ABSTRACT


The goal of this thesis was to investigate and enhance our understanding of what occurs while pre-service mathematics teachers engage in a mathematical modeling unit that is broadly based upon mathematical modeling as defined by the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA & CCSSO], 2010). Qualitative data was collected during an instructional unit on mathematical modeling in a methods course for secondary pre-service teachers. Three different vantage points were taken, resulting in two research based articles and one practitioner article.

The first article examines the mathematical modeling process used by pre-service teachers to solve mathematical modeling tasks. Results in this paper show that the cycle described and illustrated in the Common Core State Standards for Mathematics (CCSSM) do not represent the work done by these students. An alternate theory of the mathematical modeling process is suggested that recognizes free movement between the objects and actions of mathematical modeling.

The second article explores how students engaged in mathematical habits of mind while engaged in solving a realistic mathematical modeling task. The results of the study show that students engage in the habits of mind to solve the task. Additionally, the results suggest that a more thorough and interconnected use of the habits of mind create a more general solution to these types of tasks.

The final article gives a detailed account of the implementation of a mathematical modeling unit in a capstone mathematics education course for secondary mathematics
teachers and reviews the results of this implementation. Findings suggest that the unit was successful in helping students develop appropriate conceptions of mathematical modeling.

Together these articles show the complexity of mathematical modeling in the classroom and show the benefits of investigating mathematical modeling from multiple perspectives. Implications for the mathematics classroom are offered.
An Investigation of Mathematical Modeling with Pre-service Secondary Mathematics Teachers.

by
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BIOGRAPHY

Emily Nelle Plunkett was born in Texas on May 2, 1984. She graduated from Chapel Hill High School in 2002. She attended the University of North Carolina at Chapel Hill and graduated with a Bachelor of Science in Public Health majoring in Biostatistics and Anthropology in 2006. Upon graduation, Emily used her degree in biostatistics to work on health studies at the Veterans’ Medical Center in Durham, NC.

After a year of working in the biostatistics field, Emily made a career change and took a job at Horton Middle School in Pittsboro as a mathematics teacher. Afterwards, deciding that teaching was her profession, Emily pursued her Master’s of the Arts in Teaching at Duke University. She graduated in 2009 and began teaching mathematics at Jordan High School in Durham. Emily enjoyed teaching many different classes and coaching cheerleading. Also during this time she married Brad Thrasher.

In 2011, Emily left the classroom to pursue her Ph.D. in mathematics education at North Carolina State University. During her time at NCSU, Emily worked as a program manager for Noyce Mathematics Education Teaching Scholars where she worked with both pre-service and in-service teachers. This fueled her desire to continue to work with teachers to better prepare them for their classrooms. Additionally, during her time at NCSU, Emily gave birth to her two children, Evan and Lydia Thrasher.

Emily’s time in the high school classroom left her with many questions about how to best
teach mathematics to students and how to help train teachers. Today those are the questions that drive her research interests. Emily plans to continue working on these research interests and preparing teachers for the classroom.
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Chapter 1

What is Mathematical Modeling and Why Teach It?

Nationally, there is a significant and ongoing call to develop students’ critical thinking skills (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA & CCSSO], 2010; NCTM 2000; P21, n.d.). Hugh Burkhardt (2006) concisely explains the need for a focus on thinking and how this should influence the mathematics classroom:

The reason that mathematics has such a large proportion of curriculum time, historically and to this day, is its perceived utility in solving problems from outside mathematics. Historically, it was enough for mathematics to equip people with the routine skills of calculation for bookkeeping, surveying, and like occupations; these skills would give them a lifetime of employment…There is now a need for a much wider range of mathematical thinking… Society now needs thinkers, who can use their mathematics for their own and for society's purposes. Mathematics education needs to focus on developing these capabilities (emphasis added, p. 182-183).

Many argue that one way to accomplish the goal of helping students to think mathematically is to teach mathematical modeling in the classroom (Blum, 2002; Confrey & Maloney, 2007; Lesh & Doerr, 2003; NGA & CCSSO, 2010; Pollack, 2007; Zbiek & Conner, 2006). In addition to helping students learn to think, some argue that using mathematical modeling in the classroom will also help to keep students more interested in mathematics by showing students the usefulness of mathematics (Pollak, 2007) and that engaging in mathematical modeling contributes to a more robust picture of mathematics (Blum, 2011; Zbiek & Conner, 2006).
Although there is diversity in the exact definition of mathematical modeling, it can generally be defined as using mathematics to solve real world problems (Niss, Blum, & Galbraith, 2007). Beyond this simple definition, teaching mathematical modeling is a diverse endeavor because it involves a range of backgrounds and goals for its inclusion in the mathematics classroom. These goals can be captured at a macro level by considering the end reason to include mathematical modeling in the mathematics classroom. These goals range from developing mathematical content (“modeling as vehicle”) to developing the skills of modeling to solve problems (“modeling as content”) (Julie & Mudaly, 2007) or some combination of the two. These goals are evident in the types of task that each perspective designate as mathematical modeling tasks, the way the mathematical modeling process is described, and the focus of the research. Details of the different modeling perspectives in literature are discussed in Chapter 2, but a quick exposition of the research about students’ mathematical modeling follows.

**Significance of the studies**

Internationally, the enthusiasm surrounding mathematical modeling in the K-16 classroom is evident (Kaiser, Blum, Ferri, & Stillman; 2011). For example, four books on mathematics education research have been published since 2010 (Lesh, Galbraith, Haines, & Hurford, 2010; Kaiser, Blum, Ferri, & Stillman, 2011; Stillman, Kaiser, Blum, & Brown 2013; Stillman, Blum, & Biembengut, 2015) in the series titled “International Perspectives on the Teaching and Learning of Mathematical Modeling.” These books feature chapters from a wide range of countries such as Germany, China, and Singapore that explore a myriad of issues, such as how students learn to model and how curricula can include mathematical modeling (See Stillman, Kaiser, Blum, & Brown, 2013). Additionally, the Program for
International Student Assessment (PISA), which studies mathematical skills as part of its assessment, has shown that students have difficulty solving mathematical modeling tasks (OECD, 2012). These results and other arguments have motivated policy changes in many countries (e.g. Germany, South Africa, Sweden, Singapore) to increase the focus on mathematical modeling within mathematics standards (Ferri, 2013).

Many educators within the United States have incorporated mathematical modeling into their mathematics standards as well (NGA & CCSSO, 2010). These standards outline a specific definition and view of mathematical modeling. The Common Core State Standards for Mathematics (CCSSM), being used in many states across the US, define mathematical modeling as “the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions” (NGA & CCSSO, 2010, p. 72). Additionally, the CCSSM places a large focus on solving problems that come from everyday life and offers the notion that modeling problems are solved using a cyclic modeling process diagramed in Figure 1.

![Figure 1. Modeling Process in CCSSM. (NGA & CCSSO, 2010, p. 72).](image-url)
While this CCSSM view of mathematical modeling focuses on solving real world problems and the modeling cycle, research in the United States has not kept up with this new teaching focus. In fact, compared to the international effort, the United States has been relatively slow to develop a robust mathematical modeling research agenda.

Lesh, Doerr, and colleagues have conducted a significant amount of work within the United States. These publications are numerous as seen in a book published in 2003 (Lesh & Doerr) and in many other book chapters, articles, and conference proceedings since that time (e.g. Brady, Lesh, & Sevis, 2015; Doerr & O’Neil, 2011; Lesh, Middleton, Caylor, & Gupta, 2008). However, their work focuses on a view of mathematical modeling as a context to develop mathematical understanding (Lesh & Doerr, 2003). This focused view of mathematical modeling concentrates their research to the mathematical understandings that develop while engaging in mathematical modeling. In general, these studies show that modeling tasks designed to develop mathematical understanding, help students to develop conceptual mathematical understanding (e.g. Doerr & English, 2003; Doerr & O’Neil, 2011; Lesh & Harel, 2003; Lesh, Middleton, Caylor, & Gupta, 2008). These studies do not focus on other aspects of mathematical modeling such as the modeling process or the importance of messy realistic situations, which are a prominent aspect of the definition given in the CCSSM.

Apart from the works by Lesh, Doerr, and colleagues, there are a few other studies done within the United States. Legé (2005, 2007) examined 9th grade high school students using two different curricula. One curricula exposed students to many mathematical models while the other require the students to create their own mathematical models. While both groups learned about mathematical modeling, the group the created their own models
outperformed the other group on using multiple iterations of the modeling cycle and identifying the contextual features of the model.

Zbiek and Conner (2006) studied pre-service teachers as they engaged in an everyday mathematical modeling problem. They found that the task helped students connect known mathematics to new contexts. Additionally, they found that students engaged in a mathematical modeling process that was complex with a constant need to move back and forth between the real and mathematical world. More recently, Gould (2013) conducted a study that examined teachers’ understandings of mathematical modeling within the United States that incorporated the CCSSM definition of mathematical modeling. She found that teachers held many misconceptions of mathematical modeling, such as not understanding the role of the real world and not understanding the role of the mathematical modeling cycle.

While this study points to the misconceptions teachers have, it does not shed light on how teachers approach mathematical modeling tasks. Thrasher & Keene (2014) confirmed these misconceptions in a separate study of pre-service teachers.

Finally, Gould & Wasserman (2014) investigated how high school students approach mathematical modeling task in the context of the CCSSM mathematical modeling definition. They found that students tend to overcomplicate or oversimplify these tasks because of their inability to strike a good balance between making assumptions and using mathematics.

While these studies start to shed light on how students engage in mathematical modeling as defined by the CCSSM, there are still various unanswered questions about our understanding of teaching and learning of mathematical modeling in the United States, specifically with regards to pre-service teachers (Cai et. al, 2014). Three such questions are:
1. How do pre-service teachers engage in the mathematical modeling process while working on mathematical modeling tasks?

2. How is mathematical thinking supported through everyday authentic mathematical modeling tasks?

3. What do pre-service teachers learn about mathematical modeling, as defined by CCSSM, during methods courses focused on developing this content?

The following thesis addresses aspects of each of these questions.

The study and the structure of the dissertation

The overall goal of this study was to provide a broad picture of what occurred while pre-service mathematics teachers (PSTs) engage in a mathematical modeling unit that is broadly based upon mathematical modeling as defined by CCSSM. The setting for the study was an instructional unit during a methods course for secondary PSTs. Qualitative data was collected through pre- and post-surveys, digitally recorded group work on tasks, digitally recorded class discussions, and student artifacts during the unit (see appendix A for details about each data source).

From the data collected, three different paths were taken to explore what occurred during the implementation of a mathematical modeling unit in capstone for PSTs. Each path resulted in a written article (see dissertation Chapters 3, 4, and 5) geared toward a specific audience and journal. Additionally, Chapter 2 gives an overview of the diverse landscape of mathematical modeling research and Chapter 6 provides a conclusion of the whole study as well as its impact.

The first article titled, A reinterpretation of the mathematical modeling process, constitutes Chapter 3. This article answers the question: What different processes do
students use during mathematical modeling? The works of three groups, working on three different tasks, are analyzed to describe how students engaged in mathematical modeling. The results suggest that the cycle described in CCSSM does not fully represent the work done by these students and an alternate theory of the mathematical modeling process is suggested.

A study of mathematical modeling through the lens of mathematical habits of mind (Chapter 4) explores how students engaged in mathematics while engaged in a realistic mathematical modeling task. The article addresses two research questions using the habits of mind as defined by Cuoco, Goldenberg, and Mark (2010).

1. Are the mathematical habits of mind a productive way to describe students’ mathematical work during a realistic mathematical modeling task?

2. If so, how and in what ways do students practice mathematical habits of mind while engaging in a realistic mathematical modeling task?

The results of this study show that students engaged in four of the six habits of mind during the task and there is evidence that more thorough and interconnected use of the habits of mind led to more general solutions to the task.

The final article, A teaching unit on mathematical modeling in a high school methods course, is written for the mathematics teacher educator (Chapter 5). The article gives a detailed account of the implementation of a mathematical modeling unit in a capstone mathematics education course for secondary mathematics teachers. Additionally, the changing conceptions of the students from before and after the unit are analyzed. Results provide evidence that the unit helped students understand the characteristics of mathematical modeling and the characteristics of mathematical modeling tasks.
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Chapter 2

In conducting a study about how pre-service teachers (PSTs) engaged in a mathematical modeling (MM) unit that is broadly based upon mathematical modeling as defined by the Common Core State Standards for Mathematics (CCSSM), it is important to situate the study in literature on mathematical modeling. The following chapter will provide a comprehensive portrait of research in the field of MM in general, situate the CCSSM definition of mathematical modeling within this field, and more specifically review research about MM with college students and PSTs.

Even though many argue that MM should be a part of the mathematics classroom, a cohesive conceptualization for how mathematical modeling should be incorporated into K-16 mathematics has not emerged in the literature (Lesh & Fennewald, 2010, Kaiser & Sriraman, 2006). According to a classification developed by Kaiser & Sriraman (2006) there are five broad perspectives: (1) realistic modeling, (2) educational modeling, (3) modeling-eliciting activities approach, (4) socio-critical modeling, (5) epistemological modeling. While all of the perspectives assert a MM claim, the approach that researchers take to define a mathematical model and MM is influenced by the perceived goal(s) of MM and the theoretical beliefs of the researcher (Ferri, 2013). These differences lead to different frameworks, research foci, and outcomes.

Influences on Defining MM

Modeling goals. One of the defining characteristics of the perspectives is the difference in the purposes of using MM. To help categorize the goals of MM, Niss, Blum, and Galbraith (2007) and Julie and Mudaly (2007) discuss these differences in goals as a duality between goals of facilitating the learning of mathematics and exploring extra-
mathematical scenarios using mathematics as a tool. These categories are meant to represent a duality, not a dichotomy, implying that it is possible to work towards varying degrees of each goal at the same time.

The first category focuses on developing mathematics and is called “modeling for the learning of mathematics” by Niss, Blum, and Galbraith (2007) and “modeling as a vehicle” by Julie and Mudaly (2007). In this category, mathematics instruction is enhanced by placing mathematics learning in contextual situations to motivate mathematical concepts and procedures. The end goal is conceptual understanding of more advanced mathematics.

The second goal is developed slightly differently between Niss, Blum and Galbriath and Julie and Mudaly. Julie and Mudaly (2007) call this category “modeling as content” and argue that the learning goal is modeling itself without specific focus on the mathematical content. With this goal, the outcome is the mathematical model and the student’s ability to critique, adapt, and extend this model. Niss, Blum and Galbraith (2007) emphasize a learning outcome of applying mathematics to solve problems outside of mathematics. For both of these categorizations, the end goals are not specifically learning the mathematics itself but instead the ability to apply mathematics.

**Major influences.** In addition to the goal of MM, how a researcher believes that a student acquires knowledge and comes to make sense of this knowledge has a large impact on their research of MM (Ferri, 2013). While many researchers agree that modeling has roots in Piaget, Vygotsky, and the pragmatists (Confrey & Maloney, 2006; Lesh & Doerr, 2003), others show strong socio-cultural and ethno-mathematics influences (Kasier & Sriraman, 2006). These influences create both subtle and foundational differences in
definitions and frameworks within MM and help to differentiate the wealth of research perspectives in this field.

The Five Perspectives

Using these criteria of goals and background influences, Kaiser and Sriraman (2006) wrote a seminal article that established five perspectives within MM research. Also, their article establishes meta-perspectives that focus on only specific aspects on MM; however, those perspectives will not be reviewed here. The five perspectives are: (1) realistic modeling, (2) educational modeling, (3) modeling-eliciting activities approach, (4) socio-critical modeling, (5) epistemological modeling (Kaiser & Sriraman, 2006). It is important to note that some researchers have published in multiple perspectives and can work within several perspectives (Kaiser, Sriraman, Blomhoj, & Garcia, 2007).

Realistic Modeling. The most important aspect of the realistic modeling perspective is that the goal of MM is to solve real world problems (Kaiser & Sriraman, 2006). The perspective advocates that students must work with realistic, authentic, messy tasks. Authentic tasks are defined as “problems that are only a little simplified and...recognized by people working in this field as being a problem they might meet in their daily work” (Kaiser, Schwarz, & Buchholtz, 2011, p. 592). The goal is that through working on these authentic tasks, students will progress in their ability to solve real life problems. Additionally, the authentic tasks should require students go through the MM cycle because a second goal of this perspective is to foster students’ ability to use the MM cycle.

This perspective has roots in the pragmatic perspective of Pollak that emphasizes the importance of solving real life problems (Blomhoj, 2009; Kaiser, Schwarz, & Buchholtz, 2011). This perspective emphasizes the inclusion of MM in classrooms because of its
practicality. In fact, there is no focus on development of mathematics or mathematical theory.

This perspective emphasizes the use of the MM cycle. Not all cycles are exactly the same within the perspective but one example is from Blum (2011) (modified of Blum & Leiβ, 2007) and it is shown here as a prototypical example in Figure 1. This MM cycle was chosen because it is used or modified by many other researchers within this perspective (e.g. Biccard & Wessels, 2011; Frejd & Arleback, 2011; Ludwig & Reit, 2013).

In this modeling cycle, students are given a real world problem that would need to be understood which would result in a situation model. Next, students would simplify the problem and identify key aspects or variables creating a real model. This is an important step because the original modeling task must be presented within the messy real world context. From there, the real model would be transformed into a mathematical model (e.g. an equation) through mathematization (Blum, 2011). The mathematical model would allow mathematical work to be done on it (possibly solving an equation for a specific answer) where mathematical results are produced. These results must be interpreted into the real world yielding real results. Finally, validation would occur of the results, which leads to either another cycle or to exposing/reporting of the results of the real situation (Blum, 2011).
Figure 1. Modeling Process. (Blum, 2011, p. 18).

There are several key aspects of this diagram. One is that both the mathematical modeling cycle steps and sub processes (1-7 on side of the diagram) are given. The steps give products that are created from the MM cycle while the sub-processes describe the actions of modelers during the modeling process (Blomhoj & Jensen, 2003). This helps to highlight student activities that correspond to each part. From analysis of these steps, the mathematical model can be defined as the result from mathematizing some simplified real problem. MM is the process of going through these steps and sub-processes.

Although the research in this perspective covers a variety of educational levels, the primary focus is on the idea of MM competence and tools that support MM competence (Blomhoj, 2009). MM competence is “being able to autonomously and insightfully carry through all aspects of a MM process in a certain context” (Blomhoj & Jensen, 2003, p. 126). The MM competency construct is based on assumptions that students should work on full-scale modeling problems and that the challenge of teaching modeling is due to confusion.
brought on by many solution paths. The research seeks to determine the fundamental complexities in MM (Kaiser & Sriraman, 2006).

**Educational Modeling.** Similar to realistic modeling, educational modeling has a goal of developing MM competence; although, it has an equally important goal of learning mathematics (Blomhoj, 2009). This means that the goal of “modeling as content” and the goal of “modeling as a vehicle” are simultaneously advanced. This perspective can be viewed as a continuation of Freudenthal’s late work where the relationship between the real world and mathematics becomes an essential component in the structure of teaching and learning mathematics (Kaiser & Sriraman, 2006). While there is still a partial emphasis on authentic tasks, meaning that when possible the authenticity of situations are preserved, having an equal goal of reinventing mathematics can create tasks that have been more simplified that true authentic tasks. This allows for a dual focus on MM and the mathematics. In fact, often times, the perspective uses a context to develop mathematical concepts and this leads to mathematical modeling (Kaiser, Sriraman, Blomhoj & Garcia, 2007). This implies that within this perspective the goal of MM task is to motivate the need for mathematics.

Interestingly, foundational work in this perspective comes from Blum & Niss (1991), which shows the overlap of this perspective to realistic modeling. Blum and Niss (1991) define a mathematical model as “a triple (S, M, R), consisting of some real problem situation S, some collection M of mathematical entities and some relation R by which objects and relations of S are related to objects and relations M” (Blum & Niss, 1991, p. 39).

Furthermore, Blum and Niss (1991) describe that mathematical models are created through the cycle of MM where a “real problem situation” is idealized by the problem solver.
according to his/her interests. This leads to a “real model” which contains essential features of the original situation but also structured to lead to a mathematical approach. Next, the real model is mathematized (translated into mathematics) into a mathematical model. This process continues within the mathematics world where conclusions are drawn and “re-translated” into the real world to validate the model. This process may require several loops through the process (Niss & Blum, 1991).

As one would suspect, the process is very similar to the MM cycle described as the prototypical MM cycle in the realistic perspective. Considering the two perspectives share a similar goal of promoting the MM cycle, it is appropriate that the MM cycle is similar. What separates these two perspectives is that the research in the educational perspective must also focus on mathematical learning in addition to mathematical modeling.

While the above description of mathematical model and the MM cycle are considered foundational within this perspective, others have modified the cycle in the years since the early 1990s. As a second example, Zbiek and Conner (2006) use their own MM cycle (Figure 2). In this diagram, there are two worlds, the real world and the mathematical entity. The boxes represent the possible steps within the cycle while the bi-directional arrows are the sub processes.

The cycle starts in the real world. Students are exploring in the real world situation when they are obtaining more information about the real world situation. Also within the real world situation, students can be specifying (identifying the conditions and assumptions) and observing mathematically (observing what happens while exploring using mathematical
Next, students recognize and introduce mathematical ideas (properties and parameters) that relate to the mathematical entity. This process is called mathematizing and results in movement to the mathematical world from the real world (Zbiek & Conner, 2006). Once in the mathematical world, combining occurs where properties and parameters are verified as a match with the mathematical entity and mathematical entities are combined into a single entity. Additionally, analyzing (mathematical manipulation) and association (connections to real world) can occur.

Finally, the model needs to move back to the real world and three activities take place. Highlighting makes obvious any unidentified properties and parameters on the
mathematical entity. Interpreting is placing mathematical results in the real world context and examining validates real world conclusion with goal of the task.

Additionally, there are two sub processes that permeate the entire process. Aligning is the constant comparison of the current state to former ones. Communicating is any time ideas about the task are put forth between the modelers. All of these actions make up the sub processes that students engage in during the MM cycle and together with its two worlds, the real world and mathematical, create the MM cycle.

Since both the MM cycles of Niss and Blum and Zbiek and Conner are within the same perspective, it is important to compare them. Most steps coincide between the two cycles; however, Niss and Blum have an additional “real model” step before mathematization. Additionally, Zbiek and Conner have given their cycle more detail by outlining the sub processes that occur during the cycle. From these two examples, a general framework can be noted that begins in the real world, mathematizes to create a mathematical model, finds a mathematical solution, and finally transfers the solution back into the real world. Finally, this perspective also adheres to the requirement that MM tasks should be authentic and realistic real world problems.

Like the MM cycles in this perspective, the research agenda in the educational perspective is varied. As an example, Zbiek and Conner’s (2006) goal was to examine students’ mathematical understandings while working on mathematical modeling tasks, which produced the MM cycle discussed above. Other research and literature in this area discuss approaches to MM in curricula and issues of assessment with MM (Blomhoj, 2009).

**Models and Modeling Perspective.** The Models and Modeling Perspective (MMP) developed by Lesh and others develops out of problem solving research. MMP modifies the
goal of problem solving to include the development of meaning and use of mathematical concepts (Lesh & Doerr, 2003). This means that similar to the educational perspective, the goals of MMP are both “modeling as content” and “modeling as a vehicle,” but the stronger emphasis is on learning mathematics. What establishes MMP as different from the educational perspective is the foundational influences, its definition of a mathematical model, and its focus on model eliciting activities (MEAs).

MMP both defines MM and establishes a theoretical stance towards learning within mathematics that is strongly linked to the learning theories. MMP has roots in pragmatism because MMP is a framework that shapes the ways to conduct research, but also structures how to use MM in the classroom (Lesh & Doerr, 2003). In their definition of a model and the modeling process, Lesh and his colleagues have been strongly influenced by Piaget (Lesh & Doerr, 2003). Lesh and Doerr (2003) define models as,

conceptual systems (consisting of elements, relations, operations, and rules governing interactions) that are expressed using external notation systems, and that are used to construct, describe, or explain the behaviors of other system(s)—perhaps so that the other system can be manipulated or predicted intelligently (p. 10).

They restrict the model definition to say that a mathematical model “focuses on structural characteristics of the relevant systems” (Lesh & Doerr, 2003, p. 10). In Lesh and Doerr’s definition a mathematical model represents an external representation of mathematical knowledge and they state that it emphasizes development of holistic cognitive structures. While in the two prior perspectives, mathematical models are mathematizations of a specific real world problem, a mathematical model in MMP is intended to be a conceptual tool of a mathematical system that grew from a specific real world situation (Lesh et. al.,
2003).

*Model eliciting activities.* In order to achieve mathematical models as conceptual tools, the real-world problems solved in this perspective have a specific design to promote mathematical development. These problems are called model-eliciting activities (MEAs). MEAs are designed with the purpose of embodying the mathematics process in the constructed model produced. This is a key difference from the other perspectives because the design focus of an MEA is that through solving the task, the model produced represents the mathematical process.

This goal can be better understood with the six principles that guide the design of MEAs. These principles are the model construction principle, the reality principle, the self-assessment principle, the model documentation principle, the generalizability principle, and the effective prototype principle (Lesh et. al, 2000; Magiera, 2013), which can be seen in Table 1. The model construction principle states that the task must require students to describe the development of their mathematical model. The model documentation principle follows closely from the model construction principle by requiring that students produce documentation of their work. The reality principle is that the task must be plausible and take place in the real world but it should be noted that “the key to satisfying the reality principle is not for the problem to be "real" in an absolute sense” (Lesh et al., 2000, p. 616). This means that that unlike the authenticity in the realistic perspective, the problem needs to be set in a the real world but not necessarily be a task students would normally encounter in everyday life. The self-assessment principle implies that there is embedded ways for students to judge the quality of their work. The fifth and sixth principles are closely linked. The generalizability principle requires that the created model be open enough to be applied to
other similar situations while the effective prototype principle requires the model produced
to be a powerful simple (mathematical) tool for a complex situation.

Table 1.

*Six Principles of MEAs.*

<table>
<thead>
<tr>
<th>Model construction</th>
<th>Students must provide an explicit description, explanation, or prediction for a mathematically significant situation.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reality</td>
<td>Problem solvers must behave like scientists or engineers who are working for a particular client or organization.</td>
</tr>
<tr>
<td>Self-assessment</td>
<td>A group of students use criteria embedded in the activity to self-evaluate their work. In turn, they go beyond their initial ways of thinking to create a model that is more robust and more closely aligned with the needs of the client.</td>
</tr>
<tr>
<td>Model documentation</td>
<td>Students must produce documentation of their thinking that reveals their mathematical and nonmathematical interpretation of the problem situation.</td>
</tr>
<tr>
<td>Generalizability</td>
<td>Students must produce models that are generally useful and could be easily modified and applied to situations that are similar to the one being studied.</td>
</tr>
<tr>
<td>Effective prototype</td>
<td>Students use the concepts that underlie the activity to create simple but powerful models for complex situations.</td>
</tr>
</tbody>
</table>

Iterative modeling cycles (see Figure 3) are required when completing MEAs as well.

Description maps the real world (or imagined world) to the model and manipulation of the model occurs to get a result to the original situation. This is followed by prediction, which moves the results back to the real world where verification happens (Lesh & Doerr, 2003).
Research within MMP requires the use of MEA’s and tends to focus on the development and use of MEAs to promote mathematical reasoning and understanding. Specifically, Doerr and Lesh (2011) state, “we use MEAs to investigate students’ thinking as it develops among a variety of dimensions such as concrete-abstract, preoperational-operational….or specific general” (p. 254). Different from the educational perspective, research in this perspective does not focus on modeling competencies; they are only an inherent aspect of the perspective (Blomhoj, 2009).

**Socio-Critical Modeling.** Drawing from anthropology and ethnomathematics, socio-critical modeling focuses on the societal role of mathematics. Barbosa (2006) established “modeling as critic” to describe learning about the use of mathematical concepts to make decisions in society.
This perspective has goals both inside and outside of mathematics. Within mathematics, the major goal is to show the power of mathematics to make decisions. Mathematics is seen as the tool to make decisions. Although mathematics is a key component, the end goal of the framework is to make decisions about society. Additionally, a goal of this perspective is to critically understand the MM process (Kaiser et. al., 2007).

The motivation for this perspective comes from Skovsmose’s work in which he offers two important ideas. The first is that critiquing what is learned is an important part of constructing knowledge (Barbosa, 2006). The second is that mathematics strongly influences decisions made in society. From these ideas Barbosa suggests that learning of mathematics and modeling allows students the ability to criticize mathematical models in society (Barbosa, 2006). Barbosa states,

As the mathematical models are not neutral, they depend on their building processes, which include how the modeler understands the problem-situation, and how he/she projects mathematical conceptions into the situation. Following this argument, the biased nature of the mathematical models brings a closer relationship between the mathematical model and the criteria used to construct it (Barbosa, 2008, p. 134).

Furthermore, this perspective attempts to bring in a discussion about power. Since MM is used in society to make decisions, those that are capable of creating and critiquing MM, hold power. This perspective believes that students need MM skills to be independent citizens. This idea is advocated by D’Ambrosio, another foundational influence (Blomhoj, 2009).

Barbosa (2006) defines mathematical modeling as “a learning milieu where students are invited to take a problem and investigate it with reference to reality via mathematics” (p. 294). Additionally, Barbosa (2006) defines modeling as an activity within a school context
that is brought from everyday contexts. This definition of mathematical modeling is much broader than the other perspectives and does not advocate for specific MM cycle steps or subprocesses. A mathematical model is considered to be a mathematical representation of the presented situation (Barbosa, 2007).

Research in this perspective has focused on how students use mathematics to understand the world critically (Ferri, 2013). Unlike MMP where the focus is on mathematical understandings or realistic modeling which in mathematical competency is the emphasis, socio-critical modeling research often centers on the students’ ability to critique models and recognize the modeler’s power (Blomhoj, 2009).

**Epistemological modeling.** The fifth and final perspective uses MM as a lens to develop more general theories for the teaching and learning of mathematics (Blomhoj, 2009). There are three distinct features to this perspective. The first is that development of mathematical understanding is the only goal of the perspective. Secondly, this perspective does not require “real-world” phenomenon to be modeled. In fact, “every mathematical activity is identified as modeling activity for which modeling is not limited to mathematizing of non-mathematics issues” (Kaiser & Sriraman, 2006, p. 305). This is a distinct difference between this perspective and the other four. Often within this perspective, activity is seen as beginning in reality but ending in mathematics (Kaiser & Sriraman, 2006).

The third important aspect of this perspective is that another theory is driving the development of research. Two theories that are used often within this perspective are the Anthropological Theory of Didactics and the Realistic Mathematics Education Theory (RME) (Blomhoj, 2009; Kaiser et. al., 2007). Since the driving theory cannot be separated
Emergent Modeling. Coming from a RME perspective, Gravemeijer and colleagues have developed a modeling framework called Emergent Modeling. The focus of Emergent Modeling is on the learning process over time. The student develops an initial model rooted in an experience then over time and through experience, the student reorganizes the model as it moves from informal to formal mathematical understanding (Gravemeijer, 2007). These ideas stem from Freudenthal’s belief that mathematics should be rooted in human activity and stay connected to these experiences (Doorman & Gravemeijer, 2009). Additionally, this perspective was developed in response to the use of external representations in mathematics education that create representations with intrinsic meaning (Gravemeijer, 2002).

Emergent Modeling is characterized by four stages of activity. The first two stages are activity in the task setting and referential activity. As suggested by the name, activity in the task setting refers to the time when student are experiencing the task. Next during referential activity, students create models, called models-of, that rely on the specific context of the activity (Gravemeijer, 2002). In the third stage, general activity, the model-for is created. These are models that become mathematical entities in their own right and lose the bonds to context that models-of need (Gravemeijer, 2007). In the final stage, formal mathematical reasoning, the models-for are no longer needed and formal mathematical understanding is established.

Research using emergent modeling has investigated the teaching and learning of mathematical topics such as data analysis (Gravemeijer, 2002), the concept of function (Doorman et. al., 2012) and calculus (Doorman & Gravemeijer, 2009; Gravemeijer and
Doorman, 1999). Many of these studies use emergent modeling as a way to design teaching experiments.

**Discussion of the Perspectives**

From this sampling of MM frameworks and description of the five research perspectives, one can conclude that MM has a rich and diverse research program. Although the perspectives are distinct, there are also similarities in addition to the differences.

**Differences between perspectives.** These five perspectives show a range of goals for MM that directly influence the research conducted. This spectrum starts with studying the process of modeling itself, realistic modeling and ends at a purely mathematical focus with research on how to develop specific mathematical content (epistemological modeling). There is a nice balance between these two goals established in educational modeling. The socio-critical perspective adds an additional goal. This perspective aims to develop the skills needed to be a critical thinking citizen of the world. This spectrum can be seen in Table 2 where the perspectives are set up to show perspectives with more similar goals being in closer proximity. The socio-critical perspective is grouped next to the realistic perspective because these are the only two where mathematics is not a direct focus of the research.

**Differences in Task design.** There are many consequences from the range of goals for each perspective but one that is a direct result of the goals of MM within each perspective is the design of the task that the perspective demands, see design of task column in Table 2. Due to the focus of application and the MM process of realistic and educational perspectives, MM tasks in these perspectives require realistic, real world contexts (Blum, 2011). The sometimes subtle difference between the tasks of the educational and realistic perspective is the requirement of the full use of the MM cycle for the realistic perspective and need for a
focus on mathematics in addition to the MM cycle in the educational perspective. Because
the educational perspective has a competing goal of developing the mathematics, tasks in the
educational perspective can be partially simplified from their authentic real world problem to
bring forth the mathematics. For the socio-critical perspective the goal of demonstrating the
power of mathematics and its role in society, requires a strong real world, societal context for
its tasks (Barbosa, 2006). Moving toward a stronger emphasis on development of
mathematics, MMP focuses on real world contexts but MEAs need to develop a
mathematical model that is to be generalized to other contexts as opposed to focused on
solving some real world problem. Finally, Epistemological Modeling does not have a
specific requirement for its task.

Differences in mathematical modeling cycles. In addition to the goals of MM
research, these perspectives can be contrasted by looking at the MM cycles described in each.
For the epistemological perspective, the MM cycle (or in this case a better word would be
process) ends in the mathematical world. This also leads to defining a mathematical model
as being equivalent to mathematical understanding. This is drastically different than all the
other perspectives. The other four perspectives have their own nuances but at some point in
each perspective there is a cycle that begins in the real world, moves to the mathematical
world, and then ends in the real world. The work in this dissertation will look at the
breakdown of the real world and mathematical world while student work on mathematical
modeling tasks in chapter 3 to gain better understanding of these two worlds and their role
while student solve mathematical modeling tasks.
<table>
<thead>
<tr>
<th>Perspective</th>
<th>Realistic</th>
<th>Socio-critical</th>
<th>Educational</th>
<th>MMP</th>
<th>Epistemology</th>
</tr>
</thead>
<tbody>
<tr>
<td>Goal(s)</td>
<td>Develop ability to model</td>
<td>Develop MM to make decisions about society</td>
<td>Develop ability to model and mathematics</td>
<td>Develop mathematics in MM context</td>
<td>Develop mathematics</td>
</tr>
<tr>
<td>Design of Tasks</td>
<td>Authentic Tasks that include messy real life problems. Task must require use of MM cycle.</td>
<td>Tasks are set in everyday (societal) context but remain dually focused on developing specific mathematics ideas.</td>
<td>Authentic Tasks that have been simplified to also reveal specific mathematical goals.</td>
<td>MEAs focus on development of a mathematical concept set in a real world context. Require 6 MEA principles.</td>
<td>No set design</td>
</tr>
<tr>
<td>Founders/Key People</td>
<td>Pollack</td>
<td>D’Ambrosio, Skovsmose, Barbosa</td>
<td>Niss, Blum</td>
<td>Lesh, Doerr</td>
<td>Freudenthal</td>
</tr>
<tr>
<td>Definition of Mathematical Model</td>
<td>Equations and other mathematical objects that explain the real world situation.</td>
<td>Mathematical representation of a present situation</td>
<td>Mathematical entities that have a relationship to some real world situation</td>
<td>A conceptual system that focuses on structural characteristics of the relevant system</td>
<td>The results from activities on situations and mathematical entities.</td>
</tr>
<tr>
<td>MM Cycle</td>
<td>A multi-step cyclic process that starts in the real world, mathematization occurs to move into the mathematical world then goes back to the real world.</td>
<td>All parts of student involvement in investigation of a real world problem with the use of mathematics.</td>
<td>A multi-step cyclic process beginning in the real world, mathematized to a mathematical entity, and then goes back to the real world.</td>
<td>A cycle from real world to model then back to the real world. The cycle is repeated as necessary.</td>
<td>Four stages of activities where models of and models-for lead to formal mathematical reasoning.</td>
</tr>
<tr>
<td>Research Focus</td>
<td>MM Competencies</td>
<td>How students use mathematics to understand world critically</td>
<td>How mathematics is developed in MM, organizing MM in curricula</td>
<td>Development and use of MEAs to teach mathematics</td>
<td>Teaching and learning of specific mathematical content</td>
</tr>
</tbody>
</table>
Summary. Even with the differences in these perspectives, at the core of all of these perspectives lie the ideas that modeling begins with a real experience and that mathematics is a powerful way to understand its meaning. It is from this common stance that each perspective has confidence that MM has a future in mathematics education.

Common Core State Standards for Mathematics

The five perspectives addressed previously come from different research perspectives internationally. The Common Core State Standards for Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA & CCSSO], 2010) also defines MM and the MM cycle from the standpoint of standards that students are supposed to be learning but it is not clear what literature was used to develop this definition and MM cycle. The Common Core State Standards for Mathematics (CCSSM) defines MM as “the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions” (NGA & CCSSO, 2010, p. 72) and as the application of mathematics to “solve problems arising in everyday life, society, and the workplace” (p. 7). Figure 4 shows the modeling process outlined in CCSSM.

Figure 4. Modeling Process in CCSSM. (2010, p. 72).
In breaking down how the CCSSM MM standards fit into the above five perspectives, it is hard to clearly identify just one perspective. One of the main points of emphasis within the CCSSM content standard for MM is applying mathematics to real world situations, which would suggest a realistic or educational perspective. In fact the CCSSM (NGA & CCSS), 2010 states “Real-world situations are not organized and labeled for analysis; formulating tractable models, representing such models, and analyzing them is appropriately a creative process” (p. 72) which would be indicative of the realistic perspective. However, later the document says “Models can also shed light on the mathematical structures themselves, for example, as when a model of bacterial growth makes more vivid the explosive growth of the exponential function” (NGA & CCSSO, 2010, p. 72) which would suggest the MMP or the epistemological perspective. The lack of clarity within the CCSSM for the goal of MM can be interpreted as either a failure to be clear or thinking in a positive light, as an attempt to incorporate all of these different perspectives and not limit the United States standards to only one view of MM. However, by not aligning the CCSSM with one perspective, the research within these areas do not directly apply leaving a gap in understanding about the teaching and learning of MM using the CCSSM definition.

Mathematical Modeling and Learning with a focus on Pre-service Teachers

With the rich landscape of research in MM presented, it is now possible to understand that learning during MM can be viewed from many perspectives based on the goals of the perspective. But because this study took place in a methods class for PSTs, the following section explores how researchers have investigated two different types of their learning: (1) PSTs learning of the skills of MM and (2) PSTs learning of mathematics during MM to give
a thorough view of the research from the broad range of goals within the field. After extensive searches using Google Scholar with key words of mathematical modeling and pre-service teachers, examining the publications from *International Community of Teachers of Mathematical Modelling and Applications* since 2007, and examining the contents of Beyond Constructivism (Lesh & Doerr, 2003), it can be concluded that these studies are limited. However, in this section, some of the results will be highlighted.

It was found that studies either focused on mathematical learning or MM learning; therefore, the following review of the literature is divided into a section on mathematical learning and a section on MM learning. The important works will be reviewed with an emphasis on PSTs’ learning when possible.

**Research on learning the skills of Mathematical Modeling.** This section reviews the literature of PSTs knowledge and learning of MM and MM skills. It should be noted that there are other studies that explore PSTs beliefs regarding MM but since the focus of this study is not on beliefs, these studies will not be reviewed. All of these studies fall into the realistic and educational perspectives, which are the perspectives that focus on the goal of learning the skills associated with MM. It is useful to divide these works into quantitative studies and qualitative studies because the goals and findings are very different between the studies which help understand the research being conducted.

**Quantitative studies.** The majority of the quantitative studies use an instrument designed by Haines and colleagues, which attempted to measure students’ abilities to perform different parts of the MM cycle. Studies with other instruments will be reviewed as well.
Haines et. al instrument. In 2000, Haines, Crouch, and Davis reported on the development of an instrument to collect data on students’ ability to complete different sub processes of the modeling cycle. This instrument consisted of 12 multiple-choice questions that had five answer choices. Six of the questions were part of a pre-test and six of the questions made up the post-test. The questions cover the following aspects of MM: assumptions to simplify the problem, understanding the goal, formulating the mathematical problem, assigning variables, creating of mathematical statements, and selecting models (Houston & Neill, 2003). Each multiple choice question had one answer that was preferred which was assigned a score of two points, alternative appropriate answers were given a score of one point, and the other solutions that were not appropriate which were given zero points. After using the instrument with 39 undergraduate students, the authors found that the instrument provides a snapshot of modeling skills (Haines, Crouch, & Davis, 2001). For those tested, it was shown that students had trouble making assumptions and checking the model’s fit since 28% and 31% received a score of zero for each of those categories respectively (Haines et al., 2001).

Since the initial creation of this test, the multiple-choice questions have been expanded to 22 questions, which allow for the above aspects to address in a second question (Houston & Neill, 2003). Additionally, two new aspects of MM were tested: graphical representations and exploring real and mathematical world connections (Haines et al., 2001).

Many have used this instrument (either the original or one of the modified longer versions) to investigate a range of different types of students (e.g. Ikeda, Stephens, & Matsuzaki, 2007, Izard et al., 2003, Lingefjard & Holmquist, 2005) but Kaiser (2007) used the instrument with PSTs. She evaluated a seminar where German PSTs carried out MM
tasks with secondary students. The seminar contained a short introduction to MM. Then the PSTs work with groups of secondary students on MM tasks. During seminar, student work, problems, and experiences are discussed. The pre and posttest were given to 57 participants. On average, the students increased their score 1.2 points. This was a significant gain during the seminar and suggests that the PSTs’ modeling competencies improved during the seminar. The article does not examine the individual MM sub processes.

*Other instruments.* Kuntze (2011) created a questionnaire that asked questions that were about teachers views related to MM. The responses were a four-point Likert scale. Additionally, there was a sub-section that contained tasks that were viewed as having lower and higher modeling requirements. Teachers were asked to respond to the learning potential of the tasks. Lower modeling requirements were defined as task where the mathematical model is given and only one correct solution is possible. Higher modeling requirements are those that require “at least one translation step between situational context and a mathematical modeling, and that allow different solutions” (Kuntze, 2011, p. 280).

Using this instrument, Kuntze (2011) surveyed 230 PSTs and 79 in-service German teachers. Kuntze was measuring teachers understanding of the MM cycle in general and their preference for using MM tasks. It was found that PSTs favored the learning potential of lower modeling tasks but in-service teachers did not have this preference. The teachers did not report strong knowledge about MM. It should be noted that Kuntze was thorough in his analysis of the instrument and provided positive Cronbach scores to valid this instrument.

Gould (2013) on the other hand did not provide details about the reliability of the instrument she used. Even though the lack of reliability calls into question the results, Gould
does provide details of how each question was developed in her 20-question survey. Also, this is the only study that took place in the United States of the quantitative studies.

The questionnaire consisted of six questions about what constitutes a mathematical model, eight questions about aspects of MM, and six questions about the place of MM in the classroom. Additionally, this survey attempted to gain a snapshot of MM concepts of US in-service teachers and PSTs using a large sample size (274) and having representation from 35 states. The results from the survey suggest that US secondary teachers have many misconceptions about MM. It was shown that many teachers do not understand the role of making assumptions or that MM task should have real world requirements.

**Qualitative approaches.** The following studies give details of how students engage in MM with a focus on documenting MM skills of the students. Three studies provide a snapshot of how students engage with MM tasks.

In Thomas & Hart (2010) 16 primary school PSTs engage in an MEA in the United States. One group’s work was audiotaped while working on the task, work samples from all PSTs were collected, and a focus group discussion was performed to serve as data. While the primary focus of this study was to explore PSTs beliefs about MEAs, findings suggest that PSTs had struggles with the ambiguity inherent in these types of tasks and that there is a link between experiences with MM and PSTs orientation towards MM.

Winter and Venkat (2013) also examined elementary PSTs. Their data was collected at the end of a three-month unit on contextual problem solving and content development during a methods course. Data comes from a final test that the participants were given in the class. The major finding of the study was that PSTs had trouble mathematizing.
Unlike the former studies, Widjaja (2013) focused on secondary PSTs. Secondary PSTs’ were examined while working on a specific MM task (Figure 5). Data came from each group’s written report on their task. Since not all groups clearly stated their assumptions there were difficulties for the PSTs in identifying limitations and their causes.

![Re-designing Parking Space Project](image)

*Figure 5. Mathematical Task from Widjaja (2013, p. 587).*

Instead of only capturing a snapshot of students after completing a course, Tan & Ang (2013) sought to examine how PSTs’ knowledge is being shaped by their MM learning experiences. 24 PSTs experiences during a six-week unit on MM were recorded through field notes of in-class discussions, reflections, written artifacts, and a questionnaire. To analyze the data, the researchers looked at the problem areas on the students’ work. It was found that the PSTs were able to engage in the MM cycle; however, the article does not give details of how PSTs changed their knowledge of MM while engaged in the tasks.
Summary. After reviewing both the qualitative and quantitative studies, there is cause for concern that PSTs' knowledge of MM and ability to solve MM tasks is limited. This is captured in Kuntze (2011) and Gould’s (2013) surveys and through exploration of PSTs' artifacts from solving MM tasks (Hart & Thomas, 2010; Winter & Venkat, 2013; Widjaja, 2013). Specifically, only two of these studies capture information from the United States and both suggest that PSTs' knowledge of MM is limited and the ambiguity of the tasks was an issue for the PSTs. All of these studies are successful in capturing the current state of MM abilities of their participants; however, they do not explore how learning of MM can be supported.

Kaiser (2007) begins to shed light on how to support MM learning by finding that a methods course on MM can significantly impact PSTs. Tan and Ang (2013) began to explore in more detail the MM sub processes of MM used by the PSTs while engaged in MM tasks, but they are limited in their ability to investigate learning during task because they did not capture data while PSTs are engaged in the tasks. Additionally, the exploration so of the sub processes was limited and these studies were not done within the United States. Additionally, none of these studies use the definition of MM as outlined by CCSSM.

In general, it is clear from the review of these articles that there is limited understanding of how engaging in MM tasks leads to knowledge of MM modeling and MM skills. The work reviewed provides clear evidence that PSTs’ knowledge of MM is limited and there is limited evidence that methods courses can address this but more research on how PSTs learn MM is needed (Blum, 2011; Cai, 2014).

Research on Mathematical Learning through Mathematical Modeling. The following studies discuss mathematical learning from MM tasks as opposed to examining
learning of MM. These studies fall under the educational or Models and Modeling perspectives.

**Educational Perspective.** Zbeik and Conner (2006) conducted research on learning of PSTs while working on a MM task. Participants came from a course designed to teach mathematical modeling to secondary PSTs in the United States. Data was collected during the course via artifacts and task-based interviews with the goal of examining how learning mathematics occurs in the sub processes of mathematical modeling activity. The specific task examined was a MM problem about placement of a hospital (Figure 6).

![Hospital Task from Zbeik and Conner (2006, p. 94).](image)

*Figure 6. Hospital Task from Zbeik and Conner (2006, p. 94).*

Findings suggest that changes in mathematical activity occurred from working on the tasks but that these changes are not the same for all participants. For example, Zbeik and Conner show that depending on what aspects of the real world situation a modeler attends to, different mathematical concepts are developed. Additionally, the findings suggest that, “additional or novel properties and parameters can draw students’ existing conceptions to introduce new mathematical entities” (Zbeik & Conner, 2006, p. 108). Furthermore, MM can help students to connect to known procedures with new contexts; this deepens their understanding of the mathematics. While Zbeik and Conner show an example or two of these possibilities for mathematical growth, they fail to fully develop these ideas using a framework for mathematical growth.
Although the second study in this category is not about PSTs, it is about university students in a course that many PSTs take, Differential Equations. Blomhøj and Kjeldsen (2013) report on a mathematical modeling course used to develop differential equation topics. Using Sfard’s ideas of mathematical development of process and object, the authors analyze their student’s written group projects that responded to a prompt about population growth (Figure 7). The authors found that the modeling context provided a window into their students’ understanding. Specifically, Blomhøj and Kjeldsen (2013) show that “what cause difficulties for the students is shifting between viewing a differential equation as a relation between the momentary rate of change and the actual size of a certain function (here the size of the population) and viewing it as a relation between a function and its derivative” (p. 151). The modeling task’s design encouraged students to validate and interpret which helped to push students to enhance their conceptual understanding (Blomhøj & Kjeldsen, 2013).

**Explosive Population Growth**

Describe the growth of the world population in the period 1650–1960 and estimate the population in 2100. Some background for the relevance of knowing how the world population will develop, and a set of data for the world population in year 1650–1960, are provided.

To guide the students’ work with this particular modelling project, the general requirements are supplemented with the following guided questions:

- Does the given population data follow an exponential growth?
- For explosive growth, the growth rate is proportional to the square of the population size. Can the given data be described as explosive growth?
- What does the explosive model predict about the world population?
- What is your estimate of the world population in 2100?
- Try to find newer data and discuss the model’s prediction in relation to these.

*Figure 7.* Mathematical Modeling Task from Blomhøj and Kjeldsen (2013, p. 145).
**MMP.** While the previous studies do not have specific requirements for their MM tasks, the follow studies all use MEAs. Carlson, Larsen, and Lesh (2003) developed three teacher MEAs to use with pre-service elementary teachers to develop their understanding of covariational reasoning. The first activity asked students to create graphs using a motion detector and their bodies to match already constructed situations. The second activity involved a distance vs. time graph of an airplane flight and the third modeled the volume dependent on its height as a bottle is filled up with liquid. Findings showed that the negotiations between group members while working on these tasks provide “additional insights into each individual’s concept development and reasoning patterns” (Carlson et. al., 2003, p. 476). This study did not actually synthesize the understanding of its participants but instead suggested the usefulness of MEAs in conducting such a study.

Similarly, Lesh, Middleton, Caylor, and Gupta (2008) design a MEA task that asks PSTs (both elementary and secondary) to fit data of a student’s time performing a drill and their individual test scores. This data is being used to advise the school board on increasing the amount of time on the drill. From the results of the work on this activity, it was clear that participants “had difficulty thinking about the data being explained by a combination of two rules—the first being a deterministic rule that corresponds to the “curve of best fit,” and the second being a probabilistic rule that describes unexplained variation (or error)” (Lesh et. al., 2008, p. 122).

The following reviews an example that examines students getting ready to begin an engineering undergraduate degree. Doerr and O’Neil (2011) used MMP to design a model development sequence on the development of the rate of change concept. The model
development sequence began with an MEA where students created graphs using a motion
detector and their bodies to match already constructed situations. Directly following this
activity, students were involved to several exploration activities. These were designed to
help students use everyday language to explain rate of change and think about their
conceptual model developed during the MEA. Finally, students participated in two
application activities that were intended to lead to general understanding of rate of change
and encourage them to develop models of real world phenomena. Using pre- and post-tests,
it was shown that students had significant improvements on the understanding of rate of
change (Doerr & O’Neil, 2011). By analyzing student responses, it was discovered that
some students confounded changes in function values with changes in average rate of
change.

These three articles suggest that MEAs can be developed and used to help students
learn mathematics content. In both Lesh et. al. (2008) and Doerr and O’Neil (2011) there is
evidence that student reasoning can be thoroughly explored using the MMP perspective.
Doerr and O’Neil(2011) claim that this approach showed significant gains in understanding
for its participants. Clearly more work needs to be done in Carlson et. al. (2003) to
understand the impact of the MEA on the participants’ covariational reasoning. These three
articles provide a beginning to understanding how MMP can be used to help students with
understanding the concept of function but more work needs to be done to corroborate these
results and push to understand how using MMP as a design heuristic supports students’
concept development.

**Summary.** These studies about PSTs mathematical learning from MM tasks provide
a glimpse into the possibilities of using MM for mathematical concept development for PSTs
but a cohesive understanding is unclear. All of these articles agree that MM tasks, from different perspectives, provide an opportunity to reveal students’ mathematical understanding. Zbiek & Conner (2006) suggest that the MM task can even push PSTs to further develop their conceptual understanding but they fail to fully demonstrate this growth through strong examples or a framework for conceptual development. Blomhoj and Kjeldsen (2013) overcome some of these limitations but their population was undergraduate students instead of PSTs. Additionally, the studies using MEAs show how that it is possible to enhance specific mathematical content through design of specific tasks to develop that content. While these studies start to illuminate the mathematical learning potential of MM tasks, more work needs to be done to establish how mathematical learning occurs from work on MM tasks (Blum, 2011; Blomhoj & Kjeldsen, 2013). For example, how do mathematical modeling tasks that are authentic and meet the requirements for MM task from the realistic perspective support mathematical thinking and learning?

Chapter Summary

MM is a diverse and rich research paradigm that includes a range of backgrounds and goals. These goals can be captured at a macro level by considering the goals of developing mathematical content (“modeling as vehicle”) and developing the skills of modeling to solve problems (“modeling as content”). These goals are shown in the types of task that each perspective select as MM problems, the way the MM cycle described, and the focus of research.

These perspectives influence the research on PSTs’ learning while working on MM tasks. PSTs’ learning of MM come from the realistic and educational perspectives while studies about PSTs’ mathematical learning come from the educational perspective and MMP.
In both cases these research is sparse and more work needs to be done to develop a cohesive picture of the influence of the perspective, mathematical learning and MM learning from work on MM tasks. Looking specifically within the United States, the research is even further limited with no research focused on the teaching and learning of mathematical modeling while focusing on the definition provided by the CCSSM.
References


Chapter 3: Article One

A reinterpretation of the mathematical modeling process

The enthusiasm surrounding mathematical modeling in the K-16 classroom is evident (Geiger & Fredj, 2015; Kaiser, Blum, Ferri, & Stillman; 2011). In the international mathematics education research community during the last 10 years, 6 anthologies have been published with a mathematical modeling (MM) focus. These books feature articles from a wide range of countries such as Germany, China, and Singapore, articles that explore myriad issues, such as how students learn to model, and how curricula can include mathematical modeling (See Stillman, Kaiser, Blum, & Brown, 2013). The results of this literature have motivated policy changes in many countries (e.g., Germany, South Africa, Sweden, Singapore) to increase the focus on MM within mathematics standards (Ferri, 2013). With the adoption of the Common Core State Standards for Mathematics (CCSSM) in 2010 by many states in the United States, MM became both a content standard and a practice standard for many students ((National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA & CCSSO], 2010).

Although there is increased interest in mathematical modeling in the K-12 classroom, research suggests there is significant confusion about how to enact MM in the classroom. There is evidence that mathematics teachers do not have the necessary content knowledge or pedagogical understanding to implement MM in the classroom (e.g., Gould, 2013; Kuntze, 2011; Winter & Venkat, 2013; Widjaja, 2013). Additionally, five different perspectives in mathematical modeling outline different definitions of MM, ways that students might engage in the MM process, and what constitutes a MM task (Kaiser & Sriraman, 2006). Galbraith (2015) argues that these competing goals are problematic for the classroom, and Cai et. al
(2014) remind researchers that there is still a large gap between research and what is occurring in the classroom.

In the United States most classrooms have adopted the CCSSM, which includes yet another definition of MM and outlines a specific MM process. Interestingly, the definition of MM included in the CCSSM does not fit neatly into one of the research perspectives in the mathematical modeling field. In addition, research utilizing the CCSSM mathematical modeling definition is limited (see Gould, 2013; Gould & Wasserman, 2015). The aim of this study is to gain a better understanding of how students engage in MM using the CCSSM as a framework for defining MM and its tasks. Specifically, the results of this study answer the research question: How do students progress as they solve a mathematical modeling task?

**Background**

**Theoretical Lens.** This study investigates the learning that occurs from engagement in mathematical tasks assuming a broad view based in Sfard’s (2008) commognition framework. The core of this framework is the view that thinking is simply an individualized form of communication. This releases the need to view social aspects as separate from individual or cognitive aspects. Using the open view of communication, Sfard (2008) defines discourse as communication that is bound by specific rules. It follows that becoming a participant in a specific discourse means learning that discourse. This epistemological stance was adopted because it has been suggested as a potential bridge of contrasting perspectives in the mathematical modeling field (Arleback & Frejd, 2013). For this study, the students are participating in mathematical modeling discourse by solving mathematical modeling tasks and therefore learning the discourse of mathematical modeling.
Sfard defines one of the key elements of a discourse as the routines within that discourse. Routines are the repetitive patterns that are distinctive within mathematical discourse and are the heart of commognitive research (Sfard, 2008). One category of routines, called “how” routines, define the ways a specific discourse is implemented and operates. In this context, the modeling process is a “how routine” because the modeling process defines how a modeler operates to solve a mathematical modeling task. The modeling process will be the focus of analysis to examine how students participate in mathematical modeling tasks.

**Common Core Modeling Cycle.** In addition to the commognitive lens, as stated previously, this study necessitates definition of mathematical modeling and the mathematical modeling cycle for which there are many (e.g. Blum, 2011; Ferri, 2007; Zbiek & Conner, 2006). This study defines mathematical modeling and the mathematical modeling process using the CCSSM (NGA & CCSSO, 2010) because of the widespread adoption of these standards in the United States and their importance in the United States classroom. The CCSSM (NGA & CCSSO, 2010) defines mathematical modeling as “the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions” (p. 72) and as the application of mathematics to “solve problems arising in everyday life, society, and the workplace” (p. 7).

Figure 1 shows the modeling process outlined in CCSSM. The steps in this cycle will be illustrated using an example rephrased from the common core document.
Example from Common Core document:

Water and Food Relief Task: Estimate how much water and food is needed for emergency relief in a devastated city of 3 million people, and how it might be distributed (NGA & CCSSO, 2010, p. 72).

The cycle starts with the problem. These problems should have a real-world context. In this study and in the context of a classroom, the problem is a mathematical modeling task. It is in this step that the task is understood and essential features are defined. This step also includes identifying variables and assumptions. For example, in this step modelers might decide they need to know how long they need to provide water and food or that they need to know the age distribution of the city to help understand the water and food needs.

Next, the cycle moves into the mathematical world and a mathematical model is created in the formulate step. The CCSSM describes a mathematical model as a “geometric, graphical, tabular, algebraic or statistical representations that describe relationships between the variables” (NGA & CCSSO, 2010, p. 72). For the above example, this might be an algebraic equation that allows for changing scenarios.
The next step, *compute*, also occurs in the mathematical world and involves operating on the relationships. Following from the algebraic equation model for the above example, *compute* would occur when numbers are substituted for the variables and specific amounts of food and water are calculated.

*Interpret* involves understanding “the mathematics in terms of the original situation” (NGA & CCSSO, 2010, p. 72) and moves the modelers from the mathematical world back to the real world. This step would include understanding what the numbers from the computation mean in the context of the water and food example. *Validate* involves confirming the mathematical model by examining the result in the real world context. In the context of the food and water example, this would include asking whether the amount of calculated food and water made sense for the situation. Finally, *report* is communicating “the conclusions and the reasoning behind them” (NGA & CCSSO, 2010, p. 73).

This cycle has three features that we make explicit. First, all of the named steps in this model of the cycle are actions except for the beginning word, problem. This will become important to the results of this paper. Second, this mathematical modeling cycle has clear divides between the real world and the mathematical world. *Formulate* and *compute* take place in the mathematical world, while the other steps (*problem, interpret, validate, and report*) occur in the real world. Third, the arrows on the diagram imply that these steps occur in sequential order. *Validate* is the only place where there is a choice for the next step, either to finish by *reporting* the conclusions or to start a second cycle by going back to *formulate*. This creates an iterative nature to the CCSSM modeling cycle similar to other modeling cycles described in the research literature (e.g. Blomhøj & Jensen, 2003; Blum, 2011).
Three Mathematical Modeling Perspectives. In order to help bridge previous research in MM and the current study, this study used MM tasks that adhere to the definition of MM given in the CCSSM and also fit defined MM research perspectives. This study used three different types of MM tasks, one from the realistic perspective, one from the models and modeling perspective (MMP), and one from the education perspective. These three perspectives were chosen because they endorse tasks that align with the CCSSM definition of mathematical modeling and offer tasks that can be completed within a class period.

The realistic perspective advocates that students must work with realistic, authentic, messy tasks. Authentic tasks are defined as “problems that are only a little simplified and…recognized by people working in this field as being a problem they might meet in their daily work” (Kaiser, Schwarz, & Buchholtz, 2011, p. 592). The goal is for students to progress in their ability to solve real-life problems by working on these authentic tasks.

MMP modifies the goal of problem solving to include the development of the meaning and use of mathematical concepts (Lesh & Doerr, 2003). This perspective calls its tasks modeling eliciting activities (MEAs). MEAs begin with a plausible real-world problem; however, the goal of the task is primarily to develop a mathematical understanding (a mathematical model) that can be applied to other similar situations (Lesh et al., 2000). Although these models should help to find a solution to the original task, the solution is only a by-product and not the focus of the task.

Finally, the third task fits the educational perspective. The goals of educational modeling include organizing the learning process and introducing concepts for the promotion of understanding of mathematical modeling and/or mathematics (Kaiser, Sriraman, Blomhoj & Garcia, 2007).
Methods

This study follows the principles of grounded theory (Glasser, 1992; Glasser and Strauss, 1967). Because this study captures data within the social context of the preservice mathematics teacher classroom with the goal of developing a theory of how students progress as they solve the above MM tasks, grounded theory provides an excellent qualitative methodology for the study. The specific details of data collection and coding will be explained in the following section.

Participants and Setting. The study took place during Fall 2014 in the context of a capstone secondary mathematics methods course in a large southeastern university; the course is required for all middle school and high school pre-service mathematics teachers as part of the mathematics education major. The general goal of this course was to focus on the necessary mathematics for teaching secondary mathematics and to connect it to the university level courses in mathematics that the students had previously taken. The course met twice a week, each time for 75 minutes. During this course, the participants spent two weeks engaged in a mathematical modeling unit that was developed and taught by the first author with input from the second author and the course instructor. All 21 students in the course agreed to participate in the larger study; however, all data for this paper came from three specific groups of students. The first group consisted of one middle school pre-service teacher (PST) and three high school PSTs. Three of this group’s members said they had taken a course on mathematical modeling and one in the group was also majoring in mathematics (in this university, students have the option to double major in mathematics and mathematics education). All of the members in the second group were pre-service high school teachers. Two of the group members had taken a mathematical modeling course and
two were also majoring in mathematics. Finally, group three was comprised of all pre-service high school teachers. One of the group members had taken a mathematical modeling course and one was also majoring in mathematics.

**The Mathematical Modeling Unit.** The goal of the mathematical modeling unit was to engage students in mathematical modeling discourse. This unit was designed based on data collected during a study conducted in Fall 2013 (Thrasher & Keene, 2014) and suggestions from previous studies (Ferri & Blum, 2009; Mañé & Gurlitt, 2011). The unit consisted of four classes and featured two phases. The first phase was the shortest and consisted of focusing on the definitions of mathematical modeling, the mathematical modeling cycle and characteristics of a mathematical modeling task. In the second phase, students engaged in different modeling tasks and discussed their work. All data for this study came from the second phase.

**Phase one.** Before the first class, students were assigned to read one of three articles about mathematical modeling and answer some questions about mathematical modeling using the articles as a reference. These articles represented the three mathematical modeling perspectives used during the unit (realistic, educational, and MMP). On day one, the students discussed the responses to the questions in groups and created a poster and a 2-3 minute presentation about their article. Each group presented their poster and spoke about their assigned article. After their presentations, a group discussion occurred contrasting the three articles. This was followed by a presentation by the researcher on how the CCSSM defines mathematical modeling.

**Phase two.** The second phase was devoted to engaging the students in mathematical modeling tasks. First, the students completed the Empire State Building task, followed by
the Gas task, and finally the Bottle Task. Students worked in groups of 4-5 students and were asked to place their work and solutions on poster paper.

_Empire State Building Task_. This task was a Fermi problem and represents the educational perspective. Fermi problems, according to Edge and Dirks, require “sufficient understanding of the problem to decide what data might be useful in solving it, insight to conceive of useful simplifying assumptions, an ability to estimate relevant physical quantities, and some specific scientific knowledge” (1983, p. 602). It has been suggested that Fermi problems are a good entry point into mathematical modeling (Arleback, 2009). Students were not allowed to use outside resources and were asked to make relevant reasonable estimates. This requirement was placed on the students to help them focus on the assumptions being made during the process.

The prompt: The Empire State building has an information desk where the most common question asked is: How long does it take for a tourist to get to the top floor observatory? Write up a short solution to this question that includes your assumptions and your reasoning. (Arleback & Bergsten, 2013)

After each group completed the task, the groups presented their solutions. This task can be seen as representing the educational perspective because while it is set in the real world, not allowing outside resources removes some of the authenticity of the modeling activity. Additionally, the task had a very specific learning outcome of focusing on the assumptions aspect of MM.

_Gas Task_. The Gas Task met the definition of authentic (Kaiser, Schwarz, & Buchholtz, 2011) and represented the realistic mathematical modeling perspective. This was a situation that students might face in their everyday lives. The problem required making
assumptions and deciding on a model that best fit these assumptions. This task was used by the authors previously and has been shown to help engage students in mathematical modeling discourse.

The prompt: Mr. Stone lives in North Carolina in a town close to the border of Virginia. Often to fill up his car, he drives to Virginia. Gas in Virginia is around $2.55 per gallon while gas in North Carolina has been $2.85 per gallon. Is it worthwhile for Mr. Stone to drive to Virginia to fill up his car? (Blum & Leiß, 2005)

After each group found a solution, the groups shared their work through a gallery walk in which students walked around and looked at their classmates’ posters. Each student wrote feedback for each poster consisting of one idea they liked about the solution and one suggestion for improvement. The gallery walk strategy was chosen to help students understand the importance of communicating clearly, to emphasize the variability in solutions, and to promote understanding of how further iterations of the modeling cycle could improve their models. After the gallery walk, the groups read their feedback and worked to make a better model for the gas task problem.

_Bottle Task._ Carlson, Larsen, and Lesh (2003) created the Bottle Task (Figure 2) as an MEA that represents the models and modeling perspective. After the students engaged in the task, a few teacher-selected groups shared their work.
Data Collection and Coding. The sources of data for this study include digitally recorded group work on all tasks from three groups, written artifacts from the group work, and researcher field notes. Each group was recorded with one stationary camera and an audio recording device. The researcher’s field notes were used to develop an outline of what happened during each class and note interesting events that occurred.

The data was analyzed using a modified constant comparison method, which included the commognition framework conceptions to guide analysis (Merriam, 2002). We introduced the concept of modeling routes into the analysis to make visible the modeling phases that the group goes through during their work to solve a modeling task (Ferri, 2007). Specifically, a modeling route is “the individual modeling process on an internal and external level. The individual starts this process during a certain phase, according to their preferences, and then goes through different phases several times or only once, focusing on a certain phase or ignoring others” (2007, p. 2083). We used the CCSSM modeling cycle to choose the terms

Dear Math Consultants,

Dynamic Animations has just been commissioned to animate a scene in which a variety of bottles will be filled with fluid on screen. We need your help to make sure this scene appears realistic. We need a graph that shows the height of the fluid given the amount of fluid in the bottle (a height/volume graph). We have provided a drawing of one of the bottles used in the scene. Please provide a graph for this bottle and directions that tell us how to make our own graph for any bottle that may appear in this scene.

Thanks,
Dynamic Animations

Figure 2. Bottle task (Carlson et al., 2003, p. 471).
that were tagged as part of the modeling routes.

For this study, we first coded the actions of the modelers using the CCSSM steps of the modeling cycle (e.g. validate, compute). Two action codes, understanding the situation and making assumptions, were added from the description of the ‘Problem’ step in the modeling cycle to maintain consistency that all codes were actions. Understanding the situation includes trying to better understand the real world situation, simplifying the task by declaring what information is needed, and identifying variables. From the Water and Food Relief example described earlier, an example of understanding the situation is recognizing the need to know information like age distribution of the population. Making assumptions is any supposition taken by a modeler on the variables or the situation to simplify the situation. Using the Water and Food Relief Task, moving beyond recognition of the need for the age distribution to actually assuming an age distribution to help create the model would be an act of making an assumptions.

Initial coding to examine the modeling routes used by groups to solve all three mathematical modeling tasks began by coding actions taken by the group according to the stages of the modeling cycle as described above. This was done directly from the video recordings. Additionally, any actions taken that were not part of the CCSSM cycle were noted. Field notes and the student artifacts were also used to help with coding. This coding resulted in a list of the codes for each task in the order they occurred. Here is an example of the code work from Group 1 on the Gas Task:

Understanding the situation → making assumptions → compute → 
interpret → formulate → making assumptions → compute → ....

The major findings that emerged from this coding and analysis of the modeling routes will be
reviewed in the first part of the results section.

One of the findings from the original analysis was that more coding was needed. This second stage of coding required returning to the video recordings to write a “sentence” for each code, which included the objects of the actions already coded. The action code understanding the situation given in the example of Group 1’s action codes above can be used as an example. This sentence for this action was understanding the situation: task→modelers understanding of the real world situation which implied that the action understanding the situation helped to move the modeler from the modeling task to creating the modelers’ understanding of the real world situation. Similarly, the second compute in the above initial coding of Group 1’s Gas Task, comes from the group making quick calculations about how much it would cost to fill up their tanks in both North Carolina and Virginia:

Student 1: “If he stays in North Carolina he is going to spend 10 times 2.55, so $25.50, oh, ... 2.85 so $28.50. If he goes to Virginia”

Student 2: “How is he going to drive to Virginia if he doesn’t have [any gas]”

Student 1: “True”

Student 3: “Let’s say he has at least one gallon”

Student 1: “So fill up a 12 gallon tank, no an 11 gallon tank”

It can be seen from this dialogue that this computation occurred in the modelers’ real world to help the modelers start to play around with the structure of their mathematical model. This was coded as compute: modelers’ real world→mathematical model. This second level of coding created revised modeling routes that included the objects of modeling in addition to the actions. The development of these routes and the emerged theory of the mathematical
modeling process will be explicated in the second part of the results section.

Results

How can students’ mathematical modeling processes be described? This section will review the results from both the modeling routes with only the actions of modeling and the revised modeling routes that include the actions and objects of modeling.

In the first part of this section, three sub-analyses are offered. First, the representations of modeling routes that focused solely on the actions of modeling are presented. These representations were confirmation that the actions named in the CCSSM cycle are the actions that modelers’ engage in while participating in mathematical modeling discourse. Additionally in this section, one code in particular, validate, was not used often and the themes within this code are explored. Finally, two actions not mentioned in the CCSSM, reflective confirming and outside intervening are introduced.

In the second part of this section, the development and careful analysis of the revised modeling routes that include both the actions of modeling and the objects of modeling will be presented. These revised modeling routes establish a set of consistent objects within mathematical modeling discourse and culminate in a new, more accurate description for the mathematical modeling process.

Results of Analysis of Modeling Actions. The results of the first coding of the modeling routes using the actions from the CCSSM mathematical modeling cycle can be seen in Appendix B. The routes represent the actions of the students in order of occurrence. This code work shows that the students engaged in the actions of the CCSSM modeling cycle. Additionally, students participated in two other actions: reflective confirming and outside intervening which will be defined below.
**Summary of Table of Modeler’s Discourse Using Action Modeling Routes.**

Although all the actions of CCSSM cycle were used by the students, they do not engage in all of these actions equally. The *formulate code* stands out as being used most often (54 occurrences in the 9 tasks) suggesting that the students regularly engaged in the action of creating their mathematical models.

The students participated in the actions of *making assumptions, understanding situation, interpret, and compute* often as well (28-36 occurrences each). These actions were demonstrated by the students in all tasks at least once except for three instances. Group 1 did not participate in the *interpret* action in the Empire State Building Task; however, they did not finish this task so it is possible they would have engaged in interpreting their work in context if they had gotten farther along. Both Group 1 and Group 3 did not make any explicit assumptions in the Bottle Task but did make implicit assumptions. Group 1’s and Group 3’s solutions to the Bottle Task can be seen in Figure 3a and 3b, respectively. Upon inspection, it is clear that their graphs are only possible using the assumption that the water is poured at a constant rate. Therefore, both of these groups implicitly made an assumption but failed to acknowledge it. After reviewing the exceptions, they do not give any reason to believe that these modeling actions are not common to all mathematical modeling tasks.
Although the code for report does not appear in the above modeling routes often (only 14 occurrences), when it is coded, the students usually participated in the report action for an extended period of time to create their posters. This implies that although the number of occurrences was not very high, the amount of time for the action was still in line with regular participation by the students.

Finally, validate was also a code that students did not participate in very often (only 19 occurrences) and when they did participate in this action it was not for very long. Because of its limited use, this code was investigated for further patterns to better understand how students participate in the use of this action.
Uses of Validate. The code *validate* was only used after students had found a model and solution. This code identifies when students attempted to confirm their solution or examine whether their solution was realistic. Only a total of 19 instances of *validate* were observed over the nine tasks. Additionally, each group did not *validate* at all in at least one task. Groups 1 and 2 never *validated* their model during the Empire State Building task and Group 3 never *validated* their model during the bottle task.

It was found that four types of validation actions emerged from the data: Review Process, Attend to Prior Experience, Review Number Solution, and Compute an Example. The frequency of each code can be seen in Table 1.

Table 1.

*Frequency of Each Type of Validate.*

<table>
<thead>
<tr>
<th>Type of Validate</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Review Process</td>
<td>12 (63%)</td>
</tr>
<tr>
<td>Attend to Prior Experience</td>
<td>3 (16%)</td>
</tr>
<tr>
<td>Review Number Solution</td>
<td>3 (16%)</td>
</tr>
<tr>
<td>Compute an Example</td>
<td>1 (5%)</td>
</tr>
<tr>
<td>Total</td>
<td>19</td>
</tr>
</tbody>
</table>

The first and most prevalent type of action, Review Process, describes when the group reviewed the process used to make their model and solution. A majority (63%) of the occurrences were this type of validation (see Table 1). An example of Review the Process validation can be seen in the following work from Group 1 on the gas task.

Setting the scene: The group has worked through their model to determine that Mr. Stone should not go to Virginia.
Student A: "Let’s just make sure I have it"

Student B: "Okay"

Student A: "So we are saying he has one gallon and he is going to drive over there [to Virginia]. He will have zero gallons. He will fill it up but then when he goes back he will be short one gallon so he has to fill it up one gallon there. So don’t go to Virginia"

This type of validating resembles checking over your work to make sure you understand and agree with all the decisions made.

Attending to Prior Experience refers to the use of prior knowledge and experiences used to confirm a solution. This type of validating occurred a total of three times and groups used personal experiences with similar situations to the task to confirm their solution. An illustration of this can be seen in the following excerpt from Group 3 when the group uses experiences waiting for rides at Busch Gardens to compare to waiting to get to the top of the Empire State Building in the Fermi task.

Setting the scene: The group just finished calculating the amount of time they think it should take to get to the top floor of the Empire State Building and they interpreted their number to mean the time it takes in the elevator only.

Student A: "I guess that is realistic"

Student B "See I am thinking that is unrealistic. My gut tells me it is so busy there...I’m thinking like if you go to Busch Gardens to ride a ride"

Student C: "That is what I am saying"

A third method that groups used to validate developed from having a numerical solution that did not seem correct in some way. This can be seen during a validate excerpt
from Group 3. The group has just finished computing a numerical solution using their model and they question the number.

*Setting the scene: The group has computed their solutions and has started to create their poster to display their work to the Gas task.*

*Student A:* "Something doesn't feel right about this one [referring to one of their calculations]"

*Student B:* "Why?"

*Student A:* "I must have keyed it in wrong...[student grabs a calculator] So if we saved $3"

*Student B:* "3 divided by 2.55 times 28...32" *Student A:* "Okay [it was correct]. Something just didn't feel right"

As seen in this example, this type of *validate* is initiated by having a computed number that seems inaccurate. In all three of these instances, the resulting action was to recalculate their solution to make sure no calculation errors were made. This implies that the groups undergo Review Number Solution to check for computation errors but do not use it to question their models in general. This type of validating occurred three times.

The final type of validating involves checking the model by Computing an Example. This type of validating only occurred once but distinctly differs from the other three types of validating because it involves performing calculations that do not directly solve the task but are simply checking the model. Group 2 performs this type of *validate* after they had finished writing their mathematical model for the steps to draw a graph of the height vs. volume graph of any fluid (see Figure 4) for the bottle task. To check their third step, the group decides to compute an example.
Figure 4. Group 2’s work on bottle task.

Setting the scene: The group had the above steps written on their poster (see Figure 4).

Student A: “Let’s work it for a cylinder ‘cause that would be a line then [to check scenario 3]”

Student B: “Volume is constant we know that”

Student A: “So if you are half way for 2 (height)...the volume for the entire thing...”

Student B: ”$\pi r^2 h$ “

Student A: “$\pi$ times 4 squared times 4. One-half $V$ is $\pi$ [times] 4 times 2....[quiet work] change in height over change in volume...it’s a constant”

Reflective Confirming code. The researchers noticed there appeared to be something similar to validation occurring that was not being captured with the validating code. This new code was named reflective confirming and is defined as the work of modelers to confirm their work or check their work during intermediary steps leading to a model. The reflective
confirming can be seen as being similar to the validate code; however, there is a critical difference. This difference has to do with scale. While validate is considering the entire process and the cohesiveness of all work, reflective confirming focuses on the current point of work and the appropriateness of an individual part as it is being worked on. This code can be compared to monitoring or self regulation while problem solving (Schoenfeld, 1992). This code included all the instances where students questioned their work but were not specifically validating the whole model itself. This most commonly took on the form of asking the question, “Is that right?” or “Does this make sense?” when working to develop their model. Here is an example of this code from Group 2 during the Bottle Task.

Setting the scene: The group is trying to utilize the equation for the volume of a sphere using a given volume by solving for the radius and then doubling the radius to create a height. Their goal is to get ordered pairs to create their graph model.

Student B: “That is for 100 [volume]” (Student B points to calculator displaying ~2.8).

Student A: “400 is 9.14. Does that make sense?”

Student C: “9.14”

Student A: “9.142. Does that make sense though?”

Student C: “Yeah, because it’s a sphere so the further you get down, so like, the more you chop at the top the less it counts but take it to the middle”

As seen in this excerpt, the group has not created their graphical model yet; however, during this process they are questioning their work to see if it makes sense.

After coding the videos for instances of reflective confirming, it became clear that groups were confirming their work more often than the validate code suggested, just on a
smaller scale. In all but Group 1’s Bottle Task and Group 3’s Empire State Building Task there was at least one instance of *reflective confirming*.

It can also be noted that the two group/task combinations without *reflective confirming* had two instances of *validating*. This suggests that in every task/group they were either validating their work at the end or doing some reflective confirming of their process as they created their model.

**Outside Intervening Code.** In addition to reflective confirming, another action that occurred and needed to be documented during mathematical modeling was *outside intervening*. Outside intervening occurs when advice is either sought or offered from an outside modeler(s) and the advice changes some aspect of the group’s modeling route. During these tasks, outside intervening occurs in the form of teacher intervention or classmate intervention.

Outside intervening by the teacher is the most common type of this action. This occurs when the students seek advice from the teacher or when the teacher chooses to intervene. The following is an example of an outside intervening where the group adds a new assumption to their model because the teacher chose to question the group’s assumptions.
Figure 5. Group 3’s model.

Setting the scene: The group is validating their created model, as shown in Figure 5, by looking online for averages of miles per gallon (mpg) to compare with the assumptions they made in their model.

Student A: “I’m looking at the national averages [for mpg]”

Teacher: “What assumptions are y’all making?”

Student A: “This and this [pointing to mpg and tank size column on poster]”

Teacher: “So are you always filling up from empty?”

Student B: “Yes...so assuming empty tank”

Student A: “Assuming empty upon arrival”
Teacher: “Which is pretty hard to do right?”

Student B: “You would be surprised”

Student A: “Assumptions are miles per gallon, tank size, and empty tank”

This outside intervening leads the modelers to explicitly acknowledge another assumption under which their model operates.

The second method of outside intervening during these tasks comes in the form of a classmate. The only time this is explicitly shown during these tasks was during the gallery walk. Because the gallery walk required the students to critically look at other solutions to the task and leave critiques for each group, many groups used these critiques to modify their models. Additionally, the students learned from looking at the other models and explored ideas from other groups to improve their models as observed in the following scene.

Setting the Scene: Group 1 looks through their feedback and starts to question how to generalize their model

Student C: “So that is the thing, there are a lot of different things can influence the decision...if he has half a tank but gets really really high miles per gallon then it might be worth it because he gets really really high miles per gallon...there is a lot of different considerations to make it general. How are we going to make a formula that incorporates all of that?”

Student B: “Some people [other groups] had different scenarios like highway and city”

This scene points out that the group looked at their feedback and started to question their model and also used the other groups’ models as possible solutions to their own model concerns. These are forms of outside intervening because the modelers’ actions are being
influenced by other classmates.

Table 2.

*First four actions for all group/task combinations.*

<table>
<thead>
<tr>
<th>Task</th>
<th>Group</th>
<th>First Action</th>
<th>Second Action</th>
<th>Third Action</th>
<th>Forth Action</th>
</tr>
</thead>
<tbody>
<tr>
<td>Empire State Building Task</td>
<td>1</td>
<td>Making Assumptions</td>
<td>Formulate</td>
<td>Understanding Situation</td>
<td>Making Assumptions</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>Making Assumptions</td>
<td>Understanding Situation</td>
<td>Formulate</td>
<td>Making Assumptions</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>Formulate</td>
<td>Understanding Situation</td>
<td>Making Assumptions</td>
<td>Understanding Situation</td>
</tr>
<tr>
<td>Gas Task</td>
<td>1</td>
<td>Understanding Situation</td>
<td>Making Assumptions</td>
<td>Compute</td>
<td>Interpret</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>Formulate</td>
<td>Understanding Situation</td>
<td>Making Assumptions</td>
<td>Understanding Situation</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>Understanding Situation</td>
<td>Formulate</td>
<td>Making Assumptions</td>
<td>Formulate</td>
</tr>
<tr>
<td>Bottle Task</td>
<td>1</td>
<td>Understanding Situation</td>
<td>Formulate</td>
<td>Compute</td>
<td>Formulate</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>Understanding Situation</td>
<td>Formulate</td>
<td>Compute</td>
<td>Understanding Situation</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>Understanding Situation</td>
<td>Formulate</td>
<td>Understanding Situation</td>
<td>Formulate</td>
</tr>
</tbody>
</table>

**Analysis and Results for Revised Modeling Routes.** Although the actions of the CCSSM-described cycle were observed in the students’ modeling discourse, the sequential feature of the CCSSM cycle was not observed. In fact, there did not seem to be a pattern to the actions that the modelers underwent to solve these tasks. To quickly examine this, Table 2 looks at only the first four actions for each group on each task. In this very small example, only two of the nine have the same first four actions, Group 2’s Gas Task and Group 3’s Empire State Building Task. However, if you look at their entire CCSSM modeling routes (Appendix B), it is easy to see that even these two modeling routes diverge at the sixth action.
when Group 3 is understanding the situation in the Empire State Building Task and Group 2 is reflectively confirming in the Gas Task. Additionally, the order suggested by CCSSM *(Understand Situation/Make Assumptions, Formulate, Compute, Interpret, Validate, (possibly repeat), Report)* is not observed in these first four actions making this process non-sequential.

Because of the non-sequential order to the actions, the researchers sought to make the modeling routes more detailed. This led to coding the objects of the actions as described in the methods coding section. Immediately, it was clear that the specific actions did not always occur between the same objects. For instance, the action *making assumptions* can be used to simplify the task or can be used to make a certain mathematical model function properly. In the CCSSM modeling routes for Group 1’s work on the gas task, the first *making assumptions* in the list of actions is when the group was making initial assumptions to simplify the real world problem. For example, this group decided that the “car goes 20 miles to the gallon” and that “time isn’t really a factor, just money and miles.” This type of assumption clarified the boundaries of the task. This *making assumptions* was an action that took the group from the broad parameters of the mathematical modeling task to a modeler’s understanding of the real world context. While the second *making assumptions* in the group 1 Gas task modeling route occurred after the group had started to *formulate* a mathematical model and after the group had done some quick computations. In this *making assumptions* the group is “filling up a 10 gallon tank” but they decide to make this assumption because they realize from trying to create their model that they need this assumption to move forward. They choose 10 gallons because “we are just doing some nice numbers” to put into their model. Different from the first *making assumptions*, this *making assumptions* action took the
group from the real world into the mathematical world and allowed for them to start to create their model.

Because actions occurred on different objects during the modeling process, new modeling routes were created that included five objects of modeling developed from the synthesis of the codes and the actions from the CCSSM cycle. These new objects are the mathematical modeling task, the modelers’ understanding of the real world situation, the mathematical model, the solution, and the report. To help better understand the use of these objects, the Gas Task from Group 1 will be used.

The mathematical modeling task is defined as the task as first presented to the modeler. In this case, the mathematical modeling task is the Gas Task. The modeler’s understanding of the real world situation emerged as the specific ways that the group interpreted and defined the real world aspects of the mathematical modeling task. For example, Group 1 during the Gas task decide to make the real world scenario for this task that Mr. Stone’s “car averages 20 mpg” and that Mr. Stone is “20 miles from VA gas station.” These assumptions are grounded in the real world but narrow the scope of the real world situation, which helps to define the group’s understanding of the real world situation.

The mathematical model, solution, and report can be seen in Figure 6. The mathematical model is defined as the mathematical interpretation of the modelers’ real world situation. For Group 1, the mathematical model can be seen in Figure 6 in the first line of the work in purple: \( x(2.85)=(x+2)(2.55) \) where \( x \) represents the number of gallons being bought. The solution is defined as the final product, answer or conclusion to the mathematical modeling task. In Figure 6, the solution is in the pink box where it concludes that “The bigger the tanks; Go to VA. \( x >17. \)” Finally, all of Figure 6 is the report which is defined as
the way in which the work on the modeling task is formally shared. This is a change from
the original coding because report was taken from the list of actions and changed to an object
of modeling. In this case, Figure 6 is part of a poster the group put together on their work.

Figure 6. Group 1’s mathematical model and solution to Gas Task.

Revised Modeling Routes. From the second round of coding, revised modeling
routes were created. The revised modeling routes can be seen in Figure 7. In these revised
modeling routes the direction between the modeling objects is captured with an arrow, all the
actions that moved the modeler’s from one object are listed below the arrow, and the order in
the sequence of movement between objects is denoted by the number below the arrow. In an
attempt to convey information concisely, each action noted on the diagrams represents the
action occurred at least once but possibly more often between those two modeling objects.
Figure 7. All Revised Modeling Routes. KEY: MA- Making Assumptions, US- Understanding the Situation, For- Formulate, OI- Outside Intervening, Com- Compute, RC- Reflective confirming, Int- Interpret, Val- Validate
Two themes emerged after analysis of these revised routes. The first theme, which is briefly described above, is that the actions of modelers can occur between different objects, as seen with the making assumptions example. The second theme is that the revised modeling routes further show that mathematical modeling discourse is non-linear and non-cyclical. This means that the same actions do not occur in the same order or necessarily recurrently when solving these mathematical modeling tasks.

**Interactions of modeling actions and modeling objects.** From coding the actions and objects together, the resulting picture is of a discourse where many different types of modeling actions can occur on many different modeling objects. This is seen for many of the action/objects combinations. By including the objects of mathematical modeling, richer descriptions and understanding of the process of modeling can emerge. The following example shows how the actions of formulate, compute, interpret, and making assumptions occur to help the modelers go from the modeling objects of the modelers’ understanding of the real world to create a mathematical model during the Gas Task. This excerpt is represented on the revised modeling route for Group 3’s Gas Task under arrow number two. In the excerpt, the action of modeling is written at the end of each quote when applicable.

*Setting the Scene: Group 3 has narrowed the Gas Task to create their own real world understanding of the situation. In this case, they decided to look at a truck, a car, a hybrid car and a SUV. They further narrowed the real world understanding of the task by using their experiences with these vehicles to make assumptions about the average miles per gallon (mpg) and tank size of the vehicles. So far they have:*
### Modelers’ Real World Understanding of Task

<table>
<thead>
<tr>
<th>Vehicle</th>
<th>Mpg</th>
<th>Tank size</th>
</tr>
</thead>
<tbody>
<tr>
<td>Truck</td>
<td>14</td>
<td>25</td>
</tr>
<tr>
<td>Car</td>
<td>28</td>
<td>10</td>
</tr>
<tr>
<td>Hybrid</td>
<td>48</td>
<td>11</td>
</tr>
<tr>
<td>SUV</td>
<td>20</td>
<td>14</td>
</tr>
</tbody>
</table>

Student A: “Let’s make a distance for when it works…like for this car at x amount of miles it makes sense and after that it doesn’t make sense [to drive to Virginia]”

**FORMULATE**

Student B: “So greater than or equal to…so everything has to be greater than or equal to”

Student A: “So we need to figure out how much he would spend per tank. So North Carolina 2.85 times 11” **COMPUTE**

Student B: “So how many miles are you going to get per tank?”

Student A: “I’m going to do what a tank would cost and how many miles he would get. You figure out how many miles for each tank and I’ll do the gas for all of them.”

Student B: “11 gallons times [quiet work to calculate miles] so this is cost per tank and this is miles” **INTERPRET**

Student A: “How much it costs to fill up….we are assuming he is going to fill up an empty tank” **MAKING ASSUMPTIONS**

Student B: “So on the hybrid $31 [North Carolina tank cost] versus $28 [Virginia tank cost] and you get 528 miles. When is it less than or equal to, so North Carolina has to be less than or equal to Virginia to make it worth the drive. No Virginia would need to be less than or equal to North Carolina to make it worth the drive.”

**FORMULATE**
Student A: “So the difference in price is this [points to $3 difference in price between North Carolina tank and Virginia tank] so we need to see how far this will take him” COMPUTE

Closing the Scene: The group calculates for all vehicles how far the difference in price between North Carolina tanks and Virginia tanks will take them and create a table model of their results (see Figure 8).

![Table model of results](image)

*Figure 8. Group 3’s understanding of the situation.*

From this excerpt of work done by Group 3 on the Gas Task, a better understanding of the actions of formulate, compute, interpret, and making assumptions can be established. In this example, the group started to make decisions on what their mathematical model would look like by deciding to make distance the determining variable in their model and by deciding to calculate the distance using the savings from filling up in Virginia. However, to help them formulate the model, the group made multiple calculations (COMPUTE) (cost of full gas tank in North Carolina and Virginia, how many miles they drive on the savings in Virginia). These calculations were not done all at once but were used to have a better understanding of how to create the model. Also, during the computations, the group
regularly understood their calculations in terms of the real world saying “this is cost per gallon” or “this is miles” which is the action interpret. This interpretation of the calculated numbers in the context of the real world situation was needed by the groups to help them understand how to create their model and to understand their model. Finally, the group realized that they were making a new assumption while creating their model, “he is going to fill up an empty tank.” This assumption was needed to complete their model. It can be seen with this example that adding in the objects helps to better understand the actions of the modelers because it presents a more clear purpose to the actions.

Additionally, this excerpt demonstrates that the clear divide of the real world and the mathematical world evident in the CCSSM cycle is not so clear. All of these actions were performed to create a mathematical model of this task; however, throughout the excerpt and in the mathematical model, aspects of the mathematical world and aspects of the real world scenario are called upon repeatedly. For example, Student A’s motivation to compute came from understanding the real world context of “So we need to figure out how much he would spend per tank.”

Moreover, having the actions occur to move modelers from any of the mathematical modeling objects (both objects that are in the real world and those in the mathematical world) further suggests that a clean divide between the real world and mathematical world does not exist. The following example, from Group 2’s work on the Gas Task, shows how making calculations (mathematical world) help to establish assumptions in the real world.

*Setting the Scene: This group is discussing an assumption on how far Mr. Stone is from Virginia to help clarify the modelers’ understanding of the real world situation*
from the mathematical modeling task. The group has already established that Mr. Stone will be driving an average of 45 miles per hour.

Student A: “From my house to anywhere is 5 minutes at least” UNDERSTANDING SITUATION

Student B: “So what do you want to say?”

Student A: “15 [miles]?”

Student B: “How long would that take if you were driving on the highway in the city? If he is going 35 miles per hour... probably 45 miles per hour, how many miles is a 5 minute drive?...” COMPUTE

Student A: “So 5/45. 1/9 do 1/9 of an hour [calculations on a calculator] that is 6/7 miles”

Student C: “we started with 5 minutes, just convert minutes to hours”

Student D: “1/12”

Student C: “divide by 45 that should get you miles per hour, right?”

Student A: “45 miles and x is what we are trying to find [setting up a proportion of 45miles/1hour = x miles/(1/12) hour].

Student B: “So you divide 45 by 12.”

Student A: “3.75 miles”

Student B: “So let’s say 5 miles” MAKING ASSUMPTIONS

This example starts with relating the situation back to one of the modelers’ experiences of needing to drive at least 5 minutes. Then using this experience, the group computes a corresponding number of miles to establish an assumption for the miles Mr. Stone is from Virginia. Within this clip the real world situation of a 5 minute drive is motivating the need
for the mathematical world but at the same time the mathematical world’s calculations are in a constant state of being checked in the real world as seen when Student C attempts to set up a calculation saying “we started with 5 minutes, just convert minutes to hours.” What this excerpt demonstrates is the constant simultaneous checking between the real world and mathematical world. In fact, it is difficult to determine where the mathematical world and the real world begin and the modelers appear to be in both worlds at that same time. This further suggests that the divide between the real world and mathematical world while engaged in mathematical modeling is not a clean divide.

Mathematical Modeling is Not a Cycle. The revised modeling routes further confirm the results from the CCSSM modeling routes, actions only, that mathematical modeling discourse does not follow a cyclical pattern or a sequential pattern when modelers solve tasks. Looking across the nine revised modeling routes, it is clear that there is no discernable pattern to the actions and no pattern to the movement between the objects of modeling. But, this should not be interpreted as problematic on the side of the students, but instead a call to question the cyclical design of the CCSSM cycle. As evidenced by their solutions and their actions, these students engaged in mathematical modeling discourse. What these revised modeling routes suggest is that the mathematical modeling process looks different in different groups and with different tasks.

So what does the mathematical modeling process look like? What we do know from these action modeling routes and the revised action/object modeling routes is that it is possible to identify the actions and objects of mathematical modeling and know that the actions of mathematical modeling help the modeler move between the objects of mathematical modeling. What is not possible is to know in advance the order of how the
modelers will experience these different actions and objects as they work towards a solution. Figure 9, captures this idea of mathematical modeling. The mathematical modeling objects are in the box on the left and the mathematical actions are in the box on the right. The arrows represent movement between the objects with the actions but the arrows do not point to a particular action or object because it is impossible to know the order that any one modeler will experience.

\[\text{Figure 9. The emergent modeling process.}\]

In addition to our finding that modeling is not a cycle, we also found that the mathematical modeling process does not have a clear divide between the real world and the mathematical world. In fact, it appears that the students use a combination of both worlds to move toward a solution as shown in the above examples. This is captured in the above diagram of the modeling process by not dividing the modeling objects into the real world and mathematical world. This new depiction of the modeling process illustrates that at any point during the modeling process the modelers could be in the real world, the mathematical world,
or both and that the different parts of the mathematical modeling cycle do not need to be arbitrarily placed in one world or the other.

**Discussion and Conclusions**

The goal of this study was to provide insight on students’ mathematical modeling processes. By framing the study with the commogition lens (Sfard, 2008), the social and individual aspects engaged in by the students were analyzed as mathematical modeling discourse to provide a cohesive view of the actions the helped students progress while solving the MM tasks. By creating the modeling routes and the revised modeling routes, the “how routines”, one aspect of mathematical modeling discourse as define in commognition are illustrated and a richer picture of mathematical modeling discourse has emerged. From the modeling routes, this study confirmed the use of understanding the situation, making assumptions, formulate, compute, interpret, and validate as valuable to recording the actions of students while working on mathematical modeling tasks. It was not surprising to the researchers that these actions occurred from participating in modeling tasks because similar actions have been noted (Berry and Davies, 1996; Blum, 2011; Ferri, 2007; Widjaja, 2013; Zbiek & Conner, 2006).

While most of the actions occurred frequently, validate was not observed as often as the other actions. This is an interesting result when compared to results from Perrenet & Zwanevald (2012) who found that validate was present most often in self-created modeling cycles by teachers and students. However, Mousoulides, Pittalis, Christou & Sriraman (2010) also found limited participation in validating with 6th and 8th grade students’ mathematical modeling work. In addition to noting its limited use, this study sought a better understanding of the different aspects of validate. Four ways students validate their work
were identified: *review process, attend to prior experience, review number solution*, and *compute an example*. This is probably not an exhaustive list of ways to *validate* one’s mathematical model; however, identifying these methods provides a rich place to engage students in reflection upon their own use of *validate*.

Additionally, two actions were identified that are not part of the CCSSM cycle. The first, *reflective confirming* or the modeler assessing the appropriateness of his/her current work, captured the “in the moment” evaluation of one’s work. Zbiek and Conner (2006) have also suggested a similar action, *aligning*. This action is important because it helps explain one possibility for why the students did not feel the need to *validate* their work: it is possible that they felt that they had already checked their work throughout the entire process. The second additional action, *outside intervening*, captures the effects on a modeler’s work from external influences. In this study, these occurred within the context of a classroom and the outside influences came from classmates and the teacher; however, outside the classroom this could take on many forms such as new world experiences. Both of these actions are important to capturing the movements made by students while completing mathematical modeling tasks.

Finally, by identifying the actions of modeling in the students’ work, it became clear that the *actions* of modeling are, by themselves, not enough; the *objects* of modeling have an equally important place in the process. By adding the modeling objects into the mathematical modeling process, the revised modeling routes of the students clearly showed a non-linear, non-cyclical pattern. Using the revised modeling routes, a new depiction of the mathematical modeling process emerged. This new process has three prominent features.
The first feature is that the focal points of this modeling process are the actions and objects of mathematical modeling. This is really not so different from many other mathematical modeling process depictions (e.g. Blomhøj & Jensen, 2003; Blum, 2011, 2011; Ferri, 2007) thus further confirming the need for both the objects and actions of modeling. However, this is a serious critique of the CCSSM modeling cycle.

The second attribute is that this new depiction has a free flow from object to object, using the actions of modeling. Additionally, there is no predetermined cycle to the objects or actions. This is drastically different than most mathematical modeling processes including the CCSSM modeling process (Blomhøj & Jensen, 2003; Blum, 2011; CCSSM, 2010; Ferri, 2007). The lack of a cyclical nature of the mathematical modeling process is one of the most prominent features in mathematical modeling (Perrenet & Zwarense, 2012) and marks a significant departure from the mainstream illustration of the modeling process. Doerr and Pratt (2008) question the linearity as suggested by the mathematical modeling cycle, but they replace the cyclic model with one that focuses only on the actions of modeling without regard for the objects of modeling. Additionally, Ferri has also shown non-cyclical work in her work with modeling routes but fails to question the original premise of a cyclical model. Galbraith (2015) even warns that modeling routes should not be confused with the modeling cycle; however, the researchers in this paper disagree. We contend that one individual modeling route should not be confused with the modeling process; however, the similarities among modeling routes should be generalizable by the modeling process depiction. The work shown here strongly suggests that the modeling process is in fact not cyclical but the new model does provide the generalization needed.
Third, this new depiction of the mathematical modeling cycle does not emphasize the divide between real world and the mathematical world. This divide is seen in the most basic of diagrams of mathematical modeling as in Figure 10 by Niss, Galbraith & Blum (2007). As argued above, it is hard to separate out the mathematics world and the real world while students are working on mathematical modeling tasks; therefore, this divide should not be a prominent feature in the depiction of the mathematical modeling cycle. It is also important to recognize that the creation of this mathematical modeling process utilized different types of mathematical modeling tasks. This can be seen as a bridge between these different perspectives.

![Figure 10](image)

*Figure 10. Niss, Galbraith, & Blum’s (2007, p. 4) depiction of the real world and mathematics within mathematical modeling.*

Together, these three features raise concerns about the modeling cycle followed in many US mathematics classrooms and in features of other cycles being taught throughout the world. The CCSSM suggests a cyclic nature to the modeling process, which this study did not find. Additionally, the CCSSM only uses the actions of modeling; however, this study shows that is a limited view of the cycle because the objects that are being acted upon help to
provide a clearer picture of the process. At the very least, the findings from this study call for more research to ensure that the modeling cycle depicted in guiding documents for teachers, specifically the CCSSM (NGA & CCSSO, 2010), are faithful to the modeling process of students (and possibly professionals who model). Moreover, a correct understanding of the process that students undergo to solve these types of task is essential for teachers. Teachers are expected to be able to anticipate what students will do while solving these tasks (Stein & Smith, 2011) and an accurate depiction of the modeling process is a sizeable part of understanding the work students will do.

To really be able to help teachers, more research needs to be done. First, research working with the mathematical modeling process using larger numbers and varied types of students is needed to either validate the process proposed here or to improve upon it. This study is limited by a focus on only three groups of pre-service teachers. With a better understanding of the modeling process, other important research can be conducted, such as studying how the modeling cycle can help teachers implement mathematical modeling in the classroom and how understanding the modeling process can help students in solving modeling tasks.

Finally, in terms of the impact on research in mathematical modeling as a field, this study offers a glimpse of how to help build the field by bridging between the different perspectives. This study used three mathematical modeling tasks, each with characteristics that belong to a different perspective on mathematical modeling. From analyzing these tasks in one study, a general mathematical modeling process was created that spans the different perspectives. By bring the perspectives together, we can more fully understand the general field of mathematical modeling.
References


Chapter Four: Article Two

A study of mathematical modeling through the lens of mathematical habits of mind

Introduction

In mathematics education there has been ongoing emphasis on connecting the real world to mathematics (e.g., Freudenthal, 1981; Galbraith, 2013; National Council for Teachers of Mathematics, 2000). One reason for this focus is that there is research suggesting learning abstract mathematics does not transfer into using mathematics to solve problems (Niss, Blum & Galbraith, 2007; Pollack, 2007; Resnick, 1983). Utilizing mathematical modeling in the K-16 classroom represents an opportunity to recognize missing connections between mathematics commonly learned in the classroom and the rest of the world (Hestenes, 2010; Pollack, 2007) and recently, many countries are including an increased focus on mathematical modeling in their mathematical standards (Ferri, 2013).

Like the dual pull of theoretical mathematics and applied mathematics in the classroom, mathematical modeling experiences a split focus between the goal of facilitating the learning of mathematics and exploring extra-mathematical scenarios using mathematics as a tool (Julie & Mudaly, 2007; Niss et. al, 2007). Different perspectives (e.g. realistic mathematical modeling perspective, models and modeling perspective) in mathematical modeling emphasize these goals to varying degrees; however, realistic mathematical modeling is the perspective that most explicitly focuses on uses mathematics to solve real world problems. The most important feature of the realistic modeling perspective is that the goal of mathematical modeling is to solve real world problems (Kaiser & Sriraman, 2006). The perspective advocates that students must work with realistic, authentic, messy tasks. Authentic tasks are defined as “problems that are only a little simplified and…recognized by
people working in this field as being a problem they might meet in their daily work” (Kaiser, Schwarz, & Buchholtz, 2011, p. 592). The goal for students is that through working on these authentic tasks, they will progress in their ability to solve real life problems and become better mathematical modelers. This perspective emphasizes the inclusion of mathematical modeling in classrooms because of its practicality (Kaiser, Sriraman, Blomhøj, Garcia, 2007). In fact, there is little focus on the development of students’ mathematics understandings or mathematical theory (Blomhøj, 2009).

Since mathematical development is not a focus of realistic mathematical modeling, there is limited research that focuses on mathematical development while engaged in realistic mathematical modeling tasks. If fact, much of mathematics modeling research focuses on the modeling cycle and modeling competencies and few other local theories are used to drive the research (Geiger & Fredj, 2015). However, it is important to know the mathematical learning that occurs as well especially to better equip teachers to engage students in these task in the classroom (Kaiser & Maab, 2007). This study addresses this gap by using the principles of mathematical habits of mind to examine the extent to which mathematics is used while students are engaged in a realistic mathematical modeling task. Specifically, we ask:

1. Are the mathematical habits of mind a productive way to describe students’ mathematical work during a realistic mathematical modeling task?

2. If so, how and in what ways do students practice mathematical habits of mind while engaging in a realistic mathematical modeling task?
Background

**Theoretical Framing.** This study is situated in school mathematics. According to many, school mathematics is comprised of two parts, see Figure 1. The first part is mathematical content and regards subject-matter. This includes algebra, geometry, definitions, axioms, etc. While the second part has many name different names such as mathematical processes (NCTM, 2000), mathematical practices (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA & CCSSO], 2010), or Habits of mind (Cuoco, Goldenberg, & Mark, 1996, 2010; Harel, 2008) these all focuses on the actions done by individuals on and with mathematical content. Therefore, we are calling this second part of school mathematics, that can broadly be defined as the actions done by individuals on and with mathematical content, mathematical activity in honor of Freudenthal’s broad definition of mathematics as human activity (Keitel, 1993).

![School Mathematics Diagram](image)

*Figure 1.* Diagram of the components of school mathematics.

Mathematical activity has many definitions and needs further research (Moschkovich, 2013). One example from research is from Harel (2008). He uses the idea of conceptual
tools defined as the necessary understandings to generate mathematical objects to describe this component of school mathematics. From standards documents, The National Council of Teacher of Mathematics (NCTM, 2000) describes mathematical activity with its process standards, which describe the ways students should work with mathematical content and the Common Core State Standards for Mathematics (NGA & CCSSO, 2010) uses the term mathematical practices. For this study, we choose to frame mathematical activity in terms of mathematical habits of mind. In the seminal work of Cuoco, Goldenberg, and Mark (1996), the authors coin the term habits of mind as the methods and techniques used by mathematicians when doing mathematics. Cuoco and colleagues (1996) argue that habits of mind should help frame curricula, which would help students to “develop genuinely mathematical ways of thinking” (p. 401). Habits of mind were chosen because they continue to be of interest to mathematics educators; however, the research is limited on these habits (Lim & Selden, 2009). Therefore, this study hopes to develop more robust understanding of the habits of mind while simultaneously showing how mathematical activity is practiced during a realistic mathematical modeling task.

**General Mathematical Habits of Mind.** The focus of this study is on Cuoco et al.’s general mathematical habits of mind. These general habits of mind are ways of thinking that span the different content areas (e.g., algebra, geometry, number and operation). Cuoco, Goldenberg, and Mark (2010) describe six general mathematical habits of mind for high school students. They are: *performing thought experiments, finding articulating and explaining patterns, creating and using representations, generalizing from examples, articulating generality in precise language, and expecting mathematics to make sense.* We
have chosen to use this set of habits of mind because these are the latest set of habits of mind put forth by Cuoco et al. who originally developed a focus on habits of mind within the K-12 classroom. The following will review the details of each of these habits of mind. All of these details come from Cuoco et al’s two publications (1996, 2010).

**Performing Thought Experiments.** This is the ability of students to create a mental image or visualize about a context. This allows students to start to manipulate and reason about the context. Cuoco et al. (1996) argue that students should be able to visualize data, figures, relationships, processes, change, and calculations and that visualizing these ideas are key components of performing thought experiments. Cuoco et al. (2010) give the example of students visualizing a cardboard rectangle with congruent squares cut out at the corners and the sides folded up to create a three dimensional box. Then performing thought experiments might further extend to visualizing the change and relationships between the size of the squares cut and the continuum of boxes that can be created.

**Finding, articulating, and explaining patterns.** The heart of finding, articulating, and explaining patterns is looking for patterns and explaining why the patterns happen in clear language. This habit involves being on the lookout for repetition, discovering the patterns, and precisely describing them (see articulating generality in precise language). Students should seek to understand why the pattern exists. An example of this habit would be inviting students to find, describe, and explain a pattern in a multiplication table.

**Creating and using representations.** This habit of mind refers to both the creation of new representations of some context and the way this representation is used to better understand the context. This means that this habit of mind is comprised of two connected components: the representation and the affordances of the representation. Any instances of
re-presenting a context embodies a new representation. These different representations
give new affordances and lead to different understandings and approaches to solutions.

Common representations in mathematics are graphs, equations, and tables.

**Generalizing from examples.** Closely related to finding, articulating, and explaining
patterns; generalizing from examples also seeks to find and explain patterns. However, the
pattern that is being sought after specifically comes from the student experimenting with
different examples. The defining characteristic of this habit is that students are expected to
experiment, usually with numbers, to create the examples. In those examples, the students
look for structure and trends that can be generalized.

**Articulating generality in precise language.** This habit permeates through the other
habits. Articulating generality in precise language is the ability to express their mathematical
ideas into precise language. Cuoco et al. (2010) argue that students can understand
something mathematically but still not be able to express this idea in precise mathematical
language. In previous publications, Cuoco et al. argue that students need to develop
“expertise in playing the mathematics language game” (1996, p. 379) which includes giving
precise descriptions of steps in a process and invent notation. In summary, this habit of mind
is the ability of students to successfully convey their mathematical understandings.

**Expecting mathematics to make sense.** Reasoning and sense making are the key
components of expecting mathematics to make sense. Sense making is “developing
understanding of a situation, context, or concept by connecting it with existing knowledge”
(Graham, Cuoco, & Zimmermann, 2010, p.4). In the case of mathematics, sense making in
mathematics occurs when either the situation being understood is mathematical or when the
situation is better understood through mathematics. In the case of mathematical modeling,
sense making occurs from developing an understanding of a situation using mathematics. This occurs with reasoning that is “the process of drawing conclusions on the basis of evidence or stated assumptions” (Graham et. al, 2010, p. 4). Therefore, bringing these two ideas together, expecting mathematics to make sense, are instances where mathematics is used in understanding a situation and allows for conclusions to be drawn.

**Additional notes on practicing habits of mind.** These six habits of mind do not happen in isolation of each other. These habits of mind can overlap in both use and in definition. For example, articulating patterns, is a more specific version of articulating generality in precise language. Or creating representations and articulating patterns would necessarily involve expecting mathematics to make sense because creating a new representation would require the student to also be using reasoning and sense making. This implies that both their definitions overlap and the way they are practiced will overlap. Additionally, these six habits of mind do not necessarily constitute all the habits of mind that mathematicians practice; however, we agree with Cuoco and colleagues that these do comprise a great deal of what we hope our students develop and practice over time as they participate in school mathematics.

**Methods**

This research is an interpretive three-case study (Merriam, 1998). A case study allows for a rigorous way to give a rich and holistic understanding of a phenomenon (Merriam, 1998) which makes this method particularly appropriate because the goal is to develop a deep understanding of the use of habits of mind while engaged in mathematical modeling tasks. In accordance with an interpretive case study, this study focuses on the a priori theory of practicing the habits of mind and the goal is to develop a clear illustration of
the cases’ practice of these mathematical habits of mind. Each case is bounded by their group during the mathematical modeling unit in the methods course.

**Participants, setting, and the task.** This study took place in the context of a capstone secondary mathematics methods course in a southeastern university in the fall of 2014. This course is required for all middle school and high school pre-service mathematics teachers. During this course, the participants spent two weeks engaged in a mathematical modeling unit that was developed and taught by the first author with input from the second author and the course instructor. The cases for this paper came from three groups of students who agreed to be recorded while solving a mathematical modeling task. The first group consisted of three members while the second and third group had four members. All group members are majoring in mathematics education and plan to teach high school mathematics.

The groups engaged in the gas task. This task is defined as an authentic mathematical modeling task (Kaiser, Schwarz, & Buchholtz, 2011) and represents the realistic mathematical modeling perspective. As discussed previously, realistic mathematical modeling tasks require a situation that students face in their everyday lives which is true of this task. The problem requires making assumptions and deciding on a model that best fits these assumptions.

**Gas Task:** Mr. Stone lives in North Carolina in a town close to the border of Virginia. Often to fill up his car, he drives to Virginia. Gas in Virginia is around $2.55 per gallon while gas in North Carolina has been $2.85 per gallon. Is it worthwhile for Mr. Stone to drive to Virginia to fill up his car? (Blum & Leiß, 2005)

Students worked on this task for approximately 45 minutes during one class period.
Data collection and analysis. The sources of data for this study include digitally recorded work from the three groups on the gas task and student artifacts. All group work was recorded with one stationary camera and an audio recording device. The audio and video recordings were used to produce transcripts of the work completed by each group. The transcripts were used to identify all instances of each habit of mind and to create a summary of each group’s solution and process. Together the instances of each habit of mind and the summaries were used to develop the descriptions of each case that follows and were used to develop emergent themes across the cases.

Results

Not surprisingly, the three cases engaged in different habits of mind at different levels and had different solutions to the task. Table 1 shows a quick overview of the habits of mind we identified in the work of each group. In the following sections, we describe each case’s solution path, solution, and the mathematical habits of mind that were practiced. In an effort to develop the themes that cut across the groups, the first group presented (Group 2) has the least general solution to the task and the final presented group (Group 3) has a solution that is the most general. Within each group, the habits of mind we identified will be discussed separately even though there is clearly overlap between the habits of mind. This decision was made to give a clear illustration of each habit of mind within each case.

Table 1.
Table 1.

The three groups’ participation in the mathematical habits of mind.

<table>
<thead>
<tr>
<th>Habit of Mind</th>
<th>Group 1</th>
<th>Group 2</th>
<th>Group 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Performing thought experiments</td>
<td>Lack of evidence</td>
<td>Lack of evidence</td>
<td>Lack of evidence</td>
</tr>
<tr>
<td>Finding, articulating and explaining patterns</td>
<td>Lack of evidence</td>
<td>Lack of evidence</td>
<td>Lack of evidence</td>
</tr>
<tr>
<td>Creating and using representations</td>
<td>Used two different types of representations</td>
<td>Used one type of representation</td>
<td>Used three different types of representations</td>
</tr>
<tr>
<td>Generalizing from examples</td>
<td>Used sample calculations to generalize the effect of tank size.</td>
<td>Lack of evidence</td>
<td>Used sample calculations to generalize their model to all vehicles</td>
</tr>
<tr>
<td>Articulating generality in precise language</td>
<td>Lack of evidence</td>
<td>Lack of evidence</td>
<td>Articulated their understand of the relationship between the variables concisely in an equation</td>
</tr>
<tr>
<td>Expecting mathematics to make sense</td>
<td>Shown during discussion about effects of tank size (a variable) and how that affects their answer.</td>
<td>Shown during their discussions about how variables affect the gas task situation.</td>
<td>Shown while interpreting their calculations in terms of the real world and in terms of a final solution.</td>
</tr>
</tbody>
</table>

Group 2. Group 2 approached the task using cases. This group used two cases: highway or city roads and found a solution to each case. To find their solutions the group made assumptions for three variables that created a very specific, fixed situation. They made assumptions for the distance to the gas stations, amount of gas purchased, and miles per
gallon of the vehicle to calculate the price of gas at each gas station. They then chose the cheaper gas station to be their solution (Figure 2). We found evidence that Group 2 engaged in two habits of mind, creating and using representations and expecting mathematics to make sense.

**Figure 2.** Calculations made by Group 2.

**Representations.** Group 2 created and used one representation to solve the task. The representation can be seen in Figure 2. We chose to group all of these calculations together as one representation. These calculations put together their assumptions and given data in a new way to create another depiction of the information as is a requirement of this habit of mind. This new representation of the data provides the group with the ability to make a conclusion about when to go to Virginia to solve the task under a very specific set of conditions. The calculations solve for how much it would cost to fill up in North Carolina with no driving distance to the gas station and how much it would cost to fill up in Virginia with a five-mile drive to a gas station (with both highway and city driving). From
subtracting these two calculations, the difference in cost between Virginia and North Carolina was determined. From this difference, the group was able to conclude that it was cheaper to go to Virginia in both scenarios (city and highway).

**Expecting mathematics to make sense.** This group demonstrated the practice of expecting mathematics to make sense when the group was deciding on the importance of different variables in their model. This group primarily used expecting mathematics to make sense through their use of calculations to help determine how different variables should be incorporated into their approach to the task. Their use of calculations afforded the group to transform the real world situation into a mathematical problem and find a solution.

The following excerpt exemplified this group’s use of expecting mathematics to make sense. In this excerpt the group is working to better understand what variables factor into the task and they use a combination of experience with the situation, task parameters, and mathematical understanding. In this excerpt, the group is discussing the need to factor in the tank size.

*Student A:* “Find how much he needs to put in because it he only needed a little bit maybe….I don’t know if this mathematically makes sense but in my head, if he only needed a little bit its…if he is gonna need a lot…I don’t know.”

*Student C:* “Coming from my experience with cars, the more full your gas tank is the better gas mileage your gonna get. Once you get below a quarter of a tank, you actually get much worse gas mileage…”

*Student B:* “I think the bigger your gas tank is the more efficient it would be to get it cheaper now because you are saving more”

*Student D:* “Yeah”
Student A: “Right but if you just need one gallon you are going to spend however much. You are probably going to spend more than 30 cents to get there and that makes it even”

Student B: “Cars with bigger tanks get worse gas mileage...I think it is supposed to average out to about the same distance”

In this episode, the group is struggling with the influence of tank size on the decision to go to Virginia to get gas. Through their prior experience, with cars, they know that tank size may influence this issue as seen by Student C talking about her knowledge of gas mileage and Student B talking about how he “think[s] it is supposed to average out to about the same distance” for cars with bigger tanks to have worse gas mileage. The group makes sense of the need to include tank size by reasoning “I think the bigger your gas tank is the more efficient it would be to get it cheaper now because you are saving more”.

Summary of Group 2. This group used two habits of mind to create their solution based of three specific assumptions. Because this group used assumptions for the distance to the gas station, amount of gas purchased, and miles per gallon of the tank, this solution is for a very specific set of circumstances and is not considered a very general solution to the task. In their process to this solution, this group’s use of the habits of mind was sporadic as evidenced by only using two habits of mind.

Group 1. Unlike Group 2, this group approached this task by focusing on the amount of gallons that could be purchased to make it profitable to go to Virginia. This represents a more general solution than Group 2 because assumptions are only made for the distance to the gas stations and miles per gallon. This left one variable, gallons purchased, to vary. It should be noted that this group took two-thirds of their work time debating the need for tank
size as a variable. Group 1 practiced three habits of mind: *creating and using representations, generalizing from examples, and expecting mathematics to make sense.*

**Representations.** Group 1 created and used two different representations in their approach to solving the task. Similar to Group 2, the first representation was calculations (Figure 3). The group made specific assumptions about the distance, miles per gallon, and tanks size, which allowed them to make calculations for the specific scenario. From the calculations, the group concludes that it was cheaper to stay in North Carolina.

*Figure 3. Group 1’s calculations.*

Also the group used this type of representation to determine that tank size effects whether it is worth going to Virginia or not to buy gas. The following excerpt shows the group members making quick calculations with different tank sizes to see how it affects the price in both North Carolina and Virginia.
Student C: “If you have a 100 gallon tank at 30 cents per gallon [savings] you are saving tons more.”

Student A: “I don’t think so”

Student C: “So if you are filling up 100 gallons here [Virginia] and 100 gallons here [North Carolina] it’s obviously going to be cheaper here [Virginia] because it is a cheaper price per gallon”

Student A: “You are still only spending 2 gallons of gas and that is all that matters...wait...is it? Ok so let’s say he buys 100 gallons of gas at $2.85 but that he buys 102 [gallons] at $2.55 [does calculation]. You are so right!”

Student C: “So he saves tons of money [by going to Virginia]”

Student A: “Duh...of course of course.”

Figure 4. Group 1’s equation to find the break-even point.

The second representation created was an equation to solve for the break-even point in tank size, see purple writing in Figure 4. This equation still uses the assumptions of 20 miles per gallon and 20 miles from the Virginia gas station, like the calculations; however,
the equation allows the group to quickly find that when the tank size is greater than 17
gallons it is worth going to Virginia. This equation represents a quicker way to a solution for
the specific assumptions made.

**Generalizing from examples.** This group was able to determine that tank size matters
and generalized that into an equation to solve for the break-even tank size. This was
accomplished by generalizing their results from calculating different tank sizes. The group
was not convinced in the beginning of their work that tank size affected the price at all, in
fact, one group member questions, “**Doesn’t the differential matter more than the size of this
tank?**”. This conversation continues as seen in the following excerpt:

*Student A:* “But if this were a 100 gallon tank [North Carolina] then this [Virginia]
would be 101 so the differential would be $2.55” [the group is accounting for one
gallon of gas spent to get to Virginia by getting one gallon more]

*Student C:* “This is 10 $2.55 and this is 11 $2.55; This is 10 $2.85’s and this is 11
$2.85 so that 30 cent differential will be increased with every number so that [cost in
North Carolina] is going to go up.”

*Student A:* “So you are saying it does matter how big the tank is?”

*Student C:* “If you have a 100 gallon tank at 30 cents per gallon [savings] you are
saving tons more.”

During this excerpt the group starts to make sample calculations to see the effect on the price
of different tank sizes. They continue to make sample calculations, as shown in the excerpt
in the representations section of 100 gallons. This allows the group to generalize the results
into an understanding that tank size affects the price. This can be seen in its final form when
the group creates the equation $x(2.85) = (x + 2)(2.55)$ to generalize this idea and find the break-even tank size.

It should be noted that even though the group managed to develop an understanding of the effects of tank size, the group never expressed this idea in a concise mathematical manner. Instead once everyone seemed to understand that tank size affected the price, the group moved on. This means that the group stopped short of *articulating generality in precise mathematical language* and exemplified the idea that students can understand the mathematics without articulating it concisely as Cuoco et al. (2010) note.

**Expecting mathematics to make sense.** This group also used reasoning and sense making during their work. Their use of this habit is exhibited during their work to resolve whether the tank size affected the price and is very similar to group 2’s practice of *expecting mathematics to make sense*. In this excerpt, the group has already made sample calculations where the tank size is changed but the rest of the variables stay the same. The group comes to these conclusions from the sample calculations.

*Student B:* “So this specific scenario only it would make sense not to go to Virginia [referring to 10 and 12 gallon scenario].

*Student C:* “yeah”

*Student A:* “But if you had a tiny tank then if might make sense to go….not if you had a huge tank”

*Student B:* “So the bigger the tank is the more economical if is for you to go to Virginia”

*Student A:* “Right, duh”

*Student C:* “At first I didn’t know if I was right”
Student A: “You are definitely right”

Student C: “So technically, you could figure out like, at what point what gallon size you would use”

The excerpt demonstrates how the calculations help the group come to the understanding that tank size affects their solution. From this they are able to conclude how tank size affects the price.

Summary of Group 1. As compared to Group 2, this group created a marginally more general solution to the task by only making assumptions for the distance to the gas station and miles per gallon of the vehicle. In addition to the more generalized solution, this group used three habits of mind: creating and using representations, generalizing from examples, and expecting mathematics to make sense.

Group 3. Finally, Group 3 approached the modeling problem by looking at specific scenarios, similar to Group 2, and then generalizing that into model that did not make assumptions for miles per gallon or tank size. First, using specific assumptions, the group calculates the profitable distance to go to Virginia to buy gas in four different vehicles (car, hybrid, SUV, and truck). Then, using these calculations, they create their general model for any vehicle, see Figure 7. Three of the habits of mind, creating and using representations, generalizing from examples, and expecting mathematics to make sense, describe this case’s general approach to solving the task. In addition, articulating generality in precise language plays a role in the case’s final solution.

Representations. This group creates and then uses three different representations in their path to a solution for the gas task. The first representation was a table (Figure 5). The group created the table to organize their data. The group decided to work towards a solution
by examining four cases, driving a car, a hybrid, a truck, and a SUV. They determined
that they needed to know key information to work with these vehicles to create a solution:

*Student B:* “So you need to know how many miles a gallon a vehicle gets, you need to
know how many miles it is to the travel and multiple it by 2”

*Student A:* “And you need to know the size of the tank”

After deciding on these variables, they created the table to organize these variables. The
table gives the group specific affordances, namely showing the relationship between the
vehicles, their miles per gallon, and their tank size. The following excerpt shows how
creating this table furthered the groups work toward a solution by focusing them in on the
variable of distance.

*Student A:* “Now we just need the distance”

*Student B:* “That is just bogus, are we just over the state line?”

*Student A:* “Let’s make a distance for when it works...like for this car at x amount of
miles it makes sense and after that it doesn’t make sense [to drive to Virginia]”

*Figure 5.* Table representation of data for group 3.
Now that the group has organized the information through a table and decided to solve for the profitable distance, the group created its second representation, calculations for profitable distances to Virginia gas station. These calculations re-present the information from the table in a way that helps the group towards a solution. Calculations to determine the amount of money saved by filling up in Virginia are shown on their poster (Figure 6). The final calculations to determine the profitable miles can be seen in the following dialogue between the students determining how many miles away Mr. Stone can be from Virginia in a hybrid for it to be profitable to drive to Virginia.

Student B: “So on the hybrid $31 [North Carolina tank cost] verse $28 [Virginia tank cost] and you get 528 miles. When is it less than or equal to, so North Carolina has to be less than or equal to Virginia to make it work the drive. No Virginia would need to be less than or equal to North Carolina to make it worth the drive.”

Student A: “So the difference in price is this [points to $3.30 difference in price between North Carolina tank and Virginia tank] so we need to see how many miles this can take him”

Student B: “That makes sense”

Student C: “Wait 3 dollars and 30 cents?”

Student A: “In gas, that is how much more he would spend in North Carolina”

Student B: “And how many miles is that worth, right?”

Student C: “Say that again”

Student A: “This is how much he would save if he bought gas in Virginia.”

Student C: “He would save that much?”
Student B: “3 dollars and 30 cents”

Student A: “If he drove a hybrid”

Student C: “Oh, filling up, alright”

Student B: “So we need to figure out how many miles per dollar…”

Student A: “we need to know how many gallons this is first so 3.3 divided by 2.55.

Student B: “1.29 gallons”

Student A: “So that is how many gallons it is so then multiply it by the miles per gallon so 48.

Student B: “62”

Student A: “So if it is less than 62 miles, he benefits.”

From the dialog, it can be seen that the group divided the money saved per tank in a hybrid from getting gas in Virginia ($3.30) by the price per gallon in Virginia ($2.55). That gets them the number of gallons they could buy with their saved money in Virginia (1.29 gallons). They multiply that amount by the miles per gallon of a hybrid (48) to get that they could drive 62 miles or less and it would be cheaper in Virginia. Both the calculations to determine the distance and the calculated distance itself, show another representation of data. This representation is an improvement over the first representation because it provides a solution to the task for the four specific vehicles they are examining.
Finally, the group creates a third representation, when they create a general equation to find the profitable distance for any vehicle. Figure 7 shows their general equation. (Note that ptd stands for profitable distance.) This representation allows the group to move away from their four specific vehicle types and provides the group with a way to determine the profitable distance for any vehicle.
Generalizing from examples. Using the habit of mind generalizing from examples, the group comes up with their final mathematical model of this situation, (Figure 7). The following excerpt comes from their discussion while creating the general equation.

Student B: “Then we probably should put the equation, the generalized equation, that we used to get this [the distances].”

Student A: “Then we can put our general solution.”

Student B: “The general equation was North Carolina minus Virginia times tank size?”

Student A: “Yeah [looks at calculations]. That is basically what we did. We just did tank size and then we subtracted but that is the same thing. And then we did put that over VA times mpg.”

Student B: “So the difference in the cost times the tank size. Yeah that is what we did that is our general equation”

This discussion shows that the group was using the structure from their calculations (the examples) to write their general model, which is the epitome of generalizing from examples.
Articulating generality in precise language. As shown in the above section, the group was capable of generalizing their work to create an equation for any type of vehicle. However, the written version of their equation does not adhere to the traditional variable use of individual letters. Instead they use words that describe their variables. Although they did not use traditional variables, it is clear from the group dialog while creating this equation that the language they are using is understood between the members of the group. In fact, one of the group members points out that the equation they created was slightly different than their calculations, “That is basically what we did. We just did tank size and then we subtracted but that is the same thing.” Being able to show the discrepancies between the general equation and their calculation process suggests understanding of the language used in the general equation. Additionally, the group does convey the correct relationship between the variables that results in finding the profitable distance for any vehicle given miles per gallon and tank size. Because of the understanding between the group members and the correct equation, the group articulated generality in precise language.

Expecting mathematics to make sense. The final habit of mind, expecting mathematics to make sense, can be seen in the group’s ability to go from a real world situation, create a mathematical understanding of the situation, and draw conclusions about their situation. We choose to highlight two instances of reasoning and sense making from the group that highlights this habit of mind. These two instances feature the two different phases that this group practiced reasoning and sense making. The first instance shows the groups use of this habit while working mathematically toward a solution and the second shows how this group used this habit to interpret their mathematical work to come a real world solution.
The first instance comes from when the group is trying to figure out the best way to make calculations to determine the profitable distances. In this excerpt the group has determine that they believe there is a relationship between miles per gallon, tank size, and the distance to gas station. They are trying to figure out the relationship with the hybrid vehicle by using calculations.

Student A: “Let’s make a distance for when it works...like for this car at x amount of miles it makes sense and after that it doesn’t make sense [to drive to Virginia]”

Student B: “So greater than or equal to...so everything has to be greater than or equal to”

Student A: “So we need to figure out how much he would spend per tank. So North Carolina 2.85 times 11”

Student B: “So how many miles are you going to get per tank?”

Student A: “I’m going to do what a tank would cost and how many miles he would get. You figure out how many miles for each tank and I’ll do the gas for all of them.”

Student B: “11 gallons times [quiet work to calculate miles] so this is cost per tank and this is miles”

Student A: “How much it costs to fill up....we are assuming he is going to fill up an empty tank”

Student B: “So on the hybrid $31 [North Carolina tank cost] verse $28 [Virginia tank cost] and you get 528 miles. When is it less than or equal to, so North Carolina has to be less than or equal to Virginia to make it worth the drive. No Virginia would need to be less than or equal to North Carolina to make it worth the drive.”
Student A: “So the difference in price is this [points to $3 difference in price between North Carolina tank and Virginia tank] so we need to see how many miles this can take him”

This excerpt shows that the group first made several calculations, how much he spends per tank, miles per tank, difference in price. Then the group starts to think about what the calculations mean and which are helpful (see bold text). You can see that Student B states that you get “528 miles” to a tank but as you continue on this does not appear in their final decision. Essentially she determined this was not useful information to make a decision. On the other hand, her reasoning about the tank cost and “Virginia would need to be less than or equal to North Carolina [cost]” shows how she is trying to understand her calculations in terms of the real world context and use the context to determine the group’s next step of “so we need to see how many miles this can take him.”

While the first excerpt shows the sense making and reasoning needed to work towards a solution to this task, the second excerpt comes from the group trying to interpret their calculations as a solution to their real world problem.

Student A: “Correct...so it will be 31 miles, 16.57, 23.625, and 20.585 [profitable distance for each of the four vehicles]”

Student B: “So if he has a hybrid or he has an SUV he is closer to being better off but if he has just a regular car probably not so much and the truck probably not so much”

Student A: “Well truck maybe...basically if he is within 15 miles, it benefits him that is what we have come to”

Student B: “Regardless of what kind of vehicle he drives”
In this excerpt, the group goes from stating their calculated distances, to comparing the distances to give a more general solution to when Mr. Stone should drive to Virginia. The group gives a conclusion of “basically if he is within 15 miles, it benefits him” that is a conclusion from reasoning through their four example vehicles.

**Summary of Group 3.** This group created an equation as their final solution that could calculate the profitable distance to Virginia for any vehicle with any specifications. This represents the most general solution created by the three groups. In addition to having the most general solution, Group 3 used the most habits of mind (4 total).

**Cross-case themes.** From looking across the three cases, two related themes emerged about the use of the habits of mind while working on the modeling task and the type of solutions created by the groups. Both themes related to creating a more general solution to the task.

First, there appears to be a relationship between the number of habits of mind used by a group and the generality of the group’s solution appear. As the number of habits increased, so does the generality of the group’s solution. For example, Group 3 engaged in four habits of mind and their solution incorporated the most variables and required the least number of assumptions. Their final equation allowed the group to determine a profitable distance for any vehicle. The other two groups engaged in fewer habits of mind and their final solutions were for limited situations. Group 1 engaged in three habits of mind and created a solution to a scenario reliant on two key assumptions about distance and miles per gallon. Finally Group 2 required three key assumptions (distance, miles per gallon, and tank size) and only engaged in two habits on mind. The relationship shown in these groups between solution generality
and habits of mind possibly suggests that increased frequency in use of the habits of mind as increases the ability of the students to create a more general solution to the task.

The second connection to habits of mind that potentially affects the generality of the solution is how the habits were practiced. The data suggests that thorough and interconnected practice of the habits of mind led to a more general solution. Group 3 had the most general solution and exhibited the most thorough and interconnected practice of the habits of mind. In terms of using and creating representations, Group 3 created the greatest number of representations. Within their representations, the table afforded the group to organize their work, which proved to help them to quickly understand the impact of their variables and lead to calculations. Their use of calculations for four vehicles set up another habit of mind, generalizing from examples, which created their third representation, an equation to determine the profitable distance for any vehicle. Also during the calculations and creation of their final equation, this group regularly engaged in expecting mathematics to make sense. Finally, necessary for their final representation/solution, was the practice of articulating generality in precise language. In their work, the habits of mind seemed to work together and were called upon one after another seamlessly.

In contrast, the other two groups do not practice habits of mind throughout their work toward a solution and do not display the same level of interconnected use of the habits of mind. Group 2 did not engaged in their two habits of mind at the same time and failed to utilize enough habits of mind to show in-depth use. Group 1 showed interconnected use of expecting mathematics to make sense and generalizing from examples while using their representation of calculations; however, they only used one habit of mind outside of this
section of work. In both of these groups, the final solutions relied heavily on assumptions and made conclusions about a very narrow situation.

From this data it is not possible to tease out the exact impact of the number of habits of mind used verse the interconnected and thorough use of the habits of mind. But what can confidently be said is that this data suggests that, in these cases, the use of mathematical habits of mind help to generate more general solutions to the realistic mathematical modeling task.

**Discussion and Conclusion**

With limited research on the mathematics engaged in during a realistic mathematical modeling task, this paper looks to add to the discussion by analyzing school mathematical activity through mathematical habits of mind. Our analysis of three groups’ work on the same realistic mathematical modeling task shows that the mathematical habits of mind (Cuoco et al., 2010) are a productive way to discuss the mathematical thinking that occurs. Furthermore, the results imply a potential link between the more numerous and thorough use of these mathematical habits of mind and a more general solutions to the task. Each of the conclusions from the study will be discussed in light of other research and implications for future research and teacher preparation will be provided.

Providing detailed descriptions of the habits of mind for each group demonstrates how the groups participated in mathematical activity while solving the tasks. This helps to better define these habits of mind in general and in the specific context of realistic mathematical modeling task. This study shows evidence of PSTs using four of the mathematical habits of mind as presented by Cuoco et al. (2010): *creating and using representations, generalizing from examples, articulating generality in precise language, and*
expecting mathematics to make sense. All three groups used creating and using representations and expecting mathematics to make sense. Only two groups used generalizing from examples and one group engaged in articulating generality in precise language. Through describing each group’s use of each habit, a good summary of the mathematical activity of the group is constructed. This provides a productive way to breakdown the mathematical activity done by each group.

Because the habits provide a productive way to break apart the mathematical thinking that occurs, this is a new mathematical way to describe students’ approaches to mathematical modeling tasks. This is important because traditionally, the research on these types of tasks has focused how students engage in the mathematical modeling cycle (Kaiser & Sriraman, 2006). Additionally, with the limited empirical research on these habits, this study furthers the understanding of these habits of mind and provides an example of identifying these habits in student work.

While conclusions can be made about the four habits of mind seen in this case study, performing thought experiments and finding, articulating, and explaining patterns were not demonstrated during this task. We cannot include them within the above conclusions. It is not clear why these habits were not observed. It is possible that these two habits are not utilized during this mathematical modeling task, not utilized during mathematical modeling tasks in general, or that the methods of this study do not allow for observation of these habits. It is our hypothesis that different habits of mind are called up in different situations, and therefore, these two habits are not required in this specific realistic modeling task. It is possible that other tasks could be more predisposed to the use of these two habits. Further
research is needed to both confirm these two habits are used in realistic modeling tasks and to better understand how tasks have a predisposition to certain habits of mind.

Although two of the mathematical habits of mind were not seen, the habits that were observed suggested that how these habits are practices influences the solutions created. This idea is consistent with previous research. Lim (2008) would suggest that this group showed interiorized anticipation through their use of the habits of mind. According to Lim (2008), interiorized anticipation is spontaneously knowing the course of action because of an interiorized understanding of their relevance to the situation. In this case, the group has interiorized anticipation shown through their interconnected and through use of the habits to continually work towards a solution. This is also similar to Niss’s (2010) implementation anticipation, which states that students must anticipate how the mathematics relates to the real world while also anticipating how to go about solving the mathematics. More research should be conducted to investigate the connection between the way in which the habits of mind are used and solutions to these tasks.

In general, this study shows that the sole goal of realistic mathematical modeling tasks to promote learning of mathematical modeling can be broadened to include promotion of mathematical activity and therefore learning mathematics. Moreover, it is possible that more general solutions come from thorough and interconnected use of these habits and implies that mathematical modeling is in fact reliant on these habits of mind in the classroom. This refutes the concerns from teachers that these tasks do not involve mathematics reported on by Kaiser and Maaβ (2007) and helps support the need to include these tasks in our mathematics classrooms.
References


Chapter Five: Article Three

A teaching unit on mathematical modeling in a high school methods course

Introduction

Current research suggests that mathematical modeling can help students develop deeper mathematical understanding (Blomhøj and Kjeldsen, 2013; Doerr & O’Neil, 2011, Zbiek & Conner, 2006) and help students to improve their ability to think mathematically (Palhanni & Almeida, 2015). Because of this research, many educators in the United States have incorporated explicit mathematical modeling standards into guiding documents; this is very evident with the adoption of the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA & CCSSO], 2010). In the Common Core State Standards for Mathematics (CCSSM), mathematical modeling is a mathematical practice standard for all students and a mathematical content standard for high school students. While these mathematics standards encourage mathematical modeling to be taught in the K-12 schools, there is concern that teachers are not being prepared for this change. It has been documented both in the United States and abroad that teachers have limited knowledge about mathematical modeling (Gould, 2013; Kuntz, 2011; Thrasher & Keene, 2014) and lack important mathematical modeling skills (Kaiser, 2007; Thomas & Hart, 2010; Widjaja, 2013; Winter & Venkat, 2013).

One way to meet these concerns is to address these issues in teacher preparation programs. There is some evidence that methods courses for pre-service teachers (PSTs) that engage students in mathematical modeling tasks, can help remedy these deficiencies
Bukova-Guzel, 2011; Kaiser, 2010; Kaiser & Maaβ, 2007; Thomas & Hart, 2010; Widjaja, 2013; Winter & Venkat, 2013). Unfortunately, specifics on the modeling taught in these courses are not often published to help those seeking to design and implement their own teaching units. Thus, this paper will address the following questions:

1. What does implementation of a unit on mathematical modeling look like in a mathematics pre-service teachers’ capstone methods course?
2. What do pre-service teachers learn about mathematical modeling from the unit?

To answer these questions, this paper is divided into two sections after a brief introduction. The first section reviews the implementation of the unit and demonstrates student work that resulted from the activities. The second section presents results of analysis of data collected at the beginning and end of the unit in order to describe the changes in understanding of mathematical modeling (MM) of the pre-service teachers (PSTs).

**Mathematical modeling**

The field of mathematical modeling at the K-12 level is broad and diverse. Mathematical modeling can generally be defined as using mathematics to solve real world problems (Niss, Galbraith, & Blum, 2007); however, a consistent, standardized definition and description of mathematical modeling does not exist (Kaiser & Sriraman, 2006). This makes it hard to figure out what mathematical modeling should look like in the classroom (Hamson, 2003). The following paragraphs provide a brief description of some of the basic ideas of mathematical modeling that were used in this particular unit in order to set the stage for the discussion of the implementations and results.

Considering that the PSTs in this course are most likely expected to teach MM in accordance with CCSSM (NGA & CCSSO, 2010), the foundational understanding of MM
comes from this document. CCSSM defines MM as, “the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions” (NGA & CCSSO, 2010, p. 72). Additionally, CCSSM emphasize that solving a MM task requires making assumptions and approximations to simplify complicated situations and requires possible revisions to these decisions later. Ultimately, MM is a creative process.

In addition to the above definition, another focus of the literature on mathematical modeling that helped shape the developed unit was exploring the range of stated (or unstated) goals for including mathematical modeling in the mathematics classroom. The learning goals of mathematical modeling in the mathematics classroom can be expressed as a continuum ranging from promoting purely mathematical content learning goals in a real world setting to promoting purely mathematical modeling skills (see Figure 1) (Julie & Mudaly, 2002; Niss, Blum, & Galbraith, 2007).

![Figure 1. Continuum of mathematical modeling goals.](image)

One way that these different learning goals can be seen is by examining different mathematical modeling tasks. For example, mathematical modeling tasks with the focus of primarily developing mathematical modeling skills (to the right of the continuum) require real-world, messy, authentic mathematical modeling tasks that highlight the mathematical
modeling process (Kaiser, Schwarz, & Buchholtz, 2011). Authentic mathematical modeling tasks are defined as “problems that are only a little simplified and…recognized by people working in this field as being a problem they might meet in their daily work” (Kaiser, Schwarz, & Buchholtz, 2011, p. 592). Other mathematical modeling tasks that focus on developing mathematical knowledge may seek an understanding of students’ mathematical thinking and highlight mathematical structure but these tasks do not require the real world context to be authentic (Lesh & Doerr, 2003) (to the left of the continuum) (Figure 1). There are also tasks that may support goals somewhere in between. It is important for mathematics teacher educators and teachers to understand these different goals and the tasks that support them.

Interestingly, no matter what learning goals are being promoted, all mathematical modeling tasks have certain features in common. All mathematical modeling tasks are open-ended (no specific procedure to solve the task and no specific solution to the task), require a real world context, and creation of a mathematical representation of the context (Kaiser, Sriraman, Blomhøj, & Garcia, 2007; Lesh & Doerr, 2003). Within the unit, mathematical modeling tasks were chosen to represent tasks falling in different places on this continuum to provide students with the various types of mathematical modeling experiences.

**Setting**

The two-week modeling unit took place in the context of a capstone secondary mathematics methods course at a large southeastern university. This particular course was a requirement for all middle school and high school pre-service mathematics teachers and often taken the semester before their student teaching experience. The goal of this course was to focus on the necessary mathematics for teaching secondary mathematics and connect it to the
university level courses in mathematics. Because of the emphasis on mathematical content as opposed to pedagogy, this course was ideal for this unit. The course met twice a week for 75 minutes each. There were 21 students in the course.

The mathematical modeling unit

Overall design structure. The unit was designed by the authors and course instructor using content-focused design principles for teaching but modified to apply to a smaller unit of instruction (Steele & Hillen, 2012). Following Steel & Hillen, the unit was designed using three principles. The first principle was focusing on specific content relevant to secondary mathematics. In this case, the content was mathematical modeling. The second principle was adhering to a guiding inquiry to frame the unit. According to Steele and Hillen (2012), guiding inquiry is “question (similar to an essential question in K–12) designed to frame the course-long content focus” (p. 57). For this course, the guided inquiry question was: What is mathematical modeling and what makes a good mathematical modeling task? Finally, the unit needed to engage the students in activities over the spectrum from learner to teacher. The unit focused on four main types of activities to address this spectrum: mathematical modeling tasks, reading articles about teaching mathematical modeling, class discussion, and reflection. All of the tasks were chosen to help meet the following mathematical and pedagogical goals, details of which will be addressed later in the article.

Mathematical Goals

1. Develop an inclusive understanding of mathematical modeling.
2. Solve a variety of mathematical modeling tasks.

Pedagogical Goals

3. Identify attributes of mathematical modeling tasks.
4. Experience pedagogically appropriate activities to enhance the mathematical modeling classroom including supporting rich classroom discourse and individual reflection.

Table 1.

*Description of articles for PSTs to read.*

<table>
<thead>
<tr>
<th>Title</th>
<th>Author</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>The footprint problem: A pathway to modeling</td>
<td>Imm and Lorber (2013)</td>
<td>Article describes mathematical modeling with the goal of developing mathematical understanding. The article describes how students express and change their own mathematical thinking while working on a mathematical modeling task. Specific criteria for modeling tasks is outlined.</td>
</tr>
<tr>
<td>Authenticity of mathematical modeling</td>
<td>Tran and Dougherty (2014)</td>
<td>Article discusses need for authenticity in mathematical modeling tasks but also connects to mathematical content development. This article is framed within the CCSSM and gives sample student work from MM tasks.</td>
</tr>
<tr>
<td>Connected mathematics through mathematical modelling and applications</td>
<td>Stillman (2008)</td>
<td>Article outlines skills needed to be a successful mathematical modeler. The focus is on the process on mathematical modeling and does not discuss the development of mathematics.</td>
</tr>
</tbody>
</table>
Implementation. The unit consisted of four 75-minute classes and the unit consisted of two phases. These two phases came from similar phases used to develop subject matter knowledge in other studies (Ferri & Blum, 2009; Maab & Gurlitt, 2011). The first phase was the shorter of the two phases and consisted of focusing on definitions of mathematical modeling and what constituted a mathematical modeling task. In the second phase, students engaged in different modeling tasks (e.g. Thomas & Hart, 2010; Widjaja, 2013; Winter & Venkat, 2013). The first author taught the unit with help from the course instructor.

Phase one. Before the first class, students were assigned to read one of three articles about mathematical modeling. These articles were chosen because they represent the diverse range of goals included in teaching mathematical modeling and were written for practitioners. The title, author, and a brief description of each article can be found in Table 1. Each student answered the following questions about their article and brought it to class.

1. How does your article define mathematical modeling and what are its defining characteristics?
2. Does your article give a mathematical modeling cycle? If so, describe it.
3. What are the distinguishing characteristics of mathematical modeling tasks?

During the first 45 minutes of the first day of the unit, the class discussed the responses to the questions and worked in groups to create a poster and 2-3 minute presentation about their article. Each group presented their poster and the whole class discussed the similarities and differences in the articles. During the discussion, the instructor
created a running list of those similarities to help the class create a broad definition of mathematical modeling. The list included:

- open task with multiple pathways
- continuous cycle
- real world context
- give and take between real world and mathematics.

After presentations and discussion of the articles, the instructor provided a short presentation about the definition of mathematical modeling according to the CCSSM.

**Phase two.** Phase two began during the last 30 minutes of the first class when students worked in groups of four to five students to solve the Empire State Building Task. This task was a Fermi problem, which according to Edge and Dirks is a problem that requires “sufficient understanding of the problem to decide what data might be useful in solving it, insight to conceive of useful simplifying assumptions, an ability to estimate relevant physical quantities, and some specific scientific knowledge” (1983, p. 602). It has been suggested that Fermi problems are a good entry point into mathematical modeling (Arleback, 2009). This task was meant to focus students on the activity of mathematical modeling, particularly on the action of making assumptions, but the context was not really authentic because students were not allowed to use technology to solve the task.

**Empire State Building Task:** The Empire State building has an information desk where the most common question asked is: How long does it take for a tourist to get to the top floor observatory? Write up a short solution to this question that includes your assumptions and your reasoning. (Arleback & Bergsten, 2013)
At the beginning of the second class the groups presented their findings. Two poster examples can be seen in Figure 2. A short class discussion examined the following questions:

1. How are your solutions the same or different?
2. If you were a tourist, which solution would you trust/appreciate the most? Why?
3. In evaluating these solutions, what are important pieces of information that needed to be given?
4. Do you think you engaged in mathematical modeling while solving this task? Why or why not?

The first three questions led students to discuss two important aspects of their work. First was the idea of how assumptions influence the solution. One student helped to describe the differences between the solutions through different assumptions. She said “The basis of how a lot of groups did their assumptions of how the question was asked…just the starting point of what we interpreted that to mean…there was a lot of variability in our interpretation of the question.” The second notion that was explored during this discussion was how general the solution should be. As expressed in the difference of generality between the two posters in Figure 2, different groups found more and less specific solutions. A student spoke to this difference by saying, “Ours is obviously different because we didn’t get an answer (see second poster in Figure 2), but we got a solution. I think our group talked a lot about a generalized model and what does that look like”.
The fourth question elicited the different aspects of mathematical modeling that were highlighted in the articles. The way in which the students answered the question they actually appear to be answering “Was this task a mathematical modeling task? Why or why not?” For example, one student references the Stillman (2008) article, “Yes, [this was a mathematical modeling task], because just grabbing from an example in the article, we took a problem from the real world and then brought it into the classroom and tried to solve, which was a key aspect to mathematical modeling”. Another student argued that this was a mathematical modeling task using the Tran & Dougherty (2014) article because “one of the big things that separated a simple word problem with a mathematical modeling is how messy it was. All the different things you had to think of and I think this problem was really messy”. The responses pointed to the impact of the articles on the students understanding of
MM and showed that they believe that the Fermi task embodies the characteristics of a MM task.

For the second half of the second class, students engaged in the gas task. This task represents a messy real-world mathematical modeling task that has been shown in previous research to help student develop their mathematical modeling ability (Blum & Leiß, 2005). The designers chose this task to focus students on mathematical modeling skills with limited focus on specific mathematical content. Figure 3 shows the gas task as displayed for the students.

All groups had a working solution to this task at the conclusion of the second day. For homework, students were asked to reflect on their experience with the task. The reflection questions were created to focus the students on the actions used to solve the mathematical modeling task, what they were learning from engaging in the mathematical modeling task, and how their future students would engage in the mathematical modeling task.

**The Gas Task**

- Mr. Stone lives in North Carolina in a town close to the border of Virginia. Often to fill up his car, he drives to Virginia. Gas in Virginia is around $2.85 per gallon while gas in North Carolina has been $2.85 per gallon. Is it worthwhile for Mr. Stone to drive to Virginia to fill up his car?

- In your groups, solve this task. Create a poster that displays your solution as well as your reasoning. Remember to include the important pieces of information that we talked about from the Empire State Building Task.

*Figure 3.* The gas task as given to the students.
After looking over the solutions, the instructors decided the groups needed to share their work with each other and then continue to work through another iteration of mathematical modeling to improve their models. This in-the-moment decision came because most groups had only performed one iteration of the modeling to develop their solution and because many of the solutions had many assumptions that could be generalized to create a stronger model.

Instead of using group presentations, the instructors chose to use a modified gallery walk at the beginning of the third day where each group’s solution was displayed on a poster. Each student was given post-it notes to write one positive and one suggestion for improvement for the other groups’ solutions. There was no talking during the activity. Upon reflection, many students later referenced that the gallery walk was very helpful. One student said that he learned the most from the gas task because “we were able to come up with a great general form. When we walked around and wrote our thoughts about each project, it was very helpful”.

After the gallery walk was over, the instructors led a quick class discussion about what makes a good mathematical modeling solution. The discussion was opened by asking the class which solutions they liked and why. This led us to create a list of good mathematical modeling solution traits. The following is the list that this class came up with:

1. Clear communication
   a. Clear labels and variables
   b. Clear understanding of where your numbers come from
   c. Clear justification of your conclusion

2. Reasonable assumptions
3. Math that makes sense

4. More general solution with less assumptions (when possible)

Using this list as a guide and from their post-it note suggestions, each group worked to improve their gas task solutions for the remainder of the class. For homework, each student wrote up a formal solution of his/her work. This write-up included a statement of the problem, description of their group’s process, a final solution, and a reflection on their work. These served to help the students to revisit and reflect on their solution process during a modeling task.

The fourth and final day of the unit focused on the bottle task (see Figure 4). Carlson, Larsen, & Lesh (2003) created this mathematical modeling task to reveal and develop students’ understanding of covariation. With this in mind, the task represents a strong focus on mathematical goals with little focus on mathematical modeling content (left side of the continuum).

**Bottle Task**

Dear Math Consultants,

Dynamic Animations has just been commissioned to animate a scene in which a variety of bottles will be filled with fluid on screen. We need your help to make sure this scene appears realistic.

We need a graph that shows the height of the fluid given the amount of fluid in the bottle (a height/volume graph). We have provided a drawing of one of the bottles used in the scene. Please provide a graph for this bottle and directions that tells us how to make our own graph for any bottle that may appear in this scene.

Thanks,
Dynamic Animations

*Figure 4.* The bottle task representing a focus on mathematical goals.
This task took the majority of the class period to complete. Several groups needed scaffolding because they were very focused on creating an equation to create a graph. For example, one group looked up the equation of a truncated sphere and tried to work with that to create their first graph. This led the group to be stuck and struggling to solve the task. After a conversation with the instructor, the group realized that they do not need an equation to create a graph, which helped them to progress to their solution, which can be seen in Figure 5.

Figure 5. Poster of bottle task solution.

Due to time constraints, only two groups had time to share their solutions. The instructor chose two different approaches to the task to share and this ended the unit on
modeling for the class. Under ideal conditions, a comparison of solutions to this task and the former tasks would be a good way to end the unit.

**Students’ Conceptions Before and After the Unit**

The second goal of this paper is to address how students’ conceptions of mathematical modeling and mathematical modeling tasks changed from participating in the unit.

**Data collection and analysis.** The sources of data included pre- and post- surveys. The pre-surveys were given before the unit began to capture the students’ understandings before the unit. On the pre-survey and post-survey, PSTs were asked to respond to questions about the characteristics of mathematical modeling, characteristics of the mathematical modeling process, and to whether five different tasks were mathematical modeling tasks and explain why or why not (see Appendix C). This following is an example of a task from the survey where students are asked if this is a mathematical modeling task:

Local bank has noticed that customers are waiting a long time to see a bank teller at peak times. This bank has customers that have only one transaction and other that have many when they come to the bank. There is a maximum of 5 teller stations. Design away to increase process times.

The post-surveys were completed as part of the final exam and captured the students’ conceptions after the unit. Other student artifacts from the unit were used for triangulation and as supporting documentation.

Analysis occurred using the constant comparative method; a qualitative analysis method commonly used in educational research (Miles and Huberman, 2014). Each data source was open coded; the researchers created codes for the different responses and made a list where all the codes were listed and a tabulation of the numbers of each code was
recorded. A list of recurring themes was generated from the open coding that best encapsulated the most common codes. These themes are discussed in the results.

**Results.** The following sections describe the themes that emerged from analysis of the pre-survey and post-survey questions in regards to the students’ conceptions of mathematical modeling. From both of these surveys, the emergent themes were a) mathematical modeling involves representations, b) mathematical modeling involves the real world, c) mathematical modeling requires identifying variables, constraints, and making assumptions, and d) mathematical modeling is open-ended without a specific solution or solution path. Frequencies for each of these themes in the pre- and post-surveys as well as an example can be seen in Table 2. Additionally, the percent of correct responses to whether each task listed in the survey was a modeling task can be seen in Table 3.

The first part of this section is a discussion of the emergent themes from the pre-survey to provide a snapshot of the conceptions held by the students prior to specific instruction of mathematical modeling. The second section elaborates on the conceptions held by the students after experiencing this unit.

**Pre-survey Results.** While all of the themes appeared as responses by at least two students in the pre-survey, the themes of mathematical modeling involving representations and mathematical modeling involving the real world were the most prevalent. In the following section examples of the responses are provided to exemplify the beliefs of the students in each theme and other emergent patterns within the themes.

**Representations.** The most prevalent conception that emerged was that mathematical modeling involved using and creating representations. For example, Student 11 said that mathematical modeling was a “representation of some problem or solution”. In fact, 17 out
of the 21 PSTs (81%) said that mathematical modeling involved some aspect of representations.

Table 2.

**Summary of student conceptions of mathematical modeling from the pre-surveys and post-surveys.**

<table>
<thead>
<tr>
<th>Themes</th>
<th>Frequency on Pre-survey</th>
<th>Frequency on Post-survey</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Requires multiple representations</td>
<td>6 (29%)</td>
<td>1 (5%)</td>
<td>“you are using a different form other than just numbers (the given form in the problem) to solve the problem” – Student 9</td>
</tr>
<tr>
<td>Visualization of mathematics</td>
<td>14 (67%)</td>
<td>0 (0%)</td>
<td>“I would consider mathematical modeling to be the action of representing a mathematical concept, process, or relationship through either visual or physical manipulatives” – Student 8</td>
</tr>
<tr>
<td>Visualization of context</td>
<td>7 (33%)</td>
<td>0 (0%)</td>
<td>“This allows students to ‘see’ and manipulate a scenario to solve the problem” – Student 10</td>
</tr>
<tr>
<td>Real world</td>
<td>9 (43%)</td>
<td>17 (81%)</td>
<td>“approximation of real-life scenario” – Student 2</td>
</tr>
<tr>
<td>Identify variables, constraints, and making assumptions Open-ended tasks</td>
<td>2 (10%)</td>
<td>15 (71%)</td>
<td>“realize what we know and what we need to know” – Student 18</td>
</tr>
<tr>
<td></td>
<td>5 (24%)</td>
<td>17 (81%)</td>
<td>“not just formulas to substitute numbers into” – Student 12</td>
</tr>
</tbody>
</table>
Table 3.

Percentages correct student responses to whether each task was a modeling task on both the pre- and post-survey.

<table>
<thead>
<tr>
<th>Task</th>
<th>Percent of correct responses on Pre-survey</th>
<th>Percent of correct responses on Post-survey</th>
</tr>
</thead>
<tbody>
<tr>
<td>Local Bank (Task a)</td>
<td>71%</td>
<td>100%</td>
</tr>
<tr>
<td>Algebra Tiles (Task b)</td>
<td>29%</td>
<td>90%</td>
</tr>
<tr>
<td>Maximum Capacity of Classroom (Task c)</td>
<td>62%</td>
<td>90%</td>
</tr>
<tr>
<td>Quadratic Word Problem (Task d)</td>
<td>52%</td>
<td>95%</td>
</tr>
<tr>
<td>Golf Ball (Task e)</td>
<td>43%</td>
<td>90%</td>
</tr>
</tbody>
</table>

After comparing coded responses about representations, the responses fell into three categories. The first category, mentioned by six PSTs, is mathematical modeling requires multiple representations and this was simply stated without elaboration. Mathematical modeling as visualizing a mathematical concept was the second conception identified. 14 students recognized mathematical modeling in this respect. Student 8 is typical of this category, “I would consider mathematical modeling to be the action of representing a mathematical concept, process, or relationship through either visual or physical manipulatives”. Finally, the third category also has a heavy emphasis on visualization but it is on a context as opposed to a visualization of mathematics. Seven PSTs described mathematical modeling in this way. An example of this type response is, “Mathematical modeling is using “math to ‘model’ something such as weather patterns or Fibonacci’s Rabbits” (Student 19).
**Real World.** The second emergent theme was mention of the real world. 43% (9/21) mention the “real world” or “real life” at some point on their pre-survey. Of these, three PSTs require the real world as an essential component of mathematical modeling. These PSTs discuss the real world in their characteristics of mathematical modeling and make it a necessary requirement to justify their classifications of mathematical modeling tasks. For example Student 2 stated that a characteristic of mathematical modeling is “approximation of real-life scenario” and then proceeded to use this in her justifications of the mathematical modeling tasks. Three other PSTs discussed the “real world” as a requirement for mathematical modeling but are then inconsistent about if it is a necessary requirement or simply a typical requirement. Finally, the last three PSTs who mentioned the “real world” fell into the category of mentioning it as a characteristic but failed to use it at all to justify the mathematical modeling tasks.

**Other Mathematical Modeling Characteristics.** Some other mathematical modeling characteristics were mentioned by a few PSTs. Two PSTs mentioned the need to identify constraints and unknowns such as student 18 who said, “realize what we know and what we need to know” as a characteristic of mathematical modeling. Finally, five students mentioned that mathematical modeling requires more than computation, hinting that MM tasks require open-ended tasks. This type of justification usually manifested during the task classifications. For example, student 17 stated “this is simply a word problem containing a solution” as her justification for why a certain task was not a mathematical modeling task. These students’ responses focused on the idea that mathematical modeling requires high cognitive demand tasks.
Post-survey results. Many changes were seen from the pre-survey to the post-survey. In the post-survey, mathematical modeling involving the real world, mathematical modeling involving identifying variables, constraints and making assumptions, and mathematical modeling requiring open-ended tasks were identified as characteristics of mathematical modeling by most students. In fact only one student still maintained a conception of mathematical modeling involving representations (see Table 2). This student still held the view that mathematical modeling involves visualizing mathematical concepts. Since only one student maintained the theme mathematical modeling as representations, the following sections will focus on patterns that emerged from the other themes.

Real World. This category referred to any mention of the real world in both the question about characteristics of mathematical modeling and the questions about mathematical modeling tasks. 17 students (81%) mentioned the real world as a requirement for mathematical modeling. For all of these students, they mention that mathematical modeling begins with a real world problem. A prototypical response is “the modeling process begins with a real world problem” (Student 7).

Four of the 17 students that identify the real world as a requirement of mathematical modeling made a point that mathematical modeling is about using mathematics as a tool to solve real world problems. They removed the mathematical focus of mathematical modeling and instead focus on solving real world problems with mathematics as a tool. For example, Student 3 said, “Instead of trying to apply real world applications in class we want to take our knowledge and fix an actual scenario in the real world”. In these responses, the students indicated that they believe the major goal of mathematical modeling is to solve real world problems. The other thirteen students that identified the real world as a characteristic of
mathematical modeling expressed a view more consistent with mathematical modeling being “real world based” (Student 13). This view can be seen as more consistent with the view that mathematical modeling has a mathematical focus that develops out of a real world context.

*Open-ended tasks.* While only five students identified this characteristic in the pre-survey, seventeen students identified this characteristic in the post-survey. Two different conditions were used to discuss how mathematical modeling task are open-ended tasks. Ten students discussed mathematical modeling tasks as open-ended meaning that solutions for these tasks were not procedural. Many of these students noted that mathematical modeling tasks could not be solved with “plug and chug” or have a “direct” solution.

The second way students discussed mathematical modeling task as open was indicating that these tasks have multiple solution paths and solutions. Student 4 says that a mathematical modeling task is “not limited to a certain solution set” and Student 6 explains that “there should not be just one pathway or necessarily one correct solution” to mathematical modeling tasks.

*Identify variables, constraints, and making assumptions.* The emergent theme that mathematical modeling requires identifying the important variables that define the situation, deciding on the constraints, and making assumptions on the situation is noted in 71% of the responses to the questions about the characteristics of mathematical modeling and mathematical modeling tasks. Twelve of the students that provided an answer in this theme specifically use the word assumptions as a reason that a specific task is or is not a mathematical modeling task. Student 7 characterizes these responses when he explains that a task is a mathematical modeling task because “assumptions can be made”. While the
majority of responses used assumptions language, other responses can be seen such as “student would have to create their own constraints” (Student 17) that do not use the word assumptions.

In general, the responses in this theme suggest an even more robust understanding of modeling because many use the language of assumptions, constraints, and variables as the method to “create” or “define” the problem that is being solved. Student 11 plainly states this conception in her justification of a task as a mathematical modeling task, “Yes, because you would have to consider size of classrooms, different classroom styles (i.e. lecture room, chemistry lab), etc. Assumptions would need to be made and boundaries would be set”. In this response, the requirement to make assumptions in a mathematical modeling task stemmed from a larger need to define the problem to make it something that can be solved.

**Summary.** These results show the changes in conceptions of mathematical modeling for the students as they participated in the described mathematical modeling unit. In general, students moved from an undefined conception or misconception of mathematical modeling to a clearer conception of mathematical modeling that aligns with current understandings of the topic (see Table 2).

The pre-survey demonstrated that students did not have a well-defined conception of mathematical modeling as evidenced by the low numbers of students holding each type of conception. This was not a surprise and has been noted by others (Gould, 2013; Thrasher & Keene, 2014). Furthermore, many students had misconceptions of mathematical modeling as visualizing mathematics or simply visualizing a context. Specifically, the misconception of mathematical modeling as visualizing mathematics is seen in the task identification question about manipulatives where 71% of students misidentified this as a mathematical modeling
task (see Table 3). Generally, the identifications of mathematical modeling task show that students did not hold consistent or fully developed understandings of mathematical modeling because the range of correct responses was 29%-71%.

After the unit, students’ conceptions focused on a few major aspects of mathematical modeling and students showed a strong ability to identify mathematical modeling tasks. The questions identifying mathematical modeling task reveal that at least 90% of the class identified each task correctly, which is a large change from before the unit. At the end of the unit, students held appropriate understandings of mathematical modeling as requiring the real world, requiring tasks that are open-ended, and requiring identification of variables, constraints, and assumptions. Interestingly, the way in which students discussed the real world requirement also revealed that students held views about where mathematical modeling task should fall on the learning goals spectrum. One conception was that the real world was simply a context requirement (e.g., the Bottle task) while the other had a focus on solving real world tasks with mathematics as the tool (e.g. the Gas task). Future iteration of this course should work to address this issue more fully and specifically.

**Summary and Implications**

There were two goals for this paper: to detail one implementation of a mathematical modeling unit and to give evidence of what students learned about mathematical modeling from the unit. By addressing both of these aspects, this article hopes to create a space for mathematics teacher educators to build on each other’s work to create impactful coursework addressing mathematical modeling.

The unit was based on content focused design principles (Steele & Hillen, 2012) with a focus on mathematical modeling. Specifically, the unit addressed mathematical goals of
(1) developing an inclusive understanding of mathematical modeling and (2) solving a variety of mathematical modeling tasks. It addressed pedagogical goals of (1) identifying attributes of mathematical modeling tasks and (2) demonstrating pedagogically appropriate activities to enhance the mathematical modeling classroom including supporting rich classroom discourse and individual reflection. The implementation occurred over a two-week period and constituted four class periods.

Results suggest that the implementation had great success in moving students to understand the characteristics that make up mathematical modeling and to recognize mathematical modeling tasks. Specifically, implementation of the unit resulted in moving students from weak conceptions or misconceptions of mathematical modeling to a conception that includes a vision of the mathematical modeling as rooted in the real world and a view of mathematical modeling tasks as open-ended. The mathematical modeling tasks used in the unit proved effective in showing the importance and power of assumptions in the mathematical modeling process and the need to work to define the problem to begin to solve it.

The understandings gained during the unit will serve these pre-service teachers in the classroom where they will be expected to teach mathematical modeling. Furthermore, by sharing our unit, other mathematics teacher educators can use these ideas to impact the understandings held by other pre-service and classroom teachers.
References


National Governors Association Center for Best Practices & Council of Chief State


Chapter 6

The final chapter briefly summarizes the results of each of the three articles that constitute this thesis. The summaries are followed by a discussion of the three studies as a whole and their potential impact on practice and future research.

Summaries

A reinterpretation of the mathematical modeling process (Article 1) explored how pre-service teachers approached mathematical modeling tasks. Although the intention was originally to explain PSTs’ work through the lens of the Common Core State Standards for Mathematics (CCSSM) (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA & CCSSO], 2010) modeling cycle, the results suggested that the students did not actually follow the cycle as it is expressed in the CCSSM. Instead, the article proposed a new way to think about the process and provided a new depiction of the mathematical modeling process. This new depiction uses the actions named in the CCSSM cycle (with the addition of reflective confirming and outside intervening) but adds in the objects of these actions as well. These objects need to be added to the process description to fully understand the actions taking place (See Figure 1). This new theory of the mathematical modeling process diverges from previous theories because (1) it does not imply a predetermined cyclic nature to the process and (2) it does not create a division between the real world and the mathematical world. Modifying the cycle provided by CCSSM has clear implications for teachers because it calls into question the cycle that they are expected to teach.
While the work in article 1 focused on how students approach mathematical modeling from a process standpoint, *A study of mathematical modeling through the lens of mathematical habits of mind* (article 2) focused on the other primary goal of mathematical modeling, developing mathematics. This article explored the use of mathematical habits of mind (Cuoco et. al, 2010) as a way to describe the mathematics students used while solving an authentic mathematical modeling task. The results showed that students engaged in the habits of mind: creating and using representations, generalizing from examples, articulating generality in precise language, and expecting mathematics to make sense. Additionally, findings suggested that more thorough and numerous use of these habits leads to more general solutions to the task.

In addition to providing a way to explain the mathematical thinking done by students while engaged in a realistic mathematical modeling task, this article also provided empirical research on Cuoco et. al’s (2010) habits of mind. Practically, having identified these habits
of mind from students work on modeling tasks provides another way to help teachers and researchers make sense of student work and plan for how students will approach these tasks.

Finally, *A teaching unit on mathematical modeling in a high school methods course* (article 3) speaks directly to mathematics teacher educators by presenting the implementation of a unit focused on mathematical modeling as defined by CCSSM. The unit focused on developing students understanding of mathematical modeling and mathematical modeling tasks. Activities included reading practitioner articles on mathematical modeling, engaging in three mathematical modeling tasks, and reflecting on their process. Results from pre- and post-surveys suggested that students began with many misconceptions of mathematical modeling but ended with appropriate understandings of mathematical modeling as requiring the real world, requiring tasks that are open-ended, and requiring the need to define the problem through identifying variables, constraints, and assumptions. Because of the favorable results, other mathematics teacher educators could use this unit to help pre-service teachers learn about mathematical modeling.

**Discussion and Implications**

This thesis enhances our knowledge of different aspects of mathematical modeling with pre-service mathematics teachers (PSTs). By using the same data to explore questions about how students engage in the mathematical modeling process, how they think mathematically, and how their conceptions of mathematical modeling change, the rich nature of learning that comes from mathematical modeling can be understood.

This work provides further evidence related to the two primary learning goals: learning the mathematical modeling cycle and learning mathematics (Julie & Mudaly, 2007). From using the same tasks, both engagement in the modeling process and in mathematical
thinking (habits of mind) were evident while students engaged in mathematical modeling tasks. Moreover, the results from the unit described displayed how students’ conceptions of mathematical modeling change after engaging with this type of content.

One of the important confirmations from the research is that these goals are meant to represent a duality, not a dichotomy (Niss, Blum & Galbraith, 2007) showing the diversity of learning that occurred. Furthermore, these results show the value of the multiple perspectives within the field of mathematical modeling to best understand the rich learning environment that mathematical modeling in the classroom provides.

Along the same lines, the specific results of articles 1 and 2 helped to illuminate the complexities of mathematical modeling. Breaking down these complexities into different components should help teachers with their ability to utilize these types of tasks in their classrooms (Greenfrath, 2015, Thomas & Hart, 2010). In fact, the unit described in the third article could be redesigned to incorporate the proposed mathematical modeling process and talk about using habits of mind to help students while solving their tasks.

Most importantly, these three works start a research agenda for mathematical modeling using the CCSSM (2010) as a design feature. While the third article suggests that pre-service teachers are helped to understand mathematical modeling by using it as the basis for a course, there is concern about the mathematical modeling cycle as described in CCSSM. Article 1 fails to see students engage in the CCSSM (2010) mathematical modeling cycle. This questions the use of this specific cycle in the guiding document for teachers. What is confirmed is that the everyday modeling tasks called for by CCSSM can also work to develop students’ mathematical thinking because they will be engaged in mathematical habits of mind.
On a theoretical note, the benefits of using different perspectives and theories within the field of mathematical modeling can be shown within this study. While this diversity makes teaching mathematical modeling difficult, the diversity of theories shows maturity within the field (English 2002, Geiger & Fredj, 2015). Without the use of varied theories, the results from each article would not be possible.

**Future Research**

The research presented in this thesis can be extended in many respects. Introducing the mathematical habits of mind as a way to study mathematical activity during work on modeling task opens up many avenues to explore. A few of questions that come to mind are: Can habits of mind provide scaffolding for students to learn to solve modeling tasks? Can habits of mind be the key to transfer between mathematical content applying that content in the real world? How can habits of mind by incorporated into pre-service teachers understanding of how to solve mathematical modeling tasks? I intend to develop a research agenda that begins to answer some of these questions.

In general, however, the findings from these articles suggest immediate next studies. Both the mathematical modeling process presented in the first article and the habits of mind found in the second article need to be tested with other populations and mathematical modeling tasks. The findings found in article 1 and 2 need to be incorporated into the unit described in article 3 and its effectiveness needs to be tested. Most importantly, this study suggests that a stronger focus on bridging the different perspectives and theories in the field of mathematical modeling needs to be made to gain a clearer picture of mathematical modeling in the classroom.
References


Appendix A

Data Collection for the study

*Digitally recorded class discussions.* Recording all class discussions during the unit provided data to help describe the implementation of the mathematical modeling unit in chapter 5. This data was captured with a digital recording device in the back of the classroom.

*Digitally recorded group work.* Three groups were digitally recorded while engaging in all mathematical modeling tasks. The recording device was placed with each group to capture all dialog and work on artifacts. An audio recording device was also used. This data was used to in chapters 3 and 4 to answer questions about how the groups engaged in the mathematical modeling tasks using the theories of mathematical modeling processes and mathematical habits of mind.

*Student artifacts.* All posters for presentations were collected. This included posters created by groups about the articles they read about mathematical modeling, and posters made for each mathematical modeling task (the empire state building task, the gas task, and the bottle task, see chapter 5). These artifacts were de-identified. This data was used in all three articles (Chapters 3, 4, and 5) to provide evidence of the work done by each group.

*Pre- and post- surveys.* PSTs completed pre-surveys as an outside class assignment and post-surveys as part of the final exam for the course. The pre-survey was given before any part of the MM unit began. This data was used for chapter 5 to examine the impact of the mathematical modeling unit on the conceptions of mathematical modeling held by the
students. The pre-survey also collected demographic information use to describe our
participants for all three articles.

The surveys were created from prior surveys used by other researchers and from
results of an earlier study that collected data in Fall 2013 on mathematical modeling in the
same course (Thrasher & Keene, 2014). The pre- and post- surveys are the same. Table 1
discusses each question and the reason for its inclusion.
### Table 1.

**Justification of each question on pre- and post-survey.**

<table>
<thead>
<tr>
<th>Question</th>
<th>Question Number</th>
<th>Survey</th>
<th>Justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>What are the characteristics of mathematical modeling?</td>
<td>1</td>
<td>Pre- and Post-Survey</td>
<td>Comes from a survey used by MaaB and Gurlitt (2011) that also explored teachers understanding of MM.</td>
</tr>
<tr>
<td>For each of the following tasks, determine whether the task can be</td>
<td></td>
<td></td>
<td>The ability of the PSTs to recognize when MM is appropriate is an important part of participating in the MM discourse (Sfard, 2008). These questions will reveal PSTs word choice and endorsed narratives about MM tasks and MM. The different tasks given are typical of the range of types of tasks that have been identified by PSTs as MM tasks.</td>
</tr>
<tr>
<td>classified as a mathematical modeling task and state at least one reason</td>
<td></td>
<td></td>
<td>-----------------------------------------------------------------------------------------------------------------------------------------------</td>
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<tr>
<td>why or why not. (Do not solve the tasks)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Local bank has noticed that customers are waiting a long time to see a</td>
<td>2a</td>
<td>Pre- and Post-Survey</td>
<td>Represents a MM task.</td>
</tr>
<tr>
<td>bank teller at peak times. This bank has customers that have only one</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>transaction and other that have many when they come to the bank. There</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>is a maximum of 5 teller stations. Design away to increase process times</td>
<td></td>
<td></td>
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<tr>
<td>Factor the following expressions using Algebra Tiles (a manipulative that</td>
<td>2b</td>
<td>Pre- and Post-Survey</td>
<td>Represents a common misconception that manipulatives are MM (Gould, 2013; Thrasher &amp; Keene, 2014).</td>
</tr>
<tr>
<td>helps students to understand factoring).</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>The department of public safety is trying to create a way to determine the</td>
<td>2c</td>
<td>Pre- and Post-Survey</td>
<td>Represents a MM task.</td>
</tr>
<tr>
<td>safest maximum capacity for classrooms in a school. Conduct some research</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>and create a method to determine this.</td>
<td></td>
<td></td>
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<tr>
<td>An object is launched at 19.6 meters per second (m/s) from a 58.8-meter</td>
<td>2d</td>
<td>Pre- and Post-Survey</td>
<td>Represents the use of a mathematical model but does not represent a MM task because students will not engage in a modeling cycle (Lesh &amp; Doerr, 2003, Kaiser et al., 2011; Stillman, 2008).</td>
</tr>
<tr>
<td>tall platform. The equation for the object's height ( s ) at time ( t )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>seconds after launch is ( s(t) = -4.9t^2 + 19.6t + 58.8, ) where ( s )</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>is in meters. When does the object strike the ground?</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>Estimate how many golf balls it would take to fill a school bus.</td>
<td>2e</td>
<td>Pre- and Post-Survey</td>
<td>Represents most of the requirements of MM tasks such as requiring assumptions but misses one critical aspect of being a realistic context.</td>
</tr>
<tr>
<td>Question</td>
<td>Question Number</td>
<td>Survey</td>
<td>Justification</td>
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<td>------------------------------------------------------------------------</td>
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<td>-----------------------------------------------------------------------------------------------------------------------------------------------</td>
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<tr>
<td>Have you ever heard of the mathematical modeling process? If you have,</td>
<td>3</td>
<td>Pre- and Post-Survey</td>
<td>Explores the PSTs ability to participate in the discourse on the MM cycle by revealing endorsed narratives and visual mediators.</td>
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<tr>
<td>write a few sentences describing it or draw a diagram</td>
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<tr>
<td>Group</td>
<td>Task</td>
<td>Modeling Route (Actions only)</td>
<td></td>
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<td>-------</td>
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<td>------------------------------------------------------------------------------------------------------------------------</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Empire State Building</td>
<td>Making assumptions, formulate, understanding situation, making assumptions, understanding situation, making assumption, understanding situation, formulate, understanding situation, formulate, understanding situation, formulate, understanding situation, formulate, outside intervening, assumptions, formulate, compute, formulate, reflective confirming (Failed to finish task), report</td>
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<tr>
<td></td>
<td>Gas Task</td>
<td>Understanding situation, making assumptions, compute, interpret, formulate, making assumptions, compute, making assumptions, reflective confirming, formulate, making assumptions, compute, formulate, interpret, validate, compute, interpret, outside intervening, formulate, compute, interpret, report, validate, interpret, validate, interpret, validate, interpret, understanding situation, interpret, outside intervening, formulate, making assumptions, formulate, outside intervening, making assumptions, outside intervening, making assumptions, reflective confirming, interpret, making assumptions, formulate</td>
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<td></td>
<td>Bottle Task</td>
<td>Understanding situation, formulate, compute, formulate outside intervening, understanding situation, formulate, interpret, validate, report, interpret, validate, report, formulate, report, formulate, report</td>
<td></td>
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<tr>
<td>2</td>
<td>Empire State Building</td>
<td>Making assumptions, understanding situation, formulate, making assumptions, outside intervening, making assumptions, compute, formulate, compute, formulate, reflective confirming, making assumptions, formulate, compute, interpret, report</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gas Task</td>
<td>Formulate, understanding situation, making assumptions, reflective confirming, compute, formulate, making assumptions, formulate, compute, understanding situation, compute, making assumptions, compute, reflective confirming, formulate, compute, making assumptions, compute, making assumptions, compute, outside intervening, interpret, report, interpret, compute, interpret, outside intervening, formulate, making assumptions, formulate, validate, interpreting</td>
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<tr>
<td></td>
<td>Bottle Task</td>
<td>Understanding situation, formulate, compute,</td>
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<tr>
<td></td>
<td><strong>Empire State Building</strong></td>
<td>Formulate, understanding situation, making assumptions, understanding situation, making assumptions, understanding situation, making assumptions, compute, formulate, compute, making assumptions, interpret, understanding situation, making assumptions, compute, formulate, compute, making assumptions, interpret, understanding situation, validate, making assumptions, validate, understanding situation</td>
<td></td>
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<tr>
<td></td>
<td><strong>Gas Task</strong></td>
<td>Understanding situation, formulate, making assumptions, formulate, compute, making assumptions, interpret, formulate, compute, interpret, report, validate, outside intervening, validate, making assumptions, validate, compute, validate, formulate, validate, interpret, validate, outside intervening, making assumptions, understanding situation, making assumptions, interpret, outside intervening, reflective confirming, making assumptions, formulate, report, interpret, formulate, interpret, compute, formulate, interpret, formulate, outside intervening</td>
<td></td>
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<tr>
<td></td>
<td><strong>Bottle Task</strong></td>
<td>Understanding situation, formulate, understanding situation, formulate, outside intervening, formulate, outside intervening, formulate, outside intervening, formulate, outside intervening, formulate, outside intervening, formulate, outside intervening, interpret, formulate, interpret, report, formulate</td>
<td></td>
</tr>
</tbody>
</table>
Appendix C

Pre- and Post- survey questions.

1. What are the characteristics of mathematical modeling?
2. Have you ever heard of the mathematical modeling process? If you have, write a few sentences describing it or draw a diagram
3. For each of the following tasks, determine whether the task can be classified as a mathematical modeling task and state at least one reason why or why not. (Do not solve the tasks)
   a. Local bank has noticed that customers are waiting a long time to see a bank teller at peak times. This bank has customers that have only one transaction and other that have many when they come to the bank. There is a maximum of 5 teller stations. Design away to increase process times.
   b. Factor the following expressions using Algebra Tiles (a manipulative that helps students to understand factoring).
      i. $x^2+3x-18$
      ii. $x^2-16$
   c. The department of public safety is trying to create a way to determine the safest maximum capacity for classrooms in a school. Conduct some research and create a method to determine this.
   d. An object is launched at 19.6 meters per second (m/s) from a 58.8-meter tall platform. The equation for the object's height $s$ at time $t$ seconds after launch is $s(t) = -4.9t^2 + 19.6t + 58.8$, where $s$ is in meters. When does the object strike the ground?
   e. Estimate how many golf balls it would take to fill a school bus