\lambda, \theta, Z_\lambda) = g_2 Z_\lambda^2 + g_1 Z_\lambda + g_0, \quad (3.48)$$

where

$$g_2(\lambda, \theta) = \frac{(\tilde{\sigma}^2(\theta) + \rho\sigma\tilde{\sigma}(\theta)e^{-\lambda})^2}{2(1 - \gamma)q(\lambda, \theta)}, \quad g_1(\lambda, \theta) = \frac{(be^{-\lambda} + \mu - r)(\tilde{\sigma}^2(\theta) + \rho\sigma\tilde{\sigma}(\theta)e^{-\lambda})}{(1 - \gamma)q(\lambda, \theta)},$$

$$\text{and} \quad g_0(\lambda, \theta) = \frac{(be^{-\lambda} + \mu - r)^2}{2(1 - \gamma)q(\lambda, \theta)}. \quad (3.49)$$

### 3.4. EXISTENCE OF SOLUTION

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**Remark 3.4.1.** Note that  $|g_2|$ ,  $|g_1|$ , and  $|g_0|$  are all bounded. We rationalize this by first applying the bounds (3.2) and (3.13), after which we are left with continuous rational expressions in  $\lambda$ . The expressions are quotients of quadratic functions of  $y = e^{-\lambda}$ , whose denominators are bounded below by a positive constant, namely  $(1 - \gamma)q_0$ . Therefore the limits as  $\lambda$  tends to infinity and negative infinity exist. Hence  $|g_2|$ ,  $|g_1|$ , and  $|g_0|$  are bounded for all  $(\lambda, \theta) \in \mathbb{R}^2$ .

**Remark 3.4.2.** It follows from (3.48) and Remark 3.4.1 that there exist constants  $\tilde{C}_1, \tilde{C}_2 > 0$  such that

$$G(\lambda, \theta, p) \leq \tilde{C}_1 + \tilde{C}_2 p^2. \quad (3.50)$$

**Remark 3.4.3.** The coefficients of (3.24),  $\tilde{\sigma}^2(\theta)$ ,  $\tilde{\mu}(\theta)$ ,  $g_0(\lambda, \theta)$ ,  $g_1(\lambda, \theta)$ , and  $g_2(\lambda, \theta)$ , are Lipschitz continuous for all  $(\lambda, \theta) \in \mathbb{R}$ . If  $Z(\lambda, \theta) \in C^1(\bar{B}_R)$ , it follows that  $H(\lambda, \theta, Z, Z_\lambda, Z_\theta) \in C^\alpha(\bar{B}_R \times \mathbb{R} \times \mathbb{R}^2)$ .

*Proof.* By (3.2), (3.3), and the definition of  $\tilde{\mu}(\theta)$  given in (3.7), we have that  $\left| \frac{\tilde{\sigma}^2(\theta)}{d\theta} \right|$  and  $\left| \frac{d\tilde{\mu}(\theta)}{d\theta} \right|$  are bounded for all  $\theta \in \mathbb{R}$ , and so the coefficients  $\tilde{\sigma}^2(\theta)$  and  $\tilde{\mu}(\theta)$  are Lipschitz continuous.

The bounds (3.2) and (3.3) also imply that  $g_0, g_1$ , and  $g_2$  are Lipschitz continuous. To see this, we look at the derivatives of  $g_0$  with respect to  $\lambda$  and  $\theta$ . After applying the bounds on  $\tilde{\sigma}(\theta)$  and  $\tilde{\sigma}'(\theta)$ , we have that

$$|(g_0)_\lambda|, |(g_0)_\theta| \leq \frac{C_4 e^{-4\lambda} + C_3 e^{-3\lambda} + C_2 e^{-2\lambda} + C_1 e^{-\lambda} + C_0}{4(1 - \gamma)(q(\lambda, \theta))^2}, \quad (3.51)$$

for some positive constants  $C_4, C_3, C_2, C_1$ , and  $C_0$ . Take  $y = e^{-\lambda}$ . Using the same reasoning as in Remark 3.4.1, this time with a quotient of quartics in  $y$ , we see that the partial

### 3.4. EXISTENCE OF SOLUTION

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derivatives of  $g_0$  are bounded. Thus  $|Dg_0|$  is bounded, and  $g_0$  is Lipschitz continuous. We use the same reasoning to arrive at the Lipschitz continuity of  $g_1$  and  $g_2$ .

On the ball  $\bar{B}_R$ ,  $Z(\lambda, \theta)$  is bounded. Therefore the term  $e^{\frac{Z}{\tau-1}}$  is locally Lipschitz for  $(\lambda, \theta) \in \bar{B}_R$ . The Lipschitz continuity of the coefficients of  $H$  imply that  $H$  is locally Hölder continuous. That is, there exists an  $\alpha \in (0, 1)$  such that  $H(\lambda, \theta, Z, Z_\lambda, Z_\theta) \in C^\alpha(\bar{B}_R, \mathbb{R}, \mathbb{R}^2)$ .  $\square$

The next two theorems provide us with bounds on  $Z^\tau$ .

**Theorem 3.4.4.** *Suppose  $\tilde{\sigma}^2(\theta) \in C^{1,\alpha}(\bar{B}_R)$ ,  $\psi$  is continuous, and let  $\bar{Z}$  be a supersolution of (3.38) for  $\tau = 1$ . Suppose that  $Z^\tau$  is a solution of (3.38) with  $\tau \in (0, 1]$ . Then*

$$e^{Z^\tau(\lambda, \theta)} \leq \tau e^{\bar{Z}(\lambda, \theta)} + (1 - \tau)f(\lambda, \theta), \quad (3.52)$$

where  $f(\lambda, \theta)$  satisfies

$$\begin{cases} \frac{\tilde{\sigma}^2(\theta)}{2} f_{\lambda\lambda}(\lambda, \theta) + \frac{\pi^2}{2} f_{\theta\theta}(\lambda, \theta) + \tilde{\mu}(\theta) f_\lambda(\lambda, \theta) \\ \quad + a(\theta_L - \theta) f_\theta(\lambda, \theta) - \delta f(\lambda, \theta) + 1 = 0 & \text{on } B_R, \\ f(\lambda, \theta) = 1 & \text{on } \partial B_R, \end{cases} \quad (3.53)$$

and is given by

$$f(\lambda, \theta) = \frac{1}{\delta} + \left(1 - \frac{1}{\delta}\right) \mathbb{E}_{\lambda, \theta}[e^{-\delta t_R}], \quad (3.54)$$

where

$$t_R = \inf \left\{ t > 0; \sqrt{\lambda_t^2 + \theta_t^2} = R \right\}. \quad (3.55)$$



### 3.4. EXISTENCE OF SOLUTION

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Moreover, for  $0 < \gamma < 1$ ,

$$Z^\tau(\lambda, \theta) \geq -\log \left( \max \left\{ \frac{\delta}{1-\gamma}, 1 \right\} \right) - \sup_{\lambda^2 + \theta^2 = R^2} \{|\psi(\lambda, \theta)|\}. \quad (3.56)$$

*Proof.* We first prove inequality (3.56). Let

$$\hat{Z}^\tau = -\log \left( \max \left\{ \frac{\delta}{1-\gamma}, 1 \right\} \right) - \sup_{\lambda^2 + \theta^2 = R^2} \{|\psi(\lambda, \theta)|\}. \quad (3.57)$$

We will show that  $\hat{Z}^\tau$  is a subsolution of (3.38). Since  $\hat{Z}^\tau$  is a constant function, its derivatives are equal to zero. Therefore on  $B_R$ ,

$$\begin{aligned} \frac{\tilde{\sigma}^2(\theta)}{2} \hat{Z}_{\lambda\lambda}^\tau + \frac{\pi^2}{2} \hat{Z}_{\theta\theta}^\tau - H(\lambda, \theta, \hat{Z}^\tau, \hat{Z}_\lambda^\tau, \hat{Z}_\theta^\tau, \tau) &= \tau\gamma\Psi^\tau(\lambda, \theta) - \delta + \tau\gamma r \\ &+ (1 - \tau\gamma) \exp \left[ \frac{1}{1 - \tau\gamma} \log \left( \max \left\{ \frac{\delta}{1-\gamma}, 1 \right\} \right) \right] \exp \left( \frac{\sup_{\lambda^2 + \theta^2 = R} |\psi(\lambda, \theta)|}{1 - \tau\gamma} \right), \end{aligned} \quad (3.58)$$

where  $\Psi^\tau(\lambda, \theta)$  is defined in (3.36). Note that  $\Psi^\tau(\lambda, \theta) \geq 0$ , and

$$\exp \left( \frac{\sup_{\lambda^2 + \theta^2 = R} |\psi(\lambda, \theta)|}{1 - \tau\gamma} \right) \geq 1$$

### 3.4. EXISTENCE OF SOLUTION

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because the argument is nonnegative. We reduce (3.58) to the following inequality:

$$\begin{aligned}
& \frac{\tilde{\sigma}^2(\theta)}{2} \hat{Z}_{\lambda\lambda}^\tau + \frac{\pi^2}{2} \hat{Z}_{\theta\theta}^\tau - H(\lambda, \theta, \hat{Z}^\tau, \hat{Z}_\lambda^\tau, \hat{Z}_\theta^\tau, \tau) \\
& \geq -\delta + (1 - \tau\gamma) \exp \left[ \frac{1}{1 - \tau\gamma} \log \left( \max \left\{ \frac{\delta}{1 - \gamma}, 1 \right\} \right) \right] \\
& \geq -\delta + (1 - \gamma) \exp \left[ \log \left( \max \left\{ \frac{\delta}{1 - \gamma}, 1 \right\} \right) \right]. \tag{3.59}
\end{aligned}$$

Suppose  $\delta \geq 1 - \gamma$ . Then the right-hand-side simplifies to

$$-\delta + (1 - \gamma) \exp \left[ \log \frac{\delta}{1 - \gamma} \right] = -\delta + \delta = 0. \tag{3.60}$$

Now suppose  $\delta < 1 - \gamma$ . Then (3.59) simplifies to

$$-\delta + (1 - \gamma) \exp [\log 1] = -\delta + (1 - \gamma) > 0. \tag{3.61}$$

We also have that  $\hat{Z}^\tau \leq \tau\psi = Z^\tau$  on  $\partial B_R$ . Therefore  $\hat{Z}^\tau$  and  $Z^\tau$  satisfy

$$\begin{cases} \frac{\tilde{\sigma}^2(\theta)}{2} \hat{Z}_{\lambda\lambda}^\tau + \frac{\pi^2}{2} \hat{Z}_{\theta\theta}^\tau - H(\lambda, \theta, \hat{Z}^\tau, \hat{Z}_\lambda^\tau, \hat{Z}_\theta^\tau, \tau) \geq 0 & \text{on } B_R, \\ \frac{\tilde{\sigma}^2(\theta)}{2} Z_{\lambda\lambda}^\tau + \frac{\pi^2}{2} Z_{\theta\theta}^\tau - H(\lambda, \theta, Z^\tau, Z_\lambda^\tau, Z_\theta^\tau, \tau) = 0 & \text{on } B_R, \\ \hat{Z}^\tau \leq Z^\tau & \text{on } \partial B_R. \end{cases} \tag{3.62}$$

Then by Lemma 3.4.3,  $\hat{Z}^\tau \leq Z^\tau$  holds in  $\bar{B}_R$ , which gives us (3.56).

Next we prove (3.52). It is equivalent to show that

$$\frac{1}{\tau\gamma} (e^{Z^\tau} - f) \leq \frac{1}{\gamma} (e^{\bar{Z}} - f). \tag{3.63}$$

### 3.4. EXISTENCE OF SOLUTION

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Define

$$V(x, \lambda, \theta) = \frac{1}{\gamma} x^\gamma e^{\bar{Z}(\lambda, \theta)} \quad \text{and} \quad V^\tau(x, \lambda, \theta) = \frac{1}{\tau\gamma} x^{\tau\gamma} e^{Z^\tau(\lambda, \theta)}. \quad (3.64)$$

Note that

$$\begin{aligned} \max_{c \geq 0} \left[ \frac{1}{\gamma} (cx)^\gamma - cx V_x \right] &= (1 - \gamma) e^{\frac{\bar{Z}}{\gamma-1}} \cdot V, \quad \text{and} \\ \max_{c \geq 0} \left[ \frac{1}{\tau\gamma} (cx)^{\tau\gamma} - cx V_x^\tau \right] &= (1 - \tau\gamma) e^{\frac{Z^\tau}{\tau\gamma-1}} \cdot V^\tau. \end{aligned}$$

Now define

$$V_0(x, \lambda, \theta) = \frac{1}{\gamma} (x^\gamma e^{\bar{Z}(\lambda, \theta)} - f(\lambda, \theta)) \quad (3.65)$$

and

$$V_0^\tau(x, \lambda, \theta) = \frac{1}{\tau\gamma} (x^{\tau\gamma} e^{Z^\tau(\lambda, \theta)} - f(\lambda, \theta)). \quad (3.66)$$

Since  $f$  does not depend on  $x$ , we have that  $(V_0)_x = V_x$  and  $(V_0^\tau)_x = V_x^\tau$ . So

$$\max_{c \geq 0} \left[ \frac{1}{\gamma} (cx)^\gamma - cx (V_0)_x \right] = (1 - \gamma) e^{\frac{\bar{Z}}{\gamma-1}} \cdot V$$

and

$$\max_{c \geq 0} \left[ \frac{1}{\tau\gamma} (cx)^{\tau\gamma} - cx (V_0)_x^\tau \right] = (1 - \tau\gamma) e^{\frac{Z^\tau}{\tau\gamma-1}} \cdot V^\tau.$$

### 3.4. EXISTENCE OF SOLUTION

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Then on  $B_R$  we have the following:

$$\begin{aligned}
& \frac{\tilde{\sigma}^2(\theta)}{2}(V_0)_{\lambda\lambda} + \frac{\pi^2}{2}(V_0)_{\theta\theta} + rx(V_0)_x + \tilde{\mu}(\theta)(V_0)_\lambda + a(\theta_L - \theta)(V_0)_\theta - \delta V_0 \\
& + \max_{c \geq 0} \left[ \frac{1}{\gamma}(cx)^\gamma - cx(V_0)_x \right] + \max_{k \geq 0} \left\{ (be^{-\lambda} + \mu - r)kx(V_0)_x \right. \\
& + \left. \frac{k^2 x^2}{2}q(\lambda, \theta)(V_0)_{xx} + kx(\tilde{\sigma}^2(\theta) + \rho\sigma\tilde{\sigma}(\theta)e^{-\lambda})(V_0)_{x\lambda} \right\} - \frac{1}{\gamma} \\
& = V \left[ \frac{\tilde{\sigma}^2(\theta)}{2}(\bar{Z}_{\lambda\lambda} + \bar{Z}_\lambda^2) + \frac{\pi^2}{2}(\bar{Z}_{\theta\theta} + \bar{Z}_\theta^2) + \gamma r + \tilde{\mu}(\theta)\bar{Z}_\lambda \right. \\
& + a(\theta_L - \theta)\bar{Z}_\theta - \delta + (1 - \gamma)e^{\frac{\bar{Z}}{\gamma-1}} \\
& + \left. \gamma \max_{k \geq 0} \left\{ (be^{-\lambda} + \mu - r)k + \frac{k^2(\gamma - 1)}{2}q(\lambda, \theta) + k(\tilde{\sigma}^2(\theta) + \rho\sigma\tilde{\sigma}(\theta)e^{-\lambda})\bar{Z}_\lambda \right\} \right] \\
& - \frac{1}{\gamma} \left[ \frac{\tilde{\sigma}^2(\theta)}{2}f_{\lambda\lambda} + \frac{\pi^2}{2}f_{\theta\theta} + rxf_x + \tilde{\mu}(\theta)f_\lambda + a(\theta_L - \theta)f_\theta - \delta f + 1 \right] \\
& = V \left[ \frac{\tilde{\sigma}^2(\theta)}{2}(\bar{Z}_{\lambda\lambda} + \bar{Z}_\lambda^2) + \frac{\pi^2}{2}(\bar{Z}_{\theta\theta} + \bar{Z}_\theta^2) - H(\lambda, \theta, \bar{Z}, \bar{Z}_\lambda, \bar{Z}_\theta, 1) \right] - \frac{1}{\gamma}[0] \\
& \leq 0, \tag{3.67}
\end{aligned}$$

since  $f$  satisfies (3.53) and  $\bar{Z}$  is a supersolution of (3.38) for  $\tau = 1$ . We also have

$$\begin{aligned}
 & \frac{\tilde{\sigma}^2(\theta)}{2}(V_0^\tau)_{\lambda\lambda} + \frac{\pi^2}{2}(V_0^\tau)_{\theta\theta} + rx(V_0^\tau)_x + \tilde{\mu}(\theta)(V_0^\tau)_\lambda + a(\theta_L - \theta)(V_0^\tau)_\theta - \delta V_0^\tau \\
 & + \max_{c \geq 0} \left[ \frac{1}{\tau\gamma}(cx)^{\tau\gamma} - cx(V_0^\tau)_x \right] + \max_{k \geq 0} \left\{ (be^{-\lambda} + \mu - r)kx(V_0^\tau)_x \right. \\
 & \left. + \frac{k^2 x^2}{2}q(\lambda, \theta)(V_0^\tau)_{xx} + kx(\tilde{\sigma}^2(\theta) + \rho\sigma\tilde{\sigma}(\theta)e^{-\lambda})(V_0^\tau)_{x\lambda} \right\} - \frac{1}{\tau\gamma} \\
 & = V^\tau \left[ \frac{\tilde{\sigma}^2(\theta)}{2}(Z_{\lambda\lambda}^\tau + (Z_\lambda^\tau)^2) + \frac{\pi^2}{2}(Z_{\theta\theta}^\tau + (Z_\theta^\tau)^2) + \tau\gamma r + \tilde{\mu}(\theta)Z_\lambda^\tau \right. \\
 & \left. + a(\theta_L - \theta)Z_\theta^\tau - \delta + (1 - \tau\gamma)e^{\frac{Z^\tau}{\tau\gamma-1}} + \tau\gamma \max_{k \geq 0} \left\{ (be^{-\lambda} + \mu - r)k \right. \right. \\
 & \left. \left. + \frac{k^2(\tau\gamma - 1)}{2}q(\lambda, \theta) + k(\tilde{\sigma}^2(\theta) + \rho\sigma\tilde{\sigma}(\theta)e^{-\lambda})Z_\lambda^\tau \right\} \right] \\
 & - \frac{1}{\tau\gamma} \left[ \frac{\tilde{\sigma}^2(\theta)}{2}f_{\lambda\lambda} + \frac{\pi^2}{2}f_{\theta\theta} + rx f_x + \tilde{\mu}(\theta)f_\lambda + a(\theta_L - \theta)f_\theta - \delta f + 1 \right] \\
 & = V^\tau \left[ \frac{\tilde{\sigma}^2(\theta)}{2}(Z_{\lambda\lambda}^\tau + (Z_\lambda^\tau)^2) + \frac{\pi^2}{2}(Z_{\theta\theta}^\tau + (Z_\theta^\tau)^2) - H(\lambda, \theta, Z^\tau, Z_\lambda^\tau, Z_\theta^\tau, \tau) \right] - \frac{1}{\tau\gamma}[0] \\
 & = 0. \tag{3.68}
 \end{aligned}$$

Suppose that the optimal controls for the above equation are given by  $\tilde{k}$  and  $\tilde{c}$ . Then we can rewrite (3.68) as

$$\begin{aligned}
 & \frac{\tilde{\sigma}^2(\theta)}{2}(V_0^\tau)_{\lambda\lambda} + \frac{\pi^2}{2}(V_0^\tau)_{\theta\theta} + rx(V_0^\tau)_x + \tilde{\mu}(\theta)(V_0^\tau)_\lambda + a(\theta_L - \theta)(V_0^\tau)_\theta \\
 & - \delta V_0^\tau + \frac{1}{\tau\gamma}(\tilde{c}x)^{\tau\gamma} - \tilde{c}x(V_0^\tau)_x + (be^{-\lambda} + \mu - r)\tilde{k}x(V_0^\tau)_x \\
 & + \frac{\tilde{k}^2 x^2}{2}q(\lambda, \theta)(V_0^\tau)_{xx} + \tilde{k}x(\tilde{\sigma}^2(\theta) + \rho\sigma\tilde{\sigma}(\theta)e^{-\lambda})(V_0^\tau)_{x\lambda} - \frac{1}{\tau\gamma} = 0. \tag{3.69}
 \end{aligned}$$

### 3.4. EXISTENCE OF SOLUTION

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The difference between equations (3.67) and (3.68) is the function

$$g(y) = \frac{1}{y}(z^y - 1) \quad (3.70)$$

for  $z = cx > 0$  with  $y = \gamma$  in (3.67) and  $y = \tau\gamma$  in (3.68). We will show that  $g$  is nondecreasing in  $y$ , for fixed  $z > 0$ . Differentiating, we obtain

$$g'(y) = \frac{-z^y + 1 + yz^y \log z}{y^4}.$$

Let  $h(y) = -z^y + 1 + yz^y \log z$ . Then

$$h'(y) = yz^y(\log z)^2 \geq 0 \quad \text{for } y > 0.$$

So  $g'(y) \geq 0$ , which means  $g(y)$  is nondecreasing for  $y > 0$ . Since  $\tau \in (0, 1]$  and  $\gamma \in (0, 1)$ ,  $\tau\gamma \leq \gamma$ . Therefore

$$g(\tau\gamma) \leq g(\gamma), \quad (3.71)$$

or

$$\frac{1}{\tau\gamma}((cx)^{\tau\gamma} - 1) \leq \frac{1}{\gamma}((cx)^\gamma - 1). \quad (3.72)$$

Applying the left-hand-side of (3.68) to  $V_0$  and using (3.72), we have the following

### 3.4. EXISTENCE OF SOLUTION

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inequality:

$$\begin{aligned}
& \frac{\tilde{\sigma}^2(\theta)}{2}(V_0)_{\lambda\lambda} + \frac{\pi^2}{2}(V_0)_{\theta\theta} + rx(V_0)_x + \tilde{\mu}(\theta)(V_0)_\lambda + a(\theta_L - \theta)(V_0)_\theta - \delta V_0 \\
& + \max_{c \geq 0} \left[ \frac{1}{\tau\gamma}(cx)^{\tau\gamma} - cx(V_0)_x \right] + \max_{k \geq 0} \left\{ (be^{-\lambda} + \mu - r)kx(V_0)_x \right. \\
& \left. + \frac{k^2 x^2}{2}q(\lambda, \theta)(V_0)_{xx} + kx(\tilde{\sigma}^2(\theta) + \rho\sigma\tilde{\sigma}(\theta)e^{-\lambda})(V_0)_{x\lambda} \right\} - \frac{1}{\tau\gamma} \\
& \leq \frac{\tilde{\sigma}^2(\theta)}{2}(V_0)_{\lambda\lambda} + \frac{\pi^2}{2}(V_0)_{\theta\theta} + rx(V_0)_x + \tilde{\mu}(\theta)(V_0)_\lambda + a(\theta_L - \theta)(V_0)_\theta - \delta V_0 \\
& + \max_{c \geq 0} \left[ \frac{1}{\gamma}(cx)^\gamma - cx(V_0)_x \right] + \max_{k \geq 0} \left\{ (be^{-\lambda} + \mu - r)kx(V_0)_x \right. \\
& \left. + \frac{k^2 x^2}{2}q(\lambda, \theta)(V_0)_{xx} + kx(\tilde{\sigma}^2(\theta) + \rho\sigma\tilde{\sigma}(\theta)e^{-\lambda})(V_0)_{x\lambda} \right\} - \frac{1}{\gamma} \\
& \leq 0
\end{aligned} \tag{3.73}$$

by (3.67). If we evaluate (3.73) at any values of  $k$  and  $c$  that are less than optimal, the inequality will still hold. So we evaluate (3.73) at  $\tilde{k}$  and  $\tilde{c}$  to obtain the following:

$$\begin{aligned}
& \frac{\tilde{\sigma}^2(\theta)}{2}(V_0)_{\lambda\lambda} + \frac{\pi^2}{2}(V_0)_{\theta\theta} + rx(V_0)_x + \tilde{\mu}(\theta)(V_0)_\lambda + a(\theta_L - \theta)(V_0)_\theta - \delta V_0 \\
& + \frac{1}{\tau\gamma}(\tilde{c}x)^{\tau\gamma} - \tilde{c}x(V_0)_x + (be^{-\lambda} + \mu - r)\tilde{k}x(V_0)_x + \frac{\tilde{k}^2 x^2}{2}q(\lambda, \theta)(V_0)_{xx} \\
& + \tilde{k}x(\tilde{\sigma}^2(\theta) + \rho\sigma\tilde{\sigma}(\theta)e^{-\lambda})(V_0)_{x\lambda} - \frac{1}{\tau\gamma} \leq 0.
\end{aligned} \tag{3.74}$$

### 3.4. EXISTENCE OF SOLUTION

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Subtracting (3.74) from (3.69) gives us the inequality

$$\begin{aligned}
& \frac{\tilde{\sigma}^2(\theta)}{2}(V_0^\tau - V_0)_{\lambda\lambda} + \frac{\pi^2}{2}(V_0^\tau - V_0)_{\theta\theta} + rx(V_0^\tau - V_0)_x + \tilde{\mu}(\theta)(V_0^\tau - V_0)_\lambda \\
& + a(\theta_L - \theta)(V_0^\tau - V_0)_\theta - \delta(V_0^\tau - V_0) - \tilde{c}x(V_0^\tau - V_0)_x + (be^{-\lambda} + \mu - r)\tilde{k}x(V_0^\tau - V_0)_x \\
& + \frac{\tilde{k}^2 x^2}{2}q(\lambda, \theta)(V_0^\tau - V_0)_{xx} + \tilde{k}x(\tilde{\sigma}^2(\theta) + \rho\sigma\tilde{\sigma}(\theta)e^{-\lambda})(V_0^\tau - V_0)_{x\lambda} \geq 0,
\end{aligned} \tag{3.75}$$

which will be used to show a contradiction later in this proof. Note that (3.75) holds for  $x > 0, (\lambda, \theta) \in B_R$ .

To prove the estimate in (3.63), we wish to show that for  $x > 0, (\lambda, \theta) \in \bar{B}_R$ ,

$$V_0^\tau(x, \lambda, \theta) \leq V_0(x, \lambda, \theta), \tag{3.76}$$

or

$$\frac{1}{\tau^\gamma}(x^{\tau^\gamma}e^{Z^\tau} - f) \leq \frac{1}{\gamma}(x^\gamma e^{\bar{Z}} - f). \tag{3.77}$$

We then take  $x = 1$  to get the desired result. We first note that  $f(\lambda, \theta) > 0$ . This is easily seen by rearranging equation (3.54) for  $f$ :

$$\begin{aligned}
f(\lambda, \theta) &= \frac{1}{\delta} + \left(1 - \frac{1}{\delta}\right) \mathbb{E}_{\lambda, \theta}[e^{-\delta t_R}] \\
&= \frac{1}{\delta} \left(1 - \mathbb{E}_{\lambda, \theta}[e^{-\delta t_R}]\right) + \mathbb{E}_{\lambda, \theta}[e^{-\delta t_R}].
\end{aligned}$$

Since  $\delta t_R > 0$ , we have that  $0 < e^{-\delta t_R} < 1$ . Then we see that each term above is greater than zero, and hence  $f > 0$ . Therefore



### 3.4. EXISTENCE OF SOLUTION

---

$$-\frac{1}{\tau\gamma}f < -\frac{1}{\gamma}f. \quad (3.78)$$

On the boundary,  $\bar{Z} \geq \psi$ . Using (3.71) with  $z = xe^{\psi/\gamma}$ , we have that for  $x > 0, (\lambda, \theta) \in \partial B_R$ ,

$$V_0^\tau = \frac{1}{\tau\gamma}(x^{\tau\gamma}e^{\tau\psi} - 1) \leq \frac{1}{\gamma}(x^\gamma e^{\bar{Z}} - 1) = V_0. \quad (3.79)$$

Note that for  $x = 0$  and  $\tau \in (0, 1)$ ,

$$V_0^\tau(0, \lambda, \theta) = -\frac{1}{\tau\gamma} < -\frac{1}{\gamma} = V_0(0, \lambda, \theta). \quad (3.80)$$

Now consider large values of  $x$  with  $\tau \in (0, 1)$ . We can take  $x$  to be large enough such that

$$\frac{1}{\tau\gamma}x^{\tau\gamma}e^{Z^\tau} \leq \frac{1}{\gamma}x^\gamma e^{\bar{Z}}.$$

So this along with (3.78) implies that

$$V_0^\tau < V_0 \quad (3.81)$$

for large  $x$  and  $\tau \in (0, 1)$ .

Next we prove that  $V_0^\tau \leq V_0$  for  $(\lambda, \theta) \in \bar{B}_R$ . Suppose on the contrary that

$$\sup_{x>0, |(\lambda, \theta)| \leq R} \{V_0^\tau(x, \lambda, \theta) - V_0(x, \lambda, \theta)\} > 0. \quad (3.82)$$

Then the maximum is attained at some  $(x_0, \lambda_0, \theta_0)$ , where  $x_0 > 0$  and  $|(\lambda_0, \theta_0)| < R$ . So

### 3.4. EXISTENCE OF SOLUTION

---

at  $(x_0, \lambda_0, \theta_0)$  we have the following:

$$V_0(x_0, \lambda_0, \theta_0) < V_0^\tau(x_0, \lambda_0, \theta_0), \quad (3.83)$$

$$(V_0^\tau - V_0)_x(x_0, \lambda_0, \theta_0) = (V_0^\tau - V_0)_\lambda(x_0, \lambda_0, \theta_0) = (V_0^\tau - V_0)_\theta(x_0, \lambda_0, \theta_0) = 0, \quad (3.84)$$

and  $D^2(V_0^\tau - V_0)$  is negative semi-definite at  $(x_0, \lambda_0, \theta_0)$ , where  $D^2$  is the  $3 \times 3$  matrix operator of second derivatives (the Hessian). That is, for any  $\eta \in \mathbb{R}^3$ ,

$$\eta^T D^2(V_0^\tau - V_0)(x_0, \lambda_0, \theta_0) \eta \leq 0, \quad (3.85)$$

or in expanded form, for any  $\eta_1, \eta_2, \eta_3 \in \mathbb{R}$ ,

$$\begin{aligned} & (V_0^\tau - V_0)_{xx}(x_0, \lambda_0, \theta_0) \eta_1^2 + (V_0^\tau - V_0)_{\lambda\lambda}(x_0, \lambda_0, \theta_0) \eta_2^2 + (V_0^\tau - V_0)_{\theta\theta}(x_0, \lambda_0, \theta_0) \eta_3^2 \\ & + 2(V_0^\tau - V_0)_{x\lambda}(x_0, \lambda_0, \theta_0) \eta_1 \eta_2 + 2(V_0^\tau - V_0)_{x\theta}(x_0, \lambda_0, \theta_0) \eta_1 \eta_3 \\ & + 2(V_0^\tau - V_0)_{\lambda\theta}(x_0, \lambda_0, \theta_0) \eta_2 \eta_3 \leq 0. \end{aligned} \quad (3.86)$$

Note that this implies

$$(V_0^\tau - V_0)_{xx}, \quad (V_0^\tau - V_0)_{\lambda\lambda}, \quad (V_0^\tau - V_0)_{\theta\theta} \leq 0 \quad (3.87)$$

at the point  $(x_0, \lambda_0, \theta_0)$  (for any  $i$ , set  $\eta_i = 1, \eta_j = 0$  for each  $i \neq j$  in (3.86)).

### 3.4. EXISTENCE OF SOLUTION

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We also note that

$$\begin{aligned}
q(\lambda, \theta) &= \tilde{\sigma}^2(\theta) + 2\rho\sigma\tilde{\sigma}e^{-\lambda} + \sigma^2e^{-2\lambda} \\
&\geq \tilde{\sigma}^2(\theta) + 2\rho\sigma\tilde{\sigma}e^{-\lambda} + \rho^2\sigma^2e^{-2\lambda} \\
&= (\tilde{\sigma}(\theta) + \rho\sigma e^{-\lambda})^2.
\end{aligned} \tag{3.88}$$

Now we evaluate (3.75) at  $(x_0, \lambda_0, \theta_0)$  and apply conditions (3.83) and (3.84). So at the point  $(x_0, \lambda_0, \theta_0)$ , we have

$$\begin{aligned}
\frac{\tilde{\sigma}^2(\theta)}{2}(V_0^\tau - V_0)_{\lambda\lambda} + \frac{\pi^2}{2}(V_0^\tau - V_0)_{\theta\theta} + \frac{\tilde{k}^2x^2}{2}q(\lambda, \theta)(V_0^\tau - V_0)_{xx} \\
+ \tilde{k}x\tilde{\sigma}(\theta)(\tilde{\sigma}(\theta) + \rho\sigma e^{-\lambda})(V_0^\tau - V_0)_{x\lambda} > 0.
\end{aligned} \tag{3.89}$$

Applying (3.87) and (3.88) to (3.89), we have

$$\begin{aligned}
\frac{\tilde{\sigma}^2(\theta)}{2}(V_0^\tau - V_0)_{\lambda\lambda} + \frac{\tilde{k}^2x^2}{2}(\tilde{\sigma}(\theta) + \rho\sigma e^{-\lambda})^2(V_0^\tau - V_0)_{xx} \\
+ \tilde{k}x\tilde{\sigma}(\theta)(\tilde{\sigma}(\theta) + \rho\sigma e^{-\lambda})(V_0^\tau - V_0)_{x\lambda} > 0
\end{aligned} \tag{3.90}$$

at  $(x_0, \lambda_0, \theta_0)$ . Taking

$$\eta_1 = \frac{\tilde{k}x}{\sqrt{2}}(\tilde{\sigma} + \rho\sigma e^{-\lambda}), \eta_2 = \frac{\tilde{\sigma}(\theta)}{\sqrt{2}}, \eta_3 = 0$$

gives us a contradiction to (3.86). Therefore  $V_0^\tau \leq V_0$  for  $(\lambda, \theta) \in \bar{B}_R$ . Thus (3.77) holds.

### 3.4. EXISTENCE OF SOLUTION

---

We take  $x = 1$  in (3.77) to get

$$\frac{1}{\tau\gamma}(e^{Z^\tau} - f) \leq \frac{1}{\gamma}(e^{\bar{Z}} - f).$$

Thus we have proved (3.52).  $\square$

**Theorem 3.4.5.** *Let  $\tau \in (0, 1]$ . Suppose  $\tilde{\sigma}^2(\theta) \in C^{1,\alpha}(\bar{B}_R)$ , and that  $Z_\tau^0$  is a solution of*

$$\begin{cases} \frac{\tilde{\sigma}^2(\theta)}{2}Z_{\lambda\lambda} + \frac{\pi^2}{2}Z_{\theta\theta} - \tau H(\lambda, \theta, Z, Z_\lambda, Z_\theta, 0) = 0 & \text{on } B_R, \\ Z = 0 & \text{on } \partial B_R. \end{cases} \quad (3.91)$$

Then

$$-\delta \mathbb{E}[\bar{t}_R] \leq Z_\tau^0(\lambda, \theta) \leq \log(1 + \mathbb{E}[\bar{t}_R]), \quad (3.92)$$

where

$$\bar{t}_R = \inf \left\{ t > 0; \sqrt{\bar{\lambda}_t^2 + \bar{\theta}_t^2} = R \right\},$$

and  $\bar{\lambda}_t, \bar{\theta}_t$  are defined by

$$d\bar{\lambda}_t = \tau \tilde{\mu}(\bar{\theta}_t)dt + \tilde{\sigma}(\bar{\theta}_t)d\tilde{w}_t, \quad \bar{\lambda}_0 = \lambda,$$

$$d\bar{\theta}_t = \tau a(\theta_L - \bar{\theta}_t)dt + \pi d\hat{w}_t, \quad \bar{\theta}_0 = \theta.$$

*Proof.* Note that

$$H(\lambda, \theta, Z, Z_\lambda, Z_\theta, 0) = -\frac{\tilde{\sigma}^2(\theta)}{2}Z_\lambda^2 - \frac{\pi^2}{2}Z_\theta^2 - \tilde{\mu}(\theta)Z_\lambda - a(\theta_L - \theta)Z_\theta + \delta - e^{-Z}.$$

### 3.4. EXISTENCE OF SOLUTION

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Since  $Z_\tau^0$  is a solution of (3.91), we have that

$$\begin{aligned} \frac{\tilde{\sigma}^2(\theta)}{2}(Z_\tau^0)_{\lambda\lambda} + \frac{\pi^2}{2}(Z_\tau^0)_{\theta\theta} \\ + \tau \left[ \frac{\tilde{\sigma}^2(\theta)}{2}(Z_\tau^0)_\lambda^2 + \frac{\pi^2}{2}(Z_\tau^0)_\theta^2 + \tilde{\mu}(\theta)(Z_\tau^0)_\lambda + a(\theta_L - \theta)(Z_\tau^0)_\theta - \delta + e^{-Z_\tau^0} \right] = 0 \end{aligned} \quad (3.93)$$

on  $B_R$ . Subtracting some of the nonnegative terms gives us the following inequality:

$$\frac{\tilde{\sigma}^2(\theta)}{2}(Z_\tau^0)_{\lambda\lambda} + \frac{\pi^2}{2}(Z_\tau^0)_{\theta\theta} + \tau\tilde{\mu}(\theta)(Z_\tau^0)_\lambda + \tau a(\theta_L - \theta)(Z_\tau^0)_\theta - \tau\delta \leq 0. \quad (3.94)$$

Applying Ito's rule to  $Z_\tau^0(\bar{\lambda}_t, \bar{\theta}_t)$  and using (3.94), we have that for  $0 \leq t \leq \bar{t}_R$ ,

$$\begin{aligned} dZ_\tau^0(\bar{\lambda}_t, \bar{\theta}_t) &= (Z_\tau^0)_\lambda \left[ \tau\tilde{\mu}(\bar{\theta}_t)dt + \tilde{\sigma}(\bar{\theta}_t)d\tilde{w}_t \right] + (Z_\tau^0)_\theta \left[ \tau a(\theta_L - \bar{\theta}_t)dt + \pi d\hat{w}_t \right] \\ &\quad + \frac{1}{2}(Z_\tau^0)_{\lambda\lambda}\tilde{\sigma}^2(\bar{\theta}_t)dt + \frac{1}{2}(Z_\tau^0)_{\theta\theta}\pi^2dt \\ &= \left[ \frac{\tilde{\sigma}^2(\theta)}{2}(Z_\tau^0)_{\lambda\lambda} + \frac{\pi^2}{2}(Z_\tau^0)_{\theta\theta} + \tau\tilde{\mu}(\bar{\theta})(Z_\tau^0)_\lambda + \tau a(\theta_L - \theta)(Z_\tau^0)_\theta \right] dt \\ &\quad + \tilde{\sigma}(\bar{\theta}_t)(Z_\tau^0)_\lambda d\tilde{w}_t + \pi(Z_\tau^0)_\theta d\hat{w}_t \\ &\leq \tau\delta dt + \tilde{\sigma}(\bar{\theta}_t)(Z_\tau^0)_\lambda d\tilde{w}_t + \pi(Z_\tau^0)_\theta d\hat{w}_t. \end{aligned} \quad (3.95)$$

In integral form, we write

### 3.4. EXISTENCE OF SOLUTION

---

$$Z_\tau^0(\bar{\lambda}_{\bar{t}_R}, \bar{\theta}_{\bar{t}_R}) - Z_\tau^0(\lambda, \theta) \leq \int_0^{\bar{t}_R} \tau \delta dt + \int_0^{\bar{t}_R} \tilde{\sigma}(\bar{\theta}_t)(Z_\tau^0)_\lambda d\tilde{w}_t + \int_0^{\bar{t}_R} \pi(Z_\tau^0)_\theta d\hat{w}_t. \quad (3.96)$$

Taking expectations, we have  $\mathbb{E}Z_\tau^0(\bar{\lambda}_{\bar{t}_R}, \bar{\theta}_{\bar{t}_R}) = 0$  since  $Z_\tau^0 = 0$  on  $\partial B_R$ . For  $0 < t \leq \bar{t}_R$ ,  $(\bar{\lambda}_t, \bar{\theta}_t) \in \bar{B}_R$ . So for  $0 < t \leq \bar{t}_R$ , we have that  $\tilde{\sigma}(\bar{\theta}_t)(Z_\tau^0)_\lambda$  and  $\pi(Z_\tau^0)_\theta$  are bounded.

Thus

$$\mathbb{E} \left[ \int_0^{\bar{t}_R} \tilde{\sigma}^2(\bar{\theta}_t)(Z_\tau^0)_\lambda^2 dt \right] < \infty \quad \text{and} \quad \mathbb{E} \left[ \int_0^{\bar{t}_R} \pi^2(Z_\tau^0)_\theta^2 dt \right] < \infty.$$

Therefore  $\int_0^{\bar{t}_R} \tilde{\sigma}(\bar{\theta}_t)(Z_\tau^0)_\lambda d\tilde{w}_t$  and  $\int_0^{\bar{t}_R} \pi(Z_\tau^0)_\theta d\hat{w}_t$  are martingales, which gives us

$$\mathbb{E} \left[ \int_0^{\bar{t}_R} \tilde{\sigma}(\bar{\theta}_t)(Z_\tau^0)_\lambda d\tilde{w}_t \right] = \mathbb{E} \left[ \int_0^{\bar{t}_R} \pi(Z_\tau^0)_\theta d\hat{w}_t \right] = 0.$$

Then (3.102) reduces to

$$-Z_\tau^0(\lambda, \theta) \leq \tau \delta \mathbb{E}[\bar{t}_R].$$

Since  $\tau \in (0, 1]$ , we have

$$-Z_\tau^0(\lambda, \theta) \leq \delta \mathbb{E}[\bar{t}_R],$$

which proves the first inequality in (3.92).

Define

$$\phi(\lambda, \theta) = e^{Z_\tau^0(\lambda, \theta)}. \quad (3.97)$$

Then

$$\phi_\lambda = \phi(Z_\tau^0)_\lambda, \quad \phi_\theta = \phi(Z_\tau^0)_\theta,$$

### 3.4. EXISTENCE OF SOLUTION

---

$$\phi_{\lambda\lambda} = \phi((Z_\tau^0)_{\lambda\lambda} + (Z_\tau^0)_\lambda^2), \quad \text{and} \quad \phi_{\theta\theta} = \phi((Z_\tau^0)_{\theta\theta} + (Z_\tau^0)_\theta^2).$$

Now on  $B_R$  we have

$$\begin{aligned} & \frac{\tilde{\sigma}^2(\theta)}{2}\phi_{\lambda\lambda} + \frac{\pi^2}{2}\phi_{\theta\theta} - \frac{1-\tau}{2\phi}(\tilde{\sigma}^2(\theta)\phi_\lambda^2 + \pi^2\phi_\theta^2) + \tau(\tilde{\mu}(\theta)\phi_\lambda + a(\theta_L - \theta)\phi_\theta) + \tau(1 - \delta\phi) \\ &= \frac{\tilde{\sigma}^2(\theta)}{2}\phi((Z_\tau^0)_{\lambda\lambda} + (Z_\tau^0)_\lambda^2) + \frac{\pi^2}{2}\phi((Z_\tau^0)_{\theta\theta} + (Z_\tau^0)_\theta^2) \\ & \quad - \frac{1-\tau}{2\phi}(\tilde{\sigma}^2(\theta)\phi^2(Z_\tau^0)_\lambda^2 + \pi^2\phi^2(Z_\tau^0)_\theta^2) \\ & \quad + \tau\phi(\tilde{\mu}(\theta)(Z_\tau^0)_\lambda + a(\theta_L - \theta)(Z_\tau^0)_\theta) + \tau(1 - \delta\phi) \\ &= \phi \left\{ \frac{\tilde{\sigma}^2(\theta)}{2}(Z_\tau^0)_{\lambda\lambda} + \frac{\pi^2}{2}(Z_\tau^0)_{\theta\theta} + \tau \left[ \frac{\tilde{\sigma}^2(\theta)}{2}(Z_\tau^0)_\lambda^2 + \frac{\pi^2}{2}\phi(Z_\tau^0)_\theta^2 + \tilde{\mu}(\theta)(Z_\tau^0)_\lambda \right. \right. \\ & \quad \left. \left. + a(\theta_L - \theta)(Z_\tau^0)_\theta \right] - \tau\delta + \tau e^{-Z_\tau^0} \right\} \\ &= 0. \end{aligned} \tag{3.98}$$

On the boundary,  $\phi = e^0 = 1$ . Therefore  $\phi$  satisfies

$$\begin{cases} \frac{\tilde{\sigma}^2(\theta)}{2}\phi_{\lambda\lambda} + \frac{\pi^2}{2}\phi_{\theta\theta} - \frac{1-\tau}{2\phi}(\tilde{\sigma}^2(\theta)\phi_\lambda^2 + \pi^2\phi_\theta^2) \\ \quad + \tau(\tilde{\mu}(\theta)\phi_\lambda + a(\theta_L - \theta)\phi_\theta) + \tau(1 - \delta\phi) = 0 & \text{on } B_R, \\ \phi = 1 & \text{on } \partial B_R. \end{cases} \tag{3.99}$$

Since the terms  $\frac{1-\tau}{2\phi}(\tilde{\sigma}^2(\theta)\phi_\lambda^2 + \pi^2\phi_\theta^2)$  and  $\tau\delta\phi$  are nonnegative, we can add them to our expression to get an inequality:

### 3.4. EXISTENCE OF SOLUTION

---

$$\frac{\tilde{\sigma}^2(\theta)}{2}\phi_{\lambda\lambda} + \frac{\pi^2}{2}\phi_{\theta\theta} + \tau(\tilde{\mu}(\theta)\phi_{\lambda} + a(\theta_L - \theta)\phi_{\theta}) + \tau \geq 0. \quad (3.100)$$

Next, we can apply Ito's rule to  $\phi(\bar{\lambda}_t, \bar{\theta}_t)$  and use the above inequality to obtain

$$\begin{aligned} d\phi(\bar{\lambda}_t, \bar{\theta}_t) &= \phi_{\lambda} \left[ \tau \tilde{\mu}(\bar{\theta}_t) dt + \tilde{\sigma}(\bar{\theta}_t) d\tilde{w}_t \right] + \phi_{\theta} \left[ \tau a(\theta_L - \bar{\theta}_t) dt + \pi d\hat{w}_t \right] \\ &\quad + \frac{1}{2} \phi_{\lambda\lambda} \tilde{\sigma}^2(\bar{\theta}_t) dt + \frac{1}{2} \phi_{\theta\theta} \pi^2 dt \\ &= \left[ \frac{\tilde{\sigma}^2(\bar{\theta}_t)}{2} \phi_{\lambda\lambda} + \frac{\pi^2}{2} \phi_{\theta\theta} + \tau \tilde{\mu}(\bar{\theta}_t) \phi_{\lambda} + \tau a(\theta_L - \bar{\theta}_t) \phi_{\theta} \right] dt + \tilde{\sigma}(\bar{\theta}_t) \phi_{\lambda} d\tilde{w}_t + \pi \phi_{\theta} d\hat{w}_t \\ &\geq -\tau dt + \tilde{\sigma}(\bar{\theta}_t) \phi_{\lambda} d\tilde{w}_t + \pi \phi_{\theta} d\hat{w}_t. \end{aligned} \quad (3.101)$$

In integral form, we have

$$\phi(\bar{\lambda}_{\bar{t}_R}, \bar{\theta}_{\bar{t}_R}) - \phi(\lambda, \theta) \geq - \int_0^{\bar{t}_R} \tau dt + \int_0^{\bar{t}_R} \tilde{\sigma}(\bar{\theta}_t) \phi_{\lambda} d\tilde{w}_t + \int_0^{\bar{t}_R} \pi \phi_{\theta} d\hat{w}_t. \quad (3.102)$$

We can again show that the last two integrals are martingales since their arguments are bounded for  $0 < t < \bar{t}_R$ . Taking expectations, these terms vanish and we can rearrange to get

$$\begin{aligned} 1 - \phi(\lambda, \theta) &\geq -\tau \mathbb{E}[\bar{t}_R] \\ e^{Z_{\tau}^0} &\leq 1 + \tau \mathbb{E}[\bar{t}_R] \\ Z_{\tau}^0 &\leq \log(1 + \tau \mathbb{E}[\bar{t}_R]), \end{aligned}$$

and then

$$Z_{\tau}^0 \leq \log(1 + \mathbb{E}[\bar{t}_R]),$$



### 3.4. EXISTENCE OF SOLUTION

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since  $\tau \in (0, 1]$ . This proves the second inequality in (3.92).  $\square$

The following theorem gives us existence of solution to the boundary value problem (3.33) with  $\psi = \hat{Z}$ , on the closed ball  $\bar{B}_R = \{(\lambda, \theta) \in \mathbb{R}^2 : \lambda^2 + \theta^2 \leq R^2\}$ .

**Theorem 3.4.6.** *Suppose  $\tilde{\sigma}^2(\theta) \in C^{1,\alpha}(\bar{B}_R)$ , (3.24) has an ordered pair of subsolution/supersolution  $(\hat{Z}, \bar{Z})$ , and  $\hat{Z} \in C^{2,\beta}(\bar{B}_R)$  for some  $\beta \in (0, 1)$ . Suppose further that  $\bar{Z}$  is a supersolution of (3.38) for  $\tau = 1$ , and  $Z^\tau$  is a solution of (3.38) with  $\tau \in (0, 1]$ . Then the boundary value problem*

$$\begin{cases} \frac{\tilde{\sigma}^2(\theta)}{2} Z_{\lambda\lambda} + \frac{\pi^2}{2} Z_{\theta\theta} - H(\lambda, \theta, Z, Z_\lambda, Z_\theta) = 0 & \text{on } B_R, \\ Z = \hat{Z} & \text{on } \partial B_R, \end{cases} \quad (3.103)$$

has a unique solution in  $C^{2,\beta}(\bar{B}_R)$ .

*Proof.* Notice that (3.103) is equivalent to (3.33) with  $\psi = \hat{Z}$ . We use Theorem 3.4.1 to prove existence of solution. Note that conditions (a), (b), and (c) are automatically satisfied.

By Theorem 3.4.4, we have inequality (3.52):

$$e^{Z^\tau(\lambda, \theta)} \leq \tau e^{\bar{Z}(\lambda, \theta)} + (1 - \tau) f(\lambda, \theta).$$

Since  $\tau \in (0, 1]$  and  $f > 0$ , we can write

$$e^{Z^\tau(\lambda, \theta)} \leq e^{\bar{Z}(\lambda, \theta)} + f(\lambda, \theta), \quad (3.104)$$

### 3.4. EXISTENCE OF SOLUTION

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or

$$Z^\tau(\lambda, \theta) \leq \log [e^{\bar{Z}(\lambda, \theta)} + f(\lambda, \theta)] \equiv \bar{M}. \quad (3.105)$$

Also by Theorem 3.4.4, we have the following inequality:

$$Z^\tau(\lambda, \theta) \geq - \sup_{\lambda^2 + \theta^2 = R^2} \{|\hat{Z}(\lambda, \theta)|\} - \log \left( \max \left\{ \frac{\delta}{1 - \gamma}, 1 \right\} \right) \equiv \underline{M}. \quad (3.106)$$

Note that  $\underline{M}$  and  $\bar{M}$  are independent of both  $\tau$  and  $Z^\tau$ . Here we take

$$M \equiv \max\{|\underline{M}|, \bar{M}\} + C, \quad (3.107)$$

where  $C$  is any positive constant. Then we have the bound

$$|Z^\tau(\lambda, \theta)| < M, \quad (3.108)$$

where  $M$  is independent of  $\tau$  and  $Z^\tau$ . Hence condition (d) of Theorem (3.4.1) is satisfied.

For condition (e), take  $\underline{c} \equiv \min\{L^2, \pi^2\}$  and  $\bar{c} \equiv \max\{U^2, \pi^2\}$ . Then for any  $(\eta_1, \eta_2) \in \mathbb{R}^2$ ,

$$\underline{c}(\eta_1^2 + \eta_2^2) \leq \tilde{\sigma}^2(\theta)\eta_1^2 + \pi^2\eta_2^2 \leq \bar{c}(\eta_1^2 + \eta_2^2).$$

### 3.4. EXISTENCE OF SOLUTION

---

For the second part of condition (e), we have that for  $|z| \leq M$ ,

$$\begin{aligned}
& |H(\lambda, \theta, z, p, s, \tau)| + 2\tilde{\sigma}(\theta) \left| \frac{d\tilde{\sigma}(\theta)}{d\theta} \right| \\
&= \left| -\frac{\tilde{\sigma}^2(\theta)}{2} p^2 - \frac{\pi^2}{2} s^2 - \tilde{\mu}(\theta)p - \alpha(\theta_L - \theta)s - \gamma G(\lambda, \theta, p) + \delta - \gamma r - (1 - \gamma)e^{\frac{z}{\gamma-1}} \right| \\
&\quad + 2\tilde{\sigma}(\theta) \left| \frac{d\tilde{\sigma}(\theta)}{d\theta} \right| \\
&\leq \frac{U^2}{2} p^2 + \frac{\pi^2}{2} s^2 + \tilde{M}|p| + A|s| + \gamma(\tilde{C}_1 + \tilde{C}_2 p^2) + \delta + \gamma|r| + (1 - \gamma)e^{\frac{M}{\gamma-1}} + 2U\bar{U}.
\end{aligned}$$

Using the identity  $|2a| \leq a^2 + 1$ , we can bound the linear terms in  $|p|$  and  $|s|$  by expressions of the form  $C_1(1 + p^2)$  and  $C_2(1 + s^2)$ , for positive constants  $C_1$  and  $C_2$ . Therefore, for  $|z| \leq M$  and some constant  $C > 0$ ,

$$|H(\lambda, \theta, z, p, s)| + 2\tilde{\sigma}(\theta) \left| \frac{d\tilde{\sigma}(\theta)}{d\theta} \right| \leq C(1 + p^2 + s^2). \quad (3.109)$$

Thus condition (e) is satisfied, with  $\bar{k} = 2$ .

By Theorem 3.4.5, we have estimate (3.92). Thus

$$|Z_0^\tau(\lambda, \theta)| < M_1, \quad (3.110)$$

where  $M_1 = \max \{ \delta \mathbb{E}[\bar{t}_R], \log(1 + \mathbb{E}[\bar{t}_R]) \} + C_1$ , for some constant  $C_1 > 0$ . So condition (f) of Theorem 3.4.1 is satisfied.

In (3.38), take  $\psi \equiv \hat{Z}$ . For the  $\tau = 0$  case, (3.38) has the unique solution

$$Z_0^\tau = \log f, \quad (3.111)$$

### 3.4. EXISTENCE OF SOLUTION

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where  $f$  is given by (3.54). Indeed, setting  $\tau = 0$  in (3.38) and substituting (3.111) into the equation satisfied on  $B_R$ , we obtain

$$\begin{aligned}
& \frac{\tilde{\sigma}^2(\theta)}{2} \left[ -\frac{1}{f^2} f_\lambda^2 + \frac{1}{f} f_{\lambda\lambda} \right] + \frac{\pi^2}{2} \left[ -\frac{1}{f^2} + \frac{1}{f} f_{\theta\theta} \right] + \frac{\tilde{\sigma}^2(\theta)}{2} \frac{1}{f^2} f_\lambda^2 + \frac{\pi^2}{2} \frac{1}{f^2} f_\theta^2 \\
& \quad + \tilde{\mu}(\theta) \frac{1}{f} f_\lambda + a(\theta_L - \theta) \frac{1}{f} f_\theta - \delta + \frac{1}{f} \\
& = \frac{1}{f} \left[ \frac{\tilde{\sigma}^2(\theta)}{2} f_{\lambda\lambda} + \frac{\pi^2}{2} f_{\theta\theta} + \tilde{\mu}(\theta) f_\lambda + a(\theta_L - \theta) f_\theta - \delta f + 1 \right] \\
& = 0,
\end{aligned} \tag{3.112}$$

by (3.53). On the boundary  $\partial B_R$ ,  $\tau_R = 0$ . Then by (3.54) we see that  $f = 1$ , and so  $Z_0^\tau = 0$  on  $\partial B_R$ . Therefore  $Z_0^\tau = \log f$  is a solution to (3.38) for  $\tau = 0$ . Uniqueness can be proven using a method similar to that in Corollary 3.4.3.1.

This solution for  $\tau = 0$  corresponds to the solution of the *log utility* problem, the case in which the HARA utility parameter  $\gamma$  is equal to 0. We do not discuss this case in detail because our focus is on the non-log HARA utility with  $\gamma \in (0, 1)$ .

By Theorem 3.4.1, (3.103) has a solution in  $C^{2,\alpha}(\bar{B}_R)$ . For  $\tau = 1$ , (3.38) is equivalent to (3.33), which is equivalent to (3.103) when  $\psi = \hat{Z}$ . Therefore by Corollary 3.4.3.1, the solution to (3.103) is unique.  $\square$

#### 3.4.3 A Uniform Bound for $\sup_{B_R} |DZ|^2$

Before proving existence of a classical solution to (3.24), we must prove the existence of a uniform bound for  $\sup_{B_R} |DZ|^2$  for any  $R$ . We start with the following result.

**Lemma 3.4.7.** *Let  $Z(x) \in C^2(\mathbb{R}^N)$ , and  $a^{i,j}(x) \in C^2(\mathbb{R}^N)$  for all  $i, j$  ranging from 1 to*

### 3.4. EXISTENCE OF SOLUTION

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*N. Then*

$$\sum_{i,j,k=1}^N D_k a^{ij} D_k Z D_{ij} Z \leq \frac{1}{2\epsilon} \left( \sum_{i,j=1}^N |D a^{ij}|^2 \right) |DZ|^2 + \frac{\epsilon}{2} |D^2 Z|^2, \quad (3.113)$$

where  $\epsilon > 0$  is a small constant.

*Proof.* By the Cauchy-Schwarz Inequality, we have

$$\begin{aligned} \sum_{i,j,k} D_k a^{ij} D_k Z D_{ij} Z &= \sum_{i,j} \left( \sum_k D_k a^{ij} D_k Z \right) D_{ij} Z \\ &\leq \sum_{i,j} |D a^{ij}| |DZ| \cdot D_{ij} Z. \end{aligned}$$

Now using the identity  $ab \leq \frac{1}{2}(a^2 + b^2)$  and simplifying, we get

$$\begin{aligned} \sum_{i,j} |D a^{ij}| |DZ| \cdot D_{ij} Z &= \sum_{i,j} \frac{1}{\sqrt{\epsilon}} |D a^{ij}| |DZ| \cdot \sqrt{\epsilon} D_{ij} Z \\ &\leq \sum_{i,j} \left( \frac{1}{2\epsilon} |D a^{ij}|^2 |DZ|^2 + \frac{\epsilon}{2} (D_{ij} Z)^2 \right) \\ &= \frac{1}{2\epsilon} \left( \sum_{i,j} |D a^{ij}|^2 \right) |DZ|^2 + \frac{\epsilon}{2} \sum_{i,j} (D_{ij} Z)^2 \\ &= \frac{1}{2\epsilon} \left( \sum_{i,j} |D a^{ij}|^2 \right) |DZ|^2 + \frac{\epsilon}{2} |D^2 Z|^2. \end{aligned}$$

□

**Theorem 3.4.8.** *Let  $Z_{\tilde{R}}$  be a smooth function satisfying the HJB equation (3.24) in  $B_{\tilde{R}}$ .*

### 3.4. EXISTENCE OF SOLUTION

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For each  $R > 0$  and  $\tilde{R} > 2R$ , we have

$$\sup_{B_{\tilde{R}}} |DZ_{\tilde{R}}|^2 \leq C_R + C(\delta - \gamma r), \quad (3.114)$$

where  $C$  is a nonnegative constant independent of  $R$  and  $\tilde{R}$ , and  $C_R$  is a constant depending only on  $R$ .

*Proof.* We write  $G$  in its quadratic form,

$$G(\lambda, \theta, Z_\lambda) = \gamma g_2 Z_\lambda^2 + \gamma g_1 Z_\lambda + \gamma g_0,$$

where  $g_2, g_1$ , and  $g_0$  are defined in (3.49). Then  $Z_{\tilde{R}}$  satisfies

$$\begin{aligned} & \frac{\tilde{\sigma}^2(\theta)}{2} Z_{\lambda\lambda} + \frac{\pi^2}{2} Z_{\theta\theta} + \frac{1}{2} (\tilde{\sigma}^2(\theta) + 2\gamma g_2(\lambda, \theta)) Z_\lambda^2 + \frac{\pi^2}{2} Z_\theta^2 \\ & + (\tilde{\mu}(\theta) + \gamma g_1(\lambda, \theta)) Z_\lambda + a(\theta_L - \theta) Z_\theta + \gamma g_0(\lambda, \theta) + (1 - \gamma) e^{\frac{Z}{\gamma-1}} = \delta - \gamma r \end{aligned} \quad (3.115)$$

on  $B_{\tilde{R}}$ . Note that this is the case of  $be^{-\lambda} + \mu - r \geq 0$ . The more simple case of  $G = 0$  (if  $be^{-\lambda} + \mu - r < 0$ ) can be proved using the same steps as this proof, so we will not prove this case explicitly. For simplicity, we drop the subscript  $\tilde{R}$  and set  $Z \equiv Z_{\tilde{R}}$ . Differentiating (3.115) with respect to  $\lambda$  and  $\theta$ , we obtain

$$\begin{aligned}
 & \frac{\tilde{\sigma}^2(\theta)}{2} Z_{\lambda\lambda\lambda} + \frac{\pi^2}{2} Z_{\theta\theta\lambda} + (\tilde{\sigma}^2(\theta) + 2\gamma g_2) Z_{\lambda} Z_{\lambda\lambda} + \gamma(g_2)_{\lambda} Z_{\lambda}^2 + \pi^2 Z_{\theta} Z_{\theta\lambda} \\
 & + (\tilde{\mu}(\theta) + \gamma g_1) Z_{\lambda\lambda} + \gamma(g_1)_{\lambda} Z_{\lambda} + a(\theta_L - \theta) Z_{\theta\lambda} + \gamma(g_0)_{\lambda} - e^{\frac{Z}{\gamma-1}} Z_{\lambda} = 0
 \end{aligned} \tag{3.116}$$

and

$$\begin{aligned}
 & \frac{\tilde{\sigma}^2(\theta)}{2} Z_{\lambda\lambda\theta} + \tilde{\sigma}(\theta) \tilde{\sigma}'(\theta) Z_{\lambda\lambda} + \frac{\pi^2}{2} Z_{\theta\theta\theta} + (\tilde{\sigma}^2(\theta) + 2\gamma g_2) Z_{\lambda} Z_{\lambda\theta} \\
 & + (\tilde{\sigma}(\theta) \tilde{\sigma}'(\theta) + \gamma(g_2)_{\theta}) Z_{\lambda}^2 + \pi^2 Z_{\theta} Z_{\theta\theta} + (\tilde{\mu}(\theta) + \gamma g_1) Z_{\lambda\theta} \\
 & + (\tilde{\mu}'(\theta) + \gamma(g_1)_{\lambda}) Z_{\lambda} + a(\theta_L - \theta) Z_{\theta\theta} - a Z_{\theta} + \gamma(g_0)_{\theta} - e^{\frac{Z}{\gamma-1}} Z_{\theta} = 0
 \end{aligned} \tag{3.117}$$

respectively. Next, we take the sum of Equation (3.116) multiplied by  $Z_{\lambda}$ , and Equation (3.117) multiplied by  $Z_{\theta}$ . Rearranging the result (to a form that will be useful in a later step), we get

$$\begin{aligned}
 & -\frac{\tilde{\sigma}^2(\theta)}{2} Z_{\lambda\lambda\lambda} Z_{\lambda} - \frac{\pi^2}{2} Z_{\theta\theta\lambda} Z_{\lambda} - \frac{\tilde{\sigma}^2(\theta)}{2} Z_{\lambda\lambda\theta} Z_{\theta} - \frac{\pi^2}{2} Z_{\theta\theta\theta} Z_{\theta} \\
 & - (\tilde{\sigma}^2(\theta) + 2\gamma g_2) (Z_{\lambda} Z_{\lambda\lambda} Z_{\lambda} + Z_{\lambda} Z_{\lambda\theta} Z_{\theta}) - \pi^2 (Z_{\theta} Z_{\theta\lambda} Z_{\lambda} + Z_{\theta} Z_{\theta\theta} Z_{\theta}) \\
 & - (\tilde{\mu}(\theta) + \gamma g_1) (Z_{\lambda\lambda} Z_{\lambda} + Z_{\lambda\theta} Z_{\theta}) - a(\theta_L - \theta) (Z_{\theta\lambda} Z_{\lambda} + Z_{\theta\theta} Z_{\theta}) \\
 & = \tilde{\sigma}(\theta) \tilde{\sigma}'(\theta) Z_{\lambda\lambda} Z_{\theta} + \gamma(g_2)_{\lambda} Z_{\lambda}^2 Z_{\lambda} + (\tilde{\sigma}(\theta) \tilde{\sigma}'(\theta) + \gamma(g_2)_{\theta}) Z_{\lambda}^2 Z_{\theta} \\
 & + \gamma(g_1)_{\lambda} Z_{\lambda} Z_{\lambda} + (\tilde{\mu}'(\theta) + \gamma(g_1)_{\theta}) Z_{\lambda} Z_{\theta} - a Z_{\theta} Z_{\theta} + \gamma(g_0)_{\lambda} Z_{\lambda} + \gamma(g_0)_{\theta} Z_{\theta} \\
 & - e^{\frac{Z}{\gamma-1}} Z_{\lambda} Z_{\lambda} - e^{\frac{Z}{\gamma-1}} Z_{\theta} Z_{\theta}.
 \end{aligned} \tag{3.118}$$

### 3.4. EXISTENCE OF SOLUTION

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Define

$$\Phi \equiv \frac{1}{2}|DZ|^2 = \frac{1}{2}(Z_\lambda^2 + Z_\theta^2), \quad (3.119)$$

so that

$$D_\lambda \Phi = Z_\lambda Z_{\lambda\lambda} + Z_\theta Z_{\theta\lambda}, \quad D_\theta \Phi = Z_\lambda Z_{\lambda\theta} + Z_\theta Z_{\theta\theta}, \quad (3.120)$$

$$D_{\lambda\lambda} \Phi = Z_{\lambda\lambda}^2 + Z_\lambda Z_{\lambda\lambda\lambda} + Z_{\lambda\theta}^2 + Z_\theta Z_{\theta\lambda\lambda}, \quad \text{and} \quad D_{\theta\theta} \Phi = Z_{\lambda\theta}^2 + Z_\theta Z_{\lambda\theta\theta} + Z_{\theta\theta}^2 + Z_\theta Z_{\theta\theta\theta}.$$

Then by (3.118),

$$\begin{aligned} & -\frac{\tilde{\sigma}^2(\theta)}{2} D_{\lambda\lambda} \Phi - \frac{\pi^2}{2} D_{\theta\theta} \Phi - (\tilde{\sigma}^2(\theta) + 2\gamma g_2) Z_\lambda D_\lambda \Phi - \pi^2 Z_\theta D_\theta \Phi \\ & \quad - (\tilde{\mu}(\theta) + \gamma g_1) D_\lambda \Phi - a(\theta_L - \theta) D_\theta \Phi \\ & = -\frac{\tilde{\sigma}^2(\theta)}{2} Z_{\lambda\lambda\lambda} Z_\lambda - \frac{\pi^2}{2} Z_{\theta\theta\lambda} Z_\lambda - \frac{\tilde{\sigma}^2(\theta)}{2} Z_{\lambda\lambda\theta} Z_\theta - \frac{\pi^2}{2} Z_{\theta\theta\theta} Z_\theta \\ & \quad - (\tilde{\sigma}^2(\theta) + 2\gamma g_2) Z_\lambda (Z_{\lambda\lambda} Z_\lambda + Z_{\lambda\theta} Z_\theta) - \pi^2 Z_\theta (Z_{\theta\lambda} Z_\lambda + Z_{\theta\theta} Z_\theta) \\ & \quad - (\tilde{\mu}(\theta) + \gamma g_1) (Z_{\lambda\lambda} Z_\lambda + Z_{\lambda\theta} Z_\theta) - a(\theta_L - \theta) (Z_{\theta\lambda} Z_\lambda + Z_{\theta\theta} Z_\theta) \\ & \quad - \frac{\tilde{\sigma}^2(\theta)}{2} (Z_{\lambda\lambda}^2 + Z_{\lambda\theta}^2) - \frac{\pi^2}{2} (Z_{\lambda\theta}^2 + Z_{\theta\theta}^2) \\ & = \tilde{\sigma}(\theta) \tilde{\sigma}'(\theta) Z_{\lambda\lambda} Z_\theta + \gamma(g_2)_\lambda Z_\lambda^2 Z_\lambda + (\tilde{\sigma}(\theta) \tilde{\sigma}'(\theta) + \gamma(g_2)_\theta) Z_\lambda^2 Z_\theta + \gamma(g_1)_\lambda Z_\lambda Z_\lambda \\ & \quad + (\tilde{\mu}'(\theta) + \gamma(g_1)_\theta) Z_\lambda Z_\theta - a Z_\theta Z_\theta + \gamma(g_0)_\lambda Z_\lambda + \gamma(g_0)_\theta Z_\theta - e^{\frac{Z}{\gamma-1}} Z_\lambda Z_\lambda \\ & \quad - e^{\frac{Z}{\gamma-1}} Z_\theta Z_\theta - \frac{\tilde{\sigma}^2(\theta)}{2} (Z_{\lambda\lambda}^2 + Z_{\lambda\theta}^2) - \frac{\pi^2}{2} (Z_{\lambda\theta}^2 + Z_{\theta\theta}^2). \end{aligned} \quad (3.121)$$

By Lemma (3.4.7), we have that

$$\tilde{\sigma}(\theta) \tilde{\sigma}'(\theta) Z_\theta Z_{\lambda\lambda} \leq \frac{1}{4\epsilon} (2\tilde{\sigma}(\theta) \tilde{\sigma}'(\theta))^2 (Z_\lambda^2 + Z_\theta^2) + \frac{\epsilon}{4} (Z_{\lambda\lambda}^2 + 2Z_{\lambda\theta}^2 + Z_{\theta\theta}^2) \quad (3.122)$$



### 3.4. EXISTENCE OF SOLUTION

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Then

$$\begin{aligned}
(3.121) &\leq \frac{1}{4\epsilon}(2\tilde{\sigma}(\theta)\tilde{\sigma}'(\theta))^2(Z_\lambda^2 + Z_\theta^2) + \frac{\epsilon}{4}(Z_{\lambda\lambda}^2 + 2Z_{\lambda\theta}^2 + Z_{\theta\theta}^2) \\
&\quad + \tilde{\sigma}(\theta)\tilde{\sigma}'(\theta)Z_{\lambda\lambda}Z_\theta + \gamma(g_2)_\lambda Z_\lambda^2 Z_\lambda + (\tilde{\sigma}(\theta)\tilde{\sigma}'(\theta) + \gamma(g_2)_\theta)Z_\lambda^2 Z_\theta + \gamma(g_1)_\lambda Z_\lambda Z_\lambda \\
&\quad + (\tilde{\mu}'(\theta) + \gamma(g_1)_\theta)Z_\lambda Z_\theta - aZ_\theta Z_\theta + \gamma(g_0)_\lambda Z_\lambda + \gamma(g_0)_\theta Z_\theta - e^{\frac{Z}{\gamma-1}}Z_\lambda Z_\lambda \\
&\quad - e^{\frac{Z}{\gamma-1}}Z_\theta Z_\theta - \frac{\tilde{\sigma}^2(\theta)}{4}(Z_{\lambda\lambda}^2 + Z_{\lambda\theta}^2) - \frac{\pi^2}{4}(Z_{\lambda\theta}^2 + Z_{\theta\theta}^2) \\
&\quad - \frac{\tilde{\sigma}^2(\theta)}{4}(Z_{\lambda\lambda}^2 + Z_{\lambda\theta}^2) - \frac{\pi^2}{4}(Z_{\lambda\theta}^2 + Z_{\theta\theta}^2) \tag{3.123} \\
&\leq \frac{1}{\epsilon}(\tilde{\sigma}(\theta)\tilde{\sigma}'(\theta))^2(Z_\lambda^2 + Z_\theta^2) \\
&\quad + \tilde{\sigma}(\theta)\tilde{\sigma}'(\theta)Z_{\lambda\lambda}Z_\theta + \gamma(g_2)_\lambda Z_\lambda^2 Z_\lambda + (\tilde{\sigma}(\theta)\tilde{\sigma}'(\theta) + \gamma(g_2)_\theta)Z_\lambda^2 Z_\theta + \gamma(g_1)_\lambda Z_\lambda^2 \\
&\quad + (\tilde{\mu}'(\theta) + \gamma(g_1)_\theta)Z_\lambda Z_\theta - aZ_\theta^2 + \gamma(g_0)_\lambda Z_\lambda + \gamma(g_0)_\theta Z_\theta - e^{\frac{Z}{\gamma-1}}(Z_\lambda^2 + Z_\theta^2) \\
&\quad - \frac{\tilde{\sigma}^2(\theta)}{4}(Z_{\lambda\lambda}^2 + Z_{\lambda\theta}^2) - \frac{\pi^2}{4}(Z_{\lambda\theta}^2 + Z_{\theta\theta}^2), \tag{3.124}
\end{aligned}$$

where the last step uses the fact that

$$\begin{aligned}
&\frac{\epsilon}{4}(Z_{\lambda\lambda}^2 + 2Z_{\lambda\theta}^2 + Z_{\theta\theta}^2) - \frac{\tilde{\sigma}^2(\theta)}{4}(Z_{\lambda\lambda}^2 + Z_{\lambda\theta}^2) - \frac{\pi^2}{4}(Z_{\lambda\theta}^2 + Z_{\theta\theta}^2) \\
&= \frac{1}{4}\left[(\epsilon - \tilde{\sigma}^2(\theta))Z_{\lambda\lambda}^2 + (2\epsilon - \tilde{\sigma}^2(\theta) - \pi^2)Z_{\lambda\theta}^2 + (\epsilon - \pi^2)Z_{\theta\theta}^2\right] \\
&\leq 0,
\end{aligned}$$

for any constant  $\epsilon$  such that  $0 < \epsilon \leq \min\{L^2, \pi^2\}$ , and  $L$  given by (3.2).

Consider the matrix inequality  $(tr(AB))^2 \leq N\nu_2(tr(AB^2))$ , where  $A$  and  $B$  are  $N \times N$  symmetric matrices,  $A$  is positive semidefinite, and  $\nu_2$  is the maximum eigenvalue of  $A$ .

### 3.4. EXISTENCE OF SOLUTION

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We use this inequality with  $A = \begin{bmatrix} \tilde{\sigma}^2 & 0 \\ 0 & \pi^2 \end{bmatrix}$  and  $B = \begin{bmatrix} Z_{\lambda\lambda} & Z_{\lambda\theta} \\ Z_{\lambda\theta} & Z_{\theta\theta} \end{bmatrix}$  to get

$$-\frac{1}{4} \left[ \tilde{\sigma}^2(\theta) Z_{\lambda\lambda}^2 + (\tilde{\sigma}^2(\theta) + \pi^2) Z_{\lambda\theta}^2 + \pi^2 Z_{\theta\theta}^2 \right] \leq -\frac{1}{8\nu_2} (\tilde{\sigma}^2(\theta) Z_{\lambda\lambda} + \pi^2 Z_{\theta\theta})^2. \quad (3.125)$$

So then

$$\begin{aligned} & -\frac{\tilde{\sigma}^2(\theta)}{2} D_{\lambda\lambda} \Phi - \frac{\pi^2}{2} D_{\theta\theta} \Phi - (\tilde{\sigma}^2(\theta) + 2\gamma g_2) Z_{\lambda} D_{\lambda} \Phi - \pi^2 Z_{\theta} D_{\theta} \Phi \\ & \quad - (\tilde{\mu}(\theta) + \gamma g_1) D_{\lambda} \Phi - a(\theta_L - \theta) D_{\theta} \Phi \\ & \leq C_R |DZ| + C_R |DZ|^2 + C_R |DZ|^3 - \frac{1}{8\nu_2} (\tilde{\sigma}^2(\theta) Z_{\lambda\lambda} + \pi^2 Z_{\theta\theta})^2 \quad \text{in } B_{2R}. \end{aligned} \quad (3.126)$$

We use  $C_R$  to represent an arbitrary constant depending only on  $R$ ;  $C$  is a nonnegative constant independent of  $R$  and  $\tilde{R}$ .

Fix arbitrary  $\xi \in B_R$  and let  $B_R(\xi)$  be an open ball with radius  $R$  and center  $\xi$ . Let  $\phi \in C_0^\infty(\mathbb{R}^2)$  be a cutoff function such that

$$\begin{aligned} 0 \leq \phi \leq 1 \text{ in } \mathbb{R}^2, \quad \phi(\xi) = 1, \quad \phi \equiv 0 \text{ in } (B_R(\xi))^c, \\ |D\phi| \leq C\phi^{1/2}, \quad |D^2\phi| \leq C. \end{aligned} \quad (3.127)$$

Suppose the maximum of  $\phi\Phi$  in  $\bar{B}_R(\xi)$  is attained at  $(\lambda_0, \theta_0)$ . By the maximum principle, at  $(\lambda_0, \theta_0)$  we have

$$D_i(\phi\Phi) = \phi D_i \Phi + \Phi D_i \phi = 0,$$

### 3.4. EXISTENCE OF SOLUTION

---

$$\text{and } \sum_{i,j} D_{i,j}(\phi\Phi)\eta_i\eta_j \leq 0,$$

where the second condition means that the Hessian of  $(\phi\Phi)(\lambda_0, \theta_0)$  is negative semidefinite.

Then at  $(\lambda_0, \theta_0)$ , we have

$$\begin{aligned}
 0 \leq & -\frac{\tilde{\sigma}^2(\theta)}{2}D_{\lambda\lambda}(\phi\Phi) - \frac{\pi^2}{2}D_{\theta\theta}(\phi\Phi) - (\tilde{\sigma}^2(\theta) + 2\gamma g_2)Z_\lambda D_\lambda(\phi\Phi) - \pi^2 Z_\theta D_\theta(\phi\Phi) \\
 & - (\tilde{\mu}(\theta) + \gamma g_1)D_\lambda(\phi\Phi) - a(\theta_L - \theta)D_\theta(\phi\Phi)
 \end{aligned} \tag{3.128}$$

$$\begin{aligned}
 = & -\frac{\tilde{\sigma}^2(\theta)}{2}\phi D_{\lambda\lambda}\Phi - \tilde{\sigma}^2(\theta)D_\lambda\phi D_\lambda\Phi - \frac{\tilde{\sigma}^2(\theta)}{2}\Phi D_{\lambda\lambda}\phi - \frac{\pi^2}{2}\phi D_{\theta\theta}\Phi - \pi^2 D_\theta\phi D_\theta\Phi \\
 & - \frac{\pi^2}{2}\Phi D_{\theta\theta}\phi - (\tilde{\sigma}^2(\theta) + 2\gamma g_2)Z_\lambda(\phi D_\lambda\Phi + \Phi D_\lambda\phi) - \pi^2 Z_\theta(\phi D_\theta\Phi + \Phi D_\theta\phi) \\
 & - (\tilde{\mu}(\theta) + \gamma g_1)(\phi D_\lambda\Phi + \Phi D_\lambda\phi) - a(\theta_L - \theta)(\phi D_\theta\Phi + \Phi D_\theta\phi)
 \end{aligned} \tag{3.129}$$

$$\begin{aligned}
 = & \phi \left\{ -\frac{\tilde{\sigma}^2(\theta)}{2}D_{\lambda\lambda}\Phi - \frac{\pi^2}{2}D_{\theta\theta}\Phi - (\tilde{\sigma}^2(\theta) + 2\gamma g_2)Z_\lambda D_\lambda\Phi - \pi^2 Z_\theta D_\theta\Phi \right. \\
 & \left. - (\tilde{\mu}(\theta) + \gamma g_1)D_\lambda\Phi - a(\theta_L - \theta)D_\theta\Phi \right\} - \frac{\tilde{\sigma}^2(\theta)}{2}\Phi D_{\lambda\lambda}\phi - \frac{\pi^2}{2}\Phi D_{\theta\theta}\phi \\
 & - \tilde{\sigma}^2(\theta)D_\lambda\phi D_\lambda\Phi - \pi^2 D_\theta\phi D_\theta\Phi - (\tilde{\sigma}^2(\theta) + 2\gamma g_2)\Phi Z_\lambda D_\lambda\phi - \pi^2 \Phi Z_\theta D_\theta\phi \\
 & - (\tilde{\mu}(\theta) + \gamma g_1)\Phi D_\lambda\phi - a(\theta_L - \theta)\Phi D_\theta\phi
 \end{aligned} \tag{3.130}$$

$$\begin{aligned}
 \leq & \phi \left\{ -\frac{\tilde{\sigma}^2(\theta)}{2}D_{\lambda\lambda}\Phi - \frac{\pi^2}{2}D_{\theta\theta}\Phi - (\tilde{\sigma}^2(\theta) + 2\gamma g_2)Z_\lambda D_\lambda\Phi - \pi^2 Z_\theta D_\theta\Phi \right. \\
 & \left. - (\tilde{\mu}(\theta) + \gamma g_1)D_\lambda\Phi - a(\theta_L - \theta)D_\theta\Phi \right\} + C_R\Phi + C\phi^{1/2}\Phi^{3/2}
 \end{aligned} \tag{3.131}$$

$$\begin{aligned}
 \leq & \phi \left\{ C_R|DZ| + C_R|DZ|^2 + C_R|DZ|^3 - \frac{1}{8\nu_2}(\tilde{\sigma}^2(\theta)Z_{\lambda\lambda} + \pi^2 Z_{\theta\theta})^2 \right\} \\
 & + C_R\Phi + C\phi^{1/2}\Phi^{3/2}
 \end{aligned} \tag{3.132}$$

$$\begin{aligned}
 = & \phi \left\{ C_R|DZ| + C_R|DZ|^2 + C_R|DZ|^3 - \frac{1}{2\nu_2} \left[ -\frac{1}{2}(\tilde{\sigma}^2(\theta) + 2\gamma g_2)Z_\lambda^2 - \frac{\pi^2}{2}Z_\theta^2 \right. \right. \\
 & \left. \left. - (\tilde{\mu}(\theta) + \gamma g_1)Z_\lambda - a(\theta_L - \theta)Z_\theta - \gamma g_0 + \delta - \gamma r - (1 - \gamma)e^{\frac{Z}{\gamma-1}} \right]^2 \right\} \\
 & + C_R\Phi + C\phi^{1/2}\Phi^{3/2},
 \end{aligned} \tag{3.133}$$

### 3.4. EXISTENCE OF SOLUTION

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where (3.132) is true by (3.126), and (3.133) follows from (3.115).

Note that there exist  $0 < \mu_1 < \mu_2$  such that

$$\mu_1|\eta|^2 \leq (\tilde{\sigma}^2(\theta) + 2\gamma g_2)\eta_1^2 + \pi^2\eta_2^2 \leq \mu_2|\eta|^2 \quad \text{for all } (\lambda, \theta), \eta \in \mathbb{R}^2.$$

In fact, take  $\mu_1 = \min\{L^2, \pi^2\}$  and  $\mu_2 = \max\{U^2 + \gamma\tilde{C}_2\}$ , where  $\tilde{C}_2$  is the bound of the coefficient to  $p^2$  in (3.50). Then we have

$$\begin{aligned} & -\frac{1}{2}(\tilde{\sigma}^2(\theta) + 2\gamma g_2)Z_\lambda^2 - \frac{\pi^2}{2}Z_\theta^2 - (\tilde{\mu}(\theta) + \gamma g_1)Z_\lambda - a(\theta_L - \theta)Z_\theta - \gamma g_0 \\ & \quad + \delta - \gamma r - (1 - \gamma)e^{\frac{Z}{\gamma-1}} \\ & \leq -\mu_1|DZ|^2 + C_R|DZ| - \gamma g_0 + \delta - \gamma r - (1 - \gamma)e^{\frac{Z}{\gamma-1}} \\ & \leq -\kappa|DZ|^2 + C_R - \gamma g_0 + \delta - \gamma r - (1 - \gamma)e^{\frac{Z}{\gamma-1}}, \end{aligned} \tag{3.134}$$

where  $\kappa > 0$  is a constant that depends only on  $\mu_1$ , and the last step follows from the identity  $|DZ| \leq \frac{1}{2}(|DZ|^2 + 1)$ .

Now we split the problem up into cases. First, consider the case  $-\kappa|DZ|^2 + C_R - \gamma g_0 + \delta - \gamma r - (1 - \gamma)e^{\frac{Z}{\gamma-1}} \geq 0$  at  $(\lambda_0, \theta_0)$ . Then

$$\kappa|DZ|^2(\lambda_0, \theta_0) \leq C_R + \delta - \gamma r.$$

Note that

$$\frac{1}{2}|DZ|^2(\xi) = \Phi(\xi)\phi(\xi) \leq \Phi(\lambda_0, \theta_0)\phi(\lambda_0, \theta_0),$$

### 3.4. EXISTENCE OF SOLUTION

---

since  $(\lambda_0, \theta_0)$  is the maximizer of  $\phi\Phi$  in  $\bar{B}_R(\xi)$ ; and so

$$\begin{aligned}
 \kappa|DZ|^2(\xi) &\leq \kappa|DZ|^2(\lambda_0, \theta_0)\phi(\lambda_0, \theta_0) \\
 &\leq \phi(\lambda_0, \theta_0)[C_R + \delta - \gamma r] \\
 &\leq C_R + \delta - \gamma r.
 \end{aligned} \tag{3.135}$$

So in this case, we have

$$|DZ|^2(\xi) \leq C_R + \frac{1}{\kappa}(\delta - \gamma r). \tag{3.136}$$

Next, consider the case  $-\kappa|DZ|^2 + C_R - \gamma g_0 + \delta - \gamma r - (1 - \gamma)e^{\frac{Z}{\gamma-1}} \leq 0$  at  $(\lambda_0, \theta_0)$ . Then

### 3.4. EXISTENCE OF SOLUTION

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by (3.134), we obtain

$$\begin{aligned}
(3.133) &\leq \phi \left\{ C_R |DZ| + C_R |DZ|^2 + C_R |DZ|^3 - \frac{1}{2\nu_2} \left[ -\kappa |DZ|^2 + C_R - \gamma g_0 + \delta - \gamma r \right. \right. \\
&\quad \left. \left. - (1 - \gamma) e^{\frac{Z}{\gamma-1}} \right]^2 \right\} + C_R \Phi + C \phi^{1/2} \Phi^{3/2} \\
&= \phi \left\{ C_R |DZ| + C_R |DZ|^2 + C_R |DZ|^3 \right. \\
&\quad \left. - \frac{1}{2\nu_2} \left[ \kappa^2 |DZ|^4 - 2\kappa |DZ|^2 \left( C_R - \gamma g_0 + \delta - \gamma r - (1 - \gamma) e^{\frac{Z}{\gamma-1}} \right) \right. \right. \\
&\quad \left. \left. + \left( C_R - \gamma g_0 + \delta - \gamma r - (1 - \gamma) e^{\frac{Z}{\gamma-1}} \right)^2 \right] \right\} + C_R \Phi + C \phi^{1/2} \Phi^{3/2} \\
&\leq \phi \left\{ C_R \Phi^{1/2} + C_R \Phi + C_R \Phi^{3/2} - \frac{2\kappa^2}{\nu_2} \Phi^2 \right. \\
&\quad \left. + \frac{2\kappa}{\nu_2} \Phi \left( C_R - \gamma g_0 + \delta - \gamma r - (1 - \gamma) e^{\frac{Z}{\gamma-1}} \right) \right\} + C_R \Phi + C \phi^{1/2} \Phi^{3/2} \\
&= \phi \left\{ C_R \Phi^{1/2} + C_R \Phi + C_R \Phi^{3/2} \right. \\
&\quad \left. - \frac{2\kappa}{\nu_2} \Phi \left[ \kappa \Phi - \left( C_R - \gamma g_0 + \delta - \gamma r - (1 - \gamma) e^{\frac{Z}{\gamma-1}} \right) \right] \right\} + C_R \Phi + C \phi^{1/2} \Phi^{3/2},
\end{aligned} \tag{3.137}$$

at  $(\lambda_0, \theta_0)$ . If  $\kappa \Phi \leq C_R - \gamma g_0 + \delta - \gamma r - (1 - \gamma) e^{\frac{Z}{\gamma-1}} = C_R + \delta - \gamma r$ , then

$$\kappa \Phi(\xi) = \kappa \phi(\xi) \Phi(\xi) \leq \kappa \phi(\lambda_0, \theta_0) \Phi(\lambda_0, \theta_0) \leq \kappa \Phi(\lambda_0, \theta_0) \leq C_R + \delta - \gamma r, \tag{3.138}$$

so that

$$|DZ|^2(\xi) \leq C_R + \frac{2}{\kappa}(\delta - \gamma r). \tag{3.139}$$

### 3.4. EXISTENCE OF SOLUTION

---

If  $C_R \geq \Phi(\lambda_0, \theta_0)$ , we have a similar result:

$$|DZ|^2(\xi) \leq C_R \leq C_R + C(\delta - \gamma r), \quad (3.140)$$

for some nonnegative constant  $C$ .

Finally, suppose both  $\kappa\Phi \geq C_R - \gamma g_0 + \delta - \gamma r - (1 - \gamma)e^{\frac{Z}{\gamma-1}} = C_R + \delta - \gamma r$  and  $C_R \leq \Phi(\lambda_0, \theta_0)$ . Then

$$\begin{aligned} (3.137) &\leq \phi \left\{ \Phi^{3/2} + C_R \Phi + C_R \Phi^{3/2} \right\} + C_R \Phi + C \phi^{1/2} \Phi^{3/2} \\ &\leq C_1 \phi \Phi^2 + C_2 \phi^{1/2} \Phi^{3/2} + \tilde{C}_3 \Phi \quad \text{at } (\lambda_0, \theta_0), \quad \tilde{C}_3 = C_3 C_R, \end{aligned} \quad (3.141)$$

where  $C_1, C_2$  and  $C_3$  are positive constants independent of  $R, \tilde{R}$ , and  $(\delta - \gamma r)$ . (3.4.3) uses the inequality

$$C_R \Phi^{3/2} = C_R \Phi^{1/2} \Phi \leq \frac{1}{2} (C_R^2 \Phi + \Phi^2),$$

and the fact that  $\phi \leq 1$ .

Let  $Y \equiv \phi(\lambda_0, \theta_0) \Phi(\lambda_0, \theta_0)$ . Then from (3.4.3) we have

$$0 \leq -C_1 Y^2 + C_2 Y + \tilde{C}_3. \quad (3.142)$$

The quadratic on the right-hand side is concave down. Since the quadratic is greater than or equal to zero,  $Y$  is bounded by the zeros of the quadratic. This, along with the fact



### 3.4. EXISTENCE OF SOLUTION

---

that  $Y \geq 0$ , implies

$$\begin{aligned} \phi\Phi(\lambda_0, \theta_0) = Y^2 &\leq \left( \frac{C_2 + \sqrt{C_2 + 4C_1\tilde{C}_3}}{2C_1} \right)^2 \\ &\leq \frac{C_2^2}{2C_1^2} + \frac{C_2^2 + 4C_1\tilde{C}_3}{2C_1^2} \\ &= \frac{C_2^2}{C_1^2} + \frac{2C_3C_R}{C_1}. \end{aligned}$$

Again, we use the fact that  $|DZ|^2(\xi) \leq \phi(\lambda_0, \theta_0)|DZ|^2(\lambda_0, \theta_0)$ , and we obtain the bound

$$|DZ|^2(\xi) \leq C_R + C.$$

In each case, we have a bound for  $|DZ|^2$  in  $B_{\tilde{R}}$  that can be written in the form  $C_R + C(\delta - \gamma r)$ , where  $C_R$  is a constant depending only on  $R$ , and  $C$  is a positive constant.  $\square$

#### 3.4.4 Existence of Solution to the HJB Equation

Finally, we have our existence theorem.

**Theorem 3.4.9.** *Suppose  $\tilde{\sigma}^2(\theta) \in C^{1,\alpha}(\bar{B}_R)$ . Define  $\hat{Z} \equiv K_1$  and  $\bar{Z} \equiv K_2$ , where  $K_1$  and  $K_2$  are given by (3.30) and (3.32), respectively. Then there exists a solution  $\tilde{Z} \in C^{2,\beta}(\mathbb{R}^2)$  to (3.24) such that  $\hat{Z} \leq \tilde{Z}(\lambda, \theta) \leq \bar{Z}$  for all  $(\lambda, \theta) \in \mathbb{R}^2$ .*

*Proof.* By Theorem 3.4.6, there is a unique solution  $Z^l$  to

$$\begin{cases} \frac{\tilde{\sigma}^2(\theta)}{2} Z_{\lambda\lambda} + \frac{\pi^2}{2} Z_{\theta\theta} - H(\lambda, \theta, Z, Z_\lambda, Z_\theta) = 0 & \text{on } B_l, \\ Z = \hat{Z} & \text{on } \partial B_l, \end{cases} \quad (3.143)$$

### 3.4. EXISTENCE OF SOLUTION

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for  $l = 1, 2, 3, \dots$ . Since  $\hat{Z}$  and  $Z^{l+1}$  satisfy the following,

$$\begin{cases} \frac{\tilde{\sigma}^2(\theta)}{2}\hat{Z}_{\lambda\lambda} + \frac{\pi^2}{2}\hat{Z}_{\theta\theta} - H(\lambda, \theta, \hat{Z}, \hat{Z}_\lambda, \hat{Z}_\theta) \geq 0 & \forall (\lambda, \theta) \in \mathbb{R}^2, \\ \frac{\tilde{\sigma}^2(\theta)}{2}Z_{\lambda\lambda}^{l+1} + \frac{\pi^2}{2}Z_{\theta\theta}^{l+1} - H(\lambda, \theta, Z^{l+1}, Z_\lambda^{l+1}, Z_\theta^{l+1}) = 0 & \text{on } B_{l+1}, \\ \hat{Z} = Z^{l+1} & \text{on } \partial B_{l+1}, \end{cases} \quad (3.144)$$

we see that they satisfy (3.41) with  $\tau = 1$ . Thus by Lemma 3.4.3,

$$\hat{Z} \leq Z_{l+1} \quad \text{in } \bar{B}_{l+1}.$$

In particular,

$$\hat{Z} \leq Z^{l+1} \quad \text{on } \partial B_l.$$

On  $\partial B_l$ , we also have  $Z^l = \hat{Z}$ . Therefore

$$Z^l \leq Z^{l+1} \quad \text{on } \partial B_l.$$

Now we see that  $Z^l$  and  $Z^{l+1}$  satisfy

$$\begin{cases} \frac{\tilde{\sigma}^2(\theta)}{2}Z_{\lambda\lambda}^l + \frac{\pi^2}{2}Z_{\theta\theta}^l - H(\lambda, \theta, Z^l, Z_\lambda^l, Z_\theta^l) = 0 & \text{on } B_l, \\ \frac{\tilde{\sigma}^2(\theta)}{2}Z_{\lambda\lambda}^{l+1} + \frac{\pi^2}{2}Z_{\theta\theta}^{l+1} - H(\lambda, \theta, Z^{l+1}, Z_\lambda^{l+1}, Z_\theta^{l+1}) = 0 & \text{on } B_l, \\ Z^l \leq Z^{l+1} & \text{on } \partial B_l. \end{cases} \quad (3.145)$$

By Lemma 3.4.3 again,

$$Z^l \leq Z^{l+1} \quad \text{in } B_l.$$

Thus for any  $k$  such that  $|(\lambda, \theta)| \leq k$ ,  $Z^l(\lambda, \theta)$  is nondecreasing in  $l$  for  $l > k$ , and is bounded above by  $\bar{Z}$ . So by taking  $l \rightarrow \infty$ ,  $\{Z^l(\lambda, \theta)\}$  converges pointwise to some function  $\tilde{Z}(\lambda, \theta)$ .

By Theorem 3.4.8, we have a uniform bound for  $|DZ^l|^2$  on  $\bar{B}_R$  for any  $R > 0$ . By the Arzela-Ascoli Theorem,  $\{Z^l\}$  contains a subsequence that converges to a function  $Z \in C^{2,\beta}(\bar{B}_R)$  as  $l \rightarrow \infty$ . Since  $\{Z^l\}$  also converges pointwise to  $\tilde{Z}$ , then we must have  $\tilde{Z} \equiv Z$ . By (3.109) and (3.114), it follows that  $\tilde{Z}$  is a solution to (3.24) on  $\bar{B}_R$  for any  $R$ . Take  $R$  to infinity, and we have that  $\tilde{Z}$  is a solution to (3.24) on  $\mathbb{R}^2$ .  $\square$

## 3.5 Verification Theorem

In Section 3.4, we proved existence of a classical solution  $\tilde{Z}(\lambda, \theta)$  to the HJB equation (3.24). In this section, we prove that  $\tilde{V} = \frac{1}{\gamma} x^\gamma e^{\tilde{Z}}$  is equal to the value function (3.11). In effect, we will have maximized the expected discounted utility of an investor whose net worth depends on stochastic dividends and stochastic volatility of stock price.

We begin by stating a useful result.

**Lemma 3.5.1.** *Let  $\tilde{Z}(\lambda, \theta)$  be a classical solution to (3.24) such that  $K_1 \leq \tilde{Z} \leq K_2$ , where  $K_1$  and  $K_2$  are defined by (3.30) and (3.32), respectively. Define  $k^*(\lambda, \theta, \tilde{Z}_\lambda)$  and  $c^*(\lambda, \theta, \tilde{Z})$  by (3.19) and (3.17), respectively. For the processes  $\lambda_t, \theta_t$  and  $X_t$  defined in (3.6), (3.4), and (3.9), we have*

$$\mathbb{E}[\lambda_t^m], \mathbb{E}[\theta_t^m], \mathbb{E}[X_t^m] < \infty \tag{3.146}$$

for all fixed  $m > 0$ .

### 3.5. VERIFICATION THEOREM

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For a proof, refer to [Pa02], Lemma 1.1 of Chapter 1.

We now state and prove the Verification Theorem.

**Theorem 3.5.2. (Verification Theorem)** *Suppose  $0 < \gamma < 1$  and (3.31) holds. Let  $\tilde{Z}(\lambda, \theta)$  denote a classical solution of (3.24) which satisfies  $K_1 \leq \tilde{Z} \leq K_2$ . Denote*

$$\tilde{V}(x, \lambda, \theta) \equiv \frac{1}{\gamma} x^\gamma e^{\tilde{Z}(\lambda, \theta)}. \quad (3.147)$$

Then we have

$$\tilde{V}(x, \lambda, \theta) \equiv V(x, \lambda, \theta), \quad (3.148)$$

where  $V(x, \lambda, \theta)$  is the value function defined by (3.11). Moreover, the optimal control policy is

$$k^*(\lambda, \theta, \tilde{Z}_\lambda) = \left[ \frac{be^{-\lambda} + \mu - r + (\tilde{\sigma}(\theta)^2 + \rho\sigma\tilde{\sigma}(\theta)e^{-\lambda})\tilde{Z}_\lambda(\lambda, \theta)}{(1 - \gamma)q(\lambda, \theta)} \right]^+, \quad c^*(\lambda, \theta, \tilde{Z}) = e^{\frac{\tilde{Z}(\lambda, \theta)}{\gamma-1}}. \quad (3.149)$$

*Proof.* Since  $\tilde{Z}$  is a classical solution of (3.24), we have

$$\begin{aligned} \delta = & \frac{\tilde{\sigma}^2(\theta)}{2}(\tilde{Z}_\lambda^2 + \tilde{Z}_{\lambda\lambda}) + \frac{\pi^2}{2}(\tilde{Z}_\theta^2 + \tilde{Z}_{\theta\theta}) + \gamma r + \tilde{\mu}(\theta)\tilde{Z}_\lambda \\ & + a(\theta_L - \theta)\tilde{Z}_\theta + (1 - \gamma)e^{\frac{\tilde{Z}}{\gamma-1}} + \gamma G(\lambda, \theta, \tilde{Z}_\lambda). \end{aligned} \quad (3.150)$$

### 3.5. VERIFICATION THEOREM

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Using (3.150) we can show that  $\tilde{V}$  is a classical solution of (3.12). From (3.147) we get

$$\tilde{V}_x = \frac{\gamma}{x}\tilde{V}, \quad \tilde{V}_\lambda = \tilde{V}\tilde{Z}_\lambda, \quad \tilde{V}_\theta = \tilde{V}\tilde{Z}_\theta,$$

$$\tilde{V}_{xx} = \frac{\gamma(\gamma-1)}{x^2}\tilde{V}, \quad \tilde{V}_{\lambda\lambda} = \tilde{V}(\tilde{Z}_\lambda^2 + \tilde{Z}_{\lambda\lambda}), \quad \tilde{V}_{\theta\theta} = (\tilde{Z}_\theta^2 + \tilde{Z}_{\theta\theta})\tilde{V}, \quad \text{and} \quad \tilde{V}_{x\lambda} = \frac{\gamma}{x}\tilde{Z}_\lambda\tilde{V}.$$

Substituting these expressions into the right-hand side of (3.12) and simplifying, we get

$$\begin{aligned} & \frac{\tilde{\sigma}^2(\theta)}{2}\tilde{V}_{\lambda\lambda} + \frac{\pi^2}{2}\tilde{V}_{\theta\theta} + rxV_x + \tilde{\mu}(\theta)\tilde{V}_\lambda + a(\theta_L - \theta)\tilde{V}_\theta + \max_{c \geq 0} \left\{ \frac{1}{\gamma}(cx)^\gamma - cxV_x \right\} \\ & \quad + \max_{k \geq 0} \left\{ (be^{-\lambda} + \mu - r)kx\tilde{V}_x + \frac{k^2x^2}{2}q(\lambda, \theta)\tilde{V}_{xx} + kx(\tilde{\sigma}^2(\theta) + \rho\sigma\tilde{\sigma}(\theta)e^{-\lambda})\tilde{V}_{x\lambda} \right\} \\ & = \frac{\tilde{\sigma}^2(\theta)}{2}(\tilde{Z}_\lambda^2 + \tilde{Z}_{\lambda\lambda})\tilde{V} + \frac{\pi^2}{2}(\tilde{Z}_\theta^2 + \tilde{Z}_{\theta\theta})\tilde{V} + r\gamma\tilde{V} + \tilde{\mu}(\theta)\tilde{Z}_\lambda\tilde{V} + a(\theta_L - \theta)\tilde{Z}_\theta\tilde{V} \\ & \quad + \left( \frac{1}{\gamma} - 1 \right) \left( \frac{\gamma}{x}\tilde{V} \right)^{\frac{\gamma}{1-\gamma}} + \max_{k \geq 0} \left\{ (be^{-\lambda} + \mu - r)k\gamma\tilde{V} + \frac{k^2}{2}\gamma(\gamma-1)q(\lambda, \theta)\tilde{V} \right. \\ & \quad \left. + k\gamma(\tilde{\sigma}^2(\theta) + \rho\sigma\tilde{\sigma}(\theta)e^{-\lambda})\tilde{Z}_\lambda\tilde{V} \right\} \\ & = \tilde{V} \left[ \frac{\tilde{\sigma}^2(\theta)}{2}(\tilde{Z}_\lambda^2 + \tilde{Z}_{\lambda\lambda}) + \frac{\pi^2}{2}(\tilde{Z}_\theta^2 + \tilde{Z}_{\theta\theta}) + r\gamma + \tilde{\mu}(\theta)\tilde{Z}_\lambda + a(\theta_L - \theta)\tilde{Z}_\theta \right. \\ & \quad \left. + (1 - \gamma)e^{\frac{\tilde{Z}}{1-\gamma}} + \gamma G(\lambda, \theta, \tilde{Z}_\lambda) \right] \\ & = \tilde{V}\delta. \end{aligned}$$

Therefore  $\tilde{V}$  is a classical solution of (3.12).

For any admissible control  $(k_t, c_t) \in \Pi$ , using Ito's rule for  $f(t, \tilde{V}) = e^{-\delta t}\tilde{V}$ , we get

$$\begin{aligned} df &= f_t dt + f_v d\tilde{V} + \frac{1}{2}f_{vv}(d\tilde{V})^2 \\ d[e^{-\delta t}\tilde{V}(X_t, \lambda_t, \theta_t)] &= -\delta e^{-\delta t}\tilde{V}(X_t, \lambda_t, \theta_t)dt + e^{-\delta t}d\tilde{V}(X_t, \lambda_t, \theta_t). \end{aligned} \quad (3.151)$$

Using Ito's rule on  $\tilde{V}$ , we have

$$\begin{aligned}
 d\tilde{V}(X_t, \lambda_t, \theta_t) &= \tilde{V}_x dX_t + \tilde{V}_\lambda d\lambda_t + \tilde{V}_\theta d\theta_t + \frac{1}{2}\tilde{V}_{xx}(dX_t)^2 + \frac{1}{2}\tilde{V}_{\lambda\lambda}(d\lambda_t)^2 + \frac{1}{2}\tilde{V}_{\theta\theta}(d\theta_t)^2 \\
 &\quad + \tilde{V}_{x\lambda}(dX_t)(d\lambda_t) + \tilde{V}_{x\lambda}(dX_t)(d\lambda_t) + \tilde{V}_{\lambda\theta}(d\lambda_t)(d\theta_t) \\
 &= \tilde{V}_x \left[ X_t [(be^{-\lambda_t} + \mu - r)k_t + (r - c_t)] dt + k_t X_t \sigma e^{-\lambda_t} dw_t + k_t X_t \tilde{\sigma}(\theta_t) d\tilde{w}_t \right] \\
 &\quad + \tilde{V}_\lambda \left[ \tilde{\mu}(\theta_t) dt + \tilde{\sigma}(\theta_t) d\tilde{w}_t \right] + \tilde{V}_\theta \left[ a(\theta_L - \theta_t) dt + \pi d\hat{w}_t \right] \\
 &\quad + \frac{1}{2}\tilde{V}_{xx} \left[ k_t^2 X_t^2 (\tilde{\sigma}(\theta_t)^2 + 2\rho\sigma\tilde{\sigma}(\theta)e^{-\lambda_t} + \sigma^2 e^{-2\lambda_t}) dt \right] + \frac{1}{2}\tilde{V}_{\lambda\lambda} \tilde{\sigma}(\theta_t)^2 dt \\
 &\quad + \frac{1}{2}\tilde{V}_{\theta\theta} \pi^2 dt + \tilde{V}_{x\lambda} k_t X_t \left[ \tilde{\sigma}(\theta_t)^2 + \rho\sigma\tilde{\sigma}(\theta_t)e^{-\lambda_t} \right] dt \\
 &= \left[ \frac{\tilde{\sigma}(\theta)^2}{2} \tilde{V}_{\lambda\lambda} + \frac{\pi^2}{2} \tilde{V}_{\theta\theta} + \tilde{\mu}(\theta_t) \tilde{V}_\lambda + r X_t \tilde{V}_x - c_t X_t \tilde{V}_x + (be^{-\lambda_t} + \mu - r) k_t X_t \tilde{V}_x \right. \\
 &\quad \left. + a(\theta_L - \theta_t) \tilde{V}_\theta + k_t X_t (\tilde{\sigma}(\theta)^2 + \rho\sigma\tilde{\sigma}(\theta_t)e^{-\lambda_t}) \tilde{V}_{x\lambda} + \frac{k_t^2 X_t^2 q(\lambda_t, \theta_t)}{2} \tilde{V}_{xx} \right] dt \\
 &\quad + \sigma e^{-\lambda_t} k_t X_t \tilde{V}_x dw_t + \left[ \tilde{\sigma}(\theta_t) k_t X_t \tilde{V}_x + \tilde{\sigma}(\theta_t) \tilde{V}_\lambda \right] d\tilde{w}_t + \pi \tilde{V}_\theta d\hat{w}_t.
 \end{aligned}$$

In integral form,

$$\begin{aligned}
 e^{-\delta T} \tilde{V}(X_T, \lambda_T, \theta_T) - \tilde{V}(x, \lambda, \theta) &= \int_0^T e^{-\delta t} d\tilde{V}(X_t, \lambda_t, \theta_t) - \int_0^T \delta e^{-\delta t} \tilde{V}(X_t, \lambda_t, \theta_t) dt \\
 &= \int_0^T e^{-\delta t} \left[ \frac{\tilde{\sigma}(\theta)^2}{2} \tilde{V}_{\lambda\lambda} + \frac{\pi^2}{2} \tilde{V}_{\theta\theta} + \tilde{\mu}(\theta_t) \tilde{V}_\lambda + r X_t \tilde{V}_x - c_t X_t \tilde{V}_x \right. \\
 &\quad \left. + (be^{-\lambda_t} + \mu - r) k_t X_t \tilde{V}_x + a(\theta_L - \theta_t) \tilde{V}_\theta + k_t X_t (\tilde{\sigma}(\theta)^2 + \rho\sigma\tilde{\sigma}(\theta_t)e^{-\lambda_t}) \tilde{V}_{x\lambda} \right. \\
 &\quad \left. + \frac{k_t^2 X_t^2 q(\lambda_t, \theta_t)}{2} \tilde{V}_{xx} \right] dt - \int_0^T \delta e^{-\delta t} \tilde{V}(X_t, \lambda_t, \theta_t) dt + \int_0^T \sigma e^{-\lambda_t} k_t X_t \tilde{V}_x dw_t \\
 &\quad + \int_0^T \left( \tilde{\sigma}(\theta_t) k_t X_t \tilde{V}_x + \tilde{\sigma}(\theta_t) \tilde{V}_\lambda \right) d\tilde{w}_t + \int_0^T \pi \tilde{V}_\theta d\hat{w}_t. \tag{3.152}
 \end{aligned}$$

### 3.5. VERIFICATION THEOREM

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We showed above that

$$\begin{aligned} \delta \tilde{V}(X_t, \lambda_t, \theta_t) &= \frac{\tilde{\sigma}^2(\theta_t)}{2} \tilde{V}_{\lambda\lambda} + \frac{\pi^2}{2} \tilde{V}_{\theta\theta} + r X_t \tilde{V}_x + \tilde{\mu}(\theta_t) \tilde{V}_\lambda + a(\theta_L - \theta_t) \tilde{V}_\theta \\ &\quad + \max_{c_t \geq 0} \left\{ \frac{1}{\gamma} (c_t X_t)^\gamma - c_t X_t V_x \right\} + \max_{k_t \geq 0} \left\{ (b e^{-\lambda_t} + \mu - r) k_t X_t \tilde{V}_x \right. \\ &\quad \left. + \frac{k_t^2 X_t^2}{2} q(\lambda_t, \theta_t) \tilde{V}_{xx} + k_t X_t (\tilde{\sigma}^2(\theta_t) + \rho \sigma \tilde{\sigma}(\theta_t) e^{-\lambda_t}) \tilde{V}_{x\lambda} \right\}. \end{aligned} \quad (3.153)$$

Thus for arbitrary  $c_t, k_t \geq 0$ ,

$$\begin{aligned} \delta \tilde{V}(X_t, \lambda_t, \theta_t) &\geq \frac{\tilde{\sigma}^2(\theta_t)}{2} \tilde{V}_{\lambda\lambda} + \frac{\pi^2}{2} \tilde{V}_{\theta\theta} + r X_t \tilde{V}_x + \tilde{\mu}(\theta_t) \tilde{V}_\lambda + a(\theta_L - \theta_t) \tilde{V}_\theta + \frac{1}{\gamma} (c_t X_t)^\gamma \\ &\quad - c_t X_t V_x + (b e^{-\lambda_t} + \mu - r) k_t X_t \tilde{V}_x + \frac{k_t^2 X_t^2}{2} q(\lambda_t, \theta_t) \tilde{V}_{xx} \\ &\quad + k_t X_t (\tilde{\sigma}^2(\theta_t) + \rho \sigma \tilde{\sigma}(\theta_t) e^{-\lambda_t}) \tilde{V}_{x\lambda}. \end{aligned} \quad (3.154)$$

Then we have

$$\begin{aligned} &\frac{\tilde{\sigma}^2(\theta_t)}{2} \tilde{V}_{\lambda\lambda} + \frac{\pi^2}{2} \tilde{V}_{\theta\theta} + r X_t \tilde{V}_x + \tilde{\mu}(\theta_t) \tilde{V}_\lambda + a(\theta_L - \theta_t) \tilde{V}_\theta - c_t X_t V_x + (b e^{-\lambda_t} + \mu - r) k_t X_t \tilde{V}_x \\ &\quad + \frac{k_t^2 X_t^2}{2} q(\lambda_t, \theta_t) \tilde{V}_{xx} + k_t X_t (\tilde{\sigma}^2(\theta_t) + \rho \sigma \tilde{\sigma}(\theta_t) e^{-\lambda_t}) \tilde{V}_{x\lambda} \\ &\leq \delta \tilde{V}(X_t, \lambda_t, \theta_t) - \frac{1}{\gamma} (c_t X_t)^\gamma. \end{aligned} \quad (3.155)$$

### 3.5. VERIFICATION THEOREM

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Now (3.152) becomes

$$\begin{aligned}
& e^{-\delta T} \tilde{V}(X_T, \lambda_T, \theta_T) - \tilde{V}(x, \lambda, \theta) \\
& \leq \int_0^T e^{-\delta t} \left[ \delta \tilde{V}(X_t, \lambda_t, \theta_t) - \frac{1}{\gamma} (c_t X_t)^\gamma \right] dt - \int_0^T \delta e^{-\delta t} \tilde{V}(X_t, \lambda_t, \theta_t) dt \\
& \quad + \int_0^T \sigma e^{-\lambda_t} k_t X_t \tilde{V}_x dw_t + \int_0^T \left( \tilde{\sigma}(\theta_t) k_t X_t \tilde{V}_x + \tilde{\sigma}(\theta_t) \tilde{V}_\lambda \right) d\tilde{w}_t + \int_0^T \pi \tilde{V}_\theta d\hat{w}_t \\
& = - \int_0^T e^{-\delta t} \frac{1}{\gamma} (c_t X_t)^\gamma dt + m_T + \tilde{m}_T + \hat{m}_T,
\end{aligned} \tag{3.156}$$

where  $m_T = \int_0^T \sigma e^{-\lambda_t} k_t X_t \tilde{V}_x dw_t$ ,  $\tilde{m}_T = \int_0^T \left( \tilde{\sigma}(\theta_t) k_t X_t \tilde{V}_x + \tilde{\sigma}(\theta_t) \tilde{V}_\lambda \right) d\tilde{w}_t$ , and  $\hat{m}_T = \int_0^T \pi \tilde{V}_\theta d\hat{w}_t$  are local martingales. That is, they are martingales in the region  $\{x^2 + \lambda^2 + \theta^2 \leq R^2\}$ .

Rearranging (3.156) to get a bound on  $\tilde{V}$ , we have

$$\tilde{V}(x, \lambda, \theta) \geq \int_0^T e^{-\delta t} \frac{1}{\gamma} (c_t X_t)^\gamma dt + e^{-\delta T} \tilde{V}(X_T, \lambda_T, \theta_T) - m_T - \tilde{m}_T - \hat{m}_T. \tag{3.157}$$

Since  $m_T$  is a local martingale,  $\mathbb{E}[m_{T \wedge \tau_R}] = \mathbb{E}[m_0] = \int_0^0 \sigma e^{-\lambda_t} k_t X_t \tilde{V}_x dw_t = 0$ . Similarly,  $\mathbb{E}[\tilde{m}_{T \wedge \tau_R}] = \mathbb{E}[\hat{m}_0] = 0$ . Replacing  $T$  with  $T \wedge \tau_R$  in (3.157) and taking expectations, we arrive at the following:

$$\begin{aligned}
\tilde{V}(x, \lambda, \theta) & \geq \mathbb{E}_{x, \lambda, \theta} \left[ \int_0^{T \wedge \tau_R} e^{-\delta t} \frac{1}{\gamma} (c_t X_t)^\gamma dt \right] + \mathbb{E}_{x, \lambda, \theta} \left[ e^{-\delta T \wedge \tau_R} \tilde{V}(X_{T \wedge \tau_R}, \lambda_{T \wedge \tau_R}, \theta_{T \wedge \tau_R}) \right] \\
& \geq \mathbb{E}_{x, \lambda, \theta} \left[ \int_0^{T \wedge \tau_R} e^{-\delta t} \frac{1}{\gamma} (c_t X_t)^\gamma dt \right].
\end{aligned} \tag{3.158}$$

The second inequality is true because  $\tilde{V} > 0$ . We let  $R$  approach infinity and use Fatou's



### 3.5. VERIFICATION THEOREM

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lemma, giving us

$$\begin{aligned}\tilde{V}(x, \lambda, \theta) &\geq \lim_{R \rightarrow \infty} \mathbb{E}_{x, \lambda, \theta} \left[ \int_0^{T \wedge \tau_R} e^{-\delta t} \frac{1}{\gamma} (c_t X_t)^\gamma dt \right] \\ &\geq \mathbb{E}_{x, \lambda, \theta} \left[ \int_0^T e^{-\delta t} \frac{1}{\gamma} (c_t X_t)^\gamma dt \right].\end{aligned}$$

Now, letting  $T \rightarrow \infty$  and using Fatou's lemma again,

$$\begin{aligned}\tilde{V}(x, \lambda, \theta) &\geq \lim_{T \rightarrow \infty} \mathbb{E}_{x, \lambda, \theta} \left[ \int_0^T e^{-\delta t} \frac{1}{\gamma} (c_t X_t)^\gamma dt \right] \\ &\geq \mathbb{E}_{x, \lambda, \theta} \left[ \int_0^\infty e^{-\delta t} \frac{1}{\gamma} (c_t X_t)^\gamma dt \right].\end{aligned}\tag{3.159}$$

Since (3.159) holds for all arbitrary values of  $(k_t, c_t) \in \Pi$ , then

$$\begin{aligned}\tilde{V}(x, \lambda, \theta) &\geq \sup_{(k_t, c_t) \in \Pi} \mathbb{E}_{x, \lambda, \theta} \left[ \int_0^\infty e^{-\delta t} \frac{1}{\gamma} (c_t X_t)^\gamma dt \right] \\ &= V(x, \lambda, \theta).\end{aligned}\tag{3.160}$$

Next we must show the reverse inequality,  $\tilde{V}(x, \lambda, \theta) \leq V(x, \lambda, \theta)$ . We start by showing that  $(k_t^*, c_t^*) \in \Pi$ . Note that both  $k_t^*$  and  $c_t^*$  are progressively measurable because for every time  $t$  their values are determined.

We must show that  $Pr \left( \int_0^T (k_t^*)^2 dt < \infty \right) = 1$  and  $Pr \left( \int_0^T c_t^* dt < \infty \right) = 1$ . If  $k_t^* = 0$ , this is trivial. If  $k_t^* > 0$ ,

$$\int_0^T (k_t^*)^2 dt = \int_0^T \left( \frac{be^{-\lambda t} + \mu - r + (\tilde{\sigma}(\theta_t)^2 + \rho\sigma\tilde{\sigma}(\theta_t)e^{-\lambda t})\tilde{Z}_\lambda(\lambda_t, \theta_t)}{(1-\gamma)q(\lambda_t, \theta_t)} \right)^2 dt.$$

We know that  $q(\lambda_t, \theta_t) \geq q_0 > 0$ , so the denominator in the above expression does

### 3.5. VERIFICATION THEOREM

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not approach zero. This, along with the fact that  $\tilde{Z}_\lambda$  is well-defined, implies that  $k_t^*$  is well-defined for  $t \in [0, T]$ . Therefore

$$Pr \left( \int_0^T (k_t^*)^2 dt < \infty \right) = 1. \quad (3.161)$$

We also know that  $c_t^* = e^{\frac{\tilde{Z}(\lambda_t, \theta_t)}{\gamma-1}}$  is bounded because  $\tilde{Z}$  is bounded, so  $Pr \left( \int_0^T c_t^* dt < \infty \right) = 1$ . Using  $k_t^*$  and  $c_t^*$  instead of arbitrary  $k_t, c_t > 0$ , we have equality in (3.157):

$$\tilde{V}(x, \lambda, \theta) = \int_0^T e^{-\delta t} \frac{1}{\gamma} (c_t^* X_t)^\gamma dt + e^{-\delta T} \tilde{V}(X_T, \lambda_T, \theta_T) - m_T^* - \tilde{m}_T^* - \hat{m}_T^*, \quad (3.162)$$

where  $m_T^*, \tilde{m}_T^*$ , and  $\hat{m}_T^*$  are equal to the expressions for  $m_T, \tilde{m}_T$ , and  $\hat{m}_T$ , respectively, with arbitrary  $k_t$  and  $c_t$  replaced with  $k_t^*$  and  $c_t^*$ . These integrals are martingales, so these terms vanish when the expectation is taken in equation (3.162):

$$\tilde{V}(x, \lambda, \theta) = \mathbb{E} \left[ \int_0^T e^{-\delta t} \frac{1}{\gamma} (c_t^* X_t)^\gamma dt \right] + \mathbb{E} \left[ e^{-\delta T} \tilde{V}(X_T, \lambda_T, \theta_T) \right]. \quad (3.163)$$

It is apparent that

$$\mathbb{E} \left[ \int_0^\infty e^{-\delta t} \frac{1}{\gamma} (c_t^* X_t)^\gamma dt \right] \leq \tilde{V}(x, \lambda, \theta) < \infty \quad (3.164)$$

since  $V(x, \lambda, \theta) \leq \tilde{V}(x, \lambda, \theta)$ , as shown above. This implies that

$$\liminf_{T \rightarrow \infty} \mathbb{E} \left[ e^{-\delta T} \frac{1}{\gamma} (c_T^* X_T)^\gamma \right] = 0, \quad (3.165)$$

which is easily seen with a proof by contradiction. Note that  $c_t^*$  is bounded below by a

### 3.5. VERIFICATION THEOREM

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positive constant:

$$c_t^* \geq e^{\frac{K_2}{\gamma-1}} \equiv \underline{c} > 0.$$

Then we have

$$0 = \liminf_{T \rightarrow \infty} \mathbb{E} \left[ e^{-\delta T} \frac{1}{\gamma} (c_T^* X_T)^\gamma \right] \geq \underline{c}^\gamma \liminf_{T \rightarrow \infty} \mathbb{E} \left[ e^{-\delta T} \frac{1}{\gamma} X_T^\gamma \right] \geq 0,$$

which implies

$$\liminf_{T \rightarrow \infty} \mathbb{E} \left[ e^{-\delta T} \frac{1}{\gamma} X_T^\gamma \right] = 0. \quad (3.166)$$

Using this along with the fact that  $e^{\tilde{Z}} \leq e^{K_2}$ , we have that

$$\liminf_{T \rightarrow \infty} \mathbb{E} \left[ e^{-\delta T} \tilde{V}(X_T, \lambda_T, \theta_T) \right] = 0. \quad (3.167)$$

Now taking the liminf of (3.163) as  $T$  approaches infinity, we get

$$\begin{aligned} \tilde{V}(x, \lambda, \theta) &= \liminf_{T \rightarrow \infty} \mathbb{E} \left[ \int_0^T e^{-\delta t} \frac{1}{\gamma} (c_t^* X_t)^\gamma dt \right] \\ &= \mathbb{E} \left[ \int_0^\infty e^{-\delta t} \frac{1}{\gamma} (c_t^* X_t)^\gamma dt \right]. \end{aligned} \quad (3.168)$$

The second equality is true due to the monotone convergence theorem. Finally, by (3.168)

and the definition of  $V$ , we have

$$\tilde{V}(x, \lambda, \theta) \leq V(x, \lambda, \theta). \quad (3.169)$$

### 3.5. VERIFICATION THEOREM

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Combining (3.160) and (3.169), we have

$$\tilde{V}(x, \lambda, \theta) = V(x, \lambda, \theta).$$

□

In conclusion, we have proved that our solution to the HJB equation is equal to the value function. In effect, we have maximized the expected discounted HARA utility of consumption.

## Bibliography

- [BSW68] Bailey, P.B., Shampine, L.F., & Waltman, P.E. *Nonlinear Two Point Boundary Value Problems*. Academic Press, 1968.
- [BP99] Bielecki, T.R. & Pliska, S.R. (1999). “Risk-sensitive dynamic asset management,” *Applied Mathematics and Optimization*, Vol. 39, 337-360.
- [Et02] Etheridge, A. *A First Course in Financial Calculus*. Cambridge University Press, 2002.
- [FH03] Fleming, W.H. & Hernández-Hernández (2003). “An optimal consumption model with stochastic volatility,” *Finance and Stochastics*, Vol. 7, 245-262.
- [FH05] Fleming, W.H. & Hernández-Hernández (2005). “The tradeoff between consumption and investment in incomplete financial markets,” *Applied Mathematics and Optimization*, Vol. 52, No. 2, 219-235.
- [FP04] Fleming, W.H. & Pang, T. (2004). “An application of stochastic control theory to financial economics,” *SIAM Journal of Control and Optimization*, Vol. 43, No. 2, 502-531.
- [FP05] Fleming, W.H. & Pang, T. (2005). “A stochastic control model of investment, production and consumption,” *Quarterly of Applied Mathematics*, Vol. 63, 71-87.
- [FS00] Fleming, W.H. & Sheu, S.J. (2000). “Risk-sensitive control and an optimal investment model,” *Mathematical Finance*, Vol. 10, No. 2, 197-213.
- [FS93] Fleming, W.H. & Soner, H.M. *Controlled Markov Processes and Viscosity Solutions*. Springer-Verlag, 1993.
- [FPS00] Fouque, J.-P., Papanicolaou, G., Sircar, K.R. *Derivatives in Financial Markets with Stochastic Volatility*. Cambridge University Press, 2000.
- [Fr75] Friedman, A. *Stochastic Differential Equations and Applications, Volume 1* Academic Press, 1975.
- [Fr76] Friedman, A. *Stochastic Differential Equations and Applications, Volume 2* Academic Press, 1976.
- [GT77] Gilbarg, D. & Trudner, N.S. *Elliptic Partial Differential Equations of Second Order*. Springer-Verlag, 1977.

- [HS12a] Hata, H. & Sheu, S.J. (2012). “On the Hamilton-Jacobi-Bellman equation for an optimal consumption problem: I. Existence of solution,” *SIAM J. of Control and Optimization*, Vol. 50, No. 4, 2373-2400.
- [HS12b] Hata, H. & Sheu, S.J. (2012) “On the Hamilton-Jacobi-Bellman equation for an optimal consumption problem: II. Verification theorem,” *SIAM J. of Control and Optimization*, Vol. 50, No. 4, 2401-2430.
- [KS06] Kaise, H. & Sheu, S.J. (2006) “On the structure of solutions of ergodic type Bellman equation related to risk-sensitive control,” *The Annals of Probability*, Vol. 34, No. 1, 284-320.
- [KS91] Karatzas, I. & Shreve, S.E. *Brownian Motion and Stochastic Calculus*. Springer-Verlag, 1991.
- [Me69] Merton, R.C. (1969) “Lifetime portfolio selection under uncertainty: The continuous-time case,” *The MIT Press*, Vol. 51, No. 3, 247-257.
- [Na15] Nagai, H. (2015) “H-J-B equations of optimal consumption-investment and verification theorems,” *Applied Mathematics and Optimization*, Vol. 71, No. 2, 279-311.
- [NK11] Noh, E.J. & Kim, J.H. (2011) “An optimal portfolio model with stochastic volatility and stochastic interest rate,” *Journal of Mathematical Analysis and Applications*, Vol. 375, No.2, 510-522.
- [Ok07] Oksendal, B.K. *Stochastic Differential Equations: An Introduction with Applications*. Springer, 2007.
- [Pa02] Pang, T. *Stochastic Control Theory and its Applications to Financial Economics*. Ph.D Dissertation, Brown University. Providence, RI: Proquest/UMI, 2002. (Publication No. [3050947].)
- [Pa04] Pang, T. (2004) “Portfolio optimization models on infinite time horizon,” *Journal of Optimization Theory and Applications*, Vol. 122, No. 3, 573-597.
- [Pa92] Pao, C.V. *Nonlinear Parabolic and Elliptic Equations*. Plenum Press, 1992.
- [Wa98] Walter, W. *Ordinary Differential Equations*. Springer, 1998.
- [Za01] Zariphopoulou, T. (2001) “A Solution Approach to Valuation with Unhedgeable Risks,” *Finance and Stochastics*, Vol. 5, No.1, 61-82.