Abstract

RITTER, CARRIE LINEBERRY. A Study of Pre-service Elementary Teachers’ Mathematical Sophistication in a Reform-oriented Calculus Course. (Under the direction of chair Dr. Karen Keene).

Calls for better preparation of STEM teachers have been prominent in educational communities and among the public for the past several years (e.g. American Association of Colleges for Teacher Education, 2007). Some research suggests one way to improve mathematics instruction is to increase elementary pre-service teachers’ mathematical sophistication (Szydlik, Kuennen, and Seaman, 2009). Elementary teachers may lack the mathematical sophistication needed to understand mathematics from an advanced perspective, thus limiting their abilities to create fundamental mathematical understanding for students.

The current study answered the following research questions:

• How does pre-service elementary teachers’ mathematical sophistication change as they participate in a reform-oriented calculus course?
• What experiences do pre-service elementary teachers have during a reform-oriented calculus course that support their growth in mathematical sophistication?

It used a mixed methods approached to collect and analyze three sets of data: the pre- and post- administration of Seaman and Szydlik’s (2009) Mathematical Sophistication Instrument (MSI), nine semi-structured, one-on-one pre- and post- interviews including two tasks, and artifacts from three cognitively challenging tasks (paced throughout the calculus course). The data came from 33 students in a specialized calculus for elementary teachers course.

Results show that participation in a reform oriented calculus course has a positive effect on students’ mathematical sophistication. All students showed improvement in their
mathematical sophistication throughout the course, yet the results were not statistically significant. The most positive gain was for students who scored the lowest on the pre-MSI at the beginning of the course. Three of the MSF actions showed the most improvement throughout the duration of the course: definitions, justification and understanding, and precise language and fine distinctions about language.

Limited research on how to improve elementary teachers’ lack of mathematical sophistication exists. This study opens the doors and shows the value of inquiry-based learning with the use of high-level tasks on students’ mathematical sophistication.
A Study of Pre-service Elementary Teachers’ Mathematical Sophistication in a Reform-oriented Calculus Course

by

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A dissertation submitted to the Graduate Faculty of North Carolina State University in partial fulfillment of the requirements for the degree of Doctor of Philosophy

Mathematics Education

Raleigh, North Carolina

2015

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Dedication

For my grandmothers, Carrie Mae Reece and Grace Lineberry, who have loved and supported me throughout my life. Both of them passed while I was finishing my PhD in Mathematics Education. Love you both.
Biography

Carrie Lineberry Ritter knew from a very young age that she wanted to be a teacher. Mathematics has always been her passion. As a very young child she used to “play school” with her cousins in her “classroom” in an old barn. In high-school she completed an internship in a sixth-grade mathematics classroom at a nearby elementary school. This experience confirmed her passion for teaching mathematics.

She attended Appalachian State University in Boone, NC and graduated with a Bachelor of Science degree in Secondary Mathematics Education. She then started teaching high-school mathematics. After a year and a half in the classroom, she decided to continue her education. She moved to Raleigh, NC and started attending North Carolina State University. While continuing to teach high-school she completed a Master of Science degree in Mathematics Education.

She then stepped out of the high-school classroom to start pursuing a PhD in Mathematics Education from North Carolina State University. While completing her degree she had multiple opportunities to dive into mathematics education. She taught 2 years of introductory mathematics methods courses, a variety of courses at the community college level, and 2 years of reform-oriented calculus course for pre-service elementary mathematics teachers. Carrie Lineberry Ritter now teaches full-time at Shaw University in Raleigh, NC. She is revamping developmental mathematics courses for conceptual understanding of mathematics. She is also the mathematics program coordinator.
Acknowledgements

Many thanks to:

My God: For all the answered prayers, support and perseverance throughout my entire life.

My Advisor and dissertation chair: Dr. Karen Allen Keene for her continual encouragement, time, feedback, support and guidance throughout my dissertation process and completion.

Members of my committee: Dr. Allison W. McCulloch, Dr. Hollylynne Lee, Dr. Alina Duca, Dr. Jennifer Szydlik, and Dr. Carol Seamen for their leadership, guidance, support and insight throughout my dissertation process and completion.

My professors: Dr. Karen Allen Keene, Dr. Hollylynne Lee, Dr. Allison W. McCulloch, Dr. Jere Confrey, Dr. Karen Hollebrands, and Dr. Lee Stiff for all of the knowledge and experiences I have been fortunate to be apart of throughout my PhD coursework.

My husband and best friend: Dustin, without whose love, patience and support this would not have been possible.

My parents: Jeffrey Michael and Terina Lineberry: without whose love, patience and support this would not have been possible.

My grandfather: Roger Reece, whose continual love and support keeps me progressing.

My family: Wesley and Bobbie Jo Ritter; Curley and Joy Lineberry; Daniel Ritter; James and Lois Ritter; Michael, Julie, Audrey and Molly Harris; Mike, Robin, and Melanie Garner; Grandma Brown and Grandma Daniels for all of the love, patience, and support throughout my dissertation process and completion.
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Chapter 1 - Introduction

Over the past 40 years, research has shown that many elementary students in the U.S. fare poorly when asked to perform straightforward computational procedures (Ball, Thames, and Phelps, 2008). However, their performance is even poorer when asked to apply their knowledge to solve word problems. Carrying out computational procedures is an important aspect of mathematics learning; yet, pre-service elementary teachers need to push students beyond procedural understanding. They need to support students’ interest and curiosity in mathematics, their ability to solve challenging problems, their ability to think critically about mathematics, and their development in terms of building a conceptual understanding of the mathematics topics to be learned. In order for teachers to attempt implementing the reform needed for this, mathematics teacher education must change as well (Nicol, 1999; Ball, Thames, and Phelps, 2008).

The mathematical preparation of elementary teachers varies widely in the U.S.; both in the number and in the kinds of courses teachers take (NCTQ, 2008). The NRC (2001) states the importance of pre-service elementary teachers’ preparation in order to create teachers who can do more than help young children memorize basic facts and computation procedures. The Conference Board of the Mathematical Sciences (CBMS, 2001) recommends that elementary teachers’ knowledge include the fundamental ideas of elementary mathematics such as numbers and operations; algebra and functions; geometry and measurement; and data analysis, statistics and probability. This study is situated in a pre-service course that goes beyond all of these areas and provides pre-service teachers the opportunity to learn calculus and connect the concepts to elementary mathematics. The goals
of the course include teaching pre-service elementary teachers the skills needed to critically think about mathematics, solve novel tasks as well as increase their conceptual understanding of the mathematics they will eventually teach. One goal of this study will be to evaluate the mathematical growth of these pre-service teachers using a new construct, mathematical sophistication.

**Background**

The quality of mathematics education in the United States, often measured by less than stellar student achievement, has received criticism for several decades (Thames & Ball, 2013). There are many calls for improvement of K-12 teaching and learning of mathematics in order to increase student achievement (Ball, 1990; Ball, Hill & Bass, 2005; Schoenfeld, 1992; Smith & Stein, 1998). Mathematics teaching is a multidimensional endeavor requiring teachers to have suitable levels of content knowledge, pedagogical-content knowledge and pedagogical knowledge (Graham & Fennell, 2001). The National Council of Teachers of Mathematics, NCTM (2000), challenges educators to teach students through problem solving, specifically using cognitively challenging tasks. Universities have struggled with a way to provide experiences for pre-service elementary teachers that can help them develop a deep understanding of the mathematics they need to implement challenging tasks in their classrooms (Thames & Ball, 2013).

Thames and Ball (2013) list some widely publicized strategies for improving mathematics that were implemented in the past:

- Teacher-proof instruction
- Install a more challenging curriculum
- Increase accountability
• Reorganize schools
• Pay teachers more, and
• Recruit talented teachers by lowering the barriers for entry (p. 33).

It is ironic that the word “mathematics” does not appear in any of these strategies. Each of these six strategies alone cannot accomplish everything because none of these strategies gets directly at instruction. There is considerable research about what elementary mathematics teachers need to know in order to conceptually teach mathematics and ultimately improve student performance (Hill, Ball & Schilling, 2008). Yet, “there is little consensus among mathematics teacher educators about the goals, structure, or specific activities and assignments that comprise the pre-service mathematics “methods” courses” (Romagnano, Evans and Gilmore, 2008, p. 104) or pre-service elementary teachers’ mathematics content courses (Begle, 1972; Ebby, Ottinger & Silver, 2007; Eisenberg, 1977).

Thames and Ball (2013) point out that the pre-service elementary teacher education curriculum has not changed at most institutions of higher education since 1940. Simon (2013) hypothesized that elementary teacher education research lacks an underling theory of pedagogy. This may mean pre-service teachers receive little to no specific instruction on how to promote a way of understanding mathematical concepts needed for teaching children mathematics effectively at most institutions. In order to test this hypothesis, Simon looked at the titles and abstracts of all articles published in the Journal of Mathematics Teacher Education from 2008 - 2010. Simon concluded that none of the papers reviewed studied how to foster mathematical concept learning and skills. “By developing a shared consensus on students’ learning goals, by changing beliefs about how teaching improves, and by implementing a set of practices focused on teacher learning, improvements in teaching could
begin to emerge and even accumulate over time (Hiebert, 2013, p. 54). There is a need for research on how pre-service elementary teachers gain the skills needed to understand mathematical concepts needed for teaching children mathematics.

In order for pre-service teachers to reach a goal of teaching and learning consistent with the mathematics community “undergraduate courses for PSTs must offer ‘time and opportunity to think about, discuss, and explain mathematical ideas’ and the chance to learn ‘to treat mathematics as a sense-making enterprise’” (Conference Board of Mathematical Sciences, 2012, p. 24). One possibility is to help students develop mathematical habits of mind while they learn mathematics at the undergraduate level (Laursen, Hassi, & Hough, 2015). According to Laursen, Hassi, and Hough (2015), there are several mathematical habits of mind that pre-service elementary mathematics teachers must develop in order to help their students build the conceptual understanding needed to reason in mathematics. They need to have opportunities to look deep behind the mathematics and at the core concepts, represent mathematics in multiple ways and with multiple representations, noting and interpreting patterns, while posing deep conceptual questions (Laursen, Hassi, & Hough, 2015).

Cuoco, Goldenberg and Mark (1996) explain the special ways mathematicians think about mathematics as follows:

“In broad strokes, this includes learning to recognize when problems or statements that purport to be mathematical are, in truth, still quite ill-posed or fuzzy, becoming comfortable with the skilled at bringing mathematical meaning to problems and statements through definition, systemization, abstraction, or logical connections making; and seeking and developing new ways of describing situations” (p. 376).

The Mathematical Education of Teachers II (MET II) (2010) also stresses the importance of teachers having experience with the following practices common in the mathematical
community. They suggest the following practices “monitoring their own progress as they solve problems, attending to precision, constructing viable arguments, seeking and using mathematical structure, and making strategic use of appropriate tools, e.g., notations, diagrams, graphs, or procedures (whether implemented by hand or electronically)” (p. 1).

Along the same lines, Szydlik, Kuennen, and Seaman (2009) have termed the word *mathematical sophistication* to describe the “internalization of the values, practices, and habits of mind of the mathematical community that are powerful in learning mathematics” (Szydlik, Kuennen, and Seaman, 2009, p. 4). This idea is elaborated later in the study.

**Statement of the Problem**

Wu (2006) suggests the value of the collaboration between mathematicians and mathematics educators in order to provide mathematics courses for pre-service teachers that help them develop mathematical thinking skills. The MET II (2010) explains how pre-service elementary teachers should be taught the practices common within the mathematical community while learning mathematics themselves. The methods in that practice should provide teachers with a new perspective on how mathematics should be taught. This is vital since “it is not enough for teachers to rely on their past experiences as learners of mathematics” to effectively teach mathematics at the K-5 level (MET II, 2010, p. 23). The MET II (2010) also stresses the importance that courses be designed for pre-service elementary teachers that blend mathematical content with effective pedagogy.

There is a definite gap in knowledge of how pre-service elementary teachers’ mathematical skills are similar to those of mathematicians and how their mathematical sophistication may grow in a reform-oriented calculus course. Thus, a study that does this is
relevant. Such a study will provide new knowledge needed to help can improve students’ advanced mathematical thinking and possibly improve pre-service teachers’ ability to teach elementary mathematics.

Mathematicians and mathematics educators worked together to create the curriculum used in this study, “striking a balance between the rigorous mathematics of an advanced perspective while taking advantage of lessons from education research is a primary objective of our research” (Keene, Hall, Duca, 2012). This joint effort resulted in a course designed with the intent to improve pre-service elementary teachers’ mathematical knowledge as well as their mathematical sophistication.

The developers of the course content assert that current content courses for pre-service elementary teachers do not take STEM-focused elementary teachers far enough in their college-level mathematical journey as they build fundamental understanding of elementary mathematics (Ma, 1999). Thus, this course addresses a timely need for those who conceive of elementary teachers as a special group of STEM professionals who play a key role in increasing student mathematical understanding (Keene, Hall, & Duca, 2012).

**Research Questions**

Considering the current demand for mathematics education reform, a study about how to improve pre-service elementary teachers’ abilities to approach mathematical tasks from an advanced perspective is instrumental. The current study aimed to answer the following questions:

- How does pre-service elementary teachers’ mathematical sophistication change as they participate in a reform-oriented calculus course?
• What experiences do pre-service elementary teachers have during a reform-oriented calculus course that support their growth in mathematical sophistication?

The term mathematical sophistication is relatively new in the research community. The term describes the “internalization of the values, practices, and habits of mind of the mathematical community that are powerful in learning mathematics” (Szydlik, Kuennen, and Seaman, 2009, p. 4).

**Significance of the Study**

Finding the answer to these research questions will contribute to the research focused on the education of a new generation of STEM-focused elementary teachers. Consequently, this study may affect and transform STEM elementary education—from teacher preparation to student achievement in mathematics. Pre-service elementary teachers’ mathematical sophistication may improve so they may be “better equipped to engage their students in mathematical practices like those called for in the Common Core State Standards (CCSS, 2012, p. 6-8) than those with lower levels of sophistication” (Szydlik et al., unpublished, p. 23). Further, a more advanced and sophisticated mathematics course for elementary pre-service teachers that is developed around ideas of calculus can attract more and better-prepared candidates for STEM elementary education programs.

**Outline of the Dissertation**

This dissertation is organized in five chapters. Chapter 1 explains the background of the study, the purpose and research questions for the study as well as the significance of the study. Chapter 2 reviews relevant research concerning the area of focus. Chapter 3 outlines the research methodology, including: the setting of the study, the participants involved,
sources of data collection, methods of data analysis, research validity and reliability and ethical issues. Chapter 4 presents the results of both quantitative and qualitative analysis. Lastly, Chapter 5 discusses the limitations of the study, answers the research questions, and provides implications for pre-service elementary teachers’ mathematics education as well as proposed future research.
Chapter 2 - Literature Review

This study aimed to investigate how a reform-oriented calculus course affected pre-service elementary teachers’ mathematical sophistication. This chapter provides context for and justifies the study by reviewing the literature on three major topics: elementary pre-service teachers’ mathematics education and developing mathematical knowledge for teaching, habits of mind (and other constructs similar to this notion) and calculus. It also explains how each topic supports and connects to this study about improving pre-service teachers’ mathematical sophistication in a reform-oriented calculus course.

Pre-service Teachers Mathematics Education/Developing Knowledge for Teaching

Why is there such interest in pre-serviced teachers’ mathematical preparation? Recently, children studying complex situations and problems at the elementary school level have raised a need for a focus on new student abilities, such as conceptualization, collaboration, and communication (Mousoulides, 2013). Scholars have made connections between specific characteristics of teachers and the students in the teacher’s classroom regarding mathematical understanding and performance (Carpenter et al., 1996; Polly, 2008; Stigler and Hiebert, 1999). Researchers have argued that students’ understanding is positively affected by:

- teachers’ ability to effectively use cognitively challenging mathematical tasks (Polly and Hannafin, 2011; Stein et al. 2007),
- teachers’ ability to elicit students’ mathematical communication during small group and class discussions (Hufferd-Ackles et al. 2004),
- teachers’ ability to use their knowledge of students’ thinking to improve students understanding (Carpenter et al., 1996),
- teachers’ knowledge and understanding of the mathematics content they are teaching (Hill et al., 2005),
- teachers’ level of mathematical sophistication (Seaman and Szydlik, 2007).
Previous studies have shown that pre-service teachers have not participated in adequate opportunities to develop the knowledge mentioned above in order to help students understand the mathematics curriculum at all levels of mathematics education (Ball, Lubienski, and Mewborn, 2001). Research on how to help deepen pre-service teachers’ understanding of mathematics has reported several possibilities. One plausible idea is to have pre-service teachers take more mathematics courses. However, Begle (1972) found little connection between the number of mathematics courses taken by pre-service teachers’ and the performance of their students. Five years later, Eisenberg (1977) found similar results.

Researchers agree also that learning to teach mathematics for understanding is more than learning new pedagogical techniques and taking higher-level mathematics education courses (Ebby, Ottinger & Silver, 2007). In order to prepare teachers to implement reform-oriented curriculum, they must be engaged as learners and inquirers of the mathematical content, student learning, and pedagogical techniques.

Fennema and Franke (1992) explain that teachers’ knowledge of content is a combination of the content, pedagogy, students’ cognitions and teachers’ beliefs. Shulman (1986) explains that “the key to distinguishing the knowledge base of teaching lies at the intersection of content and pedagogy, in the capacity of a teacher to transform the content knowledge he or she possesses into forms that are pedagogically powerful and yet adaptive to the variations in ability and background presented by the students” (p. 15). In order for teachers to successfully transform and teach mathematics for student understanding, they must themselves deeply understand the mathematics at hand.
Research has shown that one way to help pre-service elementary teachers build mathematical content knowledge is to teach them how to think about mathematics from a different perspective, that of the mathematician. Elementary teachers often lack the mathematical behavior of mathematicians and the ability to understand mathematics from an advanced perspective. Seaman & Szydlik (2007) found that “teachers display a set of values and avenues for learning mathematics that is so different from that of the mathematical community and so impoverished, that their attempts to create fundamental mathematical understandings are often meet with little success” (p. 179). Neither mathematical sophistication nor mathematical content knowledge needed for teaching is easily achieved (Seaman & Szydlik, 2007). “Research is needed to document the processes of this enculturation of faculty in both communities as well as to describe classroom and other experiences that increase mathematical sophistication among [pre-service elementary] students” (Seaman & Szydlik, 2007, p. 180).

**Mathematical Sophistication**

The construct of mathematical sophistication is relatively new in the research community. The term describes the “internalization of the values, practices, and habits of mind of the mathematical community that are powerful in learning mathematics” (Szydlik, Kuennen, and Seaman, 2009, p. 4). Mathematical sophistication are specific actions used to understand students’ participation in mathematics. It is one construct that can be used to explain students participation in mathematics. According to Yackel & Cobb (1996), mathematical sophistication can be a result of the classroom culture and experiences. Seaman and Szydlik (2007) explain that the difference between a sophisticated mathematics student
and a naïve mathematics student “lies in her beliefs about the nature of mathematical behavior, her values concerning what it means to know mathematics, and particularly in her avenues of experiencing mathematical objects and her distinctions about language” (p. 170).

While being mathematically sophisticated can help one improve their knowledge of mathematics, it does not indicate an understanding of any specific mathematical procedure, concept, definition or object (Szydlik, Kuennen, and Seaman, 2009). It means that one possesses the values and skills of the mathematics community that allow one to develop one’s own mathematical understanding. Although the term mathematical sophistication is relatively new, significant research has been conducted on the skills needed to learn mathematics.

**Pre-Service Elementary Teachers’ Mathematical Knowledge for Teaching**

How to best teach pre-service teachers the mathematical knowledge needed for teaching is a subject of debate within the field. In the past, researchers equated teachers’ knowledge with certification and courses taken in teacher education program. As a result, one set of researchers found that when pre-service teachers take mathematics and mathematics education courses, their students benefit by about a 3% increase in their annual achievement gains (Kennedy, Ahn, & Choi, 2008). Monk (1994) contends that there is a reduction in the return for taking more and more courses and that the benefits of courses in mathematics reach a ceiling after about five mathematics courses. Additionally, Swars et al. (2007) found “little correlation between the number of higher mathematics courses a teacher takes and student learning” (p. 327). Therefore, mathematical content needed for teaching is necessary, yet it is not sufficient.
More recently, mathematics education research has moved past counting the number of courses that teachers should take and toward the study of teachers’ mathematical knowledge for teaching (Ball, Hill & Bass, 2005). As defined, this knowledge includes not only what is considered common content knowledge, i.e., knowledge of mathematics that is needed by all educated people, but also specialized content knowledge, i.e., knowledge of mathematics that is specific to the needs of teachers (Ball, Thames & Phelps, 2008). Research has shown that elementary teachers who demonstrate specialized content knowledge positively influence student achievement (Hill, Rowan & Ball, 2005). In fact, the National Mathematics Advisory Panel (NMP) noted that a critical component of their recommendation was “that teachers be given ample opportunities to learn mathematics for teaching. That is, teachers must know in detail and from a more advanced perspective the mathematical content they are responsible for teaching and the connections of that content to other important mathematics, both prior to and beyond the level they are assigned to teach” (National Mathematics Advisory Panel, 2008, p. xx).

Teachers often teach as they were taught. Ball, Thames, and Phelps (2008) report that “subject matter courses in teacher preparation programs tend to be academic in both the best and worst sense of the word, scholarly and irrelevant, either way remote from classroom teaching” while studying the nature of actual mathematics teaching (p. 404). Advanced mathematics courses are often taught procedurally and without adequate focus on the conceptual understanding of the concepts and skills pre-service teachers need to teach mathematics successfully. Pre-service elementary teachers need to learn to draw rich connections, conceptually understanding mathematical concepts, their underlying structures,
and be able to model these structures to successfully teach mathematics. Encouraging pre-service teachers to view mathematics in a different way than they experienced mathematics throughout their educational career is challenging.

If teachers teach the way they are taught, then reform in pre-service elementary teachers’ content courses need to include the mathematical skills Ball (1993) and Lampert (2000) have found to be important. Ball (1993) and Lampert (2000) stress the importance of pre-service elementary teachers’ knowledge going beyond mathematics content. They explain that teachers needs to make rich connections between different mathematical concepts, understand and use adequate mathematical notation, and model the behaviors of successful mathematics problem solving (sense making, conjecturing and reasoning). Goodson-Espy et al. (2014) explain that the discussion of both content and pedagogies must take place in a meaningful context. They say, “It is insufficient to discuss topics such as problems solving, development of students’ conceptions, and assessment of mathematical learning with PSTs [Pre-Service Teachers] in abstraction” (p. 394). Teaching a pre-service teacher the skills of those in the mathematics community could make such mathematical skills more accessible to their students.

The current study takes place in a reform oriented calculus class that integrates elementary mathematics concepts and builds on pre-service elementary teachers’ previous knowledge. It was an inquiry-based learning environment to teach pre-service teachers’ mathematics they can replicate in their future classrooms.
Inquiry-Based Learning

According to constructivist learning theories, students construct their own understanding through interaction and experiences; understanding cannot merely be passed from the instructor to the student (Saunders, 1992). In the present study, the definition of inquiry-based learning (IBL) was adopted from Linn and colleagues’ (2004) as “the intentional process of diagnosing problems, critiquing experiments, and distinguishing alternatives, planning investigations, researching conjectures, searching for information, constructing models, debating with peers, and forming coherent arguments” (p. 4). Inquiry-based learning (IBL), particularly at the university level, is one approach to teaching mathematics that offers these deep learning opportunities (Prince & Felder, 2007; Yoshinobu & Jones, 2013). Not all scholars agree with constructivist learning theories. For example, Kirschner, Sweller, and Clark (2006) argue that constructivist approaches lack proper guidance and ignore students’ long-term memory while abusing students’ working memory. However, Hmelo-Silver, Duncan, and Chinn (2007) disagree with Kirschner, Sweller, and Clark (2006) and argue that their evaluations of constructivist techniques are inadequate. Hmelo-Silver, Duncan, and Chinn (2007) explain that constructivist guided approaches such as inquiry-based instruction and problem-based learning are guided forms of instruction. They also explain that both of these approaches are similar in nature and are often considered the same approach even though different educational avenues contribute to their development.

A variety of researchers encourage the use of inquiry-based mathematics teaching in which students are challenged to solve novel mathematics tasks through investigation by
building on students’ initial ideas (Chapman, 2011; Kazempour & Amirshokoohi, 2013; Kogan & Laursen, 2014; Jaworski, 2006; Stein, Engle, Smith, & Hughes, 2008; Van de Walle, 2004). Inquiry-based learning can be linked to Dewey (1905) who supported “active, persistent, and careful consideration of any belief or supposed form of knowledge in the light of the grounds that support it, and the further conclusions to which it tends” (p. 6). More recently, Wells (1999) defines dialogical inquiry as:

A willingness to wonder, to ask questions, and to seek to understand by collaborating with others in the attempt to make answers to them. At the same time, the aim of inquiry is not “knowledge for its own sake” but the disposition and ability to use the understandings so gained to act informally and responsibly in the situations that may be encountered both now and in the future (p. 121).

Proper inquiry-based learning involves instructors facilitating a carefully scaffolded lesson in which their students are challenged to develop their own new mathematical knowledge (Hmelo-Silver, Duncan, & Chinn, 2007). Inquiry-based learning is a strongly guided learning environment that can involve direct instruction in the context of inquiry learning. According to Stonewater (2005), in inquiry-based learning, students are expected to create conjectures, analyzing conjectures, communicate, work in teams and engage in mathematical argument. Therefore, inquiry-based learning supports NCTM (2000), CCSSM (2010) and other initiatives to support mathematics education reform.

Encouraging teachers to implement inquiry-based learning approaches into their classrooms is pointless if they lack an understanding of what teaching and learning in an inquiry-based classroom looks like (Kazempour, 2013). “Therefore, it is imperative that teacher candidates, particularly elementary pre-service teachers (PSTs), gain firsthand
experience and develop a more in-depth understanding of inquiry-based learning as part of their training” (Kazempour, 2013, p. 143-144).

**Cognitively Challenging Tasks**

The association between types of tasks to which students are exposed while learning mathematics and the level of mathematics they learn has been researched for many years (Arbaugh & Brown, 2005; Doyle, 1988; Hiebert & Wearne, 1993; Jackson et. al., 2013; Smith & Stein, 1998; Stein & Lane, 1996). “A mathematical task is defined as a classroom activity, the purpose of which is to focus students’ attention on a particular mathematical idea” (Stein et al., 2000, p. 460). The importance of mathematical tasks has been emphasized in the *Principles and Standards for School Mathematics*:

> In effective teaching, worthwhile mathematical tasks are used to introduce important mathematical ideas and to engage and challenge students intellectually. Well-chosen tasks can pique students’ curiosity and draw them into mathematics. … Regardless of the context, worthwhile tasks should be intriguing; with a level of challenge that invites speculation and hard work (NCTM, 2000, p. 16-17).

Researchers have classified tasks in three different manners; by process, by cognitive demand, and by their worth. Multiple studies have supported using tasks that are higher in cognitive demand than most used in today’s classrooms (Smith & Stein, 1998; Bloom, 2007; Stanley and Sundström, 2007). Cognitively challenging tasks have many names: worthwhile tasks (NCTM, 2000), extended constructed-response tasks (Lester, 1994), extended analysis tasks (Bloom, 2007), high-level tasks (Smith & Stein, 1998), and more.

This study uses the construct of high-level tasks. “High-level tasks tend to be less structured, more difficult, and longer than the kinds of tasks to which students are typically exposed. Students often perceive these types of tasks as ambiguous and/or risky because it is
not apparent what they should do, how they should to it, and how their work will be evaluated” (Stein, 2000, p. 19). Smith and Stein (1998) created the Mathematical Task Framework (MTF) to classify tasks by their cognitive difficulty. There are four categories, the first two discussing low-level tasks and the last two describing high-level tasks: memorization, procedures without connections to concepts or meaning, procedures with connections to concepts and meaning, and doing mathematics, shown in Table 1.

**Table 1 - Mathematical Task Framework (MTF) (Smith & Stein, 1998).**

<table>
<thead>
<tr>
<th>Lower-Level Demands (Memorization)</th>
<th>Higher-Level Demands (Procedures with Connections)</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Involve reproducing previously learned facts, rules, formulas, or definitions or committing facts, rules, formulas, or definitions to memory.</td>
<td>• Focus students’ attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas.</td>
</tr>
<tr>
<td>• Cannot be solved using procedures because a procedure does not exist or because the time frame in which the task is being completed is too short to use a procedure.</td>
<td>• Suggest explicitly or implicitly pathways to follow that are broad general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms that are opaque with respect to underlying concepts.</td>
</tr>
<tr>
<td>• Are not ambiguous. Such tasks involve the exact reproduction of previously seen material, and what is to be reproduced is clearly and directly stated.</td>
<td>• Usually are represented in multiple ways, such as visual diagrams, manipulatives, symbols, and problem situations. Making connections among multiple representations helps develop meaning.</td>
</tr>
<tr>
<td>• Have no connection to the concepts or meaning that underlie the facts, rules, formulas, or definitions being learned or reproduced.</td>
<td>• Require some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly. Students need to engage in conceptual ideas that underlie the procedures to complete the task successfully and that develop understanding.</td>
</tr>
</tbody>
</table>
Table 1 - Continued.

<table>
<thead>
<tr>
<th>Lower-Level Demands (Procedures without Connections)</th>
<th>Higher-Level Demands (Doing Mathematics)</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Are algorithmic. Use of the procedure either is specifically called for is evident from prior instruction, experience, or placement of the task.</td>
<td>• Require complex and non-algorithmic thinking—a predictable well-rehearsed approach or pathway is not explicitly suggested by the task, task instructions, or a worked-out example.</td>
</tr>
<tr>
<td>• Require limited cognitive demand for successful completion. Little ambiguity exists about what is needed to be done and how to do it.</td>
<td>• Require students to explore and understand the nature of mathematical concepts, processes, or relationships.</td>
</tr>
<tr>
<td>• Have no connection to the concepts or meaning that underlie the procedure being used.</td>
<td>• Demand self-monitoring or self-regulation of one’s own cognitive processes.</td>
</tr>
<tr>
<td>• Are focused on producing adequate answers instead of developing mathematical understanding.</td>
<td>• Require students to access relevant knowledge and experiences and make appropriate use of them in working through the task.</td>
</tr>
<tr>
<td>• Require no explanations or explanations that focus solely on describing the procedure that was used.</td>
<td>• Require students to analyze the task and actively examine task constraints that may limit possible solution strategies and solutions.</td>
</tr>
</tbody>
</table>

Memorization and procedures without connections to concepts or meaning are both lower in cognitive demand. Memorization describes a task involving facts about a certain concept that can be easily recalled without having any conceptual understanding of that concept. For example, asking students to state the formula for the Pythagorean Theorem. Procedures without connections to concepts or meaning refer to tasks in which students use an algorithm or some procedure easily specified or evident from previous experience to solve the problem.
For example, continuing from the previous example, students are given the value for A and B and asked to solve for C using the Pythagorean Theorem.

*Procedures with connections to concepts or meaning* and *doing mathematics* are both higher in cognitive demand. *Procedures with connections to concepts or meaning* refer to tasks that involves the use of procedures to develop a meaningful connection to other topics while developing a deeper understanding of the mathematical concept being covered. For example, students can be given the dimensions of a building, a ladder and asked to find the missing ground length. The highest degree of cognitive demand for a task is *doing mathematics*; *Doing mathematics* requires students to use complex and non-algorithmic thinking to come up with an approach to solve the problem in a way that has not been previously rehearsed. An example for this level can be asking students to prove the Pythagorean Theorem. Rich mathematical tasks are those that ask students to use non-algorithmic thinking, make relationships among various mathematical concepts, to use considerable cognitive effort and monitor their own progress (Smith & Stein, 1998).

Doyle (1988) explained the importance of using high-level tasks in mathematics research. He explored tasks in elementary mathematics classes and found that student learning and teacher expectations were dependent upon the types of tasks assigned. He created two categories to discuss types of tasks: “familiar” (low-level) and “novel” (high-level). Hiebert and Wearne (1993) conducted a study in which six second-grade classrooms were observed with 12 weeks of instruction with high-level tasks. Students being observed were given fewer tasks and spent more time with each task to build relationships between solutions, representations and other mathematics. Students ended up analyzing and reasoning
more about mathematics in this environment. Experience with high-level tasks will ultimately affect teachers teaching abilities because they have higher-levels of pedagogical content knowledge (Bloom, 2007). This will also help student performance since research shows that students who have teachers with higher pedagogical content knowledge do in fact perform better (Karp, 2010).

“High-performing countries avoided reducing mathematics tasks to mere procedural exercises involving basic computational skills, and they placed greater cognitive demands on students by encouraging them to focus on concepts and connections among those concepts in their problem-solving” (Resnick & Zurawsky, 2006, p. 2). If students are exposed to tasks with higher cognitive demand, even in the early grades, future students will become high mathematics achievers and better reflect the country’s population. Teaching pre-service elementary teachers conceptually with the use of high-level tasks in an inquiry-based setting is supported by research and a valuable endeavor. This study provides evidence of the ways that students learn using high level tasks is useful.

**Habits of Mind, Ways of Reasoning and Mathematical Sophistication**

A habit of mind is a combination of skills, attitudes, prior knowledge, reminders, and inclinations (Costa & Kallick, 2008). A habit of mind could be one-way of solving a problem or approach to a task is preferred over another; therefore, it means that one must make a choice over which approach is appropriate in different circumstances. For example, a habit of mind could be that one must pay attention to prior knowledge that applies to different circumstances and use this knowledge whenever appropriate. It requires a level of knowledge and understanding to use and carry out needed behaviors effectively.
**Table 2 - Costa and Kallick's (2008) 16 Habits of Mind.**

<table>
<thead>
<tr>
<th>Habits of Mind</th>
<th>Summary</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Persisting</td>
<td>Stick to it. See a task through to completion, and remain focused.</td>
</tr>
<tr>
<td>2 Managing impulsivity</td>
<td>Take your time. Think before you act. Remain calm, thoughtful, and deliberate.</td>
</tr>
<tr>
<td>3 Listening with understanding and empathy</td>
<td>Seek to understand others. Devote mental energy to another person’s thoughts and ideas. Hold your own thoughts in abeyance so you can better perceive another person’s point of view and emotions.</td>
</tr>
<tr>
<td>4 Thinking flexibly</td>
<td>Look at a situation another way. Find a way to change perspectives, generate alternatives, and consider options.</td>
</tr>
<tr>
<td>5 Thinking about thinking (metacognition)</td>
<td>Know your knowing. Be aware of your own thoughts, strategies, feelings, and actions - and how they affect others.</td>
</tr>
<tr>
<td>6 Striving for accuracy</td>
<td>Check it again. Nurture a desire for exactness, fidelity, craftsmanship, and truth.</td>
</tr>
<tr>
<td>7 Questioning and posing problems</td>
<td>How do you know? Develop a questioning attitude, consider what data are needed, and choose strategies to produce those data. Find problems to solve.</td>
</tr>
<tr>
<td>8 Applying past knowledge to new situations</td>
<td>Use what you learn. Access prior knowledge, transferring that knowledge beyond the situation in which it was learned.</td>
</tr>
<tr>
<td>9 Thinking and communicating with clarity and precision</td>
<td>Be clear. Strive for accurate communication in both written and oral form. Avoid overgeneralizations, distortions, and deletions.</td>
</tr>
<tr>
<td>10 Gathering data through all senses</td>
<td>Use your natural pathways. Gather data through all the sensory paths: gustatory, olfactory, tactile, kinesthetic, auditory, and visual.</td>
</tr>
<tr>
<td>11 Creating, imagining, innovating</td>
<td>Try a different way. Generate novel ideas, and seek fluency and originality.</td>
</tr>
<tr>
<td>12 Responding with wonderment and awe</td>
<td>Let yourself be intrigued by the world’s phenomena and beauty. Find what is awesome and mysterious in the world.</td>
</tr>
<tr>
<td>13 Taking responsible risks</td>
<td>Venture out. Live on the edge of your competence.</td>
</tr>
<tr>
<td>14 Finding humor</td>
<td>Laugh a little. Look for the whimsical, incongruous, and unexpected in life. Laugh at yourself when you can.</td>
</tr>
<tr>
<td>15 Thinking interdependently</td>
<td>Work together. Truly work with and learn from others in reciprocal situations.</td>
</tr>
<tr>
<td>16 Remaining open to continuous learning</td>
<td>Learn from experiences. Be proud - and humble enough - to admit you don’t know. Resist complacency.</td>
</tr>
</tbody>
</table>
A habit of mind could be that once someone has solved a certain problem, the solution to this problem is reflected upon, evaluated, modified, and used in future situations. A set of behaviors that help define and discipline knowledge is needed to help students think powerfully about topics, be able to critique their own thinking as well as others, and be able to become thoughtful problem solvers and decision makers (Costa & Kallick, 2008). Developing powerful habits of mind is extremely important for all students because they will most likely finish school facing problems that do not exist in today’s world. Therefore, it is vital to teach students these habits since teachers cannot prepare students for problems they have not seen. Table 2 contains the 16 Habits of Mind described by Costa and Kallick (2008).

In order for teachers to teach general habits of mind to their students, they must first learn them. Teachers cannot teach skills they do not possess. Therefore, those who are creating curriculum for pre-service teachers need to incorporate habits of mind in their general education courses or work across curriculums to have them incorporated into their content courses. Teaching a curriculum around these general habits of mind may help students more than the content covered in the course itself because the course content may become obsolete but the ways in which people think about the topic will not (Cuoco, Goldenberg, & Mark, 1996).

Since this review focuses on the importance of thinking as a mathematician, the next few paragraphs will explain the General Habits of Mind within the context of mathematics as defined by Cuoco, Goldenberg and Mark (1996) (see Table 3).
Table 3 - *General Habits of Mind* (Cuoco, Goldenberg, & Mark, 1996).

<table>
<thead>
<tr>
<th>General Habits of Mind</th>
<th>Description within the Context of Mathematics</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Students Should be Pattern Sniffers</strong></td>
<td>Teachers must help students foster a desire for finding hidden patterns within mathematics. For example, teachers should create an environment for students to solve such problems as “Which primes are the sum of two squares?”.</td>
</tr>
<tr>
<td><strong>Students Should be Experimenters</strong></td>
<td>Teachers should provide students with the opportunity to experiment with mathematics. For example, teachers should have students create hypotheses, think about the problem, record results, question the experimental results and come to conclusions about novel mathematical problems.</td>
</tr>
<tr>
<td><strong>Students Should be Describers</strong></td>
<td>Teachers must give students the opportunity to create and use precise definitions, invent notation, discuss mathematics with peers, argue mathematics, and summarizing their thoughts in writing.</td>
</tr>
<tr>
<td><strong>Students Should be Tinkerers</strong></td>
<td>Students should be given the opportunity to tinker with mathematical ideas, to be able to take those ideas apart and put them back together. For example, after students have established that a triangle has 180 degrees, they should question what would happen to a triangle if certain angles measure more or less than 90 degrees. They should question the variety of triangles that exists and create their own understanding of triangle.</td>
</tr>
<tr>
<td><strong>Students Should be Inventors</strong></td>
<td>Students should be encouraged to create their own mathematics. For example, they could create rules for games, algorithms for doing things, explanations of how things work, or even axioms for a mathematical structure.</td>
</tr>
<tr>
<td><strong>Students Should be Visualizers</strong></td>
<td>Visualization in mathematics comes in many forms. Students should be able to visualize inherently visual objects or situations, visualizing models in non-visual realms (such as the area model of multiplication), and, for some people, visual accompaniments (such as actually seeing the variables in the area model of multiplication moving and joining together like some sort of play).</td>
</tr>
<tr>
<td><strong>Students Should be Conjecturers</strong></td>
<td>Students should be taught to make conjectures, which take a lot of time to develop. Ideally students will be able to make conjectures that rest on more than experimental evidence, but a combination of evidence gained from previous experience, and understanding of the underlying mathematical structure, experiments and more.</td>
</tr>
<tr>
<td><strong>Students Should be Guessers</strong></td>
<td>Guessing is a wonderful part of research; people often become more familiar with a process by being able to check guesses. For example, students should be given the answer to certain types of problems and asked to work backwards until they find an approximation for the original problem.</td>
</tr>
</tbody>
</table>
Although Costa and Kallick's (2008) Habits of Mind and Cuoco, Goldenberg and Mark’s (1996) general Habits of Mind are different, they have many ideals in common. For example, for students to be successful experimenters, they must be persistent, manage impulsivity, think flexibly, strive for accuracy, hypothesize and question, gather data through all senses, and apply previous knowledge to new situations in order to experiment with novel topics.

Cuoco, Goldenberg and Mark (1996) also describe the importance of habits of mind that are specific to mathematics. These Habits of Mind are quite common in mathematics but less common outside of the mathematical realm. Mathematical habits of mind are defined “to be the web of specialized ways of approaching mathematical problems and thinking about mathematical concepts that resemble the ways employed by mathematicians” (Matsuura et al., 2013).

**Mathematical Habits of Mind**

School mathematics has very little to do with how mathematicians perform mathematics, how mathematics is created or the application of mathematics outside of the classroom (Cuoco, Goldenberg, & Mark, 1996). One reason for this could be the public misconception that mathematics courses communicate results and methods in order to prepare students for life outside of the classroom by providing them with facts about mathematics. Students learn properties, algorithms and computational rules. Students then work some problems in which these properties, algorithms and computational rules are applied. Educators need to change the view of mathematics to explain how the skills used by mathematicians to create and solve mathematical problems are more important than specific
mathematical results (Cuoco, Goldenberg, & Mark, 1996). One goal of mathematics education is not to train students to be university mathematicians, yet to teach them some of the ways mathematicians think through mathematics. Cuoco, Goldenberg and Mark (1996) explain the important ways mathematicians think about mathematics as follows:

“In broad strokes, this includes learning to recognize when problems or statements that purpose to be mathematical are, in truth, still quite ill-posed or fuzzy, becoming comfortable with the skill at bringing mathematical meaning to problems and statements through definition, systemization, abstraction, or logical connections making; and seeking and developing new ways of describing situations” (p. 376).

Wilkerson-Jerde (2011) found that mathematicians are more likely to use definitions while explaining or questioning mathematical ideas. They are also more likely to refer to specific examples or non-examples to make sense of mathematical ideas. Focusing on how mathematicians think about mathematics, approach mathematics, and use disciplinary materials, can be extremely useful to education (Wilkerson-Jerde, 2011).

Matsura et al suggested that Mathematical Habits of Mind lie in four broad yet overlapping actions: “seeking, using, and describing mathematical structure; using mathematical language; performing purposeful experiments; and applying mathematical reasoning” (2013, p. 746). Mathematical Habits of Mind, as described by Cuoco, Goldenberg and Mark (1996), can be found in Table 4.

**Table 4 - Mathematical Habits of Mind (Cuoco, Goldenberg, & Mark, 1996).**

<table>
<thead>
<tr>
<th>Mathematical Habits of Mind</th>
<th>Description within the Context of Mathematics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mathematicians Talk Big and Think Small</td>
<td>Teachers need to implement a curriculum that starts with tasks and examples to help students build a collection of concrete understandings of mathematical topics in order to use those topics to build general theories.</td>
</tr>
<tr>
<td>Table 4 - Continued.</td>
<td></td>
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<tr>
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<tr>
<td><strong>Mathematicians Talk Small and Think Big</strong></td>
<td>Teachers need to have students tinker with small deeply understood problems and help students turn their knowledge into applications for deep mathematical ideas. Thinking big is related to abstraction and modeling. Building models and applying abstract theories comes from an understanding of diverse experiences and drawing connections in order to construct deep mathematical theories.</td>
</tr>
<tr>
<td><strong>Mathematicians Use Functions</strong></td>
<td>Teachers need to use functions to help students understand how some mathematical concepts come from a variation of another mathematical concept. To study functions means to study the actual change apparatuses that changed the things that were changed. There are three broad categories for the uses of functions in mathematics: algorithms, dependencies, and mappings. Algorithms are used to transform one thing into another. Dependencies are used to find and describing connections between different mathematical concepts. Mappings are used for counting.</td>
</tr>
<tr>
<td><strong>Mathematicians Use Multiple Points of View</strong></td>
<td>Teachers need to encourage students to look at certain mathematical concepts through the lens of different mathematics. For example, while teaching the complex number system, teachers should use both algebra and analysis approaches. Students need to be taught how to think of the same idea from different perspectives.</td>
</tr>
<tr>
<td><strong>Mathematicians Mix Deduction and Experiment</strong></td>
<td>Some mathematicians believe that the standard for truth in mathematics is stronger than in other fields of study due to the long history of proofs. Teachers need to have students establish certain conjectures using deductive reasoning and proofs. Not all topics need to be proven this way due to the availability of technology, but students need to know how this process works and how to accomplish it.</td>
</tr>
<tr>
<td><strong>Mathematicians Push the Language</strong></td>
<td>Mathematicians will take certain mathematical concepts and push them to the breaking point. They will look at certain examples of topics and find rules to apply to those examples so their definitions hold in other cases. For example, the definition of (2^0) comes from wanting the rules for positive integer exponents to hold in other cases. Teachers can have students come up with examples and non-examples of certain cases to see why certain definitions are written the way they are.</td>
</tr>
<tr>
<td><strong>Mathematicians Use Intellectual Chants</strong></td>
<td>Mathematicians spend extended periods of time thinking and writing to solve a problem. Teachers should include this process in their classroom. For example, students could be asked to interview students on how they solved a problem or have students reflect on their own work and how they approached a certain problem.</td>
</tr>
</tbody>
</table>
Seaman and Szydlik (2007) also report specific Habits of Mathematicians (similar to habits of mind) as follows: Mathematicians

- Seek to understand patterns based on underlying structure.
- Make analogies by finding the same essential structure in seemingly different mathematical objects.
- Make and test conjectures about mathematical objects and structures.
- Create mental (and physical) models for examples (and non-examples) of math objects.

Along the same lines, Harel (2008) highlights the difference between two mechanisms of knowing and doing mathematics professionally:

Mathematics consists of two complementary subsets. The first subset is a collection, or structure, of structures consisting of particular axioms, definitions, theorems, prods, problems, and solutions. This subset consists of all the institutionalized ways of understanding in mathematics throughout history. … The second subset consists of all the ways of thinking, which are characteristics of the mental acts whose products comprise the first set (p. 8). (See Figure 1)

![Figure 1 - Two Mechanisms of Knowing and Doing Mathematics Professionally.](image-url)
According to Harel (2008), “students develop ways of thinking only through the construction of ways of understanding, and the ways of understanding they produce are determined by the ways of thinking they possess” (p. 272). Figure 2 explains this relationship between mental actions, ways of understanding and ways of thinking.

![Figure 2 - Relationship Between Mental Actions, Ways of Understanding and Ways of Thinking.](image)

Looking through a mathematical lens, mental actions include interpreting, proving, structuring, generalizing, predicting, conjecturing, symbolizing, computing, formulating, searching, transforming and classifying (Harel, 2008). The ways of understanding is the product of a mental action, while the ways of knowing is a feature of the action.

Research states that teachers need to help students think desirably about mathematics, yet their thought process needs to first be evaluated to see if they are prepared to teach students to be mathematically sophisticated. Teachers need to help students move from less useful to more useful ways, those that are more sophisticated, such as the ones shown in Figure 3 (Lim, 2008).
The Conference Board of the Mathematical Sciences (2001) recommends that pre-service teachers acquire the habits of mind (although not always do they use that exact term) of a mathematician in order to help students develop refined decision-making skills. Lockhorst, Wubbels, and van Oers (2009) stress the importance of reflection and experimentation in the classroom environment to help students develop high-order thinking skills. Building students’ ability to make decisions, reflect and experiment by incorporating mathematical Habits of Mind into the classroom can help students’ appreciate their mathematics experience, see the importance of mathematics in their lives and build their own reflective nature, including students’ who are not mathematically disposed (Gordon, 2011).

**Mathematical Sophistication of Pre-Service Elementary Teachers**

Being mathematically sophisticated can help pre-service teachers increase their problem solving abilities, deeply conceive of mathematics, build relationships between
mathematical concepts, as well as other mathematical sophistication categories. Increasing teachers’ problem solving abilities will help them teach students to solve problems and to explain why certain procedures make sense. They will also be able to help students create their own mathematical problems and come up with viable solutions. Being able to deeply conceive of mathematics will help teachers build a deeper knowledge base of mathematics, helping teachers explain mathematics from different points of view. Being able to build relationships between mathematical concepts will help teachers build on student’s previous knowledge to create new mathematical concepts. Therefore, teachers with higher levels of mathematical sophistication are better equipped to implement the reform mathematics needed to positively improve elementary students’ performance.

Allen and Carifio (2007) found that students with higher levels of mathematical sophistication were better at solving problems, were able to positively self-evaluate their progress towards solutions, and expressed different emotions than those with lower levels of mathematical sophistication. As students’ mathematical sophistication increased their ability to manage their mathematical anxiety while solving problems and their negative emotions towards mathematics improved (Allen & Carifio, 2007).

Seaman and Szydlik (2007) developed the Mathematical Sophistication Framework to study how pre-service elementary teachers behave mathematically and the values they hold compared to mathematicians. Szydlik, Kuennen and Seaman (2009) developed a Mathematical Sophistication Instrument to gauge pre-service teachers’ level of mathematical sophistication.
Mathematical Sophistication Framework

The Mathematical Sophistication Framework (MSF) is not intended to be hierarchical nor exhaustive, yet research has shown that these actions would give elementary pre-service teachers enhanced abilities to learn and construct novel mathematics. The MSF is another way to frame habits of mind. The MSF used in this study can be found in Table 5 as well as Appendix B.

Mathematicians study patterns, objects, make and test conjectures, and construct structures using an assortment of techniques. While conceiving of new mathematical concepts they build mental and physical models. They use examples, non-examples and deductive arguments to prove or disprove certain mathematical claims. Mathematicians use and create definitions in a way to provide arguments for mathematical claims. As Polya states, “The mathematician is not concerned with the current meaning of his technical terms… The mathematical definition creates the mathematical meaning” (1957, p. 86).

Mathematicians study patterns, objects, make and test conjectures, and construct structures using an assortment of techniques. While conceiving of new mathematical concepts they build mental and physical models. They use examples, non-examples and deductive arguments to prove or disprove certain mathematical claims. Mathematicians use and create definitions in a way to provide arguments for mathematical claims. As Polya states, “The mathematician is not concerned with the current meaning of his technical terms… The mathematical definition creates the mathematical meaning” (1957, p. 86).
Table 5 - Mathematical Sophistication Framework (MSF) (Szydlik, Kuennen, and Seaman, 2012, p. 5 -7).

<table>
<thead>
<tr>
<th>MSF</th>
<th>MSF Action</th>
<th>MSF Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Seek to find and understand patterns.</td>
<td>Prospective elementary teachers must value patterns and regularity and have systematic ways of making sense of patterns involving number and shape.</td>
</tr>
<tr>
<td>2</td>
<td>Classify and characterize objects based on structure.</td>
<td>Prospective elementary teachers must value operational or geometric properties over mathematically superficial ones such as orientation, problem context, or labeling.</td>
</tr>
<tr>
<td>3</td>
<td>Make and test conjectures about objects and structures.</td>
<td>Prospective elementary teachers must be willing to explore problem situations and to make and test conjectures by considering extreme or divergent cases.</td>
</tr>
<tr>
<td>4</td>
<td>Create mental (and physical) models for, and examples and non-examples of, mathematical objects.</td>
<td>Prospective elementary teachers must draw or imagine models (often general or dynamic models) to help them make sense of problem situations, relationships, and novel definitions.</td>
</tr>
<tr>
<td>5</td>
<td>Value and use precise definitions of objects.</td>
<td>Prospective elementary teachers must use the mathematical definition to classify objects without regard to extraneous meanings of terms suggested by the wider culture.</td>
</tr>
<tr>
<td>6</td>
<td>Value an understanding of why relationships make sense.</td>
<td>Prospective elementary teachers must recognize that mathematics makes sense, and they must seize opportunities to explore relationships.</td>
</tr>
<tr>
<td>7</td>
<td>Value and use logical arguments and counterexamples as sources of conviction.</td>
<td>Prospective elementary teachers must understand that examples alone do not provide sufficient mathematical justification for a claim, and at the same time recognize that an example can provide the seed of a general argument (e.g., Carpenter, Franke &amp; Levi, 2003). They must value counterexamples and arguments based on structure and reasoning.</td>
</tr>
<tr>
<td>8</td>
<td>Value precise language and have fine distinctions about language.</td>
<td>Prospective elementary teachers must understand and use the mathematical culture’s normative meanings for terms such as: and, or, there exists, for each, at most, at least, always, less than, and greater than. They must also distinguish necessary from sufficient conditions.</td>
</tr>
<tr>
<td>9</td>
<td>Value and use symbolic representations of, and notation for, objects and ideas.</td>
<td>Prospective elementary teachers must understand and use the mathematically normative meanings for familiar symbols, and persevere to make sense of a new symbol or a new notation that is defined for them.</td>
</tr>
</tbody>
</table>
Throughout their exploration of pre-service elementary teachers’ ability to use a web-based teacher resource to construct their own mathematical understanding of certain elementary topics, Seaman and Szydlik (2007) found that pre-service elementary teachers lack the mathematical behaviors and values of a mathematician. Seaman and Szydlik’s “premise is that the true underlying cause of students’ lack of skill is not merely a need for knowledge refreshment, but rather is a paucity of ‘accessing skills,’ a profound lack of mathematical sophistication” (2007, p.175). They proposed to answer the question, “what facets of mathematical sophistication provide prospective elementary teachers power to make sense of elementary mathematics” (Szydlik, Kuennen, and Seaman, 2009)? To answer this question, Szydlik, Kuennen, and Seaman (2009) needed an instrument to gauge pre-service teachers’ level of mathematical sophistication. They met this need by developing the Mathematical Sophistication Instrument (MSI). Since this instrument is used in the current study, it will be described in some detail.

Mathematical Sophistication Instrument

The MSI (Szydlik, Kuennen, and Seaman, 2009) is a 25-item multiple-choice assessment designed to determine a pre-service teacher’s ability to make sense of new mathematical ideas as defined by the Mathematical Sophistication Framework. The MSI can be found in Appendix A. Szydlik, Kuennen, and Seaman (2009) used the following criteria to design the 25 items that make up the MSI:

- the items altogether must represent all nine mathematical sophistication actions found in the MSF
- the content knowledge covered in each item must only be that of elementary mathematics
each item must test mathematical practices instead of values or beliefs
the entire assessment must be able to be completed in less than an hour
the number of adequate responses will be able to distinguish between elementary pre-service teachers’ level of mathematical sophistication
the entire assessment must be given in a multiple-choice format for data analysis purposes and large-scale implementation.

Seven expert mathematics professors examined the instrument; each item was categorized using the MSF, and revised according to their suggestions. The instrument was then given to six elementary pre-service teachers in an interview setting to understand each student’s interpretations of the items. A revised MSI was then pilot tested with 111 pre-service elementary teachers in their mathematics content courses. To check the validity of the MSI, another implementation of the MSI was conducted with 56 pre-service elementary teachers, again in their mathematics content courses.

The three instructors of the mathematics content courses rated 43 of the students’ mathematical sophistication based on students’ performance (the professors did not feel confident in their ability to rate 13 of the participants). The ratings and mean MSI scores for each level can be found in Table 6.

<table>
<thead>
<tr>
<th>Level</th>
<th>Description</th>
<th>Mean MSI Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>highly unsophisticated</td>
<td>6.33</td>
</tr>
<tr>
<td>2</td>
<td>fairly unsophisticated</td>
<td>10.73</td>
</tr>
<tr>
<td>3</td>
<td>average sophistication for a prospective teacher</td>
<td>12.10</td>
</tr>
<tr>
<td>4</td>
<td>fairly sophisticated</td>
<td>15.86</td>
</tr>
<tr>
<td>5</td>
<td>highly sophisticated</td>
<td>17</td>
</tr>
</tbody>
</table>
The MSI scores and the ratings of each student were compared and the difference between each level of mathematical sophistication was found to be statistically significant at a 0.05 significance level. At this point, the authors were convinced that the MSI measured constructs important to mathematicians while assessing pre-service elementary teachers’ level of mathematical sophistication.

During a large sample pilot study of 482 students, Szydlik, Kuennen, and Seaman (2009) found that the MSI could detect gains in mathematical sophistication throughout an introductory mathematics course. This result suggests the possible usage of the MSI to evaluate courses or programs designed to develop deep conceptual understanding of mathematics and mathematical practices. The MSI can be related to mathematical knowledge needed for teaching since the MSI scores correlated with the Learning Mathematics for Teaching measures designed by Hill, Schilling and Ball (2004). Therefore, teachers having higher levels of mathematical sophistication could be more equipped to conceptually teach elementary mathematics and improve student achievement.

**Calculus**

The current study takes place in a calculus class, but students’ understanding of calculus concepts is not the focus of the study. The study focuses on helping pre-service teachers develop the skills of the mathematics community needed to successfully teach mathematics. Therefore, an extensive review of the calculus literature is not needed. Yet, the following paragraphs do review some pivotal articles discussing how calculus courses should be taught and the purpose of reform oriented calculus curricula.
Traditionally, calculus instruction has involved primarily symbolic analytic strategies and proofs to teach the concept of derivative and integral (Keene, Hall, & Duca, 2014). Calculus courses are usually highly computational with little conceptual understanding involved. For example, when students enter calculus they are reminded of slope under the disguise of average rate of change, yet most students do not know that the two are fundamentally the same (Hauger, 2000). Oehrtman, Carlson and Thompson (2008) explain how students have been asked throughout their educational career, to manipulate algebraic equations in order to compute adequate answers to a specified set of examples. They assert that this calculational emphasis on mathematics has severely undermined their abilities to build covariational conceptions. Research in this area has shown that students in reform calculus courses perform as well on standard procedural work as their peers in most traditional courses; however, students in the reform courses perform better on contextual and non-routine problems (Hurley, Koehn, & Ganter, 1999). These results have led many mathematicians to advocate for a move towards developing students’ intuitive understandings about calculus and using the integration of symbolic, numerical and graphical representation in all calculus courses.

For the past several decades, research in calculus has contributed to our understanding of students understanding, learning and teaching abilities in the areas such as limit, derivative and integral (Rasmussen, Marrongelle, & Borba, 2014). In addition, many studies have identified rate of change as one of the fundamental concepts in mathematics education and a gatekeeper to access advanced mathematical concept such as derivatives and limits (Carlson et al., 2002; Job & Schneider, 2014; Kloku, 2007; Thompson, 1994b).
Therefore, teaching these topics to pre-service teachers in all levels of mathematics education is important.

Students’ difficulties with understanding the concept of function has been well documented both at middle and high school grades (Dreyfus & Eisenberg, 1982), and at college level (Carlson, 1998; Dubinsky & Harel, 1992; Monk, 1994). Even high performing pre-calculus and calculus students have weak understandings of the concept of function (Dubinsky & Harel, 1992; Monk, 1992, 1994; Oehrtman, Carlson, & Thompson, 2008; Vinner & Dreyfus, 1989). Prospective secondary mathematics teachers encounter difficulties when dealing with functions (Breidenbach et al., 1992; Heid et al., 2006; Monk, 1992; Thompson, Carlson, & Silverman, 2007). If calculus professors do not help students make such connections then how can they truly understand the concept they are learning, much less move on to instantaneous rate of change (derivative) with a deep conceptual understanding?

To answer this question, there has been a significant movement to reform first year collegiate calculus courses over the past 35 years.

Calculus reform emphasizes the “rule of 3”, which stands for the integration of symbolic, graphical and numerical representations of calculus concepts (Hughes-Hallett et al, 1994). There is increased emphasis on students’ understanding of the concepts and less emphasis on developing procedures without connections. The development and marketing of graphics calculators, computer algebra systems on both computers and calculators, and the plunge in cost of these technologies have allowed many mathematics departments to use existing accessible technology. This technology is used to help students spend more time
understanding ideas instead of doing manipulation and make calculus more applicable to real world situations.

There is research on students’ understanding of calculus in a variety of fields (Rasmussen, Marrongelle, & Borba, 2014) and on pre-service secondary teachers’ abilities to perform, manipulation, and understand calculus (Ellis et al., 2014; Eichler & Erens, 2014). However, little to no research exists on pre-service elementary teachers’ beliefs and understandings of calculus or the connections that can be made between elementary school topics and those in the calculus curriculum.

**Chapter 2 Summary**

Present and past research on Mathematics Education, Pre-Service Mathematics Education with a concentration on inquiry-based learning and cognitively challenging tasks, Habits of Mind, Mathematical Sophistication and Calculus was synthesized in order to identify, describe and evaluate the research related to this study. This research synthesis also identifies a definite gap in knowledge of how pre-service elementary teachers’ mathematical skills are related to those of the mathematicians and how their mathematical sophistication will improve in a reform-oriented calculus course. This study investigates how a reform-oriented curriculum, designed to model inquiry-based pedagogy and to teach mathematical content from an advanced perspective affects pre-service elementary teachers’ levels of mathematical sophistication. Answering this research question will contribute to mathematics education research by making connections between tasks and mathematical sophistication. Teaching pre-service elementary teachers with high-level tasks and building their mathematical habits of mind will influence their approach to teaching mathematics.
Chapter 3 - Methods

This study utilized a mixed methods approach in order to answer the following research questions:

- How does pre-service elementary teachers’ mathematical sophistication change as they participate in a reform-oriented calculus course?
- What experiences do pre-service elementary teachers have during a reform-oriented calculus course that support their growth in mathematical sophistication?

A mixed methods approach is “the incorporation of various qualitative or quantitative strategies within a single project that may have either a qualitative or a quantitative theoretical drive” (Morse, 2003). The current study has a quantitative drive and used each piece of qualitative data to describe the quantitative analysis, shown in Figure 4.

![Mixed Methods Theoretical Drive](image)

**Figure 4 - Mixed Methods Theoretical Drive.**

The researcher chose to use a mixed method approach since it is a way to incorporate both meaning and quantity into a single project (Morse, 2010). Specifically, the quantitative data was collected and analyzed to see if students’ mathematical sophistication improved...
throughout the calculus course. The qualitative data was collected and analyzed to help explain how the calculus course impacted students’ mathematical sophistication.

This chapter presents a description of the setting, researcher, and participants involved in the study, the design of the study (including the theoretical framework employed and instruments previously used within the mathematics educational community), the data sources used for the study, the procedures used for analysis of the data, research reliability, validity, and ethical issues.

**Setting**

This study took place at a large public university located in the southeastern United States. The study was conducted in Fall 2014. The classroom is a state of the art “Smart Classroom”, including computers, DVD players, projectors, document cameras, and the ability to record all the classes for students to retrieve when needed. The participants were members of two sections of a reform oriented calculus course, CELTIC, designed for freshman level students preparing to earn a Bachelor of Science degree in elementary education.

**CELTIC**

Calculus for Elementary Teachers: an Innovative Context (CELTIC) was created and funded by a NSF grant which ran from 2010 to 2013(NSF#0942843). CELTIC is a two-semester calculus course required for elementary teachers. CELTIC was developed through a partnership among three departments, Mathematics Education, Mathematics, and Elementary Education. The team of researchers had a common goal of creating a calculus course for pre-service elementary teachers that provides them with the mathematical content of calculus
connected to elementary mathematics content, while exhibiting effective mathematics instruction based on research. “Therefore, striking a balance between the rigorous mathematics of an advanced perspective while taking advantage of lessons from education research” (Keene, Hall, Duca, 2012). Rasmussen, Marrongelle, and Borba (2014) explain the importance of this course and how studies on this course have the potential to influence post secondary institutions who wish to improve pre-service elementary school teachers’ level of conceptual mathematical understanding because it addresses the unique needs of elementary teachers in the field of mathematics education.

The CELTIC curriculum is based on four primary foundations:

- studying calculus concepts from a function and modeling perspective;
- learning targeted topics of the mathematics needed to teach elementary school through explicit connections to calculus concepts;
- using technological tools to foster understanding of mathematical concepts; and
- exploring applications to various STEM areas.

The CELTIC curriculum spreads the content of a traditional Calculus I course (derivation and integration) over two semesters to provide students with the opportunity for an innovative in-depth exploration of the fundamental ideas of elementary mathematics (algebra, number theory and geometry) simultaneously with the calculus concepts. The course integrates calculus ideas with relevant real-world applications from STEM disciplines that will help to pique students’ interest, motivate them and increase their conceptual understanding. CELTIC focuses on deep conceptual understanding and then introduces the procedural efficiency once the students understand the mathematics behind the procedures.
The first semester of CELTIC has the students studying number systems, sequences and series, functions, rate of change, and the derivative while the second semester involves related rates, optimization, geometry, measurement, integration, difference equations, and an introduction to differential equations. Each unit follows a similar format, shown below, to build conceptual understanding of each topic and model good teaching practices found in mathematics education research:

- Start with an exploratory task to introduce the overarching concepts of the unit;
- Use tasks and mini-lectures to discuss pre-calculus/calculus materials of the unit;
- Work (individually and in small groups) on the presented concepts;
- Explore tasks relating calculus concepts and elementary school mathematics;
- Facilitate discussions to deepen students’ knowledge of the elementary concepts;

It is important to note that even though CELTIC is a two-semester course, this study took place only in the first semester.

The CELTIC curriculum was not created with the Mathematical Sophistication Framework (MSF) in mind. Yet, all MSF Actions are represented in the design of the course and one purpose of this study is to see if Mathematical Sophistication is supported by this curriculum and instruction. The CELTIC curriculum uses high-level tasks to introduce each mathematical concept covered throughout the course, helps builds students’ intuitive understanding of the topic and pushes them to understand complex novel topics. The high-level introductory tasks used in the CELTIC course create an avenue for students to build on and increase personal intuition for each mathematical concept in the course. The following instructor-guided discussions continue to scaffold the students’ understanding and the homework and collaborative in-class work adds more support of the students’ mathematical development. The tasks have been implemented, analyzed and revised throughout the past
five years (Keene, Hall, Duca, 2012) to provide students with the opportunity to develop understandings of the calculus as well as concepts from elementary mathematics.

The tasks push participants to approach mathematics differently than previously experienced. Students are asked to build mathematical relationships (Action 6), come up with general formulas (Action 9) describing certain mathematical concepts, use definitions (Action 5) appropriately and even build their own definitions of new mathematical concepts working collaboratively. Each task in the course is different, and the design of each task elicits different aspects of the MSF. For example, while helping students build a conceptual understanding of sequence a series, the students looked at sets of numbers, figured out the patterns (Action 1) and create general formulas (Action 9) to represent the set of numbers. A more detailed task analysis is described later under data analysis. Once students complete a task, the instructor will strengthen their understanding with an active lecture and then the students’ build on and practice the new mathematical concept through whole group assignments and individual homework assignments.

**Instructors**

The researcher and a mathematics graduate student were the instructors of the two CELTIC . . Both instructors met on a weekly basis to ensure the participants learned from the same basic lesson plan, and were given similar homework and identical tests throughout the study.

The researcher is a professional educator with a Bachelor of Science in Secondary Mathematics Education and a Masters of Science in Mathematics Education who is currently pursuing a PhD in Mathematics Education. She has previously taught pre-curriculum
mathematics, intermediate algebra, geometry, algebra II, college algebra, technical mathematics, statistics, pre-calculus and calculus to high school, community college and university students. The researcher has also taught an introductory mathematics methods course for middle and high school pre-service teachers and elementary mathematics to pre-service elementary teachers. More recently the researcher has been one of the two teachers for the first and second semester of CELTIC.

The mathematics graduate student has a Bachelor of Arts in Mathematics, a Master of Science in Applied Mathematics, and is currently pursuing a PhD in Applied Mathematics. She has taught Pre-calculus, Introduction to Finite Mathematics with Applications and the second semester of CELTIC. Having both a mathematics educator and a mathematician teach this course continues with the goal of bridging between the two communities to bring the best knowledge available for the students taking the course.

Participants

Thirty-three pre-service elementary students in the two sections of CELTIC, 22 students in one section and 11 in the other, participated in this study. Each participant signed an Informed Consent Form (Appendix F) and an Informed Video Consent Form (Appendix G). All 33 participants were female between the ages of 18 and 20. The class was not highly diverse yet represented the population of Elementary Education majors at the university. Thirty (91%) of the students were Caucasian and three (9%) of the students were African American.

Each participant’s mathematical background can be found in the Table 7.
<table>
<thead>
<tr>
<th>Participant Name (n=33)</th>
<th>University Academic Level</th>
<th>High-School Pre-Calculus</th>
<th>High-School Calculus</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mia</td>
<td>Sophomore</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ava</td>
<td>Sophomore</td>
<td>A</td>
<td></td>
</tr>
<tr>
<td>Grace</td>
<td>Freshman</td>
<td>A</td>
<td></td>
</tr>
<tr>
<td>Leah</td>
<td>Sophomore</td>
<td>B</td>
<td></td>
</tr>
<tr>
<td>Charlotte</td>
<td>Freshman</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>Avery</td>
<td>Freshman</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Chloe</td>
<td>Freshman</td>
<td>A</td>
<td></td>
</tr>
<tr>
<td>Natalie</td>
<td>Freshman</td>
<td>A</td>
<td></td>
</tr>
<tr>
<td>Abigail</td>
<td>Sophomore</td>
<td>B</td>
<td>B</td>
</tr>
<tr>
<td>Ali</td>
<td>Sophomore</td>
<td>B</td>
<td></td>
</tr>
<tr>
<td>Riley</td>
<td>Freshman</td>
<td>C</td>
<td></td>
</tr>
<tr>
<td>Amelia</td>
<td>Sophomore</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>Lily</td>
<td>Freshman</td>
<td>B</td>
<td>C</td>
</tr>
<tr>
<td>Emma</td>
<td>Freshman</td>
<td>B</td>
<td></td>
</tr>
<tr>
<td>Isabella</td>
<td>Freshman</td>
<td>A</td>
<td></td>
</tr>
<tr>
<td>Samantha</td>
<td>Freshman</td>
<td>B</td>
<td></td>
</tr>
<tr>
<td>Faith</td>
<td>Freshman</td>
<td>B</td>
<td></td>
</tr>
<tr>
<td>Hannah</td>
<td>Freshman</td>
<td>A</td>
<td></td>
</tr>
<tr>
<td>Ella</td>
<td>Freshman</td>
<td>B</td>
<td>B</td>
</tr>
<tr>
<td>Victoria</td>
<td>Freshman</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Emily</td>
<td>Freshman</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>Evelyn</td>
<td>Freshman</td>
<td>B</td>
<td></td>
</tr>
<tr>
<td>Zoe</td>
<td>Junior</td>
<td>B</td>
<td></td>
</tr>
<tr>
<td>Sophia</td>
<td>Freshman</td>
<td>B</td>
<td></td>
</tr>
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<td>Olivia</td>
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<td>Aubrey</td>
<td>Sophomore</td>
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<td>Brooklyn</td>
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<td>B</td>
<td>B</td>
</tr>
<tr>
<td>Elizabeth</td>
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<td>B</td>
<td></td>
</tr>
<tr>
<td>Layla</td>
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<td>Lillian</td>
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<td>Addison</td>
<td>Freshman</td>
<td>A</td>
<td></td>
</tr>
<tr>
<td>Harper</td>
<td>Freshman</td>
<td>A</td>
<td></td>
</tr>
</tbody>
</table>
A summary of participant educational background can be found in Table 8.

### Table 8 - Participant educational background summary.

<table>
<thead>
<tr>
<th>Participants Previous Math Course</th>
<th>Pre-Calculus 27 (82%)</th>
<th>Calculus 1 (3%)</th>
<th>Advanced Placement Calculus AB 8 (24%)</th>
<th>Advanced Placement Calculus BC 3 (9%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of A's</td>
<td>13 (39%)</td>
<td>---</td>
<td>3 (9%)</td>
<td>2 (6%)</td>
</tr>
<tr>
<td>Number of B's</td>
<td>13 (39%)</td>
<td>1 (3%)</td>
<td>4 (12%)</td>
<td>1 (3%)</td>
</tr>
<tr>
<td>Number of C's</td>
<td>1 (3%)</td>
<td>---</td>
<td>1 (3%)</td>
<td>---</td>
</tr>
</tbody>
</table>

All students were invited to participate in two different semi-structured interviews; 17 (52%) of the students expressed interest. Of those 17 students, nine students were chosen for pre and post-interviews; participant selection is explained in the data collection section. At no time did the instructors know the students’ pseudonyms or assigned level of mathematical sophistication during the duration of the course and data collection to ensure that there would be no influence in terms of the students’ achievement in the course.

### Design of the Study

This study expands upon research conducted by Seaman and Szydlik (2007) on pre-service elementary teachers’ mathematical sophistication. Seaman and Szydlik (2007) found that pre-service elementary teachers are severely lacking mathematical sophistication. The term *mathematical sophistication* describes the “internalization of the values, practices, and habits of mind of the mathematical community that are powerful in learning mathematics” (Szydlik, Kuennen, and Seaman, 2009, p. 4). This study veers from their work as it studies pre-service elementary teachers’ mathematical sophistication in a reform oriented calculus
The design of this study included Szydlik, Kuennen and Seaman’s (2009) Mathematical Sophistication Instrument (MSI) as a pre- and post-test for all CELTIC students (See Chapter 2 for description), found in Appendix A. The study also included three tasks implemented throughout the course, found in Appendix H through Appendix J. Nine pre and post-interviews were conducted of the same subset of nine participants at the start and the end of the course. The protocols used during the pre- and post-interviews can be found in Appendices C and D respectfully.

Table 9 - A summary of the data sources and analysis.

<table>
<thead>
<tr>
<th>Research Questions</th>
<th>Sources of Data</th>
<th>Data Analysis</th>
</tr>
</thead>
</table>
| 1. How does pre-service elementary teachers’ mathematical sophistication change as they participate in a reform-oriented calculus course? | • Pre-Test (MSI)  
• Post-Test (MSI) | • Quantitative Analysis, Wilcoxon signed rank test.  
• 18 Task based Interviews (2 tasks for each of nine students with a variety of MS; interviews early and late in the semester) |  
| 2. What experiences do pre-service elementary teachers have during a reform-oriented calculus course that support their growth in mathematical sophistication? | • Artifacts (All Students Work from Class Participation):  
  o Limit (Sesame Street),  
  o Families of Functions (Function Experiments),  
• Cross-analysis of interviews and tasks  
• Student work analysis using Mathematical Sophistication Framework (MSF) and other emerging concepts |

The research questions, sources of data used to answer that question and how the data was analyzed is described in the following paragraphs and can be found in Table 9.
On the first day of the study the researcher introduced herself and the other instructor of CELTIC. Next, each student was given a recruitment form (Appendix E) including an overview of the study as well as the data courses and collection procedures. The researcher explained the study design, the students’ involvement and the rights they throughout the duration of the study. Each participant was given a consent form (Appendix F). As the students read the recruitment and consent forms, they were allowed to ask questions. Few questions were asked, some questions included if they had to do an interview (no) and what will the interviews be like (described below). All students decided to participate and signed the consent form. All students also signed a video consent form at various points throughout the semester as the need for videoing purposes arose (Appendix G). These video consent forms were used for students who participated in the interviews or any group work that was recorded for analysis.

**Data Collection**

The design of the data collection for this study can be found in Table 10.

<table>
<thead>
<tr>
<th>Activity</th>
<th>Time-length</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre-test for both classes (MSI)</td>
<td>45 minutes on Day 1</td>
</tr>
<tr>
<td>9 Pre-interviews</td>
<td>60 minutes each within two weeks of the pre-test before the instructional sequence for calculus begins</td>
</tr>
<tr>
<td>Instructional Sequence including 3 high level tasks</td>
<td>4 Months</td>
</tr>
<tr>
<td>Post-test for both classes (MSI)</td>
<td>45 minutes following the completion of the instructional sequence</td>
</tr>
<tr>
<td>9 Post-interviews</td>
<td>60 minutes each within two weeks of the completion of the differential calculus instructional sequence</td>
</tr>
</tbody>
</table>
The following section explains how data was collected for each of the following sources in the progression of the study: pre-test, pre-interview, instructional sequence, post-test, and post-interview. All student work was collected and scored in a secured location, locked, and was not analyzed until the end of the semester.

**Pre-test**

Once all signed consent forms were collected, all students were given the MSI (Mathematical Sophistication Instrument) as a pre-test. The MSI (Appendix A) is a 25-item multiple-choice assessment designed by Seaman & Szydlik (2009) to gauge pre-service elementary teachers level of mathematical sophistication. The MSI (for more information see chapter 2) does not test familiar content, but assesses students’ abilities to make sense of new mathematical ideas with attention to nine mathematical actions (the Mathematical Sophistication Framework (MSF), described in chapter 2, found in Appendix B). None of the items required classical mathematics content knowledge beyond elementary arithmetic and geometry. One example question from the MSI:

A number is called **normal** if it is less than 10 or even. According to this definition, of the numbers 5, 8, and 24,

a) Only 5 and 8 are normal.
b) Only 8 is normal.
c) Only 5 and 24 are normal.
d) All of these numbers are normal.

This question reflects Action 5 of the MSF, *Value and use precise definitions of objects* and Action 8, *Value precise language and have fine distinctions about language*. It assesses pre-service elementary teachers’ ability to apply a novel definition to a variety of whole numbers
while paying attention to language such as “less than” or “even.” The relationship between the MSI questions and the MSF actions can be found in Table 11.

Table 11 - The Relationship Between the MSI and the MSF.

<table>
<thead>
<tr>
<th>MSI Question</th>
<th>MSI Classification</th>
<th>MSF Action</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>is a definitions and language item</td>
<td>5 &amp; 8</td>
</tr>
<tr>
<td>2</td>
<td>is a logical argument item</td>
<td>7 &amp; 8</td>
</tr>
<tr>
<td>3</td>
<td>is a language item</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>is a notation item (meaning of =)</td>
<td>9</td>
</tr>
<tr>
<td>5</td>
<td>is a conjecturing item and a sense making item</td>
<td>3, 6, &amp; 7</td>
</tr>
<tr>
<td>6</td>
<td>is a logical argument item</td>
<td>7</td>
</tr>
<tr>
<td>7</td>
<td>is a pattern problem</td>
<td>1 &amp; 4</td>
</tr>
<tr>
<td>8</td>
<td>is a conjecture and pattern problem</td>
<td>1, 2, 3, &amp; 9</td>
</tr>
<tr>
<td>9</td>
<td>is a definition and language and notation item</td>
<td>5, 8, &amp; 9</td>
</tr>
<tr>
<td>10</td>
<td>is a notation and modeling problem and also a structure item</td>
<td>2, 4, &amp; 9</td>
</tr>
<tr>
<td>11</td>
<td>is a value of making sense and use of logic</td>
<td>6 &amp; 7</td>
</tr>
<tr>
<td>12</td>
<td>is a logical argument item</td>
<td>7</td>
</tr>
<tr>
<td>13</td>
<td>is a definitions item</td>
<td>5 &amp; 8</td>
</tr>
<tr>
<td>14</td>
<td>involves definitions, language (used in a new way) and logic</td>
<td>5, 7, &amp; 8</td>
</tr>
<tr>
<td>15</td>
<td>is a notations item</td>
<td>8 &amp; 9</td>
</tr>
<tr>
<td>16</td>
<td>is a definitions item</td>
<td>5 &amp; 9</td>
</tr>
<tr>
<td>17</td>
<td>is a structure item</td>
<td>2</td>
</tr>
<tr>
<td>18</td>
<td>is a structure item</td>
<td>2</td>
</tr>
<tr>
<td>19</td>
<td>is a notations item</td>
<td>5 &amp; 9</td>
</tr>
<tr>
<td>20</td>
<td>is a modeling item</td>
<td>4</td>
</tr>
<tr>
<td>21</td>
<td>is a value of sense making item</td>
<td>6</td>
</tr>
<tr>
<td>22</td>
<td>is a notations item</td>
<td>5 &amp; 9</td>
</tr>
<tr>
<td>23</td>
<td>is a modeling item</td>
<td>4</td>
</tr>
<tr>
<td>24</td>
<td>is a pattern and modeling problem</td>
<td>1, 2, 3, &amp; 4</td>
</tr>
<tr>
<td>25</td>
<td>is a pattern and modeling problem</td>
<td>1, 2, 3, &amp; 4</td>
</tr>
</tbody>
</table>
Each student was given a score based on the total number of adequate responses they provided on the pre-MSI. Each student was also given a level of mathematical sophistication based on their score on the pre-MSI, shown in Table 12.

<table>
<thead>
<tr>
<th>Level</th>
<th>Description</th>
<th>MSI Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>highly unsophisticated</td>
<td>0-5</td>
</tr>
<tr>
<td>2</td>
<td>fairly unsophisticated</td>
<td>6-10</td>
</tr>
<tr>
<td>3</td>
<td>average sophistication</td>
<td>11-15</td>
</tr>
<tr>
<td>4</td>
<td>fairly sophisticated</td>
<td>16-20</td>
</tr>
<tr>
<td>5</td>
<td>highly sophisticated</td>
<td>21-25</td>
</tr>
</tbody>
</table>

The level determined by the pre-MSI was analyzed and compared to the level the students’ scored on the post-test at the completion of the semester. Each student’s pre-MSI score and level of mathematical sophistication was entered into a database. A third party person replaced each student’s real name with a pseudonym. These results will be discussed in Chapter 4.

**Pre-Interview**

The semi-structured, one-on-one, task-based interviews used in a study by Seaman and Szydlik (2007) influenced the pre- and post-interviews in this study. Semi-structured interviews involve an interview protocol of prepared questions, guided by themes and asked in a systematic manner to elicit elaborate responses from the interviewee (Qu & Dumay, 2011). Task-based interviews involve a subject solving a problem (Koichu & Harel, 2007). Either during their solution or after their solution, the subject will discuss the problem with the interviewer. Task-based interviews allow the interviewer to gain information about the
subjects’ knowledge, problem-solving behaviors and reasoning (Koichu & Harel, 2007; Schoenfeld, 1985). The researcher chose semi-structured, one-on-one, task-based interviews to collect data because they allowed the researcher to use previously researched tasks, provide tasks for students to work on while explaining their approaches to the mathematical solutions they derive, and probe any ideas brought up by the interviewee that relate to the themes of the interview.

Seaman and Szydlik (2007) investigated eleven pre-service elementary teachers’ level of mathematical sophistication. They used a web-based teacher resource to learn how pre-service teachers use mathematical definitions, correct their own procedural errors and make sense of novel mathematical ideas. Their semi-structured interviews involved three mathematical tasks that were isomorphic to problems the students had previously answered inadequately on a 30-item multiple-choice inventory. The three mathematical tasks used in their interviews can be found in Table 13.

Table 13 - Seaman & Szydlik (2007) Interview tasks

<table>
<thead>
<tr>
<th>Task Number</th>
<th>Mathematical Problem</th>
</tr>
</thead>
</table>
| One         | What is the greatest common factor of 60 and 105?  
|             | What is the least common multiple of 60 and 105?  |
| Two         | Calculate 0.4 ÷ 0.05. |
| Three       | Brooke has a ¾ pound (12 oz.) bag of M&M’s. If she gives 1/3 of the bag to Taylor, what fraction of a pound does Taylor receive? |

During the interviews, the participants were asked to solve the previous three problems while discussing their thinking during their solution process. Then without telling each student whether they were adequate in their solutions or not, each participant was given
individual time alone with a web-based resource to study the concepts behind the three chosen problems. The web-based resource was a website coordinated with the Everyday Math (University of Chicago School Mathematics Project, 2004) curriculum, found at http://www.math.com/homeworkhelp/EverydayMath.html. After twenty minutes the interviewer re-entered the room and asked the participants if they would like to change any of their solutions. Any solution changes were recorded and the interviewer asked the students to explain why they changed their solution along with how the resources contributed to their revision of the problem.

For this study, a similar protocol was used (See Appendix C), but changes have been made to make the interviews relevant for the mathematical content covered in CELTIC. The concept of rate of change was chosen due to the numerous research articles on students’ conceptual understanding of rate of change and the limited exposure of rate of change for the majority of CELTIC students. For many years, mathematics educators and researchers have demonstrated the importance for students to understand how to conceptualize patterns of change along with having the ability to reason conceptually about functions (Moore, Carlson, and Oehrtman, 2009). Recent national curricula have included students reasoning of change as an important part of the national mathematical curricula. According to the Common Core State Standard Initiative (CCSSI, 2009), fifth graders should be able to “identify apparent relationships between corresponding terms” (p. 35). According to the National Council of Teachers of Mathematics (NCTM) “the study of change in grades 9-12 is intended to give students a deeper understanding of the ways in which changes in quantities can be represented mathematically and of the concept of rate of change” (NCTM, 2000).
Table 14 - *Mathematical tasks in the pre-interview.*

<table>
<thead>
<tr>
<th>Task Number</th>
<th>Mathematical Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>One</td>
<td>From a vertical position against the wall, a ladder is pulled away at the bottom at a constant rate. Describe the speed of the top of the ladder as it slides down the wall. Justify your claim.</td>
</tr>
<tr>
<td></td>
<td><strong>Interviewer Questions:</strong></td>
</tr>
<tr>
<td></td>
<td>Did you decide that the top of the ladder’s movement speeds up, stays constant or slows down? Do you want to change your decision?</td>
</tr>
<tr>
<td></td>
<td>Can you explain?</td>
</tr>
<tr>
<td></td>
<td>Can you show with your arm what you think will happen?</td>
</tr>
<tr>
<td></td>
<td>Is there any kind of algebraic expressions you could use to help you understand the relationship between the ladder and the wall?</td>
</tr>
<tr>
<td></td>
<td>What on the Internet helped you answer this task?</td>
</tr>
<tr>
<td></td>
<td>Do you have anything else you would like to add to this task before we move onto the next task?</td>
</tr>
<tr>
<td>Two</td>
<td>Imagine this bottle filling with water. Sketch a graph that represents the relationship between amount of water that is in the bottle and the height. Explain.</td>
</tr>
<tr>
<td></td>
<td><strong>Interviewer Questions:</strong></td>
</tr>
<tr>
<td></td>
<td>Describe how you sketched the graph.</td>
</tr>
<tr>
<td></td>
<td>Ask question about each of the different segments of the graph as you see it:</td>
</tr>
<tr>
<td></td>
<td>How does this fit with the bottle?</td>
</tr>
<tr>
<td></td>
<td>Ask question: where in the graph is this point on the bottle. (and pick different places on the bottle).</td>
</tr>
<tr>
<td></td>
<td>Ask them to follow the graph with their finger and discuss where it fits with the bottle.</td>
</tr>
<tr>
<td></td>
<td><em>Symmetric</em>: How does that affect the graph?</td>
</tr>
<tr>
<td></td>
<td><em>Middle</em>: Can you tell what happened at this point [inflection point]?</td>
</tr>
<tr>
<td></td>
<td><em>Middle</em>: What happens at the middle of the spherical portion?</td>
</tr>
<tr>
<td></td>
<td><em>Curved lines</em>: Why did you draw a smooth curve through the lines?</td>
</tr>
<tr>
<td></td>
<td>Do you have anything else to add?</td>
</tr>
</tbody>
</table>
Carlson et al. (2002) created the two tasks used in the pre-interview. The two tasks can be found in Table 14 with supporting questions developed by the researcher. The tasks are appropriate for semi-structured interviews since their open-ended nature will elicit students’ thinking and allow the interviewer to probe students thought processes, approaches and solutions to the problems.

Once all of the documents had been signed, the students were given a copy of task 1 and task 2. They were asked to start with task 1 and then go on to task 2. They were asked to read the task out loud and answer the problem. No guidance was given to the student during the interview. For example, when looking at task 2, some students asked if height should go on the x-axis or the y-axis. Both interviewers would reply with, what do you think? The interviewers would only answer questions in which the answer was simply re-wording the problem. The students were not provided feedback on their work. Once both tasks were completed, the interviewer explained to the participants that they wanted them to use the Internet, for 20 minutes, to try to understand the problems. Then the interviewer collected the student work, gave the student a blank copy of the tasks, showed them how to use the computer and left the room. While out of the room, the researcher would make a copy of the students’ original work on the tasks.

For the Internet portion of the interview, the students were working on a Mac Book Pro with two different Internet browsers, Safari and Mozilla. They were allowed to use whichever browser they were most familiar with. When the researcher left the room, the video camera was still focused on the students’ work. The students’ web searches were captured using a piece of software titled QuickTime Player. QuickTime Player is a free video
player that can also be used to create videos that capture the computer screen as well as internal and external sound. Allowing the students to freely search on the Internet allows the researcher to analyze the students’ approach to each mathematics concept.

After 20 minutes, the interviewer re-entered the room and asked the student if they needed more time. If they did, time would be granted; if not, the interviewer would continue the interview. In the third portion of the interview, the revision, students were given their tasks with their initial work, asked to revisit each problem, make any changes they feel were necessary, explain their thinking about the problem again, and explain any Internet resources that they used. Once the interview was completed, a copy of their final work was made and their pseudonym was inserted.

A subset of nine students was needed to participate in the interviews. These students were chosen in the following manner. The researchers divided the seventeen volunteers into three groups depending on their MSI scores (see Table 15).

<table>
<thead>
<tr>
<th>Group</th>
<th>Score Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>One</td>
<td>6-9</td>
</tr>
<tr>
<td>Two</td>
<td>10-13</td>
</tr>
<tr>
<td>Three</td>
<td>14-18</td>
</tr>
</tbody>
</table>

Table 15 - Participant Interview Selection Groups.

A random number generator was used to choose three students from each group. A third party person helped the researcher complete this process to ensure participant anonymity. The researcher gave a list of the students who agreed to participate in the interviews and were
chosen by the random number generator to the third party person and they in turn gave the researcher a list of the nine participants real names in which to interview.

The researcher and the other CELTIC instructor conducted nine pre-interviews. Each of the interviewers interviewed the participants from the class they were not teaching in order to ensure bias from the interviews never entered the classroom. Neither the researcher nor the CELTIC instructor knew the interviewee’s score or level of mathematical sophistication throughout the interviews or the duration of the course.

**Data Collected from Instructional Sequence: Task Artifacts**

All students participated in a four-month sequence of instruction designed to build conceptual understanding as well as mathematical sophistication. During the CELTIC instructional series, students were introduced to mathematical concepts using tasks. Each task met the requirements set by Smith & Stein (1998) as “doing mathematics” and at least one action in the MSF by Seaman & Szydlik (2007). Three tasks were chosen from the four-month series of instruction to analyze students’ mathematics sophistication throughout the course and help answer RQ2: The three tasks, the concept they introduce and the MSF actions represented within each task are provided in Table 16.

<table>
<thead>
<tr>
<th>Mathematical Concept</th>
<th>Task Title</th>
<th>Appendix Location</th>
<th>MSF Actions Represented</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Limit</strong></td>
<td>Sesame Street (Keene, Hall, &amp; Duca, 2012)</td>
<td>H</td>
<td>1, 3, 4, 6, 8, 9</td>
</tr>
<tr>
<td><strong>Families of Functions</strong></td>
<td>Function Experiments</td>
<td>I</td>
<td>1, 3, 4, 5, 6, 8, 9</td>
</tr>
<tr>
<td><strong>Direction &amp; Velocity</strong></td>
<td>How Far Can You Go with Logger Pro</td>
<td>J</td>
<td>1, 3, 4, 6, 8, 9</td>
</tr>
</tbody>
</table>
In order to categorize each task by the MSF actions needed to perform the task, the researcher, as well as another researcher, studied each task, discussing the possible mathematical information needed, how this information is found or provided and which MSF actions were used throughout the task.

During the instructional sequence students worked in groups, building on their previous knowledge in order to understand new mathematical concepts, and gaining new mathematical knowledge through analysis and discussion. One or two groups from each section were video recorded. This data was not used as part of the analysis. Once the students completed each task, the student work on the tasks was collected and copied, pseudonyms were inserted, and then their original work was returned to the students. Students’ work on each of the three tasks were not graded as part of the course; the work is part of their participation in the course. Only the student work from the nine interview students were analyzed as part of the data analysis.

**Post Test**

Following the instructional sequence, the same MSI was given to all students to determine their level of mathematical sophistication following the instructional sequence. The data from this MSI implementation was used to determine each student’s level of mathematical sophistication at the end of the study that was at the end of the first semester of the CELTIC course and used for analysis.

**Post-Interview**

Each of the same nine students chosen to participate in the pre-interview were asked to participate in a post-interview; they all agreed. The post-interview had the same structure
as the pre-interview with a focus on a different novel mathematical concept, integration and its relationship to derivative. The researcher chose this mathematics topic because the first semester of the CELTIC course ends by covering differentiation; therefore, students are familiar with differentiation and able to apply their previous knowledge to understand integration. The two tasks can be found in Table 17, both tasks are used during the second semester of CELTIC.

Table 17 - *Mathematical tasks in the post-interview.*

<table>
<thead>
<tr>
<th>Task Number</th>
<th>Mathematical Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>One</td>
<td>You and your friend start off at noon and walk in the same direction along the same path at the rates shown in the Figure below.</td>
</tr>
</tbody>
</table>

![Figure showing velocity-time graph with labels for you and friend.](image)

a) Who walks faster at 1:00 pm? Who is ahead at 1:00 pm? Explain.
b) Who walks faster at 2:00 pm? Who is ahead at 2:00 pm? Explain.
c) Who walks faster at 3:00 pm? Who is ahead a 3:00 pm? Explain.
d) How can you find the time when you and your friend will be together? Answer in words.
An orchard owner, Melinda, is trying to find out the apple production of her orchard. Unfortunately, the production notes from previous year have been lost. However, she has found a Table of the rates of production of apple bushels over the past year. Use this Table to estimate a range of how many apples were produced in the last year. To make it easier, let’s assume there are 30 days in each month.

Table:

<table>
<thead>
<tr>
<th>Month</th>
<th>Rate of production in groups of 100 apples/month</th>
</tr>
</thead>
<tbody>
<tr>
<td>January</td>
<td>20</td>
</tr>
<tr>
<td>February</td>
<td>15</td>
</tr>
<tr>
<td>March</td>
<td>10</td>
</tr>
<tr>
<td>April</td>
<td>50</td>
</tr>
<tr>
<td>May</td>
<td>70</td>
</tr>
<tr>
<td>June</td>
<td>80</td>
</tr>
<tr>
<td>July</td>
<td>90</td>
</tr>
<tr>
<td>August</td>
<td>100</td>
</tr>
<tr>
<td>September</td>
<td>200</td>
</tr>
<tr>
<td>October</td>
<td>500</td>
</tr>
<tr>
<td>November</td>
<td>300</td>
</tr>
<tr>
<td>December</td>
<td>100</td>
</tr>
</tbody>
</table>

a) How many bushels of apples did Melinda’s orchard produce last year? Show your work and explain how you did that.

b) Use the Table above to sketch a graph of the production rate on this coordinate axes. Let the horizontal axis be days starting with January 1, and let the y-axis be the production rate.

c) Look at your production rates and your graph. Is there anything that could connect the total production numbers you calculated to anything in the graph? How did you calculate the total production, does anything on the graph represent the numbers you used to calculate the production rate?

Interviewer Questions:
What connects the graph to your production rates?
Table 17 - Continued.

<table>
<thead>
<tr>
<th>Question</th>
</tr>
</thead>
<tbody>
<tr>
<td>What do you notice about the axis? Write down how you found the production rates. Now look at the numbers on the axis. Do you notice anything? Can you think of any previous math concepts that connect to these concepts? Do you have any more thoughts about this task?</td>
</tr>
</tbody>
</table>

The complete post-interview protocol can be found in Appendix D.

**Data Analysis**

Data analysis began at the completion of the course, except for grading the Pre-MSI to place students in levels of sophistication to determine the interviewees. Data Analysis consisted of three different phases: Phase 1: Quantitative, Phase 2: Qualitative, Phase 3: Triangulation.

**Phase 1**

Phase 1 consisted of the pre- and post-MSI analysis and an analysis of the total number of MSF actions exhibited during the pre- and post-interviews.

**Pre- and Post-Test**

The data from the pre-MSI was analyzed to determine each students’ level of mathematical sophistication at the start of the CELTIC course. Students’ answers to each question, their overall score and level of mathematical sophistication were logged into a database. Then the data from the post-MSI was similarly analyzed to determine each students’ level of mathematical sophistication at the end of the CELTIC course. Each student’s answer to each question, their overall score and level of mathematical
sophistication was entered into the same database as the pre-MSI data for statistical testing and comparison with the pre-MSI results.

A Wilcoxon signed rank test was used to analyze the pre- and post-MSI data as well as the MSF Actions represented from the pre- to post-interview because it is an identical set of measurements taken from the same individual at different points in time (Walker & Almond, 2010). The assumptions for a Wilcoxon signed rank test are (1) data are paired and come from the same population, (2) each pair is chosen randomly and independently, and (3) the data is measured on an ordinal scale.

The researcher aimed to answer the following questions throughout multiple Wilcoxon signed rank tests on different sets of data: Did students’ level of mathematical sophistication improve from the pre- to post-MSI? Did different groups of students show more improvement than others? Are there any questions on the MSI that all students or subsets of students consistently improved on? Did different MSF actions show more improvement over others based on the pre- and post-MSI results? Does a students’ level of MS affect their improvement? Does a students’ prior math courses affect their improvement? For example, the researcher categorized each question based on the MSF Action represented and then performed a Wilcoxon signed rank test on the number of adequate responses from the pre- to post-MSI for each MSF Action. The results from this analysis can be found in Chapter 4.

*Pre- and Post Interviews*

Following the quantitative analysis on the pre- and post-MSI data, the researcher began both quantitatively and qualitatively analyzing the pre- and post-interviews. For the
quantitative analysis, the researcher watched the videos and noted any instance of mathematical sophistication framework actions exhibited in both the pre- and post-interviews. This information was entered into a database. The researcher also identified whether a student adequately or inadequately answered each problem to help determine mathematical sophistication exhibited throughout the interview process.

A Wilcoxon signed rank test was used to see if the difference in number of MSF Actions used by each student was statistically significant from pre- to post-interview. The researcher aimed to answer questions throughout multiple Wilcoxon signed rank tests on different sets of data. For example, did participants choose to use more mathematical sophistication framework (MSF) actions from the pre- to post-interview? Analysis was also done to see if certain MSF actions showed up more often than others. The results from this analysis can be found in Chapter 4.

**Phase 2**

Phase 2 consisted of the qualitative pre- and post-interview analysis as well as the analysis of the instructional sequence.

*Pre- to Post-Interviews*

The researcher followed the subsequent steps for analyzing the video-based data. First, the researcher watched each interview, without analyzing, and wrote a summary of the events in each interview. Then the researcher watched the videos a second time and noted any instance that exemplified a mathematical sophistication action. After noting examples of each mathematical sophistication action, the researcher consulted with a mathematics educator researcher. The researcher looked at each instance for three interviews (one in each
group of mathematical sophistication described under data collection) and decided which mathematical sophistication action was represented by the example.

Once these were agreed upon, the researcher watched the videos for a third time and gathered the following information:

- Action directly before instance
- Transcription of instance
- Action directly following the instance
- Notes as to why the researcher coded the specific action

The researcher then identified which students used which MSF actions, the frequency of the MSF actions, any patterns associated with each action, any patterns associated with the instances directly before or after the instance, and any underlying patterns that emerged during data analysis. Finally, the interview created a chart of the MSF Actions exemplified by each participant during their pre-interview. Finally, the pre-interview of each student was compared to their MSI score to see if their level of mathematical sophistication was consistent.

To illustrate this procedure, the following example is how the researcher analyzed how Brooklyn answered Task 1 during the pre-interview. First the researcher watched the video and wrote summaries similar to the ones shown in Table 18. Next the researcher would watch the video a second time and note each Mathematical Sophistication Framework Action represented in the task interview response. For example, Brooklyn says that see needs to draw a picture to engage in Task 1. This instance of MSF Action would be recorded in the manner shown in Table 19.
### Table 18 - Example of Pre-Interview Analysis Step One.

<table>
<thead>
<tr>
<th>Parts of the Interview</th>
<th>Summary of Brooklyn’s Responses</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Task 1 Before Internet Portion</strong></td>
<td>She draws a picture first, she says she needs to draw a picture because she is a visual person. She has assigned variables to the unknowns. She wants to find one value, but she says that the task doesn’t give her any values in which to assign. She says that since the ladder is being pulled away at a constant rate that the ladder will slide down the wall at a constant rate. She says if we have values she could use trig to solve.</td>
</tr>
<tr>
<td><strong>Internet Portion</strong></td>
<td>She types in the problem. She finds a site explaining how to use the Pythagorean Theorem and derivatives to find the rate at which the ladder falls down the wall. Then the finds the ladder problem. This explains that the ladder does not fall down the wall at a constant rate, it must accelerate. She then writes her new ideas and solution to the problem on her notes. She looks at another site of mathematical symbols and moves quickly past it. She must not like the symbols being used. She finds an example of the ladder problem so she pauses and reads. She then starts working on the ladder problem again.</td>
</tr>
<tr>
<td><strong>Task 1 Post Internet Portion</strong></td>
<td>She finds out that when the ladder gets close to the bottom it is going to speed up. But before then, constant because it is a rigid object. So the ladder will fall at a constant rate and then speed up. She then states that if we had values for the different parts of the problem you could find the derivative of the Pythagorean Theorem and use this to solve for the rate in a related rates situation. But that wouldn't help in this situation because they are not asking for the actual rate of change, they are just asking what will the speed do as the ladder falls against the wall. Will it remain constant, speed up or slow down? How rates are related helped her. She liked examples with values in them, she completed the calculations and found that as the ladder fell in different situations, the rate would speed up.</td>
</tr>
<tr>
<td><strong>Task 1 Summary</strong></td>
<td>Before the internet, Brooklyn thinks the ladder will fall at a constant rate. Yet, after looking at multiple websites with different explanations of the ladder falling situation and calculating rates for different examples of ladder falling problems, she changed her mind. She ends the interview saying that the speed of the ladder will increase as the ladder falls down the wall.</td>
</tr>
</tbody>
</table>
Table 19 - Example of Pre-Interview Analysis Step Two.

<table>
<thead>
<tr>
<th>Brooklyn Video Excerpt</th>
<th>Takes place on Video 1</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Time Stamp</strong></td>
<td>2:29</td>
</tr>
<tr>
<td><strong>MSF Action</strong></td>
<td>MSF Action 4 - Model</td>
</tr>
<tr>
<td><strong>Summary</strong></td>
<td>Brooklyn draws a visual of the ladder falling against the wall.</td>
</tr>
<tr>
<td><strong>Transcript</strong></td>
<td>Brooklyn: I am going to draw a visual first. (She re-reads the problem quietly to herself). So here is the wall. And the ladder. The distance between the bottom of the ladder and the wall is going to increase. (She labels the height of the ladder against the wall as x). We are trying to find how fast the ladder is sliding down.</td>
</tr>
</tbody>
</table>

Interviewer: So what is x there?

Brooklyn: x is the speed that wish the ladder is moving down the wall. So, um. (She re-reads the problem quietly to herself).

Interviewer: So what are you thinking right now?

Brooklyn: I am trying to Figure out at least one value, but it doesn't give you any values at all.

Next the researcher would watch the video for a third time and note what happened in the interview before the concerned MSF Action, reviewed the transcription, what happened right after the MSF Action instance and why the researcher coded the specific action. This information is exemplified in Table 20.

Table 20 - Example of Pre-Interview Analysis Step Three.

<table>
<thead>
<tr>
<th>Brooklyn Video Excerpt</th>
<th>Takes place on Video 1</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Time Stamp</strong></td>
<td>2:29</td>
</tr>
<tr>
<td><strong>MSF Action</strong></td>
<td>MSF Action 4 - Model</td>
</tr>
<tr>
<td>Before Instance</td>
<td>Brooklyn was asked by the interviewer to read Task 1.</td>
</tr>
</tbody>
</table>
| **Transcript**         | Brooklyn: I am going to draw a visual first. (She re-reads the problem quietly to herself). So here is the wall. And the ladder. The distance
Table 20 - Continued.

between the bottom of the ladder and the wall is going to increase. (She
labels the height of the ladder against the wall as \(x\)). We are trying to
find how fast the ladder is sliding down.

Interviewer: So what is \(x\) there?

Brooklyn: \(x\) is the speed that wish the ladder is moving down the wall.
So, um. (She re-reads the problem quietly to herself).

Interviewer: So what are you thinking right now?

Brooklyn: I am trying to Figure out at least one value, but it doesn't give
you any values at all.

| After Instance | Brooklyn begins to make a model with her hands to explain how the
|                | ladder is going to wall down against the wall. |
| Justification | Brooklyn is creating a 2-D model to help her understand the information
|                | in the problem. |

Next the researcher went back to Brooklyn’s Pre-MSI data and created a chart of the
MSF Actions exemplified by Brooklyn during her pre-interview (see Table 21)

Table 21 - Example of Pre-Interview Analysis Step Four.

<table>
<thead>
<tr>
<th>MSF Action</th>
<th>Brooklyn</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Seek to find and understand patterns.</td>
<td></td>
</tr>
<tr>
<td>2. Classify and characterize objects based on structure.</td>
<td>✓</td>
</tr>
<tr>
<td>3. Make and test conjectures about objects and structures.</td>
<td>✓</td>
</tr>
<tr>
<td>4. Create mental (and physical) models for, and examples and non-examples of, mathematical objects.</td>
<td>✓</td>
</tr>
<tr>
<td>5. Value and use precise definitions of objects.</td>
<td></td>
</tr>
<tr>
<td>6. Value an understanding of why relationships make sense.</td>
<td>✓</td>
</tr>
<tr>
<td>7. Value and use logical arguments and counterexamples as sources of conviction.</td>
<td>✓</td>
</tr>
<tr>
<td>8. Value precise language and have fine distinctions about language.</td>
<td></td>
</tr>
<tr>
<td>9. Value and use symbolic representations of, and notation for, objects and ideas.</td>
<td>✓</td>
</tr>
<tr>
<td>Total</td>
<td>6</td>
</tr>
</tbody>
</table>
After watching each interview and coding the students’ mathematical sophistication, analysis extended to looking for patterns among individuals (before and after the Internet section), between students who fell in different levels of mathematical sophistication and from task 1 to task 2. For example, the researcher looked at task 1 responses from each participant and noted any patterns. Then the researcher looked at the internet portion for task 1 and noted any patterns. Then the researcher looked at all of the students who changed their mind after the internet portion and if any of the students visited the same or similar websites. Then the researcher looked across the students’ final responses to task 1. Once this was done, a summary was written for each portion of the interview and an overall summary for each interview task was compiled.

Each of these steps were followed for each of the pre- and post-interviews. Then the researcher used the data collected from each pre- and post-interview to search and find patterns among the data: Do certain actions from the MSF show up more frequently throughout each interview, each task? Do any of the students’ give similar answers? Is there any trend in adequate/inadequate answers between the students themselves and among the students as a whole? Did student responses from the pre- to post-interview differ in any way? Does the students’ level of mathematical sophistication at the start of the course affect their responses to the interview tasks? The results from this analysis can be found in Chapter 4.

**Student Work Analysis**

The researcher decided to analyze only the student work from each interviewee instead of the whole class in order to better appreciate the bigger picture of analysis that include both the interviews and the instructional sequence. The mean Pre-MSI score for the
entire class was 11.212. The mean Pre-MSI score for the interview students was 12.222. The mean Pre-MSI score is slightly higher than those of the entire class, but still the Pre-MSI to Post-MSI gain was not statistically significant. Therefore, this group was determined to be a representative subset of the entire population.

In order to analyze these tasks, the researcher first looked at the task itself and identified which MSF Actions the tasks were designed to elicit and if students did in fact use the identified MSF Actions (a more detailed description was previously provided and shown in Table 16). Next the researcher looked at the student work and identify the MSF Actions exhibited by each of the nine interview participants. For example, Sesame Street task question 8 asks students to describe the behavior of a sequence they had been working with. This question was coded to reveal a students’ ability to explain their mathematical understanding of a concept (Action 6). If you look at Emily’s work in Figure 4, it is evident that she does provide an adequate explanation of the mathematics (Action 6) along with a graphical representation (Action 4) to help explain the sequence.

**Figure 5 - Emily's Work on Sesame Street Question 8.**
Therefore, Emily’s work would be coded as representing both Action 4 and Action 6.

Next, the researcher analyzed each students’ response to each question, determine the adequacy of their response and compare this analysis to their level of mathematical sophistication. For example, looking at Figure 4 again, Emily’s work showed that she was using multiple MSF Actions simultaneously.

Finally, the researcher looked at each interviewee’s responses from the beginning of the course to the end of the course and noted any changes in adequate responses, explanations or mathematical sophistication. For example, the researcher analyzed Emily’s description of a mathematical context (as shown in Figure 4) with a question that asks her to describe another mathematical context at the end of the semester. The results from this analysis can be found in Chapter 4.

Phase 3

Phase 3 consisted of looking for emerging themes or patterns among the pre- and post-MSI, pre- and post-interviews as well as the instructional sequence. The researcher looked across all data collected to investigate he following questions: What specific aspects about the course affected students’ mathematical sophistication? Were some MSF Actions more prevalent throughout the pre- to post-MSI, pre- to post-interviews as well as the instructional sequence? Did the results from each set of data complement each other or contradict each other? Were any of the results and findings supported by other research? The results and discussion from this analysis can be found in Chapter 5.
**Research Validity and Reliability**

Validity and reliability are concerned with whether the “instruments used are measuring what we think we are observing or measuring” (Merriam, 1995, p. 53) and “whether the results of a study are consistent with the data collected” (Merriam, 1995, p. 55) respectively. When piloted, the MSI was found to be reasonably reliable and valid as a measure of mathematical sophistication (Seaman, Kuennen, and Seaman, 2012).

Several strategies have been suggested in order to strengthen the validity and reliability of a qualitative study. One of the best known is triangulation of data. Triangulation is a “strategy that qualitative researchers can employ to shore up the internal validity of a study” (Merriam, 2002, p. 25). In order to ensure triangulation of data occurs I looked at each of the data sources mentioned above and made sure the emerging findings remained consistent. I have also enlisted peer reviewers, my advisor, and my committee members to ensure that the results found were consistent with the data collected. A second person also coded portions of the data to ensure that data coding was consistent and representative of the data collected.

“The extent to which the findings of a study can be applied to other situations refers to the question of external validity, or generalizability” (Merriam, 1995, p. 57). Considering the method of choosing students, purposeful, our study cannot be generalizable to mathematical sophistication of all pre-service elementary teachers. Yet, that does not decrease the value of this research considering the “goal of qualitative research, after all, is to understand the particular in depth, rather than finding out what is generally true of many” (Merriam, 1995, p. 57). In order to strengthen this aspect of rigor the researcher has provided...
a rich-think description of the study in Chapter 4 so readers can determine how closely this research applies to their field, therefore, if the findings can be generalized.

**Ethical Issues (IRB)**

On the first day of the study the subjects were given a brief overview of the study. This overview introduced the subjects to educational research and the policies used. Their part in the study was explained to them and they were told they could stop at any time they choose without consequences. The researcher explained the pre-test, pre-interview, instructional activity, post-test, and post-interview without disclosing any information of the research questions or the purpose of the study. Each subject was given a consent form for their work (Appendix F) and a separate concept form for videoing the class, groups working on tasks and for interviews (Appendix G) all 33 students chose to sign the form.

**Chapter Summary**

This study utilized a mixed methods approach in order to answer the research questions:

- How does pre-service elementary teachers’ mathematical sophistication change as they participate in a reform-oriented calculus course?
- What experiences do pre-service elementary teachers have during a reform-oriented calculus course that support their growth in mathematical sophistication?

This study is part of a larger project titled Calculus for Elementary Teachers: an Innovative Context (CELTIC). This study included 33 participants in two different CELTIC courses at a large public university located in the southeastern United States. This study extended on research on pre-service elementary teachers’ mathematical sophistication conducted by Seaman and Szydlik (2007). The design of this study included Seaman and Szydlik’s (2009)
Mathematical Sophistication Instrument (MSI) as a pre- and post-test for all CELTIC students, pre- and post-interviews, and three tasks from a semester long instructional sequence. Data analysis began at the completion of the course since the researcher was also the instructor for one section of CELTIC. Data Analysis consisted of multiple phases: Phase 1: Quantitative, Phase 2: Qualitative, Phase 3: Triangulation. All of the results from this analysis can be found in Chapter 4. All of the discussion and conclusions from this data can be found in Chapter 5.
Chapter 4 - Results

This study used both quantitative and qualitative methods as described in Chapter 3. The quantitative component consisted of analyzing data collected from 33 students’ pre- and post-MSI, descriptive statistics and statistical tests analyzed students’ level of mathematical sophistication at the beginning of the course, the end of the first semester, and any growth throughout the semester. The qualitative component consisted of nine students’ pre- and post-interviews analyzed with two different aims. One was to triangulate information from the surveys and determine if students’ level of MSF changed from pre- to post-interview. The second was to analyze each students’ responses, find any patterns and try to explain the change from pre- to post-interview. Finally, the same nine students’ work on the three instructional tasks were analyzed to show a progression of students’ responses from the beginning of the semester to the end of the semester.

Data Analysis consisted of three different phases: Phase 1: Quantitative, Phase 2: Qualitative, Phase 3: Triangulation. Phase 1 consisted of the pre- and post-MSI analysis and an analysis of the total number of MSF actions exhibited during the pre- and post-interviews. Phase 2 consisted of the qualitative pre- and post-interview analysis as well as the analysis of the instructional sequence. Phase 3 consisted of looking for emerging themes or patterns among the pre- and post-MSI, pre- and post-interviews as well as the instructional sequence.

Phase 1

In this section, both descriptive and inferential statistics are presented. These results tell a detailed story of how students (considered as a whole and in many different subsets)
improved in their mathematical sophistication during the first semester of the CELTIC course.

**Mathematical Sophistication Instrument Implementation Analysis**

**Pre-MSI**

The pre-test used in this study was the Mathematical Sophistication Instrument (MSI) (for more information see chapter 3). Implementation of the pre-MSI happened on the first day of CELTIC. A summary of the pre-MSI results is shown in Table 22, each students’ individual score can be found in Table 22.

**Table 22 - Pre-MSI Results.**

<table>
<thead>
<tr>
<th>Level</th>
<th>Description</th>
<th>MSI Score</th>
<th>Pre-MSI</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>highly unsophisticated</td>
<td>0-5</td>
<td>3 (9%)</td>
</tr>
<tr>
<td>2</td>
<td>fairly unsophisticated</td>
<td>6-10</td>
<td>13 (39%)</td>
</tr>
<tr>
<td>3</td>
<td>average sophistication</td>
<td>11-15</td>
<td>11 (33%)</td>
</tr>
<tr>
<td>4</td>
<td>fairly sophisticated</td>
<td>16-20</td>
<td>6 (18%)</td>
</tr>
<tr>
<td>5</td>
<td>highly sophisticated</td>
<td>21-25</td>
<td>0</td>
</tr>
</tbody>
</table>

Seaman and Szydlik (2009) created the levels of mathematical sophistication, their description as well as how to categorize them by score in previous research. The mean score for the pre-MSI implementation was 11.212 with a standard deviation of 3.62; this corresponds to the low end of average sophistication. The majority, 13 (39%), of the participants scored between 6 and 10 on the pre-MSI, therefore, they are categorized as Level 2, found in Table 22. Three participants scored at Level 1, 11 scored at Level 3 and 6 scored at Level 4, found in Table 22. No one scored at Level 5.
Pre- to Post-MSI Analysis

As a group, the 33 participants showed modest improvement in their mathematical sophistication throughout the course. Individual changes can be seen in Table 23.

Table 23 - Participant Pre-MSI and Post-MSI analysis.

<table>
<thead>
<tr>
<th>Participant Name</th>
<th>Academic Level</th>
<th>Pre-Cal</th>
<th>Pre-MSI</th>
<th>Post-MSI</th>
<th>Gain</th>
<th>Pre-MSI Level</th>
<th>Post-MSI Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mia</td>
<td>Sophomore</td>
<td>5</td>
<td>4</td>
<td>-1</td>
<td></td>
<td>Level 1</td>
<td>Level 1</td>
</tr>
<tr>
<td>Ava</td>
<td>Sophomore</td>
<td>A</td>
<td>5</td>
<td>10</td>
<td>5</td>
<td>Level 1</td>
<td>Level 2</td>
</tr>
<tr>
<td>Grace</td>
<td>Freshman</td>
<td>A</td>
<td>5</td>
<td>5</td>
<td>0</td>
<td>Level 1</td>
<td>Level 1</td>
</tr>
<tr>
<td>Leah</td>
<td>Sophomore</td>
<td>B</td>
<td>7</td>
<td>8</td>
<td>1</td>
<td>Level 2</td>
<td>Level 2</td>
</tr>
<tr>
<td>Charlotte</td>
<td>Freshman</td>
<td>A</td>
<td>8</td>
<td>14</td>
<td>6</td>
<td>Level 2</td>
<td>Level 3</td>
</tr>
<tr>
<td>Avery</td>
<td>Freshman</td>
<td></td>
<td>8</td>
<td>8</td>
<td>0</td>
<td>Level 2</td>
<td>Level 2</td>
</tr>
<tr>
<td>Chloe</td>
<td>Freshman</td>
<td>A</td>
<td>8</td>
<td>5</td>
<td>-3</td>
<td>Level 2</td>
<td>Level 1</td>
</tr>
<tr>
<td>Natalie</td>
<td>Freshman</td>
<td>A</td>
<td>9</td>
<td>14</td>
<td>5</td>
<td>Level 2</td>
<td>Level 3</td>
</tr>
<tr>
<td>Abigail</td>
<td>Sophomore</td>
<td>B</td>
<td>9</td>
<td>8</td>
<td>-1</td>
<td>Level 2</td>
<td>Level 2</td>
</tr>
<tr>
<td>Ali</td>
<td>Sophomore</td>
<td>B</td>
<td>9</td>
<td>10</td>
<td>1</td>
<td>Level 2</td>
<td>Level 2</td>
</tr>
<tr>
<td>Riley</td>
<td>Freshman</td>
<td>C</td>
<td>9</td>
<td>10</td>
<td>1</td>
<td>Level 2</td>
<td>Level 2</td>
</tr>
<tr>
<td>Amelia</td>
<td>Sophomore</td>
<td>A</td>
<td>9</td>
<td>8</td>
<td>-1</td>
<td>Level 2</td>
<td>Level 2</td>
</tr>
<tr>
<td>Lily</td>
<td>Freshman</td>
<td>B</td>
<td>9</td>
<td>7</td>
<td>-2</td>
<td>Level 2</td>
<td>Level 2</td>
</tr>
<tr>
<td>Emma</td>
<td>Freshman</td>
<td>B</td>
<td>10</td>
<td>10</td>
<td>0</td>
<td>Level 2</td>
<td>Level 2</td>
</tr>
<tr>
<td>Isabella</td>
<td>Freshman</td>
<td>A</td>
<td>10</td>
<td>12</td>
<td>2</td>
<td>Level 2</td>
<td>Level 3</td>
</tr>
<tr>
<td>Samantha</td>
<td>Freshman</td>
<td>B</td>
<td>11</td>
<td>10</td>
<td>-1</td>
<td>Level 3</td>
<td>Level 2</td>
</tr>
<tr>
<td>Faith</td>
<td>Freshman</td>
<td>B</td>
<td>11</td>
<td>12</td>
<td>1</td>
<td>Level 3</td>
<td>Level 3</td>
</tr>
<tr>
<td>Hannah</td>
<td>Freshman</td>
<td>A</td>
<td>12</td>
<td>9</td>
<td>-3</td>
<td>Level 3</td>
<td>Level 2</td>
</tr>
<tr>
<td>Ella</td>
<td>Freshman</td>
<td>B</td>
<td>12</td>
<td>10</td>
<td>-2</td>
<td>Level 3</td>
<td>Level 2</td>
</tr>
<tr>
<td>Victoria</td>
<td>Freshman</td>
<td></td>
<td>12</td>
<td>12</td>
<td>0</td>
<td>Level 3</td>
<td>Level 3</td>
</tr>
<tr>
<td>Emily</td>
<td>Freshman</td>
<td>A</td>
<td>12</td>
<td>13</td>
<td>1</td>
<td>Level 3</td>
<td>Level 3</td>
</tr>
<tr>
<td>Evelyn</td>
<td>Freshman</td>
<td>B</td>
<td>12</td>
<td>13</td>
<td>1</td>
<td>Level 3</td>
<td>Level 3</td>
</tr>
<tr>
<td>Zoe</td>
<td>Junior</td>
<td>B</td>
<td>13</td>
<td>10</td>
<td>-3</td>
<td>Level 3</td>
<td>Level 2</td>
</tr>
<tr>
<td>Sophia</td>
<td>Freshman</td>
<td>B</td>
<td>13</td>
<td>15</td>
<td>2</td>
<td>Level 3</td>
<td>Level 3</td>
</tr>
<tr>
<td>Olivia</td>
<td>Freshman</td>
<td>A</td>
<td>13</td>
<td>7</td>
<td>-6</td>
<td>Level 3</td>
<td>Level 2</td>
</tr>
<tr>
<td>Aubrey</td>
<td>Sophomore</td>
<td></td>
<td>13</td>
<td>14</td>
<td>1</td>
<td>Level 3</td>
<td>Level 3</td>
</tr>
</tbody>
</table>
Table 23 - Continued.

<p>| | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Brooklyn</strong></td>
<td>Freshman</td>
<td>B</td>
<td>B</td>
<td>15</td>
<td>14</td>
<td>-1</td>
</tr>
<tr>
<td><strong>Elizabeth</strong></td>
<td>Freshman</td>
<td>B</td>
<td>16</td>
<td>16</td>
<td>0</td>
<td>Level 4</td>
</tr>
<tr>
<td><strong>Layla</strong></td>
<td>Freshman</td>
<td>A</td>
<td>16</td>
<td>17</td>
<td>1</td>
<td>Level 4</td>
</tr>
<tr>
<td><strong>Maddie</strong></td>
<td>Freshman</td>
<td>A</td>
<td>17</td>
<td>19</td>
<td>2</td>
<td>Level 4</td>
</tr>
<tr>
<td><strong>Lillian</strong></td>
<td>Freshman</td>
<td>A</td>
<td>17</td>
<td>19</td>
<td>2</td>
<td>Level 4</td>
</tr>
<tr>
<td><strong>Addison</strong></td>
<td>Freshman</td>
<td>A</td>
<td>17</td>
<td>16</td>
<td>-1</td>
<td>Level 4</td>
</tr>
<tr>
<td><strong>Harper</strong></td>
<td>Freshman</td>
<td>A</td>
<td>18</td>
<td>18</td>
<td>0</td>
<td>Level 4</td>
</tr>
<tr>
<td><strong>Mean</strong></td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>11.212</td>
<td>11.424</td>
</tr>
<tr>
<td><strong>STD</strong></td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>3.621</td>
<td>4.031</td>
</tr>
</tbody>
</table>

The data is symmetric, but does not follow a normal distribution. Thus, a non-parametric Wilcoxon signed rank test was used instead of a paired *t*-test to compare the difference in students’ level of mathematical sophistication from the start of the course until the end of the semester.

First, analysis of the overall participant improvement was conducted using a Wilcoxon signed rank test, which can be found in Table 24.

Table 24 - Pre-MSI to Post-MSI Wilcoxon signed rank test results.

<table>
<thead>
<tr>
<th>Total n</th>
<th>Mean of Pre-Test</th>
<th>Mean of Post-Test</th>
<th>Test Statistic</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>33</td>
<td>11.212</td>
<td>11.424</td>
<td>172.5</td>
<td>0.6955</td>
</tr>
</tbody>
</table>

Null Hypothesis Test Significance Level Decision
The mean of the differences between the Pre-MSI and the Post-MSI equals 0. Wilcoxon Signed Rank Test $\alpha = 0.05$ Fail to reject the null hypothesis

The conclusion of the Wilcoxon signed rank test shows that the overall participant improvement was not statistically significant although the mean did go up.
Although the improvements were not statistically significant, most groups of students’ overall mean improved from the pre- to post-MSI, which can be found in Table 25.

Table 25 - Pre-MSI to Post-MSI MSF Level Analysis.

<table>
<thead>
<tr>
<th>Level</th>
<th>Number of Participants</th>
<th>Pre-MSI Average</th>
<th>Post-MSI Average</th>
<th>Mean Growth</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 1</td>
<td>N=3</td>
<td>5</td>
<td>6.333</td>
<td>1.333</td>
</tr>
<tr>
<td>Level 2</td>
<td>N=12</td>
<td>8.75</td>
<td>9.5</td>
<td>0.75</td>
</tr>
<tr>
<td>Level 3</td>
<td>N=12</td>
<td>12.417</td>
<td>11.583</td>
<td>-0.833</td>
</tr>
<tr>
<td>Level 4</td>
<td>N=6</td>
<td>16.833</td>
<td>17.5</td>
<td>0.667</td>
</tr>
<tr>
<td>Level 5</td>
<td>N=0</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>Mean</td>
<td></td>
<td>11.212</td>
<td>11.424</td>
<td>0.212</td>
</tr>
</tbody>
</table>

Figure 6 shows the students’ average Pre-MSI score, Post-MSI Score and Average Gain throughout the study.

Figure 6 - MSI Comparison from Pre-MSI to Post-MSI by MS Level.
As you can see from Table 25 and Figure 5, every group of students’ mean, except those identified as performing at a Level 3 on the Pre-MSI, improved from the Pre-MSI to the Post-MSI. The growth is strongest for those identified as performing at a Level 1 on the Pre-MSI, followed by Level 2. Therefore, the course appears to have been most beneficial to students with the lowest level of mathematical sophistication at the beginning of the course.

**Mathematical Sophistication Instrument Question Analysis**

Next, analysis was conducted on every question on the Mathematical Sophistication Instrument (MSI). These questions are in Appendix A. An analysis of each question is in Table 26.

**Table 26 - Pre-MSI to Post-MSI questions analysis.**

<table>
<thead>
<tr>
<th>MSI Question</th>
<th>MSF Action</th>
<th>Pre-MSI Total Correct</th>
<th>Post-MSI Total Correct</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Question 1</td>
<td>5 &amp; 8</td>
<td>26</td>
<td>23</td>
<td>-3</td>
</tr>
<tr>
<td>Question 2</td>
<td>7 &amp; 8</td>
<td>10</td>
<td>13</td>
<td>3</td>
</tr>
<tr>
<td>Question 3</td>
<td>8</td>
<td>19</td>
<td>14</td>
<td>-5</td>
</tr>
<tr>
<td>Question 4</td>
<td>9</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Question 5</td>
<td>3, 6 &amp; 7</td>
<td>17</td>
<td>18</td>
<td>1</td>
</tr>
<tr>
<td>Question 6</td>
<td>7</td>
<td>6</td>
<td>4</td>
<td>-2</td>
</tr>
<tr>
<td>Question 7</td>
<td>1 &amp; 4</td>
<td>23</td>
<td>21</td>
<td>-2</td>
</tr>
<tr>
<td>Question 8</td>
<td>1, 2, 3 &amp; 9</td>
<td>7</td>
<td>14</td>
<td>7</td>
</tr>
<tr>
<td>Question 9</td>
<td>5, 8 &amp; 9</td>
<td>27</td>
<td>30</td>
<td>3</td>
</tr>
<tr>
<td>Question 10</td>
<td>2, 4 &amp; 9</td>
<td>8</td>
<td>5</td>
<td>-3</td>
</tr>
<tr>
<td>Question 11</td>
<td>6 &amp; 7</td>
<td>17</td>
<td>13</td>
<td>-4</td>
</tr>
<tr>
<td>Question 12</td>
<td>7</td>
<td>9</td>
<td>12</td>
<td>3</td>
</tr>
<tr>
<td>Question 13</td>
<td>5 &amp; 8</td>
<td>25</td>
<td>30</td>
<td>5</td>
</tr>
<tr>
<td>Question 14</td>
<td>5, 7 &amp; 8</td>
<td>5</td>
<td>4</td>
<td>-1</td>
</tr>
<tr>
<td>Question 15</td>
<td>8 &amp; 9</td>
<td>20</td>
<td>21</td>
<td>1</td>
</tr>
<tr>
<td>Question 16</td>
<td>5 &amp; 9</td>
<td>11</td>
<td>14</td>
<td>3</td>
</tr>
<tr>
<td>Question 17</td>
<td>2</td>
<td>15</td>
<td>9</td>
<td>-6</td>
</tr>
</tbody>
</table>
Table 26 - Continued.

<table>
<thead>
<tr>
<th>Question</th>
<th>Pre-MI</th>
<th>Total</th>
<th>Correct</th>
<th>Gain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Question 18</td>
<td>2</td>
<td>18</td>
<td>19</td>
<td>1</td>
</tr>
<tr>
<td>Question 19</td>
<td>5 &amp; 9</td>
<td>22</td>
<td>18</td>
<td>-4</td>
</tr>
<tr>
<td>Question 20</td>
<td>4</td>
<td>6</td>
<td>6</td>
<td>0</td>
</tr>
<tr>
<td>Question 21</td>
<td>6</td>
<td>9</td>
<td>9</td>
<td>0</td>
</tr>
<tr>
<td>Question 22</td>
<td>5 &amp; 9</td>
<td>26</td>
<td>29</td>
<td>3</td>
</tr>
<tr>
<td>Question 23</td>
<td>4</td>
<td>18</td>
<td>16</td>
<td>-2</td>
</tr>
<tr>
<td>Question 24</td>
<td>1, 2, 3 &amp; 4</td>
<td>13</td>
<td>12</td>
<td>-1</td>
</tr>
<tr>
<td>Question 25</td>
<td>1, 2, 3 &amp; 4</td>
<td>14</td>
<td>17</td>
<td>3</td>
</tr>
</tbody>
</table>

There are 25 questions on the MSI. Of those 25 questions: 12 questions showed a positive gain, 2 questions showed no change and 11 questions showed a negative gain. To analyze these differences, statistical analysis on the questions that showed either positive or negative gain, as well as those with no gain. The questions with the greatest positive gain can be found in Figure 7.

![Figure 7 - MSI questions with the most positive gain.](image-url)
Questions 2, 9, 12, 16, 22 and 25 showed some positive gain and questions 8 and 13 showed the most gain. Looking back at Table 26, all MSF actions except action 6 are represented in these questions. Actions 5, 8 and 9 are represented in at least 3 of the 8 questions above. Therefore, questions involving definitions, precise language and fine distinctions about language and symbolic notation showed the most significant gain. The questions with the most negative gain are in Figure 8.

![Pre-MSI to Post-MSI Question Negative Gain](image)

**Figure 8 - MSI questions with the most negative gain.**

Questions 1, 3, 6, 10, 11, 17 & 19 showed the most negative gain. Looking back at Table 26, all MSF actions are tested by these questions except actions 1 & 3. Questions 17 and question 3 represent the most negative gain. These questions fit in actions 2 and 8 respectively. Therefore, the most negative gain was in students’ abilities to find the meaning behind the mathematics as well as using and valuing precise language and fine distinctions
about language. Action 8 (precise language and fine distinctions about language) was in both the most positive and most negative gain. Action 8 was tested in 7 questions, only three had a negative gain. Therefore, most of the questions testing action 8 showed positive gain, of those that showed positive gain it was a high number of students who answered the question correctly on the post-MSI. The questions that showed negative gains on action 8 were lower in number and the quantity of correct answers were lower.

**Interview Analysis**

The semi-structured, one-on-one interviews used in a study by Seaman and Szydlik (2007) influenced the pre- and post-interviews in this study, as described in Chapter 2. Nine of the 33 students participated in a pre-interview and a similar post-interview in addition to the Pre-MSI and Post-MSI implementations. Each interview was analyzed and categorized as described in Chapter 3. The total number of actions of the MSF each participant used during the interview are in Table 27 under Pre-Interview Action Total, Post-Interview Action Total and Gain. The Pre-MSI Score, Post-MSI Score, Gain, (from the instrument implementation described earlier), Pre-Interview Action Total (Pre-IAT), Post-Interview Action Total (Post-IAT) and Gain (from pre-MSI to post-MSI) can be found in Table 27. Some of this data was already listed, but it is repeated here for just the nine students interviewed. The Pre-MSI score for the interview students was 12.222; the Post-MSI score for the interview students was 13.33 with an average gain of 1.111. These averages are slightly higher than those of the entire class, but still the Pre-MSI to Post-MSI gain was not statistically significant.
Table 27 - Interview Quantitative Analysis.

<table>
<thead>
<tr>
<th>Participant Name (n=9)</th>
<th>Pre-MSI Level</th>
<th>Pre-MSI Score</th>
<th>Post-MSI Score</th>
<th>MSI Gain</th>
<th>Pre-IAT</th>
<th>Post-IAT</th>
<th>IAT Gain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Charlotte</td>
<td>Level 2</td>
<td>8</td>
<td>14</td>
<td>6</td>
<td>4</td>
<td>7</td>
<td>3</td>
</tr>
<tr>
<td>Natalie</td>
<td>Level 2</td>
<td>9</td>
<td>14</td>
<td>5</td>
<td>2</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>Amelia</td>
<td>Level 2</td>
<td>9</td>
<td>8</td>
<td>-1</td>
<td>3</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>Samantha</td>
<td>Level 3</td>
<td>11</td>
<td>10</td>
<td>-1</td>
<td>4</td>
<td>7</td>
<td>3</td>
</tr>
<tr>
<td>Hannah</td>
<td>Level 3</td>
<td>12</td>
<td>9</td>
<td>-3</td>
<td>5</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>Emily</td>
<td>Level 3</td>
<td>12</td>
<td>13</td>
<td>1</td>
<td>5</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>Brooklyn</td>
<td>Level 3</td>
<td>15</td>
<td>14</td>
<td>-1</td>
<td>5</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>Maddie</td>
<td>Level 4</td>
<td>17</td>
<td>19</td>
<td>2</td>
<td>6</td>
<td>5</td>
<td>-1</td>
</tr>
<tr>
<td>Lillian</td>
<td>Level 4</td>
<td>17</td>
<td>19</td>
<td>2</td>
<td>6</td>
<td>6</td>
<td>0</td>
</tr>
<tr>
<td><strong>Mean</strong></td>
<td><strong>12.222</strong></td>
<td><strong>13.333</strong></td>
<td><strong>1.111</strong></td>
<td><strong>4.444</strong></td>
<td><strong>5.777</strong></td>
<td><strong>1.333</strong></td>
<td></td>
</tr>
</tbody>
</table>

Action 1 (find and understand patterns) was not represented in the pre-interview task since each task was using covariational reasoning to graph a representation and no pattern was needed to complete these tasks. Also, Action 7 (use logical arguments) was not represented in the post-interview tasks since the post interview tasks were focused on applying integration understanding not logic or proving integration; therefore, in order to analyze and compare the Pre-Interview Action Total and the Post-Interview Action Total (Post-IAT), Actions 1 and 7 were removed from the data analysis. An exploratory analysis was completed on all interview students’ improvement for each of the remaining 7 actions: 2 (recognizing fundamental mathematics), 3 (making and testing conjectures), 4(creating models), 5 (valuing definitions), 6 (understanding relationships), 8 (precise language and fine distinctions about language), & 9 (valuing symbolic representations).

The Pre-IAT is the total number of each MSF Action that participants used to solve the novel tasks during the pre-interview. Students with higher levels of mathematical
sophistication use more MSF actions while solving novel tasks. Therefore, the researcher looked at the total number of actions used by each student from the pre-interview to the post-interview to see if they used more MSF actions from the pre-interview to post-interview. The mean score was 4.444. The Post-IAT is the total number of each MSF Action that participants used to solve the novel tasks in the post-interview. The mean score was 5.777. The gain between Pre-IAT and Post-IAT was 1.333. This means that overall each student used more actions from the MSF to solve the tasks in the post-interview. The mean gain in the MSI (1.111) is relatively close to the mean gain in the IAT (1.333). Since the gain for the IAT was higher, a Wilcoxon signed rank test was conducted on the paired data for the nine interview students IAT results. The Pre-IAT to Post-IAT results are in Table 28.

<table>
<thead>
<tr>
<th>Total n</th>
<th>Mean of Pre-IAT</th>
<th>Mean of Post-IAT</th>
<th>Test Statistic</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>4.444</td>
<td>5.777</td>
<td>2</td>
<td>0.04858</td>
</tr>
</tbody>
</table>

**Null Hypothesis Test Significance Level Decision**

The mean of the differences between the Pre-IAT and the Post-IAT equals 0.

Wilcoxon Signed Rank Test $\alpha = 0.05$ Reject the null hypothesis, true location shift is not equal to 0

The conclusion of the Wilcoxon signed rank test shows that the improvement in the number of MSF Actions each student used from the Pre-IAT to Post-IAT was statistically significant. The results are discussed further in Chapter 5.
Next, the number of each MSF action use from the pre-interview to the post-interview was analyzed. The total of each participant who exemplified each action from the pre-interview to the post-interview can be found in Table 29.

**Table 29 - Pre-Interview to Post-Interview Action Analysis by Action.**

<table>
<thead>
<tr>
<th>Total Number of Students Who Exemplified Each Action</th>
<th>Pre-Interview Total</th>
<th>Post-Interview Total</th>
<th>Gain</th>
</tr>
</thead>
<tbody>
<tr>
<td>MSF Action 1</td>
<td>N/A</td>
<td>0</td>
<td>N/A</td>
</tr>
<tr>
<td>MSF Action 2</td>
<td>5</td>
<td>4</td>
<td>-1</td>
</tr>
<tr>
<td>MSF Action 3</td>
<td>8</td>
<td>9</td>
<td>1</td>
</tr>
<tr>
<td>MSF Action 4</td>
<td>8</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>MSF Action 5</td>
<td>3</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>MSF Action 6</td>
<td>6</td>
<td>8</td>
<td>2</td>
</tr>
<tr>
<td>MSF Action 7</td>
<td>5</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>MSF Action 8</td>
<td>2</td>
<td>9</td>
<td>7</td>
</tr>
<tr>
<td>MSF Action 9</td>
<td>8</td>
<td>8</td>
<td>0</td>
</tr>
</tbody>
</table>

The majority of the actions showed positive gain. Actions 5 (attention to definition), Action 6 (understanding and explanation) and 8 (precise language and fine distinctions about language) showed the most positive gain from Pre-Interview to Post-Interview. These results are discussed further in Chapter 5.

**Phase 1 Summary**

Analyzing the Pre-MSI to Post-MSI showed overall participant improvement in mathematical sophistication. The conclusion of the Wilcoxon signed rank test conducted on students’ Pre-MSI score as compared to their Post-MSI score was not statistically significant.
Analyzing the questions on the MSI and the Mathematical Sophistication Framework

Actions represented by each question showed important improvements on Action 5 (valuing definition), Action 8 (precise language and fine distinctions about language) and Action 9 (symbolic notation) while the most negative gain was seen through Action 2 (the meaning behind the mathematics). The other five actions showed little to no gain from Pre-MSI to Post-MSI.

To emphasize, analyzing the Pre-Interview to Post-Interview showed the total number of actions students employed while trying to solve a novel task increased. The most gain was seen in Action 5 (valuing definition), Action 6 (understanding and explanation) and Action 8 (precise language and fine distinctions about language). Therefore, action 5 (valuing definitions) and action 8 (precise language and fine distinctions about language) showed the most positive gain throughout the MSI and Interview analysis.

Phase 2

Pre-Interview

The first task, called the Ladder Problem, a modification of a problem reported by Monk (1992), prompted students to reason about a dynamic situation that includes a ladder resting on a wall (Carlson, 2002). Students were asked to describe the speed of the top of the ladder as it slides down the wall. Three answers were expected: the top of the ladder will either slow down, speed up, or remains constant as it slides down the wall.

The second task is the Bottle Problem. The problem prompted students to construct a graph of a dynamic situation with a continuously changing rate. Students were asked to describe and graph a representation of the relationship between the amount of water that is in
the bottle and the height of the water. From Carlson et al (2002) we expected students to construct a straight line with positive slope, an increasing concave-up graph, an increasing concave-down graph, or a graph where all aspects of it were acceptable. To help in the reading of the analysis, the solutions to the previous tasks are in Table 30.

**Table 30 - Mathematical Solutions to Problems in the Pre-Interview.**

<table>
<thead>
<tr>
<th>Task Number</th>
<th>Mathematical Problem and Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>One</td>
<td>From a vertical position against the wall, a ladder is pulled away at the bottom at a constant rate. Describe the speed of the top of the ladder as it slides down the wall. Justify your claim. <em>As you pull the latter away from the wall in uniform amounts the amount by which the top of the ladder falls gets bigger as it gets closer to the ground. Therefore, the rate at which the ladder is falling increases as it falls (Carlson, 2002).</em></td>
</tr>
<tr>
<td>Two</td>
<td>Imagine this bottle filling with water. Sketch a graph that represents the relationship between amount of water that is in the bottle and the height. Explain.</td>
</tr>
</tbody>
</table>

*Figure 2. Student A’s written response.*

(Carlson, 2002, p. 21).
In the following sections, discussion of some students’ solutions are presented and illustrative examples are given. Space did not permit completely discussing each of the students’ solutions to the tasks.

Task 1

Table 31 describes how each interview student responded during each phase of the pre-interview for task 1.

Table 31 - Student Responses to Pre-Interview Task 1.

<table>
<thead>
<tr>
<th>Participant/MS Level</th>
<th>Initial Task Responses</th>
<th>Internet Research</th>
<th>Concluding Task Responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>Charlotte/Level 2</td>
<td>Task 1: Stays Constant</td>
<td>Task 1: Rate definition, equation, examples</td>
<td>Task 1: Stays Constant</td>
</tr>
<tr>
<td>Natalie/Level 2</td>
<td>Task 1: Stays Constant</td>
<td>Task 1: Rate definition, equation</td>
<td>Task 1: Stays Constant</td>
</tr>
<tr>
<td>Amelia/Level 2</td>
<td>Task 1: Increases</td>
<td>Task 1: Related rate example (Seminole), creates physical model,</td>
<td>Task 1: Stays Constant - misled by her physical model</td>
</tr>
<tr>
<td>Samantha/Level 3</td>
<td>Task 1: Increases</td>
<td>Task 1: Related rate example (Seminole, Paul’s Notes &amp; Khan academy video)</td>
<td>Task 1: Stays Constant, video helped the most, she describes formulas</td>
</tr>
<tr>
<td>Hannah/Level 3</td>
<td>Task 1: Stays Constant</td>
<td>Task 1: Example, definition, explanation</td>
<td>Task 1: Stays Constant</td>
</tr>
<tr>
<td>Emily/Level 3</td>
<td>Task 1: Increases</td>
<td>Task 1: Example with solution.</td>
<td>Task 1: Increases</td>
</tr>
<tr>
<td>Brooklyn/Level 3</td>
<td>Task 1: Stays Constant</td>
<td>Task 1: Example with solution, video explanation</td>
<td>Task 1: Increase</td>
</tr>
<tr>
<td>Maddie/Level 4</td>
<td>Task 1: Increase</td>
<td>Task 1: Example with solution, video explanation</td>
<td>Task 1: Increases</td>
</tr>
<tr>
<td>Lillian/Level 4</td>
<td>Task 1: Stays Constant</td>
<td>Task 1: Definition, constant rate of change,</td>
<td>Task 1: Stays Constant</td>
</tr>
</tbody>
</table>

Four of the interviewees (Amelia, Samantha, Emily & Maddie) initially thought that the rate of the ladder would increase as it fell down the wall. Emily and Maddie both felt the ladder would still increase in speed after the Internet portion of the interview. While searching the
Internet, both Emily and Maddie found example problems online and used either algebra or calculus to prove their initial thought was adequate with the Pythagorean theorem (Action 9). Emily used algebra to explain how she knew the ladder would fall at an increasing rate (Figure 9).

**Figure 9 - Emily's work on Pre-Interview Task I.**
Emily’s description of her work can be found below:

Emily: My original assumption was somewhat adequate about it (ladder problem). It would speed up as the base of the ladder was further from the wall. And, I just found that by making up a height of the ladder and how fast you would be moving it away and using PT and showing that after 1 second it would be so far away, and as you can see, like from that, that is a short amount of distance that it is falling, but from the 4 and 5th second, it falls 6 feet.

Interviewer: So you think it is going to speed up and everything, you used PT. So why did you choose PT? You had that in your mind earlier.

Emily: Because the wall, the ladder and the base for a triangle, and I automatically think of PT when I am working with triangles.

Interviewer: Anything on the Internet that helped you answer this task?

Emily: Yeah, there was an example that had a situation and that is kind-of where I took this from [the algebra shown in Figure 8], but um, it just justified what I was thinking. And it showed like a video of the ladder falling.

Amelia and Samantha changed their minds and said that the ladder would fall at a constant rate after working on the Internet. Amelia, Samantha and Emily went to the same website titled Related Rate Problems (www2.seminolestate.edu/lvosbury/calculusl_folder/RelatedRateProblems.htm), yet neither student directly talked about this website as being of any help. The website gives a similar problem and uses Pythagorean Theorem to find the height of the ladder for different periods of time as seen in Figure 10.
When asked, Samantha attributed her change to a Khan Academy video titled Falling ladder related rates (https://www.khanacademy.org/math/differential-calculus/derivative_applications/rates_of_change/v/falling-ladder-related-rates) and Amelia attributed her change to a physical model that she made and described below:

Amelia: So I said it increased, I used my little wall right there that was a book and my pen and kept doing it. And I also looked at a video of a ladder being pulled and I changed it to constant, because, I don't know, it was just working. I was thinking that it was going to have less, like I was thinking that it was going to have less of a, like I don't know why I was thinking that the wall wasn't going to be there that much, but it is, and its going to stay on there either way, and the only change is you pulling it and if that remains constant, this is going to remain at the same rate. From what I did no my little book and from watching the videos, so I changed it to constant. So it will go at the rate that it is being pulled.

Amelia: So I was just trying to like visualize, and the reason that I was thinking that it was going to fall quick, like when I pull something away, like the ladder, it falls to the ground, but then it is not on the wall anymore, that is you pulling it off the wall, so I guess if it stays on the wall that is going to stop it from being pulled by gravity.

Interviewer: So you think the wall is actually stopping you.

Amelia: Yes.
Interviewer: What made you come up with this?

Amelia: Well I was using this book as kind-of a wall and I was using this calculator as a ladder and I was pulling it while it stayed against the wall and then I did my pen and did the same thing, pulling it form the bottom. Then I went to you tube and look at, and pictures also, I went to images, I was just looking at a ladder against the wall and the only way it is going to like fall quicker, being pulled by gravity, is if what is stopping it from being pulled by gravity is moved, and that is the wall, and the wall is not going to move.

Both Amelia and Samantha performed lower on the Pre-MSI than Emily and Maddie who kept their original adequate solution.

Five (Charlotte, Natalie, Hannah, Brooklyn, and Lillian) of the interviewees initially thought that the ladder would fall down the wall at a constant rate since the bottom of the ladder was being pulled away at a constant rate. One, Brooklyn, of these five (categorized from different levels on the pre-MSI) did change her mind after the internet search. The other four did not. All five participants who thought the ladder would stay constant looked up the definition of rate of change and found the definition of constant rate of change at some point in each of their Internet searches. Brooklyn looked up examples and found a website that explained how the rate of the ladder would speed up as it fell down the wall with the use of the Pythagorean Theorem. She originally said it would remain constant, but goes on to say that the rate will increase as the ladder falls:

Interviewer: What did you use on the Internet to help you come up with that (that the ladder falls at an increasing rate)?

Brooklyn: I looked up like how, rates are related, basically related rates, how things are related. I found examples, with values in them, that are like similar to this problem, but not exactly the same, and I just tried to Figure out if they were constant or if they were different.
Brooklyn was the only student whose response was inadequate on the ladder problem before
the Internet session and explained the solution adequately after studying on the Internet.

Three (Emily, Brooklyn, and Maddie) students’ final solutions were coded as
adequate. They all also visited the Related Rates Problems (described above) website, or a
similar, website that explained the increase algebraically.

Table 32 - Student Responses to Pre-Interview Task 2.

<table>
<thead>
<tr>
<th>Participant/ MS Level</th>
<th>Initial Task Responses</th>
<th>Internet</th>
<th>Concluding Task Responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>Charlotte/ Level 2</td>
<td>Task 2: Straight Line</td>
<td>Task 2: Examples</td>
<td>Task 2: Straight Line</td>
</tr>
<tr>
<td>Natalie/ Level 2</td>
<td>Task 2: Straight Line</td>
<td>Task 2: Equation</td>
<td>Task 2: Straight Line</td>
</tr>
<tr>
<td>Amelia/ Level 2</td>
<td>Task 2: Adequate Graph, except no top</td>
<td>Task 2: Carlson bottle paper, qualitative graph page with examples</td>
<td>Task 2: She says it gets quicker at the neck, but didn’t put this on the graph.</td>
</tr>
<tr>
<td>Samantha/ Level 3</td>
<td>Task 2: Straight Line</td>
<td>Task 2: Qualitative graph page with examples</td>
<td>Task 2: Straight line with the end tipped</td>
</tr>
<tr>
<td>Hannah/ Level 3</td>
<td>Task 2: Straight line with the end tipped</td>
<td>Task 2: Qualitative graph page with example</td>
<td>Task 2: Straight line with the end tipped</td>
</tr>
<tr>
<td>Emily/ Level 3</td>
<td>Task 2: Straight line</td>
<td>Task 2: Nothing</td>
<td>Task 2: Adequate Graph</td>
</tr>
<tr>
<td>Brooklyn/ Level 3</td>
<td>Task 2: Straight line with the end tipped</td>
<td>Task 2: Qualitative graph page with example</td>
<td>Task 2: Adequate Graph, except no top</td>
</tr>
<tr>
<td>Maddie/ Level 4</td>
<td>Task 2: Straight line with the end tipped</td>
<td>Task 2: Nothing</td>
<td>Task 2: Adequate Graph</td>
</tr>
<tr>
<td>Lillian/ Level 4</td>
<td>Task 2: Straight line, with a bend in the middle</td>
<td>Task 2: Examples with explanations,</td>
<td>Task 2: Adequate Graph</td>
</tr>
</tbody>
</table>
**Task 2**

Table 32 describes how each interview student responded during each phase of the pre-interview for task 2. No students drew a completely adequate graphical representation of the bottle problem as shown in Table 30. Five students drew a straight line and explained that the relationship between the height of the water in the bottle and the amount of water in the bottle was directly proportional. Three (Hannah, Maddie and Lillian) drew a straight line with either a bend at the tip or in the middle. These students were starting to see that the shape of the bottle matters to the graphical representation of the relationship between the height of the water in the bottle and the amount of water in the bottle. They each scored high on the pre-MSI.

Amelia, who scored low on the Pre-MSI, drew the closest representation initially. Amelia worked through many different thoughts on her work during the interview on the bottle problem. Her work can be found in Figure 11.
Her description of the graph is as follows:

Amelia: At first I was thinking I was measuring two different things, but really they are correlated with each other, one depends on the other, as the height increases so does the amount of water is in there.

At first she thought she was graphing more than one quantity, you can see that on the top left graph on her work. Then she graphed a straight-line graph; you can see that on the bottom right graph on her work. Then she was asked if the middle part (the wider part) of the bottle would have any effect on her graph. She thought about it for a while and came up with the conclusion that it would make a difference. She then looked at different portions of the bottle and drew the graph in the bottom left of her work.
Three (Charlotte, Natalie, and Hannah) of the students stuck with their initial graphical representation after spending time on the Internet, the majority of these students scored low on the Pre-MSI.

The rest of the students, six (Amelia, Samantha, Emily, Brooklyn, Maddie and Lillian) either drew a closer representation from their initial thoughts or a completely adequate graphical representation after spending time on the Internet. Emily and Maddie did not use any help from the Internet, they spent time analyzing the task and thinking about covarying quantities to correct their graphical representations. Amelia, Samantha, and Brooklyn all visited one site, titled Filling Bottles (http://www.learner.org/courses/learningmath/algebra/session1/part_c/filling.html). The website had diagrams of different bottles and asked the students to match the adequate graphical representation with the adequate bottle, these can be seen in Figure 12.
You can then click on the SOLUTION button and the website provides the adequate matching. You can click on the animated version of the solution to Problem C9(b) and it will provide a dynamic graph of the bottle being filled with water while simultaneously filling the bottle with water.

Amelia, Samantha, and Brooklyn each used different parts of this website to improve upon their graphical representation. Amelia’s drawing before visiting this site neglected to represent the neck of the bottle as being different from the body, after the website she explained how her graph was almost adequate, but that it just needed the top to be steeper. She said that the top would fill up faster. Samantha initially said that the relationship would be linear, but after visiting the Related Rate Problems website she added a steeper tip to her
linear representation. Brooklyn initially drew a linear relationship to represent the water filling the bottle. Yet, after visiting the Related Rate Problems website, she drew an adequate graphical representation; this can be seen in Figure 13.

![Figure 13 - Brooklyn's Pre-Interview Task 2 Work.](image)

Lillian visited a different website, Lab 2 - Rates and Amounts of Change (http://www.unco.edu/nhs/mathsci/facstaff/oehrtman/math131-fall2012/Lab%20Rate%20Change%20part%202.pdf). The Lab explains how the width of a bottle affects the steepness of the slope of the graphical representation. Then the website gives an example extremely similar to the bottle problem used in this study. The website also provides an explanation as well as a solution, this can be seen in Figure 14.
Figure 14 - Lab 2 - Rates and Amounts of Change Website Example 4.

Lillian used this site to correct her graphical representation from a line with a bend in the middle to a satisfactory graphical representation, shown in Figure 15.
Lillian explains how the website helped her gain a better understanding of the problem below:

Interviewer: And what did you use on the Internet?

Lillian: That one lab, I think it was a like a science lab.

Interviewer: So you followed some experiment?

Lillian: Yeah, they used different shapes and the amount of change the rate of change was changing.

Interviewer: Was it a video?

Lillian: No it was a lot of graphs.
Pre-Interview Summary

The majority of the students’ initial thoughts about task 1 were that the rate of the ladder would stay constant since it was being pulled at a constant rate. These students’ level of mathematical sophistication was varied and only one student used an example from the Internet to help change her mind and come up with the adequate solution that the rate was increasing. Two of the interviewees initially thought that the rate of the ladder would increase as it fell down the wall, yet, after working on the Internet they changed their minds and said that the ladder would fall at a constant rate. Both of these students went to the same website titled Related Rate Problems; yet all three of the students who explained that the ladder would fall down the wall at an increasing rate all went to the Related Rates Problems website (or similar) website that explained the increase algebraically. Therefore, the Related Rates website helped those with higher levels of mathematical sophistication but hindered the understanding of those with lower levels of mathematical sophistication.

When it comes to Task 2, no students drew an adequate graphical representation of the bottle problem before their individual work on the internet. Three of the students stuck with their initial graphical representation after spending time on the Internet, the majority of these students scored low on the Pre-MSI. The rest of the students, six, either drew a closer graphical representation or an adequate graphical representation after spending time on the Internet. The students who showed improvement scored higher on the Pre-MSI.

Students’ exemplification of MSF Actions throughout the pre-interview for task 1 and task 2 can be found in Table 33.
Table 33 - Total MSF Actions Represented by Each Pre-Interview Participant.

<table>
<thead>
<tr>
<th>Participant/MSF Action</th>
<th>Charlotte/Level 2</th>
<th>Natalie/Level 2</th>
<th>Amelia/Level 2</th>
<th>Samantha/Level 3</th>
<th>Hannah/Level 3</th>
<th>Emily/Level 3</th>
<th>Brooklyn/Level 3</th>
<th>Maddie/Level 4</th>
<th>Lillian/Level 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>MSF Action 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MSF Action 2</td>
<td>✔️</td>
<td>✔️</td>
<td>✔️</td>
<td>✔️</td>
<td>✔️</td>
<td>✔️</td>
<td>✔️</td>
<td>✔️</td>
<td>✔️</td>
</tr>
<tr>
<td>MSF Action 3</td>
<td>✔️</td>
<td>✔️</td>
<td>✔️</td>
<td>✔️</td>
<td>✔️</td>
<td>✔️</td>
<td>✔️</td>
<td>✔️</td>
<td>✔️</td>
</tr>
<tr>
<td>MSF Action 4</td>
<td>✔️</td>
<td>✔️</td>
<td>✔️</td>
<td>✔️</td>
<td>✔️</td>
<td>✔️</td>
<td>✔️</td>
<td>✔️</td>
<td>✔️</td>
</tr>
<tr>
<td>MSF Action 5</td>
<td>✔️</td>
<td>✔️</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>✔️</td>
</tr>
<tr>
<td>MSF Action 6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>✔️</td>
<td>✔️</td>
</tr>
<tr>
<td>MSF Action 7</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>✔️</td>
<td>✔️</td>
</tr>
<tr>
<td>MSF Action 8</td>
<td>✔️</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>✔️</td>
<td>✔️</td>
</tr>
<tr>
<td>MSF Action 9</td>
<td>✔️</td>
<td>✔️</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>✔️</td>
<td>✔️</td>
</tr>
<tr>
<td>Total</td>
<td>4</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>7</td>
<td>7</td>
</tr>
</tbody>
</table>

Action 1 (find and understand patterns) was not represented by either task 1 or task 2 in the pre-interview, therefore it cannot be seen by the student’s responses. Students’ with higher levels of mathematical sophistication use more actions from the mathematical sophistication framework. The majority of students were able to make and test a conjecture (Action 3), on a novel task and felt that drawing a picture, creating some physical model (Action 4) or using formulas (Action 9) were a great way to approach novel tasks, despite their level of mathematical sophistication. Students with higher levels of mathematical sophistication were better able to understand and explain novel mathematics (Action 6) as well as utilize counter examples (Action 7) in order to solve novel tasks. These results will be discussed further in Chapter 5.
Post-Interview

The post-interviews consisted of two novel mathematical tasks that cover integration. The mathematical topic of integration was chosen because the first semester of the CELTIC course ends by covering differentiation; therefore, students are familiar with differentiation and able to apply their previous knowledge to integration. The complete post-interview protocol is in Appendix D. The interviewees were given the two tasks to solve on paper; then they were given a period of time to search the Internet for necessary information to solve the tasks; finally, the interviewees are given their original work and asked if they would like to make any changes. See Table 34 for solutions.

Table 34 - Mathematical Tasks in Post-Interview with Solutions.

<table>
<thead>
<tr>
<th>Task Number</th>
<th>Mathematical Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>One</td>
<td>You and your friend start off at noon and walk in the same direction along the same path at the rates shown in the Figure below.</td>
</tr>
</tbody>
</table>

![Graph showing velocity over time for you and your friend.](image)

a) Who walks faster at 1:00 pm? Who is ahead at 1:00 pm? Explain.

*You because your velocity is higher, You because there is more area between the curve and the x-axis for your velocity.*
### Table 34 - Continued.

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>b)</strong></td>
<td>Who walks faster at 2:00 pm? Who is ahead at 2:00 pm? Explain.</td>
<td></td>
</tr>
<tr>
<td></td>
<td><em>Neither because you both have the same velocity, You because there is more area between the curve and the x-axis for your velocity.</em></td>
<td></td>
</tr>
<tr>
<td><strong>c)</strong></td>
<td>Who walks faster at 3:00 pm? Who is ahead at 3:00 pm? Explain.</td>
<td></td>
</tr>
<tr>
<td></td>
<td><em>Friend because their velocity is higher, You because there is more area between the curve and the x-axis for your velocity.</em></td>
<td></td>
</tr>
<tr>
<td><strong>d)</strong></td>
<td>How can you find the time when you and your friend will be together? Answer in words.</td>
<td></td>
</tr>
<tr>
<td></td>
<td><em>When the area between the curve and the x-axis for each person is equal.</em></td>
<td></td>
</tr>
</tbody>
</table>

---

**Two**

An orchard owner, Melinda, is trying to find out the apple production of her orchard. Unfortunately, the production notes from previous year have been lost. However, she has found a Table of the rates of production of apple bushels over the past year. Use this Table to estimate a range of how many apples were produced in the last year. To make it easier, let’s assume there are 30 days in each month.

1. How many bushels of apples did Melinda’s orchard produce last year?
   
   Show your work and explain how you did that.
   
   *You take the production rate for each month and multiply it by the days in the month to find the production for that month. You repeat this process and add up each month’s production to calculate the total production. 1370 bushels a day x 30 days in each month, 41,000 bushels a year.*

2. Use the Table above to sketch a graph of the production rate on this coordinate axes. Let the horizontal axis be days starting with January 1, and let the y-axis be the production rate.
3. Look at your production rates and your graph. Is there anything that could connect the total production numbers you calculated to anything in the graph? How did you calculate the total production, does anything on the graph represent the numbers you used to calculate the production rate?

The area under each production rate added together will equal the total production rate. The goal is to have the students realize that the area under the production rate equals production rate x number of days in a month. Then they can connect that to how they calculated the total production and then they can start understanding the concept of integration from a conceptual point of view.

The first task was designed to analyze students’ abilities to read a velocity time graph and connect that understanding to distance. The second task was designed to analyze students’ build a conceptual understanding of integration by using previously known knowledge.

Again, the results from the tasks are summarized and then examples are provided from only a few of the interviewees.

Task 1

Table 35 describes how each interview student responded during each phase of the post-interview for task 1.

Table 35 - Student Responses for Post-Interview Task 1.

<table>
<thead>
<tr>
<th>Participant/MS Level</th>
<th>Initial Task Responses</th>
<th>Internet</th>
<th>Concluding Task Responses</th>
</tr>
</thead>
</table>
| Charlotte/Level 2    | Task 1: Mostly inadequate  
a) Inadequate, Adequate  
b) Inadequate, Inadequate  
c) Adequate, Inadequate  
d) Lines Cross = Same Distance | Task 1: Examples, graphical representations, examples with explanations, Wikipedia | Task 1: Mostly inadequate  
a) Inadequate, Adequate  
b) Inadequate, Inadequate  
c) Inadequate, Inadequate  
d) Lines Cross = Same Distance |
<table>
<thead>
<tr>
<th>Name</th>
<th>Level</th>
<th>Task 1 Description</th>
<th>Task 1 Examples,</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>Natalie</td>
<td>Level 2</td>
<td>Mostly inadequate</td>
<td>Examples, video</td>
<td>No changes</td>
</tr>
<tr>
<td></td>
<td></td>
<td>a) Adequate, Adequate</td>
<td>explanation</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>b) Inadequate, Inadequate</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>c) Inadequate, Inadequate</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>d) Lines Cross = Same Distance</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Amelia</td>
<td>Level 2</td>
<td>Mostly Adequate</td>
<td>Definition of Velocity</td>
<td>No change</td>
</tr>
<tr>
<td></td>
<td></td>
<td>a) Adequate, Adequate</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>b) Adequate, Adequate</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>c) Adequate, Inadequate</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>d) Lines Cross = Same Distance</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Samantha</td>
<td>Level 3</td>
<td>Mostly inadequate</td>
<td>Examples, graphical</td>
<td>Completely Adequate</td>
</tr>
<tr>
<td></td>
<td></td>
<td>a) Inadequate, Adequate</td>
<td>representations, video</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>b) Inadequate, Inadequate</td>
<td>examples and explanations</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>c) Adequate, Inadequate</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>d) Lines Cross = Same Distance</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hannah</td>
<td>Level 3</td>
<td>Mostly Adequate</td>
<td>Examples, graphical</td>
<td>Completely Adequate</td>
</tr>
<tr>
<td></td>
<td></td>
<td>a) Adequate, Adequate</td>
<td>representations,</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>b) Adequate, Inadequate</td>
<td>definitions</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>c) Adequate, Inadequate</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>d) Lines Cross = Same Distance</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Emily</td>
<td>Level 3</td>
<td>Mostly Adequate</td>
<td>Video of examples and</td>
<td>Completely Adequate</td>
</tr>
<tr>
<td></td>
<td></td>
<td>a) Adequate, Adequate</td>
<td>explanations</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>b) Adequate, Adequate</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>c) Adequate, Inadequate</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>d) Adequate</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
| Brooklyn/Level 3 | Task 1: Mostly Adequate  
a) Adequate, Adequate  
b) Adequate, Inadequate  
c) Adequate, Adequate  
d) Lines Cross = Same Distance | -Examples, graphical representation, definitions | Task 1: Mostly inadequate  
a) Inadequate, Adequate  
b) Inadequate, Inadequate  
c) Adequate, Inadequate  
d) Lines Cross = Same Distance |
|----------------|-----------------------------------------------------------------|--------------------------------------------------|--------------------------------------------------|
| Maddie/Level 4 | Task 1: Mostly Adequate  
a) Adequate, Adequate  
b) Adequate, Inadequate  
c) Adequate, Inadequate  
d) Lines Cross = Same Distance | Task 1: Definition and examples with explanations | Task 1: Mostly inadequate  
a) Inadequate, Adequate  
b) Inadequate, Inadequate  
c) Adequate, Inadequate  
d) Adequate |
| Lillian/Level 4 | Task 1: Mostly Inadequate  
a) Adequate, Adequate  
b) Adequate, Inadequate  
c) Inadequate, Inadequate  
d) Lines Cross = Same Distance | Task 1: Examples with explanations | No Change |

No students gave completely adequate responses for Task 1 before the Internet portion of the interview. Four students (Charlotte, Natalie, Amelia, and Lillian) made little to no changes and three students (Samantha, Hannah, and Emily) ended up with completely appropriate responses. All three students showed average mathematical sophistication at the start of the course. Samantha and Emily watched the same video titled Position from a Velocity-Time Graph by the Sioux Falls Physics Teachers (https://www.youtube.com/watch?v=bLbU03r-n8w) and Hannah visited the site Sparknotes Applications of the Integral - Distance Traveled (http://www.sparknotes.com/math/calcbe2/applicationsoftheintegral/section5.rhtml).
Samantha, Hannah, and Emily looked up definitions of the relationship between velocity and position and examples in order to fix their misconceptions. These students also realized that the area under the curve would be equal to the distance traveled by the car. This was a novel idea to which students were not previously exposed to in CELTIC. Therefore, they all valued Action 2 (recognizing fundamental mathematics), Action 4 (models), Action 5 (definitions) and Action 6 (understanding) from the MSF.

Brooklyn and Maddie both made more mistakes on task 1 after visiting The Passing Lane website (http://www.physicsclassroom.com/mmedia/kinema/plv.cfm). Both students also scored as having average or high levels of mathematical sophistication at the start of the course. This is due to the fact that the site is showing a cars’ motion on a position-time graph. This confused the students since the task is using a velocity-time graph. This would confirm the students’ misconception that when the lines cross, that the two people are in the same spot.

Seven students found the following website: The Meaning of Slope for a V-T Graph (http://www.physicsclassroom.com/class/1DKin/Lesson-4/Meaning-of-Slope-for-a-v-t-Graph). The majority of students who visited this site made little to no improvements in their responses and two were completely confused afterwards. The Meaning of Slope for a V-T Graph site does not show the relationship between two objects on the same graph. Also, every example is represented linearly. The examples are low level and typical of those seen in introduction lessons of position/velocity/acceleration.

Eight students inadequately think that when the graphical representations of the two people (You and Friend) cross that the two people are in the same place. Three of those
students (Samantha, Hannah and Maddie) end up fixing this misunderstanding. There are no commonalities in the sites these students visited, but they did look up examples and definitions. Therefore, they valued Action 4 (models) and Action 5 (definitions) from the MSF. It is noted that looking up examples and definitions has emerged as important over the semester.

**Task 2**

Table 36 describes how each interview student responded during each phase of the post-interview for task 2.

<table>
<thead>
<tr>
<th>Participant/ MS Level</th>
<th>Initial Task Responses</th>
<th>Internet</th>
<th>Concluding Task Responses</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Charlotte/ Level 2</strong></td>
<td>Task 2: Adequate calculation, inadequate graph (she graphs connected dots)</td>
<td>Task 2: Example</td>
<td>Task 2: No Changes</td>
</tr>
<tr>
<td><strong>Natalie/ Level 2</strong></td>
<td>Task 2: Inadequate calculation (she calculated mean), inadequate graph (she graphs connected dots)</td>
<td>Task 2: Definition of Range</td>
<td>Task 2: Still inadequate, she calculated range and no change in the graph</td>
</tr>
<tr>
<td><strong>Amelia/ Level 2</strong></td>
<td>Task 2: Adequate calculation, inadequate graph (she graphs connected dots)</td>
<td>Task 2: Definition of Bushel</td>
<td>Task 2: Adequate calculation, inadequate graph (she graphs bars, but of total production not rate)</td>
</tr>
<tr>
<td><strong>Samantha/ Level 3</strong></td>
<td>Task 2: Adequate calculation, adequate graph</td>
<td>Task 2: Nothing</td>
<td>Task 2: No Change</td>
</tr>
<tr>
<td>Table 36 - Continued.</td>
<td>Hannah/Level 3</td>
<td>Emily/Level 3</td>
<td>Brooklyn/Level 3</td>
</tr>
<tr>
<td>------------------------</td>
<td>---------------</td>
<td>---------------</td>
<td>-----------------</td>
</tr>
<tr>
<td></td>
<td>Task 2: Adequate calculation, adequate graph</td>
<td>Task 2: Nothing</td>
<td>Task 2: Adequate calculation, adequate graph</td>
</tr>
<tr>
<td></td>
<td>Task 2: Inadequate calculation, adequate graph</td>
<td>Task 2: Nothing</td>
<td>Task 2: Example with explanation, graphical representations</td>
</tr>
<tr>
<td></td>
<td>Task 2: Adequate calculation, adequate graph</td>
<td>Task 2: Nothing</td>
<td>Task 2: No Change</td>
</tr>
</tbody>
</table>

Six students adequately calculated the total production for the orchard. Three students (Natalie, Emily and Lillian) did not adequately calculate the total production. Emily and Lillian made simple calculational mistakes and quickly adjusted them before the interview completed. Both Emily and Lillian scored well on the pre-MSI. One student, Natalie, never calculated the total production adequately. She was also not able to produce a satisfactory graph to represent the data either. This student scored very low on both the pre-MSI and post-MSI. She initially calculated the mean of the production rates when asked to calculate the total production. After spending time on the Internet, in which she only looked up the definition of range, she calculated the range of the production rates instead of calculating the total production.
Three students (Charlotte, Natalie, and Amelia) inadequately represented the data graphically as asked. Each student graphed a collection of dots and connected them. One example can be seen in Charlotte’s work in Figure 16.

![Graph](image)

**Figure 16 - Charlotte’s Post-Interview Task 2 Response.**

As you can see, plotting a dot in the middle of the month does not represent average rate for that entire month. Connecting the dots suggest that the data is not discrete, in which it is.

None of these students were able to correct their graphical representations after the Internet portion of the interview. One student did get closer, Amelia. Amelia’s work can be found in Figure 17.
Amelia originally graphed connected dots, just like Charlotte’s above. After spending some time on the Internet, in which she only looked up the definition of bushel, she decided to graph bars to represent the monthly total production. She was not asked to graph the total monthly production, so this was inadequate.

Six students adequately calculated the total production and created a satisfactory graphical representation of the rates by the completion of the interview. All six students always had the adequate graphical representation. Of these six students, four of them did not use the Internet to assist them with task 2. These four students were confident in their work prior to the Internet portion of the interview. Two students looked up examples of production rate and graphical representations to verify their work.
Post-Interview Summary

For task 1, no students gave completely adequate responses for task 1 before the Internet portion of the interview. Eight students inadequately think that when the graphical representations of the two people (You and Friend) cross that the two people are in the same place. Four students made little to no changes, these students primarily showed fairly unsophistication at the start of the course. Three students, who were average on the MS levels ended up with completely adequate responses. These students also realized that the area under the curve would be equal to the distance traveled by the car. This was a novel idea that the students were not previously exposed to. Therefore, they all valued Action 2 (recognizing fundamental mathematics), Action 4 (models), Action 5 (definitions) and Action 6 (understanding) from the MSF.

For Task 2, six students adequately calculated the total production for the orchard and created a satisfactory graphical representation of the rates by the completion of the interview. Charlotte, Natalie, and Amelia inadequately represented the data graphically as asked. They graphed a collection of dots and connected them. Charlotte, Natalie, and Amelia exhibited a lower level of mathematical sophistication on the pre-MSI.

Table 37 shows the total mathematical sophistication framework actions represented by each participant during the post-interview, also including the pre-interview totals for comparison.
Charlotte, Natalie, Amelia, and Samantha utilized more mathematical sophistication framework (MSF) actions on the post-interview than they did on the pre-interview. Each of them showed lower levels of mathematical sophistication on the pre-MSI. Ironically, those who scored higher on the pre-MSI scored lower on the post-MSI and also chose to utilize less actions of the MSF from the pre-Interview to the post-Interview (Hannah, Maddie and Lillian). Emily and Brooklyn both used the same amount of MSF actions, they were all classified as having average levels of mathematical sophistication on the pre-MSI

**Table 37 - Total MSF Actions Represented by Each Post-Interview Participant.**

<table>
<thead>
<tr>
<th>Participant/MSF Action</th>
<th>Level 2</th>
<th>Level 2</th>
<th>Level 2</th>
<th>Level 3</th>
<th>Level 3</th>
<th>Level 3</th>
<th>Level 4</th>
<th>Level 4</th>
<th>Level 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>MSF Action 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MSF Action 2</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MSF Action 3</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
</tr>
<tr>
<td>MSF Action 4</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
</tr>
<tr>
<td>MSF Action 5</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MSF Action 6</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
</tr>
<tr>
<td>MSF Action 7</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MSF Action 8</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
</tr>
<tr>
<td>MSF Action 9</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Post-Total</td>
<td>7</td>
<td>4</td>
<td>6</td>
<td>7</td>
<td>5</td>
<td>6</td>
<td>5</td>
<td>6</td>
<td></td>
</tr>
</tbody>
</table>

Charlotte, Natalie, Amelia, and Samantha utilized more mathematical sophistication framework (MSF) actions on the post-interview than they did on the pre-interview. Each of them showed lower levels of mathematical sophistication on the pre-MSI. Ironically, those who scored higher on the pre-MSI scored lower on the post-MSI and also chose to utilize less actions of the MSF from the pre-Interview to the post-Interview (Hannah, Maddie and Lillian). Emily and Brooklyn both used the same amount of MSF actions, they were all classified as having average levels of mathematical sophistication on the pre-MSI

**Pre-Interview to Post-Interview Comparison**

A summary of the nine students’ adequate/inadequate interview responses can be found in Table 38.
**Table 38 - Summary of Pre-Post-Interview Responses.**

<table>
<thead>
<tr>
<th>Participant/MS Level</th>
<th>Pre-Interview Task 1</th>
<th>Pre-Interview Task 2</th>
<th>Post-Interview Task 1</th>
<th>Post-Interview Task 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Charlotte/Level 2</td>
<td>No Change / Inadequate</td>
<td>No Change / Inadequate</td>
<td>No Change / Inadequate</td>
<td>No Change / Inadequate</td>
</tr>
<tr>
<td>Natalie/Level 2</td>
<td>No Change / Inadequate</td>
<td>No Change / Inadequate</td>
<td>No Change / Inadequate</td>
<td>Change / Still Inadequate</td>
</tr>
<tr>
<td>Amelia/Level 2</td>
<td>Decline</td>
<td>Improve</td>
<td>No Change / Mostly Adequate</td>
<td>Improve</td>
</tr>
<tr>
<td>Samantha/Level 3</td>
<td>Decline</td>
<td>Improve</td>
<td>Improve</td>
<td>No Change / Adequate</td>
</tr>
<tr>
<td>Hannah/Level 3</td>
<td>No Change / Inadequate</td>
<td>No Change / Inadequate</td>
<td>Improve</td>
<td>No Change / Adequate</td>
</tr>
<tr>
<td>Emily/Level 3</td>
<td>No Change / Adequate</td>
<td>Improve</td>
<td>Improve</td>
<td>Improve</td>
</tr>
<tr>
<td>Brooklyn/Level 3</td>
<td>Improve</td>
<td>Improve</td>
<td>Decline</td>
<td>No Change / Adequate</td>
</tr>
<tr>
<td>Maddie/Level 4</td>
<td>No Change / Adequate</td>
<td>Improve</td>
<td>Decline</td>
<td>Decline</td>
</tr>
<tr>
<td>Lillian/Level 4</td>
<td>No Change / Inadequate</td>
<td>Improve</td>
<td>No Change / Inadequate</td>
<td>Improve</td>
</tr>
</tbody>
</table>

When it comes to the pre-interview, two students (Charlotte and Natalie) with the lowest level of mathematical sophistication were inadequate in their responses before and after the portion of time on the Internet. When it comes to the post-interview, both Charlotte and Natalie stayed consist, they were inadequate in their responses before and after the portion of time on the Internet. They did however exhibit more MSF actions but were still unable to correct their mistakes at this point in their mathematical career.

Amelia, Samantha, Hannah and Emily showed the most improvement. They all scored around an average level of mathematical sophistication on the pre-MSI. One the pre-interview, their responses were about half-and-half. Some of them got it correct and some did not, they were not consistent from task to task. Yet, on the post-interview each of them got
the task correct and was able to explain their correct responses. Amelia and Samantha both used a much higher amount of MSF actions on the post-interview.

During the pre-interview, the majority of the students who scored the highest on the Pre-MSI did not necessarily provide an adequate solution to the tasks initially but provided adequate answers to both task 1 and task 2 after working on the Internet (Brooklyn, Maddie and Lillian). Brooklyn and Maddie both did worse on the post-interview than the pre-interview. They both declined in responses for at least one task after researching on the internet and they both used less MSF actions.

As students’ level of mathematical sophistication increases from unsophisticated to average sophistication, so does their ability to adequately solve novel tasks, fix any misunderstandings or mistakes they make, and utilize more actions of the mathematical sophistication framework. However, Table 38 shows the more sophisticated students did not necessarily follow this pattern. The results are confirmed by the results found in the pre- to post-MSI analysis.

**Instructional Tasks**

Following the pre-interviews, all students participated in a four-month sequence of instruction designed to build conceptual understanding as well as mathematical sophistication. Three high level tasks were chosen from the four-month series of instruction in order to analyze growth in students’ mathematics sophistication throughout the course. The three tasks, the concept they introduce and the MSF actions which may be utilized by students can be found in Table 39. In the following section, I describe how the nine interview students responded while working through each task and provide examples of the MSF
actions. I have chosen to focus on the nine students that participated in the interviews in order to be able to triangulate the results from the interviews and the MSIs. All student artifacts checked to be similar to the nine students.

**Table 39 - Instructional Sequence.**

<table>
<thead>
<tr>
<th>Task Title</th>
<th>Mathematical Concepts</th>
<th>Appendix Location</th>
<th>MSF Actions Represented</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sesame Street (Keene, Hall, &amp; Duca, 2014)</td>
<td>Limit</td>
<td>G</td>
<td>1, 3, 4, 6, 8, 9</td>
</tr>
<tr>
<td>Function Experiments</td>
<td>Function</td>
<td>H</td>
<td>1, 3, 4, 6, 8, 9</td>
</tr>
<tr>
<td>How Far Can You Go with Logger Pro</td>
<td>Position &amp; Velocity</td>
<td>I</td>
<td>1, 3, 4, 6, 8, 9</td>
</tr>
</tbody>
</table>

*Sesame Street*

One of the tasks used in Unit 1 of the CELTIC course is *Getting Back to Sesame Street: An Introductory Activity for Limit of a Sequence* (Sesame Street). Keene, Hall and Duca (2014) analyzed the Sesame Street tasks to show how this task builds on students’ intuition of limit to build a formal definition of limit. The Sesame Street task has two parts: Part One: Try it out! and Part Two: Big Bird & Count von Count. Part One has students model a real world situation of limits and the second part pushes them to explain this situation and gain a deeper understanding of limits. Table 40 shows the questions for each part and which MSF actions they represent.
### Table 40 - Sesame Street Questions & MSF Actions.

<table>
<thead>
<tr>
<th>Part</th>
<th>Questions</th>
<th>MSF Actions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Go and find a spot about 5-6 ft. from a wall. Try to follow the directions on the sign: Every step you take should be half the distance that is between you and the wall. Keep taking steps until you can’t take any more! Once you are done, sit down with your group and answer the following questions: 1. How many steps did you take? 2. Why did you have to stop taking steps? 3. Could you have taken more steps? If yes, how many more steps could you have taken? If not, explain why your number of steps is the maximum.</td>
<td>4 - Model 3 - Make Conjecture</td>
</tr>
<tr>
<td>2</td>
<td>Let’s help Big Bird and Count von Count Figure out just how many steps Big Bird will have to take in order to cross this strange bridge. 1. Let’s say that Zeno’s bridge is 10 meters long, how many steps do you estimate Big Bird will have to take in order to cross the bridge? Explain your estimation. 2. How far did Big Bird travel with his first step? How far has he traveled after his second step? a) After 1st Step: b) After 2nd Step: 3. Suppose the segment line below represents the bridge and suppose they travel left to right. We can indicate on this &quot;bridge&quot; the total distance left to travel after one step, $a_1$. Also, we can put on the &quot;bridge&quot; the distance traveled. Use the &quot;bridge&quot; below to represent the total distance left to travel after two steps, $a_2$. Find the distance traveled and show it on the &quot;bridge&quot;. Indicate on this &quot;bridge&quot; the total distance left to travel after three steps, $a_3$. Find the distance traveled and indicate on the &quot;bridge&quot;.</td>
<td>3 - Test Conjecture 3 - Make Conjecture 1 - Patterns 3 - Making and Testing Conjectures 4 - Modeling 8 - Language Distinctions 9 - Symbolic Representation</td>
</tr>
</tbody>
</table>
Table 40 - Continued.

Let’s let \( a_n \) be the sequence for the distance left to travel. Can you find a recursive formula for the distance left to travel after having taken \( n \) steps? What kind of sequence this is? Can you write a closed formula for the distance left to travel, \( a_n \)?

4. Let \( \{b_n\} \) be the sequence where the \( n \)th term corresponds to the distance Big Bird has traveled after his \( n \)th step. Compute the first five terms of this sequence. Find a recursive formula for the sequence. *(Hint: You computed \( a_1 \) and \( a_2 \) in question #3).* Can you create a closed formula for this sequence?

5. Would Big Bird ever be less than 0.01 meters from the end of the bridge? If yes, give support for your answer. If not, explain why not.

6. Without computing, do you know if Big Bird will ever be within 0.001 meters from the end of the bridge? How about 0.00001 meters? How do you know? When he gets to each of those places, at least how far would he have traveled?

7. Big Bird makes a shocking revelation: He claims that if Count calls out any distance, as close to 10 as he wants, if he follows Lord Zeno’s directions, after a certain step, he will have traveled that far. Do you believe this is true? Why or why not?

8. You may have noticed that the distance Big Bird has traveled seems to be getting close to 10. In your group, write a description of the behavior that the sequence exhibits as Big Bird continues his journey to the end of the bridge.

The MSF actions listed here and in the other two task descriptions were determined by the author and another researcher coming to agreement. Not surprisingly, some students exhibited more or fewer actions than those listed in the Table.

**Part one**

Part one started by asking students to stand 5-6 feet from a wall and take steps towards the wall. Each step is supposed to be half of the step before. Students were asked to model (Action 4) a real world situation of limits and then make conjectures on the number of steps they could take (Action 3). Students were then asked how many steps they take and if
they could possible take more steps. Figure 17 is an example of how Brooklyn answered the questions in Part one.

1. How many steps did you take?  
   I took 3 steps to get back to the wall 
   but one of us got 4 so between 3 & 4.

2. Why did you have to stop taking steps?  
   We couldn’t step any further. 
   because there was a wall 
   stopping our feet.

3. Could you have taken more steps? If yes, how many more steps could you have taken? If not, explain why your number of steps is the maximum.  
   We couldn’t have taken more steps 
   because your foot has length there would be no way to step half of the amount left between us & the wall.

Figure 18 - Brooklyn’s Response to Sesame Street Task Part One.

Four students said they could not take more steps, each of the four students varied in levels of mathematical sophistication. The five students who said they could take more steps had been scored in different levels of mathematical sophistication.

Part two

Part two walks students through multiple steps in order to test their conjectures (Action 3) on part one. Questions 1 and 2 ask students to make conjectures about Big Birds steps, for example, see Natalie’s work in Figure 18.
1. Let’s say that Zeno’s bridge is 10 meters long, how many steps do you estimate Big Bird will have to take in order to cross the bridge? Explain your estimation.

   About five steps because although you could continue to divide his steps they reach an unrealistic distance.

2. How far did Big Bird travel with his first step? How far has he traveled after his second step?

   a) After 1st Step: 5 meters

   b) After 2nd Step: 7.5 meters

**Figure 19 - Natalie's Responses to Sesame Street Task Part Two Questions 1-2.**

All nine students adequately answered these questions and easily moved on to questions 3-4 in which students must model (Action 4) the situation on a fictitious bridge, create multiple sets of data, and use this data to create a closed/recursive formula (Action 9) by locating patterns within the data (Action 1). Students must pay close attention to detail and notation (Action 8) in order to distinguish between the distance left to travel and the distance traveled. Lillian exemplifies this work in Figure 19.
Lillian’s work is consistent except in the second table she uses the notation for $a_n$ instead of $b_n$ to represent the distance Bid Bird has traveled. Therefore, she met all MSF Actions except action 8 the problems were designed to illicit. All nine interview students created an adequate recursive and closed formula for the distance left to travel ($a_n$), question 3 as well as the distance traveled in question 4.
Once students moved on to questions 5-8, they begin to employ different techniques. The majority of students with unsophisticated levels of mathematical sophistication simply provide rational for their answers, and some of their rationales are not satisfactory. For example, Natalie’s work on question 5 in Figure 21 shows lower levels of understanding.

**Figure 21 - Natalie's Responses to Sesame Street Task Part Two Question 5.**

Students with higher levels of mathematical sophistication implement different MSF actions to provide rational for their answers. For example, Samantha’s work on questions 5 (Figure 22) incorporates calculations to support her answers and her work on questions 6-8 (Figure 23) shows how she used action 9 (symbolic notation) and action 4 (creating models) to help explain her answers.

**Figure 22 - Samantha's Response to Sesame Street Task Part Two Question 5.**
6. Without computing, do you know if Big Bird will ever be within 0.001 meters from the end of the bridge? How about 0.00001 meters? How do you know? When he gets to each of those places, at least how far would he have traveled?

Yes, because the steps he takes are indefinite.

For 0.001, he would have travelled at least 9.998 meters \((10-0.001)\) and for 0.00001, he would have travelled at least 9.99998 meters \((10-0.00001)\).

\[
a_n = 8 + \frac{1}{2^n} \quad \text{and} \quad 0.001 = 8 + \frac{1}{2^n}
\]

\[
ln\ 0.001 = ln\ \frac{1}{2^n} \quad \text{and} \quad ln\ 0.001 = ln\ \frac{1}{2^n}
\]

7. Big Bird makes a shocking revelation: He claims that if Count calls out any distance, as close to 10 as he wants, if he follows Lord Zeno’s directions, after a certain step, he will have traveled that far. Do you believe this is true? Why or why not?

Yes, because he will be able to get as close to 10 as possible without actually reaching it with indefinite steps.

8. You may have noticed that the distance Big Bird has traveled seems to be getting close to 10. In your group, write a description of the behavior that the sequence exhibits as Big Bird continues his journey to the end of the bridge.

The sequence curves up, getting closer to 10 but not touching it. It is closest to a logarithmic graph.

Figure 23 - Samantha’s Responses to Sesame Street Task Part Two Questions 6-8.

Maddie, along with two other students, Lillian and Emily, were able to explain the mathematics being used in the task (Action 6). They also used symbolic representation (Action 9) to support their verbal claims in questions 6 and 7. Finally they used graphical
representations (Action 4) to support their verbal claims on question 8. Students’ with higher levels of mathematical sophistication incorporated multiple MSF Actions to explain their reasoning.

*Sesame Street Summary*

At the start of the Sesame Street task students’ responses were split; four said they could not take any more steps and five students said they could take more steps yet they would be extremely small and would need a tool to help measure those students. Next students started looking at the situation in a more mathematical manner by measuring the distance of each step to the wall, their understanding began to grow. All nine students were able to answer these questions adequately (Action 3 and Action 4), create multiple sets of data, and use this data to create a closed/recursive formula (Action 9) by locating patterns within the data (Action 1).

The majority of students with fairly unsophisticated levels of mathematical sophistication simply provided rational for their answers lacking a detailed explanation of their understanding, and some of their rationales are inadequate. Students’ with higher levels of mathematical sophistication incorporate multiple MSF Actions to explain their reasoning.

*Functions*

Another of the cognitively demanding tasks used in Unit 2 of the CELTIC course was Functions: Of the World or Out of the World? (Functions). The Functions task has three parts: Part One: Data Collection, Part Two: Investigate Your Functions, and Part Three: Present Your Findings. Table 41 shows the questions for each part of the Functions task and which MSF actions they represent.
### Table 41 - Functions Directions, Questions and MSF Actions.

<table>
<thead>
<tr>
<th>Part</th>
<th>Directions &amp; Questions</th>
<th>MSF Actions</th>
</tr>
</thead>
</table>
| 1    | • Read your card and identify what materials and measuring devices you may need to collect the data asked for the given relation.  
      • Collect data to create ordered pairs. You must have at least 6 ordered pairs. Write down the "quantities" you are measuring. List them in a Table.  
      • Get a piece of graph paper. On the top of the page write the name of your experiment and members in your group.  
      • Graph the ordered pairs on a piece of graph paper carefully. Be sure to mark the units and the x and y axes. Indicate which quantities are represented on your axes.  
      • On your graph: Create a continuous line or curve that connects the ordered pairs. | 3 - Make conjectures  
8 - Determine necessary and sufficient conditions  
4 - Create a model  
4 - Create a model  
8 - Precise language |
| 2    | 1. On the back of your graph paper write the name of your experiment and members in your group again, describe your experiment in 3 sentences (be clear and concise).  
2. Next identify the parent function (If you have no idea, see a list of possibilities on the board and use a graphing calculator if needed). Once you have identified the parent function: graph the function, write its domain, and write the range under the graph.  
3. Describe the family of function found in your experiment. Include a small sketch of your graph. Write its domain and range under the graph (remember it comes from the real world). Then explain how your graph is specifically related to your parent function (Figure out how to manipulate the parent function and use your graphing calculator to find a graph that matches your data).  
4. Finally, think of another real world relationship that also would generate a function from the same family. | 1 - Identify patterns  
2 - Classify objects based on structure  
3 - Make and test conjectures  
9 - Use symbolic representations  
5 - Use precise definitions of objects  
6 - Explain understanding of experiment  
2 - Classify objects based on structure  
6 - Explain understanding of experiment |
| 3    | • Be prepared to present all of the information from parts 1 and 2 to the class. Be very clear, your presentation with be critiqued. Remember you are teaching the class about the experiment, the parent function and the family of function exhibited in your experiment. | 6 - Explain understanding of experiment |
There were seven experiments and therefore seven groups within each class. Each of the seven experiments are described in Table 42.

Table 42 - *Function Experiment Descriptions & Materials Needed.*

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Students</th>
<th>Description</th>
<th>Materials Required</th>
</tr>
</thead>
<tbody>
<tr>
<td>Knot tying</td>
<td>Maddie</td>
<td>Measure the length of a rope without knots. Then tie a knot and measure again. Continue adding knots and measuring. Be careful that all the knots are the same and not on top of each other.</td>
<td>Rope that is at least 20 inch’s long and a measuring tape.</td>
</tr>
<tr>
<td>Pressure and Volume</td>
<td>Amelia</td>
<td>Use the Gas Pressure Sensor and Vernier Lab Pro to measure Volume and Pressure on at least 6 data points.</td>
<td>Gas pressure sensor, Vernier lab pro and a graphing calculator with the necessary program.</td>
</tr>
<tr>
<td>Measuring the size of a square projected</td>
<td>Brooklyn &amp; Lillian; Charlotte</td>
<td>Measure the distance an overhead projector is from the wall and the size (area) of the light on the wall.</td>
<td>Overhead projector, square, and a measuring tape.</td>
</tr>
<tr>
<td>Measure time for a Pendulum</td>
<td>Samantha</td>
<td>Use a rope with a nut tied to the end. Collect data on the length of the rope and the time it takes to do 10 swings.</td>
<td>A spool of rope, a nut, and a measuring tape.</td>
</tr>
<tr>
<td>Height of Cardboard Pile</td>
<td>Natalie &amp; Hannah</td>
<td>Take a large piece of cardboard. Measure its height. Cut it in half and make a pile. Measure its height. Cut each piece in half and make a new pile, measure its height. Continue until you can’t cut any more.</td>
<td>Large piece of cardboard, scissors, and a measuring tape.</td>
</tr>
<tr>
<td>Spring and weight</td>
<td>None</td>
<td>Look at the length of a scale with no weights. Then add weights and measure the length.</td>
<td>Scale, weights, and a measuring tape.</td>
</tr>
<tr>
<td>Displacement of ball in water</td>
<td>Emily</td>
<td>Measure the diameter or circumference of each ball and the amount of water it displaces.</td>
<td>A variety of balls, water, a measuring tape, and a large beaker.</td>
</tr>
</tbody>
</table>
Each of the experiments were completed in groups; therefore, the results cannot be used to show an individual’s growth in mathematical sophistication. However, the results can show overall students’ change/lack of change in mathematical sophistication from the beginning to the end of the semester by analyzing student responses to tasks given throughout the semester.

Part 1

For Part One, students are assigned an experiment in which they will decide what materials they need to collect data, then the students collect data to create ordered pairs, graph those ordered pairs and label necessary information. Students first have to perform the experiment and explain their experiment in writing (Action 6). All students were able to explain well (Action 6) their individual experiments and the necessary conditions (Action 8) required to complete their experiment.

Students were then asked to create a successful Table and graphical representation (Action 4) of the function the experience modeled. Eight groups were able to create adequate graphical representations (Action 4) with the necessary information adequately labeled on their graph (Action 8). Charlotte’s group, who did Measuring the Size of a Square, was not able to adequately create a Table or a graph that represented the correct function.

To exemplify the differences explained above without going through each groups work, following is a detailed analysis of the two groups who completed the Measuring the Size of a Square experiment. One group involved Brooklyn and Lillian; Charlotte was in the other group. Brooklyn and Lillian were able to verbally explain their procedure for
completing the experiment (MSF Action 6). Their steps for completing the procedure are outlined in Figure 24.

Figure 24 - Brooklyn and Lillian's Steps for Completing Their Experiment.

Brooklyn and Lillian did a good job collecting data and representing it adequately on a graph. Their graphical representation of the data can be found in Figure 25.
Looking at this graph you can see that the students adequately analyzed the experiment, were able to make a model of the data (MSF Action 4), make and test their conjectures (MSF Action 3), and adequately label and identify necessary conditions for the experiment (MSF Action 8).
Brooklyn and Lillian were able to find a pattern (MSF Action 1) within their data and relate it to the parent function represented by the experiment. Brooklyn and Lillian were able to explain their experiment to the class (MSF Action 6) and how their experiment related to the underlying parent function (MSF Action 2). They were able to explain how they found their symbolic representation (MSF Action 6) of \( y = 0.488x^2 - 8.44x + 649.71 \). While describing their work and why they knew their model was a function, they used the definition of function (MSF Action 5).

Charlotte’s group was quite different. Charlotte’s group did not adequately describe the procedure as shown in Figure 26.

![Figure 26 - Charlotte's group's Explanation of the Experiment.](image)

They did not describe how they found the area of the square or how they repeated the data collection process. Charlotte’s group’s graphical model was not too far off from Brooklyn and Lillian’s group, but the rest of their experiment had inadequacies. Charlotte’s graphical model can be found in Figure 27.
As you can see in the model, Charlotte’s group was able to create a decent model of the data they collected (MSF Action 4). They decided to measure the size of the square instead of the light around it (as Brooklyn and Lillian did) but this decision would not affect the parent function.
Next Charlotte’s group began to make and test conjectures (MSF Actions 3) in order to determine the parent function represented by their data. They did not do this adequately as shown in Figure 28.

![Figure 28 – Charlotte’s group's Inadequate Graphical Representation and Symbolic Representation.]

Their graphical representation here is completely different from the model they created in part 2. Their graph is going in a completely different manner.

**Part 2**

For Part Two, students look at the graph and the data collected in Part One and identify what parent function (basic function used in calculus) their experiment created. Then students describe the parent function in detail and explain how their function relates to the parent function. Students then create a second real life relationship that their function models.
Seven students were able to look at their data, find the linear pattern (Action 1) and discover the underlying mathematics and linear parent function (Action 2) represented by their experiment. Two groups were not able to complete part 2 adequately. One person, Charlotte, had an inadequate graphical representation and this mislead them into finding an inadequate pattern (that also did not align with their graph) and they came up with a completely different function based on their misleading data collection. One other student, Amelia, was not able to fully explain their understanding (MSF Action 6) of the parent function and the underlying mathematics (MSF Action 2).

Next, students were supposed to look at their data and create a symbolic representation of that data (Action 9). Four students were able to adequately identify their symbolic representation (Action 9) as well as explain (Action 6) how they came up with their symbolic representation. Five students were not able to come up with an adequate symbolic representation: Amelia, Charlotte, Samantha, Natalie and Hannah.

An example of the disconnect between parent function and accurate graphical representation can be seen by in Samantha’s group. Samantha was in a group who completed the Measure Time for a Pendulum experiment. The experiment procedure they created is shown in Figure 29.
The group was able to explain the experiment procedure accurately (MSF Action 6). The group was able to adequately collect data and create an adequate graphical representation (MSF Action 4) with accurate labels (MSF Action 8). The group was also able to notice a pattern in the data (MSF Action 1) and determine the parent function (MSF Action 2). Samantha’s group was also able to make conjectures (MSF Action 3), but testing those conjectures created some difficulties. This can be seen in Figure 30.

Figure 29 - Measure Time for a Pendulum Experiment Procedure.
According to their work, they identify the symbolic representation of the experiment inadequately.

Another big problem seen in part 2 was students’ ability to identify the domain and range of their parent function. Three students identified the domain and range of their function along with that of their parent function. Six students had some parts of each correct, but failed to notice the constraints within the problem for other parts. Most noticeable was students neglect in determining the maximum of their experiments. Six groups said their experimental function would continue on forever when each experiment had definite ending points. For example, the displacement of ball in water experiment could not extend forever since the size of the container holding the water would provide certain restraints for the problem.
Eight students were able to come up with a second real world situation to represent the parent function they identified. The only group who inadequately came up with one was Charlotte’s group. They inadequately represented their data and the rest of their findings were skewed due to this misunderstanding.

*Functions Summary*

All groups were able to appropriately collect data and create an adequate graphical representation (MSF Action 4) with the necessary conditions adequately labeled on the graphical representation (MSF Action 8). All groups were able to make and test conjectures (MSF Action 3) throughout the data collection phase of the experiment. Seven groups were then able to look at the patterns in the data (MSF Action 1) and accurately identify the underlying mathematics in order to Figure out the adequate parent function (MSF Action 2). Only four groups came up with the adequate symbolic representation (MSF Action 9) for their experiment. When it comes to describing their experiment, those with higher levels of mathematical sophistication did better and paid better attention to the detail in the steps. Also, the groups with higher level of mathematical sophistication were better able to explain (MSF Action 6) why their parent function was chosen and adequately represented symbolically. Something important to note was students with higher levels of MS brought in different MSF Actions than the researchers expected. For example, Brooklyn and Lillian used Action 5 (definitions) to help explain their experiment and parent function connection.

*Logger Pro*

One of the cognitively demanding tasks used in the final unit of the CELTIC course is titled Logger Pro: How Fast Will You Go. The Logger Pro task explores a foundational
aspect of mathematics and physics: the relationship between position and velocity. The task has three parts: Part A: Position vs. Time Relationship, Part B: Velocity vs. Time Relationship, and Part C: Position and Velocity Correlation. Table 43 shows the questions for each part of the Logger Pro task and which MSF actions they represent.

Table 43 - Logger Pro Directions, Questions and MSF Actions.

<table>
<thead>
<tr>
<th>Part</th>
<th>Question</th>
<th>MSF Actions</th>
</tr>
</thead>
</table>
| A    | 1. Open the file: “position.cmbl.” Have a group member click the “Collect” button while another student moves the toy. What kinds of graphs can you create? Try out a bunch of different motions and see what happens. Copy and paste an interesting graph that your group created in the file: “DistDet_Graphs.doc” under “Part A #1.” Write a brief description of the movement that created the graph below. | 3 - Make conjectures  
4 - Create a model |
|      | 2. Can you create a graph of a curve that was not a function? Explain why or why not? | 3 - Make conjectures |
|      | In this section, your group will create a specific position vs. time graph and answer some mathematical questions about the graph. | 3 - Make conjectures  
4 - Create a model |
|      | ![Figure 1-Position vs. Time Graph](image) | |
|      | 3. Use Logger Pro to create the graph using the distance detector (when you hit the Collect button again your previous graph will disappear while a new graph is created). Your group may have to conduct several trials before you attain a nice, pretty, fairly smooth graph in appearance. | |
4. Now you are going to investigate your position in detail. Go to the command bar, pull down the Analyze menu, and select Examine. This will result in a vertical line that will move across the screen of your graph. At any point, a small box in the upper left corner will give you the coordinates—time and the position from the sensor.

a. Find and record a small time interval where your toy’s distance from the detector was increasing. What was the toy’s position at each endpoint?

b. Compute the average rate of change (AROC) between these two coordinates. Provide the units for your AROC. Explain what your value represents in the context of your toy’s motion.

c. Using your graph, estimate a small interval of time during which your toy is moving the fastest. Explain what you looked at to come up with your estimate.

d. Compute the AROC between these two coordinates. How does your result compare to 3b?

e. If time, find two intervals which have the same AROC. List them here. How did you know?

We are now going to explore the connection between position and velocity. Logger Pro and the distance detectors can help us by telling us what the velocity of our toy car is. Be careful to keep your toy right in front of the detector!

5. Open the file: “velocity.cmbl.” What kinds of graphs can you create? Try out a bunch of different motions and see what happens. Copy and paste an interesting graph in the file “DistDet_Graphs” under “Part B #6.” Write a brief description of your movement below.

In this next section, your group will create a specific velocity vs. time graph and answer some mathematical questions about the graph.
6. Re-open “velocity.cml” in Logger Pro. Create the graph using the distance detector. Your group may have to conduct several trials before you attain a nice, pretty, fairly smooth graph in appearance. Copy and paste your graph in the file: “DistDet_Graphs” under “Part B #9.” In the space below, write what was the most difficult part of this.

7. Now you will plot your toy’s corresponding position vs. time graph. Right click on your graph, choose **Graph Options**, and then choose the **Axis Options** tab. Click on the box for position and a position graph will appear. Next, right click on your **Table**, click on the box for position, and your position values will appear. Copy and paste this graph (with both your velocity and position graphs) into the file: “DistDet_Graphs” under “Part B #10.”

8. Now investigate the POSITION graph like you did in the first part. Go to the command bar, pull down the **Analyze menu**, and select **Examine**. This will result in a **vertical line** that will move across the screen of your graph. At any point, a small box in the upper left corner will give you the coordinates—*time and the position from the sensor.*
   a. Find and record a time interval where your toy’s distance from the detector was **decreasing**. What was the toy’s position at each endpoint?
   b. Compute the average rate of change between these two coordinates. Provide the units for your average rate of
c. change. Explain what your value represents in the context of your toy’s motion.
d. How does the value you just computed in part (b) compare to the values on your velocity graph in the same region? Given your previous results, give your best explanation of the connection between the velocity vs. time graph and the position vs. time graph.

9. Using the distance detector and car, create a velocity graph where the velocity changes from negative to positive and then back negative over time. When you are satisfied, describe what you did below and make a rough sketch.

10. Bring up the distance graph to match you created velocity graph. Find pairs of points (close together) using ANALYZE and fill in the following Table.

<table>
<thead>
<tr>
<th>Location on graph</th>
<th>Points</th>
<th>AROC</th>
<th>Velocity Value at the same time (from the velocity graph) use t value of first point</th>
</tr>
</thead>
<tbody>
<tr>
<td>Velocity is negative at the beginning</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Velocity is near 0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Velocity is positive</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Velocity is near 0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Velocity is negative again</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

11. Compare AROC and Velocity- are they related? Looking back at your answer to 10, can you now add to your connection comments?

12. Looking at the first pair of points, how do you think you could make velocity and AROC even closer in value?
Before the students began Part one, they were asked to “take a minute to discuss the terms “velocity” and “position” with” their group. Only two students wrote about the discussion within their groups. Emily defined position and velocity as shown in Figure 31.

![Figure 31 - Emily's Description of Position and Velocity.](image)

Lillian described the terms velocity and position as shown in Figure 32.

![Figure 32 - Lillian's Description of Position and Velocity.](image)

**Part A**

Each group had a computer with Logger Pro installed. This software allowed the students to see the movement of the car collected by the distance detector (usually called a motion detector). All students were also able to save the graphs they made in their respected groups. Part A was designed to have students analyze speed and movement of a toy car in
relation to a position graph. Students are first given time to experiment (Action 3) with the technology, the movement of the car and how this relates to the position graph (Action 4). They are asked to recreate the graph (Action 4) shown in Figure 33 and explain how they were able to create their replication (Action 6).

**Figure 33 - Graph Given to Students to Replicate in Part A Number 3 on Logger Pro.**

Eight students were able to adequately recreate the graph (Action 3 & Action 4) as well as describe how (Action 6) they created the graph. To recreate this graph a student would have to start the car far away from the detector, go towards the detector at an increasing rate, change directions, and then stop and stay in a constant position. Emily was able to create the graph given in the task, shown in Figure 34.
Charlotte was not able to create an adequate replication to the graph given, shown in Figure 35.

Charlotte’s graph was extremely different than the graph she was trying to replicate. She did not start her car far away from the detector and was not able to create the curve, which the
students felt was the most difficult part. This graph also affected her ability to give adequate responses to the rest of part A.

Seven students were able to analyze their position graphs by finding average rate of change (AROC) with a symbolic representation (Action 9) as well as understand position related to the cars movement (Action 2). Two students, Natalie and Lillian, were not able to adequately pick two intervals in which the AROC was the same and use the symbolic representation to confirm their conjecture (Action 3). Looking at Natalie’s explanation, one can see that she picked two intervals that may have appeared to be horizontal graphically, but looking at the points there were not and she did not find this distinction (Figure 36).

![Figure 36 - Natalie's Response on Part A Number 4 Question e on Logger Pro.](image)

Lillian explains that the car is practically stopped and so the rate of change is the same. A little more explanation here and she may be giving similar responses to the majority of the
students by picking horizontal lines, but she only shows one interval, so there is no way to know what other interval she is talking about (Figure 37).

\[(5.4, 0.895) \quad (5.6, 0.895)\]

The car is practically stopped, therefore the rate of change is the same.

**Figure 37 - Lillian's Response on Part A Number 4 Question e on Logger Pro.**

Again, the majority of students were able to give adequate responses to all question in Part A.

**Part B**

For Part B, students investigate (Action 3) velocity and how velocity is related to position. Part B starts out similarly as Part A. Students are first given time to experiment (Action 3) with the technology and the movement of the car and how this relates to the velocity graph (Action 4). They then recreate the graph (Action 3 & Action 4) shown in Figure 38.
All students had a hard time creating this graph. Charlotte had a hard time creating a curve; she kept getting straight lines. Five students (Natalie, Amelia, Samantha, Hannah, and Brooklyn) mentioned that creating a graph with a constant speed was the hardest part of creating the graph shown in Figure 38. Two students mentioned that trying to analyze the given graph and figure out the speed and the motion needed to create the given graph was the hardest part of creating the given graph.

Despite these difficulties, eight students were able to replicate (Action 3 & Action 4) this graph shown in Figure 38. An example of an adequate representation of the graph in Figure 39 is shown in Amelia’s representation in Figure 38.
Maddie was not able to get the graph above the y-axis. She did not make the connection that being above the y-axis would be a positive distance away from the detector.

Six students provided a good analysis (Action 2 & Action 3) of velocity as it related to the car traveling to and from the distance detector in part B with the use of their graphical representation (Action 4) and AROC (Action 9). An example of this can be seen in Emily’s work in Figure 40.
c. How does the value you just computed in part (b) compare to the values on your velocity graph in the same region?

\[
\begin{align*}
(1.2, -0.95) \\
(2.6, -0.472)
\end{align*}
\]

The y-values of the velocity are the slopes of the position graph.

The velocity graph shows the rate of change/slope for every point.

d. Given your previous results, give your best explanation of the connection between the velocity vs. time graph and the position vs. time graph.

The rate of change is the average velocity. The graph of the velocity is the rate of change/slope of every point on the position graph.

**Figure 40** - *Emily's Response to Part B Question 8c and Question 8d on Logger Pro.*

Three students (Maddie, Natalie and Charlotte) were not able to compare appropriately the values on the velocity graph and the position graph. This led to a misunderstanding of how position and velocity are related. This also affected their responses in part C since they were supposed to be building a relationship between position and velocity.

**Part C**

Part C has students try to make multiple connections. Number 9 starts with students trying to connect a velocity graph (Action 4 & Action 6) and how a car would need to move to create a graph that changes from negative, to position and then back to negative (Action 3). Number 10 has students find points on the graph created in number 9 and calculate
(Action 9) the AROC and look at the Velocity value given by the software in order to find any connections. Number 11 asks students to compare AROC and Velocity and see how they are related and explain that relationship (Action 6). Number 12 asks students to push their knowledge of velocity by asking them to see how they could make velocity and AROC closer in value (Action 6).

For number 9, all students adequately created a graph in which changes from negative, to position and then back to negative. For example, Samantha describes and sketches a graph in which she states that the car was moved toward the detector, away from the detector and back toward it. This can be seen in Figure 41.

![Graph](image)

**Figure 41** - Samantha's Responses to Part C Number 9 on Logger Pro.

For number 10, seven students (Brooklyn, Emily, Charlotte, Amelia, Samantha, Hannah, Maddie, Emily, Brooklyn) adequately completed the chart. Brooklyn’s chart can be found in Figure 42.
Two students (Natalie and Lillian) inadequately filled in the chart. Lillian correctly calculated the AROC between the points she chose; yet she picked points far apart and the directions say to pick points that are close together. Picking points far apart created difficulty when trying to make connections between AROC and the calculated velocities. The AROC is going to be farther off from the AROC if you pick points that are far apart. Natalie calculated the velocity value in the last column that needed no calculation and only a recording from the software.

All of the students who created adequate charts were able to build accurate connections between AROC and velocity. Having inadequately filled out the chart affected Natalie and Lillian’s responses to the questions that pushed students to build connections between AROC and velocity. They were not able to build the necessary questions and required help from the instructor.
**Logger Pro Summary**

Eight students provided adequate responses to the investigation about position, Charlotte was not able to create an adequate replication of a position graph. Two students (Natalie and Lillian) were not able to adequately describe and explain AROC as it relates to position. Eight students were able to accurately describe the motions of the car and how it related to the velocity graph they created. Maddie, was not able to create an adequate velocity replication graph. Three students (Maddie, Natalie and Charlotte) were not able to adequately compare the values on the velocity graph and the position graph. Two students (Natalie and Lillian) inadequately drew connections between AROC and velocity. All other, seven, students were able to draw connections between AROC and velocity as well as describe the relationship between position and velocity.

**Phase 3 Triangulation & Summary**

The Wilcoxon signed rank test for the whole class difference in scores from the MSI pre-test to the post-test showed improvement, but the improvement was not statistically significant. The growth was strongest for those identified as performing at a Level 1 on the Pre-MSI. Therefore, the course appears to have been most beneficial to students with the lowest level of mathematical sophistication at the beginning of the course.

If one takes the results from the Pre- to Post-MSI and analyze by question, 8 questions had the most significant gain. This can be seen in Table 44.
Table 44 - Questions with the Most Gain from Pre- to Post-MSI.

<table>
<thead>
<tr>
<th>MSI Question</th>
<th>MSF Action</th>
<th>Pre-MSI Total Adequate</th>
<th>Post-MSI Total Adequate</th>
<th>Gain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Question 2</td>
<td>7 &amp; 8</td>
<td>10</td>
<td>13</td>
<td>3</td>
</tr>
<tr>
<td>Question 8</td>
<td>1, 2, 3 &amp; 9</td>
<td>7</td>
<td>14</td>
<td>7</td>
</tr>
<tr>
<td>Question 9</td>
<td>5, 8 &amp; 9</td>
<td>27</td>
<td>30</td>
<td>3</td>
</tr>
<tr>
<td>Question 12</td>
<td>7</td>
<td>9</td>
<td>12</td>
<td>3</td>
</tr>
<tr>
<td>Question 13</td>
<td>5 &amp; 8</td>
<td>25</td>
<td>30</td>
<td>5</td>
</tr>
<tr>
<td>Question 16</td>
<td>5 &amp; 9</td>
<td>11</td>
<td>14</td>
<td>3</td>
</tr>
<tr>
<td>Question 22</td>
<td>5 &amp; 9</td>
<td>26</td>
<td>29</td>
<td>3</td>
</tr>
<tr>
<td>Question 25</td>
<td>1, 2, 3 &amp; 4</td>
<td>14</td>
<td>17</td>
<td>3</td>
</tr>
</tbody>
</table>

Looking at the MSF Actions represented by the questions with the most gain, one can see that individual questions involving definitions (Action 5), precise language and fine distinctions about language (Action 8) and symbolic notation (Action 9) showed the most gain from pre- to post-MSI.

Looking at the gain in MSF Actions by each of the nine interview participants from the pre-interview to the post-interview also show a positive improvement. The conclusion of the Wilcoxon signed rank test shows that the MSF Actions represented by each of the nine interview participants’ improvement for solving novel tasks was statistically significant in the interviews. However, the majority of students who scored low on the Pre-MSI were not able to appropriately analyze the novel tasks in either interview session, before or after the Internet portion.

A qualitative analysis of the nine students’ responses to each task in the pre- and post-interviews showed that students’ with higher levels of mathematical sophistication use more actions from the MSF. Additionally, from pre- to post-interview students improved in
utilizing and valuing three different actions: Action 5 (valuing and using definitions), Action 6 (understanding relationships) and Action 8 (precise language and fine distinctions about language). These findings are shown in Table 45.

**Table 45 - Pre- to Post-Interview MSF Action Analysis.**

<table>
<thead>
<tr>
<th>Total Number of Students Who Exemplified Each Action</th>
<th>Pre-Interview Total</th>
<th>Post-Interview Total</th>
<th>Gain</th>
</tr>
</thead>
<tbody>
<tr>
<td>MSF Action 1</td>
<td>N/A</td>
<td>0</td>
<td>N/A</td>
</tr>
<tr>
<td>MSF Action 2</td>
<td>5</td>
<td>4</td>
<td>-1</td>
</tr>
<tr>
<td>MSF Action 3</td>
<td>8</td>
<td>9</td>
<td>1</td>
</tr>
<tr>
<td>MSF Action 4</td>
<td>8</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>MSF Action 5</td>
<td>3</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>MSF Action 6</td>
<td>6</td>
<td>8</td>
<td>2</td>
</tr>
<tr>
<td>MSF Action 7</td>
<td>5</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>MSF Action 8</td>
<td>2</td>
<td>9</td>
<td>7</td>
</tr>
<tr>
<td>MSF Action 9</td>
<td>8</td>
<td>8</td>
<td>0</td>
</tr>
</tbody>
</table>

Students already valued and exhibited Action 3 (making and testing conjectures), Action 4 (models) and Action 9 (symbolic representations) at the start of the course. This can be seen in Table 45 as well.

Students’ value of Action 3 (making and testing conjectures), Action 4 (models) and Action 9 (symbolic representations) was also shown through students’ responses to the tasks. As well as their students increase in utilizing and valuing three different actions: Action 5 (valuing and using definitions), Action 6 (understanding relationships) and Action 8 (precise language and fine distinctions about language). While explaining their reasoning, they would
incorporate graphical representations (Action 4) as well symbolic representations (Action 9) as justification. MSF Actions constantly exhibited by participants from task 1 to task 3 are shown in Table 46.

**Table 46 - MSF Actions Exhibited By Participants from Task 1 to Task 3.**

<table>
<thead>
<tr>
<th>Task</th>
<th>Participant</th>
<th>A1</th>
<th>A2</th>
<th>A3</th>
<th>A4</th>
<th>A5</th>
<th>A6</th>
<th>A7</th>
<th>A8</th>
<th>A9</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Charlotte/ Level 2</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td>Natalie/ Level 2</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td>Amelia/ Level 2</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td>Samantha/ Level 3</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td>Hannah/ Level 3</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td>Emily/ Level 3</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td>Brooklyn/ Level 3</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td>Maddie/ Level 4</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td>Lillian/ Level 4</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>9</td>
<td>0</td>
<td>9</td>
<td>0</td>
<td>5</td>
<td>0</td>
<td>6</td>
<td>9</td>
<td>9</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Task</th>
<th>Participant</th>
<th>A1</th>
<th>A2</th>
<th>A3</th>
<th>A4</th>
<th>A5</th>
<th>A6</th>
<th>A7</th>
<th>A8</th>
<th>A9</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>Charlotte/ Level 2</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td>Natalie/ Level 2</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td>Amelia/ Level 2</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td>Samantha/ Level 3</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td>Hannah/ Level 3</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td>Emily/ Level 3</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td>Brooklyn/ Level 3</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td>Maddie/ Level 4</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td>Lillian/ Level 4</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Total</td>
<td></td>
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<td>9</td>
<td>9</td>
<td>3</td>
<td>7</td>
<td>0</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>Gain</td>
<td></td>
<td>-2</td>
<td>N/A</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>-1</td>
<td></td>
</tr>
</tbody>
</table>

The gain seen for MSF Action 2 should not be considered since task 1 did not elicit MSF Action 2. Throughout the implementation of the instructional tasks, students improved in utilizing and valuing three different actions: Action 5 (valuing and using definitions), Action 6 (understanding relationships) and Action 8 (precise language and fine distinctions about
language). Action 8 showed the most noticeable gain, followed by Action 5 and then Action 6.

Action 5 (valuing and using definitions) was not directly asked for in any task but two students (Brooklyn and Lillian) decided to use it as justification in task 2 and three students (Emily, Hannah and Maddie) decided to use it as justification in task 3. Action 6 was asked for in each task but as the instructional sequence progressed, so did students’ reasoning abilities and explanations. Task 3 had students looking at two novel concepts, position and velocity, and draw connections between the two. Seven of the nine interviewees were able to do this by the end of the semester. At the beginning of the semester, the teacher had to walk the students through this reasoning. When it comes to Action 8, students paid much more attention to details as the semester progressed. On the sesame street task, students would mess up their notation and make mistakes in calculating necessary information. By the third task, these mistakes were almost non-existent.

Looking at the pre- to post-MSI implementation, students who scored the lowest scores on the pre-MSI showed the most positive gain throughout the course. This finding is consistent with the pre- to post-interview analysis. While analyzing the specific questions on the MSI, questions involving definitions (Action 5) and precise language and fine distinctions about language (Action 8) showed the most positive gain from pre- to post-MSI. This finding is also consistent with the pre- to post-interview analysis. Throughout the interview and task analysis one MSF Action showed positive gain, Action 6 (understanding relationships) which was also shown to be highly affected by students’ participation in the course.
Summarizing the data by students MS level at the start of the course follows. Students who scored at a Level 1 showed the most noticeable gain, yet, there are only three students within this category and only 1 showed noticeable gain. Therefore, limited analysis can be completed for this category. Students who were categorized by Level 2 showed the most noticeable gain (if Level 1 is eliminated due to lack of participants). Level 2 students improved across all aspects of the MSF as well as throughout all data collection throughout the entire semester. Students who scored a Level 3 at the start of the semester showed little gain. Students who scored a Level 4 showed negative gain. These results will be discussed further in Chapter 5.
Chapter 5 - Discussion

This study was based on studies on pre-service elementary teachers’ mathematical sophistication conducted by Seaman and Szydlik (2007). The design of this study included Seaman and Szydlik’s (2009) Mathematical Sophistication Framework (MSF) and the Mathematical Sophistication Instrument (MSI) used as a pre- and post-test for all CELTIC students. The study also included nine pre- and post-interviews as well as analysis of student artifacts on three tasks implemented in the classroom during the designated class time. The analysis investigated the following research questions:

- How does pre-service elementary teachers’ mathematical sophistication change as they participate in a reform-oriented calculus course?
- What experiences do pre-service elementary teachers have during a reform-oriented calculus course that support their growth in mathematical sophistication?

The analysis substantiated that participation in a reform oriented calculus course has a positive affect on students’ mathematical sophistication and provided some specifics on how the course was able to support this affect.

In the next sections, details of the results are discussed in relationship to other research. Other important findings from the analysis are presented, limitations are noted, and implications are presented. Finally, possible future research to continue investigating how curriculum design and instruction can impact pre-service elementary teachers’ levels of mathematical sophistication are offered.

Connections to Previous Research

CELTIC: Calculus for Elementary Teachers: An Innovative Context uses an inquiry-based learning approach with high-level tasks that introduces and deepens mathematical
understanding for each course concept. Participation in the course builds students’ intuitive understanding of the topic and pushes them to understand complex novel topics. Results from this study show that students’ participation in the first half of the course improved their level of mathematical sophistication. A significant amount of research on inquiry-based learning supports this finding (Chapman, 2011; Kazempour & Amirshokoohi, 2013; Kogan & Laursen, 2013; Laursen, 2014; Laursen, Hassi & Hough, 2015; Laursen, Hassi, Kogan & Weston, 2014).

**Inquiry-Based Class Environments**

Inquiry based approaches help students develop critical thinking skills (Bruner, 1961; Savin-Baden, 2004) needed for internalizing each of the MSF Actions while also building deep conceptual understanding of the topics being covered (McCann et al., 2005; Moon, 2004). The majority of students showed improvement in their mathematical sophistication throughout the course, yet the results were not statistically significant.

Szydlik et al. (Unpublished) performed a similar study, on a much bigger scale. They used the Mathematical Sophistication Instrument (MSI) to measure the extent that students’ values and ways of knowing mathematics aligned with that of the mathematics community in five different mathematics courses designed for elementary teachers. The results showed courses that implemented inquiry-based learning had a greater effect on the students and the students scored statistically significantly higher on the MSI at the end of the semester in comparison to the start of the semester. Laursen, Hassi and Hough (2015), Chapman (2011) as well as Kazempour and Amirshokoohi (2013) also found the same result that pre-service elementary teachers’ level of understanding and mathematical skills improved through IBL.
The participants in the study were freshmen and female. Although, these subgroups were not the focus of this study, it is important to note that research has shown student-learning outcomes increase significantly for both subgroups through IBL. For freshman, Laursen (2013) reports that the earlier a student is introduced to IBL the more successful they are academically in that course as well as in future courses taken in college. The same is true for women; Laursen (2013) explains that women in IBL classes improve significantly higher in all areas of cognitive development, social development and affective gain than in non-IBL courses. Women in IBL classroom perform just as well or better than their male counterparts in the same course, while women perform statistically much lower than males in non-IBL courses. “Particularly striking, the use of IBL eliminates a sizable gender gap that disfavors women students in lecture-based courses” (Laursen, Hassi, Kogan and Weston, 2014, p. 406). The current study support the proposition that IBL is a valid approach that can be utilized in mathematics departments who want to better prepare and strengthen all their pre-service elementary teacher preparation in ways supported by research.

**Use of High-Level Tasks**

The current study analyzed three different high cognitive demand tasks. Results support this type of educational strategy since pre-service elementary teachers’ level of mathematical sophistication increased throughout the course: specifically, in the areas of valuing definition, reasoning and attention to detail. The current study supports results found in similar studies. Gadanidis and Namukasa (2009) describe a course offered to pre-service elementary school teachers as well as a professional development course to in-service teachers through a series of mathematics-for-teachers courses. They use a series of high-level
tasks that are content oriented and involve teachers in deep mathematical thought as well as building relationships among concepts, such as CELTIC. Throughout the course, the pre-service elementary teachers’ perceptions of mathematics changed. They learned what doing mathematics is like and this helped change their view of mathematics for teaching and learning mathematics. Although mathematical sophistication is not mentioned, they did mention that students’ ability to reason increased throughout the course as well as their ability to build and explain relationships between mathematical concepts, one action in the MSF.

White and Mesa (2014) examined nearly 5,000 Calculus I tasks assigned by instructors of a 2-year college who all shared the same text. Only half of those tasks were of high cognitive demand as the ones used in CELTIC. Also, the majority of those high cognitive demand tasks showed up on exams. Therefore, White and Mesa (2014) found that students might not realize the importance of cognitively demanding tasks in the subject until it is too late. Therefore, it is important to do as CELTIC has done and incorporate cognitively demanding tasks into the curriculum so students can learn through the tasks and gain a deeper level of conceptual understanding of the topics.

Although student understanding of certain topics were not directly addressed in this research, all students in the course were successful and the majority of the students scored a B or higher on their final exams. This result is supported by other research that explains student learning is encouraged the most in classrooms that implement high-level tasks and encourage reasoning beyond the routinely procedural nature of traditional mathematics courses (Boaler and Staples 2008; Hiebert and Wearne 1993; Stein and Lane 1996).
Mathematical Sophistication and Habits of Mind

Pre- and post-tests, an analysis of student work on three instructional tasks, as well as pre- and post-interviews all agree that most students’ level of mathematical sophistication increases; especially when it comes to valuing and using definitions, mathematical reasoning and justification and precise language and fine distinctions about language. The current study supports results found in similar studies.

Breyfogle and Wilburne (2011) studied 55 preservice elementary teachers’ ability to solve novel mathematics tasks in an elementary content course, their cognitive levels as well as the habits of mind they used to solve mathematical tasks. As pre-service elementary teachers were solving mathematical tasks they incorporated 11 of the 16 general habits of mind identified by Costa and Kallick (2000), described in Chapter 2. For example, students learned the importance of reading problems and thinking about possible strategies before jumping straight into the mathematics (precise language and fine distinctions about language).

Hu and Hsiao (2013) studied 15 pre-service elementary teachers lack of understanding of division of fractions in an elementary education initial certification program. The certification program consisted of six 60-minute workshops that integrated the use of technology and Mathematical Habits of mind into the content teaching. The curriculum is a collection of tasks in which students were encouraged to provide in-depth explanations and reflections about their solution and the process to their solution. This study found similar results to Seaman and Szydlik (2007): pre-service elementary teachers’ conceptions were extremely limited at the start of the program. They only knew to use an
algorithm but had no means of explaining how the algorithm worked (Hu & Hsiao, 2013). Throughout the workshop, pre-service elementary teachers were better able to make connections between mathematical concepts as well as models and mathematical sentences by integrating Mathematical Habits of Mind.

The current study, as well as Hu and Hsiao’s (2013) findings, “suggest that learning and teaching Mathematical Habits of Mind improves learners’ conceptualization. Mathematical Habits of Mind promotes the development of learners’ conceptual understanding by habits such as patterning, describing, visualizing, tinkering, etc., during the problem solving process” (Hu & Hsiao, 2013). These habits of mind are strongly connected to the Mathematical Sophistication Framework (MSF) actions. Three of the MSF actions showed the most improvement throughout the duration of the course: definitions, justification and understanding, and precise language and fine distinctions about language. These three actions are vital to the teaching and learning of mathematics. The National Council of Teachers of Mathematics (2000) as well as the Common Core State Standards of Mathematics (2010) support the use of definitions, justification and understanding, and precise language and fine distinctions about language in classrooms that prepare future teachers as well as those which prepare K-12 students.

**Mathematical Sophistication Framework**

The current study supports results found in similar studies when the specific actions are considered. These include, in order of significance: Action 8, 5, and 6. Actions 3 (making and testing conjectures), 4 (mental and physical models) and 9 (symbolic notation) was noticeably high at the start of the course as well as the end of the course. Action 1 (patterns),
Action 2 (mathematics behind), and Action 7 (formal notions of proof) did not show any marketable improvements throughout the course. It is not clear why students did not show improvement in Action 1 and Action 2. It is clear why students did not show improvement in Action 7. Action 7 has to do with formal proof and CELTIC is designed to focus more on informal notions of proof instead of formal.

**Action 8: Precise language and fine distinctions about language**

Precise language and fine distinctions about language is vital for mathematicians and mathematics educators. For example, if someone were to mislabel a graph, or write the wrong word in an application problem, the information in the graph and the solution to the problem could drastically change. Seaman and Szydlik (2007) express the importance of precise language and fine distinctions about language since these distinctions are needed to communicate mathematical claims and to construct and evaluate mathematical arguments.

Throughout CELTIC, students were asked on multiple occasions to create problems to illustrate certain mathematical instances, read graphical displays as well as make their own and other instances that involve using precise language. Students would be asked to analyze each other work, present their work to the class and a class discussion on the precision, adequacy and justification of the work would be discussed as a class. Results from the current study show that the students’ ability to pay attention to detail and make necessary distinctions about language and details improved throughout the instructional sequence.

The current study supports results found in similar studies. A study by Szydlik et al. (Unpublished) used the Mathematical Sophistication Instrument (MSI) to measure the extent that students’ values and ways of knowing mathematics allied with that of the mathematics
community in five different mathematics courses designed for elementary teachers. The results showed courses that implemented inquiry-based learning had a greater effect on the students’ ability to make fine distinctions about language and pay attention to detail improved throughout the inquiry-based courses.

**Action 5: Definitions**

The current study supports findings in other research about the importance of using novel tasks (e.g., Lehrer et al., 1999; Zandieh & Rasmussen, 2010) to help students make sense of mathematics (e.g., Ambrose & Kenehan, 2009; Dahlberg & Houseman, 1997; Roth & Thom, 2009) while expanding students’ value of definitions. Definitions are vital to the world of mathematics as well as mathematics education (Zaslavsky & Shir, 2005). The value and practice of defining is normally not emphasized in traditional mathematics, students often receive definitions from the teacher in the course and do not have the ability to build, value and apply definitions as a result (Kobiela & Lehrer, 2015; Vinner; 1991). In contrast, the tasks and instructional sequence in CELTIC were designed in a way to model the practice of defining and encourage students to use, dissect and construct their own definitions of novel topics.

This study showed empirical evidence suggesting that students’ exposed to novel tasks in an inquiry-based setting improve their ability to use, value and apply definitions. A study by Zaslavsky and Shir (2005) confirmed these results as well. They found that students’ understanding of a mathematical definition was reflected and developed through tasks designed to promote students’ alternate ways of defining mathematical concepts. As students worked on the activities Zaslavsky and Shir (2005) created, their mathematical
sophistication improved: “Students’ views of the features of a mathematical definition resonated with those of the mathematical community with respect to the tendency to use minimal definitions when possible and appropriate” (p. 338).

In the current study, students’ ability to construct definitions continued to grow. Throughout the course, students began to practice defining without the assistance of the teacher. A previous study on one of the tasks used in the CELTIC curriculum, Sesame Street, found that students were able to use their own informal notion on the topic to develop a formal definition of the topic and use this definition successfully throughout the course (Keene, Hall, & Duca, 2014).

This was also seen in a study by Kobiela and Lehrer (2015) in which they studied how participation in the practice of defining influenced student’s conceptions of geometry. Their study “illustrates how the practice of defining developed for an entire class of students during a prolonged period of time” (Kobiela & Lehrer, 2015, p. 451). They found that students’ participation in a course in which the teacher models the aspects of practice of defining were important for supporting students learning of the topic as well as how to engage in defining. They also found that the teacher took on the major role of defining at the beginning of the course but as time went on, the students were able to use, value and practice the art of defining on their own. “Over time, students appeared more disposed to consider definition as an important way of establishing common mathematical ground, to hold member of the class accountable for violations of the common ground, and to generate forms of definition that relied increasingly on concise descriptions of properties and relations among these properties” (Kobiela & Lehrer, 2015, p. 451).
**Action 6: Justification**

“Justification is at the heart of mathematics” (Staples, Bartlo, & Thanheiser, 2012, p. 447). Justification has a variety of meanings, the meaning used in this study is that of understanding mathematics to the extent of being able to explain mathematics; it is a means to do and learn mathematics. A variety of research supports the fact that classrooms that support this sort of justification result in students with higher levels of mathematical thinking (Hiebert et al., 1997; Staples, Bartlo, & Thanheiser, 2012; Wood, Williams, & McNeal, 2006). Throughout CELTIC, students were asked on multiple occasions to explain, justify and informally prove their understanding of certain mathematical topics to a class of questioning peers. Results show that the students’ use of justification was limited at the start of the semester but continued to grow throughout the instructional sequence. These results are supported by a study which investigated enhancing students’ argumentation in an inquiry based multivariable calculus course (Kwon, Bae, & Oh, 2015). The study collected multiple sources of data of 18 freshman mathematics education majors in an iterative cyclic process of design research. Kwon, Bae and Oh (2015) found that students’ ability to construct valid explanations and strong arguments gradually developed into more complex forms throughout the duration of the study.

The current study supports results found in similar studies. Simon and Blume (1996) studied pre-service elementary teachers throughout a 3-year project titled Construction of Elementary Mathematics (CEM) designed to increase pre-service teachers’ mathematical knowledge as well as develop their views of mathematics similar to those of the mathematics reform documents previously mentioned and the mathematics community. Their research
supports the idea that mathematical reasoning and justification is both a cognitive and social process. Their data also suggests that active participations in a students’ education, as such done through CELTIC, increased students’ ability to practice the art of mathematical justification.

A study by Lo, Grant and Flowers (2008) of pre-service elementary teachers in a reasoning-based curriculum around number and operation found that pre-service elementary teachers’ ability to reasoning and explain their understanding grew over time. The course was designed using a variety of inquiry-based approaches and the participants were mostly first or second year students. Students had many difficulties with reasoning at the start of the course; some of them were never able to adequately reasoning and justify certain mathematical topics. Yet, the majority of students were able to make progress in both developing and justifying reasoning strategies.

**Calculus for Elementary Teachers**

This current study goes beyond any current research to show the benefit of pre-service elementary teachers building a connection between elementary mathematics and advanced mathematics, particularly as it relates to mathematical sophistication. Previous research explains that pre-service elementary teachers normally take mathematics content courses on topics directly related to the elementary mathematics that they will teach (Keene, Hall, & Duca, 2014). Results from the current study support the importance of not only teaching elementary teachers the mathematics they will teach but also advanced mathematics connected to elementary topics. Teaching students advanced mathematics situated in interesting and familiar contexts while building connections between the elementary
mathematics they will teach and the advanced mathematics should be a part of all pre-service elementary teachers’ education (Keene, Hall, & Duca, 2014).

The current study supports results found in similar studies not related to pre-service teachers Code et al. (2014) examined the effectiveness of an actively engaged research-based teaching methodology to increase conceptual and procedural aspects of calculus for undergraduate economics students. Results showed that student performance improved, especially when it came to conceptual understanding of certain mathematical topics. There was a control group, in which traditional calculus was taught, and the actively engaged classroom outperformed them on every topic. The curriculum in this study could be modified for other curricular areas and act as a model for inquiry-based learning in all calculus courses.

**Duplication Study Issues**

This study was initially designed to duplicate a study conducted by Seaman and Szydlik (2007). Due to the nature of the mathematics being covered in this course the current research made a few important changes to the design of the study. Since the current study took place in a calculus course, the tasks used in the interview process had to change from elementary topics in order to be novel tasks for the students. Also, Seaman and Szydlik (2007) use a specific online resource (discussed in chapter 4) whereas the current study allowed the students to search the internet for any sites they wished to visit to help them understand the novel tasks. These changes affected the results of the study and came up with much different results than the study conducted by Seaman and Szydlik (2007).
What started out as a duplication study became an extension of the research conducted by Seaman and Szydlik (2007). They found that pre-service teachers were severely lacking in mathematical sophistication as they defined it and that this inhibited their ability to learn mathematics to the degree needed for conceptually teaching mathematics. The current study used this framework to help analyze the results of the data collected.

They also created an instrument, Mathematical Sophistication Instrument (MSI), to gauge students’ mathematical sophistication and again, this study used this instrument to compare students’ mathematical sophistication at the start and the end. The redesign of both of these instruments allowed the current study to look at how students’ mathematical sophistication could be affected during the duration of a calculus course.

The MSI is both a reliable and valid measure of pre-service teachers’ mathematical sophistication (Szydlik, Kuennen, & Seaman, 2009). Szydlik, Kuennen, and Seaman (2009) express the need for future research on this instrument in order for it to become an assessment tool for all college mathematics programs as well as professional development. The type of course is vital to the usage of the MSI. Since the CELTIC curriculum emphasized informal notions over formal notions, some of the questions on the MSI were not appropriate for the discussion of the results for this study since none of the methods of formal logic were taught in the course. With the low number of participants in this study, questions such as these could have affected the results. Therefore, the MSI might not have been the best tool for entering freshman in a reform oriented calculus course. More research on this topic and instrument is needed to assess the instruments ability to measure students’ mathematical sophistication in this course.
Limitations of the Study

One limitation of the study was the length of time in which the study took place. The instructional activities only involved the first semester of the two-semester CELTIC course due to time constraints. Since the study only took place during the first part of a calculus course, the topics of study were limited. The results of the study may be different if time permitted giving the post-MSI at the end of the entire CELTIC course instead of the end of the first semester. Additionally, due to the lack of time, only nine pre- and post-interviews were conducted instead of interviewing all participants. The time constraint also prevented the researcher from testing the materials prior to the study and conducting a pilot study before initiating the study.

A second limitation of the study was the number and type of participants involved. Since the study needed to take place in a course in which the CELTIC curriculum was being implemented, the sample and selection of participants was limited. The sample was restricted to a large public university located in the southeastern United States. Due to the location and size of this university, there were only 33 pre-service elementary teachers and the participants lacked diversity. All 33 participants were female, 30 (91%) of the students were Caucasian and three (9%) of the students were African American. This may be representative of this particular university program, but not of the entire pre-service elementary teacher education population. Also, it is more difficult to achieve significance with the small number of participants. Additionally, the university program where the study took place recruits high-performing students in order to create teacher leaders with a deep conceptual understanding in all elementary disciplines with a special emphasis on the STEM field. Therefore,
recruitment efforts begin with students who are strong in the STEM field. Some of the
students entered the course with high levels of mathematical sophistication, so investigating
increases in Mathematical Sophistication is hampered in a four-month study.

A third limitation was the researcher and the teachers of the course. The researcher
was one of the teachers; this may be a limitation in itself. Since the researcher is the teacher,
a collaborative team was not available to work with and help analyze the data. The researcher
consulted with another researcher in her department, yet the second researcher was not in the
classroom for the implementation of the study restricting their ability to assist the researcher.
With a team of researchers, the study could have been more effective. The other teacher of
the course was a mathematics graduate student; this can also be a limitation. She had less
educational experience than the researcher. To minimize this limitation, both teachers met on
a weekly basis to ensure students were covering the same materials and the researcher
attended both sections of CELTIC throughout the entire implementation of the study.

In conclusion, there are limitations of the study, but results from this study add to the
body of research on pre-service elementary teachers’ level of mathematical sophistication as
well as research on preservice elementary teachers taking a reform oriented calculus course.

Implications

This study showed that participation in calculus courses using inquiry and high-level
tasks can improve pre-service elementary teachers’ mathematical sophistication. Having the
students construct their own understanding of complex novel mathematical topics helped
students not only gain a deep understanding of the mathematical topics being covered but
also build an array of skills they can use to approach novel mathematics when they are in the
classroom setting. It is impossible to teach teachers everything they will be teaching in an elementary teacher education program (Hammerness et al., 2005; Hiebert et al., 2007; Yeh and Santagata, 2015). However, having students leave the program with a set of skills they can employ to novel mathematics in the teacher setting can help them become better teachers and be better equipped to teach mathematics for understanding and help pre-service teachers find ways to develop their own mathematical content understanding. Advanced mathematics has roots in elementary mathematics and building those connections with pre-service elementary teachers while engaging them in skills frequently seen throughout the mathematics community may provide students learning outcomes in K-12 mathematics (Keene, Hall, & Duca, 2014).

Implementing inquiry-based learning is effective for all groups of students and can lower achievement gaps seen in previous research (Laursen, Hassi & Hough, 2015). The students who scored lowest on the pre-MSI showed the most gains, so a curriculum like CELTIC would be a useful in classrooms in which students struggle with how to approach novel mathematics topics. Alternatively, students who scored the highest on the pre-MSI showed conflicting reports, possibly because of the ceiling effect. Other research on inquiry-based learning shows that students who come in with a lower understanding make higher improvements; at the same time higher-level students also learn what they need to learn in an inquiry based classroom (Laursen, Hassi & Hough, 2015).

**Future Research**

There are multiple of ways to extend this research. CELTIC is a first year freshman course and positive improvement of students’ mathematical sophistication was shown in just
the first semester. Future research could continue to study these students or start a longitudinal study to look at how students’ mathematical sophistication changed over the course of the entire sequence of courses at this university. The researcher plans to give the students an assessment on mathematical sophistication at the end of their program and see if any of the changes are statistically significant at that point in their educational career.

There are multiple areas in which this research could improve mathematics education of elementary teachers. Research has shown that teachers with higher-levels of mathematical sophistication have higher student outcomes (Szydlik, Kuennen, & Seaman, 2009). The researcher plans to continue following the students as they become teachers, to see if their mathematical sophistication does in fact have an effect on elementary student outcomes. If it does, the researcher plans on analyzing the program to see what affected students’ mathematical sophistication.

Limited research has been done on how to improve elementary teachers’ lack of mathematical sophistication. This study opens the doors and shows the value of inquiry-based learning with the use of high-level tasks on students’ mathematical sophistication. More studies need to look at this relationship. The researcher plans to continue implementing this curriculum with a variety of students at different universities. This will add to mathematics education research by building a more complete understanding of how to improve pre-service elementary teachers’ mathematical sophistication across all levels of students. Another important aspect of continuing to teach this program at the current university as well as other universities is to see if some new patterns emerge with different groups of students.
Conclusion

Although individual finding in this study are supported by research, no other study looked at the specific participant group involved in this study; little research exists that looks at teaching calculus to pre-service elementary teachers. In other words, this study adds to what is known about improving the mathematical understandings of pre-service elementary teachers by their participation in a reform-oriented calculus course. CELTIC students were extremely good at making and testing conjectures (Action 3), using a valuing mental and physical models (Action 4) and symbolic notation (Action 9) at the start and the end of the first semester. CELTIC students showed the most improvement in using a valuing precise language and having fine distinctions about language (Action 8), using and valuing definitions (Action 5) and reasoning mathematically (Action 6).

The goal of this study was to evaluate the mathematical growth of pre-service elementary teachers using a relatively new construct similar to habits of mind, mathematical sophistication. This study implemented a mixed methods approach to see how a reform-oriented calculus course affected pre-service elementary teachers’ mathematical sophistication. The results revealed that teaching pre-service elementary teachers’ calculus through inquiry and high-level tasks is an effective way to: increase students’ abilities to use and value definitions, improve students reasoning and justification skills, as well as encouraged students’ to pay attention to detail and language. This may prove vital to improving pre-service teacher education in a STEM focused world.
References


35(2), 81-116.


to develop mathematical understanding of space. In E. Fennema & T. A. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp. 63-87). Mahwah, NJ: Lawrence Erlbaum Associates.


Appendices
Appendix A - Mathematical Sophistication Instrument

Mathematical Sophistication Instrument

For each item below, choose only the one best response. Thank you!

1) A number is called **normal** if it is less than 10 or even. According to this definition, of the numbers 5, 8, and 24,
   a) Only 5 and 8 are normal.
   b) Only 8 is normal.
   c) Only 5 and 24 are normal.
   d) All of these numbers are normal.

2) Consider the following statement about a new category of counting number called **Glick** numbers: **If a Glick number is greater than 19, then it is even.** Assuming the above statement is true, which of the following statements must also be true?
   a) If a Glick number is odd, then it is not greater than 19.
   b) If a Glick number is even, then it is greater than 19.
   c) Both of the above statements must be true
   d) None of the above statements must be true.

3) Consider the following statement: **There are at most ten people in the swimming pool.** Assuming the above statement is true, which of the following statements must also be true?
   a) There are ten people in the swimming pool.
   b) There is at least one person in the swimming pool.
   c) Both of the above statements must be true.
   d) None of the above statements must be true.

4) 3+7=____+8=____. What numbers go in the blanks?
   a) 10 and 18 respectively.
   b) 2 and 10 respectively
   c) Both of the above options work.
   d) None of the above options works.

5) Recall that the counting numbers are the numbers: 1, 2, 3, 4, 5 ... (and so on). Choose the single best response.
   a) Between any two counting numbers there is always another counting number.
   b) For each counting number, there is a counting number that is smaller.
   c) Both of the above statements are true.
d) None of the above statements is true.

6) Consider the following statement: **When you multiply any two odd numbers together, you always get an odd product.** Which of these arguments convinces you that the statement is true?

   a) $3 \times 5 = 15, 5 \times 7 = 35, 11 \times 7 = 77$ and so you can see by doing examples that the products will all be odd.
   b) Odd numbers do not have two as a factor. If neither of the numbers has two as a factor, than the product cannot have two as a factor.
   c) Both of the above are convincing.
   d) None of the above is convincing.

7) The numbers 1, 6 and 12 are called **hexagonal frame** numbers because one dot, six dots, and twelve dots can each be arranged in the shape of a hexagon frame as follows:

![Hexagonal Frame Numbers](image)

We say that 1 is the first hexagonal frame number (HF$_1$). The second hexagonal frame number (HF$_2$) is 6, and so on. What is the fourth hexagonal frame number (HF$_4$)?

   a) HF$_4$ = 18
   b) HF$_4$ = 19
   c) HF$_4$ = 24
   d) None of the above

8) What is the 101$^{st}$ hexagonal frame number (HF$_{101}$)?

   a) HF$_{101}$ = 600
   b) HF$_{101}$ = 606
   c) HF$_{101}$ = 636
   d) None of the above

9) This is how to make a “**Star**”: Draw $n$ points evenly spaced around a circle. Then start at any point and connect it with a line segment to the point $k$ spaces away. Then connect *that point* to a point $k$ spaces away (same direction). Continue until no new segments are
generated. The completed Figure is called Star(n, k).

Here is a picture of Star(7, 3):

Here is a picture of Star(12, 2):

Which of the figures below shows Star(8, 1)?

a)  

b)  

c)  

d)  

10) Please refer again to the way to make Stars as described in problem 9) above. Which of the following Stars would look the same as Star(7, 3)?

a) Star(7, 2)  

b) Star(7, 4)  

c) Star(3, 7)  

d) None of the above.

11) Please refer again to the way to make Stars as described in problem 9) above. If you needed to be sure that Star(2n, 2k) always looks the same as Star(n, k), which of the following would be most important to you?

a) Seeing several examples that work.  

b) Hearing from an instructor or reading in a text that it is true.  

c) Understanding why it makes sense.  

d) Any of the above would suffice.

12) Consider the following two statements:

All strange numbers are charming.  

Some happy numbers are charming.

If both of the above statements are true, which of the following statements must also be true?

a) Some happy numbers are strange.

b) If a number is charming, then it is strange.

c) Both a) and b) must be true.
d) Neither a) nor b) must be true.

13) Zachary has created a new category of polygons which he named isolaterals. He calls a polygon an isolateral if at least two adjacent sides are the same length. (Adjacent sides share a common endpoint.) Which of the following 4-sided polygons is/are isolateral?

I)  

II)  

III)

a) I and II only  

b) II and III only  

c) II only  

d) I and III only

14) Warm, light and dark paints are defined as follows:

A **warm** paint is a paint that contains yellow and red.  

A **light** paint is a paint that contains yellow.  

A **dark** paint is a paint that contains blue or red.

Using the definitions above, which of the following statements must be true?

a) A warm paint is always a dark paint.  

b) A dark paint cannot be a light paint.  

c) Both of the above statements must be true.  

d) None of the above statements must be true.

15) The notation \(a \sim b\) means multiply together \(a\) copies of \(b\) and then add one. For example \(3 \sim 2 = 9\). Which of the following is equivalent to 25?

a) 2~5  

b) 5~2  

c) 3~8  

d) None of the above.

16) A **graph** is set of points, called vertices, which are connected to each other by curves, called edges. Edges are not considered to intersect unless they meet at one of the vertices. The following are examples of graphs on four vertices.
A graph is called **connected** if we can get from any vertex to any other vertex by traveling along edges of the graph (we can move from one edge to another edge only at a vertex).

Which of the graphs shown above is/are connected?

a) graph IV only  
b) graphs I and IV only  
c) all except graph III  
d) none of the graphs

17) Which of the graphs below have the same mathematical structure?

a) graphs I and III only  
b) graphs II and IV only  
c) graphs I and III have the same structure and graphs II and IV have the same structure  
d) None of the graphs have the same structure

18) Which pair of story-problems below has the same mathematical structure?

I) Mark had 3 fish. He bought 6 more fish. How many does he have all together?  
II) Bari had some cookies. He ate 4, and now he has 2. How many did he have to start?  
III) Jen had some fish. She bought 6 more fish, and now she has 9. How many did she have to start?  
IV) Cal had several pencils. Now he has 20 because his teacher gave him 11 more. How many did he have to start?  

a) I and IV.  
b) I and III.  
c) III and IV.  
d) II and III.

19) The symbol Δ refers to a mathematical process with the following two properties:

For any positive numbers \( a \) and \( b \)

- \( \Delta(a \times b) = \Delta a + \Delta b \)
• \( \Delta(a \div b) = \Delta a - \Delta b \)

(Assume +, - , ×, and ÷ represent our usual arithmetic operations.)

Use the above properties to find an expression that must be equivalent to \( \Delta3 + \Delta4 \).

a) \( \Delta7 \)

b) \( \Delta12 \)

c) \( \Delta3 \times \Delta4 \)

d) None of the above.

20) Which of the following is a reasonable model for the multiplication problem: 1/2 \( \times \) 1/4?

a) 

b) 

c) Both of the above are reasonable models for 1/2 \( \times \) 1/4.

d) Neither of the above is a reasonable model for 1/2 \( \times \) 1/4.

21) You are a student learning about using lines to model data and, after the lesson, a student raises her hand and makes a guess about how other types of functions could be used to model data. Which option best reflects your view?

a) I would probably just want to know whether she was adequate or not.

b) I would probably want to spend time exploring her guess myself.

c) I would probably prefer to focus only on the material that was part of the real lesson.

d) I would probably want the instructor to Figure it out and explain it to me.

22) The notation \( a[b] \) means multiply \( a \) and \( b \), then to the product, add \( a \) and subtract \( b \).

Which of the following is equivalent to 5[2]?

a) 10[1]

b) 2[5]

c) 4[3]

d) None of the above is equivalent to 5[2].
23) Suppose that you build a $3 \times 3 \times 3$ block by gluing together 27 small cubes. Then you dip the block in paint, let it dry, and take it apart to again have 27 small cubes. How many of the small cubes will have paint on exactly three sides?
   a) 4
   b) 8
   c) 9
   d) None of the above.

24) Now we’ll repeat the experiment in 23) with a bigger block, an $n \times n \times n$ block made up of $n^3$ small cubes. Which of the following is true?
   a) The number of cubes with no paint on them will always be an odd number.
   b) The number of cubes with exactly one side painted will always be an even number.
   c) Both of the above are true.
   d) None of the above is true.

25) Now we’ll repeat the experiment in 23) with an $n \times n \times n$ block made up of $n^3$ small cubes. Which of the following statements is true?
   a) As $n$ gets bigger the number of cubes with paint on three sides gets bigger.
   b) As $n$ gets bigger, eventually most of the cubes will have no paint on them.
   c) Both of the above are true.
   d) None of the above is true.
### Appendix B - Mathematical Sophistication Framework

<table>
<thead>
<tr>
<th>MSF</th>
<th>MSF Action</th>
<th>MSF Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Seek to find and understand patterns.</td>
<td>Prospective elementary teachers must value patterns and regularity and have systematic ways of making sense of patterns involving number and shape.</td>
</tr>
<tr>
<td>2</td>
<td>Classify and characterize objects based on structure.</td>
<td>Prospective elementary teachers must value operational or geometric properties over mathematically superficial ones such as orientation, problem context, or labeling.</td>
</tr>
<tr>
<td>3</td>
<td>Make and test conjectures about objects and structures.</td>
<td>Prospective elementary teachers must be willing to explore problem situations and to make and test conjectures by considering extreme or divergent cases.</td>
</tr>
<tr>
<td>4</td>
<td>Create mental (and physical) models for, and examples and non-examples of, mathematical objects.</td>
<td>Prospective elementary teachers must draw or imagine models (often general or dynamic models) to help them make sense of problem situations, relationships, and novel definitions.</td>
</tr>
<tr>
<td>5</td>
<td>Value and use precise definitions of objects.</td>
<td>Prospective elementary teachers must use the mathematical definition to classify objects without regard to extraneous meanings of terms suggested by the wider culture.</td>
</tr>
<tr>
<td>6</td>
<td>Value an understanding of why relationships make sense.</td>
<td>Prospective elementary teachers must recognize that mathematics makes sense, and they must seize opportunities to explore relationships.</td>
</tr>
<tr>
<td>7</td>
<td>Value and use logical arguments and counterexamples as sources of conviction.</td>
<td>Prospective elementary teachers must understand that examples alone do not provide sufficient mathematical justification for a claim, and at the same time recognize that an example can provide the seed of a general argument (e.g., Carpenter, Franke &amp; Levi, 2003). They must value counterexamples and arguments based on structure and reasoning.</td>
</tr>
<tr>
<td>8</td>
<td>Value precise language and have fine distinctions about language.</td>
<td>Prospective elementary teachers must understand and use the mathematical culture’s normative meanings for terms such as: and, or, there exists, for each, at most, at least, always, less than, and greater than. They must also distinguish necessary from sufficient conditions; distinguish converse from contrapositive forms; and understand that if, for example, a person has ten pets, it is also true that she has two pets.</td>
</tr>
<tr>
<td>9</td>
<td>Value and use symbolic representations of, and</td>
<td>Prospective elementary teachers must understand and use the mathematically normative meanings for familiar...</td>
</tr>
</tbody>
</table>
notation for, objects and ideas. symbols, and persevere to make sense of a new symbol or a new notation that is defined for them.
Appendix C - Pre-Interview Protocol

Pre-Interview Protocol

Materials:
Video camera, consent form, tasks, graph paper, dark marker, and a laptop.

Preparation:
Make copies of the tasks and set up the camera.

Introduction:
Say, Hi, I’m _______. I really appreciate your taking the time to meet with me today. As you may know, I am interested in learning about how you think about mathematics. Keep in mind that what you do will not have any effect on your grade in your mathematics class and what you say will remain anonymous. You may choose to leave at anytime.

Roles:
Say, Because I am interested in how you think about mathematics, it would really help me if you would talk as much as you can. While you are working and talking, I will probably ask questions to be sure I understand what you are saying. I am asking because I am really interested. I am not asking a question because what you are doing is adequate or inadequate, but rather I just want to make sure I understand what you are saying. Chances are I probably just missed something you said or did. Likewise, if I ask a question that does not make sense to you, please tell me.

Recording:
Say, I might take a few notes to help me remember. But, because I don’t want to take too many notes, I will videotape our conversation. The video camera will be focused on what you are writing or doing. Is this ok? I need you to sign this video consent form; you can read it and ask me any questions. Do you have any questions?

Protocol:
Say, I am going to ask you a few general problems about mathematics and then to work on two different problems. Please talk out loud as much as you can. Remember whatever you say will be helpful to me. Do you have any questions?

Rate of Change Tasks:
Say, I am going to give you a couple of tasks. I would like for you to start with task 1 and then move onto task 2. Please read the task out loud and tell me everything you think about the task. I would like for you to explain anything you write on the task paper.
Task 1

From a vertical position against the wall, a ladder is pulled away at the bottom at a constant rate. Describe the speed of the top of the ladder as it slides down the wall. Justify your claim.

Did you decide that the top of the ladder’s movement speeds up, stays constant or slows down?

Can you explain?

What mathematical knowledge helped you come up with this decision?

Can you show with your arm what you think will happen?

Is there any kind of algebraic expressions you could use to help you understand the relationship between the ladder and the wall?

Task 2

Imagine this bottle filling with water. Sketch a graph that represents the relationship between amount of water that is in the bottle and the height. EXPLAIN

Describe how you sketched the graph.

Ask question about each of the different segments of the graph as you see it: How does this fit with the bottle?

Ask question: where in the graph is this point on the bottle. (and pick different places on the bottle).

Ask them to follow the graph with their finger and discuss where it fits with the bottle.

Symmetric: How does that affect the graph?
Middle: Can you tell what happened at this point [inflection point]?
Middle: What happens at the middle of the spherical portion?
Curved lines: Why did you draw a smooth curve through the lines?

Internet:
Say, Now I am going to ask you to use a computer with Internet access to study the mathematics topics covered in the tasks. The video camera will still be on and the screen will be recorded so I can learn how you learn. In 20 minutes I am going to come back in the room and we will go over the tasks one final time. Is that ok?
-Wait 20 Minutes
Say, Are you done studying? Do you want more time? (If so give more time and repeat this question until they are done)

Task Review:
Say, Now I am going to give you the tasks back. I would like for you to reread them and then explain to me if your previous solution was adequate. If you would like to make any changes please do and tell me why you made those changes.

Task 1
From a vertical position against the wall, a ladder is pulled away at the bottom at a constant rate. Describe the speed of the top of the ladder as it slides down the wall. Justify your claim.
Did you decide that the top of the ladder’s movement speeds up, stays constant or slows down? Do you want to change your decision?
Can you explain?
Can you show with your arm what you think will happen?
Is there any kind of algebraic expressions you could use to help you understand the relationship between the ladder and the wall?
What on the Internet helped you answer this task?
Do you have anything else you would like to add to this task before we move onto the next task?

Task 2
Imagine this bottle filling with water. Sketch a graph that represents the relationship between amount of water that is in the bottle and the height. EXPLAIN
Describe how you sketched the graph.
Would you like to change your sketch?
Ask question about each of the different segments of the graph as you see it: How does this fit with the bottle?
Ask question: where in the graph is this point on the bottle. (and pick different places on the bottle).
Ask them to follow the graph with their finger and discuss where it fits with the bottle.
   *Symmetric:* How does that affect the graph?
   *Middle:* Can you tell what happened at this point [inflection point]?
   *Middle:* What happens at the middle of the spherical portion?
   *Curved lines:* Why did you draw a smooth curve through the lines?
What on the Internet helped you answer this task?
Do you have anything else you would like to add to this task before we end the interview?

**Conclusion:**
Ask: Is there anything else about rate of change that you would like to tell me? Do you have any questions about the interview?
Say: Thank you for your time, this was a great interview and I really appreciate you taking the time to help me learn how you reason about rate of change.
Appendix D - Post-Interview Protocol

Post-Interview Protocol

Materials:
Video camera, consent form, tasks, graph paper, dark marker, and a laptop with Quick Time.

Preparation:
Make copies of the tasks and set up the camera.

Introduction:
Say, Hi, I’m _______. I really appreciate your taking the time to meet with me today. As you may know, I am interested in learning about how you think about mathematics. Keep in mind that what you do will not have any effect on your grade in your mathematics class and what you say will remain anonymous. You may choose to leave at anytime.

Roles:
Say, Because I am interested in how you think about mathematics, it would really help me if you would talk as much as you can. While you are working and talking, I will probably ask questions to be sure I understand what you are saying. I am asking because I am really interested. I am not asking a question because what you are doing is adequate or inadequate, but rather I just want to make sure I understand what you are saying. Chances are I probably just missed something you said or did. Likewise, if I ask a question that does not make sense to you, please tell me.

Recording:
Say, I would like to videotape our conversation so I (Mrs. Ritter) can watch it later. The video camera will be focused on what you are writing or doing. Is this ok? Do you have any questions?

Protocol:
Say, I am going to ask you to work on two different problems. Please talk out loud as much as you can. Remember whatever you say will be helpful to me. Do you have any questions?

Integration Tasks:
Say, I am going to give you a couple of tasks. I would like for you to start with task 1 and then move onto task 2. Please read the task out loud and tell me everything you think about the task. I would like for you to explain anything you write on the task paper.
Task 1
You and your friend start off at noon and walk in the same direction along the same path at the rates shown in the Figure below.

![Graph showing the rates of walking for you and your friend.](image)

a) Who walks faster at 1:00 pm? Who is ahead at 1:00 pm? Explain.
b) Who walks faster at 2:00 pm? Who is ahead at 2:00 pm? Explain.
c) Who walks faster at 3:00 pm? Who is ahead at 3:00 pm? Explain.
d) How can you find the time when you and your friend will be together? Answer in words.

What would be the same if you and your friend will be together?
Can you explain?
What does that have to do with time and velocity?

Task 2
1. An orchard owner, Melinda, is trying to find out the apple production of her orchard. Unfortunately, the production notes from previous year have been lost. However she has found a Table of the rates of production of apple bushels over the past year. Use this Table to estimate a range of how many apples were produced in the last year. To make it easier, let’s assume there are 30 days in each month.

<table>
<thead>
<tr>
<th>Month</th>
<th>Rate of production in apple bushels/day</th>
</tr>
</thead>
<tbody>
<tr>
<td>January</td>
<td>0</td>
</tr>
<tr>
<td>February</td>
<td>0</td>
</tr>
<tr>
<td>March</td>
<td>0</td>
</tr>
</tbody>
</table>
a) How many bushels of apples did Melinda’s orchard produce last year? Show your work and explain how you did that.
b) Use the Table above to sketch a graph of the production rate on this coordinate axes. Let the horizontal axis be days starting with January 1, and let the y-axis be the production rate.

c) Look at your production rates and your graph. Is there anything that could connect the total production you calculated to anything in the graph?

d) How did you calculate the total production?
What connects the graph to your production rates?
What do you notice about the axis? Write down how you found the production rates. Now look at the numbers on the axis. Do you notice anything?
Can you think of any previous math concepts that connect to these concepts?
Do you have any more thoughts about this task?

**Internet:**

_Say,_ Now I am going to ask you to use a computer with Internet access to study the mathematics topics covered in the tasks. The video camera will still be on and the screen will be recorded so I can learn how you learn. In 20 minutes I am going to come back in the room and we will go over the tasks one final time. Is that ok?

-Wait 20 Minutes

_Say,_ Are you done studying? Do you want more time? (If so give more time and repeat this question until they are done)

**Task Review:**

_Say,_ Now I am going to give you the tasks back. I would like for you to reread them and then explain to me if your previous solution was adequate. If you would like to make any changes please do and tell me why you made those changes.

**Task 1**

You and your friend start off at noon and walk in the same direction along the same path at the rates shown in the Figure below.

![Graph](image)

a) Who walks faster at 1:00 pm? Who is ahead at 1:00 pm? Explain.
b) Who walks faster at 2:00 pm? Who is ahead at 2:00 pm? Explain.
c) Who walks faster at 3:00 pm? Who is ahead at 3:00 pm? Explain.
d) How can you find the time when you and your friend will be together? Answer in words.

What would be the same if you and your friend will be together?
Can you explain?
What does that have to do with time and velocity?
Did you change your solution? Why?
What on the Internet influenced your solution?

**Task 2**

1. An orchard owner, Melinda, is trying to find out the apple production of her orchard. Unfortunately, the production notes from previous year have been lost. However she has found a Table of the rates of production of apple bushels over the past year. Use this Table to estimate a range of how many apples were produced in the last year. To make it easier, let’s assume there are 30 days in each month.

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<td>0</td>
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<tr>
<td>April</td>
<td>0</td>
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<tr>
<td>May</td>
<td>0</td>
</tr>
<tr>
<td>June</td>
<td>80</td>
</tr>
<tr>
<td>July</td>
<td>90</td>
</tr>
<tr>
<td>August</td>
<td>100</td>
</tr>
<tr>
<td>September</td>
<td>200</td>
</tr>
<tr>
<td>October</td>
<td>500</td>
</tr>
<tr>
<td>November</td>
<td>300</td>
</tr>
<tr>
<td>December</td>
<td>100</td>
</tr>
</tbody>
</table>

a) How many bushels of apples did Melinda’s orchard produce last year? Show your work and explain how you did that.
b) Use the Table above to sketch a graph of the production rate on this coordinate axes. Let the horizontal axis be days starting with January 1, and let the y-axis be the production rate.

![Graph](image)

In January the production rate was 0

c) Look at your production rates and your graph. Is there anything that could connect the total production you calculated to anything in the graph?

d) How did you calculate the total production?
What connects the graph to your production rates?
What do you notice about the axis? Write down how you found the production rates. Now look at the numbers on the axis. Do you notice anything?
Can you think of any previous math concepts that connect to these concepts?
Do you have any more thoughts about this task?
Did you change your solution? Why?
What on the Internet influenced your solution?

Conclusion:
Ask: Is there anything else you would like to tell me? Do you have any questions about the interview?
Say: Thank you for your time, this was a great interview and I really appreciate you taking the time to help me learn how you reason about mathematics.
Appendix E - Recruitment Form

Recruitment Form

Hi, I’m Carrie Lineberry Ritter. I really appreciate you taking the time to consider being a part of my study. As you may know, I am interested in learning about how you think about and approach mathematics. I am asking you to take part in a research study.

The purpose of research studies are to gain a better understanding of a certain topic or issue. You are not guaranteed any personal benefits from being in a study. I am going to ask to fill out a consent form for your work and a consent form for video recordings. In this consent form you will find specific details about the research in which you are being asked to participate. If you do not understand something in this form it is your right to ask the researcher for clarification or for more information. A copy of this consent form will be provided to you. If at any time you have questions about your participation, do not hesitate to contact me at any time.

Your participation in this study is voluntary. You have the right to be a part of this study, to choose not to participate or to stop participating at any time without penalty. Your participation in the study will be greatly appreciated, although you are not required to participate. You will not be penalized if you decide not want to be a part of the study. Your grades will not be affected by participating or not participating, anyone who is present for these lessons will get credit if they do the work, regardless if they are in the study or not. If you decide to participate and want to stop later you can, you can tell me if you want to stop your participation at any time. If you decide to decline to participate in the study, or want to stop participating, I will not use your documents for the study and will shred them after you have completed the activities.

Participation in the study will be primarily during class. You will be given an assignment at the beginning and the end of the course. All students will complete this assignment, yet only the responses to those students who consent to being in the study will be used for research purposes. This study will also include three different instructional activities that are a part of the curriculum of your MA 151 course. All students will complete each activity, yet only the responses to those students who consent to being in the study will be used for research purposes. Nine students will be asked to complete a pre-interview and a post-interview outside of class time in order to determine how you approach novel mathematics, how you approach learning a new topic and your level of mathematical sophistication at the start and end of the MA 151 course. The following Table summarizes the time needed for participation in the study:

<table>
<thead>
<tr>
<th>Activity</th>
<th>Time-length</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre-test for both classes (MSI)</td>
<td>45 minutes on Day 1, in class</td>
</tr>
<tr>
<td>9 Pre-interviews</td>
<td>60 minutes each within one week of the pre-test before calculus instruction begins, outside of class time</td>
</tr>
<tr>
<td>Instructional Sequence including 3 high level tasks</td>
<td>4 Months, each task will take place during one 75 minute class meeting.</td>
</tr>
<tr>
<td>Post-test for both classes (MSI)</td>
<td>45 minutes following the completion of the instructional sequence, in class</td>
</tr>
<tr>
<td>9 Post-interviews</td>
<td>60 minutes each within one week of the completion of the differential calculus instructional sequence, outside of class time</td>
</tr>
</tbody>
</table>
Appendix F - Informed Consent Form

North Carolina State University
INFORMED CONSENT FORM for RESEARCH

Pre-service Elementary Teachers’ Mathematical Sophistication
Principal Investigator: Carrie Lineberry Ritter
Faculty Sponsor: Dr. Karen Keene

What are some general things you should know about research studies?
You are being asked to take part in a research study. Your participation in this study is voluntary. You have the right to be a part of this study, to choose not to participate or to stop participating at any time without penalty. Your participation in the study will be greatly appreciated, although you are not required to participate and will not be penalized if you decide you do not want to be a part of the study. Your grades will not be affected by participating or not participating, anyone who is present for these lessons will get credit if they do the work, regardless if they are in the study or not. If you decide to participate and want to stop later you can, you can tell me if you want to stop your participation at any time. If you decide to decline to participate in the study, or want to stop participating, I will not use your documents for the study and will shred them after you have completed the activities.

The purpose of research studies is to gain a better understanding of a certain topic or issue. You are not guaranteed any personal benefits from being in a study. In this consent form you will find specific details about the research in which you are being asked to participate. If you do not understand something in this form it is your right to ask the researcher for clarification or for more information. A copy of this consent form will be provided to you. If at any time you have questions about your participation, do not hesitate to contact the researcher(s) named above.

What is the purpose of this study?
The intent of this project is to determine pre-service teachers’ mathematical sophistication. The importance of this work lies in its potential to increase teachers’ pedagogical content knowledge, student understanding, and achievement.

What will happen if you take part in the study?
If you agree to participate in this study, I will analyze the following class activities that you will complete due to as part of MA 151 curriculum: a pre-test, three activities, and a post-test. The study will take approximately eight days, each being 75 minute classrooms, to complete the pre-test, post-test and lessons during regular class time. You may also volunteer to participate further by completing a pre-interview and a post-interview. Each will take 60 minutes outside of class time.
Risks
There are minimal risks associated with participation in this study.

Benefits
Participating in this study could increase your knowledge and lead to deeper understandings of calculus and elementary topics in this course. No benefits are guaranteed.

What if you are a NCSU student?
Participation in this study is not a course requirement and your participation or lack thereof, will not affect your class standing or grades at NC State.

Confidentiality
The information in the study records will be kept strictly confidential. Data will be stored securely in a locked cabinet in the principle investigators office. No reference will be made in oral or written reports that could link you to the study.

Compensation
You will not receive anything for participating.

What if you have questions about this study?
If you have questions at any time about the study or the procedures, you may contact the researcher, Carrie Lineberry Ritter at calinebe@ncsu.edu or 336-466-1197.

What if you have questions about your rights as a research participant?
If you feel you have not been treated according to the descriptions in this form, or your rights as a participant in research have been violated during the course of this project, you may contact Arnold Bell, PhD, Chair of the NCSU IRB for the Use of Human Subjects in Research Committee, Box 7514, NCSU Campus (919/ 515-4420) or Mr. Matthew Ronning, Assistant Vice Chancellor, Research Administration, Box 7514, NCSU Campus (919/513-2148)

Consent To Participate
“I have read and understand the above information. I have received a copy of this form. I agree to participate in this study with the understanding that I may choose not to participate or to stop participating at any time without penalty or loss of benefits to which I am otherwise entitled.”

Subject's signature_____________________________ Date __________________
Investigator's signature_________________________ Date __________________
Appendix G - Informed Video Consent Form

North Carolina State University
INFORMED CONSENT FORM for VIDEO TAPED INTERVIEW

Pre-service Elementary Teachers’ Mathematical Sophistication
Principal Investigator: Carrie Lineberry Ritter
Faculty Sponsor: Dr. Karen Keene

What are some general things you should know about research studies?
You are being asked to take part in a research study. Your participation in this study is voluntary. You have the right to be a part of this study, to choose not to participate or to stop participating at any time without penalty. The purpose of research studies is to gain a better understanding of a certain topic or issue. You are not guaranteed any personal benefits from being in a study. In this consent form you will find specific details about the research in which you are being asked to participate. If you do not understand something in this form it is your right to ask the researcher for clarification or more information. A copy of this consent form will be provided to you. If at any time you have questions about your participation, do not hesitate to contact the researcher(s) named above.

What is the purpose of this study?
The intent of this project is to determine pre-service teachers’ mathematical sophistication. The importance of this work lies in its potential to increase teachers’ pedagogical content knowledge, student understanding, and achievement.

What will happen if you take part in the study?
If you agree to participate in this study, you will be asked to participate in two video recorded interviews to gain an understanding of your mathematical sophistication. The interview will be video recorded and transcribed. You will also complete two mathematical tasks during both interviews. The interview will take approximately 45-60 minutes outside of class time. The interviewer, Mrs. Ritter, and interviewee will be the only people in the room in order to sustain confidentiality. You may also be video recorded during class time while completing any of the three instructional tasks.

Risks
There are minimal risks associated with this research

Benefits
Participating in this study could increase your knowledge and lead to deeper understandings of calculus and elementary topics in this course. No benefits are guaranteed.
What if you are a NCSU student?
Participation in this study is not a course requirement and your participation or lack thereof, will not affect your class standing or grades at NC State.

Confidentiality
The information in the study records will be kept strictly confidential. Data will be stored securely in a locked cabinet in the principle investigators office. After the study has come to an end the videotape will be destroyed. No reference will be made in oral or written reports that could link you to the study so that no one can match your identity to the answers that you provide.

Compensation
You will not receive anything for participating.

What if you have questions about this study?
If you have questions at any time about the study or the procedures, you may contact the researcher, Carrie Lineberry Ritter at calinebe@ncsu.edu or 336-466-1197.

What if you have questions about your rights as a research participant?
If you feel you have not been treated according to the descriptions in this form, or your rights as a participant in research have been violated during the course of this project, you may contact Arnold Bell, PhD, Chair of the NCSU IRB for the Use of Human Subjects in Research Committee, Box 7514, NCSU Campus (919/ 515-4420) or Mr. Matthew Ronning, Assistant Vice Chancellor, Research Administration, Box 7514, NCSU Campus (919/513-2148)

Consent To Participate
“I have read and understand the above information. I have received a copy of this form. I agree to participate in this study with the understanding that I may choose not to participate or to stop participating at any time without penalty or loss of benefits to which I am otherwise entitled.”

Subject's signature_____________________________ Date _______________
Investigator's signature________________________ Date _______________
Annex H - Sesame Street Task

Group and Name: _____________________

Getting Back to Sesame Street:
An Introductory Activity for Limit of a Sequence

Big Bird and Count von Count are traveling back to Sesame Street when they come to a bridge. Just before the bridge there is a warning sign:

Big Bird says to Count von Count,
“Count, you’re a counting genius! I know how to get across Lord Zeno’s bridge, you count my steps and I’ll walk across!”

Unfortunately, when they get closer, Big Bird notices another sign:

Lord Zeno must be one odd fellow; this is a strange bridge indeed!
Let’s try and Figure out how Big Bird and Count von Count will get across!

*Historical Context:* Zeno was actually a real 5th century Greek philosopher. You can read more about him here: [http://www-history.mcs.st-andrews.ac.uk/Biographies/Zeno_of_Elea.html](http://www-history.mcs.st-andrews.ac.uk/Biographies/Zeno_of_Elea.html)
**TASK #1: Try it out!**

Go and find a spot about 5-6 ft. from a wall. Try to follow the directions on the sign: Every step you take should be half the distance that is between you and the wall. Keep taking steps until you can’t take any more! Once you are done, sit down with your group and answer the following questions:

4. How many steps did you take?
5. Why did you have to stop taking steps?
6. Could you have taken more steps? If yes, how many more steps could you have taken? If not, explain why your number of steps is the maximum.

**TASK #2: Big Bird & Count von Count**

Let’s help Big Bird and Count von Count Figure out just how many steps Big Bird will have to take in order to cross this strange bridge.

9. Let’s say that Zeno’s bridge is 10 meters long, how many steps do you estimate Big Bird will have to take in order to cross the bridge? Explain your estimation.

10. How far did Big Bird travel with his first step? How far has he traveled after his second step?
   a) After 1st Step:
   b) After 2nd Step:

11. Suppose the segment line below represents the bridge and suppose they travel left to right. We can indicate on this "bridge" the total distance left to travel after one step, $a_1$. Also, we can put on the "bridge" the distance traveled.

   $a_1 = 10 \times (1/2)$

   distance left to travel

   $10 \times (1/2)$

   total distance traveled
Use the "bridge" below to represent the total distance left to travel after two steps, \( a_2 \). Find the distance traveled and show it on the "bridge".

_________________________________________________________

Indicate on this "bridge" the total distance left to travel after three steps, \( a_3 \). Find the distance traveled and indicate on the "bridge".

_________________________________________________________

Let’s let \( a_n \) be the sequence for the distance left to travel. Can you find a recursive formula for the distance left to travel after having taken \( n \) steps? What kind of sequence this is? Can you write a closed formula for the distance left to travel, \( a_n \)?

12. Let \{\( b_n \)\} be the sequence where the \( n^{th} \) term corresponds to the distance Big Bird has traveled after his \( n^{th} \) step. Compute the first five terms of this sequence. Find a recursive formula for the sequence. 

(Hint: You computed \( a_1 \) and \( a_2 \) in question #3). Can you create a closed formula for this sequence?

13. Would Big Bird ever be less than 0.01 meters from the end of the bridge? If yes, give support for your answer. If not, explain why not.

14. Without computing, do you know if Big Bird will ever be within 0.001 meters from the end of the bridge? How about 0.00001 meters? How do you know? When he gets to each of those places, at least how far would he have traveled?

15. Big Bird makes a shocking revelation: He claims that if Count calls out any distance, as close to 10 as he wants, if he follows Lord Zeno’s directions, after a certain step, he will have traveled that far. Do you believe this is true? Why or why not?

16. You may have noticed that the distance Big Bird has traveled seems to be getting close to 10. In your group, write a description of the behavior that the sequence exhibits as Big Bird continues his journey to the end of the bridge.
Appendix I - Functions Task

Functions: Of the World or Out of the World?

Today we will investigate functions as they emerge from the real world. We will be collecting data, so be prepared to keep good notes. You will share your results from the activity with the class and answer questions that your classmates and/or professors may ask. In this activity we will identify families of functions that you are familiar with from Algebra. Each family has one basic “rule”/graph but members of the family may have different coefficients and graphs, but with the same basic shape.

Part I:

• Read your card and identify what materials and measuring devices you may need to collect the data asked for the given relation.
• Collect data to create ordered pairs. You must have at least 6 ordered pairs. Write down the "quantities" you are measuring. List them in a Table.
• Get a piece of graph paper. On the top of the page write the name of your experiment and members in your group.
• Graph the ordered pairs on a piece of graph paper carefully. Be sure to mark the units and the \( x \) and \( y \) axis. Indicate which quantities are represented on your axes.
• On your graph: Create a continuous line or curve that connects the ordered pairs.

Part II:

13. On the back of your graph paper write the name of your experiment and members in your group again, describe your experiment in 3 sentences (be clear and concise).
14. Next identify the parent function (If you have no idea, see a list of possibilities on the board and use a graphing calculator if needed). Once you have identified the parent function: graph the function, write its domain, and write the range under the graph.
15. Describe the family of function found in your experiment. Include a small sketch of your graph. Write its domain and range under the graph (remember it comes from the real world). Then explain how your graph is specifically related to your parent function (Figure out how to manipulate the parent function and use your graphing calculator to find a graph that matches your data).
16. Finally, think of another real world relationship that also would generate a function from the same family.

Part III:

• Be prepared to present all of the information from parts I and II to the class. Be very clear, your presentation with be critiqued. Remember you are teaching the class about the experiment, the parent function and the family of function exhibited in your experiment.
Experiment Cards

Knot tying.
• Experiment: Measure the length of a rope without knots. Then tie a knot and measure again. Continue adding knots and measuring. Be careful that all the knots are the same and not on top of each other.
• Materials: Rope that is at least 20 inch’s long and a measuring tape.

Pressure and Volume.
• Experiment: Use the Gas Pressure Sensor and Vernier Lab Pro to measure Volume and Pressure on at least 6 data point.
• Materials: Gas pressure sensor, vernier lab pro and a graphing calculator with the necessary program.

Measuring the size of a square projected.
• Experiment: Measure the distance an overhead projector is from the wall and the size (area) of the light on the wall.
• Materials: Overhead projector, square, and a measuring tape.

Measure time for a Pendulum.
• Experiment: Use a rope with a nut tied to the end. Collect data on the length of the rope and the time it takes to do 10 swings.
• Materials: A spool of rope, a nut, and a measuring tape.

Height of Cardboard Pile.
• Experiment: Take a large piece of cardboard. Measure its height. Cut it in half and make a pile. Measure its height. Cut each piece in half and make a new pile, measure its height. Continue until you can’t cut any more.
• Materials: Large piece of cardboard, scissors, and a measuring tape.

Spring and weight.
• Experiments: Look at the length of a scale with no weights. Then add weights and measure the length.
• Materials: Scale, weights, and a measuring tape.

Displacement of ball in water
• Experiments: Measure the diameter or circumference of each ball and the amount of water it displaces.
• Materials: A variety of balls, water, a measuring tape, and a large beaker.
Appendix J - Logger Pro Task

HOW FAST CAN YOU GO?

Today we are going to explore a foundational aspect of mathematics and physics: the relationship between position and velocity. In some ways this relationship is trivial, objects travel with a certain velocity and thus the position of the object changes. However, as you will see, we can be much more exact about how changes in the velocity of an object affect its position.

Before you begin, take a minute to discuss the terms “velocity” and “position” with your group. What do you know about them?

Part A. Distance Activity

We are going to explore what kinds of graphs you can create with Logger Pro and the distance detector. Use the tasks below to try and Figure out what distance the distance detector measures. Be careful to keep your toy right in front of the detector!

17. Open the file: “position.cmbl.” Have a group member click the “Collect” button while another student moves the toy. What kinds of graphs can you create? Try out a bunch of different motions and see what happens. Copy and paste an interesting graph that your group created in the file: “DistDet_Graphs.doc” under “Part A #1.” Write a brief description of the movement that created the graph below.

18. Can you create a graph of a curve that was not a function? Explain why or why not?
In this section, your group will create a specific *position vs. time* graph and answer some mathematical questions about the graph.

![Figure 1-Position vs. Time Graph](image)

19. Use Logger Pro to create the graph using the distance detector (when you hit the Collect button again your previous graph will disappear while a new graph is created). Your group may have to conduct several trials before you attain a nice, pretty, fairly smooth graph in appearance. Copy and paste your graph into the file: “DistDet_Graphs” under “Part A #4.”

20. Now you are going to investigate your position in detail. Go to the command bar, pull down the **Analyze** menu, and select **Examine**. This will result in a vertical line that will move across the screen of your graph. At any point, a small box in the upper left corner will give you the coordinates—*time and the position from the sensor*.

   a. Find and record a small time interval where your toy’s distance from the detector was **increasing**. What was the toy’s position at each endpoint?
   
   b. Compute the average rate of change (AROC) between these two coordinates. Provide the units for your AROC. Explain what your value represents in the context of your toy’s motion.
   
   c. Using your graph, estimate a small interval of time during which your toy is moving the **fastest**. Explain what you looked at to come up with your estimate.
   
   d. Compute the AROC between these two coordinates. How does your result compare to 3b?
   
   e. If time, find two intervals which have the same AROC. List them here. How did you know?
Part B. Velocity Activity

We are now going to explore the connection between position and velocity. Logger Pro and the distance detectors can help us by telling us what the velocity of our toy car is. Be careful to keep your toy right in front of the detector!


In this next section, your group will create a specific velocity vs. time graph and answer some mathematical questions about the graph.

22. Re-open “velocity.cmbl” in Logger Pro. Create the graph using the distance detector. Your group may have to conduct several trials before you attain a nice, pretty, fairly smooth graph in appearance. Copy and paste your graph in the file: “DistDet_Graphs” under “Part B #9.” In the space below, write what was the most difficult part of this.

23. Now you will plot your toy’s corresponding position vs. time graph. Right click on your graph, choose Graph Options, and then choose the Axis Options tab. Click on the box for position and a position graph will appear. Next, right click on your Table, click on the box for position, and your position values will appear. Copy and paste this graph (with both your velocity and position graphs) into the file: “DistDet_Graphs” under “Part B #10.”
24. Now investigate the POSITION graph like you did in the first part. Go to the command bar, pull down the Analyze menu, and select Examine. This will result in a vertical line that will move across the screen of your graph. At any point, a small box in the upper left corner will give you the coordinates—time and the position from the sensor.
a. Find and record a time interval where your toy’s distance from the detector was decreasing. What was the toy’s position at each endpoint?
b. Compute the average rate of change between these two coordinates. Provide the units for your average rate of change. Explain what your value represents in the context of your toy’s motion.
c. How does the value you just computed in part (b) compare to the values on your velocity graph in the same region?
d. Given your previous results, give your best explanation of the connection between the velocity vs. time graph and the position vs. time graph.

Note: Save your four graphs and upload them to Moodle. Be sure to write your names on the file.

**Part C**

25. Using the distance detector and car, create a velocity graph where the velocity changes from negative to positive and then back negative over time. When you are satisfied, describe what you did below and make a rough sketch.

26. Bring up the distance graph to match you created velocity graph. (Right click on your graph, choose Graph Options, and then choose the Axis Options tab. Select the box for position and a position graph will appear. Next, right click on your Table, select the box for position, and your position values will appear) Find pairs of points (close together) using ANALYZE and fill in the following Table.

<table>
<thead>
<tr>
<th>Location on graph</th>
<th>Points</th>
<th>AROC</th>
<th>Velocity Value at the same time (from the velocity graph) use t value of first point</th>
</tr>
</thead>
<tbody>
<tr>
<td>Velocity is negative at the beginning</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Velocity is near 0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Velocity is positive</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
27. Compare AROC and Velocity- are they related? Looking back at your answer to 10, can you now add to your connection comments?

28. Looking at the first pair of points, how do you think you could make velocity and AROC even closer in value?