ABSTRACT

BENFIELD, MICHAEL RAY. Some Geometric Aspects of Hyperbolic Conservation Laws. (Under the direction of Irina Kogan.)

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Motivated by the search for examples of blowup solutions in hyperbolic conservation laws, we investigate certain geometric features of these systems. In Chapter 2, building on work of Jenssen and Kogan, we investigate the problem of constructing a system of conservation laws with certain prescribed features, notably rarefaction curves. Chapter 3 looks at systems with *solvable interactions*; that is, systems whose interactions can be given by explicit formulas. Chapter 4 explores further geometric properties of the shock and rarefaction curves of a system of conservation laws.

Some Geometric Aspects of Hyperbolic Conservation Laws

by
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BIOGRAPHY

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Introduction

1.1 Problem and motivation

Hyperbolic conservation laws originate in the study of systems with waves propogating at finite speed, in particular in gas dynamics. Euler [7, 8] derived equations for barotropic gas flow, initiating research into continuum physics which would eventually lead to the study of general hyperbolic conservation laws. Several prominent nineteenth century mathematicians studied these and related equations, including Poisson, Stokes, and Riemann. The 1957 paper of Lax [19] is considered to have initiated the field as an important branch of partial differential equations.

Most theoretical results have applied to the case of one spatial dimension; there have been considerable difficulties in extending these results to multiple dimensions. Nevertheless, there remain interesting questions in the one dimensional case, and it is this case we examine in the present work. A hyperbolic conservation law in one space

dimension is a system of partial differential equations (PDEs) of the form

$$U_t + F(U)_x = 0$$

for an unknown function $U(x,t) \in \mathbb{R}^n$.

As in other types of partial differential equations, a central question in hyperbolic conservation laws is whether and in what sense they are well-posed. An initial value problem is well-posed if it has a unique solution that depends continuously on the initial data. Lax [19] provided a solution to the called Riemann problem, a particular type of initial value problem which will be described more fully in Section 1.6. Building on this solution, Glimm [9] developed the random choice method for constructing globally defined solutions to the Cauchy problem. Bressan, Liu, and Yang [4] used the front tracking method to prove the well-posedness of the Cauchy problem for small initial data. Both the random choice and front tracking methods rely on solving the Riemann problem, which we discuss in Section 1.6.

The initial motivation behind the present work was to explore the extent to which the Cauchy problem is ill-posed for large initial data. The general results mentioned above show that, for data sufficiently small in BV and L^∞ and under certain general conditions on F, weak solutions exist for all time. Examples for which weak solutions do not exist for all time (in the case of large initial data) have been presented by Jenssen [11], Baiti and Jenssen [2], Young [28], and Young and Szeliga [30]. One question raised by these examples is the extent to which well-posedness depends on the *uniform* hyperbolicity of the initial data and its solution: the examples in [2] and [30] both fail to exhibit uniform hyperbolicity (in a limiting sense in the case of [2]). It would be desirable to find examples of Cauchy problems which do not have global solutions despite being uniformly strictly

hyperbolic.

As a step towards this goal, we examined the geometry of the wave curves that characterize shocks and rarefactions. The example in [11] features a certain geometric arrangement of wave curves which causes solutions to blow up. A systematic approach to constructing systems with prescribed features would be helpful in creating additional examples.

Chapter 2 studies geometric features of conservation laws, continuing work of Jenssen and Kogan [12, 13, 14]. The above references describe the degrees of freedom in constructing hyperbolic conservation laws with prescribed rarefaction curves, which amounts to prescribing eigenvector fields of the flux Jacobian DF. The case of rich frames is studied for arbitrary n, while the results for non-rich frames apply to the case $n = 3$. (See Definition 1.18 for the term *rich*.) In the present work, we prove a new result (Theorem 2.2), which partly extends the earlier results on non-rich frames to arbitrary n. In Section 2.2 we look at what happens when we do not prescribe a full frame of eigenvector fields. The results largely apply to the case of prescribing two eigenvector fields when $n = 3$.

Chapter 3 explores the idea of solvable interactions, with the goal of producing examples of conservation laws whose solutions can be given explicitly. Section 3.1 presents a new result on analytic interactions, providing a new justification for the interest in coinciding shock and rarefaction curves (also explored in Chapter 4). Section 3.2 describes known results on linear interactions, while Section 3.3 gives new results on quadratic interactions: when the outgoing wave strengths are quadratic functions of the incoming wave strengths.

Chapter 4 examines the relationship between the shock and rarefaction curves that make up the wave curves. Theorem 4.2 is a new result that in one important subclass of conservation laws, the rarefaction curves determine the shock curves. The results in

Section 4.2 are all new extensions of earlier results of Temple, stated by him for the case $n = 2$ but now extended to arbitrary n.

In the rest of this chapter, we give some background in geometry and conservation laws that will be used in the rest of the work. This is largely known, however Theorem 1.12, which generalizes the well-known Frobenius theorem, appears to be new.

1.2 Geometry of vector fields on $\mathbb{R}^n$

Before beginning our study of conservation laws, we give a few definitions and results from differential geometry that will prove useful.

A basic result in smooth manifold theory is that vector fields are in a one-to-one correspondence with differential operators on smooth functions $\Omega \to \mathbb{R}$. The differential operator is the directional derivative; for a vector field R and smooth function f we denote this by $R(f)$. The value of a vector field R at $U \in \Omega$ will be denoted by R_U. An exception is when the vector field in question is a flux F (*flux* will be defined in Section 1.4), when we will follow the usual practice and write $F(U)$.

Definition 1.1. *A* smooth vector field *on an open set $\Omega \subseteq \mathbb{R}^n$ is a smooth map $R : \Omega \to \Omega \times \mathbb{R}^n$ such that $R_U \in \{U\} \times \mathbb{R}^n$ for each $U \in \Omega$.*

The intent of the preceding definition is that a vector remembers its base point in Ω.

We will use a standing assumption that all vector fields and functions are smooth (infinitely differentiable) unless otherwise specified. Also, R, S, and T, as well as $\{R_i\}$, are smooth vector fields on an open set $\Omega \subseteq \mathbb{R}^n$, and U is a coordinate system on Ω.

Definition 1.2. *A* frame *on $\Omega \subseteq \mathbb{R}^n$ is a set of n vector fields that form a basis at each $U \in \Omega$.*

Definition 1.3. *The* Lie bracket $[R, S]$ *is the vector field on* Ω *such that, for all smooth functions* f*, we have* $[R, S](f) = R(S(f)) - S(R(f))$.

In view of the correspondence between vector fields and derivations of smooth functions, this serves to define $[R, S]$ as a vector field (see [20], Chapter 3).

Definition 1.4. *Given a frame* $\{R_i\}_{i=1}^n$*, the* structure coefficients c_{ij}^k *are functions of* U*, the coefficients of the Lie bracket:*

$$[R_i, R_j] = \sum_k c_{ij}^k R_k.$$

Definition 1.5. *A parametrized curve* $a \mapsto \phi^a$ *in* $\mathbb{R}^n$ *is an* integral curve *of* R *if* $\frac{\partial}{\partial a}\phi^a = R_{\phi^a}$*. More generally, if for each* $U \in \Omega$*,* $a \mapsto \phi^a(U)$ *is an integral curve of* R *with* $\phi^0(U) = U$*, then* ϕ *is a* flow *of* R.

Elementary ODE theory guarantees that a flow exists locally at each U for smooth R ([20], Theorem 9.12). Note that if we scale R to fR for a nonvanishing scalar function f, the resulting flow has a different parametrization but traces the same curve.

The following lemma will be used in Chapter 3. It may seem to be a technical result, but it actually has an intuitive geometric meaning: it establishes that in some sense the Lie bracket $[R, S]$ measures the extent to which S varies along the flow of R.

Given a function $G : \Omega \to \mathbb{R}^m$, DG is its Jacobian matrix. For the following lemma we will also need the second order differential, which we denote D^2G.

Lemma 1.6. *If:*

1. $U \in \Omega$*;*

2. ϕ *is the flow of* R*;*

3. $Y(a) := D\phi^{-a}_{\phi^a(U)}\, S_{\phi^a(U)}$*, for* a *in some interval* $(-\epsilon, \epsilon)$*, is a curve in* $\mathbb{R}^n$*;*

then

$$Y'(a) = D\phi^{-a}_{\phi^a(U)}\, [R, S]_{\phi^a(U)}. \tag{1.1}$$

Proof. Introduce a base point parameter to Y:

$$Y(a; U) = D\phi^{-a}_{\phi^a(U)}\, S_{\phi^a(U)}. \tag{1.2}$$

Differentiate Y:

$$\begin{aligned} Y'(a; U) =& \frac{\partial}{\partial a} \left(D\phi^{-a}\right)_{\phi^a(U)}\, S_{\phi^a(U)} + D\phi^{-a}_{\phi^a(U)}\, DS_{\phi^a(U)}\, R_{\phi^a(U)} \\ &+ D^2\phi^{-a}_{\phi^a(U)} \left(S_{\phi^a(U)}, R_{\phi^a(U)}\right). \end{aligned} \tag{1.3}$$

The first term is

$$\begin{aligned} \frac{\partial}{\partial a} \left(D\phi^{-a}\right)_{\phi^a(U)}\, S_{\phi^a(U)} &= D\left(\frac{\partial}{\partial a}\phi^{-a}\right)_{\phi^a(U)} S_{\phi^a(U)} \\ &= -DR_{\phi^{-a}(U)}\, S_{\phi^a(U)}. \end{aligned}$$

We will first prove equation (1.1) for $a = 0$ only, and then indirectly prove it for other a. Note that $D\phi^0$ is the identity linear map and, consequently, $D^2\phi^0 = 0$. Thus substituting $a = 0$, equation (1.3) becomes

$$Y'(0; U) = -DR_U S_U + DS_U R_U = [R, S]_U,$$

as desired.

To establish the equation for a in general, first see that

$$\begin{aligned} Y(a+b;U) &= D\phi^{-a-b}_{\phi^{a+b}(U)} S_{\phi^{a+b}(U)} \\ &= D\phi^{-a}_{\phi^a(U)}\, D\phi^{-b}_{\phi^b(\phi^a(U))}\, S_{\phi^b\phi^a(U)} \\ &= D\phi^{-a}_{\phi^a(U)}\, Y(b;\phi^a(U)). \end{aligned} \tag{1.4}$$

This can be seen by the semigroup property of ϕ:

$$\phi^{-a}\phi^{-b}(U) = \phi^{-a-b}(U).$$

Apply D:

$$D\phi^{-a}_{\phi^{-b}(U)} D\phi^{-b}(U) = D\phi^{-a-b}(U).$$

Make the substitution $U = \phi^{a+b}$:

$$D\phi^{-a}_{\phi^a(U)} D\phi^{-b}_{\phi^b(\phi^a(U))} = D\phi^{-a-b}_{\phi^{a+b}(U)},$$

establishing equation (1.4).

Finally,

$$\begin{aligned} Y'(a;U) &= \frac{\partial}{\partial b}\Big|_{b=0} Y(a+b;U) \\ &= D\phi^{-a}_{\phi^a(U)} Y'(0;\phi^a(U)) \\ &= D\phi^{-a}_{\phi^a(U)} [R,S]_{\phi^a(U)}, \end{aligned}$$

as was to be shown. □

The special case $Y'(0) = [R,S]$ of the above lemma is sometimes taken as the defini-

tion of the Lie bracket.

The next lemma gives a clearer interpretation of the Lie bracket.

Lemma 1.7. *If:*

1. $U \in \Omega$;

2. ϕ *is the flow of* R *and* ψ *is the flow of* S;

then

$$\psi^b \phi^a U = \phi^a \psi^b U + ab[R, S] + O(a^2 b + ab^2).$$

Proof. We will find the second order expansion of

$$G(a, b) := \psi^b \phi^a U - \phi^a \psi^b U.$$

Expand the first term in b and the second in a:

$$\phi^a U + bS(\phi^a U) - \psi^b U - aR(\psi^b U) + O(a^2) + O(b^2).$$

Now expand the first two terms in a and the second two in b:

$$\begin{aligned} &U + aR(U) + bS(U) + abR(S) \\ &-U - bS(U) - aR(U) - abS(R) + O(a^2 b) + O(ab^2) + O(a^2) + O(b^2). \end{aligned}$$

Here we have used the fact that

$$R(\psi^b U) = R(U) + bDR\, S(U) + O(b^2) = R(U) + bS(R)(U) + O(b^2),$$

along with a similar expansion of $S(\phi^a U)$. Simplify to obtain

$$ab[R,S] + O(a^2 b) + O(ab^2) + O(a^2) + O(b^2).$$

Since $G(a,0) = G(0,b) = 0$, there are no terms in the expansion involving only powers of a or only powers of b. Thus $O(a^2) = O(b^2) = 0$ and the theorem is proved. □

The symbol ∇ is used in geometry for the so-called *covariant derivative*. This is a differential operator that can be defined in great generality on manifolds. For a point p on a manifold M with tangent bundle $\mathcal{X}(M)$, a covariant derivative is a map $\mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ given by $(V, R) \mapsto \nabla_V R$ satisfying the following:

1. the map $V \mapsto \nabla_V R$ is linear over $\mathbb{R}$;
2. the map $R \mapsto \nabla_V R$ is additive;
3. the operator ∇ obeys the product rule, so that, for a smooth function f, $\nabla_V(fR) = \nabla_V(f)R + f\nabla_V R$ (here $\nabla_V f$ is the ordinary directional derivative of f in the V direction).

We will not need the full generality of the covariant derivative; for us, $M = \Omega \subseteq \mathbb{R}^n$ will be the space of conserved variables U, and ∇ will represent the ordinary directional derivative. That is, $\nabla_R S = DS\,R$. Nevertheless, we will use some results from geometry that are traditionally and conveniently expressed in this notation, so we adopt it here.

Definition 1.8. *Given a frame* $\{R_i\}$*, the* Christoffel symbols *are the coefficients* Γ_{ij}^k *(*$1 \le i, j, k \le n$*) in the basis* $\{R_k\}$ *of* $R_i(R_j)$*. That is,*

$$\nabla_{R_i} R_j = \sum_k \Gamma_{ij}^k R_k.$$

Theorem 1.9 (Symmetry)**.** *The vector fields satisfy*

$$\nabla_R S - \nabla_S R = [R, S]. \tag{1.5}$$

Proof. We apply $[R, S]$ to some smooth function f:

$$[R, S](f) = R(S(f)) - S(R(f)).$$

Write out R and S in coordinates: $R = \sum_i R^i e_i$ and $S = \sum_i S^i e_i$, where $\{e_i\}$ are the standard basis vectors. Then applying the derivatives on the right hand side of our displayed equation, we have

$$R\left(\sum_i S^i \frac{\partial f}{\partial x^i}\right) - S\left(\sum_i R^i \frac{\partial f}{\partial x^i}\right).$$

We apply the next sequence of derivatives:

$$\sum_{i,j} R^j \left(\frac{\partial S^i}{\partial x^j}\frac{\partial f}{\partial x^i} + S^i \frac{\partial^2 f}{\partial x^j \partial x^i}\right) - \sum_{i,j} S^j \left(\frac{\partial R^i}{\partial x^j}\frac{\partial f}{\partial x^i} + R^i \frac{\partial^2 f}{\partial x^j \partial x^i}\right).$$

All the terms with second derivatives of f cancel and we are left with

$$\sum_{i,j} R^j \frac{\partial S^i}{\partial x^j}\frac{\partial f}{\partial x^i} - \sum_{i,j} S^j \frac{\partial R^i}{\partial x^j}\frac{\partial f}{\partial x^i} = (\nabla_R S - \nabla_S R)(f).$$

Since $[R, S]$ has the same effect when applied to smooth functions as does $\nabla_R S - \nabla_S R$, they are the same vector field. □

Note that this symmetry property implies the following relationship between the

Christoffel symbols and structure coefficients:

$$c_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k.$$

Incidentally, Theorem 1.9 serves as a proof that $[R, S]$ actually is a vector field (or, in other words, the map $f \mapsto [R, S](f) = R(S(f)) - S(R(f))$ really is a derivation on smooth functions).

Theorem 1.10 (Flatness)**.** *The vector fields satisfy*

$$\nabla_R \nabla_S T - \nabla_S \nabla_R T = \nabla_{[R,S]} T. \tag{1.6}$$

Proof. Write T in coordinates: $T = \sum_i T^i e_i$. Then the left hand side of our equation becomes

$$\nabla_R \nabla_S \sum_i T^i e_i - \nabla_S \nabla_R \sum_i T^i e_i.$$

Using additivity of ∇, as well as the fact that $\nabla e_i = 0$, this becomes

$$\sum_i \left(\nabla_R S(T^i) - \nabla_S R(T^i) \right) e_i.$$

But according to Theorem 1.9, the coefficient of e_i that appears here is just $[R, S]$ applied to T^i, so we have

$$\sum_i \left([R, S](T^i) \right) e_i = \nabla_{[R,S]} T$$

as desired. □

Definition 1.11. *A set $\mathcal{S}$ of vector fields is* involutive *if, for every pair $X, Y \in \mathcal{S}$, we have $[X, Y] \in \operatorname{span} \mathcal{S}$.*

1.3 Integrability theorems

We will be working with overdetermined systems of first order partial differential equations, particularly in Chapter 2. Here we discuss two important integrability theorems we will use. The first is a generalization of the classical Frobenius theorem that appears to be new.

There are several related theorems called the Frobenius Theorem. A common modern formulation deals with finding integral manifolds of distributions; see e.g. Theorem 19.12 of [20]. Classical versions of the Frobenius Theorem deal with overdetermined first order systems of partial differential equations. These theorems require each unknown function to appear differentiated in every direction. For our purposes, we will need a generalization of the classical Frobenius theorem which we could not find in the literature. For this version of the theorem, the unknowns $\{A^i\}$ are only differentiated in certain directions $\{R_j\}$, and initial data ψ^i for each unknown are prescribed on a manifold Λ transverse to the span of the vectors $\{R_j\}$.

The theorem reduces to the classical Frobenius theorem by taking Λ to be a single point (a 0-dimensional manifold), and $\{R_j\}$ to be a frame. In this situation, our initial data $\{\psi^j\}$ consists of $m = n$ constants.

Theorem 1.12 (Generalized Frobenius)**.** *Suppose*

(H1) $\Lambda \subseteq \Omega$ *is an embedded codimension* m *smooth submanifold;*

(H2) $\{R_j\}_{j=1}^m$ *is an involutive set of linearly independent vector fields on* Ω*, whose span is complimentary to the tangent space of* Λ *at each point.*

Consider the following system of differential equations:

$$R_j(A^i) = h^i_j(\cdot, A(\cdot)), \quad j = 1, \ldots, m, \quad i = 1, \ldots, p, \tag{1.7}$$

where the h^i_j are smooth and the $A^i : \Omega \to \mathbb{R}$ are unknown functions. If

(H3) the integrability conditions

$$R_j(R_k(A^i)) - R_k(R_j(A^i)) = \sum_{\ell=1}^{m} c^\ell_{jk} R_\ell(A^i) \tag{1.8}$$

are satisfied identically upon making substitutions from the system (1.7)*;*

(H4) $\{\psi^i\}_{i=1}^p$ are smooth functions on Λ whose images are in the domain of the functions h^i_j, so that $h^i_j(U, \psi(U))$ is well defined;

then for any $\overline{U} \in \Lambda$, there is some neighborhood $\Omega' \subseteq \Omega$ on which there exists a unique solution $\{A^i\}_{i=1}^p$ of (1.7) *satisfying*

$$A^i(U) = \psi^i(U) \text{ for } U \in \Lambda; \tag{1.9}$$

Moreover, $A = (A^1, \ldots, A^p)$ is smooth.

To keep the statement of the theorem brief, hypothesis (H3) was stated in an informal way. Before proving the theorem, we elaborate on what precisely we mean that the integrability conditions are satisfied identically. In equation (1.8), make substitutions from the system (1.7) to obtain

$$R_j(h^i_k(U, A(U))) - R_k(h^i_j(U, A(U))) = \sum_{\ell=1}^{m} c^\ell_{jk} h^i_\ell(U, A(U)).$$

Take the derivatives on the left hand side to obtain

$$D_U h_k^i(U,A)\cdot R_j + \sum_{\ell=1}^{m} \frac{\partial h_k^i}{\partial A^\ell}(U,A) h_j^\ell(U,A) - D_U h_j^i(U,A)\cdot R_k - \sum_{\ell=1}^{m} \frac{\partial h_j^i}{\partial A^\ell}(U,A) h_k^\ell(U,A) = \sum_{\ell=1}^{m} c_{jk}^\ell h_\ell^i(U,A). \quad (1.10)$$

Now treat $A = (A^1, \ldots, A^p)$ as simply coordinates rather than unknown functions, and it is equation (1.10) that must be satisfied in U and A.

Proof. We establish some notation and a new coordinate system. Let $\|\cdot\|$ be the sup norm on $\mathbb{R}^p$ or $\mathbb{R}^n$. Let ϕ_j be the flow of R_j. Shrink Λ and the domain of each h_j to $\mathcal{B}_r(\overline{U}) \times \mathcal{B}_r(\psi(\Lambda))$. Here $\mathcal{B}_r(X)$ is the open ball of radius r in the sup norm about X, and r is chosen so that each h_j is bounded by a common $M = \sup_{i,j,U,A} |h_j^i(U,A)|$. Let $T > 0$ be less than $\frac{r}{mM}$, and also small enough so that $(a_1, \ldots, a_m) \mapsto W_m^{a_m} \cdots W_1^{a_1} \overline{U}$ is well defined whenever each $|a_j| \leq T$. Put $\Lambda' = \Lambda \cap \mathcal{B}_r(\overline{U})$. Let L be a Lipschitz constant for the maps $A \mapsto h_j^i(U,A)$, uniform in i, j, and U.

Consider the map $[-T,T]^m \times \Lambda' \to \mathbb{R}^n$ given by $(a_1, \ldots, a_m, V) \mapsto W_m^{a_m} \cdots W_1^{a_1} V$. The fact this map is smooth is a difficult theorem; see [18], pages 371–379. This map is also locally invertible, which follows from the fact that $\operatorname{span}\{R_j\}$ and the tangent space of Λ' are complimentary. Thus, possibly after choosing a smaller value for T and shrinking Λ', we have a diffeomorphism $[-T,T]^m \times \Lambda' \to \Omega'$, where $\Omega' \subseteq \Omega$ is a neighborhood of $\overline{U}$ in $\mathbb{R}^n$. For any point $U \in \Omega'$, we have coordinates $(a_1, \ldots, a_m, V)$, where $V = (v_1, \ldots, v_{n-m})$ are coordinates on Λ'.

Define $\Xi_0 = \Lambda'$, and for each j with $1 \leq j \leq m$, define inductively the following

smooth manifolds:

$$\Xi_j = \{\phi_j^a(Q) : Q \in \Xi_{j-1} \text{ and } a \text{ is in the domain of } \phi_j(Q).\}$$

That is, Ξ_j is obtained by flowing along R_j from each point of Ξ_{j-1}. That these are smooth manifolds follows from the fact that $(a_1, \ldots, a_m, V)$ is a coordinate system; Ξ_j is the level set given by $a_k = 0$ for $k > j$.

Claim 1. There is a unique function $A : \Omega' \to \mathbb{R}^p$ satisfying (1.9) and

$$R_j(A^i)(U) = h_j^i(U, A(U)), \quad 1 \le j \le m,\ 1 \le i \le p,\ U \in \Xi_j. \tag{1.11}$$

That is, A satisfies our desired equations, but perhaps not everywhere in Ω'. Moreover, A is smooth.

This statement will be justified by applying the contraction mapping theorem in the style of Picard. Let $\mathcal{C}$ be the set of continuous functions $A : \Omega' \to \mathcal{B}_r(\psi(\Lambda'))$. Define a functional $\zeta : \mathcal{C} \to \mathcal{C}$, given by

$$\begin{aligned}\zeta(G)^i(U) = \psi^i(V) + \int_0^{a_1} h_1^i(\cdot, G(\cdot))\left(\phi_1^b(V)\right)\, db + \int_0^{a_2} h_2^i(\cdot, G(\cdot))\left(\phi_2^b\phi_1^{a_1}(V)\right)\, db \\ + \cdots + \int_0^{a_m} h_m^i(\cdot, G(\cdot))\left(\phi_m^b \cdots \phi_2^{a_2}\phi_1^{a_1}(V)\right)\, db.\end{aligned} \tag{1.12}$$

Here $h_j^i(\cdot, G(\cdot))(Z) = h_j^i(Z, G(Z))$. The fact that each $|a_j| \le T < \frac{r}{mM}$ ensures that $\zeta(G)(U) \in \mathcal{B}_r(\psi(\Lambda'))$ and thus $\zeta(G) \in \mathcal{C}$.

Introduce a metric on $\mathcal{C}$ given by

$$d(A, B) = \sup_{U \in \Omega'} e^{-2mL\sum_j |a_j|} \|A(U) - B(U)\|.$$

The function ζ is a contraction mapping on the complete metric space $\mathcal{C}$. To see this, observe that for each i,

$$\begin{aligned}\left|\zeta(G)^i(U) - \zeta(H)^i(U)\right| &\leq \sum_{j=1}^{m} \int_0^{a_j} \left|[h_j^i(\cdot, G(\cdot)) - h_j^i(\cdot, H(\cdot))] \left(\phi_j^b \cdots \phi_2^{a_2}\phi_1^{a_1}(V)\right)\right| \, db \\ &\leq \sum_{j=1}^{m} \int_0^{a_j} L\, d(G,H) e^{2mL\left(|b| + \sum_{k=1}^{j} |a_k|\right)} \, db \\ &\leq \frac{1}{2} d(G,H)(e^{2mL\sum_j |a_j|} - 1).\end{aligned}$$

Multiply both sides of this inequality by $e^{-2mL\sum_j |a_j|}$ to obtain

$$e^{-2mL\sum_j |a_j|}|\zeta(G)^i(U) - \zeta(H)^i(U)| \leq \frac{1}{2}\, d(G,H).$$

Thus $d(\zeta(G), \zeta(H)) \leq \frac{1}{2}\, d(G,H)$, and ζ is a contraction mapping, as was to be shown.

The fixed point A can be seen to be smooth by plugging A into (1.12) and obtaining an expression infinitely differentiable in V and each a_j.

To see that A satisfies (1.11), note that on Ξ_j, A satisfies

$$\begin{aligned}A^i(U) = \psi^i(V) + \int_0^{a_1} h_1^j(\cdot, A(\cdot)) \left(\phi_1^b(V)\right) \, db + \int_0^{a_2} h_2^j(\cdot, A(\cdot)) \left(\phi_2^b \phi_1^{a_1}(V)\right) \, db \\ + \cdots + \int_0^{a_j} h_j^j(\cdot, A(\cdot)) \left(\phi_j^b \cdots \phi_2^{a_2}\phi_1^{a_1}(V)\right) \, db. \quad (1.13)\end{aligned}$$

That is, the terms involving a_k for $k > j$ are absent. Clearly $\frac{\partial A^i}{\partial a_j}(U) = h_j^j(U, A(U))$. Since $R_j = \frac{\partial}{\partial a_j}$ on Ξ_j, equation (1.11) is established.

Claim 2. The function A established in the previous equation satisfies the desired equation (1.7).

Certainly the equation (1.7) holds for all $j = 1, \ldots, m$ on Ξ_1, since $\Xi_1 \subseteq \Xi_j$ for each

j. Inductively, suppose equation (1.7) holds on Ξ_k for all j. By construction of A, we already have that equation (1.7) holds on Ξ_{k+1} for $j > k$; we now show that it also holds on Ξ_{k+1} for $j \leq k$. Consider the following initial value problem on unknown functions q_j^i, $1 \leq j \leq k$, $1 \leq i \leq p$:

$$R_{k+1}(q_j^i)(\cdot) = R_j(h_{k+1}^i(\cdot, A(\cdot))) + \sum_{\ell=1}^{k} c_{k+1,j}^{\ell}(\cdot) q_\ell^i(\cdot) + \sum_{\ell=k+1}^{m} c_{k+1,j}^{\ell}(\cdot) h_\ell^i(\cdot, A(\cdot)); \tag{1.14}$$

$$q_j^i(U) = h_j^i(U, A(U)) \quad \text{for } U \in \Xi_k. \tag{1.15}$$

One solution to this initial value problem is given by $q_j^i = R_j(A^i)$, $1 \leq j \leq k$, $1 \leq i \leq p$. This is so since

$$R_{k+1}(R_j(A^i)) = R_j(R_{k+1}(A^i)) + [R_{k+1}, R_j]A^i.$$

From this equation we see that, on Ξ_{k+1},

$$\begin{aligned} R_{k+1}(R_j(A^i)) &= R_j(R_{k+1}(A^i)) + \sum_{\ell=1}^{m} c_{k+1,j}^{\ell} R_\ell(A^i) \\ &= R_j(R_{k+1}(A^i) + \sum_{\ell=1}^{k} c_{k+1,j}^{\ell} R_\ell(A^i) + \sum_{\ell=k+1}^{m} c_{k+1,j}^{\ell} h_\ell^i(\cdot, A(\cdot)) \end{aligned}$$

since we already have $R_\ell(A^i) = h_\ell^i(\cdot, A(\cdot))$ on Ξ_{k+1} for $\ell \geq k+1$.

Another solution to the initial value problem is given by $q_j^i = h_j^i(\cdot, A(\cdot))$, $1 \leq j \leq k$, $1 \leq i \leq p$. This is clear since the the differential equation (1.14) is exactly the integrability condition (1.8) that was assumed to hold.

Since solutions to such initial value problems are unique (see e.g. [20], Theorem 9.12), we must have $R_j(A^i) = h_j^i(\cdot, A(\cdot))$, as was to be shown. □

The following theorem is a generalization of Theorem 1.12, applicable to the case

when not every derivative $R_j(A^i)$ is specified, and the vector fields R_j are coordinate directions. The theorem appears as Theorem III in Book III, Chapter I of [6]. In our notation, $\partial_j = \frac{\partial}{\partial u^j}$.

Theorem 1.13 (Darboux)**.** *Consider a system of PDEs of the form*

$$\partial_j(A^i) = h^i_j(U, A) \tag{1.16}$$

in which each A_i appears differentiated only in certain directions ∂_j for $j \in \sigma(i)$, where $\sigma(i)$ is some set of indices. Suppose the integrability conditions

$$\partial_j h^i_k(U, A) = \partial_k h^i_j(U, A) \tag{1.17}$$

are satisfied identically and $\overline{U} = (\overline{u}^1, \ldots, \overline{u}^n) \in \Omega$. For each function A^i, let $\widetilde{U}_i = (u^j)_{j\in\sigma(i)}$ be the independent variables with respect to which A^i appears differentiated, and let $\hat{U}_i = (u^j)_{j\notin\sigma(i)}$ be the remaining independent variables. Then for each choice of initial data functions $\psi^i(\widetilde{U}_i)$ (for each i), there is a unique local solution $A = (A^1, \ldots, A^m)$ of the system 1.16 satisfying the initial condtions $A^i(\widetilde{U}, \hat{U}) = \psi^i(\widetilde{U})$ for each i.

By stating that the the integrability conditions (1.17) should be satisfied identically, we mean that only derivatives which actually appear in the system (1.16) are in the integrability equations (1.17), and upon making substitutions from the system (1.16), equations (1.17) are then identities.

We end this section with two simple observations regarding the integrability conditions that appear in Theorems 1.12 and 1.13. Sometimes our equations will naturally appear as differentiated vector fields: $\nabla_{R_j} G = H(U, G)$. To read the statements of the theorems, it would appear that we must write out each component of the differentiated vector field

to check integrability conditions. That is, we could write

$$\sum_i R_j(G^i)e_i = \sum_i H^i(U,G)e_i$$

where $\{e_i\}$ are the coordinate vector fields. Then we could consider our system to be

$$R_j(G^i) = H^i(U,G)$$

and check the integrability conditions (1.8).

But in fact, the integrability conditions are equivalent to the flatness condition

$$\nabla_{R_j}\nabla_{R_k}G - \nabla_{R_k}\nabla_{R_j}G = \nabla_{[R_j,R_k]}G. \tag{1.18}$$

Thus, in general, this is the condition we will verify.

The second observation relates only to the Darboux theorem 1.13. The theorem is stated for the case that each $R_j = \partial_j$ is a coordinate vector field. However, it is only necessary that they be coordinate vector fields in *some* coordinate system, not necessarily the coordinate system we were given. The condition that $\{R_j\}$ is a coordinate frame in some coordinate system is equivalent to each Lie bracket vanishing: $[R_j, R_k] = 0$ for each pair j, k. (This equivalence follows from the Generalized Frobenius Theorem 1.12 stated above, or see Theorem 5.14 of [24].)

1.4 Conservation laws in one space dimension

The remainder of this chapter gives the fundamental definitions and theorems in the field of hyperbolic conservation laws. More detailed treatments of this material can be found

in [5], [23], and [3].

An $n \times n$ *system of conservation laws in one space dimension* is a differential equation of the form

$$U_t + F(U)_x = 0 \tag{1.19}$$

where we assume

- the *flux* $F : \Omega \to \mathbb{R}^n$ is smooth, with $\Omega \subseteq \mathbb{R}^n$ open (the theory can be developed with F only C^3, but we will not need this level of generality);
- $U : \mathbb{R} \times [0, \infty) \to \Omega$ is an unknown function;
- (x, t) are coordinates on $\mathbb{R} \times [0, \infty)$ (typically these are space and time in applications).

We may also impose the *initial condition*

$$U(x, 0) = U_0(x), \tag{1.20}$$

where $U_0 : \mathbb{R} \to \Omega$.

Definition 1.14. *If the Jacobian DF_U is diagonalizable over $\mathbb{R}$ at each $U \in \Omega$, then equation (1.19) is* hyperbolic *in Ω.*

If there are n distinct eigenvalues, the equation is strictly hyperbolic.

In the strictly hyperbolic case we label the eigenvalues $\{\lambda_i\}_{i=1}^n$, with indices chosen so that

$$\lambda_1(U) < \ldots < \lambda_n(U) \text{ at each } U \in \Omega.$$

We label the corresponding eigenvectors $\{R_i\}_{i=1}^n$.

Definition 1.15. *A pair* (R_i, λ_i) *is a* characteristic field.

Note that the intent is to capture the *eigenspace* in question and not just the eigenvector. Thus if f is a nonvanishing function, we will consider (R_i, λ_i) and (fR_i, λ_i) to be the same characteristic field.

A ubiquitous phenomenon in nonlinear hyperbolic conservation laws is the breakdown of classical solutions. We illustrate this phenomenon with a well known example.

Example 1.16. The scalar equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \tag{1.21}$$

is called *Burgers' equation.*

We impose initial data

$$u_0(x) = e^{-x^2}.$$

We use the method of characteristics to find the solution is parametrized by (x_0, t) as:

$$(x, t, u) = (te^{-x_0^2} + x_0, t, e^{-x_0^2}).$$

This defines u as a function of (x, t) whenever $x =: f_t(x_0) = te^{-x_0^2} + x_0$ is invertible. We calculate $f_t'(x_0) = -2tx_0e^{-x_0^2} + 1$. Since f_t' is never 0 for $t \in [0, \frac{1}{\sqrt{2}})$, such f_t are invertible, but for $t > \frac{1}{\sqrt{2}}$, f_t is not invertible. In terms of characteristics, the characteristics overlap beginning at time $t > \frac{1}{\sqrt{2}}$. No continuous function is a solution to this initial value problem past this time.

The preceding example demonstrates that we must consider weak solutions in the theory of hyperbolic conservation laws. In the following definition, $L^1_{\text{loc}}(\mathbb{R} \times [0, \infty))$ is the

space of locally integrable functions, and $C_0^\infty(\mathbb{R}\times[0,\infty))$ is the space of smooth functions with compact support.

Definition 1.17. *The function $U \in L^1_{loc}(\mathbb{R}\times[0,\infty))$, with $t \mapsto U(\cdot,t)$ a continuous function $[0,\infty) \to L^1_{loc}(\mathbb{R})$, is a* weak solution *to the Cauchy problem if, for every test function $g \in C_0^\infty(\mathbb{R}\times[0,\infty))$, this holds:*

$$\int_{\mathbb{R}\times[0,\infty)} \left(g_t\, U + g_x F\circ U\right)\, dx\, dt + \int_{\mathbb{R}} g(x,0)\, U_0(x)\, dx = 0. \tag{1.22}$$

In this definition, $C_0^\infty(\mathbb{R}\times[0,\infty))$ is the space of smooth functions on $\mathbb{R}\times[0,\infty)$ with compact support and $L^1_{\text{loc}}(\mathbb{R}\times[0,\infty))$ is the space of locally integrable functions on $\mathbb{R}\times[0,\infty)$.

Finally, we define a class of systems that have been studied extensively [25, 22].

Definition 1.18. *The strictly hyperbolic system (1.19) is* rich *if $[R_i, R_j] \in \operatorname{span}\{R_i, R_j\}$ for each i, j.*

An important aspect of hyperbolic conservation laws is the fact that weak solutions are not unique, and it is necessary to use *entropy conditions* to select a solution. However, as this concept will not be necessary for our geometric study of conservation laws, we will say nothing further here; see one of the references [5], [23], or [3].

1.5 Wave curves

The wave curves which will be defined in this section will be used to construct self-similar solutions to the Riemann problem in Section 1.6. They will also will feature prominently in our geometric investigation of conservation laws.

Definition 1.19. *The* rarefaction curves *of the conservation law (1.19) are the integral curves of the eigenvector fields of DF.*

In the strictly hyperbolic case (where the eigenvectors are indexed) we let $\mathcal{R}_i$ be a parametrization of the ith rarefaction curve. Thus

$$\mathcal{R}_i^0(\overline{U}) = \overline{U} \text{ and } \frac{\partial}{\partial a}\mathcal{R}_i^a(\overline{U}) = R_i|_{\mathcal{R}_i^a\overline{U})}$$

Of course the parametrization $\mathcal{R}_i$ depends on the scaling of the eigenvector R_i.

Definition 1.20. *Given the system (1.19) and a point $\overline{U} \in \Omega$, the* Hugoniot locus *at $\overline{U}$, denoted by $\mathcal{H}(\overline{U})$, is the set of points $U \in \Omega$ that satisfy the* Rankine-Hugoniot equation

$$F(U) - F(\overline{U}) = s(U - \overline{U})$$

for some $s \in \mathbb{R}$, called the wave speed.

The following theorem describes the local structure of the Hugoniot locus. For a proof see Theorem 4.2.1 of [23].

Theorem 1.21. *If the sytem (1.19) is strictly hyperbolic and $\overline{U} \in \Omega$, then in some neighborhood of $\overline{U}$, $\mathcal{H}(\overline{U})$ consists of n curves, each one making second order contact with a distinct rarefaction curve through $\overline{U}$.*

Definition 1.22. *Each of the curves making up the Hugoniot locus through $\overline{U}$ in Theorem 1.21 is one of the* shock curves *through $\overline{U}$. We label them $\mathcal{S}_i$; we will not have a canonical parametrization of these curves, but we will always have $\mathcal{S}_i(0) = \overline{U}$, $\mathcal{S}_i'(0) = R_i|_U$, and $\mathcal{S}_i''(0) = \frac{\partial^2}{\partial a^2}\mathcal{R}_i^a(U)$ (Theorem 1.21 ensures that this is possible).*

There are two cases in which the ith rarefaction curve is a subset of the Hugoniot locus. To state them we need the following definitions.

Definition 1.23. *If λ_i is constant along $\mathcal{R}_i$ (equivalently, the directional derivative $R_i(\lambda_i)$ vanishes everywhere), then the field is* linearly degenerate *or a* contact field. *If $R_i(\lambda_i) \neq 0$ everywhere, then the field is* genuinely nonlinear.

Definition 1.24. *If $\mathcal{R}_i(\overline{U})$ is a straight line for each $\overline{U}$, then (R_i, λ_i) is a* line field.

Theorem 1.25. *If (R_i, λ_i) is a contact field or a line field, then the ith rarefaction curve through $\overline{U} \in \Omega$ is a subset of the Hugoniot locus at $\overline{U}$.*

Proof. We seek some function s such that the Rankine-Hugoniot equation is satisfied with $U = \mathcal{R}_i^a(\overline{U})$:

$$s(a)[\mathcal{R}_i^a(\overline{U}) - \overline{U}] = F(\mathcal{R}_i^a(\overline{U})) - F(\overline{U}). \tag{1.23}$$

Differentiate with respect to a to obtain

$$s'(a)[\mathcal{R}_i^a(\overline{U}) - \overline{U}] + s(a)R_i|_{\mathcal{R}_i^a(\overline{U})} = \lambda_i(\mathcal{R}_i^a(\overline{U}))R_i|_{\mathcal{R}_i^a(\overline{U})}. \tag{1.24}$$

Equation (1.23) is satisfied for any function at $a = 0$; thus the theorem is proved if equation (1.24) is satisfied.

In the case of a contact field put $s(a) = \lambda_i(\overline{U})$; the first term vanishes and, since λ_i is constant along $\mathcal{R}_i$, the equation is satisfied.

In the case of a line field, parametrize $\mathcal{R}_i$ with respect to arc length and put

$$s(a) = \frac{1}{a}\int_0^a \lambda_i(\mathcal{R}_i^b\overline{U})\,db$$

(with $s(0) = \lambda_i(\overline{U})$). Thus, s is the average value of λ_i from $\overline{U}$ to $\mathcal{R}_i^a\overline{U}$. Now note that

R_i is constant along $\mathcal{R}_i$ and $R(\mathcal{R}_i^a\overline{U}) = \overline{U} + aR_i$. Inserting these values into equation (1.24) reduces both sides to $\lambda_i(\mathcal{R}_i^a\overline{U})R_i$. □

In fact, equation (1.24) in the proof of Theorem 1.25 shows that in order for the rarefaction curve to be on the Hugoniot locus, we need $s'(a)[\mathcal{R}_i^a(\overline{U}) - \overline{U}]$ to be a multiple of $R_i(\mathcal{R}_i^a(\overline{U}))$. Barring unusual mixed cases, this will only happen if (R_i, λ_i) is a contact field (and hence $s' = 0$) or if it is a line field.

Theorem 1.25 was proved by Temple (Theorem 2 in [25]). There it is a corollary of his Theorem 1 on so-called invariant submanifolds. Here we have given a more direct proof.

With the preceding definitions, we are able to define the wave curves.

Definition 1.26. *If (R_i, λ_i) is a contact field, the the* wave curve *through $\overline{U} \in \Omega$, denoted by $W_i(\overline{U})$, is the ith rarefaction curve through $\overline{U}$.*

If the characteristic field (R_i, λ_i) is genuinely nonlinear, then the wave curve *through $\overline{U} \in \Omega$, denoted by $W_i(\overline{U})$, consists of the half of the rarefaction curve in the direction of increasing λ, together with the half of the shock curve in the direction of decreasing λ.*

The reason for choosing half of the wave curve to be part of the rarefaction curve and half to be part of the shock curve comes from the process of constructing solutins to conservation laws. This will be explained more fully in section 1.6; see especially the remarks after Definition 1.28.

In general, we may use any C^2 parametrization for the wave curves.

Since the shock and rarefaction curves make second order contact, the wave curve is C^2. We do not define wave curves for characteristic fields that are neither genuinely nonlinear nor contact fields.

We now give a simple example to illustrate these concepts.

Example 1.27. Let $\Omega = \{U = (u, v) \in \mathbb{R}^2 : v > 1\}$. We define a flux F on Ω by

$$F(u, v) = \left(u + \frac{v^2}{2}, \frac{v^2}{2}\right).$$

We find its Jacobian matrix:

$$DF = \begin{pmatrix} 1 & v \\ 0 & v \end{pmatrix}$$

and eigensystem:

$$R_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \lambda_1 = 1;$$

$$R_2 = \begin{pmatrix} \frac{v}{v-1} \\ 1 \end{pmatrix}, \qquad \lambda_2 = v.$$

The first characteristic field is a contact field, since $R_1(\lambda_1) = 0$, and the second has R_2 scaled so that $R_2(\lambda_2) = 1$. The rarefaction curves, which must satisfy the differential equations

$$\frac{\partial}{\partial a_i}\mathcal{R}_i^{a_i}(U) = R_i|_U,$$

are given by

$$\mathcal{R}_1^{a_1}(U) = \begin{pmatrix} u + a_1 \\ v \end{pmatrix}$$

$$\mathcal{R}_2^{a_2}(U) = \begin{pmatrix} u + a_2 + \log\left(\frac{a_2}{v-1} + 1\right) \\ v + b \end{pmatrix}$$

We already know by Theorem 1.25 that the shock and rarefaction curves coincide in the first characteristic field. We examine the Rankine-Hugoniot equation to find $\mathcal{S}_2$:

$$\begin{aligned} u + \frac{v^2}{2} - \overline{u} - \frac{\overline{v}^2}{2} &= s(u - \overline{u}) \\ \frac{v^2}{2} - \frac{\overline{v}^2}{2} &= s(v - \overline{v}). \end{aligned}$$

Make the substitution $v = \overline{v} + a_2$ and solve for $\overline{u}$ and s to find the second shock curve:

$$\mathcal{S}_2^{a_2}(U) = \begin{pmatrix} u + \frac{a_2(a_2+2v)}{a_2+2v-2} \\ v + a_2 \end{pmatrix}.$$

The corresponding speed is $s = v + \frac{a_2}{2}$.

1.6 The Riemann problem

The *Riemann problem* is the initial value problem for equation (1.19) with piecewise constant initial data

$$U(x,0) = \begin{cases} U_-, & \text{if } x < 0 \\ U_+, & \text{if } x > 0. \end{cases} \tag{1.25}$$

The front tracking and random choice methods for solving Cauchy problems work by approximating general initial data by piecewise constants, and solving the Riemann problem at each jump.

We now explain how to solve the Riemann problem in certain special cases, and then use those special cases to give a general solution.

Suppose equation (1.19) is strictly hyperbolic, with each characteristic field either

genuinely nonlinear or a contact field.

As a special case of the Riemann problem, suppose also that U_+ lies on the ith wave curve through U_-, so that $U_+ = W_i^a U_-$ for some a.

If the ith field is a contact field or if $a < \lambda^i(U_-)$ (so that U_+ is on the shock part of the wave curve), then the solution is

$$U(x,t) = \begin{cases} U_-, & \text{if } x < st \\ U_+, & \text{if } x > st, \end{cases} \tag{1.26}$$

where s is wave speed occurring in the Rankine-Hugoniot equation which U_- and U_+ satisfy.

We quickly illustrate that this is a weak solution, using equation (1.22). Given a test function g, let Ψ be open in $\mathbb{R} \times [0, \infty)$ containing the support of g. Let Ψ_- be the portion of Ψ to the left of the line of discontinuity $x = st$ and Ψ_+ the portion to the right. Then the first term of equation (1.22) becomes

$$\left(\int_{\Psi_-} + \int_{\Psi_+} \right) \left(g_t \, U + g_x (F \circ U) \right) \, dx \, dt.$$

On each of $\Psi_\pm$, we have

$$\begin{aligned} d(gF(U)\,dt - gU\,dx) &= \left(g_x F(U) + gF(U)_x + g_t U + gU_x \right) \, dx \wedge dt \\ &= \left(g_t \, U + g_x (F \circ U) \right) \, dx \wedge dt \end{aligned}$$

(since U is constant on each of $\Psi_\pm$). Now we apply Green's Theorem to obtain

$$\int_\gamma gF(U_-)\,dt - gU_-\,dx - gF(U_+)\,dt + gU_+\,dx - \int_{\mathbb{R}} g(x,0) U_0(x)\,dx,$$

where γ is the line of discontinuity. Since U_- and U_+ satisfy the Rankine-Hugoniot equation with $s = dx/dt$ on γ, the first term vanishes and we see that equation (1.22) is indeed satisfied.

A solution of the form (1.26) is called a *shock* in the genuinely nonlinear case and a *contact discontinuity* in the contact field case.

We now need to give a solution when the ith field is genuinely nonlinear and $a > \lambda_i(U)$ (so that U_+ is on the rarefaction part of the wave curve). Let the rarefaction parts of the wave curves for genuinely nonlinear fields be parametrized by the corresponding eigenvalue. In other words, if the ith field is genuinely nonlinear, $\lambda_i(W_i^a U) = a$ (assuming that $a > \lambda_i(U)$ and thus $W_i^a U$ is on the rarefaction part of the wave curve).

Then the solution is:

$$U(x,t) = \begin{cases} U_-, & \text{if } x < t\lambda(U_-) \\ W_i^{\frac{x}{t}} U_-, & \text{if } t\lambda(U_-) < x < t\lambda(U_+) \\ U_+, & \text{if } x > t\lambda(U_+) \end{cases}. \tag{1.27}$$

The two constant portions of U are clearly solutions; we now show that the portion given by $U = W_i^{\frac{x}{t}} U_-$ is also a solution:

$$U_t + F(U)_x = -\frac{x}{t^2} R_i + DF\,\frac{1}{t} R_i = \frac{1}{t}\left(-\frac{x}{t} + \lambda_i(W_i^{\frac{x}{t}} U_-)\right) R_i.$$

Since $\lambda_i(W_i^a U_-) = a$, this expression is 0 and U is a weak solution.

Because U is a continuous, piecewise smooth function which is a classical solution where it is smooth, it is a weak solution. A solution of the form (1.27) is called a *rarefaction wave.*

For the more general Riemann problem, when we may not have U_+ lying on some wave curve through U_-, we can nevertheless construct a solution by piecing together rarefaction waves, shock waves, and contact discontinuities. The inverse function theorem guarantees that $U_+ = W_n^{a_n} \cdots W_1^{a_1} U_-$ for some $\{a_i\}$ as long as U_+ is sufficiently close to U_- (this is a theorem of Lax [19]; see see [23], Theorem 4.6.1 for a modern exposition).

We now briefly discuss why the wave curves were chosen as they were: when the parameter a_i is positive in $W_i^{a_i} U$ we follow along $\mathcal{R}_i$, while when the parameter is negative we follow along $\mathcal{S}_i$. First note that the rarefaction solution (1.27) is only meaningful when $\lambda_i(U_-) < \lambda_i(U_+)$.

But in general, the answer requires a new concept. A solution of the form (1.26) would work even in the case that $\lambda_i(U_+) < \lambda_i(U_-)$. Indeed, there are generally many solutions to Riemann problems, and we need the following definition to choose a unique one.

Definition 1.28. *A shock (1.26) is a* Lax i-shock *if*

$$\lambda_{i-1}(U_-) < s < \lambda_i(U_-) \quad \textit{and} \quad \lambda_i(U_+) < s < \lambda_{i+1}(U_+). \tag{1.28}$$

Here we take $\lambda_0 \equiv -\infty$ *and* $\lambda_{n+1} \equiv \infty$.

Based on physical considerations, all weak shocks in solutions of hyperbolic conservation laws should be Lax shocks (see [23], section 4.3). An i-shock separating U and $\mathcal{S}_i^{a_i} U$ is a Lax i-shock if $a_i < 0$ and $|a_i|$ is small. This is why our wave curve includes only the part of $\mathcal{S}_i U$ with a negative parameter. Criteria like this which select admissible solutions are called entropy conditions.

Example 1.29. We use the same system from Example 1.27 to illustrate a solution to

the Riemann problem. Given inital values $U_- = (u_-, v_-)$ and $U_+ = (u_+, v_+)$, we seek a_1, a_2 such that

$$W_2^{a_2} W_1^{a_1}(U_-) = U_+.$$

We calculate

$$W_2^{a_2} W_1^{a_1}(U_-) = \begin{cases} \begin{pmatrix} u_- + a_1 + a_2 + \log\left(\frac{a_2}{v_- - 1} + 1\right) \\ v_- + a_2 \end{pmatrix}, & \text{if } a_2 > 0 \\ \begin{pmatrix} u_- + a_2 + \frac{a_2(a_2+2v_-)}{a_2+2v_--2} \\ v_- + a_2 \end{pmatrix}, & \text{otherwise} \end{cases}.$$

Setting this equal to (u_+, v_+), we solve and find:

$$\begin{aligned} a_2 &= v_+ - v_- \\ a_1 &= \begin{cases} u_+ - u_- - a_2 - \log\left(\frac{a_2}{v_- - 1} + 1\right), & \text{if } a_2 < 0 \\ u_+ - u_- - \frac{a_2(a_2+2v_-)}{a_2+2v_--2}, & \text{otherwise} \end{cases}. \end{aligned} \tag{1.29}$$

Now we examine a concrete case: $U_- = (3, 3)$ and $U_+ = (3, 2)$. The equations just shown reveal that $a_2 = -1$ and $a_1 = \frac{5}{3}$. Connecting U_- to $W_1^{a_1}(U_-) = \left(\frac{14}{3}, 3\right)$ will require a contact discontinuity traveling at speed $\lambda_1 = 1$. Connecting $\left(\frac{14}{3}, 3\right)$ to $(3, 2)$ will require a shock traveling at speed $s = 3 + \frac{a_2}{2} = \frac{5}{2}$. Thus the solution is:

$$U(x,t) = \begin{cases} (3,3), & \text{if } x < t \\ \left(\frac{14}{3}, 3\right), & \text{if } t < x < \frac{5}{2}t \\ (3,2), & \text{if } \frac{5}{2}t < x \end{cases}.$$

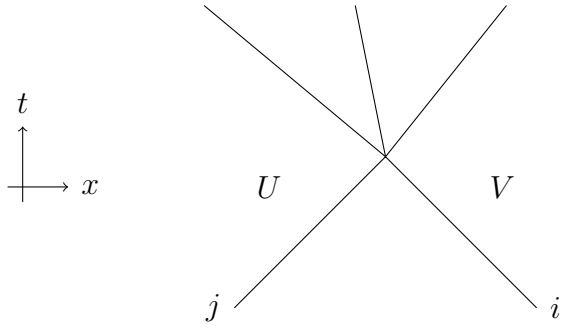

Figure 1.1: A wave interaction

1.7 Wave Interactions

In the front tracking approach to solving hyperbolic conservation laws, it is important to resolve so-called interactions of waves. For an illustration of this process, see Figure 1.1. In this figure, a wave of the jth family and a wave of the ith family have collided, and we must determine an outgoing wave fan; in effect solving the Riemann problem with left value U and right value V.

Given the ith and jth characteristic families with $j > i$, before the two waves collide we see that we travel from $\overline{U} \in \Omega$ along the jth and then the ith wave curve:

$$V = W_i^{a_i} W_j^{a_j} \overline{U}.$$

Here the parameters a_i and a_j are called the *strengths* of the i- and j-waves.

In order to determine the outgoing waves, we need to instead travel along the first wave curve, then the second, and so on, and still reach V:

$$V = W_n^{b_n} \cdots W_1^{b_1} \overline{U}.$$

The map $(b_1, \ldots, b_n) \mapsto W_n^{b_n} \cdots W_1^{b_1} \overline{U}$ has nonsingular Jacobian matrix at $(0, \ldots, 0)$ (its columns are the vectors $\{R_i(\overline{U})\}$). Thus the inverse function theorem guarantees that there is a unique solution $(b_1, \ldots, b_n)$ as a function of V (hence as a function of (a_i, a_j)), as long as the wave strengths a_i, a_j are small enough (and hence V is sufficiently close to $\overline{U}$). The parameters $\{b_i\}, 1 \leq i \leq n$ are the *outgoing wave strengths*.

The following theorem shows how the b_i are estimated up to quadratic order.

Theorem 1.30. *If:*

1. *(1.19) is a strictly hyperbolic system,*

2. *the functions $\{b_k\}$ satisfy*

$$W_n^{b_n} \cdots W_1^{b_1} U = W_i^{a_i} W_j^{a_j} U \tag{1.30}$$

for some $U \in \Omega$, with $i < j$;

then

$$b_k = \delta_{ik} a_i + \delta_{jk} a_j + c_{ji}^k a_i a_j + O(a_i a_j^2 + a_i^2 a_j).$$

Proof. Set $a_i = 0$ in equation 1.30 and see that $b_k = \delta_{jk} a_j$. Similarly set $a_j = 0$ and see that $b_k = \delta_{ik} a_i$. Thus our estimate is correct to first order. This also establishes that there are no terms of higher order involving only a_i or only a_j. To prove the theorem it will suffice to show that

$$\frac{\partial^2 b_k}{\partial a_i \partial a_j}\Big|_{a_i = a_j = 0} = c_{ji}^k(U).$$

Our approach will be to apply $\frac{\partial^2}{\partial a_i \partial a_j}\Big|_{a_i = a_j = 0}$ to both sides of equation (1.30). Since each W_i makes second order contact with ϕ_i, the flow of R_i, we will not differentiate the

left hand side of (1.30); instead we will differentiate

$$\phi_n^{b_n} \cdots \phi_1^{b_1} U. \tag{1.31}$$

Our second partial derivatives at $a_i = a_j = 0$ will be the same.

Differentiate (1.31) with respect to b_k and set $a_i = 0$.

$$D\phi_n^{b_n} \cdots D\phi_{k+1}^{b_{k+1}} R_k(\phi_k^{b_k} \cdots \phi_1^{b_1} U).$$

And if we set $a_i = 0$, we have $b_m = \delta_{jm} a_j$, and so this becomes either

$$D\phi_j^{a_j} R_k|_U$$

if $j > k$, or

$$R_k(\phi_j^{a_j} U)$$

otherwise.

Armed with this fact, we can apply the chain rule to differentiate (1.31) with respect to a_i and set $a_i = 0$ to obtain:

$$\sum_{k \geq j} \frac{\partial b_k}{\partial a_i} R_k(\phi_j^{a_j} U) + \sum_{k<j} \frac{\partial b_k}{\partial a_i} D\phi_j^{a_j} R_k|_U \tag{1.32}$$

Now differentiate with respect to a_j to obtain:

$$\sum_{k\geq j}\left(\frac{\partial^2 b_k}{\partial a_i \partial a_j}R_k(\phi_j^{a_j}U)+\frac{\partial b_k}{\partial a_i}DR_k\,R_j|_U\right)+$$
$$\sum_{k<j}\left(\frac{\partial^2 b_k}{\partial a_i \partial a_j}D\phi_j^{a_j}R_k(U)+\frac{\partial b_k}{\partial a_i}\frac{\partial}{\partial a_j}D\phi_j^{a_j}R_k|_U.\right)$$

We also have that, for any $V\in\Omega$,

$$\frac{\partial}{\partial a_j}D\phi_j^{a_j}V=D\frac{\partial}{\partial a_j}\phi_j^{a_j}V=DR_j(V).$$

Thus, when we set $a_j=0$ in equation (1.32), apply $\frac{\partial^2}{\partial a_i\partial a_j}\big|_{a_i=a_j=0}$ to the right hand side of 1.30, and set these two values equal to each other, we obtain

$$\sum_k \frac{\partial^2 b_k}{\partial a_i\partial a_j}R_{kU}+\nabla_{R_i}R_{jU}=\nabla_{R_j}R_{iU}.$$

Subtract $\nabla_{R_i}R_{jU}$ on both sides:

$$\sum_k \frac{\partial^2 b_k}{\partial a_i\partial a_j}R_k|_U=[R_j,R_i]=\sum_k c_{ji}^k(U)R_k|_U,$$

recalling the definition of the coefficients c_{ji}^k (Definition 1.8). Thus the theorem is proved.

□

Example 1.31. We now return to the system from Examples 1.27 and 1.29 to illustrate these concepts.

We find that

$$W_1^{a_1}W_2^{a_2}U = \begin{cases} \begin{pmatrix} u + a_1 + a_2 + \log\left(\frac{a_2}{v-1} + 1\right) \\ v + a_2 \end{pmatrix} & \text{if } a_2 > 0 \\ \begin{pmatrix} u + a_1 + \frac{a_2(a_2+2v)}{a_2+2v-2} \\ v + a_2 \end{pmatrix} & \text{otherwise} \end{cases}. \tag{1.33}$$

We seek b_1, b_2 such that $W_2^{b_2}W_1^{b_1}U = W_1^{a_1}W_2^{a_2}U$. To this end, we can return to equation 1.29, substituting b_i in for a_i and u, v in for u_-, v_-:

$$b_2 = v_+ - v$$

$$b_1 = \begin{cases} u_+ - u - b_2 - \log\left(\frac{b_2}{v-1} + 1\right), & \text{if } b_2 < 0 \\ u_+ - u - \frac{b_2(b_2+2v)}{b_2+2v-2}, & \text{otherwise} \end{cases}.$$

Here u_+, v_+ are the components of $W_1^{a_1}W_2^{a_2}U$ given in equation 1.33.

In general resolving interactions will involve solving a system of nonlinear algebraic and ordinary differential equations that do not admit closed form solutions.

1.8 Examples of finite time blowup

Here we present a standard theorem on the existence of global in time solutions to the Cauchy problem for (1.19), then discuss examples to which the theorem does not apply, and then spend the majority of the section on a particular class of such examples: those exhibiting finite time blow up.

Theorem 1.32. *If the system (1.19) is strictly hyperbolic, each field is genuinely nonlinear or a contact field, and $\overline{U} \in \Omega$, then there exist $\delta_0, \delta_1, c_0, c_2$ such that if*

1. $\sup_x |U_0(x) - \overline{U}| < \delta_0$,

2. $TV\, U_0 < \delta_1$,

then there exists a weak solution U to (1.19), (1.20) such that, for all $t \geq 0$,

1. $\sup_x |U(x,t)| \leq c_0 \sup_x |U_0(x)|$,

2. $TV\, U(\cdot, t) \leq c_1\, TV\, U_0$.

This theorem was proved by Glimm [9] in the case where all fields are genuinely nonlinear. The theorem was extended to cover systems which also had contact fields by Liu [21].

The primary limitation of Theorem 1.32 is the requirement that the initial data is small about some point $\overline{U}$. Examples have existed for some time to demonstrate that the estimates in the conclusion do not hold for large data. One such example was discussed in [17]. The authors constructed a system together with a collection of initial data (of small L^∞ norm but large variation) for which these estimates do not apply. Similar conclusions were reached in [27].

The examples presented here fail the conclusion of Theorem 1.32 in a still stronger way: a global in time solution does not even exist because the solution blows up in finite time. That is, at some time $t_* > 0$,

$$\lim_{t \uparrow t_*} \sup_x |U(x,t)| \to \infty$$

or

$$\lim_{t \uparrow t_*} \text{TV } U(\cdot, t) \to \infty.$$

Before turning to examples from the existing literature, we give a new example, which is useful due to its simplicity.

Example 1.33. Given the flux

$$F(u, v, w) = \begin{pmatrix} -u + \frac{w}{v-1} \\ -w \\ w \end{pmatrix},$$

one solution is

$$u = -\log(1 - t)$$

$$v = t$$

$$w = x - t.$$

We have $u \to \infty$ as $t \to 1$. The initial data can be cut off outside of some closed interval so that it is bounded; since the wave speeds are constants -1, 0, and 1, the solution will still blow up provided the interval is large enough.

The eigenvectors of Df are

$$R_1 = \begin{pmatrix} -w \\ (v-1)^2 \\ 0 \end{pmatrix}; \qquad R_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; R_3 = \begin{pmatrix} v + w - 1 \\ -2(v-1)^2 \\ 2(v-1)^2 \end{pmatrix}$$

and we see that in a limiting sense, all three vectors become collinear with $(1, 0, 0)$ as

$t \to 1$. In some sense this could be regarded as a defect in the hyperbolicity of the system if v is allowed to take values arbitrarily close to 1.

We will see that all of the examples in this section have some deficiency in their hyperbolicity similar to that of Example 1.33. In that regard, the following definition will be useful.

Definition 1.34. *System (1.19) is* uniformly strictly hyperbolic *in* Ω *if:*

1. *it is strictly hyperbolic;*
2. *for each pair* i, j, $|\lambda_i - \lambda_j|$ *is bounded away from 0;*
3. *for each index* i, $\operatorname{dist}(R_i, \operatorname{span}\{R_j\}_{j \neq i})$ *(where* R_i *is scaled to unit length) is bounded away from 0.*

Here $\operatorname{dist}(R, \Psi) = \inf\{\|R - S\| : S \in \Psi\}$, *and a function* g *being* bounded away from 0 *means that* $\inf\{g(U) : U \in \Omega\} > 0$.

The given set Ω *may larger than we want; given a solution* U, *we say the system is uniformly strictly hyperbolic* along U *if there is some open* $\Omega' \subseteq \Omega$ *containing the image of* U *such that the system meets the above criteria with* Ω *replaced by* Ω'.

Note that this term has been occasionally used but apparently not defined in the literature.

Example 1.35. This example is taken from [10]. This example cannot be placed in conservative form and thus is strictly not a member of the class of equations discussed in the current work. Nevertheless, it is an early and simple example of the same type of breakdown we are interested in, so we present it here. The equation is of the form

$$U_t + A(U)\, U_x = 0, \tag{1.34}$$

where $U = (u, v, w) \in \mathbb{R}^3$ and

$$A(u,v,w) = \begin{pmatrix} -\cosh 2v & 0 & -\sinh 2v \\ \cosh v & 0 & \sinh v \\ \sinh 2v & 0 & \cosh 2v \end{pmatrix}$$

(we have no use of the parameters α, μ, h from Jeffrey's paper and set them all to 1 for ease of presentation). This equation admits a solution

$$\begin{aligned} u &= \frac{1}{1-t} + x - 1 \\ v &= \log(1-t) \\ w &= \frac{1}{1-t} - x - 1. \end{aligned} \tag{1.35}$$

Clearly each component approaches $\pm\infty$ as $t \to 1$ from below, for any x.

The initial data is $U = (x, 0, -x)$; we could restrict its domain so that it has bounded variation and amplitude.

A has eigensystem

$$R_1 = \begin{pmatrix} -\cosh v \\ 1 \\ \sinh v \end{pmatrix}, \qquad \lambda_1 = -1;$$

$$R_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \lambda_2 = 0;$$

$$R_3 = \begin{pmatrix} -\sinh v \\ 0 \\ \cosh v \end{pmatrix}, \qquad \lambda_3 = 1.$$

The first and third eigenspaces become arbitrarily close as $v \to -\infty$.

We also mention that solution (1.35) is a blowup solution to a conservative system related to (1.34), as pointed out by Jenssen and Sinestrari [15]. This is the system with flux $F(U) = A(U)\,U$. Since A depends only on v, and in this particular solution, v does not depend on x, $F(U)_x = A(U)\,U_x$. The eigenvectors of DF are complicated enough that presenting them here would not be enlightening, but they do coalesce in the limit as we just saw in Jeffrey's system.

Example 1.36. We analyze in some depth an example from [11]. We have no need of the generality considered there, so we choose a particular example from the class considered

in that paper. We will examine the 3×3 system with flux

$$F(u,v,w) = \begin{pmatrix} uv + w \\ \frac{v^2}{16} \\ u - uv^2 - vw \end{pmatrix}.$$

This is the same example considered in Section 9.10 of [5].

We calculate

$$DF = \begin{pmatrix} v & u & 1 \\ 0 & \frac{v}{8} & 0 \\ 1 - v^2 & -2uv - w & -v \end{pmatrix}$$

and eigensystem

$$R_1 = \begin{pmatrix} 1 \\ 0 \\ -1 - v \end{pmatrix}, \qquad \lambda_1 = -1;$$

$$R_2 = \begin{pmatrix} -7uv - 8w \\ \frac{v^2}{8} - 8 \\ 8u + 6uv^2 + 7vw \end{pmatrix}, \qquad \lambda_2 = \frac{v}{8};$$

$$R_3 = \begin{pmatrix} 1 \\ 0 \\ 1 - v \end{pmatrix}, \qquad \lambda_3 = 1.$$

In any region where v remains bounded away from ± 8, this system is strictly hyperbolic.

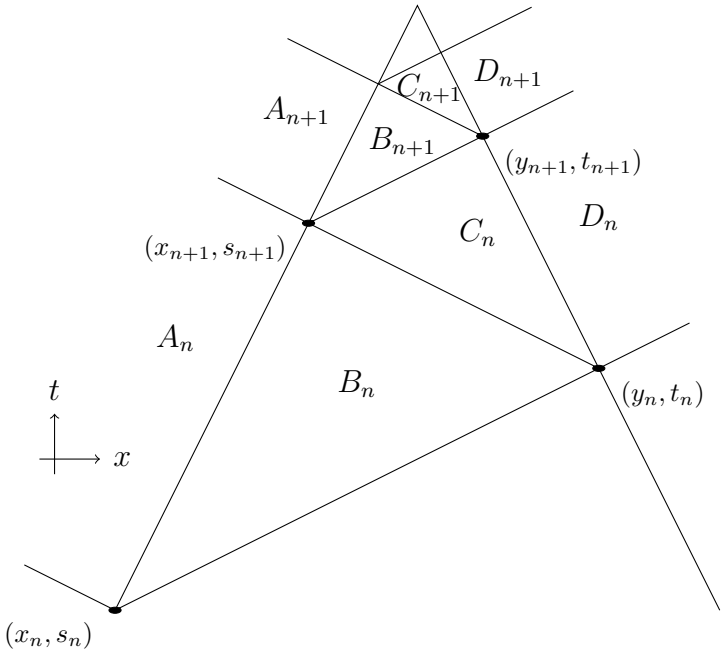

Figure 1.2: A blowup solution

The Rankine-Hugoniot equations are

$$\begin{aligned} u_+v_+ - u_-v_- + w_+ - w_- &= s(u_+ - u_-) \\ \frac{v_+^2}{16} - \frac{v_-^2}{16} &= s(v_+ - v_-) \\ u_+ - u_- - u_+v_+^2 + u_-v_-^2 - v_+w_+ + v_-w_- &= s(w_+ - w_-). \end{aligned} \tag{1.36}$$

The first and third fields are contact fields, so all self similar wave solutions in those fields are contact discontinuities with speed -1 or 1, respectively. The second field is genuinely nonlinear, but we will not need rarefaction curves for this field. Shock curves for this field have speed $s = \frac{1}{16}(v_- + v_+)$; such a shock is a Lax 2-shock provided $v_+ < v_-$.

In contrast to [11] and [5], we construct an explicit solution U. Refer to Figure 1.2

and define:

$$x_n = -\left(\frac{9}{25}\right)^n, \qquad s_n = 4 - 4\left(\frac{9}{25}\right)^n,$$
$$y_n = \frac{3}{5}\left(\frac{9}{25}\right)^n, \qquad t_n = 4 - \frac{12}{5}\left(\frac{9}{25}\right)^n,$$
$$A_n = \begin{pmatrix} -\frac{5}{19}\left[85 + 162\left(\frac{100}{81}\right)^n\right] \\ 4 \\ \frac{225}{19}\left[1 + 18\left(\frac{100}{81}\right)^n\right] \end{pmatrix}, \qquad B_n = \begin{pmatrix} 285 - 270\left(\frac{100}{81}\right)^n \\ 0 \\ -\frac{15}{19}\left[1 + 18\left(\frac{100}{81}\right)^n\right] \end{pmatrix},$$
$$C_n = \begin{pmatrix} 285 - 300\left(\frac{100}{81}\right)^n \\ 0 \\ \frac{15}{19}\left[-1 + 20\left(\frac{100}{81}\right)^n\right] \end{pmatrix}, \qquad D_n = \begin{pmatrix} -\frac{9}{19}\left[33 + 100\left(\frac{100}{81}\right)^n\right] \\ -4 \\ -\frac{225}{19}\left[-1 + 20\left(\frac{100}{81}\right)^n\right] \end{pmatrix}.$$

The figure represents the solution U in the xt plane. The line containing the points (x_n, s_n) is a 2-shock of speed $\frac{1}{4}$; the line containing the points (y_n, t_n) is a 2-shock of speed $-\frac{1}{4}$. The lines with speed -1 are of course 1-contacts, and the lines of speed 1 are 3-contacts. Each of A_n, B_n, C_n, and D_n are the values taken by U in the regions indicated by the figure. The pattern continues ad infinitum in the triangle delimited by 2-shocks. A straightforward but tedious computation reveals that each discontinuity does satisfy the Rankine-Hugoniot equation; for instance, if we substitute $U_- = B_n$, $U_+ = C_n$, and $s = -1$ into (1.36), the equation is satisfied. The solution cannot be continued past time $t = 4$ due to the amplitude blowup exhibited.

When u and w approach $\pm\infty$ but v stays bounded, as in this example, we see that the middle component of R_2 stays bounded but the first and last component become arbitrarily large. Thus the eigenspace containing R_2 becomes arbitrarily close to the space spanned by R_1 and R_3. The system is not uniformly strictly hyperbolic.

In [2] Baiti and Jenssen present a perturbation of the system just discussed, with the distinction that all three fields are genuinely nonlinear.

Example 1.37. Young, in [29], presents another system displaying finite time amplitude blowup. The 3×3 system on coordinates (u, w, S) has two equations given by

$$\begin{pmatrix} u \\ w \end{pmatrix}_t + \left(S \begin{pmatrix} \frac{u^2}{2} \\ -\frac{w^2}{2} \end{pmatrix} + (1-S)A \begin{pmatrix} u \\ w \end{pmatrix} \right)_x = 0$$

and third equation $S_t = 0$. Here A is of the form

$$A = \begin{pmatrix} a & b \\ -b & -a \end{pmatrix}.$$

The idea behind the construction is to prescribe that $S = 1$ when $0 < x < 1$, and $S = 0$ otherwise. Thus in the region $0 < x < 1$, the equations on u and w are uncoupled Burgers' equations, and shocks of any strength can easily be produced. Outside that region, the equations are linear, and there are no wave interactions. By judicious selection of values for u and w, Young produces a pattern of interactions in the region $0 < x < 1$ where forward and backward waves appear with increasingly fast speeds, so that u and w both approach ∞.

Along $S = 1$, the Jacobian of the flux for this system is

$$\begin{pmatrix} u & -au + \frac{u^2}{2} - bw & 0 \\ 0 & 0 & 0 \\ 0 & bu + aw - \frac{w^2}{2} & -w \end{pmatrix}.$$

The eigenstem along $S = 1$ is

$$R_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \lambda_1 = -w;$$

$$R_2 = \begin{pmatrix} w(2au - u^2 + 2bw) \\ 2uw \\ u(2bu + (2a - w)w) \end{pmatrix}, \qquad \lambda_2 = 0;$$

$$R_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \lambda_3 = u.$$

By Young's construction the ratio u/w is constant, and thus we see that the first and third components of R_2 grow large compared to the second component as u and w blow up. Thus R_2 approaches the span of R_1 and R_3, and the system is not uniformly strictly hyperbolic along this solution. Another deficiency in the hyperbolicity is that the wave speeds are not bounded.

The construction depends on having a region where the initial data has $S = 0$ and a region where $S = 1$, so the initial data cannot be chosen arbitrarily small.

Example 1.38. The example in [16] is a conservation law with smooth blowup solutions. For ease of presentation we discuss only the continuous version (from Jenssen and Young's

section 3.1). The 3×3 system has flux

$$F(u,v,w) = \begin{pmatrix} -\lambda\cos(2v)u + \lambda\sin(2v)w \\ \frac{v^2}{2} \\ \lambda\sin(2v)u + \lambda\cos(2v)w \end{pmatrix}$$

for a parameter $\lambda > 0$.

The eigensystem is

$$R_1 = \begin{pmatrix} -\cos v \\ 0 \\ \sin v \end{pmatrix}, \qquad \lambda_1 = -\lambda;$$

$$R_2 = \begin{pmatrix} -w\lambda + vw\cos(2v) + uv\sin(2v) \\ \frac{v^2-\lambda^2}{2\lambda} \\ u\lambda + uv\cos(2v) - vw\sin(2v) \end{pmatrix}, \qquad \lambda_2 = v;$$

$$R_3 = \begin{pmatrix} \sin v \\ 0 \\ \cos v \end{pmatrix}, \qquad \lambda_3 = \lambda.$$

The solution presented in [16], when $x = 0$, is

$$U(0,t) = \frac{1}{\lambda(1-t)} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Thus, along this line,

$$R_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \; R_2 = \begin{pmatrix} \frac{1}{t-1} \\ -\frac{\lambda}{2} \\ \frac{1}{1-t} \end{pmatrix}, \; R_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We see that as $t \to 1$ and u, w become arbitrarily large, R_2 becomes arbitrarily close to the span of R_1 and R_3, and the system is not uniformly strictly hyperbolic in our sense. The initial data takes values $v = \pm\frac{\pi}{4}$, and thus the construction does not fulfill the hypotheses of Glimm's theorem because the initial data cannot be made arbitrarily small.

Example 1.39. We now turn to the example of Young and Szeliga [30]. This example has flux

$$F(y, s, z) = \begin{pmatrix} \frac{1}{2}y^2 + \frac{1}{2}z^2 - sz \\ 0 \\ ys + yz \end{pmatrix}.$$

The eigenvalues are 0 and $y \pm \sqrt{z^2 - s^2}$. Young and Szeliga present a collection of iniital data of arbitrarily small variation. The solutions all exhibit amplitude blowup. This is not a violation of Theorem 1.32 because the statement of that theorem requires a selected point $\overline{U}$, in a neighborhood Ω of which the system is strictly hyperbolic. However, the only candidate for such a point $\overline{U}$ in Young and Szeliga's example is the origin, and the system is not hyperbolic there.

It is reasonable to conjecture that uniformly strictly hyperbolic systems with bounded eigenvalues do not admit finite time blowup. This will be an object of our future research.

Chapter 2

Prescribing geometric features

2.1 The λ-system

In [12, 13], Jenssen and Kogan considered the problem of constructing systems with a given frame of eigenvectors. Given an open $\Omega \subseteq \mathbb{R}^n$ with a frame $\{R_i\}$, this amounts to finding a vector field $F : \Omega \to \mathbb{R}^n$ such that ∇F has the vector fields $\{R_i\}$ as eigenvectors (F will be the flux of the constructed system (1.19)).

Their solutions are obtained by applying various integrability theorems for overdetermined systems, which describe the degree of freedom in the resulting solutions. Note that *trivial fluxes*, by which we mean fluxes of the form $F(U) = aU + V$, have the property that every vector is an eigenvector of DF. Thus for any frame $\{R_i\}$, we know that at least some fluxes exist.

Here we briefly review some of their results. We will make use of this material in Section 4.1.

Theorem 2.1. *If $\{R_i\} \subseteq \Omega \subseteq \mathbb{R}^n$ is a frame and $\{\lambda_i\}_{i=1}^n$ is a set of smooth functions on Ω, then the following are equivalent:*

1. *for every $U \in \Omega$, there exists some neighborhood $\Omega' \ni U$, and a vector field F on Ω', such that ∇F has eigenvector/eigenvalue pairs (R_i, λ_i);*

2. *the $n(n-1)$ differential equations*

$$R_i(\lambda_j) = \Gamma^j_{ji}(\lambda_i - \lambda_j), \quad \text{for } i \neq j \tag{2.1}$$

and the $(n-2)\binom{n}{2}$ algebraic equations

$$c^k_{ji}\lambda_k = \Gamma^k_{ji}\lambda_i - \Gamma^k_{ij}\lambda_j, \quad \text{for } i < j, i \neq k, j \neq k \tag{2.2}$$

are satisfied.

Proof. $1 \implies 2$. Write the flatness condition from Theorem 1.10 with $R = R_i$, $S = R_j$, and $T = F$:

$$\nabla_{R_i}\nabla_{R_j}F - \nabla_{R_j}\nabla_{R_i}F = \nabla_{[R_i,R_j]}F.$$

Expand:

$$\nabla_{R_i}(\lambda_j R_j) - \nabla_{R_j}(\lambda_i R_i) = \nabla_{\sum_k c^k_{ij}R_k}F. \tag{2.3}$$

Expand further:

$$R_i(\lambda_j)R_j + \lambda_j\nabla_{R_i}R_j - R_j(\lambda_i)R_i - \lambda_i\nabla_{R_j}R_i = \sum_k c^k_{ij}\lambda_k R_k.$$

Now use the definition of the Christoffel symbols Γ^k_{ij} (Definition 1.8) to see that this is equivalent to:

$$R_i(\lambda_j)R_j + \sum_k \lambda_j\Gamma^k_{ij}R_k - R_j(\lambda_i)R_i - \sum_k \lambda_i\Gamma^k_{ji}R_k = \sum_k c^k_{ij}\lambda_k R_k. \tag{2.4}$$

In this last equation, examine the coefficients of R_i or R_k to obtain equations (2.1). Examine the coefficients of R_k $(k \neq i, j)$ to obtain equations (2.2).

2 $\implies$ 1. Working backwards we see that equations (2.1) and (2.2) imply equation (2.3). Letting M be the matrix defined by $M\, R_i = \lambda_i R_i$ for each i, we see that equation (2.3) is exactly the necessary condition that M locally be a Jacobian matrix for some vector field F. □

Jenssen and Kogan called equations (2.1, 2.2) the λ-system [12].

Theorem 3.2 of [13] described the so-called IIa case: a certain category of frames in $\mathbb{R}^3$ whose λ system solutions depend on two constants. Here we present a similar result for general $\mathbb{R}^n$.

Theorem 2.2. *If:*

1. $\{R_i\}_{i=1}^n$ *is a frame on* $\Omega \subseteq \mathbb{R}^n$*;*

2. *each* λ_i *appears with nowhere vanishing coefficient in the algebraic part of the* λ *system;*

then there are at most $n - m$ *linearly independent fluxes (modulo a constant term) for systems with eigenframe* $\{R_i\}_{i=1}^n$*, where* m *is the rank of the algebraic part of the* λ *system.*

Proof. Let $\mathcal{V}$ be the vector space of fluxes on Ω with eigenframe $\{R_i\}$. Pick some point $U \in \Omega$ and define a linear map $L : \mathcal{V} \to \mathbb{R}^n$ by

$$L(F) = (\lambda_1^F(U), \ldots, \lambda_n^F(U)),$$

where λ_i^F is the eigenvalue of DF corresponding to R_i.

The kernel of L consists of the constant fluxes. To see this, note that the algebraic part of the λ system gives that each λ_i satisfies an equation of the form $\lambda_i = \alpha\lambda_j + \beta\lambda_k$. Apply R_i to obtain:

$$R_i(\lambda_i) = R_i(\alpha)\lambda_j + R_i(\beta)\lambda_k + \alpha R_i(\lambda_j) + \beta R_i(\lambda_k),$$

where $R_i(\lambda_j)$ and $R_i(\lambda_k)$ are already specified in the differential part of the λ system. Since the λ system gives $R_q(\lambda_i)$ for $q \neq i$, every directional derivative of each λ_i is determined. If each λ_i vanishes at U, they must vanish everywhere.

The image of L is of dimension no more than $n-m$ (since it satisfies a system of linear equations of rank m). Since this image is isomorphic to the space of linearly independent fluxes with eigenframe $\{R_i\}$ modulo a constant, the proof is complete. □

In a later section, we will apply Theorem 2.2 to prove a result about the relationship between the Hugoniot locus and the rarefaction curves of IIa systems (Theorem 4.2).

2.2 Prescribing fewer eigenvectors

In general, we may want to prescribe m eigenvector fields in $\Omega \subseteq \mathbb{R}^n$. The λ system of Section 2.1 applies to the case $m = n$.

First we make simple remark regarding the case $m = 1$ (that is, we seek to prescribe only one eigenvector). Theorem 1.13 clearly applies to the system $\nabla_{R_1} F = \lambda^1 R_1$ with unknowns $F = (F^1, \ldots, F^n)$ and λ^1 (there are no integrability conditions to check), and there is a flux for any choice of n functions of $n-1$ variables and one function of n variables. Actually, this integrability theorem is overkill in this case; this result follows from standard existence and uniqueness results in ODEs.

For the remainder of this section we focus on the case $m = 2, n = 3$. This is naturally treated in two subcases, depending on whether our two desired eigenvectors are in involution or not.

2.2.1 Involutive case

Proposition 2.3. *Suppose that the pair of smooth linearly independent vector fields R_1 and R_2 are in involution, i.e. $[R_1, R_2] \in \operatorname{span}\{R_1, R_2\}$. Let $\mathcal{F}$ denote the set of vector fields F on Ω with DF having R_1 and R_2 as eigenvectors. Then:*

(i) if

$$\nabla_{R_1} R_2 \in \operatorname{span}\{R_1, R_2\}, \tag{2.5}$$

then each member of $\mathcal{F}$ is determined by two functions of two variables and three functions of one variable, and $\mathcal{F}$ contains fluxes for strictly hyperbolic systems;

(ii) if

$$\nabla_{R_1} R_2 \notin \operatorname{span}\{R_1, R_2\}, \tag{2.6}$$

then each member of $\mathcal{F}$ is determined by four functions of one variable, and no member of $\mathcal{F}$ is a flux for a strictly hyperbolic system.

Remark 2.4. *In either of the cases (i) or (ii): we do not claim that all fluxes in $\mathcal{F}$ are* hyperbolic*. On the other hand, we show that $\mathcal{F}$ will contain strictly hyperbolic fluxes in case (i), while no such fluxes exist in case (ii).*

Observe that, under the assumption $[R_1, R_2] \in \operatorname{span}\{R_1, R_2\}$, the condition $\nabla_{R_1} R_2 \in \operatorname{span}\{R_1, R_2\}$ holds if and only if $\nabla_{R_2} R_1 \in \operatorname{span}\{R_1, R_2\}$. The theorem requires equation (2.5) or equation (2.6) to hold throughout Ω; it does not address mixed cases.

Proof. We start with some general considerations that apply to both case (i) and case (ii).

Scale R_1 and R_2 and relabel such that $[R_1, R_2] = 0$. Of course it immediately follows that $\Gamma^i_{12} = \Gamma^i_{21}$ for $i = 1, 2, 3$. Let S be any vector field linearly independent from R_1 and R_2, and let the Christoffel symbols Γ^k_{ij} be given in terms of this frame, so that, for instance, $\nabla_{R_1} R_2 = \Gamma^1_{12} R_1 + \Gamma^2_{12} R_2 + \Gamma^3_{12} S$.

The triple of smooth maps $(F, \lambda^1, \lambda^2)$ has the property that DF has eigenvectors R_1 (with eigenvalue λ^1) and R_2 (with eigenvalue λ^2) if and only if the following equations hold:

$$\nabla_{R_1} F = \lambda^1 R_1; \tag{2.7}$$

$$\nabla_{R_2} F = \lambda^2 R_2; \tag{2.8}$$

$$R_1(\lambda^2) = \Gamma^2_{21}(\lambda^1 - \lambda^2); \tag{2.9}$$

$$R_2(\lambda^1) = \Gamma^1_{12}(\lambda^2 - \lambda^1); \tag{2.10}$$

$$\Gamma^3_{12}\left(\lambda^2 - \lambda^1\right) = 0. \tag{2.11}$$

Of course equations (2.7) and (2.8) are exactly the desired property of F; the other equations follow as differential consequences of the first two. To elaborate, since R_1 and R_2 commute, we have

$$\nabla_{R_1} \nabla_{R_2} F = \nabla_{R_2} \nabla_{R_1} F. \tag{2.12}$$

We expand this to

$$\nabla_{R_1}\left(\lambda^2 R_2\right) = \nabla_{R_2}\left(\lambda^1 R_1\right)$$

and then to

$$R_1(\lambda^2) R_2 + \lambda^2 \nabla_{R_1} R_2 = R_2(\lambda^1) R_1 + \lambda^1 \nabla_{R_2} R_1$$

Writing this out in components of R_1, R_2, and S, using the previously mentioned property that $\Gamma^i_{12} = \Gamma^i_{21}$, gives (2.9), (2.10), and (2.11).

We now prove case (i). In this case, by hypothesis $\nabla_{R_1} R_2 \in \text{span}\{R_1, R_2\}$, which implies that $\Gamma^3_{12} = \Gamma^3_{21} = 0$ and thus equation (2.11) vanishes. The fact that R_1 and R_2 commute implies they are coordinate vector fields in some coordinate system (see Theorem 9.46 of [20]). [1] We apply Theorem 1.13 to the subsystem (2.9), (2.10). Since each unknown λ^1, λ^2 is only differentiated in one direction, there are no integrability conditions to check. We obtain the solution λ^1, λ^2 for any initial data consisting of two functions of two variables. Then, given the functions λ^1 and λ^2, we apply Theorem 1.12 to the system (2.7), (2.8), giving us a solution $F = (F^1, F^2, F^3)$ for any choice of three functions of one variable. Thus, together, the solution of the entire system (2.7), (2.8), (2.9), (2.10) depends on two functions of two variables and three functions of one variable, as was to be shown.

For case (i), it remains only to show that strictly hyperbolic systems exist. To this end, let $\overline{U} \in \Omega$. Choose any initial values for λ^1 and λ^2 such that $\lambda^1(\overline{U}) \neq \lambda^2(\overline{U})$. Let (u^1, u^2, u^3) be the affine coordinate system in $\Omega \subseteq \mathbb{R}^3$ and without loss of generality suppose $\frac{\partial}{\partial u^3}$ is linearly independent of R_1 and R_2. Now choose initial data for F so that

$$F(\overline{U}^1, \overline{U}^2, u^3) = (0, 0, \int_{\overline{U}^3}^{u^3} \lambda^3(\overline{U}^1, \overline{U}^2, v)\, dv)^T,$$

where λ^3 is any smooth function such that $\lambda^3(\overline{U}) \neq \lambda^1(\overline{U})$ and $\lambda^3(\overline{U}) \neq \lambda^2(\overline{U})$. The resulting flux F has the property that $DF_{\overline{U}}$ has distinct eigenvalues $\lambda^1(\overline{U})$, $\lambda^2(\overline{U})$, and

[1] We could apply Theorem 1.13 (Darboux's integrability theorem) directly to the system (2.7), (2.8), (2.9), (2.10); however, this would not be conducive to proving that strictly hyperbolic systems exist.

$\lambda^3(\overline{U})$. The latter is an eigenvalue since

$$DF_{\overline{U}} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{\partial F}{\partial U^3}(\overline{U}) = \lambda^3(\overline{U}) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We still need to show that DF has distinct eigenvalues in a neighborhood of $\overline{U}$. Let $g(U, \lambda) = \det(DF_U - \lambda I)$ be the characteristic polynomial of DF_U. Then $g_\lambda(\overline{U}, \lambda^3(\overline{U})) \neq 0$ (since a polynomial with distinct roots has nonzero derivative at the roots), and the implicit function theorem implies that λ^3 is a continuous function of U. Continuity imples that in a neighborhood of $\overline{U}$, the characteristic polynomial has three distinct roots.

We now turn to case (ii). Here $\nabla_{R_1} R_2 \notin \operatorname{span}\{R_1, R_2\}$, implying that $\Gamma^3_{12} = \Gamma^3_{21}$ does not vanish. Equation (2.11) then implies that $\lambda^1 = \lambda^2$. Thus the system (2.7), (2.8), (2.9), (2.10), (2.11) is equivalent to the following one:

$$\nabla_{R_1} F = \lambda R_1; \tag{2.13}$$

$$\nabla_{R_2} F = \lambda R_2; \tag{2.14}$$

$$R_1(\lambda) = 0; \tag{2.15}$$

$$R_2(\lambda) = 0; \tag{2.16}$$

where $\lambda = \lambda^1 = \lambda^2$. We apply Theorem 1.13 this time with unknowns $F = (F^1, F^2, F^3)$ and λ. The only integrability condition to check is (2.12), which is trivially satisfied. The theorem implies that members of T (hence members of S) depend on four functions of one variable.

The statement that there are no strictly hyperbolic systems follows since $\lambda^1 = \lambda^2$. $\square$

Having established that involutive pairs $\{R_1, R_2\}$ satisfying (2.5) are always eigenvectors for some strictly hyperbolic system, it is of interest whether the strictly hyperbolic systems may be rich or non-rich.

We give several examples illustrating different possibilities.

The following example is a pair having only non-rich strictly hyperbolic fluxes.

Example 2.5. Let $R_1 = (1, 0, 0)^T$ and $R_2 = (w, 1, 0)^T$. We have $[R_1, R_2] = 0$ and so we are in the case under consideration. Up to scaling, every third vector field linearly independent from R_1 and R_2 can be written in the form $R_3 = (g, h, 1)^T$, where g and h are arbitrary functions. Since $[R_3, R_2] = (1, 0, 0)^T$, we have $c^1_{32} = 1$, and therefore there is no rich system where DF has R_1 and R_2 as eigenvalues.

But there is no obstacle in finding non-rich strictly hyperbolic systems. Indeed, consider the flux

$$F = \begin{pmatrix} v - \frac{u}{w} \\ 0 \\ -\frac{1}{w} - \log w \end{pmatrix}$$

We have

$$DF = \begin{pmatrix} -\frac{1}{w} & 1 & \frac{u}{w^2} \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1-w}{w^2} \end{pmatrix}$$

This has eigenvectors

$$R_1 = (1, 0, 0)^T; \quad R_2 = (w, 1, 0)^T; \quad R_3 = (u, 0, 1)^T.$$

The eigenvalues are

$$\lambda^1 = -\frac{1}{w}; \quad \lambda^2 = 0; \quad \lambda^3 = \frac{1 - w}{w^2}.$$

The following example presents a pair which has rich strictly hyperbolic systems, but no nonrich systems.

Example 2.6. Let $R_1 = (1,0,0)^T$ and $R_2 = (0,1,0)^T$. We see that $[R_1, R_2] = 0$. As in the previous example, every linearly independent third vector field can be written $R_3 = (g,h,1)^T$, possibly after rescaling.

But now we see that $\Gamma^3_{12} = \Gamma^2_{31} = \Gamma^1_{32} = 0$. Thus every row of the algebraic part of the λ system (2.2) has at least one 0 in it, and there are thus no nonrich strictly hyperbolic systems (this follows from the classification in [13], in which it was shown that all nonrich strictly hyperbolic 3×3 systems fall into what they called case IIa, in which the algebraic part must have one row with all nonzero coefficients).

Finally, we present an example of a pair which has both rich and nonrich strictly hyperbolic systems.

Example 2.7. Consider the pair of vector fields $R_1 = (1, -\sqrt{u}, 0)^T$ and $R_2 = (1, \sqrt{u}, 0)^T$. Since the third component of $[R_1, R_2]$ is 0, we do have $[R_1, R_2] \in \text{span}\{R_1, R_2\}$.

If we adjoin the third vector field $R_3 = (0,0,1)$, then A_λ vanishes and, according to Theorem 4.2 of [13] there are certainly rich strictly hyperbolic systems for this frame.

On the other hand, we also have the flux

$$F(u,v,w) = (v, \frac{u^2}{2} + w, 0)^T.$$

We calculate

$$DF = \begin{pmatrix} 0 & 1 & 0 \\ u & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

This has eigenvectors

$$R_1 = (1, -\sqrt{u}, 0)^T; \quad R_2 = (1, \sqrt{u}, 0)^T; \quad R_3 = (1, 0, -u)^T.$$

The eigenvalues are

$$\lambda^1 = -\sqrt{u}; \quad \lambda^2 = \sqrt{u}; \quad \lambda^3 = 0.$$

In this frame, $c_{13}^2 = -\frac{1}{4u}$. Thus this is a nonrich strictly hyperbolic system.

2.2.2 Non-involutive case

In this case, there are certainly no rich systems. We categorize the pairs of vector fields $\{R_1, R_2\}$ into two subcases. The first subcase, consisting of those pairs of vector fields for which one of $\nabla_{R_1} R_2$ and $\nabla_{R_2} R_1$ is in the span of $\{R_1, R_2\}$, is treated in the following proposition.

Proposition 2.8. *Given two linearly independent vector fields R_1 and R_2 on $\Omega \subseteq \mathbb{R}^3$, if*

(H1) $[R_1, R_2] \notin \text{span}\{R_1, R_2\}$*;*

(H2) one of $\nabla_{R_1} R_2$ *or* $\nabla_{R_2} R_1$ *is in* $\text{span}\{R_1, R_2\}$*;*

then there are no strictly hyperbolic fluxes such that DF has eigenvectors R_1 and R_2.

Proof. Suppose for contradiction that F is such a flux, with third eigenvector R_3 and eigenvalue λ^3. Then the flatness condition is

$$\nabla_{R_1}(\lambda^2 R_2) - \nabla_{R_2}(\lambda^1 R_1) = \nabla_{[R_1, R_2]} F.$$

The R_3 component of this vector equation is

$$\lambda^2\Gamma^3_{12} - \lambda^1\Gamma^3_{21} = \lambda^3 c^3_{12}.$$

By hypothesis (H1), one of Γ^3_{12} or Γ^3_{21} is 0. Without loss of generality suppose $\Gamma^3_{21} = 0$. Then $c^3_{12} = \Gamma^3_{12}$ and we must have $\lambda^2 = \lambda^3$, contradicting our assumption that F is a strictly hyperbolic flux. □

The second subcase, in which neither $\nabla_{R_1} R_2$ nor $\nabla_{R_2} R_1$ is in the span of $\{R_1, R_2\}$, may admit strictly hyperbolic fluxes. The following proposition describes the types of fluxes that may result; a later proposition will count fluxes.

Proposition 2.9. *Given two linearly independent vector fields R_1 and R_2 on $\Omega \subseteq \mathbb{R}^3$, if*

(H1) $[R_1, R_2] \notin \operatorname{span}\{R_1, R_2\}$;

(H2) neither $\nabla_{R_1} R_2$ *nor* $\nabla_{R_2} R_1$ *is in* $\operatorname{span}\{R_1, R_2\}$;

(H3) F is a flux such that DF has eigenvectors R_1 and R_2;

then one of the following holds:

(C1) F is trivial;

(C2) F is strictly hyperbolic;

(C3) F is not hyperbolic;

If (C3) holds, then $\lambda^1 = \lambda^2$, the eigenspace is spanned by $\{R_1, R_2\}$, and the following equation holds:

$$\Gamma^3_{11}\Gamma^3_{22} = (\Gamma^3_{12} - 2\Gamma^3_{21})(\Gamma^3_{21} - 2\Gamma^3_{12}) \tag{2.17}$$

where Γ_{ij}^k are the Christoffel symbols with respect to a frame $\{R_1, R_2, R_3\}$, where R_3 is any vector field linearly independent from $\{R_1, R_2\}$.

Proof. Suppose neither (C1) nor (C2) holds. Suppose for contradiction that F is hyperbolic. Let R_3 be the third eigenvector with third eigenvalue λ^3. Examining the R_3 component of

$$\nabla_{R_1}\nabla_{R_2}F - \nabla_{R_2}\nabla_{R_1}F = \nabla_{[R_1,R_2]}F \tag{2.18}$$

leads to the equation

$$c_{21}^3\lambda^3 = \Gamma_{21}^3\lambda^1 - \Gamma_{12}^3\lambda^2$$

(this is one row of the algebraic part of the λ system from [13]). None of the coefficients of λ^i in this equation vanish, by hypotheses (H1) and (H2). Since by assumption F is not hyperbolic, at least two of the eigenvalues are equal. But the only solutions of this linear system where two λ^i are equal are those where all three are equal. But then F is in fact trivial (see Proposition 2.1 of [13]), a contradiction.

Note that in case (C3) there cannot be a complex eigenvalue (the characteristic equation $\det(DF - \lambda I) = 0$ is a degree 3 real polynomial with two real roots, and so cannot have a complex root), and so there must be a generalized eigenvector R_3.

Suppose for contradiction that in case (C3) we have $\lambda^1 \neq \lambda^2$. As just noted, we have a generalized eigenvector R_3; without loss of generality we assume that $\nabla_{R_3}F = R_2 + \lambda^2 R_3$. We now derive some differential consequences. Expand equation (2.18) to

$$\nabla_{R_1}(\lambda^2 R_2) - \nabla_{R_2}\lambda^1 R_2 = c_{12}^1\lambda^1 R_1 + c_{12}^2\lambda^2 R_2 + c_{12}^3(R_2 + \lambda^2 R_3).$$

Expanding once more we obtain

$$R_1(\lambda^2)R_2 + \lambda^2\nabla_{R_1}R_2 - R_2(\lambda^1)R_1 - \lambda^1\nabla_{R_2}R_1 = c_{12}^1\lambda^1 R_1 + c_{12}^2\lambda^2 R_2 + c_{12}^3(R_2 + \lambda^2 R_3).$$

The R_3 component of this vector equation leads to

$$(\lambda^2 - \lambda^1)\Gamma_{21}^3 = 0.$$

Since by assumption (H2) $\Gamma_{21}^3 \neq 0$, we must have $\lambda^2 = \lambda^1$, a contradiction.

It remains only to demonstrate that in case (C3), equation (2.17) holds. Write $\lambda := \lambda^1 = \lambda^2$. We again have a generalized eigenvector R_3, where $\nabla_{R_3}F = \beta R_1 + \gamma R_2 + \lambda R_3$ for some β, γ. At least one of β and γ must be nonzero; without loss of generality assume that $\beta \neq 0$, and scale R_3 so that in fact we have

$$\nabla_{R_3}F = R_1 + \alpha R_2 + \lambda R_3.$$

We derive some differential consequences. Taking the R_1 and R_2 components of equation (2.18) leads to

$$R_2(\lambda) = c_{21}^3 \tag{2.19}$$

and

$$R_1(\lambda) = \alpha c_{12}^3. \tag{2.20}$$

Taking the R_3 component of the equation

$$\nabla_{R_1}\nabla_{R_3}F - \nabla_{R_3}\nabla_{R_1}F = \nabla_{[R_1,R_3]}F$$

leads to the equation

$$R_1(\lambda) = -\alpha\Gamma^3_{12} - \Gamma^3_{11}. \tag{2.21}$$

Finally, taking the R_3 component of the equation

$$\nabla_{R_2}\nabla_{R_3}F - \nabla_{R_3}\nabla_{R_2}F = \nabla_{[R_2,R_3]}F$$

leads to the equation

$$R_2(\lambda) = -\alpha\Gamma^3_{22} - \Gamma^3_{21}. \tag{2.22}$$

Equations (2.20) and (2.21) give two expressions for $R_1(\lambda)$; by setting them equal to each other we obtain

$$\Gamma^3_{11} = (\Gamma^3_{21} - 2\Gamma^3_{12})\alpha. \tag{2.23}$$

Similarly, by setting (2.19) and (2.22) equal to each other we obtain

$$\Gamma^3_{12} - 2\Gamma^3_{21} = \Gamma^3_{22}\alpha. \tag{2.24}$$

Multiply equation (2.23) by Γ^3_{22} and equation (2.24) by $(\Gamma^3_{21} - 2\Gamma^3_{12})$, and then subtract, to obtain (2.17).

Equation (2.17) has been established in the case where R_3 is taken to be a generalized eigenvector, but the proposition asserts that it holds for any choice of R_3. This is easily established by noting that if we replace R_3 by $S := aR_1 + bR_2 + cR_3$, each Γ^k_{ij} that appears in (2.17) is scaled by a factor of $\frac{1}{c}$, and thus (2.17) continues to hold in the new frame. □

The next proposition counts fluxes in this subcase. We do not know whether every pair $\{R_1, R_2\}$ having nontrivial fluxes has a strictly hyperbolic flux. However, we can see

that any pair $\{R_1, R_2\}$ in this case for which (2.17) does not hold and there is a nontrivial flux (equivalently, the set of fluxes has dimension greater than four), there are strictly hyperbolic fluxes.

Proposition 2.10. *Given two linearly independent vector fields R_1 and R_2 on $\Omega \subseteq \mathbb{R}^3$, if*

(H1) $[R_1, R_2] \notin \operatorname{span}\{R_1, R_2\}$*;*

(H2) neither $\nabla_{R_1} R_2$ *nor* $\nabla_{R_2} R_1$ *is in* $\operatorname{span}\{R_1, R_2\}$*;*

(H3) F is a flux such that DF has eigenvectors R_1 and R_2;

then the set Ξ of fluxes F such that DF has eigenvectors R_1 and R_2 is a real vector space either of dimension 4, 5, 6, 7, or 8.

Proof. It is obvious that the set of fluxes is a real vector space. It is of dimension at least 4, since all trivial fluxes will have R_1 and R_2 as eigenvectors.

Put $S = [R_1, R_2]$. For each $F \in S$, we must have

$$\nabla_{R_1} F = \lambda^1 R_1; \quad \nabla_{R_2} F = \lambda^2 R_2; \quad \nabla_S F = \kappa^1 R_1 + \kappa^2 R_2 + \mu S. \tag{2.25}$$

for some functions λ^1, λ^2, κ^1, κ^2, and μ.

Let Γ_{ij}^k and c_{ij}^k be the Christoffel symbols and structure coefficients in terms of the frame $\{R_1, R_2, S\}$. For instance, $\nabla_{R_1} R_2 = \Gamma_{12}^1 R_1 + \Gamma_{12}^2 R_2 + \Gamma_{12}^3 S$. By the hypothesis in (a), neither Γ_{12}^3 nor Γ_{21}^3 vanishes. Finally, let

$$\nu = R_1(\kappa^1).$$

The proof will proceed by deriving a Frobenius system of 24 equations for the 8 unknown functions $\lambda^1, \lambda^2, \kappa^1, \kappa^2, \nu$, and $F = (F^1, F^2, F^3)$ as a differential consequence of (2.25). Here a Frobenius system is an overdetermined system in which the directional derivatives in each direction R_1, R_2, and S are specified for each unknown. The set of solutions to such a system can depend on at most 8 constants. Equations (2.25) already specify all directional derivatives for F^1, F^2, and F^3.

The set of functions satisfying this Frobenius system is a vector space of dimension no more than 8, and thus Ξ has dimension no more than 8. The fact that each dimension 4, 5, 6, 7, and 8 can actually occur is demonstrated by the examples following this proposition.

In the following we use the notation L to denote any function of $U = (u, v, w)$ and one or more of the unknown functions $\lambda^1, \lambda^2, \kappa^1, \kappa^2, \nu$, where L is linear in the latter variables.

Claim 1: As a consequence of (2.25), the following equations hold:

$$R_1(\lambda^1) = 2\frac{\Gamma^3_{12}}{\Gamma^3_{21}}\kappa^2 + L(\lambda^1, \lambda^2, \kappa^1); \tag{2.26}$$

$$R_2(\lambda^2) = -2\frac{\Gamma^3_{21}}{\Gamma^3_{12}}\kappa^1 + L(\lambda^1, \lambda^2, \kappa^1); \tag{2.27}$$

$$R_2(\lambda^1) = -\kappa^1 + L(\lambda^1, \lambda^2); \tag{2.28}$$

$$R_1(\lambda^2) = \kappa^2 + L(\lambda^1, \lambda^2); \tag{2.29}$$

$$R_1(\kappa^2) = L(\lambda^1, \lambda^2, \kappa^1, \kappa^2); \tag{2.30}$$

$$R_2(\kappa^1) = L(\lambda^1, \lambda^2, \kappa^1, \kappa^2); \tag{2.31}$$

$$R_1(\kappa^1) - S(\lambda^1) = L(\lambda^1, \lambda^2, \kappa^1, \kappa^2); \tag{2.32}$$

$$R_2(\kappa^2) - S(\lambda^2) = L(\lambda^1, \lambda^2, \kappa^1, \kappa^2). \tag{2.33}$$

Equations (2.26) through (2.31) establish six of the directional derivatives we will need in our Frobenius system. Equations (2.32) and (2.33) are not yet in the form we need, but will be used in a later claim to establish further directional derivatives.

To establish the claim, begin by expanding the following equation:

$$\nabla_{R_1}(\nabla_{R_2}F) - \nabla_{R_2}(\nabla_{R_1}F) = \nabla_{[R_1,R_2]}F = \nabla_S F.$$

We have

$$\nabla_{R_1}(\lambda^2 R_2) - \nabla_{R_2}(\lambda^1 R_1) = \kappa^1 R_1 + \kappa^2 R_2 + \mu S$$

We expand further to obtain

$$R_1(\lambda^2)R_2 + \lambda^2\nabla_{R_1}R_2 - R_2(\lambda^1)R_1 - \lambda^1\nabla_{R_2}R_1 = \kappa^1 R_1 + \kappa^2 R_2 + \mu S.$$

Now write out this equation in components. The R_1 component is:

$$\lambda^2\Gamma^1_{12} - R_2(\lambda^1) - \lambda^1\Gamma^1_{21} = \kappa^1,$$

establishing equation (2.28). The R_2 component is

$$R_1(\lambda^2) + \lambda^2\Gamma^2_{12} - \lambda^1\Gamma^2_{21} = \kappa^2,$$

establishing equation (2.29). The S component is

$$\lambda^2\Gamma^3_{12} - \lambda^1\Gamma^3_{21} = \mu, \tag{2.34}$$

giving μ explicitly.

We now proceed in a similar manner beginning with this equation:

$$\nabla_{R_1}(\nabla_S F) - \nabla_S(\nabla_{R_1} F) = \nabla_{[R_1,s]} F.$$

Expand to

$$\nabla_{R_1}(\kappa^1 R_1 + \kappa^2 R_2 + \mu S) - \nabla_S(\lambda^1 R_1) = \nabla_{c^1_{13} R_1 + c^2_{13} R_2 + c^3_{13} S} F$$

and then

$$R_1(\kappa^1)R_1 + \kappa^1 \nabla_{R_1} R_1 + R_1(\kappa^2)R_2 + \kappa^2 \nabla_{R_1} R_2 + R_1(\mu)S + \mu \nabla_{R_1} S - s(\lambda^1)R_1 - \lambda^1 \nabla_S R_1$$
$$= c^1_{13}\lambda^1 R_1 + c^2_{13}\lambda^2 R_2 + c^3_{13}(\kappa^1 R_1 + \kappa^2 R_2 + \mu S).$$

Write the R_1 component:

$$R_1(\kappa^1) + \kappa^1 \Gamma^1_{11} + \kappa^2 \Gamma^1_{12} + \mu \Gamma^1_{13} - S(\lambda^1) - \lambda^1 \Gamma^1_{31} = c^1_{13}\lambda^1 + c^3_{13}\kappa^1,$$

which, after substituting for μ using (2.34), establishes (2.32). The R_2 component is

$$\kappa^1 \Gamma^2_{11} + R_1(\kappa^2) + \kappa^2 \Gamma^2_{12} + \mu \Gamma^2_{13} - \lambda^1 \Gamma^2_{31} = c^2_{13}\lambda^2 + c^3_{13}\kappa^2,$$

establishing (2.30). The s component is

$$\kappa^1 \Gamma^3_{11} + \kappa^2 \Gamma^3_{12} + R_1(\mu) + \mu \Gamma^3_{13} - \lambda^1 \Gamma^3_{31} = \mu c^3_{13}. \tag{2.35}$$

We can use equation (2.34) to find $R_1(\mu) = R_1(\lambda^2)\Gamma^3_{12} - R_1(\lambda^1)\Gamma^3_{21} + L(\lambda^1, \lambda^2)$. Now

substitute using (2.29) to see that $R_1(\mu) = \Gamma^3_{12}\kappa^2 - R_1(\lambda^1)\Gamma^3_{21} + L(\lambda^1, \lambda^2)$. Finally, substitute this into (2.35) to establish (2.26).

There is one more immediate consequence of (2.25) we must expand:

$$\nabla_{R_2}(\nabla_S F) - \nabla_S(\nabla_{R_2} F) = \nabla_{[R_2,S]} F.$$

This becomes

$$\nabla_{R_2}(\kappa^1 R_1 + \kappa^2 R_2 + \mu S) - \nabla_S(\lambda^2 R_2) = \nabla_{c^1_{23}R_1 + c^2_{23}R_2 + c^3_{23}S} F$$

and then

$$\begin{aligned} R_2(\kappa^1)R_1 + \kappa^1 \nabla_{R_2} R_1 + R_2(\kappa^2)R_2 + \kappa^2 \nabla_{R_2} R_2 + R_2(\mu)S + \mu \nabla_{R_2} S - S(\lambda^2)R_2 - \lambda^2 \nabla_S R_2 \\ = c^1_{23}\lambda^1 R_1 + c^2_{23}\lambda^2 R_2 + c^3_{23}(\kappa^1 R_2 + \kappa^2 R_2 + \mu S). \end{aligned}$$

The R_1 component is

$$R_2(\kappa^1) + \kappa^1 \Gamma^1_{21} + \kappa^2 \Gamma^1_{22} + \mu \Gamma^1_{23} - \lambda^2 \Gamma^1_{32} = c^1_{23}\lambda^1 + c^3_{23}\kappa^1,$$

establishing (2.31). The R_2 component is

$$\kappa^1 \Gamma^2_{21} + R_2(\kappa^2) + \kappa^2 \Gamma^2_{22} + \mu \Gamma^2_{23} - S(\lambda^2) - \lambda^2 \Gamma^2_{32} = c^2_{23}\lambda^2 + c^3_{23}\kappa^2,$$

establishing (2.33). The S component is

$$\kappa^1 \Gamma^3_{21} + \kappa^2 \Gamma^3_{22} + R_2(\mu) + \mu \Gamma^3_{23} - \lambda^2 \Gamma^3_{32} = c^3_{23}\mu. \tag{2.36}$$

We can use equation (2.34) to find $R_2(\mu) = R_2(\lambda^2)\Gamma^3_{12} - R_2(\lambda^1)\Gamma^3_{21} + L(\lambda^1, \lambda^2)$. Now substitute using (2.28) to see that $R_2(\mu) = R_2(\lambda^2)\Gamma^3_{12} + \Gamma^3_{21}\kappa^1 + L(\lambda^1, \lambda^2)$ Finally, substitute this into (2.36) to establish (2.27).

Claim 2: The following additional equations hold:

$$R_1(\kappa^1) + 2\frac{\Gamma^3_{12}}{\Gamma^3_{21}}R_2(\kappa^2) + S(\lambda^1) = L(\lambda^1, \lambda^2, \kappa^1, \kappa^2); \tag{2.37}$$

$$2\frac{\Gamma^3_{21}}{\Gamma^3_{12}}R_1(\kappa^1) + R_2(\kappa^2) + S(\lambda^2) = L(\lambda^1, \lambda^2, \kappa^1, \kappa^2). \tag{2.38}$$

To see (2.37), we expand the equation

$$R_1(R_2(\lambda^1)) - R_2(R_1\lambda^1) = [R_1, R_2]\lambda^1 = S(\lambda^1).$$

First use (2.28) and (2.26):

$$R_1(-\kappa^1 + L(\lambda^1, \lambda^2)) - R_2\left(2\frac{\Gamma^3_{12}}{\Gamma^3_{21}}\kappa^2 + L(\lambda^1, \lambda^2, \kappa^1)\right) = S(\lambda^1).$$

Observe that $R_1(L(\lambda^1, \lambda^2)) = L(\lambda^1, \lambda^2, \kappa^1, \kappa^2)$. This is so because we have already established expressions for $R_1(\lambda^1)$ and $R_1(\lambda^2)$ in the form of (2.26) and (2.29). Similarly

$$R_2(L(\lambda^1, \lambda^2, \kappa^1)) = L(\lambda^1, \lambda^2, \kappa^1, \kappa^2).$$

Thus we have established (2.37).

An analogous approach, beginning with the equation

$$R_1(R_2(\lambda^2)) - R_2(R_1\lambda^2) = S(\lambda^2),$$

establishes (2.38).

Claim 3. The following equations hold:

$$S(\lambda^1) = \nu + L(\lambda^1, \lambda^2, \kappa^1, \kappa^2); \tag{2.39}$$

$$S(\lambda^2) = -\frac{\Gamma^3_{21}}{\Gamma^3_{12}}\nu + L(\lambda^1, \lambda^2, \kappa^1, \kappa^2); \tag{2.40}$$

$$R_2(\kappa^2) = -\frac{\Gamma^3_{21}}{\Gamma^3_{12}}\nu + L(\lambda^1, \lambda^2, \kappa^1, \kappa^2). \tag{2.41}$$

These equations result by substituting $R_1(\kappa^1) = \nu$ into (2.32), (2.33), and (2.37), and then solving the resulting rank 3 linear system on the 3 unknowns $s(\lambda^1)$, $s(\lambda^2)$, and $R_2(\kappa^2)$. (Note that adjoining the extra equation (2.38) still results in only a rank 3 system; thus we need to have the free parameters ν. Indeed, we do not actually need equation (2.38) in this proof, but have written it here for reference.)

Claim 4. The following equations hold:

$$R_1(\nu) = L(\lambda^1, \lambda^2, \kappa^1, \kappa^2, \nu); \tag{2.42}$$

$$R_2(\nu) = L(\lambda^1, \lambda^2, \kappa^1, \kappa^2, \nu). \tag{2.43}$$

To see (2.42), expand the equation

$$R_1(S(\lambda^1)) - s(R_1(\lambda^1)) = [R_1, S]\lambda^1$$

using (2.39) and (2.26) to obtain

$$R_1(\nu + L(\lambda^1, \lambda^2, \kappa^1, \kappa^2)) - S\left(2\frac{\Gamma^3_{12}}{\Gamma^3_{21}}\kappa^2 + L(\lambda^1, \lambda^2, \kappa^1)\right) = [R_1, S]\lambda^1.$$

All derivatives in this equation lead to expressions already known, with the exception of $R_1(\nu)$, thus giving (2.42).

Equation (2.43) is found in an analogous way.

We have now established, for the 8 unknowns, their directional derivatives in at least the R_1 and R_2 direction.

Claim 5. The derivative of each function in the s direction is also determined.

As a representative example, consider the equation

$$R_1(R_2\nu) - R_2(R_2\nu) = [R_1, R_2]\nu = S(\nu).$$

Expand the expressions on the left to obtain

$$R_1(L(\lambda^1, \lambda^2, \kappa^1, \kappa^2, \nu)) - R_2(L(\lambda^1, \lambda^2, \kappa^1, \kappa^2, \nu)) = S(\nu).$$

All the derivatives on the left are already known expressions; thus, we now have an expression for $S(\nu)$.

The equations for the other unknown functions can be found in the same way.

We have thus found a Frobenius system on our 8 unknown functions as a differential consequence of (2.25) and the proof is complete. □

The proof just given used a rank three linear system of four equations on four unknowns: equations (2.32), (2.33), (2.37), and (2.38). Of course the right hand sides of these equations must satisfy a linear relation which leads to an equation of the form

$$A\lambda^1 + B\lambda^2 + C\kappa^1 + D\kappa^2 = 0,$$

where

$$A = S(\Gamma^3_{21}) + R_2\left(\Gamma^3_{31}\right) + R_2\left(\Gamma^3_{31}\right)\Gamma^3_{21} - R_1\left(\Gamma^3_{32}\right)\Gamma^3_{21} + \Gamma^1_{13}\Gamma^3_{21} + \Gamma^1_{13}{\Gamma^3_{21}}^2 + \Gamma^2_{13}\Gamma^3_{21}\Gamma^3_{22} - \Gamma^3_{11}\Gamma^3_{21}\Gamma^1_{23} + \Gamma^3_{12}\Gamma^3_{21}\Gamma^2_{23} + \Gamma^3_{22}\Gamma^2_{31} - \Gamma^1_{21}\Gamma^3_{31} - \Gamma^3_{11}(\Gamma^1_{23} - \Gamma^1_{32}) - \Gamma^2_{21}\Gamma^3_{32} + \Gamma^3_{31}\Gamma^3_{32}$$

$$B = -S(\Gamma^3_{12}) - R_1\left(\Gamma^3_{32}\right) - R_2\left(\Gamma^3_{31}\right)\Gamma^3_{12} + R_1\left(\Gamma^3_{32}\right)\Gamma^3_{12} - \Gamma^3_{12}\Gamma^1_{13}\Gamma^3_{21} - \Gamma^3_{12}\Gamma^2_{13}\Gamma^3_{22} + \Gamma^3_{11}\Gamma^3_{12}\Gamma^1_{23} - \Gamma^3_{12}\Gamma^2_{23} + {\Gamma^3_{12}}^2\Gamma^2_{23} + \Gamma^3_{22}(\Gamma^2_{13} - \Gamma^2_{31}) + \Gamma^1_{12}\Gamma^3_{31} - \Gamma^3_{11}\Gamma^1_{32} + \Gamma^2_{12}\Gamma^3_{32} - \Gamma^3_{31}\Gamma^3_{32},$$

$$C = -R_2\left(\Gamma^3_{11}\right) + R_1\left(\Gamma^3_{21}\right) + \Gamma^3_{11}\Gamma^1_{21} + \Gamma^3_{12}\Gamma^2_{21} - \Gamma^1_{11}\Gamma^3_{21} + \Gamma^3_{13}\Gamma^3_{21} - \Gamma^2_{11}\Gamma^3_{22} - \Gamma^3_{11}\Gamma^3_{23} - \Gamma^3_{31},$$

and

$$D = -R_2\left(\Gamma^3_{12}\right) + R_1\left(\Gamma^3_{22}\right) - \Gamma^1_{12}\Gamma^3_{21} + \Gamma^3_{11}\Gamma^1_{22} + \Gamma^3_{12}\Gamma^2_{22} - \Gamma^2_{12}\Gamma^3_{22} + \Gamma^3_{13}\Gamma^3_{22} - \Gamma^3_{12}\Gamma^3_{23} - \Gamma^3_{32}.$$

The derivation of these expressions is conceptually trivial but involves quite a bit of symbolic manipulation which we do not present here (the author used the computer algebra system Mathematica to find them).

We now present examples of non-involutive pairs $\{R_1, R_2\}$ such that the corresponding set of fluxes is 4, 5, 6, 7, or 8 dimensional. (It will always be at least 4 dimensional, since for any constant $c \in \mathbb{R}$ and constant vector v, $F(u) = cu + v$ is a flux.)

The set of fluxes in the next example is only 4 dimensional.

Example 2.11. Given the two vector fields

$$R_1 = \begin{pmatrix} 0 \\ 1 \\ u \end{pmatrix}, \quad R_2 = \begin{pmatrix} w \\ 0 \\ 1 \end{pmatrix},$$

all corresponding fluxes are a multiple of the identity plus a constant vector. Note also that this example does have the property that neither $\nabla_{R_1} R_2$ and $\nabla_{R_2} R_1$ is in the span of $\{R_1, R_2\}$. Thus this condition is not sufficient to ensure the existence of nontrivial fluxes.

The set of fluxes in the next example is 5 dimensional.

Example 2.12. Given the two vector fields

$$R_1 = \begin{pmatrix} v \\ u \\ w \end{pmatrix}, \quad R_2 = \begin{pmatrix} u \\ w \\ v \end{pmatrix},$$

all corresponding fluxes are of the form

$$F(u,v,w) = \frac{c_1}{(u+v+w)^2} \begin{pmatrix} -\frac{1}{2}u(u+2v) \\ -(u^2 + (v+w)u + \frac{1}{2}v(v+2w)) \\ \frac{1}{2}w(2v+w) \end{pmatrix}$$

up to the addition of a constant vector and a multiple of $(u,v,w)^T$. The eigenvalues of

DF are

$$\lambda^1 = c_1 \frac{u - v}{(u + v + w)^2};$$
$$\lambda^2 = 0;$$
$$\lambda^3 = c_1 \frac{v - w}{(u + v + w)^2}.$$

All fluxes in this case are hyperbolic.

The set of fluxes in the next example is 6 dimensional.

Example 2.13. Given the two vector fields

$$R_1 = \begin{pmatrix} -1 \\ 0 \\ v + 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} \frac{w}{v^2 - 1} \\ -1 \\ u \end{pmatrix},$$

all corresponding fluxes are of the form

$$F(u, v, w) = c_1 \begin{pmatrix} -((v-1)u + w)e^{-1} \operatorname{Ei}(1 - v) - e^{-v}u \\ -\frac{1}{2}\left[(v-1)^2 \operatorname{Ei}(1 - v)e^{-1} + (3v + 2)e^{-v}\right] \\ (v+1)((v-1)u + w)e^{-1} \operatorname{Ei}(1 - v) + (2(v+1)u + w)e^{-v} \end{pmatrix}$$
$$+ c_2 \begin{pmatrix} uv + w \\ \frac{v^2}{2} \\ -v^2 - vw \end{pmatrix}.$$

up to the addition of a constant vector and a multiple of $(u, v, w)^T$. Here Ei is the

exponential integral:

$$\mathrm{Ei}(x) = -\int_x^\infty \frac{e^{-t}}{t}\, dt.$$

The eigenvalues are

$$\begin{aligned}
\lambda^1 &= c_1(2e^{-1}\,\mathrm{Ei}(1-v) - e^{-v}) - c_2;\\
\lambda^2 &= c_1(e^{-1}\,\mathrm{Ei}(1-v)(1-v) + e^{-v}v) + c_2 v;\\
\lambda^3 &= c_1 e^{-v} + c_2.
\end{aligned}$$

All fluxes in this case are hyperbolic.

The set of fluxes in the next example is 7 dimensional.

Example 2.14. Given the two vector fields

$$R_1 = \begin{pmatrix} 1 \\ \sqrt{w} \\ 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} u \\ 0 \\ -w \end{pmatrix},$$

all corresponding fluxes are of the form

$$F(u,v,w) = c_1 \begin{pmatrix} 3uv\sqrt{w} - v^2 - u^2 w \\ uvw \\ vw^{3/2} - uw^2 \end{pmatrix} + c_2 \begin{pmatrix} v \\ uw \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} u\sqrt{w} - v \\ 0 \\ \frac{w^{3/2}}{3} \end{pmatrix}$$

up to the addition of a constant vector and a multiple of $(u, v, w)^T$. The eigenvalues are

$$\begin{aligned}
\lambda^1 &= c_1(v\sqrt{w} + uw) + c_2\sqrt{w};\\
\lambda^2 &= c_1\left(\frac{3}{2}v\sqrt{w} - uw\right) + \frac{c_3}{2}\sqrt{w};\\
\lambda^3 &= c_1(2v\sqrt{w} - 3uw) - c_2\sqrt{w} + c_3\sqrt{w}.
\end{aligned}$$

In this case, when $c_1 = 0$ and $c_2 = \frac{1}{2}c_3$, we have $\lambda^1 = \lambda^2$ and the system is not hyperbolic.

The set of fluxes in the next example is 8 dimensional.

Example 2.15. Given the two vector fields

$$R_1 = \begin{pmatrix} 1 \\ 0 \\ \frac{v}{2} \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 \\ 1 \\ -\frac{u}{2} \end{pmatrix},$$

all corresponding fluxes are of the form

$$F(u, v, w) = c_1 \begin{pmatrix} \frac{vu^2}{2} + wu \\ -\frac{uv^2}{2} + wv \\ \frac{3}{8}v^2u^2 + \frac{w^2}{2} \end{pmatrix} + c_2 \begin{pmatrix} u^2 \\ -\frac{uv}{2} + w \\ \frac{u^2v}{2} \end{pmatrix} + c_3 \begin{pmatrix} -\frac{uv}{4} - \frac{w}{2} \\ \frac{v^2}{2} \\ -\frac{uv^2}{4} \end{pmatrix} + c_4 \begin{pmatrix} -u \\ v \\ -\frac{uv}{2} \end{pmatrix}$$

up to the addition of a constant vector and a multiple of $(u, v, w)^T$. The eigenvalues are

$$\begin{aligned}
\lambda^1 &= c_1\left(\frac{3}{2}uv + w\right) + c_2 2u - \frac{c_3}{2}v - c_4;\\
\lambda^2 &= c_1\left(-\frac{3}{2}uv + w\right) - c_2 u + \frac{c_3}{2}v + c_4;\\
\lambda^3 &= c_1 w + \frac{c_2}{2}u + c_3 v.
\end{aligned}$$

Every flux in this case is hyperbolic.

The following example illustrates a pair of vector fields for which not every corresponding flux is hyperbolic.

Example 2.16. Given the two vector fields

$$R_1 = \begin{pmatrix} 1 \\ 0 \\ 2v \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 \\ 1 \\ u \end{pmatrix},$$

all corresponding fluxes are of the form

$$F(u,v,w) = c_1 \begin{pmatrix} \frac{u^2 v}{2} - \frac{uw}{2} \\ -2uv^2 + vw \\ -3u^2v^2 + 3uvw - w^2 \end{pmatrix} + c_2 \begin{pmatrix} -\frac{u^2}{2} \\ -2uv + w \\ -u^2 v \end{pmatrix} + c_3 \begin{pmatrix} -\frac{uv}{4} + \frac{w}{4} \\ \frac{v^2}{2} \\ \frac{uv^2}{2} \end{pmatrix} + c_4 \begin{pmatrix} 0 \\ v \\ 2uv - w \end{pmatrix}$$

up to the addition of a constant vector and a multiple of $(u,v,w)^T$. The eigenvalues are

$$\begin{aligned} \lambda^1 &= \frac{1}{4}(-4c_2 + c_3 v - 2c_1 w); \\ \lambda^2 &= c_4 - c_2 u + c_3 v - 3c_1 uv + c_1 w; \\ \lambda^3 &= \frac{1}{2}(-2c_4 - 2c_2 u - c_3 v + 6c_1 uv - 4c_1 w). \end{aligned}$$

Setting $c_1 = c_3 = c_4 = 0$ and $c_2 = 1$, we obtain the flux

$$F(u,v,w) = \begin{pmatrix} \frac{u^2}{2} \\ -2uv + w \\ -u^2 v \end{pmatrix}$$

and see that all three eigenvalues are $\lambda^i = -u$. Indeed, this flux does not correspond to a hyperbolic system.

Chapter 3

Solvable interactions

3.1 Analytic interactions

It is of interest to find system of conservation laws in which the pairwise interactions are *explicitly solvable*. That is, whenever $b = (b_1, \ldots, b_n)$ are the outgoing wave strengths for incoming wave strengths a_j and a_k, we would like b to be written explicitly as a function of a_j and b_k. In general, estimating solutions to hyperbolic conservation laws is challenging. Thus, systems which can be solved explicitly are good candidates in searching for examples of finite-time blowup in amplitude or variation.

In particular, we would like the outgoing wave strengths to be polynomial or, more generally, analytic functions of the incoming wave strengths. Note that this property depends on the parametrization of the wave curves. The following theorem gives a severe restriction on this class of systems.

Theorem 3.1. *If:*

1. *the system (1.19) is strictly hyperbolic;*

2. its pairwise interactions are analytic;

then the wave curves are integral curves of the eigenvector fields.

Proof. Consider a kk interaction (two wave curves of the same family). Then $W_k^{a_k} W_k^{\alpha_k} U = W^b U$, where $b = (b_1, \ldots, b_n)$ is a function of a_k, α_k. When $a_k \geq 0$ and $\alpha_k \geq 0$ on the left hand side of the equation we are tracing along the kth rarefaction curves, and so the solution is simply $b_k = a_k + \alpha_k$ and $b_j = 0$ for $j \neq k$. Since each b_i is an analytic function of a_k and α_k, this is its value everywhere (because this is the only analytic function taking the necessary values and derivatives at $a_k = \alpha_k = 0$, since an analytic function is determined by its Taylor series). Thus we have shown that $W_k^{a_k} W_k^{\alpha_k} U = W_k^{a_k + \alpha_k} U$.

Now we show that $W_k^{a_k}(U)$ is an integral curve of R_k via a calculation:

$$\partial_{a_k} W_k^{a_k} U = \partial_{\alpha_k}\big|_{\alpha_k = 0} W_k^{\alpha_k + a_k} U = \partial_{\alpha_k}\big|_{\alpha_k = 0} W_k^{\alpha_k} W_k^{a_k} U = (R_k)_{W_k^{a_k} U}.$$

The first equality is due to the chain rule. The second is due to the semigroup property of W_k just proved. The last is due to the fact that wave curves $W_k^{a_k}(U)$ are tangent to $R_k|_U$. □

3.2 Linear interactions

The simplest interactions are linear (that is, the outgoing wave strengths are linear functions of the incoming wave strengths). These are sometimes known as Temple class systems [1], but note that the definition of Temple class sometimes varies; in fact, in the AMS Mathematical Review of [1], Denis Serre imposes the additional requirement that the rarefaction curves are straight lines.

Even the definition we have given is not the one stated in [1], but the definitions are equivalent, which is the content of Theorem 3.2, below.

Note that whether a system has linear interactions depends on the parametrization of the wave curves (equivalently, it depends on the scaling of the eigenvectors $\{R_i\}$), as does the equation $[R_i, R_j] = 0$ in the theorem below.

Theorem 3.2. *For a strictly hyperbolic system (1.19) with eigenvectors $\{R_i\}$, the following are equivalent:*

1. *the wave curves are integral curves of the eigenvectors and $[R_i, R_j] = 0$ for each pair of eigenvectors;*

2. *the system has linear interactions.*

Proof. 1 $\implies$ 2. We will show that, for indices $i < j$, we have $W_j^{a_j} W_i^{a_i} U = W_i^{a_i} W_j^{a_j} U$. To this end, fix a_j and define the curve $Y(a_i) = W_j^{a_j} W_i^{a_i} U$. Since $Y(0) = W_j^{a_j} U$, if we can show that Y is an integral curve of R_i, this will be accomplished. To this end, we calculate:

$$Y'(a_i) = DW_j^{a_j} R_i(W_i^{a_i} U). \tag{3.1}$$

We wish to show that $DW_j^{a_j} R_i(W_i^{a_i} U) = R_i(W_j^{a_j} W_i^{a_i} U)$ or, equivalently, that

$$R_i(W_i^{a_i} U) = DW_j^{-a_j} R_i(W_j^{a_j} W_i^{a_i} U). \tag{3.2}$$

The two sides are equal at $a_j = 0$; differentiate with respect to a_j and obtain 0 on both sides (applying Lemma 1.6) to see that they are equal everywhere. Thus $Y'(a_i) = R_i(W_j^{a_j} W_i^{a_i} U)$ and Y is an integral curve of R_i, as desired.

2 $\implies$ 1. By Theorem 3.1, we know the wave curves are integral curves of the

eigenvector fields. We have

$$W_j^{a_j} W_i^{a_i} U = W_i^{a_i} W_j^{a_j} U.$$

On the left hand side, apply $\frac{\partial^2}{\partial_{a_i} \partial_{a_j}}$ at $a_i = a_j = 0$ to obtain

$$\nabla_{R_j} R_i,$$

and on the right hand side, apply $\frac{\partial^2}{\partial_{a_j} \partial_{a_i}}$ at $a_i = a_j = 0$ to obtain

$$\nabla_{R_i} R_j.$$

Thus,

$$0 = \nabla_{R_j} R_i - \nabla_{R_i} R_j = [R_j, R_i],$$

as desired. □

3.3 Quadratic interactions

In the previous section, we derived the condition $[R_i, R_j] = 0$ necessary for a system to have linear interactions. We now derive an analogous condition necessary for a system to have quadratic interactions; that is, for $\{b_k\}$ in equation (1.30) to be quadaric functions of a_i, a_j. Note that according to Theorem 1.30, in this case we must have $b_k = \delta_{ik} a_i + \delta_{jk} + c_{ji}^k a_i a_j$.

Theorem 3.3. *If the system* (1.19) *with eigenvector fields* $\{R_i\}$ *has quadratic interac-*

tions, then the equations

$$[R_i, [R_i, R_j]] = 2 \sum_{k>i} c_{ij}^k [R_i, R_k]$$

hold for each pair of indices i, j.

Proof. Pick $U \in \Omega$ and consider an ij interaction ($j < i$). We have the equation

$$W_n^{b^n} W_{n-1}^{b^{n-1}} \cdots W_1^{b^1}(U) = W_j^{a_j} W_i^{a_i}(U), \tag{3.3}$$

where the outgoing wave strengths b^k are quadratic functions of the incoming wave strengths a_j, a_i. For this proof, we use upper indices as in b^k so that lower indices may be used to provide a compact notation for partial derivatives: $b_i^k = \frac{\partial b^k}{\partial a_i}$.

To prove this result we are going to differentiate equation (3.3). To demonstrate how this is done in a setting with simpler notation, we first take $n = 3, i = 2, j = 1$. Thus equation (3.3) reduces to:

$$W_3^{b^3} W_2^{b^2} W_1^{b^1}(U) = W_1^{a_1} W_2^{a_2}(U). \tag{3.4}$$

Differentiate with respect to a_1 to obtain:

$$b_1^3 R_3\big|_{W_3^{b^3} W_2^{b^2} W_1^{b^1}(U)} + b_1^2 DW_3^{b^3} R_2\big|_{W_2^{b^2} W_1^{b^1}(U)} +$$
$$b_1^1 DW_3^{b^3} DW_2^{b^2} R_1\big|_{W_1^{b^1}(U)} = R_1\big|_{W_1^{a_1} W_2^{a_2}(U)}.$$

Here $DW_k^{b^k}$ is the differential of the map $W_k^{b^k}(\cdot)$. Now set $a_1 = 0$; since $b^l = a_1\delta_{1l} + a_2\delta_{2l} + c_{21}^l a_1 a_2$, we now have $b^1 = b^3 = 0$ and $b^2 = a_2$. Also, since $W_k^0(U) = U$, we also

have that $DW_k^0 = I$, the identity map. Thus, when $a_1 = 0$ our equation becomes

$$b_1^3 R_3\big|_{W_2^{a_2}(U)} + b_1^2 R_2\big|_{W_2^{a_2}(U)} + b_1^1 DW_2^{a_2} R_1\big|_U = R_1\big|_{W_2^{a_2}(U)}.$$

Now that we have demonstrated how to carry out this differentiation in a concrete setting, we return to the general setting of equation (3.3). We will be using the interaction estimate from Theorem 1.30; according to this estimate, if b^k is quadratic in a_i, a_j, we must have $b^k = \delta_{ik} a_i + \delta_{ij} a_j + a_i a_j c_{ji}^k$.

Differentiate with respect to a_j and then set $a_j = 0$ to obtain:

$$\sum_{k \geq i} b_j^k R_k\big|_{W_i^{b^i}(U)} + \sum_{k<i} b_j^k DW_i^{a^i} R_j\big|_U = R_j\big|_{W_i^{a_i}(U)}. \tag{3.5}$$

Apply the linear operator $DW_i^{-a_i}$ to both sides of this equation. Note that $DW_i^{-a_i} R_i\big|_{W_i^{a_i}(U)} = R_i\big|_U$. (To see this, differentiate the left hand side with respect to a_i using Lemma 1.6 and see that we obtain 0; thus, this function is constant.) Thus we obtain

$$\sum_{k>i} b_j^k DW_i^{-a_i} R_k\big|_{W_i^{a_i}(U)} + \sum_{k \leq i} b_j^k R_k\big|_U = DW_i^{-a_i} R_j\big|_{W_i^{a_i}(U)}. \tag{3.6}$$

Differentiate with respect to a_i:

$$\sum_{k>i} \left(b_{ji}^k DW_i^{-a_i} R_k\big|_{W_i^{a_i}(U)} + b_j^k DW_i^{-a_i}[R_i, R_k]\big|_{W_i^{a_i}(U)} \right) + \sum_{k \leq i} b_{ji}^k R_k\big|_U \tag{3.7}$$

$$= DW_i^{-a_i}[R_i, R_j]\big|_{W_i^{a_i}(U)}. \tag{3.8}$$

Here we have applied Lemma 1.6. Differentiate with respect to a_i again:

$$\sum_{k>i}\Big(b^k_{jii}DW_i^{-a_i}R_k\big|_{W_i^{a_i}(U)}+2b^k_{ji}DW_i^{-a_i}[R_i,R_k]\big|_{W_i^{a_i}(U)} + b^k_jDW_i^{-a_i}[R_i,[R_i,R_k]]\big|_{W_i^{a_i}(U)}\Big) + \sum_{k\le i}b^k_{jii}R_k\big|_U = DW_i^{-a_i}[R_i,[R_i,R_j]]\big|_{W_i^{a_i}(U)}. \tag{3.9}$$

Set $a_i=0$ to obtain

$$\sum_{k>i}2c^k_{ij}[R_i,R_k]=[R_i,[R_i,R_j]] \tag{3.10}$$

as desired. (Here we used the fact that, at $a_i=a_j=0$, we have $b^k_j=\delta^k_j$, $b^k_{ji}=c^k_{ij}$, and $b^k_{jii}=0$.)

It remains to demonstrate that the equation also holds when $j>i$. In this case, consider a ji interaction:

$$W_n^{b^n}W_{n-1}^{b^{n-1}}\cdots W_1^{b^1}(U)=W_i^{a_i}W_j^{a_j}(U).$$

Differentiate with respect to a_j and put $a_j=0$:

$$\sum_{k\ge i}b^k_jR_k\big|_{W_i^{b^i}(U)}+\sum_{k<i}b^k_jDW_i^{b^i}R_j\big|_U=DW_i^{a_i}R_j\big|_U.$$

Apply the linear operator $DW_i^{-a_i}$ to both sides to obtain:

$$\sum_{k>i}a^k_jDW_i^{-a_i}R_k\big|_{W_i^{a_i}(U)}+\sum_{k\le i}a^k_jR_k\big|_U=R_j\big|_U.$$

The only difference between this equation and equation (3.6) is that here the right hand

side is a constant. By differentiating twice as before, we obtain

$$\sum_{k>i}\Big(b^k_{jii}DW_i^{-a_i}R_k\big|_{W_i^{a_i}(U)}+2b^k_{ji}DW_i^{-a_i}[R_i,R_k]\big|_{W_i^{a_i}(U)}$$
$$+b^k_jDW_i^{-a_i}[R_i,[R_i,R_k]]\big|_{W_i^{a_i}(U)}\Big)+\sum_{k\le i}b^k_{jii}R_k\big|_U=0.$$

This is the same as equation (3.9) except that the right hand side is 0.

Now set $a_i=0$ to obtain

$$\sum_{k>i}2c^k_{ji}[R_i,R_k]+[R_i,[R_i,R_j]]=0$$

as desired. (The reason that the term $[r_i,[r_i,r_j]]$ now appears on the left but such a term did not appear in equation 3.10 is that, since $j>i$, one of the factors b^k_j in the first summation is in fact b^j_j and thus doesn't vanish.) □

Unlike Theorem 3.2 on linear interactions, it is not clear that Theorem 3.3 gives *sufficient* conditions for a system to have quadratic interactions, although all examples we have which satisfy the conditions are indeed quadratically interacting.

We list the conditions of Theorem 3.3 for the case $n=2$:

$$[R_1,[R_1,R_2]]=2c^2_{12}[R_1,R_2],$$
$$[R_2,[R_2,R_1]]=0;$$

and for the case $n = 3$:

$$
\begin{aligned}
[R_1, [R_1, R_2]] &= 2c_{12}^2[R_1, R_2] + 2c_{12}^3[R_1, R_3], \\
[R_2, [R_2, R_1]] &= 2c_{21}^3[R_2, R_3], \\
[R_1, [R_1, R_3]] &= 2c_{13}^2[R_1, R_2] + 2c_{13}^3[R_1, R_3], \\
[R_3, [R_3, R_1]] &= 0, \\
[R_2, [R_2, R_3]] &= 2c_{23}^3[R_2, R_3], \\
[R_3, [R_3, R_2]] &= 0.
\end{aligned}
$$

It is not immediately obvious that systems which are quadratically interacting exist. We now present such a system.

Example 3.4. Consider the 3×3 system with flux

$$
F(u, v, w) = \begin{pmatrix} u \\ 2v \\ -\frac{uv}{4} + \frac{3w}{2} \end{pmatrix}.
$$

The Jacobian matrix is

$$
DF = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ -\frac{v}{4} & -\frac{u}{4} & \frac{3}{2} \end{pmatrix}
$$

with eigenvectors, eigenvalues, and wave curves

$$R_1 = \begin{pmatrix} 1 \\ 0 \\ \frac{v}{2} \end{pmatrix}, \qquad \lambda^1 = 1, \quad W_1^{a_1} U = \begin{pmatrix} u + a_1 \\ v \\ w + \frac{v}{2} a_1 \end{pmatrix}$$

$$R_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \lambda^2 = \frac{3}{2}, \quad W_2^{a_2} U = \begin{pmatrix} u \\ v \\ w + a_2 \end{pmatrix}$$

$$R_3 = \begin{pmatrix} 0 \\ 1 \\ -\frac{u}{2} \end{pmatrix}, \quad \lambda^3 = 2, \quad W_3^{a_3} U = \begin{pmatrix} u \\ v + a_3 \\ w - \frac{u}{2} a_3 \end{pmatrix}$$

(since the eigenvalues are constant, the wave curves are just the integral curves of the eigenvectors).

Since $[R_1, R_2] = [R_2, R_3] = 0$, the 2-1 and 3-2 interactions are linear, by Theorem 3.2. We now show that a 3-1 interaction is quadratic. First we compute the result of traveling along a 3-wave curve and then a 1-wave curve:

$$W_1^{a_1} W_3^{a_3} U = \begin{pmatrix} u + a_3 \\ v + a_1 \\ w + \frac{v}{2} a_3 - (u + a_3) \frac{a_1}{2} \end{pmatrix}.$$

We see that this same point is obtained as

$$W_3^{a_3} W_2^{-a_1 a_3} W_1^{a_1} U.$$

Thus $b_1 = a_1$, $b_2 = -a_1 a_3$, $b_3 = a_3$, and thus this interaction is quadratic.

We now present a family of examples of quadratically interacting systems.

Example 3.5. Consider the 3×3 systems with fluxes

$$F(u,v,w) = \begin{pmatrix} c_2 u \\ c_1 v \\ \frac{(c_1-c_2)k_1 h(v)+(c_1-c_2)k_2 g(u)+(c_1-c_2)k_1 k_2 uv+c_2 k_1 w-c_1 k_2 w}{k_1-k_2} \end{pmatrix}$$

for constants $c_1 \neq c_2$ and $k_1 \neq k_2$ and arbitrary smooth functions h, g.

We now compute the Jacobian matrix:

$$DF = \begin{pmatrix} c_2 & 0 & 0 \\ 0 & c_1 & 0 \\ \frac{k_1 k_2 (c_1-c_2)v+k_2(c_1-c_2)g'(u)}{k_1-k_2} & \frac{k_1 k_2 (c_1-c_2)u+k_1(c_1-c_2)h'(u)}{k_1-k_2} & \frac{-k_2 c_1+k_1 c_2}{k_1-k_2} \end{pmatrix}.$$

with eigenvectors, eigenvalues, and parametrized wave curves as follows:

$$R_1 = \begin{pmatrix} 1 \\ 0 \\ g'(u)+k_1 v \end{pmatrix}, \qquad \lambda^1 = c_2, \quad W_1^{a_1} U = \begin{pmatrix} u + a_1 \\ v \\ w + k_1 v a_1 + g(u+a_1) - g(u) \end{pmatrix}$$

$$R_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \lambda^2 = \frac{c_2 k_1 - c_1 k_2}{k_1 - k_2}, \quad W_2^{a_2} U = \begin{pmatrix} u \\ v \\ w + a_2 \end{pmatrix}$$

$$R_3 = \begin{pmatrix} 0 \\ 1 \\ h'(v)+k_2 u \end{pmatrix}, \qquad \lambda^3 = c_1, \quad W_3^{a_3} U = \begin{pmatrix} u \\ v + a_3 \\ w + k_2 u a_3 + h(v+a_3) - h(v) \end{pmatrix};$$

since the eigenvalues are constant, the wave curves are simply the integral curves of the eigenvector fields.

(Of course, these are labelled assuming that k_1, k_2, c_1, and c_2 were chosen such that $\lambda_1 < \lambda_2 < \lambda_3$.)

We now show that this system has quadratic interactions. Since $[R_1, R_2] = [R_2, R_3] = 0$ we have trivial 21- and 32-interactions. Finally, for a 31-interaction we have:

$$W_1^{a_1} W_3^{a_3} U = \begin{pmatrix} u + a_1 \\ v + a_3 \\ w + k_2 u a_3 + k_1 (v + a_3) a_1 + h(v + a_3) - h(v) + g(u + a_1) - g(u) \end{pmatrix}$$

and

$$W_3^{b_3} W_2^{b_2} W_1^{b_1} U = \begin{pmatrix} u + a_1 \\ v + a_3 \\ w + k_1 v a_1 + k_2 (u + a_1) a_3 + g(u + a_1) - g(u) + h(v + a_3) - h(v) + a_2 \end{pmatrix}.$$

These are equal when $b_1 = a_1, b_3 = a_3$, and $a_2 = (k_1 - k_2) a_1 a_2$. Thus these systems have quadratic interactions. These systems have only contact fields (since the eigenvalues are constant), but they are not straight line systems unless both g and h are linear functions.

Chapter 4

Geometry of shock and rarefaction curves

4.1 Relation of shock curves to rarefaction curves

It is of interest to determine the relationship between the Hugoniot loci of different systems with the same frame of eigenvectors. The following example shows that two systems with the same eigenframe may well have distinct Hugoniot loci.

Example 4.1. Consider the following flux for a 2×2 system:

$$F(u,v) = \begin{pmatrix} \frac{u^3}{3} + \frac{u(v)^2}{3} \\ \frac{v^3}{3} + \frac{v(u)^2}{3} \end{pmatrix}.$$

Its Jacobian is

$$DF = \begin{pmatrix} u^2 + \frac{v^2}{3} & \frac{2}{3}uv \\ \frac{2}{3}uv & \frac{u^2}{3} + v^2 \end{pmatrix}.$$

The flux for the second system is given by

$$G(u,v) = \begin{pmatrix} \arctan\left(\frac{u}{v}\right) \\ \ln(v) - \frac{1}{2}\ln(u^2+v^2) \end{pmatrix}.$$

Its Jacobian is

$$DG = \frac{1}{u^2+v^2}\begin{pmatrix} v & -u \\ -u & \frac{u^2}{v} \end{pmatrix}.$$

Both of these have as eigenvectors

$$R_1 = \begin{pmatrix} u \\ v \end{pmatrix}, \text{ and } R_2 = \begin{pmatrix} -v \\ u \end{pmatrix};$$

DF having eigenvalues u^2+v^2 and $\frac{1}{3}(u^2+v^2)^2$ and DG having eigenvalues 0 and $\frac{1}{u}$.

Thus for both systems $\mathcal{R}_1$ consists of radial lines meeting at the origin, and $\mathcal{R}_2$ consists of circles centered at the origin. Moreover, (R_1, λ_1) is a line field for both systems, and thus $\mathcal{S}_1$ consists of radial lines meeting at the origin for both systems. However, (R_2, λ_2) is a contact field for the F system, but neither a contact nor a line field for the G system; thus $\mathcal{S}_2$ consists of circles centered at the origin for F, but not for G.

Despite the above example, we now show that in one important situation, systems with the same eigenvectors must have the same Hugoniot locus.

In [13], Jenssen and Kogan classified frames $\{R_i\}$ in $\mathbb{R}^3$. In particular, what they call case IIa contains all non-rich frames which admit a strictly hyperbolic flux.

Theorem 4.2. *If equation (1.19) is a strictly hyperbolic non-rich 3×3 system, then any strictly hyperbolic system with the same eigenvectors (equivalently, the same rarefaction curves) has the same Hugoniot locus at each point.*

Proof. The system is not rich; thus there are two indices i and j for which $[R_i, R_j]$ is not in the span of $\{R_i, R_j\}$. Thus c_{ij}^k (where k is distinct from both i and j) is nonzero, and one row of the algebraic part of the λ system is nonzero (see Theorem 2.1). Actually, the coefficient of each λ_m must be nonzero, since there is a strictly hyperbolic solution. We now apply Theorem 2.2 and see that every strictly hyperbolic flux having these eigenvectors is of the form

$$H(U) = aF(U) + bU + C,$$

where $a, b \in \mathbb{R}$, $a \neq 0$, and $C \in \mathbb{R}^3$.

We now show that the strictly hyperbolic system with flux H has the same Hugoniot locus as the system with flux F. Let $V, W \in \Omega$ satisfy the Rankine-Hugoniot equation for H:

$$s(V - W) = H(V) - H(W) = a\,(F(V) - F(W)) + b(V - W)$$

and thus

$$\frac{(s-b)}{a}(V - W) = F(V) - F(W).$$

We see that points satisfying the Rankine Hugoniot equation for F also satisfy it for H.

Since $F(U) = \frac{1}{a}H(U) - \frac{b}{a}U$, we can use the same approach as the previous paragraph to show that points satisfying the Rankine-Hugoniot equation for H also satisfy it for F. The two systems have the same Hugoniot locus, and the proof is complete. □

The case dealt with in the above theorem, of non-rich 3×3 strictly hyperbolic systems, is the case of most interest for physical systems. For instance, the one dimensional compressible Euler equations fall into this class.

4.2 Coinciding shock and rarefaction curves

Blake Temple introduced the concept of invariant submanifolds for conservation laws [25]. These are submanifolds of U on which the conservation law reduces to a system of fewer equations. In particular, Temple showed that the one-dimensional invariant submanifolds are the rarefaction curves which are subsets of the Hugoniot locus of each of their points; that is, the shock and rarefaction curves coincide. Temple's motivation was the observation that a system arising in the study of oil recovery problems had this feature.

Rich systems with coinciding shock and rarefaction curves became known as Temple class systems, and extensive well-posedness results have been proven for this class [22]. This is one of the few classes of systems where results are available for large initial data.

Temple further showed that a rarefaction curve coincides with the corresponding shock curve if and only if the characteristic field is a line field or a contact field. This is the content of our Theorem 1.25 and the statement immediately following its proof. Temple moreover gave explicit formulas for some classes of 2×2 systems whose characteristic fields are contact fields or line fields. Here, we generalize his analysis to $n \times n$ systems.

Theorem 4.3. *If:*

1. *R and F are smooth vector fields on $\Omega \subseteq \mathbb{R}^n$;*

2. *$\lambda : \Omega \to \mathbb{R}^n$ satisfies $R(\lambda) = 0$;*

then the following are equivalent:

1. *(R, λ) is a contact field for the flux F;*

2. *there is some smooth vector field G with $\nabla_R G = 0$ and $F(U) = \lambda(U)U + G(U)$.*

Proof. 1 $\implies$ 2: Define $G(U) := F(U) - \lambda(U)U$. Then

$$\nabla_R G = \nabla_R F - \nabla_R \lambda\, U - \lambda(U)R = 0 = \lambda R - \lambda R = 0,$$

as desired.

2 $\implies$ 1: Apply $\nabla_R F = \lambda R$. □

Theorem 4.3 above is essentially Theorem 3 from [25] , but without the hypothesis $d\lambda \neq 0$, and applying to general $n \times n$ systems rather than only 2×2. If we also assume $d\lambda \neq 0$, then in the 2×2 case, the components of G are functionally dependent on λ, and we can thus locally write G as a function of λ as does Temple.

Theorem 4.4. *If $F : \Omega \to \mathbb{R}^n$ is a flux for some system and*

(H1) (R, λ) is a contact field;

(H2) (S, μ) is a line field;

(H3) $S(\lambda)$ is never 0;

(H4) $[R, S] \in span\{R, S\}$;

then

(C1) there exists smooth vector fields G and H such that $F(U) = \lambda(U)\,(U + G(U)) + H(U)$ and $\nabla_R G = \nabla_R H = \nabla_S G = \nabla_S H = 0$;

(C2) except possibly on a submanifold of codimension 1, $\lambda \neq \mu$ and the vector field S is collinear with $U + G(U)$

Proof. First we show that $\nabla_R S$ is never a multiple of S. Suppose for contradiction that

$$\nabla_R S_U = aS_U. \tag{4.1}$$

By (H4) there exist functions b and c such that $[S, R] = bR + cS$. By symmetry (Theorem 1.9) we also have

$$\nabla_S R = \nabla_R S + [S, R] = bR + (a + c)S \tag{4.2}$$

Now, by flatness (Theorem 1.10), we can write

$$\nabla_S \nabla_R F - \nabla_R \nabla_S F = \nabla_{[S,R]} F$$

or

$$S((\lambda R)) - R((\mu S)) = b\lambda R + c\mu S.$$

Expanding the left hand side by the product rule, we obtain

$$S(\lambda)\, R + \lambda \nabla_S R - R(\mu)\, S - \mu \nabla_R S = b\lambda R + c\mu S.$$

Now use the equations (4.1) and (4.2) to find

$$S(\lambda)\, R + \lambda bR + \lambda(a + c)S - R(\mu)\, S - \mu aS = b\lambda R + c\mu S.$$

After the obvious cancellation,

$$S(\lambda)\, R + \lambda(a + c)S - R(\mu)\, S - \mu aS = c\mu S.$$

Since by hypothesis $S(\lambda) \neq 0$, we have written R as a multiple of S, which is false, since we have $R(\lambda) = 0$.

Now we show that F can be written as in (C1). By Theorem 4.3, we can write $F(U) = \lambda(U)U + \overline{G}(U)$, where $\nabla_R \overline{G} = 0$. Further, we can locally write $\overline{G}(U) = J(\lambda(U), c_1(U), \ldots, c_{n-2}(U))$, where $R(c)_i = S(c)_i = 0$. This follows from the Frobenius Theorem, since $[R, S] \in \text{span}\{R, S\}$.

Let J' represent the derivative of J with respect to λ. In a series of steps we will show that $J'' = 0$.

1. $U + J'$ is a multiple of S.

   To see this, apply ∇_S to F and obtain

   $$\mu S = S(\lambda)\, U + \lambda S + S(\lambda)\, J'.$$

   By hypothesis $S(\lambda) \neq 0$, so divide through by that expression.

2. J'' is a multiple of S.

   We already have $U + J' = fS$ for some f. Apply ∇_S again to obtain

   $$S + S(\lambda)\, J'' = gS$$

   for some g, then divide through by $S(\lambda)$ again.

3. $J'' = 0$.

   We have $J'' = hS$ for some h. Apply ∇_R and obtain

   $$0 = R(h)\, S + h \nabla_R S.$$

By hypothesis $\nabla_R S$ is not a multiple of S, so we must have $h = 0$ and thus $J'' = 0$.

Since $J'' = 0$, we must have $J(\lambda, c) = \lambda K(c) + L(c)$. Write $G(U) = K(c(U))$ and $H(U) = L(c(U))$ to find F in the desired form.

It remains to demonstrate conclusion 2. Let $A(U) = U + G(U)$; since $\nabla_R A = R \neq 0$, DA has rank at least 1 everywhere. Thus the set of points where $U+G(U) = 0$ is confined to at most a codimension 1 submanifold of Ω. Since $\mu S = S(\lambda)\,(U + G(U)) + \lambda S$, some rearranging leads to $(\mu - \lambda)S = S(\lambda)\,(U + G(U))$. Thus we see that $\mu = \lambda$ on exactly the set where $U + G(U) = 0$; moreover, on the complement of that set, S is a multiple of $U + G(U)$, as was to be shown. □

Example 4.5. As an application of this theorem, suppose we wish to find fluxes having $S = (u, v, w)^T$ as a line field and $R = (u, v, 0)^T$ as a contact field. Such a flux will be of the form

$$F(U) = \lambda(U + G) + H.$$

The general solution of the system

$$S(\alpha) = 0,$$
$$R(\alpha) = 0;$$

is given by $\alpha = f(\frac{v}{u})$. Thus the components of G and H must be functions of $\frac{v}{u}$. But since also G must be a scalar multiple of $U = (u, v, w)^T$, we see than in fact $G = 0$. Finally, we must have $R(\lambda = 0)$, and thus λ is a function of $\frac{v}{u}$ and w. Thus all of our desired fluxes are of the form

$$F(U) = g\left(\frac{v}{u}, w\right) U + \left(f_1\left(\frac{v}{u}\right), f_2\left(\frac{v}{u}\right), f_3\left(\frac{v}{u}\right)\right)^T$$

for choices of functions g, f_1, f_2, and f_3.

In the 2×2 case, we can slightly weaken the hypothesis that $S(\lambda)$ is nonzero $d \cdot s$ to simply state that $d\lambda$ is nonzero. This implies $S(\lambda) \neq 0$, since otherwise the kernel of $d\lambda$ contains both R and S and hence $d\lambda$ is 0. Note also that in the 2×2 case, the vector fields G and H become constants, and thus S is a multiple of a radial vector field $U + G$.

The 2×2 case of our Theorem 4.4 is Corollary 1 from [25] (there the hypothesis that $d\lambda \neq 0$ is not stated, but seems to be implicitly carried over from Theorem 4).

Finally, we state a theorem for the case of two contact fields.

Theorem 4.6. *If $F : \Omega \to \mathbb{R}^n$ is a flux for some system and*

(H1) (R, λ) is a contact field;

(H2) (S, μ) is a contact field;

(H3) at each $U \in \Omega$, $\lambda(U) \neq \mu(U)$.

Then

$$F(U) = \frac{\lambda(U)G(U) - \mu(U)H(U)}{\lambda(U) - \mu(U)}$$

for some smooth vector fields G, H such that

(C1) $U = \frac{G(U)-H(U)}{\lambda(U)-\mu(U)}$;

(C2) $\nabla_R H = \nabla_S G = 0$.

Proof. By Theorem 4.3, we can write

$$F(U) = \lambda(U)U + H(U) = \mu(U)U + G(U)$$

where $\nabla_R H = \nabla_S G = 0$. Solve for U in this equation to obtain conclusion 1. Finally, substitute in the expression for U just found so that

$$F(U) = \lambda(U)\frac{G(U) - H(U)}{\lambda(U) - \mu(U)} + H(U);$$

upon simplifying this, we obtain the expression for F in the conclusion. □

The 2×2 case of Theorem 4.6 is Corollary 1 of Temple's later paper [26].

Theorem 4.6 can be applied in an interesting way to generate examples of systems with two contact fields. On $\mathbb{R}^3$ with coordinates $A = (a, b, c)$, choose functions $\lambda \neq \mu$, and then choose vector fields G and H such that the kernel of ∇G has nonzero intersection with the kernel of $d\mu$ and the kernel of ∇H has nonzero intersection with the kernel of $d\lambda$. Then solve the equation

$$U = \frac{G(A) - H(A)}{\lambda(A) - \mu(A)} \tag{4.3}$$

for $A = \phi(U)$. We then have our desired flux

$$F(U) = \frac{\lambda(\phi(U))G(\phi(U)) - \mu(\phi(U))H(\phi(U))}{\lambda(\phi(U)) - \mu(\phi(U))}.$$

The following example illustrates this process.

Example 4.7. Set $G(A) = (c^2, c, c + b)^T$, $H(A) = (a, b, a)^T$, $\lambda(A) = a$, and $\mu(A) = b$. Then, as suggested in the previous paragraph, solve for A in equation (4.3) to find

$$A = \left(\frac{(-2 + v - w)(-2 - u + v - w)}{(1 + w)^2}, \frac{(-1 + v - w)(-2 - u + v - 2)}{(1 + w)^2}, \frac{2 + u - v + w}{1 + w}\right)^T.$$

Then we have

$$F(U) = \begin{pmatrix} \frac{(1+u)(2-v+w)(2+u-v+w)}{(1+w)^2} \\ \frac{(2+u-v+w)(1+v-v^2+w+vw)}{(1+w)^2} \\ \frac{(-2+v-w)(-2-u+v-w)}{1+w} \end{pmatrix}$$

and, as a calculation reveals, DF does have eigenvalues a and b as expected, which correspond to contact fields.

APPENDIX

Appendix A

Maple code

This appendix contains a printout of code from the computer algebra system Maple to compute fluxes with prescribed eigenvector fields or eigenvalues, as discussed in Chapter 2. As of the time of writing, this code (in greatly expanded form, with more examples) is available on the author's web site at `http://www4.ncsu.edu/~mrbenfie/`.

The primary function, `FLambdaR`, takes as inputs the prescribed eigenvector fields and/or eigenvalues, and produces (using the Maple procedure `pdsolve`) the desired flux, given as components f_i, as well as the eigenvalues. The included examples should be sufficient to demonstrate its use.

Auxiliary procedures

```
> restart:with(LinearAlgebra):with(ArrayTools):
```

grad computes the gradient of a vector field

```
> grad:=proc(f,vars::list)
  local n;
  n:=nops(vars);
  RETURN(Vector([seq(diff(f,vars[i]),i=1..n)]));
  end proc:
```

directionalDiff computes the directional derivative of a function f

```
> directionalDiff:=proc(f,dir,vars)
  evalm(DotProduct(grad(f,vars),dir,conjugate = false));
  end proc:
```

nabla computes the covariant derivative of a vector field

```
> nabla:=proc(r1,r2,vars)
  Vector(1..NumElems(r2), (i) -> directionalDiff(r2[i],r1,vars));
  end proc:
```

F(LambdaAndR)

```
> equalComponents:=proc(vec1,vec2)
  local n;
  n:=NumElems(vec1);
  seq(vec1[i]=vec2[i],i=1..n);
  end proc:
> FLambdaR:=proc(Vars::list, Lambda::list, R::list, flux::list)
  local L,R0,F,n,m;
  n:=nops(Vars);
  m:=max(nops(Lambda), nops(R));
  if Lambda=[] then
    L:=[seq(lambda[i](op(Vars)),i=1..m)]
  else
    L:=Lambda
  end if;
  if R=[] then
    R0:=[seq(Vector([seq(r[i,j](op(Vars)),j=1..n)]),i=1..m)]
  else
    R0:=R
  end if;
  if flux=[] then
    F:=[seq(f[i](op(Vars)),i=1..n)]
  else
    F:=flux
  end if;
  [seq(equalComponents(nabla(R0[i],Vector(F),Vars),L[i]*R0[i]),i=1.
  .m)];
  end proc:
```

Two prescribed eigenvector fields in R^3

```
> r:=[Vector([1,0,0]),
      Vector([w^2,1,0])];
```

$$r := \left[\left[\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}\right], \left[\begin{matrix} w^2 \\ 1 \\ 0 \end{matrix}\right]\right] \tag{3.1}$$

```
> eqns:=FLambdaR([u,v,w],[],r,[]);
```

$$eqns := \left[\frac{\partial}{\partial u} f_1(u,v,w) = \lambda_1(u,v,w),\ \frac{\partial}{\partial u} f_2(u,v,w) = 0,\ \frac{\partial}{\partial u} f_3(u,v,w) = 0,\right. \tag{3.2}$$
$$w^2 \left(\frac{\partial}{\partial u} f_1(u,v,w)\right) + \frac{\partial}{\partial v} f_1(u,v,w) = \lambda_2(u,v,w)\, w^2,\ \left(\frac{\partial}{\partial u} f_2(u,v,w)\right) w^2$$
$$\left. + \frac{\partial}{\partial v} f_2(u,v,w) = \lambda_2(u,v,w),\ \left(\frac{\partial}{\partial u} f_3(u,v,w)\right) w^2 + \frac{\partial}{\partial v} f_3(u,v,w) = 0\right]$$

```
> pdsolve(eqns);
```

$$\left\{f_1(u,v,w) = w^2\ \_F2\left(\frac{u}{w^2} + \frac{v\,w^2 - u}{w^2}, w\right) + \_F3\left(w, \frac{v\,w^2 - u}{w^2}\right), f_2(u,v,w) = \_F2(v, \right. \tag{3.3}$$
$$w), f_3(u,v,w) = \_F1(w), \lambda_1(u,v,w) = -\frac{\mathrm{D}_2(\_F3)\left(w, \frac{v\,w^2 - u}{w^2}\right)}{w^2}, \lambda_2(u,v,w)$$
$$\left. = \frac{\partial}{\partial v}\ \_F2(v,w)\right\}$$

Three prescribed eigenvector fields in R^3

```
> r:=[Vector([1,v,0]),
      Vector([w,1,0]),
      Vector([0,0,1])];
```

$$r := \left[\left[\begin{matrix} 1 \\ v \\ 0 \end{matrix}\right], \left[\begin{matrix} w \\ 1 \\ 0 \end{matrix}\right], \left[\begin{matrix} 0 \\ 0 \\ 1 \end{matrix}\right]\right] \tag{4.1}$$

```
> eqns:=FLambdaR([u,v,w],[],r,[]);
```

$$eqns := \left[\frac{\partial}{\partial u} f_1(u,v,w) + \left(\frac{\partial}{\partial v} f_1(u,v,w)\right) v = \lambda_1(u,v,w),\ \frac{\partial}{\partial u} f_2(u,v,w)\right. \tag{4.2}$$
$$+ \left(\frac{\partial}{\partial v} f_2(u,v,w)\right) v = \lambda_1(u,v,w)\, v,\ \frac{\partial}{\partial u} f_3(u,v,w) + \left(\frac{\partial}{\partial v} f_3(u,v,w)\right) v = 0,$$
$$\left(\frac{\partial}{\partial u} f_1(u,v,w)\right) w + \frac{\partial}{\partial v} f_1(u,v,w) = \lambda_2(u,v,w)\, w,\ \left(\frac{\partial}{\partial u} f_2(u,v,w)\right) w$$
$$\left. + \frac{\partial}{\partial v} f_2(u,v,w) = \lambda_2(u,v,w),\ \left(\frac{\partial}{\partial u} f_3(u,v,w)\right) w + \frac{\partial}{\partial v} f_3(u,v,w) = 0,\right.$$

$$\frac{\partial}{\partial w} f_1(u,v,w) = 0,\ \frac{\partial}{\partial w} f_2(u,v,w) = 0,\ \frac{\partial}{\partial w} f_3(u,v,w) = \lambda_3(u,v,w)\Big]$$

```
> pdsolve(eqns);
```

$$\left\{f_1(u,v,w) = \_C1\,u + \_C2, f_2(u,v,w) = \_C1\,v + \_C3, f_3(u,v,w) = \_F1(w), \lambda_1(u,v,w)\right. \tag{4.3}$$

$$\left. = \_C1, \lambda_2(u,v,w) = \_C1, \lambda_3(u,v,w) = \frac{\mathrm{d}}{\mathrm{d}w}\,\_F1(w)\right\}$$

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