

## ABSTRACT

MUTLU, MURSIDE ASLI. What Are the Important Ideas When Teaching Rate of Change? (Under the direction of Dr. Karen Allen Keene).

When teaching the concept of rate of change (ROC), what ideas are important for educators to consider? This thesis addresses the question by mapping out the research territory on four major concepts: (a) proportional reasoning, (b) slope and rate, (c) covariational reasoning, and (d) functions and derivatives. These four constructs were selected as unifying themes for the concept of ROC because a stable consensus has emerged in the literature that they are important for students' understandings of ROC. Close examination of these four constructs reveals that they are highly related, and that together, they provide a useful framework for considering how to learn and teach the concept of ROC, and how to use ROC as an organizing construct in mathematics curriculum.

The findings of this thesis have implications for how mathematics educators understand and teach rate of change throughout the K–12 process. In particular, this study has implications for the teaching of courses in majors related to Science, Technology, Engineering, and Mathematics (STEM) because such courses focus on information about functions' behavior and shape, about the direction of functions' change, about the size of that change, and about how the change is changing. All four concepts require robust understanding and knowledge of rate of change. Moreover, in disciplines like physics, biology, and chemistry, as well as in real-life phenomena, the information that is given or known could be only about the rate of change. A modeled phenomenon can be more manageable as a derivative than as a function. The next course after high school calculus, or any other calculus class, involves modeling and solving differential equations, and as such, a

conceptual understanding of ROC is the backbone of the course. Also, after derivatives, the second focus of a calculus course is the accumulated rate of change.

Starting from the very early grades, teachers need to pay more attention to thinking across the curriculum when they teach proportions. This cross-curricular thinking can be done by integrating meaning and concepts when teaching proportions; by emphasizing rate of change in problems to prepare students for the concepts that they will encounter later in the curriculum; by fostering enriching discussions around rate problems to help students make connections regarding important concepts such as rate; and by exploring multiple representations to represent the rate and work functions, accompanied by classroom discussions to scaffold proportional reasoning, slope, covariational reasoning, and functions and derivative.

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What are the Important Ideas When Teaching Rate of Change?

by  
Murside Asli Mutlu

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APPROVED BY:

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Dr. Karen Allen Keene  
Committee Chair

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Dr. Rui Song

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Dr. Hollylynne Lee

## **BIOGRAPHY**

Asli Mutlu, born in Turkey in 1967, earned a B.S. in industrial engineering at Bogazici University. Her first job was at Citibank, in London concentration on financial markets, after which she returned to Turkey and began working as client services director at the international marketing and advertising firm called McCann Erickson World Group in Istanbul. International brands like Gillette, Coco-Cola, GM, Nestle, and L'Oreal were among those represented by this firm and she was responsible for their marketing strategies. She then moved to Antalya, Turkey, where she worked as sales and marketing director of Gloria Golf Resort, a large vacation and golf resort holding. In 2007, she moved to Raleigh, North Carolina, in United States due to her husband's job. She started volunteering in her daughter's elementary school to help her daughter to adapt to their new country and new home. Soon she became a teacher assistant for ESL. Her school's principal recommended that she use her engineering skills and become a math teacher through a lateral-entry teaching program, as this is a high-need area. Mutlu completed the lateral entry teaching program at North Carolina State University. In 2009, while still in the program, she began teaching in Granville County. Until 2016, she taught Pre-calculus, AP Calculus, and AP Statistics courses there. She started work on her MS in math education at NCSU, completing it with Statistics minor in 2016. She now teaches at North Carolina School of Science and Mathematics in Durham, North Carolina. She believes in showing the students the beauty of math and how it connects our daily lives with or without us noticing it. She is very enthusiastic about teaching mathematics. She is married and has one daughter.

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## CHAPTER 1: Introduction

When teaching the mathematical concept of rate of change, what ideas are important for educators to consider? This thesis addresses the question by mapping out the research territory on four major concepts: (a) proportional reasoning, (b) slope and rate, (c) covariational reasoning, and (d) functions and derivatives. These are the four unifying themes for the concept of rate of change. These four themes were selected because of their emergence in the literature as constructs around which a stable consensus seems to have developed regarding their importance for students' understandings of rate of change (ROC). As will be shown, when these four concepts are examined closely, it becomes apparent that they are highly related. Moreover, these concepts seem to provide a useful organization for entry into the issues of learning and teaching the concept of ROC, and for using the concept of ROC as an organizing construct in the curriculum.

These concepts do not exist in a strictly linear or sequential relationship to one another, such that they must be learned in a particular developmental order. However, the sequence of mathematics concepts presented in this work offer a way that the concepts could be developed from informal childhood experiences through formal experiences at the elementary, secondary, and college levels. In general, grade-level math curriculum is designed in such a way that the concepts do appear to students in this order, as each previous idea prepares an important step for the subsequent one. The concept of rate change starts to be incepted in a child's mind in kindergarten, and continues to develop throughout the K-12 years. In order for a student to achieve a complete understanding of the concept of rate of

change, he or she must be able to recognize and construct each of the concepts involved in understanding the rate of change in any relevant context.

The larger context of this thesis is the long-running effort by the mathematics education community (more than two decades) to reform K–12 school mathematics programs. The National Council of Teachers of Mathematics (NCTM) voiced its recommendations for how to improve mathematics education when it published the *Curriculum and Evaluation Standards for School Mathematics* (1989). This document promoted the idea that students should learn and understand important mathematics. Since 1989, mathematics educators have been working to realize the ambitious goals articulated in this document. Through extensive collaboration and commitment, the members of the NCTM have continued the dialogue regarding how best to mathematically educate children, publishing coordinated materials and updates to the original Standards. Two sets of standards, the Professional Standards for Teaching Mathematics (1991) and the Assessment Standards for School Mathematics (1995), offered information for teachers and school officials to assist them in designing and pursuing educational programs aligned with the Standards.

More recently, the NCTM released updates to its original documents; these included *Principles and Standards for School Mathematics* (PSSM) (2000), *Mathematics Teaching Today* (2007), and *Focus in High School Mathematics: Reasoning and Sense Making* (2009). In 2010, the Council of Chief State School Officers (CCSSO) and the National Governors Association (NGA) jointly published the *Common Core State Standards for Mathematics*

(CCSSM) (CCSSO/NGA, 2010). This document is the most recent effort to “define what students should understand and be able to do in their study of mathematics” (CCSSO/NGA, 2010, p. 4). It does so by placing an equal amount of emphasis on conceptual understanding and procedural fluency. The similarities in the recommendations contained in all of these volumes are many, and they consistently emphasize understanding and making sense of mathematics.

The first component of mathematics that these standards emphasize is Algebra. A key component of algebraic reasoning is a working understanding of linear functions (National Mathematics Advisory Panel, 2008; NCTM, 2009). Understanding linear function behavior is a prerequisite for learning more advanced mathematics topics such as calculus and statistics (Wilhelm & Confrey, 2003). Linear functions are characterized by slopes, or constant rates of change. Therefore, understanding the concept of slope is predominantly accepted to be one of the most important mathematical concepts students encounter. As such, a comprehensive understanding of slope is recommended in prominent mathematics standards, namely the CCSSM.

Understanding slope depends on a student’s ability to reason with ratios (Lobato & Thanheiser, 2002). Two types of reasoning with ratios are covariational reasoning and proportional reasoning (Hoffer, 1988). Covariational reasoning concerns a person’s ability to perceive and interpret relationships among quantities that vary in correspondence to one another (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). Proportional reasoning concerns a person’s ability to compare ratios (Hoffer, 1988), which is a significant milestone in a

person's mathematical development (Lamon, 1993). The simplest instances of slope are present in situations of direct variation, where the slope value equals the constant of proportionality.

If we say someone went on a “long trip,” are we referring to the distance or the duration of the trip? If we say someone “went fast,” does this refer to speed or shortness of duration? (Stroup, 2002). These ambiguities display confusions in young learners' minds for interpretations of graphs and rate of change. From early on in a child's life, then, rate of change is an important concept in mathematics. The NCTM (2000) Algebra Standards state that students should be able to “analyze change in various contexts.” Indeed, the standards emphasize the importance of rate by emphasizing this concept all the way from Kindergarten to 12<sup>th</sup> grade. In one appearance in grades 3–5, the Standards state that students should be able to “identify and describe situations with constant or varying rates of change and compare them.” Moreover, for secondary schools, the Algebra Standards offer further justification for the early inception of rate of change. These standards indicate that by the end of grade 8, students are expected to “model and solve contextualized problems using various representations, such as graphs, tables and equations,” and by the end of grade 12, they should be able to “identify essential quantitative relationships in a situation and determine the class or classes of functions that might model the relationships” (NCTM 2000, p. 395). Overall, it seems clear that NCTM recommends rate of change as a concept to teach from K–12.

Rate of change has also an important but slightly different place in CCSSM. For grade levels K–6, it makes many mentions of ratio and percentage concepts, but rate of change is not as clearly stated as it was in NCTM. The standards for grade 6 are the first place where “dependent and independent variables” are explicitly mentioned. At that point, CCSSM recommends that rate of change should be taught using real-life examples of the concept of motion (CCSS.Math.Content.6.EE.C.9). In grade 7, rate of change is not mentioned, although there are a lot of problems on interest, tax, percent increase, and percent decrease. However, in grade 8, several domain standards include proportional relationships, constant rate of change (which is the milestone for functions), linear functions, and functions. In high school algebra, the interpretation and understanding of rate of change takes its mandatory place through the functions and the modeling domains in CCSSM.

In summary, rate of change is an important concept in mathematics, but it is commonly misunderstood and often elusive to students. Finding ways to help students to understand rate of change can be very difficult or cumbersome even at the levels of Pre-Calculus and AP Calculus. The purpose of this literature review, then, is to map the territory on mathematics education research in order to clarify what ideas are important to consider when teaching rate of change. These findings will then suggest how educators might approach curriculum design and instruction related to these important ideas.

In chapter 2, I focus on proportional reasoning, the ability to find mathematical relationships between two quantities (e.g., the use of ratios to compare quantities). Then, I will explain in detail the components of the proportion as extensive variables and the value of

the proportion as an intensive variable. I show that the best approach to proportional reasoning, to support later learning, is the use of intensive and extensive variables to understand ratio: that is, noticing the importance of comparison and measurement of the two extensive variables which establishes a relationship to form an intensive variable. The ability to think about variables intensively or extensively is essential to doing higher-level mathematics.

In chapter 3, I examine slope and ratio, which can be understood as essential to a sophisticated understanding of rate of change. Slope can in fact be understood as a representation of rate of change, expressing the proportional relationship between two extensive variables, and thus expressing the value of the intensive variable.

Next, in chapter 4, I discuss literature on covariational reasoning, which is the ability to reason about the relationship between two variables that change together. Like proportional reasoning and an understanding of slope, covariational reasoning is necessary to a sophisticated understanding of rate of change. Indeed, rate of change can actually be conceptualized as covariation.

In chapter 5, I explore how the concept of rate of change is closely related to the concept of a function, specifically discussing the notion of covariational reasoning. Using covariational reasoning (envisioning two variables co-varying systematically), a student can understand the connection between the concept of slope and the concept of the first derivative of the function. After learning covariational reasoning with linear functions, the student can advance to understanding varying rates of change within non-linear functions.

Rate of change can be understood as a slope formula that belongs to both linear and nonlinear functions.

This literature review then situates the concept of rate of change in relation to other mathematical ideas: (a) proportional reasoning, (b) slope and rate, (c) covariational reasoning, and (d) function and derivatives. The findings have implications for how mathematics educators understand and teach rate of change throughout the K–12 process, as will be discussed in chapter 6.

## CHAPTER 2: Proportional Reasoning

Proportional reasoning, the use of ratios to compare quantities, is an important idea to consider when teaching rate of change, in that it is the initial step and an integrative thread in this concept. For example, measuring quantities such as speed, density, or sweetness involves reasoning about ratios of time to distance, mass to volume, and sugar to water, respectively. According to Modestou and Gagatsis (2010), proportional reasoning has three different aspects: routine proportionality, meta-analogical awareness, and analogical reasoning. Analogical reasoning is most applicable to rate of change, as it encompasses ratio, multiplicative reasoning, and extensive/intensive variables, and involves reasoning about relationships between quantities. It is used to recognize and apply patterns that are embodied in concrete objects or abstract ideas.

As mentioned in the introduction, two of the primary sets of mathematics standards in the United States acknowledge the importance of proportional reasoning: the National Council of Teachers of Mathematics (NCTM), and the Common Core State Standards for Mathematics (CCSSM). According to the NCTM, a significant amount of middle school mathematics problem-solving needs to be contextualized around students' development of number sense and computational fluency in proportional reasoning. The concept of proportional reasoning represents "an important integrative thread that connects many of the mathematics topics studied in grades 6–8" (NCTM, 2000, p. 217); thus, it "merits whatever time and effort must be expended to assure its careful development" (NCTM, 1989, p. 82). Along similar lines, the CCSSM includes ratios and proportional reasoning as one of its

eleven core standards (the CCSSM's Common Core State Standards Initiative, 2010).

According to the CCSSM, the teaching of proportional reasoning should begin in sixth grade.

The curriculum specifies that at that point, student must learn to “understand ratio concepts and use ratio reasoning to solve problems” (Math.Content.6.RP.A.1–3). In seventh grade, the skill is then reinforced and intensified as students learn to “analyze proportional relationships and use them to solve real-world and mathematical problems” (CCSS.Math.Content.

7.RP.A.1-3). The centrality assigned to proportional reasoning by both the NCTM and the CCSSM validates this study's focus on the concept. Given the importance of proportional reasoning to students' development of mathematical understanding, it is troubling that students of all ages experience difficulty with mastering proportional reasoning. When a strong conceptual understanding of proportional reasoning develops, strong comprehension of rate of change will ensue.

The difficulties that students of all ages experience with proportional reasoning are well documented (e.g. Wilson, Meyers, Edgington, & Confrey, 2012; Modestou & Gagatsis, 2010; Bryant, Howe & Nunes, 2010; Lamon, 2007; Lobato & Thanheiser, 2002; Thompson & Thompson, 1996). The main problem when teaching proportional reasoning, according to Modestou and Gagatsis (2010), is that

proportional reasoning does not coincide exclusively with success in solving a certain restricted range of proportional problems, as routine missing-value and comparison problems (routine proportionality), but it also involves handling verbal and arithmetical analogies (analogical reasoning) as well as the awareness of discerning

non-proportional situations (meta-analogical awareness), which is metacognitive in nature. . . . Thus, proportional reasoning also involves the ability to handle verbal and arithmetical analogies as well as disposition toward discerning and handling non-proportional situations. (p. 36)

Of all the topics in the school curriculum, tasks requiring proportional reasoning may hold the distinction of being the most prolonged in terms of development, the most difficult to teach, the most mathematically complex, the most cognitively challenging, the most essential to success in higher mathematics and science, and one of the most compelling research sites. Confusion about proportional reasoning has been directly implicated in students' difficulty with thinking about variables intensively or extensively: that is, with a difficulty in recognizing how some qualities depend on the object's size or amount of material (extensive variables), and other qualities depend on the relationship between extensive variables (intensive variables). Such difficulty is significant because an understanding of intensive and extensive variables is necessary for students to develop multiplicative reasoning and thus is necessary for higher-level mathematics. As will be shown later in this chapter, the concepts of intensive and extensive variables are central to the most promising approach to teaching proportional reasoning: teaching proportional reasoning as measurement.

In practical terms, proportional reasoning is the use of ratios to compare quantities. According to Confrey and Maloney (2010), proportional reasoning is the first step in students' development of some higher-level mathematical reasoning because it underpins algebra, higher mathematical reasoning, and the quantitative competence required in science.

Of particular relevance to this study, proportional reasoning is also central to students' understanding of rate of change, the changing relationship between a dependent and independent variable. Proportional reasoning is a particularly important skill to teach in the middle grades, not only because it is an integrative thread in the concepts being taught at that stage of development, but also because it is the foundation for higher-level mathematics that students learn later, such as calculus.

Various definitions of proportional reasoning are often used. All of the definitions relate the concept to rate of change, but they differ in how they frame the concept for practical purposes. Lobato and Siebert's (2002) widely accepted approach to teaching proportional reasoning in lower grades integrates it with the use of slope as a ratio. In Lobato and Siebert's (2002) approach to the instruction of rate of change, they emphasized the importance of seeing slope as the rate of change in one quantity relative to the change in another quantity, where the two quantities co-vary. That is, slope is thought of as a variable formed by the ratio of two other variables (e.g., time and distance), and as a measure of the proportions relating different connected attributes (for example, the proportional relationship between time and distance is velocity, and the proportional relationship between mass and volume is density). Another seminal definition used by Bryant et al. (2010) states that proportional reasoning is the ability to recognize "the ratio between elements within measure spaces and the functional relationship across measure spaces" (p. 392).

From another perspective, Schwartz (1988) and Lamon's (2007) approach to proportional reasoning is built on the recognition of a measure of the intensity of a

relationship. This relationship is defined as an intensive quantity, which is perceived as the relative size of the magnitudes (number of units) of measures from two different measure spaces, or of the measures of two different quantities from the same measure space, given particular units or measurement for each measure. The value of an intensive quantity may exist on its own or be calculated by finding the ratio of the value of two extensive variables. As noted above, extensive variables are quantities that depend on the object's size or amount of material. For example, to measure the intensity of the chocolate flavor of a batch of hot chocolate, we could use ratios relating the amount of chocolate packets and cups of water in a batch of hot chocolate. We measure the association amounts of extensive quantities, which are chocolate packets and cups of water. As another example, speed (intensive quantity) can be perceived as a value that exists on its own (how fast is something humans can "feel") or it is calculated using time (extensive) and distance (extensive). In this situation, then, intensive quantities may combine both direct and inverse proportionality.

In another conceptualization of proportional reasoning, Thompson and Thompson (1996) have indicated that many phenomena can be expressed as a proportional relationship between specific variables, often leading to some new, unique entity. This resonates with Lamon's extensive/intensive notion. Conceptualization and comprehension of this concept, not to mention skills and competence in using proportional reasoning, facilitate mathematical awareness.

The most comprehensive model for proportional reasoning is the three-component model proposed by Modestou and Gagatsis (2010). This model presents three different

aspects of proportional reasoning: analogical reasoning, routine proportionality, and meta-analogical awareness. Analogical reasoning is the ability to solve verbal and numerical analogies; routine proportionality is the ability to solve routine proportional tasks (commonly called “missing value problems”); and meta-analogical awareness consists of the ability to recognize proportional and non-proportional statements, as well as the ability to work with non-proportional tasks. Modestou and Gagatsis (2010) argued that the ability to construct and algebraically solve proportions is implicitly a fundamental aspect of proportional reasoning (as cited in Lamon, 1993, p. 41). According to their model, proportional reasoning involves the ability to recognize the existence of proportion between two quantities; in solving a missing value problem, a ratio is constructed. Therefore, Modestou and Gagatsis’s (2010) model of proportional reasoning includes ratio as an essential component.

In the next section, I present approaches to proportional reasoning that are used in school mathematics. In so doing, I discuss a variety of established approaches to teaching proportional reasoning. First, I review literature that establishes the relationship between proportional reasoning and rate of change. Then, I distill three main approaches to teaching proportional reasoning: ratio, multiplicative reasoning, and intensive and extensive variables. Throughout these sections, I am interested in one key question about proportional reasoning: In researchers’ studies and findings, what counts as proportional reasoning? What have become the established approaches to defining proportional reasoning? I address the question for each of the three approaches—ratio, multiplicative reasoning, and intensive and extensive variables—by reviewing literature on relevant problem types. As I will show, the most

promising approach to teaching proportional reasoning is the third approach, intensive and extensive variables, as this approach can contain the other two approaches. I argue that the approach of intensive and extensive variables provides the best integrative thread for mathematics students, particularly in the middle grades, and lays the best foundation for learning higher-level mathematics.

### **Proportional Reasoning and Rate of Change**

The connection between proportional reasoning and rate of change has been well established (Wilson, Myers, Edgington, & Confrey, 2012; Ben-Chaim, Ilany & Keret, 2012; Modestou & Gagatsis, 2010; Bryant, Howe & Nunes, 2010; Lamon, 2007; Clark, Berenson, & Cavey, 2003; Lobato & Thanheiser, 2002; Thompson & Thompson, 1996). Thompson and Saldanha (2003) expressed the integral relationship between proportional reasoning and rate of change in terms of a question posed by the mathematician d'Alembert in the 18<sup>th</sup> century: should “ $dy/dx$  ... be thought of as merely a symbol that represents one number instead of as a pair of symbols representing a ratio of two numbers?” The authors explained that d'Alembert suspected that the cultural practice of considering a rate of change as being composed of two numbers was conceptually incoherent, and that what was conventionally interpreted as a ratio of two numbers was indeed one number that was not the result of calculating” (p.10). An effective way to show students the relationship between proportional reasoning and rate of change, according to Thompson and Saldanha (2003), Lamon (1993), and Lobato and Thanheiser (2002), is through the use of problems having to do with intensive variables such as speed (whose extensive variables are time and distance), density (whose extensive

variables are mass and volume), or sweetness (whose extensive variables can be the amount of ingredients such as sugar and water).

Today, mathematics educators and researchers widely recognize that proportional reasoning cannot be separated from the idea of rate of change; to understand the latter, one must exercise the former. Yet, as early and middle-grade mathematics are currently taught, the foundations of proportional reasoning are not laid, making it more difficult for students to learn rate of change when they reach upper-level mathematics courses.

### **Established Approaches to Proportional Reasoning: A Justification**

Three main approaches to teaching proportional reasoning are teaching ratio, multiplicative reasoning, and intensive and extensive variables. In this section, I describe each of these three approaches, as well as the problem types associated with each approach—that is, the strategies that students use to solve mathematical problems that require proportional reasoning. I consider the strengths and weaknesses of each strategy and show why the third approach to proportional reasoning, intensive and extensive variables, is the most promising.

Lamon (1993) defines proportional reasoning problems based on their mathematical and semantic characteristics. Her rationale for proposing these definitions is to make it possible to consider each individual thinking process in investigating how children think and solve different proportional reasoning problems. The table below presents her detailed classification of the problems and problem types.

**Table 1. Selected problems of each semantic type. (Adapted from Lamon, 1993).**

Identifier	Problem
Well-chunked measures	
Subscription	The student is shown a subscription card from a popular magazine. It offers three plans: 1) a 6-month subscription for 3 payments of \$4.00 each; 2) a 9-month subscription for 3 payments of \$6.00 each; and 3) a 12-month subscription for 3 payments of \$8.00 each. Do you get a better deal if you buy the magazine for a longer period of time?
Travel log	The student is shown a page from a driver's log book, recording total mileage at various intervals on a long trip. After 2, 5, 7, and 8 hours of driving, distance traveled was recorded as 130 miles, 325 miles, 445 miles, and 510 miles, respectively. Was this truck driver traveling at a constant speed throughout the trip?
Part-part-whole	
Apartment	In a certain town, the demand for apartments was analyzed and it was determined that to meet the needs of the community, builders would be required to build apartments in the following way: Every time they build 3 one-bedroom apartments, they should build 4 two-bedroom apartments and 1 three-bedroom apartment. Suppose a builder is planning to build a large apartment complex containing between 35 and 45 apartments. Exactly how many apartments should he or she build to meet this regulation? How many one-bedroom, two-bedroom, and three-bedroom apartments will there be?
Eggs	The student is shown pictures of two egg cartons, one containing a dozen eggs (8 white eggs and 4 brown eggs) and the other containing 1 1/2 dozen eggs (10 white eggs and 8 brown eggs). Which carton contains more brown eggs?
Associated sets	
Cars	The student is shown two bar graphs, each depicting the number of square miles and the number of cars in two cities. In the first graph, City A has 4000 square miles and 2000 cars, and City B has 5000 square miles and 2000 cars. In the second graph, City A has 4000 square miles and 2000 cars, and City B has 3000 square miles and 2000 cars. In each case, the question is, Which city has more cars?
Pizza	The student is shown a picture of 7 girls with 3 pizzas and 3 boys with 1 pizza. Who gets more pizza, the girls or the boys?
Stretchers and shrinkers	
Trees	The student is shown a picture of two trees. Tree A is 8 feet high and tree B is 10 feet high. This picture was taken 5 years ago. Today, tree A is 14 feet high and tree B is 16 feet high. Over the last five years, which tree's height has increased most?
Rectangles	The student is shown two rectangles. The height of the first is labeled "6 feet" and its width is labeled "8 feet." The height of the second rectangle is marked "?" and its width is labeled "12 feet." These rectangles are the same shape, but one is larger than the other. Explain how you would find the height of the larger rectangle.

The proportional reasoning problem types discussed in the proceeding sections are drawn from the classifications used by Lamon (1993) in a study where she analyzed how students think while solving proportional reasoning problems of four semantic types: part-part-whole, associated sets, stretchers and shrinkers, and well-chunked measures. In the following sections, part-part-whole and associated sets problems will be discussed as instances of the ratio approach to proportional reasoning; stretchers and shrinkers will be discussed as instances of the multiplicative reasoning approach; and well-chunked measures will be discussed as examples of the intensive and extensive variables approach which prepares the foundation of covariational reasoning.

Lamon's (1993) problem-type classifications are used in this review because they closely correspond to the three approaches to proportional reasoning described in this chapter. The problem types do not correspond perfectly to the three approaches; depending on how teachers understand and present certain problems, the problems could appear in a different category than is identified here. For example, stretchers and shrinkers could be taught using any of the three approaches, even though they are discussed here as an example of the multiplicative approach. Nevertheless, I propose this framework as useful for the present purposes.

### ***The Ratio Approach***

Through my literature review, I have found that the most common approach to proportional reasoning, and that of greatest interest to both researchers and practitioners, is ratio. In calculating a ratio, the student uses proportional reasoning to compare two

quantities. Indeed, the simplest definition of ratio, according to researchers, is as a comparison of two quantities (Ben-Chaim et al., 2012; Confrey et al., 2012; Modestou & Gagatsis, 2010; Bryant, Howe & Nunes, 2010; Lamon, 2007; Clark, Berenson, & Cavey, 2003). Yet researchers have expressed concern that the term *ratio* has not been sharply distinguished from fractions and other similar concepts, and that it has been applied so broadly that the essence of its meaning may be lost. Smith (2002, p. 4, as cited in Clark et al.) argued, “Mathematics educators often talk right past one another because they are using fraction, ratio, or proportion in different ways without realizing it” (p. 297). The authors reflected upon the important differences between the concepts of *ratio* and *fraction* and agreed that fractions are just one way to understand ratio, one of the number domains in ratios. In their view,  $\pi$  is a ratio but not a fraction. In a similar argument, Confrey and Lachance (2001) discussed how students develop a broad understanding of ratio through rich and varied experiences by exploring constructs such as fraction, decimal, and percentage. According to their claim, fraction is just one of the domains to express ratio. To give a third example, Ben-Chaim et al. (2012), in their book on ratio and proportion, also defined fraction as one among several ways to represent ratio.

The significance of the concept of ratio to the understanding of rate of change is that ratio can be seen as a form of measure of rate of change. Lobato and Siebert (2002), for example, indicated that the connection between proportional reasoning and rate of change hinges upon the conceptualization of ratio as measure; and according to Lamon (2007), ratio is one of the critical aspects of understanding proportionality. Lobato and Siebert (2002)

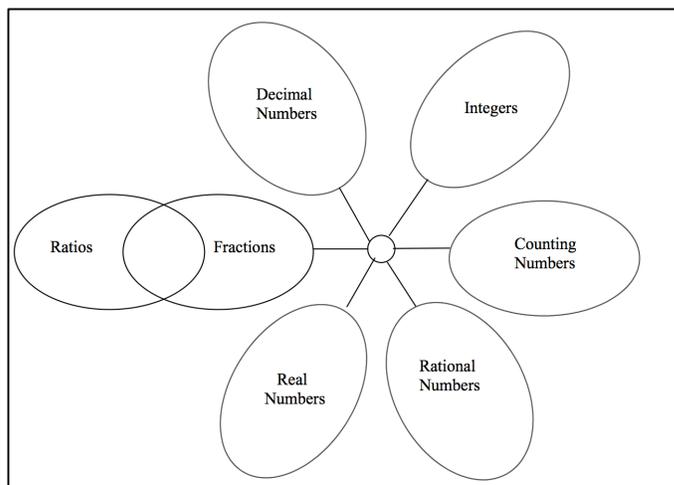
provides examples of ratio-as-measure tasks in order to “suggest ways for teachers to help students develop an understanding of the modeling and the proportional reasoning aspects of ratio-as-measure tasks, which in turn can help students develop an understanding of slope that is more general and applicable” (p. 174).

For students to understand rate of change, it is important for them to recognize that ratio is distinct from fraction. A fraction is a symbol by which to show a ratio or proportion. Not a measurement itself, a fraction is merely one way of showing this measurement. Part-part-whole problems arise from this distinction between ratio and fraction. Thompson and Saldanha (2003) describe the relationship between fractions and proportional reasoning thus:

Fractions are a synthesis. We do not claim this in a realist sense. We mean it in a cognitive sense. Fractions become “real” when people understand them through complementary schemes of conceptual operations that are each grounded in a deep understanding of proportionality. At the same time, a focus on developing these schemes will enrich students’ understanding of proportionality. (p. 37)

Clark et al. (2003) also attempted to distinguish between the terms *ratio* and *fraction* in order to elicit a definition of proportional reasoning. Ratio is defined as a comparison of two quantities, which could be either a rate as part-part comparison or could be a fraction as part-whole comparison. Because of the various interpretations of fraction that are not related to ratio, Clark et al. (2003) presented a more comprehensive model to show the relationship between ratio and many *number-type domains* (Fig. 1). This representation is similar to the image of an optometrist’s eye-examination machine in which the lenses can be rotated or

changed.



**Figure 1. Ratio with alternative number-type domains. (Adapted from Clark et al., 2003).**

The ratio lens is constant, and it is overlapped by each of the other lenses in turn. The lenses of the number-type domains can be rotated, such that each of the six domains can be made to overlap with ratio: fractions, decimal numbers, integers, counting numbers, rational numbers, and real numbers. For example, when viewing ratio through the lens of decimal numbers, a ratio can be converted to a decimal numbers form, or a counting numbers form, or rational numbers. Fractions are only one of these six lenses or forms. Each form represents a different lens, a perspective.

In another study, Howe, Nunes, and Bryant (2011) found that, surprisingly, the students who had never previously received teaching about ratios mastered the constructs of proportional relationship quickly, used them in reasoning, and through ratio teaching progressed their understanding of fractions. Even though fraction language was deliberately

avoided during the ratio lessons, the posttest scores obtained by the ratio group indicated that fraction understanding was supported here too (p. 411).

In the next section, I discuss two problem types associated with ratio, as defined by Lamon (1993): part-part-whole and associated sets. Part-part-whole involves comparing two parts of a whole, and associated sets involves comparing two unconnected quantities. (See Table 1 above.)

### ***Part-Part-Whole***

In a part-part-whole problem, an extensive measure of a single subset of a whole is given in terms of two or more subsets of which it is composed (Lamon, 1993). That is, each of the “parts” can be understood as an extensive quantity (i.e., dependent on the object’s size or amount of material), and the “whole” can be understood as an intensive quantity whose value derives from the proportions of the extensive quantities. According to Lamon (1993), examples of part-part-whole problems include “classes with a ratio of X boys to Y girls or collections of test questions with a ratio of P correct to Q incorrect answers” (p. 42). Ben-Chaim et al. (2012) conceptualized this type of ratio problem as “comparing two parts of a single whole” (p. 29). Clark et al. (2003) identified this type of ratio as *descriptive*: static counts of objects in two sets, e.g., red to blue marbles or boys to girls. In Clark et al.’s (2003) perspective, this is a definition of ratio as an adjective, which is the same as one of the four semantics in Lamon’s study (1993) as a part-part-whole. To sum up, we can say that here the ratio is an aspect of the comparison of two parts of a single whole, each part having the same units.

Wilson et al. (2012) offered a distinct approach to what are similar to part-part-whole problems that they termed “equipartitioning” (p. 483). This approach is based on the observation that multiplication is the “re-assembly of equi-partitioned collections or wholes,” an ability that, according to Confrey and Maloney (2010), “if it could readily be conceptually developed instead of and earlier than it is typically developed as repeated addition . . . would greatly strengthen children’s subsequent multiplicative reasoning” (p. 6). Wilson et al. (2012) “note[d] that equipartitioning is not part of any curriculum of which we are aware,” even though “older students who have not become proficient in equipartitioning exhibit misconceptions that could be remedied by systematic introduction to equipartitioning” (p. 7).

In closing, part-part-whole problems focus the instruction of proportional reasoning specifically around fractions, which is only one number-type domain of ratios (see Figure 1). Since part-part-whole problems (see Table 1) are defined as the comparisons of the parts with the same units (e.g., red marbles to blue marbles), as I explained previously, each of these parts is an extensive variable. Their proportion produces an intensive variable. However, this representation (intensive variable) does not contain a meaningful unit (red marbles per blue marbles), as it does in the speed example. Speed is the proportion of distance (miles) and time (hour), two extensive variables that have different units and produce an intensive variable of speed (miles per hour) as the meaningful unit. With some intensive quantities not having a meaningful unit, the question arises whether these types of problems help students develop coherent and stable understandings of proportional reasoning, making the connections we would like between ratio and rate of change.

### ***Associated Sets***

Sometimes the two parts compared in a ratio may have no commonly known connection (e.g., people and pizzas; students and birthday cake). According to Lamon (1993, 2007), this kind of ratio problem is termed “Associated Sets.” (see Table 1). Comparing associated sets of objects requires students to use qualitative proportional reasoning: to think of the ratio as a unit, apply relative thinking, and understand some numerical relationships. Lamon (1993) found that associated sets problems, when worked on by students, “had the effect of eliciting more of the language of ratio than students used in talking about other semantic types” (p. 51). Refer to Table 1 above for examples of associated sets problems. According to Wilson et al. (2012), associated sets problems present children as young as kindergarten age with opportunities to learn equipartitioning. Young children sharing a collection of toys among friends or a pie for dessert are keen on making sure that everyone gets their “fair share.” These opportunities allow young children to be successful at creating equal-size groups or parts of collections and wholes (Wilson et al., 2012; Confrey & Maloney, 2010). An example of equipartitioning is the comparison of ratios in different units, such as 7 girls and 3 pizzas versus 3 boys and 1 pizza.

Howe et al. (2010) also reported that sharing activities might facilitate the introduction of ratios to students; these activities in turn support proportional reasoning. However, the same researchers warn that sharing cannot be a sufficient means of teaching proportional reasoning: “Introducing ratios via sharing by no means guarantees linkages with

proportional reasoning. The association needs to be made explicitly” (p. 393). As Lamon (1993), Ben-Chaim et al. (2012), and Modestou and Gagatsis (2010) presented in their studies, associated set problems lead students to build divisibility relationships, integral ratios, and unit rates. Thus, students elicit more of the language of ratio in the associated set problems as sharing occurs.

In sum, the literature on proportional reasoning indicates that ratio is the most common approach to teaching this important skill. Two categories of ratio tasks described above are part-part-whole problems, which involve comparing two parts of a whole, and associated sets problems, which involve comparing two unconnected quantities. Both problem types require students to use proportional reasoning in comparing only two quantities.

### ***The Multiplicative Reasoning Approach***

A second perspective on proportional reasoning is multiplicative reasoning: the ability to think of multiplication as producing a product, and to think at the same time of the product in relation to its factors. This is an ability that, as Thompson and Saldanha (2003) pointed out, entails proportional reasoning. Ben-Chaim et al. (2012) defined multiplicative relationships in terms of ratio, explaining, “In mathematics, ‘ratio’ is the quantification of a multiplicative relationship that is calculated by dividing (or multiplying) one quantity by another. The multiplicative quantifier is determined by dividing (or multiplying) two magnitudes” (p. 25). Multiplicative reasoning is related to proportional reasoning in that, when exercising the former, the student is aware that the difference between the two

quantities (the product) is not constant. An example of multiplicative reasoning in Thompson and Thompson (1996) is that of speed:

Speed as a quantification of completed motion is made by multiplicatively comparing the distance traveled and the amount of time required to go that distance. There is a direct proportional relationship between the distance traveled and the amount of time required to travel that distance. (p. 3)

To solve a speed problem, the student must recognize that the difference between the time and the distance is not constant.

Thompson and Saldanha (2003) said that to think of multiplication as producing a product, and to think at the same time of the product in relation to its factors, entails proportional reasoning. Lamon's (2007) definition is the most comprehensive:

Proportional reasoning as providing reasons in support of claims made about the structural relationship among four quantities in a context simultaneously involving covariance of quantities and invariance of ratios and products; this would consist of the ability to discern a multiplicative relationship between two quantities. (p. 638)

The most important part of her definition is the emphasis on "discerning" the multiplicative relationship; this is done by advancing beyond the additive reasoning used in elementary school to the multiplicative reasoning required for upper-level mathematics.

In Thompson and Saldanha's (2003) detailed and powerful definition, they claim that proportional reasoning is a relationship between two quantities such that if you increase the size of one quantity by a factor  $a$ , then the other quantity's measure must increase by the

same factor to maintain the relationship. In general, when students understand multiplication in a multiplicative manner, they understand the product ( $nm$ ) as being in multiple reciprocal relationships to  $n$  and to  $m$  as:

- ...( $nm$ ) is  $n$  times as large as  $m$ ,
- ...( $nm$ ) is  $m$  times as large as  $n$ ,
- ... $m$  is  $1/n$  as large as ( $nm$ ),
- ...  $n$  is  $1/m$  as large as ( $nm$ ). (p. 26)

This definition conveys the generality inherent in thinking of multiplication in a way that entails proportionality and multiplicity.

In defining multiplicative reasoning, it is helpful to contrast it with its conceptual counterpart, additive reasoning, in which the difference between the two quantities remains constant. Early in a child's mathematics education, at the beginning of the concrete stage, the child still thinks in an additive manner. At this point, the child's problem-solving strategies are based on the differences between the sizes of the values in the problem instead of on the ratio between them, and an exercise of multiplication may be undertaken simply as repeated addition. While common sense might lead the child to a form of proportional reasoning, this reasoning is qualitative only, without any abstraction or quantitative thinking. This level of understanding represents incomplete understanding of a proportional scheme (Thompson & Thompson, 1996), and it is not adequate over the long term. As explained by Ben-Chaim (2012), "The types of problems in which addition or subtraction will not be effective are many, and a multiplicative strategy that involves understanding the concept of ratio is

required” (p. 28).

Thompson and Saldanha (2003) explained why an additive approach to multiplication is limiting: understanding multiplication as repeated addition keeps it divorced conceptually from measurement, proportionality, and fractions. This results in weak connections about various meanings of the relationship between the multiplication and fractions and conceptualizing the division, which is in the heart of proportional reasoning.

Multiplicative reasoning, unlike the simpler skill of additive reasoning, deals with abstractions, upon which the student must impose a relational structure to understand their relative values. Because the relational structure (or classification scheme) is separate from the object, event, or image in which it is exemplified, multiplicative reasoning is an act of conception rather than perception. Many researchers have emphasized that, whereas additive reasoning involves considering the sums or differences in quantities, multiplicative reasoning (and relational thinking) make possible the comparison of quantities through the use of ratios—that is, the exercise of proportional reasoning (e.g. Wilson et al. 2012; Ben-Chaim et al., 2012; Lobato & Thanheiser, 2002; Lamon, 2007; Thompson & Thompson, 1996). Thus, multiplicative reasoning is necessary to the exercise of proportional reasoning.

Several researchers have investigated students’ exercise of multiplicative reasoning in the classroom. In Lobato and Siebert’s (2002) study, students “constructed meaning for the slope as the multiplicative comparison of height to length, [and] this construction of ratio developed, not by memorizing a formula, but by mentally comparing the heights and lengths of a set of ramps that shared the same steepness” (p. 172). Also in Lobato and Siebert’s

(2002) study, students reached a breakthrough when they were able to partition (subdivide the unit) and iterate (repeat the unit) to find which character walked faster and how to measure the rate at which each character was travelling. (Distance and time were not displayed.)

To further develop an explanation of multiplicative reasoning, I now describe examples of multiplicative reasoning (ratio, fractions, and sharing) as well as a problem type that exercises multiplicative reasoning, stretchers and shrinkers (Lamon, 1993, 2007). Proportional reasoning requires multiplicative and relational thinking and involves the use of ratios in the comparison of quantities. According to Thompson and Saldanha (2003), “Students are often instructed, and therefore learn, that the fractional part is contained within the whole, so “A is some fraction of B” connotes a sense of inclusion to them, that A is a subset of B. As a result, statements like “A is  $\frac{6}{5}$  of B” make no sense to them. However, when thought of multiplicatively, “A is  $n$  as large as B” means that A is  $n$  times as large as  $\frac{1}{n}$  of B. So, from a multiplicative perspective, “A is  $\frac{6}{5}$  as large as B” does not imply that A and B have anything in common. Rather, it means that A is 6 times as large as ( $\frac{1}{5}$  of B)” (p. 31).

Sharing is another example of multiplicative reasoning. Operational understanding of division entails a conceptual isomorphism between two “types” of division settings: sharing and segmenting. Sharing (or partitioning) and segmenting (or measuring) will help students to gain mathematical power about the system of conceptual operations comprising a fraction scheme is based on conceiving two quantities as being in a reciprocal relationship of relative

size, and this will be the definition offered for proportional reasoning by Thompson and Saldanha (2003): “Amount A is  $1/n$  the size of amount B means that amount B is  $n$  times as large as amount A. Amount A being  $n$  times as large as amount B means that amount B is  $1/n$  as large as amount A.” The process of multiplication is associated with situations that involve such processes as shrinking, enlarging, scaling, duplicating, exponentiating and fair sharing (Lamon, 2007, p. 649). From a similar perspective, Wilson et al. (2012) defends concentrated engagement in fair sharing tasks to notice quantitative relationships that may mature into formal reasoning, moving beyond the realm of concrete operations.

A problem type that requires multiplicative reasoning are the “stretcher and shrinker” exercises. Refer to Table 1 above for two examples of “stretcher and shrinker” problems. These involve comparing magnitudes of two quantities that are conceptually connected but not naturally part of a common whole. An example from Ben-Chaim et al. (2012) is “the ratio of sides of two triangles is 2 to 1” (p. 29). The term “stretchers and shrinkers” comes from Lamon’s (1993) classification of four semantic types of problems. By referring the “Trees” and “Rectangles” problems in Table 1 above for Lamon’s example of stretchers and shrinkers, she describes the appellation “stretchers and shrinkers” in this way:

When a one-to-one continuous ratio-preserving mapping exists between two quantities representing a specific characteristic of an element, namely, a measure of distance such as height, length, width, circumference, and so on, or between two quantities representing two such characteristics of an element (e.g., length and width), the situation involves scaling up (stretching) or scaling down (shrinking). (p. 43)

These problems require students to think in terms of relativity. As Lamon (1993, 2007) explained, for the middle-grade students in her study, these problems are among the most difficult; the few students who did succeed at these problems were those who were able to think relatively. Lamon (2007) claimed that stretchers and shrinkers reveal the problems that students experience with proportional reasoning. Proportional reasoning involves recognition of the ratio between elements within measure spaces (extensive variable) and the functional relationship (intensive variable) across measure spaces. For example, in problems about shadow size, thinking includes recognizing that as the size difference between illuminated objects increases (the ratio between elements), so the size difference between shadows also increases (the functional relationship across measure spaces).

To summarize the approaches to proportional reasoning presented thus far (involving ratio and multiplicative reasoning), I discussed learning about ratio via three types of problems: (a) comparing two quantities that are two parts of a single whole (part-part-whole); (b) comparing two quantities that have no connection at all (associated sets); and (c) comparing the magnitudes of two quantities that are conceptually related but not as a part of a common whole (stretchers and shrinkers). All three problem types validate the necessity of understanding qualitative proportional reasoning, which requires the use of ratios as units, the use of relative thinking, and the understanding of certain numerical relationships. By contrast, quantitative proportional reasoning relies on algebraic symbols to represent proportions with full comprehension of functional and scalar relationships. Unfortunately, according to the literature, the ratio and multiplicative reasoning approaches are inadequate

for providing quantitative proportional reasoning. This was particularly evident in the discussion of stretchers and shrinkers problems.

### ***The Intensive and Extensive Variable Approach***

The two approaches to proportional reasoning, using ratios and multiplicative reasoning, are legitimate but ultimately unsatisfactory. This section describes a third, more robust approach that contains both the ratio and multiplicative reasoning approaches within it: the intensive and extensive variable approach. Proportional reasoning involves the comparison of two extensive variables (e.g., the distance that a car travels in miles and the length of time that it travels in hours) to calculate an intensive variable (e.g., speed or rate in miles per hour). This example of using proportional reasoning to calculate an intensive variable, a car's rate of travel, is described in Thompson and Saldanha (2003). Intensive quantities are those that combine direct and inverse proportionality. For example, speed is an intensive quality that depends on distance (directly proportional) and time (inversely proportional).

Understanding the nature of intensive variables requires proportional reasoning. As Howe et al. (2010) noted, intensive quantities are central to many important scientific concepts. They added, "Reasoning with intensive quantities requires understanding of rational numbers at two levels, first to represent the quantities and second to recognize the ratios on which proportional reasoning depends. Used appropriately, intensive quantities could therefore provide a rich instructional context" (p. 394).

A prime example of a problem type associated with the intensive and extensive

variable approach is the well-chunked measures problem (Lamon, 1993). (Refer to Table 1 above.) In solving such a problem, the student must compare two extensive measures to arrive at an intensive measure, or rate. Two examples of intensive measures that are composed of “chunked” extensive measures could be speed or price: speed represents the chunking of time and distance (miles per hour), while price represents the chunking of quantity and dollars per item. Lamon’s (1993, 2007) studies of well-chunked measures show that students correctly use the language of ratio and translate the necessary representations within the verbal mode: e.g., 120 miles per 2 hours, or 60 miles per hour, or 1 mile per minute. Students could grasp these verbal representations even though they were aware that the car was not going 60 at every instant during the trip. On the other hand, these students did not notice that two quantities were compared in that ratio. They were able to connect the phrase “miles per hour” with speed, but they were familiar only with the chunked term (intensive variable), not its individual components (extensive variables).

Researchers (Stroup, 2002; Lobato & Siebert, 2002) have found that asking how fast someone is traveling is ambiguous since there are at least two different types of “fastness”: (1) how fast an object moves through space, and (2) how fast the object’s legs move (Lobato & Siebert, 2002, p. 167). In Lobato and Siebert’s study (2002), when they asked the students to create a measure of how fast an object travels, several students used direct measures, such as time alone. She proposed a four-step plan to overcome this impediment: isolating the attribute that is being measured, determining the quantities that affect the attribute, understanding the characteristics of measure, and finally constructing the proportion.

Still, intensive variables are very challenging for many students. According to Howe et al. (2010), the CCSSM “make no reference to intensive quantities as such, although connecting ratio and rate is an identified aim for grade 6” (p. 394). They added that, “Likewise, whilst intensive quantities are, as noted, central to science, they are seldom recognized as a unifying construct. Rather, density, speed, temperature, intensity (and so on) are typically treated as distinct topics” (p. 394). Students’ difficulty in understanding intensive quantities has implications for their proportional reasoning at two levels: first, representing the quantities; and second, recognizing the ratios upon which proportional reasoning depends.

Clark et al. (2003) and Ben-Chaim et al. (2012) each posited that using instruction that focuses on speed and density concepts will invoke in students’ minds the intuitive and formal component of proportional reasoning. Howe et al. (2010) defended the implication is that no matter which form is used (fraction or ratio), reasoning with intensive quantities requires understanding of rational number at two levels, first to represent the quantities and second to recognize the ratios on which proportional reasoning depends. Intensive quantities can, when used appropriately, therefore provide a rich instructional context. Alternatively, Thompson and Thompson (1996) explained in detail that children first internalize the process of measuring a total distance traveled in units of speed-length. When children have internalized the measurement of a total distance in units of speed-length, they can anticipate that traveling a distance at some constant speed will produce an extensive quantity of time. This implies that children first conceive speed as a distance and time as a ratio. With this

anticipation they can reason about their image of completed motion, thinking about corresponding segmentations of accumulated distance and accumulated time. Then they will not only have a qualitative understanding of proportional reasoning (use of ratio as a unit, use of relative thinking, understanding some numerical relationships) but also a quantitative understanding (use of algebraic symbols to represent proportions with full understanding of functional and scalar relationships) (Lamon, 1993).

### Summary

In this chapter, I discussed the relationship between proportional reasoning and rate of change, reviewed literature on established approaches to teaching proportional reasoning (ratio, multiplicative reasoning, and intensive and extensive variables), and discussed problem types associated with each approach. In particular, the chapter was concerned with strategies students use to solve mathematical problems that require proportional reasoning, and what the strengths and limitations of these strategies might be. Howe et al. (2010) asserted that earlier concerns were expressed that intensive quantities might prove too challenging due to unfamiliarity and a “double dependence” upon rational numbers (p. 413). Quite apart from showing these concerns to be ill founded, the results hint of actual benefits from double dependence.

As seen in this chapter, scholars agree that as an approach to teaching proportional reasoning, fractions are inadequate. Rather, proportional reasoning is better conceptualized in terms of understanding ratios (Clark et al., 2003; Ben-Chaim et al., 2012). For example, a student may be able to solve a missing value problem by calculating that if  $x/2 = 2/4$ , then  $x =$

1. The student can recognize that the fraction  $\frac{1}{2}$  is equivalent to the fraction  $\frac{2}{4}$ . However, a missing value problem has no context: the student is presented with two ratios that are equal to each other, but no relationship is given. Students are supposed to find the variable rotely, by applying multiplication and division. The task is purely procedural. Simply calculating  $x$  in a missing value problem does not mean the student recognizes that the fractions represent proportional relationships or that the variables are proportional (Ben-Chaim et al., 2012; Modestou & Gagatsis, 2010). This gap in knowledge becomes problematic when students are asked to express fractions as ratios, for example, to understand the fraction  $\frac{1}{2}$  as representing the ratio .5 (Clark et al., 2003; Ben-Chaim et al., 2012). By understanding the proportional relationships inherent in ratios, students can come to recognize that two variables are proportional. This is an act of multiplicative reasoning, a crucial skill for a wide array of upper-level mathematics tasks (Thompson & Thompson, 1996; Thompson & Saldanda, 2003; Confrey & Maloney, 2010; Wilson et al., 2012).

One way to help students develop proportional reasoning comes from Lamon (1993; 2007), who suggested that students ought to be taught this concept in context. She identified four semantic tasks for practicing proportional reasoning: part-part-whole, associated sets, stretchers and shrinkers, and well-chunked measures. Lobato, Hohensee, and Rhodehamel (2012) agreed with Lamon (1993; 2007) that students must be taught to understand proportion in context, but they suggested that a simpler method for introducing proportion is as measurement, particularly the measurement of motion. By measuring time and distance and calculating their proportional relationships, students can learn to practice the

multiplicative reasoning aspect of proportional reasoning. In so doing, students learn the distinction between extensive variables (e.g., time and distance) and intensive variables (e.g., speed) (Lobato et al., 2012; Howe et al., 2010).

As I have shown, the intensive and extensive variables approach is the most promising approach to teaching proportional reasoning, as it can contain the other two approaches, which center on ratios and multiplicative reasoning. Students are expected to realize the existence of the relationship between two quantities, which indicates that each quantity will be an extensive variable. Then form an accurate ratio by keeping in mind that this ratio will produce a product. Consequently, the ratio of these two extensive variables will establish an intensive variable, which informs us about the intensity of the relationship of the two extensive variables. The use of intensive and extensive variables represents the most promising approach to integrating proportional reasoning into mathematics courses in the middle grades, and thus laying the best foundation for the learning of higher-level mathematics.

### CHAPTER 3: Slope and Rate

The importance of slope and rate is evident in the wide recognition that this concept has received from those responsible for developing nationwide curriculum. In the final report of the National Mathematics Advisory Panel (2008), established by President Bush, linear function, with an emphasis on slope, was identified as one of the major topics for school algebra, the Common Core State Standards for Mathematics (CCSSM), has similarly affirmed the importance of slope. Specifically, the CCSSM for middle-grades mathematics identify ratio, proportion, slope, and rate as central ideas in the curriculum (CCSSI, 2010). These ideas closely connect to themes in earlier grades (e.g., pattern-building, multiplicative reasoning, ratio, and rational number concepts) and are the foundation for understanding linear functions as well as many high-school mathematics and science topics. The concept of slope and rate is embedded throughout the nation's secondary mathematics curriculum, with the NCTM recognizing and describing multiple conceptualizations of slope in both its Principles and Standards for School Mathematics (NCTM, 2000) and more recently in its Curriculum Focal Points for Prekindergarten through Grade 12 Mathematics (NCTM, 2006).

In this chapter, I first focus on research about conceptualizations of slope, including a discussion of the close relationship between slope and rate. Then, I discuss several rate schemes present in the literature. It is necessary to discuss the rate schemes here in order to demonstrate the rationale for teaching slope as a rate. As Lobato et al. (2013) help to show, the most effective way to teach slope is as a rate of change (2013) Such an approach will prepare students for functions and derivatives, as slope is a functional property, an integral

part of learning functions, and a milestone for derivatives. Of course, it is possible for students to learn derivatives without knowing the slope or rate of change, but if the students have a firm and strong foundation of slope, they will have a robust understanding not only of function but also of derivative.

As with the concept of proportional reasoning, the importance of slope is matched only by the difficulty of understanding it. Research has documented various difficulties that students have with the concept of slope. There are misconceptions associated with the calculation of slope, and when it is represented in decimal form, students have trouble considering slope as a ratio (Stump, 2001, p. 81). Students also have difficulty interpreting linear functions, connecting graphs to linear equations, and connecting graphs to the notion of rate of change (Hattikudur, Prather, Asquith, Alibali, Knuth & Nathan, 2012; Planinic, Milin-Sipus, Katic, Susac, & Ivanjek, 2012; Stanton & Moore-Russo, 2012). Perhaps one reason for the difficulties written about—especially the difficulty in later mathematics with recognizing slope as rate of change—is that slope is presented very differently to students in their first encounter during the middle grades (pre-algebra) versus in calculus.

This chapter focuses on the concept of slope, specifically providing research about teaching slope as rate of change. The concept of slope is typically introduced in the middle grades only as the slope of a linear function. In learning slope, students exercise proportional reasoning and encounter a basic version of the concept of rate of change (Lobato & Siebert, 2002). This chapter considers how students' development of proportional reasoning in early

grades has implications for how they learn slope, which in turn has implications for students' ability to learn rate of change in upper-level mathematics courses.

Slope is important because an understanding of linearity and slope is tied to grasping other algebraic concepts like quadratic and exponential relationships, and it is considered a key prerequisite concept for calculus (Stroup, 2002). Slope has been called a “powerful linking concept” to help students understand functions and their graphs. Additionally, a fundamental understanding of the slope of a linear function (constant rate of change) is essential for students to develop the strong foundation needed for differentiation, a key concept in collegiate mathematics (Herbert & Pierce, 2010; Johnson, 2012; Lobato, Hohensee, & Rhodehamel, 2013; Stump, 2001; Stroup, 2002). Indeed, understanding the concept of slope is deemed to be one of the most important mathematical concepts that students encounter (Teuscher & Reys, 2010). Its importance lies in the fact that a myriad of functions are built upon it.

To understand the distinction between a students' ability to calculate slope and their having a conceptual understanding of slope, it is helpful to refer to Smith, Seshaiyer, Peixoto, Suh, Bagshaw, and Collins (2013), who explored how slope can be understood using steps. They found that as students worked with the idea of steepness, they developed a conceptual understanding of the rate of change and slope, which easily led into a discussion of formulas. Cheng (2013) proposed that a good way to help students with the difficulty of both proportionality and slope could be to integrate both of these concepts in this study of steepness, a physical property as the ratio of rise over run. They found that “there is a

relationship between the correctness of the participants' proportional reasoning and their steepness responses," and they called for "the development of a pre-algebraic curriculum, which introduces proportional reasoning, steepness, and slope as related concepts and focuses on students' abilities to translate between multiple representations of these concepts" (p. 39).

Furthermore, in one study, students were presented with time and distance graphs and asked to match the graphs with provided descriptions of motion. An analysis of the students' work indicated that they successfully completed this task, but despite the importance of such conceptual understanding, "mathematics teaching standards documents tend to emphasize views of slope as the ratio of rise over run; as an indicator of the increasing, decreasing, or constant behavior of a line; as determining whether two lines are parallel or perpendicular; as indicating the rate of change of a function; and as denoted by  $\text{Slope} = (y_2 - y_1)/(x_2 - x_1)$ " (Nagle & Moore-Russo, 2013, p. 275). Nagle (2013) claims that although each of these views is an important interpretation of slope, omitting its more physical interpretation as steepness may prevent students from linking the mathematical ideas with their everyday experiences.

The concept of slope is first introduced in middle-school mathematics, but it continues to be prominent throughout the secondary school mathematics curriculum, and it culminates with the derivative in calculus (Herbert & Pierce, 2010; Johnson, 2012; Stump, 2001; Stroup, 2002). In the middle grades, the first instruction in slope focuses on "ratio," and by the end of the ninth grade year, the concept of slope becomes more focused a "geometric ratio." By the end of high school, after completing pre-calculus and/or calculus

classes, the concept of slope is strongly associated with “rate” in students’ minds (Herbert & Pierce, 2011; Stanton & Moore-Russo, 2012; Stump, 2001; Thompson & Thompson, 1994). According to Hattikudur et al. (2012), the difficulty with slope begins when students move from working exclusively on linear functions to being exposed to nonlinear functions. The most common conceptualization of slope that students hold is that slope is a behavior indicator; this means that students conceptualize slope as a characteristic of a linear graph—a conceptualization that can lead to difficulties and must be changed, as this chapter argues.

According to longstanding curricula and state standards, when teaching slope to middle school students, the slope’s relationship to rate of change should not be emphasized (Stanton & Moore-Russo, 2012). This practice rests on the assumption that students are not ready to learn about rate of change until the upper grades. However, I contend that by not being taught slope as a rate of change in the early grades, students reach the level of calculus thinking that slope is only relevant to linear functions. They are thus not prepared for the requirement in calculus that they calculate the slope of a nonlinear function. Indeed, Nagle et al. (2013) found that in tertiary calculus classes, students and teachers think of slope differently; teachers with the conceptualization of slope as a rate of change can apply their understanding for nonlinear functions (i.e., using covariational reasoning), and students with only a procedural understanding of how to calculate slope only for linear functions.

Because the various interpretations of slope are emphasized at various learning stages in students’ lives, I believe that it is crucial to provide a framework to describe and guide how students learn and conceptualize slope successively from grades six to twelve. As Nagle

and Moore-Russo (2013) pointed out, “It is critical for teachers to provide opportunities for students to make connections with prior knowledge to form an integrated understanding of the concept” (p. 274). With this in mind, my intention in this chapter is to answer the following question: What does mathematics education research tell us about how students conceptualize slope from the early grades through calculus?

In addressing this question, I first establish several conceptualizations of slope in greater detail. Then, the bulk of the chapter is devoted to two sections, discussing the scholarship on slope as ratio and slope as rate. The section on slope as ratio focuses on slope in physical situations, such as measuring steepness, with attention to slope as an algebraic property (algebraic ratio, parametric coefficient, and linear constant) and as a geometrical property (geometric ratio, physical property, trigonometric conception, and determining property). In the section on slope as rate, I discuss slope in functional situations, as a measure of rate of change. Specifically, I discuss the functional property, calculus conceptions, real-world situations, and behavior indicators.

### **Conceptualizations of Slope**

Slope can be conceptualized as a ratio or as a rate. As Stump (2001) pointed out, “In both physical situations and functional situations, slope can be thought of as either a ratio or a rate, depending on the level of reflective abstraction” (p. 82). One way to understand the distinction between ratio and rate, according to Vergnaud (1983, 1988; as cited in Stump, 2001), is that “a ratio is a comparison between quantities of like nature, and a rate is a comparison of quantities of unlike nature” (p. 81). What this means in practical terms is that

slope is thought of as a ratio in physical situations (i.e., a measure of steepness), and is thought of as a rate in functional situations, as a measure of how two variables of unlike nature are related.

This section briefly introduces some slope classifications that scholars have offered. Students tend to lack deep understanding, instead displaying procedural understanding of how to calculate slope; but the various classifications and conceptualizations are important to recognize so that teachers can build on students' existing knowledge (ways of conceptualizing) and correct misconceptions where needed.

Stump (1999, 2001) was one of the pioneers to present a detailed classification of slope conceptualizations, offering seven categories: geometric ratio, algebraic ratio, physical property, functional property, parametric coefficient, trigonometric conception, and calculus conception. An eighth category, real-world representations, was included in her later work. Stump's (1999, 2001) slope classifications can be grouped into a measure of steepness (ratio) and a rate of change. By interviewing and assessing students in various ways to see how well they understood slope as ratio and as rate, in both physical and functional situations, the author developed the categorization. She found that students overall were better at understanding slope as a measure of rate of change rather than as a measure of steepness (ratio). She concluded, "The results of this investigation suggest a gap in students' understanding of slope as a measure of rate of change and imply that instruction should be focused on helping students form connections among rates involving time, rates involving other variables, and graphical representations of these relationships" (p. 87). Stump's (1999,

2001) classification of slope conceptualizations provides an important basis for other researchers' categorizations.

Later, Nagle et al. (2014) refined and extended this model from eight categories to eleven by adding determining property, behavior indicator, and linear constant. These categories used for conceptualizations of slope are briefly summarized in Table 2.

**Table 2. Slope conceptualization categories. (Adapted from Moore-Russo et al., 2011.)**

Category	Slope as . . .
Geometric ratio (G)	Rise over run, vertical displacement over horizontal displacement
Algebraic ratio (A)	Change in $y$ over change in $x$ , representation of ratio with algebraic expressions, $\frac{\Delta y}{\Delta x}$ or $\frac{y_2 - y_1}{x_2 - x_1}$
Physical property (P)	Property of line often described using expressions like "steepness" ("slant," "pitch," etc.), "how high up" or "it goes up"
Functional property (F)	(Constant) Rate of change between variables found in multiple representations including tables and verbal descriptions (e.g., when $x$ increases by 2, $y$ increases by 3)
Parametric coefficient (PC)	Coefficient $m$ in equation $y = mx + b$
Trigonometric conception (T)	Tangent of a line's angle of inclination Direction component of a vector
Calculus conception (C)	Related to the equation of a derivative; tangent line to a curve at a point
Real-world situation (R)	Static, physical situation (e.g., wheelchair ramp); dynamic, functional situation (e.g., distance vs. time)
Determining property (D)	Property that determines parallel and perpendicular lines
Behavior indicator (B)	Property with which a line can be determined, if you are also given a point Property that indicates increasing, decreasing, horizontal trends of line as well as the amount of increase/decrease of a line; property that, if nonzero, indicates line must intersect the $x$ -axis
Linear constant (L)	Property that is unaffected by translation Constant property independent of representation

Also, according to Moore-Russo et al. (2011), these categories are not discrete or mutually exclusive. The conceptualization of slope as a rate often requires thinking of slope as a functional property within a real-world situation.

Nagle et al. (2013) also presented a detailed look at 11 distinct conceptualizations of

slope, and this remains the most detailed version of conceptualized slope to date. Nagle et al.'s (2013) classification framework was based upon a survey in which they asked post-secondary students five questions: (1) What is slope? (2) List all the ways that slope can be represented. (3) How is slope used? (4) When is it appropriate to use slope? (5) Give 3 examples of slope. Nagle et al. (2013) found evidence that incoming tertiary students relied more on procedural use of slope rather than grasping some key conceptualizations of slope, meaning that even though students could calculate slope by following a procedure, and could interpret it by looking at a graph, they did not understand slope as a representation of covarying quantities or as having real-world applications to the measuring of intensive variables. This finding is significant because slope's covariational meaning and its physical applications lay the foundations for calculus, particularly the concept of the derivative.

Further, Nagle et al. (2013) found a mismatch between students' conceptualizations and those of the university instructors who taught them. The latter "demonstrate diverse, conceptual interpretations of slope but infrequently use students' most prevalent conceptualizations" (p. 1512). The findings were troubling on two counts: "that incoming tertiary students may not possess a solid foundation of covariational reasoning on which to build more advanced conceptualizations of slope" (p. 1512), and that instructors were not recognizing when or how their students' conceptualizations of slope were inadequate.

Stanton and Moore-Russo (2012) later conducted a study of state standards, finding the 11 non-discrete conceptualizations of slope (in Table 2 above) reflected across the standards. They noted that, "The most common conceptualization of slope was as a

geometric ratio (found in 45 states), followed closely by behavior indicator, determining property, functional property, and algebraic ratio conceptualizations of slope (each found in 43 states)” (p. 273). In few states did they find that state standards reflected trigonometric or calculus conceptualizations of slope or slope as a physical property. The findings suggested that “students might benefit if slope was addressed as a physical property and as a trigonometric conception related to slope being the tangent of a line’s angle of inclination” (p. 276). However, Stanton and Moore-Russo’s (2012) review of state standards did indicate “all of the slope conceptualization categories were evidenced in the standards documents, with the vast majority of state documents addressing five or more of the eleven conceptualizations” (p.270).

In the next section, I discuss those eleven conceptualizations first identified by Moore-Russo et al. (2011; see Table 1) and later elaborated by Nagle et al. (2013); however, I structure and synthesize the material in the following way. I chunk the 11 conceptualizations into two categories. The first category relates to slope as ratio and is subdivided into two sub-categories: slope as an algebraic property and slope as a geometric property. The second category represents the synthesis of geometric and algebraic slope conceptualizations: slope as rate. As I suggested earlier, the latter category of conceptualizations (slope as rate) is often skipped in mathematics courses in early grades, yet these slope-as-rate conceptualizations are important because they connect the algebraic and geometric properties of slope, and lend themselves to covariational reasoning, which is crucial to the notion of derivatives in higher-grade mathematics courses.

## **Slope as Ratio**

In this section, I discuss the conception of slope as ratio, which is common among students in middle school through early high school. Slope as ratio is a measure of steepness in physical situations. As Stump (2001) explained, “Real-world representations of slope exist in physical situations, such as mountain roads, ski slopes, and wheelchair ramps” (p. 81). Slope is fundamentally related to the idea of proportionality. However, when it is introduced to students in the middle grades, this relationship is not explored empirically, nor do curricula make the connection explicitly (Lobato & Thanheiser, 2002). Textbooks often describe slope as a measure of the direction and the steepness of a line. Slope is indeed related to the idea of steepness, a physical characteristic of a line, which can be determined visually using an angle or analytically using a proportion (Stump, 1999).

When Nagle et al. (2013) surveyed students’ and teachers’ views of slope, they found that students relied heavily on conceptualizations of slope as ratio, particularly “behavior indicator and parametric coefficient conceptualizations of slope, with algebraic ratio and geometric ratio in close proximity” (p. 1508). I borrow this distinction between algebraic and geometric conceptualizations of slope as ratio in order to subdivide the slope conceptualizations described below.

The algebraic conceptualizations are algebraic ratio, parametric coefficient, and linear constant. The geometric conceptualizations are geometric ratio, physical property, trigonometric conception, and determining property. The algebraic conceptualizations are used to find representations of slope that are numerical, while the geometric property

conceptualizations provide a context for the number in the form of a visualization that helps students to conceptualize slope as a physical property. Algebraic conceptualizations are the first learned; although they are more abstract and harder to visualize, they can be calculated with relatively little conceptual understanding by following the “rate as program” scheme described later in the chapter.

### *Algebraic Property of Slope*

Algebraic conceptualizations of slope as ratio can be used to calculate a numerical form of slope. This approach is very abstract but is learned first because it is a necessary foundation for understanding geometric visualizations of slope later on. Slope-as-ratio conceptualizations that have to do with algebraic properties are represented in formulas and equations. Having established the prominence of algebraic properties in understanding slope as ratio, I now briefly describe three examples of slope-as-ratio concepts that deal with slope as an algebraic property: algebraic ratio, parametric coefficient, and linear constant. Each of these approaches enables students to represent slope in numerical form.

The first algebraic property approach to slope-as-ratio is the algebraic ratio. The ratio representation of slope is “change in  $y$ ” over “change in  $x$ ” –for example,  $m = 4/1$ . Such a ratio can be used to represent slope as a measure of steepness. According to Nagle et al. (2013), when students and instructors were asked how slope is represented, “algebraic ratio” was one of the most common responses of both students and instructors. When asked how slope is used, “algebraic ratio” again was one of the most common responses from students, but not from instructors. Instructors’ top answers were different: instructors would rank

“functional property” first, “calculus conception” second, and “physical property” and “real-world situation” third (p. 1506). This finding suggests that although algebraic ratio is an important aspect of slope-as-ratio, students do not always place the concept in proper context for understanding slope more broadly.

The second algebraic approach to slope-as-ratio is the parametric coefficient ( $m$ ), a ratio used as a measure of steepness. According to a table adapted from Nagle et al. (2014), a parametric coefficient can be defined as, “The variable  $m$  (or its numeric value) found in  $y = mx + b$  and  $y_2 - y_1 = m(x_2 - x_1)$ ” (p. 1493). According to Nagle et al. (2013), when students and instructors were asked how slope is used, students cited “parametric coefficient” as one of the top two used, but instructors did not. Instead, instructors’ most-cited use was “functional property,” the second was “calculus conception,” and the third were “physical property” and “real-world situation” (p. 1504). This finding provides further evidence of a disconnect between instructors’ and students’ understanding of slope, with instructors using conceptions that required covariational reasoning, and students using conceptions that did not.

The third approach associated with slope as algebraic property, also enabling students to represent slope in numerical form, is the linear constant. When working with a line, any two points on the line can be used to derive the line’s slope. We can use any pair of points for the slope since it is constant. This approach is appropriate when discussing any linear (steady) phenomenon, but not appropriate when the phenomenon is nonlinear. In a table adapted from Nagle et al. (2014), the linear constant is defined as a “constant property

independent of representation; unaffected by translation of a line; reference to what makes a line ‘straight’ or the ‘straightness’ of a line” (p. 1507). The concept of linear constant is not often seen in upper-level mathematics because in fact slope is rarely if ever constant.

### *Geometric Property of Slope*

Whereas algebraic property conceptualizations of slope are used to find numerical representations of slope, geometric conceptualizations involve a visual representation of slope (i.e. a graph), enabling students to conceptualize slope as a physical property. Before hearing the term “slope” used in a mathematical sense, students have already gained every day, nonmathematical experiences of the idea of slope through encounters with steepness and inclines. For instance, children learn that sledding down a steep hill may be fun, but walking back up that same steep hill requires a great deal of effort. Other experiences, such as setting the level of incline on a treadmill or reading roadway signs that warn about steep grades, provide opportunities for students to explore the notion of slope.

In early grades, when students learn slope in terms of steepness, this directly connects with proportional reasoning. Measuring steepness requires the measurement of height and width; this introduces the rise over run concept, and hence, ratio. (See chapter 2.) The concept of steepness also helps students to connect slope to the context of real-life application, so that they can move from life to math, rather than from math to life.

The literature describes four ways of looking at the geometric properties of slope as ratio. Each provides a representation of steepness, a physical property: geometric ratio,

physical property, trigonometric conception, and determining property. All four approaches will be discussed in more detail below.

The first of the four geometric notions of slope as a ratio is the geometric ratio. Geometric ratio is a measure of steepness. Moore and Russo et al. (2011) define geometric ratio as “rise over run of a graph of a line; ratio of vertical displacement to horizontal displacement of a line’s graph” (p.1493). According to Nagle et al. (2013), when students and instructors were asked how slope is represented, both groups cited geometric ratio as one of the top three conceptualizations. Real world examples must provide opportunities to examine both physical and functional situations, and this will help students to understand slope as a measure of steepness and as a measure of rate of change.

The second geometric conceptualization of slope as ratio is another measure of steepness, physical property. Nagle et al. (2013) described this as a “property of a line often described using expressions like grade, incline, pitch, steepness, slant, tilt, and ‘how high a line goes up’ (p.1493). The authors found that when students were asked “what is slope?”, the answer “physical property” was the fifth most common answer, whereas for instructors, physical property was the second most popular answer. This result is concerning because students’ favoring of a behavior indicator over any other conceptualization suggests that students conceptualize slope as a characteristic of a linear graph. This can create problems when students encounter nonlinear functions in upper-level mathematics.

The third geometric conceptualization of slope as ratio is yet another measure of steepness: the trigonometric conception. Specifically, slope is represented as a tangent of an

angle. According to Nagle et al. (2013), this is defined as a “property related to the angle a line makes with a horizontal line; tangent of a line’s angle of inclination/decline; direction component of a vector” (p.1493). Nagle and Moore-Russo (2013) argue that introductory algebra students can strengthen their understanding of slope beyond the algebraic understanding of slope as “the rise-over-run ratio, or  $(y_2 - y_1)/(x_2 - x_1)$ ,” (p. 274) by linking algebraic concepts to trigonometric ones. Specifically, they found a way to get “students to make sense of interpretations of slope as the tangent of the angle of elevation of a line” (p. 277). This connection can then be deepened, laying the foundations for trigonometric reasoning. They point out, “Because previous research suggests that students naturally measure steepness in terms of angles, linking slope to angles provides a direct pathway by which to connect the notions of steepness and slope” (p. 277). It is important to introduce the trigonometric conception of slope because this will prepare a foundation for important future steps in the student’s mathematics education.

The final geometric conceptualization of slope as ratio is another measure of steepness: slope is a determining property of a line. According to Nagle et al. (2013), this refers to the determination of whether “lines are parallel or perpendicular; property can determine a line if a point on the line is also given” (p.1493). When students and instructors were asked how slope is used, students cited “determining property” as the top use, while instructors did not cite this use at all (p.1504). And when Nagle et al. (2013) asked students and instructors when it is appropriate to use slope, they found the following:

a larger percentage of students used behavior indicator and determining property conceptualizations than instructors, mostly to denote that slope helped (1) identify the increasing/decreasing behavior of lines and (2) determine if two lines were parallel or perpendicular. In fact, students were over three times more likely to show evidence of determining property conceptualizations and over twice as likely to show evidence of behavior indicator conceptualizations on at least one item as instructors. (p. 1505)

The findings discussed here show how differently students and teachers understand the uses of slope. In this section I have discussed these geometric properties of slope as ratio because they provide a transition into the next section, which discusses slope as rate. Slope as a geometric property plays an important role in the transition from understanding the physical situation of slope, which is ratio, to the functional situation of slope, which is rate (Johnson, 2012; Weber & Dorko, 2014).

### **Slope as Rate**

After approaching slope as ratio in the middle grades and early high school, at which time slope is chiefly understood as steepness in a physical context, it is to be hoped that students progress to understanding slope in functional (real-world) situations as a measure of rate of change. This section discusses the conception of slope that ought to develop in late high school, in honors pre-calculus or in calculus: slope as rate of change in functional situations. This is a synthesis of the two conceptualizations described above, the algebraic and geometric (slope as ratio) conceptualizations. According to Stump (2001), functional situations where slope appears include “distance versus time or quantity versus cost” (p. 81).

Again, slope is fundamentally related to the idea of proportionality, and if students do not learn this when it is introduced to them in the middle grades, difficulties will likely manifest in upper-level mathematics courses. Stump (2001) found that “students demonstrated a better understanding of slope as a measure of rate of change than as a measure of steepness,” consistent with the findings of other scholars (p. 87). Stump (2001) concluded that these results “suggest a gap in students' understanding of slope as a measure of rate of change and imply that instruction should be focused on helping students form connections among rates involving time, rates involving other variables, and graphical representations of these relationships” (p. 87). Along similar lines, Lobato et al. (2013), Weber and Dorko (2014), Confrey and Smith (1995), and Planinic et al. (2012) all defended the idea of teaching slope as rate of change from the start, rather than steepness.

In the previous section, I discussed two categories of slope-as-ratio conceptualizations: algebraic and geometric conceptualizations, which are used to find representations of slope that are numerical and visual (or physical), respectively. The slope conceptualizations discussed in this section represent a bringing-together of the algebraic and geometric conceptualizations. In conceptualizations of slope as ratio, the algebraic and geometric properties work together like two hands; in conceptualizations of slope as rate, those two hands work together like a unit. According to Stanton and Moore-Russo (2012), instructors do not currently do a good job of marrying the algebraic and geometric properties of slope as ratio; however, I maintain that such a synthesis is important.

Stump (2001) in a study of pre-calculus students' understanding of slope as a measure, found that students described slope in language that reflected "various levels of mathematical maturity," including language from everyday experience ("describing slope as steepness or an angle"), or language reflecting formal instruction in mathematical understanding ("citing a formula for slope or using the term 'rise over run'") (p. 82). The results suggested that even students whose responses reflected formal instruction might not have a mathematically mature understanding of slope or a strong conceptualization of slope as ratio or rate. Some students "cited formulas that were unrelated to slope, and others said that they could not remember how to find slope, so although it was evident that they had been exposed to formal instruction, they had not yet constructed a useful notion of slope." The students who had best "integrated their experience with formal instruction into a more useful notion of slope as measure of steepness or rate of change," were those "who used correct formulas or mentioned the term 'rise over run'"—thus exhibiting more progress toward internalizing the notion of slope as a ratio or a rate (p. 87). These results show the difficulty that students might have in transitioning to recognizing slope as a geometric ratio and eventually as a measure of rate of change.

This section of the chapter provides justification for teaching slope as rate of change. Evidence suggests that teaching slope as rate of change, rather than teaching steepness first, is more effective for students. Hattikudur et al. (2012) indicated that the problem with slope begins when students encounter nonlinear functions. Ellis et al. (2008) suggests that instructors focus on "the quantitative relationships that can be represented by both linear

functions and quadratic functions,” and help students understand “the ways in which quadratic growth differs from linear growth” (p. 299). This approach would better prepare students “to examine functional relationships that involve non-constant change” (p. 295).

As more support for using slope as rate of change, according to Lobato et al. (2013), when teachers developed lessons to help seventh graders understand slope as a measure of rate of change, it was found that the students whose teacher focused on the intensive variable speed (a rate of change) demonstrated the ability to notice and use “a composed unit of distance of time” (p.14). Thus, Lobato et al.’s (2013) findings support the idea that teaching slope as rate of change is more effective for students than teaching steepness.

I next discuss three approaches to measuring slope as a rate of change: as a calculus conception, real-world situation, and behavior indicator. These three approaches are described roughly in the order in which students will ideally encounter them.

Students first encounter slope as a functional property, at which time they may merely memorize a procedure, rather than develop a conceptual understanding. They are expected to learn how fast one thing changes as something else changes, so this is a good opportunity to introduce the notion of rate of change. Once students have encountered slope as a functional property, they can learn the calculus conception of slope. Nagle et al. (2013) defined the calculus conception of rate of change as follows: “Limit; derivative; a measure of instantaneous rate of change for any functions; tangent line to a curve at a point” (p. 1505). More about this will be discussed in later chapters.

The calculus conception of slope paves the way for understanding slope as rate of change in real-world situations, where rates of change are not constant. Nagle et al. (2013) described the real-world conception of slope as ratio as encompassing any static, physical or dynamic, functional situation (e.g. wheelchair ramp, distance versus time). Stump (2001) investigated students' understanding of slope (as ratio and rate of change) in such real-world situations and found that, while these can help students gain better understanding of slope, sufficient emphasis is not always given to teaching slope in functional situations, as a rate of change.

Nagle et al. (2013) found that when students and instructors were asked how slope is used, only instructors ranked "real-world situation" as one of the top three uses. They concluded, "Students' infrequent use of real world situations as examples of how slope is used is of some concern particularly in light of its heavy use by instructors" (p. 1506). Thus, it is possible that they do not mentally associate the real-world contexts with the mathematical concept of slope. This corroborates Planinic et al.'s (2012) finding that physics situations are harder for students to understand than pure mathematics situations.

Finally, once students have developed some understanding of slope as a rate in terms of functional properties, calculus conceptions, and real-world situations, they can learn a fourth conception of slope as a measure of rate of change: slope as a behavior indicator. As explained by Nagle, Moore-Russo, Viglietti, and Martin (2013), a behavior indicator is a "property that indicates increasing/decreasing/horizontal trends of a line or amount of increase or decrease; if nonzero, indicates intersection with  $x$ -axis" (p. 1500). In addition to

giving us the roots of the function, the “trends of the line” will also give us information about the function’s behavior as an increasing/ decreasing phenomenon and about its concavity, and will give evidence of its extreme values. In pre-algebra and algebra courses, slope will provide support for a function’s behavior and shape, and for its extreme values concepts encountered in later courses.

The multitude of ways to conceptualize slope suggests that different sets of standards might promote the development of varying ways that individuals come to understand slope. Many researchers have found that, when high school students are asked to explain slope, they may offer responses such as these: “rise over run,” “change in y over change in x,” “indicator of the increasing or decreasing behavior of a line,” “m,” “the derivative,” or “the rate of change of a function” (Stump 1999, 2001; Nagle et al., 2013; Planinic, Milin-Sipus, Katic, Susac & Ivanjek, 2012). As a result of its prominence and diversity across the curriculum, slope is particularly susceptible to changes in curricular reform also.

In the last section of this chapter, I offer different rate schemes, which can be used to teach rate in ways that may connect to and enhance the teaching of slope. There is good evidence that students have robust understanding of slope in functional situations, thus slope as rate (Lobato et. al, 2013; Johnson, 2012; Carlson et. al, 2002). Consequently, I end with one particular scheme that I consider most effective. The four rate schemes I describe are rate as a program, rate as a property of an object, rate as static comparisons of changes in a variable, and finally the scheme that I argue is most sound, rate as a measure of covariation. I

contend that this final rate scheme is most effective in helping students to develop an understanding of slope as a rate of change.

### **Rate of Change Schemes**

In the early grades (K–7), students are only introduced to linear functions; at that point, the rate of change is seen as the slope of a linear function (see earlier discussion). However, when students start to learn the polynomial, exponential functions and other necessary functions to model the real-life phenomena, they need to understand a broader concept of slope, which is rate of change. Thus, I argue that slope must be taught as rate of change.

Building on the discussion of slope as ratio and slope as rate in the previous sections, here I discuss four distinct rate schemes supported by the research. A scheme is by definition repeatable and generalizable in action. These rate schemes are important to consider in a literature review about rate of change for two reasons. First, while teachers may not explicitly mention them in the classroom, teachers may have specific rate schemes in mind as they teach and use these schemes to structure their thinking and guide their approaches to teaching. Secondly, the first three schemes I discuss—rate as program, rate as property of an object, and rate as static comparisons of changes in variable—are simply procedural techniques which students can learn and exercise without truly understanding the slope as a rate of change for any function. The fourth rate scheme I discuss, rate as a measure of covariation, is arguably the best scheme to use for that purpose.

The four rate schemes here are described in the order of progression in which students most logically encounter them. As I will show, the first three rate schemes are contained within the fourth, which I argue is the most effective rate scheme for learning slope as a rate of change. In the traditional curriculum structure through the middle grades mathematics sequence, students' learning progresses from proportional reasoning to ratio and finally to rate of change. In understanding slope as ratio, angle and steepness will only enable students to make the connection in linear functions; however, in using rate of change, students can better understand slope conceptually.

For this description, I draw on Weber and Dorko (2014), who "characterized students' and experts' schemes for rate of change and how they represent those meanings using graphical, symbolic, tabular contexts" (p. 15). It is important to note that their work specifically focused on college-level calculus students and mathematicians; the researchers examined how they understood rate of change and represented it across multiple contexts. The researchers were particularly interested in whether "students develop a single scheme for a concept specifically, rate of change and conceive of that scheme applied in multiple mediums, or if students have uncoordinated schemes for a concept that depend on the representation in which the student is operating" (p. 15). They were also interested in whether students and experts differed in the schemes they used; they did indeed find important differences.

The four schemes for rate of change that Weber and Dorko (2014) found were as follows: rate as "a program that executes on a given function and returns a value"; as "the

property of an object (graph, algebraically defined function) where the object possesses the rate”; as a “static comparison of changes in variables,” that is, “a quotient structure, composed of changes in variables, that describes a graph”; and finally, as a “measure of covariation,” specifically “the simultaneous variation of two or more variables as they vary” (p. 21). Each of these will now be described in more detail, in order to consider how students best understand rates, how rates should look in students’ minds, and how instructors can harness an understanding of students’ rate schemes. This discussion contributes to our understanding of what ideas are important to consider in teaching rate of change.

### **Rate as a Program**

The first rate scheme is rate as a program (rule). According to Weber and Dorko (2014), measuring and interpreting rate of change in this scheme “requires executing the correct program (rule) such as the quotient rule or product rule. The measurement does not entail considering changes in variables” (p. 16). The problem with this rate scheme is that students can execute certain algebraic rules without grasping the main idea(s) about rate of change, or considering changes in variables. The “program” often requires an algebraic definition (notation) of function, and the student primarily demonstrates only symbolic understandings and actions. For example, in learning slope as rise over run, no meaning is attached to it. According to Weber and Dorko (2014), “Students may think of that number as merely the result of a rule (rate as a program). There is thus no conception of measurement attached to the value, giving the student nothing upon which to base any sort of interpretation” (p. 17).

### **Rate as a Property of an Object**

The second rate scheme, which is linked to geometric property and thus allows for some visualization, is rate as a property of an object. This is again a mathematical skill that can be learned and deployed without deep mathematical understanding of rate of change. Weber and Dorko (2014), in describing the measurement and interpretation of rate of change in this scheme, noted that it “can be *directly* measured, like a weight” (p. 21); as an example, they noted that “the graph of a function possesses a rate and ‘taking the derivative’ extracts that information” (p. 21). In chapter 5, as I will be explaining in Zandieh’s (2000) framework, this is the physical concept of derivative, like velocity as a function of time.

### **Rate as Static Comparisons of Changes in Variable**

The third mathematical procedural rate scheme is one that allows the student to make comparisons: rate as static comparisons of changes in variable. According to Weber and Dorko (2014), the measurement and interpretation of rate of change in this scheme “requires finding changes in  $x$  and  $y$  (or other variables) and putting those changes into a quotient structure (e.g.  $[\Delta y / \Delta x]$ ). However, the resulting measurement does not entail the variables having covaried” (p. 21). Actions and understandings associated with this scheme include “slope, slant, rise over run, [and] steepness” (Weber & Dorko, 2014, p. 21). Johnson (2013) explains this as change independent quantitative reasoning. It does have limitations since, it involves making comparisons between static amounts of change in quantities, determining in each interval when one quantity will be increasing by a greater, lesser or the same amount of

as another quantity. Thus this scheme falls short on a more fully of dynamic aspect of the changing quantities.

### **Rate as a Measure of Covariation**

Having mastered the first three rate schemes described above, students are ready for the fourth and most compelling: rate as a measure of covariation. Thinking of rate as a function, as discussed above, enables us to see it as a measure of covariation, and thus to calculate slope in real-world situations and apply that learning to physics, biology, and other sciences. This scheme gives students meaning to attach to the concepts, and encapsulates the very purpose of calculus, which deals with changes in the rate of change. According to Weber and Dorko (2014), measurement and interpretation of ROC in this scheme “requires finding changes in variables and comparing them multiplicatively to determine how many times as large a change is with respect to another” (p. 21). Actions and understandings associated with this scheme include “difference quotient with a limit, tangent line with changes in variables illustrated, interpret  $f(x)$  as representing a unit rate” (Weber & Dorko, 2014, p.21). In Weber and Dorko’s (2014) study referred to throughout this section, a major finding was that students did not tend to use this rate scheme with the frequency that instructors did:

Students’ schemes for rate were often disconnected from variables and variation, while those ideas were central to mathematicians’ schemes. For instance, our codes indicate mathematicians’ attention was often on describing the covariation of variables as they varied (see ‘how fast’ and ‘moment’ codes). In contrast, students’

attention was on describing a property of a function using its graph or algebraic form (see ‘slope’, ‘slant’ and ‘property’ codes). Thus, even though students may have attended to variables and their changes, the majority of students did not coordinate the variables’ changes in their schemes for rate of change. (p. 23)

From this, Weber and Dorko (2014) concluded that students and instructors differ “in attention to variables and measures of their covariation in schemes for rate of change” (p. 25).

Instructors had more flexible schemes that enabled them to translate the concept of rate across contexts, whereas students were less flexible and adaptable in their rate schemes. Indeed, Nagle et al. (2013) also found evidence that students have weak covariational reasoning with respect to slope: “The use of functional property by less than one fifth of all students is significant since this conceptualization involves interpreting slope as a relationship between two covarying quantities” (p. 1502). The implication is that since covariational reasoning has been described as a critical concept for pre-calculus instruction, its absence among calculus students’ definitions of slope is concerning (Carlson et al., 2002; Confrey & Smith, 1995).

To summarize the rate of change schemes, (1) rate as a program is associated with primarily symbolic actions, like executing on a given function that returns a value; (2) rate as a property of an object is a direct measurement, which answers the question of “how much”; (3) rate as a static comparison of changes in variables involves finding the slope by communicating rise over run; and (4) rate as a measure of covariation is the simultaneous

measurement of the variation of two or more variables as they vary.

In light of the above discussion, it is clear that the most important rate scheme for students to develop in preparation for calculus is rate as a measure of covariation. This scheme provides a solid conceptual base for students, and is far more important than an easily learned and mechanically applied mathematical skill. Johnson (2013) defends this as change-dependent reasoning, qualitative reasoning which measures the intensity of the relationship of the quantities. It is a multiplicative relationship between changing quantities, which are extensive variables.

### **Summary**

This chapter has focused on the concept of slope because it is an important idea to consider in teaching rate of change. By learning slope as a rate and as a ratio, students can ultimately come to understand rate as a measure of covariation, which is a foundational understanding for calculus. Research has shown that the concept of slope can represent a point of difficulty for students in learning rate of change. Research has shown that students' knowledge of slope does not transfer between problem types, or from a mathematical to a real-world setting; students do not tend to relate slope and rate of change (Hattikudur, Prather, Asquith, Knuth, Nathan & Alibali, 2012; Planinic et al., 2012; Stump, 2001b; Lobato & Siebert, 2002; Lobato & Thanheiser, 2002; Teuscher & Reys, 2010). As this thesis shows, researchers have taken a great interest in students' ability to reason about slope and rate, and for good reason. In the next chapter, I establish the next important idea to consider in teaching rate of change: covariational reasoning, the ability to reason about the relationship

between two variables that change together. An understanding of covariation is another component of a sophisticated understanding of rate of change.

## CHAPTER 4: Covariational Reasoning

Researchers and practitioners have debated whether in the study of rate of change it is more important to focus on the question “how much?” or the question “how fast?” Taking the former approach and focusing on “how much?,” which means finding rate by forming a ratio of two quantities, is considered the “ratio-based reasoning” perspective on rate, while the latter approach of answering “how fast?” is considered the “non-ratio-based reasoning” approach to rate. One researcher, Stroup (2002), suggests that, rather than first answering the question by finding a ratio of “how much?” students might first answer the question of “how fast?” (Stroup, 2002).

Connected to this idea is that of rate of change as a proportional relationship between two extensive variables. Likewise, the difference between, or measurement of two independent variables is also an extensive variable. When we would like to know the proportional relationship between these two extensive variables, then the newly formed measurement, the final product, is an intensive variable, which is rate of change. In the paragraph above, “how fast” would be answered by an intensive variable, and “how much” would be answered as the ration of extensive variables. To illustrate, consider that we can identify two different points in time and state the difference between them in the form of a measurement, say, an elapsed time of 2 hours. Similarly, we can talk about two different points in space and find the difference between them, say, a distance of 40 miles. Two hours and 40 miles, as extensive variables, are two separate identities. But when I relate two extensive variables to one another, as when I say “20 miles per hour,” I construct an intensive

variable. An intensive variable expresses information about the proportional relationship between time and displacement. In this example, 20 miles per hour is the slope of the line if the movement is on a linear path. If the movement is not linear, but curved, we will observe 20 miles per hour as a magnified slope of the path. This notion of extensive and intensive variables supports the idea of covariation of variables.

This chapter considers how rate of change can be understood as the simultaneous and related changes in two quantities (extensive variables), that is, covariation. The simultaneous change in two coordinated quantities is a dynamic process (Carlson et al., 2002). To track the value of either quantity, with the recognition that the other quantity also has a value at every moment in time, is referred to as “covariational reasoning.” This type of reasoning was developed by researchers in the mid-1990s and became prevalent around the year 2000; since then, it has been used to help students simultaneously conceptualize the idea of rate of change and capitalize on the understanding of functions as involving the formation of links between values in a function's domain and range.

The concept of covariation is important in mathematics because it explains variations in rate of change and also unites the two (sometimes believed to be separate) concepts of slope and the first derivative of the function. This is because when students use covariational reasoning in their initial learning of rate of change in linear functions, students gain the necessary foundation for understanding varying rates of change within nonlinear functions, an important idea in the derivative.

Because change is constant in linear functions, the variables will either be directly or inversely proportional. However, in nonlinear functions, when two quantities change together nonlinearly, how can students identify and reason about the changes in quantities with respect to each other? According to Carlson et al. (2002), the quantities may be, for example, increasingly increasing or decreasingly increasing. Covariational reasoning is a set of mental actions that can enable the student to investigate and label the change in changing relationships. The concept of covariational reasoning enables us to see rate of change as a slope formula that not only belongs to linear functions, but also to nonlinear ones. Through covariational reasoning, we can express change in a dependent variable that results from a unit change in the independent variable. Rate can be variously described as constant or variable, and as average or instantaneous (Herbert & Pierce, 2011).

Mathematics education researchers (Confrey & Smith, 1994; Confrey & Smith, 1995; Thompson, 1994; Engelke & Kimani, 2012; Johnson, 2012) have proposed that students' development of covariational reasoning is imperative for understanding functions and calculus. Given the importance of covariational reasoning to upper-level mathematics, there is a need to develop students' covariational reasoning abilities and their understanding of functions prior to and during their study of calculus.

In this chapter, I address the question, "What have researchers and practitioners found to be the ways students develop and use covariational reasoning?" Specifically, I discuss research on covariational reasoning and on types of reasoning that have been found to exist when covariation is investigated: transformational reasoning, qualitative reasoning and

quantitative reasoning. Transformational reasoning is “the physical or mental enactment of an operation or set of operations on an object or set of objects that allows one to envision the transformations that these objects undergo and the set of results of these operations” (Simon, 1996, p. 201). Qualitative and quantitative reasoning are two distinct ways of reasoning about how two quantities change together (covary). As I will show, transformational reasoning, qualitative reasoning, and quantitative reasoning are all integral to students’ development and use of covariational reasoning.

Although many researchers have found covariational thinking to be helpful for developing students’ initial understanding of rate of change (Carlson et al., 2002; Thompson, 1994; Stroup, 2002; Herbert & Pierce, 2010; Johnson, 2012), some have expressed differing opinions and findings as to the optimal sequencing of students’ usage of covariational reasoning. This chapter discusses two approaches to covariational reasoning: (a) the coordination of quantities and (b) quantitative and qualitative reasoning.

### **Coordination of Quantities and Transformational Reasoning**

The coordination of quantities may begin by building a table from a function. The process of a student entering values of the function into a table is often intertwined with the construction of the variable, and basic to this process is the examination and systematization of the data values. Systematically selecting data values, ordering one’s data, and examining the patterns are all a part of this process. Through this approach, students build their image of a function as the coordination of two data columns. In other words, not only do students

describe a pattern of the values within a single column; they also can coordinate the values in two different columns in order to answer questions concerning the situation.

The covariation approach described here offers an alternative to the correspondence approach, which dominated the curriculum before the advent of CCSS-M. A correspondence approach is based on an abstract but rather narrow definition of a function as a relation between two sets, the domain and range. By contrast, the covariational approach to functions enables students to express the relationship between any two structured worlds algebraically, graphically, or in real-world situations where they are embedded.

The coordination of quantities is a foundation of covariational reasoning. In Thompson's (1994) study, he presented a similar, though not identical, framework for explaining the mature image of rate in three steps: image of change in some quantity, the coordination of images of two quantities, and finally the formation of an image of simultaneous covariation of two quantities. Thompson (1994) chose these phases to parallel Piaget's (1970) three-stage theory about children's mental operations that were involved in functional thinking about variation. Similarly, Confrey and Smith (1995) explained a covariational approach to creating and conceptualizing functions as involving the formation of links between values in a function's domain and range. In the case of tables, this approach involves coordinating the variation in two or more columns as one moves up and down the table (Confrey & Smith, 1994). For Confrey and Smith (1995) as well as Thompson (1994), coordinating is considered essential to covariational reasoning about dynamic function relationships.

Carlson et al.'s (2002) extended definition of covariational reasoning is based on the coordination of varying quantities. Specifically, Carlson et al. (2002) defined covariational reasoning as the set of cognitive activities involved in coordinating two varying quantities that change in relation to each other. The authors developed a detailed characterization of covariational reasoning in terms of the mental images that support the mental actions of coordinating the following: (i) changes in one variable with change in the other variable; (ii) the direction of change in one variable with changes in the other variable; (iii) the amount of change of one variable with changes in the other variable; (iv) the average rate-of-change of a function with changes of the independent variable; and (v) the instantaneous rate-of-change of the function with continuous changes in the input variable. These five sequential mental actions result in five different corresponding behaviors that produce five distinct developmental levels, called mental actions, of covariational reasoning ability.

The three covariation tasks in Carlson et al.'s (2002) work were the bottle problem, the temperature problem, and the ladder problem. The *bottle problem* entails giving students the task of constructing a graph of a dynamic situation to present the height as a function of volume as the bottle being filled at a constant rate. The *temperature problem* presents students with a rate of change of temperature versus time and then asks students to construct the corresponding approximate temperature-versus-time graph. The *ladder problem* prompts students to describe the speed of the top of the ladder as it slides down the wall at a constant rate. All three of these problems were used to assess students' covariational reasoning with respect to the coordination of variables.

Results of Carlson et al.'s (2002) study showed that students were consistently able to apply covariational reasoning in coordinating the quantities; that is, they successfully reasoned about changes in the direction of the dependent variable in tandem with an imagined change of the independent variable, and they could also reason about the amount of change. However, students were not observed to consistently coordinate changes in the average rate of change with constant changes in the independent variable for a function's domain. The students also had difficulty in applying their reasoning skills in coordinating the instantaneous rate of change with continuous changes in the independent variable. Finally, students had difficulty in explaining the reasoning of a curve being smooth, and in demonstrating awareness of an inflection point as being where the rate of change changes from increasing to decreasing, or from decreasing to increasing. In short, students showed weakness in their attempts to bring meaning to their constructions and graphical interpretations.

In a follow-up article to Carlson et al (2002), Johnson (2012) claimed that students with no knowledge of calculus may complete these same or similar tasks if they possessed strong covariational reasoning ability. She modified Carlson et al.'s (2002) bottle task so that instead of asking students to construct a graph of height as a function of volume, she instead asked students to construct a graph of volume as a function of height. Johnson (2012) thought that if the question were posed in this way, high school students might be less likely to treat height as if it were time.

To conduct her study, Johnson (2012) tested students who had completed at least one year of algebra, and none of them had yet been enrolled in a calculus course. Three tasks were used to support combinations of covariational reasoning: filling bottles, the changing square, and the typical high temperature. This set of tasks “incorporates dynamic and static representations of covarying quantities, with quantities varying at constant and varying rates” (Johnson, 2012, p. 317–318). The way in which a student conceives change in quantities may affect how that student makes sense of relationships that are represented by a graph (Saldanha & Thompson, 1998). A covariational interpretation of a graph, in contrast to an iconic interpretation, would involve considering a graph as representing a relationship between quantities that can change together (e.g., Carlson et al., 2002). The goal was to understand the students’ reasoning about quantities involved in constant rates of increase that varied at varying rates. However, only one of the students in Johnson’s (2012) study demonstrated reasoning that remained consistent for all three tasks. This student’s way of reasoning seemed compatible with covariational reasoning. In other words, she could answer the question, at what rate is the rate of change changing? Johnson (2012) reported that the student’s systematic variation could be discrete or continuous; and the quantity indicating the relationship between the two variables (the independent and dependent variables about which she was reasoning covariationally) could be ratio-based or non-ratio-based.

Johnson (2012) explained three reasons why this reasoning capacity is mathematically powerful: (a) it enables the student to determine variations not just in the direction of change, but also the intensity, a skill that is central to the calculus concept of

second derivative; (b) it provides a way for covariational reasoning to be integrated with transformational reasoning; and (c) it enables the student to coordinate “chunky” and “smooth” covariational reasoning in rate-of-change scenarios before being able to use ratio-based reasoning. The results of Johnson’s (2012) study suggest that specific instruction emphasizing covariational reasoning and connecting rate in a motion context to other rates is necessary in order to capitalize on students’ prior understandings of rate as speed.

Approaches to the teaching and learning of functions needs to include frequent and strong connections between alternative representations of functions with explicit emphasis on rate, thereby engaging students’ covariational reasoning activities.

Various researchers (Carlson et al., 2002; Johnson, 2014; Thompson, 1994; Stroup, 2002; Confrey & Smith, 1995) have asserted that conceptually coordinating the change in one variable with change in the other variable, while attending to how the variables change in relation to each other, involves a mental enactment of the operation of coordinating two objects. Simon (1996) explained that the mental actions involved in applying covariational reasoning in a dynamic process are characteristic of transformational reasoning. He defined the latter as “the mental enactment of an operation or set of operations on an object or set of objects that allows one to envision the transformations that these objects undergo and the set of results of these operations” (Simon, 1996, p. 201). Another way to define transformational reasoning is as a cognitive approach performed by students during the investigations of modeling dynamic events, by reasoning in a qualitative dynamic state. As Simon (1996) explained, “Central to transformational reasoning is the ability to consider, not a static state,

but a dynamic process by which a new state or continuum of states are regenerated” (p. 201). This definition will help the learner to understand and make the connections and comparisons between two concepts; the derivative at a point and the derivative of a function in the latter chapter.

Other researchers have observed that students’ transformational reasoning about rate of change is aided by the use of technology. Students can use classroom technology such as Geometer’s Sketchpad (GSP), Motion Detectors, and Fathom to model or envision the dynamic events of the given tasks or problems. Educators today have a plethora of technologies that can enable the study of dynamic real-time events, including motion detectors, microcomputer-based laboratories (MBL), computer algebra systems (CAS), geometry software, graphing calculators, and specialty software. These technologies, when properly grounded and orchestrated as collaborative inquiry by teachers’ questions, offer valuable tools for students learning to apply transformational reasoning to the analysis and interpretation of dynamic function situations.

As mentioned above, transformational and covariational reasoning are closely related. Research suggests that the two conceptualizations can be coordinated in a coherent way to provide an empirically based model for learning. Systematic variation in the independent variable allows the student to mentally run through the process as variation in the intensity of the change, while students’ reasoning also involves the coordination of both varying independent and dependent variables. In the ladder problem in Carlson et al.’s (2002) study, described above, the students make sense of how a mathematical system works, finding their

own solutions as mental or physical enactments of an operation, and this allows them to envision the transformations that these objects undergo and the set of results for these operations. In this dynamic process of transformational reasoning, the key components are objects, operations, and results. The mental or physical enactments involved in covariational reasoning in a dynamic way are in fact the characteristics of transformational reasoning (Johnson, 2012).

### **Quantitative and Qualitative Reasoning**

When students reason about quantities that change together, the way in which the students relate the changes in the quantities matters. In the tasks discussed below, students dealt with multiple representations of covarying quantities, and with their rates of change, by using both numerical (using actual number values) and non-numerical reasoning. These two styles of reasoning can be referred to as quantitative and qualitative reasoning, respectively. Quantitative reasoning (*simultaneous-independent reasoning*) involves making comparisons between static amounts of change in quantities, determining in each interval when one quantity will increase by a greater, lesser, or identical amount to another quantity. In contrast, qualitative reasoning (*change-dependent reasoning*) involves variation in how one quantity changes in relation to another. Although quantitative reasoning is useful, it does have limitations. Qualitative reasoning centers more fully on the dynamic aspect of the changing quantities.

Johnson's (2012) study indicates the importance of qualitative reasoning for students who are reasoning about covariation. Students did not need numerical amounts of change in

order to reason about the intensity of change; they indicated a high-level understanding of the (qualitative) relationship between two covarying quantities, which may provide a solid foundation for the important calculus concept of the second derivative. Johnson's (2012) study was conducted with six students from a small rural-district high school who had completed at least one year of algebra and were not currently enrolled in a calculus course. Each student performed three tasks that incorporated both covariational and transformational reasoning: the changing square task, the typical high-temperature task, and the filling bottles task.

The first task was the *changing square task*, which required students to compare the variations between two quantities, each covarying with respect to a third quantity. In this task, which was adapted from Lamon (2007), students investigated and compared the change in the area and perimeter of a square as each covaried with the side length of the square. Johnson's (2012) goal for the changing square task was to better understand students' comparison of variation between quantities involved in a constant rate of change and quantities involved in a constantly varying rate of change. The second task, the *typical high temperature task*, required students to investigate the ways in which a city's typical high temperature increased and decreased over the course of a year. The goal of this task was to better understanding students' reasoning about quantities involved in rates of change that varied at varying rates.

The third task, the *filling bottles task* (modified from Carlson et al, 2002), required students to investigate how the volume of liquid in a bottle would change as the height of the

liquid in the bottle increased, given that liquid was being dispensed into the bottle at a constant rate. Students were given a static, Cartesian graph representing the volume of liquid in a bottle as a function of the height of liquid in the bottle. In this task, some students made comparisons to interpret a graph involving changing quantities, while others coordinated the changes as they were interpreting the graph.

Johnson's (2012) results suggest that when some students interpret a graph involving changing quantities, they compare amounts of change, which could be between the magnitudes of the amounts. In this case, the quantity with the greater (or lesser) amount of change would be increasing more (or less) in that interval. Johnson (2012) called the ability to interpret this change simultaneous-independent reasoning, synonymous to quantitative reasoning discussed earlier since the comparisons that students make could be between the magnitudes of the amounts. In this case, the quantity with the greater (or lesser) amount of change would be increasing more (or less) in that interval. Students make these comparisons as if each quantity were changing independently of the other in relation to time. Related to this, Thompson (1994) used the term "extensive quantification" to refer to a ratio-based conception of quantity arising in relation to "operations of unitizing or segmenting" (p. 185).

Other students demonstrated reasoning by coordinating change in one quantity with change in another quantity. This process could involve imagining a point on a graph moving from left to right in conjunction with a point on the horizontal axis moving from left to right. Johnson (2012) called this skill of coordinating change in one quantity with change in another quantity change-dependent reasoning, synonymous to qualitative reasoning. As the

quantity represented on the horizontal axis increases, the quantity represented on the vertical axis might increase or decrease. In this second type of reasoning, there is a reversible form of embodied “whole-ness,” a dynamic structure that develops in relation to learners’ activities; this is what qualitative reasoning names the wholeness felt in common for the analyses of slope, ratio, or proportion-based forms of reasoning. This slope-, ratio- or proportion-related construct rests on a foundation of intensiveness (the notion of a rate as an intensive variable) and reversibility.

Quantitative and qualitative reasoning can be seen to significantly overlap. This overlap will be more apparent if rate-related reasoning can build on a foundation of intensification and reversibility by starting with complexity (situations where rate varies) and including constant change in the mix of possibilities. Exploring contexts where rate varies allows learners to construct an intensive sense of “how fast” that can exist in reversible relation to “how much” ideas.

The use of multiple representations has been encouraged for investigating covariational reasoning from both qualitative and quantitative perspectives. Covariational interpretations of graphs can take different forms, including chunky and smooth images of change (Castillo-Garsow, 2012). A chunky image of change would involve envisioning completed amounts of change (e.g., area changed by  $a$  units and height changed by  $b$  units), and a smooth image of change would involve envisioning continuing change (e.g., area is changing while height is changing). Thus, the covariational reasoning might happen in a “chunky” manner as the student envisions the variations in discrete chunks, or it could be

“smooth” when the student envisions the variations as occurring thorough a continuing process. How would the student sketch the shape of the bottle that the graph could represent?

We can also understand how the student coordinates “chunky” and “smooth” covariational reasoning about quantities related to rate of change prior to engaging in ratio-based reasoning. Smooth covariational reasoning will establish the fundamental of covarying quantities such that the quantities are changing simultaneously as each quantity passes through all of the intermediate values in the chunk.

### **Summary**

In this chapter, I discussed rate of change as the simultaneous change in two extensive variables (that is, covariation), and I described the process of reasoning about such simultaneous changes as covariational reasoning. The concept of covariation is important in mathematics because it explains variations in rate of change, and it shows the link between slope and the first derivative of the function. Conceptualizing the rate of change in terms of covariational reasoning enables students to develop an understanding of functions as involving the formation of links between values in a function's domain and a function's behavior and shape. The concept of a function is surely related to the understanding of rate of change. To understand rate of change, one must envision two variables co-varying systematically, which entails the idea of a functional relationship.

When students use covariational reasoning in their initial learning of rate of change in linear functions, they lay the foundation for understanding varying rates of change within nonlinear functions. Nonlinear functions are an important idea in calculus, particularly for

understanding the concept of derivative. In this chapter, in order to relate covariational reasoning to the teaching of rate of change, I discussed research on covariational reasoning and on types of reasoning that have been found to exist when covariation is investigated: transformational reasoning, qualitative reasoning and quantitative reasoning.

Although covariational reasoning is essential for understanding calculus concepts such as the derivative (Stroup, 2002), it can be nurtured long before calculus. Teachers can foster student reasoning by paying close attention to the ways in which students describe relationships between quantities; this awareness can enable teachers to better prepare students to draw on their informal ways of reasoning when the time comes to formalize relationships between varying quantities in calculus.

## CHAPTER 5: Functions and Derivatives

The overarching question of this literature review has been, “What are the important ideas when teaching rate of change?” In terms of function, rate of change can be conceptualized as the relationship between a dependent and independent variable. The concept is notoriously difficult to teach and to learn, yet it is central to students’ understanding of upper-level mathematics. The previous chapters have explored three other concepts that are important to consider when teaching rate of change: proportional reasoning, slope, and covariational reasoning. Taken together, these concepts are integral to the learning of rate of change and, thus, lay the foundation for the functions and derivatives encountered in calculus.

Chapter 2 focused on proportional reasoning, the ability to find mathematical relationships between two quantities; for example, the use of ratios to compare quantities. The approach that is most promising involves comparing two extensive variables to calculate an intensive variable. Students who do not fully understand ratio have been shown to have difficulty in thinking about variables intensively or extensively—a skill that is essential to exercising multiplicative reasoning, and thus, to doing higher-level mathematics.

In chapter 3, I discussed another step toward a sophisticated understanding of rate of change: slope. Slope expresses the proportional relationship between two extensive variables, which provides the value of the intensive variable. Next, chapter 4 turned to the subject of covariational reasoning, the ability to reason about the relationship between two variables

that change together. Rate of change can in fact be conceptualized as covariation, and covariational reasoning is necessary to a sophisticated understanding of rate of change.

In this chapter, I first discuss the importance of understanding derivatives. Next, I detail common errors in students' understanding of derivatives. After that, the remainder of the chapter discusses three distinct frameworks for understanding derivatives: an instrumental understanding of derivatives; an understanding of derivatives in terms of process object pairs (I focus on the functional layer); and an understanding of derivatives in terms of its several types of representations (I focus on the verbal representation). Finally, I conclude with a summary of how the many components in this thesis fit together into a cohesive approach to teaching rate of change from the early grades through calculus.

### **Importance of Derivatives in Calculus**

Ample literature supports the importance of learning calculus and one of its central concepts, the derivative (Sofronas, DeFranco, Vinsonhaler, Gorgievski, Schroeder, & Hamelin, 2011). For many students in, science, technology, engineering, and mathematics (STEM) fields, calculus is the entry-level, required mathematics course. Because of the importance of calculus in such a wide range of disciplines, and its large number of enrolled students, there have been many research studies on students' understanding of this subject. Sofronas et al. (2011) stated that many of the expert participants in their study noted that within the mathematics community, including their own mathematics departments, conversation with respect to articulating what the essential learning points should be for first-year calculus is often contentious; yet the experts recognized that these conversations were

important. Results from this study showed that all the participants cited student understanding of the derivative as a central skill in the mastery of basic concepts of the first-year calculus. Sofronas et al.'s (2011) study is one of the most up-to-date papers on what students should be expected to know after they complete the first-year calculus course (Calculus 1) or the 11<sup>th</sup>- or 12<sup>th</sup>-grade Calculus AB/Calculus BC course in high school.

In terms of calculus as a foundation for science, Lopez-Gay, Martinez-Torregrosa, Gras-Marti, and Torregrosa (2002), in their study of 103 high school physics teachers and their analysis of 38 physics textbooks, stressed the importance of students' understanding of differential calculus in order to understand physics. They argued that physics students do not understand the use of calculus in simple real-world problems and have difficulty in applying calculus concepts autonomously. This finding suggests that the value of calculus to other fields of study is undermined by students' lack of conceptual understanding of it.

Habre and Abboud (2006) made a strong argument for the importance of understanding derivatives and also for the perspective defended in this thesis, that students will have better conceptual understanding of derivative if they can conceive the meaning in terms of functional layer and verbal representation. As they noted, "Calls for change in calculus instruction have long been initiated in the United States and Europe," with a fundamental recent change being "an increased emphasis on visualization" in calculus. Aided by technology, teachers are able to provide students with "greater and easier access to multiple representations of concepts" (Kaput, 1992). Habre and Abboud (2006) explained that "multiple representations refer to presentations of a mathematical idea from three

different perspectives: numerical, graphical, and symbolic” (p. 57). Tyne (2014) also emphasized the importance of understanding calculus in real-life contexts. Specifically, students need to grasp how slope and derivative represent rates of change in real-life situations: slope as a graphical representation of rate of change, and derivatives as instantaneous rates of change. Both concepts are central to mathematics, yet they are little understood in connection to rate of change. Tyne (2014) added, citing Carlson et al. (2002), that “not only must students understand instantaneous rates of change, they must [also] have an understanding of the continuously changing rates, as well as strong covariational reasoning skills to interpret dynamic situations surrounding functions” (p. 299).

### **Research About Students’ Difficulties with Understanding Derivative**

A number of difficulties can be observed in students’ understanding of derivatives. First, according to researchers, students often find it challenging to distinguish between a function and its derivative function, and understand how the two are related (Nemirovsky and Rubin, 1992). This trouble arises from students’ assumption, based on intuition that the two must be similar because both have to do with change. Nemirovsky and Rubin (1992) found in a group of pre-calculus students that,

These assumptions of resemblances lead to a particular approach to problems of prediction between a function and its derivative, characterized by forcing a match of global features of the two graphs (e.g. increasing/decreasing, sign) and by focusing on one of them (function or derivative) rather than their relationship. (p. 32)

The authors concluded that the case study “supports the importance of our technique of using physical contexts to provide students with tools to explore mathematical ideas from a variety of directions,” and that the results provided “insight into how these tools help frame the interviewer/student discourse through which learning occurs” (p. 33).

Many studies with calculus students have shown that students enrolled in traditional university calculus classes understand basic calculus concepts incompletely, superficially, or not at all. Regarding rate, Herbert and Pierce (2011) found that calculus students faced a number of challenges with this concept, one significant difficulty being confusion between the rate (which is intensive) and the extensive quantities that constitute it (Thompson, 1994).

A common set of errors identified by Orton (1983) were rooted in weak understanding of integration and differentiation. He described common errors that students make with these concepts and identified which early concepts were foundational to students’ understanding of calculus. Rate of change and ratio and proportion are poorly understood by many students, as discussed in earlier chapters. Graphs, and particularly tangents to curves, are important in developing ideas of rate of change, but students’ poor understanding of graphs weakens their understanding of rate of change. According to Orton (1983), greater attention ought to be paid to ratio, proportion, graphs, and tangents throughout students’ mathematical education. Indeed, Orton (1983) believed students should do extensive work with graphs, tangents, ratio, proportion, and rates of change before encountering calculus.

Pence (1995) studied the understanding of students entering calculus and found that over half did not recognize  $2x$  as an object, representing the quantity twice as large as  $x$ . On

the other hand, they perceive  $2x$  as an instruction to multiply  $x$  by 2, to obtain another value for a given value of  $x$ . Pence found that success in calculus was strongly related to performance on this item where  $2x$  was an object. From this, Pence concluded that understanding operations at the object level is important for doing calculus, and is correlated with success in calculus. Pence (1995) considered operations to be a foundational concept, one that by a certain point in mathematics education is assumed to be familiar and well understood. Without this understanding, students will have difficulty doing calculus.

Tall (1987) also identified a difficulty found in students' understanding of calculus, one based on misunderstanding of the meaning of tangent. For Tall, this frequent misunderstanding arose from inherent in the way in which examples of tangency are typically introduced. Students first see lines tangent to circles, conclude (or are told) that the tangent touches the circle in only one place, and then incorrectly transfer this definition of tangent to their later study of tangents to functions. This misconception might be prevented by introducing other kinds of tangency with or before tangents to circles, or by altering the way in which tangents to circles are presented.

White and Mitchelmore (1996) concluded that many student errors can be traced to their holding, and drawing on, inaccurate conceptual knowledge in their attempts to solve application problems. Specifically, during the first stage of the solution process, conceptual knowledge is most likely to be used when the student is translating the application problem from the original statement or description into mathematical symbols and relationships. According to White and Mitchelmore (1996), the root cause of many errors was the students'

weak concept of variable: “students frequently treat variables as symbols to be manipulated, rather than as quantities to be related” (p. 91). Based on this finding, they concluded and recommended the following:

In the present study, the only detectable result of 24 hours of instruction intended to make the concept of rate of change more meaningful was an increase of manipulation-focused errors in symbolizing a derivative. Most students appeared to have an abstract-apart concept of variable that was blocking meaningful learning of calculus. . . . The inevitable conclusion is that a prerequisite to successful study of calculus is an abstract-general concept of a variable, at or near the point of reification. Even a concept-oriented calculus course is likely to be unsuccessful without this foundation. . . . Either entrance requirements for calculus courses should be made more stringent in terms of variable understanding, or an appropriate precalculus course should be offered at the university level. (p. 93)

As White and Mitchelmore’s (1996) findings make clear, there are no quick fixes for students who do not have the necessary concept of variable. The concept is a difficult and slow to learn, yet essential to any further progress. This is a good example of what Tall (1987) termed a “fundamental obstacle.” White and Mitchelmore (1996) have identified this obstacle, described it, and demonstrated the importance for students of surmounting it.

All of these studies—by White and Mitchelmore (1996), Pence (1995), Tall (1987), Nemirovsky and Rubin (1992), and Orton (1983)—help clarify which knowledge gaps are most consequential for calculus students. The first two found that students’ concepts of

variable are often weak, a result of poor preparation. According to Pence (1995), many students' understandings of the basic operations of multiplication and exponentiation are inadequate foundations for success in calculus. Similarly, Orton (1983) found that ratio, proportion, graphs, and tangents, basic concepts on which understanding of rate of change is constructed, are poorly understood by many students.

As one possible way to begin to address this problem, this literature review has discussed proportional reasoning, slope, covariational reasoning, and now functions and derivatives; the focus has been on rate of change as the concept that is integral to all four. Nemirovsky and Rubin (1992) found that students tend to expect a function and its derivative to resemble one another in a variety of ways, but they also found that students could make substantial progress in replacing this resemblance approach to the derivative with a more appropriate variational approach after only 150 minutes of instruction. Similarly, Tall (1987) found that, although students had a concept of a "generic tangent" that was inconsistent with the concept of tangent needed for the development of the derivative as the slope of a tangent, the students were able with appropriate instruction to develop a concept of tangency that would support study of the derivative.

These studies show us where some of the problems are and what some of the solutions might be. As Orton (1983) stated so clearly, we need to start addressing the problems of calculus instruction in grade school, giving students more opportunities to work with graphs and rates of change. From the work of Pence (1995) and White and Mitchelmore (1996), we see that concepts of variable and operations are also vital but weak, which

suggests that these topics also should receive more attention in earlier courses. Once students begin a calculus course, Tall (1987) and Nemirovsky and Rubin (1992) have shown how instruction can be designed to help students develop the necessary concepts of tangent and the relationship between a function and its derivative. Hauger (2000) helps us to begin to see how students reason about rates of change, which moves us closer to developing more effective instruction.

Given the importance of derivatives and the difficulty of understanding them, many researchers and practitioners have asked what students should learn about derivatives. In a study by Sofronos et al. (2011), the researchers asked 24 calculus experts (16 of whom are authors of respected calculus textbooks), to share perspectives on the first-year calculus. At least 33% of the experts agreed that the course has four overarching goals: a) mastery of the fundamental concepts/skills of the first-year calculus; (b) construction of connections and relationships between and among concepts and skills; (c) the ability to use the ideas of the first-year calculus; and (d) a deep sense of the context and purpose of the calculus. Understanding of the derivative is a first-tier sub-goal of the mastery of fundamental concepts/skills of the first-year calculus. Three second-tier sub-goals of the derivative emerged: understanding derivative as a rate of change, graphical understanding of the derivative, and facility with derivative computations.

### **Framework for Understanding Derivatives**

In the next few paragraphs, I focus on a framework established by Zandieh (2000) and elaborated by Roundy et al. (2014). Zandieh (2000) was among first scholars to attempt

to develop a framework for helping students understand derivatives. Her framework is now widely used in the United States and internationally, and it is highly cited and appreciated. The initial premise of Zandieh's investigation is that for a concept as multifaceted as derivative, it is not appropriate to ask simply whether or not a student understands the concept. Rather, one should ask for a *description* of the student's understanding of the concept of derivative—what aspects of the concept a student knows and the relationships a student sees between these aspects. Zandieh (2000) conducted interviews with students from the same Advanced Placement Calculus class. She asked students open-ended questions and asked them to solve problems involving derivatives. Initial indications from the case studies suggested that beginning students' preferences were not uniform, but that they became more similar as students' knowledge of the concept increased. From these data, it appeared that it was possible for students to have very different understandings of derivative even when this understanding had developed through the same class environment and homework assignments.

Zandieh (2000) described the concept of derivative as having two main components: *representations* and process object *layers*. The representations are different contexts in which derivatives are found:

The concept of derivative can be represented (a) graphically as the slope of the tangent line to a curve at a point or as the slope of the line [that] a curve seems to approach under magnification, (b) verbally as the instantaneous rate of change, (c) physically as speed or velocity, and (d) symbolically as the limit of the difference

quotient. (p. 3)

The other component of the framework, the process-object layers, describe “the similar structure of the concept of derivative in each context” (p. 107). Each of the representations in Zandieh’s (2000) framework can be used to convey the concepts behind the three process-object layers. She also likens these columns to “contexts” in the sense that each of these provides a context within which we can think about the derivative. Each of these process-object layers will be discussed in greater detail below. Zandieh’s popular framework has been modified in the last few years by Roundy, Dray, Manogue, Wagner, and Weber (2014).

The modification by Roundy et al. (2014) involved the addition of representations and instrumental understanding. They explained their additions to the framework as follows:

We have elaborated on the physical representation of the derivative; we have added a numerical representation of the derivative; and we have added space in the framework for the set of rules for finding symbolic derivatives. Each of these changes reflects an expansion of the table to incorporate additional answers to the prompt, “find the derivative.” By making use of the numerical representation of the derivative, one can answer the prompt numerically. Similarly, if the derivative is situated in a physical context, one can respond with a measurement process. Both of these responses require a conceptual understanding of the derivative in terms of ratio, limit and function, and involve a certain “thickness” in the derivative. (p. 103)

Roundy et al. (2014) also added the instrumental-understanding approach, in which the rules for symbolic derivatives are used, although this component is situated below a solid line

rather than in the matrix because, as discussed below, this approach does not require a conceptual understanding of derivative.

I have chosen to highlight the extended theoretical framework for derivative because although it has not yet been widely adapted, its greater level of detail makes it very helpful for understanding how derivatives can be conceptualized and found, and for recognizing several seemingly disparate concepts under the single heading of derivative. In the sections below, where I elaborate on both the process-object layers and the representations, I emphasize the *function* layer and the *verbal* representation because by the end of Calculus I, students need to be able to get to the *verbal* context and the *function* layer. In order to do so, they must be able to exercise covariational reasoning, as described in chapter 4. Before turning to a detailed discussion of process-object layers and representations from the Zandieh/Roundy derivative framework, I briefly discuss the instrumental/symbolic approach to finding derivatives.

### **Instrumental Understanding of Derivative**

Using the instrumental approach to finding derivatives is reasonably easy because it does not require a conceptual understanding of derivatives; the student merely needs to follow a set of rote rules for symbolic derivatives. Researchers and practitioners agree that students can readily memorize the rules and learn how to find derivatives in this way (Weber & Dorko, 2014; Zandieh, 2000; Orton, 1983). In fact, there is a real danger that students' instrumental use of the symbolic rules to find derivatives can mask their lack of understanding of the concept of derivatives. Teachers may assume that because students can

follow derivative rules, that they understand the concept of derivative. Researchers agree that the concept of derivative is so important that it requires sophisticated mathematical knowledge of rate (Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997) rather than mere “procedural competence” to follow symbolic rules (Tall, 1987; Herbert & Pierce, 2011).

Habre and Abboud’s (2006) study in a Lebanese American University focused on students’ algebraic thinking (understanding the derivative merely in terms of rules or as a function). Their study shows that if students understand derivative as a concept of rate of change (verbal), despite their resistance to new methods involving algebraic representations (i.e., applying the derivative rules solely), they will grasp the idea of derivative conceptually because, in their minds, they are clear about the definition of instantaneous rate of change. It is clear that students need a conceptual understanding of derivatives and how they relate to rate of change. In the next sections, I discuss two additional components of Zandieh (2000) and Roundy et al.’s (2014) framework, both of which lead students to a conceptual understanding: derivatives as layers of process-object pairs and derivatives as representations.

### **Derivative as Process-Object Layers**

Although Zandieh (2000) initiates the idea of constructing the first framework for derivative, she did not include the instrumental approach on her framework, and Roundy et al. (2014) relegated it to a place below a solid line, outside the matrix. Figure 2 below shows only the matrix itself, not the instrumental approach that was placed outside of it.

Process-object layer	Graphical	Verbal	Physical	Symbolic	Other
	Slope	Rate	Velocity	Difference Quotient	
Ratio					
Limit					
Function					

**Figure 2. The outline for the framework for the concept for the derivative (adapted from Figure 1 in Zandieh, 2000)**

According to Zandieh's (2000) framework, derivatives have three process-object layers: ratio, limit, and function. The derivative of a given formula can be understood as “a function whose value at any point is defined as the limit of a ratio” (p. 106; emphasis in original). The ratio layer is “the part of the derivative that deals with the average gradient” (p. 106). She added, “The ratio as a process is seen as a process of division of the numerator by the denominator, and as an object a ratio is seen as a pair of integers or as the outcome of the division process” (p. 106). The limit refers to the instantaneous rate of change and is the second layer.

The most important layer to the present discussion is the third layer, the function. According to Roundy et al.'s (2014), “the derivative as function is understood as a sequence of numerical ratios of differences, just as a function can be understood numerically as an array of numbers or set of ordered pairs” (p. 6).

In Zandieh's (2000) study, the function layer emphasizes the covariational nature of the derivative. Thus, each function has at least two process-object pairs, which she describes as follows:

(1) A process of taking an input of a function to its output paired with an object that is the value of the function at a point (noting that although the value itself is named as the object, the corresponding input is referenced). (2) A process of covariation, i.e. imagining “running through” a continuum of domain values while noting each corresponding range value, paired with an object that is the function itself, the set of ordered pairs. (p. 110)

Adding more detail to the framework, Zandieh (2000) claims that at times students will not grasp the underlying process of a layer, and will fail to develop a deep, structural conception of the object on that layer, and yet will still be able to consider the next process in the derivative structure by using what she abbreviates as *pseudo-objects*. She defined this term using Sfard’s (1992) concept of a *pseudostructural* conception, which “may be thought of as an object with no internal structure” (p. 107). For example, a student might consider the value of a limit as an object, without having an idea of the limiting process that gave rise to that value. Or, a student might see a ratio as a fraction, such as  $\frac{1}{2}$ , without perceiving that a ratio can also mean a comparison of two quantities. While pseudo-objects are not inherently harmful, as they can be used to efficiently simplify thinking about later processes, students who only have a pseudo-object understanding of the derivative may not fully grasp the concept. Still, students can use “pseudo-objects and multiple representations” to work their way toward accurate understandings of the concept.

A study of calculus students by Watson and Jones (2015) brought a slightly enhanced perspective on Zandieh’s (2000) framework in terms of the three process layers: ratio, limit,

and function. The authors presented an analysis of students' derivative-focused interviews at the end of a first-semester calculus course. Students were given two tasks in which they calculated and discussed derivatives in a pure mathematics context. Then, the students discussed the derivatives in real-world contexts. They brought a greater explanatory power to her extended framework with analogical reasoning. Their version of Zandieh's (2000) framework

allow[s] us to account for the reasons why students struggle with applying the derivative in novel contexts. Furthermore, it enables us to make reasonable predictions about how well a student will be able to apply their existing knowledge about derivatives to applications of the derivative they have never seen before. (2015, p. 1046)

The authors further concluded that their new framework suggested that calculus teachers need to spend some class time discussing "multiple derivative analogs" and "mak[ing] sure that students recognize the key causal relations, or the processes of the ratio, limit, and function layers of the derivative, that underlie those analogs" (p. 1046). Such discussions will enable students to better "extract the ratio-limit-function schema, and will [help them] be better prepared to apply their understanding of derivatives to unfamiliar situations" (p. 1046).

### **Derivative as a Representation or Context**

The second component of Zandieh's (2000) framework is the set of contexts or representations in which derivatives can appear. She explained three of the key representations as follows:

The concept of derivative can be represented (a) *graphically* as the slope of the tangent line to a curve at a point or as the slope of the line a curve seems to approach under magnification, (b) *verbally* as the instantaneous rate of change, (c) physically as speed or velocity, and (d) *symbolically* as the limit of the difference quotient. (p. 105, emphasis added)

She also discusses a physical representation. Within each type of representation, the structure of the concept is similar, as captured by the process-object layers described above. In her study, the verbal representation is most important. Students usually come into calculus with graphical and symbolic understanding of rate of change, but not verbal. Here it is worth noting a few points about each of the four representations mentioned above:

**Graphical.** At the ratio layer, this is the slope of a secant line between two points on the curve describing a function. When taking the limit, we arrive at the slope of the tangent line at a point. Finally, considering the derivative as a function requires us to recognize that the slope is different for different values of the independent variable.

**Verbal.** The verbal representation for the derivative discussed by Zandieh (2000) is the “rate of change.” At the ratio layer, this is expressed as an “average rate of change.” When taking the limit, this becomes the “instantaneous rate of change.” Understanding this verbal description as a function requires us to visualize the instantaneous rate of change for the inputs over the domain of the function.

**Physical.** The paradigmatic physical representation is velocity: average velocity, instantaneous velocity, and velocity as a function of time. These physical concepts provide a

language that we can use to understand the derivative. A large derivative means “faster,” and a varying derivative means that acceleration is occurring.

**Symbolic.** The symbolic representation of the derivative is the formal definition of the derivative in terms of the limit of a difference quotient. In this case, the distinction between the limit layer and the function layer can be subtle. The two layers differ in the recognition that the variable describing the point at which the limit is taken can be treated as the argument of a function. Zandieh (2000) expresses this difference with a notational distinction between  $x_0$  and  $x$ .

As the literature shows, if we want students to understand the multiple representations of derivative, they should be able understand the meaning of rate and, according to Weber and Dorko (2014), must be able to “imagine a process for measurement of that rate within the constraints of each representational system (e.g. a tangent line and difference quotient)” (p. 23). They explained as follows:

For instance, all mathematicians in this study moved easily between describing a tangent line and using the difference quotient with a limit. They did so because they were able to conceive of the tangent line and the result of the difference quotient as quantifying how fast variables were changing with respect to one another. They saw the measurement of rate as similar in both representations because they conceived of the tangent line as representing the result of  $\frac{f(x+h)-f(x)}{h}$  where  $h$  approaches zero. In this case, the mathematicians were expressing a meaning for rate across multiple representations. In contrast, students who relied on a tangent line or rules of

differentiation did not have a meaning that could be expressed in a different way. (p. 30)

Another consideration regarding the representations of derivative is whether there is a hierarchy inherent in the concept; that is, whether slope, limit, and function must be learned in that order. Zandieh (2000) posed this question and studied strong calculus students to see whether they could move among representations. As she explained, “Since we are considering understanding the derivative in terms of a matrix where each layer or process-object pair may be noted in many different contexts or representations, we must consider both whether the representations need to be learned in a specific order and then whether the layers of each representation are learned in a specific order” (p. 19). In a similar study, Weber and Dorko (2014) presented nearly the same findings.

Roundy et al. (2014) summarized their additions to Zandieh’s (2000) representations as falling into two categories, physical and numerical representations:

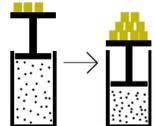
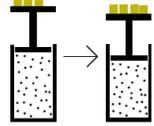
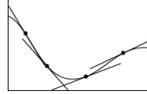
**Physical.** We suggest that although [time derivative] quantities do reside in a physical context, perhaps at least some uses of these phrases properly belong in the realm of verbal representation. We propose here a more “physical (as opposed to verbal) concept of the physical representation of the derivative.” (p. 4)

**Numerical:** The numerical representation is the one member of the Rule of Four (Hughes-Hallett et al., 1998) that was not present in the framework of Zandieh (2000). We recognize a numerical representation of the derivative that is closely allied to but distinct from the physical representation. This representation parallels the

formal symbolic concept of the derivative, but differs in ways that are of practical importance in the use of the derivative in the sciences and in numerical analysis. (p.

6)

A figure taken from Roundy et al. (2014) provides an apt summary of the process-object layers and the representations, and thus it effectively illustrates this thesis’s framework in regards to derivative.

Process-object layer	Graphical	Verbal	Symbolic	Numerical	Physical
	Slope	Rate of Change	Difference Quotient	Ratio of Changes	Measurement
Ratio		“average rate of change”	$\frac{f(x+\Delta x)-f(x)}{\Delta x}$	$\frac{y_2 - y_1}{x_2 - x_1}$ numerically	
Limit		“instantaneous ...”	$\lim_{\Delta x \rightarrow 0} \dots$	...with $\Delta x$ small	
Function		“... at any point/time”	$f'(x) = \dots$	... depends on $x$	tedious repetition
			Symbolic		
			Instrumental Understanding		
Function			rules to “take a derivative”		

**Figure 2. Roundy et al.’s extended framework (adapted from Figure 2 in Roundy et al.)**

Of the representations in their study, the most important for the purposes of this thesis is the verbal representation. This understanding of derivative as the instantaneous rate of

change can penetrate all of the other representations. This view is supported by longitudinal study of Roorda, Vos, and Goedhart (2009), who argued,

It seems important that students verbalize relationships between different words that are related to rate of change. Relationships between words such as velocity, derivatives, statements, and slope will give students opportunities to construct relationships between situations. (p. 24)

Further support for Roundy et al.'s (2014) claim that students' understanding of derivative ought to be analyzed in terms of "verbal" representation is found in Tyne's (2014) recent study with 74 differential calculus students. She studied students' understanding of the concepts of slope (as a constant rate of change) and derivative (as an instantaneous rate of change) in real-life contexts, with both linear and nonlinear relationships. According to the preliminary results, "Students have more success with slope questions than derivative questions," and "while students correctly use the slope of a linear relationship to make predictions, they do not demonstrate an understanding of the derivative as an instantaneous rate of change and an estimate of the marginal change" (p. 299). Tyne's research indicates that students can have difficulty moving beyond examining the rate of change in linear functions, to examining it in nonlinear functions (as derivative). She noted,

Without a solid of understanding of rates of change, students are not able to correctly use the slope and derivative to make valid predictions. In fact, almost two-thirds of the students incorrectly used the instantaneous rate of change as a constant rate of change, pointing out some of the misconceptions about the derivative's meaning as an

instantaneous rate of change, and incomplete across-time understanding of the derivative as a function. (p. 308)

These difficulties center on how students verbalize their understanding of the concepts in question. Tyne (2014) suggests that further research be done in the form of interviews to understand student thinking about the instantaneous rate of change at a single point, and to understand why many used the instantaneous rate of change as a constant to make predictions.

### **Summary**

This literature review has examined concepts that are important to the teaching of rate of change. This concept is foundational to upper-level mathematics, yet it is difficult to teach and learn. As discussed above, a useful theoretical framework for derivatives was initiated by Zandieh (2000) and then extended by Roundy et al. (2014). It is this extended framework that I have chosen to use in this chapter. In fact, this framework inspired the design of the entire literature review. Together, the four chapters explore the foundational elements in students' development of the concept of rate of change, and they show the relationship among proportional reasoning, the concept of slope, covariational reasoning, and the derivatives of functions. Thus, rate of change is seen to be the foundation for the functions and derivatives encountered in K–12 grade levels from pre-algebra to calculus.

## CHAPTER 6. Summary and Conclusions

To conclude, three over-arching ideas emerged in the elaboration of rate of change. First, rate and ratio do not mean the same thing, although many teachers use these two words interchangeably during their instructional time (Clark et al., 2003). As explained in chapter 2 on proportional reasoning, ratio does not require a relationship: 3 over 4 is a ratio, while the result of 3 over 4 is a rate. Ratios are formed by two discrete measurable quantities. Ratios are also the least abstract quantities, when all of the extensive quantities could be easily represented by counting, measuring or equipartitioning.

On the other hand, there are more abstract intensive quantities such as speed, stretching and shrinking, and the lemony taste of lemonade. There is a relationship between two extensive variables which are represented numerically as 3 and 4. The intensity of the relationship of 3 over 4, which is called rate. Rate and proportion do have a very close relationship; for accurate proportional reasoning, students need to conceive the relationship between two extensive quantities as a product. Furthermore, the established relationship is an intensive quantity. The rate, the result of 3 over 4 shows the intensity of this relationship. During the classroom discourse starting from an early age, students should be able to identify this relationship (proportion) as an intensive quantity.

The second key overarching ideas concerns the understanding of extensive and intensive variables in rate of change. For example, distance in miles is an extensive variable, and so is time in hours. But speed is an intensive variable, and it depicts the relationship between miles and hour. It also answers the question “how fast?” On the other hand,

“distance in miles” by itself only answers the question “how much?” The same is true for “time in hours,” or “number of lemons used for the lemonade.” The number of lemons used or amount of water used will answer the question “how much,” and will be an extensive variable. Extensive variables or quantities are measurable. However, the lemony taste of the drink or the speed are abstract and are not measured themselves but constructed as the ratio of two measurable (extensive) quantities. When we observe the lemony taste or speed, we observe the intensity of the relationship of the number of lemons per water unit or the amount of distance unit per time unit respectively in that order.

The third key point from this thesis is that derivative at a point and derivative of a function are interrelated but different concepts. We teach students derivative at a point by magnifying the curve at a point. We would like then to move the idea from average rate of change (which is slope) to instantaneous rate of change. The continuum of the instantaneous rate of change will form the derivative function. My claim is that, rather than teaching the students a traditional approach (derivative as an operator), we should instead teach derivative as rate of change. This will allow students to understand the derivative function as tracking the ratio between changes in two quantities. If students are able to attend to the derivative as a rate of change, this ability will support them in understanding the process of differentiation as an operator that measures a rate of change.

As students approach situations where they must solve novel problems about rate, they are equipped with ways of thinking, particularly about the derivative function as a rate of change, that are necessary to reason through these situations. By thinking about the

meaning of the computations, students are able to draw connections between major ideas. Yet, scholars and practitioners have long suggested the exclusion of the concept of rate from elementary-school teaching of the concept of slope. The assumption is that rate of change is too confusing for the children in grades five through nine and that only in ten through twelfth grades are students ready to learn about rate of change.

I argue, however, that the concept of rate of change needs to be introduced as early as possible, even as early as third grade, so that later they will be able to transfer what they have learned about slope to other types of problems. Stump (2001), Lobato et al. (2013), Weber and Dorko (2014), Confrey and Smith (1995), and Planinic et al. (2012) all defend the idea of teaching slope as rate of change as a start, rather than steepness.

In chapter 2 of this literature review, I examined ratio, the first process-object layer from Zandieh (2000), in terms of proportional reasoning, the ability to find mathematical relationships between two quantities (e.g., the use of ratios to compare quantities). The chapter showed that the best approach to proportional reasoning, in terms of supporting later learning, is the intensive and extensive variables approach: comparing two extensive variables to calculate an intensive variable. Understanding ratio in this way enables students to think about variables intensively or extensively, a skill that is essential to exercising multiplicative reasoning, and thus, to doing higher-level mathematics.

Next, in chapter 3, I examined slope, which although not a process-object layer, is the backbone of the concept of derivative. Slope is, after ratio, the next step toward a sophisticated understanding of rate of change. Slope can in fact be understood as a

representation of rate of change, expressing the proportional relationship between two extensive variables and thereby providing the value of the intensive variable. Next, in chapter 4, I discussed covariational reasoning, which is reasoning about the relationship between two variables that change together. This skill is necessary to a sophisticated understanding of rate of change, and rate of change can in fact be conceptualized as covariation.

In chapter 5, I discussed research surrounding the conception of derivative. Rate of change can be understood as a slope formula that belongs not only to linear functions, but also to nonlinear ones. Derivative is then the formalization of slope at a point (instantaneous rate of change) for all functions, both linear and nonlinear.

In this literature review, I explored the big ideas from research about students' conceptual understanding of rate of change. These ideas are important for the concept of a function, in that students relate a change in one variable to a corresponding change in another (covariation). More broadly, the purpose of this literature review was to synthesize the research that illuminated the ideas that are most important to consider when teaching rate of change. In future research, it may be beneficial to explore how the factors of classroom instruction and students' concept of function relate to students' understanding of rate of change.

Knowledge of Piagetian stages might help with the timing of the introduction of rate and proportional concepts. First, it is essential that students understand what is meant by the concept of change or difference in both additive and multiplicative terms. Next, students should be taught that multiplication is not just an abbreviated version of repeated addition but

also comprises scaling, magnification, and growth. Scholars and researchers (Lamon, 1993; Kaput, 1992; Simon, 1996; Saldanha & Thompson, 1998; Confrey & Smith, 1994) began emphasizing the importance of teaching ratio and rate over twenty years ago, noting that these important mathematical concepts ought to be considered by elementary and secondary teachers as they guide students' mathematical educations.

This literature review has implications for elementary and secondary math education, for teacher education, and for calculus curriculum and pedagogy. The concept of rate of change, which enables understanding of the differential calculus, cannot be built by studying the calculus itself. Rather, repeated and early exposure to the concept of rate of change is needed if students are to understand the concept of the derivative when studying calculus. Teachers at the elementary and secondary level should understand that proportional reasoning is not acquired all at once and that comparing ratios is an advanced idea. This knowledge can inform how they introduce students to ratio and slope. The time is surely ripe for providing students with a strong, early foundation for upper-level mathematics.

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