Abstract

YUAN, PEI-LUN. Game Theoretic Analysis of a Distribution System in Supply Chain. (Under the direction of Shu-Cherng Fang, and Henry L. W. Nuttle.)

We consider a distribution system in which one supplier provides a single product to several retailers at the beginning of a selling season. The supplier has infinite capacity. The customer demand at each retailer is randomly distributed. Customers who encounter a stockout at one retailer may search other retailers for the product. We study the effects of this market search behavior under both decentralized and centralized control. For the decentralized control model, we show the necessary and sufficient conditions for the existence of a Nash equilibrium, and the sufficient conditions for its uniqueness. For the centralized control model, we find that the payoff function is submodular, and thus we can only obtain allocations that are locally optimal for the entire supply chain. We also design a channel coordination mechanism to match the allocations in the decentralized control model with one of the local optimal allocations under centralized control.
Game Theoretic Analysis of a Distribution System in Supply Chain

by

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Pei-Lun Yuan was born in Taipei, Taiwan in 1979. She attended the National Taiwan University through the recommendation of the Taipei First Girl senior high school in 1997. She graduated with a Bachelor degree in Chemical Engineering in 2001. She came to North Carolina State University in the summer of 2001 to begin her graduate studies in Industrial Engineering. Her main research interests include system analysis, optimization and production systems.
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## Contents

List of Figures v

1 Introduction 1

2 Background and Literature Review 3
   2.1 Market search problem 3
   2.2 Game theory 4
   2.3 Channel coordination 6
   2.4 Product substitution problem 7

3 The model and basic notation 9

4 Decentralized Control System 13

5 Centralized Control System 26

6 Channel Coordination 35

7 Numerical Experiments 38

8 Conclusion 50

References 52
### List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Decentralized control system</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>Centralized control system</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>Allocation versus market search probability with 2 retailers</td>
<td>41</td>
</tr>
<tr>
<td>4</td>
<td>Profit of retailers versus market search probability with 2 retailers</td>
<td>43</td>
</tr>
<tr>
<td>5</td>
<td>Profit of supplier versus market search probability with 2 retailers</td>
<td>43</td>
</tr>
<tr>
<td>6</td>
<td>System profit versus market search probability with 2 retailers</td>
<td>44</td>
</tr>
<tr>
<td>7</td>
<td>Profit improvement after coordination versus market search probability with 2 retailers</td>
<td>44</td>
</tr>
<tr>
<td>8</td>
<td>Wholesale price versus market search probability with 2 retailers</td>
<td>45</td>
</tr>
<tr>
<td>9</td>
<td>Allocation versus market search probability with 3 retailers</td>
<td>45</td>
</tr>
<tr>
<td>10</td>
<td>Profit of retailers versus market search probability with 3 retailers</td>
<td>46</td>
</tr>
<tr>
<td>11</td>
<td>Profit of supplier versus market search probability with 3 retailers</td>
<td>46</td>
</tr>
<tr>
<td>12</td>
<td>System profit versus market search probability with 3 retailers</td>
<td>47</td>
</tr>
<tr>
<td>13</td>
<td>Profit improvement after coordination versus market search probability with 3 retailers</td>
<td>47</td>
</tr>
<tr>
<td>14</td>
<td>Wholesale price versus market search probability with 3 retailers</td>
<td>48</td>
</tr>
<tr>
<td>15</td>
<td>Profit percentage after coordination of supplier and retailers versus market search probability with 2 retailers</td>
<td>51</td>
</tr>
</tbody>
</table>
1 Introduction

Capacity allocation plays an important role in supply chain management. Decisions of both suppliers and retailers are significant not only because they determine the amount of flexibility the supply chain has to meet customers’ demand, but also because they affect each other. For example, allocating too much capacity to a retailer results in poor utilization and higher holding cost, but too little capacity results in poor customer satisfaction and higher penalty cost.

In many supply chains, customers who face a stockout at one retailer may search for the same product at other retailers. This behavior is called market search. (Anupindi and Bassok, 1999).

Dai (2002) considers market search in an allocation problem that involves one supplier and two retailers in the supply chain in a single selling season. Here we generalize Dai’s (2002) work to the case of one supplier and “multiple” retailers in the supply chain. We assume the supplier has infinite capacity and the customers’ demand at each retailer is randomly distributed. The market search effect is studied in two models. In the first model, assuming under decentralized control, all retailers compete with each other. Therefore, “game theory” is applied to decide their allocations. The other model assumes under centralized control, in which all retailers collaborate with each other. Thus the allocation for the whole supply chain is optimized.

In Chapter 2, some background knowledge and a literature review are given. In Chapter 3, we introduce the model and notation. In Chapter 4, we obtain the existence and uniqueness conditions for a Nash equilibrium under decentralized control.
In Chapter 5, we show that the payoff function under centralized control is submodular and thus only local optimum allocations can be found. In Chapter 6, we design a coordination mechanism between decentralized control and centralized control. In Chapter 7, through numerical experiments, we analyze the effect of coordination and the market search factor on the retailers, supplier, and whole supply chain. Finally, we give concluding remarks in Chapter 8.
2 Background and Literature Review

In this chapter, we present some important concepts used in this thesis and give a brief literature review. The concepts include market search, game theory, and channel coordination. We also introduce a related research issue called product substitution.

2.1 Market search problem

Customers who face a stockout at one retailer may keep on looking for the same product at other retailers. This phenomenon is called market search and was first introduced by Anupindi and Bassok (1999). They compare the performance of two systems: one in which the retailers hold stocks separately and the other in which the retailers centralize stocks at a single location. They find that whether centralization of stocks by retailers increases profits for the manufacturer depends on the level of market search in the supply chain. The level of market search is measured as the fraction of customers who transfer their demand from the original retailer they visit to other retailers when they encounter a stockout. They show that there exists a threshold of market search level beyond which the system profit may decrease with centralization of stocks.

In Anupindi and Bassok’s (1999) model, centralization means that all stocks in the supply chain are located at one retailer. But in our work, centralization means collaboration and information sharing. The inventory is held at different retailers based on the customer demand.

With customers’ market search behavior, we analyze the differences between two systems. A decentralized control system (Figure 1) and a centralized control system
Figure 1: Decentralized control system

(Figure 2). Under decentralized control, each retailer makes its order decision by itself and does not cooperate with other retailers. To maximize its own profit, each retailer has to determine its order strategy so as to compete with other retailers. Under centralized control, all retailers in the supply chain are willing to cooperate and make the allocation decisions together with the supplier. If customers encounter a stockout at one retailer and they want to look for the same product at other retailers, the initial retailer may tell the customers at which retailers this product is still available. Therefore, the whole supply chain will pay less penalty cost for lost customers.

2.2 Game theory

Decentralized control is often used in supply chains. Since the ordering decision of one retailer affects the demand of other retailers, they compete with each other in
trying to maximize their own profits. A strategic interaction results from retailers’ ordering decisions. Therefore, we apply game theory to analyze the allocation decisions of retailers in a decentralized control system.

Game theory has been widely used in supply chain management (Mesterton, 2000). The term “players” refers to the different parties who compete with each other in the supply chain. “Payoff function” is the profit function or utility function of each player. A strategic game is a model of interactive decision-making in which each player chooses his or her strategy once and for all, and these choices are made simultaneously. In a strategic game, the number of players has to be finite. In game theory, the players are assumed to be rational. They always try to maximize their individual profit. The so-called Nash equilibrium establishes when no player has an action yielding an outcome that he prefers to that generated when he chooses his best response,
given that every other player chooses his/her equilibrium action, i.e., no player can profitably deviate when we give other players’ strategies (Osborne, 1996).

Game theory has been applied to solve some inventory problems. Parlar (1988) studies the substitutable product problem as an extension of the classical newsvendor problem. In his two-player model, product substitution (which we will describe in detail later) occurs with a certain probability when customers encounter a stockout. For this problem, he proves the existence of a unique Nash equilibrium. Lippman and McCardle (1994) also study an extension of the classical newsvendor problem. They assume the salvage value of excess inventory and penalty for unsatisfied demand are zero. Under these assumptions, they examine the equilibrium of inventory levels and rules for reallocating excess demand. They also provide existence conditions for a Nash equilibrium when two or more newsvendors are involved.

Under decentralized control, the retailers are the players, and their individual profit functions are the payoff functions. We apply game theory to analyze the best strategies for retailers to order the inventory they need. Game theory is also used to determine the existence conditions of a Nash equilibrium for the decentralized model in Chapter 4. Furthermore, when a Nash equilibrium exists, we also want to know the conditions which support global stability of the solution.

2.3 Channel coordination

Compared to decentralized control, centralized control yields more profit for the supply chain as a whole. However, in reality, most retailers will not cooperate, share their information, and manage their inventory allocations in concert with the sup-
plier unless they are owned by the same company. To improve the performance under decentralized control, coordination mechanisms can be designed by altering the individual payoff functions. In this way, it may be possible to reach the same solution and the overall profit as with a centralized control system. This approach is called channel coordination. In a supply chain, channel coordination can be achieved through a contract between the supplier and retailers. This contract indicates benefit sharing among the supplier and retailers after coordination. To understand the effect of coordination between supplier and retailers in our scenario, we will analyze the profit ratio of supplier and retailers after channel coordination.

2.4 Product substitution problem

In addition to searching for the same product at other retailers, customers may also substitute an out-of-stock product with different products. This is called product substitution. Compared to the market search problem, product substitution problem is more general. McGillivray and Silver (1978) consider the substitution of two identical products with independent normally distributed demands and develop heuristics to obtain the optimal stocking level. Palar and Goyal (1984) analyzed this problem of two substitutable products with stochastic demands. Pasternak and Drezner (1991) provide an analytical solution to the problem of two substitutable products and conclude that substitution gives a lower revenue. Many people have followed this stream to work on similar problems. However, it is very difficult to deal with problems which have more than three substitutable products using the Leibnitz formula. In a 2001 paper, Rudi provides a more tractable expression for derivatives of high dimensional integrals. Netessine (2001) adopts Rudi’s expression to solve more complex cases of the product substitution problem. He shows that the objective function of the prod-
uct substitution problem is not generally concave, but submodular.
3 The model and basic notation

Consider one supplier distributing one product to $n$ retailers who sell this product to customers in a single period on season. If customers who visit retailer $i$ at begining face a stockout, some of them may search for the same product at a second retailer $j$ with probability $a_{ij}$. We assume that customers at most visit two retailers, i.e., if they encounter a stockout at the second visited retailer, they will not search anymore.

Since customers may transfer their demand between retailers, the total customer demand at any retailer can be separated into local demand and distant demand. Local demand is from the customers who visit this retailer at first, while distant demand is the demand transferred from other stocked out retailers. The total customer demand is called effective demand. Usually, retailers will give priority to their local demand and use the remaining inventory, if any, to satisfy their distant demand. The inventory of each retailer certainly affects other retailers’ distant demand.

We assume that the supplier has infinite capacity to provide the product. At the beginning of the season, retailers will receive whatever amount of product that they ordered. At the end of the season, a holding cost or stockout penalty is incurred by each retailer depending on whether there is unsold stock or a stockout. In the decentralized control model, the stockout penalty cost is modeled as incurred for unsatisfied local demand. In the centralized control model, the supplier and all retailers can be viewed as a whole. Therefore, the average penalty cost of all retailers is incurred by the system when customers leave the supply chain without a purchase.

In the decentralized control model, each retailer is assumed to be an independent rational “player” and aware of other retailers’ local demand distributions and the
market search probability of customers. The problem of making optimum ordering
decisions by the retailers is modeled as a game.

In the centralized control model, since the supplier and all retailers cooperate with
each other, instead of playing a game, they have to decide the total production and
allocations together so as to maximize the total expected profits of the supply chain.

Following is the notation that we use in the two models:

for the supplier:

\[ K \]: the capacity of the supplier;
\[ c \]: the unit production cost;
\[ w_i \]: the wholesale price to retailer \( i, i = 1, \ldots, n \);

for retailer \( i, i = 1, \ldots, n \):

\[ s_i \]: the unit selling price for local demand;
\[ t_i \]: the unit selling price for distant demand;
\[ h_i \]: the unit holding cost per unit of product left at the end of the season;
\[ p_i \]: the per unit stockout penalty cost;
\[ \bar{p} \]: the average unit stockout penalty cost for the \( n \) retailers;
\[ D_i \]: a continuous random variable, denoting a stochastic local demand;
\[ f_i(D_i) \]: the probability density function of \( D_i \);
$F_i(D_i)$: the cumulative distribution function of $D_i$;

$a_{ij}$: the market search probability from retailer $i$ to retailer $j$, an element of the market search matrix, $0 \leq a_{ij} \leq 1$, $a_{ii} = 0$;

$y_i$: the allocation, i.e., inventory, at the beginning of the season;

$R_i$: the effective demand. It is the summation of local demand and distant demand,

$$R_i = D_i + \sum_{j=1}^{n} a_{ji}(y_j - D_j)^-, \quad (1)$$

where $(x)^- = \max\{-x, 0\}$, and $(x)^+ = \max\{x, 0\}$;

$\pi_i(y_1, y_2, ..., y_n)$: the expected payoff given all retailers’ allocations, $y_1, y_2, ..., y_n$.

Following Pasternack and Drezner (1991), we make the following assumptions:

(A1) $s_i > w_i > 0, i = 1, ..., n$;

(A2) $t_i > w_i > 0, i = 1, ..., n$;

(A3) $s_i > t_i - p_i, i = 1, ..., n$;

Assumptions (A1) and (A2) express the idea that the product selling price should be higher than its cost. Assumptions (A3), as observed by Pasternack and Drezner, describes the phenomenon that the retailers will satisfy the local demand at first, and then use the remaining inventory, if any, to satisfy a distant demand. Assumption (A3) is relevant in the decentralized controlled model. Suppose that retailer $i$ has
only one unit of product left and faces both local and distant demand simultaneously. Because the local selling price is higher than the distant selling price less the shortage penalty, retailers will consider local customers as their priority.

If retailer \( i \) cannot distinguish the local demand from distant demand, i.e., \( s_i = t_i \), then assumption (A3) for retailer \( i \) is unnecessary. However, for other retailers who can tell the difference between local customers and distant ones, they still need assumption (A3) to give priority to their local demand.
4 Decentralized Control System

Under decentralized control, each retailer makes his/her own order decision, and does not cooperate with other retailers. To maximize his/her own profit, each retailer has to manage his/her order strategy in competition with other retailers. Here we assume the supplier has infinite capacity, i.e. all retailers can order and be allocated the amount of inventory they need. Given that retailer $i$’s order is $y_i$, the payoff for retailer $i$ consists of:

(i) Purchasing cost: $w_i y_i$;

(ii) Holding cost: $h_i (y_i - R_i)^+$;

(iii) Penalty cost: $p_i (y_i - D_i)^-$;

(iv) Selling revenue: $s_i \min\{y_i, D_i\} + t_i \min\{(y_i - D_i)^+, \sum_{j=1}^{n} a_{ji}(y_j - D_j)^-\}$.

Thus retailer $i$’s expected payoff function is

$$
\pi_i(y_1, y_2, \ldots, y_n) = E[s_i \min\{y_i, D_i\} + t_i \min\{(y_i - D_i)^+, \sum_{j=1}^{n} a_{ji}(y_j - D_j)^-\}]
$$

$$
-w_i y_i
-h_i (y_i - R_i)^+
-p_i (y_i - D_i)^-
= E[s_i y_i - w_i y_i + p_i (y_i - D_i)]
$$

$$
-h_i (y_i - R_i)^+
-(s_i + p_i - t_i)(y_i - D_i)^+
-t_i[(y_i - D_i)^+ - \sum_{j=1}^{n} a_{ji}(y_j - D_j)^-)]^+
$$

(2)
where \( \min\{a, b\} = a - (a - b)^+ \).

**Lemma 1** Given \( y_j, j = 1, \ldots, n, j \neq i, \pi_i(y_1, \ldots, y_n) \) is strictly concave in \( y_i \).

**Proof** The first derivative of \( \pi_i(y_1, \ldots, y_n) \) with respect to \( y_i \) is

\[
\frac{\partial \pi_i(y_1, y_2, \ldots, y_n)}{\partial y_i} = (s_i - w_i + p_i) - \frac{\partial E[h_i(y_i - R_i)^+]}{\partial y_i} - \frac{\partial E[(s_i + p_i - t_i)(y_i - D_i)^+]}{\partial y_i} - \frac{\partial E[t_i ((y_i - D_i)^+ - \sum_{j=1}^{n} a_{ji}(y_j - D_j)^-)^+]}{\partial y_i} \tag{3}
\]

\[
= (s_i - w_i + p_i) - (s_i + p_i - t_i) \Pr(y_i > D_i) - (h_i + t_i) \Pr(y_i > R_i). \tag{4}
\]

As noted in Netessine (2001), due to the necessity of dealing with nested integrals of high dimensionality over the regions formed by intersections of a large number of hyperplanes, it is very difficult to take derivatives by the Leibnitz formula. Thus to go from (3) to (4), we directly utilize the definition as in Rudi (2001).

The derivate of the function \( f(x) \) with respect to variable \( x \) is defined as follows,

\[
\frac{\partial f(x)}{\partial x} = \lim_{\epsilon \to 0} \frac{f(x + \epsilon) - f(x)}{\epsilon}.
\]
First, we work with the second term of (3) and determine the derivative of $E[h_i(y_i - R_i)^+]$

$$\frac{\partial E[h_i(y_i - R_i)^+]}{\partial y_i} = h_i \lim_{\epsilon \to 0} E \left[ \frac{(y_i + \epsilon - R_i)^+ - (y_i - R_i)^+}{\epsilon} \right].$$

We distinguish these cases, namely

$$(y_i + \epsilon - R_i)^+ - (y_i - R_i)^+ = \begin{cases} 
0 & \text{if } y_i - R_i \leq -\epsilon; \\
(y_i + \epsilon - R_i) & \text{if } -\epsilon < y_i - R_i \leq 0; \\
\epsilon & \text{if } 0 < y_i - R_i.
\end{cases}$$

Suppose $\epsilon > 0$ (analogously, for $\epsilon < 0$), taking the derivative for each of the three cases, we obtain,

$$\frac{\partial E[h_i(y_i - R_i)^+]}{\partial y_i} = h_i \lim_{\epsilon \to 0} \frac{0Pr(y_i - R_i < -\epsilon) + \int_{y_i}^{y_i+\epsilon} (y_i + \epsilon - x)f_R(x)dx + \epsilon Pr(y_i > R_i)}{\epsilon}.$$  \hspace{1cm} (5)

The second term on the right hand side of (5) can be re-written as:
\[
h_i \lim_{\epsilon \to 0} \frac{\int_{y_i}^{y_i+\epsilon} (y_i + \epsilon - x) f_R(x) dx}{\epsilon}
\]

\[
= h_i \lim_{\epsilon \to 0} \frac{E[y_i + \epsilon - R_i | y_{ii} < y_i + \epsilon]}{\epsilon} Pr(y_i \leq R_i < y_i + \epsilon) \leq R_i < y_i + \epsilon).
\]

Obviously,

\[
\lim_{\epsilon \to 0} \frac{E[y_i + \epsilon - R_i | y_i \leq R_i < y_i + \epsilon]}{\epsilon} < 1.
\]

Therefore,

\[
\lim_{\epsilon \to 0} \frac{\int_{y_i}^{y_i+\epsilon} (y_i + \epsilon - x) f_R(x) dx}{\epsilon} = 0.
\]

Consequently,

\[
\frac{\partial E[h_i (y_i - R_i)^+]}{\partial y_i} = h_i Pr(y_i > R_i).
\]

(6)
The third term of (3) is taken from the derivative of \( E[(s_i + p_i - t_i)(y_i - D_i)^+] \) in the same way. Hence
\[
\frac{\partial E[(s_i + p_i - t_i)(y_i - D_i)^+]}{\partial y_i} = (s_i + p_i - t_i) \text{Pr}(y_i > D_i). \tag{7}
\]

We determine the derivative of the fourth term of (3):
\[
\frac{\partial E[t_i((y_i - D_i)^+ - \sum_{j=1}^{n} a_{ji}(y_j - D_j)^-)^+]}{\partial y_i} = t_i \lim_{\epsilon \to 0} E \left[ ((y_i + \epsilon - D_i)^+ - \sum_{j=1}^{n} a_{ji}(y_j - D_j)^-)^+ - ((y_i - D_i)^+ - \sum_{j=1}^{n} a_{ji}(y_j - D_j)^-)^+ \right] / \epsilon.
\]

Here we have four cases, for the numerator of the expression under the expectation, namely
\[
((y_i + \epsilon - D_i)^+ - \sum_{j=1}^{n} a_{ji}(y_j - D_j)^-)^+ - ((y_i - D_i)^+ - \sum_{j=1}^{n} a_{ji}(y_j - D_j)^-)^+ = \begin{cases} 
\epsilon & \text{if } y_i - D_i > 0, \\
y_i + \epsilon - D_i & \text{if } -\epsilon < y_i - D_i \leq 0, \\
y_i + \epsilon - D_i - \sum_{j=1}^{n} a_{ji}(y_j - D_j)^- & \text{if } -\epsilon < y_i - D_i \leq 0, \\
0 & \text{if } y_i - D_i \leq -\epsilon.
\end{cases}
\]
This leads to

\[
\frac{\partial E[t_i((y_i - D_i)^+ - \sum_{j=1}^{n} a_{ji}(y_j - D_j)^-) +]}{\partial y_i} = t_i \lim_{\epsilon \to 0} \left[ \epsilon Pr(y_i - D_i > 0, (y_i - D_i)^+ - \sum_{j=1}^{n} a_{ji}(y_j - D_j)^-) > 0) + \int_{y_i}^{y_i+\epsilon} (y_i + \epsilon - x) f_{D_i}(x)dx + \int_{y_i}^{y_i+\epsilon} (y_i + \epsilon - x - \sum_{j=1}^{n} a_{ji}(y_j - D_j)^-) f_{D_i}(x)dx + 0 Pr(y_i - D_i \leq -\epsilon) \right] / \epsilon.
\]  

(8)

As described before, the second term of (8) can be re-written as

\[
\lim_{\epsilon \to 0} \frac{\int_{y_i}^{y_i+\epsilon} (y_i + \epsilon - x) f_{D_i}(x)dx}{\epsilon} = \lim_{\epsilon \to 0} E[y_i + \epsilon - D_i | y_i \leq D_i < y_i + \epsilon] Pr(y_i \leq D_i < y_i + \epsilon).
\]

Obviously,
\[
\lim_{\epsilon \to 0} \frac{E[y_i + \epsilon - D_i | y_i \leq D_i < y_i + \epsilon]}{\epsilon} < 1.
\]

Hence,

\[
\lim_{\epsilon \to 0} \frac{\int_{y_i}^{y_i + \epsilon} (y_i + \epsilon - x) f_{D_i}(x) dx}{\epsilon} = 0.
\]

Also, the third term of (8) can be re-written as

\[
\lim_{\epsilon \to 0} \frac{\int_{y_i}^{y_i + \epsilon} (y_i + \epsilon - x - \sum_{j=1}^{n} a_{ji}(y_j - D_j)^-) f_{D_i}(x) dx}{\epsilon} \\
= \lim_{\epsilon \to 0} \frac{E[y_i + \epsilon - D_i - \sum_{j=1}^{n} a_{ji}(y_j - D_j)^- | y_i \leq D_i < y_i + \epsilon]}{\epsilon} Pr(y_i \leq D_i < y_i + \epsilon)
\]

where

\[
\lim_{\epsilon \to 0} \frac{E[y_i + \epsilon - D_i - \sum_{j=1}^{n} a_{ji}(y_j - D_j)^- | y_i \leq D_i < y_i + \epsilon]}{\epsilon} < 1.
\]

Consequently,

\[
\lim_{\epsilon \to 0} \frac{\int_{y_i}^{y_i + \epsilon} (y_i + \epsilon - x - \sum_{j=1}^{n} a_{ji}(y_j - D_j)^-) f_{D_i}(x) dx}{\epsilon} = 0.
\]
Thus, equation (8) can be re-written as

\[
\frac{\partial E}{\partial y_i} \left[ t_i((y_i - D_i)^+ - \sum_{j=1}^{n} a_{ji}(y_j - D_j)^-)\right] = \begin{cases} 
  t_iPr(y_i - D_i > 0), \\
  (y_i - D_i)^+ - \sum_{j=1}^{n} a_{ji}(y_j - D_j)^- > 0), \\
  t_iPr(y_i - R_i > 0). 
\end{cases} \tag{9}
\]

From (6), (7), and (9), we can obtain the following result:

\[
\frac{\partial \pi_i(y_1, y_2, \ldots, y_n)}{\partial y_i} = \begin{cases} 
  [s_i - w_i + p_i] \\
  -h_iPr(y_i > R_i) \\
  -(s_i + p_i - t_i)Pr(y_i > D_i) \\
  -t_iPr(y_i > R_i) \\
  = [s_i - w_i + p_i] \\
  -(s_i + p_i - t_i)Pr(y_i > D_i) \\
  -(h_i + t_i)Pr(y_i > R_i) \\
  = [s_i - w_i + p_i] \\
  -(s_i + p_i - t_i)F_{D_i}(y_i) \\
  -(h_i + t_i)F_{R_i}(y_i) \tag{10}
\end{cases}
\]

Taking the derivative of (10) with respect to \( y_i \), we obtain the second derivative of \( \pi_i(y_1, \ldots, y_n) \) with respect to \( y_i \) as
\[
\frac{\partial^2 \pi_i}{\partial^2 y_i} = (-s_i - p_i + t_i)f_D(y_i) - (t_i + h_i)f_R(y_i)
\]

By assumptions (A2) and (A3), we know that \(t_i > 0\) and \(s_i + p_i > t_i\), thus \(\frac{\partial^2 \pi_i}{\partial^2 y_i} < 0\).

Since the second derivative of \(\pi_i(y_1, \ldots, y_n)\) with respect to \(y_i\) is less than zero, given \(y_j (j = 1, \ldots, n, j \neq i)\), \(\pi_i(y_1, \ldots, y_n)\) is strictly concave in \(y_i\). □
Lemma 2 If each player’s payoff function $\pi_i(y_1, \ldots, y_n)$ is continuous in all decision variables $(y_1, \ldots, y_n)$ and concave in its own decision variable $y_i$, then the game has at least one Nash equilibrium which is determined by letting the first partial derivative of each player’s payoff function with respect to its own decision variable be zero.

Proof The definition of a noncooperative $n$-person convex game is given below by M. Dresher and S. Karlin (1953):

(i) The $i$th player’s strategy space is a compact convex set $X_i$ of a topological linear space $E_i$;

(ii) The $i$th player’s payoff $\pi_i(x_1, \ldots, x_i, \ldots, x_n)$ is concave with respect to his own strategy variable $x_i \in X_i$;

(iii) The sum of payoffs $\sum_{i=1}^{n} \pi_i(x_1, \ldots, x_i, \ldots, x_n)$ is continuous over the cartesian product space $X_1 \otimes X_2 \otimes \cdots \otimes X_n$;

(iv) For each fixed $x_i$, $\pi_i(x_1, \ldots, x_i-1, x_i, x_{i+1}, \ldots, x_n)$ is a continuous function of the $(n-1)$-tuple $[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n] \in X_1 \otimes \cdots \otimes X_{i-1} \otimes X_{i+1} \otimes \cdots \otimes X_n$ respectively.

From Theorem 3.1 of Nikaido and Isora (1955), we know that a convex game always has at least one Nash equilibrium solution. Since the noncooperative game in our decentralized control model satisfies the conditions of a $n$-person convex game, it has at least one Nash equilibrium.

According to Moulin (1986), to compute all Nash equilibria of a given game, $G$, we need to solve the system:

$$\Pi_i(y^*) = \max_{y_i} \Pi_i(y_i, y_{-i}^*), \quad (11)$$
where $i = 1, \ldots, n$, and $y_{-i}^*$ is the vector of what strategies of the players other than $i$. Equation (11) says that given all the other player’s strategies, the Nash equilibrium strategy for player $i$ is the one that maximizes his or her profit. If $\Pi_i$ is concave in $y_i$, and $\Pi_i$ is differentiable in $y_i$, the above system is equivalent to the first order conditions

$$\frac{\partial \Pi_i}{\partial y_i}(y^*) = 0,$$

where $i = 1, \ldots, n$. ■

From Lemma 1 and Lemma 2, we can conclude the following theorem.

**Theorem 3** There exists at least one Nash equilibrium for our decentralized control model which may be obtained by solving the system of equations that results from setting the first partial derivative of each player’s payoff function with respect to its own decision variable to be zero.

Basically, at least one Nash equilibrium can be found in the decentralized control system by solving the first order conditions. We would also like to know the uniqueness conditions of a Nash equilibrium in the decentralized control system.

Theorem 3 of Chapter 6 of Moulin (1984) states that if the payoff function is twice differentiable in all of its variables with

$$\frac{\partial^2 \pi_i}{\partial y_i^2} < 0, \text{ for all } i = 1, \ldots, n.$$
The system is globally stable with a unique Nash equilibrium if

\[ \frac{\partial^2 \pi_i}{\partial y_i^2} > \sum_{j,j \neq i} \left| \frac{\partial^2 \pi_i}{\partial y_i \partial y_j} \right|, \text{ for all } i, j = 1, \ldots, n, j \neq i. \tag{12} \]

In other words, there exists a unique Nash equilibrium for the decentralized control model if its payoff function satisfies the inequalities of (12).

According to Dai (2002), when there are only two retailers in the supply chain, we can find the necessary and sufficient conditions for the existence of a unique Nash equilibrium. When we have more than two retailers, there only exists some sufficient conditions for a unique Nash equilibrium.

**Theorem 4** If \( \sum_{j=1}^n a_{ji} < 1 \), for all \( i \) and \( j, j \neq i \), the Nash equilibrium of the decentralized control model is unique.

**Proof** From assumption (A3), we know that \( s_i + p_i - t_i > 0 \). Since \( \sum_{j=1}^n a_{ji} < 1 \) for all \( i \), and \( j, j \neq i \),

\[ f_{R_i}(y_i) > \sum_{j,j \neq i} (a_{ji} f_{R_i | D_j \geq y_j}(y_i) Pr(D_j \geq y_j)). \]

Consequently,

\[ |-(s_i + p_i - t_i) f_{D_i}(y_i) - (t_i + h_i) f_{R_i}(y_i)| > \sum_{j,j \neq i} |-(t_i + h_i) a_{ji} f_{R_i | D_j \geq y_j}(y_i) Pr(D_j \geq y_j)|, \]
where $i, j = 1, \ldots, n$, and $i \neq j$.

Equivalently, we have

$$\left| \frac{\partial^2 \pi_i}{\partial y_i^2} \right| > \sum_{j, j \neq i} \left| \frac{\partial^2 \pi_i}{\partial y_i y_j} \right|,$$

where $i, j = 1, \ldots, n, j \neq i$.

Therefore, $\sum_{j=1}^{n} a_{ji} < 1$, for all $i$ and $j, j \neq i$, is a sufficient condition to guarantee the uniqueness of a Nash equilibrium in the decentralized control system.
5 Centralized Control System

In the centralized control model, we assume that all retailers in the supply chain are willing to cooperate and make the allocation decisions together with the supplier because this approach will maximize the expected total profit of the whole supply chain. Since all retailers cooperate with each other in the supply chain, when customers encounter a stockout at one retailer and decide to look for the product at other retailers, the initial retailer may tell the customers at which other retailers they may find the product. We assume that a customer may visit up to two retailers in search of the product before leaving the system. The stockout penalty is incurred only when a customer leaves the system without purchasing the product. Because the supplier and retailers can be viewed as a whole system, we assume that the unit penalty cost for losing a customer to be the average of the unit penalty costs for individual retailers in the decentralized model.

We assume the supplier has infinite capacity, and $y_i$ is the order (and allocation) for retailer $i$. The payoff function for the whole supply chain consists of:

(i) Production cost: $cK = c(y_1 + \ldots + y_n) = c \sum_{i=1}^{n} y_i$;

(ii) Holding cost: $\sum_{i=1}^{n} h_i(y_i - R_i)^+$;

(iii) Penalty cost:

$$\sum_{i=1}^{n} \bar{p}[(y_i - D_i)^- (1 - \sum_{j=1}^{n} a_{ij}) + (\sum_{j=1}^{n} a_{ji}(y_j - D_j)^{-} - (y_i - D_i)^+) +]$$

$$= \sum_{i=1}^{n} \bar{p}[(y_i - D_i)^- (1 - \sum_{j=1}^{n} a_{ij}) + \sum_{j=1}^{n} a_{ji}(y_j - D_j)^{-} - (y_i - D_i)^+] + (\sum_{j=1}^{n} a_{ji}(y_j - D_j)^{-} - (y_i - D_i)^+) -]$$

(iv) Selling revenue:
\[
\sum_{i=1}^{n} \left[ s_i \min \{ y_i, D_i \} + t_i \min \{ (y_i - D_i)^+, \sum_{j=1}^{n} a_{ji}(y_j - D_j)^- \} \right] \\
= \sum_{i=1}^{n} [s_i y_i - s_i (y_i - D_i)^+ + t_i (y_i - D_i)^+ - t_i [(y_i - D_i)^+ - \sum_{j=1}^{n} a_{ji}(y_j - D_j)^-]^+] \\
\]

Thus the expected payoff function of the whole supply chain is

\[
\pi(y_1, y_2, \ldots, y_n) = E\left[ \sum_{i=1}^{n} (s_i y_i - s_i (y_i - D_i)^+ + t_i (y_i - D_i)^+) \\
- t_i [(y_i - D_i)^+ - \sum_{j=1}^{n} a_{ji}(y_j - D_j)^-]^+] \\
- c \sum_{i=1}^{n} y_i \\
- \sum_{i=1}^{n} h_i (y_i - D_i) - \sum_{j=1}^{n} a_{ji}(y_j - D_j)^-)^+ \\
- \hat{p} \sum_{i=1}^{n} (y_i - D_i)^-(1 - \sum_{j=1}^{n} a_{ij})) \\
- \hat{p} \sum_{i=1}^{n} (\sum_{j=1}^{n} a_{ji}(y_j - D_j)^(- (y_i - D_i)^+) \\
- \hat{p} \sum_{i=1}^{n} (\sum_{j=1}^{n} a_{ji}(y_j - D_j)^- - (y_i - D_i)^+)^-) \\
= E\left[ \sum_{i=1}^{n} (s_i - c + \hat{p}) y_i \\
- \sum_{i=1}^{n} (s_i - t_i)(y_i - D_i)^+ \\
- \sum_{i=1}^{n} (t_i + \hat{p})(y_i - D_i)^+ - \sum_{j=1}^{n} a_{ji}(y_j - D_j)^-)^+ \\
- \sum_{i=1}^{n} h_i (y_i - D_i) - \sum_{j=1}^{n} a_{ji}(y_j - D_j)^-)^+] \right]. \\
\text{(13)}
\]

According to Netessine (2001), the objective function of the so-called “substitutable
product problem is submodular. In our market search problem, since the product sold by different retailers is the same, they are substitutable. In Lemma 5, we show that the payoff function is submodular.

**Lemma 5** In the centralized control model, the payoff function \( \pi(y_1, y_2, \ldots, y_n) \) is submodular in \( y_1, y_2, \ldots, y_n \).

**Proof** To show the payoff function \( \pi(y_1, y_2, \ldots, y_n) \) is submodular in \( y_1, y_2, \ldots, y_n \), we refer to Sundaram (1999). It is sufficient to demonstrate that the second order cross-partial derivatives are non-positive.

The first derivative of \( \pi(y_1, y_2, \ldots, y_n) \) with respect to \( y_i \) is

\[
\frac{\partial \pi(y_1, y_2, \ldots, y_n)}{\partial y_i} = (s_i - c + \bar{p}) - (s_i - t_i) P_r(y_i > D_i) - (t_i + h_i + \bar{p}) P_r(y_i > R_i) - \sum_{j=1, j \neq i}^n (t_j + h_j + \bar{p}) a_{ij} P_r(D_i > y_i, R_j < y_j).
\]

(14)

For (14) and (15), it is easy to see the equivalence of the first two terms in (14) and their counterparts in (15). The explanation for the last two terms in (14) is
given as follows.

First, we can separate the third term of (14) into two parts.

\[
\frac{\partial E}{\partial y_i} \left[ \sum_{i=1}^{n} (t_i + \bar{p})((y_i - D_i)^+ - \sum_{j=1}^{n} a_{ji}(y_j - D_j)^-)^+ \right]
\]

\[
= \lim_{\epsilon \to 0} \left[ \left( t_i + \bar{p} \right)((y_i + \epsilon - D_i)^+ - \sum_{j=1}^{n} a_{ji}(y_j - D_j)^-) + \right.
\]

\[
\left. - \left( t_i + \bar{p} \right)((y_i - D_i)^+ - \sum_{j=1}^{n} a_{ji}(y_j - D_j)^-) \right] / \epsilon
\]

\[
+ \lim_{\epsilon \to 0} \sum_{j=1, j \neq i}^{n} \left[ \left( t_j + \bar{p} \right)((y_j - D_j)^+ - \sum_{i=1}^{n} a_{ij}(y_i + \epsilon - D_i)^+) + \right.
\]

\[
\left. - \left( t_j + \bar{p} \right)((y_j - D_j)^+ - \sum_{i=1}^{n} a_{ij}(y_i - D_i)^-) \right] / \epsilon.
\] (16)

With \((t_i + \bar{p})\) being factored out, the numerator of the first term in (16) can be expressed as:

\[
((y_i + \epsilon - D_i)^+ - \sum_{j=1}^{n} a_{ji}(y_j - D_j)^-) + (t_i + \bar{p})((y_i - D_i)^+ - \sum_{j=1}^{n} a_{ji}(y_j - D_j)^-)
\]

\[
= \begin{cases} 
  \epsilon & \text{if } y_i - D_i > 0, \\
  y_i + \epsilon - D_i & \text{if } -\epsilon < y_i - D_i \leq 0, \\
  y_i + \epsilon - D_i - \sum_{j=1}^{n} a_{ji}(y_j - D_j)^- & \text{if } -\epsilon < y_i - D_i \leq 0, \\
  0 & \text{if } y_i - D_i \leq -\epsilon.
\end{cases}
\]
After taking the derivative for the four cases, the first term of (16) can be re-written as:

\[
\begin{align*}
\lim_{\epsilon \to 0} & \left[ (t_i + \bar{p})((y_i + \epsilon - D_i)^+ - \sum_{j=1}^{n} a_{ji}(y_j - D_j)^-) + \\
- (t_i + \bar{p})((y_i - D_i)^+ - \sum_{j=1}^{n} a_{ji}(y_j - D_j)^-) \right] / \epsilon \\
= & (t_i + \bar{p}) \lim_{\epsilon \to 0} \left[ \epsilon \Pr(y_i - D_i > 0, (y_i - D_i)^+ - \sum_{j=1}^{n} a_{ji}(y_j - D_j)^- > 0) \\
+ \int_{y_i}^{y_i + \epsilon} (y_i + \epsilon - x) f_{D_i}(x) dx \\
+ \int_{y_i}^{y_i + \epsilon} (y_i + \epsilon - x - \sum_{j=1}^{n} a_{ji}(y_j - D_j)^-) f_{D_i}(x) dx \\
+ 0 \Pr(y_i - D_i \leq -\epsilon) \right] / \epsilon, \\
= & (t_i + \bar{p}) \Pr(y_i - D_i > 0, (y_i - D_i)^+ - \sum_{j=1}^{n} a_{ji}(y_j - D_j)^- > 0), \\
= & (t_i + \bar{p}) \Pr(y_i > R_i).
\end{align*}
\]

With \((t_j + \bar{p})\) factored out, the numerator of the second term of (16) can be expressed as

\[
((y_j - D_j)^+ - \sum_{i=1}^{n} a_{ij}(y_i + \epsilon - D_i)^-) + (t_j + \bar{p})((y_j - D_j)^+ - \sum_{i=1}^{n} a_{ij}(y_i - D_i)^-)^+
\]
\[
\begin{align*}
&= \begin{cases} 
-a_{ij} \epsilon & \text{if } y_i - D_i < -\epsilon, \\
-a_{ij}(y_i + \epsilon - D_i) & \text{if } -\epsilon \leq y_i - D_i < 0, \\
0 & \text{if } 0 \leq y_i - D_i,
\end{cases}
\text{ and } (y_j - D_j)^+ - \sum_{i=1}^{n} a_{ij}(y_i - D_i)^- > 0; \\
&\text{ and } (y_j - D_j)^+ - \sum_{i=1}^{n} a_{ij}(y_i - D_i)^- > 0; \\
&\text{ and } 0 \leq (y_j - D_j)^+ - \sum_{i=1}^{n} a_{ij}(y_i - D_i)^-.
\end{align*}
\]

Then we take the derivative for these three cases,

\[
\begin{align*}
\lim_{\epsilon \to 0} \sum_{j=1, j \neq i}^{n} & \left[ (t_j + \bar{p})(y_j - D_j)^+ - \sum_{i=1}^{n} a_{ij}(y_i + \epsilon - D_i)^+ \\
&- (t_j + \bar{p})(y_j - D_j)^+ - \sum_{i=1}^{n} a_{ij}(y_i - D_i)^+ \right] / \epsilon \\
&= (t_j + \bar{p}) \lim_{\epsilon \to 0} \sum_{j=1, j \neq i}^{n} \left[ -a_{ij} \epsilon \Pr(y_i - D_i < -\epsilon, (y_j - D_j)^+ - \sum_{i=1}^{n} a_{ij}(y_i - D_i)^- > 0) \\
&\quad - \int_{y_i}^{y_i+\epsilon} a_{ij}(y_i + \epsilon - x) f_{D_i}(x) dx \\
&\quad + 0 Pr(0 \leq y_i - D_i, 0 \leq (y_j - D_j)^+ - \sum_{i=1}^{n} a_{ij}(y_i - D_i)^-) \right] / \epsilon \\
&= \sum_{j=1, j \neq i}^{n} (t_j + \bar{p}) \Pr(y_i - D_i < 0, (y_j - D_j)^+ - \sum_{i=1}^{n} a_{ij}(y_i - D_i)^- > 0) \\
&= \sum_{j=1, j \neq i}^{n} (t_j + \bar{p}) \Pr(D_i > y_i, R_j < y_j). \tag{18}
\end{align*}
\]
Similarly, for the fourth term of (14), we can show that

\[
\frac{\partial E}{\partial y_i} \sum_{i=1}^{n} h_i (y_i - D_i - \sum_{j=1}^{n} a_{ji} (y_j - D_j)^-) + \right]
= h_i \text{Pr}(y_i > R_i)
+ \sum_{j=1, j \neq i}^{n} h_j a_{ij} \text{Pr}(D_i > y_i, R_j < y_j)
\]

(19)

From equations (16) to (19), we obtain the first derivative of the payoff function with respect to \(y_i\) in centralized control system as (15).

Taking the derivative of (15), we obtain the second derivative of \(\pi(y_1, \ldots, y_n)\) with respect to \(y_i\) as

\[
\frac{\partial^2 \pi}{\partial^2 y_i} = -(s_i - t_i) f_{D_i}(y_i)
- (t_i + h_i + \bar{p}) f_{R_i}(y_i)
- \sum_{j=1, j \neq i}^{n} (t_j + h_j + \bar{p}) a_{ij}^2 f_{R_j | D_i > y_i}(y_j) \text{Pr}(D_i > y_i).
\]

(20)

The first two terms of (20) are easily seen to follow from (15). The third term of (20) requires some explanation.

\[
\frac{\partial \text{Pr}(D_i > y_i, R_j < y_j)}{\partial y_i} = \lim_{\epsilon \to 0} \left[ \text{Pr}(D_i > y_i + \epsilon, D_j + a_{ij}(y_i + \epsilon - D_i)^- < y_j) - \text{Pr}(D_i > y_i, D_j + a_{ij}(y_i - D_i)^- < y_j) \right] / \epsilon.
\]
Then we can split up the numerator into three cases:

$$Pr(D_i > y_i + \epsilon, D_j + a_{ij}(y_i + \epsilon - D_i) < y_j) - Pr(D_i > y_i, D_j + a_{ij}(y_i - D_i) < y_j)$$

$$= \begin{cases} 
Pr(D_i > y_i + \epsilon, D_j - a_{ij}(y_i + \epsilon - D_i) < y_j) \\
-Pr(D_i > y_i, D_j - a_{ij}(y_i - D_i) < y_j) & \text{if } D_i - y_i > \epsilon, \\
-Pr(D_i > y_i, D_j - a_{ij}(y_i - D_i) < y_j) & \text{if } 0 < D_i - y_i \leq \epsilon, \\
0 & \text{if } D_i - y_i \leq 0. 
\end{cases}$$

Therefore,

$$\frac{\partial Pr(D_i > y_i, R_j < y_j)}{\partial y_i} = \lim_{\epsilon \to 0} \frac{(Pr(D_i > y_i + \epsilon, D_j - a_{ij}(y_i + \epsilon - D_i) < y_j) - Pr(D_i > y_i, D_j - a_{ij}(y_i - D_i) < y_j) + Pr(D_i > y_i) - Pr(D_i > y_i, y_j))}{\epsilon}.$$

In a similar manner, we can show that the second order cross-partial derivative of $$\pi(y_1, \ldots, y_n)$$ with respect to $$y_i, y_j$$ is

33
\[
\frac{\partial^2 \pi}{\partial y_i \partial y_j} = -(t_i + h_i + \bar{p})a_{ji}f_{R_i|D_j>y_j}(y_i)P_r(D_j > y_j) \\
- (t_j + h_j + \bar{p})a_{ij}f_{R_j|D_i>y_i}(y_j)P_r(D_i > y_i) \\
- \sum_{r=1, r\neq i,j}^n (t_r + h_r + \bar{p})a_{ir}a_{jr}f_{R_r|D_i>y_i, D_j>y_j}(y_r)P_r(D_i > y_i, D_j > y_j).
\]

Following assumption (A2), since the unit selling price for distant demand is always greater than zero, the second order cross-partial derivative of \(\pi(y_1, \ldots, y_n)\) with respect to \(y_i, y_j\) is less than zero. Thus \(\pi(y_1, y_2, \ldots, y_n)\) is submodular in \(y_1, y_2, \ldots, y_n\).

Because the payoff function in the centralized control model is only submodular and not concave, a solution of the first order conditions does not guarantee global optimum in this case.
6 Channel Coordination

In reality, retailers usually do not cooperate, share their information, and manage their inventory allocations in concert with the supplier for the advantage of the whole system unless they are owned by the same company. However, through channel coordination, the supply chain operating under decentralized control still can reach the same solution and overall profit as under centralized control.

To accomplish channel coordination, there are three steps commonly used in supply chain inventory management research. First, apply game theory to the system under decentralized control and find the Nash equilibrium satisfying the first order conditions. Next, determine the optimal inventory allocation under centralized control. Third, if the solution under centralized control is better than the solution under decentralized control, alter the players’ payoffs.

In the third step, in order to alter the players’ payoffs, we have to find a parameter that only exists in the payoff function of the decentralized control model. Thus we maintain the optimal solution of the centralized control model, and adjust the payoff of the decentralized control model to yield the same solution. Because the “wholesale price” affects the Nash equilibrium in the decentralized control model, but has no effect on solutions to the centralized control model, we may utilize the wholesale price $w_i$ to provide the coordination between the two solutions.

For the decentralized control model, we know that the Nash equilibrium can be easily found from the first order conditions. Similarly, we can find a local optimal solution for the centralized control model. If we can find a better local optimal solution to the centralized control model, we can modify the wholesale prices so as to achieve
the same solution under decentralized control. The coordination is achieved by solving

$$\frac{\partial \pi_i}{\partial y_i} = \frac{\partial \pi}{\partial y_i}, \text{ where } i = 1, \ldots, n.$$ 

Then the wholesale price can be determined by

$$w_i = c + p_i - \bar{p} - p_i Pr(y_i > D_i) + \bar{p} Pr(y_i > R_i) + \sum_{j=1, j \neq i}^{n} (t_j + h_j + \bar{p}) a_{ij} Pr(D_i > y_i, R_j < y_j), \text{ where } i = 1, \ldots, n. \quad (21)$$

to achieve channel coordination between the system under centralized control and under decentralized control. But the wholesale prices have to satisfy the assumptions (A1) $s_i > w_i > 0$ and (A2) $t_i > w_i > 0$, for $i = 1, \ldots, n$.

According to the paper, “Vendor Managed Inventories and Supply Chain Coordination : The Case with One Supplier and Competing Retailers” by Bernstein, Chen, and Federgruen (2002), a strong perfect coordination mechanism can be established when the optimal solution of the whole supply chain arises as the unique Nash equilibrium.
Since the uniqueness conditions of a Nash equilibrium represent the globally stable property of the solutions in the decentralized control model, if the market search parameters satisfy the condition of $\sum_{j=1}^{n} a_{ji} < 1$ for all $i$ and $j$, where $j \neq i$, the previous coordination mechanism is stronger than without a unique Nash equilibrium under decentralized control.
7 Numerical Experiments

The customer market search behavior and channel coordination are two important factors related to retailers’ allocation policy. In the numerical experiments to be reported in this chapter, we study the effect of market search probability on the allocations, profit of the supplier and retailers, and the adjusted wholesale price after channel coordination in the one-supplier, multi-retailer supply chain. We also study the effect of channel coordination and evaluate the profit improvement of the whole supply chain resulting from coordination.

To simplify the input parameters, we assume that all retailers are identical, so customers have the same market search probabilities at each retailer. To see the effect of market search probability, we increase the market search probability from 0 to the highest possible probability (depending on the number of retailers) in increments of 0.1. We compare the allocations and profits from three different models. They are newsvendor model, decentralized control model, and centralized control model, (decentralized control model after channel coordination).

In the newsvendor model, the retailers do not consider customers’ distant demand or channel coordination. They simply apply the newsvendor formula to decide the inventory allocations $y_i, i = 1, \ldots, n$. To obtain this formula, we replace the retailer $i$’s effective customer demand $R_i$ with its local demand $D_i$ in equation (4), so that $y_i$ is determined by solving:

$$F_i(y_i) = \frac{s_i - w_i + p_i}{s_i + h_i + p_i}.$$
In the decentralized control model, we utilize the *quasi-Newton (Broyden) method* to find the Nash equilibrium solution satisfying the first order conditions:

\[
\frac{\partial \pi_i(y_1, y_2, \ldots, y_n)}{\partial y_i} = (s_i - w_i + p_i) - (s_i + p_i - t_i)Pr(y_i > D_i) - (h_i + t_i)Pr(y_i > R_i) = 0.
\]

As described in Rao (1996), the basic idea behind the *quasi-Newton method* is to approximate the Hessian matrix by another matrix. Thus we need only the first partial derivatives of the objective function to find the optimal solutions.

For a given \( y_i \), we compute \( Pr(y_i > D_i) \) by generating a large sample of values of \( D_i, i = 1, \ldots, n \), and determining the fraction which are less than \( y_i \). To compute \( Pr(y_i > R_i) \), the sample of \( R_i \) values is generated from the samples of \( D_i \) by applying equation (1). We note that this approach is applicable to situations in which the cumulative distribution functions are only known empirically.

In the centralized control model, we also use the quasi-Newton (Broyden) method to find an optimal solution of \( y_i, i = 1, \ldots, n \) satisfying the following first order condition:
\[
\frac{\partial \pi(y_1, y_2, \ldots, y_n)}{\partial y_i} = (s_i - c + \bar{p}) - (s_i - t_i) Pr(y_i > D_i) - (t_i + h_i + \bar{p}) Pr(y_i > R_i) - \sum_{j=1, j\neq i}^{n} (t_j + h_j + \bar{p}) a_{ij} Pr(D_i > y_i, R_j < y_j) = 0. \tag{22}
\]

Because the payoff function is submodular but not concave in the centralized control system, the solution may only be a local optimal solution. However, we may repeat this process several times to find better local optimal solutions. Once we find allocations that give a higher payoff for the whole supply chain than the solution given by the decentralized model, we can adjust parameters in the decentralized control model to achieve the same solution as in the centralized control model (through channel coordination).

To achieve a decided channel coordination, we modify the wholesale price \( w_i, i = 1, \ldots, n \), in the decentralized control model by setting \( y_i \) to the value obtained in (22), and solving:

\[
\frac{\partial \pi_i(y_1, y_2, \ldots, y_n)}{\partial y_i} = \frac{\partial \pi(y_1, y_2, \ldots, y_n)}{\partial y_i}, \text{ for } w_i, i = 1, \ldots, n.
\]

In our experiments, the model parameters are: the selling price for local demand \( s_i = 5 \), the selling price for distant demand \( t_i = 4 \), the holding cost \( h_i = 1 \), the stockout penalty cost \( p_i = 4 \), the average stockout penalty cost \( \bar{p} = 4 \), and the original
wholesale price $w = 2$.

Figures 3 to 8 show the results obtained for a supply chain with two retailers while Figures 9 to 14 show the corresponding results with three retailers. For each function of $Pr(y_i > D_i)$ we sample 10,000 values of $D_i$. Moreover, to reduce sample bias we replicate the calculation of each plotted value ten times and display the average value.

Figure 3 verifies that when the market search probability equals zero, the allocation before coordination is simply that from the newsvendor model. When the market search probability equals one, the allocation before coordination (i.e., decentralized control) is the same as after coordination (i.e., decentralized control). This is because when the market search probability is high, customers have a higher tendency to look for the product when they encounter a stockout. Coordination makes no difference
because the total penalty cost under decentralized control and centralized control is about the same.

As the market search probability increases, the allocation increases before coordination but decreases after coordination. The wholesale price is fixed before coordination, so higher market search probability results in higher stock levels to capture the circulating customers. The wholesale price after coordination increases with higher market search probability to finally match the fixed value before coordination (see Figure 8). The increasing wholesale price offsets the increase in circulating customers and drives the allocation requests downward. However, from equation (21), we observe that the adjusted wholesale price need not increase as the market search probability increases. Therefore, in general we will not know the relation of wholesale price to the market search probability until we simulate with all the given parameters.

In Figures 4, 5, and 6, we plot retailer, supplier, and system profits. The profit of retailers in Figure 4 is the profit of a single retailer obtained from equation (2). The profit of supplier in Figure 5 is the net profit of the supplier when production cost is subtracted from the revenue, i.e., \( \sum_{i=1}^{n} (w_i - c_i) \times y_i \). The system profit in Figure 6 is the total profit of the supplier and all retailers under decentralized control.

As the market search probability increases, the system profit increases for each of the three models. This is because more customers would stay in the supply chain to search for the same product, so that system may satisfy more customer demand and obtain more revenue. From the system viewpoint, the improved profit as a result of channel coordination is bigger when the market search probability is lower. From the retailers’ viewpoint, while coordination improves the profit all the time, the improvement is the largest when the market search probability is low and deteriorates as the
Figure 4: Profit of retailers versus market search probability with 2 retailers

Figure 5: Profit of supplier versus market search probability with 2 retailers
Figure 6: System profit versus market search probability with 2 retailers

Figure 7: Profit improvement after coordination versus market search probability with 2 retailers
Figure 8: Wholesale price versus market search probability with 2 retailers

Figure 9: Allocation versus market search probability with 3 retailers
Figure 10: Profit of retailers versus market search probability with 3 retailers

Figure 11: Profit of supplier versus market search probability with 3 retailers
Figure 12: System profit versus market search probability with 3 retailers

Figure 13: Profit improvement after coordination versus market search probability with 3 retailers
Figure 14: Wholesale price versus market search probability with 3 retailers

market search probability increases. In contrast, coordination does not benefit the supplier, and has the most negative effect when the market search probability is low. Therefore, in reality, it may be hard for the supplier to accept channel coordination. Since the coordination indeed improves the system profit, supplier and retailers may negotiate with each other to design a mutually beneficial contract.

In Figure 7, we compare the system profit improvement from coordination when the wholesale price before coordination equals to 2 and when it equals to 3. The improvement is calculated as the difference in the system profit after and before coordination divided by the system profit before coordination. Figure 7 shows that the best improvement appears when $a = 0$ and $w = 3$. Therefore, the greatest benefit from channel coordination occurs with high fixed wholesale price before coordination and low market search probability.
In Figures 9 to 14, we give the corresponding plots for the case in which the number of retailers in the supply chain equals three. Basically, the patterns in these figures are similar to those for 2 retailers. The only pattern difference is the crossing of the “before” and “after” plots at \( a = 0.4 \) in Figures 9 to 12, and Figure 14. In this case, the wholesale price to achieve coordination is higher than the original wholesale price once the market search probability reaches 0.4 (see Figure 14). When the wholesale price is higher after coordination, the allocation becomes lower. When the market search probability is close to 0.5, system profit does not improve through channel coordination which benefits the supplier and not the retailers (because of the higher wholesale price).
8 Conclusion

We generalize Dai’s (2002) 1-supplier, 2-retailer allocation problem with market search into a multi-retailer allocation problem by adopting Rudi’s expression of derivatives. We consider the system under decentralized control, centralized control, and with channel coordination. When trying to extend the number of retailers, it is very difficult to apply Leibnitz formula because of the complexity of the high dimensions. Thus we use Rudi’s expression (2001).

For the decentralized control model, we show that there exists at least one Nash equilibrium and obtain some sufficient conditions for uniqueness of the Nash equilibrium. For centralized control model, we prove the payoff function for the multi-retailer allocation problem is submodular rather than concave as in the two-retailer problem. Hence we can only obtain local optima for the centralized control model. To accomplish channel coordination, we change the wholesale price to adjust the payoff function in the decentralized control model. If the market search parameters satisfy the sufficient conditions for uniqueness of the Nash equilibrium in the decentralized control model, the coordination can be stronger than that in the case without a unique Nash equilibrium.

In our numerical experiments, we study the effects of the market search probability and channel coordination. We plot the effect of market search probability over the wholesale prices, allocations, and profits. If the wholesale price after coordination increases with the market search probability (as in our numerical experiment), the allocations become lower after coordination. However, the relation of the wholesale price to the market search factor depends on costs, prices and customer demand. Thus it needs to be obtained from the numerical simulations.
Figure 15: Profit percentage after coordination of supplier and retailers versus market search probability with 2 retailers

While the system profit increases as the market search probability rises, coordination benefits retailers more than the supplier unless the wholesale price after coordination exceeds that before coordination. In reality, this coordination could be achieved by some contract between the supplier and retailers. Since we can easily estimate the profit percentage after coordination for supplier and retailers (Figure 15), the numerical analysis may help us to design a coordination contract between the supplier and retailers, to decide the allocations according to the effects of the “market search” factor, and to estimate the ideal wholesale price for the supply chain.
References


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