Abstract

DING, YI. A New Fuzzy Set Approach To The Tolerance Analysis And Parameter Design Problems. (Under the direction of Prof. Robert E. Young and Prof. Yahya Fathi.)

Tolerance is believed to be unavoidable and necessary in engineering design and manufacturing due to universal randomness. The Tolerance Analysis and Parameter Design problems are therefore critical to quality control in the modern engineering context. Both the tolerance analysis and the parameter design problem have been studied and explored over the past decades with success. However, most of them use probability theory and statistical methods, and there are drawbacks or restrictions to these approaches.

We tackle down these problems from a new perspective using fuzzy set technology. We model the tolerance in the manufacturing process as fuzzy variables instead of using conventional random variables to represent the process uncertainty and imprecision. We develop and present a new approach to the tolerance analysis problem and also a new mathematical model to solve the parameter design problem in the fuzzy environment, respectively. Finally we present several case studies for both problems. Numerical results show that the new approach and model generate consistent and compatible results with conventional methods.

A NEW FUZZY SET APPROACH TO THE TOLERANCE ANALYSIS AND PARAMETER DESIGN PROBLEMS

by

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To Jessica, my wife and best friend

Biography

Yi Ding was born on January 24, 1974 in Wuxi, P. R. China. He completed his secondary education at Wuxi No. 2 Middle School in 1992. He received his Bachelor of Science degree with a major in manufacturing engineering from Nanjing University of Science and Technology, at Nanjing, P. R. China, in July 1996. After graduation he entered into the manufacturing industry and worked as an associate professional engineer. He began his studies at North Carolina State University in August 2000. He is in the Department of Industrial Engineering with a co-major in Operations Research where he is pursuing his Master of Science degree.

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1.   Introduction

The topic of tolerance in modern product development is discussed every day in business all over the world. Every manufactured product is toleranced, either explicitly or implicitly. Consequently, every manufacturing process used to make a product is controlled by some form of specification limits. According to the ANSI Y14.5M-1982 ASME (R1988) Standard, tolerance is formally defined as "The total amount by which a given dimension may vary, or the difference between the limits." In the engineering context, we usually interpret tolerance as the allowable variation attached to a nominal dimension. People believe that tolerances exist because we live in a probabilistic universe where randomness cannot be avoided. Because it is impossible to produce the exact dimension specified, a tolerance is used to show the acceptable variation in a dimension. Evans [9] did a historic review on statistical tolerancing to address the development of this technology. Taguchi [28] developed a theoretical and systematic off-line quality engineering technique. In practice, engineers specify tolerances to make sure parts will mate and operate satisfactorily. On the one hand, we prefer a tolerance as tight as possible for higher quality products; on the other hand, generally speaking, the tighter the tolerance the more careful the manufacturing procedures and the more rigorous the inspections raising the manufacturing cost. Tolerancing then is a compromise between higher quality and high cost, and lower quality and low cost.

In the realm of conventional statistical quality engineering, we focus on "conformance." Suppose t is a performance characteristic, and τ is the target value. We need to keep the expected value of t equal to τ and to keep its standard deviation σ as small as possible.

1.1 The Tolerance Analysis Problem

Suppose x_1, x_2, ..., x_n are the input variables (components) for a manufactured product or a manufacturing process.

We assume that x_i, $\forall$ i = 1, 2, ..., n, are random variables, with expected value $\mu(x_i) = \mu_i$, and variance $Var(x_i) = \sigma_i^2$, respectively.

Let $y = h(x_1, x_2, ..., x_n)$ be an output variable representing a performance characteristic of interest, such as temperature, speed, length, etc.

In this thesis, we assume that the transfer function $y = h(x_1, x_2, ..., x_n)$ is already known in closed form. The problem is how to evaluate or estimate the Mean and Variance of the output variable y.

So far, the methodologies for solving such problems are based on three assumptions: (1) The manufacturing process is centered; (2) The probability density functions of the components x_i are normally distributed; and (3) The assembly components are random selected and the input variables x_1 through x_n are independent random variables.

1.1.1 Linear Combinations

In some cases, the dimension of an item is a linear combination of the dimensions of the component's parts. That is to say the transfer function h is linear, i.e., $y = h(x_1, x_2, ..., x_n) = a_0 + a_1x_1 + a_2x_2 + ... + a_nx_n$. If the x_i are normally and independently distributed with mean μ_i and variance σ_i^2, then we can solve the tolerance analysis problem by knowing y is also normally distributed with its mean and variance as:

$$\mu_y = a_0 + a_1\mu_1 + a_2\mu_2 + ... + a_n\mu_n,$$

$$\sigma_y^2 = a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + ... + a_n^2\sigma_n^2.$$

1.1.2   Nonlinear Combinations

If the transfer function h is nonlinear, the problem cannot be solved using the above approach. Evans [9], [10] presented a theoretically exact method and an approximation method using numerical integration to solve this problem. But this analysis quickly becomes very complex theoretically and computationally making it unsuitable for practical use. In practice only the approximation methods are used. We briefly present several commonly used approximation methods.

Functional Approximation

If the nonlinear function $y = h(x_1, x_2, ..., x_n)$ is known, the usual approach is to approximate the nonlinear function by a linear function of the x_i in the region of interest using a Taylor Series Expansion as described in [23],

$$y = h(x_1, x_2, ..., x_n)$$

$$= h(\mu_1, \mu_2, ..., \mu_n) + \sum_{i=1}^{n}(x_i - \mu_i)\frac{\partial h}{\partial x_i}\Big|_{\mu_i} + R$$

where R represents the higher-order terms. Neglecting these higher-order terms, we have

$$\mu_y \cong h(\mu_1, \mu_2, ..., \mu_n) \tag{1.1}$$

and $$\sigma_y^2 \cong \sum_{i=1}^{n}(\frac{\partial h}{\partial x_i}\Big|_{\mu_i})^2 \sigma_i^2$$

In this method, the transfer function h has to be differentiable.

Monte Carlo Simulation

In this approach, we need to generate a set of values for $x_1, x_2, ..., x_n$ according to their given probability density functions, then compute the value of output variable y based on the function h. We repeat this process N times to obtain $y_1, y_2, ..., y_N$. From the simulated data, we use a frequency distribution histogram to observe the shape of the

distribution of output variable y and compute the $\bar{y}$ and s^2 to estimate μ_y and σ_y^2. Using these estimates, we can determine appropriate confidence intervals for μ_y and σ_y^2, and other moments of y.

The merit of this method is that the transfer function h does not have to be known or given in explicitly closed form. The drawback is that the component variables' probability density functions have to be known beforehand. In practice this knowledge is often not available, and a large volume of experiment data has to be generated. The critical issue in this method is to obtain the suitable probability density functions for x_i. Law [19] addressed this issue.

Discretization (Design of Experiments)

Consider the function $y = h(x_1, x_2, ..., x_n)$, where h is a known or given function and $x_1, x_2, ..., x_n$ are n independent random variables.

(1) If x_i (for $i = 1, 2, ..., n$) are discrete random variables with known probability mass functions (i.e., $P(x_i = d)$ is known for all possible values of d), we need to find the probability mass function of the output variable y.

Let $\{y_1, y_2, ..., y_k\}$ be the set of all possible values of y, then

$$P(y = y_k) = \sum_{d_1, ..., d_n} P(x_1 = d_1) * \cdots * P(x_n = d_n) \quad \text{for all } k.$$

Once the probability mass function of y is obtained, we can compute the mean and variance of y as follows:

$$\mu_y = \sum_k y_k * P(y = y_k), \text{ and}$$

$$\sigma_y^2 = \sum_k (y_k - \mu_y)^2 * P(y = y_k)$$

(2) For the case where $x_1, x_2, ..., x_n$ are continuous random variables, Taguchi [28] proposed to approximate x_i with a discrete random variable x_i' with the corresponding probability mass functions then proceed with the same approach. He argued that the moments of y', which are obtained from the approximated discrete components variables x_i', are good approximations of the corresponding moments of y.

However, with the increase of the dimensions of the component's variables, the computational complexity increases tremendously.

1.2   Parameter Design Problem

The *parameter design problem*, which is also known as the *robust design problem*, is a problem of determining the optimal set points for the component variables of a manufacturing process. According to Fathi [12], the problem is stated as follows.

Let $x = (x_1, ..., x_n)$ be a vector of component variables for a manufacturing product or a manufacturing process, and let $y = h(x_1, ..., x_n)$ represent an output characteristic of interest. We assume that the set points (i.e., the nominal values) for x_1 through x_n are controllable parameters (i.e., their values can be chosen or determined at the design stage). We denote the set points for x_1 through x_n by μ_1 through μ_n, respectively.

Typically, during the design stage, the designing engineers determine a set of values for μ_1 through μ_n such that

$h(\mu_1, ..., \mu_n) = \tau,$

where τ is the most desirable value of y (i.e., the target value of y).

During the manufacturing processes, the inevitable randomness in them causes the actual values of x_1 through x_n to deviate from their respective nominal values (i.e., μ_1

through μ_n), therefore causing the value of y to deviate from its target value τ. From the conventional point of view, we consider x_1 through x_n to be random variables with respective means μ_1 through μ_n. Thus, $y = h(x_1, ..., x_n)$ is also a random variable. We denote the mean and variance of y by μ_y and σ_y^2, respectively.

Normally, the probability distribution function of y and its moments depend on the form of the transfer function h as well as on the probability distribution functions of x_1 through x_n and their various parameters. In particular, when the function h is nonlinear, the value of σ_y^2 depends on the values of μ_1 through μ_n. So, now the *parameter design problem* is interpreted as to find the values of the controllable parameters μ_1 through μ_n to minimize the σ_y^2, while maintaining $h(\mu_1, ..., \mu_n) = \tau$.

This problem can be expressed in the following model, which we refer to as the *parameter design model* (PDM):

$$\underset{\mu_1, ... \mu_n}{Minimize} \quad z = \sigma_y^2(\mu_1, ..., \mu_n)$$

$$Subject\ to \quad h(\mu_1, ..., \mu_n) = \tau$$

$$(\mu_1, ..., \mu_n) \in M.$$

where M represents a set of acceptable values for μ_1 through μ_n.

This problem is essentially an optimization problem. Since it was introduced into the English literature in the early 1980's, the parameter design problem has received considerable attention, and various methodologies for solving it have been investigated and proposed during the past few decades. Among them, Design of Experiments is probably the most commonly used approach for solving this problem, as discussed in Taguchi and Wu [29], Leon, et al. [20], Box [2], Shoemaker, et al. [27], and Tsui [30]. In the case where the transfer function is either known in closed form or can be modeled by

computer simulation, the problem can be approximated and solved through appropriate use of nonlinear programming techniques, as discussed in Yang, et al. [32], Fathi [11], [12]. Recently a new methodology of *high-low tolerancing* was proposed by Fathi [13] to solve this problem as another general approach.

1.3   Issues And Problems

Both the tolerance analysis problem and the parameter design problem have been studied and explored over the past decades with success. However, there are drawbacks or restrictions to the approaches developed.

(1).   Most approaches are based on probability theory and statistics that require assumptions that the variables are normally distributed. But in practice, it is the case that the performance variables are not always normally distributed and not necessarily consistent in their shapes.

(2).   In addition, to obtain these distributions requires many repeated experiments. In many cases, these repeated experiments are too expensive to implement or the experiments themselves are nonrepeatable.

(3).   For the techniques that use the Taylor series expansion, the transfer function itself has to be differentiable, and this is also not always true. In practice, we often encounter situations in which the function is either nondifferentiable or the partial derivatives are too complex to use.

These issues and problems led us to explore tolerance problems using fuzzy sets. The following chapters present a fuzzy set approach to solve these problems preceded by a brief discussion of fuzzy set theory.

2.   Fuzzy Sets and Fuzzy Numbers

L. A. Zadeh [33] published his famous paper "Fuzzy sets" in 1965 establishing a brand new mathematical tool to enable us to deal with vagueness and ambiguity. The basic distinction between fuzzy set theory and the conventional crisp set theory is intuitive and natural: Instead of the sharp boundaries as in an conventional crisp set, a fuzzy set possesses no sharply defined boundaries characterized by a membership function with a real number value between 0 and 1. The membership function is not a matter of whether or not an element belongs to the set, but rather a matter of degree of the set membership (i.e., the degree to which extent an element belongs to the set.) Thus, the membership function is a model of the gradient existing between complete membership and no membership whatsoever.

In this section, we present some basic definitions and theorems used in other chapters, which are adapted from the classic literature discussing the fuzzy set theory by Zadeh [33], Zimmermann [34], Klir, et al. [17], Li [21], and Sakawa [26].

2.1   Basic Concepts In Fuzzy Set

Definition 2.1 (*Fuzzy sets*)

Let X be a space of points (objects), with a generic element of X denoted by x. Thus, X=$\{x\}$.

A fuzzy set (class) A in X is characterized by a membership (characteristic) function $\mu_A(x)$ which associates with each point in X a real number in the interval [0, 1], with the value of $\mu_A(x)$ at x representing the "grade of membership" of x in A [33].

Thus, the nearer the value of $\mu_A(x)$ to unity, the higher the grade of the membership of x in A. When A is a set in the ordinary sense of the term (i.e., A is a crisp set), its membership function can take only two values 0 and 1, with $\mu_A(x) = 1$ or 0 depending on whether or not x belongs to A.

We denote:

$$\mu_A(x) = f(x), \text{ where } \mu_A \in [0, 1]. \tag{2.1}$$

This is a general definition and allows any shape function to define a fuzzy set as long as it maps to [0, 1]. For instance, the membership function could have different shapes as shown in Figure 2.1. It's worth mentioning here that the membership function shape could be triangular, trapezoidal, bell-shaped, symmetric or nonsymmetrical, finitely bounded or not, etc.
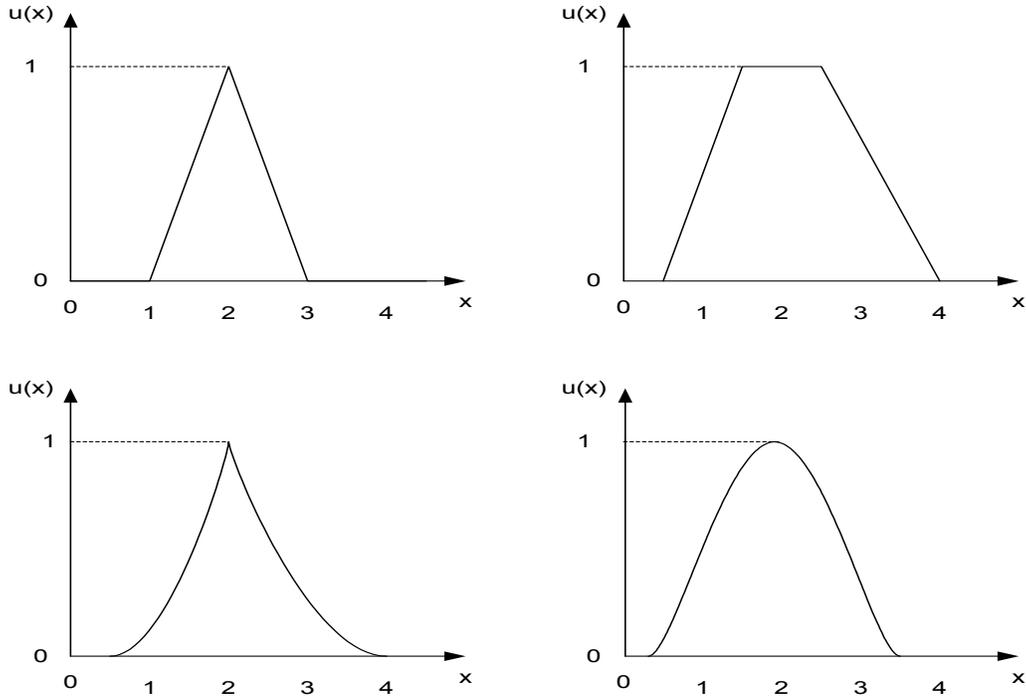
Figure 2.1. Examples of membership functions that may be used in different contexts for characterizing fuzzy sets of real numbers close to 2 [adapted from 17].

Definition 2.2 *(α-cut or α-level set)* [17]

Given a fuzzy set A defined on X and any number $\alpha \in [0, 1]$, the α-cut, A_α, and the strong α-cut $A_{\alpha+}$, respectively, are the crisp sets

$$A_\alpha = \{x \mid A(x) \geq \alpha\}; \qquad (2.2)$$

$$A_{\alpha+} = \{x \mid A(x) > \alpha\}; \qquad (2.3)$$

That is, the α-cut (or the strong α-cut) of a fuzzy set A is the set A_α (or the set of $A_{\alpha+}$) that contains all the elements of the universal set X whose membership grades in A are greater than or equal to (or greater than) the specified value of α.

The *support* of a fuzzy set A within a universal set X is the set that contains all the elements of X that have nonzero membership grades in A. The support of A is exactly the same as the strong α-cut of A for α=0. The α-cut of A when α=1 is called the *core* of A. When it is a single point it is usually called the *nominal value* or the *mode* of A.

The *height*, h(A), of a fuzzy set A is the largest membership grade obtained by any element in that set. Formally,

$$h(A) = \sup_{x \in X} A(x). \tag{2.4}$$

A fuzzy set A is called *normal* when h(A) = 1; it is called *subnormal* when h(A) < 1. Thus, the height of A is located at the core of A and for a single valued core, it is the nominal value or the mode of A.

Definition2.3 (*Convex fuzzy set*) [26]

A fuzzy set A is said to be a convex fuzzy set if and only if its α-cuts are convex. An alternative and more direct definition of a convex fuzzy set is:

A fuzzy set A is convex if and only if

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2)). \tag{2.5}$$

for all $x_1, x_2 \in X$ and $\lambda \in [0, 1]$.

Throughout this thesis, we restrict our considerations to convex fuzzy sets. The concepts of α-cuts and strong α-cuts play a fundamentally important role in the relationship between fuzzy sets and ordinary crisp sets. They serve as a bridge to connect fuzzy sets and crisp sets.

Theorem 2. 1 (*Decomposition Theorem, or Resolution Theorem*) [26]

A fuzzy set A can be represented by

$$A = \bigcup_{\alpha \in [0,1]} \alpha A_\alpha \tag{2.6}$$

where αA_α denotes the algebraic product of a scalar α with the α-cut A_α. The membership function of αA_α is given by

$$\mu_{\alpha A_\alpha}(x) = \alpha \, \mu_{A_\alpha}(x), \quad \forall \, x \in X. \tag{2.7}$$

This theorem states that a fuzzy set A can be decomposed into a series of α-cuts A_α and that A can be reconstructed from them. Thus, any fuzzy set can be represented as a family of crisp sets in which each set is an interval at α. These sets are limited intervals of A at different levels of α. As α increases or decreases in the range of [0, 1], the α-cut interval will shrink or enlarge respectively.

Theorem 2.2 (*Extension principle*)

The extension principle introduced by Zadeh provides a general method for extending the classic, traditional, and crisp, mathematical concepts and arithmetic operations to the fuzzy environment. The extension principle is a rigorously mathematical method to fuzzify traditional, crisp functions using convolutions to map between the spaces.

Let $f: X \rightarrow Y$ be a mapping from a set X to a set Y. Then the extension principle allows us to define the fuzzy set B in Y induced by the fuzzy set A in X through f as follows [26]:

$$B = \{(y, \mu_B(y)) \mid y = f(x), x \in X \} \tag{2.8}$$

with

$$\mu_B(y) \stackrel{\Delta}{=} \mu_{f(A)}(y) = \sup_{y=f(x)} \mu_A(x), \qquad f^{-1}(y) \neq \phi$$

$$= 0 \qquad\qquad f^{-1}(y) = \phi,$$

where $f^{-1}(y)$ is the inverse image of y. When sets X and Y are finite, we can replace *sup* sign with *max*.

Definition 2.4 (*Cartesian product of membership function*)

Let A_1, …, A_n be fuzzy sets in X_1, …, X_n with the corresponding membership functions $\mu_{A1}(x_1)$, …, $\mu_{An}(x_n)$, respectively. Then the Cartesian product of the fuzzy sets A_1, …, A_n, denoted by $A_1 \times \cdots \times A_n$, is defined as a fuzzy set in $X_1 \times \cdots \times X_n$ whose membership function is expressed by [26]

$$\mu_{A_1 \times \cdots \times A_n}(x_1, ..., x_n) = \min(\mu_{A_1}(x_1), ..., \mu_{A_n}(x_n)). \tag{2.9}$$

Therefore, the extension principle can be extended to Cartesian space.

Theorem 2.3 (*Extension principle in Cartesian space*)

Let $f: X_1 \times \cdots \times X_n \to Y$ be a mapping from $X_1 \times \cdots \times X_n$ to a set Y such that $y = f(x_1, ..., x_n)$. Then the extension principle allows us to define the fuzzy set B in Y induced by the fuzzy set $A_1 \times \cdots \times A_n$ in $X_1 \times \cdots \times X_n$ through f as follows [26]:

$$B = \{(y, \mu_B(y)) \mid y = f(x_1, ..., x_n), (x_1, ..., x_n) \in X_1 \times \cdots \times X_n \} \tag{2.10}$$

$$\text{with } \mu_B(y) = \sup_{\substack{(x_1, ..., x_n) \in X_1 \times \cdots \times X_n \\ y = f(x)}} \mu_{A_1 \times \cdots \times A_n}(x_1, ..., x_n), \text{ if } f^{-1}(y) \neq \phi$$

$$= 0, \qquad\qquad\qquad \text{if } f^{-1}(y) = \phi$$

where if $f^{-1}(y)$ is the inverse image of y.

Using the concept of the α-cut, H. T. Nguyen showed the following equivalent theorem.

Theorem 2.4 (*Nguyen*) [26]

If there exists $x_1, ..., x_n$ such that $\mu_B(y) = \mu_{A_1 \times \cdots \times A_n}(x_1, ..., x_n)$ for any $y \in Y$, i.e., if the supremum of the above equation is attained for some $x_1, ..., x_n$, then it holds that

$$[f(A_1, ..., A_n)]_\alpha = f(A_{1_\alpha}, ..., A_{n_\alpha}).$$ (2.11)

This theorem states that when a given function is defined on a Cartesian product, the extension principle is still applicable. The only change is that the symbol x in Theorem 2.2 represents the n-tuples $x = (x_1, x_2, ..., x_n)$.

2.2   Fuzzy Numbers And Fuzzy Arithmetic

Fuzzy Numbers

We now investigate a special fuzzy set A defined on the set $\Re$ of real numbers. Membership functions of these sets have the form A: $\Re \to [0, 1]$. This subset with its properties is called *fuzzy numbers*. We formally define a fuzzy set A on $\Re$ as a fuzzy number if and only if the set A possesses at least the following four properties: [8]

(1)   A must be a normalized fuzzy set with $\mu_A(x) \in [0, 1]$;

(2) A_α must be a closed interval for every $\alpha \in (0, 1]$;

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(3)   The support of A is finite (i.e., A_0 must be bounded);

(4)   A is monotonically increasing on the left of $h(A)$, its height, and monotonically decreasing on the right of $h(A)$.

We'd like to mention here that a fuzzy number or a fuzzy variable is not a random variable. A random variable is a model of event variation, whereas a fuzzy number is a model of imprecision. We will emphasize their difference later on in our discussion.

Fuzzy Arithmetic

There are basically two distinct methods to handle fuzzy arithmetic. One is based on interval arithmetic and is restricted to fuzzy numbers, and the second is based on the extension principle. We briefly discuss them, assuming that fuzzy numbers have continuous membership functions.

Fuzzy arithmetic is based on two properties of fuzzy numbers: (1) each fuzzy number can uniquely be represented by its α-cuts (Decomposition theorem); and (2) α-cuts of each fuzzy number are closed intervals on real line for all $\alpha \in (0, 1]$. Then the arithmetic operation on fuzzy numbers can be converted to the arithmetic operation on their α-cuts and interval analysis used for computation. For discussions on interval analysis, see Jaulin [15] or Moore [24].

Let A and B denote fuzzy numbers and let * denote any of the four basic arithmetic operations (addition, subtraction, multiplication, and division). Then we define a fuzzy set on $\Re$, $A * B$, by defining its α-cut, $(A*B)_\alpha$ we have $(A * B)_\alpha = A_\alpha * B_\alpha$ from the Ngyuen theorem. Further, from the Decomposition theorem, $A*B = \bigcup_{\alpha \in [0,1]} (A*B)_\alpha$.

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Now we present the second method based on the Extension Principle, which extends the arithmetic operations from crisp sets to fuzzy sets.

Let * denote the four basic arithmetic operations, and let *A* and *B* denote fuzzy sets. We define a fuzzy set on $\Re$, *A* * *B*, based on the theorems 2.2 and 2.3,

$$\mu_{A*B}(z) = \sup_{z=x*y} \min[\mu_A(x), \mu_B(y)] \qquad (2.12)$$

For a detailed discussion on fuzzy arithmetic, see Kaufmann [16] and Klir [17].

3.   Proposed Method

3.1   Motivations

From Chapter 1, we can see that there are basically two prevalent approaches to do the tolerance analysis and parameter design for manufacturing process uncertainty. One is the probabilistic approach, modeling the performance characteristic variables as random variables, and then using probability theory and statistical methods to evaluate and estimate the manufacturing process variation. From this point of view, the uncertainty is from the randomness of events, and these events are chance outcomes of the process. This is known as *process randomness*. We have to do repeated observations to improve our knowledge about the process randomness, and model the process randomness as a probability distribution. The success of applying the conventional probability theory and various statistical methods into modeling this randomness relies on some fundamental assumptions, such as additivity and independence to qualify these approaches as probabilistic measures, which in some cases may be unreasonable or too strict in reality. To create the probability distribution model, we usually need repeated observations from experimental data. But in some cases, repeated observations may not be possible, since such events might be unique, unrepeatable, or costly to repeat.

The second prevalent approach in engineering is *high-low tolerancing*, or the so-called *worst-case method* [5]. This approach is most often employed when several components are assembled together. It is called worst-case analysis because it is used to add or subtract all the maximum or minimum tolerances associated with the nominal set point for each component. The worst-case tolerance buildup leaves the output variable at either its largest or smallest dimension. It is the safest method, but it does not take into

consideration the laws of probability. The drawback of it is that it always examines the worst case, but in practice, it is rare that all components in an assembly will simultaneously be at their maximum or minimum tolerance levels.

These two existing dominant methods focus on two extreme facets of the process. The former focuses on the information about the process only for the frequency of the random events, and the latter focuses on the severity of the worst case.

We present here a new methodology using a fuzzy set-based approach that generates consistent and comparable results with previous work, but does not need the assumptions of statistical approaches, nor does it need to focus on the worst-case scenarios, and it also provides new information about the process that the other approaches cannot.

3.2   Modeling Tolerance In Fuzzy Environments.

Instead of modeling manufacturing process tolerances as random variables in conventional ways, we model them as fuzzy variables to represent the process uncertainty. This section examines the theoretical foundations applicable to tolerance analysis and discusses interpretations of these foundations consistent with the needs of tolerance analysis.

3.2.1   Theoretical Foundations

Fuzzy set theory treats uncertainty in an essentially different way than probability theory in the following aspects [8], [18], [25]. An important distinction between probability and fuzzy sets is that probability is the theory of the random events. The event that the length of a rod that is designed to be one meter is never exactly one meter after machining is a random event. Probability theory is concerned with the chance or the

likelihood of the relevant events. In the example, we are concerned about modeling the likelihood of the rod length being one meter. This is an *event model*. Fuzzy set theory is not concerned with modeling events. It is rather concerned with concepts, like "long", "qualified", and whether or not an object, such as a particular part, or a state, such as a particular pressure reading, matches the meaning of the concept in question. In the rod example, from the fuzzy set perspective we are concerned about whether or not the length measurement is one meter, and to what degree it is one meter long. This is the distinction between fuzziness and randomness. Fuzziness describes the vagueness or ambiguity of an event, whereas Randomness describes the chance or the likelihood of the occurrence of the event. Typically, we use probability and membership value to describe the randomness and fuzziness, respectively. If we are told the probability of the rod length being one meter after machining is 0.8, then we know there is a 0.2 probability that the rod might not be one meter long. In fact, it could be extremely short, such as 0.1 meter long. But if we are told that the membership value of the rod length of one meter is 0.8, then we would have less uncertainty for the actual length because we know if it's not one meter long exactly, it will not be extremely short. We know this because the degree of the rod to be exactly one meter long is quite high (i.e. 0.8,) its length is in the vicinity of one meter.

The second significant distinction is that probability and fuzzy sets handle different kinds of uncertainty. Probability theory deals with the expectation of a forthcoming event, based on the knowledge currently known from past events. For instance, we ask what is the probability that the next rod cut from the band saw is one meter long? To answer this question we use the concept that the rod to be cut one meter

long is cut on the band saw that matches the length distribution of rods that are cut on the universal band saw. If we know that the band saw is very new and very accurate, we will certainly have a higher expectation that the next rod cut from this saw is close to one meter long. If the machine is old and badly calibrated, we will have a lower expectation that the next rod cut is close to one meter long. Therefore, our sense of uncertainty is more related to making a prediction about a forthcoming event.

The uncertainty represented by the fuzzy sets has nothing to do with the expectation. It is rather an uncertainty resulting from the imprecision of the meaning of a concept. Suppose we are in a workshop cutting rods for one meter long, and a rod has just been cut from a typical band saw. At this point, will it make sense to ask, "What is the probability that this rod is one meter long?" Now since this rod has already been cut from the line, we find ourselves no longer in a situation of facing expectation. We know for sure that that rod has its own length and that its length is consistent to a certain degree of one meter in length. Consequently, the more suitable question might be "To what degree is that rod one meter long?" If we compare the rod cut from a typical saw with rods cut from highly accurate saws, we might say, "It is one meter long to a small degree." If we compare it with rods from poorly calibrated saws, we might say, "It is one meter long to a high degree." We are describing the conformity of a particular characteristic (the length) of an object with a given imprecision concept (one meter long). The initial discussion of this concept is made by Klir in [18].

According to Mendel [22], Bezdek and Pal also addressed the distinction between probability and fuzzy sets and came up with a third point [1]. "Suppose you had been in the desert for a week without a drink and you came upon two bottles marked C and A.

Bottle C is labeled $\mu(C \in$ drinkable water) = 0.91 and bottle A is labeled Pr[A $\in$ drinkable water] = 0.91. Confronted with this pair of bottles, and given that you must drink from the one you choose, from which one would you choose to drink? Most readers when presented with this experiment immediately see that while C could contain, say, swamp water, it would not contain oil. A membership of 0.91 means that the contents of C are to degree of 0.91 similar to pure water. On the other hand, the probability that A is drinkable of 0.91 means that over a long run of experiments, the contents of A are expected to be drinkable in about 91% of the trials; in the other 9% the contents could be any liquid. Thus, most people would opt for bottle C. There is another facet to this example and it concerns the idea of observation. Suppose we examine the contents of C and A and discover them to be drinkable water and poison, respectively. If this is the case, then, after further observation the membership value for C, the degree of drinkability is unchanged and still 0.91. However the probability value for A changes from 0.91 to 0 because it is known with certainty that A is not pure water." This example shows that the two models possess different kinds of information -- fuzzy memberships represent the degree of belongingness of objects to imprecisely defined sets, whereas probabilities convey the information about relative frequencies.

Keeping this in mind, fuzzy sets are an ideal modeling tool to represent manufacturing tolerance, which usually is an allowed interval of values instead of an exactly clear single-point value. They closely model a human's approximate reasoning process and subjective judgments on design values, especially when the prerequisite presumptions for correctly applying probability are not met, such as the distributions for

component variables are not known, or that multiple observation is too costly or impossible, etc.

To this point, it is very natural to model the tolerance as fuzzy variables. We can treat the membership function as the degree of conformity of the performance characteristic to its target value, and the degree varies within [0, 1]. On one hand, we have had theoretic bases for interpreting the representation of a graded degree of conformance with a quality standard so far as is discussed in [3], [31]; on the other hand, representative crisp values for the fuzzy measures could be found using standard methods as we will discuss in Section 3.2.3, such as fuzzy mode, α-cut midpoint, fuzzy center of gravity, fuzzy average, or fuzzy mean deviation, etc, to indicate the fuzziness, and we can use them to carry out the analysis and the evaluation of the process from the perspective of fuzzy sets theory just as we usually do in statistics using the equivalent terms, such as mean, variance, etc. So based on this new approach, we can not only get the comparable and consistent results as previous work, but also get some particular information which the previous work could not provide. We will show this in detail in the later sections.

3.2.2 Membership Function Interpretation

Suppose there is a performance characteristic variable x, which is modeled as a fuzzy number instead of a random variable. Its target value is 7, i.e., we design it to be 7. It's α-cut, when $\alpha = 0.6$, is [6, 8]. Figure 3.1 is its membership function showing the degree of conformity of x to the target value 7.
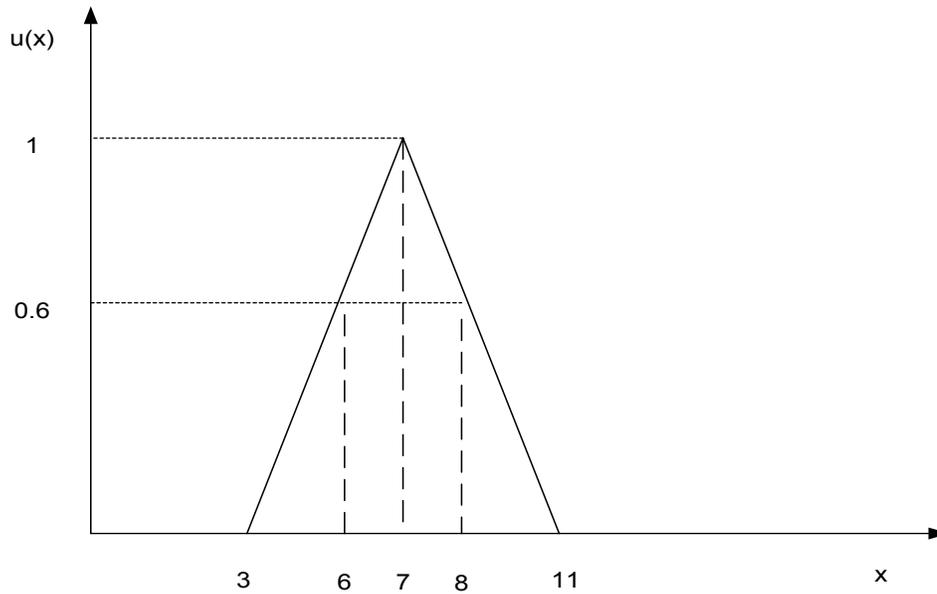
Figure 3.1. Membership function for fuzzy variable x.

Under perfect circumstances, which means the manufacturing process can guarantee producing what we design, we can machine this x to be exactly 7. In this case we say this variable x has 100% degree of conformity to the target value, i.e., the product rod is completely up to standard. But, in reality, the manufacturing process cannot be so accurate.   It will always generate more or less variations. Observing the above membership function, we say, at the level of 60% degree of conformity with respect to the target value of 7, (i.e., the membership is 60% now), the output variable x will turn out to be either 6 or 8. In other words, now the value of 6 and 8 has the membership of 60% belonging to the set (what we design it to be). Furthermore, for the interval between 6 and 8, (i.e., any numerical value between 6 and 8) has the membership greater than or equal to 60%. For instance, 6.777 might have the membership of 62%, which means it has the 62% degree of conformity to the target value of 7. It makes sense in that 6.777 is

closer to 7 than 6, so it has a higher degree of conformity with respect to the target value 7.

Here, the fuzziness, the uncertainty, or the ambiguity of the actual variable value of x is demonstrated on the degree of conformity with respect to the designed target value. From Figure 3.1, we can see that the left and right boundaries of the support are 3 and 11, respectively, so any value out of this range [3, 11] has definitely 0 degree of conformity to the target value, which means that the product is completely below standard and undesirable. Seven (7) has the highest membership value, i.e., the best conformity, and that is exactly what we desire from the point of view of quality conformance. Any other x within this range has a different membership value varying from 0 to 1. Since the membership function is convex and normal, the closer the x is to the target value, the higher membership value it has.

We should point out that it seems that while the fuzzy quality modeling is quite different from the crisp one, one can be converted into the other. It is easy to see this from Theorem 2.1 (Decomposition Theorem). From Figure 3.2, if we choose a series of α levels: $\alpha_1 < \alpha_2 < \alpha_3$…then we can get a series of corresponding α-cuts that are numerical intervals: $A_{\alpha_1}, A_{\alpha_2}, A_{\alpha_3}, \cdots$, we can easily see that $A_{\alpha_1} \supset A_{\alpha_2} \supset A_{\alpha_3} \supset \cdots$. That is to say the higher the membership value is, the narrower the corresponding α-cut will be, and the closer the variable is to its target value, if the membership function is convex.
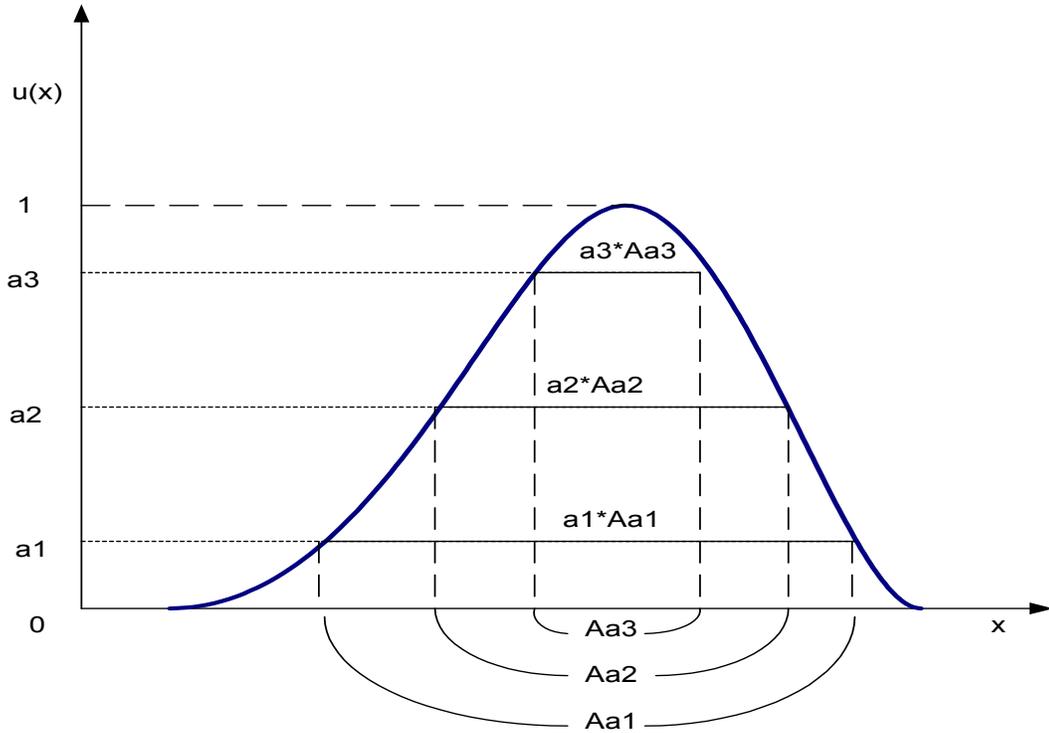
Figure 3.2. Illustration of Decomposition Theorem.

Once this interpretation of fuzzy tolerance is established, we can either carry out the tolerance analysis or the parameter design. But they are two different approaches. (1) For the parameter design problem, it means at a certain membership value (certain degree of conformity) we want the α-cut span as narrow as possible. In other words, for a narrower α-cut span, we can get a higher degree of conformity. (2) For tolerance analysis, we can see explicitly the different α-cut span at different degrees of conformity, and then do the analysis.

Now, we have several interesting observations based on the interval analysis:

(1) Since the membership function, $\mu(x)$, shows the degree of conformity (or the degree of compatibility) of the performance characteristic x to its target dimension, it's clear to see that when $\mu(x) = 1$, x is a single-point value with its highest degree of conformity equal to 1. When $\mu(x)$ drops, x becomes an interval (i.e., it deviates from its target value), the degree of conformity drops simultaneously. When $\mu(x) = 0$, the interval is widest, and it actually is the worst-case interval. Since the membership value at this interval is zero, it says any value beyond this interval will NOT have a membership value, i.e., any value out of this worst-case interval no longer belongs to this set.

(2) If every component variable is at the perfect value of its $\mu(x)=1$ (each is at its best conformity), by the transformation function, the output variable should also be at its perfect value with its $\mu(y)=1$. The input and output variables are single-point values in this case.

(3) If one of these component variables, or some, or all of them deviate from their target values (i.e. the membership function $\mu(x_i) < 1$), the degree of conformity of components drop correspondingly, as does the degree of conformity of the output variable. By applying Theorem 2.4, we can compute the α-cut of an output variable according to the corresponding α-cuts of input variables. Now, the output variable is an interval.

To close this section's exploration of the fuzzy membership function for tolerancing, we are going to adapt an example from Wang, et al. [31] to distinguish the membership function $\mu(x)$ from the probability density function $f(x)$. We consider a statement "a rod's length x is one meter," with variable x taking values from the discrete set {0.8, 0.85, 0.9, 0.95, 1, 1.05, 1.1, 1.15, 1.2}. We associate a membership function

μ(x) with x by interpreting μ(x) as the degree of conformity of x with respect to 1. We also associate a probability mass function p(x) with x by interpreting p(x) as the probability frequency of observing the occurrence of corresponding values. Table 3.1 shows some hypothetical values of μ(x) and p(x). What's worth mentioning is that typically the μ(x) is obtained subjectively, like one expert or a group of experts' judgments, whereas the f(x) is obtained objectively from observation of repeated experiments.

x	0.8	0.85	0.9	0.95	1	1.05	1.1	1.15	1.2
μ(x)	0	0.05	0.6	0.8	1	0.9	0.7	0.3	0
p(x)	0.1	0.1	0.1	0.1	0.2	0.1	0.1	0.1	0.1

Table 3.1. Membership function vs. probability mass function.

From table 3.1, we see that for x = 0.85, the membership value is 0.05, whereas it has a probability frequency of 0.1. We can conclude that they really measure two completely different aspects of the uncertainty. A higher membership value does not imply a higher probability; nor does a lower probability imply a lower membership value.

3.2.3 Representative Crisp Values In Fuzzy Measures.

After the fuzzification of the process, we still need a crisp number value to represent the fuzzy sets associated with the process for our further evaluation and computation. Different processes have different fuzzy sets associated with them to represent the processes themselves. To compare these processes, we need to convert the fuzzy sets to some crisp numbers, such as the mean, deviation, etc, and some equivalent

terms in statistics to evaluate the processes. In other words, we need to find some representative values to represent, indicate, and measure the fuzziness of the process. We can also view this step as a defuzzification process. Here we introduce some representative values from existing literature [16], [31] we will use in our approach. They are in some sense equivalent to the measures of central tendency in statistics, such as mean and variance. It should be pointed out that there is no unique criterion to say which one is best or accurate; the different choice should be made with different applications and practical context.

(1) The Fuzzy Mode, x_m or f_{mode}

The fuzzy mode of a fuzzy set A is the *nominal* value of set A, i.e., the value whose membership function is equal to 1. It is unique if the membership function is normal and unimodal.

$$x_m = \{x \mid \mu_A(\mathrm{x}) = 1\}, \ \forall \ x \in A. \tag{3.1}$$

Fuzzy mode is easy to calculate; however, the fuzzy mode will generate a biased result when the membership function is nonsymmetrical.

(2) The Fuzzy Center of Gravity, f_{cog}

The fuzzy center of gravity of a fuzzy set A is frequently used as a defuzzification method. It can be interpreted as an "expected value" of a fuzzy variable. According to Klir [17], it is defined as the value within the range of variable x for which the area under the graph of the membership function $\mu(\mathrm{x})$ is divided into two equal subareas. It is calculated by the formula:

$$f_{cog} = \frac{\int \mu_A(x) \cdot x dx}{\int \mu_A(x) dx}, \ \ \forall \ x \in A. \tag{3.2}$$

(3) The α-cut Midpoint, $f_{mp}(\alpha)$

It is defined as the midpoint of the endpoints of a certain α-cut. If a_α and b_α are the two endpoints of the α-cut, i.e., A_α, then:

$$f_{mp}(\alpha) = \frac{1}{2}(a_\alpha + b_\alpha). \qquad (3.3)$$

Obviously, the fuzzy mode is a special case of the $f_{mp}(\alpha)$, when $\alpha = 1$. Since we can take α at any value from 0 to 1, the α-cut midpoint will give us more flexibility in analysis.

(4) The Mean Deviation or Divergence for a Fuzzy Set, δ(A)

A quantitive measurement of the dispersion or span of a fuzzy set is called mean deviation, δ. Consider a convex fuzzy set with x_m as its mode, and with $x_l(\alpha)$ and $x_r(\alpha)$ as the left and right endpoints of an α-cut, respectively. δ can be obtained as the sum of the *left mean deviation*, δ_l, and the *right mean deviation*, δ_r, which, in turn, are defined as [16]:

$$\delta_l = \int_{\alpha=0}^{1}[x_m - x_l(\alpha)]d\alpha, \qquad (3.4)$$

$$\delta_r = \int_{\alpha=0}^{1}[x_r(\alpha) - x_m]d\alpha. \qquad (3.5)$$

and $\qquad \delta(A) = \delta_l(A) + \delta r(A)$

$$= \int_{\alpha=0}^{1}[x_r(\alpha) - x_l(\alpha)]d\alpha \qquad (3.6)$$

By definition, the left mean deviation and the right mean deviation are equal to the areas under the membership function curve to the left and right of the mode x_m, respectively. They can also be interpreted as the mean degree of membership for

variables on either side of the mode. The *mean deviation* $\delta(A)$ is used in our approach to represent the dispersion of a fuzzy set A.

Now let us take a triangular fuzzy number (TFN) A, as is shown in Figure3.3, with mode of x_m, left endpoint of a, and right endpoint of b. By computation,

$$\delta_l(A) = \frac{1}{2}(x_m - a), \tag{3.7}$$

$$\delta_r(A) = \frac{1}{2}(b - x_m), \tag{3.8}$$

and $\quad \delta(A) = \frac{1}{2}(b - a). \tag{3.9}$

The triangular fuzzy number's mean deviation is given by the length of α-cut at $\alpha = 0.5$; that is, half the base of the triangle.
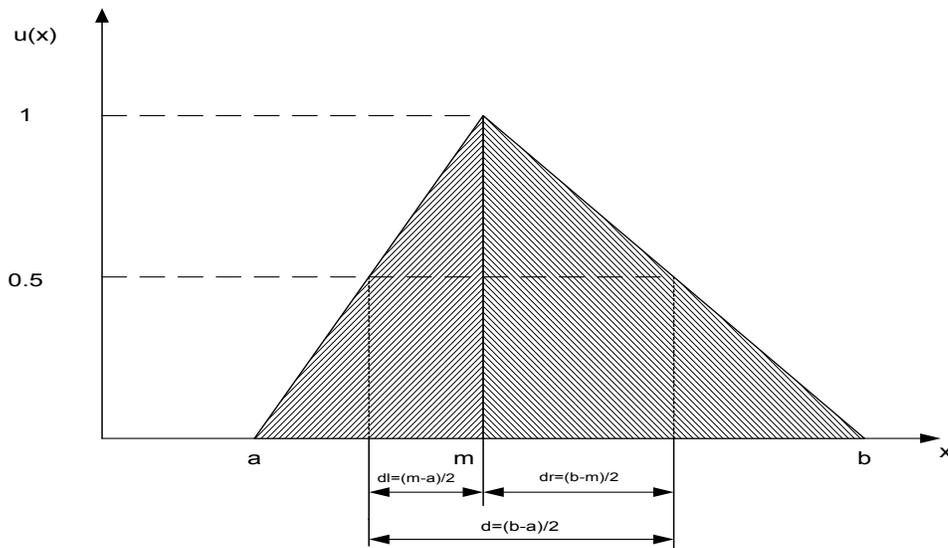
Figure 3.3. Fuzzy mean deviation.

3.3   The Proposed Approach

Here is the fuzzy set approach we are proposing. Basically, we model the manufacturing process imprecision and the engineering tolerance by fuzziness instead of the conventional randomness in such a way that we first assign each component variable a membership function to fuzzify the input variables; then we proceed with fuzzy arithmetic operations based on the transfer function; and next we can either do the tolerance analysis or solve the parameter design problem in a fuzzy environment.

3.3.1   Assign Membership Function To Each Input Variable.

Since the membership function essentially embodies all information on imprecision for a particular fuzzy set, its description and construction are the essence of a fuzzy operation and analysis. The question of determining a valid membership function closely reflecting the reality of a problem is critical to the successful application of fuzzy sets. Actually, it is the "shape" of the membership function that makes the difference here. In this section, we focus on the construction of a membership function.

Numerous methods have been proposed to construct or estimate the membership function [8], [25]. Obviously, a membership function is a matter of degree, essentially. So it is generally not absolutely defined objectively but rather assigned or decided subjectively. This assignment or estimation procedure can be intuitive or it can be based on some experience or some logical inference. These methods may be based on statistical data, relative preference, neural networks, genetic algorithms, inductive reasoning, etc [25]. Most of the methods are based on experts' judgments. For instance in an expert panel, the experts are expected to answer various kinds of questions pertaining to the construction of the membership function, and then take the weighted average of their

answers to decide the membership function. Of course, the more accurately the membership function is constructed, the better the solution that will be generated from the analysis. In our approach, we will use a simple scenario to describe the process and the idea. In practical application, we may resort to more accurate methods.

Suppose we are interested in building the membership function of the rod length cut from our workshop. It is supposed to be one meter long. We let the rod's actual length cut from the saw be the fuzzy variable x.

If the machining process is perfect, then the rod length would be 1 meter. Then we say, "1" definitely belongs to the set of rod length that is cut from that machine, i.e., "1" has membership value 1. Any other value, like "1.00001" or "9.999999" could not even be machined, so they definitely do not belong to that set, that is to say, their membership value is 0. Thus, the membership function in this circumstance is a point, a crisp set, as is shown in Figure 3.4. Here the boundary of belonging or not to the set is very clear and crisp. But in reality, this rarely happens. What is more suitable is that this boundary of dividing the belongingness to the set is fuzzy. Although it is not possible to decide the membership value for every real number point, the expert should be able to exemplify it for some representative elements of x. Then we go to the expert, who has the experience of how the process is, and consult with him.

Q: "To what degree do you think a one meter long rod cut from this machine has the conformity to one-meter length?"

A: "Of course 1."

Q: "Then what length value of the rod cut from this machine do you think has zero degree of conformity to one-meter length?"

A: "Since the rod coming out from the machine could never be longer than 1.1 meter or shorter than 0.97 meter, I would say any value less than 0.97 or greater than 1.1 has zero degree. For instance, a two-meter long rod definitely does not belong to the rods that are made from this machine, so for this process, two has zero degree belongingness or conformance."

Q: "What is the conformity degree of 0.99 with respect to the one meter?"

A: "Since 0.99 doesn't have the 100% belongingness (or conformance) to the set as 1 does, nor does it have the 0 belongingness, it has only partial conformance and has some degree between 0 and 1, like 0.8."

…

Eventually, we can plot the whole membership function by applying an appropriate curve-fitting method (see Figure 3.4). It should be pointed out that the procedure is subjective. We are dealing with a matter of degree here, not a matter of frequency. The shape of the membership function is subjectively dependent on how you recognize the process or how much knowledge you have of the process. As we discussed before, the key issue in a fuzzy application is how to determine the membership function. Whether or not you have constructed a valid membership function determines the success or failure of your application to a great extent.
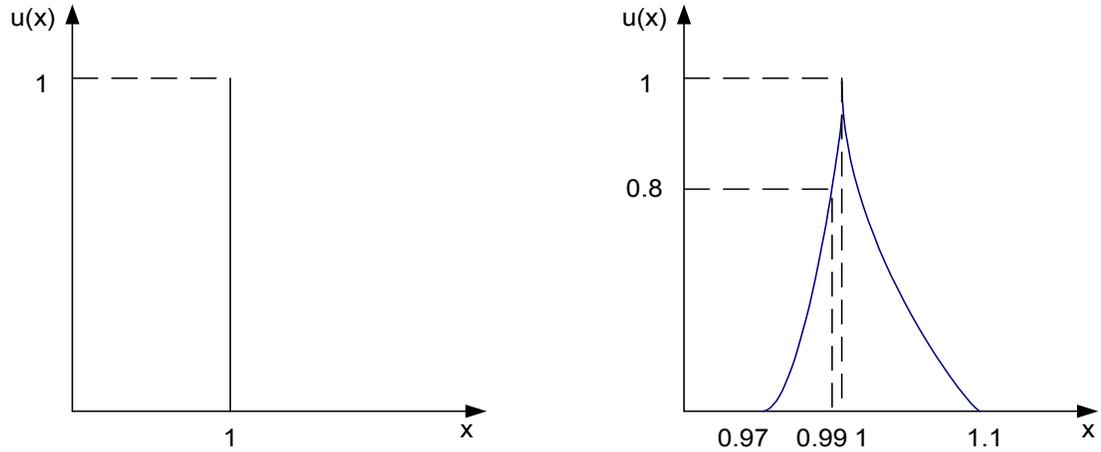
Figure 3.4. Constructing membership function.

3.3.2 Fuzzy Arithmetic Operations For The Transfer Function.

Once we get the membership function for each component variable, x_i, we substitute them into the transfer function, which typically is known or given, and then apply fuzzy arithmetic and algebraic operations to the function to get a fuzzy output variable. Theoretically, we should use the Extension Principle, which extends the standard arithmetic and algebraic operations on crisp sets to the fuzzy arithmetic and algebraic operations. But in reality, from the computational point of view, the extension principle is very difficult to apply for continuous-valued mapping. There exist some other commonly used computational algorithms as presented by Dong, et al. [7].

It is common in fuzzy analysis to discretize the continuous support into a finite number of points and approximate the fuzzy set by the one defined on the discrete domain. This practice works fairly well in general, but sometimes, there is a serious problem in propagating the fuzzy operations. This problem is the irregular and erroneous

shape of the output variable's membership function once we discretize the input variables for computational convenience. Examples are given in [6], [7]. The reason for this problem is that when we apply the extension principle, Theorem 2.3, we actually solve a nonlinear programming problem for the optimization. When we discretize the input variables we may omit some portions of the solution space. In fact, this seems to be inevitable during the discretization process.

An algorithm known as the Vertex Method, which is developed by Dong, et al. [6], [7], fixes the above problems and also serves our intention quite well. The method is based on the α-cut concept and standard interval analysis. Since α-cut is a discretization on the membership value domains of variables instead of on the variables domains themselves, the method can prevent the abnormality and irregularity in the output membership function. What is more important is that it also prevents the widening of the resulting function value set due to the multiple occurrence of input variables in the function by traditional interval analysis [15], [24].

The Vertex Method works as follows:

Consider a function $y = f(x_1, x_2, ..., x_n)$,

When the variables $x_1, x_2, ..., x_n$ are independent and have values in the intervals $(X_1, X_2, ..., X_n)$ respectively, the admissible domain in the independent-variable space is an n-dimensional rectangle. The value of the function

$$Y = f(X_1, X_2, ..., X_n),$$

is an interval defined by

$$Y = \{f(x_1, x_2, ..., x_n) \mid x_1 \in X_1, x_2 \in X_2, ..., x_n \in X_n)\}$$

By taking various α-cuts, each fuzzy component variable characterized by a convex membership function is converted into a group of crisp, classic intervals associated with a corresponding α level. According to Theorem 2.4, the intervals with the same α level value from all fuzzy component variables are processed by using standard interval analysis, resulting in an interval at the same α level for the output fuzzy variable, i.e.,

$$y_\alpha = f(x_{1_\alpha}, x_{2_\alpha}, ..., x_{n_\alpha}),$$

where

$$y_\alpha = [a_{y_\alpha}, b_{y_\alpha}], \quad x_{1_\alpha} = [a_{1_\alpha}, b_{1_\alpha}], \quad x_{2_\alpha} = [a_{2_\alpha}, b_{2_\alpha}], \quad ..., \quad x_{n_\alpha} = [a_{n_\alpha}, b_{n_\alpha}],$$

and a_i and b_i are the endpoints of each particular α-cut interval.

Now, the problem is equivalent to solving a minimization problem to get the lower bound for the output interval, and a maximization problem to get an upper bound for the output interval as follows:

$$a_{y_\alpha} = \min f(x_1, x_2, ..., x_n), \quad b_{y_\alpha} = \max f(x_1, x_2, ..., x_n).$$

Such that $x_1 \in [a_{1_\alpha}, b_{1_\alpha}]$, $x_2 \in [a_{2_\alpha}, b_{2_\alpha}]$, ..., $x_n \in [a_{n_\alpha}, b_{n_\alpha}]$.

If the function f is continuous within the domain, the value of the function can be obtained as follows:

$$y_\alpha = [\min_{j,k}(f(V_j), f(E_k)), \max_{j,k}(f(V_j), f(E_k))] \tag{3.10}$$

where $V_j, j = 1, 2, ..., 2^n$, is the coordinate of the j-th vertex (i.e., the corner point) of the rectangular domain, and E_k is the coordinate of the k-th extreme point (i.e., the point at which its partial derivatives are all equal to zero) in the domain. If there is no interior extreme point in the feasible domain, and since each component interval can only take on

one of two values (the endpoints of the interval), n component intervals can have at most 2^n different combinations. By exhausting all possible permutations of the combinations, and computing the results (the minimum and the maximum) for each, the 2^n sets of results must constitute the complete range of possible results [4]. Therefore, the lower and upper bounds for the output interval are obtained. If the extreme points are inside the feasible domain and can be identified, they are simply treated as additional vertices, E_k, in the Cartesian space. This algorithm reduces the interval analysis problem for a functional mapping to a simple procedure dealing only with the endpoints of the intervals.
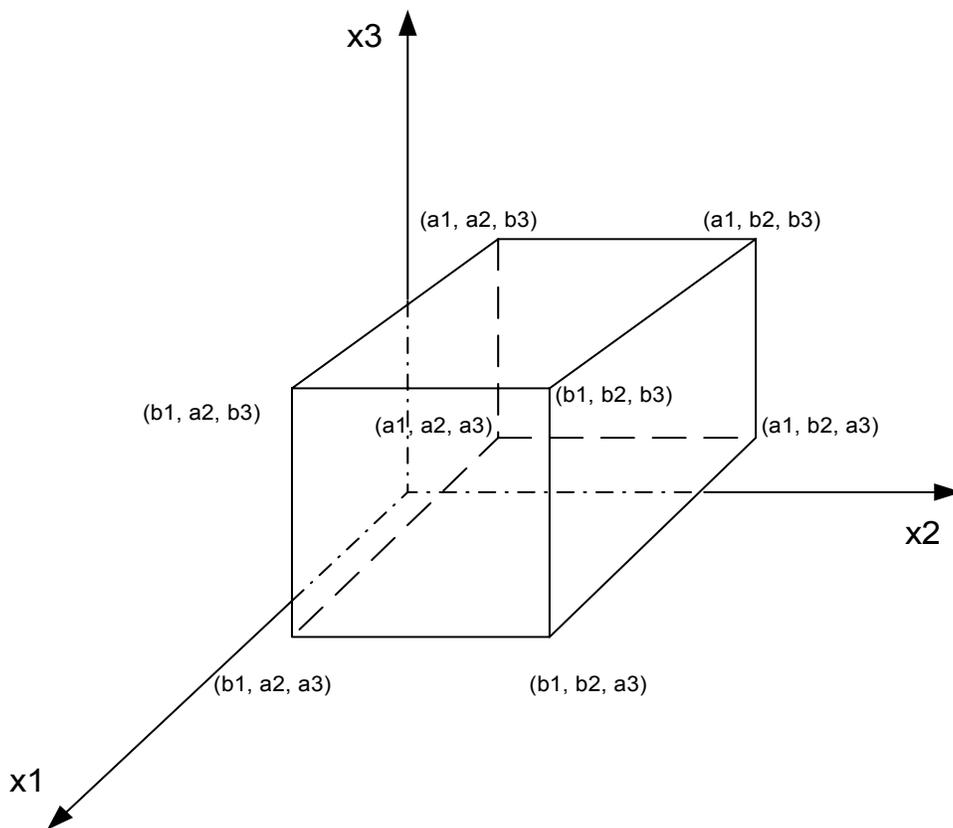
Figure 3.5. 8 vertices in 3-dimensional Cartesian space.

Figure 3.5 shows the 3-dimensional case. The ordinates of all input variables actually are all the combinations of n pairs of endpoints of the interval numbers. For this case, we have $2^3 = 8$ (i.e., $(a_1a_2a_3)$, $(a_1a_2b_3)$, $(a_1b_2a_3)$, $(a_1b_2b_3)$, $(b_1a_2a_3)$, $(b_1a_2b_3)$, $(b_1b_2a_3)$, $(b_1b_2b_3)$) combinations. We need only evaluate the function y = f(x) at these 8 corner points to get the endpoints for output variable intervals, if there are no extreme points in this domain.

In the remainder of this thesis, we further assume that transfer function is linear within the relevant feasible domain, so there is no interior extreme point in the feasible domain. Therefore, the maximum and minimum values of the function occur only at the vertices. In this case, (3.10) can be rewritten as:

$$y_\alpha = [\min_j(f(V_j)), \max_j(f(V_j))]$$ (3.11)

where V_j, $j = 1, 2, \ldots, 2^n$, is the coordinate of the j-th vertex of the rectangular domain.

Thus, we proceed our approach with,

Step (1), we take the α-cut for each component variable at the same α level at a time to get the corresponding intervals for all component variables x_i. Then by applying the vertex method (3.11), we can get the corresponding α-cut of output variable y at that particular α level, according to Theorem 2.4.

Step (2), we repeat step (1) by taking a different α level within the range of [0, 1], and based on Theorem 2.1 (Decomposition theorem), we can approximate the output membership function by adding the α-cuts from 0 to 1.

We can either stop at step (1) to study different α-cuts for the output variable, or continue to step (2) to study the whole membership function shape for the output variable, and then do further analysis.

3.3.3 Analysis or Parameter Design Based on the Output Membership Function.

Basically, at this point, we can either proceed to (1) the tolerance analysis or (2) the parameter design problem, as we need.

(1) When we get the new membership function of the output y, we can do the tolerance analysis based on the representative crisp values of fuzzy measures (i.e., fuzzy mode, α-cut midpoint, center of gravity, and mean deviation, etc.), to evaluate the degree of conformity of output variable y to the target output value τ. We will present detailed case studies to illustrate this procedure. Actually this procedure works very much like a typical fuzzy controller system, as is shown in Figure 3.6. The difference is that the controller uses a rule-based inference engine [25], whereas our method uses a vertex method to carry out the fuzzy function transformation.

One advantage of this method is that as long as we can correctly construct the input variables' membership functions, we can easily obtain the output membership function, consequently obtaining the other measurement of fuzziness, such as center of gravity and mean deviation, which are counterparts to mean and standard deviation in statistics. This approach does not need statistical assumptions, nor repeated experiments.

(2) For the fuzzy parameter design problem, we can find a closed form to represent the deviation of the fuzzy output variable at different α-cut levels. Using nonlinear programming techniques, we can minimize this deviation by finding the optimal input point set. During this procedure, we can explore alternate objectives, such as to achieve the highest symmetry of the output variable, or to obtain new information.
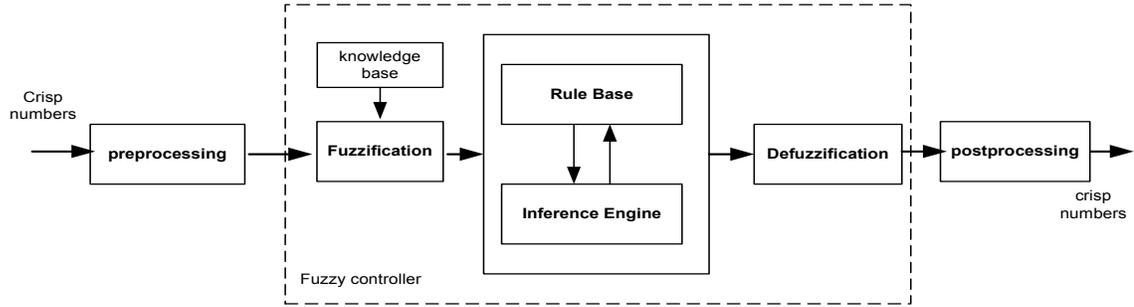
Function blocks of fuzzy controller
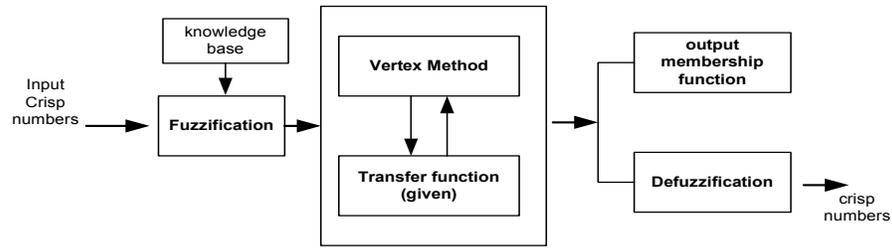
Function blocks of fuzzy tolerance analysis method

Figure 3.6. Analog of fuzzy tolerance analysis procedure.

We will use the following sections to show some case studies by applying our method. Most of these cases are from the open literature and are solved by several methods. We chose them mainly for the purpose of comparing our results with previous results.

4.   Fuzzy Tolerance Analysis

In this chapter, we present the proposed procedure to carry out our method for the tolerance analysis problem through one case study from the open literature, and compare the results with previous work discussed in Section 1.1. Then we present a new idea to achieve the output tolerance symmetry in Section 4.2.

4.1   Fuzzy Tolerance Analysis Problem Case Study

We adopt the following coil spring case from Fathi [12]. Our objective is to illustrate the proposed fuzzy tolerance analysis procedure.

Case Study 4.1: Coil Spring

For a coil spring example, we have the nonlinear transfer function as

$$y = \frac{D^3 N}{143750 d^4} \tag{4.1}$$

where, y: Deflection of the spring (inches) at 10 lb. Force

D: Base diameter (inches)

N: Number of active coils

d: Wire diameter (inches)

y is the output variable, whereas D, N, and d are input (component) variables in this case. Specification limits for the deflection y are 0.5 ± 0.2 inches.

Nominal values for D, N, and d are 0.357 inches, 11.29 coils, and 0.0517 inches, respectively. Standard deviations of D, N, and d are estimated to be 0.0197, 0.185 coils, and 0.00174 inches, respectively. Assuming that D, N, and d are centered at their respective nominal values, and the *process-capability ratio* (PCR) for each dimension is

equal to 1, determine the mean and variance of y using the appropriate approximation method. What are the nominal value and natural tolerance limits for the deflection y?

(1) First, let us present the results of using the Functional Approximation method introduced in Section 1.1.2:

As the problem stated, D, N, and d are random component variables, with expected values $\mu(D) = \mu_D = 0.357$, $\mu(N) = \mu_N = 11.29$, and $\mu(d) = \mu_d = 0.0517$; and variance $Var(D) = \sigma_D^2 = 0.0197^2$, $Var(N) = \sigma_N^2 = 0.185^2$, $Var(d) = \sigma_d^2 = 0.00174^2$, respectively. The transfer function $y = h\ (D,\ N,\ d) = \dfrac{D^3 N}{143750 d^4}$ is given. Now the problem is to compute the mean and variance of the output variable, y.

We denote the mean and the variance of output variable, y, as μ_y and σ_y^2, respectively. By applying the functional approximation method expressed in (1.1) and substituting the above known values, we can get:

$$\mu_y = \frac{\mu_D^3 \mu_N}{143750 \mu_d^4} = 0.5002 \tag{4.2}$$

$$\sigma_y^2 = (\frac{\partial h}{\partial D}\big|_{\mu_D,\mu_N,\mu_d})^2 \sigma_D^2 + (\frac{\partial h}{\partial N}\big|_{\mu_D,\mu_N,\mu_d})^2 \sigma_N^2 + (\frac{\partial h}{\partial d}\big|_{\mu_D,\mu_N,\mu_d})^2 \sigma_d^2 = 0.0117$$

$$\sigma_y = 0.1082$$

Using this method, we can estimate by computation the first and second moment of the output variable y, i.e., the mean and the variance. But we usually do not continue to the higher moments, i.e., the skew and the kurtosis. We have to assume the output y is also centered, and then use NTL = $\mu_y \pm 3\sigma_y$ to get the natural tolerance limits. So, we would answer that the expected value for the deflection y is 0.5002 inch, and the NTL is [0.1756, 0.8248].

(2). In our proposed method, we treat these input and output variables as fuzzy variables instead of the original random variables. To apply our fuzzy analysis, first we construct the membership function for each component. Since we know that D, N, and d are centered at their respective nominal values, and the PCR for each dimension is equal to 1, we conclude that the dimension outside $[\mu - 3\sigma, \mu + 3\sigma]$ is almost impossible. We choose these two points as the lower bound and upper bound for the endpoints of the membership function. For computational simplicity, we assume the membership function is triangular. Take D as an example. The nominal value is 0.357, the left and right supports both are $\pm 3\sigma_D = \pm 3*0.0197 = \pm 0.0591$. We can express this fuzzy number as (0.2979, 0.357, 0.4161), where 0.2979 stands for the lower bound, 0.357 for the nominal value, and 0.4161 for the upper bound. The fuzzy number is shown in Figure 4.1.

Next, we need to find the endpoints of this fuzzy variable D at different membership values, i.e., α levels. Suppose we take a certain α-cut as in Figure 4.1, and let a denote the lower bound, and b denote the upper bound for this α-cut. From the trigonometric relationships, we have

$$a_{D_\alpha} = 0.357 - 0.0591*(1-\alpha), \text{ and}$$

$$b_{D_\alpha} = 0.357 + 0.0591*(1-\alpha) \tag{4.3}$$

Thus, we can denote this α-cut as $D_\alpha = [a_{D_\alpha}, b_{D_\alpha}]$, or,

$$D_\alpha = [0.357 - 0.0591*(1-\alpha), 0.357 + 0.0591*(1-\alpha)]$$

In the same way, we can also generate the α-cuts for N and d as follows:

variables	nominal	lower bound	upper bound
D	0.357	0.2979	0.4161
N	11.29	10.735	11.845
d	0.0517	0.04648	0.05692
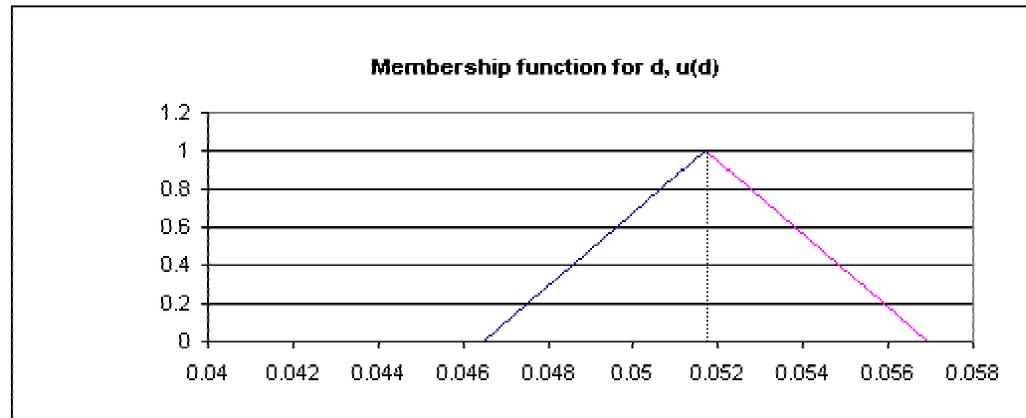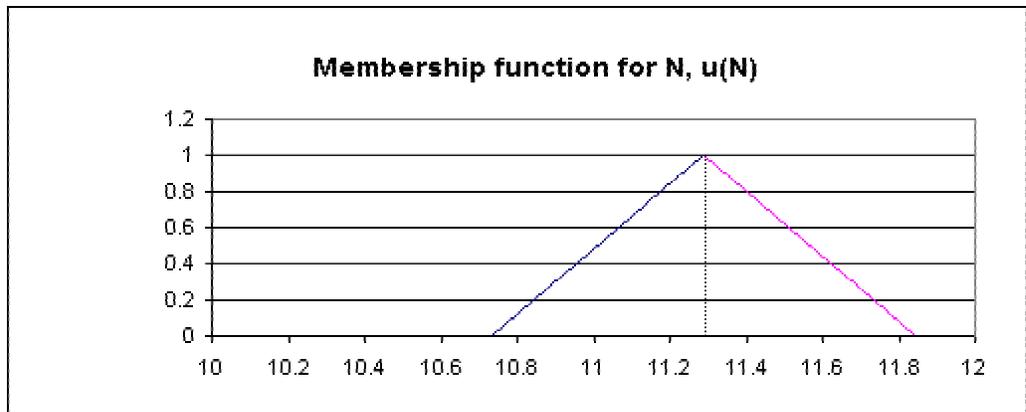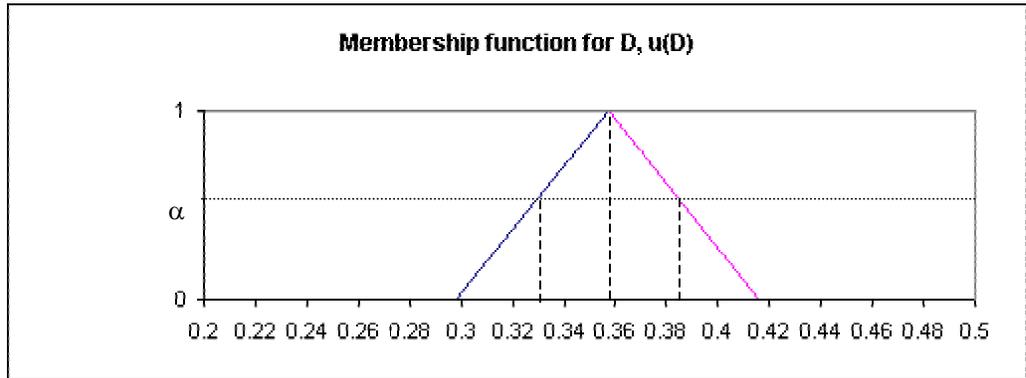
Figure 4.1. Membership functions for input variables, *D, N*, and *d*, in spring analysis case.

$$N_\alpha = [11.29 - 0.555*(1-\alpha), 11.29 + 0.555*(1-\alpha)]$$

$$d_\alpha = [0.0517 - 0.00522*(1-\alpha), 0.0517 + 0.00522*(1-\alpha)]$$

Since there are no extreme points for function $y = h\ (D,\ N,\ d)$ within the feasible domains, we apply the vertex method (3.11) by only considering $2^3 = 8$ vertices (there are 3 variables, and each of them has two endpoints) to get the corresponding α-cut for the output deflection y_α at that particular α level. Since there are 8 combinations of the input coordinates, we evaluate the function 8 times to get the minimum and maximum values as the lower bound and upper bound as the endpoints of the output variable at the same α level. By taking sufficient α cut levels ($\alpha \in [0,\ 1]$) into consideration, we can approximate the membership function of the output variable, the deflection y, as shown in Figure 4.2.

Observing the output membership function, shown in Figure 4.2, not only can we quickly compute the fuzzy mode (nominal value), mean deviation, but also we can observe that the membership function of the output variable is skewed to the right, which gives us more information about this process than the previous analysis. For the output variable, y, we apply the computation introduced in Section 3.2.3 as follows,

The fuzzy mode, $f_{mode}(y) = 0.500$,

The fuzzy mean deviation, $\delta(y) = 0.542$,

The fuzzy center of gravity, $f_{cog}(y) = 0.667$.

α-cut for output deflection

α	Lower bound	Upper bound
0	0.188	1.272
0.05	0.198	1.215
0.1	0.208	1.160
0.15	0.219	1.107
0.2	0.230	1.057
0.25	0.242	1.010
0.3	0.254	0.964
0.35	0.267	0.920
0.4	0.280	0.878
0.45	0.294	0.839
0.5	0.309	0.800
0.55	0.325	0.764
0.6	0.341	0.729
0.65	0.358	0.696
0.7	0.375	0.664
0.75	0.394	0.633
0.8	0.413	0.604
0.85	0.434	0.577
0.9	0.455	0.550
0.95	0.477	0.524
1	0.500	0.500

Membership function for the output, u(deflection)
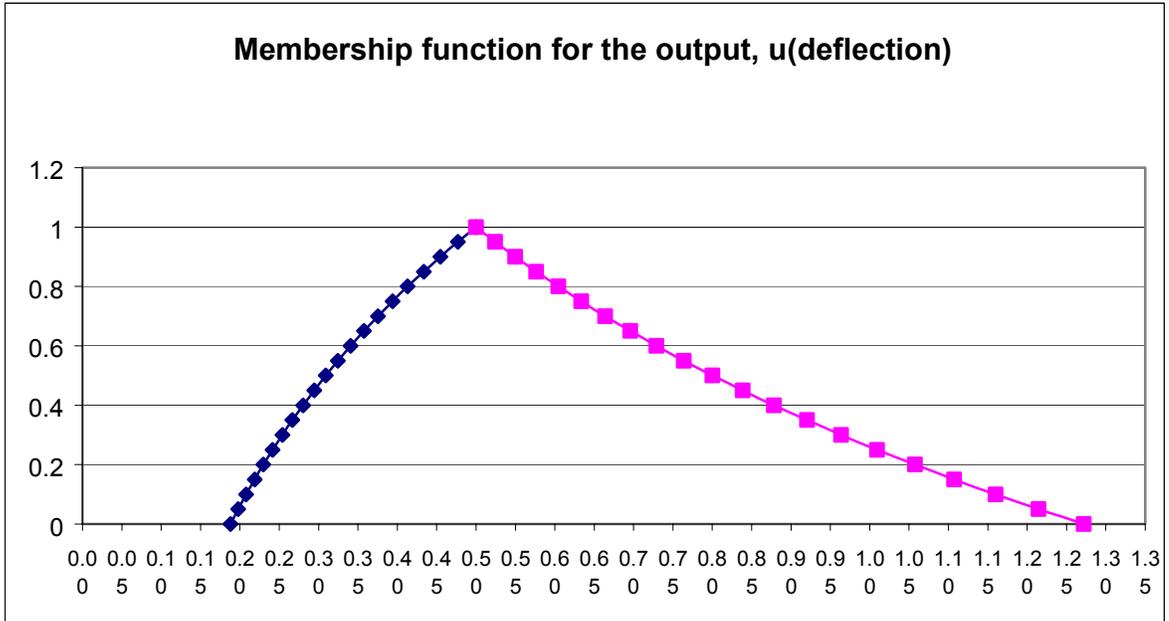
Figure 4.2. Membership function for output in spring case.

Here, we approximate the output membership function with a triangular shape and then calculate the center of gravity. Since it is not easy to express its membership function exactly, we did not calculate the exact center of gravity explicitly as it is defined in Section 3.2.3. However, it is easy to see that during this approximation, some area mass under the membership function curve to the left of the nominal value is switched to the right, the calculated $f_{cog}(y)$ by approximation is actually slightly overshot. That is to say the true center of gravity should be a little bit smaller than 0.667.

From the center of gravity, which we interpret as the equivalent term with the expected value in probability theory, we can see the process, i.e., the manufacturing process characterized by this transfer function, h, is indeed skewed to the right. This implies that even if every component is centered and symmetric, the output membership function is nonsymmetrical and skewed to the right. This means the spring deflection will tend to be greater than it is designed to be. From this point of view, fuzzy analysis gives us new information. But does this interpretation of fuzzy analysis match reality?

(3) To verify our analysis and hypothesis, we use Monte Carlo simulation to repeat this process with 1 million replicates, and then plot the frequency histogram as shown in Figure 4.3. From the simulation data, we calculate the estimated mean, μ, and the standard deviation, S, of the output, the deflection y. They are:

μ(spring deflection) = 0.5110

S(spring deflection) = 0.1120
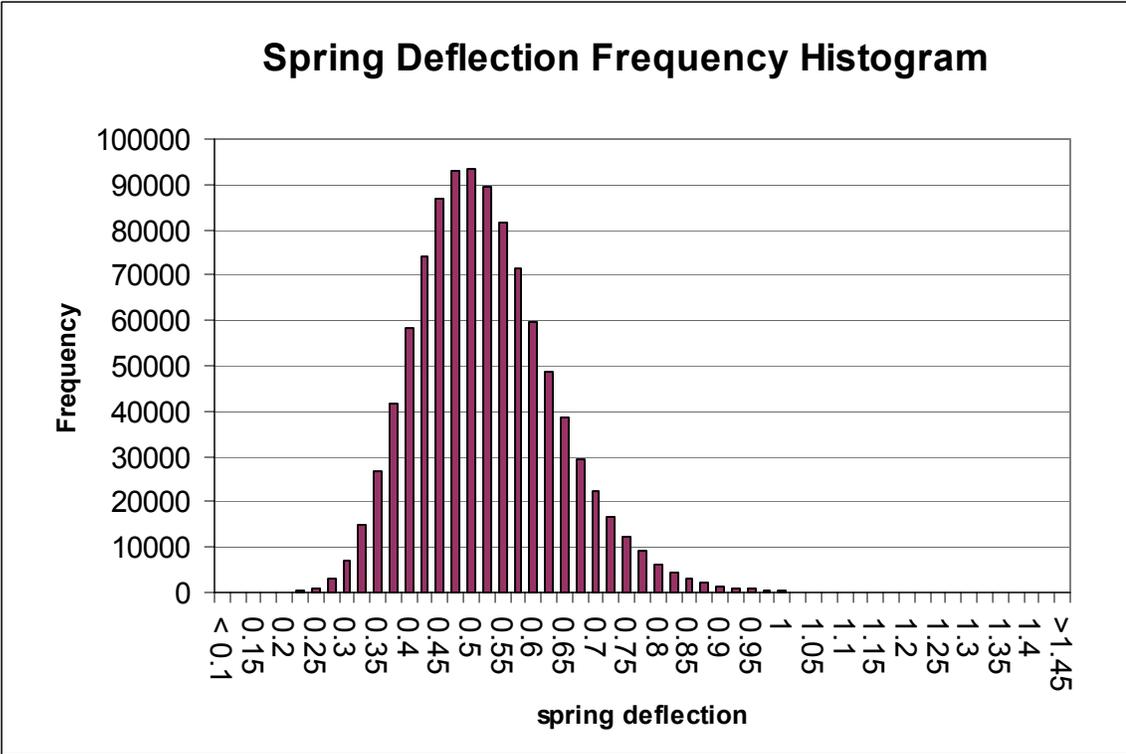
Spring Deflection Frequency Histogram

Monte Carlo Simulation results:

1000000 replicates.
Written in Matlab R11, WinNT 4.0

D is normal distributed, with mean 0.357, and STD 0.0197;
N is normal distributed, with mean 11.29, and STD 0.185;
d is normal distributed, with mean 0.0517, and STD 0.00174.

After simulation, the sample mean is 0.5110, and sample STD is 0.1120.

Figure 4.3. Spring case Simulation results.

From the mean and the distribution skewness obtained from simulation, we can see that this process is indeed skewed to the right, which is consistent with our fuzzy analysis results. So we say our method matches reality quite well in this case.

(4) As mentioned, the more accurate the membership function we construct, the more accurate our analysis will be. So, we need to put more effort on the membership function construction. In most practical cases, the fuzzy membership function will not be triangular, although triangular fuzzy number is easier to analyze. We might use some function introduced in Section 2.1. We give two examples for trapezoidal and Gauss type membership functions to see the impact of the input membership function's shape on output membership function's shape.

Figure 4.4 shows that the membership functions of input variables, *D, N*, and *d,* have been changed to a Gauss type. Figure 4.5 shows the output membership function obtained through the fuzzy analysis method.

Figure 4.6 shows that the membership functions of input variables, *D, N*, and *d,* have been changed to a trapezoidal type. Figure 4.7 shows the output membership function obtained through the fuzzy analysis method.

In above two situations, the membership function for the output variable basically still maintains the shape we obtained in the case of triangular membership functions for the inputs. We qualitatively conclude that the triangular fuzzy membership function is a good approximation in application.
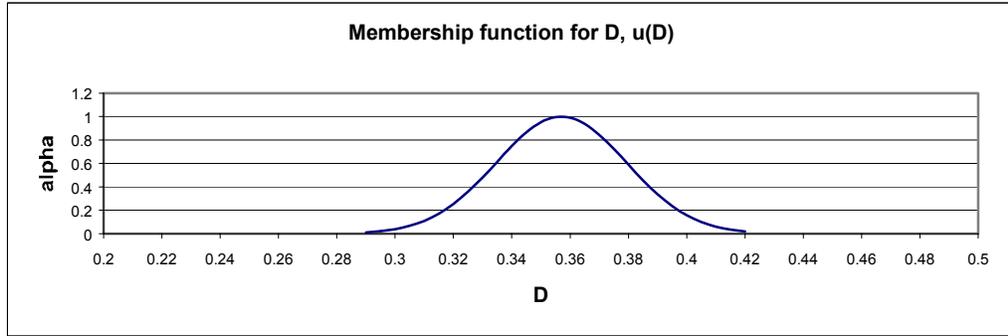
Membership function of input variable D is $\mu(D) = \dfrac{1}{e^{\frac{(D-0.357)^2}{0.001}}}$, (Gauss Type)
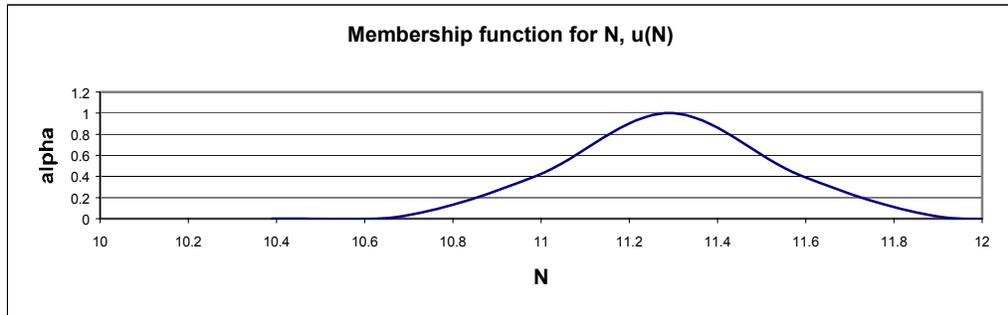
Membership function of input variable N is $\mu(N) = \dfrac{1}{e^{\frac{(N-11.29)^2}{0.1}}}$, (Gauss Type)
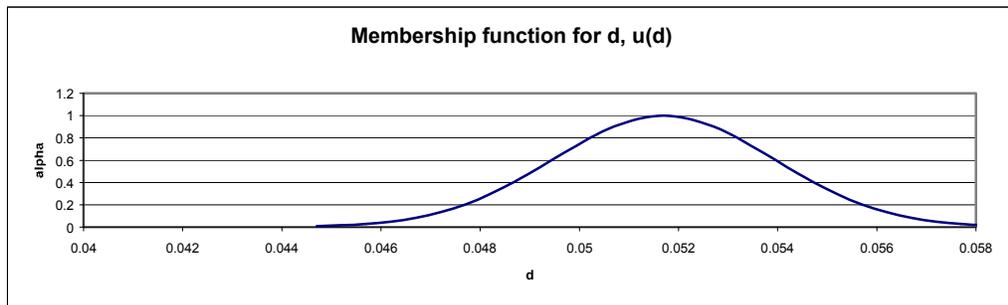
Membership function of input variable d is $\mu(d) = \dfrac{1}{e^{\frac{(d-0.0517)^2}{0.00001}}}$, (Gauss Type).

Figure 4.4. The membership functions of the input variables are Gauss Type for the Spring case.

The different α-cut for output variable corresponding to Gauss Type membership functions of input variables:

alpha	LB	UB
0	0.115	2.028
0.05	0.193	1.259
0.1	0.2178	1.124
0.15	0.235	1.043
0.2	0.25	0.98
0.25	0.263	0.938
0.3	0.275	0.899
0.35	0.286	0.865
0.4	0.297	0.834
0.45	0.3078	0.8067
0.5	0.318	0.781
0.55	0.329	0.757
0.6	0.339	0.733
0.65	0.35	0.711
0.7	0.362	0.689
0.75	0.374	0.667
0.8	0.387	0.644
0.85	0.402	0.621
0.9	0.4197	0.5954
0.95	0.443	0.565
1	0.5	0.5

Membership function for the output, u(deflection)
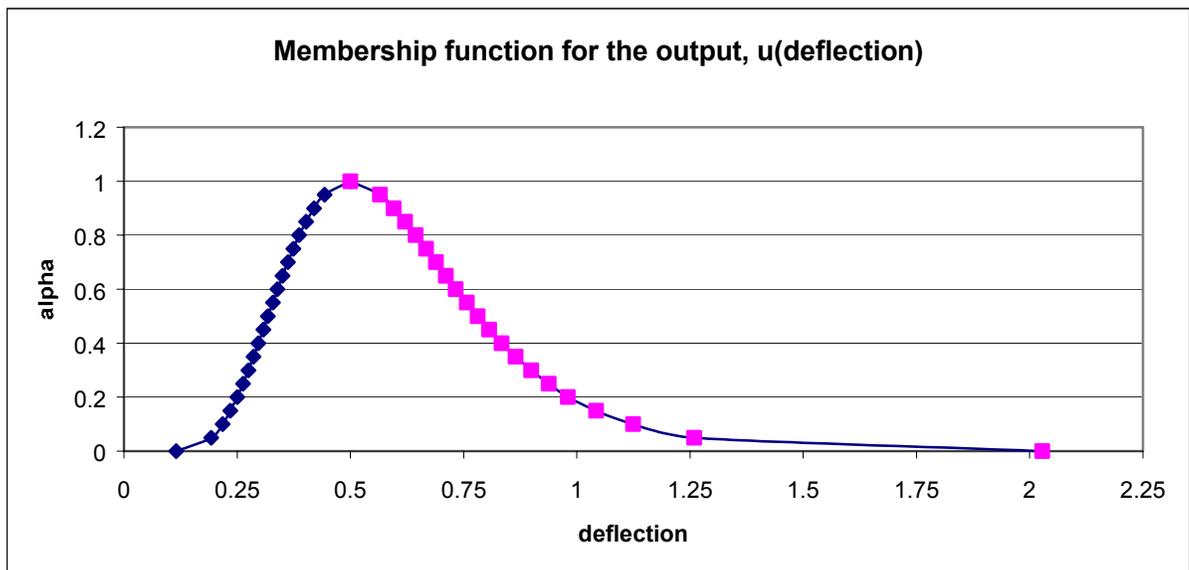
Figure 4.5. The membership function of output variable when the membership functions of input variables are Gauss type in Spring case.
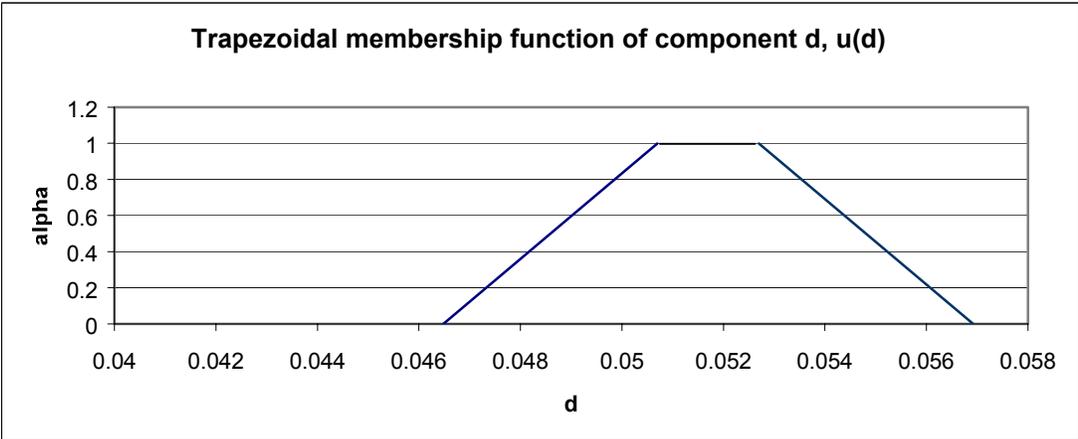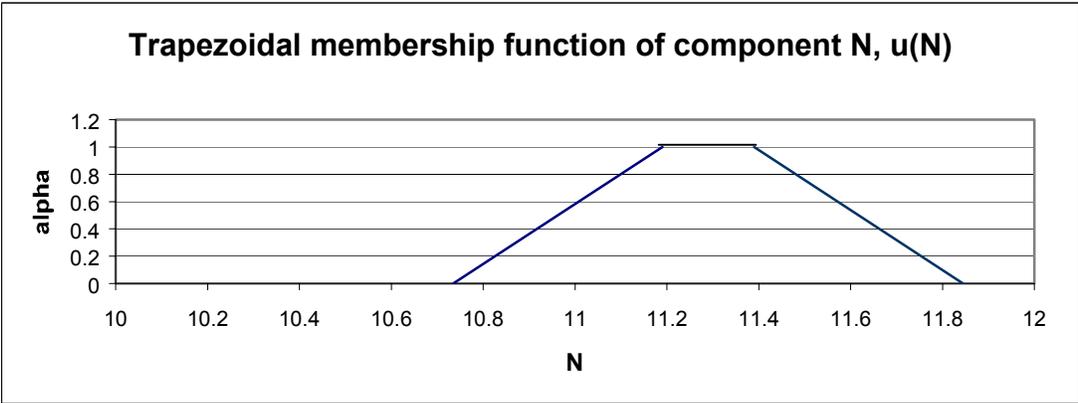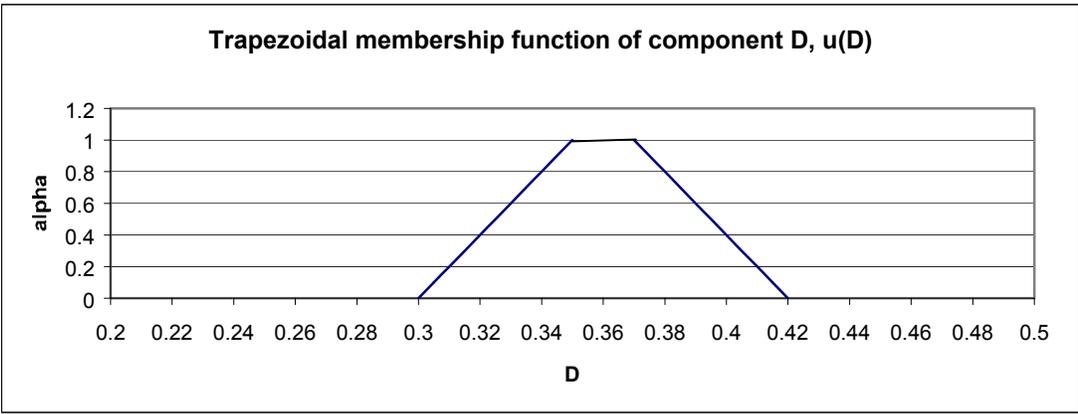
Figure 4.6. The Trapezoidal type membership functions for input variables in Spring case.

The different α-cut for output variable corresponding to Trapezoidal type membership functions of input variables:

alpha	LB	UB
0	0.19	1.3
0.1	0.21	1.21
0.2	0.226	1.124
0.3	0.246	1.041
0.4	0.267	0.965
0.5	0.29	0.894
0.6	0.3145	0.8275
0.7	0.341	0.766
0.8	0.369	0.709
0.9	0.3997	0.656
1	0.433	0.607

Membership function of output, u(deflection)
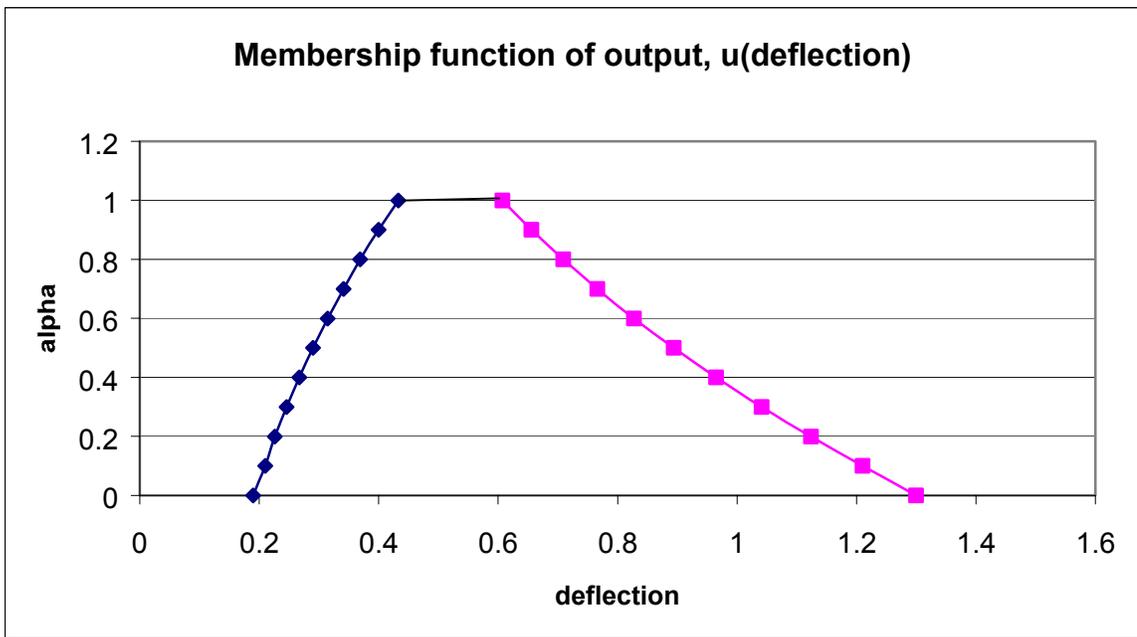
Figure 4.7. The membership function of output variable when the membership functions of input variables are trapezoidal type in Spring case.

When $\alpha = 0$, this approach actually is the High-Low tolerance analysis, or the so-called "Worst Case" analysis. When α varies from 0 to 1, we can see the corresponding α-cut and the fuzzy deviation at that α level, which gives us more flexibility and options to analyze the process. Most times, the worst case is rare. We should be able to carry out the design and analysis based on the required α levels. From this simple example, we can see that our method can generate results consistent with other approaches and also provide more information, e.g., the skewness of the output membership function.

4.2   Achieving Tolerance Symmetry

Proceeding with the previous coil spring problem, we see that the output membership function (and the output distribution) is nonsymmetrical, even though the input variables' membership functions are centered and symmetric. Suppose we are interested in achieving the highest symmetry of the output deflection, when the nominal values of the input set points are unchanged.

Suppose we can choose different membership functions for input variables. By observation, we find that the current output function in Figure 4.2 is skewed to the right at this stage. If we can find such input membership functions that they are skewed to the left, then we may possibly bring the output function back to the symmetry. Of course, it also depends on the form of the function. If this works then we need to find manufacturing processes that produce products with such membership functions.

The idea is illustrated by Figure 4.6. Suppose there are several possible manufacturing processes we can use to produce the D, the base diameter in this spring case; correspondingly, these manufacturing processes are characterized by different membership functions. On Figure 4.1, the dimension tolerance is bilateral with $D = 0.357^{+0.059}_{-0.059}$. Let us now set it as unilateral without changing the process capability with $D' = 0.357^{+0.000}_{-0.118}$. Of course, this optimization of the process symmetry is dependent on whether or not we can find such a corresponding process to make the input membership function as in Figure 4.8, but at least this fuzzy approach sheds some light on solving the problem, and also gives us some direction and ideas to make improvement. Actually, changing to a new process whose membership function has a particular shape is not impossible in practice. For instance, suppose we cut a rod. If we directly make the

cut, the output membership function is not symmetric as we examined in Section 4.1; whereas if we use some gage to restrict the longest lengths for the input variable of D, we might be able to get the unilateral shape membership function as shown in Figure 4.8. If it is the case, we can obtain a more symmetric output membership function as shown also in Figure 4.8, based on current input membership functions and our fuzzy set approach analysis.

Again, we here use Monte Carlo simulation to verify this analysis. Suppose the membership function $\mu(D)$ in Figure 4.8 implies that a probability distribution has the same shape. We construct a corresponding triangular distribution for the input variable D, and then run the simulation. The result is shown in Figure 4.9. From Figure 4.9, we see the distribution is more symmetric than that in Figure 4.3, and the distribution shape has some similarity with the membership function as in Figure 4.8. The sample mean after simulation is 0.3643, compared with the original 0.5110 discussed in Section 4.1. The mean is shifted to the left indeed. The center of gravity for the membership function in Figure 4.8 is approximately 0.466, compared with the original 0.667 discussed in Section 4.1. The fuzzy output membership function has also been shifted to the left. We conclude that this analysis matches the reality roughly.

New unilateral input Mb Fun for D, u(D)

Input membership function for N, u(N)

Input membership function for d, u(d)

More symmetric Output Membership Function
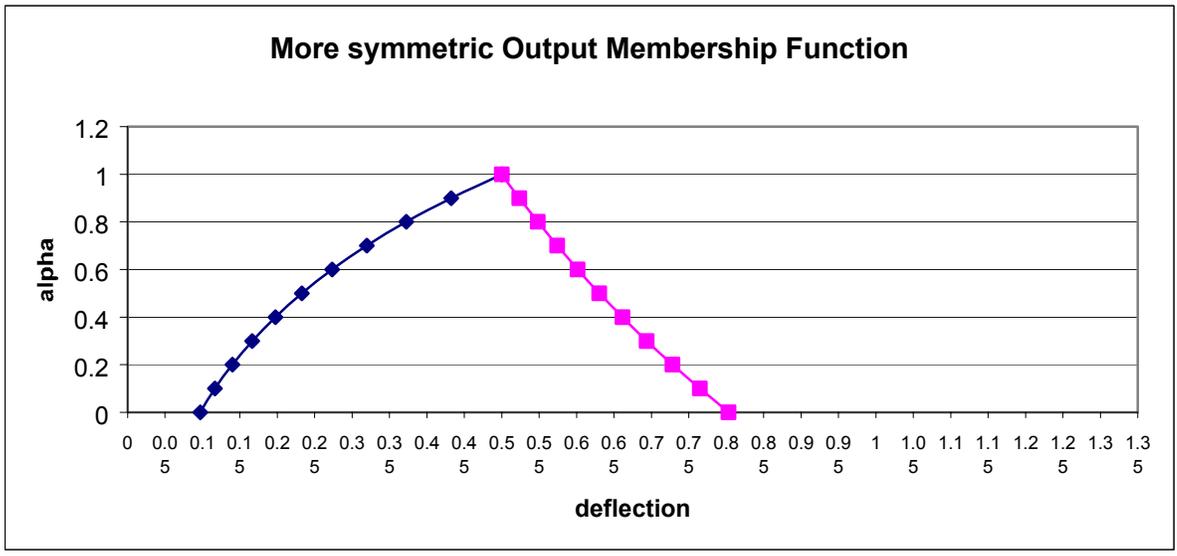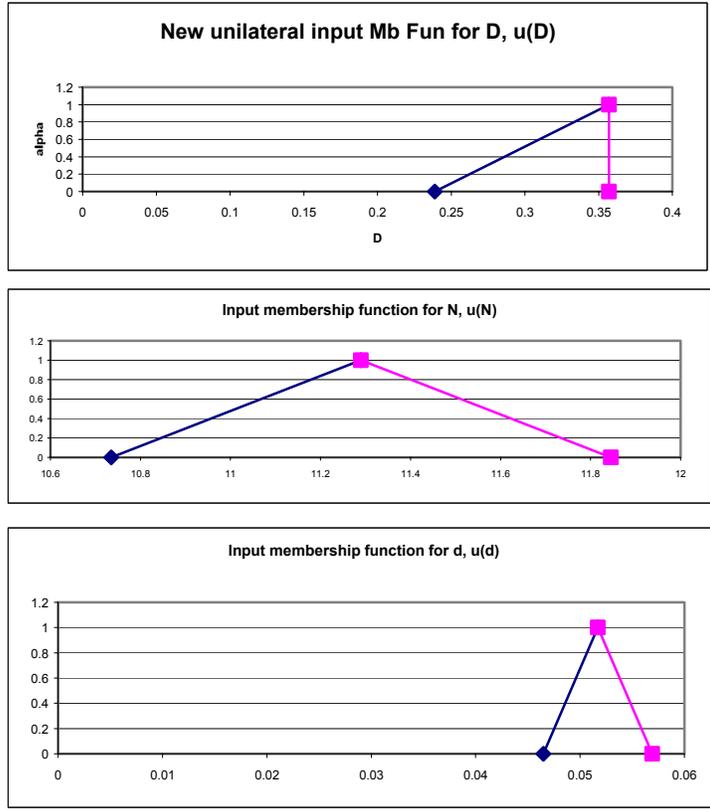
Figure 4.8. More symmetric output membership function when input membership function μ(D) changes to unilateral, μ(N) and μ(d) maintain unchanged.

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Output frequency histogram
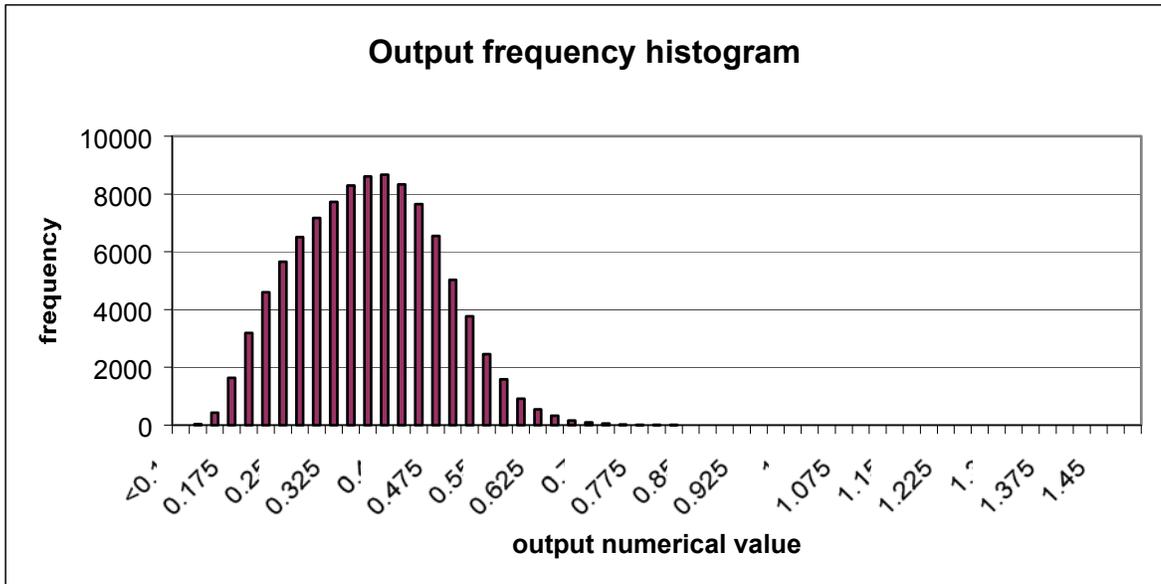
Monte Carlo Simulation results:

1000000 replicates.
Written in Matlab R11, WinNT 4.0

D is triangular distributed, a=0.2388, b=c=0.357;
N is normal distributed, with mean 11.29, and STD 0.185;
d is normal distributed, with mean 0.0517, and STD 0.00174.

After simulation, the sample mean is 0.3643, and sample STD is 0.1040.

Figure 4.9. Monte Carlo simulation results when component D has a triangular distribution, N and d are normally distributed.

5.   Fuzzy Parameter Design

5.1   Basic Fuzzy Parameter Design Model

We defined the conventional parameter design problem and parameter design model as stated by Fathi [12] in Section 1.2. Now, let us state the corresponding interpretation of the tolerance parameter design problem in the fuzzy environment. We call it the Fuzzy Parameter Design Problem and present the corresponding mathematical model, the Fuzzy Parameter Design Model.

Let $x = (x_1, ..., x_n)$ be a vector of design variables (components) for a manufacturing product or a manufacturing process, and let $y = h(x_1, ..., x_n)$ represent an output characteristic of interest. Particularly, all component variables, $x_1, ..., x_n$, are interpreted as fuzzy variables instead of as random variables. We assume that the set points (i.e., the nominal values) for x_1 through x_n are controllable parameters (i.e., their values can be chosen or determined at the design stage). We denote the set points for x_1 through x_n by μ_1 through μ_n, respectively.

Typically, during the design stage, the designing engineers determine a set of values for μ_1 through μ_n such that

$h(\mu_1, ..., \mu_n) = \tau,$

where τ is the most desirable value of y (i.e., the target value of y).

During production, the unavoidable uncertainty or ambiguity resident in the manufacturing process causes the actual values of x_1 through x_n to deviate from their respective nominal values (i.e., μ_1 through μ_n), causing the value of y to deviate from its target value τ. Since we consider x_1 through x_n to be fuzzy variables with respective

nominal values μ_1 through μ_n in the fuzzy environment, thus $y = h(x_1, ..., x_n)$ is also a fuzzy variable.

Based on the basic fuzzy concepts introduced in Section 3.2.3, we further define some notations.

- The nominal value (i.e., the mode) of a fuzzy variable y is denoted as μ_y.

- The left and right endpoints of the corresponding α-cut for the fuzzy variable y at certain α level are denoted as a_{y_α} and b_{y_α}, respectively.

- The left deviation to its nominal value at α level for fuzzy variable y is denoted as $l\delta_{y_\alpha}$, (i.e., $l\delta_{y_\alpha} = \mu_y - a_{y_\alpha}$)

- The right deviation to its nominal value at α level for fuzzy variable y is denoted as $r\delta_{y_\alpha}$, (i.e., $r\delta_{y_\alpha} = b_{y_\alpha} - \mu_y$)

- The deviation to its nominal value at α level for fuzzy variable y is denoted as δ_{y_α}, where $\delta_{y_\alpha} = l\delta_{y_\alpha} + r\delta_{y_\alpha} = b_{y_\alpha} - a_{y_\alpha}$.

The notations and relations are shown in Figure 5.1.
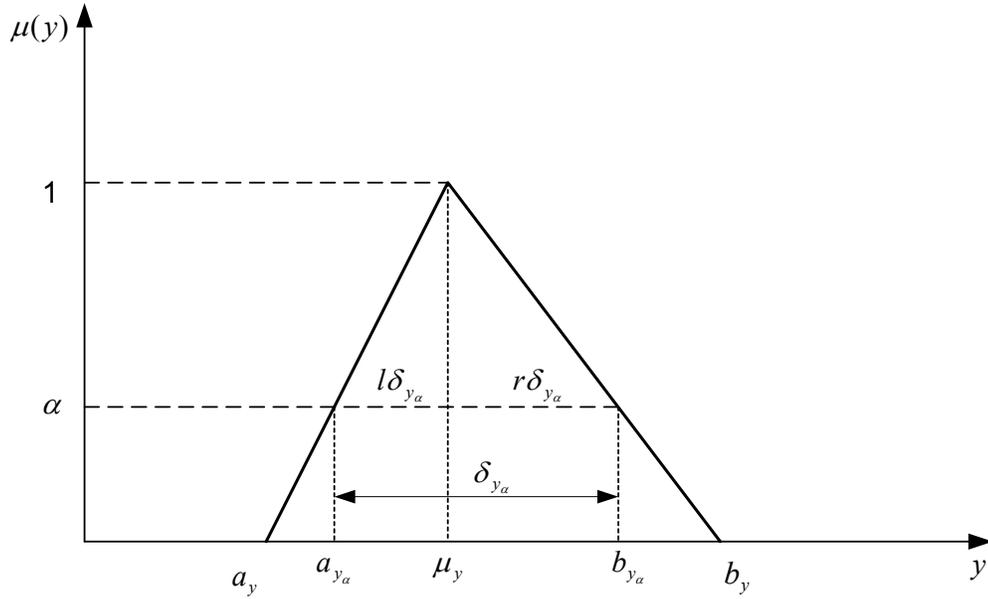
Figure 5.1. Notations related with a fuzzy variable *y* used in fuzzy parameter design model.

Since the output variable $y = h(x_1, \ldots, x_n)$ is also a fuzzy variable, our objective is to minimize the uncertainty or the ambiguity of it. In other words, we prefer this fuzzy variable as clear and crisp as possible, which is equivalent to minimizing the deviation of it. Now the original Parameter Design Problem is converted into a new problem that is to find the values of controllable parameters μ_1 through μ_n to minimize the δ_y, while maintaining $h(\mu_1, \ldots, \mu_n) = \tau$. As we have discussed in previous sections, a fuzzy set can be decomposed to a series of α-cuts at different α levels, according to the Decomposition Theorem in Section 2.1. Going one step further, the above problem can be interpreted equivalently to finding the values of the controllable parameters μ_1 through μ_n to minimize the δ_{y_α} for all α, $\alpha \in [0, 1]$, while maintaining $h(\mu_1, \ldots, \mu_n) = \tau$. Here we

assume all component variables x_1 through x_n are convex, normal, and bounded fuzzy numbers.

In Chapter 4, we have discussed the approach to obtaining the α-cut for the output variable at a certain α level when the α-cut for each component variable is known at the same α level. Now we employ the same idea to explicitly write the expression of an α-cut for the output variable at a certain α level in this scenario. The only difference is that in Chapter 4, the nominal values for components are numerical values, whereas here the nominal values for components are decision variables.

Let us illustrate the procedure and present the fuzzy parameter design model.

Step 1. Derive the α-cut expression for each component with respect to its nominal value (i.e., the set point), at certain α level.

Step 2. Obtain the corresponding α-cut for the output variable at the same α level by applying the Vertex method.

Step 3. Solve the nonlinear programming problem to minimize the output variable deviation at each α level. Notice that the optimal set points at this stage are the ones at that particular α level.

Step 4. Theoretically, we should take all α values, where α is a real number and α∈ [0, 1], to reconstruct the membership function of the output variable. As a practical approximation, we reconstruct it by taking finite and sufficient α levels. At different α levels, the optimal set points may or may not be identical, which varies case by case.

The proposed procedure can be expressed as the following model, which we call

the *Fuzzy Parameter Design Model*:

$$\underset{\mu_1,\dots,\mu_n}{Minimize} \quad z = \delta_{y_\alpha}(\mu_1,\dots,\mu_n) \qquad\qquad (5.1)$$

$$Subject\ to \quad h(\mu_1,\ \dots,\ \mu_n) = \tau,$$

$$\delta_{y_\alpha} = b_{y_\alpha} - a_{y_\alpha},$$

$$b_{y_\alpha} = \underset{j}{\max}\ (h(V_j)),$$

$$a_{y_\alpha} = \underset{j}{\min}\ (h(V_j)),$$

$$(\mu_1,\ \dots,\ \mu_n) \in M,$$

For a given value of $\alpha \in [0, 1]$.

Where a_{y_α} and b_{y_α} represent the lower and upper bounds of α-cut of output variable y at

that α level. $V_j, j = 1, 2, \dots, 2^n$, is the coordinate of jth vertex of the input variable domain

(i.e., jth combination of input variable set). M represents a set of acceptable values for μ_1

through μ_n. Note that the numerical value of the coordinates of j-th vertex will change as

α varies from 0 to 1.

This model depends on how well we can create the components' membership

functions since eventually the shape of the membership function carries the information

of the extent of variation for each component.

In this model, we convert the fuzzy set optimization problem into conventional

nonlinear programming problem by taking different α-cut levels from 0 to 1. How many

α-cut levels we use depends on how accurate the output membership function needs to

be. Generally speaking, the more α levels we use, the more information we will obtain.

By applying this model, each α "potentially" gives a different answer on the optimal solution. We can solve the problem at different α values. For each "desired" value of α, we can determine the optimal solution, and let the decision maker select a solution that is deemed to be more appropriate, if the optimal solutions for different α values differ. By taking different α-cuts, we convert the parameter design problem in the fuzzy environment into the conventional nonlinear programming problem in the crisp environment. Equivalently, it is a series of *Nonlinear Programming Problems* when α is changed at different levels from 0 to 1.

In the following section, we present several case studies of application of Fuzzy Parameter Design Model to the practical problems, and discuss certain practical issues of this method.

5.1.1 Case Study 5.1: Coil Spring

(1) We consider again the coil spring problem, which is adopted from Fathi [12] and slightly modified for brevity.

Given the transfer function as,

$$y = \frac{D^3 N}{143750 d^4},$$ (5.2)

D, N, d, and *y* have the same meaning as in Section 4.1. *D, N*, and *d* are components and *y* is the output variable. To minimize the variance of *y* as much as possible, while maintaining the output value at 0.5, determine alternative nominal values (set points) for components *D, N*, and *d* without changing their respective standard deviation (standard deviation for *D, N*, and *d* are 0.0197, 0.185, and 0.00174, respectively). For technical

reasons, the nominal values of the components D, N, and d must be within the following ranges.

$$0.25 \leq \mu_D \leq 1.30 \qquad\qquad (5.3)$$

$$2.00 \leq \mu_N \leq 15.00$$

$$0.05 \leq \mu_d \leq 0.20$$

We apply the Parameter Design Model with Linear Approximation of the Variance method [12] and solve it. The optimal solution for the set points is $\mu_D = 1.3$; $\mu_N = 15$; and $\mu_d = 0.146$. The objective function value, (the variance of the output) under optimal set points is $\sigma_y^2 = 0.00112$, with standard deviation $\sigma_y = 0.033$.

(2) Fuzzy Parameter Design Model approach.

We interpret D, N, and d as input fuzzy variables with nominal values of μ_D, μ_N, and μ_d, respectively, and y is the output fuzzy variable. Essentially, the membership function for each fuzzy variable is a different concept from the probability density function for a random variable. For Fuzzy Parameter Design Model approach, we have to be given the deviation of each fuzzy variable, so we can decide the optimal nominal values for the components, where the deviation is the dispersion of the membership function for each fuzzy component variable and the decision variables are the set points for x_1, …, x_n, and we seek a set of values for these set points to minimize the deviation of the membership function of the output variable. To compare the results obtained from Parameter Design Model and Fuzzy Parameter Design Model, we assume that the lower bound and upper bound for each component are *numerically* equal to $\mu_i - 3\sigma_i$ and $\mu_i + 3\sigma_i$ respectively. Since the standard deviations for each component in this problem have already been given, we use $[\mu_i - 3\sigma_i, \mu_i + 3\sigma_i]$ as the lower and upper bound for each

component of *D, N*, and *d*. At this stage, we further assume the membership function for each component is triangular, symmetric, and normal.

By applying the Fuzzy Parameter Design Model procedure, we first explicitly express the α-cut interval of each component at the corresponding α level as follows,

$$D_\alpha = [\mu_D - 0.0591*(1-\alpha), \mu_D + 0.0591*(1-\alpha)] \tag{5.4}$$

$$N_\alpha = [\mu_N - 0.555*(1-\alpha), \mu_N + 0.555*(1-\alpha)]$$

$$d_\alpha = [\mu_d - 0.00522*(1-\alpha), \mu_d + 0.00522*(1-\alpha)]$$

Following the Fuzzy Parameter Design Model Expression (5.1), we can have the following nonlinear programming problems corresponding to each different α value, where α is a real number between 0 and 1.

Minimize $\quad z = \delta_{y_\alpha}$ $\hfill$ (5.5)

Subject to $\quad h(\mu, \mu_N, \mu_d) = \dfrac{\mu_D{}^3 \mu_N}{143750 \mu_d{}^4} = 0.5,$

$$\delta_{y_\alpha} = b_{y_\alpha} - a_{y_\alpha},$$

$b_{y_\alpha} = \max\ (h(V_j)),\qquad \forall\, j = 1, 2, \ldots 8$

$a_{y_\alpha} = \min\ (h(V_j)),\qquad \forall\, j = 1, 2, \ldots 8$

$0.25 \leq \mu_D \leq 1.30$

$2.00 \leq \mu_N \leq 15.00$

$0.05 \leq \mu_d \leq 0.20$

For a given $\alpha \in [0, 1]$.

We solve this model using LINGO, with multiple starting points for a better chance to obtain a global optimum. Each run took less than one second of elapsed time on a low-end PC, with α varying from 0 to 1 at a 0.05 interval evenly. Each run means each NLP according to a certain α value. We find all runs at the same α value with multiple starting points converged to the same points, Thus, we consider them to be the global optimum for that particular α level.

For this particular case, the optimal set points at all examined α levels are identical, with

$$\mu_D{}^* = 1.3, \tag{5.6}$$

$$\mu_N{}^* = 15,$$

$$\mu_d{}^* = 0.146.$$

The optimal set points and corresponding output deviation *at different α levels* are shown in Figure 5.2. Since the optimal set points at all examined α levels are identical, we can interpret this to mean that no matter what degree of conformity we are thinking about for this manufacturing process, the optimal set points are all the same. This implies we should operate this manufacturing process at $(\mu_D{}^*, \mu_N{}^*, \mu_d{}^*) = (1.3, 15, 0.146)$. The output membership function in Figure 5.2 is plotted at the same set points, so we can treat it as the final output membership function under the current optimal set points. We solve this problem by using Linear Approximation of the Variance [12] and High-Low Tolerancing [13], the results of the optimal set points are exactly identical.

Comparing Figure 5.2 with Figure 4.2, we can see the deviation or the dispersion of the fuzzy output variable has been improved significantly. Numerically, the optimal $f_{cog}{}^*$ is 0.508 compared with the original 0.667; the optimal mean deviation $\delta(y^*)$ is 0.162

compared with the original 0.542. We further use Monte Carlo simulation to simulate the output distribution using the optimal set points. The simulation frequency histogram is shown in Figure 5.3. Comparing Figure 5.3 and Figure 4.3, which is the simulated distribution under nonoptimal set points, we see the variance of the process in Figure 5.3 is much better than that in Figure 4.3. These simulation results verify that the set points obtained by Fuzzy Parameter Design Model are optimal. Comparing Figure 5.2 with Figure 5.3, we see that the graph dispersion, tendency, and lower and upper bounds are similar. We conclude that the Fuzzy Parameter Design Model can obtain comparable and consistent results for the optimal set points compared with previous work for this case, even though this is solved in a completely different framework.

α	a_{y_α}	b_{y_α}	δ_{y_α}
0	0.3672	0.6916	0.3244
0.05	0.3731	0.6808	0.3077
0.10	0.3791	0.6702	0.2911
0.15	0.3852	0.6597	0.2745
0.20	0.3914	0.6494	0.2580
0.25	0.3977	0.6393	0.2416
0.30	0.4040	0.6293	0.2253
0.35	0.4105	0.6195	0.2090
0.40	0.4171	0.6098	0.1927
0.45	0.4238	0.6002	0.1764
0.50	0.4306	0.5909	0.1603
0.55	0.4375	0.5816	0.1441
0.60	0.4444	0.5725	0.1281
0.65	0.4516	0.5636	0.1120
0.70	0.4588	0.5547	0.0959
0.75	0.4661	0.5460	0.0799
0.80	0.4736	0.5375	0.0639
0.85	0.4811	0.5291	0.0480
0.90	0.4888	0.5208	0.0320
0.95	0.4966	0.5126	0.0160
1	0.5045	0.5045	0.0000
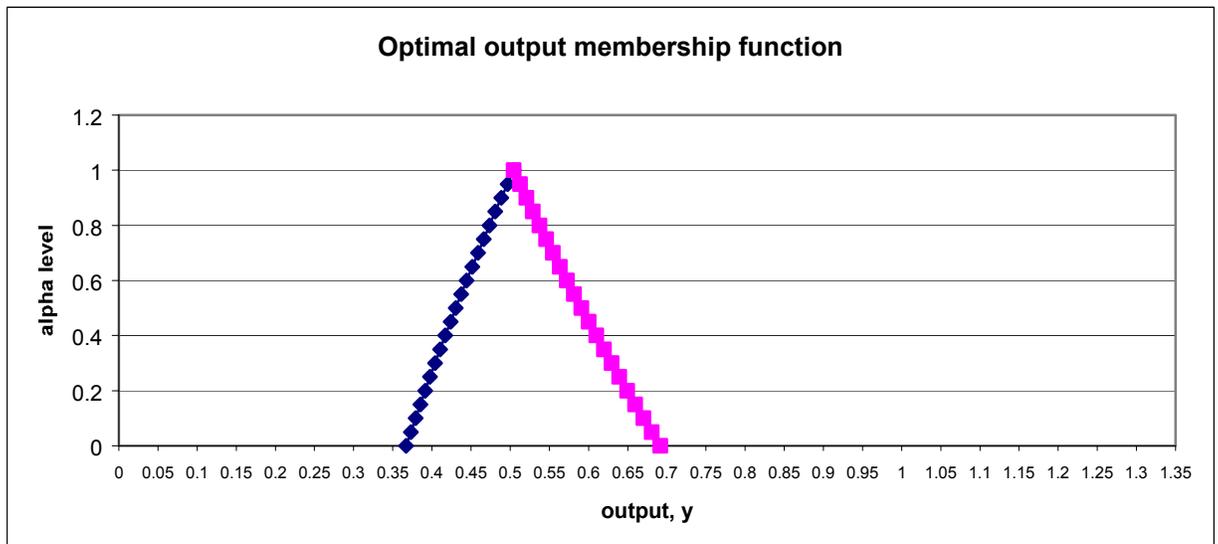
Figure 5.2. The optimal set points at different α level and the optimal output membership function obtained by Fuzzy Parameter Design Model in Spring case.
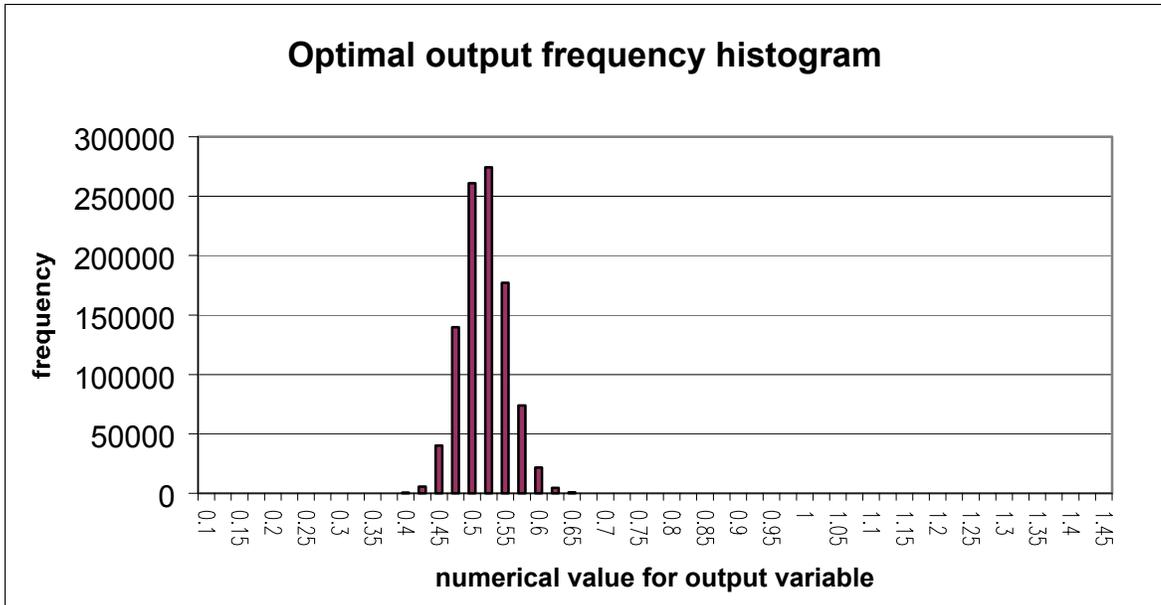
Optimal output frequency histogram

Monte Carlo Simulation results:

1000000 replicates.
Written in Matlab R11, WinNT 4.0

D is normal distributed, with mean 1.3, and STD 0.0197;
N is normal distributed, with mean 15, and STD 0.185;
d is normal distributed, with mean 0.0146, and STD 0.00174.

After simulation, the sample mean is 0.5057, and sample STD is 0.0344.

Figure 5.3. Monte Carlo simulation results for optimal set points for spring case.

5.1.2 Case Study 5.2: Wheatstone Bridge Circuit

We adopted this example from [12]. The original problem is stated as follows. The transfer function is given by

$$y = \frac{x_2 x_4}{x_3} - \frac{x_7}{x_3^2 x_5}[x_1(x_3 + x_4) + x_4(x_2 + x_3)] \times [x_2(x_3 + x_4) + x_6(x_2 + x_3)] \quad . \quad (5.7)$$

where x_1 through x_6 are the design components (5 resistors and a battery output voltage) with respective means μ_1 through μ_6, and respective standard deviations σ_1 through σ_6. In the original problem, $\sigma_i = 0.0024\mu_i$, for $i = 1, 2, 3, 4, 6$ and $\sigma_5 = 0.0408\mu_5$. Component x_7 represents the current in the galvanometer at the time of measurement. Its mean is $\mu_7 = 0$ and its standard deviation is $\sigma_7 = 0.000163$. Furthermore, we have the following restrictions on μ_1 through μ_6.

$$20 \le \mu_1 \le 500, \qquad\qquad\qquad (5.8)$$

$$\mu_2 = 2,$$

$$2 \le \mu_i \le 50, \text{ for } i = 1, 3, 4, 6.$$

$$1.2 \le \mu_5 \le 30.$$

This problem was originally solved by Taguchi in [28], and the optimal set points are $(\mu_1{}^*, \ldots, \mu_6{}^*) = (20, 2, 50, 2, 30, 2)$.

Now, we are solving it using Fuzzy Parameter Design Model procedure. Following the same arguments in the previous case, we use $\mu_i - 3\sigma_i$ and $\mu_i + 3\sigma_i$ as the lower and upper bound, respectively, for each component. Then the α cut for each component is:

$$x_{i_\alpha} = [\mu_i(1 - 0.0072(1 - \alpha)), \mu_i(1 + 0.0072(1 - \alpha))], \text{ for } i = 1, 2, 3, 4, 6.$$

$$x_{5_\alpha} = [\mu_5(1 - 0.1224(1 - \alpha)), \mu_5(1 + 0.1224(1 - \alpha))],$$

$$x_{7_\alpha} = [-0.000489(1-\alpha), 0.000489(1-\alpha)]. \tag{5.9}$$

Now we can build the Fuzzy Parameter Design Model using expression (5.1). Since the next step is just to repeat the procedure same as the case 5.1, we skip these steps and only present the results thereafter. Since the number of component variables $n =$ 6, and since each variable interval has two vertices, all combinations for the vertices are $2^6 = 64$. Therefore, using the vertex method to find the lower and upper bounds of output α-cut, we need $2^6 = 64$ evaluations for each run at every α level.

We solve this model using LINGO on a low-end PC. Elapsed runtime was one second. The optimal set points and corresponding output deviation at different α levels are shown in Figure 5.4. For this particular case, the optimal set points at all studied α levels are identical, i.e., $(\mu_1{}^*, \ldots, \mu_6{}^*) = (20, 2, 50, 2, 30, 2)$. We can interpret it to mean that no matter what degree of conformity for this manufacturing process, the optimal set points are the same. Consequently, we should operate this manufacturing process using the set points at $(\mu_1{}^*, \ldots, \mu_6{}^*) = (20, 2, 50, 2, 30, 2)$. Since the output membership function in Figure 5.4 is plotted at the same optimal set points, it is the final output membership function. We also present the optimal set points solved by several other methods from [12] in Table 5.1 for comparison.

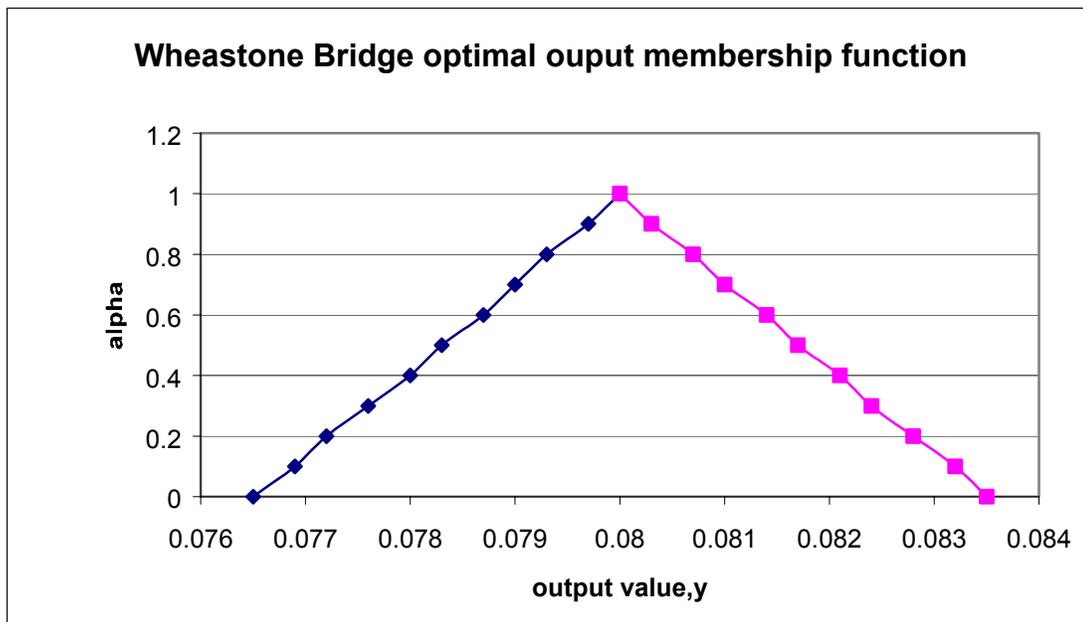| α	a_{y_α}	b_{y_α}	δ_{y_α}
0	0.0765	0.0835	0.0070
0.1	0.0769	0.0832	0.0063
0.2	0.0772	0.0828	0.0056
0.3	0.0776	0.0824	0.0048
0.4	0.0780	0.0821	0.0041
0.5	0.0783	0.0817	0.0034
0.6	0.0787	0.0814	0.0027
0.7	0.0790	0.0810	0.0020
0.8	0.0793	0.0807	0.0014
0.9	0.0797	0.0803	0.0006
1	0.0800	0.0800	0

Figure 5.4. The optimal set points at different α levels and the optimal output membership function obtained by Fuzzy Parameter Design Model for Wheatstone bridge circuit case.

Methods	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6
Fuzzy Para. Design	20.0	2.0	50.0	2.0	30.0	2.0
Linear Approx. [12]	20.0	2.0	50.0	2.0	30.0	2.0
Sequential Quadratic Variance Approx.[11]	20.0	2.0	50.0	2.0	30.0	2.0
Sigal-to-Noise Ratio [28]	20.0	2.0	50.0	10.0	30.0	2.0
High-Low Tolerancing [13]	20.0	2.0	50.0	2.0	30.0	2.0

Table 5.1. Set points for the parameters of the wheatstone bridge circuit obtained using various methods.

From Table 5.1, we see that the set points we obtained through Fuzzy Parameter Design Model are identical to all other methods except for Signal-to-Noise Ratio method. Since the objective function value in the fuzzy parameter design model, which is the optimized fuzzy deviation at different α levels, is not comparable with the variance in probability theory, we cannot compare these values directly. In the fuzzy environment we know the current output deviation for every α level is minimized and optimal, and is consistent with other methods. Although the fuzzy result is identical to High-Low tolerancing, High-Low tolerancing only studies for the worst case situation, which is a special case for Fuzzy Parameter Design Model (i.e., when $\alpha = 0$). From our model, we get the optimal set points for each α level, and we can see the fuzzy deviation at any particular α level of interest by examining the α-cut results. As an example, we know the worst case will almost never occur in practice. Let us suppose we are interested in a degree of conformity 80% (i.e., the 80% α level), then from the output membership function we see that the output value is within the range of [0.0793, 0.0807]. Other degrees of conformity ranges can be obtained by examining solutions for their corresponding α levels. Consequently, the Fuzzy Parameter Design Model gives us new information and more options and flexibility to a designer.

5.1.3 Case Study 5.3: OTL Pull-Push Circuit

We adopted this example from Fathi [12]. The original problem is stated as follows. The transfer function is given by

$$y = (V_{b1} + V_{be1})\frac{\beta R_0}{\beta R_0 + R_f} + (E_c - V_{be3})\frac{R_f}{\beta R_0 + R_f} + \frac{V_{be2} R_f \beta R_0}{(\beta R_0 + R_f)R_{c1}}. \qquad (5.10)$$

where

$$V_{b1} = \frac{E_c R_{b2}}{R_{b1} + R_{b2}},$$

$$R_0 = R_{c2} + R_L.$$

with $R_L = 9$ Ohms, $V_{be1} = V_{be3} = 0.65$ V, $V_{be2} = 0.74$ V, and $E_c = 12$ V. R_{b2}, R_{b1}, R_f, R_{c2}, and R_{c1} are resistances, and β is the current transistor gain, with respective means u_1 through u_6, and respective standard deviations σ_1 through σ_6. In the original problem, $\sigma_i = (0.05/3)\mu_i$, for $i = 1, \ldots, 5$, and $\sigma_6 = (0.5/3)\mu_6$. We have the following restrictions on u_1 through u_6:

$$25\,000 \le \mu_1 \le 70\,000, \qquad (5.11)$$

$$50\,000 \le \mu_2 \le 150\,000,$$

$$649.4 \le \mu_3 \le 2\,053.5,$$

$$237.1 \le \mu_4 \le 749.9,$$

$$1\,271.1 \le \mu_5 \le 2\,260.3,$$

$$73 \le \mu_6 \le 280.$$

The target value of y is 6 Volts.

We apply Fuzzy Parameter Design Model to this problem as we did in the previous cases and solve it. The procedure uses the steps in Section 5.1. The optimal set points and corresponding output deviation at different α levels are shown in Figure 5.5. For this particular case, the optimal set points at the α levels examined are identical with $(\mu_1^*, \ldots, \mu_6^*) = (70000, 94026, 649.4, 749.9, 2260.3, 280)$. We can interpret it to mean that no matter what degree of conformity for this manufacturing process, the optimal set points are the same and we should operate this process using them. Since the output membership function in Figure 5.5 is plotted at the optimal set points, it is the final output membership function.

The results are shown in Table 5.2 along with other optimal set points obtained by other methods [12]. Values for μ_y and σ_y for each method in Table 5.2 are obtained by using the set points as input to a Monte Carlo simulation.

Methods	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ (y)	STD(y)
Fuzzy.	70,000	94,026	649.4	749.9	2,260.3	280	6.0010	0.0690
Linear Approx. of Variance.	56,590	87,010	1,044.0	746.2	1,302.0	280	5.9998	0.0688
Sequential Elimination of Levels	58,750	100,000	1,539.9	237.1	1,467.8	280	5.9914	0.0711
Signal-to-Noise Ratio	55,835	75,000	649.4	749.9	2,260.3	280	6.0006	0.0691

** u(y) and STD(y) are obtained from Monte Carlo simulation by using the set points as input with 100,000 replicates in Matlab R11.

Table 5.2. Set points for the parameters of OTL pull-push circuit obtained using various methods.

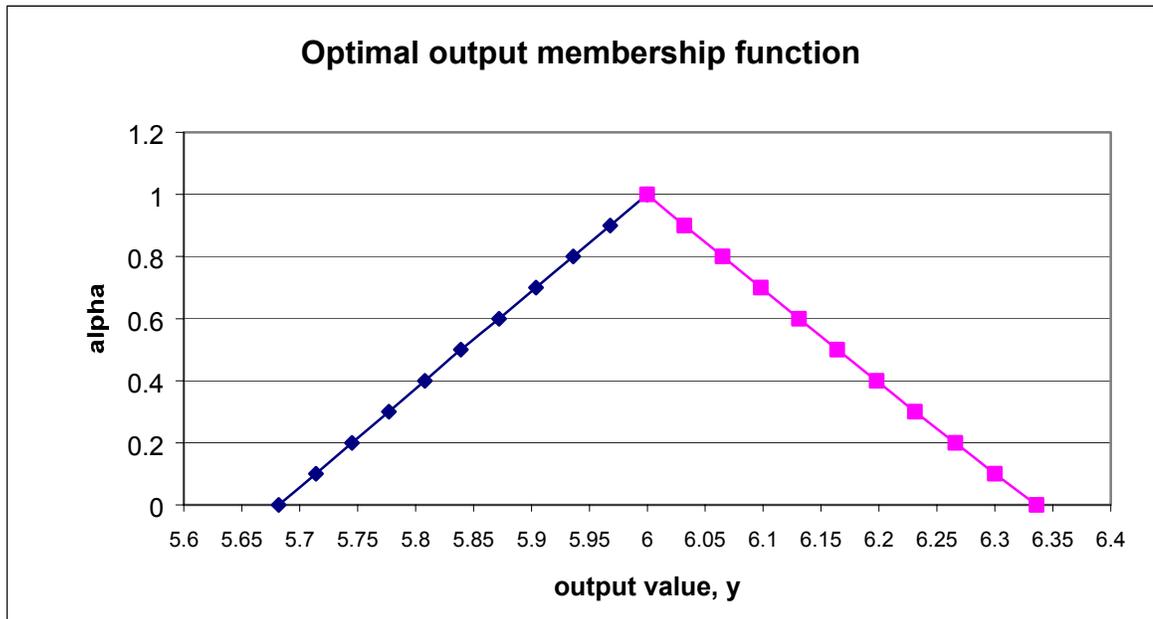| α	a_{y_α}	b_{y_α}	δ_{y_α}
0	5.682	6.336	0.654
0.1	5.714	6.300	0.586
0.2	5.745	6.266	0.521
0.3	5.777	6.231	0.454
0.4	5.808	6.198	0.390
0.5	5.839	6.164	0.325
0.6	5.872	6.131	0.259
0.7	5.904	6.098	0.194
0.8	5.936	6.065	0.129
0.9	5.968	6.032	0.064
1	6.000	6.000	0

Figure 5.5. The optimal set points at different α level and the optimal output membership function obtained by Fuzzy Parameter Design Model for OTL pull-push circuit.

From the results in Table 5.2, we see that this Fuzzy Parameter Design Model appears to work well. The standard deviation evaluated by Monte Carlo simulation using Fuzzy Parameter Design Model set points is the second best of the four methods listed in Table 5.2.

5.1.4 Case Study 5.4: Microscope Lens

This example is adopted from Fathi and Vestuti [14]. The original problem is as follows:

The magnification of a microscope can be computed from the following formula,

$$y = \frac{25L}{fo * fe} \tag{5.12}$$

where

$$fo = \frac{1}{(N-1)[(1/R_1) + (1/R_2)]} \text{ and } fe = \frac{1}{(n-1)[(1/r_1) + (1/r_2)]}$$

y: magnification of the microscope

L: Length of the microscope tube

fo: focal length of the objective lens

fe: focal length of the eyepiece lens

N: index of the refraction for glass used to make objective lens

R_1: Radius of the curvature for one side of the objective lens

R_2: Radius of the curvature for the opposite side of the objective lens

 n: index of the refraction for glass used to make eyepiece lens

r_1: Radius of the curvature for one side of the eyepiece lens

r_2: Radius of the curvature for the opposite side of the eyepiece lens

In this problem, N, R_1, R_2, n, r_1, and r_2 are the component variables, corresponding μ_N through μ_{r2}, respectively, and standard deviation σ_N through σ_{r2} as 0.02, 0.002, 0.02, 0.02, 0.002, 0.005, and 0.005, respectively. There are further restrictions on the variables.

$$2.0 \leq \mu_{R_1} \leq 2.5 \tag{5.13}$$

$$2.0 \leq \mu_{R_2} \leq 2.5$$

$$0.97 \leq \mu_{r_1} \leq 1.03$$

$$0.97 \leq \mu_{r_2} \leq 1.03$$

$$1.48 \leq \mu_N \leq 1.56$$

$$1.48 \leq \mu_n \leq 1.56$$

The target value of y is 69.

The optimal set points and corresponding output deviation at different α levels are shown in Figure 5.6. For this particular case, the optimal set points at all studied α levels are identical with $(\mu_N{}^*, \ldots, \mu_{r2}{}^*) = (1.553, 2.5, 2.5, 1.553, 1.03, 1.03)$. Table 5.3 contains the comparison with the results obtained from Sequential Quadratic variance Approximation method and Signal-to-Noise Ratio method. From the comparison results, we see that the Fuzzy Parameter Design Model also generates the consistent results to other methods in this case.

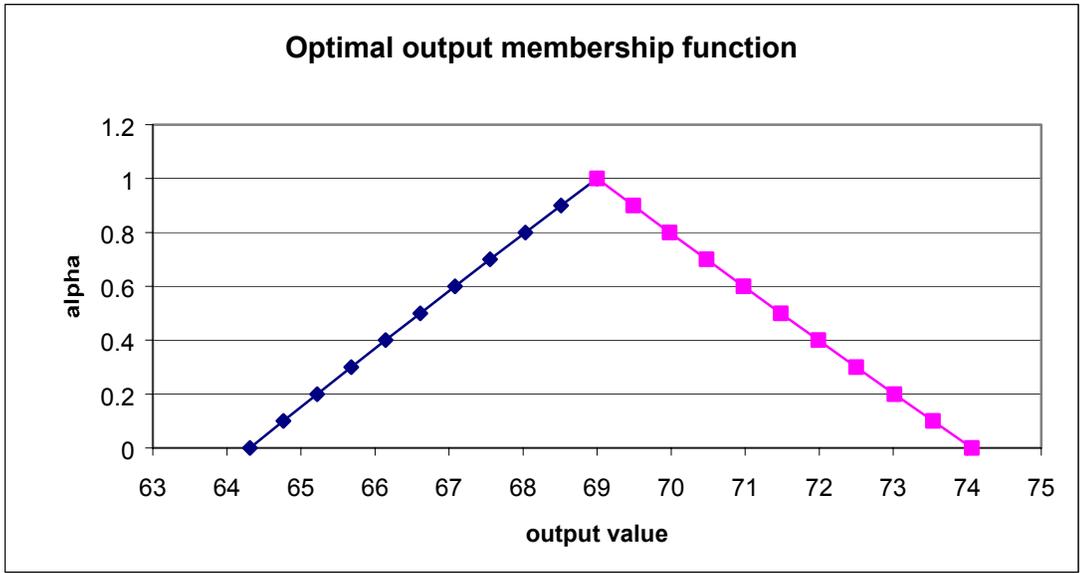| α	a_{y_α}	b_{y_α}	δ_{y_α}
0	64.311	74.064	9.753
0.1	64.764	73.539	8.775
0.2	65.221	73.019	7.798
0.3	65.68	72.503	6.823
0.4	66.144	71.991	5.847
0.5	66.611	71.483	4.872
0.6	67.081	70.979	3.898
0.7	67.555	70.478	2.923
0.8	68.033	69.982	1.949
0.9	68.515	69.489	0.974
1	69	69	0

Figure 5.6. The optimal set points at different α level and the optimal output membership function obtained by Fuzzy Parameter Design Model for Microscope case.

Methods	μ_N	μ_{R1}	μ_{R2}	μ_n	μ_{r1}	μ_{r2}	$\mu(y)$	STD(y)
Fuzzy	1.553	2.5	2.5	1.553	1.03	1.03	69.001	0.624
Sequential Quadratic Variance Approx.	1.551	2.5	2.5	1.556	1.03	1.03	69	0.624
Signal-to-Noise	1.56	2.5	2.5	1.547	1.03	1.03	68.999	0.624

Table 5.3. Set points for the parameters of Microscope obtained using various methods .

5.1.5   Conclusion

From the four cases presented in this section, we conclude that (1) within each case, the results of optimal set points at different α levels are identical; (2) that the optimal set points obtained by Fuzzy Parameter Design Model are consistent with those obtained from previous methods; (3) that the results using Monte Carlo simulation verify that Fuzzy Parameter Design Model performs quite well, since the optimal set points generated from this method are optimal.

5.2   Fuzzy Parameter Design with Alternative Objective Functions.

One of the advantages the Fuzzy Parameter Design Model has over others is that we have more options and flexibility to optimize the process through using an alternative objective function in the model. Sometimes, engineers are more interested in finding suitable set points to maintain the symmetry of the tolerance distribution, even if this means sacrificing some of the variance.

Let us explain this issue through an example. We present two modified models to demonstrate this flexibility and the robustness of the approach.

5.2.1   Fuzzy Parameter Design Model Modification I

Suppose we consider the previous coil spring case. Now our goal is not only to achieve the smallest variance (in our consideration, we need the smallest membership function dispersion, i.e., the smallest fuzzy deviation.), but also to achieve symmetry of the output function as much as possible. To achieve this goal, we need to modify the original model in expression (5.1) as follows,

Let $l\delta_{y_\alpha}$ denote the left deviation from the nominal value of fuzzy variable y at certain α level, and $r\delta_{y_\alpha}$ denote the right deviation from the nominal value of y at certain α level. Obviously, the following relations hold,

$$b_{y_\alpha} - r\delta_{y_\alpha} = \mu_y = \tau, \;\Rightarrow\; r\delta_{y_\alpha} = b_{y_\alpha} - \tau, \tag{5.14}$$

$$a_{y_\alpha} + l\delta_{y_\alpha} = \mu_y = \tau. \;\Rightarrow\; l\delta_{y_\alpha} = \tau - a_{y_\alpha}.$$

Now we can modify the original expression (5.1) by adding more constraints, such as $|r\delta_{y_\alpha} - l\delta_{y_\alpha}| \leq \xi$, (i.e., $|b_{y_\alpha} + a_{y_\alpha} - 2\tau| \leq \xi$,) to keep the output fuzzy variable as symmetric as possible. By this we mean to maintain the left deviation equal to the right

deviation of the output variable at each α level to the best extent possible, while keeping the objective function and other constraints unchanged. The modified model is as follows.

$$\underset{\mu_1,...\mu_n}{Minimize} \qquad z = \delta_{y_\alpha}(\mu_1,...\mu_n) \qquad\qquad (5.15)$$

$$Subject\ to \qquad h(\mu_1, ..., \mu_n) = \tau,$$

$$\delta_{y_\alpha} = b_{y_\alpha} - a_{y_\alpha},$$

$$-\xi \leq b_{y_\alpha} + a_{y_\alpha} - 2\tau \leq \xi,$$

$$b_{y_\alpha} = \underset{j}{\max}\ (h(V_j)),$$

$$a_{y_\alpha} = \underset{j}{\min}\ (h(V_j)),$$

$$(\mu_1, ..., \mu_n) \in M,$$

$$For\ a\ given\ \alpha \in [0, 1].$$

where ξ is a small value set by the designers. Typically, the engineer who is performing the tolerance design has a target ξ in mind, or ξ is determined by practical requirements. By controlling the value of ξ, we can control how symmetric the output process will be at optimal set points in the design stage.

We apply (5.15) to the coil spring case, and choose $\xi = 0.05$. This means the difference between the left and right deviation to its nominal value (the target value) will be no greater than 0.05. Interestingly, the optimal set points in this case are exactly identical to those in Case 5.1 at each α level. This means the original optimal set points have already achieved the maximum symmetry at the level of $\xi = 0.05$. The choice of ξ is critical. If ξ is too small then the constraint for symmetry is too strong and the model may

not have a feasible solution simultaneously satisfying both requirements of the smallest deviation and the best symmetry. In practice, we can choose suitable values of ξ by trial and error.

5.2.2 Fuzzy Parameter Design Model Modification Ⅱ

Expression 5.15 focuses more on achieving the smallest deviation than on achieving the symmetry requirement. To achieve the best symmetry, we can further modify the Expression 5.1 as:

$$\text{Minimize} \quad z = |r\delta_{y_\alpha} - l\delta_{y_\alpha}| \tag{5.16}$$

$$\text{Subject to} \quad h(\mu_1, \ ..., \ \mu_n) = \tau,$$

$$b_{y_\alpha} - r\delta_{y_\alpha} = \tau,$$

$$a_{y_\alpha} + l\delta_{y_\alpha} = \tau,$$

$$b_{y_\alpha} = \max_j (h(V_j)),$$

$$a_{y_\alpha} = \min_j (h(V_j)),$$

$$(\mu_1, \ ..., \ \mu_n) \in M,$$

For a given $\alpha \in [0, 1]$.

If we do not want the deviation to be too large then add one more constraint, $l\delta_{y_\alpha} + r\delta_{y_\alpha} \le$ *Range*. To take off the absolute value sign in the objective function, Expression 5.16 can be equivalently written without using the absolute value in the objective function as follows.

Minimize $\quad z = v_1 + v_2$ $\hspace{6cm}$ (5.17)

Subject to $\quad r\delta_{y_\alpha} - l\delta_{y_\alpha} = v_1 - v_2;$

$h(\mu_1, ..., \mu_n) = \tau,$

$b_{y_\alpha} - r\delta_{y_\alpha} = \tau,$

$a_{y_\alpha} + l\delta_{y_\alpha} = \tau,$

$b_{y_\alpha} = \max_j (h(V_j)),$

$a_{y_\alpha} = \min_j (h(V_j)),$

$(\mu_1, ..., \mu_n) \in M,$

$v_1, v_2 \geq 0.$

For a given $\alpha \in [0, 1]$.

Solving Expression 5.17 for the coil spring case, we get the identical optimal solution and the set points to those in Case 5.1. It further states that these optimal set points not only keep the output variable's deviation smallest, but also achieve its membership function best symmetry. Of course, this is not for every case. For a particular problem, a set of set points that is optimal for Fuzzy Parameter Design Model of Expression 5.1, they may not be optimal for Expression 5.15 and 5.16. Choosing the optimal set points depends on our objective. In other words, we have more options to do the parameter design using this Fuzzy Parameter Design Model approach.

In this section, we presented two modified Fuzzy Parameter Design Models to demonstrate the flexibility and robustness of the approach for the tolerance parameter design problem. Correcting asymmetry to symmetry in a tolerancing problem appears to be a practical feature for it. However, we must keep in mind that the approach is

essentially an approximation method in that we construct the components membership functions subjectively, and for computational convenience, we assumed that the extreme values occur at the corner points of the solution domain when using the vertex method. Consequently, the performance of FPDM relies greatly on the construction of the membership functions.

5.3   Fuzzy Parameter Design Model With Different α Level Considerations

In the cases we have examined we saw that the optimal solutions, the optimal set points, are identical at all α levels. In this section, we explore whether or not this is always the case. The Fuzzy Parameter Design Model is a typical nonlinear programming problem once the problem is defuzzified by decomposing it into different α levels. Since there is an interval corresponding to each α-cut associated with each component variable, it can be imagined that a component variable's feasible region is a value interval window sliding along the real number line. At each α level, the interval window associated with each component variable also changes. When $\alpha = 1$, the interval window shrinks into a single point, whereas when $\alpha = 0$, the interval window is widest for each component. Since the slopes of the membership functions for each corresponding component (i.e., the extents of the change on the interval windows), are different for these components, the feasible region changes in a different scale for different components when α changes, and we would expect that the optimal solution of set points could be different for different α levels for certain problems. This phenomenon should be more noticeable in the case where the membership functions for components are not triangular or the membership functions for components are highly out of proportion with each other.

Based on the above analytic considerations, we believe it is possible that the optimal set points differ at different α levels for certain problems. In other words, the choice of set points will also depend on which α level you choose when thinking about the process. We present a hypothetical case to show this is possible.

5.3.1 Case Study 5.5: α Level Dependent Solutions

Addressing problems when the optimal set points change at different α cut levels is another interesting feature of the fuzzy parameter design model, since neither design of experiments, linear or nonlinear programming, and other heuristic methods, nor the high-low tolerancing method, provide insight into this aspect of tolerance parameter design. To illustrate this point, let us exaggerate a hypothetical case to illustrate this point more clearly.

Suppose a system transfer function is given by,

$$y = h(x_1, x_2) = \frac{x_1 + x_2}{x_1 x_2} \ . \tag{5.18}$$

The target value of y is 66.67. Further restrictions are μ_1 and μ_2, and the nominal values of x_1 and x_2, respectively, are less than or equal to 5000. We do not know the distributions of components x_1 and x_2, but we know that the largest possible deviations from their nominal values for x_1 and x_2 are 30 and 45, respectively.

Example 1.

Now, let us suppose that with some experts' experience, the membership functions for components, x_1 and x_2, are the normal and symmetric triangular functions as shown in Figure 5.7. We can obtain the α-cut intervals for each component as,

$$x_{1_\alpha} = [\mu_1 - 30(1-\alpha), \mu_1 + 30(1-\alpha)], \tag{5.19}$$

$$x_{2_\alpha} = [\mu_2 - 45(1-\alpha), \mu_2 + 45(1-\alpha)].$$

We can obtain the output membership function discretized at certain α levels and the corresponding optimal set points using Expression 5.1. The result is shown in Figure 5.8. Notice that the optimal set points at different α levels are identical. We might conclude that no matter what degree of conformity used or considered, the set points for this problem are always $x_1 = 111.11$ and $x_2 = 166.67$.

Example 2.

It is not realistic that we always have perfect triangular membership functions as in Example 1 for output variables. As we discussed in Chapter 3, we typically construct the components membership functions based on experts' judgment or experience, which is subjective. Therefore, the function shape may deviate from the perfect triangular shape in practice. Suppose the two components' membership functions are shown as in Figure 5.9. We see that both of them are piecewise linear with a joint point at $\alpha = 0.5$. The α-cut intervals for x_1 and x_2 are as follows.

$$x_{1_\alpha} = [\mu_1 - 30 + 50\alpha, \mu_1 + 30 - 50\alpha], \text{ when } 0 \le \alpha \le 0.5 \qquad (5.20)$$

$$x_{1_\alpha} = [\mu_1 - 10 + 10\alpha, \mu_1 + 10 - 10\alpha], \text{ when } 0.5 \le \alpha \le 1$$

and

$$x_{2_\alpha} = [\mu_2 - 45 + 10\alpha, \mu_1 + 45 - 10\alpha], \text{ when } 0 \le \alpha \le 0.5$$

$$x_{2_\alpha} = [\mu_2 - 80 + 80\alpha, \mu_1 + 80 - 80\alpha], \text{ when } 0.5 \le \alpha \le 1$$

We apply Expression 5.1 to this problem, the expressions of components at the α-cut intervals are different when $\alpha > 0.5$ than $\alpha \le 0.5$.
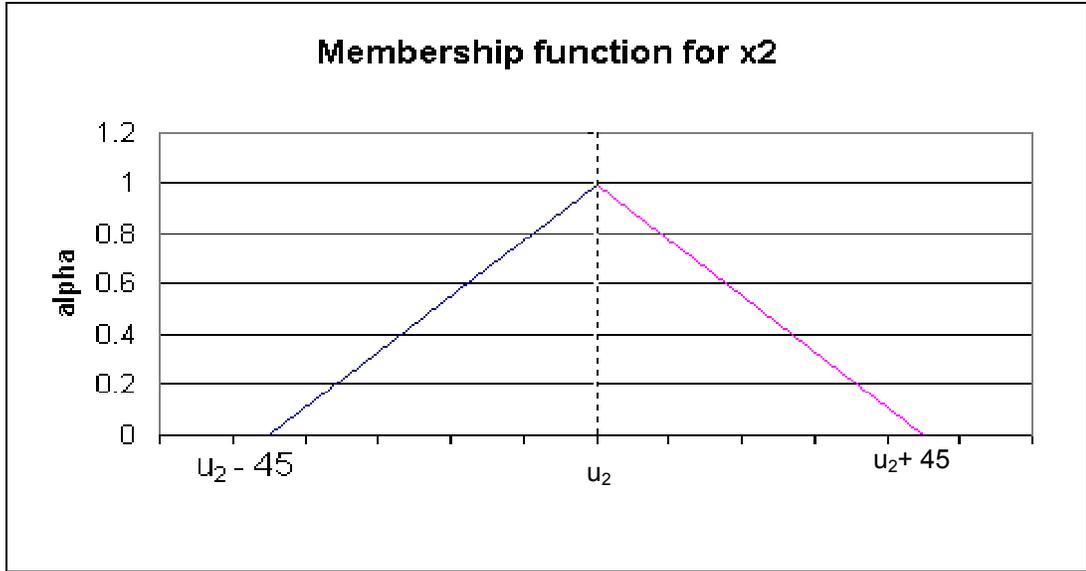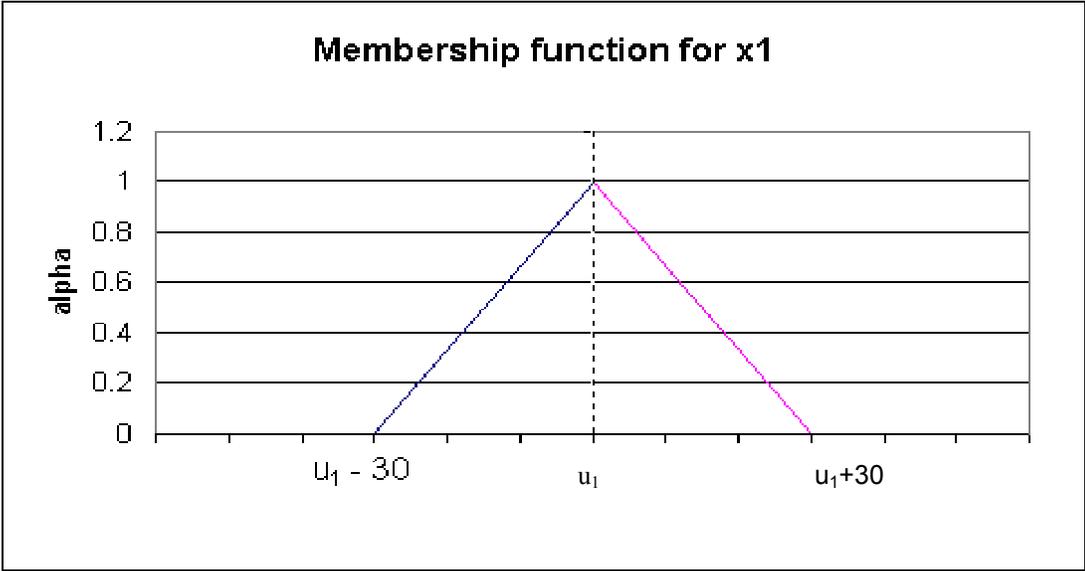
Figure 5.7. Components' membership functions for Example 1.

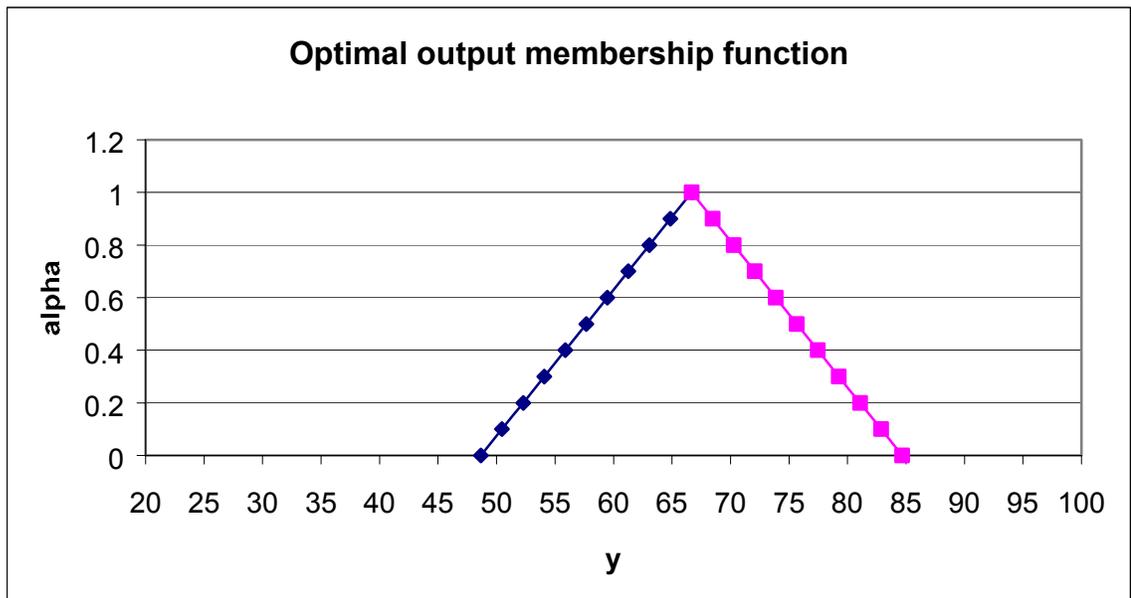| α	a_{y_α}	b_{y_α}	δ_{y_α}
0	48.67	84.67	36.00
0.1	50.47	82.87	32.40
0.2	52.27	81.07	28.80
0.3	54.07	79.27	25.20
0.4	55.87	77.47	21.60
0.5	57.67	75.67	18.00
0.6	59.47	73.87	14.40
0.7	61.27	72.07	10.80
0.8	63.07	70.27	7.20
0.9	64.87	68.47	3.60
1	66.67	66.67	0.00

Figure 5.8. The optimal set points at different α level and the optimal output membership function for Example 1.
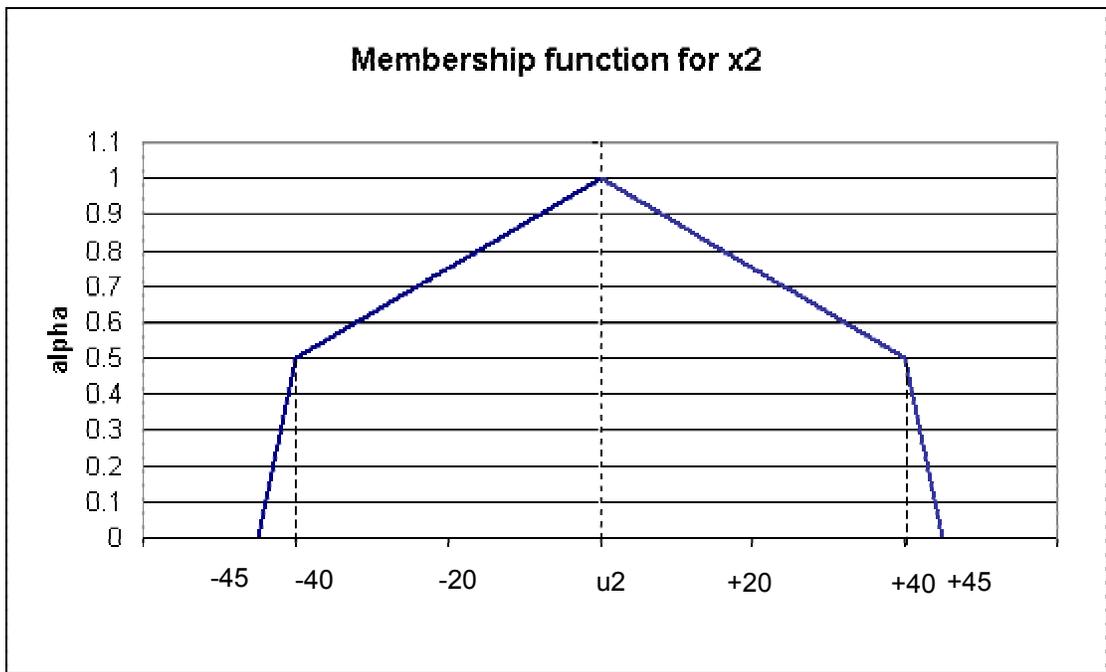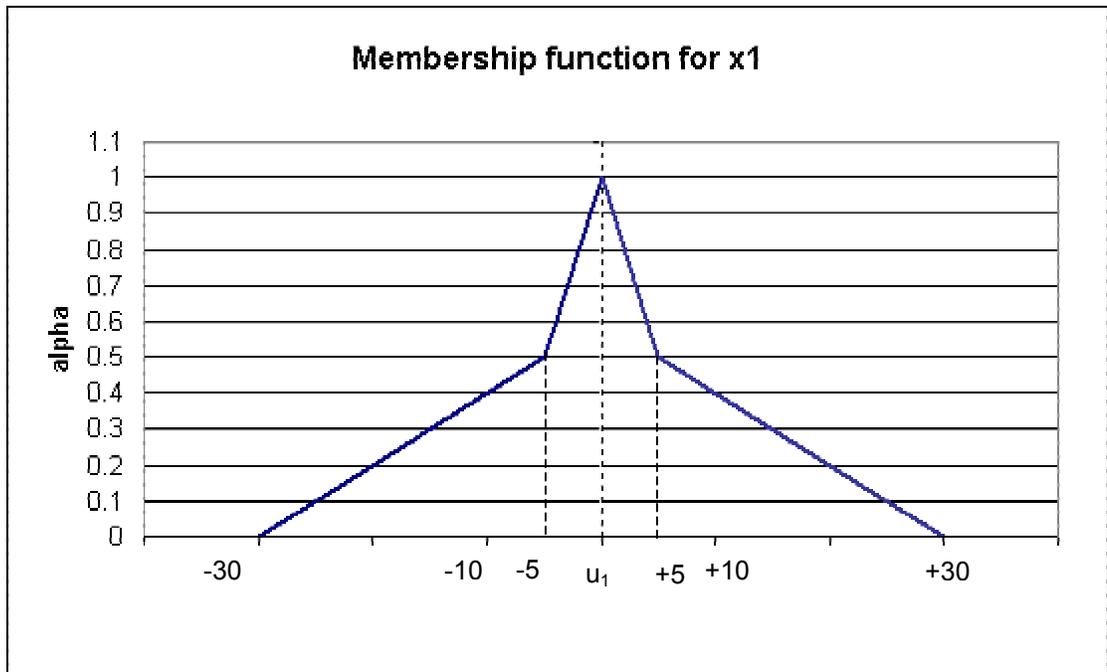
Figure 5.9. Components' membership functions for Example 2.

Solving the above problem, we get the optimal set points at each α level as shown in Table 5.4.

α	μ_1^*	μ_2^*	a_{y_α}	b_{y_α}	δ_{y_α}
0	111.11	166.67	48.67	84.67	36.00
0.1	104.54	184.00	50.72	82.61	31.88
0.2	97.67	210.00	53.02	80.32	27.30
0.3	90.48	253.34	55.61	77.72	22.11
0.4	82.94	339.79	58.63	74.71	16.08
0.45	79.02	426.58	60.34	72.99	12.66
0.5	75.01	599.26	62.22	71.11	8.89
0.55	75.01	599.63	62.67	70.67	8.00
0.6	75.02	598.91	63.11	70.22	7.11
0.7	75.06	595.95	64.00	69.33	5.33
0.8	75.06	595.90	64.89	68.44	3.56
0.9	75.16	589.84	65.78	67.56	1.78
1	76.82	504.25	66.67	66.67	0.00

*Note, the optimal set points are different at each alpha level.

Table 5.4. The optimal set points at different alpha levels for Example 2.

Examining Table 5.4 we see that the optimal set points are different at different α levels. This means that the optimal set points to achieve the minimum deviation from its nominal value for the output variable vary among different α-levels. For example, at different α-levels, the optimal set points are as follows,

$$\mu_1^* = 111.11, \; \mu_2^* = 166.67 \text{ for } \alpha = 0 \tag{5.21}$$

$$\mu_1^* = 97.67, \; \mu_2^* = 210.00 \text{ for } \alpha = 0.2$$

$$\mu_1^* = 82.94, \; \mu_2^* = 166.67 \text{ for } \alpha = 0.4$$

$$\mu_1^* = 75.01, \; \mu_2^* = 599.26 \text{ for } \alpha \geq 0.5$$

When we reconstruct the membership function of output variable y with α ranging from 0 to 1 according to the fuzzy decomposition theorem, the optimal membership function for the output variable y is a family of membership functions that are optimal at each corresponding α level. In other words, the whole optimal membership function for y is a group of linear segments assembled from the most inner segments of those individual optimal membership functions at each α level. This is shown in Figure 5.10.
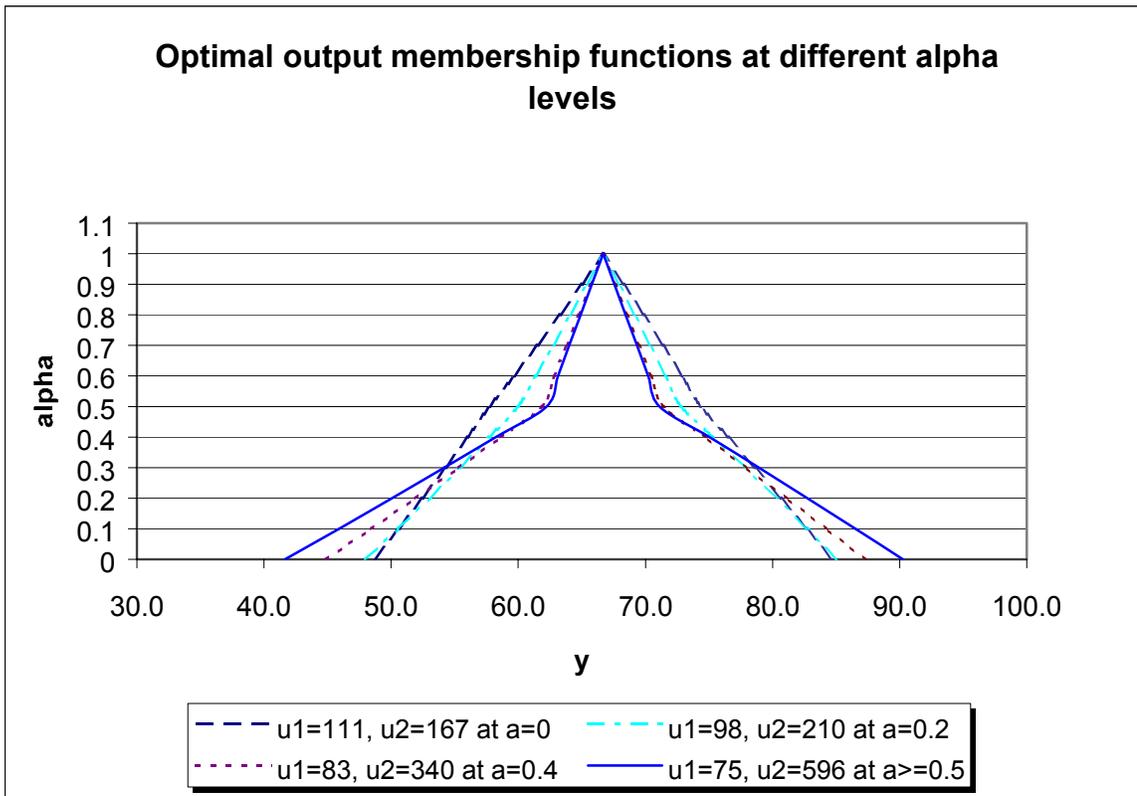

Figure 5.10. Family of optimal set points and the corresponding output membership function at different α levels in Example 2.

Figure 5.10 illustrates that the set points that are optimal at a certain α level are not necessarily still optimal at another α level. For instance, $\mu_1^* = 111.11$ and $\mu_2^* = 166.67$ are optimal set points for $\alpha = 0$, but are no longer optimal for $\alpha \geq 0.6$. If we need the degree of conformity for the output variable to be greater than or equal to 0.6, the set points should be $\mu_1^* = 75.02$ and $\mu_2^* = 598.91$.

Let us compare the difference of the deviation between these two solutions. The optimal deviation of y at $\alpha = 0.6$ is $70.22 - 63.11 = 7.11$ when $\mu_1 = 75.02$ and $\mu_2 = 598.91$, whereas the deviation is $72.88 - 59.66 = 13.22$, if we had used the set points of $\mu_1 = 111.11$ and $\mu_2 = 166.67$. This means in practice if the input components' membership functions behave as we assume, and if we use previous methods, such as High-low tolerancing methods to solve it, we will locate the set points at $\mu_1 = 111.11$ and $\mu_2 = 166.67$, and the corresponding tolerance we need to allow is 13.22. But if we believe that the system has 0.6 above degree of conformity, then the optimal set points should be $\mu_1 = 75$ and $\mu_2 = 599$, and the corresponding tolerance we need to allow is only 7.11 and which is much smaller than 13.11. The difference is $\dfrac{13.22 - 7.11}{7.11} \times 100\% = 86\%$. This means 86% of the tolerance resource is wasted if we locate the set points as at 75.02 and 598.91. This makes sense in that the high-low tolerancing is only dealing with the worst case, which is rare in practice.

We can see the difference of philosophy between Fuzzy Parameter Design Model and high-low tolerancing. High-low tolerancing only captures the worst case, and Fuzzy Parameter Design Model gives us a chance to consider systems at different levels according to our demand and/or real-world requirements. It can basically capture the

whole picture of the system behavior. In so doing, we can reduce the tolerance we need to assign or consider at a design stage by modeling reality more closely, and improve the performance of the whole system as well.

If we carry out the parameter design problem for this case by using functional approximation or other statistic approximation methods, we have to assume that the components are normally distributed. If the components' distributions are not normally distributed or they are unknown, then these methods need to be modified accordingly. But with Fuzzy Parameter Design Model, we can solve the problems even under such conditions. This is possible by modeling the uncertainty or imprecision of the manufacturing process in fuzziness instead of randomness. We use the membership function to capture and describe the fuzziness of the variables, instead of the probability density function to capture and describe the randomness of the variables. We may not be able to get the probability distribution in some situations, but we are able to get the membership function based on the experience and judgment to a certain degree of accuracy for modeling the imprecision. Now the problem is not whether we can do or we cannot do; it is rather a matter of how accurate we can model the process by using Fuzzy Parameter Design Model. As we discussed before, the key issue here is how accurately we are able to construct the membership functions to model the system.

We conclude the advantage of Fuzzy Parameter Design Model over the high-low tolerancing and functional approximation methods is that we can use a more detailed representation of the process instead of just the worst case and when the statistic methods are not applicable the Fuzzy Parameter Design Model can be used. In the situations

where the high-low tolerancing and functional approximation are applicable, Fuzzy Parameter Design Model still works and generates consistent results.

6. Conclusions and Future Work

We discussed the tolerance analysis and parameter design model problem in a new approach using fuzzy set theory and also developed a mathematical programming model for the Fuzzy Parameter Design Model. The new approach bypasses some difficulties and problems in statistical analysis on this topic by using a fuzzy membership function to model the fuzziness, instead of the randomness of the manufacturing process. This approach and model can be used to solve the problem when the transfer function is available either in closed form or through a computer simulation model, and it is continuous everywhere. The work does not require the transfer function to be differentiable, and the distributions of component variables do not need to be known. We have used this new approach either to do the tolerance analysis or to do the parameter design problem by solving several case studies from the literature. In all cases, we obtained satisfactory performance using the proposed approach. The biggest disadvantage of this approach is that when the dimension of the component variables increase, the computation will increase exponentially. Finally, we think that the fuzzy approach is an enrichment, not a replacement, to previous methods. It is not important whether we choose fuzziness or randomness for modeling; what is important is to model reality more closely.

Future work. To see if we can interpret tolerance analysis and the parameter design problem using Possibility theory, which could be used more suitably and intuitively, especially when we lack the information about the process--either the probability distribution or the fuzzy membership function.

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