ABSTRACT

HUSSAIN, AZMAT. Some Optimization Problems for Stochastic Systems with Memory. (Under the direction of Dr. Tao Pang.)

We consider portfolio optimization models of the Merton's type over finite and infinite time horizons. Unlike the classical Markov model, we study systems with delays. We consider both finite and infinite delay/memory models. The problem is formulated as a stochastic control problem and the state evolves according to a process governed by a stochastic process with delay. The goal is to choose investment and consumption controls such that the total expected discounted utility is maximized. Under certain conditions, in each model, we derive the optimal controls and explicit solutions for the associated Hamilton-Jacobi-Bellman (HJB) equations in a finite dimensional space for logarithmic, exponential and HARA utility functions. For each model, verification theorems are established to ensure that the solution obtained from HJB equation is equal to the value function.

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Some Optimization Problems for Stochastic Systems with Memory

by
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DEDICATION

To my mother, Fida Bibi, who raised eight incorrigible children and continues to support and to the loving memory of my father, Habib-ur-Rehman.

BIOGRAPHY

Azmat Hussain was born in a small town in Gilgit-Baltistan province of Pakistan. He received his BS (2008) and MS (2010) in Mathematics from COMSATS Institute of Information Technology, Islamabad. In the year 2010, he won Fulbright Scholarship for his PhD in Operations Research at North Carolina State University. Prior to starting his PhD in the year 2011, he taught in the department of Mathematics at Karakoram International University Gilgit, Gilgit-Baltistan (March 2010-December 2010). After completing his PhD, Azmat Hussain wants to go back to his home country and start a career in academia.

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Chapter 1

Introduction

In this dissertation, we study the optimal investment and consumption models when an investor makes decisions considering the past performance of the portfolio. In portfolio construction, utility function criterion is used to model the behavior of an investor. Depending on the the type of investments, an investment can be made for a finite or infinite amount of time. Investor is open to all market opportunities subject to her risk premium. The goal of the investor is to choose the investment and/or consumption such that the total discounted expected utility is maximized.

1.1 Background and Literature Review

Investment-consumption portfolio problem was first studied by Robert C. Merton in 1969, 1971 ([38, 39]). In the classical portfolio optimization model of Merton's type, the problem can be formulated as a optimal stochastic control problem, in which an investor chooses the optimal investment and/or consumption control to maximize the total discounted expected utility.

Some of the other pioneering work in modern finance theory was done by Markowitz in 1952, 1959 ([36, 37]). In Markowitz's model of mean-variance portfolio optimization construction, an investor addresses the uncertainty of an investment by constructing a portfolio such that for a given level of risk, the expected return is maximized or equivalently for a predetermined level of expected return, the variance is minimized. In Levy-Markowitz [33] and Kroll-Levy-Markowitz [30], authors have made a comparison of portfolios which maximize expected utility with mean-variance efficient portfolios.

In the Merton's type utility optimization model, a Markovian stochastic process such as geometric Brownian process is used to describe the price of the risky asset. In such a model past information is irrelevant and decisions are made only on the basis of current information. A substantial amount of work has been done on models with such settings. The case of a Merton-type model with consumption and the interest rate, which vary in a random, Markovian way, was considered by Fleming-Pang [15, 16] and Pang [44, 45]. Bielecki-Pliska [2], Fleming-Sheu [18] have considered the case where the mean returns of individual asset categories are explicitly affected by underlying economic factors such as dividends and interest rates and the goal is to maximize the long term growth rate of the utility based on the wealth with no consumption. In some other extensions of the model, stochastic volatility is taken into consideration (e.g. Fleming-Hernandez-Hernandez [14], Fouque-Papanicolaou-Sircar [20], Hata-Sheu [24, 25]). Some related literatures include [15], [16], [18], [45], [48], [49], [50] and the references therein.

Some researchers have considered stochastic systems with memory given by the following system:

$$\begin{aligned} dX(t) &= b(t, X_t, u(t))dt + \sigma(t, X_t, u(t))dW(t), \ t \in [0, T], \\ X(t) &= \varphi(t), \qquad t \in [-h, 0], \end{aligned}$$

where $h > 0$ is a fixed constant, $X_t : [-h, 0] \mapsto \mathbb{R}$ is the memory variable defined by $X_t(\theta) \equiv X(t + \theta)$, and it is the segment of the path from $t - h$ to t, $\varphi \in C[-h, 0]$ is the initial path and u is the control in some admissible control space Π. We want to point out that if T is finite, then above system is a model for an optimal stochastic control problem on a finite time horizon. If T is infinite, then the model is on an infinite time horizon. For this type of problems, the value function will depend on the initial variable φ, which is in an infinite dimensional space $C[-h, 0]$. For this type of control problems, the idea for the derivation of associated Hamilton-Jacobi-Bellman (HJB) equation was presented in Mohammed [40, 41] and the HJB equation involves the Frechét derivatives with respect to the initial variable φ. Some related works can be found in Chang-Pang-Pemy [3, 4].

The presence of the Fréchet derivative makes the stochastic control problems with memory given by X_t very complicated and working in an infinite dimensional space limits its application in practice. On the other hand, as we mentioned earlier, when investors look at the historical performance, instead of the whole path, moving average or exponential moving average is usually used. For this reason, many researcher considered stochastic system, $\forall t \in [0,T]$,

$$\begin{aligned} dX(t) &= b(t,X(t),Y(t),Z(t),u(t))dt + \sigma(t,X(t),Y(t),Z(t),u(t))dW(t),\ t \in [0,T], \\ X(t) &= \varphi(t), \qquad t \in [-h,0], \end{aligned}$$

with memory variables $Y(t), Z(t)$ given by

$$\begin{aligned} Y(t) &= \int_{-h}^{0} e^{\lambda\theta} X(t+\theta)d\theta, \\ Z(t) &= X(t-h), \end{aligned}$$

where $\lambda \geq 0$ is a parameter. As we can see, $Y(t)$ is the exponential moving average of the total wealth and $Z(t)$ is a historical value of $X(t)$ or the historical wealth. $X(t)$ over $[t-h,t]$ if $\lambda > 0$. A special case would be $\lambda = 0$, and now $Y(t)$ is just the moving average. Typically, higher historical wealth (in terms of $Y(t), Z(t)$) usually implies the investor has invested heavily on stocks rather than the risk-free asset, therefore, due to the good performance, the investor will tend to allocate more on stocks, and that will increase the stock demand. We want to point out that the state equation of $X(t)$ may not depend on $Z(t)$, as investors may not care about the historical value at a particular time point.

For the system given above, the value function V will depend on the initial variable φ through memory variables (x,y,z) that are given by

$$x \equiv X(0) = \varphi(0), \quad y \equiv Y(0) = \int_{-h}^{0} e^{\lambda\theta}\varphi(\theta)d\theta, \quad z \equiv Z(0) = \varphi(-h),$$

and we can derive the HJB equation for V in the classical sense in a finite dimensional space.

This type of models can be used to describe a stochastic system with memory or delayed information, and they can be used to model problems in finance and investment (see Elsanousi-Larssen [10] for an optimal consumptions problem, Federico [12] for a pension fund model, Chang-Pang-Yang [5] and Pang-Hussain [47, 46] for stochastic investment and consumption optimization models). In Federico [12], an optimal control problem arising in the management of a pension fund with dynamics governed by a stochastic differential equation with delay is considered. They also have applications in business management (see, e.g. Elsanousi-Øksendal-Sulem [11] for a stochastic optimal harvest problem, Gozzi-Marinelli-Savin [23] for optimal advertising models). In Federico-Goldys-Gozzi [13], the authors consider a model in economics where a class of optimal control problems with state constraints, where the state equation is a differential equation with delays. Koivo [27] studied optimal control of linear stochastic systems that have feedback loop. In Bauer-Rieder [1], the authors consider stochastic control problems with delay variable given by $Y(t) = \int_{-h}^{0} e^{\lambda\theta} f(X(t+\theta))d\theta, Z(t) = f(X(t-h))$ with three applications: linear quadratic problems with delay; the optimal consumption in a financial market with delay and a deterministic fluid problem with delay which arises from admission control in ATM communication networks.

Problems with delay also arise in modeling optimal advertising under uncertainty (see Gozzi-Marinelli [22] and Gozzi-Marinelli-Savin [23]). Some early works on optimal control for systems with delays can be found in Kolmanovskii-Maizenberg [28], Kolmanovskii-Shaikhet [29], Lindquist [34, 35] and the references therein. Further, some other applications of systems involving delays given by $Y(t)$ and/or $Z(t)$ can be found in Koivo [27], Kolmanovskii-Maizenberg [28], Kolmanovskii-Shaikhet [29], Lindquist [34, 35] and the references therein. On the other hand, some researchers have studied the maximum principles for stochastic control problems with delays (see Chen-Wu [6], Øksendal-Sulem [42], Øksendal-Sulem-Zhang [43]), but the involved backward stochastic differential equations are hard to solve. Here we still use the dynamic programming method.

As we have mentioned, $Z(t)$ may not appear in the state equation of $X(t)$. However, even for this case, z variable will appear in the associate HJB equation. On the other hand, one key requirement for the HJB equation to have a solution is that the solution V must be independent of z (see Larssen-Risebro [32] and Pang-Hussain [47] for more details). It is this condition that limits the application of the above model .

1.2 Contribution and Outline

In real world, the stock price process is not truly Markov and the historic performance of the risky asset influences the decisions of investors. As we know, investors tend to look at the moving average or exponential moving average for a stock before they make the investment decision. A stock with current price lower than its exponential moving average may signal the downward trend for the stock price, therefore it will scare away some investors. Due to the weaker demand, the price may go down further. On the other hand, if the current price is higher than its exponential moving average, it may signal an upward trend, so it will attract more investment. Due to the higher demand, the price may go up further. Therefore, it is motivated to consider a stochastic portfolio optimization model with historic information, or memory.

Stochastic control models with delay have wide range of applications and have been discussed extensively in literature. Kolmanovaskiĭ and Maizenberg [28] introduced the idea of describing the delay information like (2.3)-(2.4). Elsanousi and Larssen [10] and Larssen and Riserbro [32] have discussed model with finite delay. Elsanousi, Øksendal and Sulem [11] developed and maximum principle for optimal control problem of stochastic systems with delay. The idea of dynamic programming principle for stochastic delay differential equations appears in Gihman and Skorokhod [21] and Kolmanosvskiĭ and Shaĭkhet [29]. Larssen [31] showed dynamic programming principle for the stochastic control problems with delay.

In this dissertation, we investigate some stochastic optimization models of Merton's type with delays. In chapter 2, an optimal-investment consumption model over a finite time horizon with finite delay is investigated. In section 2.2, the problem is formulated as an optimal stochastic control problem. Newly developed functional Ito's formula is used to obtain the Hamilton-Jacobi-Bellman (HJB) equation in section 2.3. In section 2.4, the optimal investment and consumption controls and explicit solution of HJB equation is derived for logarithmic and exponential utility functions and finally, verification theorems are established to prove that the solution obtained from the HJB equation is the value function.

In chapter 3, the model discussed in chapter 2 is studied on infinite time horizon. In section 3.2, HJB equation for the value function is derived. In section 3.3, optimal controls and explicit solution of HJB equation is obtain for logarithmic, exponential and HARA utility functions. For each of the utility function, verification theorems are also established in this section.

In chapters 4 and 5, we introduce the idea of infinite delay in the model introduced in chapter 2. In chapter 4, stochastic portfolio management model is studied on a finite time horizon and the model captures all the historic performance of the portfolio. In section 4.2, HJB equation is derived. In section 4.3, for each of the exponential, logarithmic and HARA utility function, optimal controls and explicit solution of HJB equation is established followed by the verification theorems.

In each of the chapters 2, 3, 4, the volatility of the risky asset is assumed to be constant. In chapter 5, the volatility of the risky asset is assumed to be stochastic in the model. The model captures the complete historic performance of the risky asset through an infinite delay variable. In section 5.4, we use the method of sub/super solutions to obtain the classical solution of the HJB equation established in section 5.3 for logarithmic utility function. The optimal controls and verification theorems are also established in this section.

Chapter 2

A Finite Time Horizon Stochastic Portfolio Optimization Model with Bounded Memory

2.1 Introduction

In a classical portfolio of Merton's type, an investor allocates her wealth between a risky asset and a riskless asset and chooses the consumption rate to maximize the total expected utility. In this chapter we study an optimal investment-consumption model of Merton type as a stochastic control problem over a finite time horizon. The model takes into account the past performance of the risky asset. The risky asset is modeled using a delayed stochastic differential equations. This work is motivated by the model studied by Chang, Pang and Yang [5].

In this chapter, we are going to use a newly developed functional Ito's formula to derive the associated Hamilton-Jacobi-Bellman (HJB) equation. The functional Ito's formula was first initiated by Dupire [8]. Details to be given section 2.3.

We consider a stochastic control problem over a finite time horizon in which the state variable $X(t)$ is modeled by a control stochastic process with bounded memory. In particular, we consider for the following system with bounded memory:

$$\begin{cases} dX(t) &= b(t, X_t, u(t))dt + \sigma(t, X_t, u(t))dW(t), \ t \in [s, T], \\ X(t) &= \psi(t-s), \qquad t \in [s-h, s], \end{cases}$$

where h is a fixed delay and X_t is the segment of the path from $t-h$ to t, $\psi \in C[-h,0]^n$ is the initial path and u is the control in some admissible control space $\mathcal{U}$. The goal is to choose a control u to maximize the functional:

$$J(s,\psi;u) = \mathbf{E}^{s,\psi,u}\left[\int_s^T f(t,X_t,u(t))dt + g(X_T)\right].$$

The value functions is given by

$$V(s,\psi) = \sup_{u\in\mathcal{U}} J(s,\psi;u).$$

In particular, the value of the portfolio follows a stochastic process $X(t)$ (see (2.9)) that depends on following delay information:

$$\begin{aligned} Y(t) &= \int_{-h}^{0} e^{\lambda\theta} X(t+\theta)d\theta, && (2.1)\\ Z(t) &= X(t-h), \quad \forall t\in[s,T], && (2.2)\end{aligned}$$

where $\lambda > 0$ is a constant, $h > 0$ is the delay parameter and $s \in [0,T]$ is the initial time. The performance of the portfolio in $[s-h,s]$, the historic performance, gives the initial condition for $X(t)$ and is characterized as:

$$X(t) = \varphi(t-s), \quad \forall t \in [-h,s],$$

where $\varphi > 0$ is a continuous function on $[-h,0]$.

As we can see, $e^{\lambda\theta}$ in (2.1) serves as a weight function if we regard $Y(t)$ as the weighted average of the $X(t+\theta)$, $\theta \in [-h,0]$. We assume that $\lambda > 0$ because usually the most recent information will be assigned a higher weight so the weight function should be increasing with respect to θ.

The utility function comes from consumption. Let C be the consumption rate. We investigate the HJB equation for logarithmic and exponential utility functions:

$$\begin{aligned} U(C) &= \log(C)\\ U(C) &= 1-e^{\alpha c}, \quad \alpha > 0.\end{aligned}$$

Under certain conditions, for each utility function, we derive the the optimal policies and the explicit solution for associated HJB equation in a finite dimensional space.

This chapter is organized as follows. In section 2.2, the model is formulated. In section 2.3, the HJB equation is derived using the functional Ito's formula. Solution of the HJB for logarithmic and exponential utility functions and the verification theorems are given in section 2.4

2.2 Problem Formulation

Consider an investor's portfolio comprises a risky asset and a riskless asset. The riskless assets earns the investor a fixed interest rate $r > 0$. Money depositd in the bank can be considered as riskless asset. The consumption of the investor is assumed to come from riskless asset and we also assume that the investor can freely move his money between the risky and riskless asset at any time.

Assume the process $\{B(t), t \geq 0\}$ is a one-dimensional standard Brownian motion defined on a complete filtered probability space $(\Omega, \mathcal{F}, P; \mathbf{F})$, where $F = \{\mathcal{F}, t \geq 0\}$ is the P-augmented natural filtration generated by the Brownian motion $\{B(t), t \geq 0\}$. Let $K(t)$ be the amount invested in the risky asset and assume it satisfies the following stochastic differential equation

$$dK(t) = [(\mu_1 + \mu_2 Y(t) + \mu_3 Z(t))K(t) + I(t)]dt + \sigma K(t) dB(t),$$

where μ_1, μ_2, μ_3 and σ are constants, and the performance of the risky asset depends on $Y(t)$and $Z(t)$, the delay variables, given as

$$\begin{aligned} Y(t) &= \int_{-h}^{0} e^{\lambda\theta} X(t+\theta) d\theta, && (2.3) \\ Z(t) &= X(t-h), \quad \forall t \in [s, T], && (2.4) \end{aligned}$$

where $\lambda > 0$ is a constant and h is the delay parameter. Let $L(t)$ is the amount invested in the riskless asset. The price of riskless asset satisfies

$$dL(t) = [rL(t) - C(t) - I(t)]dt,$$

where $I(t)$ is the investment rate in the risky asset at t, and $C(t)$ is the consumption rate. The net wealth is given by $X(t) = K(t) + L(t)$. The equation for $X(t)$ follows by using $X(t) = K(t) + L(t)$:

$$\begin{aligned} dX(t) &= [(\mu_1 + \mu_2 Y(t) + \mu_3 Z(t))K(t) + rL(t) - C(t)]dt \\ &\quad + \sigma K(t)dB(t), \quad \forall t \in [s, T]. \end{aligned} \tag{2.5}$$

The initial condition is the information about $X(t)$ for $t \in [-h, s]$:

$$X(t) = \varphi(t - s), \quad \forall t \in [-h, s],$$

where $\varphi \in \mathbf{J}$ and $\mathbf{J}$ is defined by $\mathbf{J} \equiv C[-h, 0]$, which is the space for all continuous functions defined on $[-h, 0]$ equipped with the sup norm

$$||\varphi|| = \sup_{\theta \in [-h,0]} |\varphi(\theta)|. \tag{2.6}$$

Let $L^2(\Omega, \mathbf{J})$ be a Banach space for all $(\mathbf{F}, \mathcal{B}(\mathbf{J}))$-measurable maps $\Omega \to \mathbf{J}$ that are in L^2 in the Bochner sense, where $\mathcal{B}(\mathbf{J})$ is the Borel $\sigma-$ field on $\mathbf{J}$. For any $\phi \in L^2(\Omega, \mathbf{J})$, the Banach norm is given by

$$||\phi(\omega)||_2 = \left[\int_\Omega ||\phi(\omega)||^2 dP(\omega) \right]^{1/2}, \tag{2.7}$$

where the norm $|| \cdot ||$ is given by (2.6). The state variables are $X(t)$ and $Y(t)$. The wealth is allocated between risky and riskless asset and in order to describe it, we treat $K(t)$ and $C(t)$ as our control variables. For technical reasons, we modify the model described in (2.9) and consider the following model

$$\begin{aligned} dX(t) &= [\mu_1 K(t) + \mu_2 Y(t) + \mu_3 Z(t) + rL(t) - C(t)]dt \\ &\quad + \sigma K(t)dB(t), \quad \forall t \in [s, T] \end{aligned} \tag{2.8}$$

Remark 2.2.1. *If we assume that $K(t) > 0$ almost surely, we can use the following delay variables $\widetilde{Y}(t)$ and $\widetilde{Z}(t)$:*

$$\widetilde{Y}(t) = \frac{1}{K(t)} \int_{-h}^{0} e^{\lambda\theta} X(t + \theta) d\theta, \quad \widetilde{Z}(t) = \frac{X(t - h)}{K(t)}, \quad t \in [s, T],$$

instead of (2.3) *and* (2.4)*, so we can reach* (2.8)*.*

Instead of using $K(t)$ and $C(t)$, we use $c(t) = \frac{C(t)}{X(t)}$ and $k(t) = \frac{K(t)}{X(t)}$ as our consumption and investment controls, respectively. Using $L(t) = X(t) - K(t) = X(t)(1 - k(t))$, we can rewrite the equation for $X(t)$ as

$$\begin{aligned} dX(t) &= [((\mu_1 - r)k(t) - c(t) + r)X(t) + \mu_2 Y(t) \\ &\quad + \mu_3 Z(t)]dt + \sigma k(t)X(t)dB(t), \quad \forall t \in [s, T]. \end{aligned} \tag{2.9}$$

Initial condition is given by

$$X(t) = \varphi(t - s), \quad \forall t \in [s - h, s], \tag{2.10}$$

where $\varphi \in \mathbf{J}$ and $\varphi(\theta) > 0, \forall \theta \in [-h, 0]$.

For functional differential equations we use the following conventional notation: *If $\varphi \in C([-h, T]; \mathbf{R})$ and $t \in [0, T]$, let $\varphi_t(\theta) \in \mathbf{J}$ be defined by*

$$\varphi_t(\theta) = \varphi(t + \theta), \quad \forall \theta \in [-h, 0]. \tag{2.11}$$

Since $\varphi_t(\theta) \in \mathbf{J}$, its norm is given by the sup norm:

$$||\varphi_t(\theta)|| = \sup_{\theta \in [-h,0]} |\varphi(\theta)| = \sup_{\theta \in [-h,0]} |\varphi(t + \theta)|. \tag{2.12}$$

Using the above notation, initial condition (2.10) can be written as

$$X_s = \varphi.$$

Following definition gives the admissible control space Π for control variables $k(t)$ and $c(t)$:

Definition 2.2.1 (Admissible Control Space). *Let Π denote the admissible control space. A control policy $(k(t), c(t))$ is said to be in the admissible control space Π if it satisfies the following conditions:*

(a) $(k(t), c(t))$ is $\mathcal{F}^t$−measurable for any $t \in [0, T]$;

(b) $c(t) \geq 0, \forall t \in [0, T]$;

(c) For any $t \in [0,T]$, we have

$$\left.\begin{aligned} |k(t)X(t)| \leq \Lambda_1 |X(t) + \mu_3 Y(t)|, \\ |c(t)X(t)| \leq \Lambda_2 |X(t) + \mu_3 Y(t)|, \end{aligned}\right\} \tag{2.13}$$

where $\Lambda_1 > 0, \Lambda_2 > 0$ are constants.

Following Lemma gives the existence and uniqueness of solution of (2.9) with initial condition (2.10):

Lemma 2.2.1. *For any control $(k(t), c(t)) \in \Pi$, the equation* (2.9) *with initial condition* (2.10) *has a unique strong solution $X : [-h,T] \times \Omega \to \mathbb{R}$. Furthermore, for $t \in [s,T]$, $X_t \in$ $\mathbf{J}$ and for any positive integer n, we have*

$$\mathbf{E}\left[||X_t||^{2n}\right] \leq C_n[1 + ||\varphi||^{2n}], \quad \forall t \in [s,T], \tag{2.14}$$

where $C_n > 0$ is a constant.

Proof. Let $\|\varphi\|_2$ be the L^2 norm of φ as given in (3.7). Then, by virtue of

$$\|\varphi\|_2 \leq \|\varphi\|,$$

this Lemma is a corollary of Theorem I.2 of Mohammed [41]. □

The wealth process $X(t)$ must also satisfy the state constraint

$$X(t) > 0 \quad \forall t \in [s,T].$$

We have the following result:

Lemma 2.2.2. *The solution X(t) of the system* (2.9) -(2.10) *satisfies*

$$X(t) > 0, \quad Y(t) > 0, \quad Z(t) > 0, \quad \textit{almost surely} \quad \forall t \in [s,T].$$

Proof. By the definition of $Y(t), Z(t)$, it is sufficient to show that

$$X(t) > 0, \quad \textit{almost surely} \quad \forall t \in [s,T]. \tag{2.15}$$

First we show that

$$X(t) \geq 0, \quad a.s. \quad \forall t \in [s, T].$$

For a given intial condition $\varphi(\theta) > 0, \quad \forall \theta \in [s-h, s]$, we have

$$Y(s) > 0, \quad Z(s) > 0.$$

We note that $Y(t)$ and $Z(t)$ are non-negative as long as $X(\theta) \geq 0, \quad \forall \theta \in [t-h, t]$, and they are equal to zero if $X(t)$ has hit zero before they do. Moreover, from equation (2.9), we note that if $X(t)$ hits zero for the first time at τ, the diffusion part vanishes and the drift coefficient becomes $\mu_2 Y(\tau) + \mu_3 Z(\tau)$ and is positive. Therefore we get

$$X(t) \geq 0, \quad a.s. \quad \forall t \in [s, T].$$

So that we can get

$$Y(t) \geq 0, \quad Z(t) \geq 0, \quad a.s. \quad \forall t \in [s, T].$$

Let $\tilde{X}(t)$ be a solution of

$$d\tilde{X}(t) = [((\mu_1 - r)k(t) - c(t) + r)\tilde{X}(t)]dt + \sigma k(t)\tilde{X}(t)dB(t), \quad \forall t \in [s, T]. \tag{2.16}$$

$$\tilde{X}(t) = \varphi(t-s), \quad \forall t \in [s-h, s], \tag{2.17}$$

for same $\varphi > 0$ as in (2.10). The solution is given as

$$\begin{aligned} \tilde{X}(t) &= \varphi(s)\exp\left\{ \int_s^t \left((\mu_1 - r)k(\theta) - c(\theta) + r - \frac{\sigma^2 k^2(\theta)}{2} \right) d\theta \right. \\ &\quad \left. + \int_s^t \sigma k(\theta) dB(\theta) \right\}. \end{aligned}$$

Obviously, we have

$$\tilde{X}(t) > 0, \quad \text{a.s.} \quad \forall t \in [s, T].$$

Moreover, using Ito's formula, we get

$$\begin{aligned} d\left[\frac{1}{\tilde{X}(t)}\right] &= -\frac{d\tilde{X}(t)}{\tilde{X}(t)^2} + \frac{1}{2}\frac{2((d\tilde{X}(t))^2}{(\tilde{X}(t))^3} \\ &= -\frac{(\mu_1 - r)k(t) - c(t) + r}{\tilde{X}(t)}dt - \frac{\sigma k(t)}{\tilde{X}(t)}dB(t) + \frac{\sigma^2 k^2(t)}{\tilde{X}(t)}dt \\ &= \frac{\sigma^2 k^2(t) - [(\mu_1 - r)k(t) - c(t) + r]}{\tilde{X}(t)}dt - \frac{\sigma k(t)}{\tilde{X}(t)}dB(t). \end{aligned} \tag{2.18}$$

Consider a new stochastic process $A(t)$ defined by

$$A(t) = \frac{X(t)}{\tilde{X}(t)} = X(t) \cdot \frac{1}{\tilde{X}(t)}. \tag{2.19}$$

Then, we can get

$$\begin{aligned} dA(t) &= (dX(t)) \cdot \frac{1}{\tilde{X}(t)} + X(t) \cdot d\left[\frac{1}{\tilde{X}(t)}\right] + (dX(t)) \cdot d\left[\frac{1}{\tilde{X}(t)}\right] \\ &= A(t)\left[(\mu_1 - r)k(t) - c(t) + r\right]dt + \frac{\mu_2 Y(t)}{\tilde{X}(t)}dt + \sigma k(t)A(t)dt \\ &\quad + \left(\sigma^2 k^2(t) - [(\mu_1 - r)k(t) - c(t) + r]\right)A(t)dt \\ &\quad - \sigma k(t)A(t)dB(t) - \sigma^2 k^2(t)A(t)dt \\ &= \frac{\mu_2 Y(t)}{\tilde{X}(t)}dt \\ &\geq 0. \end{aligned} \tag{2.20}$$

Since $A(s) = \frac{X(s)}{\tilde{X}(s)} = \frac{\varphi(s)}{\varphi(s)} = 1$, therefore we have

$$A(t) \geq 1, \quad \text{a.s. } \forall t \geq s.$$

Therefore, we have

$$X(t) \geq \tilde{X}(t), \quad \text{a.s. } \forall t \in [s, T].$$

Since $\tilde{X}(t) > 0$, a.s., we have

$$X(t) > 0, \quad \text{a.s. } \forall t \in [s, T].$$

This completes the proof. $\square$

The utility function $U(C)$ is defined based on the consumption rate. We assume that Ψ is the terminal utility function that depends on both $X(T)$ and $Y(T)$. The problem under consideration is portfolio optimization problem on a finite time horizon $[0,T]$ with the objective function given by

$$\begin{aligned} J(s,\varphi,k,c) &= \mathbf{E}_{s,\varphi}\Big[\int_s^T e^{-\beta(t-s)}U(c(t)X(t))dt \\ &\qquad + e^{-\beta(T-s)}\Psi(X(T),Y(T))\Big], \quad \forall (k,c)\in\Pi. \end{aligned}$$

Then the value function is given by

$$\begin{aligned} V(s,\varphi) &= \sup_{k,c\in\Pi} J(s,\varphi,k,c) \\ &= \sup_{k,c\in\Pi} \mathbf{E}_{s,\varphi}\Big[\int_s^T e^{-\beta(t-s)}U(c(t)X(t))dt \\ &\qquad + e^{-\beta(T-s)}\Psi(X(T),Y(T))\Big]. \end{aligned} \tag{2.21}$$

Note that $V(s,\varphi)$ is a functional defined on an infinite dimensional space $[0,T]\times C[-h,0]$. We turn V into a function defined on a finite dimensional space as follows

$$V(s,\varphi) = V(s,x,y,z),$$

where $V:[0,T]\times\mathbf{R}^3\to\mathbf{R}$, and

$$x = x(\varphi) \equiv \varphi(0), \tag{2.22}$$

$$y = y(\varphi) \equiv \int_{-h}^0 e^{\lambda\varphi(\theta)}d\theta, \tag{2.23}$$

$$z = z(\varphi) \equiv \varphi(-h)., \tag{2.24}$$

Further, to derive the HJB equation, we will need that the value function V only depends on (s,x,y), i.e.

$$V(s,\varphi) = V(s,x,y,z) = V(s,x,y). \tag{2.25}$$

Actually, this is a necessary condition that we can derive a HJB equation in a finite dimensional space. See Lemma 2.3.2 for details.

2.3 Functional Ito's Formula and the HJB Equation

Recall that we use $X(t)$ to denote the current value and we use $X_t : [-h, 0] \to \mathbb{R}$ to denote the path of $X(t)$ from $t-h$ to t. We use functional Ito's formula to find the $dy(X_t)$ of the the path dependent functional $y(X_t)$. Functional Ito's formula is an extension of Ito's formula to functionals. For details about the notations and other details of functional Ito's formula see Dupire [8].

For a functional $f(X_t)$ of X_t, we have the following functional Ito's formula:

$$df(X_t) = \partial_t f(X_t)dt + \partial_x f(X_t)dX(t) + \tfrac{1}{2}\partial_{xx} f(X_t)d\langle X\rangle(t). \tag{2.26}$$

where

$$\begin{aligned}
\partial_t f(X_t) &= \lim_{\delta\to 0}\frac{f(X_{t,\delta}) - f(X_t)}{\delta}, && (2.27)\\
X_{t,\delta}(\theta) &= X_t(\delta+\theta), \quad \forall\theta \in [-h, -\delta], && (2.28)\\
X_{t,\delta}(\theta) &= X_t(0), \quad \theta \in [-\delta, 0]; && (2.29)\\
\partial_x f(X_t) &= \lim_{h\to 0}\frac{f(X_t^h) - f(X_t)}{h}, && (2.30)\\
X_t^h(\theta) &= X_t(\theta), \quad \theta < 0, && (2.31)\\
X_t^h(0) &= X_t(0) + h; && (2.32)\\
\partial_{xx} f(X_t) &= \lim_{h\to 0}\frac{\partial_x f(X_t^h) - \partial_x f(X_t)}{h}. && (2.33)
\end{aligned}$$

The above derivatives and the functional Ito's formula was initiated by Dupire [8] and was later studied in Cont and Fournié [7].

Now let us consider the following functional:

$$\begin{aligned}
y(X_t) &= \int_{-h}^{0}\phi(\theta)X_t(\theta)d\theta; && (2.34)\\
z(X_t) &= X_t(-h), && (2.35)
\end{aligned}$$

where $\phi(\theta)$ is a smooth function with a continuous first order derivative $\phi'(\theta)$.

From the above definition, we can see that the delay variable $y(X_t)$ is actually the weight average of the delayed (historic) value of $X(t+\theta)$ for $\theta \in [0,-h]$ with the weight function given by $\phi(\theta)$, $\theta \in [-h,0]$. For example, when $\phi(\theta) = e^{\lambda\theta}$, $y(X_t)$ is just the exponential moving average for $X(s)$ for $s \in [t, t-\theta]$. If $\phi(\theta) = 1$, $y(X_t)$ is just the moving average, which is a special case of the exponential moving average (with $\lambda = 0$). As we know, investors tend to look at the moving average or exponential moving average for a stock before they make the investment decision. A stock with current price lower than its exponential moving average may signal the downward trend for the stock price, therefore it will scare away some investors. Due to the weaker demand, the price may go down further. That is why we want to study the system (2.5) with delay variable given by (2.3).

On the other hand, from the following Lemma, we will see why we also include the delay variable $z(X_t)$ given by (2.35).

Lemma 2.3.1. *If $y(X_t), z(X_t)$ is given by* (2.34)-(2.35)*, then we have*

$$dy(X_t) = \left[-\int_{-h}^{0} \phi'(\theta) X_t(\theta) d\theta - \phi(-h) z(X_t) + X_t(0)\phi(0) \right] dt. \tag{2.36}$$

Proof. It is easy to see that

$$\partial_x y(X_t) = 0, \quad \partial_{xx} y(X_t) = 0. \tag{2.37}$$

On the other hand,

$$\begin{aligned} y(X_{t,\delta}) &= \int_{-h}^{0} \phi(\theta) X_{t,\delta}(\theta) d\theta \\ &= \int_{-h}^{-\delta} \phi(\theta) X_t(\delta+\theta) d\theta + \int_{-\delta}^{0} \phi(\theta) X_t(0) d\theta \\ &= \int_{-h+\delta}^{0} \phi(u-\delta) X_t(u) du + X_t(0) \int_{-\delta}^{0} \phi(\theta) d\theta. \end{aligned}$$

So we have

$$\begin{aligned}
y(X_{t,\delta}) - y(X_t) &= \int_{-h+\delta}^{0} \phi(u-\delta)X_t(u)du + X_t(0)\int_{-\delta}^{0}\phi(\theta)d\theta \\
&\quad - \int_{-h}^{0}\phi(\theta)X_t(\theta)d\theta \\
&= \int_{-h}^{0}\phi(\theta-\delta)X_t(\theta)d\theta - \int_{-h}^{-h+\delta}\phi(\theta-\delta)X_t(\theta)d\theta \\
&\quad + X_t(0)\int_{-\delta}^{0}\phi(\theta)d\theta - \int_{-h}^{0}\phi(\theta)X_t(\theta)d\theta \\
&= \int_{-h}^{0}[\phi(\theta-\delta)-\phi(\theta)]X_t(\theta)d\theta \\
&\quad - \int_{-h}^{-h+\delta}\phi(\theta-\delta)X_t(\theta)d\theta + X_t(0)\int_{-\delta}^{0}\phi(\theta)d\theta.
\end{aligned}$$

Thus, it is easy to verify that

$$\begin{aligned}
\partial_t y(X_t) &= \lim_{\delta\to 0}\frac{y(X_{t,\delta}) - y(X_t)}{\delta} \\
&= -\int_{-h}^{0}\phi'(\theta)X_t(\theta)d\theta - \phi(-h)X_t(-h) + X_t(0)\phi(0) \\
&= -\int_{-h}^{0}\phi'(\theta)X_t(\theta)d\theta - \phi(-h)z(X_t) + X_t(0)\phi(0).
\end{aligned}$$

Now by virtue of the functional Ito's formula (2.26), we can get (2.36). □

Take $\phi(\theta) = e^{\lambda\theta}$. Then we can get the initial delay variables defined by (2.22)-(2.24). It is easy to check that

$$\partial_t y(X_t) = X_t(0) - \lambda y(X_t) - e^{-\lambda h}z(X_t).$$

Therefore, we can get

$$dy(X_t) = \partial_t y(X_t)dt = [X_t(0) - \lambda y(X_t) - e^{-\lambda h}z(X_t)]dt.$$

We want to point out that Lemma 3.1 in [5] gives us the same result, but the method used there is different. In addition, in [5] and other papers, such as [32], [10], it is simply assumed that V does not depend on z but the reason was not given. The following lemma explains why this is a necessary condition.

Lemma 2.3.2. *If the value function V depends on z, there is no way to write the HJB equation in terms of (s, x, y, z).*

Proof. It is easy to check that

$$\begin{aligned}\partial_t z(X_t) &= \lim_{\delta\to 0}\frac{z(X_{t,\delta})-z(X_t)}{\delta}\\ &= \lim_{\delta\to 0}\frac{X(t-h+\delta)-X(t-h)}{\delta}.\end{aligned}$$

As we can see, $\partial_t z(X_t)$ usually does not exists since the path of Brownian motion is not differentiable. Therefore, if the value function V depends on z, there is not way to write down the Hamilton-Jacobi-Bellman equation in a finite dimensional space. □

Now assume that the value function V depends on the initial path φ only through the functionals $x(\varphi)$ and $y(\varphi)$ defined by (2.22)-(2.23) . That is,

$$V(s,\varphi) = V(s, x(\varphi), y(\varphi)) \equiv V(s, x, y). \tag{2.38}$$

We need the following dynamic programming principle to derive the HJB equation.

Lemma 2.3.3 (Dynamic Programming Principle)**.** *Assume that the value function $V(s, x, y)$ given by (2.21) and (2.38) is well defined and assume the system given by (2.9)-(2.10). Then we have*

$$V(s,x,y) = \sup_{(k,c)\in\Pi}\mathbf{E}_{s,\varphi,k,c}\left[\int_s^t e^{-\beta(\tau-s)}U(c(\tau))X(\tau)d\tau + e^{-\beta(t-s)}V(t,X(t),Y(t))\right],$$

for all $\mathcal{F}^t$-stopping time $t \in [s, T]$ and $(x, y) \in \mathbf{R}^2$, where $\varphi \in \mathbf{J}$ is such that $x = x(\varphi) = X(s)$ and $y = y(\varphi) = Y(s)$.

Proof. The proof is similar to Theorem 4.2 of [31] so it is omitted here. □

Following theorem gives the Hamilton Jacobi Bellman equation which follows from above Lemma and functional Ito's formula (2.26) and the proof is similar to Theorem 5.1 in [31]. So we omit the proof here.

Theorem 2.3.1 (HJB Equation). *Assume that* (2.38) *holds and* $V(s,x,y) \in C^{1,2,1}([0,T]\times \mathbf{R}\times\mathbf{R})$. *Then the value function value function* $V(s,x,y)$ *given by* (2.21) *and* (2.38) *satisfies the following HJB equation*

$$\begin{aligned}\beta V - V_s &= \max_k \left[\frac{(\sigma k x)^2}{2} V_{xx} + (\mu_1 - r)kxV_x\right] + (rx + \mu_2 y + \mu_3 z)V_x \\ &\quad + \max_{c\geq 0}[-cxV_x + U(cx)] + (x - e^{-\lambda h}z - \lambda y)V_y, \quad \forall z \in \mathbf{R} \end{aligned} \tag{2.39}$$

with the boundary condition

$$V(T,x,y) = \Psi(x,y). \tag{2.40}$$

We note that the HJB equation is a non-linear second order path dependent partial differential equation. The next step would be find the classical or viscosity solutions for the above equation and verify that the solution is the value function we need. Recent results on path dependent differential equations by [9] may be used to prove the existence of the viscosity solution of the above HJB equation. However, in this chapter, our focus is on the systems with bounded memory in certain forms, such that it is possible to work in a finite dimensional space and the HJB equation will become a PDE in classical sense.

2.4 The Solution of the HJB equation

2.4.1 Logarithmic Utility

In this section we consider logarithmic utility function given as

$$U(cX) = \log(cX). \tag{2.41}$$

We find explicit solution of HJB equations (2.39)-(2.40). Using the log utility we have

$$\begin{aligned}\beta V - V_s &= \max_k \left[\frac{(\sigma k x)^2}{2} V_{xx} + (\mu_1 - r)kxV_x\right] + (rx + \mu_2 y + \mu_3 z)V_x \\ &\quad + \max_{c\geq 0}[-cxV_x + \log(cx)] + (x - e^{-\lambda h}z - \lambda y)V_y, \end{aligned} \tag{2.42}$$

The candidates for optimal controls are

$$k^* = -\frac{(\mu_1 - r)V_x}{\sigma^2 x V_{xx}}, \quad c^* = \frac{1}{xV_x}.$$

The terminal utility function $\Psi(x, y)$ is assumed to be of the form

$$\Psi(x, y) = \frac{1}{\beta} \log(x + \mu_3 e^{\lambda h} y). \tag{2.43}$$

We assume the solution is of the form

$$V(s, x, y) = \psi(x, y) + Q(s), \tag{2.44}$$

where $Q(s)$ and $\psi(x, y)$ will be determined. Moreover, we have

$$\begin{aligned} V_x &= \psi_x(x, y), \qquad V_{xx} = \psi_{xx}(x, y), \\ V_y &= \psi_y(x, y), \qquad V_s = Q'(s). \end{aligned}$$

Plug k^* and c^* and solution form into (2.42), we obtain

$$\begin{aligned} \beta[\psi(x, y) + Q(s)] - Q'(s) &= -\frac{1}{2}\frac{(\mu_1 - r)^2 \psi_x^2}{\sigma^2 \psi_{xx}} + \log(\frac{1}{\psi_x}) - 1 \\ &\quad + (rx + \mu_2 y + \mu_3 z)\psi_x + (x - \lambda y - e^{-\lambda h} z)\psi_y. \end{aligned} \tag{2.45}$$

Define

$$u \equiv x + \mu_3 e^{\lambda h} y, \tag{2.46}$$

Let $\psi(x, y) = \frac{1}{\beta} \log(u)$, then we have,

$$\psi_x = \frac{1}{\beta u}, \qquad \psi_{xx} = -\frac{1}{\beta u^2}, \qquad \psi_y = \frac{\mu_3 e^{\lambda h}}{\beta u}.$$

Assume that

$$\mu_3 e^{\lambda h}(r + \mu_3 e^{\lambda h}) = \mu_2 - \mu_3 \lambda e^{\lambda h}, \tag{2.47}$$

Using assumption (2.47) and definition of u, equation (2.45) can be written as

$$Q'(s) \;=\; \beta Q(s) - \frac{(\mu_1 - r)^2}{2\beta\sigma^2} - \log(\beta) + 1 - \frac{1}{\beta}(r + \mu_3 e^{\lambda h}). \tag{2.48}$$

Let

$$\Lambda_4 \equiv \frac{(\mu_1 - r)^2}{2\beta\sigma^2} + \log\beta - 1 + \frac{1}{\beta}(r + \mu_3 e^{\lambda h}). \tag{2.49}$$

Equation (2.48) can be written as

$$\frac{d}{ds}(e^{-\beta s} Q(s)) = -\Lambda_4 e^{-\beta s}. \tag{2.50}$$

At terminal time $t = T$, we have

$$\begin{aligned} V(T, x, y) &= Q(T) + \frac{1}{\beta}\log(x + \mu_3 e^{\lambda h} y) \\ &= \Psi(x, y) \\ &= \frac{1}{\beta}\log(x + \mu_3 e^{\lambda h} y). \end{aligned} \tag{2.51}$$

So the boundary condition for $Q(s)$ at $s = T$ is given by

$$Q(T) = 0. \tag{2.52}$$

The solution for (2.50)-(4.73) is given as

$$Q(s) = \frac{\Lambda_4}{\beta}(1 - e^{-\beta(T-s)}). \tag{2.53}$$

If $\Lambda_4 > 0$, we have

$$Q(s) \geq 0, \quad \forall s \in [0, T]. \tag{2.54}$$

Therefore, the equations (2.39)-(2.40) for logarithmic utility function have the solution:

$$V(s, x, y) = Q(s) + \log(x + \mu_3 e^{\lambda h} y), \tag{2.55}$$

The optimal investment and consumption rates are given as

$$
\begin{aligned}
k(s)^* &= \frac{(\mu_1 - r)(x + \mu_3 e^{\lambda h} y)}{\sigma^2 x}, && (2.56)\\
c(s)^* &= \frac{\beta(x + \mu_3 e^{\lambda h} y)}{x}, && (2.57)
\end{aligned}
$$

where $Q(s)$ is given by (2.53), and x and y are estimated at time s as following

$$x = X(s), \quad y = Y(s) = \int_{-h}^{0} e^{\lambda\theta} X(s+\theta) d\theta.$$

To make sure that the solution given by (2.55) is satified by the value function (2.21), we need to give the verification theorem.

Theorem 2.4.1 (Verification Theorem)**.** *Assume that $X(t)$ be a strong solution of* (2.9)-(2.10) *and $Y(t)$ and $Z(t)$ are given by* (2.3) *and* (2.4) *respectively. Let $V(s,x,y) \in C^{1,2,1}([0,T] \times \mathbf{R} \times \mathbf{R})$ is a solution of the HJB equation given by* (2.39)-(2.40) *such that*

$$\mathbf{E}\left[\int_0^T [k(t)X(t)V_x(t, X(t), Y(t))]^2 dt\right] < \infty, \tag{2.58}$$

$\forall (k,c) \in \Pi$. *Then we have,*

$$V(s,x,y) \quad \geq \quad \sup_{(k,c)\in\Pi} \mathbf{E}_{s,\varphi}\left[\int_s^T e^{-\beta(T-s)} U(c(t)X(t))dt + e^{-\beta(t-s)}\Psi(X(T), Y(T))\right].$$

In addition, assume that the utility function is given by

$$U(x) = \log(x) \tag{2.59}$$

and

$$k^* = -\frac{(\mu_1 - r)V_x}{\sigma^2 x V_{xx}}, \quad c^* = \frac{1}{xV_x}.$$

If $(k^*, c^*) \in \Pi$, then (k^*, c^*) is the optimal control policy. In this case, we have

$$V(s,x,y) \quad = \quad \mathbf{E}_{s,\varphi,k^*,c^*}\left[\int_s^T e^{-\beta(t-s)} U(c(t)X(t))dt + e^{-\beta(T-s)}\Psi(X(T), Y(T))\right].$$

Proof. Rewriting equation (2.39), for log utility, we have

$$\max_{k,c\geq 0}[\mathcal{L}^{k,c}V(s,x,y)+\log(cx)]-\beta V(s,x,y)+[x-\lambda y-e^{-\lambda h}z]V_y(s,x,y)=0. \tag{2.60}$$

where $\mathcal{L}^{k,c}$ is defined as

$$\begin{aligned}\mathcal{L}^{k,c}f &= \mathcal{L}^{k,c}f(t,x,y)\\ &= f_t+(((\mu_1-r)k-c+r)x+\mu_2 y)f_x+\frac{1}{2}\sigma^2k^2x^2f_{xx},\end{aligned}$$

Assume $V(s,x,y)$ be a solution of the equation (2.60). For any given admissible control $(k,c)\in\Pi$ and for any $(s,x,y)\in[0,T]\times\mathbf{R}\times\mathbf{R}$, we have

$$\beta V(s,x,y)-\mathcal{L}^{k,c}V(s,x,y)-[x-\lambda y-e^{-\lambda h}z]V_y(s,x,y)\geq\log(cx). \tag{2.61}$$

Applying Ito's formula to $V(t,X(t),Y(t))$, we have

$$\begin{aligned}d[e^{-\beta t}V(t,X(t),Y(t))] &= e^{-\beta t}[-\beta V(t,X(t),Y(t))dt+dV(t,X(t),Y(t))]\\ &= e^{-\beta t}[-\beta V(t,X(t),Y(t))+\mathcal{L}^{k,c}V(t,X(t),Y(t))\\ &\quad +[X(t)-\lambda Y(t)-e^{-\lambda h}Z(t)]V_y(t,X(t),Y(t))dt\\ &\quad +\sigma k(t)X(t)V_x(t,X(t),Y(t))dB(t)].\end{aligned}$$

Integrating it from s to T, and using (2.61), we have

$$\begin{aligned}& e^{-\beta T}V(T,X(T),Y(T))-e^{-\beta s}V(s,x,y)\\ =& \int_s^T e^{-\beta t}(-\beta V(t,X(t),Y(t))+\mathcal{L}^{k,c}V(t,X(t),Y(t))\\ & +\left[X(t)-\lambda Y(t)-e^{-\lambda h}Z(t)\right]V_y(t,X(t),Y(t))dt\\ & +\int_s^T e^{-\beta t}\sigma k(t)X(t)V_x(t,X(t),Y(t))dB(t)\\ \leq & -\int_s^T e^{-\beta t}\log(c(t)X(t))dt\\ & +\int_s^T e^{-\beta t}\sigma k(t)X(t)V_x(t,X(t),Y(t))dB(t).\end{aligned}$$

Using the boundary condition (2.51), we have

$$
\begin{aligned}
V(s,x,y) &\geq e^{-\beta(T-s)}V(T,X(T),Y(T)) + \int_s^T e^{-\beta(T-s)} \log(c(t)X(t))dt \\
&\quad - \int_s^T e^{-\beta(T-s)}\sigma k(t)X(t)V_x(t,X(t),Y(t))dB(t) \\
&= e^{-\beta(T-s)}\Psi(X(T),Y(T)) + \int_s^T e^{-\beta(T-s)} \log(c(t)X(t))dt \\
&\quad - \int_s^T e^{-\beta(T-s)}\sigma k(t)X(t)V_x(t,X(t),Y(t))dB(t).
\end{aligned}
$$

By virtue of condition (2.58), we have

$$
\int_s^T e^{-\beta(T-s)}\sigma k(t)X(t)V_x(t,X(t),Y(t))dB(t)
$$

a martingale. Taking expection on both sides, we obtain, $\forall (k,c) \in \Pi$

$$
V(s,x,y) \geq \mathbf{E}_{s,\varphi,k,c}\left[\int_s^T e^{-\beta(T-s)} \log(c(t)X(t))dt + e^{-\beta(T-s)}\Psi(X(T),Y(T))\right].
$$

Therefore we have,

$$
V(s,x,y) \geq \sup_{k,c\geq 0} \mathbf{E}_{s,\varphi,k,c}\left[\int_s^T e^{-\beta(T-s)} \log(c(t)X(t))dt + e^{-\beta(T-s)}\Psi(X(T),Y(T))\right].
$$

By taking $(k,c) = (k^*, c^*)$, the inequalities can be replaced by equalities

$$
V(s,x,y) = \mathbf{E}_{s,\varphi,k^*,c^*}\left[\int_s^T e^{-\beta(T-s)} \log(c^*(t)X(t))dt + e^{-\beta(T-s)}\Psi(X(T),Y(T))\right].
$$

This completes the proof. $\square$

Now it remains to verify that the function defined by (2.55) is a classical solution of (2.39)-(2.40) and control policy is given by (2.56)-(2.57)

Theorem 2.4.2. *Assume that $X(t)$ be a strong solution of* (2.9)-(2.10) *and $Y(t)$ and $Z(t)$ are given by* (2.3) *and* (2.4) *respectively. Assume that the utility function is given* *by*

$$U(x) = \log(x),$$

and that the terminal function is given by

$$\Psi(x, y) = \frac{1}{\beta} \log(x + \mu_3 e^{\lambda h} y)$$

Suppose (2.47) *also holds. Then the function* $V(s, x, y)$ *given by* (2.55) *is a classical solution of the HJB equation* (2.39)-(2.40)*, and it is equal to the value of the system defined by* (2.21)-(2.25)*, that is*

$$V(s, x, y) \quad = \quad \sup_{(k,c)\in\Pi} \mathbf{E}_{s,\varphi} \left[\int_s^T e^{-\beta(T-s)} U(c(t)X(t))dt + e^{-\beta(T-s)} \Psi(X(T), Y(T)) \right].$$

In addition, the optimal control policy is given by

$$\begin{aligned} k^*(t) &= \frac{(\mu_1 - r)(X(t) + \mu_3 e^{\lambda h} Y(t))}{\sigma^2 X(t)}, \\ c^*(t) &= \frac{\beta(X(t) + \mu_3 e^{\lambda h} Y(t)}{X(t)}, \quad \forall t \in [s, T], \end{aligned}$$

where $Q(\cdot)$ *is defined by* (2.53).

Proof. It is evident from the derivation of $V(s, x, y)$ that $V(s, x, y)$ given by (2.55) is a classical solution of HJB equation (2.39)-(2.40).

To use Theorem 4.3.5, we first verify that condition (2.58) is satisfied. Using (2.55)

$$V_x(t, X(t), Y(t)) = \frac{1}{X(t) + \mu_3 e^{\lambda h} Y(t)}.$$

We have ,

$$|V_x(t, X(t), Y(t))| = \left| \frac{1}{X(t) + \mu_3 e^{\lambda h} Y(t)} \right|$$

Using the definition of admissible control space Π, we have

$$\begin{aligned}|k(t)X(t)| &\leq \Lambda_1|X(t)+\mu_3Y(t)| \\ &\leq \Lambda_1|X(t)+\mu_3e^{\lambda h}Y(t)|.\end{aligned}$$

Therefore, we have

$$\begin{aligned}|k(t)X(t)V_x(t,X(t),Y(t))| &\leq \Lambda_1|X(t)+\mu_3e^{\lambda h}Y(t)|\cdot\left|\frac{1}{X(t)+\mu_3e^{\lambda h}Y(t)}\right| \\ &= \Lambda_1.\end{aligned}$$

We have,

$$\begin{aligned}\mathbf{E}\left[\int_0^T(k(t)X(t)V_x(t,X(t),Y(t)))^2dt\right] &\leq \mathbf{E}\left[\int_0^T\Lambda_1^2dt\right] \\ &= \Lambda_1^2T<\infty.\end{aligned}$$

where $\Lambda_1>0$ is a constant independent of t. Thus condition (2.58) is verified. Using definition of (k^*,c^*), it is easy to see that $(k^*(t),c^*(t))$ is $\mathcal{F}^t$-measurable for any $t\in[0,T]$. Also, we have

$$\begin{aligned}k^*(t) &= \frac{(\mu_1-r)(X(t)+\mu_3e^{\lambda h}Y(t))}{\sigma^2X(t)}, \\ c^*(t) &= \frac{\beta(X(t)+\mu_3e^{\lambda h}Y(t)}{X(t)}, \quad \forall t\in[s,T].\end{aligned}$$

We note that $c(t)\geq 0,\forall t\in[0,T]$. The conditions given by (2.13) of admissible control space are also satisfied. Hence we get $(k^*,c^*)\in\Pi$. This completes the proof. □

2.4.2 Exponential Utility

In this section we consider the exponential utility function given as

$$U(cX)=1-e^{-\alpha cX}, \quad \alpha>0. \tag{2.62}$$

We note that maximizing the utility function with or without the the additive term 1 gives the same results. The additive term 1 in the utility function restricts the range of the function between 0 and 1 and other than that it does not have any mathematical relevance. So we drop the term 1 for technical convenience and consider the following

$$U(cX) = -e^{-\alpha c X}, \quad \alpha > 0. \tag{2.63}$$

We seek to find a solution of HJB equations (2.39)-(2.40), we have

$$\begin{aligned} \beta V - V_s &= \max_k \left[\frac{(\sigma k x)^2 V_{xx}}{2} + (\mu_1 - r) k x V_x \right] + (rx + \mu_2 y + \mu_3 z) V_x \\ &\quad + \max_{c \geq 0} \left[-c x V_x - e^{-\alpha c x} \right] + (x - e^{-\lambda h} z - \lambda y) V_y, \end{aligned} \tag{2.64}$$

The candidates for optimal controls are

$$k^* = -\frac{(\mu_1 - r) V_x}{\sigma^2 x V_{xx}}, \quad c^* = -\frac{1}{x\alpha} \log\left(\frac{V_x}{\alpha}\right).$$

Substituting k^* and c^* in equation (2.64), we obtain

$$\begin{aligned} \beta V - V_s &= -\frac{1}{2} \frac{(\mu_1 - r)^2 V_x^2}{\sigma^2 V_{xx}} + \frac{1}{\alpha} \log\left(\frac{V_x}{\alpha}\right) V_x - \frac{V_x}{\alpha} \\ &\quad + (rx + \mu_2 y + \mu_3 z) V_x + (x - \lambda y - e^{-\lambda h} z) V_y. \end{aligned} \tag{2.65}$$

We look for a solution of the form

$$V(s, x, y) = \psi(x, y) Q(s), \tag{2.66}$$

where $Q(s)$ and $\psi(x, y)$ will be determined. Moreover, we have

$$\begin{aligned} V_x &= Q(s) \psi_x(x, y), \quad V_{xx} = Q(s) \psi_{xx}(x, y), \\ V_y &= Q(s) \psi_y(x, y), \quad V_s = Q'(s) \psi(x, y). \end{aligned}$$

Substituting above into equation (2.65) yields

$$\begin{aligned}
\beta\left[\psi(x,y)Q(s)\right]-\psi(x,y)Q'(s) \quad = \quad & -\frac{1}{2}\frac{(\mu_1-r)^2Q(s)\psi_x^2}{\sigma^2\psi_{xx}}+\frac{1}{\alpha}\log\left(\frac{Q(s)\psi_x}{\alpha}\right)\psi_x \\
& -\frac{Q(s)\psi_x}{\alpha}+(rx+\mu_2 y+\mu_3 z)Q(s)\psi_x \\
& +(x-\lambda y-e^{-\lambda h}z)Q(s)\psi_y. \qquad (2.67)
\end{aligned}$$

Let

$$u \equiv x+\mu_3 e^{\lambda h}y, \qquad (2.68)$$

and

$$\psi(x,y)=-e^{-\gamma u}.$$

where γ is a constant to be determined. Now we can get,

$$\psi_x=-\gamma\psi, \quad \psi_{xx}=\gamma^2\psi, \quad \psi_y=-\gamma\mu_3 e^{\lambda h}\psi.$$

Using the above substitutions in equation (2.67), we can obtain

$$\begin{aligned}
\psi[\beta Q(s)-Q'(s)] \quad = \quad & -\frac{(\mu_1-r)^2Q(s)\psi}{2\sigma^2} \\
& +\frac{Q(s)}{\alpha}(-\gamma\psi)\left[\log\left(\frac{-\gamma\psi Q(s)}{\alpha}\right)-1\right] \\
& -\gamma(rx+\mu_2 y)Q(s)\psi-\gamma(x-\lambda y)Q(s)\mu_3 e^{\lambda h}\psi.
\end{aligned}$$

Canceling ψ and noting that $\log(-\psi) = \log e^{-\gamma u} = -\gamma u$, we can rewrite the above equation to

$$\begin{aligned}
-\beta Q(s)+Q'(s) \quad = \quad & \frac{(\mu_1-r)^2Q(s)}{2\sigma^2} \\
& +\frac{\gamma}{\alpha}Q(s)\left[\log\frac{\gamma}{\alpha}+\log Q(s)-\gamma u-1\right] \\
& +\left[(r+\mu_3 e^{\lambda h})x+(\mu_2-\lambda\mu_3 e^{\lambda h})y\right]\gamma Q(s).
\end{aligned}$$

Now let us assume that

$$\mu_3 e^{\lambda h}(r + \mu_3 e^{\lambda h}) = \mu_2 - \mu_3 \lambda e^{\lambda h}, \tag{2.69}$$

and

$$\gamma = \alpha(r + \mu_3 e^{\lambda h}). \tag{2.70}$$

Then by assumption (2.69), we can cancel u from the equation and obtain

$$\begin{aligned} -\beta Q(s) + Q'(s) \quad &= \quad \frac{(\mu_1 - r)^2 Q(s)}{2\sigma^2} \\ &\quad + (r + \mu_3 e^{\lambda h}) Q(s) \left[\log(r + \mu_3 e^{\lambda h}) + \log Q(s) - 1\right]. \end{aligned}$$

The above equation can be rewritten as

$$\begin{aligned} \frac{Q'(s)}{Q(s)} \quad &= \quad \beta + \frac{(\mu_1 - r)^2}{2\sigma^2} - (r + \mu_3 e^{\lambda h}) & (2.71) \\ &\quad + (r + \mu_3 e^{\lambda}) \log\left((r + \mu_3 e^{\lambda h}) Q(s)\right). & (2.72) \end{aligned}$$

Let

$$\Lambda_5 \quad \equiv \quad \beta + \frac{(\mu_1 - r)^2}{2\sigma^2} + (r + \mu_3 e^{\lambda h})[\log(r + \mu_3 e^{\lambda h}) - 1]. \tag{2.73}$$

Equation (2.71) can be written as

$$\frac{Q'(s)}{Q(s)} = \Lambda_5 + (r + \mu_3 e^{\lambda h}) \log Q(s). \tag{2.74}$$

At the terminal time $t = T$, we have

$$V(T, x, y) = Q(T)(-e^{-\gamma u}) = \Psi(x, y) \tag{2.75}$$

The terminal utility function $\Psi(x, y)$ is assumed to be consistent with the exponential utility function. In particular, we assume that it is of the form

$$\Psi(x, y) = -\Lambda e^{-\gamma(x + \mu_3 e^{\lambda h} y)} = -\Lambda e^{-\gamma u}, \tag{2.76}$$

where γ is given by (2.70). So the boundary condition for $Q(s)$ at $s = T$ is

$$Q(T) = \Lambda. \tag{2.77}$$

The explicit solution for (2.74)-(2.77) is given as

$$\begin{aligned} Q(s) &= \exp\left(\frac{\Lambda_5}{r + \mu_3 e^{\lambda h}} \left(e^{-(r+\mu_3 e^{\lambda h})(T-s)} - 1\right)\right. \\ &\qquad \left. + e^{-(r+\mu_3 e^{\lambda h})(T-s)} \log(\Lambda)\right). \end{aligned} \tag{2.78}$$

It is easy to see that $Q(s) > 0$ and $Q(s)$ is an increasing function for $s \in [0, T]$. Therefore, we can get

$$0 < Q(s) < \Lambda \quad \forall s \in [0, T]. \tag{2.79}$$

The HJB equations (2.39)-(2.40) have the solution

$$V(s, x, y) = -Q(s) e^{-\alpha(r+\mu_3 e^{\lambda h})(x+\mu_3 e^{\lambda h} y)}, \tag{2.80}$$

The optimal investment and consumption rates are given as

$$k^*(s) = \frac{(\mu_1 - r)}{\alpha(r + \mu_3 e^{\lambda h})\sigma^2 x}, \tag{2.81}$$

$$\begin{aligned} c^*(s) &= -\frac{1}{x\alpha}\Bigg[\log\{(r + \mu_3 e^{\lambda h}) Q(s)\} \\ &\qquad - \alpha(r + \mu_3 e^{\lambda h})(x + \mu_3 e^{\lambda h} y)\Bigg], \end{aligned} \tag{2.82}$$

where $Q(s)$ is given by (2.78), and x and y are estimated at time s as following

$$x = X(s), \quad y = Y(s) = \int_{-h}^{0} e^{\lambda\theta} X(s + \theta) d\theta.$$

By virtue of (2.79) and Lemma 2.2.2, we can see that $c^*(s) \geq 0$ as long as

$$\Lambda(r + \mu_3 e^{\lambda h}) \leq 1. \tag{2.83}$$

Now we can prove the following verification theorem.

Theorem 2.4.3 (Verification Theorem). *Assume that $X(t)$ be a strong solution of* (2.9)-(2.10) *and $Y(t)$ and $Z(t)$ are given by* (2.3) *and* (2.4) *respectively. Let $V(s,x,y) \in C^{1,2,1}([0,T] \times \mathbf{R} \times \mathbf{R})$ is a solution of the HJB equation given by* (2.39)-(2.40) *such that*

$$\mathbf{E}\left[\int_0^T (k(t)X(t)V_x(t,X(t),Y(t)))^2\,dt\right] < \infty, \tag{2.84}$$

$\forall (k,c) \in \Pi$. Then we have,

$$\begin{aligned} V(s,x,y) \;\; &\geq \sup_{(k,c)\in\Pi} \mathbf{E}_{s,\varphi}\left[\int_s^T e^{-\beta(t-s)}U(c(t)X(t))dt\right. \\ &\qquad\qquad \left. + e^{-\beta(T-s)}\Psi(X(T),Y(T))\right]. \end{aligned} \tag{2.85}$$

In addition, assume that the utility function is given by $U(x) = -e^{-\alpha x}$ and k^* and c^* are given as

$$k^* = -\frac{(\mu_1 - r)V_x}{\sigma^2 x V_{xx}}, \quad c^* \; = \; \frac{1}{xV_x}. \tag{2.86}$$

If $(k^*, c^*) \in \Pi$, then (k^*, c^*) is the optimal control policy. In this case, we have

$$V(s,x,y) \;\; = \;\; \mathbf{E}_{s,\varphi,k^*,c^*}\left[\int_s^T e^{-\beta(t-s)}U(c(t)X(t))dt + e^{-\beta(T-s)}\Psi(X(T),Y(T))\right].$$

Proof. For exponential utility function, equation (2.39) becomes

$$\max_{k,c\geq 0}[\mathcal{L}^{k,c}V(s,x,y) - e^{-\alpha cx}] - \beta V(s,x,y) + [x - \lambda y - e^{-\lambda h}z]V_y(s,x,y) = 0. \tag{2.87}$$

where $\mathcal{L}^{k,c}$ is defined as

$$\begin{aligned} \mathcal{L}^{k,c}f &= \mathcal{L}^{k,c}f(t,x,y) \\ &= f_t + (((\mu_1 - r)k - c + r)x + \mu_2 y)f_x + \frac{1}{2}\sigma^2 k^2 x^2 f_{xx}, \end{aligned}$$

Assume $V(s,x,y)$ be a solution of the equation (2.60). For any given admissible control

$(k,c) \in \Pi$ and for any $(s,x,y) \in [0,T] \times \mathbf{R} \times \mathbf{R}$, we have

$$\beta V(s,x,y) - \mathcal{L}^{k,c} V(s,x,y) - [x - \lambda y - e^{-\lambda h} z] V_y(s,x,y) \geq -e^{-\alpha c x}. \tag{2.88}$$

Applying Ito's formula to $V(t, X(t), Y(t))$, we have

$$\begin{aligned}
d[e^{-\beta t} V(t, X(t), Y(t))] &= e^{-\beta t}[-\beta V(t, X(t), Y(t)) dt + dV(t, X(t), Y(t))] \\
&= e^{-\beta t}[-\beta V(t, X(t), Y(t)) + \mathcal{L}^{k,c} V(t, X(t), Y(t)) \\
&\quad + [X(t) - \lambda Y(t) - e^{-\lambda h} Z(t)] V_y(t, X(t), Y(t)) dt \\
&\quad + \sigma k(t) X(t) V_x(t, X(t), Y(t)) dB(t)].
\end{aligned}$$

Integrating it from s to T, and using (2.88), we have

$$\begin{aligned}
& e^{-\beta T} V(T, X(T), Y(T)) - e^{-\beta s} V(s,x,y) \\
= & \int_s^T e^{-\beta t}(-\beta V(t, X(t), Y(t)) + L^{k,c} V(t, X(t), Y(t)) \\
& + [X(t) - \lambda Y(t) - e^{-\lambda h} Z(t)] V_y(t, X(t), Y(t)) dt \\
& + \int_s^T e^{-\beta t} \sigma k(t) X(t) V_x(t, X(t), Y(t)) dB(t) \\
\leq & \int_s^T e^{-\beta t} e^{-\alpha c(t) X(t)} dt \\
& + \int_s^T e^{-\beta t} \sigma k(t) X(t) V_x(t, X(t), Y(t)) dB(t).
\end{aligned}$$

Using the boundary condition (2.75), we have

$$\begin{aligned}
V(s,x,y) &\geq e^{-\beta(T-s)} V(T, X(T), Y(T)) - \int_s^T e^{-\beta(T-s)} e^{-\alpha c(t) X(t)} dt \\
&\quad - \int_s^T e^{-\beta(T-s)} \sigma k(t) X(t) V_x(t, X(t), Y(t)) dB(t) \\
&= e^{-\beta(T-s)} \Psi(X(T), Y(T)) - \int_s^T e^{-\beta(T-s)} e^{-\alpha c(t) X(t)} dt \\
&\quad - \int_s^T e^{-\beta(T-s)} \sigma k(t) X(t) V_x(t, X(t), Y(t)) dB(t).
\end{aligned}$$

By virtue of condition (2.84), we have

$$\int_s^T e^{-\beta(T-s)}\sigma k(t)X(t)V_x(t,X(t),Y(t))dB(t)$$

a martingale. Taking expection on both sides, we obtain, $\forall(k,c)\in\Pi$

$$V(s,x,y) \;\geq\; -\mathbf{E}_{s,\varphi,k,c}\Bigg[\int_s^T e^{-\beta(T-s)}e^{-\alpha c(t)X(t)}dt + e^{-\beta(T-s)}\Psi(X(T),Y(T))\Bigg].$$

We have

$$V(s,x,y) \;\geq\; \sup_{k,c\geq 0}\mathbf{E}_{s,\varphi,k,c}\Bigg[\int_s^T e^{-\beta(T-s)}\left(-e^{-\alpha c(t)X(t)}\right)dt + e^{-\beta(T-s)}\Psi(X(T),Y(T))\Bigg].$$

By taking $(k,c)=(k^*,c^*)$, the inequalities can be replaced by equalities

$$V(s,x,y) \;=\; \mathbf{E}_{s,\varphi,k^*,c^*}\Bigg[\int_s^T e^{-\beta(T-s)}(-e^{-\alpha c^*(t)X(t)})dt + e^{-\beta(T-s)}\Psi(X(T),Y(T))\Bigg].$$

This completes the proof. $\square$

Theorem 2.4.4. *Assume that the utility function is given by*

$$U(x) = -e^{-\alpha x}, \quad \alpha>0,$$

and that the terminal function is given by

$$\Psi(x,y) = -\Lambda e^{-\gamma(x+\mu_3 e^{\lambda h}y)}$$

Suppose (2.69) *and* (2.70) *also hold. . Then the function* $V(s,x,y)$ *given by* (2.80) *is a classical solution of the HJB equation* (2.39)-(2.40)*, and it is equal to the value of the system defined by* (2.21)-(2.25)*, that is*

$$V(s,x,y) \;=\; \sup_{(k,c)\in\Pi}\mathbf{E}_{s,\varphi}\Bigg[\int_s^T e^{-\beta(T-s)}U(c(t)X(t))dt + e^{-\beta(T-s)}\Psi(X(T),Y(T))\Bigg].$$

In addition, the optimal control policy is given by

$$\begin{aligned} k^*(s) &= \frac{(\mu_1 - r)}{\alpha(r + \mu_3 e^{\lambda h})\sigma^2 X(s)}, \\ c^*(s) &= -\frac{1}{X(s)\alpha}\Big[\log\{(r + \mu_3 e^{\lambda h})Q(s)\} - \alpha(r + \mu_3 e^{\lambda h})(X(s) + \mu_3 e^{\lambda h} Y(s))\Big], \end{aligned}$$

where $Q(\cdot)$ is defined by (2.78)

Proof. It is evident from the derivation of $V(s, x, y)$ that $V(s, x, y)$ given by (2.80) is a classical solution of HJB equation (2.39)-(2.40). To use theorem (2.4.3), we first verify that condition (2.84) is satisfied. Using (2.80)

$$V_x(t, X(t), Y(t)) = \gamma Q(t) e^{-\gamma(X(t) + \mu_3 e^{\lambda h} Y(t))},$$

where $\gamma > 0$ is a constant given by (2.70). In addition, it is easy to verify that

$$0 < \frac{1 + \gamma x}{e^{\gamma x}} \le 1, \quad \forall x > 0.$$

So we can get

$$|e^{-\gamma(X(t) + \mu_3 e^{\lambda h} Y(t))}| \le \left| \frac{1}{1 + \gamma(X(t) + \mu_3 e^{\lambda h} Y(t))} \right|, \quad \forall t \in [s, T].$$

Therefore, by virtue of (2.79), we can get

$$|V_x(t, X(t), Y(t))| < \left| \frac{\gamma \Lambda}{1 + \gamma(X(t) + \mu_3 e^{\lambda h} Y(t))} \right|,$$

Using the definition of admissible control space Π, we have

$$|k^*(t)X(t)| \le \Lambda_1 |X(t) + \mu_3 Y(t)| \le \Lambda_1 |X(t) + \mu_3 e^{\lambda h} Y(t)|.$$

Therefore, we can get

$$\begin{aligned}
|k^*(t)X(t)V_x(t,X(t),Y(t))| &\le \Lambda_1 \left| \frac{\gamma\Lambda(X(t)+\mu_3 e^{\lambda h}Y(t))}{1+\gamma(X(t)+\mu_3 e^{\lambda h}Y(t))} \right| \\
&\le \Lambda_1\Lambda \left| 1 - \frac{1}{1+\gamma(X(t)+\mu_3 e^{\lambda h}Y(t))} \right| \\
&\le \Lambda_1\Lambda \left(1 + \left| \frac{1}{1+\gamma(X(t)+\mu_3 e^{\lambda h}Y(t))} \right| \right) \\
&\le 2\Lambda_1\Lambda.
\end{aligned}$$

where Λ_1, Λ are positive constants independent of t. Now we can get,

$$\begin{aligned}
\mathbf{E}\left[\int_0^T \left[k^*(t)X(t)V_x(t,X(t),Y(t)) \right]^2 dt \right] &\le \mathbf{E}\left[\int_0^T 4\Lambda_1^2\Lambda^2 dt \right] \\
&= 4\Lambda_1^2\Lambda^2 T < \infty.
\end{aligned}$$

Thus condition (2.84) is verified. By definition of (k^*, c^*), it is easy to see that $(k^*(t), c^*(t))$ is $\mathcal{F}^t$-measurable for any $t \in [0, T]$. Also, by virtue of (2.79), it is easy to get that $c^*(t) \ge 0, \forall t \in [0, T]$. The conditions given by (2.13) of admissible control space are also satisfied. Hence we have $(k^*, c^*) \in \Pi$. This completes the proof. $\square$

Chapter 3

An Infinite Time Horizon Portfolio Optimization Model with Bounded Memory

3.1 Introduction

In this chapter we consider a stochastic portfolio management problem on an infinite time horizon taking into account the history of the portfolio performance. Consider an investor's portfolio consisting of a risky asset and a riskless asset. The riskless asset earns the investor a fixed interest rate $r > 0$. For example, we can treat the money in a bank as the investment on the riskless asset. We assume that the investor can freely move her money between two assets at any time and her consumption comes from the riskless asset.

In this chapter, we consider a portfolio optimization problem for stochastic systems with delay variables given by (3.1) on an infinite time horizon. The problem is formulated as a stochastic control problem. The equation of the state variable $X(t)$ is given by (3.9) and (3.10) and the objective function is given by (3.20). The goal is to choose the optimal investment control $k(t)$ and the consumption control $c(t)$ to maximize the objective function to obtain the value function defined by (3.21). The solution of a stochastic control problem with delay is assumed to depend on the initial condition φ, which is in an infinite dimensional space $C[-h, 0]$, the space of all continuous functions on $[-h, 0]$. However, if the system only depends on the delay through process $Y(t)$ and $Z(t)$ defined by (5.1), it is possible to obtain a solution in a finite dimensional space.

Let $K(t)$ be the amount invested in the risky asset and $L(t)$ be the amount invested on the riskless asset. The net wealth is given by $X(t) = K(t)+L(t)$. We consider a situation where the performance of the risky asset has memory. We assume that the performance of the risky asset depends on the following delay variables $Y(t)$ and $Z(t)$:

$$Y(t) = \int_{-h}^{0} e^{\lambda\theta} X(t+\theta)d\theta, \quad Z(t) = X(t-h), \quad \forall t \in [0,\infty), \tag{3.1}$$

where $\lambda > 0$ is a constant and h is the delay parameter.

Let $\{B(t), t \geq 0\}$ be a standard one-dimensional standard Brownian motion defined on a complete filtered probability space $(\Omega, \mathcal{F}, P; \mathbf{F})$, where $\mathbf{F} = \{\mathcal{F}^t, t \geq 0\}$ is the P-augmented natural filtration generated by the Brownian motion $\{B(t), t \geq 0\}$. We assume that $K(t)$ and $L(t)$ follow the stochastic differential equations:

$$\begin{aligned} dK(t) &= \left[(\mu_1 + \mu_2 Y(t) + \mu_3 Z(t))\, K(t) + I(t)\right] dt + \sigma K(t) dB(t), && (3.2)\\ dL(t) &= [rL(t) - C(t) - I(t)]dt, && (3.3) \end{aligned}$$

where μ_1, μ_2, μ_3 and σ are positive constants, $I(t)$ is the investment rate on the risky asset at t, and $C(t)$ is the consumption rate. We consider the modified model formulated in chapter 2 over infinite time horizon.

The equation for $X(t)$ follows by using $X(t) = K(t) + L(t)$:

$$\begin{aligned} dX(t) = &\left[(\mu_1 + \mu_2 Y(t) + \mu_3 Z(t))\, K(t) + rL(t) - C(t)\right] dt \\ &+ \sigma K(t) dB(t), \quad \forall t \in [0,\infty). \end{aligned} \tag{3.4}$$

The initial condition is the information about $X(t)$ for $t \in [-h, 0]$:

$$X(t) = \varphi(t), \quad \forall t \in [-h, 0], \tag{3.5}$$

where $\varphi \in \mathbb{J}$ and $\mathbb{J} \equiv C[-h, 0]$, which is the space for all continuous functions defined on $[-h, 0]$ equipped with the sup norm

$$||\varphi|| = \sup_{\theta \in [-h,0]} |\varphi(\theta)|. \tag{3.6}$$

Let $L^2(\Omega, \mathbb{J})$ be a Banach space for all $(\mathbf{F}, \mathcal{B}(\mathbb{J}))$-measurable maps $\Omega \to \mathbb{J}$ that are in L^2 in the Bochner sense, where $\mathcal{B}(\mathbb{J})$ is the Borel $\sigma-$ field on $\mathbb{J}$. For any $\phi \in L^2(\Omega, \mathbb{J})$, the Banach norm is given by

$$||\phi(\omega)||_2 = \left[\int_\Omega ||\phi(\omega)||^2 dP(\omega)\right]^{1/2}, \tag{3.7}$$

where the norm $||\cdot||$ is given by (3.6).

To describe the allocation between the risky asset and the riskless asset, we now treat $K(t)$ and $C(t)$ as our control variables. As we can see in equation (3.4), the change of the wealth process $X(t)$ depends on the delay variable $Y(t)$. We consider the following modified model. For details see section 2.2.

$$\begin{aligned} dX(t) = {} & [\mu_1 K(t) + \mu_2 Y(t) + \mu_3 Z(t) + rL(t) - C(t)]\, dt \\ & + \sigma K(t) dB(t), \quad \forall t \in [0, \infty). \end{aligned} \tag{3.8}$$

Instead of using $K(t)$ and $C(t)$, we use $c(t) = \frac{C(t)}{X(t)}$ and $k(t) = \frac{K(t)}{X(t)}$ as our consumption and investment controls, respectively. Using $L(t) = X(t) - K(t) = X(t)(1 - k(t))$, we can rewrite the equation for $X(t)$ as

$$\begin{aligned} dX(t) = {} & [((\mu_1 - r)k(t) - c(t) + r)X(t) + \mu_2 Y(t) + \mu_3 Z(t)]dt \\ & + \sigma k(t) X(t) dB(t), \quad \forall t \in [0, \infty). \end{aligned} \tag{3.9}$$

Initial condition is given by

$$X(t) = \varphi(t), \quad \forall t \in [-h, 0], \tag{3.10}$$

where $\varphi \in \mathbb{J}$ and $\varphi(\theta) > 0, \forall \theta \in [-h, 0]$. Using the notation given by (2.11), initial condition (3.10) can be written as

$$X_0 = \varphi.$$

Definition 3.1.1 (Admissible Control Space)**.** *Let* Π *denote the admissible control space. A control policy* $(k(t), c(t))$ *is said to be in the admissible control space* Π *if it satisfies the following conditions:*

(a) $(k(t), c(t))$ *is* $\mathcal{F}^t-$*measurable for any* $t \in [0, \infty)$*;*

(b) $c(t) \geq 0, \forall t \in [0, \infty)$*;*

(c)

$$Pr\left(\int_0^T k^2(t)dt < \infty\right) = 1, \quad \forall T > 0, \tag{3.11}$$

$$|k(t)X(t)| \leq \Lambda \left|X(t) + Y(t)\right|, \quad \forall t > 0, \tag{3.12}$$

$$|c(t)X(t)| \leq \Lambda \left|X(t) + Y(t)\right|, \quad \forall t > 0, \tag{3.13}$$

where $\Lambda > 0$ *is a constant.*

Here we assume that the investment control $k(t)$ is square integrable (equation (3.11)), to ensure that the Ito's integral $\int_0^T k(t)dB(t)$ is well defined. On the other hand, the consumption control $c(t)$ only appears in the drift part, not the diffusion part, so we do not need a similar condition for $c(t)$.

We have the following result:

Lemma 1. *For any control* $(k(t), c(t)) \in \Pi$*, the equation* (3.9) *with initial condition* (3.10) *has a unique strong solution* $X : [-h, \infty) \times \Omega \to \mathbb{R}$*. Furthermore, for* $t \in [0, \infty), X_t \in \mathbb{J}$ *and for any positive integer* n*, we have*

$$\mathbf{E}\left[||X_t||^{2n}\right] \leq C_n[1 + ||\varphi||^{2n}], \quad \forall t \in [0, \infty), \tag{3.14}$$

where $C_n > 0$ *is a constant.*

Proof. Let $\|\varphi\|_2$ be the L^2 norm of φ as given in (3.7). Then, by virtue of

$$\|\varphi\|_2 \leq \|\varphi\|,$$

this lemma is a corollary of Theorem I.2 of Mohammed [41]. □

We have the following result:

Lemma 2. *The solution X(t) of the system* (3.9) -(3.10) *satisfies*

$$X(t) > 0, \quad Y(t) > 0, \quad Z(t) > 0, \quad \textit{almost surely} \quad \forall t \in [0, \infty).$$

Proof. By the definition of $Y(t), Z(t)$, it is sufficient to show that

$$X(t) > 0, \quad \textit{almost surely} \quad \forall t \in [-h, \infty). \tag{3.15}$$

For a given initial condition $\varphi(t) > 0, \forall t \in [-h, 0]$, we have

$$X(t) > 0, \quad \textit{almost surely} \quad \forall t \in [-h, 0], \quad Y(0) > 0, \quad Z(0) > 0.$$

Define a stopping time τ as the first time $X(t)$ hits zero:

$$\tau \equiv \inf_{t \geq 0}\{t : X(t) = 0\}.$$

To show that $X(t) > 0, a.s.$, it is sufficient to show that

$$Pr(\tau < \infty) = 0.$$

Let $\tilde{X}(t)$ be a solution of

$$\begin{aligned} d\tilde{X}(t) &= [((\mu_1 - r)k(t) - c(t) + r)\tilde{X}(t)]dt + \sigma k(t)\tilde{X}(t)dB(t), \quad \forall t \in [0, \infty), && (3.16) \\ \tilde{X}(t) &= \varphi(t), \quad \forall t \in [-h, 0], && (3.17) \end{aligned}$$

for the same $\varphi > 0$ as in (3.10). The solution is given as

$$\begin{aligned} \tilde{X}(t) = \varphi(0)\exp\Big\{ &\int_0^t ((\mu_1 - r)k(\theta) - c(\theta) + r - \frac{1}{2}\sigma^2 k^2(\theta))d\theta \\ &+ \int_0^t \sigma k(\theta)dB(\theta)\Big\}. \end{aligned} \tag{3.18}$$

Apparently, we have

$$\tilde{X}(t) > 0, \quad \text{a.s.} \quad \forall t \in [0, \infty).$$

Moreover, using Ito's formula, we get

$$
\begin{aligned}
d\left[\frac{1}{\tilde{X}(t)}\right] &= -\frac{d\tilde{X}(t)}{\tilde{X}(t)^2} + \frac{1}{2}\frac{2((d\tilde{X}(t))^2}{(\tilde{X}(t))^3} \\
&= -\frac{(\mu_1 - r)k(t) - c(t) + r}{\tilde{X}(t)}dt - \frac{\sigma k(t)}{\tilde{X}(t)}dB(t) + \frac{\sigma^2 k^2(t)}{\tilde{X}(t)}dt \\
&= \frac{\sigma^2 k^2(t) - [(\mu_1 - r)k(t) - c(t) + r]}{\tilde{X}(t)}dt - \frac{\sigma k(t)}{\tilde{X}(t)}dB(t). \qquad (3.19)
\end{aligned}
$$

Consider a new stochastic process $A(t)$ as given by (2.19) in Lemma 2.2.2. It is easy to see that $A(t) \geq 1, \quad \forall t \in [0, \tau]$.

By the definition of $Y(t), Z(t)$ and τ, it is easy to see that

$$Y(t) > 0, \quad Z(t) > 0, \; a.s. \quad \forall t \in [0, \tau].$$

So we can get

$$\frac{dA(t)}{dt} > 0, \quad \forall t \in [0, \tau].$$

Since $R(0) = \frac{X(0)}{\tilde{X}(0)} = \frac{\varphi(0)}{\varphi(0)} = 1$, we can get $A(t) \geq 1, \quad \forall t \in [0, \tau]$. Therefore, we have

$$X(t) \geq \tilde{X}(t) > 0, \quad a.s. \quad \forall t \in [0, \tau].$$

By the definition of τ, if $\tau < \infty$, we must have $X(\tau) = 0$, which is a contradiction with the above inequality. So we must have that

$$Pr(\tau < \infty) = 0.$$

This completes the proof. □

The utility function $U(C)$ is defined based on the consumption rate. The problem under consideration is a portfolio optimization problem on an infinite time horizon with the objective function given by

$$J(\varphi, k, c) = \mathbf{E}_{\varphi,k,c}\left[\int_0^\infty e^{-\beta t} U(c(t)X(t))dt\right]. \qquad (3.20)$$

Then the value function is given by

$$\begin{aligned} V(\varphi) &= \sup_{k,c\in\Pi} J(\varphi,k,c) \\ &= \sup_{k,c\in\Pi} \mathbf{E}_{\varphi,k,c}\left[\int_0^\infty e^{-\beta t} U(c(t)X(t))dt\right]. \end{aligned} \tag{3.21}$$

Note that $V(\varphi)$ is a functional defined on an infinite dimensional space $C[-h,0]$. We turn V into a function defined on a finite dimensional space as follows

$$V(\varphi) = V(x,y,z),$$

where $V : \mathbb{R}^3 \to \mathbb{R}$, and

$$x = x(\varphi) \equiv \varphi(0), \quad y = y(\varphi) \equiv \int_{-h}^0 e^{\lambda\theta}\varphi(\theta)d\theta, \quad z = z(\varphi) \equiv \varphi(-h). \tag{3.22}$$

Further, we assume that the value function V only depends on (x,y) i.e.

$$V(\varphi) = V(x,y,z) = V(x,y). \tag{3.23}$$

The reason of this assumption is given in Lemma 2.3.2.

In the following section, using the dynamic programming principle, we heuristically derive the Hamilton-Jacobi-Bellman equation for the value function in terms of the initial variables $x = X(0), y = Y(0)$. In section 3.3, we derive the explicit solutions for exponential utility, log utility and power utility (HARA utility with $\gamma \neq 0$), respectively. The optimal control polices and the verification results are established correspondingly.

3.2 Hamilton-Jacobi-Bellman Equation

We will need some kind of Ito's formula with respect to the function of the delay variable $Y(t)$. There are two methods for this purpose. One is the traditional method and the other one is to use the functional Ito's formula given in chapter 2. Although we can reach the same result via both methods, by using the functional Ito's formula, we can also establish a necessary condition that the HJB equation can be derived in a finite dimensional space.

In this chapter we use the traditional method to derive the Ito's formula for functions of $(X(t), Y(t))$. Let $f \in C^{2,1}(\mathbb{R}^2)$ and define

$$G = f(X(t), y(X_t)),$$

where

$$y(\eta) \equiv \int_{-h}^{0} e^{\lambda\theta}\eta(\theta)d\theta, \quad \forall \eta \in \mathbb{J}, \quad \text{and} \quad X_t(\theta) \equiv X(t+\theta), \quad \forall \theta \in [-h, 0].$$

Lemma 3.2.1 (Ito's Formula)**.** *Consider the system given by* (3.9)-(3.10)*. We have*

$$dG = \mathcal{G}^{k,c} f dt + \sigma k x f_x dB(t), \tag{3.24}$$

where

$$\begin{aligned} \mathcal{G}^{k,c} f = \mathcal{G}^{k,c} f(x,y) = & [((\mu_1 - r)k - c + r)x + \mu_2 y + \mu_3 z] f_x \\ & + \frac{1}{2}\sigma^2 k^2 x^2 f_{xx} + f_y \cdot [x - e^{-\lambda h} z - \lambda y] dt, \end{aligned} \tag{3.25}$$

where x, y, z, k and c are evaluated as

$$\begin{aligned} x = X(t), \quad y = y(X_t) = \int_{-h}^{0} e^{\lambda\theta} X(t+\theta), \quad z = z(X_t) = X(t-h), \\ k = k(t), \quad c = c(t). \end{aligned}$$

Proof. The idea of the proof is very similar to that of Lemma 2.1 in Elsanousi-Øksendal-Sulem [11].

Consider $X_t(\theta) = X(t+\theta), \quad \forall \theta \in [-h, 0]$ and $y = y(X_t)$. We have

$$
\begin{aligned}
\frac{d}{dt} y(X_t(\cdot)) &= \frac{d}{dt}\left[\int_{-h}^{0} e^{\lambda\theta} X_t(\theta) d\theta\right] = \frac{d}{dt}\left[\int_{-h}^{0} e^{\lambda\theta} X(t+\theta) d\theta\right] \\
&= \frac{d}{dt}\left[\int_{t-h}^{t} e^{\lambda(u-t)} X(u) du\right] \quad (\text{let} \quad u \equiv \theta + t) \\
&= X(t) - e^{-\lambda h} X(t-h) - \lambda \int_{t-h}^{t} e^{\lambda(u-t)} X(u) du \\
&= X(t) - e^{-\lambda h} X(t-h) - \lambda \int_{-h}^{0} e^{\lambda\theta} X(t+\theta) d\theta \quad (\text{let} \quad \theta = u - t) \\
&= x - e^{-\lambda h} z - \lambda y.
\end{aligned} \tag{3.26}
$$

Applying classical Ito's formula to $G = f(X(t), y(X_t))$, the result follows. □

On the other hand, we can use the method of functional Ito's formula. Recall that we use $X(t)$ to denote the current value and we use $X_t : [-h, 0] \to \mathbb{R}$ to denote the path of $X(t)$ from $t-h$ to t. For details see section 2.3. Now we assume that the value function V only depends on (x, y) as given by (3.23). We will derive the HJB equation satisfied by the value function $V(x, y)$ heuristically. To derive the HJB equation, we need the following dynamic programming principle.

Lemma 3.2.2 (Dynamic Programming Principle)**.** *Assume that the value function $V(x, y)$ given by* (3.21), (3.23) *is well defined and assume the system given by* (3.9)-(3.10). *Then we have*

$$
V(x, y) = \sup_{(k,c)\in\Pi} \mathbf{E}_{x,y,k,c}\left[\int_0^t e^{-\beta\tau} U(c(\tau)) X(\tau) d\tau + e^{-\beta t} V(X(t), Y(t))\right] \tag{3.27}
$$

for all $\mathcal{F}^t$-stopping time $t \in [0, \infty)$ and $(x, y) \in \mathbb{R}^2$, where $x = x(\varphi) = \varphi(0)$ and $y = y(\varphi) = \int_{-h}^{0} e^{\lambda\theta} \varphi(\theta) d\theta$.

Proof. The proof is similar to the proof of Theorem 4.2 of Larssen [31] and we omit it here. □

Now we can use the dynamic programming principle given by Lemma 3.2.2 to heuristically derive the HJB equation. In particular, if we assume that $V(x, y)$ is smooth enough, then we can use the Ito's formula to get

$$
\begin{aligned}
d\left[e^{-\beta t}V(X(t),Y(t))\right] &= e^{-\beta t}\left[dV(X(t),Y(t)) - \beta V(X(t),Y(t))dt\right] \\
&= e^{-\beta t}\Big[\left(\mathcal{G}^{k,c}V(X(t),Y(t)) - \beta V(X(t),Y(t))\right)dt \\
&\qquad + \sigma k(t)X(t)V_x\left(t,X(t),Y(t)\right)dB(t)\Big],
\end{aligned}
$$

where $\mathcal{G}^{k,c}$ is given by (3.25). Integrate it over $[0,T]$, and we can get

$$
\begin{aligned}
e^{-\beta T}V(X(T),Y(T)) - V(x,y) &= \int_0^T e^{-\beta t}\left[\mathcal{G}^{k,c}V(X(t),Y(t)) - \beta V(X(t),Y(t))\right]dt \\
&\quad + \int_0^T e^{-\beta t}\sigma k(t)X(t)V_x\left(t,X(t),Y(t)\right)dB(t) \quad (3.28)
\end{aligned}
$$

Assume that $\int_0^T e^{-\beta t}\sigma k(t)X(t)V_x\left(t,X(t),Y(t)\right)dB(t)$ is a martingale, then we can get

$$
\begin{aligned}
\lim_{T\to 0}\frac{1}{T}\mathbf{E}_{x,y,k,c}\left[e^{-\beta t}V(X(t),Y(t)) - V(x,y)\right] &= \lim_{T\to 0}\frac{1}{T}\int_0^T e^{-\beta t}\mathbf{E}_{x,y,k,c}\Big[\mathcal{G}^{k,c}V(X(t),Y(t)) \\
&\qquad - \beta V(X(t),Y(t))\Big]dt \\
&= \mathcal{G}^{k,c}V(x,y) - \beta V(x,y). \qquad (3.29)
\end{aligned}
$$

On the other hand, from the equation (3.27), we can get

$$
\begin{aligned}
0 &= \lim_{T\to 0}\sup_{k,c\in\Pi}\mathbf{E}_{x,y,k,c}\Big[\frac{1}{T}(e^{-\beta t}V(X(t),Y(t)) - V(x,y)) \\
&\qquad + \frac{1}{T}\int_0^T e^{-\beta t}U(c(t))X(t)dt\Big] \\
&= \lim_{T\to 0}\sup_{k,c\in\Pi}\mathbf{E}_{x,y,k,c}\left[\frac{1}{T}(e^{-\beta t}V(X(t),Y(t)) - V(x,y)) + U(cx)\right]. \qquad (3.30)
\end{aligned}
$$

Together with equation (3.29), we can get the following HJB equation:

$$
\sup_{k,c}[\mathcal{G}^{k,c}V(x,y) - \beta V(x,y) + U(cx)] = 0
$$

Since $\mathcal{G}^{k,c}$ is given by (3.25), we can write the HJB equation as the following:

$$\beta V = \max_k \left[\frac{1}{2}(\sigma kx)^2 V_{xx} + (\mu_1 - r)kxV_x\right] + (rx + \mu_2 y + \mu_3 z)V_x + \max_{c\geq 0}\left[-cxV_x + U(cx)\right] + (x - \lambda y - e^{-\lambda h}z)V_y, \quad \forall z \in \mathbb{R}. \tag{3.31}$$

So we have heuristically derived the HJB equation for our stochastic control problem. For details and general theory about dynamic programming principle and HJB equations, please see Fleming-Rishel [17] or Yong-Zhou [52].

In this chapter, we will consider utility functions of exponential, logarithmic and HARA type. For each utility function, we will find an explicit solution of the corresponding HJB equation and we will verify that the solution is equivalent to the value function by establishing the verification theorem. The optimal control polices will be derived, too. The optimal control problem is then solved with the explicit value function and the optimal controls.

We want to point out that our goal is to solve the stochastic control problem to get the value function and the optimal controls while the HJB equation we derived *heuristically* just serves as an intermediate vehicle to find the explicit form of the value function and the optimal control. Therefore, we do not need to formally show that the value function is a classical or viscosity solution of the above HJB equation. Further, we do not need to show that the HJB equation has a unique solution, either.

3.3 The Solution of the HJB equation

In this section we solve the HJB equation (3.31) for each exponential, logarithamic and HARA utility function and give the verification theorems for each of those utility functions.

3.3.1 Exponential Utility

We consider the exponential utility function given as

$$U(cX) = 1 - e^{-\alpha cX}, \tag{3.32}$$

where $\alpha > 0$ is a constant. The additive term 1 has no mathematical relevance except it restricts the range of the function between 0 and 1. Therefore for technical convenience we consider

$$U(cX) = -e^{-\alpha c X}, \quad \alpha > 0. \tag{3.33}$$

For this exponential utility function, the HJB equation (3.31) now becomes

$$\begin{aligned} \beta V &= \max_k \left[\frac{1}{2}(\sigma k x)^2 V_{xx} + (\mu_1 - r)kxV_x\right] + (rx + \mu_2 y + \mu_3 z)V_x \\ &\quad + \max_{c \geq 0}\left[-cxV_x - e^{-\alpha c X}\right] + (x - e^{-\lambda h}z - \lambda y)V_y, \end{aligned} \tag{3.34}$$

First, we will solve the above equation and then we will verify that the solution is equal to the value function. By the first order condition, we can get that the candidates for optimal controls are

$$k^* = -\frac{(\mu_1 - r)V_x}{\sigma^2 x V_{xx}}, \quad c^* = -\frac{1}{\alpha x}\log\left(\frac{V_x}{\alpha}\right). \tag{3.35}$$

Substituting k^* and c^* in equation (3.34), we obtain

$$\begin{aligned} \beta V &= -\frac{(\mu_1 - r)^2 V_x^2}{2\sigma^2 V_{xx}} + \frac{V_x}{\alpha}\left[\log\left(\frac{V_x}{\alpha}\right) - 1\right] \\ &\quad + (rx + \mu_2 y + \mu_3 z)V_x + \left(x - \lambda y - e^{-\lambda h}z\right)V_y. \end{aligned} \tag{3.36}$$

The above equation can be rewritten as

$$\begin{aligned} \beta V &= -\frac{(\mu_1 - r)^2 V_x^2}{2\sigma^2 V_{xx}} + \frac{V_x}{\alpha}\left[\log\left(\frac{V_x}{\alpha}\right) - 1\right] + (rV_x + V_y)x \\ &\quad + (\mu_2 V_x - \lambda V_y)y + (\mu_3 V_x - e^{-\lambda h}V_y)z. \end{aligned} \tag{3.37}$$

Since we are looking for a solution that is independent of z, we will look for a solution in forms of

$$V(x, y) = \eta_1 e^{\eta_2 u}. \tag{3.38}$$

where η_1, η_2 are two constants to be determined, and u is defined by

$$u \equiv x + \mu_3 e^{\lambda h} y. \tag{3.39}$$

Under this assumption, we can get

$$V_x = \eta_2 V, \quad V_{xx} = \eta_2^2 V, \quad V_y = \mu_3 e^{\lambda h} \eta_2 V. \tag{3.40}$$

So it is easy to verify that

$$\mu_3 V_x - e^{-\lambda h} V_y = 0.$$

Plug (3.40) into (3.37) and we can get

$$\begin{aligned} \beta V &= -\frac{(\mu_1 - r)^2}{2\sigma^2} V + \frac{\eta_2 V}{\alpha} \left[\log\left(\frac{\eta_2 V}{\alpha}\right) - 1 \right] \\ &\quad + (r + \mu_3 e^{\lambda h}) x \eta_2 V + (\mu_2 - \lambda \mu_3 e^{\lambda h}) \eta_2 y V. \end{aligned} \tag{3.41}$$

As we can see, the term involving z has vanished. Cancel V from both sides and we can get

$$\begin{aligned} \beta &= -\frac{(\mu_1 - r)^2}{2\sigma^2} + \frac{\eta_2}{\alpha} \left[\log\left(\frac{\eta_2 V}{\alpha}\right) - 1 \right] \\ &\quad + (r + \mu_3 e^{\lambda h}) x \eta_2 + (\mu_2 - \lambda \mu_3 e^{\lambda h}) \eta_2 y. \end{aligned}$$

Plug in (3.38), and we have

$$\begin{aligned} \beta &= -\frac{(\mu_1 - r)^2}{2\sigma^2} + \frac{\eta_2}{\alpha} \left[\log\left(\frac{\eta_1 \eta_2}{\alpha}\right) + \eta_2 u - 1 \right] \\ &\quad + (r + \mu_3 e^{\lambda h}) x \eta_2 + (\mu_2 - \lambda \mu_3 e^{\lambda h}) \eta_2 y. \end{aligned} \tag{3.42}$$

Now assume that

$$\mu_2 - \lambda \mu_3 e^{\lambda h} = (r + \mu_3 e^{\lambda h}) \mu_3 e^{\lambda h} \tag{3.43}$$

Then equation (4.29) can be written as

$$\begin{aligned} \beta &= -\frac{(\mu_1 - r)^2}{2\sigma^2} + \frac{\eta_2}{\alpha} \left[\log\left(\frac{\eta_1 \eta_2}{\alpha}\right) - 1 \right] \\ &\quad + \frac{\eta_2^2}{\alpha} u + (r + \mu_3 e^{\lambda h}) \eta_2 u. \end{aligned} \tag{3.44}$$

We pick η_2 such that

$$\frac{\eta_2^2}{\alpha} + (r + \mu_3 e^{\lambda h}) \eta_2 = 0.$$

One solution is the trivial solution $\eta_2 = 0$ and the other solution is

$$\eta_2 = -\alpha(r + \mu_3 e^{\lambda h}). \tag{3.45}$$

Then equation (3.44) becomes

$$\beta = -\frac{(\mu_1 - r)^2}{2\sigma^2} + \frac{\eta_2}{\alpha}\left[\log\left(\frac{\eta_1\eta_2}{\alpha}\right) - 1\right].$$

We can solve the above equation to get η_1:

$$\begin{aligned}
\eta_1 &= \frac{\alpha}{\eta_2}\exp\left(\frac{\alpha}{\eta_2}\left[\beta + \frac{(\mu_1 - r)^2}{2\sigma^2}\right] + 1\right) \\
&= -\frac{1}{r + \mu_3 e^{\lambda h}}\exp\left(1 - \frac{\beta + \frac{(\mu_1 - r)^2}{2\sigma^2}}{r + \mu_3 e^{\lambda h}}\right).
\end{aligned} \tag{3.46}$$

Therefore the solution of the HJB equation (3.34) is given by

$$\begin{aligned}
V(x,y) &= \eta_1 e^{\eta_2 u} \\
&= -\frac{1}{r + \mu_3 e^{\lambda h}}\exp\left(1 - \frac{\beta + \frac{(\mu_1 - r)^2}{2\sigma^2}}{r + \mu_3 e^{\lambda h}}\right) e^{-\alpha(r + \mu_3 e^{\lambda h})(x + \mu_3 e^{\lambda h} y)}.
\end{aligned} \tag{3.47}$$

The optimal investment and the optimal consumption controls are

$$k^*(t) = -\frac{(\mu_1 - r)V_x}{\sigma^2 X(t) V_{xx}} = -\frac{(\mu_1 - r)}{\sigma^2 X(t)\eta_2} = \frac{(\mu_1 - r)}{\alpha\sigma^2 X(t)(r + \mu_3 e^{\lambda h})}, \tag{3.48}$$

$$\begin{aligned}
c^*(t) &= -\frac{1}{\alpha X(t)}\log\left(\frac{V_x}{\alpha}\right) = -\frac{1}{\alpha X(t)}\left[\log\left(\frac{\eta_2 V}{\alpha}\right)\right] \\
&= -\frac{1}{\alpha X(t)}\left[\log\left(\frac{\eta_1\eta_2}{\alpha}\right) + \eta_2(X(t) + \mu_3 e^{\lambda h} Y(t))\right] \\
&= \frac{\beta + \frac{(\mu_1 - r)^2}{2\sigma^2} - (r + \mu_3 e^{\lambda h})}{\alpha X(t)(r + \mu_3 e^{\lambda h})} \\
&\quad + \frac{(r + \mu_3 e^{\lambda h})\big(X(t) + \mu_3 e^{\lambda h} Y(t)\big)}{X(t)}.
\end{aligned} \tag{3.49}$$

To ensure that $c^* > 0$, we assume that

$$\beta + \frac{(\mu_1 - r)^2}{2\sigma^2} > r + \mu_3 e^{\lambda h}. \tag{3.50}$$

Next, we will establish the verification theorem to make sure that the solution given by (3.47) is equal to the value function defined by (3.21). In addition, we will verify that $k^*(t)$, $c^*(t)$ defined by (3.48)-(3.49) are the optimal policies.

Theorem 3.3.1 (Verification Theorem)**.** *Assume that* (3.43) *and* (3.50) *hold. Let* $V(x,y)$ *be defined by* (3.47)*. Then* $V(x,y) \in C^{2,1}(\mathbb{R} \times \mathbb{R})$ *and it is a solution of* (3.34) *such that*

$$\mathbf{E}\left[\int_0^T \Big(k(t)X(t)V_x(X(t),Y(t))\Big)^2 dt\right] < \infty, \quad \forall (k,c) \in \Pi, \quad \forall T > 0. \tag{3.51}$$

In addition, we have

(a) $V(x,y) \geq J(x,y;k,c)$ *for any admissible progressively measurable control process* $(k(t),c(t)) \in \Pi$.

(b) Assume that $k^*(t)$, $c^*(t)$ *are defined by* (3.48)-(3.49)*. Then* $k^*(t), c^*(t) \in \Pi$ *and* $V(x,y) = J(x,y;k^*,c^*)$.

Proof. We first verify that condition (3.51) is satisfied. By the definition of $V(x,y)$, we can get

$$V_x(X(t),Y(t)) = \eta_1\eta_2 e^{\eta_2(X(t)+\mu_3 e^{\lambda h}Y(t))},$$

where η_1, η_2 are defined by (3.46) and (3.45), respectively. By the definitions of η_1 and η_2, we can see that $\eta_1\eta_2 > 0$. In addition, by virtue of the fact that $X(t) > 0, Y(t) > 0$ and $\eta_2 < 0$, it is not hard to get that

$$e^{\eta_2(X(t)+\mu_3 e^{\lambda h}Y(t))} \leq \frac{1}{1 + |\eta_2|(X(t) + \mu_3 e^{\lambda h}Y(t))}, \quad \forall t \in [0,\infty).$$

Therefore we have,

$$|V_x(X(t),Y(t))| \leq \eta_1\eta_2 \left|\frac{1}{1 + |\eta_2|(X(t) + \mu_3 e^{\lambda h}Y(t))}\right|.$$

Using the definition of admissible control space Π, we have

$$|k(t)X(t)| \leq \Lambda|X(t)+Y(t)| \leq \Lambda_1|X(t)+\mu_3 e^{\lambda h}Y(t)|,$$

where

$$\Lambda_1 = \Lambda \max\left\{\frac{1}{\mu_3 e^{\lambda h}}, 1\right\}.$$

Therefore, we have

$$\begin{aligned}
|k(t)X(t)V_x(X(t),Y(t))| &\leq \Lambda_1\eta_1\eta_2\left|\frac{X(t)+\mu_3 e^{\lambda h}Y(t)}{1+|\eta_2|(X(t)+\mu_3 e^{\lambda h}Y(t))}\right| \\
&\leq \Lambda_1\eta_1\eta_2\left|\frac{1}{|\eta_2|}\left(1-\frac{1}{1+|\eta_2|(X(t)+\mu_3 e^{\lambda h}Y(t))}\right)\right| \\
&\leq \frac{\Lambda_1\eta_1\eta_2}{|\eta_2|} = -\Lambda_1\eta_1,
\end{aligned}$$

where $-\Lambda_1\eta_1 > 0$ is a constant independent of t. Thus it is easy to check that condition (3.51) holds.

Let $\mathcal{G}^{k,c}$ defined by (3.25) be the generator of the process $(X(t), Y(t))$ for any control process $(k(t), c(t)) \in \Pi$. Then by using Ito's rule

$$\begin{aligned}
d\left[e^{-\beta t}V(X(t),Y(t))\right] &= e^{-\beta t}\left[dV(X(t),Y(t)) - \beta V(X(t),Y(t))dt\right] \\
&= e^{-\beta t}\Big[\left(\mathcal{G}^{k,c}V(X(t),Y(t)) - \beta V(X(t),Y(t))\right)dt \\
&\qquad + \sigma k(t)X(t)V_x\left(t, X(t), Y(t)\right)dB(t)\Big].
\end{aligned}$$

Integrating it on $[0,T]$ and noting that $V(x,y)$ is a classical solution of (3.34), we have

$$\begin{aligned}
& e^{-\beta T}V(X(T),Y(T)) - V(x,y) \\
\leq & \int_0^T e^{-\beta t}e^{-\alpha c X(t)}dt + \int_0^T e^{-\beta t}\sigma k(t)X(t)V_x\left(t, X(t), Y(t)\right)dB(t).
\end{aligned} \tag{3.52}$$

By virtue of (3.51), $\int_0^T e^{-\beta t}\sigma k(t)X(t)V_x\left(t, X(t), Y(t)\right)dB(t)$ is a martingale. Then by taking expectation on both sides, for every finite T we have

$$V(x,y) \geq -\mathbf{E}\left[\int_0^T e^{-\beta t}e^{-\alpha c X(t)}dt\right] + \mathbf{E}\left[e^{-\beta T}V(X(T),Y(T))\right], \tag{3.53}$$

where $V(\cdot,\cdot)$ is given by (3.47). By (3.47) and noting that $\eta_1 < 0, \eta_2 < 0$, we have

$$\begin{aligned} \mathbf{E}\left[\left|e^{-\beta T}V(X(T),Y(T))\right|\right] &= \mathbf{E}\left[\left|e^{-\beta T}\eta_1 e^{\eta_2(X(T)+\mu_3 Y(T))}\right|\right] \\ &\leq \mathbf{E}\left[|\eta_1|e^{-\beta T}\right] = |\eta_2|e^{-\beta T}. \end{aligned} \tag{3.54}$$

Therefore, by taking $T \to \infty$, we have $\mathbf{E}\left[e^{-\beta T}V(X(T),Y(T))\right] \to 0$. Then we have

$$V(x,y) \geq \mathbf{E}\left[\int_0^\infty e^{-\beta t}\left(-e^{-\alpha c X(t)}\right)dt\right] \tag{3.55}$$

for any admissible control $(k(t),c(t)) \in \Pi$. This proves (a).

On the other hand, for $k^*(t), c^*(t)$ defined by (3.48) and (3.49), we can easily verify that $\mathcal{F}^t$ measurable. In addition, from Lemma 2, we know that $X(t) > 0$, *a.s.*, so we can get that $k^*(t), c^*(t)$ are well defined and and $c^*(t) \geq 0$. In addition, it is not hard to check that (3.12) and (3.13) are true for $k^*(t)$ and $c^*(t)$. Finally, since $X(t) > 0$ are well defined, we can get that

$$Pr(k^*(t) < \infty) = 1, \quad \forall t \geq 0.$$

Therefore, we can get

$$Pr\left(\int_0^T (k^*(t))^2 dt < \infty\right) = 1, \quad \forall T > 0.$$

So we can get that $(k^*(t), c^*(t)) \in \Pi$. With $(k(t),c(t)) = (k^*(t),c^*(t))$ all the inequalities up to inequality (3.53) in the first part can be replaced by equalities. Then instead of (3.53), we have

$$V(x,y) = -\mathbf{E}\left[\int_0^T e^{-\beta t}e^{-\alpha c^*(t)X^*(t)}dt\right] + \mathbf{E}\left[e^{-\beta T}V(X^*(T),Y^*(T))\right]. \tag{3.56}$$

Taking $T \to \infty$ and using (3.54), we have

$$V(x,y) = \mathbf{E}\left[\int_0^\infty e^{-\beta t}\left(-e^{-\alpha c^* X^*(t)}\right)dt\right].$$

Therefore we have $V(x,y) = J(x,y;k^*,c^*)$. This proves (b). $\square$

3.3.2 Logarithmic Utility

We consider a logarithmic utility function given by

$$U(cX) = \log\big(cX\big). \tag{3.57}$$

Now the HJB equation (3.31) becomes

$$\begin{aligned} \beta V &= \max_k \left[\frac{1}{2}(\sigma k x)^2 V_{xx} + (\mu_1 - r)kxV_x\right] + (rx + \mu_2 y + \mu_3 z)V_x \\ &\quad + \max_{c\geq 0}\left[-cxV_x + \log(cx)\right] + \left(x - e^{-\lambda h}z - \lambda y\right)V_y, \end{aligned} \tag{3.58}$$

The candidates for optimal controls are

$$k^* = -\frac{(\mu_1 - r)V_x}{\sigma^2 x V_{xx}}, \quad c^* = \frac{1}{xV_x}. \tag{3.59}$$

Substituting k^* and c^* in (3.58), we obtain

$$\begin{aligned} \beta V &= -\frac{1}{2}\frac{(\mu_1 - r)^2 V_x^2}{\sigma^2 V_{xx}} + \log\left(\frac{1}{V_x}\right) - 1 \\ &\quad + (rx + \mu_2 y + \mu_3 z)V_x + \left(x - \lambda y - e^{-\lambda h}z\right)V_y, \end{aligned} \tag{3.60}$$

To cancel the items involving z, we assume that

$$u \equiv x + \mu_3 e^{\lambda h} y, \tag{3.61}$$

and we look for the solution of the form

$$V(x, y) = \eta_1 \log(u) + \eta_2 \tag{3.62}$$

where η_1, η_2 are two constants to be determined. It is easy to see that

$$V_x = \frac{\eta_1}{u}, \quad V_{xx} = -\frac{\eta_1}{u^2}, \quad V_y = \frac{\eta_1 \mu_3 e^{\lambda h}}{u}. \tag{3.63}$$

By plugging (3.62) and (3.63) into (5.25) we get

$$\begin{aligned} \beta(\eta_1 \log u + \eta_2) &= \frac{1}{2}\frac{(\mu_1 - r)^2 \eta_1}{\sigma^2} - \log \eta_1 + \log u - 1 \\ &\quad + (rx + \mu_2 y)\frac{\eta_1}{u} + (x - \lambda y)\,\frac{\mu_3 e^{\lambda h}\eta_1}{u} \end{aligned} \tag{3.64}$$

Assume that

$$\eta_1 = \frac{1}{\beta} \tag{3.65}$$

and

$$\mu_3 e^{\lambda h}(r + \mu_3 e^{\lambda h}) = \mu_2 - \mu_3 \lambda e^{\lambda h}. \tag{3.66}$$

Then we can plug (3.65) into (3.64) and use (3.66) to cancel u. The explicit formula for η_2 is

$$\eta_2 = \frac{1}{\beta^2}\left[\frac{(\mu_1 - r)^2}{2\sigma^2} + \beta \log \beta - \beta + \left(r + \mu_3 e^{\lambda h}\right)\right]. \tag{3.67}$$

Therefore (3.58) has a solution

$$V(x, y) = \eta_2 + \frac{1}{\beta}\log\left(x + \mu_3 e^{\lambda h} y\right), \tag{3.68}$$

where η_2 is given by (5.26). The optimal investment and consumption control policies are

$$k^*(t) = \frac{(\mu_1 - r)(X(t) + \mu_3 e^{\lambda h} Y(t))}{\sigma^2 X(t)}, \quad c^*(t) = \frac{\beta(X(t) + \mu_3 e^{\lambda h} Y(t))}{X(t)}. \tag{3.69}$$

Now we verify that $V(x, y)$ is the maximum expected discounted utility and k^*, c^* are the optimal policies.

Theorem 3.3.2 (Verification Theorem)**.** *Assume that condition* (3.66) *holds and let* $V(x, y)$ *be given by* (3.68). *Then* $V(x, y) \in C^{2,1}(\mathbb{R} \times \mathbb{R})$ *and it is a solution of* (3.58) *such that*

$$\mathbf{E}\left[\int_0^T \left(k(t)X(t)V_x(X(t), Y(t))\right)^2 dt\right] < \infty \quad \forall k \in \Pi, \quad \forall T > 0. \tag{3.70}$$

Moreover, we have

(a) $V(x, y) \geq J(x, y; k, c)$ *for any admissible control process* $(k(t), c(t)) \in \Pi$.

(b) If k^*, c^* are given by (3.69)*, then $k^*(t), c^*(t) \in \Pi$ and $V(x,y) = J(x,y;k^*,c^*)$.*

Proof. By the construction of the function $V(x,y)$, it is easy to check that $V(x,y) \in C^{2,1}(\mathbb{R}\times\mathbb{R})$ and it is a solution of (3.58). Next we verify the condition (3.70). Using (3.68), we have

$$|V_x(X(t),Y(t))| = \frac{1}{\beta}\left|\frac{1}{X(t)+\mu_3 e^{\lambda h}Y(t)}\right|$$

Using the definition of admissible control space Π, we have

$$|k(t)X(t)| \le \Lambda|X(t)+Y(t)| \le \Lambda_1|X(t)+\mu_3 e^{\lambda h}Y(t)|,$$

where

$$\Lambda_1 = \Lambda \max\left\{\frac{1}{\mu_3 e^{\lambda h}}, 1\right\}.$$

Therefore, we have

$$|k(t)X(t)V_x(X(t),Y(t))| \le \frac{\Lambda_1}{\beta}|X(t)+\mu_3 e^{\lambda h}Y(t)|\cdot\left|\frac{1}{X(t)+\mu_3 e^{\lambda h}Y(t)}\right| = \frac{\Lambda_1}{\beta}.$$

Then we have,

$$\mathbf{E}\left[\int_0^T [k(t)X(t)V_x(t,X(t),Y(t))]^2 dt\right] \le \mathbf{E}\left[\int_0^T \frac{\Lambda_1^2}{\beta^2}dt\right] = \frac{\Lambda_1^2 T}{\beta^2} < \infty.$$

Thus condition (3.70) is verified.

Let $\mathcal{G}^{k,c}$ be the generator of the process $(X(t),Y(t))$ for any control process $(k(t),c(t)) \in \Pi$. Then by using Ito's rule

$$\begin{aligned} d\left[e^{-\beta t}V(X(t),Y(t))\right] &= e^{-\beta t}\left[dV(X(t),Y(t)) - \beta V(X(t),Y(t))dt\right] \\ &= e^{-\beta t}\left[\mathcal{G}^{k,c}V(X(t),Y(t)) - \beta V(X(t),Y(t))\right]dt \\ &\quad + e^{-\beta t}\sigma k(t)X(t)V_x(X(t),Y(t))dB(t). \end{aligned}$$

Integrating above on $[0,T]$ and also noting that $V(x,y)$ is classical solution of (3.58), we have

$$
\begin{aligned}
e^{-\beta T} V(X(T), Y(T)) - V(x, y) \quad &\leq \quad - \int_0^T e^{-\beta t} \log(c(t)X(t)) dt \\
&\quad + \int_0^T e^{-\beta t} \sigma k(t) X(t) V_x(X(t), Y(t)) dB(t).
\end{aligned}
$$

By virtue of (3.70), $\int_0^T e^{-\beta t} \sigma k(t) X(t) V x(t, X(t), Y(t)) dB(t)$ is a martingale. Then we have

$$
V(x, y) \geq \mathbf{E}\left[\int_0^T e^{-\beta t} \log(c(t)X(t)) dt\right] + \mathbf{E}\left[e^{-\beta T} V(X(T), Y(T))\right]. \tag{3.71}
$$

Using (3.68), we have

$$
\begin{aligned}
& \lim_{T \to \infty} \mathbf{E}\left[e^{-\beta T} V(X(T), Y(T))\right] \\
= \quad & \lim_{T \to \infty} \mathbf{E}\left[e^{-\beta T}\left(\frac{1}{\beta} \log\left(X(T) + \mu_3 e^{\lambda h} Y(T)\right) + \eta_2\right)\right],
\end{aligned}
$$

where η_2 is given by (5.26). To show that

$$
\lim_{T \to \infty} \mathbf{E}\left[e^{-\beta T} V(X(T), Y(T))\right] = 0,
$$

it is sufficient to show

$$
\lim_{T \to \infty} \mathbf{E}\left[e^{-\beta T} \log\left(X(T) + \mu_3 e^{\lambda h} Y(T)\right)\right] = 0. \tag{3.72}
$$

Let

$$
S(t) = X(t) + \mu_3 e^{\lambda h} Y(t). \tag{3.73}
$$

Then using (3.66), we have

$$
\begin{aligned}
dS(t) \quad &= \quad dX(t) + \mu_3 e^{\lambda h} dY(t) \\
&= \quad \left[\left((\mu_1 - r)k(t) - c(t)\right) X(t) + (r + \mu_3 e^{\lambda h}) S(t)\right] dt + \sigma k(t) X(t) dB(t).
\end{aligned}
$$

Using Ito's rule, we have

$$\begin{aligned} d\log S(t) &= \frac{dS(t)}{S(t)} - \frac{1}{2}\frac{(dS(t))^2}{(S(t))^2} \\ &= \left[((\mu_1 - r)k(t) - c(t))\,\frac{X(t)}{S(t)} + (r + \mu_3 e^{\lambda h}) \right. \\ &\quad \left. - \frac{1}{2}\sigma^2 k^2(t)\left(\frac{X(t)}{S(t)}\right)^2 \right] dt + \sigma k(t)\frac{X(t)}{S(t)} dB(t). \end{aligned} \tag{3.74}$$

Since $(k(t), c(t)) \in \Pi$, it is easy to verify that $\int_0^T \sigma k(t)\frac{X(t)}{S(t)} dB(t)$ is a martingale. Therefore, we can get

$$\begin{aligned} \mathbf{E}\left[\log S(T)\right] &= \log(S(0)) + \mathbf{E}\left[\int_0^T ((\mu_1 - r)k(t) - c(t))\,\frac{X(t)}{S(t)} dt\right] \\ &\quad + \int_0^T (r + \mu_3 e^{\lambda h}) dt - \mathbf{E}\left[\int_0^T \frac{1}{2}\sigma^2 k^2(t)\left(\frac{X(t)}{S(t)}\right)^2 dt\right]. \end{aligned} \tag{3.75}$$

For any $(k(t), c(t)) \in \Pi$, by virtue of (3.12) and (3.13), we have

$$\begin{aligned} \left| k(t)\frac{X(t)}{S(t)} \right| &\le \frac{\Lambda\,|X(t) + Y(t)|}{|X(t) + \mu_3 e^{\lambda h} Y(t)|} \le \Lambda_1, \\ \left| c(t)\frac{X(t)}{S(t)} \right| &\le \frac{\Lambda\,|X(t) + Y(t)|}{|X(t) + \mu_3 e^{\lambda h} Y(t)|} \le \Lambda_1, \end{aligned}$$

where

$$\Lambda_1 = \Lambda \max\left\{ \frac{1}{\mu_3 e^{\lambda h}}, 1 \right\}.$$

Then we can get

$$\begin{aligned} &\lim_{T\to\infty} e^{-\beta T}\mathbf{E}\left[\int_0^T ((\mu_1 - r)k(t))\,\frac{X(t)}{S(t)} dt\right] = 0, \\ &\lim_{T\to\infty} e^{-\beta T}\int_0^T \left(r + \mu_3 e^{\lambda h}\right) dt = 0, \\ &\lim_{T\to\infty} e^{-\beta T}\mathbf{E}\left[\int_0^T (-c(t)\frac{X(t)}{S(t)}) dt\right] = 0, \\ &\lim_{T\to\infty} e^{-\beta T}\mathbf{E}\left[\int_0^T \left(-\frac{1}{2}\sigma^2 k^2(t)\left(\frac{X(t)}{S(t)}\right)^2\right) dt\right] = 0. \end{aligned}$$

Then from (3.75), we can get

$$\lim_{T\to\infty} e^{-\beta T}\mathbf{E}[\log(S(T))] = 0.$$

Thus, we have

$$\lim_{T\to\infty} \mathbf{E}\left[e^{-\beta T} V\left(X(T), Y(T)\right)\right] = 0, \quad \forall (k(t), c(t)) \in \Pi.$$

Combined with (3.71), this implies (a).

Now assume that k^*, c^* are given by (3.69). We can easily verify that they are $\mathcal{F}^t$-measurable. In addition, from Lemma 2, we know that $X(t) > 0,\ a.s.$, so we can get that $k^*(t), c^*(t)$ are well defined and $c^*(t) \geq 0$. In addition, it is not hard to check that (3.12) and (3.13) are true for $k^*(t)$ and $c^*(t)$. Finally, since $X(t) > 0$ and $k^*(t)$ is well defined, we can get that

$$Pr(k^*(t) < \infty) = 1, \quad \forall t \geq 0.$$

Therefore, we can get

$$Pr\left(\int_0^T (k^*(t))^2 dt < \infty\right) = 1, \quad \forall T > 0.$$

So we can get that $(k^*(t), c^*(t)) \in \Pi$. Moreover, the equation for $X(t)$ now is

$$\begin{aligned} dX^*(t) &= \Bigg[\left(\frac{(\mu_1 - r)^2}{\sigma^2} - \beta\right)(X^*(t) + \mu_3 e^{\lambda h} Y^*(t)) + rX^*(t) + \mu_2 Y^*(t) \\ &\quad + \mu_3 Z^*(t)\Bigg] dt + \frac{\mu_1 - r}{\sigma}(X^*(t) + \mu_3 e^{\lambda h} Y^*(t)) dB(t). \qquad (3.76) \end{aligned}$$

Let

$$S^*(t) = X^*(t) + \mu_3 e^{\lambda h} Y^*(t). \qquad (3.77)$$

Using assumption (3.66), we have

$$\begin{aligned} dS^*(t) &= dX^*(t) + \mu_3 e^{\lambda h} dY^*(t) \\ &= \Bigg[\left(\frac{(\mu_1 - r)^2}{\sigma^2} - \beta \right) (X^*(t) + \mu_3 e^{\lambda h} Y^*(t)) + rX^*(t) + \mu_2 Y^*(t) \\ &\qquad + \mu_3 Z^*(t) \Bigg] dt + \frac{\mu_1 - r}{\sigma} (X^*(t) + \mu_3 e^{\lambda h} Y^*(t)) dB(t) \\ &\quad + \mu_3 e^{\lambda h} \left[X^*(t) - \lambda Y^*(t) - e^{-\lambda h} Z^*(t) \right] dt \\ &= \left[\left(\frac{(\mu_1 - r)^2}{\sigma^2} - \beta \right) S^*(t) + rX^*(t) + \mu_2 Y^*(t) + \mu_3 Z^*(t) \right] dt \\ &\quad + \frac{\mu_1 - r}{\sigma} S^*(t) dB(t) + \mu_3 e^{\lambda h} \left[X^*(t) - \lambda Y^*(t) - e^{-\lambda h} Z^*(t) \right] dt \\ &= \left[\frac{(\mu_1 - r)^2}{\sigma^2} - \beta + r + \mu_3 e^{\lambda h} \right] S^*(t) dt + \frac{\mu_1 - r}{\sigma} S^*(t) dB(t). \end{aligned} \tag{3.78}$$

The solution is

$$\begin{aligned} S^*(t) &= S^*(0) e^{\left[\frac{(\mu_1 - r)^2}{\sigma^2} - \beta + (r + \mu_3 e^{\lambda h}) - \frac{(\mu_1 - r)^2}{2\sigma^2} \right] t + \frac{\mu_1 - r}{\sigma} B(t)} \\ &= (x + \mu_3 e^{\lambda h} y) e^{\left[\frac{(\mu_1 - r)^2}{2\sigma^2} - \beta + r + \mu_3 e^{\lambda h} \right] t + \frac{\mu_1 - r}{\sigma} B(t)}. \end{aligned} \tag{3.79}$$

Using $(k^*(t), c^*(t))$ as controls, similar to (3.71), now we can get

$$V(x, y) = \mathbf{E} \left[\int_0^T e^{-\beta t} \log(c^*(t) X^*(t)) dt \right] + \mathbf{E} \left[e^{-\beta T} V(X^*(T), Y^*(T)) \right]. \tag{3.80}$$

By virtue of (3.68) and (3.79), we can get

$$\begin{aligned} V(X^*(T), Y^*(T)) &= \frac{1}{\beta} \log \Big(X^*(T) + \mu_3 e^{\lambda h} Y^*(T) \Big) + \eta_2, \\ &= \frac{1}{\beta} \log(S^*(T)) + \eta_2 \\ &= \frac{1}{\beta} \log(x + \mu_3 e^{\lambda h} y) + \left[\frac{(\mu_1 - r)^2}{2\sigma^2} - \beta + r + \mu_3 e^{\lambda h} \right] T \\ &\quad + \frac{\mu_1 - r}{\sigma} B(T) + \eta_2. \end{aligned} \tag{3.81}$$

Then it is easy to show that

$$\lim_{T\to\infty} \mathbf{E}\left[e^{-\beta T} V(X^*(T), Y^*(T)\right] = 0. \tag{3.82}$$

Hence we have $V(x,y) = J(x,y;k^*,c^*)$. Therefore (b) is proved. This completes the proof. $\square$

3.3.3 HARA Utility

In this section we consider HARA utility function

$$U(cX) = \frac{1}{\gamma}(cX)^{\gamma}, \quad 0 < \gamma < 1. \tag{3.83}$$

The HJB equation (3.31) can now be written as

$$\begin{aligned} \beta V &= \max_k \left[\frac{1}{2}(\sigma k x)^2 V_{xx} + (\mu_1 - r) k x V_x\right] + (rx + \mu_2 y + \mu_3 z) V_x \\ &\quad + \max_{c\geq 0}\left[-cxV_x + \frac{1}{\gamma}(cx)^{\gamma}\right] + \left(x - e^{-\lambda h} z - \lambda y\right) V_y, \end{aligned} \tag{3.84}$$

The candidates for optimal controls are

$$k^* = -\frac{(\mu_1 - r)V_x}{\sigma^2 x V_{xx}}, \quad c^* = \frac{1}{x} V_x^{\frac{1}{\gamma-1}}. \tag{3.85}$$

Substituting k^* and c^* in (3.84), we obtain

$$\begin{aligned} \beta V &= -\frac{1}{2}\frac{(\mu_1 - r)^2 V_x^2}{\sigma^2 V_{xx}} + \left(\frac{1}{\gamma} - 1\right) V_x^{\frac{\gamma}{\gamma-1}} + (rx + \mu_2 y + \mu_3 z) V_x \\ &\quad + \left(x - \lambda y - e^{-\lambda h} z\right) V_y, \end{aligned} \tag{3.86}$$

Suppose solution is of the form

$$V(x,y) = \frac{1}{\gamma}\eta u^{\gamma} \tag{3.87}$$

where $u \equiv x + \mu_3 e^{\lambda h} y$. Now we have

$$V_x = \eta u^{\gamma-1}, \quad V_{xx} = (\gamma - 1)\eta u^{\gamma-2}, \quad V_y = \mu_3 e^{\lambda h} \eta u^{\gamma-1}. \tag{3.88}$$

By substituting (3.87) and (3.88) in (3.86) we get

$$\begin{aligned}\frac{\beta}{\gamma}\eta u^{\gamma} &= -\frac{1}{2}\frac{(\mu_1-r)^2}{(\gamma-1)\,\sigma^2}\eta u^{\gamma}+\left(\frac{1}{\gamma}-1\right)\eta^{\frac{\gamma}{\gamma-1}}u^{\gamma}\\ &\quad+(rx+\mu_2 y)\eta u^{\gamma-1}+(x-\lambda y)\,\mu_3 e^{\lambda h}\eta u^{\gamma-1}\end{aligned}\tag{3.89}$$

Assume that

$$\mu_3 e^{\lambda h}(r+\mu_3 e^{\lambda h})=\mu_2-\mu_3\lambda e^{\lambda h}.\tag{3.90}$$

Then the explicit formula for η is

$$\eta=\left(\frac{\gamma}{1-\gamma}\left(\frac{\beta}{\gamma}-\frac{(\mu_1-r)^2}{2\sigma^2(1-\gamma)}-\left(r+\mu_3 e^{\lambda h}\right)\right)\right)^{\gamma-1}.\tag{3.91}$$

Assume that

$$\frac{\beta}{\gamma}>\frac{(\mu_1-r)^2}{2\sigma^2(1-\gamma)}+(r+\mu_3 e^{\lambda h}).\tag{3.92}$$

Then it is easy to check that $\eta>0$. Therefore the solution of the HJB equation (3.84) is given by

$$V(x,y)=\frac{1}{\gamma}\eta\left(x+\mu_3 e^{\lambda h}y\right)^{\gamma},\tag{3.93}$$

The optimal investment and consumption control policies are

$$k^*(t)=\frac{(\mu_1-r)(X(t)+\mu_3 e^{\lambda h}Y(t))}{(1-\gamma)\sigma^2 X(t)},\quad c^*(t)=\frac{\eta^{\frac{1}{\gamma}}\left(X(t)+\mu_3 e^{\lambda h}Y(t)\right)}{X(t)},\tag{3.94}$$

where η is given by (3.91). It remains to verify that $V(x,y)$ is equal to the value function and k^*, c^* are the optimal policies. We give the verification in the following theorem.

Theorem 3.3.3 (Verification Theorem)**.** *Assume that the condition* (3.90) *and* (3.92) *hold. Let* $V(x,y)$ *be given by* (3.93)*. Then* $V(x,y)\in C^{2,1}(\mathbb{R}\times\mathbb{R})$ *and it is a solution of* (3.84) *such that*

$$\mathbf{E}\left[\int_0^T\Big(k(t)X(t)V_x(t,X(t),Y(t))\Big)^2dt\right]<\infty,\quad\forall k\in\Pi,\quad\forall T>0.\tag{3.95}$$

Further, we have

(a) $V(x,y)\geq J(x,y;k,c)$ *for any admissible progressively measurable control process*

$(k(t), c(t)) \in \Pi$.

(b) If $k^*(t), c^*(t)$ *are given by* (3.94)*, then* $k^*(t), c^*(t) \in \Pi$ *and*

$$V(x, y) = J(x, y; k^*, c^*). \tag{3.96}$$

Proof. By the construction of $V(x, y)$, it is easy to check that $V(x, y) \in C^{2,1}(\mathbb{R} \times \mathbb{R})$ and it is a solution of (3.84). Next we verify the condition (3.95). Using (3.93), we have

$$V_x(X(t), Y(t)) = \eta(X(t) + \mu_3 e^{\lambda h} Y(t))^{\gamma-1}$$

where $\eta > 0$ is a constant. Using $\eta > 0$, we can get

$$|V_x(X(t), Y(t))| = \eta \left|(X(t) + \mu_3 e^{\lambda h} Y(t))^{\gamma-1}\right|.$$

Using the definition of admissible control space Π, we have

$$|k(t)X(t)| \le \Lambda |X(t) + Y(t)| \le \Lambda_1 |X(t) + \mu_3 e^{\lambda h} Y(t)|,$$

where

$$\Lambda_1 = \Lambda \max \left\{ \frac{1}{\mu_3 e^{\lambda h}}, 1 \right\}.$$

Therefore, we have

$$|k(t)X(t)V_x(t, X(t), Y(t))| \;\; \le \;\; \Lambda_1 \eta \left|X(t) + \mu_3 e^{\lambda h} Y(t)\right|^{\gamma}$$

where $\Lambda_1 \eta > 0$ is a constant. Now using the definition of $Y(t)$,

$$\begin{aligned} Y(t) &= \int_{-h}^{0} e^{\lambda h} X(t+\theta) d\theta \le \int_{-h}^{0} X(t+\theta) d\theta \\ &\le h \max_{\theta \in [-h, 0]} |X(t+\theta)| \le h \|X_t\|. \end{aligned}$$

Therefore, noting that $|X(t)| \le \|X_t\|$, we have

$$\begin{aligned} |k(t)X(t)V_x(t, X(t), Y(t))| &\le \Lambda_1 \eta \left|X(t) + \mu_3 e^{\lambda h} Y(t)\right|^{\gamma} \\ &\le \Lambda_2 \|X_t\|^{\gamma}. \end{aligned}$$

where $\Lambda_2 > 0$ is a constant independent of t. Therefore, by (3.14) and noting that $2\gamma < 2$, we can get

$$
\begin{aligned}
& \mathbf{E}\left[\int_0^T \Big(k(t)X(t)V_x(t, X(t), Y(t))\Big)^2 dt\right] \\
\leq\ & \mathbf{E}\left[\int_0^T \Lambda_2^2 \|X_t\|^{2\gamma} dt\right] \leq \mathbf{E}\left[\int_0^T \Lambda_2^2 (1 + \|X_t\|)^{2\gamma} dt\right] \\
\leq\ & \mathbf{E}\left[\int_0^T \Lambda_2^2 (1 + \|X_t\|)^2 dt\right] \leq \mathbf{E}\left[\int_0^T \Lambda_2^2 (2 + 2\|X_t\|^2) dt\right] \\
\leq\ & \Lambda_2^2 \left[\int_0^T \mathbf{E}[2 + 2\|X_t\|^2] dt\right] \\
\leq\ & \Lambda_2^2 \left(2T + \int_0^T 2C_2(1 + \|\varphi\|^2) dt\right) < \infty.
\end{aligned}
$$

This verifies condition (3.95).

Let $\mathcal{G}^{k,c}$ be the generator of the process $(X(t), Y(t))$ for any control process $(k(t), c(t)) \in \Pi$. Then by using Ito's rule

$$
\begin{aligned}
d\left[e^{-\beta t} V(X(t), Y(t))\right] &= e^{-\beta t}\left[dV(X(t), Y(t)) - \beta V(X(t), Y(t)) dt\right] \\
&= e^{-\beta t}\left[\mathcal{G}^{k,c} V(X(t), Y(t)) - \beta V(X(t), Y(t))\right] dt \\
&\quad + e^{-\beta t} \sigma k(t) X(t) V_x(t, X(t), Y(t)) dB(t).
\end{aligned}
$$

Integrating above on $[0, T]$ and also noting that $V(x, y)$ is classical solution of (3.84), we have

$$
\begin{aligned}
& e^{-\beta T} V(X(T), Y(T)) - V(x, y) \\
\leq\ & \int_0^T e^{-\beta t} \sigma k(t) X(t) V_x(t, X(t), Y(t)) dB(t) - \int_0^T e^{-\beta t} \frac{1}{\gamma}(cx)^\gamma dt,
\end{aligned}
$$

where, by virtue of (3.95), $\int_0^T e^{-\beta t} \sigma k(t) X(t) V_x(t, X(t), Y(t)) dB(t)$ is local martingale under P. Then we have

$$
V(x, y) \geq \mathbf{E}\left[\int_0^T e^{-\beta t} \frac{1}{\gamma}(cx)^\gamma dt\right] + \mathbf{E}\left[e^{-\beta T} V(X(T), Y(T))\right] \tag{3.97}
$$

Since $V(x,y) \geq 0$, we have

$$\limsup_{T\to\infty} e^{-\beta T}\mathbf{E}\left[V(X(T),Y(T))\right] \geq 0.$$

Then, taking $T \to \infty$ in (3.97), we can get

$$V(x,y) \geq \mathbf{E}\left[\int_0^\infty e^{-\beta t}\frac{1}{\gamma}(cx)^\gamma dt\right] \tag{3.98}$$

Hence $V(x,y) \geq J(x,y;k,c)$ for all admissible $(k(t),c(t))$. This proves (a).

Now assume that k^*, c^* are given by (3.94). We note that $k^*(t)$ and $c^*(t)$ are $\mathcal{F}^t$-measurable. In addition, from Lemma 2, we know that $X(t) > 0,\ a.s.$, so we can get that $k^*(t), c^*(t)$ are well defined and $c^*(t) \geq 0$. In addition, it is not hard to check that (3.12) and (3.13) are true for $k^*(t)$ and $c^*(t)$. Finally, since $X(t) > 0$ and $k^*(t)$ is well defined, we get

$$Pr(k^*(t) < \infty) = 1, \quad \forall t \geq 0.$$

Therefore, we can get

$$Pr\left(\int_0^T (k^*(t))^2 dt < \infty\right) = 1, \quad \forall T > 0.$$

So we can get that $(k^*(t), c^*(t)) \in \Pi$.

Using $(k^*(t), c^*(t))$ as controls, instead of (3.97), now we can get

$$V(x,y) = \mathbf{E}\left\{\frac{1}{\gamma}\int_0^T e^{-\beta t}(c^*(t)X^*(t))^\gamma dt + e^{-\beta T}V\left(X^*(T),Y^*(T)\right)\right\}. \tag{3.99}$$

Next we will show that

$$\lim_{T\to\infty}\mathbf{E}\left[e^{-\beta T}V^*(X^*(T),Y^*(T))\right] = 0. \tag{3.100}$$

Let

$$S^*(t) = X^*(t) + \mu_3 e^{\lambda h}Y^*(t). \tag{3.101}$$

Using (3.93), we have

$$V(X^*(T),Y^*(T)) = \frac{1}{\gamma}\eta\left(X^*(T) + \mu_3 e^{\lambda h}Y^*(T)\right)^\gamma = \frac{1}{\gamma}\eta\left(S^*(T)\right)^\gamma. \tag{3.102}$$

where η is given by (3.91). To show equation (3.100), it is sufficient to show

$$\lim_{T\to\infty} \mathbf{E}\left[e^{-\beta T}\left(S^*(T)\right)^\gamma\right] = 0. \tag{3.103}$$

Using assumption (3.90), we have

$$\begin{aligned}
dS^*(t) &= dX^*(t) + \mu_3 e^{\lambda h} dY^*(t) \\
&= \Bigg[\left(\frac{(\mu_1 - r)^2}{(1-\gamma)\sigma^2} - \eta^{\frac{1}{\gamma}}\right)\left(X^*(t) + \mu_3 e^{\lambda h} Y^*(t)\right) \\
&\qquad + rX^*(t) + \mu_2 Y^*(t) + \mu_3 Z^*(t)\Bigg] dt \\
&\quad + \frac{\mu_1 - r}{(1-\gamma)\sigma}\left(X^*(t) + \mu_3 e^{\lambda h} Y^*(t)\right) dB(t) \\
&\quad + \mu_3 e^{\lambda h}\left[X^*(t) - \lambda Y^*(t) - e^{-\lambda h} Z^*(t)\right] dt \\
&= \left[\left(\frac{(\mu_1 - r)^2}{(1-\gamma)\sigma^2} - \eta^{\frac{1}{\gamma}}\right) S^*(t) + rX^*(t) + \mu_2 Y^*(t) + \mu_3 Z^*(t)\right] dt \\
&\quad + \frac{\mu_1 - r}{(1-\gamma)\sigma} S^*(t) dB(t) + \mu_3 e^{\lambda h}\left[X^*(t) - \lambda Y^*(t) - e^{-\lambda h} Z^*(t)\right] dt \\
&= \left[\frac{(\mu_1 - r)^2}{(1-\gamma)\sigma^2} - \eta^{\frac{1}{\gamma}} + r + \mu_3 e^{\lambda h}\right] S^*(t) dt + \frac{\mu_1 - r}{(1-\gamma)\sigma} S^*(t) dB(t).
\end{aligned}$$

The solution is

$$\begin{aligned}
S^*(t) &= S^*(0) e^{\left[\frac{(\mu_1 - r)^2}{(1-\gamma)\sigma^2} - \eta^{\frac{1}{\gamma}} + (r + \mu_3 e^{\lambda h})\right] t} e^{\left[-\frac{(\mu_1 - r)^2}{2(1-\gamma)^2\sigma^2}\right] t + \frac{\mu_1 - r}{(1-\gamma)\sigma} B(t)} \\
&= (x + \mu_3 e^{\lambda h} y) e^{\left[\frac{(\mu_1 - r)^2}{(1-\gamma)\sigma^2} - \eta^{\frac{1}{\gamma}} + (r + \mu_3 e^{\lambda h})\right] t} e^{\left[-\frac{(\mu_1 - r)^2}{2(1-\gamma)^2\sigma^2}\right] t + \frac{\mu_1 - r}{(1-\gamma)\sigma} B(t)}.
\end{aligned}$$

Then we have

$$\begin{aligned}
[S^*(t)]^\gamma &= (x + \mu_3 e^{\lambda h} y)^\gamma \exp\left(\gamma\left[\frac{(\mu_1 - r)^2}{(1-\gamma)\sigma^2} - \eta^{\frac{1}{\gamma}} + (r + \mu_3 e^{\lambda h})\right] t\right) \\
&\quad \cdot \exp\left(\gamma\left[-\frac{(\mu_1 - r)^2}{2(1-\gamma)^2\sigma^2}\right] t + \frac{\gamma(\mu_1 - r)}{(1-\gamma)\sigma} B(t)\right).
\end{aligned} \tag{3.104}$$

Take the expectation, and we can get

$$
\begin{aligned}
\mathbf{E}\left[(S^*(t))^\gamma\right] &= (x+\mu_3 e^{\lambda h}y)^\gamma \exp\left(\gamma\left[\frac{(\mu_1-r)^2}{(1-\gamma)\sigma^2}-\eta^{\frac{1}{\gamma}}+r+\mu_3 e^{\lambda h}\right]t\right) \\
&\quad \cdot \exp\left(\gamma\left[-\frac{(\mu_1-r)^2}{2(1-\gamma)^2\sigma^2}\right]t+\frac{1}{2}\frac{\gamma^2(\mu_1-r)^2}{(1-\gamma)^2\sigma^2}t\right) \\
&= (x+\mu_3 e^{\lambda h}y)^\gamma \exp\left(\gamma\left[\frac{(\mu_1-r)^2}{2(1-\gamma)\sigma^2}-\eta^{\frac{1}{\gamma}}+(r+\mu_3 e^{\lambda h})\right]t\right).
\end{aligned}
$$

By virtue of (3.92), we can get that $\eta > 0$ and

$$
-\beta+\gamma\left[\frac{(\mu_1-r)^2}{2(1-\gamma)\sigma^2}-\eta^{\frac{1}{\gamma}}+(r+\mu_3 e^{\lambda h})\right] < 0.
$$

Therefore, we can get

$$
\begin{aligned}
&\lim_{T\to\infty} e^{-\beta T}\mathbf{E}\left[(S^*(T))^\gamma\right] \\
&= (x+\mu_3 e^{\lambda h}y)^\gamma \lim_{T\to\infty}\left[e^{\left[-\beta+\gamma\left[\frac{(\mu_1-r)^2}{2(1-\gamma)\sigma^2}-\eta^{\frac{1}{\gamma}}+(r+\mu_3 e^{\lambda h})\right]\right]T}\right] \\
&= 0.
\end{aligned}
$$

Thus (3.103) is established. So we can get (3.100). Then, by virtue of (3.99), we can get that $V(x,y) = J(x,y;k^*,c^*)$. This completes the proof of (b). $\square$

Chapter 4

A Stochastic Portfolio Optimization Model with Complete Memory

4.1 Introduction

In chapter 2 and chapter 3, we considered stochastic systems with memory given by the following system:

$$\begin{cases} dX(t) &= b(t, X_t, u(t))dt + \sigma(t, X_t, u(t))dW(t), \ t \in [0, T], \\ X(t) &= \varphi(t), \qquad t \in [-h, 0], \end{cases}$$

where $h > 0$ is a fixed constant, $X_t : [-h, 0] \mapsto \mathbb{R}$ is the memory variable defined by $X_t(\theta) \equiv X(t + \theta)$, and it is the segment of the path from $t - h$ to t, $\varphi \in C[-h, 0]$ is the initial path and u is the control in some admissible control space Π. The presence of the Fréchet derivative makes the stochastic control problems with memory given by X_t very complicated and working in an infinite dimensional space limits its application in practice. On the other hand, as we mentioned earlier, when investors look at the historical performance, instead of the whole path, moving average or exponential moving average is usually used.

$$\begin{cases} dX(t) &= b(t, X(t), Y(t), Z(t), u(t))dt + \sigma(t, X(t), Y(t), Z(t), u(t))dW(t), \\ X(t) &= \varphi(t), \qquad t \in [-h, 0], \end{cases}$$

with memory variables $Y(t), Z(t)$ given by

$$Y(t) = \int_{-h}^{0} e^{\lambda\theta} X(t+\theta) d\theta, \quad Z(t) = X(t-h), \tag{4.1}$$

where $\lambda \geq 0$ is a parameter. As we can see, $Y(t)$ is the exponential moving average of $X(t)$ over $[t-h, t]$ if $\lambda > 0$ and it is the moving average if $\lambda = 0$. $Z(t)$ is a historical value of $X(t)$. For the system given above, the value function V will depend on the initial variable φ through memory variables (x, y, z) that are given by

$$x \equiv X(0) = \varphi(0), \quad y \equiv Y(0) = \int_{-h}^{0} e^{\lambda\theta} \varphi(\theta) d\theta, \quad z \equiv Z(0) = \varphi(-h),$$

and we can derive the HJB equation for V in the classical sense in a finite dimensional space.

In this chapter (and chapter 5), we introduce idea of complete memory or infinite delay in a stochastic portfolio management model. In particular, we consider a stochastic portfolio management model on a finite time horizon and the model incorporates all the historic performance of the portfolio. The state equation describing the model is a stochastic delay differential equation. In this model, investor's portfolio comprises a risky and a riskless asset. The value of the portfolio follows a stochastic process $X(t)$ which satisfies a delay equation of the form

$$\begin{aligned} dX(t) &= f(t, X(t), Y(t), u(t))dt + \sigma(t, X(t), u(t)) \, dB(t), \quad \forall t \in [s, T] \\ X(t) &= \varphi(t), \quad \forall t \in (-\infty, s], \end{aligned}$$

where $u(t)$ is the control in some control space Π applied at time t, and the $Y(t)$ is defined by

$$Y(t) \equiv \int_{-\infty}^{0} e^{\lambda\theta} X(t+\theta) d\theta. \tag{4.2}$$

where $\lambda > 0$ is a constant, and φ is a function which gives the initial path. We will give a more precise formulation of the problem in the next section. The main contribution in our model is that we deal with infinite delay i.e. the model captures the complete memory. One motivation to consider the complete memory variable $Y(t)$ given by (4.2) is that investors tend to consider all available historical information of a stock instead of the information for the past 10 or 50 days. Another motivation is that with the complete memory, we can skip the memory variable $Z(t)$ and z will not appear in the HJB equation any more. This will make the model more feasible and applicable. Another thing we want to point out is that in reality, the historical information does not go back to $-\infty$, but we still use $Y(t)$ given by (4.2) for convenience. See Remark 5.2.1.

Consider an investor's portfolio consisting of a risky asset and a riskless asset. An example of riskless asset is money deposited in a bank account that earns a fixed interest rate $r > 0$. In this model the complete performance of the risky asset is taken into account. We also assume that the investor can freely move his money between two assets at any time and his consumption comes from the riskless asset.

Let $K(t)$ be the amount invested in the risky asset and $L(t)$ is the amount invested on the riskless asset. The total wealth is given by $X(t) = K(t) + L(t)$. We assume that the performance of the risky asset depends on the following delay variable $Y(t)$:

$$Y(t) = \int_{-\infty}^{0} e^{\lambda\theta} X(t+\theta) d\theta, \tag{4.3}$$

where $\lambda > 0$ is a constant.

Remark 4.1.1. *From the definition of $Y(t)$, its value is actually the exponential average of all the historical values of the total wealth $X(t)$. We want to point out that in reality, there is always a historical time that the value of X is available. In other words, the initial path $\varphi(t)$ will start at some point $t = -M$ instead of $t = -\infty$. In this case, we simply assume that $\varphi(t) = 0, \forall t \in (\infty, -M)$, and we can still use the formula (4.3) to define the delay variable $Y(t)$. In addition, $Y(t)$ is still a continuous function with respect to t for $t \geq s$.*

Let $\{B(t), t \geq 0\}$ be a one-dimensional standard Brownian motion defined on a complete filtered probability space $(\Omega, \mathcal{F}, P; \mathbf{F})$, where $\mathbf{F} = \{\mathcal{F}^t, t \geq 0\}$ is the P-augmented natural filtration generated by the Brownian motion $\{B(t), t \geq 0\}$. We assume that $K(t)$ and $L(t)$ follow the stochastic differential equations:

$$\begin{aligned} dK(t) &= [(\mu_1 + \mu_2 Y(t))K(t) + I(t)]dt + \sigma K(t) dB(t), & (4.4) \\ dL(t) &= [rL(t) - C(t) - I(t)]dt, & (4.5) \end{aligned}$$

where μ_1, μ_2, and σ are positive constants, $I(t)$ is the investment rate on the risky asset at t, and $C(t)$ is the consumption rate. The equation for $X(t)$ follows by using

$X(t) = K(t) + L(t)$:

$$\begin{aligned} dX(t\quad) = & \quad [(\mu_1 + \mu_2 Y(t))\, K(t) + rL(t) - C(t)]\, dt \\ & + \sigma K(t) dB(t), \quad \forall t \in [s, T], \end{aligned} \tag{4.6}$$

with the initial condition given by the information about $X(t)$ for $t \in (-\infty, s]$:

$$X(t) = \varphi(t), \quad \forall t \in (-\infty, s], \tag{4.7}$$

where φ is a positive, bounded function which is integrable on $(-\infty, s]$.

As we can see, we introduce a delay (or memory) variable $Y(t)$ in the state equations for $K(t)$ and $X(t)$. The reason is that an investor usually will look at her portfolio performance to determine whether to allocate more money to the risky asset. While all the investors act this way, a higher value of $Y(t)$, the historical moving average, will increase the risky asset demand, so it tends to push the price of the risky asset even higher.

To describe the allocation between the risky asset and the riskless asset, we now treat $K(t)$ and $C(t)$ as our control variables. The state variables are $X(t)$ and $Y(t)$. As we can see in Equation (4.6), the change of the wealth process $X(t)$ depends on the delay variable $Y(t)$. For technical reasons, we modify the model described in (4.6)

$$dX(t) = [\mu_1 K(t) + \mu_2 Y(t) + rL(t) - C(t)]\, dt + \sigma K(t) dB(t), \quad \forall t \in [s, T]. \tag{4.8}$$

Remark 4.1.2. *If we assume that $K(t) > 0$ almost surely, we can use the following delay variable $\widetilde{Y}(t)$:*

$$\widetilde{Y}(t) = \frac{1}{K(t)} \int_{-\infty}^{0} e^{\lambda\theta} X(t+\theta) d\theta, \quad t \in [s, T],$$

instead of (4.3)*, so we can reach* (4.8)*.*

Instead of using $K(t)$ and $C(t)$, we use $c(t) = \frac{C(t)}{X(t)}$ and $k(t) = \frac{K(t)}{X(t)}$ as our consumption and investment controls, respectively. In Lemma 4.1.1, we will show that $X(t) > 0$, *a.s.*, so that $c(t), k(t)$ are well defined.

Using $L(t) = X(t) - K(t) = X(t)(1 - k(t))$, the equation for $X(t)$ can be written as

$$\begin{aligned} dX(t) &= [((\mu_1 - r)k(t) - c(t) + r)X(t) + \mu_2 Y(t)]dt \\ &\quad + \sigma k(t)X(t)dB(t), \forall t \in [s, T]. \end{aligned} \tag{4.9}$$

Recall that the initial condition is given by

$$X(t) = \varphi(t), \quad \forall t \in (-\infty, s], \tag{4.10}$$

where φ is a bounded function and $\varphi(\theta) > 0, \forall \theta \in (-\infty, s]$. The above initial condition implies the following initial conditions:

$$X(s) = x \equiv \varphi(s), \quad Y(s) = y \equiv \int_{-\infty}^{0} e^{\lambda\theta}\varphi(s + \theta)d\theta. \tag{4.11}$$

$X(t)$ stands for the total wealth, and it should remain positive all the time. Actually, we can show that the solution $X(t)$ of (4.9)-(4.11) is strictly positive almost surely (see Lemma 4.1.1). Before we show the result, let us first define the admissible control space.

Definition 4.1.1 (Admissible Control Space)**.** *Let* Π *denote the admissible control space. A control policy* $(k(t), c(t))$ *is said to be in the admissible control space* Π *if it satisfies the following conditions:*

(a) $(k(t), c(t))$ *is* $\mathcal{F}^t-$*measurable for any* $t \in [0, T]$*;*

(b)

$$Pr\left(\int_0^T k^2(t)dt < \infty\right) = 1.$$

(c) $c(t) \geq 0, \forall t \in [0, T]$*;*

(d)

$$|k(t)X(t)| \leq \Lambda_0 |X(t) + Y(t)|, \quad \forall t > 0, \tag{4.12a}$$

$$|c(t)X(t)| \leq \Lambda_0 |X(t) + Y(t)|, \quad \forall t > 0, \tag{4.12b}$$

where $\Lambda_0 > 0$ *is a constant.*

We have the following lemma.

Lemma 4.1.1. *The solution $X(t)$ of the system* (4.9)-(4.10) *satisfies*

$$X(t) > 0, \quad Y(t) > 0 \quad \textit{almost surely} \quad \forall t \in [0, T].$$

Proof. First, for $t \in [0, s]$, noting that the initial condition $\varphi(\theta) > 0, \forall \theta \in -\infty, s]$, it is easy to see that

$$X(t) = \varphi(t) > 0, \quad Y(t) = \int_{-\infty}^{0} e^{\lambda\theta} X(t+\theta) d\theta = \int_{-\infty}^{0} e^{\lambda\theta} \varphi(t+\theta) d\theta > 0. \tag{4.13}$$

On the other hand, for any $t \in [s, T]$, by the definition of $Y(t)$, it is easy to see that $Y(t) \geq 0$ as long as $X(t+\theta) \geq 0, \forall \theta \in (-\infty, 0]$ and $Y(t)$ reaches zero only after $X(t)$ has reached zero. The rest of the proof is similar to proof of Lemma 2.2.2. □

Next, let us derive the equation for $Y(t)$. We have the following lemma:

Lemma 4.1.2. *The process $Y(t)$ defined by* (4.3) *satisfies the following equation:*

$$dY(t) = (X(t) - \lambda Y(t))dt. \tag{4.14}$$

Proof. By virtue of the definition of $Y(t)$ (4.2), we can get

$$\begin{aligned}
\frac{d}{dt} Y(t) &= \frac{d}{dt}\left[\int_{-\infty}^{0} e^{\lambda\theta} X_t(\theta) d\theta\right] = \frac{d}{dt}\left[\int_{-\infty}^{0} e^{\lambda\theta} X(t+\theta) d\theta\right] \\
&= \frac{d}{dt}\left[\int_{-\infty}^{t} e^{\lambda(u-t)} X(u) du\right] \quad (\text{let} \quad u \equiv \theta + t) \\
&= \frac{d}{dt}\left[\lim_{\tau \to -\infty} \int_{\tau}^{t} e^{\lambda(u-t)} X(u) du\right] \\
&= \lim_{\tau \to -\infty}\left[\frac{d}{dt} \int_{\tau}^{t} e^{\lambda(u-t)} X(u) du\right] \\
&= \lim_{\tau \to -\infty}\left[-\lambda \int_{\tau}^{t} e^{\lambda(u-t)} X(u) du + X(t)\right] \\
&= -\lambda \int_{-\infty}^{t} e^{\lambda(u-t)} X(u) du + X(t) \\
&= -\lambda \int_{-\infty}^{0} e^{\lambda\theta} X(t+\theta) d\theta + X(t) \quad (\text{let} \quad \theta = u - t) \\
&= X(t) - \lambda Y(t).
\end{aligned}$$

So we can get (4.14). □

Using the above result, we can see that the system given by (4.9), (4.10) and (4.3) can be described equivalently by

$$\begin{aligned} dX(t) &= [((\mu_1 - r)k(t) - c(t) + r)X(t) + \mu_2 Y(t)]dt \\ &\quad + \sigma k(t)X(t)dB(t), && (4.15) \\ dY(t) &= (X(t) - \lambda Y(t))dt, \qquad\qquad t \in [s, T], && (4.16) \end{aligned}$$

with the initial conditions

$$X(s) = x, \quad Y(s) = y, \tag{4.17}$$

where x, y are given by (4.11).

We have the following lemma for existence and uniqueness of solution of (4.15)-(4.17).

Lemma 4.1.3. *For any admissible control* $(k(t), c(t)$ *in* Π*,* (4.15)-(4.17) *has a unique strong solution* $(X(t), Y(t))$ *and it satisfies*

$$\mathbf{E}\left[|X(t)|^2 + |Y(t)|^2\right] \leq \Lambda(x^2 + y^2)e^{Ct}, \quad \forall t \in [s, T], \tag{4.18}$$

where Λ *and* C *are constants that may depend on* T*.*

Proof. It is easy to check that if $(k(t), c(t)) \in \Pi$, then the drift coefficients and the diffusion coefficients satisfy global Lipschitz conditions. Then the result follows from Theorem 2.9 of Karataz and Shreve [26] (page 289). □

From Lemma 4.1.2, we note that any solution of (4.9)-(4.11) satisfies (4.15)-(4.17) and vice versa. Given the existence and uniqueness result for solution of (4.15)-(4.17), we can get the existence and uniqueness result for the solution of (4.9)-(4.11).

From now on, we will consider the system with state equations given by (4.15)-(4.17) instead. The utility function $U(C)$ is defined based on the consumption rate. The problem under consideration is on a finite time horizon $[s, T]$. The terminal utility function Ψ depends on both $X(T)$ and $Y(T)$. Assume the expected total discounted utility $J(s,x,y,k,c)$ be given by

$$J(s,x,y,k,c) \;=\; \mathbf{E}_{s,x,y}\bigg[\int_s^T e^{-\beta(t-s)}U(c(t)X(t))dt + e^{-\beta(T-s)}\Psi(X(T),Y(T))\bigg], \;\forall (k,c)\in\Pi.$$

Then the value function is given by

$$\begin{aligned} V(s,x,y) &= \sup_{k,\,c\in\Pi} J(s,x,y,k,c) \\ &= \sup_{k,\,c\in\Pi} \mathbf{E}_{s,x,y}\bigg[\int_s^T e^{-\beta(t-s)}U(c(t)X(t))dt \\ &\qquad + e^{-\beta(T-s)}\Psi(X(T),Y(T))\bigg]. \end{aligned} \tag{4.19}$$

The rest of the chapter is organized as follows. In Section 4.2, we derive the HJB equation for the value function in a classical sense. Exponential, logarithm and Hyperbolic Absolute Risk Averse (HARA) utility functions are considered in Section 4-6, respectively. For each utility function, we derive the explicit solutions for the associated HJB equation and establish the verification results.

4.2 Hamilton-Jacobi-Bellman Equation

In this section, we will derive the HJB equation satisfied by the value function $V(s,x,y)$. First, we have the following dynamic programming principle.

Lemma 4.2.1 (Dynamic Programming Principle)**.** *Assume that the value function* $V(s,x,y)$ *is given by* (4.19) *and assume the system given by* (4.15)-(4.17)*. Then we have*

$$\begin{aligned} V(s,x,y) &= \sup_{(k,c)\in\Pi} \mathbf{E}_{s,x,y,k,c}\bigg[\int_s^t e^{-\beta(\tau-s)}U(c(\tau))X(\tau)d\tau \\ &\qquad + e^{-\beta(t-s)}V(t,X(t),Y(t))\bigg], \end{aligned} \tag{4.20}$$

for all $\mathcal{F}^t$*-stopping time* $t\in[0,T]$ *and* $(x,y)\in\mathbb{R}^2$.

The proof is straight forward and we omit it here. For details, please see Section III.7

of Fleming and Soner [19].

Next, we will need some kind of Ito's formula with respect to the function of the delay variable $Y(t)$. Let $f \in C^{1,2,1}([0,T] \times \mathbb{R}^2)$ and define

$$G(t) = f(t, X(t), Y(t)).$$

Lemma 4.2.2 (Ito's Formula)**.** *Consider the system given by* (4.9)-(4.10)*. We have*

$$dG(t) = \mathcal{L}^{k,c} f dt + \sigma k x f_x dB(t) + f_y \cdot (x - \lambda y) dt, \tag{4.21}$$

where

$$\begin{aligned} \mathcal{L}^{k,c} f &= \mathcal{L}^{k,c} f(t,x,y) \\ &= f_t + (((\mu_1 - r)k - c + r)x + \mu_2 y) f_x + \frac{1}{2}\sigma^2 k^2 x^2 f_{xx}, \end{aligned} \tag{4.22}$$

where x, y, k and c are evaluated as

$$x = x(t), \quad y = Y(t) = \int_{-\infty}^{0} e^{\lambda\theta} X(t+\theta) d\theta, \quad k = k(t), \quad c = c(t). \tag{4.23}$$

Proof. Applying classical Ito's formula to $G(t) = f(t, X(t), Y(t))$ and using the result of Lemma 4.1.2, we can easily get the result. □

Now we can give the HJB equation for the value function $V(s,x,y)$.

Theorem 4.2.1. (HJB Equation) *Assume that $V(s,x,y) \in C^{1,2,1}([0,T] \times \mathbb{R} \times \mathbb{R})$ then the value function $V(s,x,y)$ given by* (4.19) *satisfies the following Hamilton-Jacobi-Bellman partial differential equation*

$$\begin{aligned} \beta V - V_s \;=\; & \max_k \left[\frac{1}{2}(\sigma k x)^2 V_{xx} + (\mu_1 - r) k x V_x \right] + (rx + \mu_2 y) V_x \\ & + \max_{c \ge 0} [-cxV_x + U(cx)] + (x - \lambda y) V_y, \end{aligned} \tag{4.24a}$$

with the boundary condition

$$V(T,x,y) = \Psi(x,y). \tag{4.24b}$$

The proof is standard so we omit it here. For more details, please see Section IV.3 of Fleming and Soner [19].

4.3 The Solution of the HJB Equation

4.3.1 Exponential Utility

In this section we consider the exponential utility function which is given as

$$U(cX) = 1 - e^{-\alpha c X}, \quad \alpha > 0.$$

We note that maximizing the utility function with or without the additive term 1 gives the same results. The additive term 1 in the utility function restricts the range of the function between 0 and 1 and other than that it does not have any mathematical relevance. So we drop the term 1 for technical convenience and consider the following

$$U(cX) = -e^{-\alpha c X}, \quad \alpha > 0. \tag{4.25}$$

In this case, the utility function is negative. To be consistent, we will also assume that the terminal utility function Ψ is negative. More conditions on Ψ will be specified later. Under those assumption, it is easy to see that $V(s, x, y) < 0$.

Now the HJB equation is

$$\begin{aligned} \beta V - V_s \quad &= \quad \max_k \left[\frac{1}{2}(\sigma k x)^2 V_{xx} + (\mu_1 - r) k x V_x \right] + (rx + \mu_2 y) V_x \\ &\quad + \max_{c \geq 0} \left[-cxV_x - e^{-\alpha c x} \right] + (x - \lambda y) V_y. \end{aligned} \tag{4.26}$$

The candidates for optimal controls are

$$k^* = -\frac{(\mu_1 - r) V_x}{\sigma^2 x V_{xx}}, \quad c^* = -\frac{1}{x\alpha} \log \frac{V_x}{\alpha}.$$

Substituting k^* and c^* in equation (4.26) , we obtain

$$\begin{aligned} \beta V - V_s \quad &= \quad -\frac{(\mu_1 - r)^2 V_x^2}{2\sigma^2 V_{xx}} + \frac{V_x}{\alpha} \left[\log \left(\frac{V_x}{\alpha} \right) - 1 \right] + (rx + \mu_2 y) V_x \\ &\quad + (x - \lambda y) V_y. \end{aligned} \tag{4.27}$$

Suppose the solution is of the form

$$V(s,x,y) = \psi(x,y)Q(s), \tag{4.28}$$

where $Q(s)$ and $\psi(x,y)$ will be determined. For convenience, we assume that $\psi(x,y) < 0$ and $Q(s) > 0$.

Now, we have

$$\begin{aligned} V_x &= Q(s)\psi_x(x,y), \quad V_{xx} = Q(s)\psi_{xx}(x,y), \\ V_y &= Q(s)\psi_y(x,y), \quad V_s = Q'(s)\psi(x,y). \end{aligned}$$

Substituting above equations into equation (4.27) and noting that $Q(s) > 0$, we can get

$$\begin{aligned} (\beta Q(s) - Q'(s))\psi(x,y) &= -\frac{(\mu_1 - r)^2 Q(s)\psi_x^2}{2\sigma^2\psi_{xx}} + (x - \lambda y)Q(s) \\ &\quad + \frac{Q(s)\psi_x}{\alpha}\left(\log(Q(s)) + \log\left(\frac{\psi_x}{\alpha}\right) - 1\right) \\ &\quad + (rx + \mu_2 y)Q(s)\psi_x\psi_y. \end{aligned} \tag{4.29}$$

Define $u \equiv x + \eta y$ and assume that

$$\psi(x,y) = -e^{\eta_1 u} = -e^{\eta_1(x+\eta y)},$$

where η, η_1 are constants to be determined. It is easy to see that

$$\psi_x(x,y) = \eta_1\psi(x,y), \quad \psi_{xx}(x,y) = \eta_1^2\psi(x,y), \quad \psi_y(x,y) = \eta\eta_1\psi(x,y). \tag{4.30}$$

By plugging (4.30) into (4.29) we get

$$\begin{aligned}\beta Q(s) - Q'(s) &= -\frac{(\mu_1 - r)^2}{2\sigma^2} Q(s) + (rx + \mu_2 y)\eta_1 Q(s) + (x - \lambda y)\eta\eta_1 Q(s) \\ &\quad + \frac{\eta_1}{\alpha} Q(s) \left(\log\left(Q(s)\right) + \log\left(\frac{\eta_1 \psi(x,y)}{\alpha}\right) - 1 \right) \\ &= -\frac{(\mu_1 - r)^2}{2\sigma^2} Q(s) + \eta_1 [(r+\eta)x + (\mu_2 - \lambda\eta)y] Q(s) \\ &\quad + \frac{\eta_1}{\alpha} Q(s) \left(\log\left(Q(s)\right) + \log\left(\frac{\eta_1 \psi(x,y)}{\alpha}\right) - 1 \right). \end{aligned} \tag{4.31}$$

We pick a constant η such that

$$\mu_2 - \lambda\eta = (r+\eta)\eta, \tag{4.32}$$

then we can get

$$(r+\eta)x + (\mu_2 - \lambda\eta)y = (r+\eta)(x + \eta y) = (r+\eta)u. \tag{4.33}$$

Remark 4.3.1. *From the equation* (4.32), *we can get*

$$\eta = \frac{1}{2}\left(\pm\sqrt{(r+\lambda)^2 + 4\mu_2} - (r+\lambda) \right).$$

We will pick the solution

$$\eta = \frac{1}{2}\left(\sqrt{(r+\lambda)^2 + 4\mu_2} - (r+\lambda) \right), \tag{4.34}$$

so that $\eta > 0$ *and* $r + \eta > 0$.

Plugging the equation (4.33) and $\psi(x,y)$ into equation (4.31), we can get

$$\begin{aligned}\beta Q(s) - Q'(s) &= -\frac{(\mu_1 - r)^2}{2\sigma^2} Q(s) + \frac{\eta_1}{\alpha} Q(s) \left(\log\left(Q(s)\right) - 1\right) \\ &\quad + \frac{\eta_1^2}{\alpha} u Q(s) + \eta_1 (r+\eta) u Q(s) \\ &\quad + \frac{\eta_1}{\alpha} Q(s) \log\left(-\frac{\eta_1}{\alpha}\right) \end{aligned} \tag{4.35}$$

We pick η_1 such that

$$\frac{\eta_1^2}{\alpha} + (r+\eta)\eta_1 = 0.$$

$\eta_1 = 0$ is the trivial solution and the other solution is

$$\eta_1 = -\alpha(r+\eta). \tag{4.36}$$

Then equation (4.35) becomes

$$\begin{aligned} \beta Q(s) - Q'(s) &= -\frac{(\mu_1 - r)^2}{2\sigma^2} Q(s) - (r+\eta)Q(s)\left(\log\left(Q(s)\right) - 1\right) \\ &\quad - (r+\eta)\log((r+\eta))Q(s) \end{aligned} \tag{4.37}$$

Rewriting the above equation

$$\begin{aligned} Q'(s) &= \left(\beta + \frac{(\mu_1 - r)^2}{2\sigma^2} - (r+\eta) + (r+\eta)\log((r+\eta)\right) Q(s) \\ &\quad + (r+\eta)Q(s)\log\left(Q(s)\right) \end{aligned} \tag{4.38}$$

Define

$$\Lambda_1 \equiv \beta + \frac{1}{2}\frac{(\mu_1 - r)^2}{\sigma^2} - (r+\eta) + (r+\eta)\log(r+\eta). \tag{4.39}$$

Then equation (4.38) can be written as

$$\frac{Q'(s)}{Q(s)} = \Lambda_1 + (r+\eta)\log\left(Q(s)\right) \tag{4.40}$$

At the terminal time $s = T$, we have

$$V(T, x, y) = Q(T)(-e^{-\eta_1 u}) = \Psi(x, y)$$

The terminal utility function $\Psi(x, y)$ is assumed to be consistent with the exponential utility function. In particular, we assume that it is of the form

$$\Psi(x, y) = -\Lambda e^{-\eta_1(x+\eta y)} = -\Lambda e^{-\alpha(r+\eta)(x+\eta y)}, \tag{4.41}$$

where $\Lambda > 0$ is a constant, η_1 is given by (4.36) and η is given by (4.34).

So the boundary condition for $Q(s)$ at $s = T$ is

$$Q(T) = \Lambda. \tag{4.42}$$

The explicit solution for (4.40)-(4.42) is given as

$$Q(s) = \exp\left(-\frac{\Lambda_1}{r+\eta} + \left(\frac{\Lambda_1}{r+\eta} + \log \Lambda \right) e^{-(r+\eta)(T-s)} \right). \tag{4.43}$$

We note that $Q(s) > 0$. Moreover, it is easy to verify that, if $\Lambda_1 > 0$, we have

$$0 < Q(s) \leq \Lambda \quad \forall s \in [0, T]. \tag{4.44}$$

The HJB equations (4.24a)-(4.24b) have the solution

$$V(s, x, y) = -Q(s)e^{-\alpha(r+\eta)(x+\eta y)} \tag{4.45}$$

The optimal investment and consumption rates are given as

$$k^*(s) = \frac{(\mu_1 - r)}{\alpha(r+\eta)\sigma^2 x}, \tag{4.46}$$

$$c^*(s) = -\frac{1}{\alpha x}\left[\log\{(r+\eta)Q(s)\} - \alpha(r+\eta)(x+\eta y) \right], \tag{4.47}$$

where $Q(s)$ is given by (4.43), and x and y are estimated at time s as following

$$x = X(s), \quad y = Y(s) = \int_{-\infty}^{0} e^{\lambda\theta} X(s+\theta) d\theta.$$

We can see that $c^*(s) \geq 0$ as long as

$$\Lambda(r+\eta) \leq 1. \tag{4.48}$$

Next, we will prove the verification results for exponential utility function case in two steps. First, in Theorem 4.3.1, we will show that, under certain conditions, a classical solution of the HJB equation (4.24a)-(4.24b) will equal to the value function defined by (4.19). Secondly, in Theorem 4.3.2, we will show that the function defined by (4.45) is a classic solution of (4.24a)-(4.24b) and it satisfies the conditions in Theorem 4.3.1. In addition, the explicit formula for the optimal controls will be given in Theorem 4.3.2.

Theorem 4.3.1. (Verification Theorem) *Let* $(X(t), Y(t))$ *be a strong solution of* (4.15)-(4.17)*. Assume that* $V(s, x, y) \in C^{1,2,1}([0, T] \times \mathbb{R} \times \mathbb{R})$ *is a solution of the HJB equations* (4.24a)-(4.24b) *such that*

$$\mathbf{E}\left[\int_0^T (k(t)X(t)V_x(t, X(t), Y(t)))^2 \, dt\right] < \infty, \quad \forall (k, c) \in \Pi. \tag{4.49}$$

Then we have

$$\begin{aligned} V(s, x, y) \quad \geq \quad & \sup_{(k,c)\in\Pi} \mathbf{E}_{s,x,y,k,c}\left[\int_s^T e^{-\beta(t-s)} U(c(t)X(t))dt \right. \\ & \left. \qquad + e^{-\beta(T-s)}\Psi(X(T), Y(T))\right]. \end{aligned}$$

In addition, assume that the utility function and (k^*, c^*) *are given as*

$$\begin{aligned} & U(x) = 1 - e^{-\alpha x}; \\ & k^* = -\frac{(\mu_1 - r)V_x}{\sigma^2 x V_{xx}}, \quad c^* = -\frac{1}{x\alpha}\log\frac{V_x}{\alpha}. \end{aligned} \tag{4.50}$$

If $(k^*, c^*) \in \Pi$*, then* (k^*, c^*) *is the optimal control policy. In this case, we have*

$$V(s, x, y) = \mathbf{E}_{s,x,y,k^*,c^*}\left[\int_s^T e^{-\beta(t-s)} U(c(t)X(t))dt + e^{-\beta(T-s)}\Psi(X(T), Y(T))\right].$$

Proof. Equation (4.24a) can be written using the notation $\mathcal{L}^{k,c}$ defined by (4.22) as

$$\max_{k,c\geq 0}\left[\mathcal{L}^{k,c}V(s, x, y) + U(cx)\right] + (x - \lambda y)\, V_y(s, x, y) - \beta V(s, x, y) = 0. \tag{4.51}$$

Assume that $V(s, x, y)$ is a classic solution of the equation (4.51). For any given admissible control $(k, c) \in \Pi$ and for any $(s, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$, we have

$$\beta V(s, x, y) - \mathcal{L}^{k,c}V(s, x, y) - (x - \lambda y)\, V_y(s, x, y) \geq U(cx). \tag{4.52}$$

Applying Ito's formula to $V(t, X(t), Y(t))$, we have

$$
\begin{aligned}
& d\left[e^{-\beta t} V(t, X(t), Y(t))\right] \\
= \quad & e^{-\beta t}\left[-\beta V(t, X(t), Y(t)) dt + dV(t, X(t), Y(t))\right] \\
= \quad & e^{-\beta t}[-\beta V(t, X(t), Y(t)) + \mathcal{L}^{k,c} V(t, X(t), Y(t)) \\
& + (X(t) - \lambda Y(t))\, V_y(t, X(t), Y(t)] dt \\
& + e^{-\beta t}[\sigma k(t) X(t) V_x(t, X(t), Y(t)) dB(t)].
\end{aligned}
$$

Integrating it from s to T, and using (4.52), we have

$$
\begin{aligned}
& e^{-\beta T} V(T, X(T), Y(T)) - e^{-\beta s} V(s, x, y) \\
= \quad & \int_s^T e^{-\beta t}(-\beta V(t, X(t), Y(t)) + \mathcal{L}^{k,c} V(t, X(t), Y(t)) \\
& + (X(t) - \lambda Y(t))\, V_y(t, X(t), Y(t)) dt \\
& + \int_s^T e^{-\beta t} \sigma k(t) X(t) V_x(t, X(t), Y(t)) dB(t) \\
\leq \quad & \int_s^T e^{-\beta t} \sigma k(t) X(t) V_x(t, X(t), Y(t)) dB(t) \\
& - \int_s^T e^{-\beta t} U(c(t) X(t)) dt.
\end{aligned}
$$

Using the boundary condition (4.24b) we have

$$
\begin{aligned}
V(s, x, y) \quad \geq \quad & e^{-\beta(T-s)} V(T, X(T), Y(T)) + \int_s^T e^{-\beta(t-s)} U(c(t) X(t)) dt \\
& - \int_s^T e^{-\beta(t-s)} \sigma k(t) X(t) V_x(t, X(t), Y(t)) dB(t) \\
= \quad & e^{-\beta(T-s)} \Psi(X(T), Y(T)) + \int_s^T e^{-\beta(t-s)} U(c(t) X(t)) dt \\
& - \int_s^T e^{-\beta(t-s)} \sigma k(t) X(t) V_x(t, X(t), Y(t)) dB(t).
\end{aligned}
$$

By virtue of condition (4.49) we have

$$
\int_s^T e^{-\beta(t-s)} \sigma k(t) X(t) V_x(t, X(t), Y(t)) dB(t)
$$

a martingale. Taking expectation on both sides we obtain

$$V(s,x,y) \geq \mathbf{E}_{s,x,y,k,c}\left[\int_s^T e^{-\beta(t-s)}U(c(t)X(t))dt + e^{-\beta(T-s)}\Psi(X(T),Y(T))\right], \quad \forall (k,c) \in \Pi.$$

Then for any $(k,c) \in \Pi$ we have

$$\begin{aligned} V(s,x,y) &\geq \sup_{(k,c)\in\Pi} \mathbf{E}_{s,x,y,k,c}\left[\int_s^T e^{-\beta(t-s)}U(c(t)X(t))dt \right. \\ &\qquad \left. + e^{-\beta(T-s)}\Psi(X(T),Y(T))\right]. \end{aligned}$$

Now let us take $(k,c) = (k^*,c^*)$ which are given by (4.50). If $(k^*,c^*) \in \Pi$, the inequality (4.52) can be replaced by the corresponding equality. Further, all the following inequalities can be replaced by the corresponding inequalities. Using the exponential utility function, we can eventually get

$$V(s,x,y) = \mathbf{E}_{s,x,y,k^*,c^*}\left[\int_s^T e^{-\beta(t-s)}U(c^*(t)X(t))dt + e^{-\beta(T-s)}\Psi(X(T),Y(T))\right].$$

This completes the proof. □

Theorem 4.3.2. *Assume that the utility function is given by* $U(x) = -e^{-\alpha x}$ *and the terminal function is given by* (4.41)*. In addition, assume that* (4.48) *holds and* $\Lambda_1 > 0$*, where* Λ_1 *is defined by* (4.39)*. Then the function* $V(s,x,y)$ *given by* (4.45) *is a classical solution of the HJB equation* (4.24a)-(4.24b)*, and it is equal to the value of the system defined by* (4.19) *that is*

$$V(s,x,y) = \sup_{(k,c)\in\Pi} J(s,x,y,k,c).$$

In addition, the optimal control policy is given by

$$c^*(t) = -\frac{1}{X(t)\alpha}\left[\log\{(r+\eta)Q(t)\} - \alpha(r+\eta)(X(t)+\eta Y(t))\right], \tag{4.53}$$

$$k^*(t) = \frac{(\mu_1 - r)}{\alpha(r+\eta)\sigma^2 X(t)}, \qquad \forall t \in [s,T]. \tag{4.54}$$

Proof. From the derivation of $V(s,x,y)$, it is easy to check that $V(s,x,y)$ given by (4.45) is a classical solution of HJB equation (4.24a)-(4.24b).

To use Theorem 4.3.1 we first verify that condition (4.49) is satisfied. Using (4.45) we have

$$V_x(t,X(t),Y(t)) = \frac{Q(t)}{\alpha(r+\eta)} e^{-\alpha(r+\eta)(X(t)+\eta Y(t))}.$$

From the inequality $e^x > 1+x$, we can get $e^{-x} < \frac{1}{1+x}$ for $x>0$. Therefore, using that $X(t)>0, Y(t)>0, \eta>0, \alpha>0, r>0$, we can get

$$|e^{-\alpha(r+\eta)(X(t)+\eta Y(t))}| \le \left| \frac{1}{\alpha(r+\eta)(X(t)+\eta Y(t))+1} \right|, \quad \forall t \in [s,T].$$

Therefore, noting that $0 < Q(t) \le \Lambda$, we can get,

$$|V_x(t,X(t),Y(t))| \le \frac{\Lambda}{\alpha(r+\eta)} \left| \frac{1}{\alpha(r+\eta)(X(t)+\eta Y(t))+1} \right|,$$

Using the definition of admissible control space Π, we have

$$|k^*(t)X(t)| \le \Lambda_0 |X(t)+Y(t)| \le \bar{\Lambda}_0 |X(t)+\eta Y(t)|,$$

where $\bar{\Lambda}_0 = \Lambda_0 \max\{\frac{1}{\eta}, 1\}$.

Therefore, we have

$$\begin{aligned}
|k^*(t)X(t)V_x(t,X(t),Y(t))| \le & \quad \frac{\Lambda\bar{\Lambda}_0}{\alpha^2(r+\eta)^2} \left| \frac{\alpha(r+\eta)(X(t)+\eta Y(t)}{\alpha(r+\eta)(X(t)+\eta Y(t))+1} \right| \\
\le & \quad \frac{\Lambda\bar{\Lambda}_0}{\alpha^2(r+\eta)^2} \left(\left| 1 - \frac{1}{\alpha(r+\eta)(X(t)+\eta Y(t))+1} \right| \right) \\
\le & \quad \frac{\Lambda\bar{\Lambda}_0}{\alpha^2(r+\eta)^2} \left(1 + \left| \frac{1}{\alpha(r+\eta)(X(t)+\eta Y(t))+1} \right| \right) \\
\le & \quad 2\frac{\Lambda\bar{\Lambda}_0}{\alpha^2(r+\eta)^2} = \bar{\Lambda}
\end{aligned}$$

where $\bar{\Lambda} > 0$ is a constant independent of t. We have,

$$\mathbf{E}\left[\int_0^T \left[k^*(t)X(t)V_x(t,X(t),Y(t))\right]^2 dt \right] \le \ \mathbf{E}\left[\int_0^T \bar{\Lambda}^2 dt \right] = \bar{\Lambda}T < \infty.$$

Thus condition (4.49) is verified. Using definition of (k^*, c^*), it is easy to see that $(k^*(t), c^*(t))$ is $\mathcal{F}^t$-measurable for any $t \in [0, T]$. Also, by virtue of (4.46), (4.47) and (4.44), it is easy to get that $k^*(t) \geq 0, c^*(t) \geq 0, \forall t \in [0, T]$. The conditions given by (4.12a) and (4.12b) of admissible control space are also satisfied. Hence we get $(k^*, c^*) \in \Pi$. This completes the proof. $\square$

4.3.2 Log Utility

In this section, we consider log utility function given as

$$U(cX) = \log(cX). \tag{4.55}$$

Now the HJB equation (4.24a) become

$$\begin{aligned} \beta V - V_s &= \max_k \left[\frac{1}{2}(\sigma k x)^2 V_{xx} + (\mu_1 - r)kxV_x\right] + (rx + \mu_2 y)V_x \\ &\quad + \max_{c\geq 0}\left[-cxV_x + \log(cx)\right] + (x - \lambda y)V_y, \end{aligned} \tag{4.56}$$

The candidates for optimal controls are

$$k^* = -\frac{(\mu_1 - r)V_x}{\sigma^2 x V_{xx}}, \qquad c^* = \frac{1}{xV_x}.$$

Substituting k^* and c^* in equation (5.21), we get

$$\beta V - V_s = -\frac{1}{2}\frac{(\mu_1 - r)^2 V_x^2}{\sigma^2 V_{xx}} + \log\left(\frac{1}{V_x}\right) - 1 + (rx + \mu_2 y)V_x + (x - \lambda y)V_y, \tag{4.57}$$

Suppose solution is of the form

$$V(s, x, y) = \psi(x, y) + Q(s), \tag{4.58}$$

where $Q(s)$ and $\psi(x, y)$ will be determined. Moreover, we have

$$V_x = \psi_x(x, y), \quad V_{xx} = \psi_{xx}(x, y), \quad V_y = \psi_y(x, y), \quad V_s = Q'(s). \tag{4.59}$$

By plugging (4.58) and (4.59) into equation (4.57) we get

$$\begin{aligned}\beta[\psi(x,y)+Q(s)]-Q'(s) &= -\frac{1}{2}\frac{(\mu_1-r)^2\psi_x^2}{\sigma^2\psi_{xx}}+\log\left(\frac{1}{\psi_x}\right)-1 \\ &\quad +(rx+\mu_2 y)\psi_x+(x-\lambda y)\psi_y. \end{aligned} \tag{4.60}$$

Define $u \equiv x+\eta y$ and assume that

$$\psi(x,y)=\eta_1\log(u)=\eta_1\log(x+\eta y). \tag{4.61}$$

where η, η_1 are constants to be determined. It is easy to see that

$$\psi_x=\frac{\eta_1}{u}, \qquad \psi_{xx}=-\frac{\eta_1}{u^2}, \qquad \psi_y=\frac{\eta_1\eta}{u}. \tag{4.62}$$

By plugging (4.61) and (4.62) into (4.62) we get

$$\begin{aligned} & \beta[\eta_1\log(u)+Q(s)]-Q'(s) \\ = & \frac{1}{2}\frac{(\mu_1-r)^2}{\sigma^2}\eta_1+\log\left(\frac{u}{\eta_1}\right)-1+(rx+\mu_2 y)\frac{\eta_1}{u}+(x-\lambda y)\frac{\eta\eta_1}{u} \\ = & \frac{1}{2}\frac{(\mu_1-r)^2}{\beta\sigma^2}\eta_1+\log u-\log\eta_1-1+\frac{\eta_1}{u}[(r+\eta)x+(\mu_2-\lambda\eta)y]. \end{aligned} \tag{4.63}$$

We pick a constant η such that

$$\mu_2-\lambda\eta=(r+\eta)\eta. \tag{4.64}$$

In particular, we pick the positive solution of the above equation(see remark (4.3.1)).

$$\eta=\frac{1}{2}\left(\sqrt{(r+\lambda)^2+4\mu_2}-(r+\lambda)\right). \tag{4.65}$$

Then we can get

$$(r+\eta)x+(\mu_2-\lambda\eta)y=(r+\eta)(x+\eta y)=(r+\eta)u. \tag{4.66}$$

Assume that

$$\eta_1=\frac{1}{\beta}. \tag{4.67}$$

Then, by virtue of (4.66), we have

$$\beta Q(s) - Q'(s) = \frac{1}{2}\frac{(\mu_1 - r)^2}{\beta\sigma^2} + \log(\beta) - 1 + \frac{1}{\beta}(r + \eta). \tag{4.68}$$

Rewriting the above equation

$$Q'(s) = \beta Q(s) - \Lambda_2, \tag{4.69}$$

where

$$\Lambda_2 \equiv \frac{1}{2}\frac{(\mu_1 - r)^2}{\beta\sigma^2} + \log(\beta) - 1 + \frac{1}{\beta}(r + \eta). \tag{4.70}$$

At terminal time $s = T$ we have

$$V(T, x, y) = Q(T) + \frac{1}{\beta}\log(x + \eta y). \tag{4.71}$$

The terminal utility function $\Psi(x, y)$ is assumed to be consistent with the log utility function. In particular, for technical reasons, we assume that the function $\Psi(x, y)$ is of the form

$$\Psi(x, y) = \frac{1}{\beta}\log(x + \eta y). \tag{4.72}$$

Then, by virtue of (4.71) and (4.72), we can get the boundary condition for $Q(s)$ at $s = T$

$$Q(T) = 0. \tag{4.73}$$

The solution for (5.27)-(4.73) is given as

$$Q(s) = \frac{\Lambda_2}{\beta}(1 - e^{-\beta(T-s)}). \tag{4.74}$$

Assume $\frac{1}{2}\frac{(\mu_1 - r)^2}{\sigma^2} + (r + \eta) > \beta - \beta\log(\beta)$, we have

$$0 \le Q(s) \le \frac{\Lambda_2}{\beta}, \quad \forall s \in [0, T]. \tag{4.75}$$

Therefore, the HJB equations (4.24a)-(4.24b) have the solution

$$V(s,x,y) = Q(s) + \log(x+\eta y), \tag{4.76}$$

The optimal investment and consumption rates are given as

$$k^*(s) = \frac{(\mu_1 - r)(x+\eta y)}{\sigma^2 x}, \tag{4.77}$$

$$c^*(s) = \frac{\beta(x+\eta y)}{x}, \tag{4.78}$$

where $Q(s)$ and η are given by (4.74) and (4.65) respectively and x and y are estimated at time s as following

$$x = X(s), \quad y = Y(s) = \int_{-\infty}^{0} e^{\lambda\theta} X(s+\theta)d\theta.$$

Theorem 4.3.3. (Verification Theorem) *Let* $(X(t), Y(t))$ *be a strong solution of* (4.15)-(4.17). *Assume that* $V(s,x,y) \in C^{1,2,1}([0,T]\times\mathbb{R}\times\mathbb{R})$ *is a solution of the HJB equations* (4.24a)-(4.24b) *such that*

$$\mathbf{E}\left[\int_0^T (k(t)X(t)V_x(t,X(t),Y(t)))^2\, dt\right] < \infty, \quad \forall (k,c)\in\Pi. \tag{4.79}$$

Then we have

$$\begin{aligned} V(s,x,y) \geq \sup_{(k,c)\in\Pi} \mathbf{E}_{s,x,y,k,c}\Bigg[&\int_s^T e^{-\beta(t-s)}U(c(t)X(t))dt \\ &+ e^{-\beta(T-s)}\Psi(X(T),Y(T))\Bigg]. \end{aligned}$$

In addition, assume that the utility function and (k^*, c^*) *are given as*

$$U(x) = \log(x)$$
$$k^* = -\frac{(\mu_1 - r)V_x}{\sigma^2 x V_{xx}}, \quad c^* = \frac{1}{xV_x}.$$

If $(k^*, c^*) \in \Pi$, *then* (k^*, c^*) *is the optimal control policy. In this case, we have*

$$V(s,x,y) = \mathbf{E}_{s,x,y,k^*,c^*}\left[\int_s^T e^{-\beta(t-s)}U(c(t)X(t))dt + e^{-\beta(t-s)}\Psi(X(T),Y(T))\right].$$

Proof. The proof is very similar to the proof of Theorem 4.3.1, so we omit it here. □

Theorem 4.3.4. *Let* $(X(t), Y(t))$ *be a strong solution of* (4.15)-(4.17)*. Assume that the utility function is given by* $U(x) = \log(x)$ *and that the terminal function is given by* $\Psi(x,y) = \frac{1}{\beta}\log(x + \eta y)$*. Also assume that* η *is given by* (4.65)*. Then the function* $V(s,x,y)$ *given by* (4.76) *is a classical solution of the HJB equation* (4.24a)-(4.24b)*, and it is equal to the value of the system defined by* (4.19)*, that is*

$$V(s,x,y) \quad = \sup_{(k,c)\in\Pi} J(s,x,y,k,c).$$

In addition, the optimal control policy is given by

$$k^*(t) = \frac{(\mu_1 - r)(X(t) + \eta Y(t))}{\sigma^2 X(t)}, \tag{4.80}$$

$$c^*(t) = \frac{\beta(X(t) + \eta Y(t))}{X(t)}, \quad \forall t \in [s,T], \tag{4.81}$$

where $Q(\cdot)$ *is defined by (4.74).*

Proof. It is evident from the derivation of $V(s,x,y)$ that $V(s,x,y)$ given by (4.76) is a classical solution of HJB equation (4.24a)-(4.24b). To use Theorem 4.3.3, we first verify that condition (4.79) is satisfied. Using (4.76), we can get

$$|V_x(t, X(t), Y(t))| = \left|\frac{1}{X(t) + \eta Y(t)}\right|$$

Using the definition of admissible control space Π, we have

$$|k^*(t)X(t)| \le \Lambda_0|X(t) + Y(t)| \le \bar{\Lambda}_0|X(t) + \eta Y(t)|,$$

where $\bar{\Lambda}_0 = \Lambda_0 \max\{\frac{1}{\eta}, 1\}$. Therefore, we have

$$\begin{aligned} |k^*(t)X(t)V_x(t, X(t), Y(t))| &\leq \bar{\Lambda}_0 |X(t) + \eta Y(t)| \cdot \left| \frac{1}{X(t) + \eta Y(t)} \right| \\ &= \bar{\Lambda}_0. \end{aligned}$$

We have,

$$\mathbf{E}\left[\int_0^T [k^*(t)X(t)V_x(t, X(t), Y(t))]^2 dt\right] \leq \mathbf{E}\left[\int_0^T \bar{\Lambda}_0^2 dt\right] = \bar{\Lambda}_0^2 T < \infty.$$

where $\bar{\Lambda}_0 > 0$ is a constant independent of t. Thus condition (4.79) is verified. Using definition of (k^*, c^*), it is easy to see that $(k^*(t), c^*(t))$ is $\mathcal{F}^t$-measurable for any $t \in [0, T]$. Also, by virtue of (4.77) and (4.78), it is easy to get that $k^*(t) \geq 0, c^*(t) \geq 0, \forall t \in [0, T]$. The conditions given by (4.12b) and (4.12a) of admissible control space are also satisfied. □

4.3.3 HARA Utility

In this section we consider HARA utility function

$$U(cX) = \frac{1}{\gamma}(cX)^\gamma, \quad \gamma \in (-\infty, 1), \quad \gamma \neq 0.$$

We seek to find explicit solution of HJB equations (4.24a)-(4.24b). For HARA utility function, the HJB equation is given as

$$\begin{aligned} \beta V - V_s &= \max_k \left[\frac{1}{2}(\sigma k x)^2 V_{xx} + (\mu_1 - r)kxV_x\right] + (rx + \mu_2 y)V_x \\ &\quad + \max_{c \geq 0} \left[-cxV_x + \frac{1}{\gamma}(cx)^\gamma\right] + (x - \lambda y)V_y, \end{aligned} \tag{4.82}$$

The candidates for maximum over c and k are given as

$$k^* = -\frac{(\mu_1 - r)V_x}{\sigma^2 x V_{xx}}, \quad c^* = \frac{1}{x} V_x^{\frac{1}{\gamma - 1}}.$$

Substituting k^* and c^* in (4.82), we get

$$\beta V - V_s = -\frac{1}{2}\frac{(\mu_1 - r)^2 V_x^2}{\sigma^2 V_{xx}} + \left(\frac{1}{\gamma} - 1\right) V_x^{\frac{\gamma}{\gamma-1}} + (rx + \mu_2 y)V_x + (x - \lambda y)V_y. \quad (4.83)$$

Suppose solution is of the form

$$V(s,x,y) = Q(s)\psi(x,y), \quad (4.84)$$

where $Q(s)$ and $\psi(x,y)$ will be determined. Moreover, we have

$$\begin{aligned} V_x &= Q(s)\psi_x(x,y), \quad V_{xx} = Q(s)\psi_{xx}(x,y), \\ V_y &= Q(s)\psi_y(x,y), \quad V_s = Q'(s)\psi(x,y). \end{aligned} \quad (4.85)$$

Substituting (4.85) into (4.83) yields

$$\begin{aligned} [\beta Q(s) - Q'(s)]\psi(x,y) &= -\frac{1}{2}\frac{(\mu_1 - r)Q(s)\psi_x^2}{\sigma^2\psi_{xx}} + \left(\frac{1}{\gamma} - 1\right)[Q(s)\psi_x]^{\frac{\gamma}{\gamma-1}} \\ &\quad + (rx + \mu_2 y)Q(s)\psi_x + (x - \lambda y)Q(s)\psi_y. \end{aligned} \quad (4.86)$$

Define $u \equiv x + \eta y$ where η is a constant to be determined. Assume that

$$\psi(x,y) = \frac{1}{\gamma}u^\gamma. \quad (4.87)$$

Then we have

$$\psi_x = u^{\gamma-1}, \qquad \psi_{xx} = (\gamma-1)u^{\gamma-2}, \qquad \psi_y = \eta u^{\gamma-1}. \quad (4.88)$$

By plugging (4.87) and (4.88) into (4.86) we have

$$\begin{aligned} \frac{1}{\gamma}[\beta Q(s) - Q'(s)]u^\gamma &= -\frac{1}{2}\frac{(\mu_1 - r)^2 Q(s)u^\gamma}{\sigma^2(\gamma-1)} + \left(\frac{1}{\gamma} - 1\right)[Q(s)]^{\frac{\gamma}{\gamma-1}}u^\gamma \\ &\quad + [(r+\eta)x + (\mu_2 - \lambda\eta)y]Q(s)u^{\gamma-1}. \end{aligned} \quad (4.89)$$

We pick a constant η such that

$$\mu_2 - \lambda\eta = (r+\eta)\eta. \quad (4.90)$$

In particular, we pick the following positive solution of the above equation(see remark (4.3.1))

$$\eta = \frac{1}{2}\left(\sqrt{(r+\lambda)^2 + 4\mu_2} - (r+\lambda)\right). \tag{4.91}$$

Then we can get

$$(r+\eta)x + (\mu_2 - \lambda\eta)y = (r+\eta)(x+\eta y) = (r+\eta)u. \tag{4.92}$$

Now the equation (4.89) can be written as

$$\begin{aligned}\frac{1}{\gamma}[\beta Q(s) - Q'(s)]u^\gamma &= -\frac{1}{2}\frac{(\mu_1 - r)^2 Q(s) u^\gamma}{\sigma^2(\gamma-1)} + \left(\frac{1}{\gamma} - 1\right)[Q(s)]^{\frac{\gamma}{\gamma-1}} u^\gamma \\ &\quad + (r+\eta)Q(s)u^\gamma\end{aligned}$$

Canceling the term u^γ on both sides of the equation (4.89) we get

$$\begin{aligned}\frac{1}{\gamma}[\beta Q(s) - Q'(s)] &= -\frac{(\mu_1 - r)^2 Q(s)}{2\sigma^2(\gamma-1)} + \left(\frac{1}{\gamma} - 1\right)[Q(s)]^{\frac{\gamma}{\gamma-1}} \\ &\quad + (r+\eta)Q(s).\end{aligned}$$

The above equation can be re-written as

$$Q'(s) = \Lambda_3 Q(s) + (\gamma - 1)[Q(s)]^{\frac{\gamma}{\gamma-1}} \tag{4.93}$$

where

$$\Lambda_3 \equiv \beta + \frac{1}{2}\frac{\gamma(\mu_1 - r)^2}{\sigma^2(\gamma-1)} - \gamma(r+\eta). \tag{4.94}$$

At terminal time $s = T$ we have

$$V(T, x, y) = \frac{1}{\gamma} Q(T)(x + \eta y)^\gamma \tag{4.95}$$

The terminal utility function $\Psi(x, y)$ is assumed to be consistent with the HARA utility function. In particular, for technical reasons, we assume that the function $\Psi(x, y)$ is of the form

$$\Psi(x, y) = \frac{\Lambda}{\gamma}(x + \eta y)^\gamma, \tag{4.96}$$

where Λ is a constant. Then, by virtue of (4.95) and (4.96), we can get the boundary condition for $Q(s)$ at $s = T$

$$Q(T) = \Lambda. \tag{4.97}$$

The solution for (4.93)-(4.97) is given as

$$Q(s) = \left[\frac{1-\gamma}{\Lambda_3}(1 - e^{-\frac{\Lambda_3(T-s)}{1-\gamma}}) + \Lambda^{\frac{1}{1-\gamma}} e^{-\frac{\Lambda_3(T-s)}{1-\gamma}} \right]^{1-\gamma}. \tag{4.98}$$

In order for $Q(s) > 0$, we assume

$$\frac{\beta}{\gamma} > \frac{1}{2} \frac{(\mu_1 - r)^2}{\sigma^2(1-\gamma)} + (r + \eta), \tag{4.99}$$

Therefore, the HJB equations (4.24a)-(4.24b) have the solution

$$V(s, x, y) = \frac{1}{\gamma} Q(s)(x + \eta y)^{\gamma}. \tag{4.100}$$

The optimal investment and consumption rates are given as

$$k^*(s) = \frac{(\mu_1 - r)(x + \eta y)}{(1-\gamma)\sigma^2 x}, \tag{4.101a}$$

$$c^*(s) = \frac{x + \eta y}{x} Q(s)^{\frac{1}{\gamma - 1}}, \tag{4.101b}$$

where η and $Q(s)$ are respectively given by (4.90) and (4.98), and x and y are estimated at time s as following

$$x = X(s), \quad y = Y(s) = \int_{-\infty}^{0} e^{\lambda\theta} X(s + \theta) d\theta.$$

We give the verification theorem for the classical solution we obtained above.

Theorem 4.3.5. (Verification Theorem) *Let* $(X(t), Y(t))$ *be a strong solution of* (4.15)-(4.17). *Assume that* $V(s, x, y) \in C^{1,2,1}([0, T] \times \mathbb{R} \times \mathbb{R})$ *is a solution of the HJB equations* (4.24a)-(4.24b) *such that*

$$\mathbf{E}\left[\int_0^T (k(t)X(t)V_x(t, X(t), Y(t)))^2 \, dt \right] < \infty, \quad \forall (k, c) \in \Pi. \tag{4.102}$$

Then we have

$$V(s,x,y) \;\geq\; \sup_{(k,c)\in\Pi} \mathbf{E}_{s,x,y,k,c}\Bigg[\int_s^T e^{-\beta(t-s)}U(c(t)X(t))dt + e^{-\beta(T-s)}\Psi(X(T),Y(T))\Bigg].$$

In addition, assume that the utility function is given by

$$U(x) = \frac{1}{\gamma}x^{\gamma}, \quad \gamma \in (-\infty, 1), \quad \gamma \neq 0, \tag{4.103}$$

and

$$k^* = -\frac{(\mu_1 - r)V_x}{\sigma^2 x V_{xx}}, \quad c^* = \frac{1}{x}V_x^{\frac{1}{\gamma-1}}.$$

If $(k^*, c^*) \in \Pi$, *then* (k^*, c^*) *is the optimal control policy. In this case, we have*

$$V(s,x,y) = \mathbf{E}_{s,x,y,k^*,c^*}\left[\int_s^T e^{-\beta(t-s)}U(c(t)X(t))dt + e^{-\beta(T-s)}\Psi(X(T),Y(T))\right].$$

Proof. The proof is very similar to the proof of Theorem 4.3.1, so we omit it here. □

Now it remains to verify that the function defined by (4.100) is a classical solution of (4.24a)- (4.24b) and control policy is given by (4.101a)-(4.101b) Assume that $X(t)$ is a strong solution of (4.9)-(4.10) and $Y(t)$ be given by (4.2).

Theorem 4.3.6. *Assume that the utility function is given by*

$$U(x) = \frac{1}{\gamma}x^{\gamma}, \quad \gamma \in (-\infty, 1), \quad \gamma \neq 0,$$

and that the terminal function is given by

$$\Psi(x,y) = \frac{1}{\gamma}(x + \eta y)^{\gamma}$$

Also assume that η *satisfies* (4.90). *Then the function* $V(s,x,y)$ *given by* (4.100) *is a classical solution of the HJB equation* (4.24a)-(4.24b) *and it is equal to the value of the* *system defined by* (4.19)*, that is*

$$V(s,x,y) = \sup_{(k,c)\in\Pi} \mathbf{E}_{s,x,y}\left[\int_s^T e^{-\beta(T-s)}U(c(t)X(t))dt + e^{-\beta(T-s)}\Psi(X(T),Y(T))\right].$$

In addition, the optimal control policy is given by

$$k^*(t) = \frac{(\mu_1 - r)(X(t)+\eta Y(t))}{(1-\gamma)\sigma^2 X(t)}, \quad c^*(t) = \frac{X(t)+\eta Y(t)}{X(t)} Q(t)^{\frac{1}{\gamma-1}}, \quad \forall t \in [s,T]$$

where $Q(\cdot)$ *is defined by* (4.98)*.*

Proof. It is evident from the derivation of $V(s,x,y)$ that $V(s,x,y)$ given by (4.100) is a classical solution of HJB equation (4.24a)-(4.24b).
To use theorem 4.3.5 we first verify that condition (4.102) is satisfied. Using (4.100) we have

$$V_x(t,X(t),Y(t)) = Q(t)(X(t)+\eta Y(t))^{\gamma-1}.$$

Using $Q(t) > 0$, we have

$$|V_x(t,X(t),Y(t))| = Q(t)|X(t)+\eta Y(t)|^{\gamma-1}. \tag{4.104}$$

Using the definition of admissible control space Π, we have

$$|k(t)X(t)| \le \Lambda_0|X(t)+Y(t)| \le \bar{\Lambda}_0|X(t)+\eta Y(t)|$$

where $\bar{\Lambda}_0 = \Lambda_0 \max\{\frac{1}{\eta},1\}$.

Since $Q(t)$ is bounded on $[0,T]$ we have

$$\begin{aligned} |k(t)X(t)V_x(t,X(t),Y(t))| &\le \bar{\Lambda}_0 Q(t)|X(t)+\eta Y(t)|^{\gamma} \\ &\le \Lambda_3|X(t)+\eta Y(t)|^{\gamma} \end{aligned}$$

where $\Lambda_3 > 0$ is a constant that does not depend on t. Using lemma 4.1.3 and noting that $2\gamma < 2$, we can show

$$\mathbf{E}\left[\int_0^T (k(t)X(t)V_x(t,X(t),Y(t)))^2\, dt\right] < \infty$$

where $\Lambda_3 > 0$ is a constant independent of t. Thus condition (4.102) is verified. The optimal controls are

$$k^*(t) = \frac{(\mu_1 - r)(X(t) + \eta Y(t))}{(1-\gamma)\sigma^2 X(t)}, \quad c^*(t) = \frac{X(t) + \eta Y(t)}{X(t)} Q(t)^{\frac{1}{\gamma-1}}, \quad \forall t \in [s, T].$$

It is easy to see that $(k^*(t), c^*(t))$ is $\mathcal{F}^t$-measurable for any $t \in [0, T]$. Moreover, $c^*(t) \geq 0$. Hence we get $(k^*, c^*) \in \Pi$. This completes the proof. $\square$

Chapter 5

A Stochastic Portfolio Model with Complete Memory and Stochastic Volatility

5.1 Introduction

In this chapter, an optimal investment-consumption model is considered, in which the historical performance of the model is accounted for and the volatility of the risky asset is considered to be stochastic. In chapters 2 - 4 the volatility of the risky asset is assumed to be constant. In this chapter, we study the model for the case when the volatility of the risky asset is stochastic. In this model, investor's portfolio comprises a risky and a riskless asset. The economic factor Θ_t of the coefficient $\sigma(\Theta_t)$ in the risky asset dynamics (5.4) is an ergodic diffusion process which satifies (5.5). The value of the portfolio follows a stochastic process $X(t)$ (see (5.6)) that depends on following exponential delay information:

$$Y(t) = \int_{-\infty}^{0} e^{\lambda\tau} X(t+\tau)d\tau, \tag{5.1}$$

where $\lambda > 0$ is a constant.

The historic performance from $-\infty$ to s is the initial condition for $X(t)$ where $s \in [0, T]$ is the initial time.

$$X(t) = \varphi(t-s), \quad \forall t \in (-\infty, s],$$

where $\varphi > 0$ is a bounded function which is integrable $(-\infty, 0]$.

The goal is to choose investment and consumption controls such that the total expected discounted log utility of consumption is maximized. Like in chapter 4, the model incorporates the complete past information. This is done by introducing the complete memory variable (5.1). Unlike in chapters 2 and 3, the memory variable $\Theta(t)$ (see (2.2)/(3.1)) does not appear in the HJB equation for this model.

The rest of the chapter is organized as follows. In Section 5.2, we formulate the model for infinite delay and stochastic volatility. In Section 5.3, HJB equation is established. In Section 5.4, the method of sub and super solutions is used to prove the existence of the solution of the HJB equation for log utility function and the verification theorem is also established.

5.2 Stochastic Volatility Model

Consider an investor's portfolio consisting of a risky asset and a riskless asset. An example of riskless asset is money deposited in a bank account that earns a fixed interest rate $r > 0$. In this model the complete performance of the risky asset is taken into account. We also assume that the investor can freely move his money between two assets at any time and his consumption comes from the riskless asset.

Let $K(t)$ be the amount invested in the risky asset and $L(t)$ is the amount invested on the riskless asset. The total wealth is given by $X(t) = K(t) + L(t)$. We assume that the performance of the risky asset depends on the following delay variable $Y(t)$:

$$Y(t) = \int_{-\infty}^{0} e^{\lambda\tau} X(t+\tau) d\tau, \tag{5.2}$$

where $\lambda > 0$ is a constant.

Remark 5.2.1. *From the definition of $Y(t)$, its value is actually the exponential average of all the historical values of the total wealth $X(t)$. We want to point out that in reality, there is always a historical time that the value of X is available. In other words, the initial path $\varphi(t)$ will start at some point $t = -M$ instead of $t = -\infty$. In this case, we simply assume that $\varphi(t) = 0, \forall t \in (\infty, -M)$, and we can still use the formula (5.2) to define the delay variable $Y(t)$. In addition, $Y(t)$ is still a continuous function with respect to t for $t \geq s$.*

Assume the processes $B_1(t)$ and $B_2(t)$ are one-dimensional standard Brownian motions defined on a complete filtered probability space $(\Omega, \mathcal{F}, P)$. The Brownian motions are correlated with correlation $\rho \in (-1, 1)$. The price of the riskless asset $L(t)$ satisfies the stochastic differential equations

$$dL(t) = [rL(t) - C(t) - I(t)]dt, \tag{5.3}$$

where $C(t)$ is the consumption rate. The price of the risky asset is modeled as a process $K(t)$ satisfying

$$dK(t) = [(\mu_1 + \mu_2 Y(t))K(t) + I(t)]dt + \sigma(\Theta(t))K(t)dB_1(t), \tag{5.4}$$

where μ_1 and μ_2 are constants, $I(t)$ is the investment rate and $0 < \sigma_1 \leq \sigma(\theta) \leq \sigma_2$. The process $\Theta(t)$ is referred to as the stochastic factor and it satisfies

$$d\Theta(t) = a(\Theta(t))dt + b(\Theta(t))dB_2(t), \tag{5.5}$$

The drift and diffusion coefficients $a(\theta)$ and $b(\theta)$ are such that equation (5.5) has a unique strong solution. In particular, in this chapter, we consider the case where $a(\theta) = c(\bar{\theta} - \theta)$, where $c, \bar{\theta} > 0$ and $b(\theta) = b$ is a constant.

If we assume that $K(t) > 0$ almost surely, we can use the following delay variable $\widetilde{Y}(t)$:

$$\widetilde{Y}(t) = \frac{1}{K(t)} \int_{-\infty}^{0} e^{\lambda \tau} X(t + \tau) d\tau,$$

instead of (5.2). Using the above assumption, the equation for $K(t)$ is rewritten as

$$\begin{aligned} dK(t) &= [(\mu_1 + \mu_2 \tilde{Y}(t))K(t) + I(t)]dt + \sigma(\Theta(t))K(t)dB_1(t) \\ &= [\mu_1 K(t) + \mu_2 Y(t) + I(t)]dt + \sigma(\Theta(t))K(t)dB_1(t), \end{aligned}$$

where $Y(t)$ is given by (5.2). The process followed by the net wealth $X(t) = K(t) + L(t)$ is given as

$$dX(t) = [\mu_1 K(t) + \mu_2 Y(t) + rL(t) - C(t)]dt + \sigma(\Theta(t))K(t)dB_1(t), \quad \forall t \in [0, \infty). \tag{5.6}$$

In order to describe the distribution of wealth between the risky asset and the riskless asset, the control variables are defined using $C(t)$ and $K(t)$ as $c(t) = \frac{C(t)}{X(t)}$ and $k(t) = \frac{K(t)}{X(t)}$ as consumption and investment controls, respectively. $X(t)$ and $Y(t)$ are state variables. As we shall prove $X(t) > 0$ in the following section so that the controls are well defined. Using $L(t) = X(t) - K(t) = X(t)(1 - k(t))$, equation for $X(t)$ can be written as

$$dX(t) = [((\mu_1 - r)k(t) - c(t) + r)X(t) + \mu_2 Y(t)]dt + \sigma(\Theta(t))k(t)X(t)dB(t), \quad (5.7)$$

The initial condition is given by

$$X(t) = \varphi(t - s), \quad \forall t \in (-\infty, s], \quad (5.8)$$

where φ is a bounded function and $\varphi(\tau) > 0, \forall \tau \in (-\infty, s]$.

Definition 5.2.1 (Admissible Control Space)**.** *Let* Π *denote the admissible control space. A control policy* $(k(t), c(t))$ *is said to be in the admissible control space* Π *if it satisfies the following conditions:*

(a) $(k(t), c(t))$ *is* $\mathcal{F}^t-$*measurable for any* $t \in [0, \infty)$*;*

(b) $c(t) \geq 0, \forall t \in [0, \infty)$*;*

(c)

$$Pr\left(\int_0^T k^2(t)dt < \infty\right) = 1, \quad \forall T > 0, \quad (5.9)$$

$$|k(t)X(t)| \leq \Lambda\,|X(t) + Y(t)|\,, \quad \forall t > 0, \quad (5.10)$$

$$|c(t)X(t)| \leq \Lambda\,|X(t) + Y(t)|\,, \quad \forall t > 0, \quad (5.11)$$

where $\Lambda > 0$ *is a constant.*

Lemma 5.2.1. *The solution X(t) of equation* (5.7) *with initial condition* (5.8) *is almost surely positive.*

Proof. The proof is similar to the proof of Lemma 4.1.1. □

The utility function $U(C)$ is defined based on the consumption rate. The problem under consideration is on an infinite time horizon $[0, \infty)$. Assume the expected total discounted utility $J(\varphi, \theta, k, c)$ be given by

$$J(\varphi, \theta, k, c) = \mathbf{E}_\varphi \left[\int_0^\infty e^{-\beta t} U(c(t)X(t))dt \right], \quad \forall (k, c) \in \Pi.$$

Then the value function is given by

$$\begin{aligned} V(\varphi, \theta) &= \sup_{k,\, c \in \Pi} J(\varphi, k, c) \\ &= \sup_{k,\, c \in \Pi} \mathbf{E}_\varphi \left[\int_0^\infty e^{-\alpha t} U(c(t)X(t))dt \right]. \end{aligned} \tag{5.12}$$

We note that the initial data has infinite dimension. We will show that for (5.7) the value function and optimal controls depend on the initial data as given below,

$$x = x(\varphi) \equiv \varphi(0),$$

$$y = y(\varphi) \equiv \int_{-\infty}^0 e^{\lambda \varphi(\tau)} d\tau.$$

Taking advantage of this we can write

$$V(\varphi, \theta) = V(x, y, \theta),$$

where $V : \mathbb{R}^3 \to \mathbb{R}$.

5.3 Hamilton-Jacobi-Bellman Equation

In this section HJB equation satisfied by the value function $V(\varphi, \theta)$ is derived. Let $f \in C^{1,2,1}(\mathbb{R}^3)$ and define

$$G(t) = f(t, X(t), y(X_t)),$$

where

$$y(\eta) \equiv \int_{-\infty}^0 e^{\lambda \tau} \eta(\tau) d\tau,$$

and

$$X_t(\tau) \equiv X(t+\tau), \quad \forall \tau \in (\infty, 0].$$

Lemma 5.3.1 (Ito's Formula)**.** *Consider the system given by* (5.7)-(5.8)*. We have*

$$dG(t) = \mathcal{L}^{k,c} f dt + \sigma k x f_x dB(t) + f_y \cdot [x - \lambda y] dt, \tag{5.13}$$

where

$$\begin{aligned} \mathcal{L}^{k,c} f &= \mathcal{L}^{k,c} f(t,x,y) \\ &= f_t + (((\mu_1 - r)k - c + r)x + \mu_2 y) f_x + \frac{1}{2}\sigma^2 k^2 x^2 f_{xx}, \end{aligned} \tag{5.14}$$

where x, y, k and c are evaluated as

$$x = x(t), \quad y = y(X_t) = \int_{-\infty}^{0} e^{\lambda\tau} X(t+\tau) d\tau, \quad k = k(t), \quad c = c(t). \tag{5.15}$$

Proof. See Lemma 4.1.2. □

By virtue of Lemma 5.3.1, we obtain the following formula for $dY(t)$:

$$dY(t) = (X(t) - \lambda Y(t))dt.$$

We assume that the initial path φ depends on the value function V only through the functionals $x(\varphi)$ and $y(\varphi)$ as defined by (5.15) . That is,

$$V(\varphi, \theta) = V(x(\varphi), y(\varphi), \theta) \equiv V(x, y, \theta). \tag{5.16}$$

Theorem 5.3.1 (HJB Equation)**.** *Assume that* (5.16) *holds and $V(x,y,\theta) \in C^{1,2,1}(\mathbb{R} \times \mathbb{R} \times \mathbb{R})$ then the value function value function $V(x,y,\theta)$ given by* (5.12) *and* (5.16) *satisfies the following Hamilton-Jacobi-Bellman partial differential equation*

$$\begin{aligned} \beta V \;\; = \;\; & \max_k \left[\frac{1}{2}(\sigma(\theta)kx)^2 V_{xx} + (\mu_1 - r)kxV_x + \rho k x \sigma(\theta) b(\theta) V_{x\theta}\right] + (rx + \mu_2 y)V_x \\ & + \max_{c \geq 0} \left[-cxV_x + U(cx)\right] + (x - \lambda y)V_y + a(\theta)V_\theta + \frac{1}{2} b^2(\theta) V_{\theta\theta}. \end{aligned} \tag{5.17}$$

Proof. The proof is standard so we omit it here. $\square$

5.4 Logarithmic Utility

In this section we study the HJB equation (5.17) for log utility function. We use subsolution and supersolution method to establish the classical solution for HJB equation. Log utility function is given by

$$U(cX) = \log(cX). \tag{5.18}$$

Substituting in equation (5.17), we have

$$\begin{aligned}\beta V &= \max_k \left[\frac{1}{2}(\sigma(\theta)kx)^2 V_{xx} + (\mu_1 - r)kxV_x + \rho kx\sigma(\theta)b(\theta)V_{x\theta}\right] + (rx + \mu_2 y)V_x \\ &\quad + \max_{c\geq 0}[-cxV_x + \log(cx)] + (x - \lambda y)V_y + a(\theta)V_\theta + \frac{1}{2}b^2(\theta)V_{\theta\theta}, \end{aligned}\tag{5.19}$$

The candidates for optimal controls are

$$k^* = \frac{-(\mu_1 - r)V_x - \rho b(\theta)\sigma(\theta)V_{x\theta}}{\sigma^2(\theta)xV_{xx}}, \quad c^* = \frac{1}{xV_x} \tag{5.20}$$

Plugging optimal control candidates in (5.25), we have

$$\begin{aligned}\beta V &= -\frac{1}{2}\frac{[(\mu_1 - r)V_x + \rho b(\theta)\sigma(\theta)V_{x\theta}]^2}{\sigma^2(\theta)V_{xx}} + (rx + \mu_2 y)V_x + (x - \lambda y)V_y \\ &\quad + \log\left(\frac{1}{V_x}\right) - 1 + a(\theta)V_\theta + \frac{1}{2}b^2(\theta)V_{\theta\theta}. \end{aligned}\tag{5.21}$$

We assume a solution is of the form

$$V(x, y, \theta) = \eta_1 \log(u) + W(\theta) \tag{5.22}$$

where η_1 is a constant to be determined, $u \equiv x + \eta y$ and $\eta = \eta(r, \lambda, \mu_2) > 0$ satisfies

$$\eta(r + \eta) = \mu_2 - \eta\lambda. \tag{5.23}$$

Moreover, we have

$$\begin{aligned} V = \eta_1 \log(u) + W(\theta), \quad V_x = \tfrac{\eta_1}{u}, \quad V_{xx} = -\tfrac{\eta_1}{u^2}, \quad V_{x\theta} = 0, \\ V_y = \tfrac{\eta_1 \eta}{u}, \quad V_\theta = W_\theta, \quad V_{\theta\theta} = W_{\theta\theta} \end{aligned} \tag{5.24}$$

Choose $\eta_1 = \frac{1}{\beta}$ then (5.21) becomes

$$\begin{aligned} \log(u) + \beta W(\theta) &= \frac{1}{2}\frac{(\mu_1 - r)^2}{\beta\sigma^2(\theta)} + (rx + \mu_2 y)\frac{1}{\beta u} + (x - \lambda y)\frac{\eta}{\beta u} \\ &\quad + \log(\beta) - 1 + a(\theta)W_\theta + \frac{1}{2}b^2(\theta)W_{\theta\theta}. \end{aligned} \tag{5.25}$$

Using the assumption (5.23), equation (5.25) becomes

$$\beta W(\theta) = \frac{1}{2}\frac{(\mu_1 - r)^2}{\beta\sigma^2(\theta)} + \log(\beta) - 1 + \frac{1}{\beta}(r + \eta) + a(\theta)W_\theta + \frac{1}{2}b^2(\theta)W_{\theta\theta}. \tag{5.26}$$

The above equation can be rewritten as

$$\frac{1}{2}b^2(\theta)W_{\theta\theta}(\theta) + a(\theta)W_\theta(\theta) - \beta W(\theta) + Q(\theta) = 0, \tag{5.27}$$

where

$$Q(\theta) = \frac{1}{2}\frac{(\mu_1 - r)^2}{\beta\sigma^2(\theta)} + \frac{1}{\beta}(r + \eta) + \log(\beta) - 1. \tag{5.28}$$

Now we use the subsolution-supersolution method to get a classical solution $W(\theta)$ of equation (5.21).

Define

$$\begin{aligned} LW &= \frac{1}{2}b^2(\theta)W_{\theta\theta}(\theta) + a(\theta)W_\theta(\theta) &\quad (5.29) \\ f(\theta, W) &= Q(\theta) - \beta W(\theta). &\quad (5.30) \end{aligned}$$

Using above notations equation (5.27) can be written as

$$-LW = f(\theta, W). \tag{5.31}$$

5.4.1 Subsolution and Supersolution

Now we define subsolutions and supersolutions.

Definition 5.4.1. *A function* $\underline{W}(\theta)$ *is called a subsolution of equation* (5.31) *if*

$$-L\underline{W}(\theta) \leq f(\theta, \underline{W}(\theta)), \qquad \forall \theta \in \mathbb{R}. \tag{5.32}$$

While a function $\overline{W}(\theta)$ *is called a supersolution of equation* (5.31) *if*

$$-L\overline{W}(\theta) \geq f(\theta, \overline{W}(\theta)), \qquad \forall \theta \in \mathbb{R}. \tag{5.33}$$

In addition, $(\underline{W}(\theta), \overline{W}(\theta))$ *is called a pair of subsolution and supersolution if*

$$\underline{W}(\theta) \leq \overline{W}(\theta), \quad \forall \theta \in \mathbb{R}. \tag{5.34}$$

Lemma 5.4.1. *Assume*

$$\frac{(\mu_1 - r)^2}{2\sigma_2^2} + r + \eta \geq \beta(1 - \log(\beta)) \tag{5.35}$$

where $0 < \sigma_1 \leq \sigma(\theta) \leq \sigma_2$*. Any constant* $K < K_1$ *is a subsolution of* (5.31) *where* K_1 *is given as*

$$K_1 \equiv \frac{1}{\beta^2}\left[\left(\frac{(\mu_1 - r)^2}{2\sigma_2^2} + r + \eta\right) + \beta(\log(\beta) - 1)\right]. \tag{5.36}$$

Proof. Since K_1 is a constant, we have $-LK_1 = 0$.

Now consider

$$\begin{aligned} f(\theta, K_1) &= Q(\theta) - \beta K_1 && (5.37)\\ &= \frac{(\mu_1 - r)^2}{2\beta}\left[\frac{1}{\sigma^2(\theta)} - \frac{1}{\sigma_2^2}\right] && (5.38)\\ &\geq 0 && (5.39) \end{aligned}$$

Therefore, we get

$$-LK_1 \leq f(\theta, K_1). \tag{5.40}$$

Thus K_1 is a subsolution of (5.31). □

Lemma 5.4.2. *Assume* (5.35) *holds. Any constant* $K > K_2$ *is a supersolution of* (5.31) *where* K_2 *is given as*

$$K_2 \equiv \frac{1}{\beta^2}\left[\left(\frac{(\mu_1 - r)^2}{2\sigma_1^2} + r + \eta\right) + \beta(\log(\beta) - 1)\right]. \tag{5.41}$$

Proof. Since K_2 is a constant, we have $-LK_2 = 0$.

Now consider

$$\begin{aligned} f(\theta, K_2) &= Q(\theta) - \beta K_2 && (5.42)\\ &= \frac{(\mu_1 - r)^2}{2\beta}\left[\frac{1}{\sigma^2(\theta)} - \frac{1}{\sigma_1^2}\right] && (5.43)\\ &\leq 0, && (5.44)\end{aligned}$$

Therefore, we get

$$-LK_1 \geq f(\theta, K_2). \tag{5.45}$$

Thus K_2 is a supersolution of (5.31). □

We note that $K_1 \leq K_2$ for all $\theta \in \mathbb{R}$ follows from $\sigma_1 \leq \sigma_2$. Therefore, (K_1, K_2) is an ordered pair of subsolution and supersolution.

5.4.2 Existence of Solution

Following theorem establishes the existence result.

Theorem 5.4.1. *The equation* (5.31) *possesses a classical solution* $\hat{W}(\theta)$ *such that*

$$K_1 \leq \hat{W}(\theta) \leq K_2, \tag{5.46}$$

where K_1 *and* K_2 *are the subsolution and supersolution given by equations* (5.36) *and* (5.41) *respectively.*

Proof. By virtue of Lemma 5.4.1 and 5.4.2, K_1 and K_2 are ordered subsolution and supersolution. Also note that equation (5.31) has a positive subsolution and $f(\theta, 0) \geq 0$. The result follows using theorem 7.5.2 in [51] □

Lemma 5.4.3. *If* $\hat{W}(\theta)$ *is a classical solution of* (5.31) *such that* (5.46) *is satisfied, then* $\hat{W}_\theta(\theta)$ *is bounded.*

Proof. Let $a(\theta) = c(\bar{\theta} - \theta)$, where $c, \bar{\theta}$ are positive constants. From (5.46) we note that $\hat{W}(\theta)$ is bounded. To show that $\hat{W}_\theta(\theta)$ is bounded, it is enough to show that $\hat{W}_\theta(\theta)$ is bounded at its local maxima or minima. Without any loss of generality, assume that $\hat{W}_\theta(\theta)$ has a local maxima or minima at θ_0. Then we have $\hat{W}_{\theta\theta}(\theta_0) = 0$. Therefore at $\theta = \theta_0$, we can get

$$a(\theta)\hat{W}_\theta(\theta) - \beta\hat{W}(\theta) + Q(\theta) = 0, \tag{5.47}$$

For any $\Lambda > 0$, let $\theta \in [-\Lambda, \Lambda]$, we have $|W_\theta(\theta)| \le C_\Lambda$, where C_Λ is a constant that depends on Λ. Let $\theta \in [\Lambda, \infty)$. Assume that $a(\theta) \le -\breve{C}_\Lambda < 0$ where $\breve{C}_\Lambda$ is a constant. We can write equation (5.47) as

$$\hat{W}_\theta(\theta) = \frac{1}{a(\theta)}[\beta\hat{W}(\theta) - Q(\theta)] \tag{5.48}$$

Then we have

$$\begin{aligned}
|\hat{W}_\theta(\theta)| &= \frac{1}{|a(\theta)|}[|\beta\hat{W}(\theta) - Q(\theta)|] \\
&\le \frac{1}{\breve{C}_\Lambda}\left[\beta|\hat{W}(\theta)| + |Q(\theta)|\right] \\
&\le \frac{1}{\breve{C}_\Lambda}\left[\beta K_2 + \bar{Q}\right] < \infty
\end{aligned}$$

where $\bar{Q} = \frac{1}{2}\frac{(\mu_1 - r)^2}{\beta\sigma_1^2} + \frac{1}{\beta}(r+\eta) + \log(\beta) - 1$.

Similarly when $\theta \in (-\infty, -\Lambda]$, Assume that $a(\theta) \ge \hat{C}_\Lambda > 0$ where $\hat{C}_\Lambda$ is a constant. We can write equation (5.47) as

$$\hat{W}_\theta(\theta) = \frac{1}{a(\theta)}[\beta\hat{W}(\theta) - Q(\theta)] \tag{5.49}$$

Then we have

$$\begin{aligned}
|\hat{W}_\theta(\theta)| &= \frac{1}{|a(\theta)|}[|\beta\hat{W}(\theta) - Q(\theta)|] \\
&\le \frac{1}{\hat{C}_\Lambda}\left[\beta|\hat{W}(\theta)| + |Q(\theta)|\right] \\
&\le \frac{1}{\hat{C}_\Lambda}\left[\beta K_2 + \bar{Q}\right] < \infty
\end{aligned}$$

Therefore for $\theta \in (-\infty, -\Lambda) \cup (\Lambda, \infty)$, we have

$$|\hat{W}_\theta(\theta)| \leq \frac{1}{\min\{\hat{C}_\Lambda, \breve{C}_\Lambda\}}\left[\beta K_2 + \bar{Q}\right] < \infty \tag{5.50}$$

Since $\theta = \theta_0$ is an arbitrary minimum or maximum point of $\hat{W}_\theta(\theta)$, therefore $\hat{W}_\theta(\theta)$ is bounded at its minima and maxima. Hence $\hat{W}_\theta(\theta)$ is bounded.

□

5.4.3 Verification Theorem

Now we give the verification theorem.

Theorem 5.4.2. *(Verification Theorem) Suppose that $\hat{W}(\theta)$ (such that (5.46) holds) is a classical solution of equation (5.31). Define*

$$\hat{V}(x, y, \theta) = \frac{1}{\beta}\log(u) + \hat{W}(\theta). \tag{5.51}$$

such that

$$\mathbf{E}\left[\int_0^T \Big(\sigma(\theta)k(t)X(t)\hat{V}_x(X(t), Y(t), \Theta(t))\Big)^2 dt\right] < \infty, \quad \forall (k, c) \in \Pi. \tag{5.52}$$

then we have

(a) For every admissible control process

$$\hat{V}(x, y, \theta) \geq \mathbf{E}\int_0^\infty e^{-\beta t}\log(c_t x_t)dt, \tag{5.53}$$

(b) If

$$k^* = \frac{(\mu_1 - r)u}{\sigma^2(\theta)x}, \quad c^* = \frac{\beta u}{x}. \tag{5.54}$$

Then $(k^*, c^*) \in \Pi$ and

$$\hat{V}(x, y, \theta) = \mathbb{E}\int_0^\infty e^{-\beta t}\log(c_t^* x_t^*)dt. \tag{5.55}$$

Proof. First we verify the condition (5.52). Using (5.51), we have

$$|\hat{V}_x(X(t), Y(t), \Theta(t))| = \frac{1}{\beta}\left|\frac{1}{X(t)+\eta Y(t)}\right|$$

Using the definition of admissible control space Π, we have

$$|k(t)X(t)| \leq \Lambda|X(t)+Y(t)| \leq \Lambda_1|X(t)+\eta Y(t)|,$$

where

$$\Lambda_1 = \Lambda \max\left\{\frac{1}{\eta}, 1\right\}.$$

Therefore, we have

$$|\sigma(\theta)k(t)X(t)\hat{V}_x(X(t), Y(t), \Theta(t))| \leq \frac{\sigma_2\Lambda_1}{\beta}|X(t)+\eta Y(t)| \cdot \left|\frac{1}{X(t)+\eta Y(t)}\right| = \frac{\sigma_2\Lambda_1}{\beta}.$$

Then we have,

$$\mathbf{E}\left[\int_0^T [\sigma(\theta)k(t)X(t)\hat{V}_x(t, X(t), Y(t), \Theta(t))]^2 dt\right] \leq \mathbf{E}\left[\int_0^T \frac{\sigma_2^2\Lambda_1^2}{\beta^2} dt\right] = \frac{\sigma_2^2\Lambda_1^2 T}{\beta^2} < \infty.$$

Thus condition (5.52) is verified.

Using equation (5.17) we have

$$\begin{aligned}\beta\hat{V} - U(cx) \quad &\geq \quad \frac{1}{2}(\sigma(\theta)kx)^2\hat{V}_{xx} + (((\mu_1 - r)k - c + r)x + \mu_2 y)\,\hat{V}_x + \rho kx\sigma(\theta)b(\theta)\hat{V}_{x\theta} \\ &\quad + (x - \lambda y)\hat{V}_y + a(\theta)\hat{V}_\theta + \frac{1}{2}b^2(\theta)\hat{V}_{\theta\theta}. \qquad (5.56)\end{aligned}$$

Applying Ito's rule to $e^{-\beta t}\hat{V}(X(t), Y(t), \Theta(t))$, we have

$$d\left(e^{-\beta t}\hat{V}(X(t), Y(t), \Theta(t))\right) = e^{-\beta t}d\hat{V}(X(t), Y(t), \Theta(t)) - \beta e^{-\beta t}\hat{V}(X(t), Y(t), \Theta(t))dt \qquad (5.57)$$

where

$$\begin{aligned}d\hat{V}(X(t), Y(t), \Theta(t)) \quad &= \quad \Bigg(\frac{1}{2}(\sigma(\theta)kx)^2\hat{V}_{xx} + (((\mu_1 - r)k - c + r)x + \mu_2 y)\hat{V}_x + \rho kx\sigma(\theta)b(\theta)\hat{V}_{x\theta} \\ &\quad + (x - \lambda y)\hat{V}_y + a(\theta)\hat{V}_\theta + \frac{1}{2}b^2(\theta)\hat{V}_{\theta\theta}\Bigg)dt + \sigma(\theta)xkV_x dB_1(t) + b(\theta)V_\theta dB_2(t).\end{aligned}$$

Combining equations (5.56), (5.57) and (5.58), we have

$$d\left(e^{-\beta t}\hat{V}(X(t),Y(t),\Theta(t))\right) \leq e^{-\beta t}\log(c(t)X(t)) + \sigma(\theta)xkV_x dB_1(t) + b(\theta)V_\theta dB_2(t). \tag{5.58}$$

By virtue of (5.52) and Lemma 5.4.3 it is easy to verify that $\int_0^T \sigma(\theta)xkV_x dB_1(t)$ and $\int_0^T b(\theta)V_\theta dB_2(t)$ are martingales. Therefore using equation (5.58), we have

$$\hat{V}(x,y) \geq \mathbf{E}\left[\int_0^T e^{-\beta t}\log(c(t)X(t))dt\right] + \mathbf{E}\left[e^{-\beta T}\hat{V}(X(T),Y(T))\right]. \tag{5.59}$$

Using (5.51), we have

$$\begin{aligned}
& \lim_{T\to\infty}\mathbf{E}\left[e^{-\beta T}\hat{V}(X(T),Y(T))\right] \\
= & \lim_{T\to\infty}\mathbf{E}\left[e^{-\beta T}\left(\frac{1}{\beta}\log\left(X(T)+\eta Y(T)\right)+\hat{W}(\theta)\right)\right],
\end{aligned}$$

From equation (5.46) we note that

$$\lim_{T\to\infty}\mathbf{E}\left[e^{-\beta T}\hat{W}(\theta)\right] \geq 0.$$

To show that

$$\lim_{T\to\infty}\mathbf{E}\left[e^{-\beta T}\hat{V}(X(T),Y(T))\right] = 0,$$

it is sufficient to show

$$\lim_{T\to\infty}\mathbf{E}\left[e^{-\beta T}\log\left(X(T)+\eta Y(T)\right)\right] = 0. \tag{5.60}$$

Let

$$S(t) = X(t) + \eta Y(t). \tag{5.61}$$

Then using (5.23), we have

$$\begin{aligned}
dS(t) &= dX(t) + \eta dY(t) \\
&= \left[\left((\mu_1 - r)k(t) - c(t)\right)X(t) + (r+\eta)S(t)\right]dt + \sigma(\theta)k(t)X(t)dB(t).
\end{aligned}$$

Using Ito's rule, we have

$$\begin{aligned}
d\log S(t) &= \frac{dS(t)}{S(t)} - \frac{1}{2}\frac{(dS(t))^2}{(S(t))^2} \\
&= \Bigg[\left((\mu_1 - r)k(t) - c(t)\right)\frac{X(t)}{S(t)} + (r+\eta) \\
&\qquad - \frac{1}{2}\sigma^2(\theta)k^2(t)\left(\frac{X(t)}{S(t)}\right)^2 \Bigg] dt + \sigma(\theta)k(t)\frac{X(t)}{S(t)}dB(t). \qquad (5.62)
\end{aligned}$$

Since $(k(t), c(t)) \in \Pi$, it is easy to verify that $\int_0^T \sigma(\theta)k(t)\frac{X(t)}{S(t)}dB(t)$ is a martingale. Therefore, we can get

$$\begin{aligned}
\mathbf{E}\left[\log S(T)\right] &= \log(S(0)) + \mathbf{E}\left[\int_0^T \left((\mu_1 - r)k(t) - c(t)\right)\frac{X(t)}{S(t)}dt\right] \\
&\quad + \int_0^T (r+\eta)dt - \mathbf{E}\left[\int_0^T \frac{1}{2}\sigma^2(\theta)k^2(t)\left(\frac{X(t)}{S(t)}\right)^2 dt\right]. \qquad (5.63)
\end{aligned}$$

For any $(k(t), c(t)) \in \Pi$, by virtue of (5.10) and (5.11), we have

$$\begin{aligned}
\left|k(t)\frac{X(t)}{S(t)}\right| &\le \frac{\Lambda\left|X(t)+Y(t)\right|}{\left|X(t)+\eta Y(t)\right|} \le \Lambda_1, \\
\left|c(t)\frac{X(t)}{S(t)}\right| &\le \frac{\Lambda\left|X(t)+Y(t)\right|}{\left|X(t)+\eta Y(t)\right|} \le \Lambda_1,
\end{aligned}$$

where

$$\Lambda_1 = \Lambda \max\left\{\frac{1}{\eta}, 1\right\}.$$

And also $\sigma^2(\theta) \le \sigma^2$. Then we can get

$$\begin{aligned}
&\lim_{T\to\infty} e^{-\beta T}\mathbf{E}\left[\int_0^T \left((\mu_1 - r)k(t)\right)\frac{X(t)}{S(t)}dt\right] = 0, \\
&\lim_{T\to\infty} e^{-\beta T}\int_0^T (r+\eta)\,dt = 0, \\
&\lim_{T\to\infty} e^{-\beta T}\mathbf{E}\left[\int_0^T (-c(t)\frac{X(t)}{S(t)})dt\right] = 0, \\
&\lim_{T\to\infty} e^{-\beta T}\mathbf{E}\left[\int_0^T \left(-\frac{1}{2}\sigma^2(\theta)k^2(t)\left(\frac{X(t)}{S(t)}\right)^2\right)dt\right] = 0
\end{aligned}$$

Then from (5.63), we can get

$$\lim_{T\to\infty} e^{-\beta T}\mathbf{E}[\log(S(T))] = 0.$$

Thus, we have

$$\lim_{T\to\infty} \mathbf{E}\left[e^{-\beta T}\hat{V}\left(X(T),Y(T)\right)\right] = 0, \quad \forall (k(t),c(t)) \in \Pi.$$

combined with (5.59), part 1 of the theorem is established.

Now assume that k^*, c^* are given by (5.54). We can easily verify that they are $\mathcal{F}^t$-measurable. In addition, from Lemma 5.2.1, we know that $X(t) > 0,\ a.s.$, so we can get that $k^*(t), c^*(t)$ are well defined and $c^*(t) \geq 0$. In addition, it is not hard to check that (5.10) and (5.11) are true for $k^*(t)$ and $c^*(t)$. Finally, since $X(t) > 0$ and $k^*(t)$ is well defined, we can get that

$$Pr(k^*(t) < \infty) = 1, \quad \forall t \geq 0.$$

Therefore, we can get

$$Pr\left(\int_0^T (k^*(t))^2 dt < \infty\right) = 1, \quad \forall T > 0.$$

So we can get that $(k^*(t), c^*(t)) \in \Pi$. For $(k^*(t), c^*(t)) \in \Pi$, instead of (5.59), we have

$$\hat{V}(x,y,\theta) = \mathbf{E}\left[\int_0^T e^{-\beta t}\log(c^*(t)X^*(t))dt\right] + \mathbf{E}\left[e^{-\beta T}\hat{V}(X(T),Y(T))\right]. \tag{5.64}$$

To show

$$\hat{V}(x,y,\theta) = \mathbb{E}\int_0^\infty e^{-\beta t}\log(c_t^* x_t^*)dt, \tag{5.65}$$

By virtue of (5.51), it is sufficient to show

$$\lim_{T\to\infty} \mathbf{E}\left[e^{-\beta T}\log\left(X^*(T) + \eta Y^*(T)\right)\right] = 0. \tag{5.66}$$

Using k^* and c^* equation (5.62) becomes

$$d\log S^*(t) \quad = \quad \left[\left(\frac{1}{2}\frac{(\mu_1 - r)^2}{\sigma^2(\theta)} - \beta\right) + (r+\eta)\right]dt + \frac{(\mu_1 - r)}{\sigma^2(\theta)}dB(t) \tag{5.67}$$

Since $\sigma(\theta)$ is assumed to be bounded, therefore we have

$$\mathbf{E}\left[\log S^*(T)\right] \quad = \quad \log(S^*(0)) + \mathbf{E}\left[\int_0^T \left(\frac{1}{2}\frac{(\mu_1 - r)^2}{\sigma^2(\theta)} - \beta + r + \eta\right) dt\right]$$

It immediately follows that

$$\lim_{T\to\infty} e^{-\beta T}\left[\int_0^T \log(S(0))dt\right] = 0,$$
$$\lim_{T\to\infty} e^{-\beta T}\mathbf{E}\int_0^T \left(\frac{1}{2}\frac{(\mu_1 - r)^2}{\sigma^2(\theta)} - \beta + r + \eta\right) dt = 0,$$

Therefore we have

$$\lim_{T\to\infty} \mathbf{E}\left[e^{-\beta T} V(X^*(T), Y^*(T), \Theta^*(t))\right] = 0. \tag{5.68}$$

Hence we have $V(x, y) = J(x, y; k^*, c^*)$. This completes the proof. □

Chapter 6

Conclusion and Future Work

In this dissertation, we consider some stochastic portfolio optimization models with delays. In chapters 2 and 3, models with finite delay for finite and infinite time horizons have been investigated. In chapters 4 and 5, models with infinite delay have been studied. In chapter 5, the volatility for the price process of risky asset is assumed to be stochastic.

We consider utility functions that have been widely used, such as exponential, log and HARA utility functions. Due to the delay variables $Y(t)$ and $Z(t)$, the system is no longer a Markovian system. Under certain conditions, we have derived the explicit formulas for the value functions as well as the optimal investment and consumption controls.

We want to point out that a crucial condition for both log utility case and the non-log HARA utility case is (see (2.47) and (3.66))

$$\mu_3 e^{\lambda h}(r + \mu_3 e^{\lambda h}) = \mu_2 - \mu_3 \lambda e^{\lambda h}. \tag{6.1}$$

This condition is necessary to ensure that the HJB equations have solutions that are independent of z. As we have showed in Lemma 2.3.2, the independence of z is a necessary condition that we can solve the delay problem in a finite dimensional space. Actually, this is the main reason that stochastic control problems with delays are very challenging (see Larssen-Risebro [32]). More discussions about the condition (6.1) can be found in Chang-Pang-Yang [5] (Section 5 and 6). Finally, in this dissertation, we only consider the delay variables that depend on the total wealth process $X(t)$.

In chapters 4 and 5, we have some stochastic portfolio optimization models where the price of the risky asset is no more Markovian because of the infinite delay variable $Y(t)$.

In chapter 4, the volatility of risky asset is considered to be constant. In chapter 5, the volatility of the risky asset is no more constant. The dynamics of the model are

$$\begin{aligned} dX(t) &= [((\mu_1 - r)k(t) - c(t) + r)X(t) + \mu_2 Y(t)]dt + \sigma(\Theta)k(t)X(t)dB_1(t), \\ d\Theta(t) &= a(\Theta)dt + b(\Theta)dB_2(t) \\ dY(t) &= (X(t) - \lambda Y(t))dt, \end{aligned}$$

with the initial conditions

$$X(s) = x, \quad Y(s) = y$$

where x, y are given as

$$X(s) = x \equiv \varphi(s), \quad Y(s) = y \equiv \int_{-\infty}^{0} e^{\lambda\theta}\varphi(s+\theta)d\theta.$$

By taking the volatility of the risky asset to be constand and by taking $\mu_2 = 0$, above model reduces to the classical Merton's portfolio optimization problem.

In chapters 4 and 5, for each of the utility functions (exponential, logarithm and HARA), we obtained the explicit solution in terms of functions of $u \equiv x + \eta y$ such that

$$\eta = \frac{1}{2}\left(\sqrt{(r+\lambda)^2 + 4\mu_2} - (r+\lambda)\right).$$

For instance, consider the case in chapter 4. As we can see, if we take $\mu_2 = 0$, then we get $\eta = 0$. Therefore, the value function and the optimal controls are independent of the memory (delay) variable $Y(t)$. Further, when $\mu_2 = 0$, the process for $X(t)$ does not depend on $Y(t)$ and it reduces to

$$dX(t) = [((\mu_1 - r)k(t) - c(t) + r)X(t)]dt + \sigma k(t)X(t)dB(t), \quad \forall t \in [s, T],$$

with the initital condition

$$X(s) = x.$$

For this reduced model, for each of the utility functions, we note the following

- Exponential Utility

The value function is given by

$$V(s,x,y) = -Q(s)e^{-\alpha r x}.$$

The optimal investment and consumption rates for exponential utlity are given as

$$\begin{aligned} k^*(s) &= \frac{(\mu_1 - r)}{\alpha r \sigma^2 x}, \\ c^*(s) &= -\frac{1}{\alpha x}\Bigg[\log\left(rQ(s)\right) - \alpha r x\Bigg], \end{aligned}$$

where

$$Q(s) = \exp\left(-\frac{\Lambda_1}{r} + \left(\frac{\Lambda_1}{r} + \log \Lambda\right) e^{-r(T-s)}\right)$$

and

$$\Lambda_1 \equiv \beta + \frac{1}{2}\frac{(\mu_1 - r)^2}{\sigma^2} - r + r\log(r).$$

- Logarithmic Utility

  The value function is given by

$$V(s,x,y) = Q(s) + \log(x). \tag{6.2}$$

The optimal investment and consumption rates are given as

$$\begin{aligned} k^*(s) &= \frac{(\mu_1 - r)}{\sigma^2} \\ c^*(s) &= \beta, \end{aligned}$$

where

$$Q(s) = \frac{\Lambda_2}{\beta}(1 - e^{-\beta(T-s)}),$$

and

$$\Lambda_2 \equiv \frac{1}{2}\frac{(\mu_1 - r)^2}{\beta\sigma^2} + \log(\beta) - 1 + \frac{r}{\beta}.$$

- HARA Utility

The value function is given by

$$V(s,x,y)=\frac{1}{\gamma}Q(s)x^{\gamma}.$$

The optimal investment and consumption rates are given as

$$\begin{aligned} k^*(s) &= \frac{(\mu_1-r)}{(1-\gamma)\sigma^2},\\ c^*(s) &= [Q(s)]^{\frac{1}{\gamma-1}}, \end{aligned}$$

where

$$Q(s)=\left[\frac{1-\gamma}{\Lambda_3}(1-e^{-\frac{\Lambda_3(T-s)}{1-\gamma}})+\Lambda^{\frac{1}{1-\gamma}}e^{-\frac{\Lambda_3(T-s)}{1-\gamma}}\right]^{1-\gamma},$$

and

$$\Lambda_3\equiv\beta-\frac{1}{2}\frac{\gamma(\mu_1-r)^2}{\sigma^2(1-\gamma)}-\gamma r.$$

In each of the cases discussed above, the value function and optimal controls are solutions of the Merton's classical model on a finite time horizon with the objective function given by

$$J(s,x)=\mathbf{E}_{s,x}\left[\int_s^T e^{-\beta(\tau-s)}U(c(\tau))X(\tau)d\tau+e^{-\beta(T-s)}\Psi(X(T))\right].$$

Similarly it is easy to verify that it is true for models discussed in chapters 2, 3 and 5.

Our future plans for research are driven from the models discussed in this dissertation. The models discussed in this dissertation consider single risky asset. We are interested in extending these models to multiple risky assets.

The model considered in chapter 5 investigates an infinite time horizon model for logarithmic utility function. One of our future works include extension of the model for exponential and HARA utility functions for this model. We also plan to study this model on finite time horizon.

In practice the interest rate may not be constant. We also plan on extending the models for stochastic interest rates. The models considered in this dissertation do not consider risky assets that produce stochastic dividends. These models can be studied for risky assets which produce stochastic dividends.

Moreover, the consumption and investment controls are assumed to be Markovian in this dissertation. However, in practice, an investor would consider the historic data of consumption and investment. The models studied in this dissertation can be investigated for non-Markovian control processes where an investor makes his/her decisions by incorporating the past information of the control processes.

To use the HJB framework for solving these class of models, we use functional Ito's calculus to obtain the modified HJB equation. The HJB is too complicated in this case. Obtaining explicit solution of HJB equation might not be possible. In that case method of sub/super solutions and viscosity solutions can be used to prove the existence of the solution of the HJB equation. These extensions will be our future research topics.

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