ABSTRACT

HANSEN, BRITTANY LOUISE. The Hyperbolic Kac-Moody Lie Algebra of Type $G_2^{(1)}$ and its Root Multiplicities. (Under the direction of Dr. Kailash Misra.)

In 1968 V. Kac and R. Moody independently defined Kac-Moody algebras to be an infinite dimensional analog to finite-dimensional semisemple Lie algebras. Kac-Moody algebras are classified by symmetrizable indecomposable matrices known as generalized Cartan matrices (GCM), which fall into three categories: finite type, affine type, and indefinite type.

An important problem in the study of Kac-Moody algebras is to determine root multiplicities. Roots are classified as either real or imaginary. For Kac-Moody algebras of finite and affine type, the root multiplicities are known in generality. For indefinite type Kac-Moody algebras, however, it is still an important and open problem to compute the multiplicities of imaginary roots. In this thesis our primary interest is in the hyperbolic Kac-Moody algebra $HG_2^{(1)}$ of indefinite type.

First we give a realization of $HG_2^{(1)}$ following the construction given in [Ben93]. We construct $HG_2^{(1)}$ as a $\mathbb{Z}$-graded Lie algebra with $G_2^{(1)}$ being the zero component. Utilizing this construction we apply Kang's multiplicity formula to obtain multiplicities of roots of $HG_2^{(1)}$ using the multiplicities of certain weights of some integrable highest weight $G_2^{(1)}$-modules. In order to determine such weight multiplicities, we utilize the combinatorics of the crystal base for the $G_2^{(1)}$-modules.

Let $\{\alpha_0, \alpha_1, \alpha_2\}$ and $\{\alpha_{-1}, \alpha_0, \alpha_1, \alpha_2\}$ denote the simple roots of $G_2^{(1)}$ and $HG_2^{(1)}$, respectively, and let δ be the canonical null root of $G_2^{(1)}$. In this thesis we provide some general results for imaginary roots of $HG_2^{(1)}$. Using these results, Kang's multiplicity formula, and crystal base theory, we determine the multiplicities of roots $-\ell\alpha_{-1} - \alpha - k\delta$, for $\ell \leq 3$ and $\alpha = m_0\alpha_0 + m_1\alpha_1 + m_2\alpha_2 < \delta$ ($m_i \in \mathbb{Z}_{\geq 0}$). We also determine the energy function for the level 1 perfect crystal of $U_q(G_2^{(1)})$ and use it to determine a bound on the fluctuations of level 1 paths. In addition, we observe that the root multiplicities for $HG_2^{(1)}$ obtained in this thesis satisfy Frenkel's conjectured bound.

The Hyperbolic Kac-Moody Lie Algebra of Type $G_2^{(1)}$
and its Root Multiplicities

by
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DEDICATION

To my loving husband, Andrew Hansen.

BIOGRAPHY

Brittany Hansen was born in Tampa, Florida, to parents Brooke Gentile and Catherine LeCroy and has three younger siblings Ashley Gentile, Austin Drum, and Savannah Drum. She grew up in several different small towns outside of Charlotte, North Carolina, during which time she discovered a love for mathematics. It was because of the experience of frequently moving that she learned to appreciate the universality of mathematics and it became the subject in which she was most focused. Shortly after starting high school in North Carolina, Brittany moved back to Florida where she attended Lakeland Senior High School and received several opportunities to advance in mathematics. She graduated high school with honors in the top percent of her class.

Brittany pursued her interest in mathematics at Carson-Newman University in Jefferson City, Tennessee. While attending Carson-Newman, she discovered another love: teaching mathematics. This was fostered in part by the support of her professors and by numerous opportunities to tutor others, both at Carson-Newman and at the online high school Palmetto State E-cademy. Brittany finished strong, graduating Summa Cum Laude and earning a Bachelors degree in Mathematics with a minor in Computer Science.

With a renewed desire to expand her current knowledge of mathematics, Brittany started graduate school at North Carolina State University. At NC State, she became interested in Lie algebras and representation theory after taking a course in that area with Dr. Bojko Bakalov and participating in a summer research program with Dr. Kailash Misra. After passing her qualifier exams, she began research with Dr. Misra, which culminated in this dissertation. In addition to research, Brittany was able to gain valuable experience teaching undergraduates in subjects such as calculus and abstract algebra. She is excited to start her next chapter in teaching and research at Campbell University in Buies Creek, North Carolina.

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INTRODUCTION

Introduction

Lie groups and Lie algebras arose out of the study of symmetries of algebraic and geometric objects [Kac90]. In the 19th century W. Killing and E. Cartan completely classified finite-dimensional simple Lie Algebras. In the 1960's interest in infinite-dimensional Lie algebras led to several classifications of infinite-dimensional Lie groups and Lie algebras. In 1968 V. Kac and R. Moody independently defined an infinite dimensional analog to finite-dimensional semisemple Lie algebras, which are known as Kac-Moody algebras. Representation theory of Kac-Moody algebras is rich with many connections to mathematics and physics such as combinatorics, topology, finite simple groups, modular forms and theta functions, soliton equations, and quantum field theory [Kac90].

Kac-Moody algebras are classified by special indecomposable matrices known as generalized Cartan matrices (GCM). These matrices fall into three categories: finite type, affine type, and indefinite type, and their corresponding Kac-Moody algebras are classified in the same manner. Briefly speaking, finite type GCMs are positive definite and as the name suggests, produce finite dimensional Kac-Moody algebras. The other types correspond to positive semidefinite and negative definite GCMs, respectively, and yield infinite dimensional Kac-Moody algebras. To each GCM we associate an oriented diagram, called a Dynkin diagram, that encodes the data of the GCM. An important subset of the indefinite GCMs are the hyperbolic type GCMs which have Dynkin diagrams whose proper subdiagrams are of finite or affine type only.

An important question in the study of Kac-Moody algebras is to determine the multiplicities of roots. Informally speaking, a root space of a Kac-Moody algebra $\mathfrak{g}$ is a generalized eigenspace under the adjoint action of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Formally speaking, a root space of $\alpha \in Q$ is the set $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$, where Q is the root lattice of $\mathfrak{g}$. We say $\alpha \neq 0$ is a root if $\mathfrak{g}_\alpha \neq 0$. We define the multiplicity of root α to be the dimension of $\mathfrak{g}_\alpha$ and denote it by $mult(\alpha)$. Roots are classified as either real or imaginary (see Definition 2.2.20). For Kac-Moody algebras of all types, the real roots are known to have multiplicity equal to 1. All of the roots of finite type Kac-Moody algebras are real roots and thus have multiplicity equal to 1. For affine type, the imaginary roots have multiplicity equal to the rank of the associated GCM and hence are completely known.[1] For indefinite type Kac-Moody algebras, however, it is still an important and open problem to compute the multiplicities of imaginary roots. Although this problem has been studied for Kac-Moody algebras of types $HD_3^{(4)}$ and $HX_n^{(1)}, X = A, B, C, D$ for example, the multiplicities have not been completely determined for any of the aforementioned algebras. See for example, [Erb12], [KM94], [HM02], [Ben94], [Wil03], and [Wil12].

Our primary focus in this thesis is to determine the multiplicities of imaginary roots of the hyperbolic Kac-Moody algebra $HG_2^{(1)}$, whose GCM is given by

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -3 & 2 \end{pmatrix}$$

In Chapter 3, we give a realization of $HG_2^{(1)}$ which follows the construction given in [Ben93]. We begin with the affine Kac-Moody algebra $G_2^{(1)}$ having GCM equal to that of $HG_2^{(1)}$ with the first column and row removed, and construct $HG_2^{(1)}$ as a $\mathbb{Z}$-graded Lie algebra with $G_2^{(1)}$ as the degree zero component. Utilizing this construction we apply Kang's multiplicity formula, which we review in Chapter 4, to determine the root multiplicities $HG_2^{(1)}$ using the weight multiplicities of certain integrable highest weight $G_2^{(1)}$-modules. In order to determine such weight multiplicities, we utilize crystal base theory.

[1] Technically, this is only true for untwisted affine Kac-Moody algebras. However, we only work with untwisted Kac-Moody algebras in this thesis and drop the untwisted qualifier.

Quantum groups were introduced by V. G. Drenfel'd [Dre85] and M. Jimbo [Jim88] independently in 1985. In short, a quantum group $U_q(\mathfrak{g})$ is a q-deformation of the universal enveloping algebra $U(\mathfrak{g})$ of a Kac-Moody algebra $\mathfrak{g}$. It is known [Lus85] that for generic "q" the $U_q(\mathfrak{g})$ representation theory parallels that of $U(\mathfrak{g})$ representation theory and hence provides the structure for studying representations of Kac-Moody algebras. Crystal bases are important combinatorial tools used in the study of quantum groups and are used for determining weight multiplicities of $U_q(G_2^{(1)})$-modules, hence $G_2^{(1)}$-modules, in this thesis. Developed by M. Kashiwara [Kas90; Kas91] and G. Lusztig [Lus90] independently in 1990, crystal bases can be viewed as a basis at $q = 0$ and have nice combinatorial structures. In Chapter 5, we review the theory of quantum groups and crystal bases and outline the procedure for computing root multiplicities of $HG_2^{(1)}$.

Let $\{\alpha_0, \alpha_1, \alpha_2\}$ and $\{\alpha_{-1}, \alpha_0, \alpha_1, \alpha_2\}$ denote the simple roots of $G_2^{(1)}$ and $HG_2^{(1)}$, respectively. Let $\delta = \alpha_0 + 2\alpha_1 + 3\alpha_2$ be the canonical null root of $G_2^{(1)}$, and denote the set of positive roots of $G_2^{(1)}$ by Δ_S^+. In Chapter 4, we provide some general results for imaginary roots of $HG_2^{(1)}$. For example, when $\ell \geq 1$ and $\alpha = m_1\alpha_1 + m_2\alpha_2 \in \Delta_S^+$, we show $mult(-\ell\alpha_{-1} - \ell\delta) = 2$ and $mult(-\ell\alpha_{-1} - \alpha - \ell\delta) = 1$. In Chapter 6, we use these results, Kang's multiplicity formula, and crystal base theory to determine the root multiplicities of $-\ell\alpha_{-1} - \alpha - k\delta$, where $\ell = 1, 2, 3$ and $\alpha = m_0\alpha_0 + m_1\alpha_1 + m_2\alpha_2 < \delta$ ($m_i \in \mathbb{Z}_{\geq 0}$). We also determine the energy function associated with the level 1 perfect crystal of $U_q(G_2^{(1)})$ in Chapter 5 and use it and the affine weight formula to determine a bound on the fluctuations on level 1 paths in Chapter 6. In addition, we verify that all of the root multiplicities in Chapter 6 satisfy Frenkel's conjectured bound [Fre85].

CHAPTER

2

PRELIMINARIES

In this chapter we survey the basic definitions and theorems associated with Lie algebras and Kac-Moody algebras that are found in the literature. For more details on the subject see, for example [Kan94], [Kac90], [Hum72], and [Mis12]. Unless otherwise stated we assume the field $F = \mathbb{C}$.

2.1 Lie Algebras

Definition 2.1.1. A vector space L over a field F is a **Lie algebra** if there is a product $[\cdot,\cdot] : L \times L \to L$, called the **bracket**, such that

(1) $[\cdot,\cdot]$ is bilinear,

(2) $[x, x] = 0$ for all $x \in L$,

(3) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ (Jacobi identity), for all $x, y, z \in L$.

Observe that identity (2) implies the bracket is anticommutative: $[x, y] = -[y, x]$.

Theorem 2.1.2. *Let $\{x_i \mid i \in I\}$ be a basis of a vector space L with bracket $[\cdot,\cdot]$. L is a Lie algebra if and only if the relations in Definition 2.1.1 are satisfied on the basis elements.*

Therefore it is enough to define a Lie algebra on the basis elements and extend it by linearity. Next we review some notable examples of Lie algebras.

Example 2.1.3. Let A be an associative algebra over F with associative product '$\cdot$'. Then A is a Lie algebra under the bracket defined by $[x, y] = x \cdot y - y \cdot x$, called the **commutator bracket**. Some particular associative algebras of interest are as follows.

1. Let $M_n(F)$ denote the set of $n \times n$ matrices with entries in F. Then $M_n(F)$ is an associative algebra under matrix multiplication. We denote the Lie algebra $M_n(F)$ with commutator bracket as $\mathfrak{gl}(n, F)$.

2. Let V be a vector space and denote the set of linear transformations from V to V by $g\ell(V)$. Then $g\ell(V)$ is an associative algebra with associative product equal to the composition of maps. If $dimV = n$, then $g\ell(V) \cong \mathfrak{gl}(n, F)$.

Definition 2.1.4. A subspace M of a Lie algebra L is a **subalgebra** if $[x, y] \in M$ for all $x, y \in M$. A subalgebra I of L is an **ideal** if $[x, y] \in I$ for all $x \in L, y \in I$. If I is an ideal, the quotient space L/I is a Lie algebra with bracket defined by

$$[x + I, y + I] = [x, y] + I \text{ for all } x, y \in L.$$

Example 2.1.5. The subset $Z(L) = \{x \in L \mid [x, y] = 0 \text{ for all } y \in L\}$ is an ideal of L, called the **center** of L. If $Z(L) = L$ then we say L is **abelian**.

Definition 2.1.6. A Lie algebra L is **simple** if the only ideals of L are $\{0\}$ and L itself. L is **semisimple** if it can be expressed as a direct sum of simple ideals.

Definition 2.1.7. A **derivation** of L is a linear transformation $\partial : L \to L$ satisfying

$$\partial([x, y]) = [\partial(x), y] + [x, \partial(y)] \text{ for all } x, y \in L.$$

Example 2.1.8. The map $ad_x : L \to L$ defined as $ad_x(y) = [x, y]$ for all $y \in L$ is a derivation.

2.1.1 Representations and Modules of Lie Algebras

In this section, we introduce Lie algebra homomorphisms which give rise to a field of study known as representation theory.

Definition 2.1.9. Let L_1 and L_2 be Lie algebras. A **homomorphism** of Lie algebras is a linear transformation $\varphi : L_1 \to L_2$ satisfying

$$\varphi([x, y]) = [\varphi(x), \varphi(y)] \text{ for all } x, y \in L_1.$$

A one-to-one and onto homomorphism is called an **isomorphism**. The **kernal** of φ is defined as ker$\varphi = \{x \in L_1 \mid \varphi(x) = 0\}$.

Definition 2.1.10. Let L be a Lie algebra over field F. Let V be a vector space over F.

(1) A Lie algebra homomorphism $\varphi : L \to g\ell(V)$ is called a **representation** of L on V.

(2) An L**-module** is a vector space V with bilinear map $L \times V \to V$, denoted by $(x, v) \mapsto x \cdot v$, which satisfies

$$[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v) \text{ for all } x, y \in L, v \in V.$$

(3) A **submodule** of an L-module V is a subspace U of V such that $x \cdot u \in U$ for all $x \in L, u \in U$.

(4) An L-module V is **irreducible** if the only submodules of V are $\{0\}$ and V itself.

(5) Let V be an L-module. Then $x \in L$ is **locally nilpotent** on V if there exists an integer N such that $x^N \cdot v = 0$ for any $v \in V$.

(6) If U is a submodule of V, then the quotient V/U is an L-module under the action

$$x \cdot (v + U) = x \cdot v + U \text{ for all } x \in L, v \in V.$$

Observe that any representation of $\varphi : L \to g\ell(V)$ induces an L-module structure on V given by $x \cdot v = \varphi(x)v$ for all $x \in L, v \in V$ and conversely.

Example 2.1.11. The following are notable L-modules.

1. A Lie algebra L is an L-module under the adjoint action $x \cdot y = ad_x(y)$. We call ad_x the **adjoint representation**.

2. Let V and W be L-modules, then $V \oplus W$ is an L-module with action given by

$$x \cdot (v + w) = x \cdot v + x \cdot w \text{ for all } x \in L, v \in v, w \in W.$$

3. Let V and W be L-modules, then $V \otimes W$ is an L-module with action given by

$$x \cdot (v \otimes w) = (x \cdot v) \otimes w + v \otimes (x \cdot w) \text{ for all } x \in L, v \in v, w \in W.$$

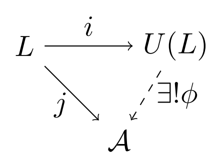
4. Let V be an L-module over field F, and consider the dual space $V^* = \{f : V \to F \mid f \text{ is linear}\}$. Then V^* is an L-module under the action $x \cdot f$ satisfying

$$(x \cdot f)(v) = -f(x \cdot v) \text{ for all } v \in V, \text{ where } x \in L, f \in V^*.$$

Definition 2.1.12. Let $U(L)$ be an associative algebra with unity over field F. Let $i : L \to U(L)$ be a linear map such that

$$i([x, y]) = i(x)i(y) - i(y)i(x) \text{ for all } x, y \in L.$$

We say the pair $(U(L), i)$ is the **universal enveloping algebra** of L if it has the universal property represented by the diagram:



for any associative algebra $\mathcal{A}$ and linear map $j : L \to A$ satisfying

$$j([x, y]) = j(x)j(y) - j(y)j(x) \text{ for all } x, y \in L.$$

Theorem 2.1.13. *[HK02] For Lie algebra L and universal enveloping algebra $(U(L), i)$, we have the following.*

1. *$U(L)$ is unique.*

2. *$i : L \to U(L)$ is injective.*

Theorem 2.1.13 allows us to view L as a subspace of $U(L)$ and the Poincaré-Birkhoff-Witt Theorem provides a basis for $U(L)$.

Theorem 2.1.14 (Poincaré-Birkhoff-Witt (PBW))**.** *Let L be a Lie algebra with index set Ω and ordered basis $\{x_\alpha | \alpha \in \Omega\}$. For $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$, consider elements of the form $x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_n} \in L$. Together with the identity, 1, all such elements form a basis for $U(L)$.*

Suppose V is an L-module. Then we can extend the L-module structure on V naturally to an $U(L)$-module structure on V by defining the $U(L)$ action on V as follows

$$(x_1 x_2 \cdots x_r) \cdot v = x_1 \cdot ((x_2 \cdots x_r) \cdot v) = x_1 \cdot (x_2 \cdots (x_r \cdot v)),$$

for all $x_1, \cdots, x_r \in L, v \in V$. Therefore a representation of L extends to a representation on $U(L)$. Conversely, the PBW theorem implies $U(L)$ contains L, and so a representation of $U(L)$ is a representation of L. Therefore the representation theory of Lie algebras and universal enveloping algebras is essentially the same.

2.2 Kac-Moody Algebras

In this section we discuss Kac-Moody algebras which are generalizations of finite-dimensional semisimple Lie algebras and may or may not be infinite dimensional. The study of Kac-Moody Lie algebras starts with a matrix known as a generalized Cartan matrix, whose entries determine the structure of the algebra.

Definition 2.2.1. Let I be an index set of size n. A matrix $A = (a_{ij})_{i,j \in I}$ of rank l is called a **generalized Cartan matrix (GCM)** if it satisfies:

- $a_{ii} = 2$ for all $i \in I$
- $a_{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j \in I$
- $a_{ij} = 0 \iff a_{ji} = 0$ for all $i, j \in I$

Definition 2.2.2. A GCM is **symmetrizable** if there exists some invertible diagonal matrix $d = diag(s_1, s_2, \cdots, s_n \mid s_i \in \mathbb{Z}_{>0})$ such that DA is symmetric.

Definition 2.2.3. A GCM is **indecomposable** if every decomposition of I into two disjoint nonempty subsets I_1, I_2 has the property that $a_{ij} \neq 0$ for some $i \in I_1$ and $j \in I_2$.

Theorem 2.2.4. *[HK02] Let A be an indecomposable GCM. Let $u = (u_1 \ u_2 \ \cdots \ u_n)^T$ be a column vector in $\mathbb{R}^n$. If $u_i > 0$ (resp. $u_i \geq 0$) for all $i = 1, \cdots, n$, we say $u > 0$ (resp. $u \geq 0$). Then A has one (and only one) of the following types:*

- *(Finite type) if there exists a $u > 0$ such that $Au > 0$,*
- *(Affine type) if there exists a $u > 0$ such that $Au = 0$,*

- *(Indefinite type) if there exists a $u > 0$ such that $Au < 0$.*

Definition 2.2.5. Let A be an indecomposable GCM. The **Dynkin diagram** of A is the connected diagram with nodes indexed by I and edges defined using the following rules for $i \neq j$:

(1) If $a_{ij}a_{ji} \leq 4$ and $|a_{ij}| \geq |a_{ji}|$, then the ith and jth nodes are connected by $|a_{ij}|$ edges with an arrow pointing toward the ith node if $|a_{ij}| > 1$.

(2) If $a_{ij}a_{ji} > 4$, then the nodes are connected with a bold-faced edge and the ordered pair $(|a_{ij}|, |a_{ji}|)$.

A **subdiagram** of a Dynkin diagram is a subset of nodes and connecting edges taken from the original Dynkin diagram.

Theorem 2.2.6. *[HK02] Let A be an indecomposable GCM with an associated Dynkin diagram. Then,*

1. *A is of finite type if and only if all of the subdiagrams are of finite type.*
2. *A is of affine type if and only if all of the proper subdiagrams are of finite type and $detA = 0$.*

Definition 2.2.7. Let A be an indecomposable GCM of indefinite type with an associated Dynkin diagram. We say A is **hyperbolic** if every proper subdiagram is of finite type or affine type.

Definition 2.2.8. The **Cartan datum** associated with the GCM $A = (a_{ij})_{i,j \in I}$ is a quintuple $(A, \Pi, \check{\Pi}, P, \check{P})$ where

- $\check{P} = \text{span}_{\mathbb{Z}}\{\{h_1, \cdots, h_n\} \cup \{d_s \mid s = 1, \cdots, n - l\}\}$ is a free abelian group of rank $2|I| - l$, called the **dual weight lattice**,
- $P = \{\lambda \in \mathfrak{h}^* | \lambda(\check{P}) \subset \mathbb{Z}\}$ is called the **weight lattice**,
- $\check{\Pi} = \{h_1, \cdots, h_n\} \subset \mathfrak{h}$ is the set of **simple coroots**, and
- $\Pi = \{\alpha_1, \cdots, \alpha_n\} \subset \mathfrak{h}^*$ is the set of **simple roots** which satisfy:

$$\alpha_j(h_i) = a_{ij}$$

$$\alpha_j(d_s) = 0 \text{ or } 1$$

for all $i, j \in I$ and $s \in \{1, \cdots, n - \ell\}$.

We let $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} \check{P}$ be the complex extension of $\check{P}$, called the **Cartan subalgebra**.

Definition 2.2.9. The **Kac-Moody** algebra $\mathfrak{g} = \mathfrak{g}(A)$ associated with the *Cartan datum* $(A, \Pi, \check{\Pi}, P, \check{P})$ is a Lie algebra with generators e_i, f_i $(i \in I)$ and $h \in \check{P}$ satisfying:

(1) $[h, h'] = 0$ for $h, h' \in \check{P}$,

(2) $[e_i, f_i] = \delta_{ij} h_i$,

(3) $[h, e_i] = \alpha_i(h) e_i$ for $h \in \check{P}$,

(4) $[h, f_i] = -\alpha_i(h) f_i$ for $h \in \check{P}$,

(5) $(ade_i)^{1-a_{ij}}(e_j) = 0$ for $i \neq j$,

(6) $(adf_i)^{1-a_{ij}}(f_j) = 0$ for $i \neq j$.

The generators e_i and f_i are called **Chevalley generators** and $\mathfrak{g}_+ = \langle e_i \mid i \in I \rangle$ and $\mathfrak{g}_- = \langle f_i \mid i \in I \rangle$ are the subalgebras of $\mathfrak{g}$. The relations 1-4 are known as the **Chevalley relations** and relations 5-6 are known as the **Serre relations**.

Theorem 2.2.10. *Let $\mathfrak{g} = \mathfrak{g}(A)$ be a Kac-Moody algebra. Then,*

1. *Suppose $\mathfrak{g}_+ = \langle e_i \mid i \in I \rangle$ and $\mathfrak{g}_- = \langle f_i \mid i \in I \rangle$ are as in Definition 2.2.9. Then $\mathfrak{g}$ has* ***triangular decomposition***

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+.$$

2. *The center of $\mathfrak{g}$ is given by*

$$Z(\mathfrak{g}) = \{h \in \mathfrak{h} \mid \alpha_i(h) = 0 \text{ for all } i \in I\}.$$

Furthermore, the dimension of the center is $dim \mathfrak{h} - |I|$.

Definition 2.2.11. Let $\mathfrak{g} = \mathfrak{g}(A)$ be an affine Kac-Moody algebra with associated affine GCM $A = (a_{ij})_{i,j \in I}$ indexed by $I = \{0, 1, \cdots, n\}$. The **canonical null vector** is the vector $\delta = (a_0 \; a_1 \; \cdots \; a_n)^T$ such that $A\delta = 0$. In this case we decorate the nodes of the Dynkin diagram of A with the labels $a_0, a_1, \cdots, a_n$. In addition, we find a dual vector $v = (\check{a}_0 \; \check{a}_1 \; \cdots \; \check{a}_n)^T$ such that $A^T v = 0$. One can show that the center of $\mathfrak{g}$ is spanned by $c = \check{a}_0 h_0 \; \check{a}_1 h_1 + \cdots \check{a}_n h_n \in \mathfrak{h}$, called the **canonical central element**.

Example 2.2.12. The affine Kac-Moody Lie algebra $\mathfrak{g}_0 = \mathfrak{g}(A_0)$ corresponding to the GCM below is called $G_2^{(1)}$ and has Dynkin diagram given in Fig. 2.1. In this case, we use index $I = \{0, 1, 2\}$.

$$A_0 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 2 \end{pmatrix}$$

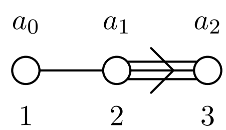


Figure 2.1 Dynkin diagram for $G_2^{(1)}$

Example 2.2.13. The hyperbolic Kac-Moody Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$ corresponding to the GCM below is called $HG_2^{(1)}$ and has Dynkin diagram given in Fig. 2.2. In this case, we use index $I = \{-1, 0, 1, 2\}$ and note that $HG_2^{(1)}$ can be constructed from $G_2^{(1)}$ by adding an additional row and column with entries: $(2, -1, 0, 0)$.

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -3 & 2 \end{pmatrix}$$

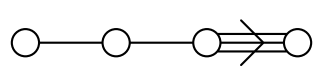


Figure 2.2 Dynkin diagram for $HG_2^{(1)}$

2.2.1 Roots and the Weyl Group

For simple roots α_i $(i \in I)$, the **root lattice** is defined to be the free abelian group $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$. We denote the **positive root lattice** by $Q_+ = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$ and the **negative root lattice** as $Q_- = -Q_+$. Having defined the positive root lattice we endow $\mathfrak{h}^*$ with the partial ordering given by $\lambda \geq \mu$ if and only if $\lambda - \mu \in Q_+$ for $\lambda, \mu \in \mathfrak{h}^*$. We say $\lambda > \mu$ if and only if $\lambda - \mu \in Q_+$ and $\lambda - \mu \neq 0$.

Definition 2.2.14. For $\alpha \in Q$ define the **root space of** α to be

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}.$$

For $\alpha \neq 0$, if $\mathfrak{g}_\alpha \neq 0$ we say α is a **root** of $\mathfrak{g}$ and define its **multiplicity** as $mult(\alpha) = dim\mathfrak{g}_\alpha$. We denote the set of all roots by Δ. Then $\Delta^+ = \Delta \cap Q_+$ (resp. $\Delta^- = \Delta \cap Q_-$) denotes the set of positive roots (resp. negative roots). It is possible to view every root as either positive or negative, therefore, we have the decomposition $\Delta = \Delta^- \cup \Delta^+$.

Example 2.2.15. Consider simple roots $\Pi = \{\alpha_i \mid i \in I\}$. Then $\mathfrak{g}_{\alpha_i} = Fe_i$ and $\mathfrak{g}_{-\alpha_i} = Ff_i$ for all $i \in I$.

Theorem 2.2.16. *[HK02] For $\mathfrak{g}_- = \bigoplus_{\alpha \in \Delta^-} \mathfrak{g}_\alpha$ and $\mathfrak{g}_+ \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$, we can write the triangular decomposition as*

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$$

Theorem 2.2.17. *In addition to a triangular decomposition, $\mathfrak{g}$ has **root space decomposition***

$$\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha$$

and furthermore $dim\mathfrak{g}_\alpha < \infty$ for all $\alpha \in Q$.

Definition 2.2.18. The **simple reflections** are reflections of $\beta \in \mathfrak{h}^*$ defined by

$$r_i(\beta) = \beta - \beta(h_i)\alpha_i \text{ for all } i \in I.$$

Let $g\ell(\mathfrak{h}^*)$ be the group of invertible linear transformations from $\mathfrak{h}^*$ to $\mathfrak{h}^*$. The subgroup W of $g\ell(\mathfrak{h}^*)$ generated by the set of simple reflections is called the **Weyl group**.

Definition 2.2.19. Let $w \in W$ and choose t to be the smallest integer such that $w = r_{i_1} r_{i_2} \cdots r_{i_t}$. Then the expression $w = r_{i_1} r_{i_2} \cdots r_{i_t}$ is called the **reduced expression** of w and we denote the length of w as $\ell(w) = t$.

Definition 2.2.20. A root $\alpha \in \Delta$ is a **real root** if it is Weyl group conjugate to a simple root, i.e. if there exists a $w \in W$ and α_i such that $\alpha = w(\alpha_i)$. Otherwise, α is called an imaginary root.

Denote the set of real roots as Δ^{re} and the imaginary roots as Δ^{im}. Then, $\Delta = \Delta^{re} \cup \Delta^{im}$ and we have the following properties.

1. For $\alpha \in \Delta, w \in W$, $mult(\alpha) = mult(w\alpha)$. In particular, $mult(\alpha) = mult(-\alpha)$.

2. If $\alpha \in \Delta^{re}$, then $mult(\alpha) = 1$.

3. If $\alpha \in \Delta^{re}$, then $\pm\alpha$ are the only multiples of α in Δ^{re}.

4. Suppose $\mathfrak{g}(A)$ is of affine type. If $\alpha \in \Delta^{im}$, then $mult(\alpha) = \text{rank}(A)$.

Hence we can restrict our focus to either positive or negative roots. For convenience we focus on negative roots in this thesis.

Suppose $\mathfrak{g}(A)$ is of finite type, then all roots are Weyl group conjugate to simple roots, and hence have multiplicity equal to 1.

Suppose $\mathfrak{g}(A)$ is of affine type, then the multiplicities of real roots are equal to 1. For imaginary roots we recall the canonical null vector $\delta = (a_0\ a_1\ \cdots\ a_n)^T$ and define the **canonical null root** of $\mathfrak{g}(A)$ as the root $\delta = a_0\alpha_0 + a_1\alpha_0 + \cdots + a_n\alpha_n$ where α_i $(i \in I)$ are the simple roots of $\mathfrak{g}(A)$. Observe that $k\delta$ $(k \in \mathbb{Z}_{\geq 1})$ is not Weyl group conjugate to any simple root and hence is an imaginary root. The imaginary roots of affine Kac-Moody algebras are nonzero multiples of δ and have multiplicity equal to the rank of A.

For $\mathfrak{g}(A)$ of indefinite type, the multiplicities of real roots are equal to 1, as with the finite and affine cases. However, the problem of determining multiplicities of imaginary roots is still an open problem in general for indefinite Kac-Moody algebras. This topic is the primary focus of this thesis.

2.2.2 Weights and Modules of Kac-Moody Algebras

Definition 2.2.21. Let $\mathfrak{g} = \mathfrak{g}(A)$ be a Kac-Moody algebra associated with $A = (a_{ij})_{i,j\in I}$ of rank l. For $i \in I$, let $\Lambda_i \in \mathfrak{h}^*$ be the linear functional satisfying

$$\Lambda_i(h_i) = \delta_{ij} \text{ and } \Lambda_i(d_s) = 0 \ (s = 1, \cdots, |I| - l),$$

then $\{\Lambda_i \mid i \in I\} \subset \mathfrak{h}^*$ is called the set of **fundamental weights** of $\mathfrak{g}$.

As with roots we define the weight space and weight space decomposition for a $\mathfrak{g}$-module V.

Definition 2.2.22. Let V be a $\mathfrak{g}$-module and define the **weight space** of $\mu \in \mathfrak{h}^*$ as

$$V_\mu = \{v \in V \mid h \cdot v = \mu(h)v \text{ for all } h \in \mathfrak{h}\}.$$

If $V_\mu \neq 0$, we call μ a **weight** of V and define the **multiplicity** of μ as $mult(\mu) = dimV_\mu$. Vectors in V_μ are called **weight vectors** and a weight vector $v \in V_\mu$ satisfying $e_i \cdot v = 0$ for all $i \in I$ is called a **maximal vector**. We say V is a **weight module** if it has **weight space decomposition**: $V = \bigoplus_{\mu \in h^*} V_\mu$. We denote the set of weights of V as $wt(V)$. If $dimV_\mu < \infty$ for all $\mu \in \mathfrak{h}^*$, we define the **character** of V to be

$$ch\ V = \sum_{\mu \in \mathfrak{h}^*} dimV_\mu e^\mu,$$

where e^μ are formal basis elements of the group algebra $F[\mathfrak{h}^*]$ such that $e^\lambda e^\mu = e^{\lambda+\mu}$.

Definition 2.2.23. An **integrable** module V over $\mathfrak{g}$ is a weight module for which e_i and f_i $(i \in I)$ are locally nilpotent on V.

Next we give a presentation of the universal enveloping algebra for a Kac-Moody algebra $\mathfrak{g}$ [HK02].

The **universal enveloping algebra** $U(\mathfrak{g})$ is the associative algebra with unity over field F generated by e_i, f_i $(i \in I)$ and $\mathfrak{h}$ satisfying relations:

(1) $hh' = h'h$ for $h \in \mathfrak{h}$,

(2) $e_i f_j - f_j e_i = \delta_{ij} h_i$ for $i, j \in I$,

(3) $he_i - e_i h = \alpha_i(h) e_i$ for $h \in \mathfrak{h}, i \in I$,

(4) $hf_i - f_i h = -\alpha_i(h) f_i$ for $h \in \mathfrak{h}, i \in I$,

(5) $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{pmatrix} 1-a_{ij} \\ k \end{pmatrix} e_i^{1-a_{ij}-k} e_j e_i^k = 0$ for $i \neq j$,

(6) $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{pmatrix} 1-a_{ij} \\ k \end{pmatrix} f_i^{1-a_{ij}-k} f_j f_i^k = 0$ for $i \neq j$.

Now that we have defined the universal enveloping algebra for Kac-Moody algebras, we define an important weight module, called a highest weight module, and its restricted dual, called a lowest weight module.

Definition 2.2.24. We say V is a **highest weight module** of **highest weight** $\lambda \in \mathfrak{h}^*$ if V is a weight module such that there exists a **highest weight vector** $0 \neq v_\lambda \in V$ satisfying

$$V = U(\mathfrak{g})v_\lambda,$$
$$e_i \cdot v_\lambda = 0 \text{ for all } i \in I,$$
$$h \cdot v_\lambda = \lambda(h)v_\lambda \text{ for all } h \in \mathfrak{h}.$$

Observe, $dimV_\lambda = 1$ and $dimV_\mu < \infty$ for all $\mu \in wt(V)$. Therefore, $V = \bigoplus_{\mu \in h^*} V_\mu$, and V is indeed a weight module.

Definition 2.2.25. We say V^* is a **lowest weight module** of **lowest weight** $\lambda \in \mathfrak{h}^*$ if V^* is a weight module such that there exists a **lowest weight vector** $0 \neq v_\lambda^* \in V^*$ satisfying

$$V^* = U(\mathfrak{g})v_\lambda^*,$$
$$f_i \cdot v_\lambda^* = 0 \text{ for all } i \in I,$$
$$h \cdot v_\lambda^* = -\lambda(h)v_\lambda^* \text{ for all } h \in \mathfrak{h}.$$

For a fixed $\lambda \in \mathfrak{h}^*$, consider the left ideal of $U(\mathfrak{g})$ generated by e_i and $h - \lambda(h)1$ $(i \in I, h \in \mathfrak{h})$, denoted by $J(\lambda)$. The **Verma module** is the quotient $M(\lambda) = U(\mathfrak{g})/J(\lambda)$ with $U(\mathfrak{g})$-module structure given by left multiplication. Let $N(\lambda)$ denote the unique maximal submodule of $M(\lambda)$. Then $V(\lambda) = M(\lambda)/N(\lambda)$ is the **irreducible highest weight module** of weight λ.

We now define category $\mathcal{O}$. Let V be weight module and define

$$D(\lambda) = \{\mu \in \mathfrak{h}* \mid \mu \leq \lambda\} \text{ for } \lambda \in \mathfrak{h}^*\}.$$

Definition 2.2.26. [**Category** $\mathcal{O}$]

- Objects: weight modules V over $\mathfrak{g}$ with $dimV_\lambda < \infty$ for all $\lambda \in \mathfrak{h}^*$ and for which there exists a finite number of weights $\lambda_1, \cdots, \lambda_s \in \mathfrak{h}^*$ such that $wt(V) \subset D(\lambda_1) \cup \cdots \cup D(\lambda_s)$

- Morphisms: $\mathfrak{g}$-module homomorphisms

- Closure: the category is closed under finite direct sums or finite tensor products

Proposition 2.2.27. *The following are true of weight modules V in category $\mathcal{O}$.*

1. *Let V be a highest weight module, then $V \in \mathcal{O}$.*

2. *Let V be an irreducible module in $\mathcal{O}$, then V is isomorphic to $V(\lambda)$ for some $\lambda \in \mathfrak{h}^*$.*

We now define a subcategory of $\mathcal{O}$ in which all modules are integrable and have integral weights.

Definition 2.2.28. [**Category $\mathcal{O}_{int}$**]
The objects in this category are integrable $\mathfrak{g}$-modules in category $\mathcal{O}$ such than $wt(V) \subset P$.

As a consequence of Definition 2.2.28, every weight module V in category $\mathcal{O}_{int}$ has a weight space decomposition $V = \bigoplus_{\lambda \in P} V_\lambda$ where $V_\lambda = \{v \in V \mid h \cdot v = \mu(h)v \text{ for all } h \in \check{P}\}$.

Theorem 2.2.29. *Let V be a weight module in category $\mathcal{O}_{int}$ and let $w \in W$, the Weyl group. Then for weight $\lambda \in wt(V)$,*

$$dimV_\lambda = dimV_{w\lambda}.$$

Recall the weight lattice $P = \{\lambda \in \mathfrak{h}^* \mid \lambda(\check{P}) \subset \mathbb{Z}\}$. We say weight $\lambda \in P$ is an **integral weight** since $\lambda(h_i) \in \mathbb{Z}$ for all $i \in I$. The set of **dominant integral weights** is denoted by

$$P^+ = \{\lambda \in P \mid \lambda(h_i) \in \mathbb{Z}_{\geq 0} \text{ for all } i \in I\}.$$

Proposition 2.2.30. *The following hold for weight modules V in category $\mathcal{O}_{int}$.*

1. *Let $V(\lambda)$ be the irreducible highest weight module with highest weight $\lambda \in \mathfrak{h}^*$, then $V(\lambda) \in \mathcal{O}_{int}$ if and only if $\lambda \in P^+$.*

2. *Every irreducible module in $\mathcal{O}_{int}$ is isomorphic to $V(\lambda)$ for some $\lambda \in P^+$.*

Observe that part 2 of Proposition 2.2.30 is an immediate consequence of parts 1 and 2 from Proposition 2.2.30 and Proposition 2.2.27 respectively. Finally, by Theorem 2.2.31 below we have that V in $\mathcal{O}_{int}$ is completely reducible.

Theorem 2.2.31. *Let V be any $\mathfrak{g}$-module in category $\mathcal{O}_{int}$. Then, $V \cong \bigoplus_{\lambda \in P^+} V(\lambda)$.*

CHAPTER

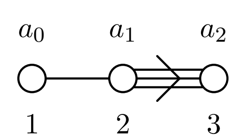
3

CONSTRUCTION OF $HG_2^{(1)}$

In this chapter we realize $HG_2^{(1)}$ using the construction outlined in [Ben93]. Starting with the affine Kac-Moody Lie algebra $\mathfrak{g}_0 = G_2^{(1)}$, we first define a $\mathfrak{g}_0$-module homomorphism $\Psi : V^*(\Lambda_0) \otimes V(\Lambda_0) \to \mathfrak{g}_0$, where $V^*(\Lambda_0)$ and $V(\Lambda_0)$ are irreducible highest weight $G_2^{(1)}$-modules. Using this homomorphism, we then build the maximal (resp. minimal) $\mathbb{Z}$-graded Lie algebra $\hat{\mathfrak{g}}$ (resp. $\mathfrak{g}$) with local part $V(\Lambda_0) \oplus \mathfrak{g}_0 \oplus V^*(\Lambda_0)$. In Theorem 3.0.1, we show that $\mathfrak{g}$ is isomorphic to $HG_2^{(1)}$, completing our construction.

We begin with the affine Kac-Moody Lie algebra $\mathfrak{g}_0 = G_2^{(1)}$, having the generalized Cartan matrix A_0 and corresponding Dynkin diagram presented in Section 2.2 and reproduced below. Then we denote the Cartan subalgebra, central canonical element, simple roots, and fundamental weights of $\mathfrak{g}_0$ below.

$$A_0 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 2 \end{pmatrix}$$



- Cartan subalgebra: $\mathfrak{h}_0 = \mathrm{span}_{\mathbb{C}}\{H_0, H_1, H_2\} \cup \{d\}$

- central canonical element: $c = H_0 + 2H_1 + H_2$
- simple roots: $\Pi = \{\alpha_0, \alpha_1, \alpha_2\} \subset \mathfrak{h}_0^*$, satisfying

$$\alpha_j(H_i) = a_{ij} \text{ and } \alpha_j(d) = \delta_{0j} \ (i, j = 0, 1, 2)$$

- fundamental weights: $\{\Lambda_0, \Lambda_1, \Lambda_2\} \subset \mathfrak{h}_0^*$, satisfying

$$\Lambda_i(H_i) = \delta_{ij} \text{ and } \Lambda_i(d) = 0 \ (i, j = 0, 1, 2)$$

We focus on Λ_0 and recall that $V(\Lambda_0)$ is the irreducible highest weight module with highest weight Λ_0 and highest weight vector v_0 satisfying the following relations for $i \in \{0, 1, 2\}$.

$$H_i \cdot v_0 = \Lambda_0(H_i) \cdot v_0 = \delta_{0i} \cdot v_0$$

$$E_i \cdot v_0 = 0$$

$$d \cdot v_0 = \Lambda_0(d) \cdot v_0 = 0$$

Similarly, $V^*(\Lambda_0)$ is the irreducible lowest weight module with lowest weight $-\Lambda_0$ lowest weight vector v_0^* and satisfies the following for $i \in \{0, 1, 2\}$.

$$H_i \cdot v_0^* = -\Lambda_0(H_i) \cdot v_0 = -\delta_{0i} \cdot v_0$$

$$F_i \cdot v_0^* = 0$$

$$d \cdot v_0^* = -\Lambda_0(d) \cdot v_0 = 0$$

In general, the module action of $\mathfrak{g}_0$ is given by $\langle g \cdot v^*, v\rangle = -\langle v, g \cdot v^*\rangle$, equivalently $(g \cdot v^*)(v) = -v^*(g \cdot v)$, for every $g \in \mathfrak{g}_0, v \in V(\Lambda_0),$ and $v^* \in V^*(\Lambda_0)$.

Next we define a $\mathfrak{g}_0$-module homomorphism from $V^*(\Lambda_0) \otimes V(\Lambda_0)$ to $\mathfrak{g}_0$, which is key in our construction of $HG_2^{(1)}$. For this purpose we recall from [Kac90] the standard, symmetric, bilinear form on $\mathfrak{h}_0$, $(\cdot|\cdot)$, defined as:

$$(H_i|H_j) = a_j \check{a_j}^{-1} a_{ij}, \ i, j \in \{0, 1, 2\}$$

$$(d|d) = 0$$

$$(H_i|d) = \delta_{i0}, \ i \in \{0, 1, 2\}$$

where a_{ij} corresponds to the ij-th entry of A_0, a_i corresponds to the labels of the Dynkin diagram for $G_2^{(1)}$, and $\check{a}_j$ corresponds to the labels of the Dynkin diagram for the dual algebra, $D_3^{(4)}$, given in Fig. 3.1.

$D_4^{(3)}$

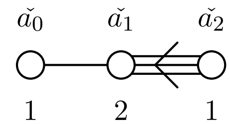


Figure 3.1 Dynkin diagram for $D_4^{(3)}$

Specifically, $(\cdot|\cdot)$ acting on the basis elements of $\mathfrak{h}_0$ is computed as:

$$(H_0|d) = \delta_{0,0} = 1$$
$$(H_0|H_0) = (1)(\frac{1}{1})(2) = 2$$
$$(H_0|H_1) = (2)(\frac{1}{2})(-1) = -1$$
$$(H_0|H_2) = (3)(\frac{1}{1})(0) = 0$$
$$(H_1|d) = \delta_{1,0} = 0$$
$$(H_1|H_1) = (2)(\frac{1}{2})(2) = 2$$
$$(H_1|H_2) = (3)(\frac{1}{1})(-1) = -3$$
$$(H_2|d) = \delta_{2,0} = 0$$
$$(H_2|H_2) = (3)(\frac{1}{1})(2) = 6$$
$$(d|d) = 0.$$

Then $(\cdot|\cdot)$ is nondegenerate and can be uniquely extended to a nondegenerate, symmetric, bilinear form on the elements of $\mathfrak{g}_0$. See Theorem 2.2 of [Kac90] for more details. Therefore, $\mathfrak{g}_0$ has an orthonormal basis with respect to $(\cdot|\cdot)$, which we denote as $\{x_i|i \in \mathcal{I}\}$ and use in our homomorphism from $V^*(\Lambda_0) \otimes V(\Lambda_0)$ to $\mathfrak{g}_0$.

Define the map $\Psi : V^*(\Lambda_0) \otimes V(\Lambda_0) \longrightarrow \mathfrak{g}_0$ by

$$\Psi(v^* \otimes v) = -\sum_{i\in\mathcal{I}} \langle v^*|x_i \cdot v\rangle x_i - 2\langle v^*|v\rangle c \tag{3.1}$$

where $v \in V(\Lambda_0), v^* \in V^*(\Lambda_0)$, and $c = H_0 + 2H_1 + H_2$ is the canonical central element of $\mathfrak{g}_0$. Then Ψ is a $\mathfrak{g}_0$-module homomorphism by [Ben93].

Having defined a $\mathfrak{g}_0$-module homomorphism Ψ, we construct the $\mathbb{Z}$-graded Lie algebra $\hat{\mathfrak{g}}$ with local part $V(\Lambda_0) \oplus \mathfrak{g}_0 \oplus V^*(\Lambda_0)$, having local Lie algebra structure given by

$$[v^*, v] = \Psi(v^* \otimes v)$$

$$[g, v] = g \cdot v$$

$$[g, v^*] = g \cdot v^*$$

for $v \in V(\Lambda_0), v^* \in V^*(\Lambda_0)$, and $g \in \mathfrak{g}_0$ that can be extended to $\hat{\mathfrak{g}}$ [Ben93]. For this purpose we define the spaces:

$$\hat{\mathfrak{g}}_{-1} = V(\Lambda_0)$$

$$\hat{\mathfrak{g}_0} = \mathfrak{g}_0$$

$$\hat{\mathfrak{g}}_1 = V^*(\Lambda_0)$$

For $i > 1$, let $\hat{\mathfrak{g}}_i$ (resp. $\hat{\mathfrak{g}}_{-i}$) be the space spanned by all products of i vectors from $\hat{\mathfrak{g}}_1$ (resp. $\hat{\mathfrak{g}}_{-1}$). In particular,

$$\hat{\mathfrak{g}}_{-i} = \mathrm{span}_{\mathbb{C}}\{[y_1, [y_2 \ldots, [y_{i-1}, y_i]] \ldots] \mid y_1, \cdots, y_i \in \hat{\mathfrak{g}}_{-1}\}$$

$$\hat{\mathfrak{g}}_i = \mathrm{span}_{\mathbb{C}}\{[y_1^*, [y_2^* \ldots, [y_{i-1}^*, y_i^*]] \ldots] \mid y_1^*, \cdots, y_i^* \in \hat{\mathfrak{g}}_1\}.$$

Then $\hat{\mathfrak{g}}_- = \bigoplus_{i\geq 1} \hat{\mathfrak{g}}_{-i}$ and $\hat{\mathfrak{g}}_+ = \bigoplus_{i\geq 1} \hat{\mathfrak{g}}_i$ are free Lie algebras generated by $\hat{\mathfrak{g}}_{-1}$ and $\hat{\mathfrak{g}}_1$, respectively. In addition,

$$\begin{aligned} \hat{\mathfrak{g}} &= \bigoplus_{i\in\mathbb{Z}} \hat{\mathfrak{g}}_i \\ &= \left(\bigoplus_{i>1} \hat{\mathfrak{g}}_{-i}\right) \oplus V(\Lambda_0) \oplus \mathfrak{g}_0 \oplus V^*(\Lambda_0) \oplus \left(\bigoplus_{i>1} \hat{\mathfrak{g}}_i\right) \end{aligned} \tag{3.2}$$

is the maximal graded Lie algebra with local part $V(\Lambda_0) \oplus \mathfrak{g}_0 \oplus V^*(\Lambda_0)$. To find the minimal graded Lie algebra with local part $V(\Lambda_0) \oplus \mathfrak{g}_0 \oplus V^*(\Lambda_0)$, we define the subspaces for $k > 1$:

$$J_k = \{x \in \hat{\mathfrak{g}}_k \mid [y_1, [\dots, [y_{k-1}, x]] \dots] = 0 \ \forall y_1, \dots, y_{k-1} \in V(\Lambda_0)\}$$

$$J_{-k} = \{x \in \hat{\mathfrak{g}}_{-k} \mid [y_1^*, [\dots, [y_{k-1}^*, x]] \dots] = 0 \ \forall y_1^*, \dots, y_{k-1}^* \in V^*(\Lambda_0)\}$$

$$J_\pm = \sum_{k>1} J_{\pm k}$$

$$J = J_- \oplus J_+$$

Since $\hat{\mathfrak{g}}_- = \bigoplus_{i \geq 1} \hat{\mathfrak{g}}_{-i}$ and $\hat{\mathfrak{g}}_+ = \bigoplus_{i \geq 1} \hat{\mathfrak{g}}_i$ are free Lie algebras, $J_\pm$ are ideals of $\hat{\mathfrak{g}}$ and J is the largest graded ideal of $\hat{\mathfrak{g}}$ which intersects the local part of $\hat{\mathfrak{g}}$ trivially. Define,

$$\begin{aligned} \mathfrak{g} &= \hat{\mathfrak{g}}/J \\ &= \left(\bigoplus_{i>1} \mathfrak{g}_{-i} \right) \oplus V(\Lambda_0) \oplus \mathfrak{g}_0 \oplus V^*(\Lambda_0) \oplus \left(\bigoplus_{i>1} \mathfrak{g}_i \right) \end{aligned} \tag{3.3}$$

where $\mathfrak{g}_{\pm i} = \hat{\mathfrak{g}}_{\pm i}/J_{\pm i}$ for $i > 1$. Then $\mathfrak{g}$ is the minimal graded Lie algebra with local part $V(\Lambda_0) \oplus \mathfrak{g}_0 \oplus V^*(\Lambda_0)$. In Theorem 3.0.1, we prove $\mathfrak{g}$ is isomorphic to $HG_2^{(1)}$.

Theorem 3.0.1. *Consider the map $\Phi : HG_2^{(1)} \to \mathfrak{g}$ defined by*

$$e_{-1} \mapsto v_0^*, \quad f_{-1} \mapsto v_0, \quad h_{-1} \mapsto -d - 2c,$$

$$e_i \mapsto E_i, \quad f_i \mapsto F_i, \quad h_i \mapsto H_i$$

for $i = 0, 1, 2$ where $\{e_{-1}, e_0, e_1, e_2\}$ and $\{f_{-1}, f_0, f_1, f_2\}$ are the Chevalley generators of $HG_2^{(1)}$, and $\mathfrak{h} = span_{\mathbb{C}}\{h_{-1}, h_0, h_1, h_2\}$ is the Cartan subalgebra of $HG_2^{(1)}$. Then, Φ is a Lie algebra isomorphism.

Proof. We begin by reviewing $HG_2^{(1)}$, with Cartan matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -3 & 2 \end{pmatrix}$$

Then $HG_2^{(1)}$ is the Kac-Moody algebra generated by $\{e_{-1}, e_0, e_1, e_2\}, \{f_{-1}, f_0, f_1, f_2\}$, and the

Cartan subalgebra $\mathfrak{h}$ with the following relations for $i, j = \{-1, 0, 1, 2\}$, and $h, h' \in \mathfrak{h}$, and $a_{ij} \in A$:

$$
\begin{aligned}
&[e_i, f_j] = \delta_{ij} h_i \\
&[h, h'] = 0 \\
&[h, e_i] = \alpha_i(h) e_i \\
&[h, f_i] = -\alpha_i(h) f_i \\
&(ad_{e_i})^{1-a_{ij}}(e_j) = 0, i \neq j \\
&(ad_{f_i})^{1-a_{ij}}(f_j) = 0, i \neq j
\end{aligned}
$$

where α_i is a simple root of $HG_2^{(1)}$ defined by $\alpha_i(h_j) = a_{ji}$ as with $\mathfrak{g}_0$.

From the previous section, we have that J is the maximal graded ideal of $\hat{\mathfrak{g}}$ which intersects the local part of $\hat{\mathfrak{g}}$ trivially. In addition, it is clear that Φ is a linear bijection. Thus, we need only show the following relations hold for $i, j = \{-1, 0, 1, 2\}$, and $h, h' \in \mathfrak{h}$, and $a_{ij} \in A$:

$$
\begin{aligned}
&[\Phi(e_i), \Phi(f_j)] = \delta_{ij} \Phi(h_i) \\
&[\Phi(h), \Phi(h')] = 0 \\
&[\Phi(h), \Phi(e_i)] = \alpha_i(h) \Phi(e_i) \\
&[\Phi(h), \Phi(f_i)] = -\alpha_i(h) \Phi(f_i)
\end{aligned}
$$

Consider the above relations for $i, j \in \{0, 1, 2\}$. We observe that A is identical to A_0 with an extra row and column in the (-1)-position. Thus, A and A_0 are identical when we restrict to $i, j \in \{0, 1, 2\}$. Now for $i, j \in \{0, 1, 2\}$, we have $\Phi(e_i) = E_i, \Phi(f_i) = F_i$, and $\Phi(h_i) = H_i$, as defined by our homomorphism. Thus, the above relations hold for $i, j \in \{0, 1, 2\}$. We need only to show:

$$[\Phi(e_{-1}), \Phi(f_{-1})] = \Phi(h_{-1}) \tag{3.4}$$

$$[\Phi(e_{-1}), \Phi(f_i)] = 0, (i = 0, 1, 2) \tag{3.5}$$

$$[\Phi(e_i), \Phi(f_{-1})] = 0, (i = 0, 1, 2) \tag{3.6}$$

$$[\Phi(h), \Phi(h')] = 0 \tag{3.7}$$

$$[\Phi(h), \Phi(e_{-1})] = \alpha_{-1}(h) \Phi(e_{-1}) \tag{3.8}$$

$$[\Phi(h), \Phi(f_{-1})] = -\alpha_{-1}(h) \Phi(f_{-1}) \tag{3.9}$$

To show (3.4), we give a change of basis for $\mathfrak{h}$ so that the new basis is orthonormal with respect to the bilinear form $(\cdot|\cdot)|_{\mathfrak{h}}$. Choose $y_{-1} = \frac{1}{\sqrt{-2}}(d-c)$ and $y_0 = \frac{1}{\sqrt{2}}(d+c)$, where $c = H_0 + 2H_1 + H_2$ and d is as defined in the previous section. Then y_{-1} and y_0 are orthonormal:

$$\begin{aligned}(y_{-1}|y_{-1}) &= -\frac{1}{2}(d-c|d-c)\\ &= -\frac{1}{2}[(d|d)-(d|c)-(c|d)+(c|c)]\\ &= -\frac{1}{2}[0-1-1+0]\\ &= 1\end{aligned}$$

$$\begin{aligned}(y_0|y_0) &= \frac{1}{2}(d+c|d+c)\\ &= \frac{1}{2}[(d|d)+(d|c)+(c|d)+(c|c)]\\ &= \frac{1}{2}[0+1+1+0]\\ &= 1\end{aligned}$$

$$\begin{aligned}(y_{-1}|y_0) &= \frac{1}{2i}(d-c|d+c)\\ &= \frac{1}{2i}[(d|d)+(d|c)-(c|d)-(c|c)]\\ &= \frac{1}{2i}[0-1+1-0]\\ &= 0\end{aligned}$$

Now let $\{y_1, y_2\}$ be an orthonormal basis for $\mathcal{H} = \text{span}\{H_1, H_2\}$. This basis exists because $(\cdot|\cdot)|_{\mathcal{H}}$ is an inner product. Then $\{y_{-1}, y_0, y_1, y_2\}$ is an orthonormal basis for $\mathfrak{h}$ and can be extend to form an orthonormal basis for $HG_2^{(1)}$, denoted by $\{x_i | i \in I\}$ and ordered so that $x_i = y_i$ for $i \in \{-1, 0, 1, 2\}$. It follows that

$$\begin{aligned}[\Phi(e_{-1}), \Phi(f_{-1})] &= [v_0^*, v_0]\\ &= \Psi(v_0^* \otimes v_0)\\ &= -\sum_{i\in\mathcal{I}} \langle v_0^* | x_i \cdot v_0\rangle x_i - 2\langle v_0^* | v_0\rangle c\end{aligned}$$

$$= -\sum_{i\in\mathcal{I}} v_0^*(x_i \cdot v_0)x_i - 2v_0^*(v_0)c. \tag{3.10}$$

Clearly x_i is a linear combination of E_i and F_i for $x_i \neq y_i$. Therefore the expression $v_0^*(x_i \cdot v_0) = -(x_i \cdot v_0^*)(v_0)$ in (3.10) vanishes for $x_i \neq y_i$ since $E_i \cdot v_0 = 0$ and $F_i \cdot v_0^* = 0$. Therefore,

$$\begin{aligned}[\Phi(e_{-1}), \Phi(f_{-1})] &= -v_0^*(y_{-1} \cdot v_0)y_{-1} - v_0^*(y_0 \cdot v_0)y_0 - \sum_{i=1}^{2} v_0^*(x_i \cdot v_0)x_i - 2v_0^*(v_0)c \\ &= -v_0^*(y_{-1} \cdot v_0)y_{-1} - v_0^*(y_0 \cdot v_0)y_0 - \sum_{i=1}^{2} v_0^*(x_i \cdot v_0)x_i - 2c. \end{aligned} \tag{3.11}$$

We now argue that summation portion of equation (3.11) equals 0. Recall $\{x_1, x_2\} = \{y_1, y_2\}$ forms a basis for $\mathcal{H} = \{H_1, H_2\}$. As such, x_1 and x_2 are linear combinations of H_1 and H_2 only and thus the action $H_i \cdot v_0 = \delta_{i0} \cdot v_0$ implies $x_1 \cdot v_0 = x_2 \cdot v_0 = 0$. Therefore,

$$\begin{aligned}[\Phi(e_{-1}), \Phi(f_{-1})] &= -v_0^*(y_{-1} \cdot v_0)y_{-1} - v_0^*(y_0 \cdot v_0)y_0 - 2c \\ &= -v_0^*\left(\frac{1}{\sqrt{-2}}(d-c) \cdot v_0\right)\frac{1}{\sqrt{-2}}(d-c) - v_0^*\left(\frac{1}{\sqrt{2}}(d+c) \cdot v_0\right)\frac{1}{\sqrt{2}}(d+c) - 2c \\ &= \frac{1}{2}v_0^*(d \cdot v_0 - c \cdot v_0)(d-c) - \frac{1}{2}v_0^*(d \cdot v_0 + c \cdot v_0)(d+c) - 2c \\ &= -\frac{1}{2}v_0^*(v_0)(d-c) - \frac{1}{2}v_0^*(v_0)(d+c) - 2c \\ &= -\frac{1}{2}(d-c) - \frac{1}{2}(d+c) - 2c \\ &= -d - 2c \\ &= \Phi(h_{-1}). \end{aligned}$$

Next we prove (3.5) and (3.6) below, simultaneously for $i = 0, 1, 2$.

$$\begin{aligned}[\Phi(e_{-1}), \Phi(f_i)] &= [v_0^*, F_i] \\ &= -[F_i, v_0^*] \\ &= -F_i \cdot v_0^* \\ &= 0 \end{aligned} \qquad \begin{aligned}[\Phi(e_i), \Phi(f_{-1})] &= [E_i, v_0] \\ &= E_i \cdot v_0 \\ &= 0 \end{aligned}$$

To prove (3.7) we observe that $\Phi(h_{-1}) = -d - 2H_0 - 4H_1 - 2H_2$ and $\Phi(h_i) = H_i$ are elements of $\mathfrak{h}_0$ for $i = 0, 1, 2$ and recall that $[H, H'] = 0$ for all $H, H' \in \mathfrak{h}_0$. Therefore, the linearity of Φ

guarantees that the images of h and h' will be in $\mathfrak{h}_0$ and hence $[\Phi(h), \Phi(h')] = 0$ for all $h, h' \in \mathfrak{h}$.

It remains to prove (3.8) and (3.9). For convenience, suppose $h = c_{-1}h_{-1}+c_0h_0+c_1h_1+c_2h_2 \in \mathfrak{h}$ for complex scalars c_{-1}, c_0, c_1, c_2. Then,

$$\begin{aligned}[\Phi(h), \Phi(e_{-1})] &= [c_{-1}(-d-2c) + c_0H_0 + c_1H_1 + c_2H_2, v_0^*] \\ &= (2c_{-1} - c_0)v_0^*\end{aligned}$$

and

$$\begin{aligned}\alpha_{-1}(h)\Phi(e_{-1}) &= \alpha_{-1}(c_{-1}h_{-1} + c_0h_0 + c_1h_1 + c_2h_2) \cdot v_0^* \\ &= (c_{-1}a_{-1,-1} + c_0a_{0,-1} + c_1a_{1,-1} + c_2a_{2,-1}) \cdot v_0^* \\ &= (2c_{-1} - c_0)v_0^*\end{aligned}$$

implies $[\Phi(h), \Phi(e_{-1})] = \alpha_{-1}(h)\Phi(e_{-1})$. Similarly,

$$\begin{aligned}[\Phi(h), \Phi(f_{-1})] &= [c_{-1}(-d-2c) + c_0H_0 + c_1H_1 + c_2H_2, v_0] \\ &= (-2c_{-1} + c_0)v_0 \\ &= -(2c_{-1} - c_0)v_0\end{aligned}$$

and

$$\begin{aligned}\alpha_{-1}(h)\Phi(f_{-1}) &= -\alpha_{-1}(c_{-1}h_{-1} + c_0h_0 + c_1h_1 + c_2h_2) \cdot v_0 \\ &= -(c_{-1}a_{-1,-1} + c_0a_{0,-1} + c_1a_{1,-1} + c_2a_{2,-1}) \cdot v_0 \\ &= -(2c_{-1} - c_0)v_0\end{aligned}$$

implies $[\Phi(h), \Phi(f_{-1})] = \alpha_{-1}(h)\Phi(f_{-1})$. □

CHAPTER

4

GENERAL RESULTS

In this chapter we present some well known theorems of general hyperbolic Kac-Moody algebras, as well as some new results. Together these results are instrumental in our exploration of level 1, 2, and 3 root multiplicities in Chapter 6. Specifically, we provide some general results regarding the multiplicities of certain level ℓ roots in Section 4.1. In Section 4.2 we review Kang's multiplicity formula, which provides an algorithm for computing root multiplicities of $HG_2^{(1)}$ using weight multiplicities of irreducible highest weight $G_2^{(1)}$-modules. In Section 4.3 we review the multiplicity bound given by Frenkel's conjecture, which are verified for all roots in Chapter 6.

4.1 Level ℓ Root Multiplicities

In this section we use the simple Weyl group reflections r_i $(i = -1, 0, 1, 2)$ defined by

$$r_i(\beta) = \beta - \beta(h_i)\alpha_i, \text{ for } \beta \in \mathfrak{h}$$

to prove several multiplicity theorems for general roots of level ℓ. These theorems are utilized in Chapter 6 to find multiplicities of specific roots of level $\ell \leq 3$.

Proposition 4.1.1. *Let $-\ell\alpha_{-1} - m_0\alpha_0 - m_1\alpha_1 - m_2\alpha_2$ be a root of $HG_2^{(1)}$, then*

$$r_{-1}(-\ell\alpha_{-1} - m_0\alpha_0 - m_1\alpha_1 - m_2\alpha_2) = -(m_0 - \ell)\alpha_{-1} - m_0\alpha_0 - m_1\alpha_1 - m_2\alpha_2$$
$$r_0(-\ell\alpha_{-1} - m_0\alpha_0 - m_1\alpha_1 - m_2\alpha_2) = -\ell\alpha_{-1} - (\ell - m_0 + m_1)\alpha_0 - m_1\alpha_1 - m_2\alpha_2$$
$$r_1(-\ell\alpha_{-1} - m_0\alpha_0 - m_1\alpha_1 - m_2\alpha_2) = -\ell\alpha_{-1} - m_0\alpha_0 - (m_0 - m_1 + m_2)\alpha_1 - m_2\alpha_2$$
$$r_2(-\ell\alpha_{-1} - m_0\alpha_0 - m_1\alpha_1 - m_2\alpha_2) = -\ell\alpha_{-1} - m_0\alpha_0 - m_1\alpha_1 - (3m_1 - m_2)\alpha_2$$

Proof.

$$\begin{aligned} r_{-1}(-\ell\alpha_{-1} - m_0\alpha_0 - m_1\alpha_1 - m_2\alpha_2) &= -\ell\alpha_{-1} - m_0\alpha_0 - m_1\alpha_1 - m_2\alpha_2 \\ &\quad - (-\ell\alpha_{-1} - m_0\alpha_0 - m_1\alpha_1 - m_2\alpha_2)(h_{-1})\alpha_{-1} \\ &= -\ell\alpha_{-1} - m_0\alpha_0 - m_1\alpha_1 - m_2\alpha_2 - (-2\ell + m_0)\alpha_{-1} \\ &= -(m_0 - \ell)\alpha_{-1} - m_0\alpha_0 - m_1\alpha_1 - m_2\alpha_2 \end{aligned}$$

$$\begin{aligned} r_0(-\ell\alpha_{-1} - m_0\alpha_0 - m_1\alpha_1 - m_2\alpha_2) &= -\ell\alpha_{-1} - m_0\alpha_0 - m_1\alpha_1 - m_2\alpha_2 \\ &\quad - (-\ell\alpha_{-1} - m_0\alpha_0 - m_1\alpha_1 - m_2\alpha_2)(h_0)\alpha_0 \\ &= -\ell\alpha_{-1} - m_0\alpha_0 - m_1\alpha_1 - m_2\alpha_2 - (\ell - 2m_0 + m_1)\alpha_0 \\ &= -\ell\alpha_{-1} - (\ell - m_0 + m_1)\alpha_0 - m_1\alpha_1 - m_2\alpha_2 \end{aligned}$$

$$\begin{aligned} r_1(-\ell\alpha_{-1} - m_0\alpha_0 - m_1\alpha_1 - m_2\alpha_2) &= -\ell\alpha_{-1} - m_0\alpha_0 - m_1\alpha_1 - m_2\alpha_2 \\ &\quad - (-\ell\alpha_{-1} - m_0\alpha_0 - m_1\alpha_1 - m_2\alpha_2)(h_1)\alpha_1 \\ &= -\ell\alpha_{-1} - m_0\alpha_0 - m_1\alpha_1 - m_2\alpha_2 - (m_0 - 2m_1 + m_2)\alpha_1 \\ &= -\ell\alpha_{-1} - m_0\alpha_0 - (m_0 - m_1 + m_2)\alpha_1 - m_2\alpha_2 \end{aligned}$$

$$\begin{aligned} r_2(-\ell\alpha_{-1} - m_0\alpha_0 - m_1\alpha_1 - m_2\alpha_2) &= -\ell\alpha_{-1} - m_0\alpha_0 - m_1\alpha_1 - m_2\alpha_2 \\ &\quad - (-\ell\alpha_{-1} - m_0\alpha_0 - m_1\alpha_1 - m_2\alpha_2)(h_2)\alpha_2 \\ &= -\ell\alpha_{-1} - m_0\alpha_0 - m_1\alpha_1 - m_2\alpha_2 - (3m_1 - 2m_2)\alpha_2 \\ &= -\ell\alpha_{-1} - m_0\alpha_0 - m_1\alpha_1 - (3m_1 - m_2)\alpha_2 \end{aligned}$$

□

Theorem 4.1.2. *If $\ell, k \in \mathbb{Z}_{>0}$ and $-\ell\alpha_{-1} - k\delta$ is a root of $HG_2^{(1)}$, then $k \geq \ell$. Furthermore,*

$$mult(-\ell\alpha_{-1} - \ell\delta) = 2.$$

Proof. Suppose $-\ell\alpha_{-1} - k\delta$ is a root. By Proposition 4.1.1,

$$r_{-1}(-\ell\alpha_{-1} - k\delta) = -(k-\ell)\alpha_{-1} - k\delta$$

is a root as well. Therefore $k \geq \ell$. If $k = \ell$, then

$$\begin{aligned} mult(-\ell\alpha_{-1} - \ell\delta) &= mult(r_{-1}(-\ell\alpha_{-1} - \ell\delta)) \\ &= mult(-\ell\delta) \\ &= \text{rank of } A_0 \\ &= 2 \end{aligned}$$

□

Let $\Delta^{\pm}$ be the set of positive (respectively, negative) roots of $\mathfrak{g} = HG_2^{(1)}$ and $\Delta_S^{\pm}$ be the set of positive (respectively, negative) roots of $\mathfrak{g}_0 = G_2^{(1)}$, where $S = \{0, 1, 2\}$.

Theorem 4.1.3. *For $\ell, k \in \mathbb{Z}_{>0}$ and $\alpha = m_1\alpha_1 + m_2\alpha_2 \in \Delta_S^+ \cup \{0\}$,*

$$mult(-\ell\alpha_{-1} - \alpha - k\delta) = mult(-(k-\ell)\alpha_{-1} - \alpha - k\delta).$$

Proof. We use Proposition 4.1.1 to show $-\ell\alpha_{-1} - \alpha - k\delta$ and $-(k-\ell)\alpha_{-1} - \alpha - k\delta$ are Weyl group conjugate:

$$\begin{aligned} r_{-1}(-\ell\alpha_{-1} - \alpha - k\delta) &= r_{-1}(-\ell\alpha_{-1} - k\alpha_0 - (2k+m_1)\alpha_1 - (3k+m_2)\alpha_2) \\ &= -(k-\ell)\alpha_{-1} - k\alpha_0 - (2k+m_1)\alpha_1 - (3k+m_2)\alpha_2 \\ &= -(k-\ell)\alpha_{-1} - \alpha - k\delta \end{aligned}$$

Therefore $mult(-\ell\alpha_{-1} - \alpha - k\delta) = mult(-(k-\ell)\alpha_{-1} - \alpha - k\delta)$.

□

Corollary 4.1.4. *For $\ell \in Z_{>0}$ and $\alpha = m_1\alpha_1 + m_2\alpha_2 \in \Delta_S^+$,*

$$mult(-\ell\alpha_{-1} - \alpha - \ell\delta) = 1.$$

Proof. By Theorem 4.1.3,

$$\begin{aligned} mult(-\ell\alpha_{-1} - \alpha - \ell\delta) &= mult(-(\ell - \ell)\alpha_{-1} - \alpha - \ell\delta) \\ &= mult(-\alpha - \ell\delta). \end{aligned}$$

If $\alpha \in \Delta_S^+$, then $-\alpha - \ell\delta$ is a real root and hence has multiplicity 1. Therefore,

$$mult(-\ell\alpha_{-1} - \alpha - \ell\delta) = 1.$$

□

Theorem 4.1.5. *For $\ell, k \in \mathbb{Z}_{>0}$ and $\alpha = m_1\alpha_1 + m_2\alpha_2 \in \Delta_S^+ \cup \{0\}$,*

$$mult(-\ell\alpha_{-1} - \ell\alpha_0 - \alpha - k\delta) = mult(-k\alpha_{-1} - \ell\alpha_0 - \alpha - k\delta).$$

Proof. We use Proposition 4.1.1 to show $-\ell\alpha_{-1} - \ell\alpha_0 - \alpha - k\delta$ and $-k\alpha_{-1} - \ell\alpha_0 - \alpha - k\delta$ are Weyl group conjugate:

$$\begin{aligned} r_{-1}(-\ell\alpha_{-1} - \ell\alpha_0 - \alpha - k\delta) &= r_{-1}(-\ell\alpha_{-1} - (\ell + k)\alpha_0 - (2k + m_1)\alpha_1 - (3k + m_2)\alpha_2) \\ &= -k\alpha_{-1} - (\ell + k)\alpha_0 - (2k + m_1)\alpha_1 - (3k + m_2)\alpha_2 \\ &= -k\alpha_{-1} - \ell\alpha_0 - \alpha - k\delta \end{aligned}$$

Therefore $mult(-\ell\alpha_{-1} - \ell\alpha_0 - \alpha - k\delta) = mult(-k\alpha_{-1} - \ell\alpha_0 - \alpha - k\delta)$.

□

In the next sections we provide a means for computing additional root multiplicities of $HG_2^{(1)}$, as well as a conjectured bound on the multiplicities.

4.2 Kang's multiplicity Formula

In Chapter 3 we constructed $HG_2^{(1)}$ as a graded Lie algebra with degree zero component $G_2^{(1)}$. Utilizing this construction, Kang's multiplicity formula allows us to determine the root multiplicities of $HG_2^{(1)}$ using the weight multiplicities of certain $G_2^{(1)}$-modules. In this section, we recall Kang's multiplicity formula and outline specifically the algorithm for computing the root multiplicities of $HG_2^{(1)}$. For more details, see [Kan94].

Theorem 4.2.1 (Kang's Multiplicity formula for $HG_2^{(1)}$ [Kan94])**.** *For Kac-Moody algebras $\mathfrak{g} = HG_2^{(1)}$ and $\mathfrak{g}_0 = G_2^{(1)}$, let $\Delta^{\pm}$ be the set of positive (respectively, negative) roots of $\mathfrak{g}$ and $\Delta_S^{\pm}$ be the set of positive (respectively, negative) roots of $\mathfrak{g}_0$. Define $\Delta^{\pm}(S) = \Delta^{\pm}/\Delta_S^{\pm}$. Then, for $\alpha \in \Delta^-(S)$:*

$$dim(\mathfrak{g}_\alpha) = \sum_{\tau|\alpha} \mu(\frac{\alpha}{\tau}) \binom{\tau}{\alpha} \mathcal{B}(\tau)$$

where,

- *$\tau|\alpha$ if $\alpha = k\tau$ for some positive integer k,*
- *μ is the classical Möbius function, $\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^s & \text{if } n \text{ is a product of } s \text{ distinct primes}, \\ 0 & \text{otherwise} \end{cases}$*
- *$\mathcal{B}(\tau) = \sum_{(n_i,\tau_i)\in T(\tau)} \frac{(\sum n_i - 1)!}{\prod (n_i)!} \prod K_{\tau_i}^{n_i}$,*
- *$T(\tau) = \{(n_i, \tau_i) | n_i \in \mathbb{Z}_{\geq 0},\ \tau_i \in P,\ \sum n_i \tau_i = \tau\}$ is the finite set of all partitions of τ,*
- *$K_{\tau_i} = \sum_{\substack{w \in W(S) \\ \ell(w) \geq 1}} (-1)^{\ell(w)+1} dim V(w\rho - \rho)_{\tau_i}$,*
- *$W(S) = \{w \in W | w\Delta^- \cap \Delta^+ \subseteq \Delta^+(S)\}$, and*
- *$V(w\rho - \rho)$ is the highest weight $\mathfrak{g}_0$-module with highest weight $w\rho - \rho$.*
- *$\rho \in \mathfrak{h}^*$ such that $\rho(h_i) = 1$ for all $i \in I$.*

Note Theorem 4.2.1 provides a nice algorithm for computing root multiplicities for negative roots. First, we sum over the τ that divide the negative root of interest, α. For each τ, we compute $\mathcal{B}(\tau)$ by summing over the partitions of τ, but observe that we only need to consider the partitions for which K_{τ_i} is nonzero for each τ_i. To determine K_{τ_i}, we sum over the Weyl group reflections satisfying $w\Delta^- \cap \Delta^+ \subseteq \Delta^+(S)$, i.e. the set of reflections that send negative roots to positive roots containing an (α_{-1})-term. Finally, we compute weight multiplicities of the $G_2^{(1)}$-modules, $V(w\rho - \rho)$. In the next chapter, we provide the framework for computing weight multiplicities of the $G_2^{(1)}$-modules using the combinatorial structure of crystal bases and path realizations.

4.3 Frenkel's Conjecture

In his paper on representations of Kac-Moody algebras and dual resonance models [Fre85], Frenkel conjectured a bound for the root multiplicities of indefinite Kac-Moody Lie algebras. Although this bound has been disproven for some algebras such as $HC_n^{(1)}$ in [Wil03], it has remained valid for other algebras such as $HA_n^{(1)}$, $HD_n^{(1)}$, and $HD_4^{(3)}$ (see [KM94], [HM02], [Wil12], and [Erb12]) and provides some insight into the behavior of hyperbolic root multiplicities. In order to consider Frenkel's bound, we let $\mathfrak{g}(A)$ be any indefinite Kac-Moody Lie algebra with corresponding Cartan matrix A of rank l and standard invariant bilinear form, denoted by $(\cdot|\cdot)$, on the set of roots Δ. Then for $\alpha \in \Delta$, Frenkel's conjecture states [Fre85]:

$$mult(\alpha) \leq p^{(l-2)}\left(1 - \frac{(\alpha|\alpha)}{2}\right),$$

where $p^{(l-2)}(k)$ are the formal coefficients in the infinite series satisfying

$$\sum_{k=0}^{\infty} p^{(l-2)}(k)q^k = \prod_{i=1}^{\infty}(1-q^i)^{-(l-2)}.$$

Now suppose $\mathfrak{g}(A)$ is a Kac-Moody Lie algebra with symmetrizable GCM, $A = (a_{ij})_{n\times n}$, and diagonal matrix, $D = diag(s_i | i \in \{1, \cdots, n\})$, such that DA is symmetric. Then $(\cdot|\cdot)$ defined on the simple roots of $\mathfrak{g}(A)$ by $(\alpha_i|\alpha_j) = s_i a_{ij}$ is a standard invariant bilinear form [Kac90]. For our algebra $HG_2^{(1)}$ we let $D = diag(3, 3, 3, 1)$ so that DA is a symmetric matrix. Then for simple roots $\{\alpha_{-1}, \alpha_0, \alpha_1, \alpha_2\}$ we have:

$$\begin{array}{llll}
(\alpha_{-1}|\alpha_{-1}) = 6 & (\alpha_{-1}|\alpha_0) = -3 & (\alpha_{-1}|\alpha_1) = 0 & (\alpha_{-1}|\alpha_2) = 0 \\
(\alpha_0|\alpha_{-1}) = -3 & (\alpha_0|\alpha_0) = 6 & (\alpha_0|\alpha_1) = -3 & (\alpha_0|\alpha_2) = 0 \\
(\alpha_1|\alpha_{-1}) = 0 & (\alpha_1|\alpha_0) = -3 & (\alpha_1|\alpha_1) = 6 & (\alpha_1|\alpha_2) = -3 \\
(\alpha_2|\alpha_{-1}) = 0 & (\alpha_2|\alpha_0) = 0 & (\alpha_2|\alpha_1) = -3 & (\alpha_2|\alpha_2) = 2
\end{array}$$

Consider level ℓ roots of the form $\alpha = -\ell\alpha_{-1} - m_0\alpha_0 - m_1\alpha_1 - m_2\alpha_2 \in \Delta^-$, then

$$\begin{aligned}
(\alpha|\alpha) &= (-\ell\alpha_{-1} - m_0\alpha_0 - m_1\alpha_1 - m_2\alpha_2 | -\ell\alpha_{-1} - m_0\alpha_0 - m_1\alpha_1 - m_2\alpha_2) \\
&= 6\ell^2 - 3\ell m_0 - 3\ell m_0 + 6m_0^2 - 3m_0m_1 - 3m_0m_1 + 6m_1^2 - 3m_1m_2 - 3m_1m_2 + 2m_2^2 \\
&= 6(\ell^2 + m_0^2 + m_1^2 - \ell m_0 - m_0m_1 - m_1m_2) + 2m_2^2
\end{aligned}$$

and Frenkel's bound implies

$$\begin{aligned} mult(\alpha) &\le p^{(2)}\left(1 - \frac{(\alpha|\alpha)}{2}\right) \\ &= p^{(2)}\left(1 - 3(\ell^2 + m_0^2 + m_1^2 - \ell m_0 - m_0 m_1 - m_1 m_2) - m_2^2\right). \end{aligned} \tag{4.1}$$

In the remainder of the thesis we use (4.1) and our C# program to compute $p^{(2)}\left(1 - \frac{(\alpha|\alpha)}{2}\right)$, specifically for the imaginary root $\alpha \in \Delta^-$ appearing in Chapter 6.

CHAPTER

5

CRYSTALS

In Section 4.2, we outlined a procedure for determining root multiplicities of $HG_2^{(1)}$ using weight multiplicities of certain highest weight $G_2^{(1)}$-modules. In this chapter we recall combinatorial objects known as crystals, which can be used to determine the weight multiplicities of highest weight modules of the associated affine quantum group $U_q(G_2^{(1)})$. Because the characters of the integrable modules $U_q(\mathfrak{g})$ and $U(\mathfrak{g})$ the same, the weight multiplicities $U_q(G_2^{(1)})$-modules are equivalent to those of $U(G_2^{(1)})$ and hence to those of $G_2^{(1)}$ (see Theorem 5.1.5). Therefore, we focus on finding weight multiplicities of $U_q(G_2^{(1)})$-modules. Unless otherwise stated we assume the field $F = \mathbb{C}$.

5.1 Quantum Groups

In this section we present the basic definitions and theorems of quantum groups which we need to define the quantum affine algebra $U_q(G_2^{(1)})$ in Section 5.3.

Definition 5.1.1. Fix an indeterminate q such that $q^m \neq 1$ for all $m \in \mathbb{Z}$. For $n \in \mathbb{Z}$, we define the q**-number** $[n]_q$ by

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}},$$

and q-**binomial coefficient** by

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[n]_q![m-n]_q!},$$

where $0 \le n \le m$ and $[0]_q! = 1$ and $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$. Note that as $q \to 1$, $[n]_q \to n$ and $\begin{bmatrix} m \\ n \end{bmatrix}_q \to \begin{pmatrix} m \\ n \end{pmatrix}$, the standard binomial coefficient.

Now suppose $A = (a_{ij})_{i,j \in I}$ is a symmetrizable GCM with symmetrizing diagonal matrix $D = diag(s_i \in \mathbb{Z}_{>0} \mid i \in I)$ and Cartan datum $(A, \Pi, \check{\Pi}, P, \check{P})$. We now define the quantum group of $\mathfrak{g} = \mathfrak{g}(A)$, also known as the quantized universal enveloping algebra of $\mathfrak{g} = \mathfrak{g}(A)$.

Definition 5.1.2. The associative algebra with unity over $F(q)$ generated by e_i, f_i $(i \in I)$ and q^h $(h \in \check{P})$ satisfying the following relations is known as the **quantum group** or **quantized universal enveloping algebra** of $\mathfrak{g} = \mathfrak{g}(A)$ and is denoted $U_q(\mathfrak{g})$.

(1) $q^0 = 1, q^h q^{h'} = q^{h+h'}$ for $h \in \check{P}$,

(2) $q^h e_i q^{-h} = q^{\alpha_i(h)} e_i$ for $h \in \check{P}, i \in I$,

(3) $q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i$ for $h \in \check{P}, i \in I$,

(4) $e_i f_j - f_j e_i = \delta_{ij} \dfrac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$ for $i, j \in I$,

(5) $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} e_i^{1-a_{ij}-k} e_j e_i^k = 0$ for $i \neq j$,

(6) $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} f_i^{1-a_{ij}-k} f_j f_i^k = 0$ for $i \neq j$,

where $q_i = q^{s_i}$ and $K_i = q^{s_i h_i}$.

Observe that in Definition 5.1.2 we recover $U(\mathfrak{g})$ when $q \to 1$. Therefore the representation theory for $U_q(\mathfrak{g})$ is analogous to that of $U(\mathfrak{g})$ and hence to that of $\mathfrak{g}$. Specifically, $U_q(\mathfrak{g})$ has triangular and root space decompositions, as well as weight modules V_q, categories $\mathcal{O}^q$ and $\mathcal{O}^q_{int}$, and other theorems that parallel those for $\mathfrak{g}$. For this reason we omit all but a few definitions and theorems here. For more details see [HK02].

Let V^q be an element in category $\mathcal{O}^q_{int}$. Then the following are true of V^q:

(1) V^q can be decomposed into weight spaces $V^q = \bigoplus_{\lambda \in P} V^q_\lambda$ where $dim_{F(q)} V^q_\lambda < \infty$ for all $\lambda \in P$.

(2) $wt(V^q) \subset D(\lambda_1) \cup \cdots \cup D(\lambda_s)$ for finitely many $\lambda_1, \cdots, \lambda_s \in P$.

(3) e_i and f_i are locally nilpotent on V^q for all $i \in I$.

We recall the following important theorems from [HK02].

Theorem 5.1.3. *Let $V^q(\lambda)$ denote the irreducible highest weight $U_q(\mathfrak{g})$-module of highest weight $\lambda \in P$. Then $V^q(\lambda) \in \mathcal{O}^q_{int}$ if and only if $\lambda \in P^+$.*

Theorem 5.1.4. *Let V^q be a $U_q(\mathfrak{g})$-module in $\mathcal{O}^q_{int}$. Then V^q is completely reducible, i.e. is isomorphic to a direct sum of irreducible highest weight modules $V^q(\lambda)$ with highest weight $\lambda \in P^+$.*

Theorem 5.1.5. *Let $V^q(\lambda)$ denote the irreducible highest weight $U_q(\mathfrak{g})$-module of highest weight $\lambda \in P$. Then $V^1(\lambda)$ is isomorphic to the highest weight $U(\mathfrak{g})$-module $V(\lambda)$. Moreover,*

$$ch\ V^q(\lambda) = ch\ V(\lambda).$$

Therefore the multiplicity of weights in $V^q(\lambda)$ is equivalent to the multiplicity in $V(\lambda)$.

5.2 Crystal Bases

In this section, we consider $U_q(\mathfrak{g})$-modules $V^q = \bigoplus_{\lambda \in P} V^q_\lambda$ with $V^q_\lambda \neq 0$ in category $\mathcal{O}^q_{int}$. We define Kashiwara operators on V^q and use them to develop the notion of a crystal base for V^q. In short, crystal bases can be viewed as a basis for V^q at $q = 0$. The combinatorics of crystal bases makes it simple to study the combinatorial structure of integrable representations of quantum groups in category $\mathcal{O}^q_{int}$.

To begin, we recall that for every $i \in I$ we can write every weight vector $v \in V^q_\lambda$ as

$$v = v_0 + f_i^{(1)} v_1 + \cdots + f_i^{(N)} v_N,$$

where $N \in \mathbb{Z}_{\geq 0}$, $v_k \in V^q_{\lambda + k\alpha_i} \cap \ker e_i$, and $f_i^{(k)} = \dfrac{f_i^k}{[k]_q!}$ for all $k = 0, 1, \cdots, N$. Then each v_k has the property that $v_k \neq 0$ if and only if $\lambda(h_i) + k \geq 0$ and is uniquely determined by v [HK02]. Now we are able to define the Kashiwara operators on V^q and give some of their properties.

Definition 5.2.1. Define the **Kashiwara operators** $\tilde{e}_i$ and $\tilde{f}_i$ ($i \in I$) on V^q by

$$\tilde{e}_i v = \sum_{k=1}^{N} f_i^{(k-1)} v_k, \qquad \tilde{f}_i v = \sum_{k=0}^{N} f_i^{(k+1)} v_k.$$

Theorem 5.2.2. *For all $i \in I$ and $\lambda \in P$, we have*

$$\tilde{e}_i V_\lambda^q = e_i V_\lambda^q \subset V_{\lambda+\alpha_i}^q, \qquad \tilde{f}_i V_\lambda^q = f_i V_\lambda^q \subset V_{\lambda-\alpha_i}^q.$$

Next, we consider the principal ideal domain, whose field of fractions is $F(q)$:

$$\mathcal{A}_0 = \left\{ \frac{f(q)}{g(q)} \;\middle|\; f(q), g(q) \in F[q], g(0) \neq 0 \right\}.$$

Definition 5.2.3. A **crystal lattice** is a free $\mathcal{A}_0$-submodule $\mathcal{L}$ of V^q that satisfies

(1) For all $\mu \in wt(V^q)$, we have $V_\mu^q \cong \mathcal{L}_\mu \otimes_{\mathcal{A}_0} F(q)$. In other words, V_q is generated by $\mathcal{L}$ as a vector space over $F(q)$.

(2) For all $\lambda \in P$, define $\mathcal{L}_\lambda = \mathcal{L} \cap V_\lambda^q$, then $\mathcal{L} = \bigoplus_{\lambda \in P} \mathcal{L}_\lambda$.

(3) For all $i \in I$, $\tilde{e}_i$ and $\tilde{f}_i$ satisfy $\tilde{e}_i \mathcal{L} \subset \mathcal{L}$ and $\tilde{f}_i \mathcal{L} \subset \mathcal{L}$.

Now consider the unique maximal ideal of $\mathcal{A}_0$, $\mathcal{J}_0 = \langle q \rangle$. Then the field isomorphism $\mathcal{A}_0/\mathcal{J}_0 \xrightarrow{\sim} F$ given by $f(q) + \mathcal{J}_0 \mapsto f(0)$ induces an isomorphism $F \otimes_{\mathcal{A}_0} \mathcal{L} \xrightarrow{\sim} \mathcal{L}/\mathcal{J}_0\mathcal{L} = \mathcal{L}/q\mathcal{L}$. Specifically, q gets mapped to 0 in the quotient space $\mathcal{L}/q\mathcal{L}$. The mapping from $\mathcal{L}$ into $\mathcal{L}/q\mathcal{L}$ is referred to as taking the **crystal limit**. By Definition 5.2.3 we observe $\tilde{e}_i$ and $\tilde{f}_i$ preserve $\mathcal{L}$ and are well defined in $\mathcal{L}/q\mathcal{L}$, therefore no change of notation is necessary. We now define a crystal base for integrable weight module $V^q \in \mathcal{O}_{int}^q$.

Definition 5.2.4. A **crystal base** for V^q is a pair $(\mathcal{L}, \mathcal{B})$ such that

(1) $\mathcal{L}$ is a crystal lattice of V^q,

(2) $\mathcal{B}$ is an F-basis of $\mathcal{L}/q\mathcal{L}$,

(3) $\mathcal{B}$ is the disjoint union of subspaces $\mathcal{B}_\lambda = \mathcal{B} \cap (\mathcal{L}_\lambda/q\mathcal{L}_\lambda)$:

$$\mathcal{B} = \bigsqcup_{\lambda \in P} \mathcal{B}_\lambda,$$

(4) for all $i \in I$, $\tilde{e}_i\mathcal{B} \subset \mathcal{B} \cup \{0\}$ and $\tilde{f}_i\mathcal{B} \subset \mathcal{B} \cup \{0\}$,

(5) $\mathcal{B}$ has the property that for any $b, b' \in \mathcal{B}$ and $i \in I$, $\tilde{f}_i b = b'$ if and only if $\tilde{e}_i b' = b$.

One important application of crystal bases comes from the following theorem. This theorem allows us to determine weight multiplicities of $U_q(\mathfrak{g})$-modules using the nice combinatorics of the crystal bases for such modules.

Theorem 5.2.5. *Let $V^q(\lambda)$ be an irreducible highest weight $U_q(\mathfrak{g})$-module and let $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$ be its crystal base. For every weight $\mu \in V^q(\lambda)$, the multiplicity of μ is equal to the cardinality of $\mathcal{B}(\lambda)_\mu$.*

The set $\mathcal{B}$ admits a graph structure called a **crystal graph** because we can depict the set as an i-colored, oriented graph. To do so, we define an ***i*-colored** arrow:

$$b \xrightarrow{i} b' \text{ if and only if } \tilde{f}_i b = b' \text{ for all } i \in I,$$

and construct an oriented graph with one node for every $b \in \mathcal{B}$, connected by i-colored arrows. Thus the crystal graph depicts the internal structure of $\mathcal{B}$. In addition, the crystal graph allows us a nice way to interpret the maps $\varepsilon_i, \varphi_i : \mathcal{B} \to \mathbb{Z}$ defined by,

$$\varepsilon_i(b) = \max\{k \geq 0 \mid \tilde{e}_i^k b \in \mathcal{B}\},$$

$$\varphi_i(b) = \max\{k \geq 0 \mid \tilde{f}_i^k b \in \mathcal{B}\}.$$

Therefore, $\varepsilon_i(b)$ (resp. $\varphi_i(b)$) represents the number of i-colored arrows coming into (resp. going out of) b and $\varphi_i(b) - \varepsilon_i(b) = \lambda(h_i)$ represents the length of the i-string through b. Finally, observe that for $\tilde{e}_i b \in \mathcal{B}$ and $\tilde{f}_i b \in \mathcal{B}$,

$$\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1, \quad \varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1,$$

$$\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1, \quad \varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1,$$

which implies $\tilde{e}_i\mathcal{B}_\lambda \subset \mathcal{B}_{\lambda+\alpha_i} \cup \{0\}$ and $\tilde{f}_i\mathcal{B}_\lambda \subset \mathcal{B}_{\lambda-\alpha_i} \cup \{0\}$. We use these maps to abstract the notion of crystal bases, but before doing so we discuss how crystal bases behave with respect to direct sums and tensor products.

Theorem 5.2.6 (Direct Sums and Tensor Products)**.** *[HK02] Let V_1^q and V_2^q be $U_q(\mathfrak{g})$-modules in category $\mathcal{O}_{int}^q$ having crystal bases $(\mathcal{L}_1, \mathcal{B}_1)$ and $(\mathcal{L}_2, \mathcal{B}_2)$, respectively.*

1. *The crystal base for* $V^q = V_1^q \oplus V_2^q$ *is given by* $(\mathcal{L}, \mathcal{B})$*, where* $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2$ *and* $\mathcal{B} = \mathcal{B}_1 \sqcup \mathcal{B}_2$*. The Kashiwara operators and crystal graph for* V^q *are defined in the natural way.*

2. *The crystal base for* $V^q = V_1^q \otimes_{F(q)} V_2^q$ *is given by* $(\mathcal{L}, \mathcal{B})$*, where* $\mathcal{L} = \mathcal{L}_1 \otimes_{\mathcal{A}_0} \mathcal{L}_2$ *and* $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2$*. For* $b_1 \otimes b_2 \in \mathcal{B}_1 \times \mathcal{B}_2$*, the Kashiwara operators* $\tilde{e}_i$ *and* $\tilde{f}_i$ $(i \in I)$ *are given by the* ***tensor product rule****,*

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}$$

Notice we use the notation $b_1 \otimes b_2$ *in place of* $(b_1, b_2) \in \mathcal{B}_1 \times \mathcal{B}_2$ *and set* $b_1 \otimes 0 = 0 \otimes b_2 = 0$*. Moreover, we have*

$$\begin{aligned} wt(b_1 \otimes b_2) &= wt(b_1) + wt(b_2), \\ \varepsilon_i(b_1 \otimes b_2) &= max\ \{\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, wt(b_1)\rangle\}, \\ \varphi_i(b_1 \otimes b_2) &= max\ \{\varphi_i(b_2), \varphi_i(b_1) - \langle h_i, wt(b_2)\rangle\}. \end{aligned}$$

The crystal graph of $\mathcal{B}_1 \times \mathcal{B}_2$ *is defined in the same manner as before and is denoted as* $\mathcal{B}_1 \otimes \mathcal{B}_2$*.*

The tensor product rule yields a convenient combinatorial way to describe the Kashiwara operators on multifold tensor products of crystals. Let $\mathcal{B}_1, \cdots, \mathcal{B}_k$ be crystals and fix $i \in I$. Consider $b = b_1 \otimes \cdots \otimes b_k \in \mathcal{B}_1 \otimes \cdots \otimes \mathcal{B}_k$. For each $b_j \in \mathcal{B}_j$ assign $\varepsilon_i(b_j)$ many $-$'s and $\varphi_i(b_j)$ many $+$'s:

$$b = b_1 \otimes b_2 \otimes \cdots\cdots \otimes b_k \longmapsto$$

$$(\underbrace{-, \cdots, -}_{\varepsilon_i(b_1)}, \underbrace{+, \cdots, +}_{\varphi_i(b_1)}, \underbrace{-, \cdots, -}_{\varepsilon_i(b_2)}, \underbrace{+, \cdots, +}_{\varphi_i(b_2)}, \cdots\cdots, \underbrace{-, \cdots, -}_{\varepsilon_i(b_k)}, \underbrace{+, \cdots, +}_{\varphi_i(b_k)})$$

Next cancel any $(+, -)$-pairs in the sequence generated by b. This yields a reduced sequence

$$i\text{-}sgn(b) := (-, -, \cdots, -, +, +, \cdots, +),$$

called the **reduced** i**-signature** of b. Then the tensor product rule implies $\tilde{e}_i$ acts on b by applying $\tilde{e}_i$ to b_j corresponding to the right-most $-$ in i-$sgn(b)$. Similarly, $\tilde{f}_i$ acts on b by applying $\tilde{f}_i$ to b_j corresponding to the left-most $+$ in i-$sgn(b)$. Therefore,

$$\tilde{e}_i b = b_1 \otimes \cdots \otimes \tilde{e}_i b_j \otimes \cdots \otimes b_k,$$

$$\tilde{f}_i b = b_1 \otimes \cdots \otimes \tilde{f}_i b_j \otimes \cdots \otimes b_k.$$

We now abstract the definition of crystal bases by defining what are called crystals.

Definition 5.2.7. [HK02] Let $A = (a_{ij})_{i,j \in I}$ be a GCM with associated Cartan datum $(A, \Pi, \check{\Pi}, P, \check{P})$ and let $\mathfrak{g} = \mathfrak{g}(A)$ be a Kac-Moody algebra. A $(U^q(\mathfrak{g})-)$**crystal** associated with Cartan datum $(A, \Pi, \check{\Pi}, P, \check{P})$ is a set $\mathcal{B}$, together with the maps $wt : \mathcal{B} \to P$, $\tilde{e}_i, \tilde{f}_i : \mathcal{B} \to \mathcal{B} \cup \{0\}$, and $\varepsilon_i, \varphi_i : \mathcal{B} \to \mathbb{Z} \cup \{-\infty\}$ $(i \in I)$, if $\mathcal{B}$ satisfies the following properties:

(1) for all $i \in I$, $\varphi_i(b) - \varepsilon_i(b) = \langle h_i, wt(b) \rangle$,

(2) if $\tilde{e}_i b \in \mathcal{B}$, then $wt(\tilde{e}_i b) = wt(b) + \alpha_i$,

(3) if $\tilde{f}_i b \in \mathcal{B}$, then $wt(\tilde{f}_i b) = wt(b) - \alpha_i$,

(4) if $\tilde{e}_i b \in \mathcal{B}$, then $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1, \quad \varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$,

(5) if $\tilde{f}_i b \in \mathcal{B}$, then $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1, \quad \varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$,

(6) for $b, b' \in \mathcal{B}$ and $i \in I$, $\tilde{f}_i b = b'$ if and only if $\tilde{e}_i b' = b$,

(7) for $b \in \mathcal{B}$, if $\varphi_i(b) = -\infty$, then $\tilde{e}_i b = \tilde{f}_i b = 0$.

We denote $\mathcal{B}_\lambda = \{b \in \mathcal{B} \mid wt(b) = \lambda\}$ so that $\mathcal{B} = \bigsqcup_{\lambda \in P} \mathcal{B}_\lambda$. A crystal $\mathcal{B}$ is called **semiregular** if we have

$$\varepsilon_i(b) = \max\{k \geq 0 \mid \tilde{e}_i^k b \in \mathcal{B}\},$$

$$\varphi_i(b) = \max\{k \geq 0 \mid \tilde{f}_i^k b \in \mathcal{B}\}.$$

Example 5.2.8. Let V^q be a $U_q(\mathfrak{g})$-module in category $\mathcal{O}_{int}$.

1. The crystal graph $\mathcal{B}$ of V^q is a $U_q(\mathfrak{g})$-crystal.

2. Similarly, if $V^q(\lambda)$ is a highest weight $U_q(\mathfrak{g})$-module in category $\mathcal{O}^q_{int}$, then its crystal graph $\mathcal{B}(\lambda)$ is a $U_q(\mathfrak{g})$-crystal.

Theorem 5.2.9. *An* ***isomorphism*** *between two crystals $\mathcal{B}_1, \mathcal{B}_2$ associated with Cartan datum $(A, \Pi, \check{\Pi}, P, \check{P})$ is a bijection $\Psi : \mathcal{B}_1 \cup \{0\} \to \mathcal{B}_2 \cup \{0\}$ satisfying:*

1. $\Psi(0) = 0$,
2. *If* $b \in \mathcal{B}_1$, *then* $wt(\Psi(b)) = wt(b)$, $\varepsilon_i(\Psi(b)) = \varepsilon_i(b)$, *and* $\varphi_i(\Psi(b)) = \varphi_i(b)$ *for all* $i \in I$.
3. *If* $b, b' \in \mathcal{B}_1$ *and* $\tilde{f}_i b = b'$, *then* $\tilde{f}_i \Psi(b) = \Psi(b')$ *and* $\tilde{e}_i \Psi(b') = \Psi(b)$ *for all* $i \in I$.

The following theorem from [HK02] provides the foundation for the existence and uniqueness of crystal bases for $U_q(\mathfrak{g})$-modules in category $\mathcal{O}^q_{int}$.

Theorem 5.2.10. *Let* $V^q(\lambda)$ *denote the irreducible highest weight* $U_q(\mathfrak{g})$*-module with highest weight* $\lambda \in P^+$ *and highest weight vector* v_λ. *Define* $\mathcal{L}(\lambda)$ *to be the free* $\mathcal{A}_0$*-submodule of* $V^q(\lambda)$ *spanned by* $\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} v_\lambda$ *for* $r \geq 0$ *and* $i_j \in I$. *Set*

$$\mathcal{B}(\lambda) = \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} v_\lambda + q\mathcal{L}(\lambda) \in \mathcal{L}(\lambda)/q\mathcal{L}(\lambda) \mid r \geq 0,\ i_j \in I\}/\{0\},$$

then the pair $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$ *is a crystal base of* $V^q(\lambda)$.

5.3 Perfect Crystals

In this section we consider special quantum groups known as quantum affine algebras and present the notion of perfect crystals. Specifically, we recall path realizations of crystals using perfect crystals. With this technique we determine the weight multiplicities of irreducible highest weight $U_q(G_2^{(1)})$-modules and hence $G_2^{(1)}$-modules.

We begin with some basic definitions. Suppose $A = (a_{ij})_{i,j \in I}$ is an affine GCM with index $I = \{0, 1, \cdots, n\}$. Then $\check{P} = \mathbb{Z}h_0 \oplus \mathbb{Z}h_1 \oplus \cdots \mathbb{Z}h_n \oplus \mathbb{Z}d$ is called the **affine dual weight lattice**, $P = \{\lambda \in \mathfrak{h}^* | \lambda(\check{P}) \subset \mathbb{Z}\}$ is called the **affine weight lattice**, and $(A, \Pi, \check{\Pi}, P, \check{P})$ is called the **affine Cartan datum**. For affine Kac-Moody algebras, recall that the center of $\mathfrak{g}$ is one dimensional and spanned by the canonical central element $c = \check{a}_0 h_0 + \check{a}_1 h_1 + \cdots + \check{a}_n h_n$, where h_i is an element of the Cartan subalgebra $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} \check{P}$. In addition, recall the canonical null root $\delta = a_0\alpha_0 + a_1\alpha_1 + \cdots + a_n\alpha_n$, where α_i are simple roots defined by $\alpha_i(h_j) = a_{ji}$ and $\alpha_i(d) = \delta_{0i}$. Using the null root we can rewrite P in terms of the fundamental weights given by $\Lambda_i(h_j) = \delta_{ij}$ and $\Lambda_i(d) = 0$:

$$P = \mathbb{Z}\Lambda_0 + \mathbb{Z}\Lambda_1 + \cdots + \mathbb{Z}\Lambda_n \oplus \mathbb{Z}\frac{1}{a_0}\delta.$$

We call elements of P **affine weights** and denote the set of **affine dominant integral weights** by $P^+ = \{\lambda \in P \mid \lambda(h_i) \in \mathbb{Z}_{\geq 0}$ for all $i \in I\}$. The **level** of an affine dominant integral weight $\lambda \in P^+$ is the integer $\lambda(c) \geq 0$.

Example 5.3.1. *Consider the affine Kac-Moody algebra* $\mathfrak{g}_0 = G_2^{(1)}$ *with simple roots* $\{\alpha_0, \alpha_1, \alpha_2\} \subset \mathfrak{h}^*$ *and fundamental weights* $\{\Lambda_0, \Lambda_1, \Lambda_2\} \subset \mathfrak{h}^*$ *satisfying:*

$$\alpha_i(h_j) = a_{ji}, \quad \alpha_i(d) = \delta_{0i},$$

$$\Lambda_i(h_j) = \delta_{ij}, \quad \Lambda_i(d) = 0,$$

and recall the null root and canonical central elements are

$$\delta = \alpha_0 + 2\alpha_1 + 3\alpha_2,$$

$$c = h_0 + 2h_1 + h_2.$$

We can write the simple roots in terms of fundamental weights and the null vector as follows:

$$\alpha_0 = 2\Lambda_0 - \Lambda_1 + \delta$$

$$\alpha_1 = -\Lambda_0 + 2\Lambda_1 - 3\Lambda_2$$

$$\alpha_2 = -\Lambda_1 + 2\Lambda_2$$

Definition 5.3.2. Let $\mathfrak{g} = \mathfrak{g}(A)$ be an affine Kac-Moody algebra with affine Cartan datum $(A, \Pi, \check{\Pi}, P, \check{P})$. In this case we call the quantized universal enveloping algebra $U_q(\mathfrak{g})$ the **quantum affine algebra**. We also call $U_q'(\mathfrak{g})$ the subalgebra of $U_q(\mathfrak{g})$ generated by $e_i, f_i, K_i^{\pm 1}$ $(i \in I)$ the quantum affine algebra. Then $U_q'(\mathfrak{g})$ can be viewed as the quantum group associated with the **classical Cartan datum** $(A, \Pi, \Pi^\vee, \bar{P}, \bar{P}^\vee)$:

- $\bar{P}^\vee = \mathbb{Z}h_0 \oplus \mathbb{Z}h_1 \oplus \cdots \oplus \mathbb{Z}h_n$ and $\bar{\mathfrak{h}} = \mathbb{C} \otimes_{\mathbb{Z}} \bar{P}^\vee$,
- $\bar{P} = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \cdots \oplus \mathbb{Z}\Lambda_n$,
- Π and $\Pi^\vee$ are the same as in the affine case.

We call elements in $\bar{P}$ **classical weights** and denote the set of **classical dominant weights** as

$$\bar{P}^+ = \{\lambda \in \bar{P} \mid \lambda(h_i) \in \mathbb{Z}_{\geq 0} \text{ for all } i \in I\}.$$

We say a classical dominant weight $\lambda \in \bar{P}^+$ is of level l if $\lambda(c) = l$, same as with affine weights.

Definition 5.3.3. An abstract crystal associated with affine Cartan datum $(A, \Pi, \check{\Pi}, P, \check{P})$ is called an **affine crystal**. An abstract crystal associated with classical Cartan datum $(A, \Pi, \Pi^\vee, \bar{P}, \bar{P}^\vee)$ is called a **classical crystal**.

Let $l > 0$ be an integer and $\mathcal{B}$ be a classical crystal. For $b \in \mathcal{B}$, recall

$$\varepsilon_i(b) = \max\{k \geq 0 \mid \tilde{e}_i^k b \in \mathcal{B}\},$$

$$\varphi_i(b) = \max\{k \geq 0 \mid \tilde{f}_i^k b \in \mathcal{B}\}.$$

and define

$$\varepsilon(b) = \sum_{i \in I} \varepsilon_i(b)\Lambda_i, \quad \text{and} \quad \varphi(b) = \sum_{i \in I} \varphi_i(b)\Lambda_i.$$

Then $\overline{wt}(b) = \varphi(b) - \varepsilon(b)$, where $\overline{wt}$ dentoes the classical weights. Let

$$\bar{P}_l^+ = \{\lambda \in \bar{P}^+ \mid \langle c, \lambda \rangle = l\}$$

denote the subset of the classical dominant weights having level $l > 0$. We are now ready to define perfect crystals.

Definition 5.3.4. [HK02] Let $\mathcal{B}_l$ be a finite classical crystal with maps defined as above. Then $\mathcal{B}_l$ is a **perfect crystal of level** l, if it satisfies the conditions:

(1) there exists a finite dimensional $U_q'(\mathfrak{g})$-module whose crystal is isomorphic to $\mathcal{B}_l$,

(2) $\mathcal{B}_l \otimes \mathcal{B}_l$ is connected,

(3) there exists a classical weight $\lambda_0 \in \bar{P}$ such that

$$\overline{wt}(\mathcal{B}_l) \subset \lambda_0 + \sum_{i \neq 0} \mathbb{Z}_{\leq 0}\alpha_i \text{ and } dim(\mathcal{B}_l)_{\lambda_0} = 1,$$

(4) $\langle c, \varepsilon(b) \rangle = l$ for any $b \in \mathcal{B}_l$,

(5) for each $\lambda \in \bar{P}_l^+$, there exists unique vectors $b^\lambda \in \mathcal{B}_l$ and $b_\lambda \in \mathcal{B}_l$ such that $\varepsilon(b^\lambda) = \lambda$ and $\varphi(b_\lambda) = \lambda$.

Theorem 5.3.5 (Level l perfect crystal for $U_q(G_2^{(1)})$)**.** *[Mis10] The $U_q'(G_2^{(1)})$-crystal $\mathcal{B}_l$ given below is a perfect crystal of level $l \geq 1$.*

$$B_l = \{b = (x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \in (\mathbb{Z}_{\geq 0}/3)^6 \mid x_1, \bar{x}_1, x_2 - x_3 \in \mathbb{Z},\ 3x_3 \equiv 3\bar{x}_3 \ (\textit{mod 2}),$$
$$\textit{and } s(b) = x_1 + x_2 + \frac{x_3 + \bar{x}_3}{2} + \bar{x}_2 + \bar{x}_1 \leq l\}$$

equipped with the Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$ $(i = 0, 1, 2)$ given by:

$$\tilde{e}_0(b) = \begin{cases} (x_1-1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) & \textit{if } (E_1), \\ (x_1, x_2, x_3-1, \bar{x}_3-1, \bar{x}_2, \bar{x}_1+1) & \textit{if } (E_2), \\ (x_1, x_2-\frac{2}{3}, x_3-\frac{2}{3}, \bar{x}_3+\frac{4}{3}, \bar{x}_2+\frac{1}{3}, \bar{x}_1) & \textit{if } (E_3) \textit{ and } \bar{x}_3 - x_3 = -\frac{2}{3}, \\ (x_1, x_2-\frac{1}{3}, x_3-\frac{4}{3}, \bar{x}_3+\frac{2}{3}, \bar{x}_2+\frac{2}{3}, \bar{x}_1) & \textit{if } (E_3) \textit{ and } \bar{x}_3 - x_3 = -\frac{4}{3}, \\ (x_1, x_2, x_3-2, \bar{x}_3, \bar{x}_2+1, \bar{x}_1) & \textit{if } (E_3) \textit{ and } \bar{x}_3 - x_3 \neq -\frac{4}{3}, -\frac{2}{3}, \\ (x_1, x_2-1, x_3, \bar{x}_3+2, \bar{x}_2, \bar{x}_1) & \textit{if } (E_4), \\ (x_1-1, x_2, x_3+1, \bar{x}_3+1, \bar{x}_2, \bar{x}_1) & \textit{if } (E_5), \\ (x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1+1) & \textit{if } (E_6), \end{cases}$$

$$\tilde{f}_0(b) = \begin{cases} (x_1+1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) & \textit{if } (F_1), \\ (x_1, x_2, x_3+1, \bar{x}_3+1, \bar{x}_2, \bar{x}_1-1) & \textit{if } (F_2), \\ (x_1, x_2, x_3+2, \bar{x}_3, \bar{x}_2-1, \bar{x}_1) & \textit{if } (F_3), \\ (x_1, x_2+\frac{1}{3}, x_3+\frac{4}{3}, \bar{x}_3-\frac{2}{3}, \bar{x}_2-\frac{2}{3}, \bar{x}_1) & \textit{if } (F_4) \textit{ and } \bar{x}_3 - x_3 = \frac{2}{3}, \\ (x_1, x_2+\frac{2}{3}, x_3+\frac{2}{3}, \bar{x}_3-\frac{4}{3}, \bar{x}_2-\frac{1}{3}, \bar{x}_1) & \textit{if } (F_4) \textit{ and } \bar{x}_3 - x_3 = \frac{4}{3}, \\ (x_1, x_2+1, x_3, \bar{x}_3-2, \bar{x}_2, \bar{x}_1) & \textit{if } (F_4) \textit{ and } \bar{x}_3 - x_3 \neq \frac{4}{3}, \frac{2}{3}, \\ (x_1+1, x_2, x_3-1, \bar{x}_3-1, \bar{x}_2, \bar{x}_1) & \textit{if } (F_5), \\ (x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1-1) & \textit{if } (F_6), \end{cases}$$

$$\tilde{e}_1(b) = \begin{cases} (x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1-1) & \textit{if } (\bar{x}_2 - \bar{x}_3) \geq (x_2 - x_3)_+, \\ (x_1, x_2, x_3+1, \bar{x}_3-1, \bar{x}_2, \bar{x}_1) & \textit{if } (\bar{x}_2 - \bar{x}_3) < 0 \leq (x_3 - x_2), \\ (x_1+1, x_2-1, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) & \textit{if } (\bar{x}_2 - \bar{x}_3)_+ < (x_2 - x_3), \end{cases}$$

$$\tilde{f}_1(b) = \begin{cases} (x_1-1, x_2+1, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) & \textit{if } (\bar{x}_2 - \bar{x}_3)_+ \leq (x_2 - x_3), \\ (x_1, x_2, x_3-1, \bar{x}_3+1, \bar{x}_2, \bar{x}_1) & \textit{if } (\bar{x}_2 - \bar{x}_3) \leq 0 < (x_3 - x_2), \\ (x_1, x_2, x_3, \bar{x}_3, \bar{x}_2-1, \bar{x}_1+1) & \textit{if } (\bar{x}_2 - \bar{x}_3) > (x_2 - x_3)_+, \end{cases}$$

$$\tilde{e}_2(b) = \begin{cases} (x_1, x_2, x_3, \bar{x}_3+\frac{2}{3}, \bar{x}_2-\frac{1}{3}, \bar{x}_1) & \textit{if } \bar{x}_3 - x_3 \geq 0, \\ (x_1, x_2+\frac{1}{3}, x_3-\frac{2}{3}, \bar{x}_3, \bar{x}_2, \bar{x}_1) & \textit{if } \bar{x}_3 - x_3 < 0, \end{cases}$$

$$\tilde{f}_2(b) = \begin{cases} (x_1, x_2-\frac{1}{3}, x_3+\frac{2}{3}, \bar{x}_3, \bar{x}_2, \bar{x}_1) & \textit{if } \bar{x}_3 - x_3 \leq 0, \\ (x_1, x_2, x_3, \bar{x}_3-\frac{2}{3}, \bar{x}_2+\frac{1}{3}, \bar{x}_1) & \textit{if } \bar{x}_3 - x_3 > 0, \end{cases}$$

where $(F_1) - (F_6)$ are given below and $(E_1) - (E_6)$ are defined by replacing $>$ with $\geq$ and $\leq$ with $<$ in the definition of $(F_1) - (F_6)$. If $b \in B_l$ and $\tilde{e}_i(b)$ or $\tilde{f}_i(b)$ is not in B_l we take it to be 0.

$$(F_1) = \begin{cases} 2\bar{x}_1 - 2x_1 + 2\bar{x}_2 - 2x_2 + \bar{x}_3 - x_3 \le 0 \\ 2\bar{x}_1 - 2x_1 + 2\bar{x}_2 + \bar{x}_3 - 3x_3 \le 0 \\ \bar{x}_1 - x_1 + \bar{x}_2 - \bar{x}_3 \le 0 \\ \bar{x}_1 - x_1 \le 0 \end{cases}$$

$$(F_2) = \begin{cases} 2\bar{x}_1 - 2x_1 + 2\bar{x}_2 - 2x_2 + \bar{x}_3 - x_3 \le 0 \\ 2\bar{x}_2 + \bar{x}_3 - 3x_3 \le 0 \\ \bar{x}_2 - \bar{x}_3 \le 0 \\ \bar{x}_1 - x_1 > 0 \end{cases}$$

$$(F_3) = \begin{cases} 2\bar{x}_1 - 2x_1 - 2x_2 + 3\bar{x}_3 - x_3 \le 0 \\ 3\bar{x}_3 - x_3 - 2x_2 \le 0 \\ \bar{x}_3 - x_3 \le 0 \\ \bar{x}_2 - \bar{x}_3 > 0 \\ \bar{x}_1 - x_1 + \bar{x}_2 - \bar{x}_3 > 0 \end{cases}$$

$$(F_4) = \begin{cases} 2\bar{x}_1 - 2x_1 + 2\bar{x}_2 + \bar{x}_3 - 3x_3 > 0 \\ 2\bar{x}_2 + \bar{x}_3 - 3x_3 > 0 \\ \bar{x}_3 - x_3 > 0 \\ x_3 - x_2 \le 0 \\ \bar{x}_1 - x_1 + x_3 - x_2 \le 0 \end{cases}$$

$$(F_5) = \begin{cases} 2\bar{x}_1 - 2x_1 + 2\bar{x}_2 - 2x_2 + \bar{x}_3 - x_3 > 0 \\ 3\bar{x}_3 - x_3 - 2x_2 > 0 \\ x_3 - x_2 > 0 \\ \bar{x}_1 - x_1 \le 0 \end{cases}$$

$$(F_6) = \begin{cases} 2\bar{x}_1 - 2x_1 + 2\bar{x}_2 - 2x_2 + \bar{x}_3 - x_3 > 0 \\ 2\bar{x}_1 - 2x_1 - 2x_2 + 3\bar{x}_3 - x_3 > 0 \\ \bar{x}_1 - x_1 + x_3 - x_2 > 0 \\ \bar{x}_1 - x_1 > 0 \end{cases}$$

Example 5.3.6 (Level 1 perfect crystal for $U_q(G_2^{(1)})$). Using the set defined in Theorem 5.3.5 we determine the elements in B_1, determine the classical weights for each element, and construct the crystal graph of B_1 in Fig. 5.1. Please note in Fig. 5.1 we denote the elements $b_j \in \mathcal{B}_1$ by the number j encased in a rectangle and represent the $\tilde{f}_i$ actions by i-arrows ($i = 0, 1, 2$).

We have $\mathcal{B}_1 = \{b_i \mid 0 \le i \le 14\}$ where

$b_0 = (0,0,0,0,0,0)$ $b_4 = (0,0,0,\frac{4}{3},\frac{1}{3},0)$ $b_8 = (0,0,2,0,0,0)$ $b_{12} = (0,\frac{2}{3},\frac{2}{3},0,0,0)$
$b_1 = (0,0,0,0,0,1)$ $b_5 = (0,0,0,2,0,0)$ $b_9 = (0,\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},0)$ $b_{13} = (0,1,0,0,0,0)$
$b_2 = (0,0,0,0,1,0)$ $b_6 = (0,0,1,\frac{1}{3},\frac{1}{3},0)$ $b_{10} = (0,\frac{1}{3},\frac{1}{3},1,0,0)$ $b_{14} = (1,0,0,0,0,0)$
$b_3 = (0,0,0,\frac{2}{3},\frac{2}{3},0)$ $b_7 = (0,0,1,1,0,0)$ $b_{11} = (0,\frac{1}{3},\frac{4}{3},0,0,0)$

$\overline{wt}(b_0) = 0$ $\overline{wt}(b_4) = -(\alpha_1+\alpha_2)$ $\overline{wt}(b_8) = \alpha_1$ $\overline{wt}(b_{12}) = \alpha_1+2\alpha_2$
$\overline{wt}(b_1) = -(2\alpha_1+3\alpha_2)$ $\overline{wt}(b_5) = -\alpha_1$ $\overline{wt}(b_9) = 0$ $\overline{wt}(b_{13}) = \alpha_1+3\alpha_2$
$\overline{wt}(b_2) = -(\alpha_1+3\alpha_2)$ $\overline{wt}(b_6) = -\alpha_2$ $\overline{wt}(b_{10}) = \alpha_2$ $\overline{wt}(b_{14}) = 2\alpha_1+3\alpha_2$
$\overline{wt}(b_3) = -(\alpha_1+2\alpha_2)$ $\overline{wt}(b_7) = 0$ $\overline{wt}(b_{11}) = \alpha_1+\alpha_2$

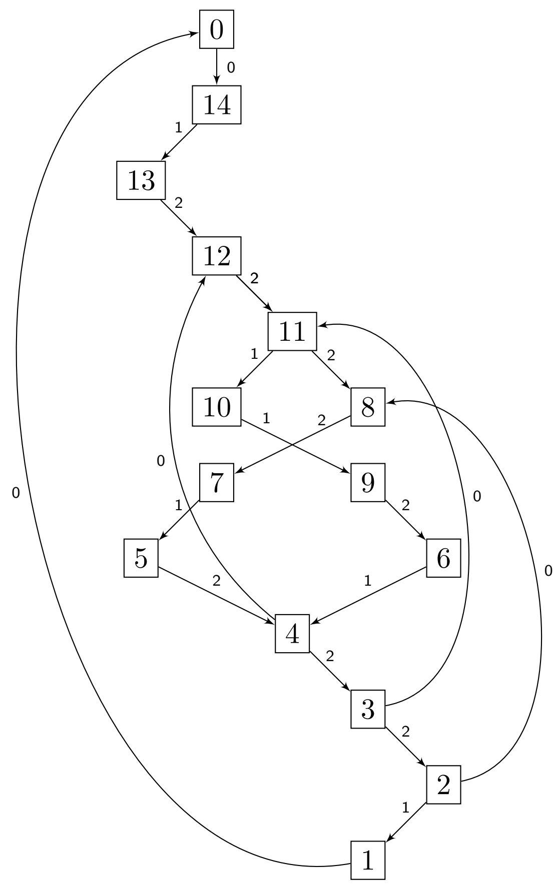


Figure 5.1 Crystal graph of $\mathcal{B}_1$

Example 5.3.7. Using the set defined in Theorem 5.3.5 and our C# program, we determine the elements in $\mathcal{B}_2$ and list them in Table A.1 of the appendix. Because the crystal graph of $\mathcal{B}_2$ is quite large, we do not depict the crystal graph pictorially. Instead we provide in Appendix A the necessary information from the crystal graph that is needed in the examples to follow. The interested reader can refer to [Mis10] for a picture of $\mathcal{B}_2$. Similarly, we use our C# program to find the elements in $\mathcal{B}_3$ and the desired information. See Table A.2 of the appendix.

5.3.1 Path Realizations

Theorem 5.3.8. *[HK02] Let $\mathcal{B}_l$ be a perfect crystal of level $l > 0$ and let λ be any classical dominant integral weight in $\bar{P}_l^+$. Then, there exists a crystal isomorphism*

$$\Psi : \mathcal{B}(\lambda) \to \mathcal{B}(\varepsilon(b_\lambda)) \otimes \mathcal{B} \text{ given by } v_\lambda \mapsto v_{\varepsilon(b_\lambda)} \otimes b_\lambda,$$

where $b_\lambda \in \mathcal{B}$ is the unique vector such that $\varphi(b_\lambda) = \lambda$, and v_λ, $v_\varepsilon(b_\lambda)$ are the highest weight vectors of $\mathcal{B}(\lambda)$ and $\mathcal{B}(\varepsilon(b_\lambda))$, respectively.

We use Theorem 5.3.8 to inductively construct a sequence of isomorphism as follows. Set $\lambda_0 = \lambda$ and define $\lambda_{k+1} = \varepsilon(b_{\lambda_k})$. Similarly, set $b_0 = b_\lambda$ and $b_{k+1} = b_{\lambda_k+1}$. Then Theorem 5.3.8 implies that there exists a crystal isomorphism

$$\Psi : \mathcal{B}(\lambda_k) \to \mathcal{B}(\lambda_{k+1}) \otimes \mathcal{B} \text{ given by } v_{\lambda_k} \mapsto v_{\lambda_{k+1}} \otimes b_k,$$

such that $\varphi(b_k) = \lambda_k$, and $\varepsilon(b_k) = \lambda_{k+1}$. Composing such isomorphisms for $k \geq 1$, we obtain the sequence

$$\mathcal{B}(\lambda) \xrightarrow{\sim} \mathcal{B}(\lambda_1) \otimes \mathcal{B} \xrightarrow{\sim} \mathcal{B}(\lambda_2) \otimes \mathcal{B} \otimes \mathcal{B} \xrightarrow{\sim} \cdots,$$

given by

$$v_\lambda \mapsto v_{\lambda_1} \otimes b_0 \mapsto v_{\lambda_2} \otimes b_1 \otimes b_0 \mapsto \cdots$$

Therefore we obtain the following infinite sequences

$$p_\lambda = (b_k)_{k=0}^{\infty} = \cdots \otimes b_{k+1} \otimes b_k \otimes \cdots \otimes b_1 \otimes b_0 \in \mathcal{B}^{\otimes\infty}.$$

Using the fact that there is only a finite number of elements in $\bar{P}_l^+$ and in $\mathcal{B}$, it is straightforward to show that p_λ is periodic with some period $N > 0$. See [HK02] for more details.

Definition 5.3.9.

(1) $p_\lambda = (b_k)_{k=0}^\infty = \cdots \otimes b_{k+1} \otimes b_k \otimes \cdots \otimes b_1 \otimes b_0$ is called the **ground-state path** of weight λ.

(2) A sequence $p = (p_k)_{k=0}^\infty = \cdots \otimes p_{k+1} \otimes p_k \otimes \cdots \otimes p_1 \otimes p_0$ with $p_k \in \mathcal{B}$ such that $p_k = b_k$ for all $k \gg 0$ is called a **λ-path** in $\mathcal{B}$.

(3) Define $\mathcal{P}(\lambda)$ to be the set of all λ-paths in $\mathcal{B}$.

For $p = (p_k)_{k=0}^\infty \in \mathcal{P}(\lambda)$ such that $p_k = b_k$ for all $k \geq N > 0$ and for all $i \in I$ we define maps

$$
\begin{aligned}
\overline{wt}(p) &= \lambda_N + \sum_{k=0}^{N-1} \overline{wt}(p_k), \\
\tilde{e}_i p &= \cdots \otimes p_{N+1} \otimes \tilde{e}_i(p_N \otimes \cdots \otimes p_0), \\
\tilde{f}_i p &= \cdots \otimes p_{N+1} \otimes \tilde{f}_i(p_N \otimes \cdots \otimes p_0), \\
\varepsilon_i(p) &= \max\{\varepsilon_i(p_{N-1} \otimes \cdots \otimes p_0) - \varphi_i(b_N), 0\}, \\
\varphi_i(p) &= \varphi_i(p_{N-1} \otimes \cdots \otimes p_0) + \max\{\varphi_i(b_N) - \varepsilon_i(p_{N-1} \otimes \cdots \otimes p_0), 0\}.
\end{aligned}
$$

Observe that ε_i and φ_i can be determined by using the i-signature rule in Section 5.2. Then $\mathcal{P}(\lambda)$, together with the above maps, is a $U_q'(\mathfrak{g})$-crystal [HK02]. By Theorem 5.3.10 below we have $\mathcal{B}(\lambda) \cong \mathcal{P}(\lambda)$. In this case, we say $\mathcal{P}(\lambda)$ is a **path realization** of $\mathcal{B}(\lambda)$. We sometimes use the language “path crystal” to describe $\mathcal{P}(\lambda)$.

Theorem 5.3.10. *Let $\mathcal{B}(\lambda)$ be a highest weight $U_q'(\mathfrak{g})$-module and let $\mathcal{P}(\lambda)$ be the set of λ-paths in $\mathcal{B}(\lambda)$. Then there exists an $U_q'(\mathfrak{g})$-crystal isomorphism*

$$\Psi : \mathcal{B}(\lambda) \to \mathcal{P}(\lambda) \text{ given by } v_\lambda \mapsto p_\lambda.$$

Example 5.3.11. Let $V^q(\Lambda_0)$ be the irreducible highest weight $U_q(G_2^{(1)})$-module. Consider the perfect crystal of level 1 $\mathcal{B}_1$ given in Example 5.3.6 and recall that $b_0 = (0,0,0,0,0,0) \in \mathcal{B}_1$. We consider $\lambda = \Lambda_0$ and determine the ground-state path of $\mathcal{P}(\Lambda_0)$. Set $\lambda_0 = \Lambda_0$ and observe

$$
\begin{aligned}
\varphi(b_0) &= \varphi_0(b_0)\Lambda_0 + \varphi_1(b_0)\Lambda_1 + \varphi_2(b_0)\Lambda_2 \\
&= \Lambda_0.
\end{aligned}
$$

Therefore $b_0 = (0,0,0,0,0,0)$ is the unique vector in $\mathcal{B}_1$ such that $\varphi(b_{\lambda_0}) = \lambda_0$.

Next we find $\lambda_1 = \varepsilon(b_{\lambda_0})$:

$$\begin{aligned}\varepsilon(b_{\lambda_0}) &= \varepsilon(b_0)\\ &= \varepsilon_0(b_0)\Lambda_0 + \varepsilon_1(b_0)\Lambda_1 + \varepsilon_2(b_0)\Lambda_2\\ &= \Lambda_0.\end{aligned}$$

Therefore $\lambda_1 = \Lambda_0$ and hence $b_{\lambda_1} = b_0 = (0,0,0,0,0,0)$. Thus the ground-state path for $\mathcal{P}(\Lambda_0)$ is

$$p_{\Lambda_0} = \cdots \otimes b_0 \otimes b_0.$$

Using the signature rule and the perfect crystal $\mathcal{B}_1$, we obtain a partial depiction of $\mathcal{P}(\Lambda_0)$, the path realization of $\mathcal{B}(\Lambda_0)$ given in Fig. 5.2. By Theorem 5.2.5 we know the multiplicities of weights $\mu \in wt(V^q(\Lambda_0))$ are equal to the cardinality of $\mathcal{B}(\Lambda_0)_\mu$. Then by Theorem 5.1.5 we have $mult_{V^q(\Lambda_0)}(\mu) = mult_{V(\Lambda_0)}(\mu)$. Therefore, we say $\mathcal{P}(\Lambda_0)$ the path realization of $V(\Lambda_0)$.

Example 5.3.12. Let $V^q(\Lambda_1)$ be the irreducible highest weight $U_q(G_2^{(1)})$-module. Consider the perfect crystal of level 2 $\mathcal{B}_2$ given in Example 5.3.7 and recall that $b_{78} = (1,0,0,0,0,1) \in \mathcal{B}_2$. We consider $\lambda = \Lambda_1$ and determine the ground-state path of $\mathcal{P}(\Lambda_1)$. Set $\lambda_0 = \Lambda_1$ and observe

$$\begin{aligned}\varphi(b_{78}) &= \varphi_0(b_{78})\Lambda_0 + \varphi_1(b_{78})\Lambda_1 + \varphi_2(b_{78})\Lambda_2\\ &= \Lambda_1.\end{aligned}$$

Therefore $b_{78} = (1,0,0,0,0,1)$ is the unique vector in $\mathcal{B}_2$ such that $\varphi(b_{\lambda_0}) = \lambda_0$.

Next we find $\lambda_1 = \varepsilon(b_{\lambda_0})$:

$$\begin{aligned}\varepsilon(b_{\lambda_0}) &= \varepsilon(b_{78})\\ &= \varepsilon_0(b_{78})\Lambda_0 + \varepsilon_1(b_{78})\Lambda_1 + \varepsilon_2(b_{78})\Lambda_2\\ &= \Lambda_1.\end{aligned}$$

Therefore $\lambda_1 = \Lambda_1$ and hence $b_{\lambda_1} = b_{78} = (1,0,0,0,0,1)$. Thus the ground-state path for $\mathcal{P}(\Lambda_1)$ is

$$p_{\Lambda_1} = \cdots \otimes b_{78} \otimes b_{78}.$$

Using the signature rule and the perfect crystal $\mathcal{B}_2$, we obtain a partial depiction of $\mathcal{P}(\Lambda_1)$, the path realization of $\mathcal{B}(\Lambda_1)$ given in Fig. 5.3. As in Example 5.3.11 we call $\mathcal{P}(\Lambda_1)$ the path realization of $V(\Lambda_1)$.

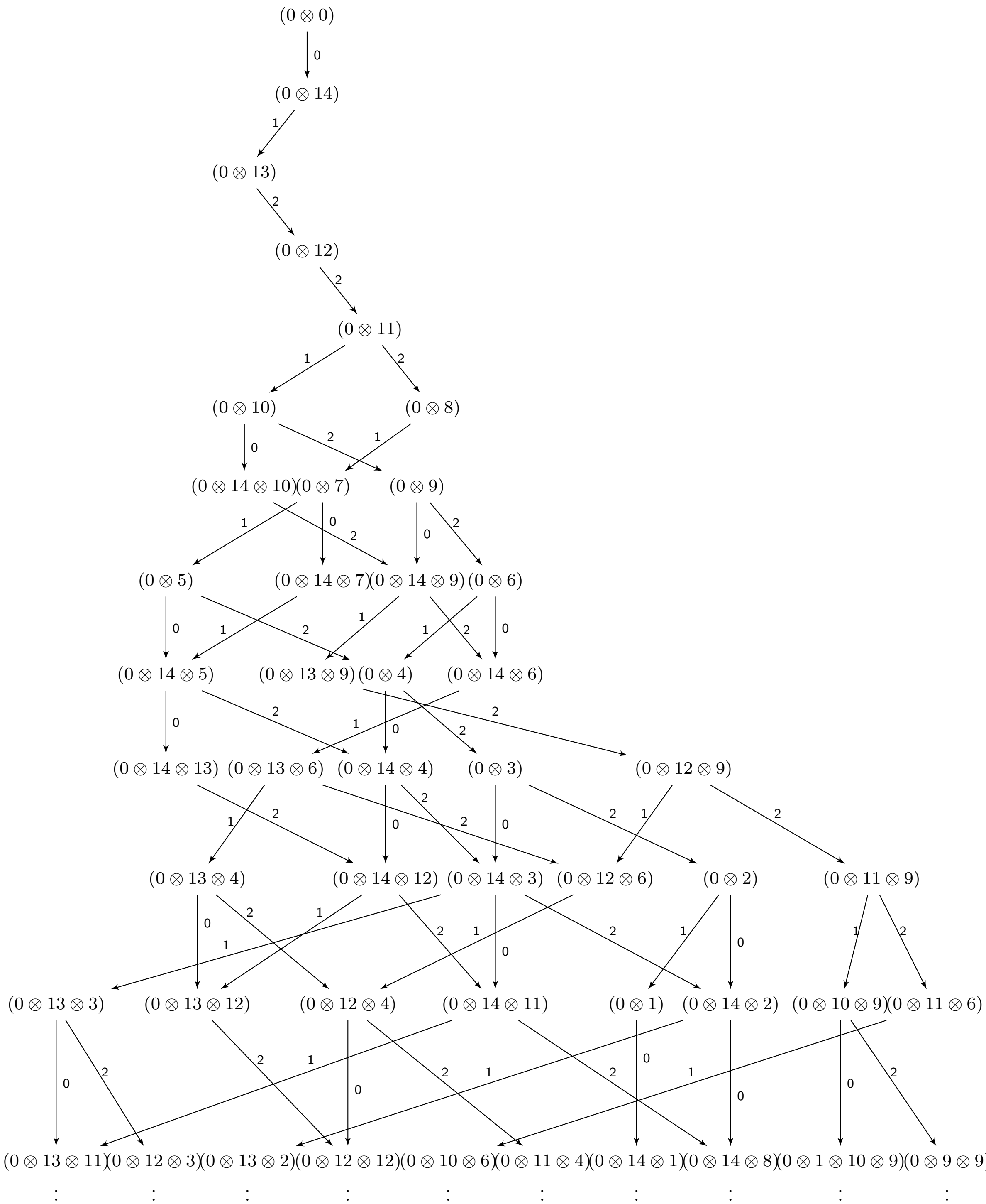


Figure 5.2 Partial graph of $\mathcal{P}(\Lambda_0)$

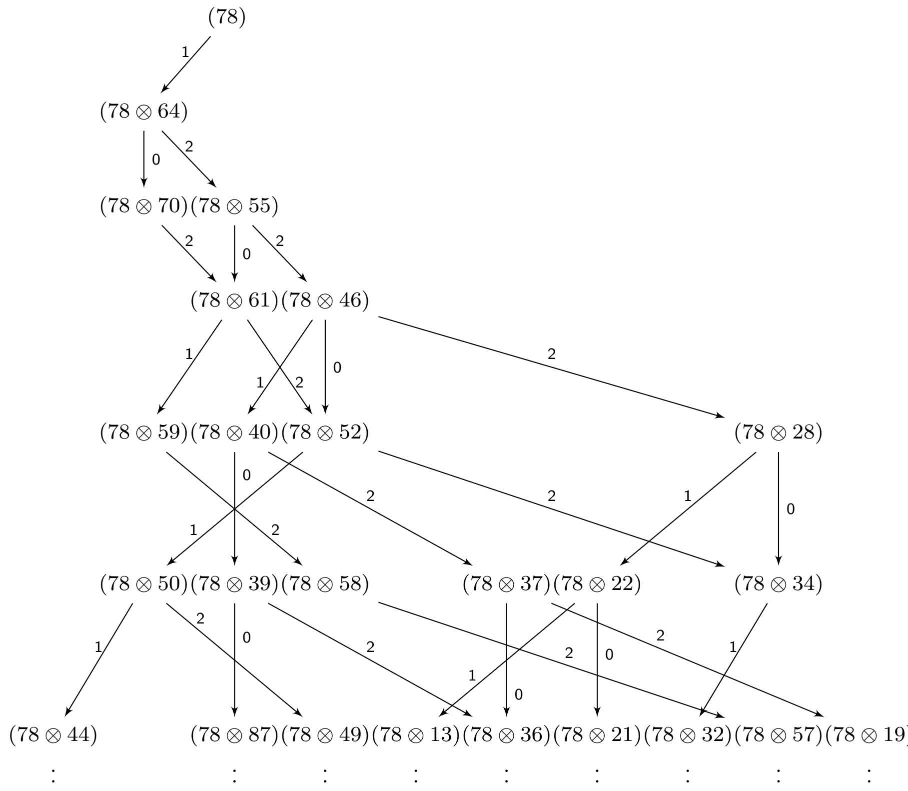


Figure 5.3 Partial graph of $\mathcal{P}(\Lambda_1)$

Example 5.3.13. Let $V^q(3\Lambda_2)$ be the irreducible highest weight $U_q(G_2^{(1)})$-module. Consider the perfect crystal of level 3 $\mathcal{B}_3$ and recall that $b_{219} = (0,1,1,1,1,0) \in \mathcal{B}_3$. The ground-state path for $\mathcal{P}(3\Lambda_2)$ is

$$p_{3\Lambda_2} = \cdots \otimes b_{219} \otimes b_{219}.$$

Using the signature rule and the perfect crystal $\mathcal{B}_3$, we a obtain partial depiction of $\mathcal{P}(3\Lambda_2)$, the path realization of $\mathcal{B}(3\Lambda_2)$ given in Fig. 5.4.

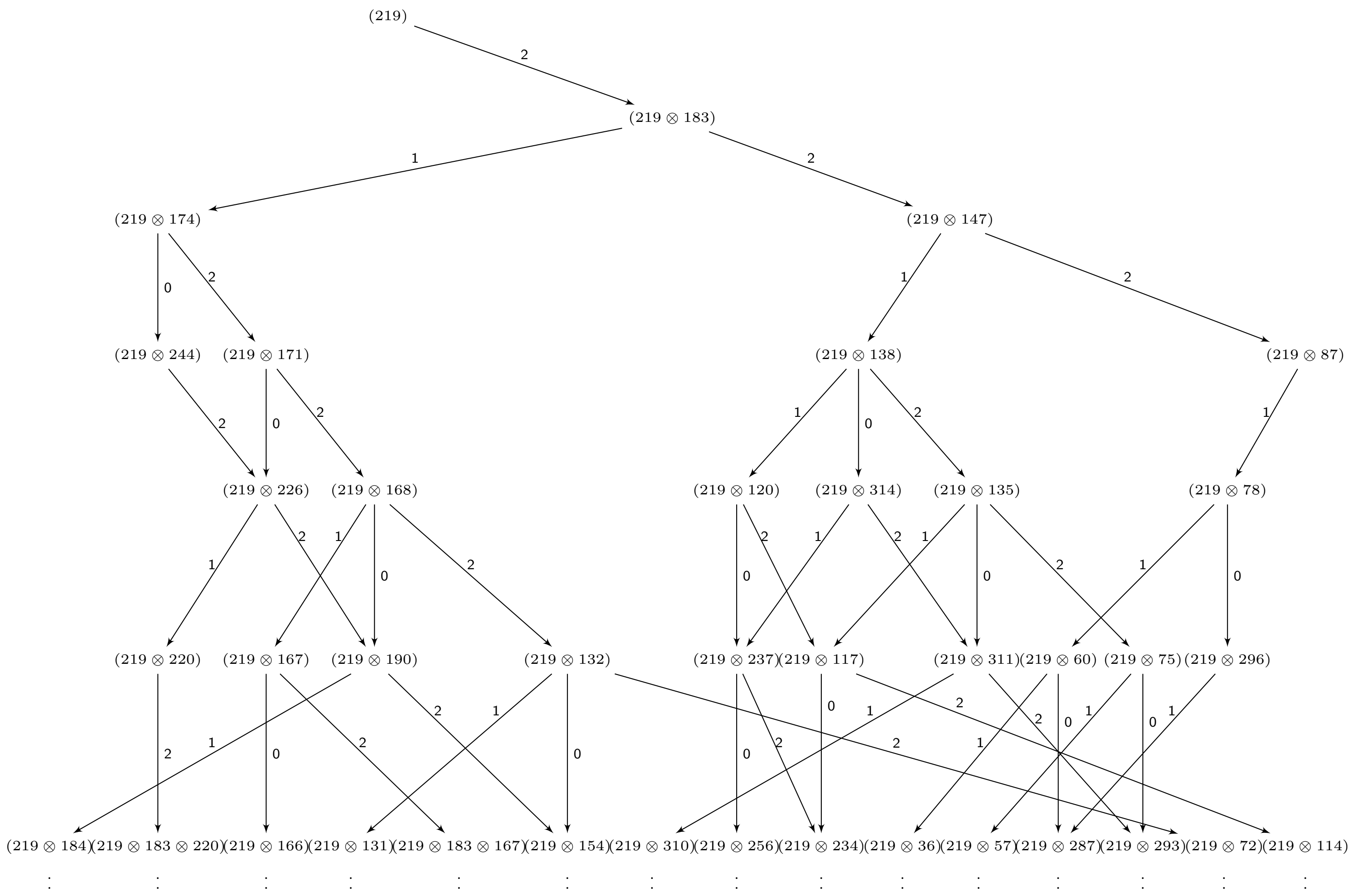


Figure 5.4 Partial graph of $\mathcal{P}(3\Lambda_2)$

5.3.2 Energy Functions and Applications

Let $(\mathcal{L}, \mathcal{B})$ be a crystal base for a finite dimensional $U_q^{'}(\mathfrak{g})$-module V^q. In this section we define the energy function from $\mathcal{B} \otimes \mathcal{B}$ to $\mathbb{Z}$ and use it to determine the affine weight of paths in the path realizations of perfect crystals.

Definition 5.3.14. An **energy function** on $\mathcal{B}$ is a function $H : \mathcal{B} \otimes \mathcal{B} \to \mathbb{Z}$ satisfying:

$$H(\tilde{e}_i(b_1 \otimes b_2)) = \begin{cases} H(b_1 \otimes b_2) & \text{if } i \neq 0, \\ H(b_1 \otimes b_2) + 1 & \text{if } i = 0, \varphi_0(b_1) \geq \varepsilon_0(b_2), \\ H(b_1 \otimes b_2) - 1 & \text{if } i = 0, \varphi_0(b_1) < \varepsilon_0(b_2), \end{cases}$$

for all $i \in I$, $b_1 \otimes b_2 \in \mathcal{B} \otimes \mathcal{B}$ with $\tilde{e}_i(b_1 \otimes b_2) \in \mathcal{B} \otimes \mathcal{B}$.

Lemma 5.3.15. *[Jim91] Let $\lambda, \mu \in P^+$ and $u \in \mathcal{B}(\lambda), v \in \mathcal{B}(\mu)$. Then $\tilde{e}_i(u \otimes v) = 0$ for all $i \in I$ if and only if $\tilde{e}_i u = 0$ and $\tilde{e}_i^{\lambda(h_i)+1} v = 0$ for all $i \in I$.*

Theorem 5.3.16. *Let $\mathcal{B}_1$ be the perfect crystal of level 1 for $U_q(G_2^{(1)})$ from Example 5.1 and define $H : \mathcal{B}_1 \otimes \mathcal{B}_1 \to \mathbb{Z}$ below. Then H is an energy function on $\mathcal{B}_1$.*

$$H(b_i \otimes b_j) = \begin{cases} 0 & \textit{if} \quad i = 0 \textit{ and } j = 0 \\ & \quad\; i = 6-8 \textit{ and } j = 1 \\ & \quad\; i = 9-13 \textit{ and } j = 1-4, 6, 9 \\ & \quad\; i = 14 \textit{ and } j = 1-7, 9, 10 \\ \\ 1 & \textit{if} \quad i = 0 \textit{ and } j = 1-14 \\ & \quad\; i = 1 \textit{ and } j = 0 \\ & \quad\; i = 2 \textit{ and } j = 0, 1 \\ & \quad\; i = 3-5 \textit{ and } j = 0-4, 6, 9 \\ & \quad\; i = 6-8 \textit{ and } j = 0, 2-7, 9, 10 \\ & \quad\; i = 9-13 \textit{ and } j = 0, 5, 7, 8, 10-12 \\ & \quad\; i = 14 \textit{ and } j = 0, 8, 11-13 \\ \\ 2 & \textit{if} \quad i = 1 \textit{ and } j = 1-14 \\ & \quad\; i = 2 \textit{ and } j = 2-14 \\ & \quad\; i = 3-5 \textit{ and } j = 5, 7, 8, 10-14 \\ & \quad\; i = 6-8 \textit{ and } j = 8, 11-14 \\ & \quad\; i = 9-13 \textit{ and } j = 13, 14 \\ & \quad\; i = 14 \textit{ and } j = 14 \end{cases}$$

Proof. It follows by direct verification. □

We remark on how the energy of $\mathcal{B}_1$ was determined.

Remark 5.3.17. *Recall that $\mathcal{B}_1 = \mathcal{B}(0) \oplus \mathcal{B}(\Lambda_1)$ [Mis10]. Then,*

$$\begin{aligned}\mathcal{B}_1 \otimes \mathcal{B}_1 &\cong (\mathcal{B}(0) \otimes \mathcal{B}(0)) \oplus (\mathcal{B}(0) \otimes \mathcal{B}(\Lambda_1)) \oplus (\mathcal{B}(\Lambda_1) \otimes \mathcal{B}(0)) \oplus (\mathcal{B}(\Lambda_1) \otimes \mathcal{B}(\Lambda_1)) \\ &\cong \mathcal{B}(0) \oplus \mathcal{B}(\Lambda_1) \oplus \mathcal{B}(\Lambda_1) \oplus (\mathcal{B}(\Lambda_1) \otimes \mathcal{B}(\Lambda_1))\end{aligned}$$

Then the crystal graph of $\mathcal{B}_1 \otimes \mathcal{B}_1$ can be viewed as union of crystal graphs $\mathcal{B}(\Lambda_1) \otimes \mathcal{B}(\Lambda_1)$, $\mathcal{B}(0)$, and two copies of $\mathcal{B}(\Lambda_1)$, connected by $\tilde{e}_0$-action. Now observe that the definition of an energy function implies that the $\tilde{e}_0$-action requires a change of $H(b_1 \otimes b_2)$ by ± 1 and all other actions ($\tilde{e}_1$ and $\tilde{e}_2$) preserve the value of $H(b_1 \otimes b_2)$. Therefore, it is enough to determine the connected components of $\mathcal{B}_1 \otimes \mathcal{B}_1$ and assign values for $H(b_1 \otimes b_2)$ which are in accordance with the $\tilde{e}_0$-actions that connect these components. For this purpose, we now determine the connected components of $\mathcal{B}(\Lambda_1) \otimes \mathcal{B}(\Lambda_1)$ using Lemma 5.3.15 to identify the highest weights and weight vectors of $\mathcal{B}(\Lambda_1) \otimes \mathcal{B}(\Lambda_1)$. Observe that the following vectors satisfy the conditions of Lemma 5.3.15 and are therefore highest weight vectors of $\mathcal{B}(\Lambda_1) \otimes \mathcal{B}(\Lambda_1)$ with corresponding highest weights:

$$\begin{aligned}&b_{14} \otimes b_{14}, \quad \overline{wt}(b_{14} \otimes b_{14}) = 2\Lambda_1,\\ &b_{14} \otimes b_{13}, \quad \overline{wt}(b_{14} \otimes b_{13}) = 3\Lambda_2,\\ &b_{14} \otimes b_{10}, \quad \overline{wt}(b_{14} \otimes b_{10}) = 2\Lambda_2,\\ &b_{14} \otimes b_{7}, \quad \overline{wt}(b_{14} \otimes b_{7}) = \Lambda_1,\\ &b_{14} \otimes b_{1}, \quad \overline{wt}(b_{14} \otimes b_{1}) = 0,\end{aligned}$$

Then,

$$\begin{aligned}\mathcal{B}_1 \otimes \mathcal{B}_1 &\cong \mathcal{B}(0) \oplus \mathcal{B}(\Lambda_1) \oplus \mathcal{B}(\Lambda_1) \oplus (\mathcal{B}(\Lambda_1) \otimes \mathcal{B}(\Lambda_1)) \\ &\cong \mathcal{B}(0) \oplus \mathcal{B}(\Lambda_1) \oplus \mathcal{B}(\Lambda_1) \oplus \mathcal{B}(2\Lambda_1) \oplus \mathcal{B}(3\Lambda_2) \oplus \mathcal{B}(2\Lambda_2) \oplus \mathcal{B}(\Lambda_1) \oplus \mathcal{B}(0).\end{aligned}$$

Now assign the following values to the elements in each connected component of $\mathcal{B}_1 \otimes \mathcal{B}_1$:

0 *to the elements of* $\mathcal{B}(0) \cong \{b_0 \otimes b_0\}$

1 *to the elements of* $\mathcal{B}(\Lambda_1) \cong \{b_0 \otimes b_i\}$ $(i = 1 - 14)$

1 *to the elements of* $\mathcal{B}(\Lambda_1) \cong \{b_i \otimes b_0\}$ $(i = 1 - 14)$

2 *to the elements of* $\mathcal{B}(2\Lambda_1)$*, which are generated by* $b_{14} \otimes b_{14}$

1 *to the elements of* $\mathcal{B}(3\Lambda_1)$*, which are generated by* $b_{14} \otimes b_{13}$

0 *to the elements of* $\mathcal{B}(2\Lambda_2)$, *which are generated by* $b_{14} \otimes b_{10}$
0 *to the elements of* $\mathcal{B}(\Lambda_1)$, *which are generated by* $b_{14} \otimes b_7$
0 *to the elements of* $\mathcal{B}(0) \cong \{b_{14} \otimes b_1\}$

Let $V^q(\lambda)$ be an irreducible highest weight $U_q'(\mathfrak{g})$-module with crystal base $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$. Now that we have defined the energy function for crystal bases, we recall the affine weight formula for $\mathcal{P}(\lambda)$, the path realization $\mathcal{B}(\lambda)$, which is given in [HK02]. In later sections we derive many results using this weight formula. For example, in Section 6.1 we use it and the energy function in Theorem 5.3.16 to find certain level 1 root multiplicities of $HG_2^{(1)}$.

Theorem 5.3.18 (Affine Weight Formula)**.** *[HK02] Let* $p = (p_k)_{k=1}^{\infty}$ *be a path in* $\mathcal{P}(\lambda)$ *having ground-state path* $p_\lambda = (b_k)_{k=1}^{\infty}$. *Then the affine weight of* p *is given by*

$$\begin{aligned} wt(p) = \lambda + &\sum_{k=0}^{\infty} \left(\overline{wt}(p_k) - \overline{wt}(b_k)\right) \\ &- \left(\sum_{k=0}^{\infty} (k+1)\left(H(p_{k+1} \otimes p_k) - H(b_{k+1} \otimes b_k)\right)\right)\delta. \end{aligned}$$

CHAPTER

6

ROOT MULTIPLICITIES OF $HG_2^{(1)}$

In this chapter we use the path realizations of $G_2^{(1)}$ irreducible highest weight modules from Section 5.3, our general results from Section 4.1, and Kang's multiplicity formula from Section 4.2 to find multiplicities of imaginary roots of $HG_2^{(1)}$. The focus is on roots of the form $-\ell\alpha_{-1}-\alpha-k\delta$, where $\ell \leq 3$ corresponds to the level of the root, $\delta = \alpha_0+2\alpha_1+3\alpha_2$ is the canonical imaginary root of $G_2^{(1)}$, and $\alpha = m_0\alpha_0 + m_1\alpha_1 + m_2\alpha_2 < \delta$ for $m_i \in \mathbb{Z}_{\geq 0}$. We also verify that the multiplicities of these roots satisfy Frenkel's conjecture bound from Section 4.3.

6.1 Level 1 Roots

Consider level 1 roots of the form $-\alpha_{-1} - \alpha - k\delta$, where $\alpha = m_0\alpha_0 + m_1\alpha_1 + m_2\alpha_2 < \delta$ for $m_i \in \mathbb{Z}_{\geq 0}$. In Theorem 6.1.1 and Theorem 6.1.2 we show that for all $\alpha < \delta$ the multiplicity of $-\alpha_{-1} - \alpha - k\delta$ is equal to the multiplicity of $-\alpha_{-1} - k\delta$ or $-\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2 - k\delta$ with a possible shift of the root string by at most -4δ. Therefore we only compute multiplicities of level 1 roots of these forms.

By our construction in Chapter 3 the simple root $-\alpha_{-1}$ of $HG_2^{(1)}$ corresponds to the fundamental weight Λ_0 of $G_2^{(1)}$ and hence the multiplicity of level 1 roots are equal to the dimension of the corresponding weight space of $V(\Lambda_0)$. For small multiples of δ, we determine the $dimV(\Lambda_0)_{-\alpha_{-1}-\alpha-k\delta}$ by counting the number of paths of weight $\Lambda_0 - \alpha - k\delta$ in $\mathcal{P}(\Lambda_0)$, the path realization of $V(\Lambda_0)$, and collect the results in Table 6.1 and Table 6.3. For large multiples of δ, we utilize the computer programming language $C\#$ to write a program to count the paths in $\mathcal{P}(\Lambda_0)$ and display the multiplicities in Table 6.2 and Table 6.4. In addition to determining the multiplicity of roots $\beta = -\alpha_{-1} - \alpha - k\delta$, we compute the multiplicity bound given by Frenkel's conjecture: $p^{(2)}(1 - \frac{(\beta|\beta)}{2})$. In each case, we observe that this bound is satisfied.

In Theorem 6.1.3 we show that paths of weight $\Lambda_0 - k\delta$ have a maximum of k terms which differ from the ground-state path. Moreover, we show that there is at least one path which will attain this bound. Recall that there are at most 15 possibilities for each fluctuation from the ground-state path corresponding to the 15 elements in the perfect crystal of level 1. Therefore, Theorem 6.1.3 gives us a concrete bound on the multiplicity of roots $\alpha_{-1} - k\delta$:

$$mult(\alpha_{-1} - k\delta) \leq 15^k.$$

Lastly, we provide an alternative way to determine level 1 root multiplicities using the energy function from Section 5.3.2, Theorem 6.1.3, and the weight formula from Section 5.3.2, for roots $-\alpha_{-1} - 4\delta$ and $-\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2 - 4\delta$. Although we only use this method to calculate the multiplicity of two roots, it can be extended to find the multiplicity of any level 1 root.

Theorem 6.1.1. *If $-\alpha_{-1} - k\delta$ is a root of $HG_2^{(1)}$, then*

$$\begin{aligned} mult(-\alpha_{-1} - k\delta) &= mult(\beta_0 - k\delta) \\ &= mult(\beta_1 - (k+1)\delta) \\ &= mult(\beta_2 - (k+2)\delta) \\ &= mult(\beta_3 - (k+3)\delta) \\ &= mult(\beta_4 - (k+4)\delta) \end{aligned}$$

where β_i is as follows:

β_0	β_1	β_2	β_3	β_4
$-\alpha_{-1} - \alpha_0$	$-\alpha_{-1} - \alpha_1$	$-\alpha_{-1} - \alpha_0 - 2\alpha_1$	$-\alpha_{-1} - 3\alpha_2$	$-\alpha_{-1} - 2\alpha_1$
$-\alpha_{-1} - \alpha_0 - \alpha_1$	$-\alpha_{-1} - \alpha_1 - 3\alpha_2$		$-\alpha_{-1} - \alpha_0 - 3\alpha_2$	
$-\alpha_{-1} - \alpha_0 - \alpha_1 - 3\alpha_2$	$-\alpha_{-1} - 2\alpha_1 - 3\alpha_2$			

Proof. It is enough to show each $\beta_i - (k+i)\delta$ is Weyl group conjugate to $-\alpha_{-1} - k\delta$.

For each β_0 we choose reflection, w, as follows and apply Proposition 4.1.1:

- $w = r_0 \Longrightarrow w(-\alpha_{-1} - k\delta) = -\alpha_{-1} - \alpha_0 - k\delta$
- $w = r_1 r_0 \Longrightarrow w(-\alpha_{-1} - k\delta) = -\alpha_{-1} - \alpha_0 - \alpha_1 - k\delta$
- $w = r_2 r_1 r_0 \Longrightarrow w(-\alpha_{-1} - k\delta) = -\alpha_{-1} - \alpha_0 - \alpha_1 - 3\alpha_2 - k\delta$

For each β_1 we choose reflection, w, as follows and apply Proposition 4.1.1:

- $w = r_1 r_2 r_1 r_0 \Longrightarrow w(-\alpha_{-1} - k\delta) = -\alpha_{-1} - \alpha_1 - (k+1)\delta$
- $w = r_2 r_1 r_2 r_1 r_0 \Longrightarrow w(-\alpha_{-1} - k\delta) = -\alpha_{-1} - \alpha_1 - 3\alpha_2 - (k+1)\delta$
- $w = r_1 r_2 r_1 r_2 r_1 r_0 \Longrightarrow w(-\alpha_{-1} - k\delta) = -\alpha_{-1} - 2\alpha_1 - 3\alpha_2 - (k+1)\delta$

For each β_2 we choose reflection, w, as follows and apply Proposition 4.1.1:

- $w = r_1 r_0 r_2 r_1 r_2 r_1 r_0 \Longrightarrow w(-\alpha_{-1} - k\delta) = -\alpha_{-1} - \alpha_0 - 2\alpha_1 - (k+2)\delta$

For each β_3 we choose reflection, w, as follows and apply Proposition 4.1.1:

- $w = r_2 r_1 r_0 r_2 r_1 r_2 r_1 r_0 \Longrightarrow w(-\alpha_{-1} - k\delta) = -\alpha_{-1} - 3\alpha_2 - (k+3)\delta$
- $w = r_2 r_1 r_0 r_1 r_2 r_1 r_2 r_1 r_0 \Longrightarrow w(-\alpha_{-1} - k\delta) = -\alpha_{-1} - \alpha_0 - 3\alpha_2 - (k+3)\delta$

For each β_4 we choose reflection, w, as follows and apply Proposition 4.1.1:

- $w = r_1 r_2 r_1 r_0 r_1 r_2 r_1 r_2 r_1 r_0 \Longrightarrow w(-\alpha_{-1} - k\delta) = -\alpha_{-1} - 2\alpha_1 - (k+4)\delta$

□

Theorem 6.1.2. *If $-\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2 - k\delta$ is a root of $HG_2^{(1)}$, then*

$$\begin{aligned} mult(-\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2 - k\delta) &= mult(\beta_0 - k\delta) \\ &= mult(\beta_1 - (k+1)\delta) \\ &= mult(\beta_2 - (k+2)\delta) \\ &= mult(\beta_3 - (k+3)\delta) \end{aligned}$$

where β_i is as follows:

β_0	β_1	β_2	β_3
$-\alpha_{-1}-\alpha_0-\alpha_1-2\alpha_2$	$-\alpha_{-1}-\alpha_0-\alpha_2$	$-\alpha_{-1}-2\alpha_2$	$-\alpha_{-1}-2\alpha_1-\alpha_2$
$-\alpha_{-1}-\alpha_0-2\alpha_1-2\alpha_2$	$-\alpha_{-1}-\alpha_0-2\alpha_1-\alpha_2$	$-\alpha_{-1}-\alpha_0-2\alpha_2$	
	$-\alpha_{-1}-\alpha_2$	$-\alpha_{-1}-2\alpha_1-2\alpha_2$	
	$-\alpha_{-1}-\alpha_1-\alpha_2$		
	$-\alpha_{-1}-\alpha_1-2\alpha_2$		

Proof. It is enough to show each $\beta_i-(k+i)\delta$ is Weyl group conjugate to $-\alpha_{-1}-\alpha_0-\alpha_1-\alpha_2-k\delta$.

For each β_0 we choose reflection, w, as follows and apply Proposition 4.1.1:

- $w=r_2 \Longrightarrow w(-\alpha_{-1}-\alpha_0-\alpha_1-\alpha_2-k\delta)=-\alpha_{-1}-\alpha_0-\alpha_1-2\alpha_2-k\delta$
- $w=r_1r_2 \Longrightarrow w(-\alpha_{-1}-\alpha_0-\alpha_1-\alpha_2-k\delta)=-\alpha_{-1}-\alpha_0-2\alpha_1-2\alpha_2-k\delta$

For each β_1 we choose reflection, w, as follows and apply Proposition 4.1.1:

- $w=r_2r_0r_1r_2 \Longrightarrow w(-\alpha_{-1}-\alpha_0-\alpha_1-\alpha_2-k\delta)=-\alpha_{-1}-\alpha_0-\alpha_2-(k+1)\delta$
- $w=r_1r_2r_0r_1r_2 \Longrightarrow w(-\alpha_{-1}-\alpha_0-\alpha_1-\alpha_2-k\delta)=-\alpha_{-1}-\alpha_0-2\alpha_1-\alpha_2-(k+1)\delta$
- $w=r_2r_1r_2 \Longrightarrow w(-\alpha_{-1}-\alpha_0-\alpha_1-\alpha_2-k\delta)=-\alpha_{-1}-\alpha_2-(k+1)\delta$
- $w=r_1r_2r_1r_2 \Longrightarrow w(-\alpha_{-1}-\alpha_0-\alpha_1-\alpha_2-k\delta)=-\alpha_{-1}-\alpha_1-\alpha_2-(k+1)\delta$
- $w=r_2r_1r_2r_1r_2 \Longrightarrow w(-\alpha_{-1}-\alpha_0-\alpha_1-\alpha_2-k\delta)=-\alpha_{-1}-\alpha_1-2\alpha_2-(k+1)\delta$

For each β_2 we choose reflection, w, as follows and apply Proposition 4.1.1:

- $w=r_2r_0r_2r_1r_2 \Longrightarrow w(-\alpha_{-1}-\alpha_0-\alpha_1-\alpha_2-k\delta)=-\alpha_{-1}-2\alpha_2-(k+2)\delta$
- $w=r_0r_2r_0r_2r_1r_2 \Longrightarrow w(-\alpha_{-1}-\alpha_0-\alpha_1-\alpha_2-k\delta)=-\alpha_{-1}-\alpha_0-2\alpha_2-(k+2)\delta$
- $w=r_1r_2r_0r_2r_1r_2 \Longrightarrow w(-\alpha_{-1}-\alpha_0-\alpha_1-\alpha_2-k\delta)=-\alpha_{-1}-2\alpha_1-2\alpha_2-(k+2)\delta$

For each β_3 we choose reflection, w, as follows and apply Proposition 4.1.1:

- $w=r_1r_2r_1r_0r_2r_1r_2r_1r_2 \Longrightarrow w(-\alpha_{-1}-\alpha_0-\alpha_1-\alpha_2-k\delta)=-\alpha_{-1}-2\alpha_1-\alpha_2-(k+3)\delta$

□

Table 6.1 Multiplicity table for $-\alpha_{-1} - k\delta$ when $1 \leq k \leq 4$

$-\alpha_{-1} - k\delta$	Multiplicity	Paths	Frenkel Bound
$k = 1$	2	$\cdots b_0 \otimes b_7$	2
		$\cdots b_0 \otimes b_9$	
$k = 2$	6	$\cdots b_0 \otimes b_9 \otimes b_9$	20
		$\cdots b_0 \otimes b_{10} \otimes b_6$	
		$\cdots b_0 \otimes b_{11} \otimes b_4$	
		$\cdots b_0 \otimes b_{12} \otimes b_3$	
		$\cdots b_0 \otimes b_{13} \otimes b_2$	
		$\cdots b_0 \otimes b_{14} \otimes b_1$	
$k = 3$	14	$\cdots b_0 \otimes b_6 \otimes b_{10}$	110
		$\cdots b_0 \otimes b_7 \otimes b_0$	
		$\cdots b_0 \otimes b_7 \otimes b_7$	
		$\cdots b_0 \otimes b_7 \otimes b_9$	
		$\cdots b_0 \otimes b_8 \otimes b_5$	
		$\cdots b_0 \otimes b_9 \otimes b_0$	
		$\cdots b_0 \otimes b_9 \otimes b_7$	
		$\cdots b_0 \otimes b_9 \otimes b_9 \otimes b_9$	
		$\cdots b_0 \otimes b_{10} \otimes b_9 \otimes b_6$	
		$\cdots b_0 \otimes b_{11} \otimes b_9 \otimes b_4$	
		$\cdots b_0 \otimes b_{12} \otimes b_9 \otimes b_3$	
		$\cdots b_0 \otimes b_{13} \otimes b_9 \otimes b_2$	
		$\cdots b_0 \otimes b_{14} \otimes b_7 \otimes b_1$	
		$\cdots b_0 \otimes b_{14} \otimes b_9 \otimes b_1$	
$k = 4$	32	$\cdots b_0 \otimes b_1 \otimes b_{14}$	481
		$\cdots b_0 \otimes b_2 \otimes b_{13}$	
		$\cdots b_0 \otimes b_3 \otimes b_{12}$	
		$\cdots b_0 \otimes b_4 \otimes b_{11}$	
		$\cdots b_0 \otimes b_5 \otimes b_8$	
		$\cdots b_0 \otimes b_9 \otimes b_6 \otimes b_{10}$	
		$\cdots b_0 \otimes b_9 \otimes b_9 \otimes b_0$	
		$\cdots b_0 \otimes b_9 \otimes b_9 \otimes b_7$	
		$\cdots b_0 \otimes b_{10} \otimes b_6 \otimes b_0$	
		$\cdots b_0 \otimes b_{10} \otimes b_6 \otimes b_7$	

Table 6.1 (continued)

		$\cdots b_0 \otimes b_{10} \otimes b_6 \otimes b_9$	
		$\cdots b_0 \otimes b_{11} \otimes b_4 \otimes b_0$	
		$\cdots b_0 \otimes b_{11} \otimes b_4 \otimes b_9$	
		$\cdots b_0 \otimes b_{11} \otimes b_6 \otimes b_5$	
		$\cdots b_0 \otimes b_{12} \otimes b_3 \otimes b_0$	
		$\cdots b_0 \otimes b_{12} \otimes b_3 \otimes b_9$	
		$\cdots b_0 \otimes b_{12} \otimes b_4 \otimes b_6$	
		$\cdots b_0 \otimes b_{12} \otimes b_6 \otimes b_4$	
		$\cdots b_0 \otimes b_{13} \otimes b_2 \otimes b_0$	
		$\cdots b_0 \otimes b_{13} \otimes b_3 \otimes b_6$	
		$\cdots b_0 \otimes b_{13} \otimes b_6 \otimes b_3$	
		$\cdots b_0 \otimes b_{14} \otimes b_1 \otimes b_0$	
		$\cdots b_0 \otimes b_{14} \otimes b_3 \otimes b_4$	
		$\cdots b_0 \otimes b_{14} \otimes b_4 \otimes b_3$	
		$\cdots b_0 \otimes b_{14} \otimes b_5 \otimes b_2$	
		$\cdots b_0 \otimes b_9 \otimes b_9 \otimes b_9 \otimes b_9$	
		$\cdots b_0 \otimes b_{10} \otimes b_9 \otimes b_9 \otimes b_6$	
		$\cdots b_0 \otimes b_{11} \otimes b_9 \otimes b_9 \otimes b_4$	
		$\cdots b_0 \otimes b_{12} \otimes b_9 \otimes b_9 \otimes b_3$	
		$\cdots b_0 \otimes b_{13} \otimes b_9 \otimes b_9 \otimes b_2$	
		$\cdots b_0 \otimes b_{14} \otimes b_9 \otimes b_9 \otimes b_1$	
		$\cdots b_0 \otimes b_{14} \otimes b_{10} \otimes b_6 \otimes b_1$	

Table 6.2 Multiplicity table for $-\alpha_{-1} - k\delta$ when $5 \leq k \leq 20$

$-\alpha_{-1} - k\delta$	$mult(-\alpha_{-1} - k\delta)$	Frenkel Bound
$k = 5$	66	1770
$k = 6$	135	5822
$k = 7$	258	17490
$k = 8$	486	49010
$k = 9$	878	129512
$k = 10$	1559	326015
$k = 11$	2696	786814
$k = 12$	4589	1831065
$k = 13$	7652	4126070
$k = 14$	12584	9035539
$k = 15$	20370	19283830
$k = 16$	32570	40210481
$k = 17$	51406	82088400
$k = 18$	80280	164363280
$k = 19$	124004	323275512
$k = 20$	189764	625425005

Table 6.3 Multiplicity table for $-\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2 - k\delta$ when $1 \leq k \leq 4$

$-\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2 - k\delta$	Multiplicity	Paths	Frenkel Bound
$k = 1$	3	$\cdots b_0 \otimes b_{12} \otimes b_9$	10
		$\cdots b_0 \otimes b_{13} \otimes b_6$	
		$\cdots b_0 \otimes b_{14} \otimes b_4$	
$k = 2$	9	$\cdots b_0 \otimes b_9 \otimes b_{12}$	65
		$\cdots b_0 \otimes b_{10} \otimes b_{11}$	
		$\cdots b_0 \otimes b_{11} \otimes b_{10}$	
		$\cdots b_0 \otimes b_{12} \otimes b_0$	
		$\cdots b_0 \otimes b_{12} \otimes b_7$	
		$\cdots b_0 \otimes b_{12} \otimes b_9 \otimes b_9$	
		$\cdots b_0 \otimes b_{13} \otimes b_9 \otimes b_6$	
		$\cdots b_0 \otimes b_{14} \otimes b_9 \otimes b_4$	
		$\cdots b_0 \otimes b_{14} \otimes b_{10} \otimes b_3$	
$k = 3$	21	$\cdots b_0 \otimes b_4 \otimes b_{14}$	300
		$\cdots b_0 \otimes b_6 \otimes b_{13}$	
		$\cdots b_0 \otimes b_7 \otimes b_{12}$	

Table 6.3 (continued)

		$\cdots b_0 \otimes b_9 \otimes b_9 \otimes b_{12}$	
		$\cdots b_0 \otimes b_{10} \otimes b_9 \otimes b_{11}$	
		$\cdots b_0 \otimes b_{11} \otimes b_9 \otimes b_{10}$	
		$\cdots b_0 \otimes b_{12} \otimes b_6 \otimes b_{10}$	
		$\cdots b_0 \otimes b_{12} \otimes b_9 \otimes b_0$	
		$\cdots b_0 \otimes b_{12} \otimes b_9 \otimes b_7$	
		$\cdots b_0 \otimes b_{13} \otimes b_6 \otimes b_0$	
		$\cdots b_0 \otimes b_{13} \otimes b_6 \otimes b_7$	
		$\cdots b_0 \otimes b_{13} \otimes b_6 \otimes b_9$	
		$\cdots b_0 \otimes b_{14} \otimes b_4 \otimes b_0$	
		$\cdots b_0 \otimes b_{14} \otimes b_4 \otimes b_9$	
		$\cdots b_0 \otimes b_{14} \otimes b_5 \otimes b_6$	
		$\cdots b_0 \otimes b_{14} \otimes b_6 \otimes b_5$	
		$\cdots b_0 \otimes b_{14} \otimes b_7 \otimes b_4$	
		$\cdots b_0 \otimes b_{12} \otimes b_9 \otimes b_9 \otimes v_9$	
		$\cdots b_0 \otimes b_{13} \otimes b_9 \otimes b_9 \otimes v_6$	
		$\cdots b_0 \otimes b_{14} \otimes b_9 \otimes b_9 \otimes v_4$	
		$\cdots b_0 \otimes b_{14} \otimes b_{10} \otimes b_9 \otimes v_3$	
$k = 4$	48	$\cdots b_0 \otimes b_9 \otimes b_4 \otimes b_{14}$	1165
		$\cdots b_0 \otimes b_9 \otimes b_6 \otimes b_{13}$	
		$\cdots b_0 \otimes b_9 \otimes b_{12} \otimes b_9$	
		$\cdots b_0 \otimes b_{10} \otimes b_3 \otimes b_{14}$	
		$\cdots b_0 \otimes b_{10} \otimes b_6 \otimes b_{12}$	
		$\cdots b_0 \otimes b_{10} \otimes b_{11} \otimes b_9$	
		$\cdots b_0 \otimes b_{10} \otimes b_{12} \otimes b_6$	
		$\cdots b_0 \otimes b_{11} \otimes b_3 \otimes b_{13}$	
		$\cdots b_0 \otimes b_{11} \otimes b_4 \otimes b_{12}$	
		$\cdots b_0 \otimes b_{11} \otimes b_{10} \otimes b_9$	
		$\cdots b_0 \otimes b_{11} \otimes b_{12} \otimes b_4$	
		$\cdots b_0 \otimes b_{12} \otimes b_0 \otimes b_0$	
		$\cdots b_0 \otimes b_{12} \otimes b_1 \otimes b_{14}$	
		$\cdots b_0 \otimes b_{12} \otimes b_2 \otimes b_{13}$	
		$\cdots b_0 \otimes b_{12} \otimes b_3 \otimes b_{12}$	

Table 6.3 (continued)

		$\cdots b_0 \otimes b_{12} \otimes b_4 \otimes b_{11}$	
		$\cdots b_0 \otimes b_{12} \otimes b_{10} \otimes b_6$	
		$\cdots b_0 \otimes b_{12} \otimes b_{11} \otimes b_4$	
		$\cdots b_0 \otimes b_{12} \otimes b_{12} \otimes b_3$	
		$\cdots b_0 \otimes b_{13} \otimes b_2 \otimes b_{12}$	
		$\cdots b_0 \otimes b_{13} \otimes b_3 \otimes b_{11}$	
		$\cdots b_0 \otimes b_{13} \otimes b_4 \otimes b_8$	
		$\cdots b_0 \otimes b_{13} \otimes b_{11} \otimes b_3$	
		$\cdots b_0 \otimes b_{13} \otimes b_{12} \otimes b_2$	
		$\cdots b_0 \otimes b_{14} \otimes b_1 \otimes b_{12}$	
		$\cdots b_0 \otimes b_{14} \otimes b_3 \otimes b_{10}$	
		$\cdots b_0 \otimes b_{14} \otimes b_4 \otimes b_7$	
		$\cdots b_0 \otimes b_{14} \otimes b_{12} \otimes b_1$	
		$\cdots b_0 \otimes b_9 \otimes b_9 \otimes b_9 \otimes b_{12}$	
		$\cdots b_0 \otimes b_{10} \otimes b_9 \otimes b_9 \otimes b_{11}$	
		$\cdots b_0 \otimes b_{11} \otimes b_9 \otimes b_9 \otimes b_{10}$	
		$\cdots b_0 \otimes b_{12} \otimes b_9 \otimes b_6 \otimes b_{10}$	
		$\cdots b_0 \otimes b_{12} \otimes b_9 \otimes b_9 \otimes b_0$	
		$\cdots b_0 \otimes b_{12} \otimes b_9 \otimes b_9 \otimes b_7$	
		$\cdots b_0 \otimes b_{13} \otimes b_9 \otimes b_6 \otimes b_0$	
		$\cdots b_0 \otimes b_{13} \otimes b_9 \otimes b_6 \otimes b_7$	
		$\cdots b_0 \otimes b_{13} \otimes b_9 \otimes b_6 \otimes b_9$	
		$\cdots b_0 \otimes b_{14} \otimes b_9 \otimes b_4 \otimes b_0$	
		$\cdots b_0 \otimes b_{14} \otimes b_9 \otimes b_4 \otimes b_9$	
		$\cdots b_0 \otimes b_{14} \otimes b_9 \otimes b_6 \otimes b_5$	
		$\cdots b_0 \otimes b_{14} \otimes b_{10} \otimes b_3 \otimes b_0$	
		$\cdots b_0 \otimes b_{14} \otimes b_{10} \otimes b_3 \otimes b_9$	
		$\cdots b_0 \otimes b_{14} \otimes b_{10} \otimes b_4 \otimes b_6$	
		$\cdots b_0 \otimes b_{14} \otimes b_{10} \otimes b_6 \otimes b_4$	
		$\cdots b_0 \otimes b_{12} \otimes b_9 \otimes b_9 \otimes b_9 \otimes b_9$	
		$\cdots b_0 \otimes b_{13} \otimes b_9 \otimes b_9 \otimes b_9 \otimes b_6$	
		$\cdots b_0 \otimes b_{14} \otimes b_9 \otimes b_9 \otimes b_9 \otimes b_4$	
		$\cdots b_0 \otimes b_{14} \otimes b_{10} \otimes b_9 \otimes b_9 \otimes b_3$	

Table 6.4 Multiplicity table for $\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2 - k\delta$ when $5 \leq k \leq 20$

$-\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2 - k\delta$	$mult(-\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2 - k\delta)$	Frenkel Bound
$k = 5$	99	3956
$k = 6$	199	12230
$k = 7$	378	35002
$k = 8$	702	94235
$k = 9$	1258	240840
$k = 10$	2211	589128
$k = 11$	3789	1386930
$k = 12$	6388	3157789
$k = 13$	10566	6978730
$k = 14$	17235	15018300
$k = 15$	27691	31551450
$k = 16$	43962	64854575
$k = 17$	68931	130673928
$k = 18$	106967	258508230
$k = 19$	164259	502810130
$k = 20$	249957	962759294

In the next theorem we use the energy function for the perfect crystal $\mathcal{B}_1$ of level 1 and the affine weight formula (see Theorem 5.3.18) to provide a bound on the number of fluctuations a path of $\mathcal{P}(\Lambda_0)$ can have from the ground-state path. Then we use these tools to find the root multiplicity of $-\alpha_{-1} - 4\delta$ and $-\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2 - 4\delta$ in Example 6.1.4 and Example 6.1.5. For convenience we recall the weight of each $b_i \in \mathcal{B}_1$ and the energy function $H : \mathcal{B}_1 \otimes \mathcal{B}_1 \to \mathbb{Z}$ from sections 5.3 and 5.3.2.

$$
\begin{array}{lll}
\overline{wt}(b_1) = -(2\alpha_1 + 3\alpha_2) & \overline{wt}(b_{14}) = 2\alpha_1 + 3\alpha_2 & \overline{wt}(b_0) = 0 \\
\overline{wt}(b_2) = -(\alpha_1 + 3\alpha_2) & \overline{wt}(b_{13}) = \alpha_1 + 3\alpha_2 & \overline{wt}(b_7) = 0 \\
\overline{wt}(b_3) = -(\alpha_1 + 2\alpha_2) & \overline{wt}(b_{12}) = \alpha_1 + 2\alpha_2 & \overline{wt}(b_9) = 0 \\
\overline{wt}(b_4) = -(\alpha_1 + \alpha_2) & \overline{wt}(b_{11}) = \alpha_1 + \alpha_2 & \\
\overline{wt}(b_5) = -\alpha_1 & \overline{wt}(b_8) = \alpha_1 & \\
\overline{wt}(b_6) = -\alpha_2 & \overline{wt}(b_{10}) = \alpha_2 &
\end{array}
$$

$$H(b_i \otimes b_j) = \begin{cases} 0 & \text{if} \quad i = 0 \text{ and } j = 0 \\ & \quad i = 6-8 \text{ and } j = 1 \\ & \quad i = 9-13 \text{ and } j = 1-4, 6, 9 \\ & \quad i = 14 \text{ and } j = 1-7, 9, 10 \\ \\ 1 & \text{if} \quad i = 0 \text{ and } j = 1-14 \\ & \quad i = 1 \text{ and } j = 0 \\ & \quad i = 2 \text{ and } j = 0, 1 \\ & \quad i = 3-5 \text{ and } j = 0-4, 6, 9 \\ & \quad i = 6-8 \text{ and } j = 0, 2-7, 9, 10 \\ & \quad i = 9-13 \text{ and } j = 0, 5, 7, 8, 10-12 \\ & \quad i = 14 \text{ and } j = 0, 8, 11-13 \\ \\ 2 & \text{if} \quad i = 1 \text{ and } j = 1-14 \\ & \quad i = 2 \text{ and } j = 2-14 \\ & \quad i = 3-5 \text{ and } j = 5, 7, 8, 10-14 \\ & \quad i = 6-8 \text{ and } j = 8, 11-14 \\ & \quad i = 9-13 \text{ and } j = 13, 14 \\ & \quad i = 14 \text{ and } j = 14 \end{cases}$$

Theorem 6.1.3. *Let $p = (\cdots b_0 \otimes x_m \otimes x_{m-1} \otimes \cdots \otimes x_1 \otimes x_0)$ be a path in $\mathcal{P}(\Lambda_0)$ with weight $wt(p) = -\alpha_{-1} - k\delta$, where m is the smallest positive integer such that $x_j = b_0$ for all $j > m$. Then,*

$$m \leq k - 1 \text{ for all } k \geq 1.$$

Furthermore, $k-1$ is the least upper bound of m.

Proof. Clearly m is finite, since k is finite. Suppose to the contrary $m > k-1$. Then Theorem 5.3.18 yields,

$$\begin{aligned} -\alpha_{-1} - k\delta &= wt(\cdots b_0 \otimes x_m \otimes x_{m-1} \otimes \cdots \otimes x_0) \\ &= \Lambda_0 + \overline{wt}(x_m) + \overline{wt}(x_{m-1}) + \cdots + \overline{wt}(x_0) \\ &\quad - [(m+1)H(b_0 \otimes x_m) + \cdots + kH(x_k \otimes x_{k-1}) + \cdots + H(x_1 \otimes x_0)]\,\delta \end{aligned} \tag{6.1}$$

Observe $\overline{wt}(b_i)$ is a linear combination of α_1 and α_2 for all $0 \leq i \leq 14$ and thus cannot form a δ when summed. Therefore equation (6.1) implies

$$\overline{wt}(x_m) + \overline{wt}(x_{m-1}) + \cdots + \overline{wt}(x_0) = 0 \tag{6.2}$$

$$(m+1)H(b_0 \otimes x_m) + \cdots + kH(x_k \otimes x_{k-1}) + \cdots + H(x_1 \otimes x_0) = k. \tag{6.3}$$

By the definition of the energy function, we have $H(b_i \otimes b_j) \geq 0$ for every $0 \leq i \leq 14$ and $0 \leq j \leq 14$. Since we assumed $m+1 > k$, equation (6.3) implies $H(b_0 \otimes x_m) = 0$. Using the defined energy function, we conclude $x_m = b_0$, which is a contradiction. Therefore, $m \leq k-1$.

We now show that for every k the bound on m is obtained:

$k = 1$: Consider $p = (\cdots b_0 \otimes b_9)$, which has weight $\Lambda_0 - \delta$ and $m = 0$.

$k = 2$: Consider $p = (\cdots b_0 \otimes b_{14} \otimes b_1)$, which has weight $\Lambda_0 - 2\delta$ and $m = 1$.

$k = 3$: Consider $p = (\cdots b_0 \otimes b_{14} \otimes b_9 \otimes b_1)$, which has weight $\Lambda_0 - 3\delta$ and $m = 2$.

$k = 4$: Consider $p = (\cdots b_0 \otimes b_{14} \otimes b_{10} \otimes b_6 \otimes b_1)$, which has weight $\Lambda_0 - 4\delta$ and $m = 3$.

$k > 4$: Consider $p = (\cdots b_0 \otimes b_{14} \otimes b_{10} \otimes \underbrace{b_9 \otimes \cdots \otimes b_9}_{k-4 \text{ times}} \otimes b_6 \otimes b_1) \in P(\Lambda_0)$ with $m = k-1$.

Using the defined energy function, we compute the weight of p:

$$\begin{aligned} wt(p) &= \Lambda_0 + \overline{wt}(b_{14}) + \overline{wt}(b_{10}) + (k-4)\overline{wt}(b_9) + \overline{wt}(b_6) + \overline{wt}(b_1) \\ &\qquad - [kH(b_0 \otimes b_{14}) + (k-1)H(b_{14} \otimes x_{10}) + (k-2)H(b_{10} \otimes b_9) + \\ &\qquad\qquad \cdots + 2H(b_9 \otimes x_6) + H(b_6 \otimes b_1)]\,\delta \\ &= \Lambda_0 - [k + 0 + 0 + \cdots + 0 + 0]\,\delta \\ &= \Lambda_0 - k\delta \end{aligned}$$

Therefore, $k-1$ is the least upper bound of m.

□

Example 6.1.4 (Multiplicity of $-\alpha_{-1} - 4\delta$). To find the multiplicity of $-\alpha_{-1} - 4\delta$, we use the affine weight formula to determine the paths of weight $-\alpha_{-1} - 4\delta$ in $\mathcal{P}(\Lambda_0)$. Let $p \in \mathcal{P}(\Lambda_0)$. Theorem 6.1.3 implies p has at most 4 fluctuations from the ground-state path. Then the weight formula becomes,

$$\begin{aligned}-\alpha_{-1} - 4\delta &= wt(\cdots b_0 \otimes x_3 \otimes x_2 \otimes x_1 \otimes x_0) \\ &= \Lambda_0 + \overline{wt}(x_3) + \overline{wt}(x_2) + \overline{wt}(x_1) + \overline{wt}(x_0) \\ &\quad - \left[4H(b_0 \otimes x_3) + 3H(x_3 \otimes x_2) + 2H(x_2 \otimes x_1) + H(x_1 \otimes x_0)\right]\delta\end{aligned}$$

Since $\overline{wt}(x_i)$ contains no α_0's for all $0 \le i \le 14$, we have

$$\overline{wt}(x_3) + \overline{wt}(x_2) + \overline{wt}(x_1) + \overline{wt}(x_0) = 0$$

$$[4H(b_0 \otimes x_3) + 3H(x_3 \otimes x_2) + 2H(x_2 \otimes x_1) + H(x_1 \otimes x_0)] = 4.$$

This yields the 4 cases below, which we examine in turn.

Case (1): $H(b_0 \otimes x_3) = 1$ and $H(x_3 \otimes x_2) = H(x_2 \otimes x_1) = H(x_1 \otimes x_0) = 0$

Case (2): $H(x_3 \otimes x_2) = H(x_1 \otimes x_0) = 1$ and $H(b_0 \otimes x_3) = H(x_2 \otimes x_1) = 0$

Case (3): $H(x_2 \otimes x_1) = 2$ and $H(b_0 \otimes x_3) = H(x_3 \otimes x_2) = H(x_1 \otimes x_0) = 0$

Case (4): $H(x_2 \otimes x_1) = 1$, $H(x_1 \otimes x_0) = 2$, and $H(b_0 \otimes x_3) = H(x_3 \otimes x_2) = 0$

Case (1): Using the definition of the energy function we narrow down the possible choices for x_3, x_2, x_1, x_0. First, observe $H(b_0 \otimes x_3) = 1$ and $H(x_3 \otimes x_2) = 0$ imply $x_3 = b_6 - b_{14}$.

If $x_3 = b_6 - b_8$, then $H(x_3 \otimes x_2) = 0$ implies $x_2 = b_1$, which is impossible when $H(x_2 \otimes x_1) = 0$.
If $x_3 = b_9 - b_{13}$, then $H(x_3 \otimes x_2) = 0$ and $H(x_2 \otimes x_1) = 0$ imply $x_2 = b_6, b_9$.
  If $x_2 = b_6$, then $H(x_2 \otimes x_1) = 0$ implies $x_1 = b_1$, which is impossible when $H(x_1 \otimes x_0) = 0$.
  If $x_2 = b_9$, then $H(x_2 \otimes x_1) = 0$ and $H(x_1 \otimes x_0) = 0$ imply $x_1 = b_6, b_9$.
    If $x_1 = b_6$, then $H(x_1 \otimes x_0) = 0$ implies $x_0 = b_1$.
    If $x_1 = b_9$, then $H(x_1 \otimes x_0) = 0$ implies $x_0 = b_1 - b_4, b_6, b_9$.
If $x_3 = b_{14}$, then $H(x_3 \otimes x_2) = 0$ and $H(x_2 \otimes x_1) = 0$ imply $x_2 = b_6, b_7, b_9, b_{10}$.
  If $x_2 = b_6, b_7$, then $H(x_2 \otimes x_1) = 0$ implies $x_1 = b_1$, which is impossible if $H(x_1 \otimes x_0) = 0$.
  If $x_2 = b_9, b_{10}$, then $H(x_2 \otimes x_1) = 0$ and $H(x_1 \otimes x_0) = 0$ imply $x_1 = b_6, b_9$.
    If $x_1 = b_6$, then $H(x_1 \otimes x_0) = 0$ implies $x_0 = b_1$.
    If $x_1 = b_9$, then $H(x_1 \otimes x_0) = 0$ implies $x_0 = b_1 - b_4, b_6, b_9$.

For each of the above choices of x_3, x_2, x_1, x_0, we determine which paths satisfy $\overline{wt}(x_3)+\overline{wt}(x_2)+\overline{wt}(x_1)+\overline{wt}(x_0)=0$:

$$\begin{array}{rr} \cdots b_0\otimes b_9\otimes b_9\otimes b_9\otimes b_9 & \cdots b_0\otimes b_{13}\otimes b_9\otimes b_9\otimes b_2 \\ \cdots b_0\otimes b_{10}\otimes b_9\otimes b_9\otimes b_6 & \cdots b_0\otimes b_{14}\otimes b_9\otimes b_9\otimes b_1 \\ \cdots b_0\otimes b_{11}\otimes b_9\otimes b_9\otimes b_4 & \cdots b_0\otimes b_{14}\otimes b_{10}\otimes b_6\otimes b_1 \\ \cdots b_0\otimes b_{12}\otimes b_9\otimes b_9\otimes b_3 & \end{array}$$

Case (2): Using the definition of the energy function, we again narrow down the possible choices for x_3, x_2, x_1, x_0. Observe $H(b_0\otimes x_3)=0$ implies $x_3=b_0$. Then $H(x_3\otimes x_2)=1$ and $H(x_2\otimes x_1)=0$ imply $x_2=b_6-b_{14}$.

If $x_2=b_6-b_8$, then $H(x_2\otimes x_1)=0$ implies $x_1=b_1$. Then $H(x_1\otimes x_0)=1$ implies $x_0=b_0$.
If $x_2=b_9-b_{13}$, then $H(x_2\otimes x_1)=0$ implies $x_1=b_1-b_4,b_6,b_9$.
  If $x_1=b_1$, then $H(x_1\otimes x_0)=1$ implies $x_0=b_0$.
  If $x_1=b_2$, then $H(x_1\otimes x_0)=1$ implies $x_0=b_0,b_1$.
  If $x_1=b_3,b_4$, then $H(x_1\otimes x_0)=1$ implies $x_0=b_0-b_4,b_6,b_9$.
  If $x_1=b_6$, then $H(x_1\otimes x_0)=1$ implies $x_0=b_0,b_2-b_7,b_{10}$.
  If $x_1=b_9$, then $H(x_1\otimes x_0)=1$ implies $x_0=b_0,b_5,b_7,b_8,b_{10}-b_{12}$.
If $x_2=b_{14}$, then $H(x_2\otimes x_1)=0$ implies $x_1=b_1-b_7,b_9,b_{10}$.
  If $x_1=b_1$, then $H(x_1\otimes x_0)=1$ implies $x_0=b_0$.
  If $x_1=b_2$, then $H(x_1\otimes x_0)=1$ implies $x_0=b_0,b_1$.
  If $x_1=b_3-b_5$, then $H(x_1\otimes x_0)=1$ implies $x_0=b_0-b_4,b_6,b_9$.
  If $x_1=b_6,b_7$, then $H(x_1\otimes x_0)=1$ implies $x_0=b_0,b_2-b_7,b_{10}$.
  If $x_1=b_9,b_{10}$, then $H(x_1\otimes x_0)=1$ implies $x_0=b_0,b_5,b_7,b_8,b_{10}-b_{12}$.

For each of the above choices of x_2, x_1, x_0, we determine which paths satisfy $\overline{wt}(x_2)+\overline{wt}(x_1)+\overline{wt}(x_0)=0$:

$$\begin{array}{rrr} \cdots b_0\otimes b_9\otimes b_6\otimes b_{10} & \cdots b_0\otimes b_{11}\otimes b_4\otimes b_9 & \cdots b_0\otimes b_{13}\otimes b_3\otimes b_6 \\ \cdots b_0\otimes b_9\otimes b_9\otimes b_0 & \cdots b_0\otimes b_{11}\otimes b_6\otimes b_5 & \cdots b_0\otimes b_{13}\otimes b_6\otimes b_3 \\ \cdots b_0\otimes b_9\otimes b_9\otimes b_7 & \cdots b_0\otimes b_{12}\otimes b_3\otimes b_0 & \cdots b_0\otimes b_{14}\otimes b_1\otimes b_0 \\ \cdots b_0\otimes b_{10}\otimes b_6\otimes b_0 & \cdots b_0\otimes b_{12}\otimes b_3\otimes b_9 & \cdots b_0\otimes b_{14}\otimes b_3\otimes b_4 \\ \cdots b_0\otimes b_{10}\otimes b_6\otimes b_7 & \cdots b_0\otimes b_{12}\otimes b_4\otimes b_6 & \cdots b_0\otimes b_{14}\otimes b_4\otimes b_3 \\ \cdots b_0\otimes b_{10}\otimes b_6\otimes b_9 & \cdots b_0\otimes b_{12}\otimes b_6\otimes b_4 & \cdots b_0\otimes b_{14}\otimes b_5\otimes b_2 \\ \cdots b_0\otimes b_{11}\otimes b_4\otimes b_0 & \cdots b_0\otimes b_{13}\otimes b_2\otimes b_0 & \end{array}$$

Case (3): Observe $H(b_0 \otimes x_3) = 0$ and $H(x_3 \otimes x_2) = 0$ imply $x_3 = x_2 = b_0$. However, x_2 cannot be b_0 when $H(x_2 \otimes x_1) = 2$.

Case (4): Again we use the definition of the energy function to narrow down the possible choices for x_3, x_2, x_1, x_0. Observe $H(b_0 \otimes x_3) = 0$ and $H(x_3 \otimes x_2) = 0$ imply $x_3 = x_2 = b_0$. Then $H(x_2 \otimes x_1) = 1$ and $H(x_1 \otimes x_0) = 2$ imply $x_1 = b_1 - b_{14}$.

If $x_1 = b_1$, then $H(x_1 \otimes x_0) = 2$ implies $x_0 = b_1 - b_{14}$.
If $x_1 = b_2$, then $H(x_1 \otimes x_0) = 2$ implies $x_0 = b_2 - b_{14}$.
If $x_1 = b_3 - b_5$, then $H(x_1 \otimes x_0) = 2$ implies $x_0 = b_5, b_7, b_8, b_{10} - b_{14}$.
If $x_1 = b_6 - b_8$, then $H(x_1 \otimes x_0) = 2$ implies $x_0 = b_8, b_{11} - b_{14}$.
If $x_1 = b_9 - b_{13}$, then $H(x_1 \otimes x_0) = 2$ implies $x_0 = b_{13}, b_{14}$.
If $x_1 = b_{14}$, then $H(x_1 \otimes x_0) = 2$ implies $x_0 = b_{14}$.

For each of the above choices of x_1, x_0, we determine which paths satisfy $\overline{wt}(x_1) + \overline{wt}(x_0) = 0$:

$$\begin{gathered} \cdots b_0 \otimes b_1 \otimes b_{14} \\ \cdots b_0 \otimes b_2 \otimes b_{13} \\ \cdots b_0 \otimes b_3 \otimes b_{12} \\ \cdots b_0 \otimes b_4 \otimes b_{11} \\ \cdots b_0 \otimes b_5 \otimes b_8 \end{gathered}$$

Combining the results from cases (1)-(4) in Table 6.1, we have

$$mult(-\alpha_{-1} - 4\delta) = 32.$$

Example 6.1.5 (Multiplicity of $-\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2 - 4\delta$)**.** Similar to the previous example, we use the affine weight formula to determine the paths of weight $-\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2 - 4\delta$ in $\mathcal{P}(\Lambda_0)$. Let $p \in \mathcal{P}(\Lambda_0)$. We cannot use Theorem 6.1.3, but using similar logic we conclude that p has at most 5 fluctuations from the ground-state path. Then the weight formula becomes,

$$\begin{aligned} -\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2 - 4\delta &= wt(\cdots b_0 \otimes x_4 \otimes x_3 \otimes x_2 \otimes x_1 \otimes x_0) \\ &= \Lambda_0 + \overline{wt}(x_4) + \overline{wt}(x_3) + \overline{wt}(x_2) + \overline{wt}(x_1) + \overline{wt}(x_0) \\ &\quad - [5H(b_0 \otimes x_4) + 4H(x_4 \otimes x_3) + 3H(x_3 \otimes x_2) \\ &\quad + 2H(x_2 \otimes x_1) + H(x_1 \otimes x_0)]\,\delta \end{aligned} \tag{6.4}$$

Since $\overline{wt}(x_i)$ contains no α_0's for all $0 \leq i \leq 14$, we have

$$\overline{wt}(x_4) + \overline{wt}(x_3) + \overline{wt}(x_2) + \overline{wt}(x_1) + \overline{wt}(x_0) = 0$$

$$[5H(b_0 \otimes x_4) + 4H(x_4 \otimes x_3) + 3H(x_3 \otimes x_2) + 2H(x_2 \otimes x_1) + H(x_1 \otimes x_0)] = 5.$$

This yields the 5 cases below, which we examine in turn.

Case (1): $H(b_0 \otimes x_4) = 1$ and $H(b_4 \otimes x_3) = H(x_3 \otimes x_2) = H(x_2 \otimes x_1) = H(x_1 \otimes x_0) = 0$

Case (2): $H(x_4 \otimes x_3) = H(x_1 \otimes x_0) = 1$ and $H(b_0 \otimes x_4) = H(x_3 \otimes x_2) = H(x_2 \otimes x_1) = 0$

Case (3): $H(x_3 \otimes x_2) = H(x_2 \otimes x_1) = 1$ and $H(b_0 \otimes x_4) = H(x_4 \otimes x_3) = H(x_1 \otimes x_0) = 0$

Case (4): $H(x_3 \otimes x_2) = 1$, $H(x_1 \otimes x_0) = 2$, and $H(b_0 \otimes x_4) = H(x_4 \otimes x_3) = H(x_2 \otimes x_1) = 0$

Case (5): $H(x_2 \otimes x_1) = 2$, $H(x_1 \otimes x_0) = 1$, and $H(b_0 \otimes x_4) = H(x_3 \otimes x_2) = H(x_1 \otimes x_0) = 0$

As in the previous example, we use the energy function to narrow down the choices of x_4, x_3, x_2,x_1, x_0 for cases (1)-(5). This process parallels Example 6.1.4, so we omit some justification.

Case (1): The conditions of (6.4) for this case imply $x_4 = b_6 - b_{14}$.

If $x_4 = b_6 - b_8$, then $x_3 = b_1$, which is impossible.

If $x_4 = b_9 - b_{13}$, then $x_3 = b_6, b_9$.

If $x_3 = b_6$, then $x_2 = 1$, which is impossible.

If $x_3 = b_9$, then $x_2 = b_6 - b_9$.

If $x_2 = b_6$, then $x_1 = b_1$, which is impossible.

If $x_2 = b_9$, then $x_1 = b_6, b_9$.

If $x_1 = b_6$, then $x_0 = b_1$.

If $x_1 = b_9$, then $x_0 = b_1 - b_4, b_6, b_9$.

If $x_4 = b_{14}$, then $x_3 = b_6, b_7, b_9, b_{10}$.

If $x_3 = b_6 - b_7$, then $x_2 = b_1$, which is impossible.

If $x_3 = b_9 - b_{10}$, then $x_2 = b_6, b_9$.

If $x_2 = b_6$, then $x_1 = b_1$, which is impossible.

If $x_2 = b_9$, then $x_1 = b_6, b_9$.

If $x_1 = b_6$, then $x_0 = b_1$.

If $x_1 = b_9$, then $x_0 = b_1 - b_4, b_6, b_9$.

For each of the above choices of x_4, x_3, x_2, x_1, x_0, we determine which paths satisfy $\overline{wt}(x_4) + \overline{wt}(x_3) + \overline{wt}(x_2) + \overline{wt}(x_1) + \overline{wt}(x_0) = 0$:

$$\cdots b_0 \otimes b_{12} \otimes b_9 \otimes b_9 \otimes b_9 \otimes b_9$$
$$\cdots b_0 \otimes b_{13} \otimes b_9 \otimes b_9 \otimes b_9 \otimes b_6$$
$$\cdots b_0 \otimes b_{14} \otimes b_9 \otimes b_9 \otimes b_9 \otimes b_4$$
$$\cdots b_0 \otimes b_{14} \otimes b_{10} \otimes b_9 \otimes b_9 \otimes b_3$$

Case (2): The conditions of (6.4) for this case imply $x_4 = b_0$ and $x_3 = b_6 - b_{14}$.

If $x_3 = b_6 - b_8$, then $x_2 = b_1$, which is impossible.

If $x_3 = b_9 - b_{13}$, then $x_2 = b_6, b_9$.

If $x_2 = b_6$, then $x_1 = b_1$ and $x_0 = b_0$.

If $x_2 = b_9$, then $x_1 = b_1 - b_4, b_6, b_9$.

If $x_1 = b_1$, then $x_0 = b_0$.

If $x_1 = b_2$, then $x_0 = b_0, b_1$.

If $x_1 = b_3 - b_4$, then $x_0 = b_0 - b_4, b_6, b_9$.

If $x_1 = b_6$, then $x_0 = b_0, b_2 - b_7, b_9, b_{10}$.

If $x_1 = b_9$, then $x_0 = b_0, b_5, b_7, b_8, b_{10} - b_{12}$.

If $x_3 = b_{14}$, then $x_2 = b_6, b_7, b_9, b_{10}$.

If $x_2 = b_6 - b_7$, then $x_1 = b_1$ and $x_0 = b_0$.

If $x_2 = b_9 - b_{10}$, then $x_1 = b_1 - b_4, b_6, b_9$.

If $x_1 = b_1$, then $x_0 = b_0$.

If $x_1 = b_2$, then $x_0 = b_0, b_1$.

If $x_1 = b_3 - b_4$, then $x_0 = b_0 - b_4, b_6, b_9$.

If $x_1 = b_6$, then $x_0 = b_0, b_2 - b_7, b_9, b_{10}$.

If $x_1 = b_9$, then $x_0 = b_0, b_5, b_7, b_8, b_{10} - b_{12}$.

For each of the above choices of x_3, x_2, x_1, x_0, we determine which paths satisfy $\overline{wt}(x_3) + \overline{wt}(x_2) + \overline{wt}(x_1) + \overline{wt}(x_0) = 0$:

$$\cdots b_0 \otimes b_9 \otimes b_9 \otimes b_9 \otimes b_{12} \quad \cdots b_0 \otimes b_{13} \otimes b_9 \otimes b_6 \otimes b_0 \quad \cdots b_0 \otimes b_{14} \otimes b_9 \otimes b_6 \otimes b_5$$
$$\cdots b_0 \otimes b_{10} \otimes b_9 \otimes b_9 \otimes b_{11} \quad \cdots b_0 \otimes b_{13} \otimes b_9 \otimes b_6 \otimes b_7 \quad \cdots b_0 \otimes b_{14} \otimes b_{10} \otimes b_3 \otimes b_0$$
$$\cdots b_0 \otimes b_{11} \otimes b_9 \otimes b_9 \otimes b_{10} \quad \cdots b_0 \otimes b_{13} \otimes b_9 \otimes b_6 \otimes b_9 \quad \cdots b_0 \otimes b_{14} \otimes b_{10} \otimes b_3 \otimes b_9$$
$$\cdots b_0 \otimes b_{12} \otimes b_9 \otimes b_6 \otimes b_{10} \quad \cdots b_0 \otimes b_{14} \otimes b_9 \otimes b_4 \otimes b_0 \quad \cdots b_0 \otimes b_{14} \otimes b_{10} \otimes b_4 \otimes b_6$$
$$\cdots b_0 \otimes b_{12} \otimes b_9 \otimes b_9 \otimes b_0 \quad \cdots b_0 \otimes b_{14} \otimes b_9 \otimes b_4 \otimes b_9 \quad \cdots b_0 \otimes b_{14} \otimes b_{10} \otimes b_6 \otimes b_4$$
$$\cdots b_0 \otimes b_{12} \otimes b_9 \otimes b_9 \otimes b_7$$

Case (3): The conditions of (6.4) for this case imply $x_4 = x_3 = b_0$ and $x_2 = b_1 - b_{14}$.

If $x_2 = b_1$, then $x_1 = b_0$ and $x_0 = b_0$.

If $x_2 = b_2$, then $x_1 = b_0$ and $x_0 = b_0$.

If $x_2 = b_3 - b_5$, then $x_1 = b_0, b_6, b_9$.

If $x_1 = b_0$, then $x_0 = b_0$.

If $x_1 = b_6$, then $x_0 = b_1$.

If $x_1 = b_9$, then $x_0 = b_1 - b_4, b_6, b_9$.

If $x_2 = b_6 - b_8$, then $x_1 = b_0, b_6, b_7, b_9, b_{10}$.

If $x_1 = b_0$, then $x_0 = b_0$.

If $x_1 = b_6 - b_7$, then $x_0 = b_1$.

If $x_1 = b_9 - b_{10}$, then $x_0 = b_1 - b_4, b_6, b_9$.

If $x_2 = b_9 - b_{13}$, then $x_1 = b_0, b_7, b_8, b_{10} - b_{12}$.

If $x_1 = b_0$, then $x_0 = b_0$.

If $x_1 = b_7 - b_8$, then $x_0 = b_1$.

If $x_1 = b_{10} - b_{12}$, then $x_0 = b_1 - b_4, b_6, b_9$.

If $x_2 = b_{14}$, then $x_1 = b_0, b_8, b_{11} - b_{13}$.

If $x_1 = b_0$, then $x_0 = b_0$.

If $x_1 = b_8$, then $x_0 = b_1$.

If $x_1 = b_{11} - b_{13}$, then $x_0 = b_1 - b_4, b_6, b_9$.

For each of the above choices of x_2, x_1, x_0, we determine which paths satisfy $\overline{wt}(x_2)+\overline{wt}(x_1)+\overline{wt}(x_0) = 0$:

$$
\begin{array}{rrr}
\cdots b_0 \otimes b_9 \otimes b_4 \otimes b_{14} & \cdots b_0 \otimes b_{11} \otimes b_4 \otimes b_{12} & \cdots b_0 \otimes b_{13} \otimes b_2 \otimes b_{12} \\
\cdots b_0 \otimes b_9 \otimes b_6 \otimes b_{13} & \cdots b_0 \otimes b_{12} \otimes b_1 \otimes b_{14} & \cdots b_0 \otimes b_{13} \otimes b_3 \otimes b_{11} \\
\cdots b_0 \otimes b_{10} \otimes b_3 \otimes b_{14} & \cdots b_0 \otimes b_{12} \otimes b_2 \otimes b_{13} & \cdots b_0 \otimes b_{13} \otimes b_4 \otimes b_8 \\
\cdots b_0 \otimes b_{10} \otimes b_6 \otimes b_{12} & \cdots b_0 \otimes b_{12} \otimes b_3 \otimes b_{12} & \cdots b_0 \otimes b_{14} \otimes b_1 \otimes b_{12} \\
\cdots b_0 \otimes b_{11} \otimes b_3 \otimes b_{13} & \cdots b_0 \otimes b_{12} \otimes b_4 \otimes b_{11} & \cdots b_0 \otimes b_{14} \otimes b_3 \otimes b_{10} \\
 & & \cdots b_0 \otimes b_{14} \otimes b_4 \otimes b_7
\end{array}
$$

Case (4): The conditions of (6.4) for this case imply $x_4 = x_3 = b_0$ and $x_2 = b_6 - b_{14}$.

If $x_2 = b_6 - b_8$, then $x_1 = b_1$ and $x_0 = b_1 - b_{14}$.

If $x_2 = b_9 - b_{13}$, then $x_1 = b_1 - b_4, b_6, b_9$.

If $x_1 = b_1$, then $x_0 = b_1 - b_{14}$.

If $x_1 = b_2$, then $x_0 = b_2 - b_{14}$.

If $x_1 = b_3 - b_4$, then $x_0 = b_5, b_7, b_8, b_{10} - b_{14}$.

If $x_1 = b_6$, then $x_0 = b_8, b_{11} - b_{14}$.
If $x_1 = b_9$, then $x_0 = b_{13}, b_{14}$.
If $x_2 = b_{14}$, then $x_1 = b_1 - b_7, b_9, b_{10}$.
If $x_1 = b_1$, then $x_0 = b_1 - b_{14}$.
If $x_1 = b_2$, then $x_0 = b_2 - b_{14}$.
If $x_1 = b_3 - b_5$, then $x_0 = b_5, b_7, b_8, b_{10} - b_{14}$.
If $x_1 = b_6 - b_7$, then $x_0 = b_8, b_{11} - b_{14}$.
If $x_1 = b_9, b_{10}$, then $x_0 = b_{13}, b_{14}$.

For each of the above choices of x_2, x_1, x_0, we determine which paths satisfy $\overline{wt}(x_2)+\overline{wt}(x_1)+\overline{wt}(x_0) = 0$:

$$
\begin{array}{lll}
\cdots b_0 \otimes b_9 \otimes b_{12} \otimes b_9 & \cdots b_0 \otimes b_{11} \otimes b_{12} \otimes b_4 & \cdots b_0 \otimes b_{12} \otimes b_{12} \otimes b_3 \\
\cdots b_0 \otimes b_{10} \otimes b_{11} \otimes b_9 & \cdots b_0 \otimes b_{12} \otimes b_0 \otimes b_0 & \cdots b_0 \otimes b_{13} \otimes b_{11} \otimes b_3 \\
\cdots b_0 \otimes b_{10} \otimes b_{12} \otimes b_6 & \cdots b_0 \otimes b_{12} \otimes b_{10} \otimes b_6 & \cdots b_0 \otimes b_{13} \otimes b_{12} \otimes b_2 \\
\cdots b_0 \otimes b_{11} \otimes b_{10} \otimes b_9 & \cdots b_0 \otimes b_{12} \otimes b_{11} \otimes b_4 & \cdots b_0 \otimes b_{14} \otimes b_{12} \otimes b_1
\end{array}
$$

Case (5): The conditions of (6.4) for this case imply $x_4 = x_3 = x_2 = b_0$, but $x_2 = b_0$ is impossible.

Combining the results from cases (1)-(5) in Table 6.3, we have

$$mult(-\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2 - 4\delta) = 48.$$

Lastly, we compile in Table 6.5 those multiplicities that were calculated in [Hon98] using different techniques than ours. Please note that after a change of α index, we obtain the same results.

Table 6.5 Recovered level 1 multiplicities

α	$\delta-$factored α	$mult(\alpha)$	Frenkel's Bound
$-\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - 4\alpha_2$	$-\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2 - \delta$	3	10
$-\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - 5\alpha_2$	$-\alpha_{-1} - \alpha_0 - \alpha_1 - 2\alpha_2 - \delta$	3	10
$-\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - 6\alpha_2$	$-\alpha_{-1} - \alpha_0 - \alpha_1 - 3\alpha_2 - \delta$	2	2
$-\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - 6\alpha_2$	$-\alpha_{-1} - 2\delta$	6	20
$-\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - 7\alpha_2$	$-\alpha_{-1} - \alpha_2 - 2\delta$	3	10
$-\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - 8\alpha_2$	$-\alpha_{-1} - 2\alpha_2 - 2\delta$	1	1
$-\alpha_{-1} - 2\alpha_0 - 5\alpha_1 - 8\alpha_2$	$-\alpha_{-1} - \alpha_1 - 2\alpha_2 - 2\delta$	3	10
$-\alpha_{-1} - 3\alpha_0 - 5\alpha_1 - 7\alpha_2$	$-\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2 - 2\delta$	9	65
$-\alpha_{-1} - 3\alpha_0 - 5\alpha_1 - 8\alpha_2$	$-\alpha_{-1} - \alpha_0 - \alpha_1 - 2\alpha_2 - 2\delta$	9	65

6.2 Level 2 Roots

Consider level 2 roots of the form $-2\alpha_{-1} - \alpha - k\delta$, where $\alpha = m_0\alpha_0 + m_1\alpha_1 + m_2\alpha_2 < \delta$ and $m_i \in \mathbb{Z}_{\geq 0}$. In Theorem 6.2.1 through Theorem 6.2.4 we show that for all $\alpha < \delta$ the multiplicity of $-2\alpha_{-1} - \alpha - k\delta$ is equal to that one of the following with a possible shift of the root string by at most -2δ.

$$-2\alpha_{-1} - k\delta$$

$$-2\alpha_{-1} - \alpha_0 - k\delta$$

$$-2\alpha_{-1} - \alpha_0 - \alpha_2 - k\delta$$

$$-2\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2 - k\delta$$

Therefore we primarily consider level 2 roots of these forms.

From Section 4.1 we recall Theorem 4.1.2: If $-\ell\alpha_{-1} - k\delta$ is a root of $HG_2^{(1)}$, then $k \geq \ell$. Furthermore, $mult(-\ell\alpha_{-1} - \ell\delta) = 2$. Therefore,

$$dim(\mathfrak{g}_{-2\alpha_{-1}-\delta}) = 0$$

$$dim(\mathfrak{g}_{-2\alpha_{-1}-2\delta}) = 2.$$

For $k > 2$, we use Kang's multiplicity formula and the path realizations of the highest weight modules $V(\Lambda_0)$ and $V(\Lambda_1 - \delta)$ to find the multiplicities. When k is small, we compute the multiplicities explicitly as in Example 6.2.5 and Example 6.2.7. When k is large we use our C# program for level 2 roots to compute these multiplicities, as well as the additional multiplicities and give the results in Table 6.9 through Table 6.12.

In Proposition 6.2.8 we give a specialized version of Kang's multiplicity formula for roots of the form $-2\alpha_{-1} - \beta$ where $\beta = \sum_{i=0}^{2} m_i\alpha_i$ and $m_i \in \mathbb{Z}_{\geq 0}$. This specialization gives the outline of the algorithm used in our C# program for finding multiplicities of level 2 roots.

Theorem 6.2.1. *If $-2\alpha_{-1} - k\delta$ is a root of $HG_2^{(1)}$, then*

$$\begin{aligned} mult(-2\alpha_{-1} - k\delta) &= mult(\beta_1 - (k+1)\delta) \\ &= mult(\beta_2 - (k+2)\delta) \end{aligned}$$

where β_i is as follows:

β_1	β_2
$-2\alpha_{-1} - \alpha_0 - 3\alpha_2$	$-2\alpha_{-1} - 2\alpha_1$

Proof. It is enough to show each $\beta_i - (k+i)\delta$ is Weyl group conjugate to $-2\alpha_{-1} - k\delta$.

For each β_1 we choose reflection, w, as follows and apply Proposition 4.1.1:

- $w = r_2 r_1 r_0 \Longrightarrow w(-2\alpha_{-1} - k\delta) = -2\alpha_{-1} - \alpha_0 - 3\alpha_2 - (k+1)\delta$

For each β_2 we choose reflection, w, as follows and apply Proposition 4.1.1:

- $w = r_1 r_2 r_1 r_0 \Longrightarrow w(-2\alpha_{-1} - k\delta) = -2\alpha_{-1} - 2\alpha_1 - (k+2)\delta$

□

Theorem 6.2.2. *If $-2\alpha_{-1} - \alpha_0 - k\delta$ is a root of $HG_2^{(1)}$, then*

$$\begin{aligned} mult(-2\alpha_{-1} - \alpha_0 - k\delta) &= mult(\beta_0 - k\delta) \\ &= mult(\beta_1 - (k+1)\delta) \\ &= mult(\beta_2 - (k+2)\delta) \end{aligned}$$

where β_i is as follows:

β_0	β_1	β_2
$-2\alpha_{-1} - \alpha_0 - \alpha_1$	$-2\alpha_{-1} - \alpha_1$	$-2\alpha_{-1} - 3\alpha_2$
$-2\alpha_{-1} - \alpha_0 - \alpha_1 - 3\alpha_2$	$-2\alpha_{-1} - \alpha_1 - 3\alpha_2$	
	$-2\alpha_{-1} - 2\alpha_1 - 3\alpha_2$	
	$-2\alpha_{-1} - \alpha_0 - 2\alpha_1$	

Proof. It is enough to show each $\beta_i - (k+i)\delta$ is Weyl group conjugate to $-2\alpha_{-1} - \alpha_0 - k\delta$.

For each β_0 we choose reflection, w, as follows and apply Proposition 4.1.1:

- $w = r_1 \Longrightarrow w(-2\alpha_{-1} - \alpha_0 - k\delta) = -2\alpha_{-1} - \alpha_0 - \alpha_1 - k\delta$
- $w = r_2 r_1 \Longrightarrow w(-2\alpha_{-1} - \alpha_0 - k\delta) = -2\alpha_{-1} - \alpha_0 - \alpha_1 - 3\alpha_2 - k\delta$

For each β_1 we choose reflection, w, as follows and apply Proposition 4.1.1:

- $w = r_1r_2r_1 \Longrightarrow w(-2\alpha_{-1} - \alpha_0 - k\delta) = -2\alpha_{-1} - \alpha_1 - (k+1)\delta$
- $w = r_2r_1r_2r_1 \Longrightarrow w(-2\alpha_{-1} - \alpha_0 - k\delta) = -2\alpha_{-1} - \alpha_1 - 3\alpha_2 - (k+1)\delta$
- $w = r_1r_2r_1r_2r_1 \Longrightarrow w(-2\alpha_{-1} - \alpha_0 - k\delta) = -2\alpha_{-1} - 2\alpha_1 - 3\alpha_2 - (k+1)\delta$
- $w = r_1r_0r_2r_1 \Longrightarrow w(-2\alpha_{-1} - \alpha_0 - k\delta) = -2\alpha_{-1} - \alpha_0 - 2\alpha_1 - (k+1)\delta$

For each β_2 we choose reflection, w, as follows and apply Proposition 4.1.1:

- $w = r_2r_1r_0r_2r_1 \Longrightarrow w(-2\alpha_{-1} - \alpha_0 - k\delta) = -2\alpha_{-1} - 3\alpha_2 - (k+2)\delta$

□

Theorem 6.2.3. *If $-2\alpha_{-1} - \alpha_0 - \alpha_2 - k\delta$ is a root of $HG_2^{(1)}$, then*

$$\begin{aligned} mult(-2\alpha_{-1} - \alpha_0 - \alpha_2 - k\delta) &= mult(\beta_0 - k\delta) \\ &= mult(\beta_1 - (k+1)\delta) \end{aligned}$$

where β_i is as follows:

β_0	β_1
$-2\alpha_{-1} - \alpha_0 - 2\alpha_1 - \alpha_2$	$-2\alpha_{-1} - 2\alpha_2$
	$-2\alpha_{-1} - 2\alpha_1 - 2\alpha_2$

Proof. It is enough to show each $\beta_i - (k+i)\delta$ is Weyl group conjugate to $-2\alpha_{-1} - \alpha_0 - \alpha_2 - k\delta$.

For each β_0 we choose reflection, w, as follows and apply Proposition 4.1.1:

- $w = r_1 \Longrightarrow w(-2\alpha_{-1} - \alpha_0 - \alpha_2 - k\delta) = -2\alpha_{-1} - \alpha_0 - 2\alpha_1 - \alpha_2 - k\delta$

For each β_1 we choose reflection, w, as follows and apply Proposition 4.1.1:

- $w = r_2r_1 \Longrightarrow w(-2\alpha_{-1} - \alpha_0 - \alpha_2 - k\delta) = -2\alpha_{-1} - 2\alpha_2 - (k+1)\delta$
- $w = r_1r_2r_1 \Longrightarrow w(-2\alpha_{-1} - \alpha_0 - \alpha_2 - k\delta) = -2\alpha_{-1} - 2\alpha_1 - 2\alpha_2 - (k+1)\delta$

□

Theorem 6.2.4. *If $-2\alpha_{-1}-\alpha_0-\alpha_1-\alpha_2-k\delta$ is a root of $HG_2^{(1)}$, then*

$$\begin{aligned} mult(-2\alpha_{-1}-\alpha_0-\alpha_1-\alpha_2-k\delta) &= mult(\beta_0-k\delta) \\ &= mult(\beta_1-(k+1)\delta) \\ &= mult(\beta_2-(k+2)\delta) \end{aligned}$$

where β_i is as follows:

β_0	β_1	β_2
$-2\alpha_{-1}-\alpha_0-\alpha_1-2\alpha_2$	$-2\alpha_{-1}-\alpha_2$	$-2\alpha_{-1}-2\alpha_1-\alpha_2$
$-2\alpha_{-1}-\alpha_0-2\alpha_1-2\alpha_2$	$-2\alpha_{-1}-\alpha_1-\alpha_2$	
	$-2\alpha_{-1}-\alpha_1-2\alpha_2$	
	$-2\alpha_{-1}-\alpha_0-2\alpha_2$	

Proof. It is enough to show each $\beta_i-(k+i)\delta$ is Weyl group conjugate to $-2\alpha_{-1}-\alpha_0-\alpha_1-\alpha_2-k\delta$.

For each β_0 we choose reflection, w, as follows and apply Proposition 4.1.1:

- $w=r_2 \Longrightarrow w(-2\alpha_{-1}-\alpha_0-\alpha_1-\alpha_2-k\delta)=-2\alpha_{-1}-\alpha_0-\alpha_1-2\alpha_2-k\delta$
- $w=r_1r_2 \Longrightarrow w(-2\alpha_{-1}-\alpha_0-\alpha_1-\alpha_2-k\delta)=-2\alpha_{-1}-\alpha_0-2\alpha_1-2\alpha_2-k\delta$

For each β_1 we choose reflection, w, as follows and apply Proposition 4.1.1:

- $w=r_2r_1r_2 \Longrightarrow w(-2\alpha_{-1}-\alpha_0-\alpha_1-\alpha_2-k\delta)=-2\alpha_{-1}-\alpha_2-(k+1)\delta$
- $w=r_1r_2r_1r_2 \Longrightarrow w(-2\alpha_{-1}-\alpha_0-\alpha_1-\alpha_2-k\delta)=-2\alpha_{-1}-\alpha_1-\alpha_2-(k+1)\delta$
- $w=r_2r_1r_2r_1r_2 \Longrightarrow w(-2\alpha_{-1}-\alpha_0-\alpha_1-\alpha_2-k\delta)=-2\alpha_{-1}-\alpha_1-2\alpha_2-(k+1)\delta$
- $w=r_2r_1r_0 \Longrightarrow w(-2\alpha_{-1}-\alpha_0-\alpha_1-\alpha_2-k\delta)=-2\alpha_{-1}-\alpha_0-2\alpha_2-(k+1)\delta$

For each β_2 we choose reflection, w, as follows and apply Proposition 4.1.1:

- $w=r_1r_2r_1r_2r_0 \Longrightarrow w(-\alpha_{-1}-\alpha_0-\alpha_1-\alpha_2-k\delta)=-2\alpha_{-1}-2\alpha_1-\alpha_2-(k+2)\delta$

□

In the examples to follow, we demonstrate the algorithm corresponding to Kang's multiplicity formula for the level 2 roots $-2\alpha_{-1} - 3\delta$ and $-2\alpha_{-1} - 4\delta$.

Example 6.2.5 (Multiplicity of $-2\alpha_{-1} - 3\delta$)**.** Recall that Kang's formula sums over the τ that divide α:

$$dim(\mathfrak{g}_\alpha) = \sum_{\tau|\alpha} \mu(\frac{\alpha}{\tau}) \left(\frac{\tau}{\alpha}\right) \mathcal{B}(\tau)$$

When $k = 3$ the only divisor of α is α itself and hence

$$\begin{aligned} dim(\mathfrak{g}_{-2\alpha_{-1}-3\delta}) &= \mu\left(\frac{-2\alpha_{-1} - 3\delta}{-2\alpha_{-1} - 3\delta}\right)\left(\frac{-2\alpha_{-1} - 3\delta}{-2\alpha_{-1} - 3\delta}\right)\mathcal{B}(-2\alpha_{-1} - 3\delta) \\ &= \mathcal{B}(-2\alpha_{-1} - 3\delta). \end{aligned}$$

Next we recall $\mathcal{B}(\tau)$ sums over the partitions of τ:

$$\mathcal{B}(\tau) = \sum_{(n_i,\tau_i)\in T(\tau)} \frac{(\sum n_i - 1)!}{\prod (n_i)!} \prod K_{\tau_i}^{n_i}$$

where $K_{\tau_i} = \sum_{\substack{w\in W(S) \\ \ell(w)\geq 1}} (-1)^{\ell(w)+1} dimV(w\rho - \rho)_{\tau_i}$ and $V(w\rho - \rho)$ is the highest weight module of weight $w\rho - \rho$ in $G_2^{(1)}$. For τ of level 2, each τ_i will be a root of level less than or equal to 2. Therefore $dimV(w\rho - \rho)_{\tau_i} = 0$ for weight $w\rho - \rho$ with level greater than 2. To find the $w \in W(S)$ satisfying this criteria, we utilize the following lemma from [Kan94].

Lemma 6.2.6. *For $w = w'r_j$ and $\ell(w) = \ell(w') + 1$,*

$$w \in W(S) \Longleftrightarrow w' \in W(S) \text{ and } w'(\alpha_j) \in \Delta^+(S).$$

where $r_j : \mathfrak{h}^* \to \mathfrak{h}^*$ are the simple Weyl group reflections defined as $r_j(\beta) = \beta - \beta(h_j)\alpha_j$.

We start with the identity reflection in $W(S)$ of length zero and use Lemma 6.2.6 to induce on the length of w in $W(S)$. Lemma 6.2.6 implies

$$\begin{aligned} r_j \in W(S) &\Longleftrightarrow \alpha_j \in \Delta^+(S) \\ &\Longleftrightarrow j = -1 \end{aligned}$$

Therefore, the only length one reflection is r_{-1}. Letting $w' = r_{-1}$ we find the length two reflections. Lemma 6.2.6 implies

$$\begin{aligned} r_{-1}r_j \in W(S) &\iff r_{-1}(\alpha_j) \in \Delta^+(S) \\ &\iff \alpha_j - \alpha_j(H_{-1})\alpha_{-1} \in \Delta^+(S) \\ &\iff j = 0 \end{aligned}$$

Therefore, $r_{-1}r_0$ is the only length two reflection in $W(S)$. Table 6.6 summarizes these findings and verifies the level of $w\rho - \rho$ is less than or equal to 2 for our w of length one and two.

Table 6.6 Length one and two refections $w \in W(S)$

k	$\ell(w) = k$	$w\rho - \rho$	level of $w\rho - \rho$
1	r_{-1}	$r_{-1}\rho - \rho = [\rho - \rho(h_{-1})\alpha_{-1}] - \rho$ $= -\alpha_{-1}$ $= \Lambda_0$	$\Lambda_0(K) = 1$
2	$r_{-1}r_0$	$r_{-1}r_0(\rho) - \rho = r_{-1}(\rho - \rho(h_0)\alpha_0) - \rho$ $= r_{-1}(\rho - \alpha_0) - \rho$ $= [(\rho - \alpha_0) - (\rho - \alpha_0)(h_{-1})\alpha_{-1}] - \rho$ $= [(\rho - \alpha_0) - 2\alpha_{-1}] - \rho$ $= -2\alpha_{-1} - \alpha_0$ $= \Lambda_1 - \delta$	$(\Lambda_1 - \delta)(K) = \Lambda_1(K)$ $= 2$

Summing over the above w from Table 6.6, we simplify the expression for K_{τ_i}.

$$\begin{aligned} K_{\tau_i} &= \sum_{\substack{w \in W(S) \\ \ell(w) \geq 1}} (-1)^{\ell(w)+1} dimV(w\rho - \rho)_{\tau_i} \\ &= dimV(\Lambda_0)_{\tau_i} - dimV(\Lambda_1 - \delta)_{\tau_i} \end{aligned}$$

For $dimV(\Lambda_0)_{\tau_i}$ to be nonzero, we require the τ_i to be of form

$$\Lambda_0 - \sum_{j=0}^{2} m_{ij}\alpha_j = -\alpha_{-1} - \sum_{j=0}^{2} m_{ij}\alpha_j \tag{6.5}$$

where $m_{ij} \in \mathbb{Z}_{\geq 0}$. Therefore, we partition τ into $\tau_1 + \tau_2$, where τ_1 and τ_2 are level 1 roots of form (6.5). Observe that since $k = 3$ is odd, all unique partitions of τ are such that $\tau_1 < \tau_2$. To find the $dimV(\Lambda_0)_{\tau_i}$, recall $mult(\tau_i) = dimV(\Lambda_0)_{\tau_i}$ for level 1 roots and refer to the methods and results presented in Section 6.1. Since $\mathcal{B}(\tau)$ requires we sum over K_{τ_i}, we consider only $\tau_1 < \tau_2$ for which $dimV(\Lambda_0)_{\tau_1} \neq 0$ and $dimV(\Lambda_0)_{\tau_2} \neq 0$ and collect them in Table 6.7.

Table 6.7 Partitions of $\tau = -2\alpha_{-1} - 3\delta$ into $\tau_1 + \tau_2$ with $\tau_i = -\alpha_{-1} - \sum_{j=0}^{2} m_{ij}\alpha_j$ and $m_{ij} \in \mathbb{Z}_{\geq 0}$

τ_1	$dimV(\Lambda_0)_{\tau_1}$	τ_2	$dimV(\Lambda_0)_{\tau_2}$
$-\alpha_{-1}$	1	$-\alpha_{-1} - 3\alpha_0 - 6\alpha_1 - 9\alpha_2$	14
$-\alpha_{-1} - \alpha_0$	1	$-\alpha_{-1} - 2\alpha_0 - 6\alpha_1 - 9\alpha_2$	2
$-\alpha_{-1} - \alpha_0 - \alpha_1$	1	$-\alpha_{-1} - 2\alpha_0 - 5\alpha_1 - 9\alpha_2$	2
$-\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 5\alpha_1 - 8\alpha_2$	3
$-\alpha_{-1} - \alpha_0 - \alpha_1 - 2\alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 5\alpha_1 - 7\alpha_2$	3
$-\alpha_{-1} - \alpha_0 - \alpha_1 - 3\alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 5\alpha_1 - 6\alpha_2$	2
$-\alpha_{-1} - \alpha_0 - 2\alpha_1 - 2\alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - 7\alpha_2$	3
$-\alpha_{-1} - \alpha_0 - 2\alpha_1 - 3\alpha_2$	2	$-\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - 6\alpha_2$	6
$-\alpha_{-1} - \alpha_0 - 2\alpha_1 - 4\alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - 5\alpha_2$	3
$-\alpha_{-1} - \alpha_0 - 3\alpha_1 - 3\alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - 6\alpha_2$	2
$-\alpha_{-1} - \alpha_0 - 3\alpha_1 - 4\alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - 5\alpha_2$	3
$-\alpha_{-1} - \alpha_0 - 3\alpha_1 - 5\alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - 4\alpha_2$	3
$-\alpha_{-1} - \alpha_0 - 3\alpha_1 - 6\alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - 3\alpha_2$	2
$-\alpha_{-1} - \alpha_0 - 4\alpha_1 - 6\alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 2\alpha_1 - 3\alpha_2$	2

For $dimV(\Lambda_1 - \delta)_{\tau_i}$ to be nonzero, we require τ_i to be of form

$$\Lambda_1 - \delta - \sum_{j=0}^{2} m_{ij}\alpha_j = -2\alpha_{-1} - \alpha_0 - \sum_{j=0}^{2} m_{ij}\alpha_j \tag{6.6}$$

$$= -2\alpha_{-1} - \sum_{j=0}^{2} \overline{m}_{ij}\alpha_j \tag{6.7}$$

where $m_{ij}, \overline{m}_{ij} \in \mathbb{Z}_{\geq 0}$. Then $\tau = -2\alpha_{-1} - 3\delta$ is the only partition of τ with form (6.7). Using the isomorphism $V(\Lambda_1) \cong V(\Lambda_1 - \delta) \otimes V(\delta)$, we have $dimV(\Lambda_1 - \delta)_\tau = dimV(\Lambda_1)_{\tau+\delta}$. Thus we determine the $dimV(\Lambda_1 - \delta)_{-2\alpha_{-1}-3\delta}$, by counting the paths in $P(\Lambda_1) \cong V(\Lambda_1)$ of weight $-2\alpha_{-1} - 2\delta$ (viewed in terms of the fundamental weights of $G_1^{(2)}$) below.

$$\begin{aligned} dimV(\Lambda_1 - \delta)_{-2\alpha_{-1}-3\delta} &= dimV(\Lambda_1)_{-2\alpha_{-1}-2\delta} \\ &= dimV(\Lambda_1)_{\Lambda_1+\alpha_0-3\delta} \\ &= dimV(\Lambda_1)_{\Lambda_1-2\alpha_0-6\alpha_1-9\alpha_2} \\ &= 42 \end{aligned}$$

We are now ready to find $dim(\mathfrak{g}_{-2\alpha_{-1}-3\delta}) = \mathcal{B}(-2\alpha_{-1}-3\delta)$:

$$\begin{aligned}
\mathcal{B}(-2\alpha_{-1}-3\delta) &= \sum_{(n_i,\tau_i)\in T(-2\alpha_{-1}-3\delta)} \frac{(\sum n_i - 1)!}{\prod n_i!} \prod K_{\tau_i}^{n_i} \\
&= \left(\sum_{\substack{\tau_1+\tau_2=-2\alpha_{-1}-3\delta \\ \tau_i=-\alpha_{-1}-\sum m_{ij}\alpha_j \\ \tau_1<\tau_2}} \frac{(2-1)!}{1!\cdot 1!} \prod_{i=1}^{2} K_{\tau_i} \right) + \frac{(1-1)!}{1!} K_{-2\alpha_{-1}-3\delta} \\
&= \left(\sum_{\substack{\tau_1+\tau_2=-2\alpha_{-1}-3\delta \\ \tau_i=-\alpha_{-1}-\sum m_{ij}\alpha_j \\ \tau_1<\tau_2}} dimV(\Lambda_0)_{\tau_1} \cdot dimV(\Lambda_0)_{\tau_2} \right) - dimV(\Lambda_1-\delta)_{\tau} \\
&= 56 - 42 \\
&= 14
\end{aligned}$$

Recall Theorem 4.1.3 from Section 4.1 which states $mult(-\ell\alpha_{-1} - \alpha - k\delta) = mult(-(k-\ell)\alpha_{-1} - \alpha - k\delta)$ for $\ell, k \in \mathbb{Z}_{>0}$ and $\alpha = m_1\alpha_1 + m_2\alpha_2 \in \Delta_S^+ \cup \{0\}$. When $\ell = 2$ and $k = 3$, this theorem implies $mult(-2\alpha_{-1} - 3\delta) = mult(-\alpha_{-1} - 3\delta) = 14$. Observe that this result agrees with our calculation of $mult(-2\alpha_{-1} - 3\delta)$ using Kang's Formula in Example 6.2.5.

Example 6.2.7 (Multiplicity of $-2\alpha_{-1} - 4\delta$)**.** Consider now $\alpha = -2\alpha_{-1} - 4\delta$. We find the multiplicity of α using Kang's multiplicity formula which sums over all τ that divide α. Clearly, $\tau = \alpha$ divides α. In addition, $\tau = -\alpha_{-1} - 2\delta$ divides α. Applying Kang's formula, we obtain

$$\begin{aligned}
dim(\mathfrak{g}_{-2\alpha_{-1}-4\delta}) &= \mu\left(\frac{-2\alpha_{-1}-4\delta}{-2\alpha_{-1}-4\delta}\right)\left(\frac{-2\alpha_{-1}-4\delta}{-2\alpha_{-1}-4\delta}\right)\mathcal{B}(-2\alpha_{-1}-4\delta) \\
&+ \mu\left(\frac{-2\alpha_{-1}-4\delta}{-2\alpha_{-1}-2\delta}\right)\left(\frac{-2\alpha_{-1}-2\delta}{-2\alpha_{-1}-4\delta}\right)\mathcal{B}(-\alpha_{-1}-2\delta) \\
&= \mathcal{B}(-2\alpha_{-1}-4\delta) - \frac{1}{2}\mathcal{B}(-\alpha_{-1}-2\delta)
\end{aligned}$$

We start by calculating $\mathcal{B}(-\alpha_{-1} - 2\delta)$ using the level 1 result for $-\alpha_{-1} - 2\delta$.

$$\begin{aligned}
\mathcal{B}(-\alpha_{-1}-2\delta) &= \frac{(1-1)!}{1!} K_{-\alpha_{-1}-2\delta} \\
&= dimV(\Lambda_0)_{-\alpha_{-1}-2\delta} \\
&= 6 \qquad (6.8)
\end{aligned}$$

Now consider the formula for $\mathcal{B}(-2\alpha_{-1}-4\delta)$, which sums over the partitions of $-2\alpha_{-1}-4\delta$.

$$\mathcal{B}(-2\alpha_{-1}-4\delta) = \sum_{(n_i,\tau_i)\in T(-2\alpha_{-1}-4\delta)} \frac{(\sum n_i - 1)!}{\prod n_i!} \prod K_{\tau_i}^{n_i}$$

As in the previous example, we consider only τ_i for which K_{τ_i} is nonzero and hence we consider only the following partitions of $\tau = -2\alpha_{-1}-4\delta$: $\tau = \tau_1 + \tau_2$ where τ_1 and τ_2 are level 1 roots of form (6.5) and the level 2 partition $\tau = \tau$.

For $\tau = \tau_1 + \tau_2$, observe that since $k = 4$ is even there is one partition where $\tau_1 = \tau_2 = \frac{\tau}{2}$. Therefore, the unique partitions $\tau = \tau_1 + \tau_2$ are such that $\tau_1 \leq \tau_2$. Since τ_1 and τ_2 are level 1 roots, we again use the level 1 methods from Section 6.1 to find their multiplicities and collect the τ_1 and τ_2 that have nonzero multiplicity in Table 6.8 .

For the partition $\tau = -2\alpha_{-1}-4\delta$, we use the path realization of $V(\Lambda_1)$ and the isomorphism $V(\Lambda_1) \cong V(\Lambda_1 - \delta) \otimes V(\delta)$ to determine $dimV(\Lambda_1-\delta)_\tau$ below.

$$\begin{aligned} dimV(\Lambda_1-\delta)_{-2\alpha_{-1}-4\delta} &= dimV(\Lambda_1)_{-2\alpha_{-1}-3\delta} \\ &= dimV(\Lambda_1)_{\Lambda_1+\alpha_0-4\delta} \\ &= dimV(\Lambda_1)_{\Lambda_1-3\alpha_0-8\alpha_1-12\alpha_2} \\ &= 141 \end{aligned} \tag{6.9}$$

We are now ready to use the results from Table 6.8 and (6.9) to calculate $\mathcal{B}(-2\alpha_{-1}-4\delta)$.

$$\begin{aligned} \mathcal{B}(-2\alpha_{-1}-4\delta) &= \sum_{(n_i,\tau_i)\in T(-2\alpha_{-1}-4\delta)} \frac{(\sum n_i - 1)!}{\prod n_i!} \prod K_{\tau_i}^{n_i} \\ &= \left(\sum_{\substack{\tau_1+\tau_2=-2\alpha_{-1}-4\delta \\ \tau_i=-\alpha_{-1}-\sum m_{ij}\alpha_j \\ \tau_1<\tau_2}} \frac{(2-1)!}{1!\cdot 1!} \prod_{i=1}^{2} K_{\tau_i} \right) \\ &\quad + \frac{(2-1)!}{2!} K_{-\alpha_{-1}-2\delta}^2 \\ &\quad\quad + \frac{(1-1)!}{1!} K_{-2\alpha_{-1}-4\delta} \end{aligned}$$

$$= \left(\sum_{\substack{\tau_1+\tau_2=-2\alpha_{-1}-4\delta \\ \tau_i=-\alpha_{-1}-\sum m_{ij}\alpha_j \\ \tau_1<\tau_2}} dimV(\Lambda_0)_{\tau_1} \cdot dimV(\Lambda_0)_{\tau_2} \right)$$
$$+ \frac{1}{2}(dimV(\Lambda_0)_{-\alpha_{-1}-2\delta})^2$$
$$- dimV(\Lambda_1 - \delta)_{-2\alpha_{-1}-4\delta}$$
$$= 192 + \frac{1}{2}(6)^2 - 141$$
$$= 69 \tag{6.10}$$

Table 6.8 Partitions of $\tau = -2\alpha_{-1} - 4\delta$ into $\tau_1 + \tau_2$ with $\tau_i = -\alpha_{-1} - \sum_{j=0}^{2} m_{ij}\alpha_j$ and $m_{ij} \in \mathbb{Z}_{\geq 0}$

τ_1	$dimV(\Lambda_0)_{\tau_1}$	τ_2	$dimV(\Lambda_0)_{\tau_2}$
$-\alpha_{-1}$	1	$-\alpha_{-1} - 4\alpha_0 - 8\alpha_1 - 12\alpha_2$	32
$-\alpha_{-1} - \alpha_0$	1	$-\alpha_{-1} - 3\alpha_0 - 8\alpha_1 - 12\alpha_2$	6
$-\alpha_{-1} - \alpha_0 - \alpha_1$	1	$-\alpha_{-1} - 3\alpha_0 - 7\alpha_1 - 12\alpha_2$	6
$-\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2$	1	$-\alpha_{-1} - 3\alpha_0 - 7\alpha_1 - 11\alpha_2$	9
$-\alpha_{-1} - \alpha_0 - \alpha_1 - 2\alpha_2$	1	$-\alpha_{-1} - 3\alpha_0 - 7\alpha_1 - 10\alpha_2$	9
$-\alpha_{-1} - \alpha_0 - \alpha_1 - 3\alpha_2$	1	$-\alpha_{-1} - 3\alpha_0 - 7\alpha_1 - 9\alpha_2$	6
$-\alpha_{-1} - \alpha_0 - 2\alpha_1 - 2\alpha_2$	1	$-\alpha_{-1} - 3\alpha_0 - 6\alpha_1 - 10\alpha_2$	9
$-\alpha_{-1} - \alpha_0 - 2\alpha_1 - 3\alpha_2$	2	$-\alpha_{-1} - 3\alpha_0 - 6\alpha_1 - 9\alpha_2$	14
$-\alpha_{-1} - \alpha_0 - 2\alpha_1 - 4\alpha_2$	1	$-\alpha_{-1} - 3\alpha_0 - 6\alpha_1 - 8\alpha_2$	9
$-\alpha_{-1} - \alpha_0 - 3\alpha_1 - 3\alpha_2$	1	$-\alpha_{-1} - 3\alpha_0 - 5\alpha_1 - 9\alpha_2$	6
$-\alpha_{-1} - \alpha_0 - 3\alpha_1 - 4\alpha_2$	1	$-\alpha_{-1} - 3\alpha_0 - 5\alpha_1 - 8\alpha_2$	9
$-\alpha_{-1} - \alpha_0 - 3\alpha_1 - 5\alpha_2$	1	$-\alpha_{-1} - 3\alpha_0 - 5\alpha_1 - 7\alpha_2$	9
$-\alpha_{-1} - \alpha_0 - 3\alpha_1 - 6\alpha_2$	1	$-\alpha_{-1} - 3\alpha_0 - 5\alpha_1 - 6\alpha_2$	6
$-\alpha_{-1} - \alpha_0 - 4\alpha_1 - 6\alpha_2$	1	$-\alpha_{-1} - 3\alpha_0 - 4\alpha_1 - 6\alpha_2$	6
$-\alpha_{-1} - 2\alpha_0 - 2\alpha_1 - 2\alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 6\alpha_1 - 10\alpha_2$	1
$-\alpha_{-1} - 2\alpha_0 - 2\alpha_1 - 3\alpha_2$	2	$-\alpha_{-1} - 2\alpha_0 - 6\alpha_1 - 9\alpha_2$	2
$-\alpha_{-1} - 2\alpha_0 - 2\alpha_1 - 4\alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 6\alpha_1 - 8\alpha_2$	1
$-\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - 3\alpha_2$	2	$-\alpha_{-1} - 2\alpha_0 - 5\alpha_1 - 9\alpha_2$	2
$-\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - 4\alpha_2$	3	$-\alpha_{-1} - 2\alpha_0 - 5\alpha_1 - 8\alpha_2$	3
$-\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - 5\alpha_2$	3	$-\alpha_{-1} - 2\alpha_0 - 5\alpha_1 - 7\alpha_2$	3
$-\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - 6\alpha_2$	2	$-\alpha_{-1} - 2\alpha_0 - 5\alpha_1 - 6\alpha_2$	2
$-\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - 4\alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - 8\alpha_2$	1
$-\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - 5\alpha_2$	3	$-\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - 7\alpha_2$	3
$-\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - 6\alpha_2$	6	$-\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - 6\alpha_2$	6

Combining results (6.8) and (6.10), we determine $dim(\mathfrak{g}_{-2\alpha_{-1}-4\delta})$.

$$\begin{aligned} dim(\mathfrak{g}_{-2\alpha_{-1}-4\delta}) &= \mathcal{B}(-2\alpha_{-1}-4\delta) - \frac{1}{2}\mathcal{B}(-\alpha_{-1}-2\delta) \\ &= 69 - \frac{1}{2}(6) \\ &= 66 \end{aligned}$$

For simplicity of calculations and for use in our C# code, we now present Kang's multiplicity formula in generality for roots of the form $-2\alpha_{-1}-\beta$ where $\beta = \sum_{i=0}^{2} m_i\alpha_i$ and $m_i \in \mathbb{Z}_{\geq 0}$. For these equations to hold we use the fact that we have one divisor, $\tau = \alpha$, of α when $2 \nmid \beta$ and two divisors, $\tau = \alpha$ and $\tau = \frac{\alpha}{2}$, of α when $2|\beta$. We also observe that only if $2|\beta$ do we have a partition of $\tau = \alpha$ into $\tau_1 + \tau_2$ such that $\tau_1 = \tau_2$.

Proposition 6.2.8. *Let $-2\alpha_{-1}-\beta$ be a root of $HG_2^{(1)}$, then the $dim(\mathfrak{g}_{-2\alpha_{-1}-\beta})$ is computed as follows.*

If $2 \nmid \beta$, then

$$\begin{aligned} dim(\mathfrak{g}_{-2\alpha_{-1}-\beta}) &= \mathcal{B}(-2\alpha_{-1}-\beta) \\ &= \left(\sum_{\substack{\tau_1+\tau_2=-2\alpha_{-1}-\beta \\ \tau_i=-\alpha_{-1}-\sum m_{ij}\alpha_j \\ \tau_1<\tau_2}} dimV(\Lambda_0)_{\tau_1} \cdot dimV(\Lambda_0)_{\tau_2} \right) - dimV(\Lambda_1)_{\Lambda_1+\alpha_0-\beta} \end{aligned} \quad (6.11)$$

If $2|\beta$, then

$$\begin{aligned} dim(\mathfrak{g}_{-2\alpha_{-1}-\beta}) &= \mathcal{B}(-2\alpha_{-1}-\beta) - \frac{1}{2}\mathcal{B}(-\alpha_{-1}-\frac{\beta}{2}) \\ &= \left(\sum_{\substack{\tau_1+\tau_2=-2\alpha_{-1}-\beta \\ \tau_i=-\alpha_{-1}-\sum m_{ij}\alpha_j \\ \tau_1<\tau_2}} dimV(\Lambda_0)_{\tau_1} \cdot dimV(\Lambda_0)_{\tau_2} \right) \\ &\quad + \frac{1}{2}(dimV(\Lambda_0)_{-\alpha_{-1}-\frac{\beta}{2}})^2 - dimV(\Lambda_1)_{\Lambda_1+\alpha_0-\beta} \\ &\quad - \frac{1}{2}dimV(\Lambda_0)_{-\alpha_{-1}-\frac{\beta}{2}} \end{aligned} \quad (6.12)$$

In the next example we use formula (6.12) to find the multiplicity of $-2\alpha_{-1} - 2\alpha_0 - 2\alpha_1 - 2\alpha_2 - 2\delta = -2\alpha_{-1} - 4\alpha_0 - 6\alpha_1 - 8\alpha_2$. In [Hon98], this root was shown to have multiplicity 21 utilizing $\mathfrak{sl}_2(\mathbb{C})$ theory. We verify the calculation using our methods.

Example 6.2.9 (Multiplicity of $-2\alpha_{-1} - 4\alpha_0 - 6\alpha_1 - 8\alpha_2$)**.**

$$\begin{aligned}
dim(\mathfrak{g}_{-2\alpha_{-1}-4\alpha_0-6\alpha_1-8\alpha_2}) &= \mathcal{B}(-2\alpha_{-1} - 4\alpha_0 - 6\alpha_1 - 8\alpha_2) - \frac{1}{2}\mathcal{B}(-\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - 4\alpha_2) \\
&= \left(\sum_{\substack{\tau_1+\tau_2=-2\alpha_{-1}-4\alpha_0-6\alpha_1-8\alpha_2 \\ \tau_i=-\alpha_{-1}-\sum m_{ij}\alpha_j \\ \tau_1<\tau_2}} dimV(\Lambda_0)_{\tau_1} \cdot dimV(\Lambda_0)_{\tau_2} \right) \\
&\quad + \frac{1}{2}(dimV(\Lambda_0)_{-\alpha_{-1}-2\alpha_0-3\alpha_1-4\alpha_2})^2 - dimV(\Lambda_1)_{\Lambda_1-3\alpha_0-6\alpha_1-8\alpha_2} \\
&\quad - \frac{1}{2}dimV(\Lambda_0)_{-\alpha_{-1}-2\alpha_0-3\alpha_1-4\alpha_2} \\
&= 78 + \frac{1}{2}(3)^2 - 60 - \frac{1}{2}(3) \\
&= 21
\end{aligned}$$

We now use our C# program to calculate additional multiplicities and collect them in Table 6.9 through Table 6.12.

Table 6.9 Multiplicity table for $-2\alpha_{-1} - k\delta$ when $2 \le k \le 12$

$-2\alpha_{-1} - k\delta$	$mult(-2\alpha_{-1} - k\delta)$	Frenkel Bound
$k = 2$	2	2
$k = 3$	14	110
$k = 4$	66	1770
$k = 5$	258	17490
$k = 6$	878	129512
$k = 7$	2692	786814
$k = 8$	7626	4126070
$k = 9$	20244	19283830
$k = 10$	50930	82088400
$k = 11$	122417	323275512
$k = 12$	282934	1191580872

Table 6.10 Multiplicity table for $-2\alpha_{-1} - \alpha_0 - k\delta$ when $1 \leq k \leq 12$

$-2\alpha_{-1} - \alpha_0 - k\delta$	$mult(-2\alpha_{-1} - \alpha_0 - k\delta)$	Frenkel Bound
$k = 1$	1	1
$k = 2$	6	20
$k = 3$	32	481
$k = 4$	135	5822
$k = 5$	485	49010
$k = 6$	1558	326015
$k = 7$	4578	1831065
$k = 8$	12524	9035539
$k = 9$	32320	40210481
$k = 10$	79396	164363280
$k = 11$	186985	625425005
$k = 12$	424550	2238075315

Table 6.11 Multiplicity table for $-2\alpha_{-1} - \alpha_0 - \alpha_2 - k\delta$ when $1 \leq k \leq 12$

$-2\alpha_{-1} - \alpha_0 - \alpha_2 - k\delta$	$mult(-2\alpha_{-1} - \alpha_0 - \alpha_2 - k\delta)$	Frenkel Bound
$k = 1$	0	1
$k = 2$	3	10
$k = 3$	21	300
$k = 4$	99	3956
$k = 5$	378	35002
$k = 6$	1257	240840
$k = 7$	3781	1386930
$k = 8$	10518	6978730
$k = 9$	27489	31551450
$k = 10$	68200	130673928
$k = 11$	161937	502810130
$k = 12$	370176	1816715170

Table 6.12 Multiplicity table for $-2\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2 - k\delta$ when $1 \leq k \leq 12$

$-2\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2 - k\delta$	$mult(-2\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2 - k\delta)$	Frenkel Bound
$k = 1$	1	1
$k = 2$	9	65
$k = 3$	48	1165
$k = 4$	199	12230
$k = 5$	702	94235
$k = 6$	2208	589128
$k = 7$	6367	3157789
$k = 8$	17134	15018300
$k = 9$	43570	64854575
$k = 10$	105647	258508230
$k = 11$	245949	962759294
$k = 12$	552674	3381689157

6.3 Level 3 Roots

Consider level 3 roots of the form $-3\alpha_{-1} - \alpha - k\delta$, where $\alpha = m_0\alpha_0 + m_1\alpha_1 + m_2\alpha_2 < \delta$ and $m_i \in \mathbb{Z}_{\geq 0}$. In Theorem 6.3.1 through Theorem 6.3.5 we show that for all $\alpha < \delta$ the multiplicity of $-3\alpha_{-1} - \alpha - k\delta$ is equal to that of the following with a possible shift of the roots string by at most -2δ.

$$-3\alpha_{-1} - k\delta$$

$$-3\alpha_{-1} - \alpha_0 - k\delta$$

$$-3\alpha_{-1} - \alpha_0 - \alpha_2 - k\delta$$

$$-3\alpha_{-1} - \alpha_0 - 2\alpha_2 - k\delta$$

$$-3\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2 - k\delta$$

$$-3\alpha_{-1} - \alpha_0 - 2\alpha_1 - k\delta$$

Therefore we only consider level 3 roots of these forms.

Recall once more Theorem 4.1.2: If $-\ell\alpha_{-1} - k\delta$ is a root of $HG_2^{(1)}$, then $k \geq \ell$. Furthermore, $mult(-\ell\alpha_{-1} - \ell\delta) = 2$. Therefore,

$$dim(\mathfrak{g}_{-3\alpha_{-1}-\delta}) = 0$$

$$dim(\mathfrak{g}_{-3\alpha_{-1}-2\delta}) = 0$$

$$dim(\mathfrak{g}_{-3\alpha_{-1}-3\delta}) = 2.$$

When $k > 3$ we use Kang's multiplicity formula and the path crystal representations of $V(\Lambda_0)$, $V(\Lambda_1)$, and $V(3\Lambda_2)$ from Section 5.3 to calculate the multiplicities of $-3\alpha_{-1} - \alpha - k\delta$. Specifically, for $-3\alpha_{-1} - k\delta$ ($k = 4, 5, 6$) we apply Kang's multiplicity formula explicitly. For brevity, however, we omit some details of the calculations when $k = 5$ and $k = 6$. For larger values of k, we use our C# program for level 3 roots to compute the multiplicities. These are displayed in Table 6.16 through Table 6.21.

Theorem 6.3.1. *If $-3\alpha_{-1} - \alpha_0 - k\delta$ is a root of $HG_2^{(1)}$, then*

$$\begin{aligned} mult(-3\alpha_{-1} - \alpha_0 - k\delta) &= mult(\beta_0 - k\delta) \\ &= mult(\beta_1 - (k+1)\delta) \\ &= mult(\beta_2 - (k+2)\delta) \end{aligned}$$

where β_i is as follows:

β_0	β_1	β_2
$-3\alpha_{-1} - \alpha_0 - \alpha_1$	$-3\alpha_{-1} - \alpha_0 - 3\alpha_2$	$-3\alpha_{-1} - 2\alpha_1$
$-3\alpha_{-1} - \alpha_0 - \alpha_1 - 3\alpha_2$	$-3\alpha_{-1} - \alpha_1$	
	$-3\alpha_{-1} - \alpha_1 - 3\alpha_2$	
	$-3\alpha_{-1} - 2\alpha_1 - 3\alpha_2$	

Proof. It is enough to show each $\beta_i - (k+i)\delta$ is Weyl group conjugate to $-3\alpha_{-1} - \alpha_0 - k\delta$.

For each β_0 we choose reflection, w, as follows and apply Proposition 4.1.1:

- $w = r_1 \Longrightarrow w(-3\alpha_{-1} - \alpha_0 - k\delta) = -3\alpha_{-1} - \alpha_0 - \alpha_1 - k\delta$
- $w = r_2r_1 \Longrightarrow w(-3\alpha_{-1} - \alpha_0 - k\delta) = -3\alpha_{-1} - \alpha_0 - \alpha_1 - 3\alpha_2 - k\delta$

For each β_1 we choose reflection, w, as follows and apply Proposition 4.1.1:

- $w = r_2r_1r_0 \Longrightarrow w(-3\alpha_{-1} - \alpha_0 - k\delta) = -3\alpha_{-1} - \alpha_0 - 3\alpha_2 - (k+1)\delta$
- $w = r_1r_2r_1 \Longrightarrow w(-3\alpha_{-1} - \alpha_0 - k\delta) = -3\alpha_{-1} - \alpha_1 - (k+1)\delta$
- $w = r_2r_1r_2r_1 \Longrightarrow w(-3\alpha_{-1} - \alpha_0 - k\delta) = -3\alpha_{-1} - \alpha_1 - 3\alpha_2 - (k+1)\delta$
- $w = r_1r_2r_1r_2r_1 \Longrightarrow w(-3\alpha_{-1} - \alpha_0 - k\delta) = -3\alpha_{-1} - 2\alpha_1 - 3\alpha_2 - (k+1)\delta$

For each β_2 we choose reflection, w, as follows and apply Proposition 4.1.1:

- $w = r_1r_2r_1r_0 \Longrightarrow w(-\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2 - k\delta) = -3\alpha_{-1} - 2\alpha_1 - (k+2)\delta$

□

Theorem 6.3.2. *If $-3\alpha_{-1} - \alpha_0 - \alpha_2 - k\delta$ is a root of $HG_2^{(1)}$, then*

$$\begin{aligned} mult(-3\alpha_{-1} - \alpha_0 - \alpha_2 - k\delta) &= mult(\beta_0 - k\delta) \\ &= mult(\beta_1 - (k+1)\delta) \end{aligned}$$

where β_i is as follows:

β_0	β_1
$-3\alpha_{-1}-\alpha_0-2\alpha_1-\alpha_2$	$-3\alpha_{-1}-2\alpha_2$
	$-3\alpha_{-1}-2\alpha_1-2\alpha_2$

Proof. It is enough to show each $\beta_i-(k+i)\delta$ is Weyl group conjugate to $-3\alpha_{-1}-\alpha_0-\alpha_2-k\delta$.

For each β_0 we choose reflection, w, as follows and apply Proposition 4.1.1:

- $w=r_1 \Longrightarrow w(-3\alpha_{-1}-\alpha_0-\alpha_2-k\delta)=-3\alpha_{-1}-\alpha_0-2\alpha_1-\alpha_2-k\delta$

For each β_1 we choose reflection, w, as follows and apply Proposition 4.1.1:

- $w=r_2r_1 \Longrightarrow w(-3\alpha_{-1}-\alpha_0-\alpha_2-k\delta)=-3\alpha_{-1}-2\alpha_2-(k+1)\delta$
- $w=r_1r_2r_1 \Longrightarrow w(-3\alpha_{-1}-\alpha_0-\alpha_2-k\delta)=-3\alpha_{-1}-2\alpha_1-2\alpha_2-(k+1)\delta$

□

Theorem 6.3.3. *If $-3\alpha_{-1}-\alpha_0-2\alpha_2-k\delta$ is a root of $HG_2^{(1)}$, then*

$$mult(-3\alpha_{-1}-\alpha_0-2\alpha_2-k\delta)=mult(-3\alpha_{-1}-2\alpha_1-\alpha_2-(k+1)\delta).$$

Proof. Use Proposition 4.1.1 to apply $w=r_1r_2r_1r_2r_1r_2$ to $-3\alpha_{-1}-\alpha_0-2\alpha_2-k\delta$ to obtain $-3\alpha_{-1}-2\alpha_1-\alpha_2-(k+1)\delta$. □

Theorem 6.3.4. *If $-3\alpha_{-1}-\alpha_0-\alpha_1-\alpha_2-k\delta$ is a root of $HG_2^{(1)}$, then*

$$\begin{aligned} mult(-3\alpha_{-1}-\alpha_0-\alpha_1-\alpha_2-k\delta) &= mult(\beta_0-k\delta) \\ &= mult(\beta_1-(k+1)\delta) \end{aligned}$$

where β_i is as follows:

β_0	β_1
$-3\alpha_{-1}-\alpha_0-\alpha_1-2\alpha_2$	$-3\alpha_{-1}-\alpha_2$
$-3\alpha_{-1}-\alpha_0-2\alpha_1-2\alpha_2$	$-3\alpha_{-1}-\alpha_1-\alpha_2$
	$-3\alpha_{-1}-\alpha_1-2\alpha_2$

Proof. It is enough to show each $\beta_i-(k+i)\delta$ is Weyl group conjugate to $-3\alpha_{-1}-\alpha_0-\alpha_1-\alpha_2-k\delta$.

For each β_0 we choose reflection, w, as follows and apply Proposition 4.1.1:

- $w = r_2 \Longrightarrow w(-3\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2 - k\delta) = -3\alpha_{-1} - \alpha_0 - \alpha_1 - 2\alpha_2 - k\delta$
- $w = r_1r_2 \Longrightarrow w(-3\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2 - k\delta) = -3\alpha_{-1} - \alpha_0 - 2\alpha_1 - 2\alpha_2 - k\delta$

For each β_1 we choose reflection, w, as follows and apply Proposition 4.1.1:

- $w = r_2r_1r_2 \Longrightarrow w(-3\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2 - k\delta) = -3\alpha_{-1} - \alpha_2 - (k+1)\delta$
- $w = r_1r_2r_1r_2 \Longrightarrow w(-3\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2 - k\delta) = -3\alpha_{-1} - \alpha_1 - \alpha_2 - (k+1)\delta$
- $w = r_2r_1r_2r_1r_2 \Longrightarrow w(-3\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2 - k\delta) = -3\alpha_{-1} - \alpha_1 - 2\alpha_2 - (k+1)\delta$

□

Theorem 6.3.5. *If $-3\alpha_{-1} - \alpha_0 - 2\alpha_1 - k\delta$ is a root of $HG_2^{(1)}$, then*

$$mult(-3\alpha_{-1} - \alpha_0 - 2\alpha_1 - k\delta) = mult(-3\alpha_{-1} - 3\alpha_2 - (k+1)\delta).$$

Proof. Use Proposition 4.1.1 to apply $w = r_2$ to $-3\alpha_{-1} - \alpha_0 - 2\alpha_1 - k\delta$ to obtain $-3\alpha_{-1} - 3\alpha_2 - (k+1)\delta$. □

In the examples to follow, we demonstrate the algorithm corresponding to Kang's multiplicity formula for the level 3 roots $-3\alpha_{-1} - 4\delta$, $-3\alpha_{-1} - 5\delta$, and $-3\alpha_{-1} - 6\delta$.

Example 6.3.6 (Multiplicity of $-3\alpha_{-1} - 4\delta$)**.** Consider the level 3 root $\alpha = -3\alpha_{-1} - 4\delta$. Applying Kang's multiplicity formula, we sum over the τ that divide α. In this case, the only divisor of α is $\tau = \alpha$ since $\ell = 3$ is prime and does not divide $k = 4$. Therefore,

$$\begin{aligned} dim(\mathfrak{g}_{-3\alpha_{-1}-4\delta}) &= \sum_{\tau|\alpha} \mu(\frac{\alpha}{\tau}) \left(\frac{\tau}{\alpha}\right) \mathcal{B}(\tau) \\ &= \mu\left(\frac{-3\alpha_{-1} - 4\delta}{-3\alpha_{-1} - 4\delta}\right) \left(\frac{-3\alpha_{-1} - 4\delta}{-3\alpha_{-1} - 4\delta}\right) \mathcal{B}(-3\alpha_{-1} - 4\delta) \\ &= \mathcal{B}(-3\alpha_{-1} - 4\delta). \end{aligned}$$

For convenience we once again recall that $\mathcal{B}(\tau)$ sums over the partitions of τ:

$$\mathcal{B}(\tau) = \sum_{(n_i,\tau_i)\in T(\tau)} \frac{(\sum n_i - 1)!}{\prod (n_i)!} \prod K_{\tau_i}^{n_i}$$

where $K_{\tau_i} = \sum_{\substack{w\in W(S)\\ \ell(w)\geq 1}} (-1)^{\ell(w)+1} dim V(w\rho - \rho)_{\tau_i}$ and $V(w\rho-\rho)$ is the highest weight module of weight $w\rho - \rho$ in $G_2^{(1)}$. As with level 2 roots, we need only consider the τ_i and reflections $w \in W(S)$ such that the $dim V(w\rho - \rho)_{\tau_i}$ is nonzero. For level 3 roots, this means K_{τ_i} sums over the reflections for which $w\rho - \rho$ has level less than or equal to 3. In Section 6.2 we used Lemma 6.2.6 to show r_{-1} and $r_{-1}r_0$ are the only length one and two reflections, respectively. See Table 6.6. Similarly, we use Lemma 6.2.6 to show that $r_{-1}r_0r_1$ is the only length three reflection in $W(S)$:

$$\begin{aligned}
r_{-1}r_0r_j \in W(S) &\iff r_{-1}r_0(\alpha_j) \in \Delta^+(S)\\
&\iff r_{-1}(\alpha_j - \alpha_j(H_0)\alpha_0) \in \Delta^+(S)\\
&\iff r_{-1}(\alpha_j) - \alpha_j(H_0) r_{-1}(\alpha_0) \in \Delta^+(S)\\
&\iff (\alpha_j - \alpha_j(H_{-1})\alpha_{-1}) - \alpha_j(H_0)(\alpha_0 - \alpha_0(H_{-1})\alpha_{-1}) \in \Delta^+(S)\\
&\iff (\alpha_j - \alpha_j(H_{-1})\alpha_{-1}) - \alpha_j(H_0)(\alpha_0 + \alpha_{-1}) \in \Delta^+(S)\\
&\iff \alpha_j - \alpha_j(H_0)\alpha_0 - \alpha_j(H_{-1} + H_0)\alpha_{-1} \in \Delta^+(S)\\
&\iff j = 1
\end{aligned}$$

In Table 6.13 we write $w\rho - \rho$ in terms of the fundamental weights of $G_2^{(1)}$ and verify $w\rho - \rho$ is a level 3 weight. Substituting in our length one, two, and three reflections K_{τ_i} becomes

$$\begin{aligned}
K_{\tau_i} &= \sum_{\substack{w\in W(S)\\ \ell(w)\geq 1}} (-1)^{\ell(w)+1} dim V(w\rho - \rho)_{\tau_i}\\
&= dim V(\Lambda_0)_{\tau_i} - dim V(\Lambda_1 - \delta)_{\tau_i} + dim V(3\Lambda_2 - 2\delta)_{\tau_i}
\end{aligned}$$

For $dim V(\Lambda_0)_{\tau_i}$ to be nonzero, we need τ_i to be a level 1 root of form

$$\Lambda_0 - \sum_{j=0}^{2} m_{ij}\alpha_j = -\alpha_{-1} - \sum_{j=0}^{2} m_{ij}\alpha_j \tag{6.13}$$

Table 6.13 Length three reflections $w \in W(S)$

k	$\ell(w) = k$	$w\rho - \rho$	level of $w\rho - \rho$
3	$r_{-1}r_0r_1$	$r_{-1}r_0r_1(\rho) - \rho = r_{-1}r_0(\rho - \rho(h_1)\alpha_1) - \rho$ $= r_{-1}r_0(\rho - \alpha_1) - \rho$ $= r_{-1}[(\rho - \alpha_1) - (\rho - \alpha_1)(h_0)\alpha_0] - \rho$ $= r_{-1}[\rho - \alpha_1 - 2\alpha_0] - \rho$ $= [(\rho - \alpha_1 - 2\alpha_0) - (\rho - \alpha_1 - 2\alpha_0)(h_{-1})\alpha_{-1}] - \rho$ $= [(\rho - \alpha_1 - 2\alpha_0) - 3\alpha_{-1}] - \rho$ $= -3\alpha_{-1} - 2\alpha_0 - \alpha_1$ $= 3\Lambda_2 - 2\delta$	$(3\Lambda_2 - 2\delta)(K) = 3\Lambda_2(K)$ $= 3$

where $m_{ij} \in \mathbb{Z}_{\geq 0}$. One option is to partition $\tau = -3\alpha_{-1} - 4\delta$ into the sum of three roots of form (6.13). In order to prevent repeated partitions, we consider only the partitions for which $\tau_1 \leq \tau_2 \leq \tau_3$. Because this list becomes quite large, even when $k = 4$, we prove Theorem 6.3.7 below, which determines the form of the largest τ_1 satisfying $\tau_1 \leq \tau_2 \leq \tau_3$. Therefore, we only have to consider the partitions with τ_1 less than or equal the maximum τ_1, thus minimizing the time needed to find the total number of distinct partitions with $\tau_1 \leq \tau_2 \leq \tau_3$.

Theorem 6.3.7. *Let $\tau = -3\alpha_{-1} - x_0\alpha_0 - x_1\alpha_1 - x_2\alpha_2$ be a level 3 root of $HG_2^{(1)}$. Let $\tau_1 = -\alpha_{-1} - a_0\alpha_0 - a_1\alpha_1 - a_2\alpha_2$, $\tau_2 = -\alpha_{-1} - b_0\alpha_0 - b_1\alpha_1 - b_2\alpha_2$, and $\tau_3 = -\alpha_{-1} - c_0\alpha_0 - c_1\alpha_1 - c_2\alpha_2$ be a partition of τ satisfying $\tau = \tau_1 + \tau_2 + \tau_3$ and $\tau_1 \leq \tau_2 \leq \tau_3$. Define $s = x_0 \bmod 3$ and $t = x_1 \bmod (3 - s)$. Then the largest such τ_1 has form*

$$\tau_1 = -\alpha_{-1} - \left\lfloor \frac{x_0}{3} \right\rfloor \alpha_0 - \left\lfloor \frac{x_1}{3 - s} \right\rfloor \alpha_1 - \left\lfloor \frac{x_2}{3 - s - t} \right\rfloor \alpha_2.$$

To prove Theorem 6.3.7, observe the ordering $\tau_1 \leq \tau_2 \leq \tau_3$ is a Lexicographic ordering. For the reader's convenience we recall the definition of Lexicographic ordering below.

Definition 6.3.8. Let $\tau_1 = -\alpha_{-1} - a_0\alpha_0 - a_1\alpha_1 - a_2\alpha_2$ and $\tau_2 = -\alpha_{-1} - b_0\alpha_0 - b_1\alpha_1 - b_2\alpha_2$. We define $\tau_1 = \tau_2$ if and only if $a_i = b_i$ for all $i = \{0, 1, 2\}$. We define $\tau_1 < \tau_2$ if and only if there exists an integer $m > 0$ such that for all $i < m$ $a_i = b_i$ and $a_m < b_m$. Finally, we define $\tau_1 \leq \tau_2$ if and only if $\tau_1 = \tau_2$ or $\tau_1 < \tau_2$.

Proof. We examine the three cases for $s = x_0 \bmod 3$ in turn.

Case 1: $s = 0$. By the Lexicographic ordering, we have $a_0 \leq b_0 \leq c_0$ and thus the largest possible a_0 is $\lfloor \frac{x_0}{3} \rfloor$, which equals $\frac{x_0}{3}$ since $3|x_0$. Consequently, $a_0 = b_0 = c_0 = \frac{x_0}{3}$ and $a_1 \leq b_1 \leq c_1$ by the Lexicographic ordering. Then the largest possible a_1 is $\lfloor \frac{x_1}{3} \rfloor$, which is determined by $t = x_1 \bmod (3 - s) = x_1 \bmod 3$. We consider the three cases for t in turn.

Case 1 (a): $t = 0$. Then $3|x_1$ and $a_1 = b_1 = c_1 = \frac{x_1}{3}$ by the Lexicographic ordering. Consequently, $a_2 \leq b_2 \leq c_2$ and the largest possible a_2 is $a_2 = \left\lfloor \frac{x_2}{3} \right\rfloor$. Therefore, the largest τ_1 for $s = 0$ and $t = 0$ has form

$$\begin{aligned}\tau_1 &= -\alpha_{-1} - \frac{x_0}{3}\alpha_0 - \frac{x_1}{3}\alpha_1 - \left\lfloor \frac{x_2}{3} \right\rfloor \alpha_2 \\ &= -\alpha_{-1} - \left\lfloor \frac{x_0}{3} \right\rfloor \alpha_0 - \left\lfloor \frac{x_1}{3-s} \right\rfloor \alpha_1 - \left\lfloor \frac{x_2}{3-s-t} \right\rfloor \alpha_2.\end{aligned}$$

Case 1 (b): $t = 1$. Then $\frac{x_1}{3} = \left\lfloor \frac{x_1}{3} \right\rfloor + 1$. In order to maximize a_1 and preserve the Lexicographic order, $a_1 \leq b_1 \leq c_1$, we must let $a_1 = b_1 = \left\lfloor \frac{x_1}{3} \right\rfloor$ and $c_1 = \left\lfloor \frac{x_1}{3} \right\rfloor + 1$. Since $a_1 = b_1 < c_1$, we have $\tau_1 \leq \tau_2 < \tau_3$ for any choice of c_2. To maximize a_2, let $c_2 = 0$ so that $a_2 + b_2 = x_2$. To preserve the Lexicographic ordering, $a_2 \leq b_2$ and hence the largest possible a_2 is $\left\lfloor \frac{x_2}{2} \right\rfloor$. Therefore, the largest τ_1 for the case when $s = 0$ and $t = 1$ has form

$$\begin{aligned}\tau_1 &= -\alpha_{-1} - \frac{x_0}{3}\alpha_0 - \left\lfloor \frac{x_1}{3} \right\rfloor \alpha_1 - \left\lfloor \frac{x_2}{2} \right\rfloor \alpha_2 \\ &= -\alpha_{-1} - \left\lfloor \frac{x_0}{3} \right\rfloor \alpha_0 - \left\lfloor \frac{x_1}{3-s} \right\rfloor \alpha_1 - \left\lfloor \frac{x_2}{3-s-t} \right\rfloor \alpha_2.\end{aligned}$$

Case 1 (c): $t = 2$. Then $\frac{x_1}{3} = \left\lfloor \frac{x_1}{3} \right\rfloor + 2$ and there are two options satisfying the condition $a_1 \leq b_1 \leq c_1$. We show that the first option maximizes τ_1.

1. $a_1 = \left\lfloor \frac{x_1}{3} \right\rfloor$ and $b_1 = c_1 = \left\lfloor \frac{x_1}{3} \right\rfloor + 1$
2. $a_1 = b_1 = \left\lfloor \frac{x_1}{3} \right\rfloor$ and $c_1 = \left\lfloor \frac{x_1}{3} \right\rfloor + 2$.

Option 1: Since $a_1 < b_1 = c_1$, we have $\tau_1 < \tau_2 \leq \tau_3$ for any choice of b_2 and c_2. To maximize a_2, let $b_2 = c_2 = 0$. Then $a_2 = x_2$. Therefore, the largest τ_1 for this option has form

$$\tau_1 = -\alpha_{-1} - \frac{x_0}{3}\alpha_0 - \left\lfloor \frac{x_1}{3} \right\rfloor \alpha_1 - x_2\alpha_2.$$

Because the cases differ only by their choice of a_2 and since option 1 yields the largest possible a_2, we need not consider option 2. Therefore, the largest τ_1 for the case when $s = 1$ and $t = 2$ has form

$$\begin{aligned}\tau_1 &= -\alpha_{-1} - \frac{x_0}{3}\alpha_0 - \left\lfloor \frac{x_1}{3} \right\rfloor \alpha_1 - x_2\alpha_2 \\ &= -\alpha_{-1} - \left\lfloor \frac{x_0}{3} \right\rfloor \alpha_0 - \left\lfloor \frac{x_1}{3-s} \right\rfloor \alpha_1 - \left\lfloor \frac{x_2}{3-s-t} \right\rfloor \alpha_2.\end{aligned}$$

Case 2: $s = 1$. Then $\frac{x_0}{3} = \left\lfloor \frac{x_0}{3} \right\rfloor + 1$. By the Lexicographic ordering, we require $a_0 \leq b_0 \leq c_0$ and hence the largest possible choice for a_0 is $\left\lfloor \frac{x_0}{3} \right\rfloor$. Then $a_0 = b_0 = \left\lfloor \frac{x_0}{3} \right\rfloor$ and $c_0 = \left\lfloor \frac{x_0}{3} \right\rfloor + 1$. Since $a_0 = b_0 < c_0$, we have $\tau_1 \leq \tau_2 < \tau_3$ for any choice of c_1 and c_2. To maximize a_1, let $c_1 = 0$ so that $a_1 + b_1 = x_1$. To preserve the Lexicographic order, $a_1 \leq b_1$ and hence the largest possible a_1 is $\left\lfloor \frac{x_1}{2} \right\rfloor$, which is determined by $t = x_1 \bmod (3 - s) = x_1 \bmod 2$. We consider the two cases for t in turn.

Case 2 (a): $t = 0$. Then $2|x_1$ and $a_1 = b_1 = \frac{x_1}{2}$. To maximize a_2, let $c_2 = 0$ so that $a_2 + b_2 = x_2$. This is possible since we have shown $\tau_1 \leq \tau_2 < \tau_3$ for any choice of c_2. To preserve Lexicographic ordering, $a_2 \leq b_2$ and hence $a_2 = \left\lfloor \frac{x_2}{2} \right\rfloor$ is the largest possible a_2. Therefore, the largest τ_1 for the case when $s = 1$ and $t = 0$ has form

$$\begin{aligned}\tau_1 &= -\alpha_{-1} - \left\lfloor \frac{x_0}{3} \right\rfloor \alpha_0 - \left\lfloor \frac{x_1}{2} \right\rfloor \alpha_1 - \left\lfloor \frac{x-2}{2} \right\rfloor \alpha_2 \\ &= -\alpha_{-1} - \left\lfloor \frac{x_0}{3} \right\rfloor \alpha_0 - \left\lfloor \frac{x_1}{3-s} \right\rfloor \alpha_1 - \left\lfloor \frac{x_2}{3-s-t} \right\rfloor \alpha_2.\end{aligned}$$

Case 2 (b): $t = 1$. Then $\frac{x_1}{2} = \left\lfloor \frac{x_1}{2} \right\rfloor + 1$ and hence $a_1 = \left\lfloor \frac{x_1}{2} \right\rfloor$ and $b_1 = \left\lfloor \frac{x_1}{2} \right\rfloor + 1$. Since $a_1 < b_1 < c_1$, we have $\tau_1 < \tau_2 < \tau_3$ for any choice of b_2 and c_2. To maximize a_2, let $b_2 = c_2 = 0$. Then, $a_2 = x_2$ and hence the largest τ_1 for the case when $s = 1$ and $t = 1$ has form

$$\begin{aligned}\tau_1 &= -\alpha_{-1} - \left\lfloor \frac{x_0}{3} \right\rfloor \alpha_0 - \left\lfloor \frac{x_1}{2} \right\rfloor \alpha_1 - x_2\alpha_2 \\ &= -\alpha_{-1} - \left\lfloor \frac{x_0}{3} \right\rfloor \alpha_0 - \left\lfloor \frac{x_1}{3-s} \right\rfloor \alpha_1 - \left\lfloor \frac{x_2}{3-s-t} \right\rfloor \alpha_2.\end{aligned}$$

Case 3: $s = 2$. Then $t = x_1 \bmod 1 = 0$ and $\frac{x_0}{3} = \left\lfloor \frac{x_0}{3} \right\rfloor + 2$. Therefore, only options two options satisfying the condition $a_0 \leq b_0 \leq c_0$. We show that the first option maximizes τ_1.

1. $a_0 = \left\lfloor \frac{x_0}{3} \right\rfloor$ and $b_0 = c_0 = \left\lfloor \frac{x_0}{3} \right\rfloor + 1$
2. $a_0 = b_0 = \left\lfloor \frac{x_0}{3} \right\rfloor$ and $c_0 = \left\lfloor \frac{x_0}{3} \right\rfloor + 2$.

Option 1: Since $a_0 < b_0 = c_0$, we have $\tau_1 < \tau_2 \leq \tau_3$ for any choice of b_1, b_2, c_1, and c_2 satisfying $\tau_2 \leq \tau_3$. To maximize a_1, let $b_1 = c_1 = 0$ so that $a_1 = x_1$. To maximize a_2, let $b_2 = c_2 = 0$ so that $a_2 = x_2$. Therefore, the largest τ_1 for this option has form

$$\tau_1 = -\alpha_{-1} - \left\lfloor \frac{x_0}{3} \right\rfloor \alpha_0 - x_1\alpha_1 - x_2\alpha_2.$$

Because the cases differ only by their choice of a_1 and a_2 and since option 1 yields the largest possible a_1 and a_2, we need not consider option 2. Therefore, the largest τ_1 for the case when $s = 2$ has form

$$\begin{aligned}\tau_1 &= -\alpha_{-1} - \left\lfloor \frac{x_0}{3} \right\rfloor \alpha_0 - x_1\alpha_1 - x_2\alpha_2 \\ &= -\alpha_{-1} - \left\lfloor \frac{x_0}{3} \right\rfloor \alpha_0 - \left\lfloor \frac{x_1}{3-s} \right\rfloor \alpha_1 - \left\lfloor \frac{x_2}{3-s-t} \right\rfloor \alpha_2.\end{aligned}$$

In all cases, the largest possible τ_1 has form

$$\tau_1 = -\alpha_{-1} - \left\lfloor \frac{x_0}{3} \right\rfloor \alpha_0 - \left\lfloor \frac{x_1}{3-s} \right\rfloor \alpha_1 - \left\lfloor \frac{x_2}{3-s-t} \right\rfloor \alpha_2.$$

□

For our example, the maximum τ_1 satisfying $\tau_1 \leq \tau_2 \leq \tau_3$ for the partition of $\tau = -3\alpha_{-1} - 4\delta$ is $-\alpha_{-1} - \alpha_0 - 4\alpha_1 - 6\alpha_2$. In addition to the constraint on τ_1 given by Theorem 6.3.7, we need only consider the partitions of τ such that $dimV(\Lambda_0)_{\tau_i} \neq 0$ for all three τ_i. Since each τ_i is a level 1 root, we use the results and methods from Section 6.1 to find $dimV(\Lambda_0)_{\tau_i}$ and collect them in Table 6.14.

For $dimV(\Lambda_1 - \delta)_{\tau_i}$ to be nonzero, we need τ_i to be a level 2 root of form

$$\Lambda_1 - \delta - \sum_{j=0}^{2} m_{ij}\alpha_j = -2\alpha_{-1} - \alpha_0 - \sum_{j=0}^{2} m_{ij}\alpha_j \tag{6.14}$$

$$= -2\alpha_{-1} - \sum_{j=0}^{2} \overline{m}_{ij}\alpha_j \tag{6.15}$$

where $m_{ij}, \overline{m}_{ij} \in \mathbb{Z}_{\geq 0}$. To involve τ_i of form (6.15) we partition $\tau = -3\alpha_{-1} - 4\delta$ into $\tau_1 + \tau_2$ where τ_1 is a level 1 root of the form (6.13) and τ_2 is a level 2 root of the form (6.15). To find the $dimV(\Lambda_0)_{\tau_1}$ we use the level 1 results from Section 6.1. For $dimV(\Lambda_1 - \delta)_{\tau_2}$, we use the isomorphism $V(\Lambda_1) \cong V(\Lambda_1 - \delta) \otimes V(\delta)$ and our C# program to count the paths in the path crystal $\mathcal{P}(\Lambda_1) \cong V(\Lambda_1)$, whose procedure is outlined in Section 6.2. The partitions of this form and their corresponding weight multiplicities are given in Table 6.15.

For $dimV(3\Lambda_2 - 2\delta)_{\tau_i}$ to be nonzero, τ_i must be of the form

$$3\Lambda_2 - 2\delta - \sum_{j=0}^{2} m_{ij}\alpha_j = -3\alpha_{-1} - 2\alpha_0 - \alpha_1 - \sum_{j=0}^{2} m_{ij}\alpha_j \tag{6.16}$$

$$= -3\alpha_{-1} - \sum_{j=0}^{2} \hat{m}_{ij}\alpha_j \tag{6.17}$$

where $m_{ij}, \hat{m}_{ij} \in \mathbb{Z}_{\geq 0}$. Then $\tau = -3\alpha_{-1} - 4\delta$ is the only partition of τ satisfying (6.17). Utilizing the path crystal realization of $V(3\Lambda_2)$, from Chapter 5 and the isomorphism $V(3\Lambda_2 - 2\delta) \cong V(3\Lambda_2) \otimes V(2\delta)$, we obtain the $dimV(3\Lambda_2 - 2\delta)_\tau$ below.

$$\begin{aligned} dimV(3\Lambda_2 - 2\delta)_{-3\alpha_{-1}-4\delta} &= dimV(3\Lambda_2)_{-3\alpha_{-1}-2\delta} \\ &= dimV(3\Lambda_2)_{3\Lambda_2+2\alpha_0+\alpha_1-4\delta} \\ &= dimV(3\Lambda_2)_{3\Lambda_2-2\alpha_0-7\alpha_1-12\alpha_2} \\ &= 173 \end{aligned} \tag{6.18}$$

Table 6.14 Partitions of $\tau = -3\alpha_{-1} - 4\delta$ into $\tau_1 + \tau_2 + \tau_3$ with $\tau_i = -\alpha_{-1} - \sum_{j=0}^{2} m_{ij}\alpha_j$ and $m_{ij} \in \mathbb{Z}_{\geq 0}$

τ_1	$dimV(\Lambda_0)_{\tau_1}$	τ_2	$dimV(\Lambda_0)_{\tau_2}$	τ_3	$dimV(\Lambda_0)_{\tau_3}$
$-\alpha_{-1}$	1	$-\alpha_{-1}$	1	$-\alpha_{-1} - 4\alpha_0 - 8\alpha_1 - 12\alpha_2$	32
$-\alpha_{-1}$	1	$-\alpha_{-1} - \alpha_0$	1	$-\alpha_{-1} - 3\alpha_0 - 8\alpha_1 - 12\alpha_2$	6
$-\alpha_{-1}$	1	$-\alpha_{-1} - \alpha_0 - \alpha_1$	1	$-\alpha_{-1} - 3\alpha_0 - 7\alpha_1 - 12\alpha_2$	6
$-\alpha_{-1}$	1	$-\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2$	1	$-\alpha_{-1} - 3\alpha_0 - 7\alpha_1 - 11\alpha_2$	9
$-\alpha_{-1}$	1	$-\alpha_{-1} - \alpha_0 - \alpha_1 - 2\alpha_2$	1	$-\alpha_{-1} - 3\alpha_0 - 7\alpha_1 - 10\alpha_2$	9
$-\alpha_{-1}$	1	$-\alpha_{-1} - \alpha_0 - \alpha_1 - 3\alpha_2$	1	$-\alpha_{-1} - 3\alpha_0 - 7\alpha_1 - 9\alpha_2$	6
$-\alpha_{-1}$	1	$-\alpha_{-1} - \alpha_0 - 2\alpha_1 - 2\alpha_2$	1	$-\alpha_{-1} - 3\alpha_0 - 6\alpha_1 - 10\alpha_2$	9
$-\alpha_{-1}$	1	$-\alpha_{-1} - \alpha_0 - 2\alpha_1 - 3\alpha_2$	2	$-\alpha_{-1} - 3\alpha_0 - 6\alpha_1 - \alpha_2$	14
$-\alpha_{-1}$	1	$-\alpha_{-1} - \alpha_0 - 2\alpha_1 - 4\alpha_2$	1	$-\alpha_{-1} - 3\alpha_0 - 6\alpha_1 - 8\alpha_2$	9
$-\alpha_{-1}$	1	$-\alpha_{-1} - \alpha_0 - 3\alpha_1 - 3\alpha_2$	1	$-\alpha_{-1} - 3\alpha_0 - 5\alpha_1 - 9\alpha_2$	6
$-\alpha_{-1}$	1	$-\alpha_{-1} - \alpha_0 - 3\alpha_1 - 4\alpha_2$	1	$-\alpha_{-1} - 3\alpha_0 - 5\alpha_1 - 8\alpha_2$	9
$-\alpha_{-1}$	1	$-\alpha_{-1} - \alpha_0 - 3\alpha_1 - 5\alpha_2$	1	$-\alpha_{-1} - 3\alpha_0 - 5\alpha_1 - 7\alpha_2$	9
$-\alpha_{-1}$	1	$-\alpha_{-1} - \alpha_0 - 3\alpha_1 - 6\alpha_2$	1	$-\alpha_{-1} - 3\alpha_0 - 5\alpha_1 - 6\alpha_2$	6
$-\alpha_{-1}$	1	$-\alpha_{-1} - \alpha_0 - 4\alpha_1 - 6\alpha_2$	1	$-\alpha_{-1} - 3\alpha_0 - 4\alpha_1 - 6\alpha_2$	6
$-\alpha_{-1}$	1	$-\alpha_{-1} - 2\alpha_0 - 2\alpha_1 - 2\alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 6\alpha_1 - 10\alpha_2$	1
$-\alpha_{-1}$	1	$-\alpha_{-1} - 2\alpha_0 - 2\alpha_1 - 3\alpha_2$	2	$-\alpha_{-1} - 2\alpha_0 - 6\alpha_1 - 9\alpha_2$	2
$-\alpha_{-1}$	1	$-\alpha_{-1} - 2\alpha_0 - 2\alpha_1 - 4\alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 6\alpha_1 - 8\alpha_2$	1
$-\alpha_{-1}$	1	$-\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - 3\alpha_2$	2	$-\alpha_{-1} - 2\alpha_0 - 5\alpha_1 - 9\alpha_2$	2
$-\alpha_{-1}$	1	$-\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - 4\alpha_2$	3	$-\alpha_{-1} - 2\alpha_0 - 5\alpha_1 - 8\alpha_2$	3
$-\alpha_{-1}$	1	$-\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - 5\alpha_2$	3	$-\alpha_{-1} - 2\alpha_0 - 5\alpha_1 - 7\alpha_2$	2
$-\alpha_{-1}$	1	$-\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - 6\alpha_2$	2	$-\alpha_{-1} - 2\alpha_0 - 5\alpha_1 - 6\alpha_2$	2
$-\alpha_{-1}$	1	$-\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - 4\alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - 8\alpha_2$	1
$-\alpha_{-1}$	1	$-\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - 5\alpha_2$	3	$-\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - 7\alpha_2$	3
$-\alpha_{-1}$	1	$-\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - 6\alpha_2$	6	$-\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - 6\alpha_2$	6
$-\alpha_{-1} - \alpha_0$	1	$-\alpha_{-1} - \alpha_0 - 2\alpha_1 - 2\alpha_2$	1	$-\alpha_{-1} - 2\alpha_0 - 6\alpha_1 - 10\alpha_2$	1
$-\alpha_{-1} - \alpha_0$	1	$-\alpha_{-1} - \alpha_0 - 2\alpha_1 - 3\alpha_2$	2	$-\alpha_{-1} - 2\alpha_0 - 6\alpha_1 - 9\alpha_2$	2

Table 6.14 (continued)

τ_1	$dimV(\Lambda_0)_{\tau_1}$	τ_2	$dimV(\Lambda_0)_{\tau_2}$	τ_3	$dimV(\Lambda_0)_{\tau_3}$
$-\alpha_{-1}-\alpha_0$	1	$-\alpha_{-1}-\alpha_0-2\alpha_1-4\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-6\alpha_1-8\alpha_2$	1
$-\alpha_{-1}-\alpha_0$	1	$-\alpha_{-1}-\alpha_0-3\alpha_1-3\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-5\alpha_1-9\alpha_2$	2
$-\alpha_{-1}-\alpha_0$	1	$-\alpha_{-1}-\alpha_0-3\alpha_1-4\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-5\alpha_1-8\alpha_2$	3
$-\alpha_{-1}-\alpha_0$	1	$-\alpha_{-1}-\alpha_0-3\alpha_1-5\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-5\alpha_1-7\alpha_2$	3
$-\alpha_{-1}-\alpha_0$	1	$-\alpha_{-1}-\alpha_0-3\alpha_1-6\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-5\alpha_1-6\alpha_2$	2
$-\alpha_{-1}-\alpha_0$	1	$-\alpha_{-1}-\alpha_0-4\alpha_1-6\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-4\alpha_1-6\alpha_2$	6
$-\alpha_{-1}-\alpha_0-\alpha_1$	1	$-\alpha_{-1}-\alpha_0-\alpha_1-2\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-6\alpha_1-10\alpha_2$	1
$-\alpha_{-1}-\alpha_0-\alpha_1$	1	$-\alpha_{-1}-\alpha_0-\alpha_1-3\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-6\alpha_1-9\alpha_2$	2
$-\alpha_{-1}-\alpha_0-\alpha_1$	1	$-\alpha_{-1}-\alpha_0-2\alpha_1-3\alpha_2$	2	$-\alpha_{-1}-2\alpha_0-5\alpha_1-9\alpha_2$	2
$-\alpha_{-1}-\alpha_0-\alpha_1$	1	$-\alpha_{-1}-\alpha_0-2\alpha_1-4\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-5\alpha_1-8\alpha_2$	3
$-\alpha_{-1}-\alpha_0-\alpha_1$	1	$-\alpha_{-1}-\alpha_0-3\alpha_1-4\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-4\alpha_1-8\alpha_2$	1
$-\alpha_{-1}-\alpha_0-\alpha_1$	1	$-\alpha_{-1}-\alpha_0-3\alpha_1-5\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-4\alpha_1-7\alpha_2$	3
$-\alpha_{-1}-\alpha_0-\alpha_1$	1	$-\alpha_{-1}-\alpha_0-3\alpha_1-6\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-4\alpha_1-6\alpha_2$	6
$-\alpha_{-1}-\alpha_0-\alpha_1$	1	$-\alpha_{-1}-\alpha_0-4\alpha_1-6\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-3\alpha_1-6\alpha_2$	2
$-\alpha_{-1}-\alpha_0-\alpha_1-\alpha_2$	1	$-\alpha_{-1}-\alpha_0-\alpha_1-\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-6\alpha_1-10\alpha_2$	1
$-\alpha_{-1}-\alpha_0-\alpha_1-\alpha_2$	1	$-\alpha_{-1}-\alpha_0-\alpha_1-2\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-6\alpha_1-9\alpha_2$	2
$-\alpha_{-1}-\alpha_0-\alpha_1-\alpha_2$	1	$-\alpha_{-1}-\alpha_0-\alpha_1-3\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-6\alpha_1-8\alpha_2$	1
$-\alpha_{-1}-\alpha_0-\alpha_1-\alpha_2$	1	$-\alpha_{-1}-\alpha_0-2\alpha_1-2\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-5\alpha_1-9\alpha_2$	2
$-\alpha_{-1}-\alpha_0-\alpha_1-\alpha_2$	1	$-\alpha_{-1}-\alpha_0-2\alpha_1-3\alpha_2$	2	$-\alpha_{-1}-2\alpha_0-5\alpha_1-8\alpha_2$	3
$-\alpha_{-1}-\alpha_0-\alpha_1-\alpha_2$	1	$-\alpha_{-1}-\alpha_0-2\alpha_1-4\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-5\alpha_1-7\alpha_2$	3
$-\alpha_{-1}-\alpha_0-\alpha_1-\alpha_2$	1	$-\alpha_{-1}-\alpha_0-3\alpha_1-3\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-4\alpha_1-8\alpha_2$	1
$-\alpha_{-1}-\alpha_0-\alpha_1-\alpha_2$	1	$-\alpha_{-1}-\alpha_0-3\alpha_1-4\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-4\alpha_1-7\alpha_2$	3
$-\alpha_{-1}-\alpha_0-\alpha_1-\alpha_2$	1	$-\alpha_{-1}-\alpha_0-3\alpha_1-5\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-4\alpha_1-6\alpha_2$	6
$-\alpha_{-1}-\alpha_0-\alpha_1-\alpha_2$	1	$-\alpha_{-1}-\alpha_0-3\alpha_1-6\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-4\alpha_1-5\alpha_2$	3
$-\alpha_{-1}-\alpha_0-\alpha_1-\alpha_2$	1	$-\alpha_{-1}-\alpha_0-4\alpha_1-6\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-3\alpha_1-5\alpha_2$	3
$-\alpha_{-1}-\alpha_0-\alpha_1-2\alpha_2$	1	$-\alpha_{-1}-\alpha_0-\alpha_1-2\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-6\alpha_1-8\alpha_2$	1

Table 6.14 (continued)

τ_1	$dimV(\Lambda_0)_{\tau_1}$	τ_2	$dimV(\Lambda_0)_{\tau_2}$	τ_3	$dimV(\Lambda_0)_{\tau_3}$
$-\alpha_{-1}-\alpha_0-\alpha_1-2\alpha_2$	1	$-\alpha_{-1}-\alpha_0-2\alpha_1-2\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-5\alpha_1-8\alpha_2$	3
$-\alpha_{-1}-\alpha_0-\alpha_1-2\alpha_2$	1	$-\alpha_{-1}-\alpha_0-2\alpha_1-3\alpha_2$	2	$-\alpha_{-1}-2\alpha_0-5\alpha_1-7\alpha_2$	3
$-\alpha_{-1}-\alpha_0-\alpha_1-2\alpha_2$	1	$-\alpha_{-1}-\alpha_0-2\alpha_1-4\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-5\alpha_1-6\alpha_2$	2
$-\alpha_{-1}-\alpha_0-\alpha_1-2\alpha_2$	1	$-\alpha_{-1}-\alpha_0-3\alpha_1-3\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-4\alpha_1-7\alpha_2$	3
$-\alpha_{-1}-\alpha_0-\alpha_1-2\alpha_2$	1	$-\alpha_{-1}-\alpha_0-3\alpha_1-4\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-4\alpha_1-6\alpha_2$	6
$-\alpha_{-1}-\alpha_0-\alpha_1-2\alpha_2$	1	$-\alpha_{-1}-\alpha_0-3\alpha_1-5\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-4\alpha_1-5\alpha_2$	3
$-\alpha_{-1}-\alpha_0-\alpha_1-2\alpha_2$	1	$-\alpha_{-1}-\alpha_0-3\alpha_1-6\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-4\alpha_1-4\alpha_2$	1
$-\alpha_{-1}-\alpha_0-\alpha_1-2\alpha_2$	1	$-\alpha_{-1}-\alpha_0-4\alpha_1-6\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-3\alpha_1-4\alpha_2$	3
$-\alpha_{-1}-\alpha_0-\alpha_1-3\alpha_2$	1	$-\alpha_{-1}-\alpha_0-2\alpha_1-2\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-5\alpha_1-7\alpha_2$	3
$-\alpha_{-1}-\alpha_0-\alpha_1-3\alpha_2$	1	$-\alpha_{-1}-\alpha_0-2\alpha_1-3\alpha_2$	2	$-\alpha_{-1}-2\alpha_0-5\alpha_1-6\alpha_2$	2
$-\alpha_{-1}-\alpha_0-\alpha_1-3\alpha_2$	1	$-\alpha_{-1}-\alpha_0-3\alpha_1-3\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-4\alpha_1-6\alpha_2$	6
$-\alpha_{-1}-\alpha_0-\alpha_1-3\alpha_2$	1	$-\alpha_{-1}-\alpha_0-3\alpha_1-4\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-4\alpha_1-5\alpha_2$	3
$-\alpha_{-1}-\alpha_0-\alpha_1-3\alpha_2$	1	$-\alpha_{-1}-\alpha_0-3\alpha_1-5\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-4\alpha_1-4\alpha_2$	1
$-\alpha_{-1}-\alpha_0-\alpha_1-3\alpha_2$	1	$-\alpha_{-1}-\alpha_0-4\alpha_1-6\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-3\alpha_1-3\alpha_2$	2
$-\alpha_{-1}-\alpha_0-2\alpha_1-2\alpha_2$	1	$-\alpha_{-1}-\alpha_0-2\alpha_1-2\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-4\alpha_1-8\alpha_2$	1
$-\alpha_{-1}-\alpha_0-2\alpha_1-2\alpha_2$	1	$-\alpha_{-1}-\alpha_0-2\alpha_1-3\alpha_2$	2	$-\alpha_{-1}-2\alpha_0-4\alpha_1-7\alpha_2$	3
$-\alpha_{-1}-\alpha_0-2\alpha_1-2\alpha_2$	1	$-\alpha_{-1}-\alpha_0-2\alpha_1-4\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-4\alpha_1-6\alpha_2$	6
$-\alpha_{-1}-\alpha_0-2\alpha_1-2\alpha_2$	1	$-\alpha_{-1}-\alpha_0-3\alpha_1-4\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-3\alpha_1-6\alpha_2$	2
$-\alpha_{-1}-\alpha_0-2\alpha_1-2\alpha_2$	1	$-\alpha_{-1}-\alpha_0-3\alpha_1-5\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-3\alpha_1-5\alpha_2$	3
$-\alpha_{-1}-\alpha_0-2\alpha_1-2\alpha_2$	1	$-\alpha_{-1}-\alpha_0-3\alpha_1-6\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-3\alpha_1-4\alpha_2$	3
$-\alpha_{-1}-\alpha_0-2\alpha_1-2\alpha_2$	1	$-\alpha_{-1}-\alpha_0-4\alpha_1-6\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-2\alpha_1-4\alpha_2$	1
$-\alpha_{-1}-\alpha_0-2\alpha_1-3\alpha_2$	2	$-\alpha_{-1}-\alpha_0-2\alpha_1-3\alpha_2$	2	$-\alpha_{-1}-2\alpha_0-4\alpha_1-6\alpha_2$	6
$-\alpha_{-1}-\alpha_0-2\alpha_1-3\alpha_2$	2	$-\alpha_{-1}-\alpha_0-2\alpha_1-4\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-4\alpha_1-5\alpha_2$	3
$-\alpha_{-1}-\alpha_0-2\alpha_1-3\alpha_2$	2	$-\alpha_{-1}-\alpha_0-3\alpha_1-3\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-3\alpha_1-6\alpha_2$	2
$-\alpha_{-1}-\alpha_0-2\alpha_1-3\alpha_2$	2	$-\alpha_{-1}-\alpha_0-3\alpha_1-4\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-3\alpha_1-5\alpha_2$	3
$-\alpha_{-1}-\alpha_0-2\alpha_1-3\alpha_2$	2	$-\alpha_{-1}-\alpha_0-3\alpha_1-5\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-3\alpha_1-4\alpha_2$	3

Table 6.14 (continued)

τ_1	$dimV(\Lambda_0)_{\tau_1}$	τ_2	$dimV(\Lambda_0)_{\tau_2}$	τ_3	$dimV(\Lambda_0)_{\tau_3}$
$-\alpha_{-1}-\alpha_0-2\alpha_1-3\alpha_2$	2	$-\alpha_{-1}-\alpha_0-3\alpha_1-6\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-3\alpha_1-3\alpha_2$	2
$-\alpha_{-1}-\alpha_0-2\alpha_1-3\alpha_2$	2	$-\alpha_{-1}-\alpha_0-4\alpha_1-6\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-2\alpha_1-3\alpha_2$	2
$-\alpha_{-1}-\alpha_0-2\alpha_1-4\alpha_2$	1	$-\alpha_{-1}-\alpha_0-2\alpha_1-4\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-4\alpha_1-4\alpha_2$	1
$-\alpha_{-1}-\alpha_0-2\alpha_1-4\alpha_2$	1	$-\alpha_{-1}-\alpha_0-3\alpha_1-3\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-3\alpha_1-5\alpha_2$	3
$-\alpha_{-1}-\alpha_0-2\alpha_1-4\alpha_2$	1	$-\alpha_{-1}-\alpha_0-3\alpha_1-4\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-3\alpha_1-4\alpha_2$	3
$-\alpha_{-1}-\alpha_0-2\alpha_1-4\alpha_2$	1	$-\alpha_{-1}-\alpha_0-3\alpha_1-5\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-3\alpha_1-3\alpha_2$	2
$-\alpha_{-1}-\alpha_0-2\alpha_1-4\alpha_2$	1	$-\alpha_{-1}-\alpha_0-4\alpha_1-6\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-2\alpha_1-2\alpha_2$	1
$-\alpha_{-1}-\alpha_0-3\alpha_1-3\alpha_2$	1	$-\alpha_{-1}-\alpha_0-3\alpha_1-5\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-2\alpha_1-4\alpha_2$	1
$-\alpha_{-1}-\alpha_0-3\alpha_1-3\alpha_2$	1	$-\alpha_{-1}-\alpha_0-3\alpha_1-6\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-2\alpha_1-3\alpha_2$	2
$-\alpha_{-1}-\alpha_0-3\alpha_1-4\alpha_2$	1	$-\alpha_{-1}-\alpha_0-3\alpha_1-4\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-2\alpha_1-4\alpha_2$	1
$-\alpha_{-1}-\alpha_0-3\alpha_1-4\alpha_2$	1	$-\alpha_{-1}-\alpha_0-3\alpha_1-5\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-2\alpha_1-3\alpha_2$	2
$-\alpha_{-1}-\alpha_0-3\alpha_1-4\alpha_2$	1	$-\alpha_{-1}-\alpha_0-3\alpha_1-6\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-2\alpha_1-2\alpha_2$	1
$-\alpha_{-1}-\alpha_0-3\alpha_1-5\alpha_2$	1	$-\alpha_{-1}-\alpha_0-3\alpha_1-5\alpha_2$	1	$-\alpha_{-1}-2\alpha_0-2\alpha_1-2\alpha_2$	1

Table 6.15 Partitions of $\tau = -3\alpha_{-1} - 4\delta$ into $\tau_1 + \tau_2$ with $\tau_1 = -\alpha_{-1} - \sum_{j=0}^{2} m_{ij}\alpha_j$ and $\tau_2 = -2\alpha_{-1} - \sum_{j=0}^{2} \overline{m}_{ij}\alpha_j$ for $m_{ij}, \overline{m}_{ij} \in \mathbb{Z}_{\geq 0}$

τ_1	$dimV(\Lambda_0)_{\tau_1}$	τ_2	$dimV(\Lambda_1 - \delta)_{\tau_2}$
$-\alpha_{-1}$	1	$-2\alpha_{-1} - 4\alpha_0 - 8\alpha_1 - 12\alpha_2$	141
$-\alpha_{-1} - \alpha_0$	1	$-2\alpha_{-1} - 3\alpha_0 - 8\alpha_1 - 12\alpha_2$	22
$-\alpha_{-1} - \alpha_0 - \alpha_1$	1	$-2\alpha_{-1} - 3\alpha_0 - 7\alpha_1 - 12\alpha_2$	22
$-\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2$	1	$-2\alpha_{-1} - 3\alpha_0 - 7\alpha_1 - 11\alpha_2$	31
$-\alpha_{-1} - \alpha_0 - \alpha_1 - 2\alpha_2$	1	$-2\alpha_{-1} - 3\alpha_0 - 7\alpha_1 - 10\alpha_2$	31
$-\alpha_{-1} - \alpha_0 - \alpha_1 - 3\alpha_2$	1	$-2\alpha_{-1} - 3\alpha_0 - 7\alpha_1 - 9\alpha_2$	22
$-\alpha_{-1} - \alpha_0 - 2\alpha_1 - 2\alpha_2$	1	$-2\alpha_{-1} - 3\alpha_0 - 6\alpha_1 - 10\alpha_2$	31
$-\alpha_{-1} - \alpha_0 - 2\alpha_1 - 3\alpha_2$	2	$-2\alpha_{-1} - 3\alpha_0 - 6\alpha_1 - 9\alpha_2$	42
$-\alpha_{-1} - \alpha_0 - 2\alpha_1 - 4\alpha_2$	1	$-2\alpha_{-1} - 3\alpha_0 - 6\alpha_1 - 8\alpha_2$	31
$-\alpha_{-1} - \alpha_0 - 3\alpha_1 - 3\alpha_2$	1	$-2\alpha_{-1} - 3\alpha_0 - 5\alpha_1 - 9\alpha_2$	22
$-\alpha_{-1} - \alpha_0 - 3\alpha_1 - 4\alpha_2$	1	$-2\alpha_{-1} - 3\alpha_0 - 5\alpha_1 - 8\alpha_2$	31
$-\alpha_{-1} - \alpha_0 - 3\alpha_1 - 5\alpha_2$	1	$-2\alpha_{-1} - 3\alpha_0 - 5\alpha_1 - 7\alpha_2$	31
$-\alpha_{-1} - \alpha_0 - 3\alpha_1 - 6\alpha_2$	1	$-2\alpha_{-1} - 3\alpha_0 - 5\alpha_1 - 6\alpha_2$	22
$-\alpha_{-1} - \alpha_0 - 4\alpha_1 - 6\alpha_2$	1	$-2\alpha_{-1} - 3\alpha_0 - 4\alpha_1 - 6\alpha_2$	22
$-\alpha_{-1} - 2\alpha_0 - 2\alpha_1 - 2\alpha_2$	1	$-2\alpha_{-1} - 2\alpha_0 - 6\alpha_1 - 10\alpha_2$	3
$-\alpha_{-1} - 2\alpha_0 - 2\alpha_1 - 3\alpha_2$	2	$-2\alpha_{-1} - 2\alpha_0 - 6\alpha_1 - 9\alpha_2$	5
$-\alpha_{-1} - 2\alpha_0 - 2\alpha_1 - 4\alpha_2$	1	$-2\alpha_{-1} - 2\alpha_0 - 6\alpha_1 - 8\alpha_2$	3
$-\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - 3\alpha_2$	2	$-2\alpha_{-1} - 2\alpha_0 - 5\alpha_1 - 9\alpha_2$	5
$-\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - 4\alpha_2$	3	$-2\alpha_{-1} - 2\alpha_0 - 5\alpha_1 - 8\alpha_2$	7
$-\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - 5\alpha_2$	3	$-2\alpha_{-1} - 2\alpha_0 - 5\alpha_1 - 7\alpha_2$	7
$-\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - 6\alpha_2$	2	$-2\alpha_{-1} - 2\alpha_0 - 5\alpha_1 - 6\alpha_2$	5
$-\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - 4\alpha_2$	1	$-2\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - 8\alpha_2$	3
$-\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - 5\alpha_2$	3	$-2\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - 7\alpha_2$	7
$-\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - 6\alpha_2$	6	$-2\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - 6\alpha_2$	11
$-\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - 7\alpha_2$	3	$-2\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - 5\alpha_2$	7
$-\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - 8\alpha_2$	1	$-2\alpha_{-1} - 2\alpha_0 - 4\alpha_1 - 4\alpha_2$	3
$-\alpha_{-1} - 2\alpha_0 - 5\alpha_1 - 6\alpha_2$	2	$-2\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - 6\alpha_2$	5
$-\alpha_{-1} - 2\alpha_0 - 5\alpha_1 - 7\alpha_2$	3	$-2\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - 5\alpha_2$	7
$-\alpha_{-1} - 2\alpha_0 - 5\alpha_1 - 8\alpha_2$	3	$-2\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - 4\alpha_2$	7
$-\alpha_{-1} - 2\alpha_0 - 5\alpha_1 - 9\alpha_2$	2	$-2\alpha_{-1} - 2\alpha_0 - 3\alpha_1 - 3\alpha_2$	5
$-\alpha_{-1} - 2\alpha_0 - 6\alpha_1 - 8\alpha_2$	1	$-2\alpha_{-1} - 2\alpha_0 - 2\alpha_1 - 4\alpha_2$	3
$-\alpha_{-1} - 2\alpha_0 - 6\alpha_1 - 9\alpha_2$	2	$-2\alpha_{-1} - 2\alpha_0 - 2\alpha_1 - 3\alpha_2$	5
$-\alpha_{-1} - 2\alpha_0 - 6\alpha_1 - 10\alpha_2$	1	$-2\alpha_{-1} - 2\alpha_0 - 2\alpha_1 - 2\alpha_2$	3
$-\alpha_{-1} - 3\alpha_0 - 4\alpha_1 - 6\alpha_2$	6	$-2\alpha_{-1} - \alpha_0 - 4\alpha_1 - 6\alpha_2$	1
$-\alpha_{-1} - 3\alpha_0 - 5\alpha_1 - 6\alpha_2$	6	$-2\alpha_{-1} - \alpha_0 - 3\alpha_1 - 6\alpha_2$	1
$-\alpha_{-1} - 3\alpha_0 - 5\alpha_1 - 7\alpha_2$	9	$-2\alpha_{-1} - \alpha_0 - 3\alpha_1 - 5\alpha_2$	1
$-\alpha_{-1} - 3\alpha_0 - 5\alpha_1 - 8\alpha_2$	9	$-2\alpha_{-1} - \alpha_0 - 3\alpha_1 - 4\alpha_2$	1
$-\alpha_{-1} - 3\alpha_0 - 5\alpha_1 - 9\alpha_2$	6	$-2\alpha_{-1} - \alpha_0 - 3\alpha_1 - 3\alpha_2$	1
$-\alpha_{-1} - 3\alpha_0 - 6\alpha_1 - 8\alpha_2$	9	$-2\alpha_{-1} - \alpha_0 - 2\alpha_1 - 4\alpha_2$	1
$-\alpha_{-1} - 3\alpha_0 - 6\alpha_1 - 9\alpha_2$	14	$-2\alpha_{-1} - \alpha_0 - 2\alpha_1 - 3\alpha_2$	2
$-\alpha_{-1} - 3\alpha_0 - 6\alpha_1 - 10\alpha_2$	9	$-2\alpha_{-1} - \alpha_0 - 2\alpha_1 - 2\alpha_2$	1
$-\alpha_{-1} - 3\alpha_0 - 7\alpha_1 - 9\alpha_2$	6	$-2\alpha_{-1} - \alpha_0 - \alpha_1 - 3\alpha_2$	1
$-\alpha_{-1} - 3\alpha_0 - 7\alpha_1 - 10\alpha_2$	9	$-2\alpha_{-1} - \alpha_0 - \alpha_1 - 2\alpha_2$	1
$-\alpha_{-1} - 3\alpha_0 - 7\alpha_1 - 11\alpha_2$	9	$-2\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2$	1
$-\alpha_{-1} - 3\alpha_0 - 7\alpha_1 - 12\alpha_2$	6	$-2\alpha_{-1} - \alpha_0 - \alpha_1$	1
$-\alpha_{-1} - 3\alpha_0 - 8\alpha_1 - 12\alpha_2$	6	$-2\alpha_{-1} - \alpha_0$	1

Putting result (6.18) together with the results from Table 6.14 and Table 6.15, we determine the multiplicity of $-3\alpha_{-1} - 4\delta$.

$$\begin{aligned}
dim(\mathfrak{g}_{-3\alpha_{-1}-4\delta}) &= \mathcal{B}(-3\alpha_{-1} - 4\delta) \\
&= \sum_{(n_i,\tau_i)\in T(-3\alpha_{-1}-4\delta)} \frac{(\sum n_i - 1)!}{\prod n_i!} \prod K_{\tau_i}^{n_i} \\
&= \left(\sum_{\substack{\tau_1+\tau_2+\tau_3=-3\alpha_{-1}-4\delta \\ \tau_i=-\alpha_{-1}-\sum m_{ij}\alpha_j \\ \tau_1\le\tau_2\le\tau_3}} \frac{(\sum n_i - 1)!}{\prod n_i!} \prod K_{\tau_i}^{n_i} \right) \\
&\quad + \left(\sum_{\substack{\tau_1+\tau_2=-3\alpha_{-1}-4\delta \\ \tau_1=-\alpha_{-1}-\sum m_j\alpha_j \\ \tau_2=-2\alpha_{-1}-\sum \overline{m}_j\alpha_j}} \frac{(2-1)!}{1!\cdot 1!} \prod_{i=1}^{2} K_{\tau_i} \right) \\
&\quad + \frac{(1-1)!}{1!} K_{-3\alpha_{-1}-4\delta} \\
&= \left(\sum_{\substack{\tau_1+\tau_2+\tau_3=-3\alpha_{-1}-4\delta \\ \tau_i=-\alpha_{-1}-\sum m_{ij}\alpha_j \\ \tau_1\le\tau_2\le\tau_3}} \frac{(\sum n_i - 1)!}{\prod n_i!} \prod (dimV(\Lambda_0)_{\tau_i})^{n_i} \right) \\
&\quad - \left(\sum_{\substack{\tau_1+\tau_2=-3\alpha_{-1}-4\delta \\ \tau_1=-\alpha_{-1}-\sum m_j\alpha_j \\ \tau_2=-2\alpha_{-1}-\sum \overline{m}_j\alpha_j}} dimV(\Lambda_0)_{\tau_1} dimV(\Lambda_1 - \delta)_{\tau_2} \right) \\
&\quad + dimV(3\Lambda_2 - 2\delta)_{-3\alpha_{-1}-4\delta} \\
&= 790 - 931 + 173 \\
&= 32
\end{aligned}$$

Example 6.3.9 (Multiplicity of $-3\alpha_{-1} - 5\delta$)**.** In this example we determine the multiplicity of $\alpha = -3\alpha_{-1} - 5\delta$. Since $3 \nmid 5$, we only have one divisor of α, which is $\tau = \alpha$. Then Kang's formula implies

$$\begin{aligned}
dim(\mathfrak{g}_{-3\alpha_{-1}-5\delta}) &= \sum_{\tau|\alpha} \mu(\frac{\alpha}{\tau}) \left(\frac{\tau}{\alpha}\right) \mathcal{B}(\tau) \\
&= \mu\left(\frac{-3\alpha_{-1}-5\delta}{-3\alpha_{-1}-5\delta}\right)\left(\frac{-3\alpha_{-1}-5\delta}{-3\alpha_{-1}-5\delta}\right)\mathcal{B}(-3\alpha_{-1}-5\delta) \\
&= \mathcal{B}(-3\alpha_{-1}-5\delta).
\end{aligned}$$

In order to calculate $\mathcal{B}(-3\alpha_{-1}-5\delta)$, we sum over the partitions of $\tau = -3\alpha_{-1}-5\delta$ for which K_{τ_i} is nonzero for each τ_i. As in Example 6.3.6 the partitions fall into three categories:

1. a partition of τ into three level 1 roots of the form $\tau_i = -\alpha_{-1} - \sum_{j=0}^{2} m_{ij}\alpha_j$ where $m_{ij} \in \mathbb{Z}_{\geq 0}$,
2. a partition of τ into a level 1 of form $\tau_1 = -\alpha_{-1} - \sum_{j=0}^{2} m_{ij}\alpha_j$ and a level 2 root of form $\tau_2 = -2\alpha_{-1} - \sum_{j=0}^{2} \overline{m}_{ij}\alpha_j$ where $m_{ij}, \overline{m}_{ij} \in \mathbb{Z}_{\geq 0}$, and
3. the partition $\tau = -3\alpha_{-1} - 5\delta$.

For brevity, the table of partitions for categories 1 and 2 are omitted. The level 1 and 2 roots for these partitions were calculated using our C# program in the same manner as in Example 6.3.6 and are later used to find the multiplicity of $-3\alpha_{-1}-5\delta$. For the partition $\tau = -3\alpha_{-1}-5\delta$, we use our C# program to count the paths of $\mathcal{P}(3\Lambda_2) \cong V(3\Lambda_2)$ and the isomorphism $V(3\Lambda_2 - 2\delta) \cong V(3\Lambda_2) \otimes V(2\delta)$ to calculate $dimV(3\Lambda_2 - 2\delta)_\tau$ below.

$$\begin{aligned}
dimV(3\Lambda_2 - 2\delta)_{-3\alpha_{-1}-5\delta} &= dimV(3\Lambda_2)_{-3\alpha_{-1}-3\delta} \\
&= dimV(3\Lambda_2)_{3\Lambda_2+2\alpha_0+\alpha_1-5\delta} \\
&= dimV(3\Lambda_2)_{3\Lambda_2-3\alpha_0-9\alpha_1-15\alpha_2} \\
&= 686
\end{aligned}$$

Putting this result together with the results from category 1 and 2 partitions, we determine the multiplicity of $-3\alpha_{-1}-5\delta$.

$$\begin{aligned}
dim(\mathfrak{g}_{-3\alpha_{-1}-5\delta}) &= \mathcal{B}(-3\alpha_{-1}-5\delta) \\
&= \sum_{(n_i,\tau_i)\in T(-3\alpha_{-1}-5\delta)} \frac{(\sum n_i - 1)!}{\prod n_i!} \prod K_{\tau_i}^{n_i}
\end{aligned}$$

$$= \left(\sum_{\substack{\tau_1+\tau_2+\tau_3=-3\alpha_{-1}-5\delta \\ \tau_i=-\alpha_{-1}-\sum m_{ij}\alpha_j \\ \tau_1\leq\tau_2\leq\tau_3}} \frac{(\sum n_i - 1)!}{\prod n_i!} \prod K_{\tau_i}^{n_i} \right)$$
$$+ \left(\sum_{\substack{\tau_1+\tau_2=-3\alpha_{-1}-5\delta \\ \tau_1=-\alpha_{-1}-\sum m_j\alpha_j \\ \tau_2=-2\alpha_{-1}-\sum \overline{m}_j\alpha_j}} \frac{(2-1)!}{1!\cdot 1!} \prod_{i=1}^{2} K_{\tau_i} \right)$$
$$+ \frac{(1-1)!}{1!} K_{-3\alpha_{-1}-5\delta}$$
$$= \left(\sum_{\substack{\tau_1+\tau_2+\tau_3=-3\alpha_{-1}-5\delta \\ \tau_i=-\alpha_{-1}-\sum m_{ij}\alpha_j \\ \tau_1\leq\tau_2\leq\tau_3}} \frac{(\sum n_i - 1)!}{\prod n_i!} \prod (dimV(\Lambda_0)_{\tau_i})^{n_i} \right)$$
$$- \left(\sum_{\substack{\tau_1+\tau_2=-3\alpha_{-1}-5\delta \\ \tau_1=-\alpha_{-1}-\sum m_j\alpha_j \\ \tau_2=-2\alpha_{-1}-\sum \overline{m}_j\alpha_j}} dimV(\Lambda_0)_{\tau_1} dimV(\Lambda_1-\delta)_{\tau_2} \right)$$
$$+ dimV(3\Lambda_2 - 2\delta)_{-3\alpha_{-1}-5\delta}$$
$$= 3628 - 4056 + 686$$
$$= 258$$

Recall once more Theorem 4.1.3 from Section 4.1: For $\ell, k \in \mathbb{Z}_{>0}$ and $\alpha = m_1\alpha_1 + m_2\alpha_2 \in \Delta_S^+ \cup \{0\}$, we have $mult(-\ell\alpha_{-1} - \alpha - k\delta) = mult(-(k-\ell)\alpha_{-1} - \alpha - k\delta)$. Applying this theorem to $-3\alpha_{-1} - 4\delta$ and $-3\alpha_{-1} - 5\delta$, we obtain

$$mult(-3\alpha_{-1} - 4\delta) = mult(-\alpha_{-1} - 4\delta) = 32$$

$$mult(-3\alpha_{-1} - 5\delta) = mult(-2\alpha_{-1} - 5\delta) = 258.$$

Observe that these results agree with our calculations of $mult(-3\alpha_{-1}-4\delta)$ and $mult(-3\alpha_{-1}-5\delta)$ using Kang's formula in Example 6.3.6 and Example 6.3.9.

Example 6.3.10 (Multiplicity of $-3\alpha_{-1}-6\delta$)**.** Lastly, we find the multiplicity of $\alpha = -3\alpha_{-1}-6\delta$. In this case, $3|6$ and so α has two divisors: $\tau = -\alpha_{-1} - 3\delta$ and $\tau = -3\alpha_{-1} - 6\delta$. Using Kang's multiplicity formula, the $dim(\mathfrak{g}_{-3\alpha_{-1}-6\delta})$ has two terms:

$$\begin{aligned} dim(\mathfrak{g}_{-3\alpha_{-1}-6\delta}) &= \sum_{\tau|\alpha} \mu(\frac{\alpha}{\tau}) \left(\frac{\tau}{\alpha}\right) \mathcal{B}(\tau) \\ &= \mu\left(\frac{-3\alpha_{-1}-6\delta}{-3\alpha_{-1}-6\delta}\right)\left(\frac{-3\alpha_{-1}-6\delta}{-3\alpha_{-1}-6\delta}\right)\mathcal{B}(-3\alpha_{-1}-6\delta) \\ &\quad + \mu\left(\frac{-3\alpha_{-1}-6\delta}{-\alpha_{-1}-2\delta}\right)\left(\frac{-\alpha_{-1}-2\delta}{-3\alpha_{-1}-6\delta}\right)\mathcal{B}(-\alpha_{-1}-2\delta) \\ &= \mathcal{B}(-3\alpha_{-1}-6\delta) - \frac{1}{3}\mathcal{B}(-\alpha_{-1}-2\delta) \end{aligned}$$

We start by calculating $\mathcal{B}(-\alpha_{-1}-2\delta)$ using the level 1 result for $-\alpha_{-1}-2\delta$.

$$\begin{aligned} \mathcal{B}(-\alpha_{-1}-2\delta) &= \frac{(1-1)!}{1!}K_{-\alpha_{-1}-2\delta} \\ &= dimV(\Lambda_0)_{-\alpha_{-1}-2\delta} \\ &= 6 \end{aligned} \tag{6.19}$$

As in the previous example, we calculate $\mathcal{B}(-3\alpha_{-1}-6\delta)$ by summing over the partitions of $\tau = -3\alpha_{-1}-6\delta$ falling into three categories:

1. a partition of τ into three level 1 roots of the form $\tau_i = -\alpha_{-1} - \sum_{j=0}^{2} m_{ij}\alpha_j$ where $m_{ij} \in \mathbb{Z}_{\geq 0}$,
2. a partition of τ into a level 1 of form $\tau_1 = -\alpha_{-1} - \sum_{j=0}^{2} m_{ij}\alpha_j$ and a level 2 root of form $\tau_2 = -2\alpha_{-1} - \sum_{j=0}^{2} \overline{m}_{ij}\alpha_j$ where $m_{ij}, \overline{m}_{ij} \in \mathbb{Z}_{\geq 0}$, and
3. the partition $\tau = -3\alpha_{-1} - 6\delta$.

The tables for categories 1 and 2 are omitted and we compute the $dimV(3\Lambda_2 - 2\delta)_\tau$ using our C# program, with results shown below.

$$\begin{aligned} dimV(3\Lambda_2-2\delta)_{-3\alpha_{-1}-6\delta} &= dimV(3\Lambda_2)_{-3\alpha_{-1}-4\delta} \\ &= dimV(3\Lambda_2)_{3\Lambda_2+2\alpha_0+\alpha_1-6\delta} \\ &= dimV(3\Lambda_2)_{3\Lambda_2-4\alpha_0-11\alpha_1-18\alpha_2} \\ &= 2355 \end{aligned}$$

Then,

$$
\begin{aligned}
\mathcal{B}(-3\alpha_{-1}-6\delta) &= \sum_{(n_i,\tau_i)\in T(-3\alpha_{-1}-6\delta)} \frac{(\sum n_i - 1)!}{\prod n_i!} \prod K_{\tau_i}^{n_i} \\
&= \left(\sum_{\substack{\tau_1+\tau_2+\tau_3=-3\alpha_{-1}-6\delta \\ \tau_i=-\alpha_{-1}-\sum m_{ij}\alpha_j \\ \tau_1\le\tau_2\le\tau_3}} \frac{(\sum n_i - 1)!}{\prod n_i!} \prod K_{\tau_i}^{n_i} \right) \\
&\quad + \left(\sum_{\substack{\tau_1+\tau_2=-3\alpha_{-1}-6\delta \\ \tau_1=-\alpha_{-1}-\sum m_j\alpha_j \\ \tau_2=-2\alpha_{-1}-\sum \overline{m}_j\alpha_j}} \frac{(2-1)!}{1!\cdot 1!} \prod_{i=1}^{2} K_{\tau_i} \right) \\
&\quad + \frac{(1-1)!}{1!} K_{-3\alpha_{-1}-6\delta} \\
&= \left(\sum_{\substack{\tau_1+\tau_2+\tau_3=-3\alpha_{-1}-6\delta \\ \tau_i=-\alpha_{-1}-\sum m_{ij}\alpha_j \\ \tau_1\le\tau_2\le\tau_3}} \frac{(\sum n_i - 1)!}{\prod n_i!} \prod (dimV(\Lambda_0)_{\tau_i})^{n_i} \right) \\
&\quad - \left(\sum_{\substack{\tau_1+\tau_2=-3\alpha_{-1}-6\delta \\ \tau_1=-\alpha_{-1}-\sum m_j\alpha_j \\ \tau_2=-2\alpha_{-1}-\sum \overline{m}_j\alpha_j}} dimV(\Lambda_0)_{\tau_1} dimV(\Lambda_1-\delta)_{\tau_2} \right) \\
&\quad + dimV(3\Lambda_2-2\delta)_{-3\alpha_{-1}-6\delta} \\
&= 14739 - 15533 + 2355 \\
&= 1561
\end{aligned} \tag{6.20}
$$

Combining results from (6.19) and (6.20), we determine the multiplicity of $-3\alpha_{-1}-6\delta$.

$$
\begin{aligned}
dim(\mathfrak{g}_{-3\alpha_{-1}-6\delta}) &= \mathcal{B}(-3\alpha_{-1}-6\delta) - \frac{1}{3}\mathcal{B}(-\alpha_{-1}-2\delta) \\
&= 1561 - \frac{1}{3}(6) \\
&= 1559
\end{aligned}
$$

In Table 6.16 through Table 6.21 to follow, we list some additional level 3 multiplicities, which were calculated using our C# program for level 3 roots.

Table 6.16 Multiplicity table for $-3\alpha_{-1} - k\delta$ when $3 \leq k \leq 10$

$-3\alpha_{-1} - k\delta$	$mult(-3\alpha_{-1} - k\delta)$	Frenkel Bound
$k = 3$	2	2
$k = 4$	32	481
$k = 5$	258	17490
$k = 6$	1561	326015
$k = 7$	7638	4126070
$k = 8$	32397	40210481
$k = 9$	122760	323275512
$k = 10$	425788	2238075315

Table 6.17 Multiplicity table for $-3\alpha_{-1} - \alpha_0 - k\delta$ when $1 \leq k \leq 10$

$-3\alpha_{-1} - \alpha_0 - k\delta$	$mult(-3\alpha_{-1} - \alpha_0 - k\delta)$	Frenkel Bound
$k = 1$	0	1
$k = 2$	1	1
$k = 3$	14	110
$k = 4$	135	5822
$k = 5$	878	129512
$k = 6$	4584	1831065
$k = 7$	20287	19283830
$k = 8$	79608	164363280
$k = 9$	283746	1191580872
$k = 10$	936048	7592053897

Table 6.18 Multiplicity table for $-3\alpha_{-1} - \alpha_0 - \alpha_2 - k\delta$ when $1 \le k \le 10$

$-3\alpha_{-1} - \alpha_0 - \alpha_2 - k\delta$	$mult(-3\alpha_{-1} - \alpha_0 - \alpha_2 - k\delta)$	Frenkel Bound
$k = 1$	0	1
$k = 2$	0	1
$k = 3$	9	65
$k = 4$	99	3956
$k = 5$	702	94235
$k = 6$	3785	1386930
$k = 7$	17169	15018300
$k = 8$	68379	130673928
$k = 9$	246656	962759294
$k = 10$	820932	6214880700

Table 6.19 Multiplicity table for $-3\alpha_{-1} - \alpha_0 - 2\alpha_2 - k\delta$ when $1 \le k \le 10$

$-3\alpha_{-1} - \alpha_0 - 2\alpha_2 - k\delta$	$mult(-3\alpha_{-1} - \alpha_0 - 2\alpha_2 - k\delta)$	Frenkel Bound
$k = 1$	0	1
$k = 2$	0	1
$k = 3$	3	10
$k = 4$	48	1165
$k = 5$	378	35002
$k = 6$	2210	589128
$k = 7$	10537	6978730
$k = 8$	43677	64854575
$k = 9$	162389	502810130
$k = 10$	554261	3381689157

Table 6.20 Multiplicity table for $-3\alpha_{-1} - \alpha_0 - 2\alpha_1 - k\delta$ when $1 \le k \le 10$

$-3\alpha_{-1} - \alpha_0 - 2\alpha_1 - k\delta$	$mult(-3\alpha_{-1} - \alpha_0 - 2\alpha_1 - k\delta)$	Frenkel Bound
$k = 1$	0	1
$k = 2$	0	1
$k = 3$	2	2
$k = 4$	32	481
$k = 5$	258	17490
$k = 6$	1559	326015
$k = 7$	7637	4126070
$k = 8$	32395	40210481
$k = 9$	122755	323275512
$k = 10$	425777	2238075315

Table 6.21 Multiplicity table for $-3\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2 - k\delta$ when $1 \leq k \leq 10$

$-3\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2 - k\delta$	$mult(-3\alpha_{-1} - \alpha_0 - \alpha_1 - \alpha_2 - k\delta)$	Frenkel Bound
$k = 1$	0	1
$k = 2$	1	1
$k = 3$	21	300
$k = 4$	199	12230
$k = 5$	1258	240840
$k = 6$	6377	3157789
$k = 7$	27551	31551450
$k = 8$	105939	258508230
$k = 9$	371252	1816715170
$k = 10$	1207170	11285536125

BIBLIOGRAPHY

[Ben93] Benkart, G. et al. "Graded Lie Algebras of Kac-Moody Type". *Advances In Mathematics* **97**.2 (1993), pp. 154–190.

[Ben94] Benkart, G. et al. "Indefinite Kac-Moody Algebras of Classical Type". *Advances In Mathematics* **105** (1994), pp. 76–110.

[Dre85] Drenfel'd, V. G. "Hopf algebras and the quantum Yang-Baxter equation". *Soviet Mathematics* **32**.1 (1985), pp. 254–258.

[Erb12] Erbacher, C. E. "Root Multiplicities of the Indefinite Kac-Moody Algebra $HD_4^{(3)}$". PhD thesis. North Carolina State University, 2012.

[Fre85] Frenkel, I. "Representations of Kac-Moody algebras and dual resonance models". *Applications of group theory in physics and mathematical physics*. Ed. by Flato, M. et al. Providence, R.I.: American Mathematical Society, 1985. Chap. 4, pp. 325–353.

[HK02] Hong, J. & Kang, S.-J. *Introduction to Quantum Groups and Crystal Bases*. American Mathematical Society, 2002.

[HM02] Hontz, J. & Misra, K. C. "On root multiplicities of $HA_n^{(1)}$". *International Journal of Algebra and Computation* **12**.3 (2002), pp. 477–508.

[Hon98] Hontz, J. M. "Root multiplicities of some Kac-Moody lie algebras of indefinite type". PhD thesis. North Carolina State University, 1998.

[Hum72] Humphreys, J. E. *Introduction to Lie Algebras and Representation Theory*. Springer Science+Business Media, Inc., 1972.

[Jim88] Jimbo, M. "A q-difference analogue of $U(\mathfrak{g})$ and the Yang-Baxter equation". *Advances In Mathematics* **70** (1988), pp. 237–249.

[Jim91] Jimbo, M. et al. "Combinatorics of Representations of $U_q(\hat{\mathfrak{sl}}(n))$ at $q = 0$". *Communications in Mathematical Physics* **136** (1991), pp. 543–566.

[Kac90] Kac, V. G. *Infinite dimensional Lie algebras*. 3rd Ed. Cambridge University Press, 1990.

[Kan94] Kang, S.-J. "Root multiplicities of Kac-Moody algebras". *Duke Math* **J.74** (1994), pp. 635–666.

[KM94] Kang, S.-J. & Melville, D. J. "Root multiplicities of the Kac-Moody Algebras $HA_n^{(1)}$". *Journal of Algebra* **170** (1994), pp. 277–299.

[Kas90] Kashiwara, M. “Crystallizing the q-analogue of universal enveloping algebras”. *Communications in Mathematical Physics* **133** (1990), pp. 249–260.

[Kas91] Kashiwara, M. “On crystal bases of the q-analogue of universal enveloping algebras”. *Duke Mathematical Journal* **63** (1991), pp. 465–516.

[Lus85] Lusztig, G. “Quantum deformation of certain simple modules over enveloping algebras”. *Letters in Mathematical Physics* **10**.1 (1985), pp. 68–70.

[Lus90] Lusztig, G. “Canonical bases arising from quantized enveloping algebras”. *Journal of the American Mathematical Society* **3**.2 (1990), pp. 447–498.

[Mis10] Misra, K. C. et al. “Zero Action on Perfect Crystals for $U_q(G_2^{(1)})$”. *Symmetry, integrability and geometry, methods and applications* **6** (2010), pp. 18–30.

[Mis12] Misra, K. C. “Lecture Notes in Lie Algebra”. unpublished. 2012.

[Wil03] Williams, V. “Root Multiplicities of the indefinite Kac-Moody algebras $HC_n^{(1)}$”. PhD thesis. North Carolina State University, 2003.

[Wil12] Wilson, E. A. “Root Multiplicities of the Indefinite Type Kac-Moody Algebra $HD_n^{(1)}$”. PhD thesis. North Carolina State University, 2012.

APPENDIX

APPENDIX

A

DATA FOR $\mathcal{B}_2$ AND $\mathcal{B}_3$

Table A.1 Data from the level 2 crystal graph $\mathcal{B}_2$

b	$\left(\tilde{f}_0(b), \tilde{f}_1(b), \tilde{f}_2(b)\right)$	$\left(\varepsilon_0(b), \varepsilon_1(b), \varepsilon_2(b)\right)$	$\left(\varphi_0(b), \varphi_1(b), \varphi_2(b)\right)$
$b_0 = (0,0,0,0,0,0)$	$(b_{77},0,0)$	$(2,0,0)$	$(2,0,0)$
$b_1 = (0,0,0,0,0,1)$	$(b_0,0,0)$	$(1,1,0)$	$(3,0,0)$
$b_2 = (0,0,0,0,0,2)$	$(b_1,0,0)$	$(0,2,0)$	$(4,0,0)$
$b_3 = (0,0,0,0,1,0)$	$(b_{27},b_1,0)$	$(1,0,3)$	$(2,1,0)$
$b_4 = (0,0,0,0,1,1)$	$(b_3,b_2,0)$	$(0,1,3)$	$(3,1,0)$
$b_5 = (0,0,0,0,2,0)$	$(b_{29},b_4,0)$	$(0,0,6)$	$(2,2,0)$
$b_6 = (0,0,0,\frac{2}{3},\frac{2}{3},0)$	$(b_{45},0,b_3)$	$(1,0,2)$	$(2,0,1)$
$b_7 = (0,0,0,\frac{2}{3},\frac{2}{3},1)$	$(b_6,0,b_4)$	$(0,1,2)$	$(3,0,1)$
$b_8 = (0,0,0,\frac{2}{3},\frac{5}{3},0)$	(b_{47},b_7,b_5)	$(0,0,5)$	$(2,1,1)$
$b_9 = (0,0,0,\frac{4}{3},\frac{1}{3},0)$	$(b_{54},0,b_6)$	$(1,1,1)$	$(2,0,2)$
$b_{10} = (0,0,0,\frac{4}{3},\frac{1}{3},1)$	$(b_9,0,b_7)$	$(0,2,1)$	$(3,0,2)$
$b_{11} = (0,0,0,\frac{4}{3},\frac{4}{3},0)$	$(b_{56},0,b_8)$	$(0,0,4)$	$(2,0,2)$
$b_{12} = (0,0,0,2,0,0)$	$(b_{63},0,b_9)$	$(1,2,0)$	$(2,0,3)$
$b_{13} = (0,0,0,2,0,1)$	$(b_{12},0,b_{10})$	$(0,3,0)$	$(3,0,3)$
$b_{14} = (0,0,0,2,1,0)$	$(b_{65},0,b_{11})$	$(0,1,3)$	$(2,0,3)$
$b_{15} = (0,0,0,\frac{8}{3},\frac{2}{3},0)$	$(b_{66},0,b_{14})$	$(0,2,2)$	$(2,0,4)$
$b_{16} = (0,0,0,\frac{10}{3},\frac{1}{3},0)$	$(b_{67},0,b_{15})$	$(0,3,1)$	$(2,0,5)$

Table A.1 (continued)

$b_{17} = (0,0,0,4,0,0)$	$(b_{68},0,b_{16})$	$(0,4,0)$	$(2,0,6)$
$b_{18} = (0,0,1,\frac{1}{3},\frac{1}{3},0)$	$(b_{83},b_9,0)$	$(1,0,2)$	$(1,1,0)$
$b_{19} = (0,0,1,\frac{1}{3},\frac{1}{3},1)$	$(b_{18},b_{10},0)$	$(0,1,2)$	$(2,1,0)$
$b_{20} = (0,0,1,\frac{1}{3},\frac{4}{3},0)$	$(b_{33},b_{19},0)$	$(0,0,5)$	$(1,2,0)$
$b_{21} = (0,0,1,1,0,0)$	$(b_{84},b_{12},0)$	$(1,1,0)$	$(1,1,0)$
$b_{22} = (0,0,1,1,0,1)$	$(b_{21},b_{13},0)$	$(0,2,0)$	$(2,1,0)$
$b_{23} = (0,0,1,1,1,0)$	$(b_{79},b_{14},0)$	$(0,0,3)$	$(1,1,0)$
$b_{24} = (0,0,1,\frac{5}{3},\frac{2}{3},0)$	(b_{80},b_{15},b_{23})	$(0,1,2)$	$(1,1,1)$
$b_{25} = (0,0,1,\frac{7}{3},\frac{1}{3},0)$	(b_{81},b_{16},b_{24})	$(0,2,1)$	$(1,1,2)$
$b_{26} = (0,0,1,3,0,0)$	(b_{82},b_{17},b_{25})	$(0,3,0)$	$(1,1,3)$
$b_{27} = (0,0,2,0,0,0)$	$(b_{85},b_{21},0)$	$(2,0,3)$	$(1,2,0)$
$b_{28} = (0,0,2,0,0,1)$	$(b_{34},b_{22},0)$	$(0,1,3)$	$(1,2,0)$
$b_{29} = (0,0,2,0,1,0)$	$(b_{35},b_{28},0)$	$(1,0,6)$	$(1,3,0)$
$b_{30} = (0,0,2,\frac{2}{3},\frac{2}{3},0)$	$(0,b_{24},0)$	$(0,0,4)$	$(0,2,0)$
$b_{31} = (0,0,2,\frac{4}{3},\frac{1}{3},0)$	$(0,b_{25},0)$	$(0,1,2)$	$(0,2,0)$
$b_{32} = (0,0,2,2,0,0)$	$(0,b_{26},0)$	$(0,2,0)$	$(0,2,0)$
$b_{33} = (0,0,3,\frac{1}{3},\frac{1}{3},0)$	$(0,b_{31},0)$	$(1,0,5)$	$(0,3,0)$
$b_{34} = (0,0,3,1,0,0)$	$(0,b_{32},0)$	$(1,1,3)$	$(0,3,0)$
$b_{35} = (0,0,4,0,0,0)$	$(0,b_{34},0)$	$(2,0,6)$	$(0,4,0)$
$b_{36} = (0,\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},0)$	$(b_{86},0,b_{18})$	$(1,0,1)$	$(1,0,1)$
$b_{37} = (0,\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},1)$	$(b_{36},0,b_{19})$	$(0,1,1)$	$(2,0,1)$
$b_{38} = (0,\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{4}{3},0)$	(b_{51},b_{37},b_{20})	$(0,0,4)$	$(1,1,1)$
$b_{39} = (0,\frac{1}{3},\frac{1}{3},1,0,0)$	$(b_{87},0,b_{36})$	$(1,1,0)$	$(1,0,2)$
$b_{40} = (0,\frac{1}{3},\frac{1}{3},1,0,1)$	$(b_{39},0,b_{37})$	$(0,2,0)$	$(2,0,2)$
$b_{41} = (0,\frac{1}{3},\frac{1}{3},1,1,0)$	$(b_{60},0,b_{38})$	$(0,0,3)$	$(1,0,2)$
$b_{42} = (0,\frac{1}{3},\frac{1}{3},\frac{5}{3},\frac{2}{3},0)$	$(b_{69},0,b_{41})$	$(0,1,2)$	$(1,0,3)$
$b_{43} = (0,\frac{1}{3},\frac{1}{3},\frac{7}{3},\frac{1}{3},0)$	$(b_{72},0,b_{42})$	$(0,2,1)$	$(1,0,4)$
$b_{44} = (0,\frac{1}{3},\frac{1}{3},3,0,0)$	$(b_{73},0,b_{43})$	$(0,3,0)$	$(1,0,5)$
$b_{45} = (0,\frac{1}{3},\frac{4}{3},0,0,0)$	(b_{88},b_{39},b_{27})	$(2,0,2)$	$(1,1,1)$
$b_{46} = (0,\frac{1}{3},\frac{4}{3},0,0,1)$	(b_{52},b_{40},b_{28})	$(0,1,2)$	$(1,1,1)$
$b_{47} = (0,\frac{1}{3},\frac{4}{3},0,1,0)$	(b_{53},b_{46},b_{29})	$(1,0,5)$	$(1,2,1)$
$b_{48} = (0,\frac{1}{3},\frac{4}{3},\frac{2}{3},\frac{2}{3},0)$	$(0,b_{42},b_{30})$	$(0,0,3)$	$(0,1,1)$
$b_{49} = (0,\frac{1}{3},\frac{4}{3},\frac{4}{3},\frac{1}{3},0)$	$(0,b_{43},b_{31})$	$(0,1,1)$	$(0,1,1)$
$b_{50} = (0,\frac{1}{3},\frac{4}{3},2,0,0)$	$(0,b_{44},b_{49})$	$(0,2,0)$	$(0,1,2)$
$b_{51} = (0,\frac{1}{3},\frac{7}{3},\frac{1}{3},\frac{1}{3},0)$	$(0,b_{49},b_{33})$	$(1,0,4)$	$(0,2,1)$
$b_{52} = (0,\frac{1}{3},\frac{7}{3},1,0,0)$	$(0,b_{50},b_{34})$	$(1,1,2)$	$(0,2,1)$
$b_{53} = (0,\frac{1}{3},\frac{10}{3},0,0,0)$	$(0,b_{52},b_{35})$	$(2,0,5)$	$(0,3,1)$
$b_{54} = (0,\frac{2}{3},\frac{2}{3},0,0,0)$	$(b_{89},0,b_{45})$	$(2,0,1)$	$(1,0,2)$
$b_{55} = (0,\frac{2}{3},\frac{2}{3},0,0,1)$	$(b_{61},0,b_{46})$	$(0,1,1)$	$(1,0,2)$
$b_{56} = (0,\frac{2}{3},\frac{2}{3},0,1,0)$	(b_{62},b_{55},b_{47})	$(1,0,4)$	$(1,1,2)$

Table A.1 (continued)

$b_{57} = (0, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, 0)$	$(0, 0, b_{48})$	$(0, 0, 2)$	$(0, 0, 2)$
$b_{58} = (0, \frac{2}{3}, \frac{2}{3}, \frac{4}{3}, \frac{1}{3}, 0)$	$(0, 0, b_{57})$	$(0, 1, 1)$	$(0, 0, 3)$
$b_{59} = (0, \frac{2}{3}, \frac{2}{3}, 2, 0, 0)$	$(0, 0, b_{58})$	$(0, 2, 0)$	$(0, 0, 4)$
$b_{60} = (0, \frac{2}{3}, \frac{5}{3}, \frac{1}{3}, \frac{1}{3}, 0)$	$(0, b_{58}, b_{51})$	$(1, 0, 3)$	$(0, 1, 2)$
$b_{61} = (0, \frac{2}{3}, \frac{5}{3}, 1, 0, 0)$	$(0, b_{59}, b_{52})$	$(1, 1, 1)$	$(0, 1, 2)$
$b_{62} = (0, \frac{2}{3}, \frac{8}{3}, 0, 0, 0)$	$(0, b_{61}, b_{53})$	$(2, 0, 4)$	$(0, 2, 2)$
$b_{63} = (0, 1, 0, 0, 0, 0)$	$(b_{90}, 0, b_{54})$	$(2, 1, 0)$	$(1, 0, 3)$
$b_{64} = (0, 1, 0, 0, 0, 1)$	$(b_{70}, 0, b_{55})$	$(0, 2, 0)$	$(1, 0, 3)$
$b_{65} = (0, 1, 0, 0, 1, 0)$	$(b_{71}, 0, b_{56})$	$(1, 0, 3)$	$(1, 0, 3)$
$b_{66} = (0, 1, 0, \frac{2}{3}, \frac{2}{3}, 0)$	$(b_{74}, 0, b_{65})$	$(1, 1, 2)$	$(1, 0, 4)$
$b_{67} = (0, 1, 0, \frac{4}{3}, \frac{1}{3}, 0)$	$(b_{75}, 0, b_{66})$	$(1, 2, 1)$	$(1, 0, 5)$
$b_{68} = (0, 1, 0, 2, 0, 0)$	$(b_{76}, 0, b_{67})$	$(1, 3, 0)$	$(1, 0, 6)$
$b_{69} = (0, 1, 1, \frac{1}{3}, \frac{1}{3}, 0)$	$(0, 0, b_{60})$	$(1, 0, 2)$	$(0, 0, 3)$
$b_{70} = (0, 1, 1, 1, 0, 0)$	$(0, 0, b_{61})$	$(1, 1, 0)$	$(0, 0, 3)$
$b_{71} = (0, 1, 2, 0, 0, 0)$	$(0, b_{70}, b_{62})$	$(2, 0, 3)$	$(0, 1, 3)$
$b_{72} = (0, \frac{4}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$	$(0, 0, b_{69})$	$(1, 1, 1)$	$(0, 0, 4)$
$b_{73} = (0, \frac{4}{3}, \frac{1}{3}, 1, 0, 0)$	$(0, 0, b_{72})$	$(1, 2, 0)$	$(0, 0, 5)$
$b_{74} = (0, \frac{4}{3}, \frac{4}{3}, 0, 0, 0)$	$(0, 0, b_{71})$	$(2, 0, 2)$	$(0, 0, 4)$
$b_{75} = (0, \frac{5}{3}, \frac{2}{3}, 0, 0, 0)$	$(0, 0, b_{74})$	$(2, 1, 1)$	$(0, 0, 5)$
$b_{76} = (0, 2, 0, 0, 0, 0)$	$(0, 0, b_{75})$	$(2, 2, 0)$	$(0, 0, 6)$
$b_{77} = (1, 0, 0, 0, 0, 0)$	$(b_{91}, b_{63}, 0)$	$(3, 0, 0)$	$(1, 1, 0)$
$b_{78} = (1, 0, 0, 0, 0, 1)$	$(0, b_{64}, 0)$	$(0, 1, 0)$	$(0, 1, 0)$
$b_{79} = (1, 0, 0, 0, 1, 0)$	$(0, b_{78}, 0)$	$(1, 0, 3)$	$(0, 2, 0)$
$b_{80} = (1, 0, 0, \frac{2}{3}, \frac{2}{3}, 0)$	$(0, b_{66}, b_{79})$	$(1, 0, 2)$	$(0, 1, 1)$
$b_{81} = (1, 0, 0, \frac{4}{3}, \frac{1}{3}, 0)$	$(0, b_{67}, b_{80})$	$(1, 1, 1)$	$(0, 1, 2)$
$b_{82} = (1, 0, 0, 2, 0, 0)$	$(0, b_{68}, b_{81})$	$(1, 2, 0)$	$(0, 1, 3)$
$b_{83} = (1, 0, 1, \frac{1}{3}, \frac{1}{3}, 0)$	$(0, b_{81}, 0)$	$(2, 0, 2)$	$(0, 2, 0)$
$b_{84} = (1, 0, 1, 1, 0, 0)$	$(0, b_{82}, 0)$	$(2, 1, 0)$	$(0, 2, 0)$
$b_{85} = (1, 0, 2, 0, 0, 0)$	$(0, b_{84}, 0)$	$(3, 0, 3)$	$(0, 3, 0)$
$b_{86} = (1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$	$(0, b_{72}, b_{83})$	$(2, 0, 1)$	$(0, 1, 1)$
$b_{87} = (1, \frac{1}{3}, \frac{1}{3}, 1, 0, 0)$	$(0, b_{73}, b_{86})$	$(2, 1, 0)$	$(0, 1, 2)$
$b_{88} = (1, \frac{1}{3}, \frac{4}{3}, 0, 0, 0)$	$(0, b_{87}, b_{85})$	$(3, 0, 2)$	$(0, 2, 1)$
$b_{89} = (1, \frac{2}{3}, \frac{2}{3}, 0, 0, 0)$	$(0, b_{75}, b_{88})$	$(3, 0, 1)$	$(0, 1, 2)$
$b_{90} = (1, 1, 0, 0, 0, 0)$	$(0, b_{76}, b_{89})$	$(3, 1, 0)$	$(0, 1, 3)$
$b_{91} = (2, 0, 0, 0, 0, 0)$	$(0, b_{90}, 0)$	$(4, 0, 0)$	$(0, 2, 0)$

Table A.2 Data from the level 2 crystal graph $\mathcal{B}_3$

b	$\left(\tilde{f}_0(b), \tilde{f}_1(b), \tilde{f}_2(b)\right)$	$\left(\varepsilon_0(b), \varepsilon_1(b), \varepsilon_2(b)\right)$	$\left(\varphi_0(b), \varphi_1(b), \varphi_2(b)\right)$
$b_0 = (0,0,0,0,0,0)$	$(b_{273}, 0, 0)$	$(3,0,0)$	$(3,0,0)$
$b_1 = (0,0,0,0,0,1)$	$(b_0, 0, 0)$	$(2,1,0)$	$(4,0,0)$
$b_2 = (0,0,0,0,0,2)$	$(b_1, 0, 0)$	$(1,2,0)$	$(5,0,0)$
$b_3 = (0,0,0,0,0,3)$	$(b_2, 0, 0)$	$(0,3,0)$	$(6,0,0)$
$b_4 = (0,0,0,0,1,0)$	$(b_{64}, b_1, 0)$	$(2,0,3)$	$(3,1,0)$
$b_5 = (0,0,0,0,1,1)$	$(b_4, b_2, 0)$	$(1,1,3)$	$(4,1,0)$
$b_6 = (0,0,0,0,1,2)$	$(b_5, b_3, 0)$	$(0,2,3)$	$(5,1,0)$
$b_7 = (0,0,0,0,2,0)$	$(b_{67}, b_5, 0)$	$(1,0,6)$	$(3,2,0)$
$b_8 = (0,0,0,0,2,1)$	$(b_7, b_6, 0)$	$(0,1,6)$	$(4,2,0)$
$b_9 = (0,0,0,0,3,0)$	$(b_{69}, b_8, 0)$	$(0,0,9)$	$(3,3,0)$
$b_{10} = (0,0,0,\frac{2}{3},\frac{2}{3},0)$	$(b_{124}, 0, b_4)$	$(2,0,2)$	$(3,0,1)$
$b_{11} = (0,0,0,\frac{2}{3},\frac{2}{3},1)$	$(b_{10}, 0, b_5)$	$(1,1,2)$	$(4,0,1)$
$b_{12} = (0,0,0,\frac{2}{3},\frac{2}{3},2)$	$(b_{11}, 0, b_6)$	$(0,2,2)$	$(5,0,1)$
$b_{13} = (0,0,0,\frac{2}{3},\frac{5}{3},0)$	(b_{127}, b_{11}, b_7)	$(1,0,5)$	$(3,1,1)$
$b_{14} = (0,0,0,\frac{2}{3},\frac{5}{3},1)$	(b_{13}, b_{12}, b_8)	$(0,1,5)$	$(4,1,1)$
$b_{15} = (0,0,0,\frac{2}{3},\frac{8}{3},0)$	(b_{129}, b_{14}, b_9)	$(0,0,8)$	$(3,2,1)$
$b_{16} = (0,0,0,\frac{4}{3},\frac{1}{3},0)$	$(b_{160}, 0, b_{10})$	$(2,1,1)$	$(3,0,2)$
$b_{17} = (0,0,0,\frac{4}{3},\frac{1}{3},1)$	$(b_{16}, 0, b_{11})$	$(1,2,1)$	$(4,0,2)$
$b_{18} = (0,0,0,\frac{4}{3},\frac{1}{3},2)$	$(b_{17}, 0, b_{12})$	$(0,3,1)$	$(5,0,2)$
$b_{19} = (0,0,0,\frac{4}{3},\frac{4}{3},0)$	$(b_{163}, 0, b_{13})$	$(1,0,4)$	$(3,0,2)$
$b_{20} = (0,0,0,\frac{4}{3},\frac{4}{3},1)$	$(b_{19}, 0, b_{14})$	$(0,1,4)$	$(4,0,2)$
$b_{21} = (0,0,0,\frac{4}{3},\frac{7}{3},0)$	$(b_{165}, b_{20}, b_{15})$	$(0,0,7)$	$(3,1,2)$
$b_{22} = (0,0,0,2,0,0)$	$(b_{196}, 0, b_{16})$	$(2,2,0)$	$(3,0,3)$
$b_{23} = (0,0,0,2,0,1)$	$(b_{22}, 0, b_{17})$	$(1,3,0)$	$(4,0,3)$
$b_{24} = (0,0,0,2,0,2)$	$(b_{23}, 0, b_{18})$	$(0,4,0)$	$(5,0,3)$
$b_{25} = (0,0,0,2,1,0)$	$(b_{199}, 0, b_{19})$	$(1,1,3)$	$(3,0,3)$
$b_{26} = (0,0,0,2,1,1)$	$(b_{25}, 0, b_{20})$	$(0,2,3)$	$(4,0,3)$
$b_{27} = (0,0,0,2,2,0)$	$(b_{201}, 0, b_{21})$	$(0,0,6)$	$(3,0,3)$
$b_{28} = (0,0,0,\frac{8}{3},\frac{2}{3},0)$	$(b_{202}, 0, b_{25})$	$(1,2,2)$	$(3,0,4)$
$b_{29} = (0,0,0,\frac{8}{3},\frac{2}{3},1)$	$(b_{28}, 0, b_{26})$	$(0,3,2)$	$(4,0,4)$
$b_{30} = (0,0,0,\frac{8}{3},\frac{5}{3},0)$	$(b_{204}, 0, b_{27})$	$(0,1,5)$	$(3,0,4)$
$b_{31} = (0,0,0,\frac{10}{3},\frac{1}{3},0)$	$(b_{205}, 0, b_{28})$	$(1,3,1)$	$(3,0,5)$
$b_{32} = (0,0,0,\frac{10}{3},\frac{1}{3},1)$	$(b_{31}, 0, b_{29})$	$(0,4,1)$	$(4,0,5)$
$b_{33} = (0,0,0,\frac{10}{3},\frac{4}{3},0)$	$(b_{207}, 0, b_{30})$	$(0,2,4)$	$(3,0,5)$
$b_{34} = (0,0,0,4,0,0)$	$(b_{208}, 0, b_{31})$	$(1,4,0)$	$(3,0,6)$
$b_{35} = (0,0,0,4,0,1)$	$(b_{34}, 0, b_{32})$	$(0,5,0)$	$(4,0,6)$
$b_{36} = (0,0,0,4,1,0)$	$(b_{210}, 0, b_{33})$	$(0,3,3)$	$(3,0,6)$
$b_{37} = (0,0,0,\frac{14}{3},\frac{2}{3},0)$	$(b_{211}, 0, b_{36})$	$(0,4,2)$	$(3,0,7)$

Table A.2 (continued)

$b_{38} = (0,0,0,\frac{16}{3},\frac{1}{3},0)$	$(b_{212},0,b_{37})$	$(0,5,1)$	$(3,0,8)$
$b_{39} = (0,0,0,6,0,0)$	$(b_{213},0,b_{38})$	$(0,6,0)$	$(3,0,9)$
$b_{40} = (0,0,1,\frac{1}{3},\frac{1}{3},0)$	$(b_{291},b_{16},0)$	$(2,0,2)$	$(2,1,0)$
$b_{41} = (0,0,1,\frac{1}{3},\frac{1}{3},1)$	$(b_{40},b_{17},0)$	$(1,1,2)$	$(3,1,0)$
$b_{42} = (0,0,1,\frac{1}{3},\frac{1}{3},2)$	$(b_{41},b_{18},0)$	$(0,2,2)$	$(4,1,0)$
$b_{43} = (0,0,1,\frac{1}{3},\frac{4}{3},0)$	$(b_{82},b_{41},0)$	$(1,0,5)$	$(2,2,0)$
$b_{44} = (0,0,1,\frac{1}{3},\frac{4}{3},1)$	$(b_{43},b_{42},0)$	$(0,1,5)$	$(3,2,0)$
$b_{45} = (0,0,1,\frac{1}{3},\frac{7}{3},0)$	$(b_{84},b_{44},0)$	$(0,0,8)$	$(2,3,0)$
$b_{46} = (0,0,1,1,0,0)$	$(b_{294},b_{22},0)$	$(2,1,0)$	$(2,1,0)$
$b_{47} = (0,0,1,1,0,1)$	$(b_{46},b_{23},0)$	$(1,2,0)$	$(3,1,0)$
$b_{48} = (0,0,1,1,0,2)$	$(b_{47},b_{24},0)$	$(0,3,0)$	$(4,1,0)$
$b_{49} = (0,0,1,1,1,0)$	$(b_{276},b_{25},0)$	$(1,0,3)$	$(2,1,0)$
$b_{50} = (0,0,1,1,1,1)$	$(b_{49},b_{26},0)$	$(0,1,3)$	$(3,1,0)$
$b_{51} = (0,0,1,1,2,0)$	$(b_{278},b_{50},0)$	$(0,0,6)$	$(2,2,0)$
$b_{52} = (0,0,1,\frac{5}{3},\frac{2}{3},0)$	(b_{279},b_{28},b_{49})	$(1,1,2)$	$(2,1,1)$
$b_{53} = (0,0,1,\frac{5}{3},\frac{2}{3},1)$	(b_{52},b_{29},b_{50})	$(0,2,2)$	$(3,1,1)$
$b_{54} = (0,0,1,\frac{5}{3},\frac{5}{3},0)$	(b_{281},b_{30},b_{51})	$(0,0,5)$	$(2,1,1)$
$b_{55} = (0,0,1,\frac{7}{3},\frac{1}{3},0)$	(b_{282},b_{31},b_{52})	$(1,2,1)$	$(2,1,2)$
$b_{56} = (0,0,1,\frac{7}{3},\frac{1}{3},1)$	(b_{55},b_{32},b_{53})	$(0,3,1)$	$(3,1,2)$
$b_{57} = (0,0,1,\frac{7}{3},\frac{4}{3},0)$	(b_{284},b_{33},b_{54})	$(0,1,4)$	$(2,1,2)$
$b_{58} = (0,0,1,3,0,0)$	(b_{285},b_{34},b_{55})	$(1,3,0)$	$(2,1,3)$
$b_{59} = (0,0,1,3,0,1)$	(b_{58},b_{35},b_{56})	$(0,4,0)$	$(3,1,3)$
$b_{60} = (0,0,1,3,1,0)$	(b_{287},b_{36},b_{57})	$(0,2,3)$	$(2,1,3)$
$b_{61} = (0,0,1,\frac{11}{3},\frac{2}{3},0)$	(b_{288},b_{37},b_{60})	$(0,3,2)$	$(2,1,4)$
$b_{62} = (0,0,1,\frac{13}{3},\frac{1}{3},0)$	(b_{289},b_{38},b_{61})	$(0,4,1)$	$(2,1,5)$
$b_{63} = (0,0,1,5,0,0)$	(b_{290},b_{39},b_{62})	$(0,5,0)$	$(2,1,6)$
$b_{64} = (0,0,2,0,0,0)$	$(b_{300},b_{46},0)$	$(3,0,3)$	$(2,2,0)$
$b_{65} = (0,0,2,0,0,1)$	$(b_{85},b_{47},0)$	$(1,1,3)$	$(2,2,0)$
$b_{66} = (0,0,2,0,0,2)$	$(b_{65},b_{48},0)$	$(0,2,3)$	$(3,2,0)$
$b_{67} = (0,0,2,0,1,0)$	$(b_{91},b_{65},0)$	$(2,0,6)$	$(2,3,0)$
$b_{68} = (0,0,2,0,1,1)$	$(b_{92},b_{66},0)$	$(0,1,6)$	$(2,3,0)$
$b_{69} = (0,0,2,0,2,0)$	$(b_{93},b_{68},0)$	$(1,0,9)$	$(2,4,0)$
$b_{70} = (0,0,2,\frac{2}{3},\frac{2}{3},0)$	$(b_{303},b_{52},0)$	$(1,0,4)$	$(1,2,0)$
$b_{71} = (0,0,2,\frac{2}{3},\frac{2}{3},1)$	$(b_{70},b_{53},0)$	$(0,1,4)$	$(2,2,0)$
$b_{72} = (0,0,2,\frac{2}{3},\frac{5}{3},0)$	$(b_{94},b_{71},0)$	$(0,0,7)$	$(1,3,0)$
$b_{73} = (0,0,2,\frac{4}{3},\frac{1}{3},0)$	$(b_{304},b_{55},0)$	$(1,1,2)$	$(1,2,0)$
$b_{74} = (0,0,2,\frac{4}{3},\frac{1}{3},1)$	$(b_{73},b_{56},0)$	$(0,2,2)$	$(2,2,0)$
$b_{75} = (0,0,2,\frac{4}{3},\frac{4}{3},0)$	$(b_{293},b_{57},0)$	$(0,0,5)$	$(1,2,0)$
$b_{76} = (0,0,2,2,0,0)$	$(b_{305},b_{58},0)$	$(1,2,0)$	$(1,2,0)$
$b_{77} = (0,0,2,2,0,1)$	$(b_{76},b_{59},0)$	$(0,3,0)$	$(2,2,0)$

Table A.2 (continued)

$b_{78} = (0, 0, 2, 2, 1, 0)$	$(b_{296}, b_{60}, 0)$	$(0, 1, 3)$	$(1, 2, 0)$
$b_{79} = (0, 0, 2, \frac{8}{3}, \frac{2}{3}, 0)$	$(b_{297}, b_{61}, b_{78})$	$(0, 2, 2)$	$(1, 2, 1)$
$b_{80} = (0, 0, 2, \frac{10}{3}, \frac{1}{3}, 0)$	$(b_{298}, b_{62}, b_{79})$	$(0, 3, 1)$	$(1, 2, 2)$
$b_{81} = (0, 0, 2, 4, 0, 0)$	$(b_{299}, b_{63}, b_{80})$	$(0, 4, 0)$	$(1, 2, 3)$
$b_{82} = (0, 0, 3, \frac{1}{3}, \frac{1}{3}, 0)$	$(b_{306}, b_{73}, 0)$	$(2, 0, 5)$	$(1, 3, 0)$
$b_{83} = (0, 0, 3, \frac{1}{3}, \frac{1}{3}, 1)$	$(b_{95}, b_{74}, 0)$	$(0, 1, 5)$	$(1, 3, 0)$
$b_{84} = (0, 0, 3, \frac{1}{3}, \frac{4}{3}, 0)$	$(b_{97}, b_{83}, 0)$	$(1, 0, 8)$	$(1, 4, 0)$
$b_{85} = (0, 0, 3, 1, 0, 0)$	$(b_{307}, b_{76}, 0)$	$(2, 1, 3)$	$(1, 3, 0)$
$b_{86} = (0, 0, 3, 1, 0, 1)$	$(b_{96}, b_{77}, 0)$	$(0, 2, 3)$	$(1, 3, 0)$
$b_{87} = (0, 0, 3, 1, 1, 0)$	$(0, b_{78}, 0)$	$(0, 0, 6)$	$(0, 3, 0)$
$b_{88} = (0, 0, 3, \frac{5}{3}, \frac{2}{3}, 0)$	$(0, b_{79}, 0)$	$(0, 1, 4)$	$(0, 3, 0)$
$b_{89} = (0, 0, 3, \frac{7}{3}, \frac{1}{3}, 0)$	$(0, b_{80}, 0)$	$(0, 2, 2)$	$(0, 3, 0)$
$b_{90} = (0, 0, 3, 3, 0, 0)$	$(0, b_{81}, 0)$	$(0, 3, 0)$	$(0, 3, 0)$
$b_{91} = (0, 0, 4, 0, 0, 0)$	$(b_{308}, b_{85}, 0)$	$(3, 0, 6)$	$(1, 4, 0)$
$b_{92} = (0, 0, 4, 0, 0, 1)$	$(b_{98}, b_{86}, 0)$	$(1, 1, 6)$	$(1, 4, 0)$
$b_{93} = (0, 0, 4, 0, 1, 0)$	$(b_{99}, b_{92}, 0)$	$(2, 0, 9)$	$(1, 5, 0)$
$b_{94} = (0, 0, 4, \frac{2}{3}, \frac{2}{3}, 0)$	$(0, b_{88}, 0)$	$(1, 0, 7)$	$(0, 4, 0)$
$b_{95} = (0, 0, 4, \frac{4}{3}, \frac{1}{3}, 0)$	$(0, b_{89}, 0)$	$(1, 1, 5)$	$(0, 4, 0)$
$b_{96} = (0, 0, 4, 2, 0, 0)$	$(0, b_{90}, 0)$	$(1, 2, 3)$	$(0, 4, 0)$
$b_{97} = (0, 0, 5, \frac{1}{3}, \frac{1}{3}, 0)$	$(0, b_{95}, 0)$	$(2, 0, 8)$	$(0, 5, 0)$
$b_{98} = (0, 0, 5, 1, 0, 0)$	$(0, b_{96}, 0)$	$(2, 1, 6)$	$(0, 5, 0)$
$b_{99} = (0, 0, 6, 0, 0, 0)$	$(0, b_{98}, 0)$	$(3, 0, 9)$	$(0, 6, 0)$
$b_{100} = (0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$	$(b_{309}, 0, b_{40})$	$(2, 0, 1)$	$(2, 0, 1)$
$b_{101} = (0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1)$	$(b_{100}, 0, b_{41})$	$(1, 1, 1)$	$(3, 0, 1)$
$b_{102} = (0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 2)$	$(b_{101}, 0, b_{42})$	$(0, 2, 1)$	$(4, 0, 1)$
$b_{103} = (0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{4}{3}, 0)$	$(b_{142}, b_{101}, b_{43})$	$(1, 0, 4)$	$(2, 1, 1)$
$b_{104} = (0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{4}{3}, 1)$	$(b_{103}, b_{102}, b_{44})$	$(0, 1, 4)$	$(3, 1, 1)$
$b_{105} = (0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{7}{3}, 0)$	$(b_{144}, b_{104}, b_{45})$	$(0, 0, 7)$	$(2, 2, 1)$
$b_{106} = (0, \frac{1}{3}, \frac{1}{3}, 1, 0, 0)$	$(b_{312}, 0, b_{100})$	$(2, 1, 0)$	$(2, 0, 2)$
$b_{107} = (0, \frac{1}{3}, \frac{1}{3}, 1, 0, 1)$	$(b_{106}, 0, b_{101})$	$(1, 2, 0)$	$(3, 0, 2)$
$b_{108} = (0, \frac{1}{3}, \frac{1}{3}, 1, 0, 2)$	$(b_{107}, 0, b_{102})$	$(0, 3, 0)$	$(4, 0, 2)$
$b_{109} = (0, \frac{1}{3}, \frac{1}{3}, 1, 1, 0)$	$(b_{178}, 0, b_{103})$	$(1, 0, 3)$	$(2, 0, 2)$
$b_{110} = (0, \frac{1}{3}, \frac{1}{3}, 1, 1, 1)$	$(b_{109}, 0, b_{104})$	$(0, 1, 3)$	$(3, 0, 2)$
$b_{111} = (0, \frac{1}{3}, \frac{1}{3}, 1, 2, 0)$	$(b_{180}, b_{110}, b_{105})$	$(0, 0, 6)$	$(2, 1, 2)$
$b_{112} = (0, \frac{1}{3}, \frac{1}{3}, \frac{5}{3}, \frac{2}{3}, 0)$	$(b_{214}, 0, b_{109})$	$(1, 1, 2)$	$(2, 0, 3)$
$b_{113} = (0, \frac{1}{3}, \frac{1}{3}, \frac{5}{3}, \frac{2}{3}, 1)$	$(b_{112}, 0, b_{110})$	$(0, 2, 2)$	$(3, 0, 3)$
$b_{114} = (0, \frac{1}{3}, \frac{1}{3}, \frac{5}{3}, \frac{5}{3}, 0)$	$(b_{216}, 0, b_{111})$	$(0, 0, 5)$	$(2, 0, 3)$
$b_{115} = (0, \frac{1}{3}, \frac{1}{3}, \frac{7}{3}, \frac{1}{3}, 0)$	$(b_{232}, 0, b_{112})$	$(1, 2, 1)$	$(2, 0, 4)$
$b_{116} = (0, \frac{1}{3}, \frac{1}{3}, \frac{7}{3}, \frac{1}{3}, 1)$	$(b_{115}, 0, b_{113})$	$(0, 3, 1)$	$(3, 0, 4)$
$b_{117} = (0, \frac{1}{3}, \frac{1}{3}, \frac{7}{3}, \frac{4}{3}, 0)$	$(b_{234}, 0, b_{114})$	$(0, 1, 4)$	$(2, 0, 4)$

Table A.2 (continued)

$b_{118} = (0, \frac{1}{3}, \frac{1}{3}, 3, 0, 0)$	$(b_{235}, 0, b_{115})$	$(1, 3, 0)$	$(2, 0, 5)$
$b_{119} = (0, \frac{1}{3}, \frac{1}{3}, 3, 0, 1)$	$(b_{118}, 0, b_{116})$	$(0, 4, 0)$	$(3, 0, 5)$
$b_{120} = (0, \frac{1}{3}, \frac{1}{3}, 3, 1, 0)$	$(b_{237}, 0, b_{117})$	$(0, 2, 3)$	$(2, 0, 5)$
$b_{121} = (0, \frac{1}{3}, \frac{1}{3}, \frac{11}{3}, \frac{2}{3}, 0)$	$(b_{238}, 0, b_{120})$	$(0, 3, 2)$	$(2, 0, 6)$
$b_{122} = (0, \frac{1}{3}, \frac{1}{3}, \frac{13}{3}, \frac{1}{3}, 0)$	$(b_{239}, 0, b_{121})$	$(0, 4, 1)$	$(2, 0, 7)$
$b_{123} = (0, \frac{1}{3}, \frac{1}{3}, 5, 0, 0)$	$(b_{240}, 0, b_{122})$	$(0, 5, 0)$	$(2, 0, 8)$
$b_{124} = (0, \frac{1}{3}, \frac{4}{3}, 0, 0, 0)$	$(b_{318}, b_{106}, b_{64})$	$(3, 0, 2)$	$(2, 1, 1)$
$b_{125} = (0, \frac{1}{3}, \frac{4}{3}, 0, 0, 1)$	$(b_{145}, b_{107}, b_{65})$	$(1, 1, 2)$	$(2, 1, 1)$
$b_{126} = (0, \frac{1}{3}, \frac{4}{3}, 0, 0, 2)$	$(b_{125}, b_{108}, b_{66})$	$(0, 2, 2)$	$(3, 1, 1)$
$b_{127} = (0, \frac{1}{3}, \frac{4}{3}, 0, 1, 0)$	$(b_{151}, b_{125}, b_{67})$	$(2, 0, 5)$	$(2, 2, 1)$
$b_{128} = (0, \frac{1}{3}, \frac{4}{3}, 0, 1, 1)$	$(b_{152}, b_{126}, b_{68})$	$(0, 1, 5)$	$(2, 2, 1)$
$b_{129} = (0, \frac{1}{3}, \frac{4}{3}, 0, 2, 0)$	$(b_{153}, b_{128}, b_{69})$	$(1, 0, 8)$	$(2, 3, 1)$
$b_{130} = (0, \frac{1}{3}, \frac{4}{3}, \frac{2}{3}, \frac{2}{3}, 0)$	$(b_{321}, b_{112}, b_{70})$	$(1, 0, 3)$	$(1, 1, 1)$
$b_{131} = (0, \frac{1}{3}, \frac{4}{3}, \frac{2}{3}, \frac{2}{3}, 1)$	$(b_{130}, b_{113}, b_{71})$	$(0, 1, 3)$	$(2, 1, 1)$
$b_{132} = (0, \frac{1}{3}, \frac{4}{3}, \frac{2}{3}, \frac{5}{3}, 0)$	$(b_{154}, b_{131}, b_{72})$	$(0, 0, 6)$	$(1, 2, 1)$
$b_{133} = (0, \frac{1}{3}, \frac{4}{3}, \frac{4}{3}, \frac{1}{3}, 0)$	$(b_{322}, b_{115}, b_{73})$	$(1, 1, 1)$	$(1, 1, 1)$
$b_{134} = (0, \frac{1}{3}, \frac{4}{3}, \frac{4}{3}, \frac{1}{3}, 1)$	$(b_{133}, b_{116}, b_{74})$	$(0, 2, 1)$	$(2, 1, 1)$
$b_{135} = (0, \frac{1}{3}, \frac{4}{3}, \frac{4}{3}, \frac{4}{3}, 0)$	$(b_{311}, b_{117}, b_{75})$	$(0, 0, 4)$	$(1, 1, 1)$
$b_{136} = (0, \frac{1}{3}, \frac{4}{3}, 2, 0, 0)$	$(b_{323}, b_{118}, b_{133})$	$(1, 2, 0)$	$(1, 1, 2)$
$b_{137} = (0, \frac{1}{3}, \frac{4}{3}, 2, 0, 1)$	$(b_{136}, b_{119}, b_{134})$	$(0, 3, 0)$	$(2, 1, 2)$
$b_{138} = (0, \frac{1}{3}, \frac{4}{3}, 2, 1, 0)$	$(b_{314}, b_{120}, b_{135})$	$(0, 1, 3)$	$(1, 1, 2)$
$b_{139} = (0, \frac{1}{3}, \frac{4}{3}, \frac{8}{3}, \frac{2}{3}, 0)$	$(b_{315}, b_{121}, b_{138})$	$(0, 2, 2)$	$(1, 1, 3)$
$b_{140} = (0, \frac{1}{3}, \frac{4}{3}, \frac{10}{3}, \frac{1}{3}, 0)$	$(b_{316}, b_{122}, b_{139})$	$(0, 3, 1)$	$(1, 1, 4)$
$b_{141} = (0, \frac{1}{3}, \frac{4}{3}, 4, 0, 0)$	$(b_{317}, b_{123}, b_{140})$	$(0, 4, 0)$	$(1, 1, 5)$
$b_{142} = (0, \frac{1}{3}, \frac{7}{3}, \frac{1}{3}, \frac{1}{3}, 0)$	$(b_{324}, b_{133}, b_{82})$	$(2, 0, 4)$	$(1, 2, 1)$
$b_{143} = (0, \frac{1}{3}, \frac{7}{3}, \frac{1}{3}, \frac{1}{3}, 1)$	$(b_{155}, b_{134}, b_{83})$	$(0, 1, 4)$	$(1, 2, 1)$
$b_{144} = (0, \frac{1}{3}, \frac{7}{3}, \frac{1}{3}, \frac{4}{3}, 0)$	$(b_{157}, b_{143}, b_{84})$	$(1, 0, 7)$	$(1, 3, 1)$
$b_{145} = (0, \frac{1}{3}, \frac{7}{3}, 1, 0, 0)$	$(b_{325}, b_{136}, b_{85})$	$(2, 1, 2)$	$(1, 2, 1)$
$b_{146} = (0, \frac{1}{3}, \frac{7}{3}, 1, 0, 1)$	$(b_{156}, b_{137}, b_{86})$	$(0, 2, 2)$	$(1, 2, 1)$
$b_{147} = (0, \frac{1}{3}, \frac{7}{3}, 1, 1, 0)$	$(0, b_{138}, b_{87})$	$(0, 0, 5)$	$(0, 2, 1)$
$b_{148} = (0, \frac{1}{3}, \frac{7}{3}, \frac{5}{3}, \frac{2}{3}, 0)$	$(0, b_{139}, b_{88})$	$(0, 1, 3)$	$(0, 2, 1)$
$b_{149} = (0, \frac{1}{3}, \frac{7}{3}, \frac{7}{3}, \frac{1}{3}, 0)$	$(0, b_{140}, b_{89})$	$(0, 2, 1)$	$(0, 2, 1)$
$b_{150} = (0, \frac{1}{3}, \frac{7}{3}, 3, 0, 0)$	$(0, b_{141}, b_{149})$	$(0, 3, 0)$	$(0, 2, 2)$
$b_{151} = (0, \frac{1}{3}, \frac{10}{3}, 0, 0, 0)$	$(b_{326}, b_{145}, b_{91})$	$(3, 0, 5)$	$(1, 3, 1)$
$b_{152} = (0, \frac{1}{3}, \frac{10}{3}, 0, 0, 1)$	$(b_{158}, b_{146}, b_{92})$	$(1, 1, 5)$	$(1, 3, 1)$
$b_{153} = (0, \frac{1}{3}, \frac{10}{3}, 0, 1, 0)$	$(b_{159}, b_{152}, b_{93})$	$(2, 0, 8)$	$(1, 4, 1)$
$b_{154} = (0, \frac{1}{3}, \frac{10}{3}, \frac{2}{3}, \frac{2}{3}, 0)$	$(0, b_{148}, b_{94})$	$(1, 0, 6)$	$(0, 3, 1)$
$b_{155} = (0, \frac{1}{3}, \frac{10}{3}, \frac{4}{3}, \frac{1}{3}, 0)$	$(0, b_{149}, b_{95})$	$(1, 1, 4)$	$(0, 3, 1)$
$b_{156} = (0, \frac{1}{3}, \frac{10}{3}, 2, 0, 0)$	$(0, b_{150}, b_{96})$	$(1, 2, 2)$	$(0, 3, 1)$
$b_{157} = (0, \frac{1}{3}, \frac{13}{3}, \frac{1}{3}, \frac{1}{3}, 0)$	$(0, b_{155}, b_{97})$	$(2, 0, 7)$	$(0, 4, 1)$

Table A.2 (continued)

$b_{158} = (0, \frac{1}{3}, \frac{13}{3}, 1, 0, 0)$	$(0, b_{156}, b_{98})$	$(2, 1, 5)$	$(0, 4, 1)$
$b_{159} = (0, \frac{1}{3}, \frac{16}{3}, 0, 0, 0)$	$(0, b_{158}, b_{99})$	$(3, 0, 8)$	$(0, 5, 1)$
$b_{160} = (0, \frac{2}{3}, \frac{2}{3}, 0, 0, 0)$	$(b_{327}, 0, b_{124})$	$(3, 0, 1)$	$(2, 0, 2)$
$b_{161} = (0, \frac{2}{3}, \frac{2}{3}, 0, 0, 1)$	$(b_{181}, 0, b_{125})$	$(1, 1, 1)$	$(2, 0, 2)$
$b_{162} = (0, \frac{2}{3}, \frac{2}{3}, 0, 0, 2)$	$(b_{161}, 0, b_{126})$	$(0, 2, 1)$	$(3, 0, 2)$
$b_{163} = (0, \frac{2}{3}, \frac{2}{3}, 0, 1, 0)$	$(b_{187}, b_{161}, b_{127})$	$(2, 0, 4)$	$(2, 1, 2)$
$b_{164} = (0, \frac{2}{3}, \frac{2}{3}, 0, 1, 1)$	$(b_{188}, b_{162}, b_{128})$	$(0, 1, 4)$	$(2, 1, 2)$
$b_{165} = (0, \frac{2}{3}, \frac{2}{3}, 0, 2, 0)$	$(b_{189}, b_{164}, b_{129})$	$(1, 0, 7)$	$(2, 2, 2)$
$b_{166} = (0, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, 0)$	$(b_{330}, 0, b_{130})$	$(1, 0, 2)$	$(1, 0, 2)$
$b_{167} = (0, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, 1)$	$(b_{166}, 0, b_{131})$	$(0, 1, 2)$	$(2, 0, 2)$
$b_{168} = (0, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{5}{3}, 0)$	$(b_{190}, b_{167}, b_{132})$	$(0, 0, 5)$	$(1, 1, 2)$
$b_{169} = (0, \frac{2}{3}, \frac{2}{3}, \frac{4}{3}, \frac{1}{3}, 0)$	$(b_{331}, 0, b_{166})$	$(1, 1, 1)$	$(1, 0, 3)$
$b_{170} = (0, \frac{2}{3}, \frac{2}{3}, \frac{4}{3}, \frac{1}{3}, 1)$	$(b_{169}, 0, b_{167})$	$(0, 2, 1)$	$(2, 0, 3)$
$b_{171} = (0, \frac{2}{3}, \frac{2}{3}, \frac{4}{3}, \frac{4}{3}, 0)$	$(b_{226}, 0, b_{168})$	$(0, 0, 4)$	$(1, 0, 3)$
$b_{172} = (0, \frac{2}{3}, \frac{2}{3}, 2, 0, 0)$	$(b_{332}, 0, b_{169})$	$(1, 2, 0)$	$(1, 0, 4)$
$b_{173} = (0, \frac{2}{3}, \frac{2}{3}, 2, 0, 1)$	$(b_{172}, 0, b_{170})$	$(0, 3, 0)$	$(2, 0, 4)$
$b_{174} = (0, \frac{2}{3}, \frac{2}{3}, 2, 1, 0)$	$(b_{244}, 0, b_{171})$	$(0, 1, 3)$	$(1, 0, 4)$
$b_{175} = (0, \frac{2}{3}, \frac{2}{3}, \frac{8}{3}, \frac{2}{3}, 0)$	$(b_{253}, 0, b_{174})$	$(0, 2, 2)$	$(1, 0, 5)$
$b_{176} = (0, \frac{2}{3}, \frac{2}{3}, \frac{10}{3}, \frac{1}{3}, 0)$	$(b_{254}, 0, b_{175})$	$(0, 3, 1)$	$(1, 0, 6)$
$b_{177} = (0, \frac{2}{3}, \frac{2}{3}, 4, 0, 0)$	$(b_{255}, 0, b_{176})$	$(0, 4, 0)$	$(1, 0, 7)$
$b_{178} = (0, \frac{2}{3}, \frac{5}{3}, \frac{1}{3}, \frac{1}{3}, 0)$	$(b_{333}, b_{169}, b_{142})$	$(2, 0, 3)$	$(1, 1, 2)$
$b_{179} = (0, \frac{2}{3}, \frac{5}{3}, \frac{1}{3}, \frac{1}{3}, 1)$	$(b_{191}, b_{170}, b_{143})$	$(0, 1, 3)$	$(1, 1, 2)$
$b_{180} = (0, \frac{2}{3}, \frac{5}{3}, \frac{1}{3}, \frac{4}{3}, 0)$	$(b_{193}, b_{179}, b_{144})$	$(1, 0, 6)$	$(1, 2, 2)$
$b_{181} = (0, \frac{2}{3}, \frac{5}{3}, 1, 0, 0)$	$(b_{334}, b_{172}, b_{145})$	$(2, 1, 1)$	$(1, 1, 2)$
$b_{182} = (0, \frac{2}{3}, \frac{5}{3}, 1, 0, 1)$	$(b_{192}, b_{173}, b_{146})$	$(0, 2, 1)$	$(1, 1, 2)$
$b_{183} = (0, \frac{2}{3}, \frac{5}{3}, 1, 1, 0)$	$(0, b_{174}, b_{147})$	$(0, 0, 4)$	$(0, 1, 2)$
$b_{184} = (0, \frac{2}{3}, \frac{5}{3}, \frac{5}{3}, \frac{2}{3}, 0)$	$(0, b_{175}, b_{148})$	$(0, 1, 2)$	$(0, 1, 2)$
$b_{185} = (0, \frac{2}{3}, \frac{5}{3}, \frac{7}{3}, \frac{1}{3}, 0)$	$(0, b_{176}, b_{184})$	$(0, 2, 1)$	$(0, 1, 3)$
$b_{186} = (0, \frac{2}{3}, \frac{5}{3}, 3, 0, 0)$	$(0, b_{177}, b_{185})$	$(0, 3, 0)$	$(0, 1, 4)$
$b_{187} = (0, \frac{2}{3}, \frac{8}{3}, 0, 0, 0)$	$(b_{335}, b_{181}, b_{151})$	$(3, 0, 4)$	$(1, 2, 2)$
$b_{188} = (0, \frac{2}{3}, \frac{8}{3}, 0, 0, 1)$	$(b_{194}, b_{182}, b_{152})$	$(1, 1, 4)$	$(1, 2, 2)$
$b_{189} = (0, \frac{2}{3}, \frac{8}{3}, 0, 1, 0)$	$(b_{195}, b_{188}, b_{153})$	$(2, 0, 7)$	$(1, 3, 2)$
$b_{190} = (0, \frac{2}{3}, \frac{8}{3}, \frac{2}{3}, \frac{2}{3}, 0)$	$(0, b_{184}, b_{154})$	$(1, 0, 5)$	$(0, 2, 2)$
$b_{191} = (0, \frac{2}{3}, \frac{8}{3}, \frac{4}{3}, \frac{1}{3}, 0)$	$(0, b_{185}, b_{155})$	$(1, 1, 3)$	$(0, 2, 2)$
$b_{192} = (0, \frac{2}{3}, \frac{8}{3}, 2, 0, 0)$	$(0, b_{186}, b_{156})$	$(1, 2, 1)$	$(0, 2, 2)$
$b_{193} = (0, \frac{2}{3}, \frac{11}{3}, \frac{1}{3}, \frac{1}{3}, 0)$	$(0, b_{191}, b_{157})$	$(2, 0, 6)$	$(0, 3, 2)$
$b_{194} = (0, \frac{2}{3}, \frac{11}{3}, 1, 0, 0)$	$(0, b_{192}, b_{158})$	$(2, 1, 4)$	$(0, 3, 2)$
$b_{195} = (0, \frac{2}{3}, \frac{14}{3}, 0, 0, 0)$	$(0, b_{194}, b_{159})$	$(3, 0, 7)$	$(0, 4, 2)$
$b_{196} = (0, 1, 0, 0, 0, 0)$	$(b_{336}, 0, b_{160})$	$(3, 1, 0)$	$(2, 0, 3)$
$b_{197} = (0, 1, 0, 0, 0, 1)$	$(b_{217}, 0, b_{161})$	$(1, 2, 0)$	$(2, 0, 3)$

Table A.2 (continued)

$b_{198} = (0,1,0,0,0,2)$	$(b_{197},0,b_{162})$	$(0,3,0)$	$(3,0,3)$
$b_{199} = (0,1,0,0,1,0)$	$(b_{223},0,b_{163})$	$(2,0,3)$	$(2,0,3)$
$b_{200} = (0,1,0,0,1,1)$	$(b_{224},0,b_{164})$	$(0,1,3)$	$(2,0,3)$
$b_{201} = (0,1,0,0,2,0)$	$(b_{225},b_{200},b_{165})$	$(1,0,6)$	$(2,1,3)$
$b_{202} = (0,1,0,\frac{2}{3},\frac{2}{3},0)$	$(b_{241},0,b_{199})$	$(2,1,2)$	$(2,0,4)$
$b_{203} = (0,1,0,\frac{2}{3},\frac{2}{3},1)$	$(b_{242},0,b_{200})$	$(0,2,2)$	$(2,0,4)$
$b_{204} = (0,1,0,\frac{2}{3},\frac{5}{3},0)$	$(b_{243},0,b_{201})$	$(1,0,5)$	$(2,0,4)$
$b_{205} = (0,1,0,\frac{4}{3},\frac{1}{3},0)$	$(b_{250},0,b_{202})$	$(2,2,1)$	$(2,0,5)$
$b_{206} = (0,1,0,\frac{4}{3},\frac{1}{3},1)$	$(b_{251},0,b_{203})$	$(0,3,1)$	$(2,0,5)$
$b_{207} = (0,1,0,\frac{4}{3},\frac{4}{3},0)$	$(b_{252},0,b_{204})$	$(1,1,4)$	$(2,0,5)$
$b_{208} = (0,1,0,2,0,0)$	$(b_{259},0,b_{205})$	$(2,3,0)$	$(2,0,6)$
$b_{209} = (0,1,0,2,0,1)$	$(b_{260},0,b_{206})$	$(0,4,0)$	$(2,0,6)$
$b_{210} = (0,1,0,2,1,0)$	$(b_{261},0,b_{207})$	$(1,2,3)$	$(2,0,6)$
$b_{211} = (0,1,0,\frac{8}{3},\frac{2}{3},0)$	$(b_{262},0,b_{210})$	$(1,3,2)$	$(2,0,7)$
$b_{212} = (0,1,0,\frac{10}{3},\frac{1}{3},0)$	$(b_{263},0,b_{211})$	$(1,4,1)$	$(2,0,8)$
$b_{213} = (0,1,0,4,0,0)$	$(b_{264},0,b_{212})$	$(1,5,0)$	$(2,0,9)$
$b_{214} = (0,1,1,\frac{1}{3},\frac{1}{3},0)$	$(b_{342},0,b_{178})$	$(2,0,2)$	$(1,0,3)$
$b_{215} = (0,1,1,\frac{1}{3},\frac{1}{3},1)$	$(b_{227},0,b_{179})$	$(0,1,2)$	$(1,0,3)$
$b_{216} = (0,1,1,\frac{1}{3},\frac{4}{3},0)$	$(b_{229},b_{215},b_{180})$	$(1,0,5)$	$(1,1,3)$
$b_{217} = (0,1,1,1,0,0)$	$(b_{343},0,b_{181})$	$(2,1,0)$	$(1,0,3)$
$b_{218} = (0,1,1,1,0,1)$	$(b_{228},0,b_{182})$	$(0,2,0)$	$(1,0,3)$
$b_{219} = (0,1,1,1,1,0)$	$(0,0,b_{183})$	$(0,0,3)$	$(0,0,3)$
$b_{220} = (0,1,1,\frac{5}{3},\frac{2}{3},0)$	$(0,0,b_{219})$	$(0,1,2)$	$(0,0,4)$
$b_{221} = (0,1,1,\frac{7}{3},\frac{1}{3},0)$	$(0,0,b_{220})$	$(0,2,1)$	$(0,0,5)$
$b_{222} = (0,1,1,3,0,0)$	$(0,0,b_{221})$	$(0,3,0)$	$(0,0,6)$
$b_{223} = (0,1,2,0,0,0)$	$(b_{344},b_{217},b_{187})$	$(3,0,3)$	$(1,1,3)$
$b_{224} = (0,1,2,0,0,1)$	$(b_{230},b_{218},b_{188})$	$(1,1,3)$	$(1,1,3)$
$b_{225} = (0,1,2,0,1,0)$	$(b_{231},b_{224},b_{189})$	$(2,0,6)$	$(1,2,3)$
$b_{226} = (0,1,2,\frac{2}{3},\frac{2}{3},0)$	$(0,b_{220},b_{190})$	$(1,0,4)$	$(0,1,3)$
$b_{227} = (0,1,2,\frac{4}{3},\frac{1}{3},0)$	$(0,b_{221},b_{191})$	$(1,1,2)$	$(0,1,3)$
$b_{228} = (0,1,2,2,0,0)$	$(0,b_{222},b_{192})$	$(1,2,0)$	$(0,1,3)$
$b_{229} = (0,1,3,\frac{1}{3},\frac{1}{3},0)$	$(0,b_{227},b_{193})$	$(2,0,5)$	$(0,2,3)$
$b_{230} = (0,1,3,1,0,0)$	$(0,b_{228},b_{194})$	$(2,1,3)$	$(0,2,3)$
$b_{231} = (0,1,4,0,0,0)$	$(0,b_{230},b_{195})$	$(3,0,6)$	$(0,3,3)$
$b_{232} = (0,\frac{4}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},0)$	$(b_{345},0,b_{214})$	$(2,1,1)$	$(1,0,4)$
$b_{233} = (0,\frac{4}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},1)$	$(b_{245},0,b_{215})$	$(0,2,1)$	$(1,0,4)$
$b_{234} = (0,\frac{4}{3},\frac{1}{3},\frac{1}{3},\frac{4}{3},0)$	$(b_{247},0,b_{216})$	$(1,0,4)$	$(1,0,4)$
$b_{235} = (0,\frac{4}{3},\frac{1}{3},1,0,0)$	$(b_{346},0,b_{232})$	$(2,2,0)$	$(1,0,5)$
$b_{236} = (0,\frac{4}{3},\frac{1}{3},1,0,1)$	$(b_{246},0,b_{233})$	$(0,3,0)$	$(1,0,5)$
$b_{237} = (0,\frac{4}{3},\frac{1}{3},1,1,0)$	$(b_{256},0,b_{234})$	$(1,1,3)$	$(1,0,5)$

Table A.2 (continued)

$b_{238} = (0, \frac{4}{3}, \frac{1}{3}, \frac{5}{3}, \frac{2}{3}, 0)$	$(b_{265}, 0, b_{237})$	$(1, 2, 2)$	$(1, 0, 6)$
$b_{239} = (0, \frac{4}{3}, \frac{1}{3}, \frac{7}{3}, \frac{1}{3}, 0)$	$(b_{268}, 0, b_{238})$	$(1, 3, 1)$	$(1, 0, 7)$
$b_{240} = (0, \frac{4}{3}, \frac{1}{3}, 3, 0, 0)$	$(b_{269}, 0, b_{239})$	$(1, 4, 0)$	$(1, 0, 8)$
$b_{241} = (0, \frac{4}{3}, \frac{4}{3}, 0, 0, 0)$	$(b_{347}, 0, b_{223})$	$(3, 0, 2)$	$(1, 0, 4)$
$b_{242} = (0, \frac{4}{3}, \frac{4}{3}, 0, 0, 1)$	$(b_{248}, 0, b_{224})$	$(1, 1, 2)$	$(1, 0, 4)$
$b_{243} = (0, \frac{4}{3}, \frac{4}{3}, 0, 1, 0)$	$(b_{249}, b_{242}, b_{225})$	$(2, 0, 5)$	$(1, 1, 4)$
$b_{244} = (0, \frac{4}{3}, \frac{4}{3}, \frac{2}{3}, \frac{2}{3}, 0)$	$(0, 0, b_{226})$	$(1, 0, 3)$	$(0, 0, 4)$
$b_{245} = (0, \frac{4}{3}, \frac{4}{3}, \frac{4}{3}, \frac{1}{3}, 0)$	$(0, 0, b_{227})$	$(1, 1, 1)$	$(0, 0, 4)$
$b_{246} = (0, \frac{4}{3}, \frac{4}{3}, 2, 0, 0)$	$(0, 0, b_{245})$	$(1, 2, 0)$	$(0, 0, 5)$
$b_{247} = (0, \frac{4}{3}, \frac{7}{3}, \frac{1}{3}, \frac{1}{3}, 0)$	$(0, b_{245}, b_{229})$	$(2, 0, 4)$	$(0, 1, 4)$
$b_{248} = (0, \frac{4}{3}, \frac{7}{3}, 1, 0, 0)$	$(0, b_{246}, b_{230})$	$(2, 1, 2)$	$(0, 1, 4)$
$b_{249} = (0, \frac{4}{3}, \frac{10}{3}, 0, 0, 0)$	$(0, b_{248}, b_{231})$	$(3, 0, 5)$	$(0, 2, 4)$
$b_{250} = (0, \frac{5}{3}, \frac{2}{3}, 0, 0, 0)$	$(b_{348}, 0, b_{241})$	$(3, 1, 1)$	$(1, 0, 5)$
$b_{251} = (0, \frac{5}{3}, \frac{2}{3}, 0, 0, 1)$	$(b_{257}, 0, b_{242})$	$(1, 2, 1)$	$(1, 0, 5)$
$b_{252} = (0, \frac{5}{3}, \frac{2}{3}, 0, 1, 0)$	$(b_{258}, 0, b_{243})$	$(2, 0, 4)$	$(1, 0, 5)$
$b_{253} = (0, \frac{5}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, 0)$	$(0, 0, b_{244})$	$(1, 1, 2)$	$(0, 0, 5)$
$b_{254} = (0, \frac{5}{3}, \frac{2}{3}, \frac{4}{3}, \frac{1}{3}, 0)$	$(0, 0, b_{253})$	$(1, 2, 1)$	$(0, 0, 6)$
$b_{255} = (0, \frac{5}{3}, \frac{2}{3}, 2, 0, 0)$	$(0, 0, b_{254})$	$(1, 3, 0)$	$(0, 0, 7)$
$b_{256} = (0, \frac{5}{3}, \frac{5}{3}, \frac{1}{3}, \frac{1}{3}, 0)$	$(0, 0, b_{247})$	$(2, 0, 3)$	$(0, 0, 5)$
$b_{257} = (0, \frac{5}{3}, \frac{5}{3}, 1, 0, 0)$	$(0, 0, b_{248})$	$(2, 1, 1)$	$(0, 0, 5)$
$b_{258} = (0, \frac{5}{3}, \frac{8}{3}, 0, 0, 0)$	$(0, b_{257}, b_{249})$	$(3, 0, 4)$	$(0, 1, 5)$
$b_{259} = (0, 2, 0, 0, 0, 0)$	$(b_{349}, 0, b_{250})$	$(3, 2, 0)$	$(1, 0, 6)$
$b_{260} = (0, 2, 0, 0, 0, 1)$	$(b_{266}, 0, b_{251})$	$(1, 3, 0)$	$(1, 0, 6)$
$b_{261} = (0, 2, 0, 0, 1, 0)$	$(b_{267}, 0, b_{252})$	$(2, 1, 3)$	$(1, 0, 6)$
$b_{262} = (0, 2, 0, \frac{2}{3}, \frac{2}{3}, 0)$	$(b_{270}, 0, b_{261})$	$(2, 2, 2)$	$(1, 0, 7)$
$b_{263} = (0, 2, 0, \frac{4}{3}, \frac{1}{3}, 0)$	$(b_{271}, 0, b_{262})$	$(2, 3, 1)$	$(1, 0, 8)$
$b_{264} = (0, 2, 0, 2, 0, 0)$	$(b_{272}, 0, b_{263})$	$(2, 4, 0)$	$(1, 0, 9)$
$b_{265} = (0, 2, 1, \frac{1}{3}, \frac{1}{3}, 0)$	$(0, 0, b_{256})$	$(2, 1, 2)$	$(0, 0, 6)$
$b_{266} = (0, 2, 1, 1, 0, 0)$	$(0, 0, b_{257})$	$(2, 2, 0)$	$(0, 0, 6)$
$b_{267} = (0, 2, 2, 0, 0, 0)$	$(0, 0, b_{258})$	$(3, 0, 3)$	$(0, 0, 6)$
$b_{268} = (0, \frac{7}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$	$(0, 0, b_{265})$	$(2, 2, 1)$	$(0, 0, 7)$
$b_{269} = (0, \frac{7}{3}, \frac{1}{3}, 1, 0, 0)$	$(0, 0, b_{268})$	$(2, 3, 0)$	$(0, 0, 8)$
$b_{270} = (0, \frac{7}{3}, \frac{4}{3}, 0, 0, 0)$	$(0, 0, b_{267})$	$(3, 1, 2)$	$(0, 0, 7)$
$b_{271} = (0, \frac{8}{3}, \frac{2}{3}, 0, 0, 0)$	$(0, 0, b_{270})$	$(3, 2, 1)$	$(0, 0, 8)$
$b_{272} = (0, 3, 0, 0, 0, 0)$	$(0, 0, b_{271})$	$(3, 3, 0)$	$(0, 0, 9)$
$b_{273} = (1, 0, 0, 0, 0, 0)$	$(b_{350}, b_{196}, 0)$	$(4, 0, 0)$	$(2, 1, 0)$
$b_{274} = (1, 0, 0, 0, 0, 1)$	$(b_{351}, b_{197}, 0)$	$(1, 1, 0)$	$(1, 1, 0)$
$b_{275} = (1, 0, 0, 0, 0, 2)$	$(b_{274}, b_{198}, 0)$	$(0, 2, 0)$	$(2, 1, 0)$
$b_{276} = (1, 0, 0, 0, 1, 0)$	$(b_{352}, b_{274}, 0)$	$(2, 0, 3)$	$(1, 2, 0)$
$b_{277} = (1, 0, 0, 0, 1, 1)$	$(b_{301}, b_{275}, 0)$	$(0, 1, 3)$	$(1, 2, 0)$

Table A.2 (continued)

$b_{278} = (1, 0, 0, 0, 2, 0)$	$(b_{302}, b_{277}, 0)$	$(1, 0, 6)$	$(1, 3, 0)$
$b_{279} = (1, 0, 0, \frac{2}{3}, \frac{2}{3}, 0)$	$(b_{353}, b_{202}, b_{276})$	$(2, 0, 2)$	$(1, 1, 1)$
$b_{280} = (1, 0, 0, \frac{2}{3}, \frac{2}{3}, 1)$	$(b_{319}, b_{203}, b_{277})$	$(0, 1, 2)$	$(1, 1, 1)$
$b_{281} = (1, 0, 0, \frac{2}{3}, \frac{5}{3}, 0)$	$(b_{320}, b_{280}, b_{278})$	$(1, 0, 5)$	$(1, 2, 1)$
$b_{282} = (1, 0, 0, \frac{4}{3}, \frac{1}{3}, 0)$	$(b_{354}, b_{205}, b_{279})$	$(2, 1, 1)$	$(1, 1, 2)$
$b_{283} = (1, 0, 0, \frac{4}{3}, \frac{1}{3}, 1)$	$(b_{328}, b_{206}, b_{280})$	$(0, 2, 1)$	$(1, 1, 2)$
$b_{284} = (1, 0, 0, \frac{4}{3}, \frac{4}{3}, 0)$	$(b_{329}, b_{207}, b_{281})$	$(1, 0, 4)$	$(1, 1, 2)$
$b_{285} = (1, 0, 0, 2, 0, 0)$	$(b_{355}, b_{208}, b_{282})$	$(2, 2, 0)$	$(1, 1, 3)$
$b_{286} = (1, 0, 0, 2, 0, 1)$	$(b_{337}, b_{209}, b_{283})$	$(0, 3, 0)$	$(1, 1, 3)$
$b_{287} = (1, 0, 0, 2, 1, 0)$	$(b_{338}, b_{210}, b_{284})$	$(1, 1, 3)$	$(1, 1, 3)$
$b_{288} = (1, 0, 0, \frac{8}{3}, \frac{2}{3}, 0)$	$(b_{339}, b_{211}, b_{287})$	$(1, 2, 2)$	$(1, 1, 4)$
$b_{289} = (1, 0, 0, \frac{10}{3}, \frac{1}{3}, 0)$	$(b_{340}, b_{212}, b_{288})$	$(1, 3, 1)$	$(1, 1, 5)$
$b_{290} = (1, 0, 0, 4, 0, 0)$	$(b_{341}, b_{213}, b_{289})$	$(1, 4, 0)$	$(1, 1, 6)$
$b_{291} = (1, 0, 1, \frac{1}{3}, \frac{1}{3}, 0)$	$(b_{356}, b_{282}, 0)$	$(3, 0, 2)$	$(1, 2, 0)$
$b_{292} = (1, 0, 1, \frac{1}{3}, \frac{1}{3}, 1)$	$(0, b_{283}, 0)$	$(0, 1, 2)$	$(0, 2, 0)$
$b_{293} = (1, 0, 1, \frac{1}{3}, \frac{4}{3}, 0)$	$(0, b_{292}, 0)$	$(1, 0, 5)$	$(0, 3, 0)$
$b_{294} = (1, 0, 1, 1, 0, 0)$	$(b_{357}, b_{285}, 0)$	$(3, 1, 0)$	$(1, 2, 0)$
$b_{295} = (1, 0, 1, 1, 0, 1)$	$(0, b_{286}, 0)$	$(0, 2, 0)$	$(0, 2, 0)$
$b_{296} = (1, 0, 1, 1, 1, 0)$	$(0, b_{287}, 0)$	$(1, 0, 3)$	$(0, 2, 0)$
$b_{297} = (1, 0, 1, \frac{5}{3}, \frac{2}{3}, 0)$	$(0, b_{288}, b_{296})$	$(1, 1, 2)$	$(0, 2, 1)$
$b_{298} = (1, 0, 1, \frac{7}{3}, \frac{1}{3}, 0)$	$(0, b_{289}, b_{297})$	$(1, 2, 1)$	$(0, 2, 2)$
$b_{299} = (1, 0, 1, 3, 0, 0)$	$(0, b_{290}, b_{298})$	$(1, 3, 0)$	$(0, 2, 3)$
$b_{300} = (1, 0, 2, 0, 0, 0)$	$(b_{358}, b_{294}, 0)$	$(4, 0, 3)$	$(1, 3, 0)$
$b_{301} = (1, 0, 2, 0, 0, 1)$	$(0, b_{295}, 0)$	$(1, 1, 3)$	$(0, 3, 0)$
$b_{302} = (1, 0, 2, 0, 1, 0)$	$(0, b_{301}, 0)$	$(2, 0, 6)$	$(0, 4, 0)$
$b_{303} = (1, 0, 2, \frac{2}{3}, \frac{2}{3}, 0)$	$(0, b_{297}, 0)$	$(2, 0, 4)$	$(0, 3, 0)$
$b_{304} = (1, 0, 2, \frac{4}{3}, \frac{1}{3}, 0)$	$(0, b_{298}, 0)$	$(2, 1, 2)$	$(0, 3, 0)$
$b_{305} = (1, 0, 2, 2, 0, 0)$	$(0, b_{299}, 0)$	$(2, 2, 0)$	$(0, 3, 0)$
$b_{306} = (1, 0, 3, \frac{1}{3}, \frac{1}{3}, 0)$	$(0, b_{304}, 0)$	$(3, 0, 5)$	$(0, 4, 0)$
$b_{307} = (1, 0, 3, 1, 0, 0)$	$(0, b_{305}, 0)$	$(3, 1, 3)$	$(0, 4, 0)$
$b_{308} = (1, 0, 4, 0, 0, 0)$	$(0, b_{307}, 0)$	$(4, 0, 6)$	$(0, 5, 0)$
$b_{309} = (1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$	$(b_{359}, b_{232}, b_{291})$	$(3, 0, 1)$	$(1, 1, 1)$
$b_{310} = (1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1)$	$(0, b_{233}, b_{292})$	$(0, 1, 1)$	$(0, 1, 1)$
$b_{311} = (1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{4}{3}, 0)$	$(0, b_{310}, b_{293})$	$(1, 0, 4)$	$(0, 2, 1)$
$b_{312} = (1, \frac{1}{3}, \frac{1}{3}, 1, 0, 0)$	$(b_{360}, b_{235}, b_{309})$	$(3, 1, 0)$	$(1, 1, 2)$
$b_{313} = (1, \frac{1}{3}, \frac{1}{3}, 1, 0, 1)$	$(0, b_{236}, b_{310})$	$(0, 2, 0)$	$(0, 1, 2)$
$b_{314} = (1, \frac{1}{3}, \frac{1}{3}, 1, 1, 0)$	$(0, b_{237}, b_{311})$	$(1, 0, 3)$	$(0, 1, 2)$
$b_{315} = (1, \frac{1}{3}, \frac{1}{3}, \frac{5}{3}, \frac{2}{3}, 0)$	$(0, b_{238}, b_{314})$	$(1, 1, 2)$	$(0, 1, 3)$
$b_{316} = (1, \frac{1}{3}, \frac{1}{3}, \frac{7}{3}, \frac{1}{3}, 0)$	$(0, b_{239}, b_{315})$	$(1, 2, 1)$	$(0, 1, 4)$
$b_{317} = (1, \frac{1}{3}, \frac{1}{3}, 3, 0, 0)$	$(0, b_{240}, b_{316})$	$(1, 3, 0)$	$(0, 1, 5)$

Table A.2 (continued)

$b_{318} = (1, \frac{1}{3}, \frac{4}{3}, 0, 0, 0)$	$(b_{361}, b_{312}, b_{300})$	$(4, 0, 2)$	$(1, 2, 1)$
$b_{319} = (1, \frac{1}{3}, \frac{4}{3}, 0, 0, 1)$	$(0, b_{313}, b_{301})$	$(1, 1, 2)$	$(0, 2, 1)$
$b_{320} = (1, \frac{1}{3}, \frac{4}{3}, 0, 1, 0)$	$(0, b_{319}, b_{302})$	$(2, 0, 5)$	$(0, 3, 1)$
$b_{321} = (1, \frac{1}{3}, \frac{4}{3}, \frac{2}{3}, \frac{2}{3}, 0)$	$(0, b_{315}, b_{303})$	$(2, 0, 3)$	$(0, 2, 1)$
$b_{322} = (1, \frac{1}{3}, \frac{4}{3}, \frac{4}{3}, \frac{1}{3}, 0)$	$(0, b_{316}, b_{304})$	$(2, 1, 1)$	$(0, 2, 1)$
$b_{323} = (1, \frac{1}{3}, \frac{4}{3}, 2, 0, 0)$	$(0, b_{317}, b_{322})$	$(2, 2, 0)$	$(0, 2, 2)$
$b_{324} = (1, \frac{1}{3}, \frac{7}{3}, \frac{1}{3}, \frac{1}{3}, 0)$	$(0, b_{322}, b_{306})$	$(3, 0, 4)$	$(0, 3, 1)$
$b_{325} = (1, \frac{1}{3}, \frac{7}{3}, 1, 0, 0)$	$(0, b_{323}, b_{307})$	$(3, 1, 2)$	$(0, 3, 1)$
$b_{326} = (1, \frac{1}{3}, \frac{10}{3}, 0, 0, 0)$	$(0, b_{325}, b_{308})$	$(4, 0, 5)$	$(0, 4, 1)$
$b_{327} = (1, \frac{2}{3}, \frac{2}{3}, 0, 0, 0)$	$(b_{362}, b_{250}, b_{318})$	$(4, 0, 1)$	$(1, 1, 2)$
$b_{328} = (1, \frac{2}{3}, \frac{2}{3}, 0, 0, 1)$	$(0, b_{251}, b_{319})$	$(1, 1, 1)$	$(0, 1, 2)$
$b_{329} = (1, \frac{2}{3}, \frac{2}{3}, 0, 1, 0)$	$(0, b_{328}, b_{320})$	$(2, 0, 4)$	$(0, 2, 2)$
$b_{330} = (1, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, 0)$	$(0, b_{253}, b_{321})$	$(2, 0, 2)$	$(0, 1, 2)$
$b_{331} = (1, \frac{2}{3}, \frac{2}{3}, \frac{4}{3}, \frac{1}{3}, 0)$	$(0, b_{254}, b_{330})$	$(2, 1, 1)$	$(0, 1, 3)$
$b_{332} = (1, \frac{2}{3}, \frac{2}{3}, 2, 0, 0)$	$(0, b_{255}, b_{331})$	$(2, 2, 0)$	$(0, 1, 4)$
$b_{333} = (1, \frac{2}{3}, \frac{5}{3}, \frac{1}{3}, \frac{1}{3}, 0)$	$(0, b_{331}, b_{324})$	$(3, 0, 3)$	$(0, 2, 2)$
$b_{334} = (1, \frac{2}{3}, \frac{5}{3}, 1, 0, 0)$	$(0, b_{332}, b_{325})$	$(3, 1, 1)$	$(0, 2, 2)$
$b_{335} = (1, \frac{2}{3}, \frac{8}{3}, 0, 0, 0)$	$(0, b_{334}, b_{326})$	$(4, 0, 4)$	$(0, 3, 2)$
$b_{336} = (1, 1, 0, 0, 0, 0)$	$(b_{363}, b_{259}, b_{327})$	$(4, 1, 0)$	$(1, 1, 3)$
$b_{337} = (1, 1, 0, 0, 0, 1)$	$(0, b_{260}, b_{328})$	$(1, 2, 0)$	$(0, 1, 3)$
$b_{338} = (1, 1, 0, 0, 1, 0)$	$(0, b_{261}, b_{329})$	$(2, 0, 3)$	$(0, 1, 3)$
$b_{339} = (1, 1, 0, \frac{2}{3}, \frac{2}{3}, 0)$	$(0, b_{262}, b_{338})$	$(2, 1, 2)$	$(0, 1, 4)$
$b_{340} = (1, 1, 0, \frac{4}{3}, \frac{1}{3}, 0)$	$(0, b_{263}, b_{339})$	$(2, 2, 1)$	$(0, 1, 5)$
$b_{341} = (1, 1, 0, 2, 0, 0)$	$(0, b_{264}, b_{340})$	$(2, 3, 0)$	$(0, 1, 6)$
$b_{342} = (1, 1, 1, \frac{1}{3}, \frac{1}{3}, 0)$	$(0, b_{265}, b_{333})$	$(3, 0, 2)$	$(0, 1, 3)$
$b_{343} = (1, 1, 1, 1, 0, 0)$	$(0, b_{266}, b_{334})$	$(3, 1, 0)$	$(0, 1, 3)$
$b_{344} = (1, 1, 2, 0, 0, 0)$	$(0, b_{343}, b_{335})$	$(4, 0, 3)$	$(0, 2, 3)$
$b_{345} = (1, \frac{4}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$	$(0, b_{268}, b_{342})$	$(3, 1, 1)$	$(0, 1, 4)$
$b_{346} = (1, \frac{4}{3}, \frac{1}{3}, 1, 0, 0)$	$(0, b_{269}, b_{345})$	$(3, 2, 0)$	$(0, 1, 5)$
$b_{347} = (1, \frac{4}{3}, \frac{4}{3}, 0, 0, 0)$	$(0, b_{270}, b_{344})$	$(4, 0, 2)$	$(0, 1, 4)$
$b_{348} = (1, \frac{5}{3}, \frac{2}{3}, 0, 0, 0)$	$(0, b_{271}, b_{347})$	$(4, 1, 1)$	$(0, 1, 5)$
$b_{349} = (1, 2, 0, 0, 0, 0)$	$(0, b_{272}, b_{348})$	$(4, 2, 0)$	$(0, 1, 6)$
$b_{350} = (2, 0, 0, 0, 0, 0)$	$(b_{364}, b_{336}, 0)$	$(5, 0, 0)$	$(1, 2, 0)$
$b_{351} = (2, 0, 0, 0, 0, 1)$	$(0, b_{337}, 0)$	$(2, 1, 0)$	$(0, 2, 0)$
$b_{352} = (2, 0, 0, 0, 1, 0)$	$(0, b_{351}, 0)$	$(3, 0, 3)$	$(0, 3, 0)$
$b_{353} = (2, 0, 0, \frac{2}{3}, \frac{2}{3}, 0)$	$(0, b_{339}, b_{352})$	$(3, 0, 2)$	$(0, 2, 1)$
$b_{354} = (2, 0, 0, \frac{4}{3}, \frac{1}{3}, 0)$	$(0, b_{340}, b_{353})$	$(3, 1, 1)$	$(0, 2, 2)$
$b_{355} = (2, 0, 0, 2, 0, 0)$	$(0, b_{341}, b_{354})$	$(3, 2, 0)$	$(0, 2, 3)$
$b_{356} = (2, 0, 1, \frac{1}{3}, \frac{1}{3}, 0)$	$(0, b_{354}, 0)$	$(4, 0, 2)$	$(0, 3, 0)$
$b_{357} = (2, 0, 1, 1, 0, 0)$	$(0, b_{355}, 0)$	$(4, 1, 0)$	$(0, 3, 0)$

Table A.2 (continued)

$b_{358} = (2, 0, 2, 0, 0, 0)$	$(0, b_{357}, 0)$	$(5, 0, 3)$	$(0, 4, 0)$
$b_{359} = (2, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$	$(0, b_{345}, b_{356})$	$(4, 0, 1)$	$(0, 2, 1)$
$b_{360} = (2, \frac{1}{3}, \frac{1}{3}, 1, 0, 0)$	$(0, b_{346}, b_{359})$	$(4, 1, 0)$	$(0, 2, 2)$
$b_{361} = (2, \frac{1}{3}, \frac{4}{3}, 0, 0, 0)$	$(0, b_{360}, b_{358})$	$(5, 0, 2)$	$(0, 3, 1)$
$b_{362} = (2, \frac{2}{3}, \frac{2}{3}, 0, 0, 0)$	$(0, b_{348}, b_{361})$	$(5, 0, 1)$	$(0, 2, 2)$
$b_{363} = (2, 1, 0, 0, 0, 0)$	$(0, b_{349}, b_{362})$	$(5, 1, 0)$	$(0, 2, 3)$
$b_{364} = (3, 0, 0, 0, 0, 0)$	$(0, b_{363}, 0)$	$(6, 0, 0)$	$(0, 3, 0)$