ABSTRACT

MASON, SARAH BETH. Conjugacy Classes of Maximal k-split Tori Invariant Under an Involution of SL(n,k). (Under the direction of Dr. Aloysius Helminck.)

Given an involution θ and a reductive algebraic group G, we define a symmetric space as the space G/H, where H is the fixed point group of θ. One can extend the applications of symmetric spaces to symmetric k-varieties over an arbitrary field k. To study the representation theory of symmetric k-varieties, it is important to first understand their structure. The action of minimal parabolic k-subgroups on symmetric k-varieties helps in identifying the structure of these symmetric k-varieties. Maximal k-split tori invariant under an involution are of fundamental importance in the characterization of minimal parabolic k-subgroups acting on symmetric k-varieties. In this dissertation, symmetric k-varieties for the special linear group $\mathrm{SL}(n,k)$ are considered and a classification of standard tori is given. From the standard tori, one can study the H- and H_k-conjugacy classes where H_k is the set of k-rational points of H.

Conjugacy Classes of Maximal k-split Tori Invariant
Under an Involution of SL(n,k)

by
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DEDICATION

To my parents.

BIOGRAPHY

Sarah Beth Mason was born on March 17th, 1986 in Newport News, Virginia. When she was very young, her family moved to North Carolina and she has lived here ever since. In 2004, she started her undergraduate studies at Western Carolina University. She completed her B.S. at WCU in 2008 and went on to receive her M.S. in applied mathematics from WCU in 2010. She started her studies at North Carolina State University in 2011.

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Introduction

1.1 Symmetric Spaces

Given an involution θ and a reductive algebraic group G, we define a *symmetric space* as the space G/H, where H is the fixed point group of θ. A symmetric space is also known as a *symmetric variety*. Symmetric spaces were originally studied by Cartan and arose in the context of Riemannian manifolds and Lie groups. The globally Riemannian symmetric spaces of differential geometry are a special case of the algebraic definition common in Lie theory.

One can also consider a more generalized version of a symmetric space. When considering a symmetric space over a non-algebraically closed field k, we call this a *symmetric k-variety*, and it is defined by G_k/H_k, where G_k and H_k are the k-rational points of G and H, respectively. Real symmetric k-varieties are also called *real reductive symmetric spaces*. The p-adic symmetric k-varieties are also known as reductive p-adic symmetric spaces or as simply p-adic symmetric spaces.

Symmetric k-varieties over fields other than the real numbers occur in a number of areas of mathematics. These areas include representation theory (see [6], [37] and [38], geometry ([10],[11],[3]), singularity theory ([28],[24]), the study of character sheaves ([14], [29]), and the study of cohomology of arithmetic subgroups ([36]). These symmetric k-varieties are most well known in the area of representation theory.

Example 1.1.1. Let $M_n(k)$ denote the set of $n \times n$ matrices with entries in k. Then

$$\mathrm{GL}(n,k) = \{A \in M_n(k) \mid \det(A) \neq 0\}$$

is the general linear group. Let $G = \mathrm{GL}(n,k)$ and define an involution $\theta : G \to G$ by $\theta(g) = (g^T)^{-1}$. Then $H = \{g \in G \mid g^T = g^{-1}\}$, the set of $n \times n$ orthogonal matrices. The quotient G/H is then contained within the set of symmetric matrices, giving the motivation for the name symmetric space. If $k = \bar{k}$, then G/H is the set of symmetric matrices. When $k \neq \bar{k}$, then the extended symmetric space $\tilde{Q} = \{A \in \mathrm{GL}(n,k) \mid \theta(A) = A^{-1}\}$ is the set of symmetric matrices.

Representations associated with real reductive symmetric spaces have been studied by many people over the past few decades. Much of the early work was done by Harish-Chandra. Symmetric k-varieties were first introduced in the late 1980's as a way of generalizing the concept of these real reductive symmetric spaces to similar spaces over the p-adic numbers and as a way to study the representations that are associated with these spaces. A number of interesting results have arisen from this research (see [17], [27], [33]). Another case of interest is the representations associated with symmetric k-varieties defined over a finite field (see [29] and [14]).

Expanding on Harish-Chandra's ideas, the representation theory for the general real reductive symmetric spaces has been carried out by a number of mathematicians including Flensted-Jensen, Ōshima, Sekiguchi, Matsuki, Brylinski, Delorme, Schlichtkrul, and van den Ban (see [5], [9], [12],[13], [15], [31], [32]).

For $\mathbb{Q}_p$, the p-adic numbers, it is natural to study the harmonic analysis of these p-adic symmetric spaces. The main aim of harmonic analysis is to decompose unitary representations as explicitly as possible into irreducible components, which is called finding the *Plancherel decomposition*. Most of the representations occurring in this decomposition are representations induced from a parabolic k-subgroup. This means that the one essential tool in the study of symmetric k-varieties and their representations is a description of the geometry of the orbits of a minimal parabolic k-subgroup acting on the symmetric k-variety. A description of these orbits naturally leads to a description of most of the fine structure of these symmetric k-varieties including the restricted root system of the symmetric k-variety.

1.2 Orbits of Parabolic Subgroups on Symmetric Spaces

There are several ways of classifying the orbits of a minimal parabolic k-subgroup P_k on a symmetric k-variety. One way of classifying these orbits is to consider the P_k-orbits acting on the symmetric k-variety G_k/H_k by a process called θ-twisted conjugation. For $x, g \in G_k$, we define $g * x := gx\theta(g)^{-1}$. Another way of viewing these orbits is to identify G_k/P_k with the set of parabolic k-subgroups of G that are G_k-conjugate with P, i.e. $G_k/P_k = \{xPx^{-1} \mid x \in G_k\}$, and look at the H_k-orbits on this. Here H_k acts by conjugation. Lastly, we can consider these orbits as the set $P_k \backslash G_k/H_k$ of (P_k, H_k)-double cosets in G_k. This last characterization is the same as the set of $P_k \times H_k$-orbits on the set G_k.

When we have that $k = \overline{k}$, we use a Borel subgroup B in place of a minimal parabolic k-subgroup, because in the case of algebraic closure we have that $P = B$. For this case, these orbits were characterized by Springer [35]. Several characterizations of the double cosets $B \setminus G/H$ were proven by Springer, including the orbits of H, B, and $B \times H$. For details, see [35]. For algebraically closed k and P a general parabolic subgroup, these orbits were characterized by Brion and Helminck in [8] and [21]. For $k = \mathbb{R}$ and P a minimal parabolic k-subgroup, characterizations were given by Matsuki [30] and Rossmann [34]. For general fields, these orbits were characterized by Helminck and Wang [22].

Let P_k be a minimal parabolic k-subgroup of G. The double coset $P_k \setminus G_k/H_k$ has been characterized in several ways by Helminck and Wang [22], including the orbits of H_k, P_k, and $P_k \times H_k$.

We will first consider the H_k-orbits on G_k/P_k. If we let A be a torus of G_k, then we can denote by $N_{G_k}(A)$ the normalizer of A in G_k, $Z_{G_k}(A)$ denotes the centralizer of A in G_k, $W_{G_k}(A) = N_{G_k}(A)/Z_{G_k}(A)$ the Weyl group of A in G_k, and $W_{H_k}(A) = N_{H_k}(A)/Z_{H_k}(A) = \{w \in W_{G_k}(A) \mid w \text{ has a representative in } N_{H_k}(A)\}$, the Weyl group of A in H_k. All of the above sets are θ-stable if A is θ-stable.

Let A' be a θ-stable maximal k-split torus of G_k which is contained in the minimal parabolic k-subgroup P'. Then we will denote by C_k the set of pairs of (P', A'). $\mathcal{P}_k$ will be the space of all minimal parabolic k-subgroups of G. The fixed point group H_k acts on C_k and $\mathcal{P}_k$ by conjugation. We will denote the set of H_k-orbits in C_k (respectively $\mathcal{P}_k$)

by $H_k \setminus C_k$ (respectively $H_k \setminus \mathcal{P}_k$).

The H_k-orbits in C_k can be broken up into two parts. We can first consider the H_k-conjugacy classes of θ-stable maximal k-split tori in G_k. For each θ-stable maximal k-split torus that is a representative of these conjugacy classes, we can determine the minimal parabolic k-subgroups containing this torus that are not H_k-conjugate. Thus, if we define the set $\{A_i \mid i \in I\}$ as the set of representatives of the H_k-conjugacy classes of θ-stable maximal k-split tori in G_k, then the H_k-orbits in C_k can be identified with the union of Weyl group quotients $\bigcup_{i\in I} W_{G_k}(A_i)/W_{H_k}(A_i)$.

For the P_k-orbits, we identify G_k/H_k with $Q_k = \{g\theta(g)^{-1} \mid g \in G_k\}$. P_k acts on Q_k by the θ-twisted action previously described. We will denote the set of θ-twisted P_k-orbits on Q_k by $P_k \setminus Q_k$.

In the characterization of the $P_k \times H_k$-orbits in G_k, let A be a θ-stable maximal k-split torus of P_k and $\mathcal{V}_k = \{x \in G_k \mid \tau(x) \in N_{G_k}(A)\}$, where $\tau(x)$ is defined as follows.

$$\tau(x) = x\theta(x)^{-1}$$

The group $Z_{G_k}(A) \times H_k$ acts on $\mathcal{V}_k$ by $(x, z) \cdot y = xyz^{-1}$, $(x, z) \in Z_{G_k}(A) \times H_k, y \in \mathcal{V}_k$. Let V_k be the set of $(Z_{G_k}(A) \times H_k)$-orbits on $\mathcal{V}_k$.

Theorem 1.2.1. *[22] Let P be a minimal parabolic k-subgroup of G and let $\{A_i \mid i \in I\}$ be representatives of the H_k-conjugacy classes of θ-stable maximal k-split tori in G_k. Then*

$$P_k\backslash G_k/H_k \simeq H_k \setminus \mathcal{P}_k \simeq \bigcup_{A_i\in I} W_{G_k}(A_i)/W_{H_k}(A_i) \simeq H_k \setminus C_k \simeq P_k \setminus Q_k \simeq V_k.$$

From the previous theorem, we can see that there are several ways to classify the double cosets $P_k\backslash G_k/H_k$. Throughout this thesis, we will be interested in the classification

$$P_k\backslash G_k/H_k \simeq \bigcup_{A_i\in I} W_{G_k}(A_i)/W_{H_k}(A_i).$$

This classificiation will help us in studying the orbits of minimal parabolic k-subgroups acting on the symmetric k-variety. From this classification, we see that in order to classify the double coset $P_k\backslash G_k/H_k$, we will first need to classify the set of H_k-conjugacy classes of θ-stable, maximal k-split tori. This will give us the set $\{A_i \mid i \in I\}$. Once we have done this, we will need to determine the coset $W_{G_k}(A_i)/W_{H_k}(A_i)$ for each θ-stable, maximal

k-split tori representative A_i.

In an algebraically closed field, i.e. $k = \bar{k}$, we have the double cosets $B \setminus G/H$. When considering G over a non-algebraically closed field, we look at minimal parabolic k-subgroups instead of Borels and we get $P_k \setminus G_k/H_k$. One method of classifying these double cosets is to consider $P \setminus G/H$ and determining how the algebraically closed orbits break up over the k-rational points. An alternate method is to instead reverse this process via an embedding map $P_k \backslash G_k/H_k \hookrightarrow P \backslash G/H$ which we call generalized complexification. The surjectivity of the generalized complexification map is then equivalent to all orbits over the algebraic closure contributing to the k-orbits. Given a group N, k-rank(N) denotes the dimension of a maximal k-split torus of N. When the group G is k-split we have a characterization of the surjectivity:

Theorem 1.2.2. *[26] Let G be a k-split group, H the set of fixed points of an involution θ and P a minimal parabolic k-subgroup. Then the generalized complexification map*

$$\varphi : P_k \setminus G_k/H_k \to P \setminus G/H, \ \varphi(P_k x H_k) = PxH$$

is surjective if and only if k-rank(H) = k-rank(G).

This embedding process will be the motivation for considering the H-conjugacy classes of θ-stable, maximal k-split tori as well as the H_k-conjugacy classes of θ-stable, maximal k-split tori.

1.3 H_k-conjugacy classes of θ-stable maximal k-split tori

The classification that we are interested in centers around tori, so we will use the following definitions and results. If we have a torus T of G that is defined over the field k, then there are subtori T_a and T_s of T, where T_a is the largest anisotropic subtorus of T and T_s is the largest k-split subtorus of T defined over k. We have that $T = T_a \cdot T_s$ and $T_a \cap T_s$ is finite.

Given an involution θ of G, we have similar definitions and results for θ-stable tori. Let T be a maximal k-split torus. We call T θ-stable if $\theta(T) = (T)$. If we let $T^+ = \{t \in T \mid \theta(t) = t\}^\circ$ and $T^- = \{t \in T \mid \theta(t) = t^{-1}\}^\circ$, then we have that $T = T^+T^-$ and

$T^+ \cap T^-$ is finite. We call the torus T^- θ-split. A torus T is called (θ, k)-split if it is both k-split and θ-split.

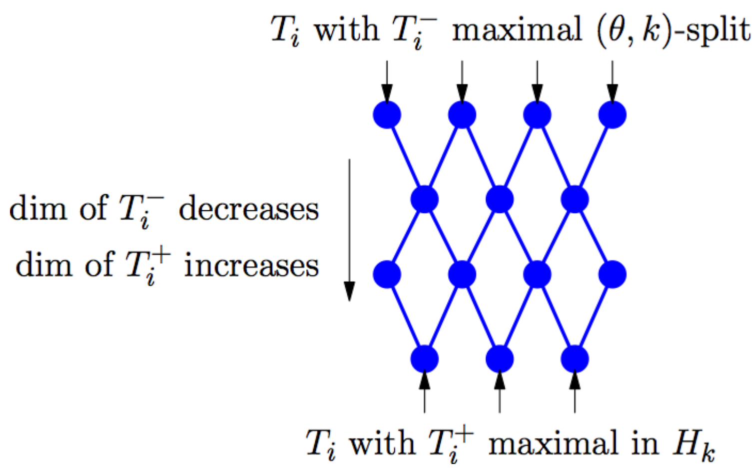

Figure 1.1: The poset of H_k-conjugacy classes of θ-stable maximal k-split tori

As we are classifying the H_k-conjugacy classes of θ-stable, maximal k-split tori, we will obtain a poset of tori. Each node in the poset will represent one of the H_k-conjugacy classes of θ-stable, maximal k-split tori. As we have stated previously, each of these representatives T_i can be written as the product of two subtori, $T_i = T_i^+ T_i^-$. As we look across each level of the poset, the dimension of T_i^+ for each torus T_i on that level will be equivalent. The same is true for the dimension of T_i^- across each level of the poset. We construct the poset in such a way that the top level has tori with a maximal θ-split part, i.e. T^- is maximal. Also, we will have the tori with a maximal T^+ part at the bottom level of the poset. As we move down the poset, we decrease the dimension of the θ-split portion of the torus by one and increase the dimension of the T^+ part by one. A representation of this poset can be seen in Figure 1.1.

Once we have constructed the poset of H_k-conjugacy classes of θ-stable, maximal k-split tori, we can get a second poset by expanding each node. Our motivation for creating the first poset is to classify the Weyl group quotients $W_{G_k}(T_i)/W_{H_k}(T_i)$. Since each node represents one of our T_i's, we can expand the poset to give us this quotient. The number of nodes in this second lattice will give us $\mid P_k \backslash G_k / H_k \mid$. Additionally, the lines connecting the nodes in the second lattice correspond to the Bruhat ordering.

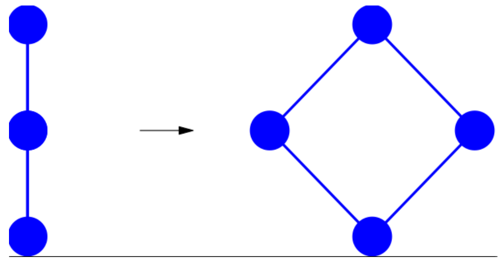

Figure 1.2: Posets of tori related to $G = \mathrm{SL}(4, \mathbb{C})$ with $\theta(A) = (A^T)^{-1}$

Example 1.3.1. Let $G = \mathrm{SL}(4, \mathbb{C})$ with $\theta(A) = (A^T)^{-1}$. Then the first poset has three levels with one H_k-conjugacy class of θ-stable k-split tori in each level. In the top and bottom levels, we have that $\mid W_{G_k}(T_i)/W_{H_k}(T_i) \mid = 2$. Then $\mid P_k \setminus G_k/H_k \mid = 4$. This is shown graphically in Figure 1.2.

In order to determine the poset of H_k-conjugacy classes of θ-stable, maximal k-split tori and the expansion at each node to $W_{G_k}(T_i)/W_{H_k}(T_i)$, we will need to consider each algebraic group G and its involutions individually. Throughout this thesis, we will be considering the group $G = \mathrm{SL}(n, k)$ and the involutions associated with it. We will break up our problem into various subproblems in order to help classify $P_k \setminus G_k/H_k$.

1. Classify the H_k-conjugacy classes of the maximal (θ, k)-split tori.

2. Classify the H_k-conjugacy classes of maximal k-split tori containing a maximal (θ, k)-split torus. These tori are θ-stable and the number of conjugacy classes will tell us the number of nodes in the top level of the first poset, whose nodes correspond to the H_k-conjugacy classes of θ-stable, maximal k-split tori.

3. Determine the maximal k-split tori in H_k and classify the H_k-conjugacy classes of θ-stable maximal k-split tori which contain a maximal k-split torus in H_k. This will give us the bottom level of the first poset, whose nodes are the H_k-conjugacy classes of θ-stable maximal k-split tori.

4. Classify the H_k-conjugacy classes of θ-stable maximal k-split tori. This is where we will determine all of the middle levels of the first poset, whose nodes are the H_k-conjugacy classes of θ-stable maximal k-split tori.

In order to determine the middle levels of our poset, we will first consider the poset of standard tori and see how this poset collapses down under H-conjugacy and then how it expands when we consider H_k-conjugacy.

Once the poset of H_k-conjugacy classes of θ-stable, maximal k-split tori has been constructed, we are able to started classifying the Weyl group quotients $W_{G_k}(T_i)/W_{H_k}(T_i)$ in order to characterize the double cosets $P_k \setminus G_k/H_k$.

1.4 Summary of Results for $\mathrm{SL}(2, k)$

When looking to classify the H_k-conjugacy classes of θ-stable, maximal k-split tori for $G = \mathrm{SL}(n, k)$, it is natural to start with the case $n = 2$. For $G = \mathrm{SL}(2, k)$, we have that tori of G are one-dimensional; thus, $T_i = T_i^+$ or $T_i = T_i^-$. There are at most two levels to the poset for $\mathrm{SL}(2, k)$.

Remark 1.4.1. $\mathrm{SL}(2, k)$ is a k-split group. This means that $\mathrm{SL}(2, k)$ contains a maximal torus that is also maximal k-split, namely $T = \{\text{diagonal matrices}\} = \{\text{diag}\} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$. In this case, minimal parabolic k-subgroups are also Borel subgroups. Thus, if we let B be a Borel subgroup, $T \subseteq B$, the classification of $B_k \setminus G_k/H_k$ is identical to the classification of $P_k \setminus G_k/H_k$.

The classification of involutions of $\mathrm{SL}(2, k)$ is given by Helminck and Wu. They show all involutions come from bilinear forms. In particular, we have the following theorem for involutions of $\mathrm{SL}(2, k)$. Note that $\theta(X) = \mathrm{Inn}_A(X) = A^{-1}XA$ for any $X \in G$.

Theorem 4.1.1 *[23] The number of isomorphy classes of involutions over $G = \mathrm{SL}(2, k)$ equals the order of $k^*/(k^*)^2$. Furthermore, the involutions over G are of the form $\theta = \mathrm{Inn}\left(\begin{smallmatrix} 0 & 1 \\ m & 0 \end{smallmatrix}\right)$ for all $m \in k^*/(k^*)^2$. Additionally, if m and q are in the same square class, then $\mathrm{Inn}\left(\begin{smallmatrix} 0 & 1 \\ m & 0 \end{smallmatrix}\right)$ is isomorphic to the involution $\mathrm{Inn}\left(\begin{smallmatrix} 0 & 1 \\ q & 0 \end{smallmatrix}\right)$.*

Since all involutions of $\mathrm{SL}(2, k)$ come from bilinear forms, the orbits of minimal parabolic k-subgroups acting on the symmetric k-variety related to the involution can be classified using quadratic forms.

A summary of the main results for $\mathrm{SL}(2, k)$ is found in the following theorem. Note that we use $\bar{m}$ to denote the entire square class of m and we use $m \in k^*/(k^*)^2$ to denote that m is the representative of the square class $\bar{m}$ of k. Also, T is the set of diagonal matrices in $\mathrm{SL}(2, k)$.

Theorem 1.4.2. *[1] Let $G_k = \mathrm{SL}(2,k)$ and let $\theta = \mathrm{Inn}\left(\begin{smallmatrix} 0 & 1 \\ m & 0 \end{smallmatrix}\right)$.*

1. *H_k is k-anisotropic if and only if $m \notin \bar{1}$. If $m \in \bar{1}$, then H_k is a maximal k-split torus.*

2. *Let $U = \{q \in k^*/(k^*)^2 \mid x_1^2 - m^{-1}x_2^2 = q^{-1}$ has a solution in k $\}$. Then the number of H_k-conjugacy classes of (θ,k)-split maximal tori is $\mid U/\{1,-m\} \mid$.*

3. *For $y \in U$, let $r, s \in k$ such that $r^2 - m^{-1} = y^{-1}$ and let $g = \left(\begin{smallmatrix} r & sym^{-1} \\ s & ry \end{smallmatrix}\right)$. Then $\{T_y = g^{-1}Tg \mid y \in U\}$ is a set of representatives of the H_k-conjugacy classes of maximal (θ,k)-split tori in G_k.*

4. *Let T_i be a (θ,k)-split maximal torus. Then*

   (a) $\mid W_{H_k}(T_i) \mid = 2$ when $m \in \bar{1}$ and $-1 \in (k^*)^2$.

   (b) $\mid W_{H_k}(T_i) \mid = 2$ when $m \in \overline{-1}$ and $-1 \notin (k^*)^2$.

   (c) $\mid W_{H_k}(T_i) \mid = 1$ otherwise.

Parts (1) and (2) give the means of counting the H_k-conjugacy classes of θ-stable maximal k-split tori. Additionally, part (3) provides the necessary information to find the tori representatives of each H_k-conjugacy class, allowing the calculation of $W_{H_k}(T_i)$ in part (4). In the cases that H_k is a maximal k-split torus, we clearly have $W_{H_k} = \mathrm{id}$, thus $\mid W_{G_k}(H_k)/W_{H_k}(H_k) \mid = 2$.

The above results give us a detailed description of the double cosets $B_k \setminus G_k/H_k$ for $G = \mathrm{SL}(2,k)$. Our goal is to do this in general for $\mathrm{SL}(n,k)$. The following chapters will help us in this classification.

1.4.3 Summary of Results for $\mathrm{SL}(n,k)$

When classifying the H_k-conjugacy classes of θ-stable maximal k-split tori of $\mathrm{SL}(n,k)$ for various involutions, we will consider $\mathrm{SL}(2,k)$ blocks inside of the larger $n \times n$ matrices. This will reduce the problem to something we have already considered and make our calculations simpler.

The involutions of G are separated into two groups-the inner involutions and the outer involutions. We will start with outer involutions, specifically $\theta(X) = (X^T)^{-1}$. All outer involutions can be written as $\theta(X) = \mathrm{Inn}_M(X^T)^{-1} = M^{-1}(X^T)^{-1}M$ for all X in G, where M comes from a bilinear form. Depending on the field k and the size of the matrices we are considering, this could give us a large number of involutions to characterize. For example, if $k = \mathbb{Q}_p$ with $p \cong 1 \mod 4$ and n is even, then there are a total of $\frac{n}{2} + 9$ isomorphism classes of involutions.

Remark 1.4.4. Like $\mathrm{SL}(2,k)$, $\mathrm{SL}(n,k)$ is a k-split group, meaning that $\mathrm{SL}(n,k)$ contains a maximal torus that is also maximal k-split, namely the set of diagonal matrices T. In this case, minimal parabolic k-subgroups are Borel subgroups. Therefore, if we let B be a Borel subgroup containing T, the classification of $B_k \setminus G_k/H_k$ is the exact same as $P_k \setminus G_k/H_k$.

We will build posets of standard tori in order to classify the conjugacy classes of θ-stable, maximal k-split tori. We will first need to consider the H_k-conjugacy classes of maximal (θ,k)-split tori. This will tell us the number of tori on the top level of our poset. The results for this are found in Chapter 5. A summary of the results is below.

Theorem 5.2.1 *Let $\theta = \mathrm{Inn}^*_{I_{n-i,i}}$ such that $i < n/2$. Then there exists only one H_k-conjugacy class of maximal (θ,k)-split tori with representative*

$$A_1 = \{\mathrm{diag}(a_1, \dots, a_i, a_i^{-1}, \dots, a_1^{-1}, 1, \dots, 1) \mid a_i \in k^*\}.$$

Theorem 5.2.2 *Let $\theta = \mathrm{Inn}^*_{I_{n-i,i}}$ such that $i = n/2$. Note that n must be even. Then the number of H_k-conjugacy classes of maximal (θ,k)-split tori is at most $\mid \frac{k^*/(k^*)^2}{\pm 1} \mid$.*

Corollary 5.2.3 *Let $\theta = \mathrm{Inn}^*_{I_{n-i,i}}$ such that $i = n/2$. Note that n must be even. Additionally, let k be $\mathbb{C}, \mathbb{R}, \mathbb{F}_p$ with $p \neq 2$, or $\mathbb{Q}_p$. Then the number of H_k-conjugacy classes of maximal (θ,k)-split tori equals $\mid \frac{k^*/(k^*)^2}{\pm 1} \mid$.*

Theorem 5.2.4 *Let $\theta = \mathrm{Inn}_{n,x}$. Not that n must be even and $x \in k^*/(k^*)^2, x \neq 1 mod (k^*)^2$. Then the number of H_k-conjugacy classes of maximal (θ,k)-split tori is at most $\mid \frac{k^*/(k^*)^2}{\pm 1} \mid$.*

Theorem 5.2.5 *Let $\theta = \mathrm{Inn}_{J_{2m}}(A^T)^{-1}$ for all $A \in G$. Then there exists one H_k-conjugacy class of maximal (θ,k)-split tori with representative*

$$A_2 = \{\mathrm{diag}(a_1, a_2, \cdots, a_n) \mid a_i \in k \textit{ and } a_1 = a_{n/2+1}, a_2 = a_{n/2+2}, \dots, a_{n/2} = a_n\}.$$

Chapter 6 gives the results of classifying the (θ, k)-singular roots for each involution θ of $\mathrm{SL}(n, k)$. The set of (θ, k)-singular roots will help in constructing the poset of standard tori that are θ-stable and maximal k-split. A summary of results for this follows.

Theorem 6.1.4 *Let $G = \mathrm{SL}(n, k)$ and $\theta = (A^T)^{-1}$. Then Φ_θ^+, the set of positive (θ, k)-singular roots, is the set of all positive roots if $-1 \in (k^*)^2$ and Φ_θ^+ is the empty set if $-1 \notin (k^*)^2$.*

Theorem 6.1.6 *Let $G = \mathrm{SL}(n, k)$ and $\theta(A) = \mathrm{Inn}_{J_{2m}}(A^T)^{-1}$. Then $\Phi_\theta^+ = \emptyset$.*

Theorem 6.1.9 *Let $G = \mathrm{SL}(n, \mathbb{R})$ and $\theta = \mathrm{Inn}_{M_i}(A^T)^{-1}$. Then $\Phi_\theta^+ = \{\alpha_{n-i}$, roots containing α_{n-i} in their sum $\}$.*

Theorem 6.1.13 *Let $G = \mathrm{SL}(n, \mathbb{F}_p)$ and $\theta = \mathrm{Inn}_{M_i}(A^T)^{-1}$. Then $\Phi_\theta^+ = \{\alpha_{n-i}$, roots containing α_{n-i} in their sum $\}$.*

Theorem 6.2.4 *Let $G = \mathrm{SL}(n, k)$ and $\tilde{\theta} = \mathrm{Inn} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & \iddots & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$. Then $\Phi_\theta^+ = \Phi_{\tilde{A}_{n-i,i}}$.*

Theorem 6.2.6 *Let $G = \mathrm{SL}(n, k)$ and $\theta = \mathrm{Inn}(L_{n,-1})$. Then $\Phi_\theta^+ = \Phi(\tilde{A}_{n-i,i})$.*

From there, we will check for H_k-conjugacy and H-conjugacy among the rows of the poset to help in obtaining representatives of the conjugacy classes. This will get us one step closer to classifying the Weyl group quotients $W_{G_k}(A_i)/W_{H_k}(A_i)$ and thus the double cosets $P_k \setminus G_k/H_k$. We will use the following result to help in checking for conjugacy.

Theorem 7.2.3 *For $\theta = \mathrm{Inn} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we have*

$$\begin{pmatrix} a+b & 0 \\ 0 & a-b \end{pmatrix} = \mathrm{Inn} \begin{pmatrix} x & -y \\ x & y \end{pmatrix} \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

where $a, b, x \in k$ and $a^2 + b^2 = 1, 2x^2 = 1$.

Chapter 2

Preliminaries

2.1 Preliminaries

Throughout this thesis, we will let k be a field with $\text{char}(k) \neq 0$. G will be a reductive algebraic group defined over k. In particular, we will be interested in the special linear group $\text{SL}(n, k)$.

Definition 2.1.1. Let $M_n(k)$ denote the set of $n \times n$ matrices with entries in k. Then

$$\text{SL}(n, k) = \{A \in M_n(k) \mid \det(A) = 1\}$$

is the *special linear group.*

Let θ be an involution, i.e. $\theta \in \text{Aut}(G)$ and $\theta^2 = \text{Id}$. When $k = \bar{k}$, we denote by T_i the tori of G. When $k \neq \bar{k}$, we denote by A_i the tori of G_k.

Definition 2.1.2. $T \subset G$ is a *torus* if it is connected, abelian, and consists of semisimple elements.

The set $\Phi(T)$ (respectively $\Phi(A)$) is the set of roots of T in G (respectively A in G). When $T = \{\text{diag}\}$, then $\Phi(T)$ is the set of all positive roots.

Definition 2.1.3. T is *θ-stable* if $\theta(T) = T$. Then we have $T = T^+T^-$, where

1. $T^+ = \{t \in T \mid \theta(t) = t\}^\circ$

2. $T^- = \{t \in T \mid \theta(t) = t^{-1}\}^\circ$. T^- is called θ-split.

Moreover, $T^+ \cap T^-$ is finite.

For the case when $n = 2$, we have that all maximal tori in G are one-dimensional. Thus, for a θ-stable maximal torus T in $G = \mathrm{SL}(2, k)$, $T = T^+$ or $T = T^-$.

We call a torus k-split if the torus is able to be diagonalized over the field k. For $k = \bar{k}$, all tori are k-split.

Definition 2.1.4. T a torus is called (θ, k)-*split* if T is k-split and θ-split.

We are interested in two types of splitting of a torus T. The first type of splitting involves the involution θ and will split the torus T into $T_\theta^+ = \{t \in T \mid \theta(t) = t\}^\circ$ and $T_\theta^- = \{t \in T \mid \theta(t) = t^{-1}\}^\circ$. The second type of splitting involves the field k and will divide the torus T into T_s and T_a, the k-split part of T and the k-anisotropic part of T, respectively. Note that if $k = \bar{k}$, then $T = T_s$. Given a maximal (θ, k)-split torus A, we can find a maximal torus T containing A in which T_θ^- is maximal θ-split and T_s is maximal k-split.

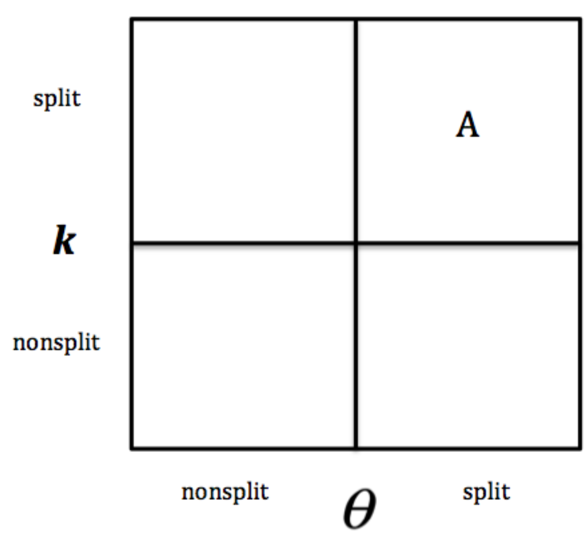

Figure 2.1: Illustration of k-split and θ-split parts of tori

Figure 2.1 illustrates the k-split and θ-split parts of tori. The top two boxes represent

T_s. The two boxes on the right represent T_θ^-. A torus A that is (θ, k)-split is represented by the upper right box.

It is interesting to look at how H_k acts on all of these different tori. There are several questions to consider.

1. H_k-conjugacy classes of maximal (θ, k)-split tori.

2. H_k-conjugacy classes of k-split tori.

3. H_k-conjugacy classes of θ-split tori.

Note that if $k = \bar{k}$, then the H-conjugacy classes and H_k-conjugacy classes are equivalent and all tori are k-split. Our interest lies in the first two questions.

2.2 Generalized Symmetric Spaces

We will now cover some basics of symmetric spaces. We will let $V = \mathbb{R}^n$ be a Euclidean vector space. $\mathrm{GL}(V)$ is the space of invertible matrices and $\mathrm{GL}(V) \simeq \mathrm{GL}(n, \mathbb{R})$. We define a symmetric bilinear form $B(x, y) = x^T M y$ where M is an $n \times n$ matrix and $M = M^T$. If $M = \mathrm{Id}$, then $B(x, y) = x^T y$ is called the dot product. By one of the conditions of the bilinear form, we have that $B(x, x) > 0$ if $x \neq 0$. We denote by A' the adjoint of $A \in \mathrm{GL}(V)$ with respect to B. Then

$$B(A(x), y) = B(x, A'(y)) \; \forall \; x, y \in V$$

Remark 2.2.1. If $M = \mathrm{Id}$, then $A' = A^T$.

This can be seen as follows.

$$B(A(x), y) = B(x, A'(y))$$

$$(A(x))^T M y = x^T A^T M y = x^T M A' y$$

$$\Rightarrow A^T M = M A'$$

Thus, if $M = \mathrm{Id}$, then $A^T = A'$.

Let $\theta(A) = (A')^{-1}$, $\theta : \mathrm{GL}(V) \to \mathrm{GL}(V)$ and $\theta^2 = \mathrm{Id}$. Then θ is an involution. Let $H = \{A \in \mathrm{GL}(V) \mid B(A(x), A(y)) = B(x, y)\}$. Then $H = \mathrm{O}(V, B) = \{A \in \mathrm{GL}(V) \mid \theta(A) = A\}$, the set of orthogonal transformations.

Definition 2.2.2. Let the set $Q = \{A\theta(A)^{-1} \mid A \in \mathrm{GL}(V)\} = \{AA' \mid A \in \mathrm{GL}(V)\}$. Then Q is a *symmetric space*.

If $M = \mathrm{Id}$, then $Q = \{AA^T\} \subsetneq \{\text{symmetric matrices}\}$.

Definition 2.2.3. Let $\tilde{Q} = \{A \in \mathrm{GL}(V) \mid \theta(A) = A^{-1}\}$. Then $\tilde{Q}$ is called the extended symmetric space.

Lemma 2.2.4. *Let Q be a symmetric space and $\tilde{Q}$ the extended symmetric space, as defined above. Then $Q \subset \tilde{Q}$.*

Proof. Let $AA' \in Q$. Then

$$\theta(AA') = (A')^{-1}((A')')^{-1} = (A')^{-1}(A^{-1}) = (AA')^{-1}$$

Thus $AA' \in \tilde{Q}$ and $Q \subset \tilde{Q}$. □

A torus A is defined to be a subgroup consisting of commuting semisimple elements. As stated previously, we call a torus $A \subset G$ θ-split if $\theta(a) = a^{-1}$ for all $a \in A$. In fact, all maximal θ-split tori are conjugate under the fixed point group $H = G^\theta$.

Definition 2.2.5. Let G be a real group. Then $\theta \in \mathrm{Aut}(G)$ is a *Cartan involution* if

1. $\theta^2 = \mathrm{id}$
2. $H = G^\theta = \{x \in G \mid \theta(x) = x\}$ is a (maximal) compact subgroup.

Remark 2.2.6. In this case, compact implies maximal compact, so compact is sufficient.

Any reductive real Lie group has a unique (up to conjugation) Cartan involution. For our cases, in $\mathrm{SL}(n, k)$, the Cartan involution is $\theta = (A^T)^{-1}$.

In the study of symmetric spaces, the two most important questions are

1. What is the action of the fixed point group H_k on G_k/H_k? What kind of orbits do we get?
2. What is the action of the parabolic k-subgroup on G_k/H_k?

We are interested in the second question. For this, we first need to define Borel subgroups and parabolic subgroups.

Definition 2.2.7. A subgroup $B \subset G$ is called a Borel subgroup if

1. B is a maximal solvable subgroup.

2. B is connected.

Example 2.2.8. Let $G = \mathrm{GL}(n, k)$. Then B is the set of upper triangular matrices.

For $H \subset G$ with H solvable, we have that H can be triangularized simultaneously. In other words, there exists $x \in G$ such that $xHx^{-1} \subset \{\text{upper triangular matrices}\}$.

Definition 2.2.9. A closed subgroup P of G is a *parabolic subgroup* if the quotient variety G/P is complete.

A closed subgroup of G is parabolic if and only if it contains a Borel subgroup B. Moreover, parabolic subgroups are connected and self-normalizing: $N_G(P) = P$.

Example 2.2.10. Let $G = \mathrm{GL}(2, \mathbb{R})$. Then we have $G \supset H = \mathrm{O}(2, \mathbb{R})$ is not a connected subgroup and thus not parabolic. If we take the connected component of H, $H^\circ = \mathrm{SO}(2, \mathbb{R})$, then we have a parabolic subgroup of G.

Example 2.2.11. We have stated that for $G = \mathrm{GL}(n, k)$, B is the group of upper triangular matrices. P the group of matrices with blocks on the diagonal and zeros beneath the blocks. The following matrix is an example of a matrix in the parabolic subgroup but not in the Borel subgroup.

$$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{pmatrix}$$

For the case where $k = \bar{k}$, we instead look at the action of the Borel subgroups on G/H, since $B = P$ in the algebraically closed case.

We have that $A \subset B \subset P \subset G$, where G is an algebraic group defined over the field k, P is a minimal parabolic k-subgroup of G, B is the Borel contained in P and A is a maximal k-split torus.

We will define the centralizer of A in G in the following way:

$$Z_G(A) = \{g \in G \mid ag = ga \,\forall\, a \in A\}.$$

We will define the normalizer of A in G in the following way:

$$N_G(A) = \{g \in G \mid Ag = gA\}.$$

We have the following facts about Borels and parabolics:

1. $N_G(P) = P$ and $N_G(B) = B$.
2. For $B \subset G$ a Borel, then there exists a maximal torus $T \subset B$.
3. B determines the positive roots.
4. Let $\Phi(T)$ denote the root system of T and let $\Delta \subset \Phi(T)$ be a basis determined by B. Then there exists a one-to-one correspondence between subsets of Δ and parabolic subgroups $P \supset B$.
5. For $B \subset G$, B a Borel subgroup, then we have the Bruhat decomposition $G = \bigcup_{w \in W(T)} BwB$. Also, $B \setminus G/B \simeq W(T)$.

From the second fact above, it follows that there is a one-to-one correspondence between Borel subgroups containing T and $W(T) = N_G(T)/Z_G(T) = N_G(T)/T$.

If θ is the Cartan involution and $P \subset G$ is a minimal parabolic $\mathbb{R}$-subgroup, then we get

$$\mid P_{\mathbb{R}} \setminus G_{\mathbb{R}}/H_{\mathbb{R}} \mid = 1.$$

Let $\tau : G \to G, g \mapsto g\theta(g)^{-1}$. Then $\mathrm{Im}(\tau) = Q \simeq G/H$. We have $\mathcal{V} = \{g \in G \mid \tau(g) \in N_G(A)\}$. $Z_G(A) \times H$ acts on $\mathcal{V}$ in the following way: $(a, h) * g = agh$. Note that $agh \in \mathcal{V}$:

$$\begin{aligned} agh\theta(agh)^{-1} &= agh\theta(h)^{-1}\theta(g)^{-1}\theta(a)^{-1} \\ &= aghh^{-1}\theta(g)^{-1}\theta(a)^{-1} \\ &= ag\theta(a)^{-1}\theta(t)^{-1} \end{aligned}$$

Since $g\theta(g)^{-1} \in N_G(A)$, by definition of $\mathcal{V}$, we have that $\tau(agh) \in N_G(A)$ and thus $agh \in \mathcal{V}$.

We will denote by V the set of $Z_G(A) \times H$ -orbits in $\mathcal{V}$.

If the torus A is θ-stable, then $Z_G(A)$ and $N_G(A)$ are also θ-stable.

Definition 2.2.12. Let $w \in W(A)$. We call w a *twisted involution* if $\theta(w) = w^{-1}$. We denote by $\mathfrak{I}$ the set of twisted involutions.

2.3 Algebraically Closed vs. Non-algebraically Closed

There are some important differences between the cases of $k = \bar{k}$ and $k \neq \bar{k}$.

When $k = \bar{k}$, all maximal tori are conjugate. When $k \neq \bar{k}$, we get two types of tori: those that are diagonalizable over k and those that are not diagonalizable over k.

Example 2.3.1. Let $G = \mathrm{SL}(2, \mathbb{C})$ and $G_{\mathbb{R}} = \mathrm{SL}(2, \mathbb{R})$. We have T_1 and T_2 defined in the following way:

$$T_1 = \{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in k^*\}$$

$$T_2 = \{\begin{pmatrix} x & y \\ -y & x \end{pmatrix} \mid x^2 + y^2 = 1\}$$

T_1 is $\mathbb{R}$-split and T_2 is $\mathbb{R}$-anisotropic. When we calculate the eigenvalues for T_2, we get $x \pm iy$, which is not in $\mathbb{R}$.

If we consider a different field, it is possible that T_2 will not be k-anisotropic. For example, when $k = \mathbb{F}_5$, we have that -1 is a square because $-1 = 4 = 2^2$. Thus, we do not get imaginary eigenvalues and we are able to diagonalize T_2, making T_2 a k-split torus. When considering $\mathbb{F}_p$, we have that -1 is a square when $p \equiv 1 \bmod 4$ and -1 is not a square when $p \equiv 3 \bmod 4$.

When $k = \bar{k}$, we have a reductive group G, involution θ, fixed point group H, and a Borel subgroup B that contains a torus T. We will let $\mathfrak{T}^\theta$ be the set of θ-stable maximal tori, $\mathcal{V} = \{g \mid g\theta(g)^{-1} \in N_G(T)\}$, and $V = \{T \times H \text{ orbits on } \mathcal{V}\} \simeq B \setminus G/H$. We have the map $\varphi : V \to \mathfrak{T}^\theta/H$. We are interested in characterizing $\mathfrak{T}^\theta/H$ in order to classify the double cosets $B \setminus G/H$.

Lemma 2.3.2. *[18] Assume $T_1, T_2 \in \mathfrak{T}^\theta$ with T_i^+ maximal, i.e. T_i^+ is a maximal torus of H. Then there exists an $h \in H$ such that $hT_1h^{-1} = T_2$.*

Lemma 2.3.3. *[18] Assume $T_1, T_2 \in \mathfrak{T}^\theta$ with T_i^- maximal, i.e. T_i^- is a maximal θ-split torus. Then there exists an $h \in H$ such that $hT_1h^{-1} = T_2$.*

When $k \neq \bar{k}$, we have a reductive group G_k, involution θ, fixed point group H_k, and a minimal parabolic subgroup P_k that contains a torus A. We will let $\mathcal{A}^\theta$ be the set of θ-stable maximal k-split tori, $\mathcal{V} = \{g \mid g\theta(g)^{-1} \in N_G(A)\}$, and $V = \{Z_G(A) \times H \text{ orbits on } \mathcal{V}\} \simeq P_k \setminus G_k/H_k$. We have the map $\varphi : V \to \mathcal{A}^\theta/H_k$. We are interested in characterizing $\mathcal{A}^\theta/H_k$ in order to classify the double cosets $P_k \setminus G_k/H_k$.

We have the following modified lemmas for the non-algebraically closed case.

Lemma 2.3.4. *[18] Assume $A_1, A_2 \in \mathcal{A}^\theta$ with A_i^+ maximal, i.e. A_i^+ is a maximal torus of H. Then there exists an $h \in H$ such that $hT_1h^{-1} = T_2$. Thus A_1 and A_2 are H-conjugate but not necessarily H_k-conjugate.*

Lemma 2.3.5. *[18] Assume $A_1, A_2 \in \mathcal{A}^\theta$ with A_i^- maximal, i.e. A_i^- is a maximal (θ, k)-split torus. Then there exists an $x \in (H.Z_G(A_1))_k$ such that $xT_1x^{-1} = T_2$.*

From Lemma 2.3.5, we have that there are usually infinitely many (θ, k)-split tori but in some cases there is only one H_k-conjugacy class of (θ, k)-split tori. If there is only one H_k-conjugacy class of (θ, k)-split tori, then the lemma for $k = \bar{k}$ holds for $k \neq \bar{k}$.

For algebraically closed G, $B \subset G$ a Borel subgroup, $\theta \in \mathrm{Aut}(G)$, $\theta^2 = \mathrm{id}$, and $H = G^\theta$ the fixed point group, we can classify $B \setminus G/H$ in the following ways:

1. H-orbits on $G/B \simeq \mathcal{B} = \{gBg^{-1} \mid g \in G\}$. Then $B \setminus G/H \simeq \bigcup_{i \in I} W_G(T_i)/W_H(T_i)$ where $\{T_i \mid i \in I\}$ are the H-conjugacy classes of θ-stable maximal tori.

2. $B \times H$-orbits on G.

3. B-orbits on $G/H \simeq Q = \{x\theta(x)^{-1} \mid x \in G\}$

If we consider G defined over k with $k \neq \bar{k}$, we have G_k the k-rational points on G, P_k the minimal parabolic subgroup defined over k, $P_k \subset G_k$, and A a θ-stable maximal k-split torus with $A \subset P_k$. Then we can classify the double cosets $P_k \setminus G_k/H_k$ in the following way:

$$P_k \setminus G_k/H_k \simeq \bigcup_{i \in I} W_{G_k}(A_i)/W_{H_k}(A_i)$$

with $\{A_i \mid i \in I\}$ the set of representatives of H_k-conjugacy classes of θ-stable maximal k-split tori.

Example 2.3.6. Let $G = \mathrm{SL}(2, \mathbb{C})$, $\theta(g) = (g^T)^{-1}$, $H = \{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a^2 + b^2 = 1\}$. Then $G_{\mathbb{R}} = \mathrm{SL}(2, \mathbb{R})$. We have that $P_k = B = \{$upper triangular matrices$\}$ is the minimal parabolic $\mathbb{R}$-subgroup. Then

$$B \setminus G/H \simeq \bigcup_{i \in I} W_G(T_i)/W_H(T_i)$$

$$B_{\mathbb{R}} \setminus G_{\mathbb{R}}/H_{\mathbb{R}} \simeq \bigcup_{i \in I} W_{G_{\mathbb{R}}}(A_i)/W_{H_{\mathbb{R}}}(A_i)$$

The set $\{T_i \mid i \in I\} = \{T_1, T_2\}$ and $\{A_i \mid i \in I\} = \{T_1\}$. The set $\{A_i \mid i \in I\}$ does not contain T_2 because T_2 cannot be diagonalized over $\mathbb{R}$ and thus is not $\mathbb{R}$-split. The Weyl group $W_G(T_1)$ has representatives in $H_{\mathbb{R}}$ and since $H = T_2$, the group $W_H(T_2) = \{\mathrm{id}\}$. So we have

$$B \setminus G/H \simeq \{W_H(T_1)/W_H(T_1), W_G(T_2)\}$$

$$B_{\mathbb{R}} \setminus G_{\mathbb{R}}/H_{\mathbb{R}} \simeq \{\mathrm{id}\}$$

We get three orbits in the algebraically closed case and only one orbit in the non-algebraically closed case.

If we consider $k = \mathbb{Q}$ in the previous example, we have that

$$B_{\mathbb{Q}} \setminus G_{\mathbb{Q}}/H_{\mathbb{Q}} \simeq \bigcup_{i \in I} W_{G_{\mathbb{Q}}}(A_i)/W_{H_{\mathbb{Q}}}(A_i).$$

There are infinitely many $H_{\mathbb{Q}}$-conjugacy classes of θ-split tori. This can be seen in the following example.

Example 2.3.7. Let $G = \mathrm{SL}(2, \mathbb{Q})$, $\theta(x) = (x^T)^{-1}$, $B =$ the Borel subgroup of upper triangular matrices and A the group of diagonal matrices. In this case the computations work out nicer if we let $H = G^\theta$ act from the left and define $\tau(g) = g^{-1}\theta(g)$ and $\mathcal{V}_{\mathbb{Q}} = \{g \in G_{\mathbb{Q}} \mid \tau(g) \in N_{G_{\mathbb{Q}}}(A)\}$. If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Q})$, then

$$g^{-1}\theta(g) = \begin{pmatrix} b^2 + d^2 & -ab - cd \\ -ab - cd & a^2 + c^2 \end{pmatrix} \in N_{G_{\mathbb{Q}}}(A)$$

if and only if $ab + cd = 0$. So $\tau(\mathcal{V}_{\mathbb{Q}}) \subset A_{\mathbb{Q}}$ and it coincides with the set consisting of $\begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}$ with $r = x^2 + y^2$, $(x, y) \in \mathbb{Q}^2 - \{(0,0)\}$. If $z = \begin{pmatrix} v & u \\ 0 & v^{-1} \end{pmatrix} \in B$ and $\tau(g) = \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}$, then

$$z^{-1}\tau(g)\theta(z) = \begin{pmatrix} rv^{-2} + u^2r^{-1} & -uvr^{-1} \\ -uvr^{-1} & v^2r^{-1} \end{pmatrix} \in A_{\mathbb{Q}}$$

if and only if $u = 0$. It follows that $\begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}$ and $\begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}$ are in the same twisted $B_{\mathbb{Q}}$ orbit if and only if $r^{-1}s \in (\mathbb{Q}^*)^2$. Hence $V_{\mathbb{Q}} \cong \oplus_{\substack{p\equiv 1(4) \\ \text{prime}}} \mathbb{Z}/2\mathbb{Z}$ and the set $H_{\mathbb{Q}}\backslash G_{\mathbb{Q}}/B_{\mathbb{Q}}$ is infinite.

2.4 Lie Algebras

The work done throughout this thesis focuses on Lie groups. It is possible to do similar work with Lie algebras with some adjustments. In fact, some of the calculations are easier when dealing with Lie algebras so it can be beneficial to go between the two.

Let $\mathfrak{g}$ be a Lie algebra. We will denote by $\mathfrak{t}$ a maximal toral subalgebra and $\mathfrak{b}$ a Borel subalgebra, i.e. $\mathfrak{b}$ is a maximal solvable subalgebra. To go between the Lie algebras and the Lie groups, we take the exponential of the Lie algebra. Note that for tori the exponential map Exp is surjective, $T = \mathrm{Exp}(\mathfrak{t})$, but for most semisimple Lie groups Exp is not surjective.

Since the operation used in the Lie algebra is addition, we will use the additive inverse in place of the multiplicative inverse used in the Lie group. For example, if we let θ be an involution of $\mathfrak{g}$, we call $\mathfrak{t}$ θ-split if $\theta(t) = -t$ for all $t \in \mathfrak{t}$. This change will apply to all definitions and involutions.

Example 2.4.1. Let $\theta(A) = (A^T)^{-1}$ be an involution of G. Then for $\mathfrak{g}$, $\theta(A) = -A^T$. Thus, we have that $\mathfrak{h} = \{X \in \mathfrak{g} \mid -X = X^T\}$. This fixed point group is much easier to calculate than the fixed point group $H = \{X \in G \mid A^{-1} = A^T\}$.

Many of the results from the Lie group hold for the Lie algebra as well. We have that $\mathfrak{t} = \mathfrak{t}_s \oplus \mathfrak{t}_a$, where $\mathfrak{t}_s$ is the k-split part of $\mathfrak{t}$ and $\mathfrak{t}_a$ is the k-anisotropic part of $\mathfrak{t}$. We also have similar definitions for $\mathfrak{t}^+$ and $\mathfrak{t}^-$.

2.5 $P_k \setminus G_k/H_k$

The motivation behind this thesis focuses on classifying the double cosets $P_k \setminus G_k/H_k$. We have the following definitions.

Definition 2.5.1. Given a group G and an involution θ, we define the *fixed point group* H as the group $H = G^\theta = \{g \in G \mid \theta(g) = g\}$.

We will let G_k and H_k denote the k-rational points of G and H, respectively. We let $N_G(H)$ (respectively $N_{G_k}(H_k)$) denote the normalizer of H in G (respectively H_k in G_k) and $Z_G(H)$ (respectively $Z_{G_k}(H_k)$) denote the centralizer of H in G (respectively H_k in G_k).

Example 2.5.2. If $G = \mathrm{SL}(2, \mathbb{C})$, then $G_\mathbb{R}$ will be the 2×2 matrices of determinant one with entries in $\mathbb{R}$.

Now that we have established a group G, an involution θ of G, and the fixed point group $H = G^\theta$ associated with this involution, we can define a symmetric variety.

Definition 2.5.3. A *symmetric variety* (or *symmetric space*) is the set $Q = \{g\theta(g)^{-1} \mid g \in G\} \simeq G/H$. A *symmetric k-variety* is the set $Q_k = \{g\theta(g)^{-1} \mid g \in G_k\} \simeq G_k/H_k$.

Helminck and Wang gave the following characterization of $P_k \backslash G_k/H_k$:

Theorem 2.5.4. *[22] $P_k \backslash G_k/H_k \simeq \bigcup_{A_i \in I} W_{G_k}(A_i)/W_{H_k}(A_i)$, where I is the set of H_k-conjugacy classes of θ-stable maximal $\Bbbk$-split tori and the A_i are representatives of these $H_\Bbbk$-conjugacy classes.*

Thus, to classify $P_k \backslash G_k/H_k$, we first need to classify the H_k-conjugacy classes of θ-stable maximal k-split tori.

For maximal (θ, k)-split tori, Helminck and Wang gave the following result:

Theorem 2.5.5. *[22] Let A_1 and A_2 be maximal $(\theta, \Bbbk)$-split tori. Let $T \supset A_1$ be maximal $\Bbbk$-split. Then there exists a $g \in (\overline{H} \cdot \overline{Z_G(T)})_\Bbbk$ such that $gA_1g^{-1} = A_2$.*

Example 2.5.6. Let $G = \mathrm{SL}(2,k), k = \bar{k}$, $\theta(g) = (g^T)^{-1}$. Then

$$H = G^\theta = \{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a^2 + b^2 = 1\} = SO(2,k).$$

Let $T_1 = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$. We have that T_1 is θ-split; for $t \in T_1, \theta(t) = (t^T)^{-1} = t^{-1}$. Thus $T_1 = (T_1)^-_\theta$. Let $T_2 = H = \mathrm{SO}(2)$. Then for $t \in T_2$, we have that $\theta(t) = t$ since T_2 is the fixed point group. This means that $T_2 = (T_2)^+_\theta$. Any θ-stable maximal torus is H-conjugate to either T_1 or T_2. Then

$$B \setminus G/H \simeq W_G(T_1)/W_H(T_1) \cup W_G(T_2)/W_H(T_2)$$

and we have the following:

$$W_G(T_1) = \{\mathrm{id}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\} = W_H(T_1)$$

$$\{\mathrm{id}\} = W_H(H) \subsetneq W_G(T_2)$$

Chapter 3

Standard Tori

In the process of classifying the H_k-conjugacy classes of θ-stable maximal k-split tori, it is sufficient to consider only standard tori. We will first consider standard k-split tori and then look at quasi k-split tori and H-quasi k-split tori.

3.1 Standard k-split tori

Let G be a connected reductive algebraic group, θ an involution of G defined over k and $H = G^\theta$, the fixed point group. Given a θ-stable torus A, we will let $A^+ = A_\theta^+ = \{X \in A \mid \theta(X) = X\}$ and $A^- = A_\theta^- = \{X \in A \mid \theta(X) = X^{-1}\}$. Let $\mathcal{A}_k^\theta$ denote the set of θ-stable maximal k-split tori of G, $\mathcal{A}^\theta$ denote the set of θ-stable maximal tori of G and $\mathcal{A}_0^\theta$ the set of θ-stable quasi k-split tori of G, which are H-conjugate to a θ-stable maximal k-split torus.

The H_k-conjugacy classes of θ-stable maximal k-split tori are determined by the image and fibers of the map $\zeta : \mathcal{A}_k^\theta / H_k \to \mathcal{A}^\theta / H$. The image of ζ consists of the H-conjugacy classes of θ-stable maximal k-split tori.

To be able to give a characterization of these in terms of conjugacy classes of involutions in a Weyl group, we will first show in this section that every conjugacy class contains a standard torus. We will first start with some definitions.

Definition 3.1.1. A *quasi k-split torus* is a torus that is conjugate under G to a k-split torus, i.e. A is quasi k-split if

$$A = gTg^{-1}$$

with $g \in G$ and T a k-split torus.

For quasi k-split tori we define singular roots with respect to the involution. Note that k-split tori are also quasi k-split tori, with $g = \mathrm{id}$. If A is a quasi k-split torus and $\alpha \in \Phi(A)$, then we write $G_\alpha = Z_G(\ker(\alpha)^\circ)$.

Definition 3.1.2. The roots of $\Phi(A)$ can be divided in four subsets, related to the action of θ, as follows.

(a) $\theta(\alpha) \neq \pm\alpha$. Then α is called *complex* (relative to θ).

(b) $\theta(\alpha) = -\alpha$. Then α is called *real* (relative to θ).

(c) $\theta(\alpha) = \alpha$ and $\theta|G_\alpha \neq \mathrm{id}$. Then α is called *non-compact imaginary* (relative to θ).

(d) $\theta(\alpha) = \alpha$ and $\theta|G_\alpha = \mathrm{id}$. Then α is called *compact imaginary* (relative to θ).

Definition 3.1.3. If α is either a real or non-compact imaginary root, then we call it a *θ-singular root.*

For k-split tori we have to combine the idea of θ-singular with the k-structure of the group itself. The θ-singular roots can be divided in those which are singular with respect to the k-structure and those which are not. These are defined as follows.

Definition 3.1.4. A real θ-singular root $\alpha \in \Phi(A)$ is called *(θ, k)-singular* (respectively *θ-singular anisotropic*) if $\overline{G}_\alpha \cap H$ is isotropic (respectively $\overline{G}_\alpha \cap H$ is anisotropic).

Similarly an imaginary θ-singular root is called *(θ, k)-singular* (respectively *θ-singular anisotropic*) if $\overline{G}_\alpha$ has a non-trivial (θ, k)-split torus (respectively $\overline{G}_\alpha$ has no non-trivial (θ, k)-split tori).

Similar as for quasi k-split tori we have the following result.

Proposition 3.1.5. *[19] Let A be a θ-stable maximal k-split torus of G. Then we have the following.*

(1) *A^+ is a maximal k-split torus of H if and only if $\Phi(A)$ has no (θ, k)-singular real roots.*

(2) *A^- is a maximal (θ, k)-split torus of G if and only if $\Phi(A)$ has no (θ, k)-singular imaginary roots.*

Let $\mathcal{A}^\theta$ be the set of θ-stable maximal tori of G.

Definition 3.1.6. For $A_1, A_2 \in \mathcal{A}^\theta$, the pair (A_1, A_2) is called *standard* if $A_1^- \subset A_2^-$ and $A_1^+ \supset A_2^+$. In this case, we also say that A_1 is standard with respect to A_2.

The θ-stable maximal tori of G can be put into standard position. Since every conjugacy class contains a standard torus, it suffices to look at the θ-stable, maximal k-split standard tori.

Lemma 3.1.7. *Let $A_1, A_2 \in \mathcal{A}$ such that $A_1^+ \supset A_2^+$ (respectively $A_1^- \subset A_2^-$). Then there exists $x \in Z_H(A_2^+)$ (respectively $Z_H(A_1^-)$) such that (A_1, xA_2x^-1) is standard. In particular, if A_1^+ and A_2^+ (respectively A_1^- and A_2^-) are H-conjugate, so are A_1 and A_2.*

Proof. Let $M\ Z_G(A_2^+)$. Then A_1^- and A_2^- are θ-split tori of M. Let $A \subset M$ be a maximal θ-split torus with $A \supset A_1^-$. Since A_2^- is a maximal θ-split torus of M, there exists an $x \in (M \cap H)^\circ$ such that $xAx^{-1} = A_2^-$ and hence $xA_1^-x^{-1} \subset A_2^-$. The proof for the second statement is similar. □

Any θ-stable maximal k-split torus is standard with respect to one containing a maximal (θ, k)-split torus (respectively a maximal k-split torus of H).

Lemma 3.1.8. *[19] Let A_1 be a θ-stable maximal k-split torus. Then there exists a standard pair (A, S) of θ-stable maximal k-split tori, with A^- maximal (θ, k)-split, S^+ a maximal k-split torus of H and A_1 is standard with respect to A and S.*

Notation 3.1.9. Let $A_0 \in \mathcal{A}_0^\theta$ (respectively $S_0 \in \mathcal{A}_0^\theta$) be θ-stable maximal k-split tori of G, such that A_0^- (respectively S_0^+) is a maximal (θ, k)-split torus of G (respectively a maximal k-split torus of H). We can choose S_0 to be standard with respect to A_0. In the following we fix such a standard pair (S_0, A_0).

It remains to show that any θ-stable maximal k-split torus is H-conjugate with a θ-stable maximal k-split torus standard with respect to A_0 and S_0.

Proposition 3.1.10. *[19] Let (S_0, A_0) be a standard pair as above and let $A_1 \in \mathcal{A}_0^\theta$. Then A_1 is H-conjugate with a θ-stable maximal k-split torus, which is standard with respect to A_0 and S_0.*

A standard pair (A_1, A_2) of θ-stable maximal tori of G gives rise to an involution in $W(T_1)$ (respectively $W(T_2)$).

Lemma 3.1.11. *Let (A_1, A_2) be a standard pair of θ-stable maximal k-split tori of G. Then we have the following conditions:*

(*i*) *There exists $g \in Z_G(A_1^- A_2^+)$ such that $gA_1g^{-1} = A_2$.*

(*ii*) *If $n_1 = \theta(g)^{-1}g$ and $n_2 = \theta(g)g^{-1}$, then $n_1 \in N_G(A_1)$ and $n_2 \in N_G(A_2)$.*

(*iii*) *Let w_1 and w_2 be the images of n_1 and n_2 in $W(A_1)$ and $W(A_2)$ respectively. Then $w_1^2 = \mathrm{id}$, $w_2^2 = \mathrm{id}$, and $(A_1)_{w_1}^+ = (A_2)_{w_2}^+ = A_1^- A_2^+$, which characterizes w_1 and w_2.*

Proof. Since A_1 and A_2 are maximal k-split tori of $Z_G(A_1^- A_2^+)$, the first statement is clear. For (*ii*) and (*iii*), consider first n_1. Since $g \in Z_G(A_1^- A_2^+)$ and $A_1 = A_1^- A_2^+ (A_1^+ \cap g^{-1} A_2^- g)$ it suffices to look at $A_1^+ \cap g^{-1} A_2^- g$. So let $x \in A_1^+ \cap g^{-1} A_2^- g$ and write $x = g^{-1}tg$ with $t \in A_2^-$. Then $n_1 x n_1^{-1} = \theta(g^{-1}t^{-1}g) = \theta(x)^{-1} = x^{-1}$. It follows that

$$\mathrm{Inn}(n_1)|A_1^- A_2^+ = \mathrm{id},\ \mathrm{Inn}(n_1)|A_1^+ \cap g^{-1} A_2^- g = -\,\mathrm{id}$$

which implies that $n_1 \in N_G(A_1)$, $w_1^2 = \mathrm{id}$ and $(A_1)_{w_1}^+ = A_1^- A_2^+$.

The assertion for n_2 and w_2 follows with a similar argument. □

Remark 3.1.12. By (*iii*) of Lemma 3.1.11, w_1 and w_2 are independent of the choice of the element $g \in Z_G(A_1^- A_1^+)$ with $gA_1g^{-1} = A_2$.

The above leads to the following definition of standard involutions:

Definition 3.1.13. Let A_1, A_2, $w_1 \in W(A_1)$ and $w_2 \in W(A_2)$ be as in Lemma 3.1.11. We call w_1 (respectively w_2) the A_2-*standard involution* (respectively A_1-*standard involution*) of $W(A_1)$ (resp. $W(A_2)$). Moreover we will call an involution $w \in W(A_2)$ a k-*standard involution*, if there exists a θ-stable maximal k-split torus A_3 standard with respect to A_2 such that w is the A_3-standard involution in $W(A_2)$.

In practice, standard involutions are written as the product of Weyl group elements s_α's, as we will see in Chapter 7.

Remark 3.1.14. To show that the H-conjugacy classes in $\mathcal{A}_0^\theta$ correspond to conjugacy classes of the k-standard involutions, we need to prove first a similar result for $\mathcal{A}^\theta/H$, the H-conjugacy classes of θ-stable maximal quasi k-split tori. Namely if A_1, $A_2 \in \mathcal{A}_0^\theta$ standard with respect to A_0 and $h \in H$ such that $hA_1h^{-1} = A_2$, then $A_3 = hA_0h^{-1}$ and A_0 are θ-stable maximal quasi k-split tori with A_3^- and A_0^- maximal. To prove that the

A_1-standard involution and the A_2-standard involution are conjugate under $W(A_0)$ we will need to show that A_3 and A_0 are actually maximal quasi k-split tori of $Z_G(A_2^-)$. This will be shown in the next section, where we have a closer look at the H-conjugacy classes of θ-stable maximal quasi k-split tori.

3.2 Standard Quasi k-split tori

In this section we show that, similar as in the case of k-split tori, every H-conjugacy class of θ-stable maximal quasi k-split tori contains a standard torus. The conjugacy classes in $\mathcal{A}^\theta/H$ are not only of importance for a characterization of the double cosets $P\backslash G/H$, but also for a classification of the subset $\mathcal{A}_0^\theta/H$ of $\mathcal{A}^\theta/H$. The characterization of $\mathcal{A}^\theta/H$ is more complicated then that of $\mathcal{A}_0^\theta/H$.

Let G be a connected reductive algebraic group, θ an involution of G defined over k. Let $H = G^\theta$ be the fixed point group. Let $\mathcal{A}^\theta$ denote the set of θ-stable maximal quasi k-split tori of G. Let A_0 denote a θ-stable maximal k-split torus with A_0^- a maximal (θ, k)-split torus of G and $T \supset A_0$ a maximal torus of G, such that T_θ^- is a maximal θ-split torus of G. We write $\mathcal{V}$ for $\tau^{-1}(N_G(A_0))$.

For k-split tori, we had the concept of two tori being standard with respect to one another. When dealing with quasi k-split tori, we must introduce the idea of almost standard.

Definition 3.2.1. For A_1, $A_2 \in \mathcal{A}^\theta$, the pair (A_1, A_2) is called *almost standard* if $A_1^- \subset A_2^-$. An almost standard pair (A_1, A_2) is called *standard* if $A_1^+ \supset A_2^+$. In these cases, we also say that A_1 is *almost standard* (respectively *standard*) with respect to A_2.

Only the -1 eigenspace is able to be put in standard position when we have that two tori are almost standard. We only get standard if we take the maximal quasi k-split tori in H to be conjugate.

All θ-stable maximal quasi k-split tori of G with A^- maximal are conjugate under H^0. This conjugacy of the θ-stable maximal quasi k-split tori A with A^- maximal enables us to show that any θ-stable maximal quasi k-split torus is conjugate to one almost standard with respect to A_0.

Proposition 3.2.2. *Let A_1 be a θ-stable maximal quasi k-split torus of G. Then there exists $h \in H^0$ such that $hA_1^-h^{-1} \subset A_0^-$.*

Proof. We use induction with respect to $\dim A_0^- - \dim A_1^-$. If $\dim A_0^- - \dim A_1^- = 1$, then since A_1^- is not maximal there exists $\lambda \in \Phi(A_1)$ a θ-singular imaginary root. There exists $g \in Z_G((\ker \lambda)^0)$ such that $A_2 = gA_1g^{-1}$ is θ-stable and $\dim A_2^- = \dim A_1^- + 1 = \dim A_0^-$. But then A_2^- is maximal, thus there exists $h \in H^0$ such that $hA_2h^{-1} = A_0$. Since $A_1^- \subset (\ker \lambda)^0 \subset A_2$ it follows that $hA_1^-h^{-1} \subset A_0^-$.

Assume now that $\dim A_0^- - \dim A_1^- = k > 0$. Let $\lambda \in \Phi(A_1)$ be a θ-singular imaginary root and $g \in Z_G((\ker \lambda)^0)$ such that $A_2 = gA_1g^{-1}$ is θ-stable and $\dim A_2^- = \dim A_1^- + 1 = \dim A_0^-$. Since $\dim A_0^- - \dim A_2^- = k - 1$ it follows from the induction hypothesis that there exists $h \in H^0$ such that $hA_2^- \subset A_0^-$. Since $A_1^- \subset A_2^-$ the result follows. $\square$

Lemma 3.2.3. *[19] Let A_1 be a θ-stable maximal quasi k-split torus almost standard with respect to A_0. There exists an involution $w \in W(A_0)$ with $(A_0)_w^- \subset A_0^-$ and such that $A_1^- = ((A_0)_w^- \cap A_0^-)^0$.*

The above Lemma enables us to obtain the following result.

Proposition 3.2.4. *[19] Let A_0 be as above and let $A_1 \in \mathcal{A}^\theta$ be a θ-stable maximal quasi k-split torus, almost standard with respect to A_0. Then we have the following conditions:*

(i) There exists $g \in Z_G(A_1^-)$ such that $gA_1g^{-1} = A_0$.

(ii) If A_1 is standard with respect to A_0, then there exists $g \in Z_G(A_1^-A_0^+)$ such that $gA_1g^{-1} = A_0$.

Corollary 3.2.5. *Any θ-stable maximal quasi k-split torus is H^0-conjugate to one standard with respect to A_0.*

Proof. Let A_1 be a θ-stable maximal quasi k-split torus of G. From Proposition 3.2.2 it follows that we may assume that A_1 is almost standard with respect to A_0. Similar as in the proof of Lemma 3.2.3 there exist strongly orthogonal θ-singular imaginary roots $\alpha_1, \ldots, \alpha_n \in \Phi(A_1)$ ($n = \dim A_0^- - \dim A_1^-$) and $x \in Z_G((A_1)_w^+)$ such that $A_2 = xA_1x^{-1}$ is θ-stable and $\dim A_2^- = \dim A_1^- + n = \dim A_0^-$. Here $w = s_{\alpha_1} \cdots s_{\alpha_n}$. By Corollary **??** there exists $h \in H^0$ such that $hA_2h^{-1} = A_0$. Let $A_3 = hA_1h^{-1}$. Since $A_1^- \subset (A_1)_w^+ \subset A_2^-$ and $A_2^+ \subset A_1^+$ it follows that $A_3^- = hA_1^-h^{-1} \subset A_0^-$ and $A_3^+ = hA_1^+h^{-1} \supset hA_2^+h^{-1} = A_0^+$. This proves the result. $\square$

This result reduces the study of H^0-conjugacy classes of θ-stable maximal quasi k-split tori to H^0-conjugacy classes of θ-stable maximal quasi k-split tori standard with respect to A_0. As in Lemma 3.1.11 we can associate an involution with a θ-stable maximal quasi k-split torus, standard with respect to A_0.

Lemma 3.2.6. *[19] Let A_0 be as above and let $A_1 \in \mathcal{A}^\theta$ be a θ-stable maximal quasi k-split torus, standard with respect to A_0 and $g \in Z_G(A_1^- A_0^+)$ such that $gA_1g^{-1} = A_0$. Then we have the following conditions:*

- *(i) If $n = \theta(g)g^{-1}$, then $n \in N_G(A_0)$.*
- *(ii) Let w be the image of n in $W(A_0)$. Then $w^2 = e$ and $(A_0)_w^+ = A_1^- A_0^+$, which characterizes w.*

This result follows using a similar argument as in Lemma 3.1.11.

Remark 3.2.7. By (ii) of Lemma 3.2.6, w is independent of the choice of the element $g \in Z_G(A_1^- A_0^+)$ such that $gA_1g^{-1} = A_0$.

This leads to the following definition of standard involutions for the θ-stable maximal quasi k-split torus, standard with respect to A_0:

Definition 3.2.8. Let $A_1 \in \mathcal{A}^\theta$ and $w \in W(A_0)$ be as in Lemma 3.2.6. We call w the A_1-standard involution of $W(A_0)$. Moreover we will call an involution $w_0 \in W(A_0)$ a *standard involution*, if there exists a θ-stable maximal quasi k-split torus A_1 standard with respect to A_0 such that w_0 is the A_1-standard involution in $W(A_0)$.

For a θ-stable k-torus A_0 of G, write $W(A_0, H)$ for $N_H(A_0)/Z_H(A_0)$. We have now the following result.

Proposition 3.2.9. *Assume that A_1, $A_2 \in \mathcal{A}^\theta$ such that they are standard with respect to A_0. Let w_1 and w_2 be the A_1-standard and A_2-standard involutions in $W(A_0)$ respectively. If A_1 and A_2 are conjugate under H, then w_1 and w_2 are conjugate under $W(A_0, H)$.*

Proof. Assume $h \in H^0$ such that $hA_1h^{-1} = A_2$. Then $A_3 = hA_0h^{-1}$ is θ-stable with A_3^- maximal. It suffices to show that A_3 and A_0 are conjugate under $H^0 \cap Z_G(A_1^- A_0^+)$.

Let $\alpha \in \Phi(A_1)$ with $\theta(\alpha) = -\alpha$ and β the corresponding root in $\Phi(A_2)$. If A_1^- does not contain any real roots, then A_i^+ $(i = 1, 2)$ is maximal and in this case both w_1 and w_2 are maximal involutions in $\Phi_1 = \{\alpha \in \Phi(A_0) \mid \theta(\alpha) = -\alpha\}$ and hence conjugate under $W(\Phi_1) \subset W(A_0, H)$.

Since both α and β correspond to roots in $\Phi(A_0)$, there exists $w \in W(A_0, H^0)$ such that $w(\alpha) = \beta$. Let $h_1 \in H^0$ be a representative of w and $A_4 = h_1 A_1 h_1^{-1}$. Then $hh_1^{-1} A_4 h_1 h^{-1} = hA_1 h^{-1} = A_2$, $hh_1^{-1} A_0 h_1 h^{-1} = A_3$ and $hh_1^{-1} \in Z_H(\beta^\vee)$. Using induction it follows that there exists $h_2 \in Z_G(A_2^- A_0^+) \cap H^0$ such that $h_2 A_3 h_2^{-1} = A_0$. Then $g = h_2 h \in N_H(A_0)$ maps w_1 to w_2. $\square$

Remark 3.2.10. The θ-stable maximal quasi k-split tori A_1 with A_1^+ maximal are not necessarily conjugate under H can be seen as follows. If T is a θ-stable maximal torus with T^+ a maximal torus of H, then $N_G(T) \neq N_H(T)$. This means that T can contain several H-conjugacy classes of θ-stable maximal quasi k-split tori. These are basically conjugates of the A_0^+ part of these tori.

A consequence of this is that there is not a unique minimal element in the set of H-conjugacy classes of θ-stable maximal quasi k-split tori. For the minimal element all we can show is the following:

Lemma 3.2.11. *[19] Let S be a θ-stable maximal quasi k-split torus with S^+ maximal. There exists a θ-stable maximal torus T of G with $T \supset S$ and T^+ is a maximal torus of H.*

3.3 Standard H-Quasi k-split tori

Many of the results for quasi k-split tori are also true for H-quasi k-split tori.

Definition 3.3.1. An *H-quasi k-split torus* is a torus that is H-conjugate to a k-split torus, i.e. A is H-quasi k-split if

$$A = hTh^{-1}$$

with $h \in H$ and T a k-split torus.

When dealing with maximal H-quasi k-split tori, we no longer have to use the definition for almost standard. This is because all maximal H-quasi tori A with A^+ maximal are conjugate. This gives us a unique bottom to the poset of standard tori.

Since we have a unique top and a unique bottom in the poset of θ-stable, maximal H-quasi k-split standard tori, we are in a much simpler case than in the maximal quasi k-split tori case, where we are not guaranteed a unique bottom to the poset.

Chapter 4

$\mathrm{SL}(2, k)$

4.1 Introduction to $\mathrm{SL}(2, k)$

In order to help us understand what happens in $\mathrm{SL}(n, k)$, we will first work on fully understanding $\mathrm{SL}(2, k)$. Since $\mathrm{SL}(n, k)$ is made up of multiple blocks of $\mathrm{SL}(2, k)$, our understanding of the $\mathrm{SL}(2, k)$ case will be very beneficial. We have the following theorem to classify the involutions of $\mathrm{SL}(2, k)$.

We will let $\mathrm{Inn}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be conjugation by the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, i.e.

$$\mathrm{Inn}\begin{pmatrix} a & b \\ c & d \end{pmatrix}(X) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}(X)\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}.$$

Theorem 4.1.1. *[23] The number of isomorphy classes of involutions over G equals the order of $k^*/(k^*)^2$. Furthermore, the involutions of G are of the form $\theta = \mathrm{Inn}\left(\begin{smallmatrix} 0 & 1 \\ m & 0 \end{smallmatrix}\right)$ for all $m \in k^*/(k^*)^2$. Additionally, if m and q are in the same square class, then $\mathrm{Inn}\left(\begin{smallmatrix} 0 & 1 \\ m & 0 \end{smallmatrix}\right)$ is isomorphic to the involution $\mathrm{Inn}\left(\begin{smallmatrix} 0 & 1 \\ q & 0 \end{smallmatrix}\right)$.*

For $k = \mathbb{C}$, we have that there is exactly one involution for $\mathrm{SL}(2, k)$, which is $\theta = \mathrm{Inn}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. For $k = \mathbb{R}$, we have that there are two involutions for $\mathrm{SL}(2, k)$, namely $\theta_1 = \mathrm{Inn}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\theta_2 = \mathrm{Inn}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. For $k = \mathbb{F}_p$, we will have two cases. If $-1 \in (k^*)^2$, then for $k = \mathbb{F}_p$, there is exactly one involution for $\mathrm{SL}(2, k)$, which is $\theta = \mathrm{Inn}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. If $-1 \notin (k^*)^2$, then for $k = \mathbb{F}_p$, we have that there are two involutions for $\mathrm{SL}(2, k)$, namely $\theta_1 = \mathrm{Inn}\left(\begin{smallmatrix} 0 & 1 \\ s_p & 0 \end{smallmatrix}\right)$ where s_p is the smallest nonsquare in F_p and $\theta_2 = \mathrm{Inn}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Note that if -1 is not a square in F_p, we can use $s_p = -1$ because they are in the same square class.

4.1.2 $k = \mathbb{C}$

Let us first consider $k = \mathbb{C}$. Note that $\theta(A) = \mathrm{Inn}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)(A) \simeq (A^T)^{-1}$. We have that the maximal (θ, k)-split torus for this involution is $T = \{\mathrm{diag}\} = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$.

Theorem 4.1.3. *Let $G = \mathrm{SL}(2, k)$, $\theta = (A^T)^{-1}$. If $-1 \in (k^*)^2$, then $T_1 = \left(\begin{smallmatrix} x & 0 \\ 0 & x^{-1} \end{smallmatrix}\right)$ and $T_2 = \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$ are the two H_k-conjugacy classes of θ-stable, maximal k-split tori. If $-1 \notin (k^*)^2$, then T_1 is the only H_k-conjugacy class of θ-stable, maximal k-split tori.*

Proof. For $\theta = (A^T)^{-1}$, the maximum (θ, k)-split torus is $T_1 = \left(\begin{smallmatrix} x & 0 \\ 0 & x^{-1} \end{smallmatrix}\right)$. $T_2 = H_k = \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$ is a θ-stable torus, but we need to check to see if T_2 is k-split. To check to see if T_2 is diagonalizable, we find its eigenvalues. We get the characteristic equation

$$(a - \lambda)^2 + b^2 = 0$$

$$a^2 - 2a\lambda + \lambda^2 + b^2 = 0$$

$$\lambda^2 - 2a\lambda + 1 = 0$$

$$\lambda = a \pm \sqrt{a^2 - 1}$$

$$\lambda = a \pm \sqrt{-b^2}$$

$$\lambda = a \pm b\sqrt{-1}$$

If $-1 \in (k^*)^2$, we have that T_2 is k-split. If $-1 \notin (k^*)^2$, then we are not able to go from T_1 to T_2 because T_2 would not be k-split. □

From this theorem, we have that there are two H_k-conjugacy classes of θ -stable, k-split tori for $k = \mathbb{C}$. The two representatives are T_1 and T_2.

The process of going from T_1 to T_2 is called flipping down the poset. When we move one level down the poset, we are increasing the dimension of the T^+ part of the torus by one and decreasing the dimension of the T^- part of the torus by one. In the $\mathrm{SL}(2, k)$ case, we are only able to flip a maximum of one time, since there is only one possible root to flip over, namely α_1. When flipping from T_1 to T_2, we are flipping over the α_1 position. This terminology will help in future examples. In cases of $\mathrm{SL}(n, k)$ where n is larger than two, it could be possible to flip down the poset more than once. This will be seen in later chapters. For now, we will move on to the case for $\mathrm{SL}(2, \mathbb{R})$.

4.1.4 $k = \mathbb{R}$

If we consider $k = \mathbb{R}$, then we have two involutions because there are two square classes for the real numbers. Our two involutions are $\theta_1 = \mathrm{Inn}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\theta_2 = \mathrm{Inn}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Note that $\theta_1 \simeq (A^T)^{-1}$. From Theorem 4.1.3, we have that there is one H_k-conjugacy class of θ_1 -stable, k-split tori for $k = \mathbb{R}$ with representative T_1 because -1 is not a square in $\mathbb{R}$.

For θ_2, we have that there are two H_k-conjugacy classes of θ -stable, k-split tori for $k = \mathbb{R}$. The two representatives are $T = \{\text{diag}\}$ and $H = G^{\theta_2}$. If we work through the characteristic equation, we get the following, noting that $a^2 - b^2 = 1$:

$$(a - \lambda)^2 - b^2 = 0$$

$$a^2 - 2a\lambda + \lambda^2 - b^2 = 0$$

$$\lambda^2 - 2a\lambda + 1 = 0$$

$$\lambda = a \pm \sqrt{a^2 - 1}$$

$$\lambda = a \pm \sqrt{b^2}$$

$$\lambda = a \pm b$$

Thus we get eigenvalues $a + b$ and $a - b$, both contained in $\mathbb{R}$, so H is $\mathbb{R}$-split.

For $k = \mathbb{R}$, we see that for θ_1, we are not able to flip down the poset and for θ_2 we are able to flip. If we are unable to flip, then we get that H_k is k-anisotropic.

Definition 4.1.5. H_k is *k-anisotropic* if it does not contain any non-trivial k-split tori.

For $\theta = \mathrm{Inn}\left(\begin{smallmatrix} 0 & 1 \\ m & 0 \end{smallmatrix}\right)$, we have that H_k is k-anisotropic if and only if $m \notin \bar{1}$. If $m \in \bar{1}$, then H_k contains a maximal k-split torus.

4.1.6 $k = \mathbb{F}_p$

When considering $\mathrm{SL}(2, \mathbb{F}_p)$, we have two cases. If -1 is a square, then $\mathrm{SL}(2, \mathbb{F}_p)$ is similar to the case of $\mathrm{SL}(2, \mathbb{C})$. If -1 is not a square, then we are in a similar case to $\mathrm{SL}(2, \mathbb{R})$. We have that -1 is a square when $p = 1 \mod 4$ and -1 is not a square when $p = 3 \mod 4$.

Example 4.1.7. For $\mathrm{SL}(2, \mathbb{F}_5)$, we have that $-1 \in (\mathbb{F}_5^*)^2$. This gives us only one involution and we are able to flip down the poset for $\theta(A) = (A^T)^{-1}$. For $\mathrm{SL}(2, \mathbb{F}_3)$, we have that

$-1 \notin (\mathbb{F}_3^*)^2$. This will give us two involutions. We are not able to flip down the poset for $\theta(A) = (A^T)^{-1}$, but we are able to flip down the poset when $\theta = \mathrm{Inn}\left(\begin{smallmatrix} 0 & 1 \\ s_p & 0 \end{smallmatrix}\right)$.

From these results, we get an idea of what the poset of θ-stable, H_k-conjugate tori looks like for $\mathrm{SL}(2, k)$ given an involution. For $k = \mathbb{C}$, the poset will contain two levels, with $T = \{\mathrm{diag}\}$ at the top and $H = G^\theta$ at the bottom. For $k = \mathbb{R}$, for θ_1, we have that there is only one level to the poset, with $T = \{\mathrm{diag}\}$ the only torus in the poset. For θ_2, we have two levels to the poset, with $T = \{\mathrm{diag}\}$ at the top and $H = G^\theta$ at the bottom. For $k = \mathbb{F}_p$, we get a similar situation to either $k = \mathbb{C}$ or $k = \mathbb{R}$, depending on whether or not -1 is a square.

4.2 Orbit Decompositions of $\mathrm{SL}(2, k)$

We have considered three fields for $\mathrm{SL}(2, k)$. The full classification of the double cosets $P_k \backslash G_k / H_k$ for $G = \mathrm{SL}(2, k)$ is given in [16]. Since the dimension of a maximal torus is one, any θ-stable maximal k-split torus is either contained in H or is (θ, k)-split. Before we characterize the (θ, k)-split tori we need some more notation. For $m \in k$ we will use $\overline{m}$ to denote the entire square class of m in $k^*/(k^*)^2$. By abuse of notation, we will use $m \in k^*/(k^*)^2$ to denote that m is the representative of the square class $\overline{m}$ of k. Recall from Theorem 4.1.1 that we may assume that $\theta = \mathrm{Inn}(A)$ with $A = \left(\begin{smallmatrix} 0 & 1 \\ m & 0 \end{smallmatrix}\right)$.

Theorem 4.2.1. *[16] Let $G = \mathrm{SL}(2, k)$, T the diagonal matrices in G, and $U = \{q \in k^*/(k^*)^2 \mid x_1^2 - m^{-1}x_2^2 = q^{-1}$ has a solution in $k\}$. Then we have the following.*

1. *The number of H_k-conjugacy classes of (θ, k)-split maximal tori is $|U/\{1, -m\}|$.*
2. *For $y \in U$, let $r, s \in k$ such that $r^2 - m^{-1}s^2 = y^{-1}$ and let $g = \left(\begin{smallmatrix} r & sym^{-1} \\ s & ry \end{smallmatrix}\right)$. Then $\{T_y = g^{-1}Tg \mid y \in U\}$ is a set of representatives of the H_k-conjugacy classes of maximal (θ, k)-split tori in G.*

As for the maximal k-split tori contained in H we have:

Proposition 4.2.2. *[1] Let G, θ be as above. Then H_k is k-anisotropic if and only if $m \notin \overline{1}$. If $m \in \overline{1}$, then H is a maximal k-split torus of G.*

It remains to determine which Weyl group elements of θ-stable maximal k-split tori have representatives in H_k. For this we have the following results.

Lemma 4.2.3. *[1] Let T_i be a (θ, k)-split maximal torus. Then,*

1. $|W_{H_k}(T_i)| = 2$ *when* $m \in \overline{1}$ *and* $-1 \in (k^*)^2$.
2. $|W_{H_k}(T_i)| = 2$ *when* $m \in \overline{-1}$ *and* $-1 \notin (k^*)^2$.
3. $|W_{H_k}(T_i)| = 1$ *otherwise.*

Finally in the case that H is a maximal k-split torus we clearly have $W_{H_k}(H) = \{\mathrm{id}\}$, thus $|W_{G_k}(H)/W_{H_k}(H)| = 2$. The above results give us a detailed description of the double cosets $P_k \backslash G_k / H_k$ for $G = \mathrm{SL}(2, k)$. We illustrate the results with the following example.

Example 4.2.4. Let $G_k = \mathrm{SL}(2, k)$ with $k = \mathbb{Q}_p$, $p \neq 2$. Then $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2 = \{1, s_p, p, ps_p\}$ where s_p is the smallest nonsquare in $\mathbb{F}_p$. Let $m \in k^*/(k^*)^2$ such that $\theta = \mathrm{Inn}(A)$ with $A = \left(\begin{smallmatrix} 0 & 1 \\ m & 0 \end{smallmatrix}\right)$. The number of H_k-conjugacy classes of (θ, k)-split maximal tori is equal to

1. 1 for $m = p, ps_p$.
2. 1 for $p \cong 1 \mod 4$ and $m = s_p$.
3. 2 for $p \cong 3 \mod 4$ and $m = s_p$.
4. 2 for $p \cong 3 \mod 4$ and $m = 1$.
5. 4 for $p \cong 1 \mod 4$ and $m = 1$.

Moreover for $m = 1$ we also have a maximal k-split torus in H. So this leads to either 2 or 6 orbits:

1. $|P_k \backslash G_k / H_k| = 2$ for $m = s_p$, $m = p$, and $m = ps_p$.
2. $|P_k \backslash G_k / H_k| = 6$ for $m = 1$.

This example illustrates that maximal (θ, k)-split tori in G do not need to be conjugate under H_k and that for the open orbit HP in G the set of k-rational points $(HP)_k$ consist of multiple k-orbits.

We have now considered the tori of $\mathrm{SL}(2, k)$ of various involutions and over various fields as well as calculating the orbits of parabolic subgroups on the symmetric k-variety G_k/H_k. Using this information, we can now consider what will happen in $\mathrm{SL}(n, k)$ in cases where $n > 2$.

Chapter 5

Building the Poset of Tori

We will now start the process of building the posets of standard tori. In order to build our standard posets, we will need to find the maximal (θ, k)-split tori that will make up the top of our poset.

5.1 Computing the maximal (θ, k)-split tori

For each of the different types of involutions over the algebraically closed field $\bar{k}$ we will compute the maximal (θ, k)-split torus A. In the following let T be the maximal k-split torus consisting of all the diagonal matrices. Since T is a nice torus to work with, we would like our representative maximal (θ, k)-split torus to be contained in T for each involution θ.

We will first need to define the following matrices. J_{2m} is the $(2m) \times (2m)$ matrix, where $n = 2m$ and

$$J_{2m} = \begin{pmatrix} 0 & I_{m\times m} \\ -I_{m\times m} & 0 \end{pmatrix}.$$

We will denote by $I_{n-i,i}$ the matrix

$$I_{n-i,i} = \begin{pmatrix} I_{n-i\times n-i} & 0 \\ 0 & -I_{i\times i} \end{pmatrix}.$$

We will denote by $\operatorname{diag} a_1, a_2, \cdots a_n$ the matrix

$$\begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}.$$

We have the following results from [2].

1. If σ is the Cartan involution, then the maximal (σ, k)-split torus is $S_1 = T_\sigma^- = \{t \in T \mid \sigma(t) = t^{-1}\} = T$.

2. If $\theta = \sigma \operatorname{Inn}_{J_{2m}}$, where σ is the Cartan involution, then let $T' = B^{-1}TB$ with $B \in \mathrm{SL}(n,k)$. We need to choose B such that $S_2 = T_\theta^- = \{B^{-1}tB \mid t \in T, \theta(B^{-1}tB) = (B^{-1}tB)^{-1}\}^\circ$ has maximal dimension. Note that

$$\begin{aligned} \theta(B^{-1}tB) = (B^{-1}tB)^{-1} &\Rightarrow \theta \operatorname{Inn}_{J_{2m}}(B^{-1}tB) = B^{-1}t^{-1}B \\ &\Rightarrow J_{2m}^{-1}(B^{-1}tB)J_{2m} = \theta(B^{-1}t^{-1}B) = B^T t (B^T)^{-1} \\ &\Rightarrow BJ_{2m}B^T t = tBJ_{2m}B^T. \end{aligned}$$

   For $B = I$, the dimension of A_2 is maximal and equal to $\frac{n}{2}$. In particular the maximal (θ, k)-split torus is:

$$A_2 = \{\operatorname{diag}(a_1, a_2, \ldots, a_n) \mid a_1 = a_2, a_3 = a_4, \ldots a_{n-1} = a_n\}.$$

3. If $\theta = \operatorname{Inn}_A$ with A one of $I_{n-i,i}$, $i = 1, 2, \ldots, \lceil \frac{n-1}{2} \rceil$, then let $T' = B^{-1}TB$ with $B \in \mathrm{SL}(n,k)$. We need to choose B such that $S_{n-i,i} = T_\sigma^- = \{B^{-1}tB \mid t \in T, \theta(B^{-1}tB) = (B^{-1}tB)^{-1}\}^0$ has maximal dimension.

For the maximal (θ, k)-split torus and their dimensions, we have

Lemma 5.1.1. *[2]*

The maximal (θ, k)-split torus for $I_{n-i,i}$, $i = 1, 2, \ldots, \lceil \frac{n-1}{2} \rceil$ can be chosen as:

$$A_{n-i,i} = \{B^{-1} \operatorname{diag}(a_1, \ldots, a_i, a_i^{-1}, \ldots, a_1^{-1}, 1, \ldots, 1)B\},$$

where B satisfies $BAB^{-1} = \begin{pmatrix} J & 0 \\ 0 & I_{n-2i} \end{pmatrix}$. The dimension of the maximal (θ, k)-split torus is of course i.

For the Cartan involution σ and $\theta = \sigma\,\mathrm{Inn}_{J_{2m}}$, we have that the maximal (θ, k)-split torus for these involutions is contained in $T = \{\mathrm{diag}\}$, the torus of diagonal matrices.

For the third category of involutions mentioned above, we will modify the involution in order to change the maximal (θ, k)-split torus into something contained within $T = \{\mathrm{diag}\}$.

Proposition 5.1.2. *[1] Let* $I^*_{n-i,i} = \begin{pmatrix} A & 0 \\ 0 & I_{(n-2i)\times(n-2i)} \end{pmatrix}$, *where*

$$A = \begin{pmatrix} 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \\ \vdots & & & & \vdots \\ 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \text{ a } 2i \times 2i \text{ matrix.}$$

Let $\theta = \mathrm{Inn}_{I_{n-i,i}}$ *and* $\tilde{\theta} = \mathrm{Inn}^*_{I_{n-i,i}}$. *Then* $\theta \approx \tilde{\theta}$. *Moreover,*

$$A_1 = \{\mathrm{diag}(a_1, \ldots, a_i, a_i^{-1}, \ldots, a_1^{-1}, 1, \ldots, 1) \mid a_i \in k^*\}$$

is a maximal $(\tilde{\theta}, k)$*-split torus.*

We will now use $\tilde{\theta} = \mathrm{Inn}^*_{I_{n-i,i}}$ in place of $\theta = \mathrm{Inn}_{I_{n-i,i}}$ since it gives a representative maximal $(\tilde{\theta}, k)$-split torus contained in $T = \{\mathrm{diag}\}$.

We will define the matrix $L_{n,x}$ as follows:

$$L_{n,x} = \begin{pmatrix} 0 & 1 & \ldots & 0 & 0 \\ x & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & x & 0 \end{pmatrix}.$$

Proposition 5.1.3. *[1] Let* $\theta = \mathrm{Inn}_{L_{n,x}}$. *Then*

$$A_1 = \{\mathrm{diag}(a_1, a_1^{-1}, \cdots, a_{n/2}, a_{n/2}^{-1}) \mid a_i \in k^*\}$$

is a maximal (θ, k)*-split torus.*

Let $M_{n,x,y,z}$ be the following matrix.

$$M_{n,x,y,z} = \begin{pmatrix} I_{n-3\times n-3} & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & z \end{pmatrix}$$

Proposition 5.1.4. *[1] Let θ be one of the following:*

1. $\theta(A) = (A^T)^{-1}$ *for all* $A \in G$
2. $\theta(A) = \mathrm{Inn}_{I_{n-i,i}}(A^T)^{-1}$ *for all* $A \in G$
3. $\theta(A) = \mathrm{Inn}_{M_{n,x,y,z}}(A^T)^{-1}$ *for all* $A \in G$

Then T is a maximal (θ, k)-split torus.

Theorem 5.1.5. *[1] Let $\theta = \mathrm{Inn}_{J_{2m}}(A^T)^{-1}$ for all $A \in G$. Then a maximal (θ, k)-split torus is*

$$A_2 = \{\mathrm{diag}(a_1, a_2, \cdots, a_n) \mid a_i \in k \text{ and } a_1 = a_{n/2+1}, a_2 = a_{n/2+2}, \dots, a_{n/2} = a_n\}.$$

5.2 H_k-conjugacy classes of (θ, k)-split maximal tori

Now that we know how to find the maximal (θ, k)-split tori for each involution of $SL(n, k)$, we can consider the H_k-conjugacy classes for each of these maximal (θ, k)-split tori. We have the following results from [1].

Theorem 5.2.1. *[1] Let $\theta = \mathrm{Inn}^*_{I_{n-i,i}}$ such that $i < n/2$. Then there exists only one H_k-conjugacy class of maximal (θ, k)-split tori with representative*

$$A_1 = \{\mathrm{diag}(a_1, \dots, a_i, a_i^{-1}, \dots, a_1^{-1}, 1, \dots, 1) \mid a_i \in k^*\}.$$

Theorem 5.2.2. *[1] Let $\theta = \mathrm{Inn}^*_{I_{n-i,i}}$ such that $i = n/2$. Note that n must be even. Then the number of H_k-conjugacy classes of maximal (θ, k)-split tori is at most $\mid \frac{k^*/(k^*)^2}{\pm 1} \mid$.*

Corollary 5.2.3. *[1] Let $\theta = \mathrm{Inn}^*_{I_{n-i,i}}$ such that $i = n/2$. Note that n must be even. Additionally, let k be $\mathbb{C}, \mathbb{R}, \mathbb{F}_p$ with $p \neq 2$, or $\mathbb{Q}_p$. Then the number of H_k-conjugacy classes of maximal (θ, k)-split tori equals $\mid \frac{k^*/(k^*)^2}{\pm 1} \mid$.*

Theorem 5.2.4. *[1] Let $\theta = \text{Inn}_{L_{n,x}}$. Note that n must be even and $x \in k^*/(k^*)^2, x \neq 1$* mod $(k^*)^2$. *Then the number of H_k-conjugacy classes of maximal (θ, k)-split tori is at most $\mid \frac{k^*/(k^*)^2}{\pm 1} \mid$.*

Theorem 5.2.5. *[1] Let $\theta = \text{Inn}_{J_{2m}}(A^T)^{-1}$ for all $A \in G$. Then there exists one H_k-conjugacy class of maximal (θ, k)-split tori with representative*

$$A_2 = \{\text{diag}(a_1, a_2, \cdots, a_n) \mid a_i \in k \text{ and } a_1 = a_{n/2+1}, a_2 = a_{n/2+2}, \ldots, a_{n/2} = a_n\}.$$

Knowing the number of H_k-conjugacy classes of maximal (θ, k)-split tori will tell us the number of tori in the top level of the poset.

5.3 Restricting θ to G_α

Now that we know what happens in the $\text{SL}(2, k)$ case, we can use that information to help us understand $\text{SL}(n, k)$. We will do this by restricting the involution to blocks of $\text{SL}(2, k)$ inside of $\text{SL}(n, k)$ and seeing how these blocks behave. We will also make use of our knowledge of the maximal (θ, k)-split tori for each involution and their conjugacy classes.

First, we will need to define the matrix G_α.

For $\alpha \in \Phi$, let $A_\alpha = \{a \in A \mid s_\alpha(a) = a\}^\circ$. Then we have the following definition.

Definition 5.3.1. $G_\alpha = Z_G(A_\alpha)$ and $\bar{G}_\alpha = [G_\alpha, G_\alpha]$.

If α is either real or imaginary, then G_α is θ-stable.

When applying this definition to $G = \text{SL}(n, k)$ for the root α_i, we have that G_{α_i} is the $n \times n$ matrix with an $\text{SL}(n, k)$ block in the α_i position along the diagonal and all other diagonal entries equal to 1. All non-diagonal entries equal 0. When considering $G_{\alpha_i + \alpha_j}$, we will again have an $\text{SL}(2, k)$ block, but it will be more broken up. We can see this more explicitly in the following example.

Example 5.3.2. $G = \text{SL}(3, k)$, $\Phi = \{\alpha_1,\ \alpha_2,\ \alpha_1 + \alpha_2\}$. Then we have

$$G_{\alpha_1} = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}, G_{\alpha_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}, \text{ and } G_{\alpha_1+\alpha_2} = \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix}.$$

Example 5.3.3. $G = \mathrm{SL}(4,k)$, $\Phi = \{\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_1+\alpha_2, \alpha_2+\alpha_3,\ \alpha_1+\alpha_2+\alpha_3\}$. Then we have $G_{\alpha_1} = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $G_{\alpha_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $G_{\alpha_3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix}$,

$G_{\alpha_1+\alpha_2} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $G_{\alpha_2+\alpha_3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix}$, and $G_{\alpha_1+\alpha_2+\alpha_3} = \begin{pmatrix} a & 0 & 0 & b \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ c & 0 & 0 & d \end{pmatrix}$.

Restricting θ to these matrices will help us determine if α is a (θ,k)-singular root. Once we determine the set of (θ,k)-singular roots, Φ_θ^+, we will be one step closer to constructing the poset of standard tori.

We will only consider roots that are in $\Phi(A) \cap \Phi(A_1)$, where A is the maximal (θ,k)-split torus and A_1 is the maximal k-split torus. In the case where $A = \{\mathrm{diag}\}$, we have that $\Phi(A) \cap \Phi(A_1)$ is the set of all positive roots.

Example 5.3.4. For

$$A = \begin{pmatrix} a & & & \\ & b & & \\ & & c & \\ & & & d \end{pmatrix}$$

we have that $\Phi(A) \cap \Phi(A_1) = \{\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_1+\alpha_2,\ \alpha_2+\alpha_3,\ \alpha_1+\alpha_2+\alpha_3\}$.

For

$$A = \begin{pmatrix} a & & & \\ & b & & \\ & & b^{-1} & \\ & & & a^{-1} \end{pmatrix}$$

we have that $\Phi(A) \cap \Phi(A_1) = \{\alpha_2,\ \alpha_1+\alpha_2+\alpha_3\}$.

From the set $\Phi(A) \cap \Phi(A_1)$, we will only be interested in the (θ,k)-singular roots.

The roots of $\Phi(A)$ can be divided in four subsets, related to the action of θ, as follows.

(a) $\theta(\alpha) \neq \pm\alpha$. Then α is called *complex* (relative to θ).

(b) $\theta(\alpha) = -\alpha$. Then α is called *real* (relative to θ).

(c) $\theta(\alpha) = \alpha$ and $\theta|G_\alpha \neq \mathrm{id}$. Then α is called *non-compact imaginary* (relative to θ).

(d) $\theta(\alpha) = \alpha$ and $\theta|G_\alpha = \mathrm{id}$. Then α is called *compact imaginary* (relative to θ).

If α is either a real or non-compact imaginary, then G_α is θ-stable and we call α a θ-singular root.

Definition 5.3.5. Let A be a θ-stable maximal k-split torus of G. For $\alpha \in \Phi(A)$ let $A_\alpha = \{a \in A \mid s_\alpha(a) = a\}^0$, $G_\alpha = Z_G(A_\alpha)$ and $\overline{G}_\alpha = [G_\alpha, G_\alpha]$. A root $\alpha \in \Phi(A)$ with $\theta(\alpha) = \pm\alpha$ is called *θ-singular* (respectively *θ-compact*) if $\overline{G}_\alpha \not\subset H$ (respectively $\overline{G}_\alpha \subset H$).

If α is a complex root, then we have that $\theta \mid_{G_\alpha} \simeq \mathrm{id}$.

The set of θ-singular roots can be divided into two subsets: those that are singular with respect to the k-structure and those that are not.

Definition 5.3.6. A real θ-singular root $\alpha \in \Phi(A)$ is called *(θ, k)-singular* (respectively *θ-singular anisotropic*) if $\overline{G}_\alpha \cap H$ is isotropic (respectively $\overline{G}_\alpha \cap H$ is anisotropic).

Similarly an imaginary θ-singular root is called *(θ, k)-singular* (respectively *θ-singular anisotropic*) if $\overline{G}_\alpha$ has a non-trivial (θ, k)-split torus (respectively $\overline{G}_\alpha$ has no non-trivial (θ, k)-split tori).

What we get from this definition is that if α is a (θ, k)-singular root, then the torus that results after flipping over the $\mathrm{SL}(2, k)$ block in the α position will be a k-split torus. Flipping over roots that are only θ-singular but not k-singular will result in a torus that does not split over k.

Corollary 5.3.7. *For $G_k = \mathrm{SL}(n, k)$, if $\theta \mid_{G_\alpha} \simeq \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, then $T_1 = \{\mathrm{diag}\}$ can not be flipped over α unless $-1 \in (k^*)^2$. For $G_k = \mathrm{SL}(n, k)$, if $\theta \mid_{G_\alpha} \simeq \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, then $T_1 = \{\mathrm{diag}\}$ can be flipped over α.*

Proof. This follows from the $\mathrm{SL}(2, k)$ case. If -1 is not a square, once we flip down the poset, we get a torus that is not k-split. If -1 is a square, then the new torus obtained by flipping is k-split. □

Theorem 5.3.8. *For $G = \mathrm{SL}(n, k)$, $\theta = (A^T)^{-1}$, if -1 is a square, then $T_1 = \{\mathrm{diag}\}$ can be flipped over each root α_i to obtain a torus with a T^+ part that is one dimension larger than T_1^+. If -1 is not a square, then T_1 cannot be flipped.*

Proof. Because $\theta\mid_{G_\alpha}\simeq \left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$, this fact follows from the previous corollary. $\square$

By examining the $\mathrm{SL}(2,k)$ blocks inside of $\mathrm{SL}(n,k)$ and comparing how they behave to what happened in the previous chapter, we can get an idea of which roots we can flip over and which roots do not allow us to move down the poset. We call the set of roots that we can flip over Φ_θ^+, the (θ,k)-singular roots.

We have that the (θ,k)-singular roots are precisely the roots α such that $\theta\mid_{G_\alpha}\simeq \mathrm{Inn}\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$.

Once we have the set of (θ,k)-singular roots Φ_θ^+, we need to find the maximal orthogonal subset of Φ_θ^+, $(\Phi_\theta^+)^\perp$. This will tell us how many levels there are in the poset of tori. For example, if there is one root in $(\Phi_\theta^+)^\perp$, then we can flip one time and there will be two levels to the poset. If there are n roots in $(\Phi_\theta^+)^\perp$, then we can flip n times and there will be $n+1$ levels to the poset.

In Figure 5.1, we can see an illustration of what is happening when we flip down the poset. Note that A is standard with respect to A_j. We wish to take a one-dimensional slice of A_j^- and flip it down to A^+. In this way, as we move down the poset, the dimension of the θ-split part of the torus will decrease by one while the dimension of the A_j^+ part will increase by one. We continue to flip down the poset until we get a torus with a maximal A_i^+ part.

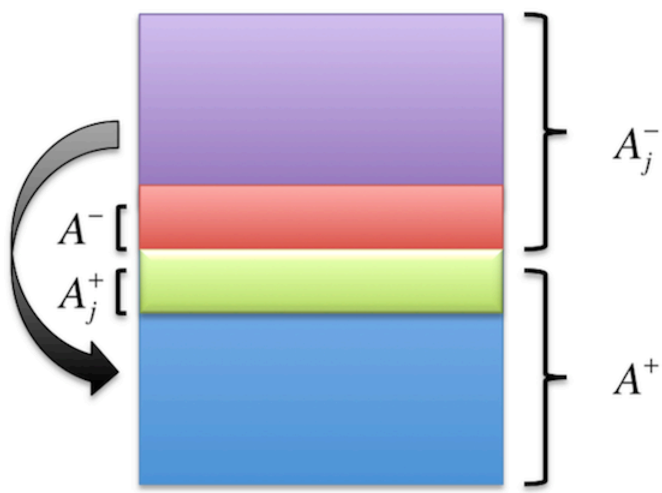

Figure 5.1: Flipping one dimension of a torus

Chapter 6

(θ, k)-singular roots of Involutions of $\mathrm{SL}(n, k)$

We will now start our classification of (θ, k)-singular roots of involutions of $\mathrm{SL}(n, k)$. Using our knowledge of the behavior of $\mathrm{SL}(2, k)$, we will be able to create posets of standard tori for each involution.

6.1 Outer Involutions

We will first consider outer involutions of $\mathrm{SL}(n, k)$ with $n > 2$. The tables for outer involutions over various fields are included in the following pages for completeness.

6.1.1 $(A^T)^{-1}$

The first involution we will consider is $\theta = (A^T)^{-1}$, which is the Cartan involution for $\mathrm{SL}(n, k)$. All other outer involutions are based around $(A^T)^{-1}$, so it is logical to start with this involution.

We will first consider the case where $k = \mathbb{C}$. Our maximal (θ, k)-split torus is the diagonal matrix T. To create our set Φ_θ^+, we will restrict θ to each root and see how the involution behaves, as explained in the previous chapter.

Example 6.1.2. Let $G = \mathrm{SL}(3, \mathbb{C})$, $\theta = (A^T)^{-1}$. If we restrict θ to G_{α_1}, we have that $\theta\mid_{G_{\alpha_1}} \simeq \mathrm{Inn}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Since we are in the field $k = \mathbb{C}$, we know that we are able to flip down the poset over α_1. Similarly, we get that we are able to flip over α_2 and $\alpha_1 + \alpha_2$.

Table 6.1: Outer Involutions of G over $k = \bar{k}$, $\mathbb{R}$ and $\mathbb{F}_p$ with $p \neq 2$.

Field	Semi-Congruence Class M	Involution Class on G
$k = \bar{k}$		
n odd	$M = I_{n\times n}$	$\theta(A) = (A^T)^{-1}$
n even	$M_1 = I_{n\times n}$ $M_2 = J_{2m}$ $(n = 2m)$	$\theta_1(A) = (A^T)^{-1}$ $\theta_2(A) = \mathrm{Inn}_{J_{2m}}(A^T)^{-1}$
$k = \mathbb{R}$		
n odd	$M_i = I_{n-i,i}$ $i = 0, 1, 2, \ldots, \frac{n-1}{2}$	$\theta_i(A) = \mathrm{Inn}_{M_i}(A^T)^{-1}$
n even	$M_i = I_{n-i,i}$ $i = 0, 1, 2, \ldots, \frac{n}{2}$ $M_{\frac{n}{2}+1} = J_{2m}$	$\theta_i(A) = \mathrm{Inn}_{M_i}(A^T)^{-1}$ $\theta_{\frac{n}{2}+1}(A) = \mathrm{Inn}_{J_{2m}}(A^T)^{-1}$
$k = \mathbb{F}_p$ $p \neq 2$		
n odd	$M_1 = I_{n\times n}$ $M_2 = \begin{pmatrix} I_{(n-1)\times(n-1)} & 0 \\ 0 & S_p \end{pmatrix}$	$\theta_1(A) = (A^T)^{-1}$ $\theta_2(A) = \mathrm{Inn}_{M_2}(A^T)^{-1}$
n even	$M_1 = I_{n\times n}$ $M_2 = \begin{pmatrix} I_{(n-1)\times(n-1)} & 0 \\ 0 & S_p \end{pmatrix}$ $M_3 = J_{2m}$	$\theta_1(A) = (A^T)^{-1}$ $\theta_2(A) = \mathrm{Inn}_{M_2}(A^T)^{-1}$ $\theta_3(A) = \mathrm{Inn}_{M_3}(A^T)^{-1}$

Table 6.2: Outer Involutions of G over $k = \mathbb{Q}_2$

$\dim(V)$	Dual Value $d_{\alpha,c}$ of the Semi-Congruence Class	Matrix M such that $\theta(A) = \mathrm{Inn}_M(A^T)^{-1}$
$n = 4k$	$d_{1,1}$	$M_1 = I_{n\times n}$
	$d_{1,-1}$	$M_{n,2,3,6}$
	$d_{-1,1}$	$M_{n,1,1,-1}$
	$d_{2,1}$	$M_{n,1,1,2}$
	$d_{-2,1}$	$M_{n,1,1,-2}$
	$d_{3,1}$	$M_{n,1,1,3}$
	$d_{-3,1}$	$M_{n,1,1,-3}$
	$d_{6,1}$	$M_{n,1,1,6}$
	$d_{-6,1}$	$M_{n,1,1,-6}$
		$M = J_{2m}$
$n = 4k+1$	$d_{1,1}$	$M = I_{n\times n}$
$n = 4k+2$	$d_{1,1}$	$M_1 = I_{n\times n}$
	$d_{-1,1}$	$M_{n,1,1,-1}$
	$d_{-1,-1}$	$M_{n,2,3,-6}$
	$d_{2,1}$	$M_{n,1,1,2}$
	$d_{-2,1}$	$M_{n,1,1,-2}$
	$d_{3,1}$	$M_{n,1,1,3}$
	$d_{-3,1}$	$M_{n,1,1,-3}$
	$d_{6,1}$	$M_{n,1,1,6}$
	$d_{-6,1}$	$M_{n,1,1,-6}$
		$M = J_{2m}$
$n = 4k+3$	$d_{1,1}$	$M = I_{n\times n}$
	$d_{1,-1}$	$M_{n,2,3,6}$

Table 6.3: Outer Involutions of $\mathrm{SL}(n, \mathbb{Q}_p)$ ($p \neq 2$)

	$\dim(V)$	Dual Value $d_{\alpha,c}$ of the Semi-Congruence Class	Matrix M s.t. $\theta(A) = \mathrm{Inn}_M(A^T)^{-1}$
$-1 \notin (\mathbb{Q}_p^*)^2$			
	$n = 4k$	$d_{1,1}$	$M_1 = I_{n\times n}$
		$d_{1,-1}$	M_{n,p,S_p,pS_p}
		$d_{p,1}$	$M_{n,1,1,p}$
		$d_{S_p,1}$	$M_{n,1,1,S_p}$
		$d_{pS_p,1}$	$M_{n,1,1,pS_p}$
			$M = J_{2m}$
	$n = 4k+1$	$d_{1,1}$	$M = I_{n\times n}$
	$n = 4k+2$	$d_{1,1}$	$M_1 = I_{n\times n}$
		$d_{p,1}$	$M_{n,1,1,p}$
		$d_{S_p,1}$	$M_{n,1,p,pS_p}$
		$d_{S_p,-1}$	$M_{n,1,1,S_p}$
		$d_{pS_p,1}$	$M_{n,1,1,pS_p}$
			$M = J_{2m}$
	$n = 4k+3$	$d_{1,1}$	$M = I_{n\times n}$
		$d_{1,-1}$	M_{n,p,S_p,pS_p}
$-1 \in (\mathbb{Q}_p^*)^2$			
	$n = 4k$	$d_{1,1}$	$M_1 = I_{n\times n}$
		$d_{1,-1}$	$M_{n,1,p,p}$
		$d_{p,1}$	$M_{n,1,1,p}$
		$d_{S_p,1}$	$M_{n,1,1,S_p}$
		$d_{pS_p,1}$	$M_{n,1,1,pS_p}$
			$M = J_{2m}$
	$n = 4k+2$	$d_{1,1}$	$M_1 = I_{n\times n}$
		$d_{1,-1}$	M_{n,p,S_p,pS_p}
		$d_{p,1}$	$M_{n,1,1,p}$
		$d_{S_p,1}$	$M_{n,1,1,S_p}$
		$d_{pS_p,1}$	$M_{n,1,1,pS_p}$
			$M = J_{2m}$
	n odd	$d_{1,1}$	$M = I_{n\times n}$
		$d_{1,-1}$	M_{n,p,S_p,pS_p}

Thus, $\Phi_\theta^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$. The largest orthogonal subset of Φ_θ^+ contains only one root, so we are only able to flip down the poset one time, giving us two levels. Thus, the standard poset for $G = \mathrm{SL}(3, \mathbb{C})$, $\theta = (A^T)^{-1}$ has two levels with $T = \{\text{diag}\}$ as the top level and three posets on the second level obtained from flipping over α_1, α_2, and $\alpha_1 + \alpha_2$, respectively.

Example 6.1.3. In Figure 6.1, we have the poset of standard tori for $G = \mathrm{SL}(4, \mathbb{C}), \theta = (A^T)^{-1}$. We are able to flip over each positive root because $-1 \in (k^*)^2$ in $\mathbb{C}$. Each subset $(\Phi_\theta^+)^\perp$ of perpendicular $(\theta, k)-$singular roots has cardinality two. Therefore, we are able to flip twice down the poset and we have three levels to the poset. The numbers in the poset correspond to the roots we are flipping over, with the poset at the top of the poset $T = \{\text{diag}\}$. For example, the 1 signifies the torus obtained by flipping over α_1, 12 signifies the torus obtained by flipping over $\alpha_1 + \alpha_2$, and so on.

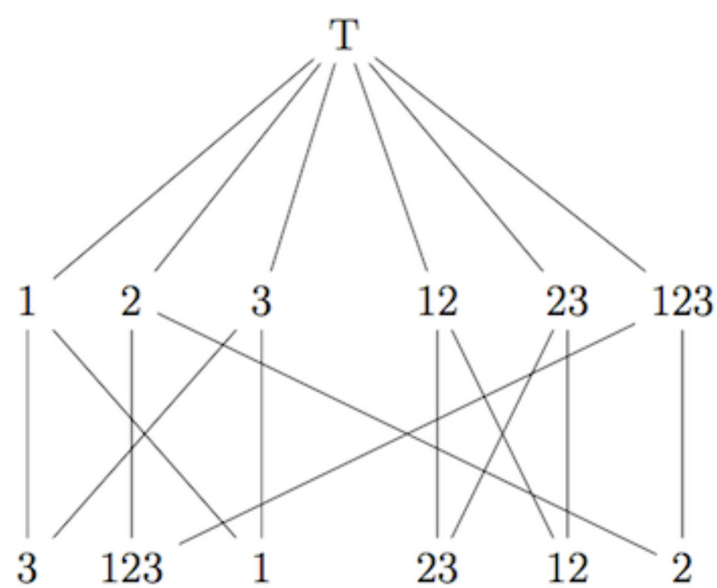

Figure 6.1: Standard Poset for $G = \mathrm{SL}(4, \mathbb{C}), \theta = (A^T)^{-1}$

As seen in the previous example, we are able to flip over every root for $G = \mathrm{SL}(n, \mathbb{C})$

and $\theta = (A^T)^{-1}$. This is because $-1 \in (k^*)^2$ when $k = \mathbb{C}$. As we saw in the chapter on $\mathrm{SL}(2,k)$, when $-1 \in (k^*)^2$ and $\theta = (A^T)^{-1}$, we are able to flip down the poset. This will hold true for any n. Thus, for $\mathrm{SL}(n,\mathbb{C})$ and $\theta = (A^T)^{-1}$, every positive root is contained within Φ_θ^+.

When we consider the field $k = \mathbb{R}$, then we have that -1 is not a square. Thus, we are unable to flip down the poset over any root, since all G_α's restricted to θ will be isomorphic to $(A^T)^{-1}$. Posets of standard tori for $\mathrm{SL}(n,\mathbb{R})$ will only contain one torus, the maximal (θ,k)-split torus $T = \{\mathrm{diag}\}$.

When considering $k = \mathbb{F}_p$, we have a similar situation to our previous two cases. If -1 is a square, then we are able to flip over all positive roots, so we would have a situation similar to $k = \mathbb{C}$. If -1 is not a square, then we are in a situation like $k = \mathbb{R}$ and we are unable to flip over any roots, leaving us with a poset of only one torus. Recall that for $\mathbb{F}_p$, if $p \equiv 1 \mod 4$, then $-1 \in (k^*)^2$ and if $p \equiv 3 \mod 4$, then $-1 \notin (k^*)^2$.

Theorem 6.1.4. *Let $G = \mathrm{SL}(n,k)$ and $\theta = (A^T)^{-1}$. Then Φ_θ^+ is the set of all positive roots if $-1 \in (k^*)^2$ and Φ_θ^+ is the empty set if $-1 \notin (k^*)^2$.*

Let us first define the matrix X_α. Let X_α be the $n \times n$ matrix with the 2×2 matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in the α position, all other diagonal entries $= 1$ and all other non-diagonal entries $= 0$.

Proof. We will consider $\theta \mid_{G_\alpha}$. We have that $(A^T)^{-1} \simeq \mathrm{Inn}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ for $\mathrm{SL}(2,k)$. If we first consider α_1, we have

$$\theta \mid_{G_{\alpha_1}} \simeq \mathrm{Inn}\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$$

Thus, $\theta \mid_{G_{\alpha_1}}$ is conjugation by the matrix X_{α_1}, where X_{α_1} is the $n \times n$ matrix with the 2×2 matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in the α_1 position, all other diagonal entries $= 1$ and all other non-diagonal entries $= 0$. From this, it is clear to see that

$$\theta \mid_{G_{\alpha_1}} \simeq \operatorname{Inn} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Applying this to all positive roots α, we see that $\theta \mid_{G_\alpha} \simeq \operatorname{Inn} X_\alpha \simeq \operatorname{Inn} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Thus, as stated in the $\mathrm{SL}(2,k)$ case, we have that α is (θ,k)-singular if $-1 \in (k^*)^2$ and α is not (θ,k)-singular if $-1 \notin (k^*)^2$. Therefore, Φ_θ^+ is the set of all positive roots if $-1 \in (k^*)^2$ and Φ_θ^+ is the empty set if $-1 \notin (k^*)^2$. □

6.1.5 $\operatorname{Inn}_{J_{2m}}(A^T)^{-1}$

We will now consider the involution $\theta(A) = \operatorname{Inn}_{J_{2m}}(A^T)^{-1}$ where J_{2m} is the $(2m)\times(2m)$ matrix, where $n = 2m$ and

$$J_{2m} = \begin{pmatrix} 0 & I_{m\times m} \\ -I_{m\times m} & 0 \end{pmatrix}.$$

We will first look at the maximal (θ,k)-split torus for this involution. In a result from [2], we have that the maximal (θ,k)-split torus A is defined as

$$A = \{\operatorname{diag}(a_1, a_2, \cdots, a_n) \mid a_1 = a_2, a_3 = a_4, \cdots a_{n-1} = a_n\}.$$

This torus has dimension $\frac{n}{2}$. If we consider $\Phi(A)$, we have that there are no roots contained within this torus. Thus $\Phi(A) \cap \Phi(T) = \emptyset$. Since there are no roots in this intersection, we do not have any roots available to restrict θ to. Thus, we are unable to flip down the poset with this involution for any field.

Theorem 6.1.6. *Let $G = \mathrm{SL}(n,k)$ and $\theta(A) = \operatorname{Inn}_{J_{2m}}(A^T)^{-1}$. Then $\Phi_\theta^+ = \emptyset$.*

Proof. Since $\Phi(A) = \emptyset$, there are no roots to consider for Φ_θ^+. Thus $\Phi_\theta^+ = \emptyset$. □

6.1.7 $\operatorname{Inn}_{M_i}(A^T)^{-1}$

We will define $M_i = I_{n-i,i}$, with

$$I_{n-i,i} = \begin{pmatrix} I_{n-i\times n-i} & 0 \\ 0 & -I_{i\times i} \end{pmatrix}.$$

We will first consider $\theta = \mathrm{Inn}_{M_i}(A^T)^{-1}$ over $\mathbb{R}$. When restricting θ to G_α, we will use the fact that $(A^T)^{-1} = \mathrm{Inn}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in the $\mathrm{SL}(2,k)$ case, as we did previously. Let us start with an example.

Example 6.1.8. Let $G = \mathrm{SL}(3,k)$, $\theta = \mathrm{Inn}_{M_1}(A^T)^{-1}$. Then $A = T = \{\mathrm{diag}\}$ and $\Phi(T)^+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2\}$. First we will consider α_1 as a possible (θ,k)-singular root.

$$\theta\mid_{G_{\alpha_1}} = \mathrm{Inn}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathrm{Inn}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore, $\alpha_1 \notin \Phi_\theta^+$. We can do a similar calculation for the remaining positive roots to see that $\Phi_\theta^+ = \{\alpha_2, \alpha_1+\alpha_2\}$. Since conjugating by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ will only affect the last row and last column, it will only affect the root α_2 and any roots that contain α_2 in its sum.

The roots α_2 and $\alpha_1+\alpha_2$ are not orthogonal, so there will only be two levels to the standard poset, the top containing T and the bottom containing the two tori obtained by flipping over α_2 and $\alpha_1+\alpha_2$.

We can extend this process to larger n to see that for $\theta = \mathrm{Inn}_{M_i}(A^T)^{-1}$, the (θ,k)-singular roots will be α_{n-i} and any roots containing α_{n-i} in its sum.

Theorem 6.1.9. *Let $G = \mathrm{SL}(n,\mathbb{R})$ and $\theta = \mathrm{Inn}_{M_i}(A^T)^{-1}$. Then $\Phi_\theta^+ = \{\alpha_{n-i}$, roots containing α_{n-i} in their sum $\}$.*

Proof. Let us first note that $\Phi(A) = \Phi$, all positive roots. We will first consider $\theta =$

$\mathrm{Inn}_{M_1}(A^T)^{-1}$. Then

$$\theta\,|_{G_{\alpha_1}} = \mathrm{Inn}\begin{pmatrix}1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & -1\end{pmatrix}\begin{pmatrix}0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1\end{pmatrix} = \mathrm{Inn}\begin{pmatrix}0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & -1\end{pmatrix}.$$

Thus, we have that $\theta\,|_{G_{\alpha_1}} \simeq \mathrm{Inn}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$. It is easy to see that $\theta\,|_{G_\alpha} \simeq \mathrm{Inn}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ for any root α that is not in the lower right position. Let us consider α_{n-1}.

$$\theta\,|_{G_{\alpha_{n-1}}} = \mathrm{Inn}\begin{pmatrix}1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & -1\end{pmatrix}\begin{pmatrix}1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0\end{pmatrix}$$

$$= \mathrm{Inn}\begin{pmatrix}1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & 1 & 0\end{pmatrix} \simeq \mathrm{Inn}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

Because the α_{n-1} block is in a position that is affected by the -1 in the n, n entry, we get that $\theta\,|_{G_{\alpha_{n-1}}} \simeq \mathrm{Inn}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$. Thus, we have that α_{n-1} is a $(\theta, k)-$singular root.

Any other root that has a component in the lower right position will also be affected. This will include α_{n-1} and all roots that contain α_{n-1} in their sum. Thus, the set of $(\theta, k)-$singular roots will be α_{n-1} and all roots that contain α_{n-1} in their sum.

We can easily extend this to $\theta = \mathrm{Inn}_{M_i}(A^T)^{-1}$. The root α that is in the position affected by the $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ block will be a $(\theta, k)-$singular root, as well as all roots containing α

in their sum. $\square$

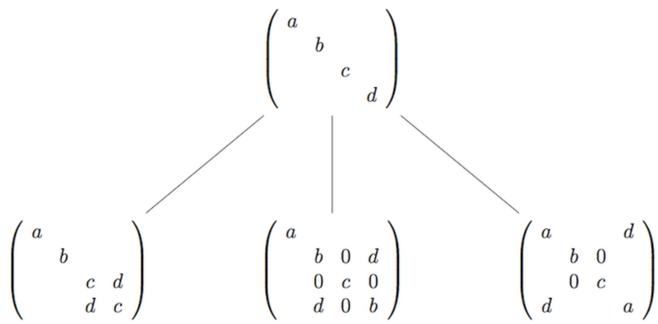

Figure 6.2: Poset of standard tori related to $G = \mathrm{SL}(4, \mathbb{R})$ with $\theta(A) = \mathrm{Inn}_{M_1}(A^T)^{-1}$

Example 6.1.10. For $\mathrm{SL}(5, \mathbb{R}), \mathrm{Inn}(M_1)(A^T)^{-1}$, we have $\Phi_s^\theta = \{\alpha_4, \alpha_3 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\}$.

Example 6.1.11. For $\mathrm{SL}(5, \mathbb{R}), \mathrm{Inn}(M_2)(A^T)^{-1}$, we have $\Phi_s^\theta = \{\alpha_3, \alpha_3 + \alpha_4, \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\}$.

Let us now consider $k = \mathbb{F}_p$. For $k = \mathbb{F}_p$, we will look at two cases. First, we redefine $M_i = I_{n-i,i}$, as

$$I_{n-i,i} = \begin{pmatrix} I_{n-i \times n-i} & 0 \\ 0 & s_p I_{i \times i} \end{pmatrix}$$

where s_p is the smallest nonsquare in $\mathbb{F}_p$. If -1 is not a square, then we can let $s_p = -1$. This gives us a similar case to $k = \mathbb{R}$ and have $\Phi_\theta^+ = \{\alpha_{n-i},$ roots containing α_{n-i} in their sum $\}$.

If -1 is a square, then we have to consider what happens with the s_p.

Example 6.1.12. Let $G = \mathrm{SL}(4, F_p)$, $\theta = \mathrm{Inn}_{M_1}(A^T)^{-1}$. Then we have

$$\theta\mid_{G_{\alpha_1}} = \text{Inn}\left(\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&s_p\end{pmatrix}\begin{pmatrix}0&1&0&0\\s_p&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right) = \begin{pmatrix}0&1&0&0\\s_p&0&0&0\\0&0&1&0\\0&0&0&s_p\end{pmatrix} \simeq \text{Inn}\begin{pmatrix}0&1\\s_p&0\end{pmatrix}$$

Thus, we are unable to flip over the positive root α_1 because s_p is not a square.

If we consider the root α_3, we get

$$\theta\mid_{G_{\alpha_3}} = \text{Inn}\left(\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&s_p\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&s_p&0\end{pmatrix}\right) = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&(s_p)^2&0\end{pmatrix} \simeq \text{Inn}\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

because $(s_p)^2$ is a square. Thus, we are able to flip over the root α_3.

Through similar calculations, we will see that the (θ, k)-singular roots are $\alpha_3, \alpha_2 + \alpha_3$, and $\alpha_1 + \alpha_2 + \alpha_3$.

From the previous example, we see that we get similar results for $\mathbb{F}_{\text{\tiny |}}$ when -1 is not a square as we get for $\mathbb{R}$.

Theorem 6.1.13. *Let $G = \text{SL}(n, \mathbb{F}_p)$ and $\theta = \text{Inn}_{M_i}(A^T)^{-1}$. Then $\Phi^+_\theta = \{\alpha_{n-i}$, roots containing α_{n-i} in their sum $\}$.*

Proof. The proof is similar to the $\mathbb{R}$ case. $\square$

6.2 Inner Involutions

We will now classify (θ, k)-singular roots for inner involutions of $\text{SL}(n, k)$. A table containing the list of inner involutions for $\text{SL}(n, k)$ over various fields is shown in Figure 6.4. There are two inner involutions to consider.

6.2.1 $\text{Inn}(I_{n-i,i})$

When considering $\theta = \text{Inn}(I_{n-i,i})$, we first need to find the maximal (θ, k)-split torus.

Table 6.4: Isomorphism Classes of Inner Involution of G

Field	Number of Inner Involutions	Representative Matrix Y such that $\theta = \mathrm{Inn}_Y$
n odd, $k =$ any field	$\frac{n-1}{2}$	$Y = I_{n-i,i} \quad i = 1, 2, \ldots, \frac{n-1}{2}$
n even		
$k = \bar{k}$	$\frac{n}{2}$	$Y = I_{n-i,i} \quad i = 1, 2, \ldots, \frac{n}{2}$
$k = \mathbb{R}$	$\frac{n}{2} + 1$	$Y = I_{n-i,i} \quad i = 1, 2, \ldots, \frac{n}{2}$ $Y = L_{n,-1}$
$k = \mathbb{Q}$	∞	$Y = I_{n-i,i} \quad i = 1, 2, \ldots, \frac{n}{2}$ $Y = L_{n,\alpha} \quad \alpha \neq 1 \mod (\mathbb{Q}^*)^2$
$k = \mathbb{F}_p \;\; p \neq 2$	$\frac{n}{2} + 1$	$Y = I_{n-i,i} \quad i = 1, 2, \ldots, \frac{n}{2}$ $Y = L_{n,S_p}$
$k = \mathbb{Q}_2$	$\frac{n}{2} + 7$	$Y = I_{n-i,i} \quad i = 1, 2, \ldots, \frac{n}{2}$ $Y = L_{n,\alpha} \quad \alpha \in \{-1, {}^{\pm}2, {}^{\pm}3, {}^{\pm}6\}$
$k = \mathbb{Q}_p \;\; p \neq 2$	$\frac{n}{2} + 3$	$Y = I_{n-i,i} \quad i = 1, 2, \ldots, \frac{n}{2}$ $Y = L_{n,\alpha} \quad \alpha \in \{p, S_p, pS_p\}$

Lemma 6.2.2. *[[2]] The maximal (θ, k)-split torus for $I_{n-i,i}$, $i = 1, 2, \ldots, \lceil \frac{n-1}{2} \rceil$ can be chosen as:*

$$A_{n-i,i} = \{B^{-1} \operatorname{diag}(a_1, \ldots, a_i, a_i^{-1}, \ldots, a_1^{-1}, 1, \ldots, 1)B\},$$

where B satisfies $BAB^{-1} = \begin{pmatrix} J & 0 \\ 0 & I_{n-2i} \end{pmatrix}$. The dimension of the maximal (θ, k)-split torus is of course i.

Proof. Note that

$$\begin{aligned} \theta(B^{-1}tB) = (B^{-1}tB)^{-1} &\Rightarrow \mathrm{Inn}_A(B^{-1}tB) = B^{-1}t^{-1}B \\ &\Rightarrow A^{-1}(B^{-1}tB)A = B^{-1}t^{-1}B \\ &\Rightarrow tBAB^{-1} = BAB^{-1}t^{-1}. \end{aligned}$$

Since t is conjugate to t^{-1}, then highest possible dimension can only be less or equal to $\frac{n}{2}$. Also, if $t = \operatorname{diag}(a_1, \ldots, a_i, a_i^{-1}, \ldots, a_1^{-1}, 1, \ldots, 1)$, and $tY = Yt^{-1}$, then we have

$Y = \begin{pmatrix} J & 0 \\ 0 & Y_{n-2i} \end{pmatrix}$, therefore, if the (θ, k)-split tori has dimension of i, the corresponding $I_{n-j,j}$ has to be conjugate to $\begin{pmatrix} J & 0 \\ 0 & Y_{n-2i} \end{pmatrix}$. Hence a (θ, k)-split torus has dimension i if and only if the corresponding $I_{n-j,j}$ such that $j \geq i$, i.e. the maximal (θ, k)-split tori has dimension j for $I_{n-j,j}$. □

We will consider the maximal (θ, k)-split torus $\tilde{A}_{n-i,i}$ as defined below:

$$\tilde{A}_{n-i,i} = \mathrm{diag}(a_1, \ldots, a_i, a_i^{-1}, \ldots, a_1^{-1}, 1, \ldots, 1)$$

Then our new involution, $\tilde{\theta}$ becomes Inn $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & \iddots & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, i.e. the matrix with 1's on the reverse diagonal and all other entries equaling zero. This will make our calculations simpler.

When considering the maximal (θ, k)-split torus $\tilde{A}_{n-i,i}$, we have that $\Phi(\tilde{A}_{n-i,i}) = \{\alpha_i, \alpha_{i-1} + \alpha_i + \alpha_{i+1}, \cdots \alpha_1 + \alpha_2 + \ldots + \alpha_i + \ldots \alpha_{n-1}\}$. Essentially, $\Phi(\tilde{A}_{n-i,i})$ contains the positive root α_i and all roots with α_i in the center of the sum.

When we restrict G_α with $\alpha \in \Phi(\tilde{A}_{n-i,i})$ to $\tilde{\theta}$, we see that all roots in $\Phi(\tilde{A}_{n-i,i})$ are (θ, k)-singular. We get the following result

$$\tilde{\theta} \mid_{G_\alpha} \simeq \mathrm{Inn} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

for all $\alpha \in \Phi(\tilde{A}_{n-i,i})$. Thus we are able to flip over all positive roots in $\Phi(\tilde{A}_{n-i,i})$.

Example 6.2.3. Let $G = \mathrm{SL}(4, k)$ and $\theta = \mathrm{Inn}(I_{2,2})$. We will consider

$$\tilde{\theta} = \mathrm{Inn} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Our maximal (θ, k)-split torus is

$$\tilde{A}_{2,2} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b^{-1} & 0 \\ 0 & 0 & 0 & a^{-1} \end{pmatrix}$$

Thus, the only roots in $\Phi(\tilde{A}_{2,2})$ are α_2 and $\alpha_1 + \alpha_2 + \alpha_3$.

For G_{α_2}, we have

$$\tilde{\theta}(G_{\alpha_2}) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & d & c & 0 \\ 0 & b & a & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \simeq \mathrm{Inn} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Thus we are able to flip over the $\mathrm{SL}(2,k)$ block in the α_2 position. We get similar results when we consider $\alpha_1 + \alpha_2 + \alpha_3$:

$$\tilde{\theta}(G_{\alpha_1+\alpha_2+\alpha_3}) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 & 0 & b \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ c & 0 & 0 & d \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} d & 0 & 0 & c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ d & 0 & 0 & a \end{pmatrix} \simeq \mathrm{Inn} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

From this, we see that we are able to flip over both of the roots in $\Phi(\tilde{A_{2,2}})$. Since the two roots α_2 and $\alpha_1 + \alpha_2 + \alpha_3$ are perpendicular to each other, we are able to flip down the poset twice, obtaining 3 levels to the poset of standard tori.

An illustration of the poset from the previous example can be seen in Figure 6.3 .

Theorem 6.2.4. *Let* $G = \mathrm{SL}(n,k)$ *and* $\tilde{\theta} = \mathrm{Inn}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & \iddots & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$. *Then* $\Phi_{\theta}^{+} = \Phi(\tilde{A}_{n-i,i})$.

Proof. We have that $\Phi(\tilde{A}_{n-i,i})$ is the set of positive roots α_i and all positive roots containing α_i at the center of its sum. This is easy to compute since we have that

$$\tilde{A}_{n-i,i} = \begin{pmatrix} a & & & & \\ & b & & & \\ & & \ddots & & \\ & & & b^{-1} & \\ & & & & a^{-1} \end{pmatrix}.$$

The $\mathrm{SL}(2,k)$ block in the α position, with $\alpha \in \Phi(\tilde{A}_{n-i,i})$, will be a square block that shares a center with the center of the larger $n \times n$ matrix. When we restrict $\tilde{\theta}$ to G_α with $\alpha \in \Phi(\tilde{A}_{n-i,i})$, we have

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & \iddots & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} G_\alpha \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & \iddots & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The matrix G_α will have an $\mathrm{SL}(2,k)$ block in a central position, as stated previously. Thus

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & \iddots & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} G_\alpha \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & \iddots & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \simeq \mathrm{Inn}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

$\square$

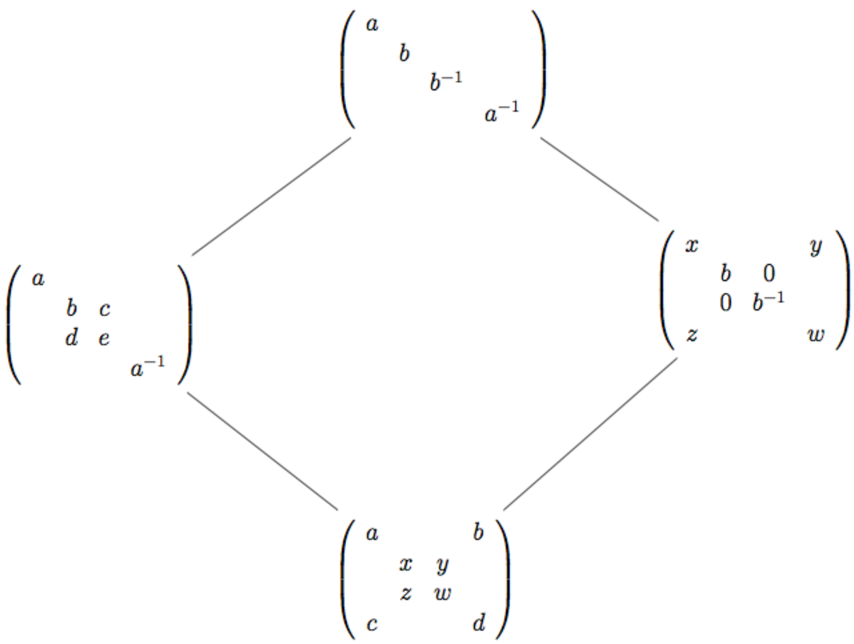

Figure 6.3: Poset of standard tori for $\mathrm{SL}(4, k)$ and $\theta = \mathrm{Inn}(I_{2,2})$

6.2.5 $\mathrm{Inn}(L_{n,-1})$

We will define the matrix $L_{n,x}$ as follows:

$$L_{n,x} = \begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ x & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & x & 0 \end{pmatrix}.$$

For $x = -1$, we have that $\mathrm{Inn}(J_{2m}) \simeq \mathrm{Inn}(L_{n,-1})$. (Similarly, if $x = s_p$, we have that $\mathrm{Inn}(J_{2m}) \simeq \mathrm{Inn}(L_{n,s_p})$. If we consider the case $n = 2$, we have that $n = 2m$ and $m = 1$. Thus we have

$$J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$L_{2,-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

If we diagonalize J_2, we get eigenvalues $\lambda = \pm i$. Thus $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \simeq \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. When we consider $\mathrm{Inn}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, this is equivalent to $\mathrm{Inn}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ because the scalar i will cancel out.

From the previous calculations, we have that $\mathrm{Inn}(L_{2,-1}) \simeq \mathrm{Inn}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \simeq \mathrm{Inn}_{I_{1,1}}$. We can expand this from $n = 2$ to a general n and obtain

$$\mathrm{Inn}_{L_{n,-1}} = \mathrm{Inn}_{J_{2m}} \simeq \mathrm{Inn}_{I_{n-i,i}}$$

We have previously done the calculations for $\mathrm{Inn}_{I_{n-i,i}}$. We get the same results for $\mathrm{Inn}_{L_{n,-1}}$. We will again be able to flip over all positive roots in $\Phi(\tilde{A}_{n-i,i})$.

Theorem 6.2.6. *Let $G = \mathrm{SL}(n, k)$ and $\theta = \mathrm{Inn}(L_{n,-1})$. Then $\Phi^+_\theta = \Phi(\tilde{A}_{n-i,i})$.*

Proof. This follows from the fact that $\mathrm{Inn}_{L_{n,-1}} = \mathrm{Inn}_{J_{2m}} \simeq \mathrm{Inn}_{I_{n-i,i}}$. $\square$

Chapter 7

Conjugacy Classes of Maximal θ-stable, k-split Tori

Now that we have a method for buildling the posets of standard tori for each of the involutions of $\mathrm{SL}(n, k)$ over various fields, we can now check for conjugacy among the rows of the poset. We can check for H-conjugacy first to see how the poset collapses and then we can check for H_k-conjugacy to see how the collapsed poset expands.

7.1 Quasi k-split tori

Given a maximal (θ, k)-split torus A, we have a maximal k-split torus A_1 such that $A \subset A_1$. We also have T, a maximal torus, with $A_1 \subset T$. We have been considering positive roots $\alpha \in \Phi(A) \cap \Phi(A_1)$ and determining if α is a (θ, k)-singular root. If α is a (θ, k)-singular root, we are able to flip down the poset over α and obtain a new torus with a $+$-part that is one dimension larger. If α is not a (θ, k)-singular root, we are unable to flip down the poset over α.

If we consider the roots $\alpha \in \Phi(A) \cap \Phi(A_1)$ which are not (θ, k)-singular, we are able to flip over the algebraic closure of k. The resulting torus will not be k-split. In fact, it will contain a one-dimensional anisotropic subtorus. The process of flipping over a non-(θ, k)-singular root will result in a quasi k-split torus.

Definition 7.1.1. A quasi k-split torus is a torus that is G-conjugate to a k-split torus, i.e. A is quasi k-split if

$$A = gTg^{-1}$$

$$
\begin{tikzcd}[column sep=small]
 & \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} \arrow[dl, dash] \arrow[dr, dash] & \\
\begin{pmatrix} x & & \\ & a & b \\ & c & d \end{pmatrix} & & \begin{pmatrix} a & & b \\ & x & \\ c & & d \end{pmatrix}
\end{tikzcd}
$$

Figure 7.1: Poset of standard tori for $\mathrm{SL}(3, \mathbb{R})$

with $g \in G$ and T a k-split torus.

Quasi k-split tori split over G but not over G_k with $k \neq \bar{k}$. If we consider quasi k-split tori, we will be able to flip down the poset more than if we only consider k-split tori, obtaining more tori for our poset.

Example 7.1.2. Let $G = \mathrm{SL}(3, \mathbb{R})$, $\theta = \mathrm{Inn}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}(A^T)^{-1}$. We have that $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$. From previous calculations, we know that the set of $(\theta, k)-$singular roots are $\Phi^+_\theta = \{\alpha_2, \alpha_1 + \alpha_2\}$. Thus we get the poset as seen in Figure 7.1.

If we want to create a poset of quasi k-split tori, then we are able to flip over all three roots in Φ^+. Thus we get the poset as seen in Figure 7.2. As we can see from these two posets, we are able to get more tori when allowing quasi k-split tori.

7.2 H_k-Conjugacy

We already discused the H_k conjugacy classes of maximal (θ, k)-split tori, which will make up the top of the poset. For tori A_1 and A_2 with A_i^- maximal (θ, k)-split, there exists $g \in (H.Z_G(A_1))_k$ such that

$$gA_1g^{-1} = A_2$$

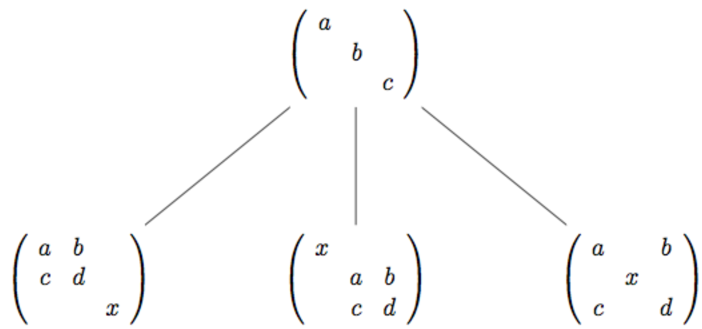

Figure 7.2: Poset of quasi k-split tori for $\mathrm{SL}(3,\mathbb{R})$

.

Corollary 7.2.1. *Let $g = h.z$ such that $gA_1g^{-1} = A_2$. Then $hA_1h^{-1} = A_2$.*

Proof. Since $z \in Z_G(A_1)$, we have

$$gA_1g^{-1} = h.zA_1z^{-1}h^{-1} = hA_1h^{-1} = A_2$$

. $\square$

Thus, all maximal (θ, k)-split tori are H-conjugate, which gives us a single element at the top of our poset of H-conjugacy classes of tori.

Given the above, we now have the following result when considering the bottom level of the H-conjugate poset:

Theorem 7.2.2. *Let A_1 and A_2 be θ-stable, k-split tori with A_i^+ maximal. Then A_1 and A_2 are H-conjugate, i.e. the bottom of the poset of standard tori collapses to one H-conjugacy class.*

Proof. A_1^+ and A_2^+ are maximal k-split in H. Then $\exists\, h \in H_k$ such that

$$hA_1^+h^{-1} = A_2^+$$

because all maximal k-split tori are H-conjugate. In $Z_G(A_1^+)$, both A_1^- and A_2^- are maximal (θ, k)-split. Thus A_1^- and A_2^- are conjugate under $H \cap Z_G(A_1^+)$, which implies A_1 and A_2 are H-conjugate. $\square$

Our process of checking for conjugacy among each row of the poset is shown graphically in Figure 7.3. If we wish to check for conjugacy among T_{α_1} and T_{α_2}, we will look at the matrix $(c_1)^{-1}hc_2$. If this matrix is an element of H, then T_{α_1} and T_{α_2} is H-conjugate. If $(c_1)^{-1}hc_2$is an element of H_k, then T_{α_1} and T_{α_2} is H_k-conjugate. We use the following theorem to help us identify c_i.

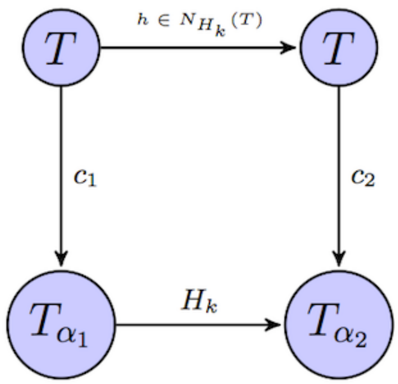

Figure 7.3: Graph demonstrating how to check for conjugacy

Theorem 7.2.3. *For* $\theta = \mathrm{Inn}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, *we have*

$$\begin{pmatrix} a-b & 0 \\ 0 & a+b \end{pmatrix} = \mathrm{Inn}\begin{pmatrix} x & -x \\ x & x \end{pmatrix}\begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

with $a^2 - b^2 = 1$ *and* $2x^2 = 1$ *and* $a, b, x \in k$.

Proof. Since $2x^2 = 1$, we have the following:

$$\mathrm{Inn}\begin{pmatrix} x & -x \\ x & x \end{pmatrix}\begin{pmatrix} a & b \\ b & a \end{pmatrix} = \begin{pmatrix} x & -x \\ x & x \end{pmatrix}\begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} x & x \\ -x & x \end{pmatrix}$$

$$= \begin{pmatrix} (ax^2 - bx^2 - bx^2 + ax^2) & (ax^2 - bx^2 + bx^2 - ax^2) \\ (ax^2 + bx^2 - bx^2 - ax^2) & (ax^2 + bx^2 + bx^2 + ax^2) \end{pmatrix}$$
$$= \begin{pmatrix} a - b & 0 \\ 0 & a + b \end{pmatrix}$$

□

We can choose the matrix $\begin{pmatrix} x & -x \\ x & x \end{pmatrix}$ depending on our field.

For example, if $k = \mathbb{R}$, we choose the matrix $\frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

Theorem 7.2.4 ([19])**.** *If A is standard with respect to A_j, then there exists an involution $\omega \in W(A_j)$ that is (θ, k)-singular such that*

$$(A_j)^-_\omega \centerdot A^- = A^-_j$$

Moreover, we can move from A to A_j in r-steps if

$$\omega = s_{\alpha_1} s_{\alpha_2} ... s_{\alpha_r}$$

$$\alpha_1 \perp \alpha_2 \perp ... \perp \alpha_r$$

where α_i are perpendicular (θ, k)-singular roots.

The previous theorem gives us an explicit method for checking for conjugacy.

Example 7.2.5. Let $G = \mathrm{SL}(3, \mathbb{F}_3), \theta = \mathrm{Inn}\begin{pmatrix} 1 & & \\ & 1 & \\ & & 2 \end{pmatrix}(A^T)^{-1}$. Then we have that $\Phi^+_\theta = \{\alpha_2, \alpha_1 + \alpha_2\}$. We get the poset of standard tori as seen in Figure 7.4.

Now we need to check for conjugacy among the two tori on the bottom level of the poset. We will consider $\mathrm{Inn}(c_2 N_{\alpha_1} c_1^{-1})$. From Theorem 7.2.3, we will choose the matrix c_1 and c_2 as follows.

$$c_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}, c_2 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

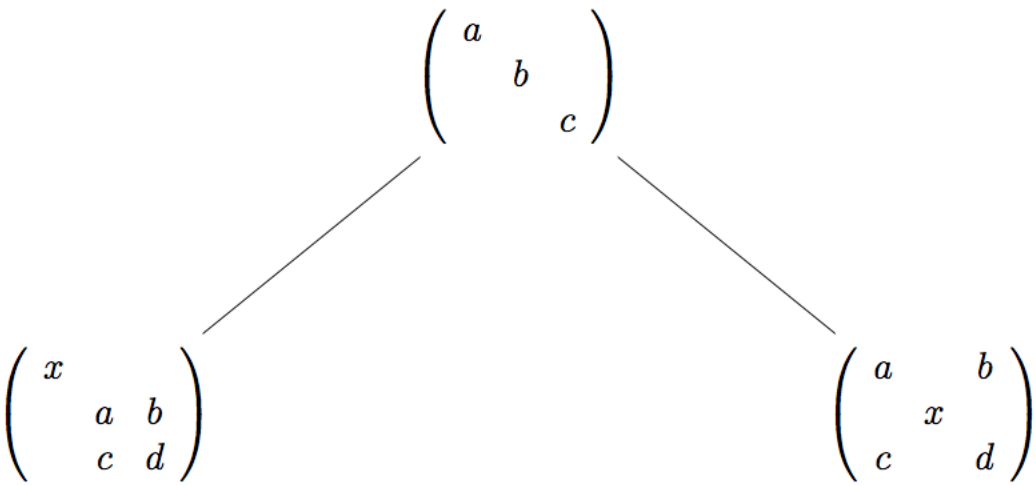

Figure 7.4: Poset of Standard tori for $\mathrm{SL}(3, \mathbb{F}_3), \theta = \mathrm{Inn}_{M_1}(A^T)^{-1}$

The matrix c_1 takes the diagonal matrix T to the matrix $\begin{pmatrix} x & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}$ and c_2 takes T to the matrix $\begin{pmatrix} a & 0 & b \\ 0 & x & 0 \\ c & 0 & d \end{pmatrix}$.

We find N_{α_1} by first noting that $s_{\alpha_1}(\alpha_2) = \alpha_1 + \alpha_2$. Then we have the matrix N_{α_1} as follows

$$N_{\alpha_1} = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

From this, we have that

$$\mathrm{Inn}(c_2 N_{\alpha_1} c_1^{-1}) = \mathrm{Inn} \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Through simple calculations, we can see that the matrix $\begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is in H_k (and thus in H). Thus the bottom of the poset collapses to a single torus and we get two conjugacy classes for $G = \mathrm{SL}(3, \mathbb{F}_3), \theta = \mathrm{Inn}\begin{pmatrix} 1 & & \\ & 1 & \\ & & 2 \end{pmatrix}(A^T)^{-1}$. This can be seen in Figure 7.5.

Figure 7.5: Poset of H_k-conjugacy classes of θ-stable, maximal k-split tori for $\mathrm{SL}(3, \mathbb{F}_3), \theta = \mathrm{Inn}_{M_1}(A^T)^{-1}$

Example 7.2.6. Let $G = \mathrm{SL}(4, \mathbb{C})$, $\theta(X) = (X^T)^{-1}$. We have $A = \{\mathrm{diag}\}$. Then $\Phi_\theta^+ = \Phi(A)$, all positive roots. We get the poset of standard tori seen in Figure 7.6. When we check for H_k-conjugacy among the rows of the poset, through similar calculations from the previous example, this poset collapses down to the poset seen in Figure 7.7. Thus we have three H_k-conjugacy classes of θ-stable, maximal k-split tori.

It is not always the case that the poset of standard tori will collapse completely. Sometimes the poset of standard tori will collapse and sometimes it will not. Once we have checked for conjugacy, we are able to compute the Weyl group quotients $W_{G_k}(A_i)/W_{H_k}(A_i)$ in order to classify the double cosets $P_k \setminus G_k/H_k$.

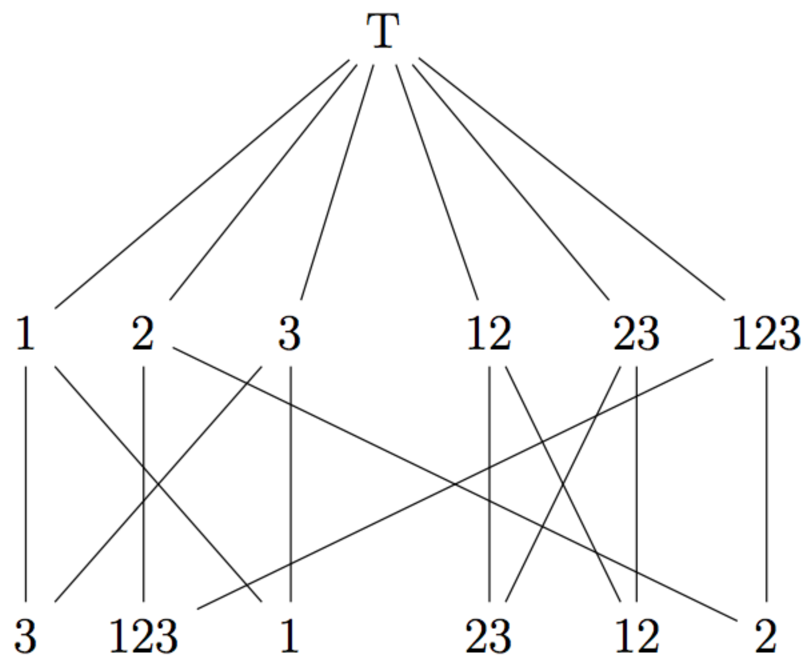

Figure 7.6: Poset of standard θ-stable, maximal k-split tori for $\mathrm{SL}(4,\mathbb{C}), \theta(X) = (X^T)^{-1}$

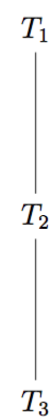

Figure 7.7: Poset of conjugacy classes of θ-stable, maximal k-split tori for $\mathrm{SL}(4,\mathbb{C}), \theta(X) = (X^T)^{-1}$

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