ABSTRACT

ROBB, JAMES LAWRENCE IV. Design and Simulation of an Active Load Balancing System for High-Speed, Magnetically Supported Rotors. (Under the direction of Dr. Gregory D. Buckner.)

Active magnetic bearings (AMBs) are being increasingly employed in the development of oil-free turbo machinery. One disadvantage of AMB systems, particularly AMB thrust bearings, is their limited dynamic load capacity relative to fluid film bearings. For centrifugal compressors, the most significant transient axial loads are associated with compressor surge, dictating that some of the AMB’s load capacity be preserved to handle dynamic loads in this operating region. For other regions of operation, however, the AMB’s load capacity may not be fully utilized, compromising compressor efficiency. One common solution to this problem involves the use of static balance pistons to keep thrust loads sufficiently small. Static balance pistons, however, employ seals that leak process gas flow and reduce machine performance. For these reasons, an active thrust load management system is sought.

The active thrust balancing design proposed in this thesis seeks to improve the performance of AMB-supported turbo machines by maximizing load capacity and minimizing leakage across the machine’s operating space. This design specifically targets high pressure ratio, single-overhung compressor systems that use magnetic thrust bearings. Detailed modeling and simulations are utilized to illustrate the limitations of magnetic thrust bearings and to discuss the pertinent design issues and benefits of regulating thrust loads. The modeling process addresses realistic dynamic effects such as amplifier saturation, magnetic flux saturation, and eddy currents. Simulation results are used to design an active thrust balancing system, and axial force and leakage flow characteristics of this active device are compared to a stationary design. The proposed active design is shown to offer average leakage reductions of 9.0% to 26.4% relative to static balancing devices. Finally, an observer-based controller is designed, and a gain-scheduling methodology is proposed to cover the compressor’s full operating map.
Design and Simulation of an Active Load Balancing System for High-Speed, Magnetically Supported Rotors

by
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A thesis submitted to the Graduate Faculty of North Carolina State University in partial fulfillment of the requirements for the Degree of Master of Science in Mechanical Engineering, Raleigh, North Carolina 02-05-08

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Chair of Advisory Committee
DEDICATION

I would like to dedicate this work to my family who have supported and encouraged me in all my academic and professional endeavors.
BIOGRAPHY

James Lawrence Robb IV is a Master of Science candidate in the Department of Mechanical and Aerospace Engineering at North Carolina State University. He has focused his academic study in the areas of dynamic systems and control. Mr. Robb is also a practicing engineer, with 6 years experience in the field of centrifugal compressor design and development. He has spent 4 years working in a research and development role with active magnetic bearing supported direct drive turbo compressors. Prior to this, Mr. Robb spent 8 years in the electronic connector industry. Mr. Robb holds a Bachelor of Science in Mechanical Engineering (Summa Cum Laude) from North Carolina State University (1992) and Master of Business Administration from Wake Forest University (2000).
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LIST OF SYMBOLS

\[ A_i = [1562, 885]_{i=1,2} \text{; AMB pole area (i), } \text{mm}^2 \]
\[ A_m = \text{Mean seal annular area, in}^2 \text{ (mm}^2) \]
\[ B = \text{flux density, Tesla} \]
\[ c_{max} = 0.254 \text{ (0.010); maximum radial clearance of seal, mm (inch)} \]
\[ c_{min} = 0.127 \text{ (0.005); minimum radial clearance of seal, mm (inch)} \]
\[ C = 2.4961 \times 10^{-5}; \text{ coefficient of AMB actuator, } \text{Nm}^2/\text{A}^2 \]
\[ C_i = 3.4954 \times 10^{-6} \text{ (5.3131} \times 10^{-5}); \text{ capacitance of chamber volume (i=1,2), } \text{(kg/kPa) lbm/psi} \]
\[ C_{vp} = 1.0; \text{ 100% flow coefficient of control valve, cfm/psi} \]
\[ C_{bp}(s) = \text{closed loop transfer function of balance piston servo system} \]
\[ d_2 = 6.5 \text{ (165); Impeller diameter, inch (mm)} \]
\[ d_{bp\_stat} = 4.20 \text{ (106.68); stationary balancing seal diameter, static balancing system, inch (mm)} \]
\[ d_{bp\_dyn} = 2.20 \text{ (55.88); stationary balancing seal diameter, active balancing system, inch (mm)} \]
\[ d_b = 4.724 \text{ (120); balance piston actuator disk diameter, inch (mm)} \]
\[ d_s = 2.0 \text{ (50.8); shaft diameter, inch (mm)} \]
\[ F = \text{Force, N} \]
\[ f(x) = \text{valve trim function} \]
\[ f' = \text{closed loop pole placement factor} \]
\[ g_0 = 0.5; \text{ air gap, mm} \]
\[ G_r = \text{seal leakage mass flow, kg/min (lbm/min)} \]
\[ G_{ac}(s) = \text{AMB current amplifier transfer function} \]
\[ G_{ac}(s) = \text{actuator transfer function} \]
\[ k_p = -1.1502 \times 10^6; \text{ negative position stiffness, N/m} \]
\[ k_i = 479.253; \text{ current stiffness, N/A; integrator gain of servo} \]
\[ k_{ip} = 2,752; \text{ current controller integral gain} \]
\[ k_{pc} = 21,234; \text{ AMB controller proportional gain, A/m} \]
\[ k_{it} = 1.2523 \times 10^6; \text{ AMB controller integral gain, A/m-s} \]
\[ k_j = [0.473 \ 1.699 \ 3.499]_{j=1-3} \text{ ([0.306 1.099 2.264]_{j=1-3}); seal (j) design coefficient, cfm/psi}^{0.5} \text{ (m}^3/\text{hr} / \text{kPa}^{0.5}) \]
\[ k_v = 0.8632 \text{ (0.212); valve flow coefficient, cfm/psi (m}^3/\text{hr} / \text{kPa}) \]
\[ K_{sf\_srv} = \text{state feedback gain of observer controller} \]
\[ K_{i\_srv} = \text{observer controller integrator gain} \]
\[ K_{0,1} = \text{modified Bessel functions second kind, } 0^{th} \text{ and } 1^{st} \text{ order} \]
\[ L_{coil} = 0.399; \text{ inductance, Henrie} \]
\[ m = 10 \text{ (22.03); rotor mass, kg (lbm)} \]
\[ M_{s1} = 7.5e-4; \text{ surge margin value when control effort is maximum} \]
\[ M_{s0} = 0.003; \text{ surge margin value when no control effort is required} \]
\[ N = 375; \text{ no. turns of AMB coil} \]
$N_{max} = 60,000$; maximum rotational speed, rev/min

$n_i = 5$; no. teeth of labyrinth seal stationary seal

$N_{bi} = [40, 40, 5], i=1,2,3$; no. teeth of labyrinth seal (actuator) i

$P_{2b} =$ pressure upstream of stationary seal, kPa(a) (psi(a))

$P_1 =$ pressure in chamber 1 of balance piston actuator, kPa(a) (psi(a))

$P_2 =$ pressure in chamber 2 of balance piston actuator, kPa(a) (psi(a))

$P_f =$ vent pressure (atmospheric), kPa(a) (psi(a))

$P_s =$ supply pressure, kPa(a) (psi(a))

$R_c =$ 4.526; coil resistance, Ohm

$R_T =$ 39.22; tuning resistor, Ohm

$R_{eff,i} =$ effective reluctance of element (i), A/Wb

$R_v =$ 9; Rangeability of valve

$R_{H1} =$ 60; Relative humidity of inlet air, %

$r_i =$ [25, 33.5, 38.5, 42, 78.5, 92.5, 42,] i=1-7 radial location (i), mm

$S_{hp}(s) =$ sensitivity transfer function of servo

$T_d =$ 27.752; AMB Controller derivative time constant, A-s/m

$T_i =$ 35 (95); Inlet Temperature of Compressor, deg C (deg F)

$T_s =$ 15.6 (60); Supply Temperature to the Balance Piston Actuator, deg C (deg F)

$T_{lp}(s) =$ loop transfer function of servo

$V_{dc} =$ 175; saturation voltage of the current amplifier, V

$W =$ airgap energy, J

$x_{bb} =$ 190e-6; auxiliary bearing clearance, m

$z_i =$ [8.5, 6.5, 10.5, 14.5], i=1-4 axial location (i), mm

$\alpha =$ 1e-5; lead compensator factor; frequency parameter, 1/m

$\alpha_1 =$ modified frequency parameter, 1/m

$\alpha_f =$ seal (j)flow coefficient

$\delta =$ skin depth, m

$\Phi =$ flux, Wb

$\lambda_c =$ closed loop pole of servo controller

$\mu_0 =$ $4\pi e^{-7}$; permeability of free space, N/A²

$\mu_r =$ 10000; relative permeability

$\rho =$ density, kg/m³ (lbm/ft³)

$\sigma =$ 2.703e6; electrical conductivity, S/m

$\sigma_r =$ real part of closed loop pole, rad/s

$\zeta =$ closed loop damping ratio

$\omega_i =$ imaginary part of closed loop pole, rad/s

$\omega_{nc} =$ closed loop undamped natural frequency, rad/s

$\nu =$ specific volume, m³/kg (ft³/lbm)
INTRODUCTION AND LITERATURE REVIEW

Within the turbomachinery industry there is a trend toward using bearing technologies that require no oil for lubrication. This trend is due to two primary factors: 1) Increasing energy costs that encourage continual improvements in the efficiency of turbomachines, and 2) Environmental concerns regarding the storage and handling of lubricating oils. Furthermore, some applications of turbomachines are subject to design or regulatory constraints that may not permit the use of some or all types of oil (e.g. food and pharmaceutical industries).

For those cases where oil-free support of the rotor is required, the Active Magnetic Bearing (AMB) is particularly attractive, offering such benefits as:

1) Contact-free support of the rotor (even when it is not turning)
2) High rotational speed capabilities
3) Tolerance of shaft misalignments
4) Inherent diagnostic and health monitoring capabilities
5) High load capacities (relative to other oil-free bearing technologies)

One disadvantage of AMB systems, particularly AMB thrust bearings, is their limited load capacity relative to fluid film bearings. For compressor designers, this limitation imposes unique challenges on the design of the machine. For AMBs with high pressure ratio stages (e.g. PR > 2.0), thrust loads must be accounted for in the design process to conform to the limits of the bearing. One common solution to this problem involves the design of a balance piston system to keep the thrust loads sufficiently small. In most cases, however, balance pistons employ seals that leak some of the process gas flow and reduce the overall machine performance.

In the design of turbo compressors, the highest thrust loads are experienced during peak operating pressure ratios, which are typically encountered near compressor surge. Some of the AMB’s load capacity must be preserved to handle dynamic loads in these operating regions. For other regions of operation, however, the AMB’s load capacity may not be fully utilized due to these conservative design margins.
Devices for controlling or adjusting thrust loads on rotating machinery have been proposed in the literature. One idea described in a US patent application [1] is a concept for the control of axial load on a turbo machine rotor using a piston device. A signal indicative of the axial load is used in a feedback loop to regulate the load on the machine through a change of machine speed. The patent is written in a general way, but appears to be applicable to a low speed, center hung, multistage centrifugal compressor. Wang and Schill [2] proposed a self-adjusting balancing device for use on centrifugal boiler feed water pumps. In their paper, the authors cite a relative lack of focus on improving machine efficiency by reducing leakage losses, as compared to design efforts focused on improving aerodynamic performance. Their proposed device is passive in nature; it does not rely on feedback control to regulate thrust loads. Instead, their design is based on a flexible element that can elastically deform inside a thrust piston. As the load increases on the piston, the elastic deformation of the flexible element provides a restoring force to offset the force acting on the piston. Another device for thrust load regulation, called the “Automatic Thrust Equalizer”, is described by Jumonville [3]. Axial movement of the rotor triggers the transport of fluid in a pneumatic system to adjust the net axial forces acting on the rotor.

The active thrust balancing design presented in this paper, like those proposed in the literature, is focused on improving the performance of AMB-supported turbo machines by minimizing losses across the full operating space. However, this design specifically targets high pressure ratio, single-overhung compressor systems that use magnetic thrust bearings. Furthermore, the intent of this paper is to illustrate the limitations of magnetic thrust bearings through analysis and simulation and to discuss the pertinent design issues and benefits of regulating thrust loads. The modeling process addresses realistic dynamic effects such as amplifier saturation, magnetic flux saturation, and eddy currents. Simulation results are used to design an active thrust balancing system, and axial force and leakage flow characteristics of this active device are compared to a stationary design. The proposed active design is shown to offer average leakage reductions of 9.0% to 26.4% relative to static balancing devices. Finally, an observer-based controller is designed, and a gain-scheduling methodology is proposed to cover the compressor’s full operating map.

Active Magnetic Thrust Bearings and Turbomachinery

Many factors influence the design of rotors for high speed centrifugal compressors. For high-pressure (e.g. 1034kPa (150psi) discharge), multi-stage centrifugal compressors, the stage pressure ratios
dictate that stage rotational speeds be in the range of 25,000-80,000 rpm. These speeds depend on factors such as the number of stages, flow rates, pressure ratios in each stage, material strength limits, rotordynamic constraints, and driver (motor) magnetic or thermal limits. Loss mechanisms in compressors that employ gears and fluid-film journal bearings are primarily shearing losses in the journal and thrust bearings and gear mesh losses in the transmission. For low power machines (e.g. < 372kW (500 hp)), these losses constitute a significant portion of the overall machine losses and tend to drive the efficiency downward. For this class of turbomachines in particular, the argument for “oil-free” bearing support can be very compelling.

Active magnetic bearings are a logical choice for support of the rotors of this class of machinery as they have higher specific load capacity (i.e., load / projected area) relative to other oil free alternatives such as air foil bearings. The disparity in specific load capacity of magnetic and air foil bearings is most acute for thrust bearings. Specific load capacities for foil thrust bearings have been reported to be 241-310kPa (35-45psi) for more recent designs [4]. Others have reported peak capacities as low as 103kPa (15psi), depending on cooling air flow [5]. In contrast, magnetic bearings can readily have specific load capacities above 345kPa (50psi) [6]. Furthermore, the magnetic bearing capacity is not speed dependent, to the extent that rotation or thermal growth effects do not change the operating air gap. Nevertheless, the load capacities of both oil free alternatives are considerably smaller than those available from oil-film bearings, which can approach 2068-3447kPa (300-500psi) [7]. This relative disadvantage of the AMB is most pronounced when considering the application of a magnetic thrust bearing in a single-overhung centrifugal compressor rotor (meaning a single compressor impeller on the shaft, not one impeller on each end) such as that shown in the simplified depiction of Figure 1. For such a compressor operating with ambient pressure at the inlet and a pressure ratio of 3.5 and impeller diameter of 165mm (6.5in), the net axial load can be significant, with net loads reaching over 2,224N (500lbf) in steady conditions.
Figure 1. Single overhung compressor with stationary balance piston, active balance piston and axial magnetic bearing

Another limitation imposed by the use of magnetic bearing technologies relative to oil-film bearings is that of dynamic capacity. Relative to oil-film bearings, magnetic bearings tend to have high static stiffness, but low dynamic stiffness. Furthermore, magnetic bearings only provide positive effective damping within a specific range of frequencies. Because magnetic bearing systems are inherently unstable, they require feedback control of rotor gap position. The static capacity of the magnetic bearing is mainly limited by magnetic flux saturation of the core material. However, many of the factors limiting the dynamic characteristics of magnetic bearings are associated with feedback control.

1) Frequency response characteristics of sensors, amplifiers and controllers
2) Frequency response characteristics of AMB coils
3) Amplifier voltage/current saturation
4) Flux (force) delays due to eddy current effects

Centrifugal compressors operating at high speed can exert large dynamic forces on the rotor when a surge occurs, in some cases adding to already high static loads. These considerations place constraints on the design of single-overhung compressors using magnetic bearings. One strategy often employed
is to use a double overhung design (Figure 2) so that respective stages on each side of the shaft can “offset” each other’s net thrust load, in part or in full. In so doing, the required thrust capacity of each magnetic bearing is reduced.

Figure 2. Example of offsetting by design: double overhung turbocharger [8]

This strategy is not always feasible, however, due to economic and technical considerations. For the case of a single-overhung compressor as shown in Figure 1, the thrust load is typically managed by including a stationary balancing device, sometimes referred to as a balance piston as shown in Figure 3.
Figure 3. Free body diagram of a rotor with a fixed thrust piston showing aerodynamic loads

This free body diagram illustrates axial pressure distributions in a machine with a stationary balance piston represented by \( d_{2bp} \). This seal throttles the gas from a high pressure \( P_{2b} \) (assumed to equalize with discharge pressure \( P_{2f} \) in steady-state conditions) to a low pressure \( P_3 \) (ambient pressure), thus reducing the net axial load \( F_{net} \). By properly specifying the diameter of this seal (\( d_{2bp} \)), the target net load can be achieved. Typically for this seal, a labyrinth design is chosen based on simplicity and cost considerations. Balancing devices such as this contribute to system losses, however, as the process gas is bled through seals. Because the leak flow rate through these seals is related to their effective diameters, a common design approach is to tune the seal dimensions to optimize the competing performance objectives:

1) Providing static loads low enough for a magnetic thrust bearing to manage (in all operating regimes of the compressor), including disturbance forces associated with surge
2) Minimizing the process gas leak flow, as this is detrimental to the efficiency of the compressor
Figure 4. Comparison of net thrust load and leak flow rate vs. stationary balance piston diameter (P2=414kPa(a), P3=117kPa(a), Nt=5)

Figure 4 illustrates the tradeoff between force and leak flow. As the balance piston diameter is increased, the net thrust load decreases, but the leakage increases. The nature of these competing objectives motivates the work presented here: to design a magnetic thrust balancing system that utilizes available bearing capacity to minimize leak flow, yet preserves the bearing’s load capacity while operating close to surge conditions.

AMBI MODELING

In order to design the proposed thrust balancing device, it is important to understand the performance limitations of AMBs. For this reason, a prototype magnetic bearing is designed and analyzed in the following sequence:

1. AMB geometry and design specifications
2. AMB controller design and simulation
3. Effects of dynamic limitations (actuator, saturation, current amplifier)
   a. Step responses with unit force disturbances
b. Relative Stability  
c. Surge responses  

4. Specifications for maximum static load and capacity margin for surge

**AMB Actuator Design**

The AMB actuator considered for this study is shown in Figure 1, with the geometry of each actuator half illustrated in Figure 5. The actuators and disk are solid, un-laminated, non-oriented electrical steel. The maximum excitation field is 1500 AT for each actuator, and the nominal air gap is 0.5 mm. The parameters of the design are shown in Table 1.

![Figure 5. AMB actuator geometry](image)

**Table 1 – AMB actuator design parameters**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>r₁</td>
<td>25mm</td>
<td>z₁</td>
<td>8.5mm</td>
<td>µᵣ</td>
<td>10000</td>
</tr>
<tr>
<td>r₂</td>
<td>33.5mm</td>
<td>z₂</td>
<td>6.5mm</td>
<td>σ</td>
<td>2.703e6 S/m</td>
</tr>
<tr>
<td>r₃</td>
<td>38.5mm</td>
<td>z₃</td>
<td>10.5mm</td>
<td>L</td>
<td>0.399 H</td>
</tr>
<tr>
<td>r₄</td>
<td>42mm</td>
<td>z₄</td>
<td>14.5mm</td>
<td>m</td>
<td>10 kg</td>
</tr>
<tr>
<td>r₅</td>
<td>78.5mm</td>
<td>N</td>
<td>375 turns</td>
<td></td>
<td></td>
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<tr>
<td>r₆</td>
<td>92.5mm</td>
<td>Iₘₐₓ</td>
<td>4.0A</td>
<td></td>
<td></td>
</tr>
<tr>
<td>r₇</td>
<td>42mm</td>
<td>g₀</td>
<td>0.5mm</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
In order to evaluate the static capacity of this AMB, two finite element analyses of force vs. DC current were performed using FEMM [9]. The results of this analysis are shown in Figure 6. The first curve represents an FEA model containing only the non-linear BH characteristics of the actuator iron, while the FEA model includes leakage and stray flux effects due to air and surrounding parts (rotor and twin actuator). A third curve represents the analytical calculation of static force using a relationship which can be derived for an actuator consisting of two airgaps with areas $A_1$ and $A_2$:

$$F_{\text{static}} = \frac{\mu_0 N^2 I^2}{2g_0} \left( \frac{A_1 A_2}{A_1 + A_2} \right)$$

Figure 6 shows that the “no leakage FEA model” and the analytical calculations are in reasonable agreement until material saturation starts to affect the air gap flux (approximately 3.0 A). The “full FEA model”, which includes leakage and stray flux effects, reveals important limitations on maximum actuator force, which begins to level out at higher currents (above 3.0 A).

**AMB Control**

A block diagram of the axial magnetic bearing is shown in Figure 7. The primary elements of this system include controller, amplifier, bearing, and sensor. For this study, the sensor dynamics are assumed to be insignificant to the response characteristics, (i.e., the sensor bandwidth is assumed to
be much greater than the bandwidths of other system components... therefore unity feedback is assumed), so $H_s = 1$.

**Figure 7. Axial AMB closed loop system block diagram**

**AMB Plant Dynamics**

A linearized model of the AMB can be derived from the analytical force expression (Equation 1) and the relevant equation of motion about the nominal point, $x_0 = 0$:

$$
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
0 & -k_p \\
-k_i & m
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
0 \\
-k_i
\end{bmatrix} i_c;
\quad y = \begin{bmatrix}
1 \\
x_2
\end{bmatrix}
$$

where:

$\dot{x}_1 = x$ (position); $\dot{x}_2 = \dot{x}$ (velocity);

$$k_i = \frac{4CI}{g_o^2}; \quad k_p = -\frac{4C(I^2_i + i_c^2)}{g_o}; \quad C = \frac{\mu_b N^2}{4} \left( \frac{A_1A_2}{A_1 + A_2} \right)$$

Here $k_i$ represents the current stiffness, $k_p$ is the negative position stiffness, and $C$ is the bearing coefficient for the electromagnet [10]. This system model is not stable in the open loop due to the pole located at $\sqrt{-k_p/m}$. Although it is generally important to include the effects of actuator dynamic limitations when simulating the performance of AMB systems, these effects (arising from eddy
currents in the un-laminated bearing structure, etc.) are intentionally excluded from the model of Equation 2. Such effects will be considered later in this paper.

**Current Amplifier**

The architecture of the amplifier specified for this design study is one commonly used for magnetic bearings: a Proportional + Integral (PI) linear current controlled amplifier [6]. A block diagram of the amplifier is shown in Figure 8. Parameters for this controller are presented in Table 2, and the current sensor and the power electronic dynamics are neglected. The amplifier’s output is saturated to +/-175 volts; a tuning resistor in series with the actuator coil limits the current output to +/-4 amps. The closed loop transfer function of the amplifier is given in Figure 8, and the unit step response of the amplifier is shown in Figure 9.

![Current feedback amplifier closed loop block diagram](image)

**Figure 8. Current feedback amplifier closed loop block diagram**

**Table 2 – Design parameters of the current amplifier**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_{DC}$</td>
<td>175 V</td>
<td>$R_{coil}$</td>
<td>4.526 Ω</td>
<td>$K_{ip}$</td>
<td>2,752 V/A</td>
</tr>
<tr>
<td>$I_{max}$</td>
<td>4.0 A</td>
<td>$R_{same}$</td>
<td>39.22 Ω</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$L_{coil}$</td>
<td>0.399 H</td>
<td>$K_{ii}$</td>
<td>9.9844e6 V/A-s</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Figure 9. Step response of the current amplifier

**AMB Controller**

One source reports that over half of the industrial controllers in use are either PID or modified PID types [11]. Since many AMB systems employ similar controllers, the PID architecture will be utilized for this study (Figure 10). To simulate the practical implementation of derivative control action, which is required for closed-loop stabilization [6], a lead compensator is utilized:

\[
G_c = K_p + \frac{K_i}{s} + \frac{T_ds + 1}{aT_ds + 1}
\]

Figure 10. AMB controller (PID) block diagram
Table 3 – Design parameters of the AMB controller

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_{pc}$</td>
<td>21,234 A/m</td>
<td>$T_d$</td>
<td>27.752 A-s/m</td>
<td>$\omega_{nc}$</td>
<td>950 rad/s</td>
</tr>
<tr>
<td>$k_{in}$</td>
<td>1.2523e6 1/m-s</td>
<td>$f$</td>
<td>0.1</td>
<td>$\zeta$</td>
<td>0.7</td>
</tr>
</tbody>
</table>

Design parameters for the AMB PID controller are presented in Table 3. The controller gains were determined by first selecting proportional and derivative gains that provided adequate speed of response and acceptable stability margins (gain margin > 6 dB and phase margin between 30 and 60 Degrees [11]). These gains were tuned by considering the locations of closed-loop system poles, natural frequencies and damping ratios (sensor and actuator not considered: $H(s) = G_a(s) = 1$):

$$k_p = \frac{m\omega_n^2 - k_p}{H G_s k_i} = \frac{m\omega_n^2 - k_p}{k_i}$$  \hspace{1cm} (3)$$

$$T_d = \frac{2m\zeta\omega_n}{H G_s k_i} = \frac{2m\zeta\omega_n}{k_i}$$  \hspace{1cm} (4)$$

$$k_m = \frac{-f \sigma (\sigma^2 + \omega_n^2)m}{H G_s k_i} = \frac{-f \sigma (\sigma^2 + \omega_n^2)m}{k_i}, \quad \sigma = -\zeta \omega_n, \quad \omega_n = \omega_n \sqrt{1 - \zeta^2}$$  \hspace{1cm} (5)$$

Next, integral control action was incorporated to remove steady-state error. The integral gain was adjusted so that the closed-loop unit disturbance response yielded approximately 0.1 $\mu$m peak displacement and the response settled in less than 100 ms. The step responses to unit reference and disturbance inputs are shown in Figure 11.
Figure 11. (a) Unit step response of AMB system. (b) Unit disturbance response of AMB system

Figure 11a shows that the closed-loop response is well-damped with moderate overshoot and settles in approximately 35 ms. Since the position reference for the AMB is normally zero, the disturbance response (Figure 11b) is of greater interest to the designer. Here, it is apparent that the peak displacement is approximately 0.12 μm and that the rotor displacement settles in approximately 70 ms.

Next, the loop transfer function of the system was evaluated for adequate stability margins (Figure 12).
The gain and phase margins are -16.19 dB (212.8 rad/s) and 43.92 deg (1515.7 rad/s), respectively, which is acceptable (negative GM since plant system contains an unstable pole). Figure 12a shows that the closed-loop compliance transfer function (disturbance force input) peak is well damped, and that the low frequency stiffness is relatively high. Figure 12b shows that the initial condition response of the rotor against the auxiliary bearing with the controller gain limited to 25% of the error signal. Without such a gain limitation for this initial condition, the error signal would be too large, with saturation of the current amplifier resulting in an unstable response. Figure 12b shows that the reference position is reached quickly (<10 ms).

These results, which do not include the effects of actuator dynamics, serve as baseline for investigating the effects of such effects as eddy currents and saturation. After the inclusion of a model for the actuator dynamics and performance simulation thereof, the design constraints governing the balancing system can be defined.

*Actuator Transfer Function Model*

The simulation results of Figure 12, which do not include the effects of actuator dynamic limitations and saturation, serve as baseline for comparing several “real world” phenomena. Indeed, an
important consideration in the design of axial AMBs is that their dynamic response characteristics are limited due to the presence of eddy currents. Eddy currents are induced in any conductive material that is subject to a time-varying magnetic field. Eddy currents are a major loss mechanism and are particularly important to consider in axial AMB actuators, as the structural components do not employ laminated electrical steel. It is simply not practical in most cases to use laminated materials in the construction of axi-symmetric cylindrical components.

There are several works in the literature describing methods for modeling the dynamic characteristics of AMB actuators in the presence of eddy currents. In an early paper, Feely models a rectangular C-shaped actuator coil using a 2D Cartesian coordinate system [12]. Feely solved the relevant field equations and proposed an approximation to the hyperbolic tangent function that appears in the flux solution. From this simplification, expressions for the effective reluctance of the actuator were developed as functions of frequency and a simplified transfer function model of the actuator was proposed. More recently, a significant advancement of this work was made by Zhu, Knopse, and Maslen [13], whereby a 2D axi-symmetric C-shaped actuator was subdivided to obtain simplified element-level expressions of effective reluctance. These effective reluctances were then summed to obtain an expression for the entire actuator. Using this work as a primary reference, a description of the development of an actuator transfer function for the proposed axial magnetic bearing follows.

Figure 13. AMB actuator element divisions
A schematic of the bearing geometry and defined elements is shown in Figure 13. This geometry differs from the one considered by Zhu, et al., who considered a solid core stator without a center hole (i.e. for Element 1 of Figure 13, the inside radius \( r_1 = 0 \)). Also, the basic shape of the stator modeled in by Zhu, et al. was a C-shape. However, the axial space required by this type of actuator is not acceptable for many practical rotor assemblies. Furthermore, solid stator parts without a center bore are only practical for actuators located outside of the rotor assembly, since the shaft cannot pass through such actuator types. Nevertheless, by direct extension a simplified dynamic model for the actuator of interest can be derived.

Before describing the derivation of the model, it is useful to investigate the behavior of the magnetic field inside the actuator for different excitation frequencies. The FEA actuator model presented previously can be used for this purpose. Figure 14 shows the magnetic flux distribution inside the actuator core for a 1.2A DC and at 1 Hz AC sinusoidal current excitation of equivalent peak to peak amplitudes.

![Figure 14. Actuator flux contour plot comparison: (a) 1.2A DC, (b) 1.2A (pk-pk) 1Hz AC](image)

These contour plots reveal that the flux distribution changes dramatically with time varying excitation. The magnetic field concentrates in surface regions of the actuator that are enclosed by the excitation current field. This discrepancy in magnetic flux distribution as the exciting current frequency increases forms the basis of the argument for the model development henceforth.
The primary assumption guiding the derivation of an actuator model is that the actuator geometry can be discretized into a number of small elements in such a way the magnetic field distribution in each element can be considered one dimensional. This 1D field approximation enables a series of effective reluctances to be computed for each element. In this way, a magnetic circuit model developed for the problem can yield a transfer function between the time-dependent control current and the resulting force produced by the actuator. The adequacy of this 1D approximation varies with frequency. As the frequency increases, the field concentrates closer to the surface of the conductor, which improves the one dimensional approximation. Conversely, the 1D field approximation is less adequate at lower frequencies (< 1Hz). Nevertheless, this approximation is necessary for the development of a simple model with sufficient accuracy for control design and simulation.

Figure 5 shows the eight discretized elements defined for analysis of the actuator model. A careful inspection of the 1 Hz flux contours in Figure 14 reveals that the 1D field assumption appears to be valid for all elements except perhaps Element 5. For the non-airgap elements 1, 3, 6, 7, and 8, deriving the 1D effective reluctances can proceed in a fairly straightforward manner by considering the elements to be axi-symmetric. Then, applying Maxwell’s equations in a circular-cylindrical coordinate system, one can derive the effective reluctance for each element by solving the relevant boundary-value problem. Assuming each element is a linear material (μ=constant) of constant conductivity (σ=constant), Maxwell’s equations can be combined to produce the following Poisson’s equation [14]:

$$\nabla^2 \mathbf{H} = \sigma \mu \frac{\partial}{\partial t} \mathbf{H}$$

(6)

Where $\mathbf{H}$ is the magnetic field, $\sigma$ is the conductivity, and $\mu$ is the permeability.

For the elements comprising the airgaps (elements 2 and 4), a reluctance network is derived similarly to [13]. For Element 5, an alternative approach using parallel combinations of 1D solutions in the r and z directions can be adopted to determine the effective reluctance. Please refer to Appendix A for a derivation of the effective reluctance of each element, which are summarized in Equations 7-14.

$$R_{eff} = \frac{\alpha (z_1 + z_2 + z_3)}{2 \pi \mu \mu_0 r_z} \left[ \frac{K_r(\alpha r_1)I_r(\alpha r_2) + I_r(\alpha r_1)K_r(\alpha r_2)}{K_r(\alpha r_2)I_r(\alpha r_1) - I_r(\alpha r_2)K_r(\alpha r_1)} \right]$$

(7)
\[ R_{e_1} = \frac{g\alpha_i}{2\pi\mu_0 r_i} \left[ \frac{K_i(\alpha, r_i) I_s(\alpha, r_i) + I_s(\alpha, r_i) K_i(\alpha, r_i)}{K_i(\alpha, r_i) I_s(\alpha, r_i) - I_s(\alpha, r_i) K_i(\alpha, r_i)} \right] \]  

(8)

\[ R_{e_2} = \frac{\alpha \ln \left( \frac{r_1}{r_2} \right)}{2\pi\mu_0 \mu_t \tanh(\alpha z_2)} \]  

(9)

\[ R_{e_3} = \frac{g\alpha_i}{2\pi\mu_0 r_i} \left[ \frac{K_i(\alpha, r_i) I_s(\alpha, r_i) + I_s(\alpha, r_i) K_i(\alpha, r_i)}{K_i(\alpha, r_i) I_s(\alpha, r_i) - I_s(\alpha, r_i) K_i(\alpha, r_i)} \right] \]  

(10)

\[ R_{e_4} = \frac{\alpha \left( \frac{z_1}{r_1} \right) \ln \left( \frac{r_1}{r_3} \right) \left[ \frac{K_i(\alpha, r_i) I_s(\alpha, r_i) + I_s(\alpha, r_i) K_i(\alpha, r_i)}{K_i(\alpha, r_i) I_s(\alpha, r_i) - I_s(\alpha, r_i) K_i(\alpha, r_i)} \right] + \ln \left( \frac{r_2}{r_3} \right)}{2\pi\mu_0 \mu_t \tanh(\alpha z_2)} \]  

(11)

\[ R_{e_5} = \frac{\alpha}{2\pi\mu_0 \mu_t \tanh(\alpha z_2)} \]  

(12)

\[ R_{e_6} = \frac{\alpha \left( z_1 + z_2 + z_3 \right) \left[ \frac{K_i(\alpha, r_i) I_s(\alpha, r_i) + I_s(\alpha, r_i) K_i(\alpha, r_i)}{K_i(\alpha, r_i) I_s(\alpha, r_i) - I_s(\alpha, r_i) K_i(\alpha, r_i)} \right]}{2\pi\mu_0 \mu_t r_s} \]  

(13)

\[ R_{e_7} = \frac{\alpha \ln \left( \frac{r_1}{r_3} \right)}{2\pi\mu_0 \mu_t \tanh(\alpha z_2)} \]  

(14)

Where \( \delta = \sqrt{\frac{2}{\omega\mu_0\mu_t}} \) is the skin depth and the frequency parameters are: \( \alpha = \frac{1 + j}{\delta} \); \( \alpha_i = \frac{2\alpha}{\mu g} \). \( I_0, I_1 \) are modified Bessel functions, first kind, \( 0^{th} \) and \( 1^{st} \) order respectively, and \( K_0, K_1 \) are modified Bessel functions, second kind, \( 0^{th} \) and \( 1^{st} \) order respectively.

The force that the actuator can develop is related to the air gap energy which is given by [15]:

\[ W_g = \frac{1}{2\mu_0} \int B^2 dv \quad \text{(energy - for each pole face)} \]  

(15)

\[ W_{g\text{tot}} = W_{g1} + W_{g2} = \frac{1}{2\mu_0} \left[ \int B_1^2 dv_1 + \int B_2^2 dv_2 \right] \quad \text{(for both pole faces)} \]  

(16)
The force is related to the change in energy that results from displacing the floating element over the
distance of the airgap, \( g_0 \):

\[
F_{tot} g_0 = W_{g_0}
\]  

(17)

Specifying the differential volume element \( (dV = 2\pi g_0 r dr) \) in Equation 15, the force becomes:

\[
F_{tot} = \frac{\pi}{\mu_0} \left[ \int B_1^2 r dr + \int B_2^2 r dr \right]
\]  

(18)

Axial magnetic bearings are frequently powered using DC bias currents (which contribute to the
current stiffness and negative position stiffness) summed with time-varying control currents. This
control current is typically linearized about a nominal point of operation such that:

\[
I_{tot} = I_s + I_c(s) \quad (s = \text{Laplace variable})
\]  

(19)

Using this excitation, the resulting expressions for the flux density in each pole area are:

\[
B_1 = B_{1b} + B_{1c}(s)
\]  

(20)

\[
B_2 = B_{2c} + B_{2c}(s)
\]  

(21)

If the control flux is assumed to be small relative to the bias flux given that \( I_c(s) \ll I_b \):

\[
B_1^2 = B_{1b}^2 + 2B_{1b}B_{1c} + B_{1c}^2 \approx B_{1b}^2 + 2B_{1b}B_{1c}
\]  

(22)

\[
B_2^2 = B_{2s}^2 + 2B_{2s}B_{2c} + B_{2c}^2 \approx B_{2s}^2 + 2B_{2s}B_{2c}
\]  

(23)

By substitution of these squared flux density expressions into Equation 18:

\[
F_{tot} = F_b + F_c(s) = \frac{\pi}{\mu_0} \left[ \int B_{1b}^2 r dr + \int B_{2c}^2 r dr + \int 2B_{1b}B_{1c} r dr + \int 2B_{2s}B_{2c} r dr \right]
\]  

(24)
This force expression can be broken into 2 terms, one for the DC bias force and one for the control force:

\[
F_s = \frac{\pi}{\mu_0} \left[ \int_0^2 B_{sz} r dr + \int_0^2 B_{sz} r dr \right]
\]

(25)

\[
F_c(s) = \frac{\pi}{\mu_0} \left[ \int_0^2 2B_{sz} r dr + \int_0^2 2B_{sz} B_{sz} r dr \right]
\]

(26)

The bias and control flux densities can be expressed on a differential basis:

\[
[B_s B_c](r, s) = \frac{d[\phi_s \phi_c]}{dA}; \quad dA = 2\pi dr
\]

(27)

Therefore, by substitution of Equation 27 into Equation 26:

\[
F_s(s) = \frac{1}{\mu_0} \left[ \int_0^1 \frac{d(\phi_s \phi_s)}{dA_1} + \int_0^1 \frac{d(\phi_s \phi_c)}{dA_2} \right]
\]

(28)

\[
F_c(s) = \frac{1}{\mu_0} \left[ \int_0^1 \frac{d(\phi_s \phi_s)}{dA_1} \rightarrow F_c(s) \int dA_1 = \frac{1}{\mu_0} \int d(\phi_s \phi_s) \right]
\]

(29)

\[
F_c(s) = \frac{1}{\mu_0} \left[ \int_0^1 \frac{d(\phi_s \phi_c)}{dA_2} \rightarrow F_c(s) \int dA_2 = \frac{1}{\mu_0} \int d(\phi_s \phi_c) \right]
\]

(30)

And after integration, the total control force is (letting \( \phi_s = \phi_c = \phi \) and \( \phi_s = \phi_c = \phi \) ) :

\[
F_c(s) = \frac{\phi_s \phi_c}{\mu_0} \left[ \frac{1}{A_1} + \frac{1}{A_2} \right]
\]

(31)

According to the magnetic circuit model, the bias flux and control flux expressions are [13]:
\[ \phi_b = \frac{N I_b}{\sum R_i^b} \] (bias flux - 0Hz) \quad (32)

\[ \phi_c(s) = \frac{N I_c(s)}{\sum R_c(s)} \] (control flux) \quad (33)

Where \( R_i^b \) is the effective DC reluctance of the \( i^{th} \) subscript and \( R_i(s) \) is the

\[ G_i(s) = \frac{F_i(s)}{I_i(s)} = \frac{\phi_i N}{\mu_0} \left[ \frac{1}{A_1} + \frac{1}{A_2} \right] \sum \frac{1}{R_i(s)} ; \quad G_{in}(s) = \frac{G_i(s)}{G_i(0)} \] \quad (34)

Equation 34 is the transfer function for the actuator. The critical elements of this transfer function are the effective elemental reluctances (Equations 7-14), which are complicated by natural logarithms, hyperbolic and modified Bessel functions. In order to use these reluctance expressions in a practical model framework, they can be simplified by considering their asymptotic behavior [12, 13]. The total reluctance of each element can be thought of as the sum of a DC term and a frequency dependent term:

\[ R_i^c = R_i^0 + R_i(s) \] \quad (35)

For the frequency dependent terms, the following approximations for Bessel functions with large arguments are useful (when \( \alpha \) is large) [16]:

\[ I_0(\alpha r) \approx I_0(\alpha r) \approx \frac{e^{-\alpha r}}{\sqrt{2\pi \alpha r}} \] \quad (36)

\[ K_0(\alpha r) \approx K_0(\alpha r) \approx \sqrt{\frac{\alpha r}{2\pi}} e^{-\alpha r} \] \quad (37)

And for hyperbolic tangent functions, the approximation for large arguments (when \( \alpha \) is large):

\[ \tanh(\alpha r) \approx 1 \] \quad (38)
Substituting these expressions into the effective reluctances of Equations 7-14 yields simplified expressions for the reluctance of each element:

\[
R_{\text{eff.approx.1}} = \frac{(z_3 + z_3 + z_4)}{\pi \mu_0 \mu_z (r_0^2 - r_i^2)} + \frac{(z_2 + z_2 + z_4)}{2 \pi z_2} \sqrt{\frac{\sigma}{\mu_0}} \sqrt{s}
\]

\[
R_{\text{eff.approx.2}} = \frac{g_0}{\pi \mu_0 (r_0^2 - r_i^2)} + \frac{1}{\pi z_2} \left[ \frac{g_0}{2} \right] \left( \frac{\sigma}{\mu_0} \right) \sqrt{\frac{\sigma}{\mu_0}} \sqrt{s}
\]

\[
R_{\text{eff.approx.3}} = \frac{\ln \left( \frac{r_o}{r_i} \right)}{2 \pi \mu_0 z_1} + \frac{\ln \left( \frac{r_o}{r_i} \right)}{2 \pi \mu_0} \sqrt{\frac{\sigma}{\mu_0}} \sqrt{s}
\]

\[
R_{\text{eff.approx.4}} = \frac{\ln \left( \frac{r_o}{r_i} \right)}{2 \pi \mu_0 z_1} + \frac{\ln \left( \frac{r_o}{r_i} \right)}{2 \pi \mu_0} \sqrt{\frac{\sigma}{\mu_0}} \sqrt{s}
\]

When substituted into the transfer function model (Equation 34), these approximate reluctance expressions yield a model that is of fractional order in s, specifically \(s^{1/2}\) and \(s^{1/4}\). In order to assess the accuracy of this simplified transfer function, a comparison between the two dimensional, axi-symmetric, finite element solution of the actuator frequency response and the one dimensional models (Equation 34 - normalized) was made. Frequency response plots for the analytical solution (Equation 34) using the exact reluctance expressions (Equations 7-14) and the approximate reluctance expressions (Equations 39-46) are compared with FEA results in Figure 15. The three FEA models differ in their constitutive flux density models: the first assumes non-linear B-H behavior, the second
assumes linear B-H behavior, and the third assumes a linear characteristic with the conductivity of the material reduced by one order of magnitude in order to illustrate the positive impact on the actuator bandwidth that the sometimes-used technique of slotting the actuator can have [17].

Figure 15. Comparison of actuator transfer function frequency response

Figure 15 reveals that when the actuator material is assumed to behave linearly, the one dimensional analytical model and FEA results agree fairly well. When the fractional order approximation to the analytical model is utilized, significant error is introduced. The difference between the two analytical models is effectively zero at 0 Hz and increases with frequency, where the errors become significant due to the inadequacy of the large value assumption on the frequency parameter $\alpha$. However, as the frequency increases the error on the transfer function magnitude and phase decreases. These differences between exact and approximate analytical models can be seen more readily in Figure 16. The maximum error is approximately -12% on the magnitude (150Hz), and approximately -1.8 degrees for the phase (30Hz).
At higher frequencies the approximate analytical solution exhibits a phase shift toward 45 degrees which is the anticipated result given the dominance of the $s^{1/2}$ term in the approximated transfer function expression, shown in Equation 47.

**Approximation of Actuator Transfer Function Useful for LTI Modeling**

The normalized actuator transfer function resulting from the approximate reluctance expressions is in the form of a fractional order rational expression is:

$$G_{av}(s) = \frac{23854}{236.9904 \left[ s^{0.43} + 4.3144 s^{0.20} + 100.6548 \right]}$$  \hspace{1cm} (47)

Although the study of fractional order systems is addressed in non-linear control theory, most systems in industrial use today are based on linear time invariant (LTI) system theory. To use the derived model (Equation 47) within the framework of LTI simulation and control synthesis, it must first be approximated by an integer order model. Xue, Zhao, and Chen discuss a method for making this approximation based on a technique known as Oustaloup’s Approximation Algorithm [18]. Using this technique, an approximation of the fractional order terms of Equation 47 is developed over the frequency range $10^{-5}$ to $10^{5}$ rad/s. The approximation technique involves fitting the fractional order function of the Laplace variable, $s$, in the frequency domain by a series of filters such that:
\[
\begin{align*}
s' & \approx K \left[ \frac{ds^\tau + b \omega_k s}{d(1 - \tau)s^\tau + b \omega_k s + b \tau} \right] \prod_{j=k}^N \frac{s + \omega_j}{s + \omega_j}; \quad \omega_j = \left( \frac{b \omega_k}{b} \right)^{\frac{j-2}{N-1}}; \quad \omega_k = \left( \frac{b \omega_k}{b} \right)^{\frac{-2}{N-1}}; \\
N &= \left( \frac{b \omega_k}{b} \right)^{\frac{1}{N-1}}; \quad K = \left( \frac{b \omega_k}{b} \right)^{\frac{N-1}{N-1}} (48)
\end{align*}
\]

Where \([\omega_b, \omega_k]\) is the frequency range to fit and \(0 < \tau < 1\), \(\tau\) is the fractional order of \(s\). \(N\) is the integer order of the approximation in the frequency range of the fit. The parameters \(b\) and \(d\) are fitting parameters obtained from numerical experimentation. Xue et al. recommend using \(b=10\) and \(d=9\).

Notice that the fitting is only for the fractional order differentiator, \(s^\tau\). Therefore, to fit the fractional order rational transfer function of Equation 47, the \(s^{0.5}\) and the \(s^{0.25}\) terms must be fit and then combined in the rational fractional order expression to form the transfer function. The end result of this effort is a rational transfer function of integer order, but with the penalty of high order. The normalized transfer function of the actuator approximated using order \(N=5\) integer approximations for \(s^{0.5}\) and \(s^{0.25}\), respectively, in the frequency range \(10^{-5}\) to \(10^5\) is:

\[
G_{\text{col}}, (s) = 0.11538(s + 2.222e5)(s + 1.481e5)(s + 6.552e4)(s + 5.032e4)(s + 7925)(s + 6086) \\
(s + 958.5)(s + 736.1)(s + 115.9)(s + 89.02)(s + 14.02)(s + 10.77)(s + 1.696)(s + 1.302) \\
(s + 0.2051)(s + 0.1575)(s + 0.0248)(s + 0.1995)(s + 0.0032)(s + 0.002304)(s + 0.0003628) \\
(s + 0.0002786)(s + 3.37e - 5) \\
(s + 1.576e5)(s + 1.171e5)(s + 5.328e4)(s + 3.175e4)(s + 6700)(s + 4644)(s + 862.7) \\
(s + 645.7)(s + 110.3)(s + 83.29)(s + 13.74)(s + 10.38)(s + 1.683)(s + 1.275) \\
(s + 0.2045)(s + 0.1555)(s + 0.02478)(s + 0.01891)(s + 0.002999)(s + 0.02294) \\
(s + 0.0003628)(s + 0.0002779)(s + 3.365e - 5) \\
\]

Even after the collocated poles and zeros are cancelled, the order of this approximated normalized transfer function remains very high (23rd order). For the purpose of developing a controller that would include such an actuator transfer function model, it would be desired to reduce the order of this model. One approach to model order reduction is to use the Hankel singular values of the state space representation of Equation 49 [19, 20]. Hankel singular values (\(\sigma_i\)) are the square root of the eigenvalues (\(\lambda_i\)) of the product of the controllability and observability gramians \((P,Q)\) of the state space realization of the system [20].

\[
\sigma_i^2 = \lambda_i(PQ); \quad P = \int_0^\infty e^{at}BB^Te^{aT}dt; \quad Q = \int_0^\infty e^{cT}Ce^{aT}dt;
\]

(50)
Hankel singular values represent the energy captured by the states of the system. Therefore, states with small Hankel singular values can be removed from the model with little change in the dynamic characteristics (input vs. output) [19, 20]. Using this approach, an appropriate lower order for the model can be estimated by eliminating states that contribute little to the input-output behavior of the LTI system.

Figure 17. Plot of Hankel singular values of integer high order system

Figure 17 shows the singular values of the Hankel matrix formulation of the LTI system represented by Equation 49. From this figure, the model order can be selected arbitrarily by eliminating states with low energy transfer. For the case shown, if states with energy less than 0.005 were eliminated, then the retained model order is 7.

Using Figure 17 as a guide for the selection of model order, the high order model was reduced using the MATLAB™ function “balred” to derive a balanced model order reduction for orders N=4 and 7 based on the Hankel singular value distribution plotted in Figure 17. The frequency response of these low order models were plotted along with the fractional order model and the full order integer approximation model (Equations 47 and 49 respectively). Figure 18 compares the frequency responses of all four of these models over a frequency band of $10^{-5}$ to $10^{5}$ rad/s, which is sufficiently large to capture the relevant closed-loop dynamics. It is evident that the magnitude and phase
compare favorably except near the end of the frequency range (starting at approximately $4 \times 10^4$ rad/s). Furthermore, it is clear that while the 4th order integer model may be insufficient, the 7th order model compares well to the high order model, while achieving a substantial reduction in order relative to the system represented by Equation 49.

![Figure 18. Comparison of integer order approximations and fractional order transfer function frequency response](image)

For the sake of clarity, the reduced order model (N=7) is shown in Equation 51.

$$G_{av,s}(s) = \frac{0.13425 (s + 1.06e005) (s + 5840) (s + 553.9) (s + 53.24) (s + 5.168)(s + 0.3654)(s + 0.00693)}{(s + 3.413e4)(s + 3702) (s + 444.3) (s + 48.06) (s + 4.923) (s + 0.3549)(s + 0.006817)} \quad (51)$$

**AMB PERFORMANCE**

In order to design a thrust balancing system with a given load capacity, it is important to understand the performance limitations of magnetic thrust bearing. Therefore, the performance of a magnetic bearing with the actuator transfer function model (Equation 49) will be simulated using a proposed surge model.

*Magnetic Bearing Performance*

The active magnetic bearing design discussed above was simulated with the addition of actuator dynamics (Equation 49). A Simulink diagram of this simulation is shown in Figure 19. In this
model, the behavior of the rotor system against the auxiliary bearings is included with stiffness of 40e6 N/m and damping constant 2500 N-s/m.

Unit force disturbance responses of the AMB system with and without actuator dynamics are compared in Figure 20. The effect on the relative stability of the closed loop system is presented in the frequency response of Figure 21.
Figure 20. Unit step disturbance force response comparison

Figure 21. Effect of actuator dynamics on system: (a) Frequency response of the open loop transfer function, (b) Closed loop disturbance transfer function of AMB system

Figure 20 reveals a significant increase in the displacement response (approximately 60%) when the actuator dynamic limitations are included, though both responses settle in approximately the same time (70 ms). Figure 21a shows that the additional phase lag from the actuator introduces a second crossing of the -180 degree phase line. This implies that the loop can be driven unstable from either
gain increases or decreases. Therefore, the effect of the actuator dynamics is to reduce the overall stability margins of the system (gain margin (dB) decreases by 20.9% and the phase margin (degrees) decreases by 15.6 degrees relative to the system without actuator dynamics included). Finally, from the closed loop compliance transfer function frequency response (Figure 21b), it is clear that there is some slight additional amplification at the damped natural frequency for the case when actuator dynamics are present in the system model.

**Surge Force Transients**

The next step in the analysis is to evaluate the performance of the AMB system during a surge transient.

The characteristics of surge transients in centrifugal compressors depend on many factors. Some of the most relevant factors are pressure ratios, impeller/volute/diffuser geometry, static pressure loads, downstream volumes (due to coolers, receivers, etc.), number of stages (upstream and downstream), gas properties, and other loads on the rotor. There is really no “typical” magnitude and duration for a surge force transient, as the properties depend greatly on the specific stage and installation. Nevertheless, there are some basic characteristics of surge that generally hold. First, there is commonly a transient axial force associated with the compressor instability that characterizes a surge event. Furthermore, the magnitude of the surge force transient is generally proportional to the static loading; thus for higher pressure ratios, the magnitude of the force change is higher.

For the sake of this study, an ad-hoc surge transient model for the axial force is proposed. While the model is based on the author’s experience with centrifugal compressors characteristic of the design discussed herein, there is also support in the literature for many of its components. Specifically, the proposed rise time of the surge pulse is 50 ms, which agrees with test data presented by Moore [21]. Furthermore, the basic characteristic of the hold time and fall time proposed are similar to the data of Moore. The magnitudes of the pressure ratio changes published by Moore are difficult to translate into axial loads because local pressure distributions around the impeller are not part of the test data collected in this work. In the absence of this data, the ad-hoc model is derived based on the author’s experience that for diffuser-separation induced surges there is a drop in outlet pressure and increase in inlet pressure. This is consistent with Moore’s data citing a drop in both pressure ratio and differential pressure. Since the magnitudes of the respective drops are volume dependent and design
dependent, the ad-hoc model simply assigns the transient force magnitudes to be equivalent to the static values. The axial force - time history is shown in Figure 22.

Several observations can be made from this time history of Figure 22. First is the presence of the static load $F_0$ prior to the surge. This static force is reached slowly (in approximately 100 ms), possibly representing a compressor start or opening of a suction-side control valve. Secondly, the rise time and magnitude of the surge are assumed to be 50 ms and $F_0$, respectively. The hold time and fall time of the surge “pulse” are comparably slower than the rise time, and have been set to values that keep them from limiting the actuator response. Again, these assumptions are supported in the literature and have been observed in practice.

Surge transient simulations were conducted using the Simulink diagram of Figure 19. Figure 23 compares the simulation results with and without the inclusion of the actuator transfer function model (dynamics) for two different static loads (600 N and 660 N) that are on opposing sides of the stability boundary.
Figure 23. Surge responses of the AMB model with and without actuator dynamics: (a) F0=660 N, (b) F0=600 N

In Figure 23a, the steady force prior to surge was 660 N, as this value corresponds to the maximum value that the model not including eddy current effects could manage without becoming unstable due to saturation of the current amplifier (saturation current is 4A total). When actuator dynamics are included, the response becomes unstable, causing the rotor to oscillate in the gap between the two auxiliary bearings. The dynamic limitations of the actuator cause the saturation limits of the current amplifier to be reached at lower loads (10% lower) relative to the model not including these effects. Figure 23b shows the results for each case when the steady force prior to surge is 600 N. Here, the response of both models is stable, but with slightly more peak displacement for the model including the actuator dynamics.

To investigate the effect of surge force rise time on the response of the model (with actuator dynamics included), the rise time was varied from the nominal value of 50 ms to 5 ms. Figure 24a shows the results of this simulation.
Figure 24. Surge response of the AMB model subject to parameter variation (with and without actuator dynamics): (a) Ramp time, (b) Surge Force

Figure 24a shows that the force rise time has a large impact on the resulting peak displacement. There is approximately a threefold increase in the peak displacement of the rotor when the surge pulse rise time is varied from 50 ms to 5 ms. Another observation is that actuator limitations have a greater impact when rise times are small. Indeed, Figure 24a shows that the peak rotor displacement associated with a 5 ms force rise time is approximately 45 μm without the actuator dynamics included ($G_{aN}=1$) and almost 60 μm with these limitations imposed. Therefore, the peak displacement increased approximately 33%. Figure 24b shows the effect of surge pulse magnitude on displacement response. Here, the surge pulse magnitude was varied from 1xF₀ (value of the steady force) to 1.35xF₀ (the stability limit), holding the rise time at 50 ms. This figure reveals that the peak displacement increased by approximately 43% when the surge force magnitude was increased by 35% for a constant load.
Figure 25. Closed loop disturbance transfer function – values at near 125 rad/s (20Hz)

The higher surge displacement for the case of the model including the actuator dynamics can be understood further by considering the closed loop disturbance transfer function (Figure 21b). Figure 25 shows this transfer function in the frequency range near 125 rad/s, which represents the surge rise time dynamics of 50ms. The difference in surge force displacement can be understood from calculation of the effective stiffness at this frequency:

$$K_{\text{eff}} = \frac{1}{D} \cos(\Phi_D) + m \omega^2$$  \hspace{1cm} (52)

$K_{\text{eff}}$, $D$, $\Phi_D$, $m$, $\omega$ are the effective stiffness, disturbance function value, disturbance phase angle, mass, and frequency respectively. Referencing Figure 25, the effective stiffness at approximately 125 rad/s is 7.259e6 N/m for the system with the actuator dynamics and 9.498e6 N/m for the system without the actuator dynamics. This lower effective stiffness (-31%) of the system without the actuator dynamics explains the additional displacement of the rotor subjected to a disturbance load. Furthermore, the additional error due to excess displacement introduces additional control action, requiring more force from the actuator to correct the rotor position.

From these simulations, clearly the dynamics of the actuator have a significant impact on the performance of the magnetic bearing system subject to a surge disturbance. The impact is
significantly larger for faster surge rise times and larger surge magnitudes. Furthermore, the actuator
dynamics lead to amplifier saturation at lower disturbance loads, and current saturation then leads to
loss of stability during surges of sufficient magnitude. Another interesting observation is that the
dynamic actuator capacity is much lower than the static capacity. For example, the maximum steady
load and surge disturbance load that the model including actuator dynamics could accept were 600 N
and 600 N, respectively. This means that the compressor design would have to ensure that the
actuator steady load is no greater than 600 N, a value much lower than the analytical static capacity of
the actuator (1566N) shown in Figure 6 when the control current is 2.8A (maximum when I_{bias} =
1.2A).

To remedy this dramatic reduction in load capacity, the AMB system could be improved by taking
steps to improve the dynamic capacity:

1) Slot the actuator to reduce eddy current effects by reducing the effective conductivity of the
iron
2) Increase the capacity of the current amplifier to raise the saturation threshold
3) Reduce the inductance of the coil with lower number of turns
4) Change the bias current to increase the current stiffness of the bearing.

The remainder of this study will explore ways to more effectively use the given AMB design’s
capacity to eliminate degradation in compressor performance.

**DESIGN OF THE ACTIVE BALANCING SYSTEM**

*Active Balance Piston Design*

A compressor designer must account for capacity limitations of the magnetic bearing in the loading of
the thrust bearing. Most balancing devices employ some type of seal arrangement to minimize
leakages. For this study, it is assumed that the seals employed are labyrinth type. There are many
empirical and analytical studies that have been made on labyrinth seals to understand leak flow rates
and other fluid behavior. Probably the most recognized work describing the leak flow rates in
labyrinth seals was by Egli, who proposed the following relationships for the leakage flows based on
an ideal labyrinth that exhibits complete removal of the associated gas kinetic energy of each
throttling (tooth) [22].

\[
G_s = \frac{A_m}{144} \alpha_s \Phi_n \sqrt{\frac{(32.2)(144)P_t}{\nu_h}} 60(0.454) \tag{53}
\]

\[
\Phi_n = \sqrt{1 - \left(\frac{P_{t1}}{P_{t2}}\right)^2} \quad \frac{N_i + \log\left(\frac{P_{n+}}{P_t}\right)} \tag{54}
\]

Where \(G_s\) is the leakage flow in kg/min, \(A_m\) is the annular area of the seal, \(\alpha_s\) is the flow coefficient
which is related to tooth geometry and spacing relative to clearance, \(\Phi_n\) is the expansion function, \(N_i\)
is the number of throttlings (seal teeth), and \(\nu_h\) is the specific volume.

There are two primary means by which the balancing force can be adjusted: 1) Radial adjustment of
the seal location to change the effective area of the balancing pressure; 2) Adjustment of the fluid
pressure. It seems that while the concept of a movable seal could offer a relative benefit in terms of
compactness, this advantage is more than offset by the complexity of actuator designs that this author
has considered. Therefore, the second concept of using a controlled chamber pressure will be
explored.

The active balancing concept and actuator are shown in Figure 26.
Figure 26 details a T-shaped disk with a close-clearance seal located on the rotor just to the left of the AMB actuator. This disk is part of a two-chamber actuator with leak controlling seals that operate at a specific close clearance against the shaft. A supply of pressurized air is injected into Chamber 1, with flow throttled by a control valve. Flow through the device is through Seal 0 and Seal 1, which have clearances tuned to obtain the desired fully open valve force capacity. Seal 2 is located at the exit of Chamber 2 and can be used to tune the nominal pressure of Chamber 2. This seal is generally characterized by a large clearance to keep the pressure from becoming too high. Also, it serves to direct air into the AMB which can be used to cool the actuator. This seal may or may not be present in the actual design, but can enter the mathematical modeling of the system in order to simulate any downstream restriction imposed by the magnetic bearing components. Connected to the high pressure side of the actuator is a control valve that supplies pressurized air.
Table 4 – Balance piston system and compressor design parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_v$</td>
<td>1.0</td>
<td>$d_{2bp, stat}$</td>
<td>106.68mm (4.2in)</td>
<td>$P_0$</td>
<td>99.3kPa(a) (14.4psia)</td>
</tr>
<tr>
<td>$C_{1,2}$</td>
<td>5.3131e-5 kg/kPa</td>
<td>$d_{2bp, dyn}$</td>
<td>55.88mm (2.2in)</td>
<td>$RH_1$</td>
<td>60%</td>
</tr>
<tr>
<td>$d_{1, avg}$</td>
<td>60mm (2.36in)</td>
<td>$n_t$</td>
<td>5 teeth</td>
<td>$R_t$</td>
<td>9</td>
</tr>
<tr>
<td>$d_2$</td>
<td>165mm (6.5in)</td>
<td>$N_{i=1,2,3}$</td>
<td>[40, 40, 5] teeth</td>
<td>$T_f$</td>
<td>35C (95F)</td>
</tr>
<tr>
<td>$d_t$</td>
<td>50.8mm (2in)</td>
<td>$N_{max}$</td>
<td>60,000rpm</td>
<td>$T_s$</td>
<td>15.6C (60F)</td>
</tr>
</tbody>
</table>

Table 4 shows the design parameters for the compressor and balance piston actuator considered in this study. Figure 27 shows the operating concept for this active balancing device. In this concept, the magnetic bearing serves as a force transducer, providing feedback to the servo controller. The compressor controller receives the pressure and temperature signals required to determine the net aerodynamically-produced thrust load. By subtracting the measured AMB force from the computed aerodynamic force, a force reference (that the servo balancing device must produce) is generated.
This force set point is converted to a valve signal by a separate servo loop controller residing on the compressor controller. The valve position determines the pressures of Chambers 1 and 2 of the actuator, and therefore the net force produced on the rotor.

**Balance Piston Actuator**

A critical element in the design of the active thrust balancing system is the piston actuator, which must be designed in concert with the stationary balancing seal. Referencing Figure 26, the actuator consists of a T shaped disk mounted to the left of the AMB actuator on the compressor shaft opposite the impeller. This disk runs against a labyrinth seal on the stationary part (Seal 1). Mounted on the housing left of the disk is a shaft seal for controlling the pressure in the high pressure chamber (Seal 0). Mounted on the low pressure side of the disk, below the magnetic bearing actuator, is a low pressure seal (Seal 2).

An analytical model of this pneumatic actuator has been developed to determine its force capabilities and its air leakage characteristics. Furthermore, this mathematical is used for controller design. Referring to Figure 26 and assuming air to be an incompressible, isothermal, and ideal gas, the equations of motion for each chamber in terms of the state variable pressure are derived from the continuity relationship:

\[
\frac{dP_i}{dt} = \frac{\rho}{C}(q_{i,\text{in}} - q_{i,\text{out}})
\]  

(55)

Where \( q_i \) represents the volume flow into and out of the \( i \)th chamber, and \( C \) is the capacitance of the chamber volume [11]:

\[
C = \frac{V}{nRT}
\]  

(56)

Where \( n \) is the polytropic exponent of air, \( R \) is the gas constant, and \( T \) is the temperature. Using flow relationships for labyrinth seals given by Equation 53 together with a relationship for sub-critical flow through the control valve [23], the first order non-linear dynamic equations of motion are found by substitution into Equation 55:
\[
\frac{dP_1}{dt} = f_1 = \frac{P}{C_v} \left[ k_v f(x) \left( \frac{P_2 - P_1}{P_1} \right)^{0.5} - k_i \left( \frac{P_1^2 - P_i^2}{P_1^2} \right)^{0.5} \right] \tag{57}
\]

\[
\frac{dP_2}{dt} = f_2 = \frac{P}{C_v} \left[ k_i \left( \frac{P_1^2 - P_2^2}{P_1^2} \right)^{0.5} - k_i \left( \frac{P_2^2 - P_0^2}{P_2^2} \right)^{0.5} \right] \tag{58}
\]

\[
k_v = \frac{1360}{60} C_v \frac{1}{\sqrt{SG(T + 459)}}; \quad k_i = 28.373 A_\sigma \frac{1}{\sqrt{\rho N_j}}; \quad j = 0, 1, 2
\]

\[
f(x) = R^{(x-1)}
\]

Here, \(f(x)\) represents the valve characteristic (also known as the “trim” of the valve), and \(R_v\) is the rangeability of the valve defined as a ratio of the maximum to minimum flow that the valve can control.

It is apparent that the rate of chamber pressure change is related to the position of the valve, \(x\), by the control valve characteristic, \(f(x)\). This rate is also related to the valve design coefficients \(k_v\) and the seal constants \(k_i\). The control valve is characterized by a flow coefficient, \(C_v\), and a functional characteristic (trim) dictated by its design, \(f(x)\) [23]. To assess the performance of the actuator and static seal system, it is important to understand the steady-state pressures, forces, and leak rates as functions of the control valve position, \(x\). Steady-state pressures \(P_1\) and \(P_2\) are obtained from Equations 57\&58 by setting \(\frac{dP}{dt} = 0\). This requires the use of a numerical root solver, in this case MATLAB’s™ “fsolve” command, that solves the simultaneous system of non-linear equations for the chamber pressures inside the range \([P_v, P_s]\) where \(P_v\) is the vent pressure (assumed to be ambient pressure) and \(P_s\) is the supply pressure (compressor discharge pressure). For the case of fixed supply and vent pressures, Equations 57\&58 have been solved simultaneously for various valve positions to obtain net forces (from the pressure results). These forces and air consumption results also have been compared to finite element based solutions of the steady pressure driven flow problem through the device [24]. The results of these comparisons are shown in Figure 28.
Figure 28. Analytical and finite element force calculations: (a) axial force, (b) leakage flow

For the finite element solutions, two results are shown: the first based on a laminar, incompressible fluid assumption with no rotation of the shaft considered. The second finite element solution considers shaft rotation (swirl) at maximum compressor speed, compressibility effects, and turbulence based on a $k$-$\varepsilon$ model. For each case, the problem is assumed to be isothermal. The net force and leakage flow results of the analytical and finite element model agree fairly well (less than 10% difference in force, less than 3% difference in leakage) for a valve range of 0.7-1.0 and $P_s=60$ psia (414 kPa(a)). However, for valve positions less than 0.7 the differences are more significant. These differences could result from higher pressure drops across the valve at low settings, leading to weaknesses in the sub-critical valve flow assumptions. Additionally, due to element size limitations the finite element solution does not directly model the flow field in the regions of the valve and seals, but instead relies on resistance coefficient models similar to those used in the analytical models to capture the element behavior in these regions. Therefore, the finite element results could likely be improved using direct models of the seals and valve to capture the behavior more directly. Also, there are slight variations in the net force when the assumptions regarding fluid behavior are modified to include swirl, density variations, and turbulence.

Nevertheless, at lower supply pressures the analytical and finite element models exhibit similar trends in net force. The leakage results also compare favorably in the low valve setting regions. Another encouraging result of the finite element analyses is that the behavior of the net force with valve position appears more linear than the analytical model predicts. Therefore, it can be expected that for
a given supply pressure, the plant dynamics may exhibit less variation than predicted by the analytical model, making the controller design task simpler.

Axisymmetric contour plots from the finite element simulations (Figure 29) clearly illustrate the pressure and velocity distributions for each of two cases (Case 1: laminar, incompressible flow with no swirl, Case 2: turbulent, compressible flow with swirl). Higher pressure gradients are observed for Case 2, but the net force results shown comparable. In deriving the simplified model of the system (Equations 57,58), the air flow was considered incompressible. It is clear from Figure 29c that the maximum velocity of the air (555.6 in/s (14.112 m/s)) is sufficiently small relative to sonic velocity (100 m/s) to justify this assumption for the flow [25]. Furthermore, even when compressibility effects are introduced, the effect on the net force result is small (Figure 28a).

Figure 29. Finite element model contour plots (Ps=60psia (414kPa(a)), x0=100% open) - (a) Case 1 pressure, (b) Case 2 pressure, (c) Case 1 velocity, (d) Case 2 velocity
Active Balance System Performance (Steady)

The compressor characteristics used for this design study, derived from an example found in Aungier [26], is shown in Figure 30. This compressor is characterized by a relatively high pressure ratio stage with design flow coefficient of 0.095. The polytropic head coefficient and efficiency specifications assume that a vaned diffuser and an “open” (no shroud) impeller are used. The speed and diameter of the impeller were selected to provide a discharge pressure of approximately 414kPa(a) (60psia) near the surge point at maximum speed. Because higher pressure ratio compressors are characterized by high seal leak flow rates and static thrust loads, this type of machine is ideally suited to the study.

![Figure 30. Compressor design characteristics (Aungier)](image)

To evaluate actuator performance (i.e., leakage and force) over the compressor operating map, simulations were conducted based on the actuator model (Equations 57&58). Indeed, such simulations are necessary to properly size the following system components:

1) Stationary balancing seal on the impeller
2) Balance piston thrust disk
3) Leakage control seals on the balance piston
4) Control valve

The optimal design parameters were determined iteratively by considering some important constraints. First, the minimum radial clearance of all seal is limited to 0.127mm (0.005in), since
most AMB supported rotors have auxiliary bearings that warrant such clearance [27]. Second, the maximum clearance at the largest rotating radial location is limited to 0.254mm (0.010in) to account for high speed inertial deformations. All allowable clearances are linear interpolations of these values between radial locations \( r_s \) (shaft radius) and \( r_2 \) (impeller radius). Third, it is assumed that the seal on the impeller body is fixed in design (number of teeth, geometry, etc.) with the exception of its locating radius. For the stationary seal, further reductions in leak flow rates could be realized if the seal geometry were allowed to change to increase the tooth count. It is assumed that since the stationary seal resides in the hot portion of the compressor, it is subject to size limitations due to thermo-structural constraints. Finally, it is assumed that the balancing actuator is supplied with air after-cooled from the compressor to a constant temperature of 15.6 °C (60 °F). Indeed, one can see that a possible secondary benefit of this design is to use bleed air to cool the magnetic thrust bearing. Therefore, for cooling purposes one would need to have a lower air supply temperature than could be found at the discharge of the compressor. Furthermore, the use of cooled supply air would facilitate the use of lighter materials such as aluminum from which to construct the actuator.

In addition to these design constraints, inherent limitations of the magnetic bearing need to be considered. Results of the thrust bearing simulations presented previously indicated that dynamic capacity limited the static bearing load to 600 N in regions close to. Also, the static bearing capacity is limited to 1550 N (due to the current limitation with 30% bias current). Based on these constraints, the operating logic of the controller should position the valve in order to maximize the use of the bearing static capacity when compressor operation is away from surge but tend to unload the bearing as the operating point moves close to surge. Therefore, the compressor controller would have to assess the position of the compressor operating point on the map, and based on a measurement of the bearing force and map position, determine the balance system servo-loop set-point. Based on the limits described above (with 5% safety margin), the operating point valve positions required to meet these objectives were computed using Equations 57&58. The reference loading of the magnetic bearing is defined by Equation 59 which shows the set-point generating function that would be implemented on the compressor controller.

\[
F_{AMB \_ref} = \begin{cases} 
0.95(1500), & M_s \geq M_{s1} \\
(0.95) \left[ \frac{1500 - 600}{M_{s1} - M_{s2}} \right] (M_s - M_{s2}) + 600, & M_{s2} < M_s < M_{s1} \\
0.95(600), & M_s \leq M_{s2}
\end{cases}
\]  
(59)
Where $M_s$ is the surge margin and $M_s1 (=0.003)$ is the surge margin value when control actions starts. $M_{s0} (=7.5e-4)$ is the value of the surge margin corresponding to surge (offset slightly from 0, the ideal value). This set point generating function will result in loading the magnetic bearing close to the static capacity when the compressor is operating away from surge, but as the compressor approaches surge as measured by surge margin, $M_s$, the set-point for the loading of the magnetic bearing shifts toward the previously defined value of 600 N close to surge.

There are many ways to determine the surge margin. This value can come from measurements of drive power, such as current from a motor, or from compressor parameters such as flow or pressure. Here it is envisioned that the compressor controller is implemented on a microcontroller that facilitates calculation of the surge margin from compression process measurements (temperature and pressure). Furthermore, for this case the surge margin is computed from calculation of the non-dimensional head coefficient, relative to the surge head coefficient (approximately 0.61 in Figure 30).

\[ M_s = \mu - \mu_s; \quad \mu_s = \text{surge head coefficient} \]  \hspace{1cm} (60)

$M_s$, $\mu$, $\mu_s$ are the surge margin, head coefficient, and surge head coefficient respectively. Using these simulation parameters, the compressor’s operating map can be computed (Figure 31). The map is plotted between the compressor surge and choke lines, and 5 speed lines are shown from 100% to 50% of the maximum speed.
Figure 31. Compressor operating map with balancing device leakage effect: (a) Static, (b) Active

Figure 31a represents the design case using only a stationary balance seal on the impeller. For this case, the stationary seal shown in Figure 1 diameter is 106.68mm (4.2in) to prevent loads from exceeding 95% of the allowable bearing load close to surge (i.e., 600 N). The seal leakage is directly related to the upstream (compressor discharge) pressure, and is represented by a leftward shift in the compressor map outline (inlet vs. outlet flow). The map for the same compressor using the automatic balancing device is shown in Figure 31b. For portions of the operating map away from surge, the shift for the active system is less than that of the static system. Closer to surge, however, the shift is more when the pressure is high. The dotted line on each map represents the surge control line, located 15% from the surge line.

Indeed many centrifugal compressors use an anti-surge controller to keep the compressor out of surge and use control margins between 10% and 20% (flow) from the surge point [23]. Based on this consideration, it is more appropriate to compare the relative leakage performance of the static balance and dynamic balance systems, not over the entire map range, but in regions where anti-surge measures are not being taken. In such regions, all of the compressor discharge air is no longer supplied to the plant system, and as a result, seal leak flow is not significant compared to the compressor bleed air. Therefore, use of a surge control line is a more appropriate way to evaluate the performance implications of the active balancing system.
Figure 32 compares leakage flow rates of the static and active balance piston systems. Figure 32a shows leakage flow when the stationary seal has 5 teeth. In this case, the active system exhibits lower total leak flows in all areas of the compressor map, even in the region close to surge (when leakage is compromised in order to decrease the steady thrust load). Figure 32b shows leakage flow when the stationary seal has 10 teeth. In this case, the relative advantages of the active system are diminished compared to the 5 tooth stationary seal. Here, the leakage of the active system is lower in the normal operating range than the stationary system on every speed line except in the high pressure region of maximum speed. In the region close to surge, the active system tends to leak more, except at low speed.

Figure 32 makes it clear that leakage rates depend on the compressor’s operating point. In order to evaluate the relative performance of each system, a consistent comparison of leakage is needed. One method is to compute the centroid of the leakage map boundary using the following formula:
\[ G_n = \frac{\sum_{i=1}^{4} G_i A_i}{A_T} \quad ; \quad \bar{G}_i = \frac{\int G_i(m) dA}{A_i} \]

Where \( \bar{G}_i \) is the centroid location of the \( i \)th curve of the leakage map, \( m \) is the mass flow (abscissa) of the \( i \)th curve, \( A_i \) is the area under the \( i \)th curve, and \( A_T \) is the total area of the leakage map. The map regions (1,2,3,4) are illustrated in Figure 31a.

Strictly speaking, the leakage flow is not an explicit function of the mass flow, but the two are related through the compressor’s outlet pressure, a dependency which can be incorporated into the leakage equation (Equation 61). Table 5 compares the static and active balancing device average leak flow rates, computed using this centroidal measure of the average leakage.

**Table 5 – Comparison of static and active balancing device average leak flow rates**

<table>
<thead>
<tr>
<th>( G_n ) – Average (Centroidal) Leakage (kg/min)</th>
<th>Static Balancing</th>
<th>Active Balancing</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reference to Surge Line (( N_t=5 ))</td>
<td>1.1545</td>
<td>0.8721 (-24.46%)</td>
</tr>
<tr>
<td>Reference to Anti-Surge Control Line (( N_t=5 ))</td>
<td>1.1451</td>
<td>0.8434 (-26.35%)</td>
</tr>
<tr>
<td>Reference to Surge Line (( N_t=10 ))</td>
<td>0.8163</td>
<td>0.7425 (-11.73%)</td>
</tr>
<tr>
<td>Reference to Anti-Surge Control Line (( N_t=10 ))</td>
<td>0.8097</td>
<td>0.7147 (-11.73%)</td>
</tr>
</tbody>
</table>

Table 5 shows that the active system results in 24.5% less leakage than the static balancing device when the comparison is made over the entire map, and results in 26.4% less leakage over the normal operating range of the compressor (up to the 15% surge control line). Also, increasing the tooth count on the stationary seal from 5 to 10 reduces the relative advantages of the active system.
Figure 33. Difference in total leakage of balance system (static – dynamic): (a) Leak flow difference, (b) Percentage reduction in leak flow using active system

Figure 33 illustrates more clearly that the operating point of the compressor will greatly determine the actual performance improvement that is realizable from the use of the active balancing concept. For example, if the compressor is operated most of the time at low mass flow (low speed), the improvement in leakage realized can be between approximately 22-50% (Figure 33b). However, if the compressor is operated at high mass flow (full speed) and away from surge the improvement is between 11 to 37%. Likewise, if the compressor is operated at full speed and close to surge (high pressure ratio), the same relative improvement range is between -10 to 1% (no improvement).

Figure 34 shows that active system’s leak flow is composed of flows emanating from the stationary seal and from the actuator. Due to the smaller radius of the stationary seal used with the active balance piston (55.88 vs. 106.68mm (2.2 vs. 4.2 in)), the impeller seal leakage is lower than that of the static balancing device. As shown in Figure 34, the other leakage component of the active system is due to the controlled operation of the system using the set point logic described in Equation 59.
Figure 34. Composition of leak flow over operating map for active balancing system

Figure 35 shows the axial rotor forces over the range of the compressor map for each system.

The net forces of the static system, shown in Figure 35a, can be managed via design not to exceed 600 N to ensure adequate surge performance. For the active system (Figure 35b), the regulation of axial force follows the set point generation function of Equation 59. The controller for the balance piston acts to keep the axial force below 1500 N (with a 5% margin). As the system approaches surge, the set point is varied so that the net load carried by the magnetic bearing does not exceed 600
Figure 36 illustrates valve position for the same compressor map. The fact that the peak valve position is approximately 92% open suggests that there exists some excess capacity in the actuator which would be valuable to account for uncertainties.

The previous discussion serves to illustrate the potential benefits of the active balance piston. Significant improvements in compressor seal leakage can be realized by more effective use of the available axial AMB capacity. These simulation results, however, are limited to steady-state operating performance. In order to fully investigate the regulating capabilities of the active balance piston, its transient response characteristics must be explored.

Active Balance Piston Control Design and Dynamic Performance

The non-linear system equations for the active balance piston actuator have been derived and are presented in Equations 57&58. These equations can be linearized about a specific operating point \( x=x_0, P=P_0= [P_{10}, P_{20}] \) to obtain the state space representation:

\[
\begin{bmatrix}
\dot{P}_1 \\
\dot{P}_2
\end{bmatrix} =
\begin{bmatrix}
A_1 & A_2 \\
B_1 & B_2
\end{bmatrix}
\begin{bmatrix}
P_1 \\
P_2
\end{bmatrix}
+ \begin{bmatrix}
A_1 \\
0
\end{bmatrix} x;
F = \begin{bmatrix}
a_1 & a_2
\end{bmatrix}
\begin{bmatrix}
P_1 \\
P_2
\end{bmatrix}
\]

(62)

Where:
\[ A_1 = \frac{\partial f}{\partial x} \bigg|_{x=0} ; A_2 = \frac{\partial f}{\partial P_1} \bigg|_{x=0} ; A_3 = \frac{\partial f}{\partial P_2} \bigg|_{x=0} ; B_1 = \frac{\partial f}{\partial P_1} \bigg|_{x=0} ; B_2 = \frac{\partial f}{\partial P_2} \bigg|_{x=0} \]

Where \( x \) represents the system input (valve position) and \( F \) represents the system output (net force generated by the actuator, so \( a_{1,2} \) represent the areas of the actuator over which the pressures act).

Step responses of this linearized plant model were simulated for 5% valve steps about three different nominal operating points. These results were compared to the non-linear transient analyses using the incompressible, laminar solution to the finite element model of the actuator previously discussed (Figure 37).

![Figure 37. Comparison of actual step responses: linearized plant model vs. non-linear FEA model](image)

This figure reveals that the linearized model and the finite element model compare favorably when the valve is almost fully open (\( x=0.95 \) or 95% open). Furthermore, the responses from the 0% and 50% nominal points show similar initial slopes between the two models, but the steady state values are quite different. These differences in response can be explained by the steady-state solution differences that were observed in Figure 28. Therefore, controller synthesis should address parameter uncertainty in the model, and should seek to optimize robustness of the control system.
Active Balance Piston Servo Observer Based Controller

Given these concerns with model uncertainty, a Type 1 servo using a state feedback control law was designed (Figure 38) [11]. To avoid the need for full-state measurement (due to difficulties in measuring chamber pressures), a full-order state observer was also designed.

![Figure 38. Block diagram of controller and observer (Simulink®)](image)

**Table 6 – Parameters of the servo observer-controller system**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Parameter</th>
<th>Description</th>
<th>Parameter</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r )</td>
<td>Reference force</td>
<td>( A )</td>
<td>System matrix of plant</td>
<td>( e )</td>
<td>State estimate error</td>
</tr>
<tr>
<td>( y )</td>
<td>Output force</td>
<td>( B )</td>
<td>Input vector of plant</td>
<td>( K )</td>
<td>State feedback gain</td>
</tr>
<tr>
<td>( \xi )</td>
<td>Integrated error</td>
<td>( C )</td>
<td>Output vector of plant</td>
<td>( L )</td>
<td>Observer gain</td>
</tr>
<tr>
<td>( k_i )</td>
<td>Integrator gain</td>
<td>( x )</td>
<td>State vector ((p_1, p_2))</td>
<td>(~)</td>
<td>Estimate</td>
</tr>
<tr>
<td>( u )</td>
<td>Plant input</td>
<td>( X_f )</td>
<td>State vector full system with observer</td>
<td>( X_e )</td>
<td>State vector modified system</td>
</tr>
</tbody>
</table>

The feedback control law for this SISO system is:

\[
u = k_i \xi - K \ddot{x}
\] (63)
\begin{equation}
\dot{\xi} = r - y = r - Cx
\end{equation}

For the observer, the error between actual and observed states is:

\begin{equation}
e = x - \tilde{x}
\end{equation}

These observer error and plant dynamics are governed by:

\begin{align*}
\dot{e} &= (A - LC)e \\
\dot{x} &= (A - BK)x + Bke + Bk_i \xi
\end{align*}

Combining equations 64, 66, 67 yields the system representation in state vector form:

\begin{equation}
\begin{bmatrix}
\dot{x} \\
\dot{e} \\
\xi
\end{bmatrix} = \begin{bmatrix}
(A - BK) & BK & Bk_i \\
0 & (A - LC) & 0 \\
-C & 0 & 0
\end{bmatrix} \begin{bmatrix}
x \\
e \\
\xi
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} + \begin{bmatrix}
x \\
e \\
\xi
\end{bmatrix}
\end{equation}

Equation 68 relates the output axial force to the input reference command for the servo controller, and reveals that the closed-loop system is 5th order. Therefore, in order to define the controller gains, \( K \), \( k_i \), and \( L \) five desired pole locations must be specified. Two poles relate to the state feedback gain, two relate to the observer gain, and one relates to the integrator. The controllability and observability matrices for this system are given by:

\begin{equation}
M_{ad} = \begin{bmatrix}
B_i & A_i B_i & (A_i)^T B_i & (A_i)^T B_i & (A_i)^T B_i
\end{bmatrix} \rightarrow rank(M_{ad}) = 3
\end{equation}

\begin{equation}
M_{od} = \begin{bmatrix}
C_i^T & A_i C_i^T & (A_i)^T C_i^T & (A_i)^T C_i^T & (A_i)^T C_i^T
\end{bmatrix} \rightarrow rank(M_{od}) = 5
\end{equation}

Given that the system of Equation 68 has 5 states, the condition for controllability is not met (since the state estimates are inside the state vector of this system), so the problem must be reduced to assign the poles. In order to fully specify these poles, the problem was broken up to first specify the poles of the servo loop under a full state feedback assumption. Then, using the separation principle for
observer design, the observer poles were separately placed [11]. The full-state system poles can be realized by removing the state observer from Figure 38 and feeding back the states directly:

\[
\begin{bmatrix}
\dot{x} \\
\dot{\xi}
\end{bmatrix} =
\begin{bmatrix}
A & 0 \\
-C & 0
\end{bmatrix}
\begin{bmatrix}
x \\
\xi
\end{bmatrix} +
\begin{bmatrix}
B & \text{0} \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
0 \\
u + \text{0}
\end{bmatrix}
\tag{71}
\]

To eliminate the reference input from Equation 71, a step input is considered and the states are redefined as deviations from steady state values:

\[
\begin{bmatrix}
\dot{x}_e \\
\dot{\xi}_e
\end{bmatrix} =
\begin{bmatrix}
A & 0 \\
-C & 0
\end{bmatrix}
\begin{bmatrix}
x_e \\
\xi_e
\end{bmatrix} +
\begin{bmatrix}
B & \text{0} \\
0 & 1
\end{bmatrix}
u_e \quad \longrightarrow \quad \dot{X}_e = AX_e + B_u e
\tag{72}
\]

Where:

\[
\begin{align*}
x_e &= x(t) - x(\infty) \\
\xi_e &= \xi(t) - \xi(\infty) \\
u_e &= u(t) - u(\infty) \\
r_e &= r(t) - r(\infty) = 0 \quad \text{(for unit step)}
\end{align*}
\]

The feedback control law thus becomes:

\[
u_e = -\hat{k}_e \begin{bmatrix}
x_e \\
\xi_e
\end{bmatrix}; \quad \hat{k}_e = \begin{bmatrix}
K \\
k_i
\end{bmatrix}
\tag{73}
\]

The first two elements of the feedback gain matrix correspond to the actuator plant state feedback (pressure) gains, and the third element is the forward path integrator gain. It is shown in the reference [11] that the system represented in Equation 72 is controllable given that the underlying plant system:

\[
\dot{x} = Ax + Bu; \quad y =Cx
\tag{74}
\]

is controllable. After substitution of the control law into Equation 72, the eigenvalue problem from which the state feedback gains can be computed becomes:

\[
\begin{bmatrix}
\dot{x}_e \\
\dot{\xi}_e
\end{bmatrix} =
\begin{bmatrix}
A - BK & BK_i \\
-C & 0
\end{bmatrix}
\begin{bmatrix}
x_e \\
\xi_e
\end{bmatrix}
\tag{75}
\]
Since the system of Equation 72 is single input, single output, Ackermann’s formula can be used for the pole placement:

\[ \hat{\mathbf{k}}_r = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{B} & \mathbf{A} \mathbf{B} & \mathbf{A}^2 \mathbf{B} & \ldots \mathbf{A}^n \mathbf{B} \end{bmatrix}^* \Phi_x \]
\[ \Phi_x = \alpha_1 \mathbf{A}^1_x + \alpha_2 \mathbf{A}^2_x + \alpha_3 \mathbf{A}^3_x + \mathbf{I} \]

(76)

Where, \( \alpha_{ei} \) is the \( i^{th} \) coefficient of the characteristic polynomial of the system matrix, \( \mathbf{A}_c \).

Since it is often impractical to have direct measurement of the plant states in this design, an observer can be designed for the purpose of estimating the plant states from the output. Referencing Figure 38 the following equations are evident from the system diagram:

\[ \dot{\mathbf{x}} = \mathbf{A} \mathbf{x} + \mathbf{B} \mathbf{u} + \mathbf{L} (\mathbf{y} - \tilde{\mathbf{y}}) \]
\[ \mathbf{e} = \mathbf{x} - \dot{\mathbf{x}}; \quad \tilde{\mathbf{y}} = \mathbf{C} \mathbf{x} \]
\[ \dot{\mathbf{e}} = [\mathbf{A} - \mathbf{L} \mathbf{C}] \mathbf{e} \]

(77)  (78)  (79)

And the corresponding pole placement and resulting observer gain calculation can proceed using Ackermann’s formula and the observability matrix.

\[ \mathbf{L} = \Phi_x \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} & \mathbf{1} \end{bmatrix} \]
\[ \Phi_x = \alpha_1 \mathbf{A}^1_x + \alpha_2 \mathbf{A}^2_x + \mathbf{I} \]

(80)

Guiding the pole placement scheme was a desire to have well-damped poles that result in closed loop dynamics with generous stability margins to account for the model uncertainty. The primary poles relating to the state feedback gain, \( \mathbf{K} \), were placed closest to the origin. The integrator pole was located to the left of this open loop pole. In this way, the closed-loop system was designed to behave like a low-order system, with the poles closer to the origin dominating the closed loop response. The observer poles were placed in the left half plane at least 3 times faster than the fastest state feedback pole to ensure that the state estimate error converged quickly. The convergence of the observer state estimates to the actual states can be seen from the initial condition response of Figure 39.
Figure 39. Initial condition response showing observer state estimates vs. actual values.

The closed loop system comprising the servo controller and actuator plant is shown in Figure 40.

Using this block diagram and the controller outlined above, the controller transfer function (K_bp in Figure 40) is:

\[
k_{\infty}(s) = \frac{-U(s)}{Y(s)} = \frac{k_i + K \left[ sI - A + BK + LC \right]}{s} \left[ sL - Bk \right]
\]  

\[\text{(81)}\]

Figure 40. Servo observer controller block diagram (Simulink®)
As demonstrated from the finite element simulations, variations between the model and the actual plant are inevitable, so it is desirable to have small values of the sensitivity transfer function in the low frequency range, inside the bandwidth of the actuator. In this way, the closed loop performance of the control loop will be less influenced by plant variations. Furthermore, with the addition of a loop gain to the loop of Figure 40, the closed loop properties can be adjusted to decrease the sensitivity. The properties of the closed loop system can be evaluated using the transfer functions defining the open and closed loop diagram that is referenced in Figure 40:

\[
T_{bp}(s) = k_p k_a G_{bp} 
\]  

(82)

\[
C_{bp}(s) = \frac{T_{bp}}{1 + T_{bp}}
\]  

(83)

\[
S_{bp}(s) = \frac{1}{1 + T_{bp}}
\]  

(84)

\( T_{bp}(s), C_{bp}(s), S_{bp}(s) \) are the loop, closed loop, and sensitivity functions respectively. The controller design outlined here assumes that the plant dynamics are constant. However, as previously demonstrated in the actuator performance analysis, the plant dynamics change significantly as the valve position advances from 0 to 1 (0-100%) and as the supply pressure varies from 101kPa(a) (14.7psia) (ambient pressure) to 414kPa(a) (60psia) (maximum pressure). To illustrate this point, consider the variation of eigenvalues of the plant model’s system (A) matrix over the range of these two variables (\( p_{so} = [138 207 276 345 414](kPa(a)) \), \( x_{so} = [0 0.25 0.50 0.75 1.0] \)) as shown in Figure 41.
Considering these significant variations in the actuator plant dynamics, a family of observer based controllers has been designed over the range of nominal plants described by \((p_{s0}, x_{s0})\) in Equation 62. Although the number and distribution of points about which to specify controllers is up to the discretion of the designer, enough should be included to avoid interpolation problems. For this analysis, the valve increments and supply pressures for which controllers are shown above.

This selection of nominal points leads to a 5 x 5 matrix of controllers to be specified through the eigenvalue assignment procedure outlined above. Figure 42 shows the closed loop step responses of the systems for 5 nominal points at the maximum and minimum pressures for which the controllers are designed \((138kPa, x_{s0})\) and \((414kPa, x_{s0})\).
Notice that all of the responses are well damped and settle within 0.3s. These response characteristics are significantly slower than the magnetic bearing, but should be sufficient for the purpose of the servo controller, given its purpose to adjust the static axial loading of the rotor. The controller is intended to impart high stability margins to the family of controllers so that adjustments for plant modeling errors later can be accommodated. This is reflected in the step response plots, but can be seen as well from the Nyquist plots of the loop transfer functions of these respective systems, shown in Figure 43.

Figure 42. Closed loop step responses for two supply pressures: (a) Ps=138 kPa(a), (b) Ps=414 kPa(a)

Figure 43. Nyquist plot of open loop transfer function of servo system for two supply pressures:
(a) Ps=138 kPa(a), (b) Ps=414 kPa(a)
The loop transfer functions are all stable, without encirclements of the critical point (s=-1). Furthermore, the systems are all minimum phase, exhibit infinite gain margins and high phase margins (typically >75 degrees). The complete set of controller pole specifications and controller gains is provided in Appendix A2.

The primary effect of the observer on the closed loop dynamics is increased bandwidth relative to the full state feedback case. This bandwidth increase tends to reduce the sensitivity of the system to plant model changes. However, in order to make further sensitivity reductions, a loop gain was added (ref. Figure 40). To illustrate this increase in bandwidth, Figure 44 compares frequency responses of the open loop plant and the closed loop observer based controller. Also plotted is the sensitivity function.

![Figure 44. Frequency response of closed loop servo system, sensitivity of closed loop, plant system, and closed loop and sensitivity with loop gain increase (K_p=1 to K_p=5). Point: x_0 = 100%, P_s = 414 kPa(a)](image)

The sensitivity relates changes in output to plant perturbations by [29]:

$$\frac{dy}{y} = S \frac{dG_r}{G_p}$$

(85)
Therefore, given an expectation of significant plant variations, \( dG_p \), a closed loop system with small values of magnitude of \( S(s) \) over the relevant frequency range of the actuator is sought. In Figure 44, for the bandwidth of the actuator (taken at the roll off point around 35 rad/s), the sensitivity magnitude is approximately \(-6\)dB, so an accommodation of a 2X change in the nominal plant with no change in the output is expected. However, if the loop gain is increased (as shown for \( k_p = 5 \)), then the value of the sensitivity reduces to approximately \(-20\)dB at the same point, and thus improves the robustness by accommodating 10X changes in the plant. However, taking this action also reduces the phase margin, thus affecting the relative stability and imparting overshoot in the step response. Raising the loop gain also increases the closed loop system bandwidth, so sensor noise could become a problem. Furthermore, with non-unity loop gain, the closed loop poles will no longer lie at the original specification.

Implied in the notion of the design of a family of controllers described above is that the overall system is not time invariant. Indeed, given the control strategy described by the set point generation of Equation 59, and the non-linearity of the actuator, the existence of time-dependence of the plant is anticipated. Therefore, it is presumed here that this family of controllers would be implemented along with a gain selection protocol that would allow proper controller selection/interpolation based on the compressor operation point and valve position information (either by inference or measurement of the valve position). For gain scheduling controllers to be effective, the time-dependence of the underlying system must be sufficiently slow relative to the closed loop dynamics of each of the “frozen time” linear systems given by the respective bounding nominal points \((p_{s0-1}, x_{s0-1}), (p_{s0-2}, x_{s0-2})\) [28]. In order to evaluate the suitability of the time-dependent system for implementation of an interpolated gain scheduling procedure, one reference suggests the following upper bound on the rate of change of the scheduling parameter [30]:

\[
|\zeta(t)| \leq \min_{i=1 \ldots q-1} \left\{ \frac{|c_i - b_i|}{\|W_{i+1} - W_i\|} \right\} ; \quad \zeta(t) = \text{scheduling parameter} \tag{86}
\]

Where \(i = 1 \ldots q-1\) are the divisions of the range of the scheduling parameter, over a range and \([b_i, c_i]\) is an interval over which to evaluate consecutive linear, stable, time invariant systems, \((A + BK_i)\). The symmetric, positive definite matrices, \(W_i\), and \(W_{i+1}\) are found from the following theorem [30]:

\[
W_i (A_x(\zeta) + B_x(\zeta)K_i)^T + (A_x(\zeta) + B_x(\zeta)K_i)W_i < I \quad (\text{state feedback gain}) \tag{87}
\]
Since the compressor is implemented typically as a pressure controlling device along with a reservoir appropriately sized to satisfy the compressed air system demand, it is reasonable to conclude that the dynamics of the compressed air system will govern the rates at which the scheduling variables change. Taking this as a guiding assumption, it is logical to conclude that the discharge pressure of the compressor will be the appropriate scheduling variable that governs the rate of change of the system parameters defining the nominal operation point of the compressor. Consider an example test case of operating the control valve at \( x = 0.5 \) in the region of supply pressure between \( p_{s0-1} = 138 \text{kPa}(a) \) (20 psia), and \( p_{s0-2} = 276 \text{kPa}(a) \) (40 psia). This is a region where the controller gains had to be adjusted to satisfy the desired dynamic properties of the servo system. For this case, using Equations 86, 87, 88 the resulting maximum allowable rate of change of the scheduling variable becomes:

\[
|\dot{z}(t)| = |\dot{p}| \leq \min \left[ \frac{40 - 20}{0.3853}, \frac{40 - 20}{0.0002} \right] \]

\[
|\dot{p}| \leq 51.9 \text{ psi/s (358 kPa/s)}
\]

This maximum rate of change value can be compared to the expected pressure “slew” rate of the compressed air plant system which is calculated using a basic rule of thumb for compressed air systems to have as a reservoir approximately minimum size of 1.0 gal/cfm (2228 cm\(^3\)/(m\(^3\)/hr)) [31]:

\[
\frac{dp}{dt} = \frac{\rho(\Delta \dot{Q})}{C}, \quad C = \frac{V_{reservoir}}{nR_{air}(T + 460)}
\]

\[
V_{reservoir} = 1.0 \Delta \dot{Q} = \frac{1.0 \text{gal/cfm}(55 \text{kg/min})(1 \text{lbf} / 0.454 \text{kg})}{0.075 \text{lbf/ft}^2} = 1,615 \text{ gal} = 216 \text{ ft}^3 = 6,113,440 \text{ cm}^3
\]

\[
C = \frac{(216 \text{ ft}^3)(144 \text{ in}^2 / \text{ft}^2)}{(1.0)(53.356 \text{ ft-lbf/lbm-R}(520 \text{R}))} = 1.121 \text{lbf-in/} \text{lbf (72.3 kg-mm}^2 \text{/ N)}
\]

\[
\frac{dp}{dt} = \frac{(55 \text{kg/min})(1 \text{lbf} / 0.454 \text{kg})}{(1.121 \text{lbf-in/} \text{lbf}(60 \text{s/min})} \approx 1.8 \text{psi/s (12.4 kPa/s)}
\]

Therefore, using a typical mass flow for the compressor of 55kg/min, since the expected compressor dynamics are significantly slower (12.4 kPa/s vs. 358 kPa/s) than the maximum allowable rate of change of the scheduling variable, a reasonable chance of successful implementation of a gain
scheduling procedure is anticipated. Furthermore, the storage rule of thumb is a lower bound, and most practical systems would have more capacity required in order to adequately control the compressed air system pressure. It is not intended to design and simulate the gain scheduled controller here, but rather to indicate it’s prognosis for success.

CONCLUSION

The motivation for this work was to design and simulate an alternative thrust load management system for turbo machines using active magnetic thrust bearings. Because the static capacities of AMBs are generally much larger than their dynamic capacities, imposing the dynamic capacity restrictions in the design of stationary thrust piston balancing devices leads to under-utilization of the AMB and over-penalizes the efficiency of the compressor. For these reasons, an active thrust load management system is highly desirable.

In this thesis, a centrifugal compressor rotor supported on axial AMBs was investigated. The AMB system was modeled using appropriate dynamic limitations. An analytical model was developed to study the phase lag associated with eddy currents in the actuator. Finite element modeling was used to validate the adoption of a fractional order actuator transfer function model. Approximation techniques were utilized to convert this fractional order transfer function to a rational, high order transfer function more suitable to simulations. Current saturation, RL lag effects and flux saturation were included in the system model to capture realistic dynamic response characteristics.

The transient response of the AMB model to a surge disturbance was simulated to determine the limits of performance. Simulation of the closed loop system with the actuator dynamics included reveals a 60% increase in step disturbance response peak relative to the same system without these included. Likewise, the gain margin decreased by 20.9% and the phase margin decreased by 15.6 degrees relative to the system without the actuator dynamic limitation. From the surge simulations, it was found that current saturation and actuator dynamic limitations were the most important factors limiting the surge response of the bearing. The conclusions of this study were incorporated into the design of a thrust balancing system. The force and leakage performance characteristics of two prototype thrust management systems were compared. The first system employed a stationary balancing ring seal. The second system comprised of a stationary thrust piston actuator to which a control valve is connected, and a method for operating the valve based on compressor map and AMB
loading conditions is suggested. Using an example compressor characteristic, the proposed active
system was shown to offer an average leakage rate reduction of 24.5% to 26.4% relative to the static
balancing device for a 5 tooth stationary seal and 9% to 11.7% for a 10 tooth stationary seal. Relative
performance of these two systems is of course related to actual compressor operating characteristics.

An observer-based servo controller was designed to operate the control valve. The balancing device
actuator was designed and simulated with a one dimensional continuity equation based model. The
results of this model were compared to an axi-symmetric, finite element based flow model.
Comparisons of static and transient performance of the actuator were made. Results show good
agreement between the 1D and FEA models for valve positions in the valve position range 0.7 – 1.0
at high supply pressures, and less agreement at lower valve position and pressures. Therefore, one
has to be concerned with plant model fluctuations below approximately 70% open valve positions.
To address this issue, the observer controller gains are set relatively high to reduce sensitivity.
Proceeding with the analytical model, a family of controllers was designed around the range of supply
pressures and control valve positions. Although the controller interpolation was not specified, the
required rate of change of the scheduling variable (supply pressure) was found to exceed by over two
times its expected rate of change. Therefore, there is some evidence that the servo controller could be
implemented with gain scheduling.

**RECOMMENDATIONS FOR FUTURE WORK**

There are clearly areas where this research and development could be extended. First, the AMB
system model utilized was relatively basic, and could be improved by including position sensor
dynamics, a flux feedback model, and other enhancements that are typical of modern, commercial
bearing systems. Furthermore, other controller architectures could be considered. Second, testing
should be conducted to validate the actuator transfer function proposed. The surge model used for the
bearing simulations could be improved to capture the compressor system dynamics more accurately.
Such a model needs to incorporate the downstream and upstream plena geometries, and adopt some
dynamic model of the compressor and throttle characteristics. Greitzer provides one of the most
recognized dynamic models of a dynamic compressor system, and thus could be a good starting point
[32]. Another subject of future work is to improve the balance piston modeling to address the
deficiencies in accuracy at low valve position settings. Some improvements could include
modification of the numerical minimization procedures to achieve better mass flow balance in the
actuator and to verify the seal flow models with finite element models matching the envisioned geometry. Finally, a prototype of the device should be constructed to validate the actuator plant dynamic models and to test the servo controller performance, with an ultimate objective to test in an axial AMB supported compressor.

There are many other factors that influence the design of a centrifugal compressor, but have not been treated explicitly in this study. For example, thermal and structural effects must be considered for the rotating bodies. Also, the shaft lateral rotor-dynamics and bearings have not been addressed. Finally, detailed geometry of the aerodynamic components required to construct the compressor is not developed. The design of any turbomachine is an inherently iterative process, requiring many converging factors to be addressed through the normal engineering tradeoffs and compromises. This study is focused on just one aspect (albeit an important aspect) of this process.
REFERENCES


Appendix A: Derivation of Effective Reluctances Used in AMB Actuator Model

A1) Effective Reluctance of Element 7 (Also applies to Element 1)

Assuming a linear, conducting magnetic material, of constant conductivity and permeability, the starting point for the derivation is the field equation (Maxwell’s Equation):

\[ \nabla^2 \vec{H} = \sigma \mu \mu_0 \frac{\partial}{\partial t} (\vec{H}) \]  

(90)

Where \( \vec{H} \) is the magnetic field vector, \( \sigma \) is the conductivity, and \( \mu \) is the permeability. For an axially directed field that has only dependence on radius \( r \), for this coordinate frame reduces to:

\[ \frac{\partial^2 H}{\partial r^2} + \frac{1}{r} \frac{\partial H}{\partial r} = \sigma \mu \mu_0 \frac{\partial H}{\partial t}; \quad H = H_z \]  

(91)

If the excitation is sinusoidal, then adopting a frequency response approach, the following form is derived:
\[ \frac{d^2 H}{dr^2} + \frac{1}{r} \frac{dH}{dr} = \alpha^2 H; \quad \frac{\partial}{\partial t} = j \omega; \quad \alpha = \frac{1 + j}{\delta}; \quad \delta = \sqrt{\frac{2}{\omega \sigma \mu_0}}; \quad \alpha^2 = j \omega \sigma \mu_0 \] 

(92)

The problem above has a mixed Neumann, Dirichlet boundary as shown in Figure 45. The solution to the 2nd order differential equation is given by:

\[ H(r, \omega) = A_1 * I_0(\alpha r) + A_2 * K_0(\alpha r) \] 

(93)

After applying the boundary conditions, the solution for the field:

\[ H(r, \omega) = H_0 \left[ \frac{K_1(\alpha r) I_0(\alpha r) + I_1(\alpha r) K_0(\alpha r)}{K_1(\alpha r) I_0(\alpha r) + I_1(\alpha r) K_0(\alpha r)} \right] \] 

(94)

After integrating this expression over the surface, expressions for the flux density and flux become:

\[ B(r, \omega) = \mu \mu_0 H(r, \omega) \] 

(95)

\[ \Phi(\omega) = 2\pi \mu \mu_0 \int_0^\infty rB(r, \omega)dr \] 

(96)

After integrating, the exact solution for the flux becomes:

\[ \Phi(\omega) = \frac{2\pi \mu \mu_0 H_0}{\alpha} \left[ \frac{K_1(\alpha r) I_0(\alpha r) - I_1(\alpha r) K_0(\alpha r)}{K_1(\alpha r) I_0(\alpha r) + I_1(\alpha r) K_0(\alpha r)} \right] \] 

(97)

After application of Ohm’s Law for Magnetic Circuits, an expression for the effective reluctance for the element with length \( \ell \) becomes:

\[ R_{eff} = \frac{\alpha \ell}{2\pi \mu \mu_0} \left[ \frac{K_1(\alpha r) I_0(\alpha r) - I_1(\alpha r) K_0(\alpha r)}{K_1(\alpha r) I_0(\alpha r) + I_1(\alpha r) K_0(\alpha r)} \right] \] 

(98)
A2) Effective Reluctance of Element 3 (Also applies to Element 6 and 8)

After applying the scalar Laplacian in cylindrical coordinates to the field directed radially, with dependence on r and z (The r dependence arises due to the r dependence of the surface field), the appropriate form of Maxwell’s Equation is:

\[
\frac{\partial^2 H}{\partial r^2} + \frac{1}{r} \frac{\partial H}{\partial r} + \frac{\partial^2 H}{\partial z^2} = \sigma \mu \frac{\partial H}{\partial t} \tag{99}
\]

Using separation of variables and noting that the z variation of the field exhibits the time-dependence since the r dependence of the field is due to the geometry and not wave propagation, two equations are obtained:

\[
\frac{\partial^2 H}{\partial r^2} = 0 \tag{100}
\]

\[
\frac{\partial^2 H}{\partial z^2} = \sigma \mu \epsilon \frac{\partial H}{\partial t} = \alpha^2 H \tag{101}
\]
When these equations are solved together for the case of the mixed boundary shown, then the resulting field distribution is:

$$H(r, z, \omega) = H_s(r) \left[ \frac{e^{\alpha(z-z_0)} + e^{-\alpha(z-z_0)}}{e^{\alpha_0} + e^{-\alpha_0}} \right]$$ (102)

Given the radial-directed field exhibits \( r \) and \( z \) spatial variation, integration over the entire boundary of the element to obtain the expression for the effective reluctance reveals the following starting from a differential element, \( dr \), with reluctance contribution, \( dR_{\text{eff}} \):

$$dR_{\text{eff}} = \frac{H_s(r) dr}{d\phi}$$ (103)

After integrating, the solution for the flux becomes:

$$R_{\text{eff}} = \frac{1}{2\pi\mu_0\mu_\alpha} \int_{r_0}^{r_0} \left[ \frac{dr}{e^{\alpha(z-z_0)} + e^{-\alpha(z-z_0)}} \right] dz$$ (104)

$$R_{\text{eff}} = \frac{\alpha \ln \left( \frac{r_0}{r_z} \right)}{2\pi\mu_0\mu_\alpha \tanh(\alpha z)}$$ (105)
A3) Effective Reluctance of Element 2 (Also applies to Element 4)

![Element 2 geometry sketch]

This element is characterized by two transition regions, where the field orientation changes to coincide with the \( r \) direction, while it is oriented in the \( z \) direction when crossing the air gap. This is confirmed by the plots of Figure 14. These regions and the air gap comprise the region of interest in defining the effective reluctance of this element.

In the derivation of the element effective reluctance, an argument based on wave propagation theory is proposed [13]. The argument is that the penetration of the time-varying field attenuates exponentially from the value at the surface. Specifically, that the penetration of the field into the solid is such that the field strength is 0 at depth \( d \) (\( \delta \) is the skin depth):

\[
d = 3\delta; \quad \delta = \frac{1}{\sqrt{\pi f \mu_0 \mu \sigma}}
\] (106)

So the effective permeability in the transition zones is:

\[
\mu_{\text{eff}} = \mu_0 \mu e^{-\frac{d}{\delta}}
\] (107)
In the region of the transition zones and the air gap, the field is clearly not one dimensional. However, it is nonetheless assumed that the field is one dimensional oriented along $r$ in the transition regions and along $z$ in the air gap. A subdivision of the transition zones and air gap is presented as shown in Figure 48.

![Figure 48 – Reluctance model for differential element subdivision of element 2 [13]](image)

Using the effective permeability expression and integrating over the region from $z=0$ to $z=d$, the effective reluctance of the differential elements can be found:

$$
\frac{1}{\Delta R_z} = \frac{1}{\varepsilon} \int_0^d 2\pi \mu_0 \mu r e^{-at} dz = \frac{2\pi \mu_0 \mu r}{\varepsilon} \left[ \frac{1}{\alpha} - \frac{e^{-at}}{\alpha} \right] \approx \frac{2\pi \mu_0 \mu}{\alpha \varepsilon} \quad (\alpha \text{ large})
$$

(108)

And therefore the differential element reluctances are:

$$
R_{1e} = \frac{\alpha \varepsilon}{2\pi \left( r - \frac{e}{2} \right) \mu_0 \mu},
$$

(109)
\[ R_{2e} = \frac{\alpha \varepsilon}{2\pi \left( r + \frac{\varepsilon}{2} \right) \mu_r \mu_0} \]  

(110)

And the differential air gap reluctance is:

\[ R_s = \frac{g}{2\pi \varepsilon \mu_b} \]  

(111)

Using an equivalent circuit approach, a network of reluctances that represents the behavior in the regions of the air gap and the transition zones is shown in Figure 49:

**Figure 49 – Reluctance network model of air gap and near air gap regions of element 2** [13]

From continuity of flux at point A and from the balance in the network:

\[ F_s(r + \varepsilon) - 0 = F_s - F_s(r + \varepsilon) \]  

(112)

\[ F_s(r) - 0 = F_s - F_s(r) \]  

(113)

\[ \frac{F_s(r + \varepsilon) - F_s(r)}{R_{2e}} = \frac{2F_s(r) - F_s(r + \varepsilon)}{R_s} + \frac{F_s(r) - F_s(r - \varepsilon)}{R_{1e}} \]  

(114)
Now, proceeding as below the differential equation for the MMF across the network is derived:

$$\Delta F_s(r) = F_s(r) - F_s(r - \varepsilon)$$ \hspace{1cm} (115)$$

By linearized expansion:

$$\Delta F_s(r + \varepsilon) = \Delta F_s(r) + \frac{d\Delta F_s}{dr} \varepsilon = F_s(r + \varepsilon) - F_s(r)$$ \hspace{1cm} (116)$$

Thus, by substitution into Equation 114

$$\frac{1}{R_{se}} \frac{d\Delta F_s}{dr} \varepsilon + \left[ \frac{1}{R_{se}} - \frac{1}{R_{sw}} \right] \Delta F_s(r) = \frac{2F_s(r) - F_s}{R_s}$$ \hspace{1cm} (117)$$

After substitution of the differential reluctance terms yields:

$$\frac{d\Delta F_s}{dr} + \frac{1}{2r} \frac{d\Delta F_s}{dr} \varepsilon + \frac{1}{\varepsilon} \frac{\Delta F_s}{r} = \frac{\alpha}{\mu, g} \left[ 2F_s(r) - F_s \right]$$ \hspace{1cm} (118)$$

Now, by taking the limit as $\varepsilon$ approaches 0:

$$\frac{d^2 F_s}{dr^2} + \frac{1}{r} \frac{dF_s}{dr} - \alpha_i F_s(r) = \frac{\alpha_i^2}{2} F_s; \hspace{1cm} \alpha_i = \frac{2\alpha}{\mu, g}$$ \hspace{1cm} (119)$$

Equation 119 is a modified Bessel Equation similar to what would be obtained for the one dimensional problems in terms of the magnetic field, $H$. Applying the mixed boundary conditions:

$$F_s(r_z) = F_s; \hspace{1cm} \frac{dF_s}{dr} \bigg|_{r_z} = 0$$ \hspace{1cm} (120)$$

The solution is:
By symmetry of the network:

$$F_r(r, \omega) = \frac{F}{2} + \frac{F}{2} \left[ \frac{K_s(a, r_i) I_s(a, r) + I_1(a, r_i) K_s(a, r)}{K_s(a, r_i) I_s(a, r) + I_1(a, r_i) K_s(a, r)} \right]$$

(121)

The air gap flux becomes:

$$\phi_r(\omega) = \int \frac{F(r) - F_r(r)}{R_r(r)} dr = \frac{2 \pi \mu_0}{g} \int r(F(r) - F_r(r)) dr$$

(123)

And the effective reluctance is therefore:

$$R_{eff} = \frac{F}{\phi_r(\omega)} = \frac{g \alpha}{2 \pi \mu_0 r_2} \left[ \frac{K_s(a, r_i) I_s(a, r) + I_1(a, r_i) K_s(a, r)}{K_s(a, r_i) I_s(a, r) + I_1(a, r_i) K_s(a, r)} \right]$$

(124)

A4) Effective Reluctance of Element 5
Element 5 is unique among the elements comprising the specified division of the actuator in the sense that the field distribution is difficult to argue is one dimensional. A 2D treatment indeed would be more precise to represent the field distribution as:

\[ \tilde{H} = H_r(r, z) \hat{a}_r + H_z(r) \hat{a}_z, \]  

(125)

In order to solve this type of problem for the two dimensional field distribution, it is useful to introduce the vector potential function [14], defined as:

\[ \nabla^2 \times \vec{A} = \vec{B} \quad \text{and} \quad \nabla \cdot \vec{A} = 0 \]  

(126)

Where \( \vec{A} = A(r, z) \hat{a}_r \) is the vector potential. The vector potential is aligned with the direction of the current density vector \((\vec{J})\) in the solid.

The Maxwell equation for a linear conductor of constant permeability expressed in terms of the vector potential becomes:

\[ \nabla^2 \vec{A} = \sigma \mu \left( \frac{\partial \vec{A}}{\partial t} + \nabla V \right) \]  

(127)

Here, \( V \) is the scalar electric field potential. Taking the vector Laplacian of \( \vec{A} = A(r, z) \hat{a}_r \) and the gradient of \( V \) in cylindrical coordinates and substituting into Equation 127:

\[ \frac{\partial^2 \vec{A}}{\partial r^2} + \frac{1}{r} \frac{\partial \vec{A}}{\partial r} - \frac{A}{r^2} + \frac{\partial^2 \vec{A}}{\partial z^2} = \sigma \mu \left( \frac{\partial \vec{A}}{\partial t} + \frac{1}{r} \frac{\partial V}{\partial \theta} \right) \]  

(128)

Considering the excitation to be sinusoidal and making the substitution [14]:

\[ \vec{A}' = A + j \frac{1}{\omega r} \frac{\partial V}{\partial \theta} \]  

(129)

It is possible to obtain the appropriate differential equation to solve for the vector potential:
\[
\frac{\partial^2 A}{\partial r^2} + \frac{1}{r} \frac{\partial A}{\partial r} - \frac{A}{r^2} + \frac{\partial^2 A}{\partial z^2} = \alpha^2 A
\]  

(130)

According to the above equation, in order to solve the behavior of the vector potential along the boundary must be known. Alternatively, given the distribution of the current density within the confines of the bounded element is known, the following relationship between the vector potential and the current density can be used \([14]\):

\[
A = \int \frac{\mu \mu_o}{4 \pi \rho_r} \nu dV
\]  

(131)

Where \(\rho_r\) is the distance from the point P in the field to the differential volume element, \(\nu\). However, to make use of this relationship, the current density distribution in the solid must be known. In this case, the current density distribution is not known since to obtain it, one must know the magnetic field distribution, which is what ultimately are seeking. Therefore, to solve Equation 130, it is best to start from an assumption of the magnetic field at the surfaces of the element. In the case of conducting rectangular bars in a Cartesian frame, the equivalent to Equation 130 in two dimensions has been solved \([14]\). However, it is noted that the requirement to specify the surface field, simply leads to a linear superposition of the one dimensional solutions for the vector potential, \(A\). The magnetic field is then found from the definition of the vector potential function:

\[
\vec{H} = \frac{1}{\mu, \mu_o} (\nabla \times A)
\]  

(132)

This approach should lead to a 2D field solution in this element. However, given that the solution depends on the specification of the field (or derivative of the field) at the boundary, it seems that this approach is no more adequate than consideration of a combination of one-dimensional solutions.

Since the ultimate goal of this analysis is the development of a simplified actuator transfer function, a magnetic circuit concept must convey a relationship between the actuator flux and the frequency of excitation. Implied in a magnetic circuit analysis is the adoption of a one-dimensional model of the flux through each element. Since the field distribution is actually two-dimensional in \(r\) and \(z\), a
simplifying assumption must be made. One option would be linear superposition of each one-dimensional mixed boundary solution of the field equation. However, this approach would not be consistent with the equivalent circuit approach that used for elements 2 and 4 and would result in an effective series combination of reluctances for element 5. Instead, a parallel combination of each one-dimensional effective reluctance is more appropriate to account for the flux to take a preferential direction that is inversely proportional to the reluctance of that direction.

On a differential basis, using the same assumption adopted in [13] regarding the attenuation of the magnetic field away from a surface subject to a known magnetic field, Hs, expressions for the differential element reluctances shown in Figure 51 are derived.

![Diagram](image)

**Figure 51 – Element 5 reluctance network and equivalent magnetic circuit**

Specifically, the following is obtained assuming $\alpha$ to be large:

$$dR_{zr} = \frac{\epsilon \alpha}{2\pi \mu \mu_r r_z}$$  \hspace{1cm} (133)
\[
dR_{r\alpha}(r) = \frac{\alpha \epsilon}{2\pi \mu \mu_r r}
\]  

(134)

Finding the equivalent component total reluctances from:

\[
R_z = \int_{z_i}^{z_f} \frac{dR_{z\alpha}}{\epsilon} \frac{dz}{dz} = \int_{z_i}^{z_f} \frac{\alpha}{2\pi \mu \mu_r \mu_{r_0} r_0^2} d\mu = \frac{\alpha z_i}{2\pi \mu \mu_r r_0^2}
\]  

(135)

\[
R_r = \int_{r_i}^{r_f} \frac{dR_{r\alpha}}{\epsilon} \frac{dr}{dr} = \int_{r_i}^{r_f} \frac{\alpha}{2\pi \mu \mu_r \mu_{r_0} r_0} \ln \left( \frac{r_i}{r_0} \right)
\]  

(136)

Using a parallel combination to develop the expression for the effective reluctance of element 5 as:

\[
\frac{1}{R_{\text{eff}}} = \frac{1}{R_z} + \frac{1}{R_r}
\]  

(137)

\[
R_{\text{eff}} = \frac{\alpha \left( \frac{z_z}{r_i} \right) \ln \left( \frac{r_i}{r_f} \right)}{2\pi \mu \mu_r \left[ \frac{z_z}{r_i} + \ln \left( \frac{r_i}{r_f} \right) \right]}
\]  

(138)

Equation 138 is simplified since large values of \(\alpha\) were assumed along with an exponentially decaying field distribution. Instead, solving Maxwell’s equation in one dimension for two separate boundary value problems in \(r\) and \(z\) on element 5, will yield an expression similar to Equation 138 via a parallel combination of these two one-dimensional expressions. In appendices A1 and A2, the methods and form of these reluctance expressions were identified, so simply stating the result here:

\[
R_z = \frac{\alpha \ln \left( \frac{r_i}{r_f} \right)}{2\pi \mu \mu_r \ln(\alpha z_i)}
\]  

(139)

And
\[ R_i = \frac{\alpha z_i}{2\pi \mu_i \mu_r r_i} \left[ \frac{K_i(\alpha r_i)I_i(\alpha r_i) + I_i(\alpha r_i)K_i(\alpha r_i)}{K_i(\alpha r_i)I_i(\alpha r_i) - I_i(\alpha r_i)K_i(\alpha r_i)} \right] \]  

(140)

And so the precise expression for the effective reluctance becomes:

\[
\frac{1}{R_{\text{eff}}} = \frac{1}{R_i} + \frac{1}{R_r} 
\]

(141)

\[
R_{\text{eff}} = \frac{\alpha \left( \frac{z_i}{r_i} \right) \ln \left( \frac{r_i}{r_1} \right) \left[ \frac{K_i(\alpha r_i)I_i(\alpha r_i) + I_i(\alpha r_i)K_i(\alpha r_i)}{K_i(\alpha r_i)I_i(\alpha r_i) - I_i(\alpha r_i)K_i(\alpha r_i)} \right]}{2\pi \mu_i \mu_r \tanh(\alpha z_i)} 
\]

(142)

A5) Effective Reluctance of Remaining Elements

Since the effective reluctance expressions of the remaining elements of the actuator (elements 1,4,6,8) are analogous to previously determined expressions, these expressions are shown directly:

A5.1) Element 1 (Analogous to Element 7)

\[
R_{\text{eff}} = \frac{\alpha (z + z_i + z_2)}{2\pi \mu_i \mu_r r_i} \left[ \frac{K_i(\alpha r_i)I_i(\alpha r_i) + I_i(\alpha r_i)K_i(\alpha r_i)}{K_i(\alpha r_i)I_i(\alpha r_i) - I_i(\alpha r_i)K_i(\alpha r_i)} \right] 
\]

(143)

A5.2) Element 4 (Analogous to Element 2)

\[
R_{\text{eff}} = \frac{g \alpha_i}{2\pi \mu \mu_r r_i} \left[ \frac{K_i(\alpha r_i)I_i(\alpha r_i) + I_i(\alpha r_i)K_i(\alpha r_i)}{K_i(\alpha r_i)I_i(\alpha r_i) - I_i(\alpha r_i)K_i(\alpha r_i)} \right] 
\]

(144)
A5.3) Element 6 and 8 (Analogous to Element 3)

Element 6:

\[
R_{\text{eff}} = \frac{\alpha \ln \left(\frac{r_5}{r_4}\right)}{2\pi \mu_0 \mu_r \tanh(\alpha z_r)}
\]  

(145)

Element 8:

\[
R_{\text{eff}} = \frac{\alpha \ln \left(\frac{r_5}{r_2}\right)}{2\pi \mu_0 \mu_r \tanh(\alpha z_4)}
\]  

(146)
Appendix B: Servo Controller Pole Specification and Controller Gains

B1) Closed Loop Pole Specifications of Servo System

Nominal Operating Points:

\[ p_{oa} = [20 \ 30 \ 40 \ 50 \ 60] \text{ (psia)}; \quad x_o = [0 \ 0.25 \ 0.50 \ 0.75 \ 1.0] \text{ (fractional position)} \]

Pole Specifications:

\[
\begin{align*}
\lambda_{c1}(x_o, p_{oa}) &= \begin{bmatrix}
-80 & -60 & -60 & -60 & -60 \\
-50 & -50 & -50 & -50 & -50 \\
-40 & -20 & -20 & -20 & -20 \\
-30 & -20 & -20 & -20 & -20 \\
-20 & -20 & -20 & -20 & -20
\end{bmatrix} \quad \lambda_{c2}(x_o, p_{oa}) &= \begin{bmatrix}
-80 & -60 & -60 & -60 & -60 \\
-50 & -50 & -50 & -50 & -50 \\
-40 & -20 & -20 & -20 & -20 \\
-30 & -20 & -20 & -20 & -20 \\
-20 & -20 & -20 & -20 & -20
\end{bmatrix} \\
\lambda_{c3}(x_o, p_{oa}) &= \begin{bmatrix}
-600.0 & -450.0 & -450.0 & -450.0 \\
-375.0 & -375.0 & -375.0 & -375.0 \\
-300.0 & -150.0 & -150.0 & -150.0 \\
-225.0 & -150.0 & -150.0 & -150.0 \\
-150.0 & -150.0 & -105.0 & -112.5 & -150.0
\end{bmatrix}
\end{align*}
\]

B2) Observer-Controller Gains

State Feedback Gains:

\[
\begin{align*}
K_{obs1}(x_o, p_{oa}) &= \begin{bmatrix}
-7.5377 & -2.1593 & -0.9002 & -0.4401 & -0.2218 \\
-2.3956 & -0.2647 & 0.0547 & 0.1478 & 0.1791 \\
-0.5513 & -0.2181 & -0.0554 & 0.0008 & 0.0241 \\
-0.1602 & -0.0078 & 0.0337 & 0.0398 & 0.0381 \\
-0.1922 & -0.0045 & -0.0073 & -0.0007 & 0.0095
\end{bmatrix} \\
K_{obs2}(x_o, p_{oa}) &= \begin{bmatrix}
33.0459 & 10.0388 & 4.8288 & 2.8834 & 1.9245 \\
10.6520 & 2.0245 & 0.7015 & 0.2686 & 0.0825 \\
2.7856 & 1.1001 & 0.4230 & 0.1756 & 0.0630 \\
0.8763 & 0.1723 & 0.0049 & -0.0285 & -0.0328 \\
0.5773 & 0.0129 & 0.0039 & -0.0172 & -0.0335
\end{bmatrix}
\end{align*}
\]
Integrator Gains:

\[ K_{tob}(x_t, p_u) = \begin{bmatrix} 3.4197 & 1.3074 & 1.1672 & 1.0457 & 0.9440 \\ 0.8372 & 0.7628 & 0.6850 & 0.6179 & 0.5620 \\ 0.4319 & 0.0500 & 0.0459 & 0.0426 & 0.0401 \\ 0.1853 & 0.0529 & 0.0508 & 0.0490 & 0.0474 \\ 0.0562 & 0.0560 & 0.0382 & 0.0395 & 0.0507 \end{bmatrix} \]

Observer Gains:
