ABSTRACT

BARHAM, OLIVER M. A Mathematical Model for the Mechanics of a Mosquito Bite with Applications to Microneedle Design (Under the direction of Associate Professor Dr. M. K. Ramasubramanian).

How does the female mosquito push it’s fascicle, a flexible tube the size of a single human hair, through human skin so that she may suck blood? This process is not very well understood, and this paper attempts to take a mathematical approach, approximating the fascicle as a long Euler column to explain its mechanics. It is believed that the mosquito is using a combination of conservative and non-conservative forces during its attempt to pierce the skin, which leads to some forms of instability before it succeeds. The mosquitoes fascicle is also covered in a sheath called the labium, whose purpose is not know to any certainty, and is seen in this paper as a support mechanism providing the fascicle with lateral support while it attempts to pierce the skin. The hypotheses presented are based upon high-speed video of the mosquito during feeding and Scanning Electron Microscopy pictures of its anatomy, so they are believed to be reasonably accurate.

A new equation is derived based on classical column theory, with both Beck and Euler forces present, and a foundation stiffness factor is added to represent the effects of the labium. This equation is verified to be accurate; solved numerically for different values of foundation stiffness to determine its relative importance to the critical load of failure, compared to increases in non-conservative forces.

The results indicate that by increasing foundation stiffness it is theoretically possible to raise the critical buckling load of such a structure significantly compared to the buckling load when only conservative and nonconservative forces are present. Microneedles taking advantage of such added support could possibly be much smaller in size than microneedles without any lateral support.
A Mathematical Model for the Mechanics of a Mosquito Bite
with Applications to Microneedle Design

by

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Dedication

This paper is dedicated to my parents and all their hard work over the years that helped me to get to this point.
Biography

Oliver was born in Belgrade, Yugoslavia (now Serbia) on November 17, 1981, to James Barham and Celica Milovanovic. When Oliver was a small boy the family moved to Philadelphia, Pa, and shortly thereafter to the small town of Lancaster, where Oliver grew up. He attended the University of Pittsburgh where he obtained his undergraduate degree in Mechanical Engineering, and found that BioMechanical Engineering also appealed to him through projects involving modeling of the Human ACL (Anterior Cruciate Ligament) and the Human Eye. He also Co-Oped with Curtiss-Wright Electro Mechanical Division, where he gained valuable experience in an engineering and manufacturing environment and developed interest in motors, generators and other electro-mechanical systems. He then pursued a masters degree in mechanical engineering at North Carolina State University in Raleigh, NC. His future plans are to join the real (working) world after 7 years of being a student. He is apprehensive...
Acknowledgements

I’d like to thank all the members of my board, each of whom had me as a student and helped further my knowledge of engineering and mathematics. Thank you all for making my time here at State enjoyable as well as fulfilling. I am also in debt to all my other professors and great classmates whom I have had the pleasure to get to know along the way. In addition, I would like to thank Dr. Ram for allowing me to work on this project and guiding me along with patience and enthusiasm.
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Chapter 1

Introduction

Research into developing painless needles is ongoing and there has recently been a continuous flow of research publications [7][17][18] in this area. Many of these needles have dimensions on the micro (µ) scale, much smaller than traditional hypodermic needles; these can be referred to as microneedles. Among the most important reasons for this research is the possibility that, once made correctly, the needles could cause little or no pain to the subject. This would benefit diabetics, who extract blood samples multiple times daily, or anyone who cannot tolerate traditional hypodermic needles. Unfortunately, there are a number of inherent design challenges when considering a device of this size. The approximate dimensions of length and diameter are on the scale of several mm and tens of µmeters, respectively, so of foremost consideration is the structure and design of the needle. It must be small enough to not be felt by the patient, but at the same time strong enough to withstand the forces applied to it when it punctures the skin. The forces themselves then have to be taken into account. Should a hollow needle be pushed in a direct perpendicular fashion through the skin, or at some angle?

In order to find answers to some of these questions the current research has taken a biomimetic approach and turned to Mother Nature. Nature’s very own blood-reaper, the female mosquito, was observed, with the intention of understanding how she is able to penetrate human skin without buckling of her fascicle, and to offer our observations and hypothesis on the mechanics of penetration.
Chapter 2

Literature

Some of the relevant literature in the area of microneedle design and fabrication has been gathered and is summarized here. In addition, background is also given on column buckling theory and mosquito anatomy.

2.1 Microneedles

The conception of microneedles happened fairly recently, with a US Patent in the mid 1970s by Gerstel and Place [6], but there has been much research done on the topic of, and topics relating to, microneedles within the last decade.

Henry et al fabricated solid conical microneedle arrays with needle lengths of 150 µm, 80 µm diameters at the base, and tip radii of 1 µm. These needles have been shown to penetrate painlessly into the stratum corneum (the external tough layer of the skin) and access the epidermis layer [7].

A serrated microneedle (1 mm long, roughly 40 micrometers in diameter, with a wall thickness of 1.6 µm) has been fabricated and tested by Kazunari et al [9]. They conclude that a 1.5 gram force (gf) is sufficient to penetrate the skin while the silicon microneedle they fabricated had a strength of 6 gf in compression. The needle, unfortunately, underwent brittle fracture upon insertion. Brittle fracture can be a problem when such a needle is used in live tissue, and small fragments of silicon could be carried away in the blood stream,
possibly lodging themselves in vital organs. A ductile material is preferable for this reason. These researchers looked at a mosquito fascicle tip under a scanning electron microscope (SEM) and concluded that a barbed edge would mimic the mosquito bite phenomenon. No attention to the mechanics of the biting process was given, however.

Giorgio et al describe a hollow microneedle in their work with a height of 450 $\mu$m, a base length of 300 $\mu$m, a base width of 100 $\mu$m, hole diameter of 50 $\mu$m, and an orifice aperture range between 300 and 375 $\mu$m above the base [5]. They claim that the hole diameter of 50 $\mu$m is sufficient to draw blood without external pumps or suction devices. However, no experimental results were presented in the paper. Furthermore, the name e-mosquito was simply used to indicate that it is a microneedle penetration concept; penetration mechanics were not studied.

Acupuncture needles are long and slender with a very sharp tip. The dimensions have been reported as 4 cm in length and 200 $\mu$m diameter, with sharp tips having a radius of curvature of about 2 $\mu$m. These needles were used to deliver drugs by coating the needle surface with the agent and penetrating the skin. Once inside the skin, the agents were transported by fluids beneath the skin [10].

Oki et al describe a hardened stainless steel microneedle used in their work. The external diameter was 100 $\mu$m, internal diameter was 50 $\mu$m; length more than 600 $\mu$m. The tip was chemi-mechanically polished to a bevel of 10 degrees. This needle was claimed to be painless by the researchers [18].

Martanto et al inserted hollow glass microneedles with and without beveled ends and studied the effect of microneedle design on the infusion of fluid into cadaver skin [13]. Needles with effective tip opening radii of 22-48 $\mu$m and a tip bevel angle of 35-38 degrees were inserted to a depth of 1080 $\mu$m. Once infusion was initiated, the needle was retracted to 180 $\mu$m in steps and the flow was measured. They concluded that the dermal tissue compressed at the needle tip, causing resistance to flow. Having a sharp needle with a side opening almost increased the flow rate by three fold compared to that of a blunt tipped needle. A beveled needle delivers fluid to less dense regions adjacent the dense region directly below the needle tip, thus retracting the needle partially provided the least resistance and increased flow almost 12x.

Despite the advances in microneedle fabrication and experiments with infusion into the skin, limited work has been done to understand the mechanics of penetration into the skin.
Yang and Zohn claim that a mosquito uses vibratory cutting at a frequency of 200-400 Hz, but no further information is provided [11]. They state that current designs of needles do not tolerate forces associated with insertion and intact removal and are typically either too fragile or too ductile. Vibratory techniques are proposed as a method for inserting the needle into the skin. A silicon microneedle (100 µm diameter and 6 mm long) was used in their experiments. They show that there is a 70% decrease in insertion force when the needle is vibrated compared to quasi-static insertion. The vibration applied was parallel to the skin surface with an amplitude of 0.6 mm, vibrating in the kHz range. The static peak insertion force was found to be 0.3 N; upon inserting the needle, the force drops to 0.05 N. Further increase in the displacement causes the force to rise rapidly in a linear fashion.

Another study by Newton drives a microneedle array into the skin and reports that the use of ultrasonic vibration of the needle patch reduces the insertion force [1].

A mechanical insertion study was done by Davis et al, who measured the insertion force versus displacement for 720 µm length, 30-80 µm tip radius microneedles and concluded that the insertion force ranged from 0.08 N to 3.04 N for the range of tip radii considered [19]. The needle compressive strength was measured to be about 4N. It was concluded that the needle will penetrate the skin without failing, however, the geometry of the needle was wider at the root tapering to a smaller diameter at the tip. The needle was also flat cut without a bevel. These idealizations prevent the needle from buckling and the only mode of failure is compressive crushing by fracture. In other words, the structure is over designed.

All the preceding mechanics investigations have been assessing the force required to insert the needle and the force required for fracture. Due to the rigid nature of silicon needles, brittle fracture was the only mode of failure identified. Ductile needles have not been addressed in the preceding articles, possibly because they were deemed as not strong enough to penetrate the skin without bucking.

Some have also applied their own fabricated microneedles as a component in a prototype medical device [16][17], but introduced the needle directly into the skin without any non-conservative forces. One of the previous papers also considers mosquito mechanics briefly, stating that the female mosquito moves its head like a hammer at 6 to 7 hertz [16], but does not explore the mechanics of the mosquito in great depth.
2.2 Column Buckling Theory

This section summarizes relevant theories on buckling, elastic force, Beck and Beck/Euler combined loading in order to delineate the roles of each in the penetration of the fascicle into the skin. These results could then be applied to microneedle design.

Mathematically, the fascicle can be approximated as a slender column [24], which, generally speaking, is a long, straight, symmetric bar subjected to compressive, axial loading. Columns of this type often fail through buckling, and not due to material compressive stresses being exceeded. Buckling can be defined as “...the sudden large deformation of a structure due to a slight increase of an existing load under which the structure exhibited little, if any, deformation before the load was increased [23].” During this type of failure, the stress is smaller than that which the material is capable of withstanding, but because of the geometry of the column it begins to “bend” in one of several different shapes, called modes, and then fails from effectively bowing out too far and collapsing catastrophically upon itself. Columns used to support large buildings and structures often fail this way, which baffled engineers hundreds of years ago before the exact cause was known. Columns generally conform to buckling laws put forth in the 18th century by Leonhard Euler, that are still used by engineers today. These laws give a maximum force, or “critical load,” which can be applied to a column before it will fail by buckling. In the twentieth century work was done by Max Beck to determine what effect could be had on the critical load of a column if the force causing it to fail remained normal to the deformed surface, instead of remaining normal to the base [3]. This type of loading can be referred to as non-conservative or Beck loading. A comparison of Euler and Beck loading is shown in figure 2.1. The problem of Beck loading cannot be solved through static considerations like the Euler problem, and so a more dynamic formulation is needed.

2.2.1 Classical Column Theory

The topic of column stability, as opposed to microneedles, is much older and dates back (formally) to the 18th century. The problem of column stability per se, however, has been around since humans began building structures. A classic text on column stability was written by Timoshenko and Gere [25] in the early 20th century (with subsequent updates, which cover Euler’s equations from the 18th century, and also tackle the addition of an
elastic foundation to a buckling bar, but does not do so in conjunction with nonconservative forces [25]). The basic equations of motion for a beam can be derived as follows:

Consider figure 2.2, where \( q \) is the distributed load over the beam - having a constant intensity over distance \( dx \), \( V \) is the shearing force, \( M \) is the moment acting on either side of the element and \( P \) is the axial load.

Element (b) is in equilibrium, and summing forces in the y-direction (downward) yields

\[
-V + q \, dx + (V + dV) = 0
\]

\[
q = -\frac{dV}{dx}
\]  \hspace{1cm} (2.1)

Summing moments about point n and assuming a small angle of deflection gives us

\[
M + q \, dx \frac{dx}{2} + (V + dV) \, dx - (M + dM) + P \, \frac{dy}{dx} \, dx = 0
\]

and by neglecting second order terms,

\[
V = \frac{dM}{dx} - P \, \frac{dy}{dx}
\]  \hspace{1cm} (2.2)

If shearing effects and shortening of the beam axis are neglected (Zahn showed that in microneedles the value of shear force required for failure was higher than buckling force [28]), equation 2.2 becomes

\[
EI \, \frac{d^2y}{dx^2} = -M
\]  \hspace{1cm} (2.3)
Figure 2.2: Simply Supported Beam (a) and Cross Section (b)

Where $EI$ is the flexural rigidity of the beam in the plane of bending. Combining equations 2.3, 2.2 and 2.1, the differential equation can take on 2 forms:

$$EI \frac{d^3y}{dx^3} + P \frac{dy}{dx} = -V$$  \hspace{1cm} (2.4) \\

$$EI \frac{d^4y}{dx^4} + P \frac{d^2y}{dx^2} = q$$  \hspace{1cm} (2.5) \\

The next step is to determine the critical load in an ideal column (a perfectly straight column compressed by a centrally applied axial load). Again, the critical load is the maximum force that can be applied to the column to cause it to retain it’s bent shape without becoming unstable; upon the removal of which the column will return to it’s undisturbed shape.

Considering figure 2.2,

$$M = -P (\delta - y)$$  \hspace{1cm} (2.6)
from equation 2.3,

\[ EI \frac{d^2 y}{dx^2} = -P (\delta - y) \]  \hspace{1cm} (2.7)

Adopting the notation \( k^2 = \frac{P}{EI} \),

\[ \frac{d^2 y}{dx^2} + k^2 y = k^2 \delta \]  \hspace{1cm} (2.8)

Which has a general solution, with constants \( A \) & \( B \)

\[ y = A \cos kx + B \sin kx + \delta \]  \hspace{1cm} (2.9)

Using the end conditions on the built-in side of the column, \( y = \frac{dy}{dx} = 0 \) at \( x = 0 \):

\[ y = \delta (1 - \cos kx) \]  \hspace{1cm} (2.10)

and on the upper end using \( y = \delta \) at \( x = l \) gives

\[ \delta \cos kl = 0 \]

\[ \therefore kl = (2n - 1) \frac{\pi}{2}, \ (n = 1, 2, 3...) \]

And since the smallest value is desired, \( n = 1 \) is used

\[ kl = l \sqrt{\frac{P}{EI}} = \frac{\pi}{2} \]

therefore the smallest critical load for this bar in question is

\[ P_{cr} = \frac{\pi^2 EI}{4l^2} \]  \hspace{1cm} (2.11)

### 2.2.2 Beck or Nonconservative Forces

When considering the mathematics behind non-conservative forces as applied to columns, one must look back to Beck’s classic paper on the subject [3]. While the original paper was published in German, there have been subsequent translations, and the aforementioned Timoshenko book does a good job of covering the topic [25].

When conservative (Euler) loading of columns was considered, only the initial and final positions of the column were considered, since the path between them is irrelevant (like the case of an object falling under the pull of gravity; only the lowering of its center of gravity is of interest). Nonconservative loading, however, requires a different approach than
standard static analysis - a more dynamic approach. Following the analysis on pp.152-155 [25], the final frequency equation is given below

\[ 2a\omega^2 + k^4 + 2a\omega^2 \cosh \lambda_1 l \cos \lambda_2 l + k^2 \sqrt{a\omega^2} \sinh \lambda_1 l \sin \lambda_2 l = 0 \] (2.12)

where \( \omega \) is the radian frequency at which the bar vibrates, \( a \) is the mass per unit length divided by the flexural rigidity \( a = \frac{q}{gEI} \), \( l \) is the length of the column, \( k^2 \) is defined as before and

\[
\lambda_1 = \sqrt{\sqrt{4a\omega^2 + k^4} - \frac{k^2}{2}}
\]
\[
\lambda_2 = \sqrt{\sqrt{4a\omega^2 + k^4} + \frac{k^2}{2}}
\] (2.13)

Observing equation 2.12, if \( \omega = 0 \) (the static condition of the column), there is no value of \( k^2 \) that can satisfy the equation. Therefore, there is no value of \( P \) at which the column can stay slightly bent, as was the case in the static analysis. The critical value of the tangential (nonconservative) force is shown in equation 2.14, but the concept of the critical force as it applies to this kind of loading is still unclear. Table 2.1 shows a relationship between non-dimensional parameters, where the effect of increasing load \( P \) can be seen on the first two frequencies of the beam. This will be referenced later.

\[
P_{cr} = \frac{2.0316\pi^2EI}{l^2}
\] (2.14)

<table>
<thead>
<tr>
<th>( \frac{Pl^2}{\pi^2EI} )</th>
<th>0</th>
<th>0.5</th>
<th>1.0</th>
<th>1.5</th>
<th>2.0</th>
<th>2.032</th>
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<td>0.272</td>
<td>0.441</td>
<td>0.992</td>
<td>1.246</td>
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<tr>
<td>( \frac{\omega_2^2a^4}{\pi^4} )</td>
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<td>4.296</td>
<td>3.567</td>
<td>2.750</td>
<td>1.542</td>
<td>1.246</td>
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</tbody>
</table>

2.2.3 Buckling of a Bar with Elastic Foundation

Timoshenko’s book also covers a more advanced topic in column theory: the addition of elastic supports to a bar; the subsequent effect on critical buckling load. The author introduces the term \( \beta \), the modulus of foundation, which represents the magnitude
of the elastic foundation reaction. Energy methods are used (see [25] page 95) to derive the critical load for a Euler column with pinned ends (as opposed to the fixed/free condition considered in previous derivations, which results in a factor of 4 difference of critical load between the two), and equation 2.15 is determined,

\[ P_{cr} = \frac{\pi^2 EI}{l^2} \left( m^2 + \frac{\beta l^4}{m^2 \pi^4 EI} \right) \]  

(2.15)

where \( m \) is an integer representing the number of half sine waves (i.e. the mode) in which the column subdivides upon buckling. Depending on the beam properties \( (\beta, l, EI) \) the lowest critical load could occur with \( m = 1, 2, 3, \ldots \) etc. Based on the mosquito feeding video, we are only interested in critical loads where \( m = 1 \), which constricts the modulus to the following equation 2.16

\[ \frac{\beta l^4}{\pi^4 EI} = 4 \]  

(2.16)

The results of this derivation are the following critical load and table. Table 2.2 shows the effect of increasing foundation stiffness on the quantity \( L_r/l \), which is related to the critical load. \( L_r \) is the reduced length of the column, and substituted here for the actual length \( l \).

\[ P_{cr} = \frac{\pi^2 EI}{L_r^2} \]  

(2.17)

<table>
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<tr>
<th>( \frac{\beta l^4}{16 EI} )</th>
<th>0</th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
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<tr>
<td>( \frac{L_r}{l} )</td>
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<td>0.819</td>
<td>0.741</td>
<td>0.615</td>
<td>0.537</td>
<td>0.438</td>
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<td>0.351</td>
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<td>( \frac{L_r}{l} )</td>
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<td></td>
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</tr>
<tr>
<td>( \frac{L_r}{l} )</td>
<td>0.140</td>
<td>0.132</td>
<td>0.117</td>
<td>0.110</td>
<td></td>
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</table>
2.2.4 Combination of Beck and Euler Loads

After the development of the concepts of conservative and nonconservative forces on columns, it was a natural progression to consider the effects of combining the loads into a single case, as Celep did in 1977 [4]. This concept was revisited in 2001 when Rao and Singh published a paper showing interaction curves between Euler and Beck loads and explaining their physical significance [21].

Rao describes the two typical instabilities observed in a Beck column as divergence (when one of the frequencies becomes zero) and flutter (when the two frequencies coalesce and subsequently become complex). Divergence can be visualized by thinking about a column which has started to vibrate ($\omega \neq 0$), where the frequency of vibration goes to zero as the load is increased. This column would essentially be toppling over from the weight of the applied force, and hence not vibrating any more back and forth. Flutter, on the other hand, can be visualized as the classic example of a cantilever beam which has approached one of it’s natural frequencies. When the two converging $\omega$s coalesce as load is increased, the column vibrates instantaneously between the two frequencies and becomes very unstable. A numerical example of a column reaching flutter instability can be seen in table 2.1; notice that as load is increased the two frequencies approach one another in value.

Among the main focuses of this paper is showing the interactions between Beck and Euler forces when used simultaneously, using “...user friendly non-dimensional parameters and explanations [21].”

Rao follows the aforementioned Timoshenko as well as Celep and comes up with the following non-dimensional governing equation of a column subjected to both Beck and Euler loading, with $w$ being the deflection of the column,

$$\frac{\partial^4 w}{\partial \xi^4} + k^2 \frac{\partial^2 w}{\partial \xi^2} + a \frac{\partial^2 w}{\partial t^2} = 0$$

(2.18)

with boundary conditions $w = \frac{\partial w}{\partial \xi} = 0$ at $\xi = 0$, $\frac{\partial^3 w}{\partial \xi^3} = 0$ and $\frac{\partial^3 w}{\partial \xi^3} + \lambda_e \frac{\partial w}{\partial \xi} = 0$ at $\xi = 1$. 

The non-dimensional parameters are

\[
\begin{align*}
\xi &= \frac{x}{l} \\
k^2 &= \frac{P_e + P_b}{EI} L^2 \\
\lambda_e &= \frac{P_e}{EI} L^2 \\
k^2 &= \lambda_e + \lambda_b \\
a &= \frac{mL^4}{EI}
\end{align*}
\]  
(2.19)

From these equations a general solution is assumed of the form

\[w = e^{i\omega t} (A \cosh \lambda_1 \xi + B \sinh \lambda_1 \xi + C \cos \lambda_2 \xi + D \sin \lambda_2 \xi)\]

with \(\omega\) being the frequency of oscillation of the column. A final transcendental equation is arrived upon:

\[
\left[(\lambda_1^2 + \lambda_1 \lambda_e) \sinh \lambda_1 - (\lambda_2^2 - \lambda_2 \lambda_e) \sin \lambda_2\right] (\lambda_1 \sinh \lambda_1 + \lambda_2 \sin \lambda_2) - \\
\left[(\lambda_1^2 + \lambda_e) \cosh \lambda_1 + (\lambda_2^2 - \lambda_e) \cos \lambda_2\right] (\lambda_1^2 \cosh \lambda_1 + \lambda_2^2 \cos \lambda_2) = 0
\]  
(2.20)

To validate the above equation, consider two cases: \(\lambda_e = 0\); \(\lambda_e = \lambda_b = 0\) (\(\lambda_1 = \lambda_2 = \omega \sqrt{a}\)). The first result,

\[k^4 + 2\omega^2 a + k^2 \sqrt{\omega^2 a} \sinh \lambda_1 \sin \lambda_2 + 2\omega^2 a \cosh \lambda_1 \cos \lambda_2 = 0\]

is Beck’s classic result (equation 2.12). The second result,

\[1 + \cosh \omega \sqrt{a} \cos \omega \sqrt{a} = 0\]

is the equation for free vibration of a cantilever beam (pg 162 [15]).

Rao solved the transcendental equation using Maple-V©, for 2 cases. The first case, called flutter instability, called for \(\lambda_b\) to be increased, for a given \(\lambda_e\), from zero until the two frequencies coalesce. The point of coalescing gives a critical value for both Beck and Euler, which together make 1 critical point. The divergence analysis calls for \(\lambda_e\) to be increased, for a given \(\lambda_b\), from zero until the first frequency becomes zero, and again these two make another critical point. In this way, plot 2.3 was created. The two non-dimensional critical loads used in the plot are defined as

\[
\begin{align*}
\lambda_b^* &= \frac{P_b L^2}{\pi^2 EI} \\
\lambda_e^* &= \frac{4P_e L^2}{\pi^2 EI}
\end{align*}
\]  
(2.21)
The lines on the plot are not continuous, but actually represent the trend of individual critical pair points, and the plot is not intuitive on its own, so Rao offers some analysis. Zone CO on the plot shows that for negative values of $\lambda^*_b$, $\lambda^*_e$ is very close to 0. In other words, if a negative Beck load (here positive loads are compressive and negative are tensile) were to be applied to a column, the effect would be tremendously de-stabilizing. Almost any non zero Euler force would cause the column to fail catastrophically. Zone OA shows that for even a small increase in positive beck force, a large increase in critical Euler force is seen. The Beck force has some stabilizing effect on the column, and allows it to sustain higher Euler loads than without the Beck force. For the flutter type of instability seen in Zone AB, $\lambda^*_b$ decreases with an increase in the positive $\lambda^*_e$ load. During this type of instability, increasing Beck load might not have the same desired effect of increasing the critical Euler load of the column. Should the Euler load be increased beyond point A, the “second” and “third” divergences take place, which are purely of theoretical interest with no direct practical application, and therefore are not considered in our study.
2.3 Mosquito

The mosquito feeds through a tubular structure known as the proboscis. The proboscis extends from the mouth and projects in front like the trunk of an elephant. The proboscis is the name for a collective structure that consists of the outer sheath known as the labium, and a bunch of feeding stylets, known as the fascicle. The fascicle consists of a labrum tube which is the primary path for blood flow, mandibles (thin tubes), maxilla with saw toothed tips, and a flat hypopharynx with a central salivary duct. In the current work, the mosquito head and mouth parts were observed using SEM in order to analyze the anatomy from a mechanics point of view. Figure 2.4 shows a SEM picture of the mosquito head, with the proboscis covered by the feather like outer sheath with a hair-like structure. The fascicle tip is shown protruding from the labium at the end (figure 2.5); normally, it is fully withdrawn. All SEM images and high speed video (HSV) stills are of the common type of mosquito Aedes aegypti, courtesy of Swaminathan [24].

![Figure 2.4: Aedes aegypti Anatomy](image)

Conversely, figures 2.6 and 2.7 show how the male mosquitos equipment is more suited for drinking sugary fluids such as fruit nectar for sustenance, and so is not considered in this study.
Figure 2.5: Fascicle and Labium
Figure 2.6: Male Mosquito Anatomy

Figure 2.7: Male Mosquito Tip Closeup
Chapter 3

Experimental Observations

The mosquito feeding process was filmed using a HSV camera, the Fastcam-APX 120K, manufactured by Photron USA, San Diego, CA, which is capable of filming 2000 full frames per second (fps) [24].

A sequence of events from the HSV is shown in figure 3.1. It was initially observed that as soon as the mosquito settles on its host, there was a period of time where it was continuously attempting to poke into the skin. If the proboscis buckled, then the mosquito rapidly moved on to the next spot and tried this again, possibly looking for a "soft spot" to penetrate. The fascicle is still inside the labium with just its tip out during this "poking process," and the mosquito keeps applying sudden dynamic impulse loads through a
combination of pushing the head down and kicking up the rear legs. Due to this applied load, the proboscis is compressed at its ends, supported by the skin surface and the head. If the force exceeds the buckling load for the proboscis geometry, then the proboscis begins to buckle. If it buckles, the mosquito seems to understand that it is a failed attempt and jumps to another location. Essentially, buckling of the proboscis seems to be the feedback for the mosquito to move on, and is something which can inherently be felt physically by the mosquito. Figure 3.1 shows the mosquito beginning to push down the proboscis onto the skin surface which initiates buckling and rapidly grows to a fully buckled condition. The head and the whole body tilt to accommodate the buckling and attempt to recover from it. From the final still image it is apparent that the fascicle exhibits the first fundamental mode of buckling (no inflection points). Given our hypothesis that the labium provides lateral support along the whole length of the fascicle, this is analogous to buckling of a beam on an elastic foundation.

![Figure 3.2: HSV of Mosquito Fascicle Penetration](image)

When the critical load of the proboscis is higher than the penetration failure load of the skin at the probing spot, penetration occurs. It is at this time that the risk of buckling failure is the maximum because resistance of the skin is a maximum at its outermost layers.
The sequence in figure 3.2 shows the process of skin penetration once a soft spot in the skin has been found and initial penetration has occurred. Clearly visible is the decrease in free length of the fascicle and the folding of the labium, showing that it is not completely rigid. When compared with the earlier images, it appears that the head is not tilted at the same angle, and the applied force is predominantly axial to the direction of penetration. Since the initial penetration has occurred, the resistive forces of the skin reduce dramatically and hence there is no observed buckling mode in the fascicle [24].

3.1 Summary of Experimental Observations

From direct observation of female mosquito behavior captured on HSV, some hypotheses can be formulated. Firstly, the labium covering the fascicle is in contact with the majority of the fascicle which is not yet inserted into the skin. The exact role of the labium is not known, but we will surmise here that it provides lateral support to the needle while it tries to penetrate the skin (after penetrating the tough outermost layer of the skin, further penetration through softer subsequent tissue would be easier). The second observation from the film is that the head of the mosquito is clearly not pushing straight down onto the shaft (such as traditional Euler column loading), but rather is pushing at an angle (very reminiscent of Beck or combined Beck/Euler loading). Therefore it is hypothesized that she is applying some follower force during insertion. Lastly, the fascicle appears to vibrate back and forth while it is being pushed into the skin, which we attribute to the increase of Euler/Beck loads until the column (fascicle) reaches the state of “flutter” instability. At this point the mosquito, since it is a living organism, can sense the inherent instability and will not greatly increase the applied loads in fear of causing a structural collapse. This flutter could also possibly be intentional, as a way of helping the fascicle penetrate the skin’s surface.
Chapter 4

Theory and Implementation

In this chapter a new equation describing the forces at work on a mosquito fascicle is derived. Subsequently, numerical analysis techniques were used to evaluate this equation and are explained in detail.

4.1 Derivation of Theory

The equations of motion of a column subjected to Euler and Beck loads \( P_e \) and \( P_b \), respectively, can be seen starting with equation 4.1. \( q(x) \) is a distributed force per unit length, \( y \) is the deflection at \( x \), \( V \) is the shear force, and \( m \) is the mass per unit length of the beam. \( \beta \), the foundation modulus, has been added here to represent the inherent stiffness of the member in question (the mosquito’s labium in our hypothesis). Figure 4.1 is a conceptual drawing of the forces at work in this scenario.

\[
(q - \beta y) = -\frac{\partial V}{\partial x} \quad (4.1)
\]

\[
V = \frac{dM}{dx} - P\frac{dy}{dx} \quad (4.2)
\]

\[
EI\frac{d^2y}{dx^2} = -M \quad (4.3)
\]

\[
EI\frac{d^3y}{dx^3} = -\frac{dM}{dx} = -\left(V + P\frac{dy}{dx}\right) \quad (4.4)
\]
Figure 4.1: Mosquito Head Pushing Fascicle; Forces Present

\[ EI \frac{d^3 y}{dx^3} + P \frac{dy}{dx} = -V \quad (4.5) \]

\[ EI \frac{d^4 y}{dx^4} + P \frac{d^2 y}{dx^2} = (q - \beta y) \quad (4.6) \]

\[ EI \frac{d^4 y}{dx^4} + P_e \frac{d^2 y}{dx^2} + \beta y = q \quad (4.7) \]

Now, using D’Alembert’s Principle to replace \( q \) with an inertial force

\[ q = -m \frac{\partial^2 y}{\partial t^2} \quad (4.8) \]

and combining the Beck and Euler loads in the equation gives

\[ \frac{\partial^4 y}{\partial x^4} + \left( \frac{P_e + P_b}{EI} \right) \frac{\partial^2 y}{\partial x^2} + \frac{m}{EI} \frac{\partial^2 y}{\partial t^2} + \frac{\beta}{EI} y = 0 \quad (4.9) \]

The boundary conditions are as follows, with \( l \) being the length of the column

\[ y(0, t) = 0 \]
\[ y'(0, t) = 0 \]
\[ y''(l, t) = 0 \]
\[ y'''(l, t) + \frac{P_e}{EI} y'(l, t) = 0 \quad (4.10) \]
Next, we let $\xi = \xi$ and $dx = l \, d\xi$, which changes some terms as follows

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial \xi} \frac{1}{l} \frac{\partial \xi}{\partial x}$$

(4.11)

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial \xi} \left( \frac{1}{l} \frac{\partial \xi}{\partial x} \right) \frac{\partial y}{\partial \xi}$$

(4.12)

and gives

$$\frac{1}{l^4} \frac{\partial^4 y}{\partial \xi^4} + \left( \frac{P_e + P_b}{EI} \right) \frac{\partial^2 y}{\partial \xi^2} + \frac{m}{EI} \frac{\partial^2 y}{\partial t^2} + \frac{\beta y}{EI} = 0$$

(4.13)

$$\frac{\partial^4 y}{\partial \xi^4} + \left( \frac{P_e + P_b l^2}{EI} \right) \frac{\partial^2 y}{\partial \xi^2} + \frac{ml^4}{EI} \frac{\partial^2 y}{\partial t^2} + \frac{\beta l^4}{EI} y = 0$$

(4.14)

Substituting non-dimensional numbers,

$$k^2 = \frac{P_e + P_b}{EI} l^2$$

$$\lambda_e = \frac{P_e}{EI} l^2$$

$$\lambda_b = \frac{P_b}{EI} l^2$$

$$k^2 = \lambda_e + \lambda_b$$

$$a = \frac{ml^4}{EI}$$

$$\alpha = \frac{\beta l^4}{EI}$$

(4.15)

the following non-dimensional equation is

$$\frac{\partial^4 y}{\partial \xi^4} + k^2 \frac{\partial^2 y}{\partial \xi^2} + a \frac{\partial^2 y}{\partial t^2} + \alpha y = 0$$

(4.16)

Letting $y = W(\xi)e^{i\omega t}$,

$$\frac{\partial y}{\partial \xi} = W'(\xi)e^{i\omega t}$$

$$\frac{\partial^2 y}{\partial \xi^2} = W''(\xi)e^{i\omega t}$$

$$\frac{\partial^3 y}{\partial \xi^3} = W'''(\xi)e^{i\omega t}$$

$$\frac{\partial^4 y}{\partial \xi^4} = W^4(\xi)e^{i\omega t}$$

$$\frac{\partial y}{\partial t} = W(\xi)i\omega e^{i\omega t}$$

$$\frac{\partial^2 y}{\partial t^2} = -W(\xi)\omega^2 e^{i\omega t}$$

(4.17)
\[
\left( \frac{d^4W}{d\xi^4} + k^2 \frac{d^2W}{d\xi^2} - a\omega^2W + \alpha W \right) e^{i\omega t} = 0 \quad (4.18)
\]

\[
\frac{d^4W}{d\xi^4} + k^2 \frac{d^2W}{d\xi^2} - (a\omega^2 - \alpha) W = 0 \quad (4.19)
\]

Assume the solution \( W(\xi) = A \cosh \lambda_1 \xi + B \sinh \lambda_1 \xi + C \cos \lambda_2 \xi + D \sin \lambda_2 \xi \), and for the sin and cos terms the characteristic equation is

\[
\lambda_1^4 - \lambda_2^2 k^2 - (a\omega^2 - \alpha) = 0 \quad (4.20)
\]

\[
\lambda_2^2 = \frac{k^2 + \sqrt{k^4 + 4(a\omega^2 - \alpha)}}{2} \quad (4.21)
\]

The characteristic equation for the hyperbolic terms is

\[
\lambda_1^4 + \lambda_2^2 k^2 - (a\omega^2 - \alpha) = 0 \quad (4.22)
\]

\[
\lambda_1^2 = \frac{-k^2 + \sqrt{k^4 + 4(a\omega^2 - \alpha)}}{2} \quad (4.23)
\]

To develop a final transcendental equation, \( W(\xi) \) is substituted into the boundary conditions.

1) \( W = 0 \) at \( \xi = 0 \):

\[
A + C = 0 \quad (4.24)
\]

2) \( \frac{\partial W}{\partial \xi} = 0 \) at \( \xi = 0 \)

\[
\lambda_1 A \sinh \lambda_1 \xi + \lambda_1 B \cosh \lambda_1 \xi - \lambda_2 C \sin \lambda_2 \xi + \lambda_2 D \cos \lambda_2 \xi = 0
\]

\[
B\lambda_1 + D\lambda_2 = 0 \quad (4.25)
\]

3) \( \frac{\partial^2 W}{\partial \xi^2} = 0 \) at \( \xi = 1 \)

\[
\lambda_1^3 A \cosh \lambda_1 \xi + \lambda_1^2 B \sinh \lambda_1 \xi - \lambda_2^2 C \cos \lambda_2 \xi - \lambda_2^2 D \sin \lambda_2 \xi = 0
\]

\[
\lambda_1^2 A \cosh \lambda_1 + \lambda_1^2 B \sinh \lambda_1 - \lambda_2^2 C \cos \lambda_2 - \lambda_2^2 D \sin \lambda_2 = 0 \quad (4.26)
\]

4) \( \frac{\partial^3 W}{\partial \xi^3} + \lambda_e \frac{\partial W}{\partial \xi} = 0 \) at \( \xi = 1 \)
\[
\lambda_1^2 A \sinh \lambda_1 + \lambda_1^2 B \cosh \lambda_1 + \lambda_2^2 C \sin \lambda_2 - \lambda_2^2 D \cos \lambda_2 \\
+ \lambda_2 (\lambda_1 A \sinh \lambda_1 + \lambda_1 B \cosh \lambda_1 - \lambda_2 C \sin \lambda_2 + \lambda_2 D \cos \lambda_2) = 0 \quad (4.27)
\]

Putting the coefficients from the 4 boundary conditions into a matrix and applying Cramer’s rule:

\[
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & \lambda_1 & 0 & \lambda_2 \\
\lambda_1^2 \cosh \lambda_1 & \lambda_1^2 \sinh \lambda_1 & -\lambda_2^2 \cos \lambda_2 & -\lambda_2^2 \sin \lambda_2 \\
\lambda_1 \sinh \lambda_1 + \lambda_2 \lambda_1 \sinh \lambda_1 & \lambda_1^2 \cosh \lambda_1 + \lambda_2 \lambda_1 \cosh \lambda_1 & \lambda_2^2 \sin \lambda_2 - \lambda_2 \lambda_2 \sin \lambda_2 & -\lambda_2^2 \cos \lambda_2 + \lambda_2 \lambda_2 \cos \lambda_2
\end{bmatrix} = 0
\]

\[
(4.28)
\]

\[
\begin{bmatrix}
\lambda_1 & 0 & \lambda_2 \\
\lambda_1^2 \sinh \lambda_1 & -\lambda_2^2 \cos \lambda_2 & -\lambda_2^2 \sin \lambda_2 \\
\lambda_1^3 \cosh \lambda_1 + \lambda_2 \lambda_1 \cosh \lambda_1 & \lambda_2^2 \sin \lambda_2 - \lambda_2 \lambda_2 \sin \lambda_2 & -\lambda_2^2 \cos \lambda_2 + \lambda_2 \lambda_2 \cos \lambda_2 \\
0 & \lambda_1 & \lambda_2 \\
\lambda_1^2 \cosh \lambda_1 & \lambda_1^2 \sinh \lambda_1 & -\lambda_2^2 \sin \lambda_2 \\
\lambda_1^3 \sinh \lambda_1 + \lambda_2 \lambda_1 \sinh \lambda_1 & \lambda_1^2 \cosh \lambda_1 + \lambda_2 \lambda_1 \cosh \lambda_1 & -\lambda_2^2 \cos \lambda_2 + \lambda_2 \lambda_2 \cos \lambda_2
\end{bmatrix} = 0
\]

\[
(4.29)
\]
\[
\begin{align*}
\lambda_1 \begin{pmatrix}
-\lambda_2^2 \cos \lambda_2 & -\lambda_2^2 \sin \lambda_2 \\
\lambda_2^2 \sin \lambda_2 - \lambda_1 \lambda_2 \sin \lambda_2 & -\lambda_2^2 \cos \lambda_2 + \lambda_1 \lambda_2 \cos \lambda_2
\end{pmatrix} \\
+ \lambda_2 \begin{pmatrix}
\lambda_1^2 \sinh \lambda_1 & -\lambda_2^2 \cos \lambda_2 \\
\lambda_1^2 \cosh \lambda_1 + \lambda_1 \lambda_2 \cosh \lambda_1 & \lambda_2^2 \sin \lambda_2 - \lambda_1 \lambda_2 \sin \lambda_2
\end{pmatrix} \\
- \lambda_1 \begin{pmatrix}
\lambda_1^2 \cosh \lambda_1 & -\lambda_2^2 \sin \lambda_2 \\
\lambda_1^2 \sinh \lambda_1 + \lambda_1 \lambda_2 \sinh \lambda_1 & -\lambda_2^2 \cos \lambda_2 + \lambda_1 \lambda_2 \cos \lambda_2
\end{pmatrix} \\
+ \lambda_2 \begin{pmatrix}
\lambda_1^2 \cosh \lambda_1 & \lambda_1^2 \sinh \lambda_1 \\
\lambda_1^2 \sinh \lambda_1 + \lambda_1 \lambda_2 \sinh \lambda_1 & \lambda_1 \lambda_2 \cosh \lambda_1 + \lambda_1 \lambda_2 \cosh \lambda_1
\end{pmatrix} = 0 \quad (4.30)
\end{align*}
\]

\[
\begin{align*}
\lambda_1 \left[ (-\lambda_2^2 \cos \lambda_2) (-\lambda_2^2 \cos \lambda_2 + \lambda_1 \lambda_2 \cos \lambda_2) + (\lambda_2^2 \sin \lambda_2) (\lambda_2^2 \sin \lambda_2 - \lambda_1 \lambda_2 \sin \lambda_2) \right] \\
+ \lambda_2 \left[ (\lambda_1^2 \sinh \lambda_1) (\lambda_2^2 \sin \lambda_2 - \lambda_1 \lambda_2 \sin \lambda_2) + (\lambda_2^2 \cos \lambda_2) (\lambda_1^2 \cosh \lambda_1 + \lambda_1 \lambda_2 \cosh \lambda_1) \right] \\
- \lambda_1 \left[ (\lambda_1^2 \cosh \lambda_1) (-\lambda_2^2 \cos \lambda_2 + \lambda_1 \lambda_2 \cos \lambda_2) + (\lambda_2^2 \sin \lambda_2) (\lambda_1^2 \sinh \lambda_1 + \lambda_1 \lambda_2 \sinh \lambda_1) \right] \\
+ \lambda_2 \left[ (\lambda_1^2 \cosh \lambda_1) (\lambda_1^2 \cosh \lambda_1 + \lambda_1 \lambda_2 \cosh \lambda_1) - (\lambda_1^2 \sinh \lambda_1) (\lambda_1^2 \sinh \lambda_1 + \lambda_1 \lambda_2 \sinh \lambda_1) \right] = 0 \quad (4.31)
\end{align*}
\]

\[
\begin{align*}
\lambda_1 \left[ \lambda_1^5 \cos^2 \lambda_2 - \lambda_2^5 \lambda_1 \cos^2 \lambda_2 + \lambda_2^5 \sin^2 \lambda_2 - \lambda_2^2 \lambda_1 \sin^2 \lambda_2 \right] \\
+ \lambda_2 \left[ \lambda_1^5 \lambda_2^2 \sin \lambda_1 \sin \lambda_2 - \lambda_1^3 \lambda_2 \lambda_1 \sin \lambda_1 \sin \lambda_2 + \lambda_1^3 \lambda_2^2 \cos \lambda_2 \cos \lambda_2 \cosh \lambda_1 + \lambda_1 \lambda_2^2 \lambda_1 \cos \lambda_2 \cosh \lambda_1 \right] \\
- \lambda_1 \left[ -\lambda_1^3 \lambda_2^2 \cosh \lambda_1 \cos \lambda_2 + \lambda_1^3 \lambda_2 \lambda_1 \cosh \lambda_1 \cos \lambda_2 + \lambda_1^3 \lambda_2^2 \sin \lambda_2 \sin \lambda_2 \sinh \lambda_1 + \lambda_1 \lambda_2^2 \lambda_1 \sin \lambda_2 \sin \lambda_1 \right] \\
+ \lambda_2 \left[ \lambda_1^3 \cosh^2 \lambda_1 + \lambda_1^3 \lambda_1 \cosh^2 \lambda_1 - \lambda_1^3 \sinh^2 \lambda_1 - \lambda_1^3 \lambda_1 \sinh^2 \lambda_1 \right] = 0 \quad (4.32)
\end{align*}
\]
\[
\lambda_2^5 \lambda_1 - \lambda_1 \lambda_2^3 \lambda_e + (\lambda_1^2 \lambda_2^2 - \lambda_1^2 \lambda_2^2 \lambda_e) (\sinh \lambda_1 \sin \lambda_2) \\
+ (\lambda_1^2 \lambda_2^2 + \lambda_1 \lambda_2^3 \lambda_e) (\cos \lambda_2 \cosh \lambda_1) + (\lambda_1^2 \lambda_2^2 - \lambda_1^2 \lambda_2^2 \lambda_e) (\cosh \lambda_1 \cos \lambda_2) \\
- (\lambda_1^4 \lambda_2^2 + \lambda_1^4 \lambda_2^2 \lambda_e) (\sinh \lambda_1 \sin \lambda_2) + \cosh^2 \lambda_1 (\lambda_0^5 \lambda_2 + \lambda_1 \lambda_2 \lambda_e) - \sinh^2 \lambda_1 (\lambda_0^5 \lambda_2 + \lambda_1 \lambda_2 \lambda_e) = 0 \\
(4.33)
\]

\[
\lambda_1 \lambda_2^5 - \lambda_1 \lambda_2^3 \lambda_e + [\lambda_1^2 \lambda_2^2 - \lambda_1^2 \lambda_2^2 \lambda_e - 2 (\lambda_1^2 \lambda_2^2 \lambda_e)] (\sinh \lambda_1 \sin \lambda_2) \\
+ (2 (\lambda_1^3 \lambda_2^3) - \lambda_1 \lambda_2 \lambda_e + \lambda_1 \lambda_2^3 \lambda_e) (\cosh \lambda_1 \cos \lambda_2) + (\lambda_1^5 \lambda_2 + \lambda_1^3 \lambda_2 \lambda_e) (\cosh^2 \lambda_1 - \sin^2 \lambda_1) = 0 \\
(4.34)
\]

After dividing by \( \lambda_1 \lambda_2 \)

\[
\lambda_1^2 \lambda_e + \lambda_1^4 + \lambda_2^4 - \lambda_2^2 \lambda_e + [-\lambda_1^3 \lambda_2 + \lambda_1 \lambda_2^3 - 2 (\lambda_1 \lambda_2 \lambda_e)] (\sinh \lambda_1 \sin \lambda_2) \\
+ [2 (\lambda_1^3 \lambda_2^3) + \lambda_e (-\lambda_1^2 + \lambda_2^2)] (\cosh \lambda_1 \cos \lambda_2) = 0 \\
(4.35)
\]

As a method of checking the work, only the Beck load is considered, with no foundation stiffness or Euler load

\[
\lambda_1^4 + \lambda_2^4 + (\lambda_1^3 \lambda_2 + \lambda_1 \lambda_2^3) (\sinh \lambda_1 \sin \lambda_2) + 2 (\lambda_1^2 \lambda_2^2) (\cosh \lambda_1 \cos \lambda_2) = 0 \\
(4.36)
\]

Grouping non-sinusoidal terms

A)

\[
\lambda_1^4 + \lambda_2^4 = \left[ \frac{k^2 + \sqrt{k^4 + 4a \omega^2}}{2} \right]^2 + \left[ \frac{-k^2 + \sqrt{k^4 + 4a \omega^2}}{2} \right]^2 \\
= 2 \left( \frac{k^4}{4} + 2a \omega^2 \right) \\
= k^4 + 2a \omega^2 \\
(4.37)
\]

B)

\[
-\lambda_1^3 \lambda_2 + \lambda_1 \lambda_2^3 = \lambda_1 \lambda_2 (-\lambda_1^2 + \lambda_2^2) \\
= k^2 \left( \sqrt{a \omega^2} \right) \\
(4.38)
\]
Therefore for zero foundation stiffness and Euler load, the pure Beck case equals

\[ k^4 + 2a\omega^2 + k^2\sqrt{a\omega^2} (\sinh \lambda_1 \sin \lambda_2) + 2a\omega^2 (\cosh \lambda_1 \cos \lambda_2) = 0 \]  

(4.40)

Which is the exact result from Beck’s classic paper (with \( \xi = \frac{l}{l} = 1 \)), (equation 2.12).

Now, if the Beck load is also taken to equal zero - \( k = 0 \) (i.e. in addition to \( \alpha = 0, \lambda_e = 0, \) now \( \lambda_b = 0 \)):

\[ \lambda_2^2 = \sqrt{\frac{4a\omega^2}{2}} = \lambda_1^2 = \sqrt{a\omega^2} = \omega\sqrt{a} \]  

(4.41)

And therefore, from equation 4.40, we have the free bending vibration of a bar

\[ 2a\omega^2 + 2a\omega^2 (\cosh \lambda_1 \cos \lambda_2) = 0 \]

\[ \therefore 1 + \cosh \lambda_1 \cos \lambda_2 = 0 \]  

(4.42)

A comparison is also done with Celep’s equations; for the sake of redundancy this can be found in Appendix A.

So, after checking our equation’s validity, the final transcendental form is

\[
\begin{align*}
(\lambda_1^4 + 2\lambda_1^2) + \lambda_e (\lambda_1^2 - \lambda_2^2) &+ \lambda_1 \lambda_2 (-\lambda_1^2 + \lambda_2^2 - 2\lambda_e) (\sinh \lambda_1 \sin \lambda_2) \\
&+ \left[ 2\lambda_1^2 \lambda_2^2 + \lambda_e (-\lambda_1^2 + \lambda_2^2) \right] (\cosh \lambda_1 \cos \lambda_2) = 0
\end{align*}
\]  

(4.43)

Where, from equations 4.15, 4.21 and 4.23 we have

\[
\begin{align*}
\lambda_1^2 &= \frac{-k^2 + \sqrt{k^4 + 4(a\omega^2 - \alpha)}}{2} \\
\lambda_2^2 &= \frac{k^2 + \sqrt{k^4 + 4(a\omega^2 - \alpha)}}{2} \\
k^2 &= \frac{P_e + P_b l^2}{EI} \\
\lambda_e &= \frac{P_e l^2}{EI} \\
\lambda_b &= \frac{P_b l^2}{EI} \\
k^2 &= \lambda_e + \lambda_b \\
a &= \frac{ml^4}{EI} \\
\alpha &= \frac{\beta l^4}{EI}
\end{align*}
\]  

(4.44)
Doing a unit check, with $F$ = force, $m$ = mass, $l$ = length, $t$ = time and $\beta$ = foundation stiffness

\[
k^2 = \frac{Fl^2}{Fl^2} = \frac{Fl^2}{Fl^2} \\
\therefore \lambda_c = \lambda_b = \text{Unitless} \\
a\omega^2 = \frac{(ml^{-1}l^4)}{(mt^{-2})(l^{-2})l^4} = \frac{l}{l} \\
\alpha = \frac{Fl^{-2}l^4}{Fl^{-2}l^4} \\
\beta = \frac{F}{l^2} \tag{4.45}
\]

4.2 Numerical Implementation

Once equation 4.43 was developed, numerical analysis was used to solve the transcendental equation for the radian frequency $a\omega^2$. Matlab® programs were written (see appendices) as m-files® to accomplish this. In keeping with Rao’s terminology [21], programs were developed to solve the distinct cases of flutter and divergence instability. Figure 4.2 shows a flow chart of the algorithms.

The m-file Column_Flutter_Rao.m works by setting values for $P_e$, $\alpha$ and $P_b$, and evaluating the equation between an upper and lower limit of the frequency $a\omega^2$. The equation is plotted with frequency on the x-axis and the value of the equation on the y-axis. The user is then able to graphically determine the general whereabouts of the zeros (where the graph crosses the x-axis). In order to get the exact location of these points, the code interpolates between all of the data points to create an array with far more data points (and therefore more accuracy than the original plotted equation) and then picks out the points that are equal to zero (or closest to zero in that set). The program then compares the first 2 zeros of the function and stores the difference between them as the value “diff.” The loop runs until the variable $P_b$ is stepped up to its maximum value, at which point the program stops if there are no errors and produces several output. First, it graphs each set of points which correspond to each Beck load iterated; next it displays in the command window a table of the Beck loads which it ran through and the corresponding values of diff - i.e. in each case how far apart the first two zeros were. Often, the program will run for a
Set
Variables
α, Pe, Pb,
$\omega^2$ range

&
Set accuracy
of
interpolation

Begin

Set
Equation 4.43

OR

Plot 4.43 vs. $\omega^2$ for each step

Step Pb with constant Pe
(Flutter Instability)

Step Pe with constant Pb
(Divergence Instability)

Observe plots and output
to determine if a solution
is within the P range set

Use interpolation for a
high degree of accuracy
when determining the
critical value that
causes:

The first two frequencies to coalesce
(flutter)

OR

The first frequency to approach 0
(divergence)

Obtain Critical Pair

Figure 4.2: Flow Chart of Algorithms
period of time and then exit with an error. This happens when the program tries to record zeros that do not exist. For example, if several lines of data consecutively each intersect the x-axis and have zero points, they will be recorded and the program will run correctly, but once the line of data has moved above or below the x-axis completely, or doesn’t intersect in the interval which we are addressing (set by the variable x), the program will not find a zero value and will exit with an error. When this happens, the operator should: a) examine the plots, if any were generated, and decide whether to change the range x; b) step back the upper limit of the Beck load to try and find the last value to give zeros. This is a continuous process to try and find the Beck load which produces its zeros closest together. Both of these m-files concepts will be further explained through examples in the following sections.

The m-file Column_Diverge_Rao.m also works by setting values for $P_e$, $\alpha$ and $P_b$, and evaluating the equation between an upper and lower limit of the frequency $a\omega^2$. The difference is that here the Beck load is constant, and the Euler load is increased, until the first zero of the function approaches $x = 0$. This code runs from a lower Euler limit to an upper limit, and once the operator visually confirms the first zero is approaching the origin (also through generated plots and command window output), then the same interpolation procedure followed to find the x-intercept points (from the Coalescence code) can be used again to pinpoint the $P_e$ value that drives the column to divergence.

4.2.1 Validation

Before equation 4.43 could be solved, the Matlab programs had to be validated; for this table 2-15 from Timoshenko [25] was first referenced (table 2.1). This table refers to the case of a column subjected to pure Beck loading (non-conservative force only). Equation 4.43 was used as the basis for both the coalescence and divergence m-files, with constants added to the nondimensional quantities where needed for correspondence with Timoshenko and Rao’s values.

This table shows that as the load $P$ (in this case Beck load) is increased, the first two frequencies of the bar get closer in value until they converge to a single number. This result will be duplicated using the flutter m-file. In the first run below: $P_e = 0; \alpha = 0; P_b = 0$ and to make the results easier to follow the file will not be set up to loop through until the values coalesce.

The data points in figure 4.3 represent the values of equation 4.43 with no founda-
tion stiffness or Euler force, and with Beck also equal to zero. The equation is evaluated at x intervals of 0.1 (the variable “x” runs from 0 to 5 in 0.1 increments - x-values corresponding to \(a\omega^2\)), and the values of the equation (the y-values) are not relevant except when they become zero. The solid line connecting each data point is the representation that will be used in subsequent plots for ease of estimation of the zeros’ location. Increased accuracy will come in during the interpolation process, where no matter the distance between successive data points, the interpolation process will make its own set of data in intervals ranging from 0.1 to 0.001 units (as designated by the user). As will soon be apparent, this method gives quite accurate results. In figures 4.4 and 4.5 the previous plot has been refined and zoomed in to the areas where the values of the equation equal zero. In addition to the plot generated, if the variable “zero1” is called, the output in the command window reads: “zero1 = 0 0.1267 4.9843.” zero1 is the matrix where the zeros of the equation are stored once they are found through interpolation, and in this case the first zero is a trivial solution owing to the fact that taking out the Beck and Euler forces together and evaluating at \(a\omega^2 = 0, \alpha = 0\) completely wipes out equation 4.43 (for this reason values of zero are

Figure 4.3: Solution to Equation 4.43 with \(P_b = 0, P_e = 0\) and \(\alpha = 0\)
typically left out of the zero1 matrix). Comparing to Timoshenko’s data in table 2.1, there is exact numerical correlation to 3 decimal places.

Figure 4.4: Solution to Equation 4.43 with $P_b = 0, P_e = 0$ and $\alpha = 0$

Figure 4.5: Solution to Equation 4.43 with $P_b = 0, P_e = 0$ and $\alpha = 0$

Now that the method of solving is apparent, the next step is to run one loop, starting at the above prescribed conditions, but now stepping the Beck load in 0.5 increments
until a coalescence is observed. For comparison again with Timoshenko’s Table 2-15, the Beck load $P$ will be stepped from 0 to 2.5 in increments of 0.5. When the code is run with these parameters, it exits with an error but produces a plot. The plot can be seen in figure 4.6, with added colors and labels to make it easier to decipher. $P$ values from 0 to 2.0 were plotted with no problem, but an error occurred during the plotting of $P = 2.5$. This can most readily be explained by observing what happens when all 6 values of $P$ are plotted without use of the interpolation code determining the exact zero locations - this can be seen in figure 4.7. In this figure it is easy to see that the last set of points does not cross the $x$-axis in the range of $a\omega^2$ which we have set, so therefore when the interpolation program looks for the zeros and does not find any there is no information for zero1, and when the program tries to subtract the first two entries of zero1 an error occurs.

Figure 4.6: Solution to Equation 4.43 with $P_b = 0 : 0.5 : 2.0$, $P_c = 0$ and $\alpha = 0$

Now that the error has been explained, we can examine the results of the program. Figure 4.8 shows a zoomed in version of figure 4.6, where one can observe the zeros of the function coalescing, and it appears that shortly after a $P$ value of 2.0 the solution will occur.
The final solution for this set of parameters could be obtained, for example, as outlined below:

1) Adjust the $P$ to run from 2 to 2.5 in 0.1 steps (written as $2:0.1:2.5$), and observe that a $P$ value of 2.1 produces an error, signifying the solution is between 2.0 and 2.1.

2) Adjust the $P$ to run $2.00:0.01:2.1$, and observe that a $P$ value of 2.04 produces an error, signifying the solution is between 2.03 and 2.04.

3) Follow same procedure; determine answer lies between 2.031 and 2.032; set step to 0.0001.

4) After narrowing it down this far, we have produced an accurate enough result, and we find that the final value that produces zeros is 2.0315, with a difference among the zeros of 0.0280. If the last 4 lines of code are copied from the m-file and run in the command window (since the error keeps them from being run on their own), table 4.1 is output:

From table 4.1, one can see that coalescence might occur somewhere just above 2.0315, were the user to spend enough time running through scenarios with increased res-
Figure 4.8: Solution to Equation 4.43 with $P_b = 0 : 0.5 : 2$, $P_e = 0$ and $\alpha = 0$

Table 4.1: $P_b$ iterations and Corresponding Difference Between Zeros

<table>
<thead>
<tr>
<th>$P_b$</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0310</td>
<td>0.0744</td>
</tr>
<tr>
<td>2.0311</td>
<td>0.0679</td>
</tr>
<tr>
<td>2.0312</td>
<td>0.0600</td>
</tr>
<tr>
<td>2.0313</td>
<td>0.0520</td>
</tr>
<tr>
<td>2.0314</td>
<td>0.0410</td>
</tr>
<tr>
<td>2.0315</td>
<td>0.0281</td>
</tr>
</tbody>
</table>
olution. 2.0315 rounds up to 2.032, which is the exact Beck load causing coalescence in Timoshenko (table 2.1) and page 126 of Rao [21].

After understanding the coalescence code fully, the divergence code should be more easily understood. The m-file Column_Diverge_Rao.m works on the same principle, also solving equation 4.43, but for different quantities.

Let us take the case of pure Euler load ($P_e = 0$), with $\alpha = 0$, and iterate starting at zero and going to, say, 4 ($P_e = 0 : 1 : 4$) with medium accuracy ($\text{interval}\_\text{step}=.01$). The results are the following text delivered in the command window (table 4.2) and figure 4.9 (colors are again added to the figure for ease of understanding and the numbers on the graphs mark the order in which they were graphed, with the first being the $P_e = 0$ case, the second being $P_e = 1$, etc.).

<table>
<thead>
<tr>
<th>$P_e$</th>
<th>Zero</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1.0000</td>
<td>4.1597</td>
</tr>
<tr>
<td>2.0000</td>
<td>3.3342</td>
</tr>
<tr>
<td>3.0000</td>
<td>2.5256</td>
</tr>
<tr>
<td>4.0000</td>
<td>1.7724</td>
</tr>
</tbody>
</table>

In this case, instead of finding the load that causes coalescence, we are looking for the frequency to be pushed to $x=0$. If we examine the table output as well as zoom in towards the origin of the plot (figure 4.10), we can see what is happening.

At first glance, it is easy to see that plots 3,4 and 5 do not have zeros in this area (close to $x=0$), but it is harder to tell what is going on with the first 2 graphs (corresponding to $P_e = 0\&1$). By looking at the table spit out by the m-file in the command window, we cannot tell what is happening between the first two points because of the trivial solution at $P_e = 0$, so we will focus on this area to find out what is happening. The next iteration will run $P_e = 0 : 0.2 : 1.4$. Table 4.3 contains the command window output.

One can notice that (disregarding the trivial first case) the zeros seem to be approaching $x = 0$, right up until $P_e = 1$, where the zero jumps up to a higher number (meaning the first zero of the function is no longer close to zero, but at some higher value). To investigate further, we will run $P_e = 0.9 : 0.01 : 1.05$ (table 4.4). At this point the graphical usefulness of the program is less important, since it is difficult to zoom in enough
Figure 4.9: Solution to Equation 4.43 with $P_b = 0$, $P_e = 0 : 1 : 4$ and $\alpha = 0$

Table 4.3: $P_e$ iterations and Corresponding Zeros

<table>
<thead>
<tr>
<th>$P_e$</th>
<th>Zero</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.2000</td>
<td>0.1029</td>
</tr>
<tr>
<td>0.4000</td>
<td>0.0783</td>
</tr>
<tr>
<td>0.6000</td>
<td>0.0529</td>
</tr>
<tr>
<td>0.8000</td>
<td>0.0264</td>
</tr>
<tr>
<td>1.0000</td>
<td>4.1597</td>
</tr>
<tr>
<td>1.2000</td>
<td>3.9943</td>
</tr>
<tr>
<td>1.4000</td>
<td>3.8290</td>
</tr>
</tbody>
</table>
to tell exactly what is happening close to zero; the tables are more beneficial.

Following these results, we will now increase our accuracy (interval_step=.001; considerably more processor time is required), and run $P_e = 0.98 : 0.001 : 1.00$ (table 4.5).

Looking at the results, it is easiest to think of them as a limit: $\lim_{P_e \to 0} P_e = 1$. The solution $P_e = 1$ is the exact solution from page 126 of Rao [21].

Next, the implementation of $\alpha$ in our derivation was checked using table 2.2. A new table was created (table 4.6) showing how changing $\beta$ affects our loads for the basic case of simple Euler loading (Beck load equals 0) where the critical divergence load equals 1 (the first row shows the default case). In column A the values of foundation stiffness from table 2.2 were directly copied. Column B shows the effect of the change in foundation stiffness on the critical load, based on equation 2.17. Column C shows the foundation stiffness values multiplied by $\frac{16}{\pi^4}$ to make them match up with our formulation of $\alpha$. In column D calculated critical Euler loads based on the foundation stiffness values in column C are displayed. Columns E and F show the ratio of Timoshenko foundation stiffness to the author’s foundation stiffness, and the value of the newly gathered data multiplied
by the constant 24.35, respectively. It was found that if the new data is multiplied by 24.35 and plotted against column B data, the correlation is quite strong for most of the points (figure 4.11). 24.35 is the result of taking the ratio $\frac{16}{\pi^4}$ and multiplying it by 4 (the difference between the critical loads because of the author’s use of fixed-free end conditions and Timoshenko’s use of pinned-pinned). This strong correlation appears to trail off for large foundation stiffness values, but without more data points it is hard to be sure. The data might be showing larger variation, but were more values able to be plotted, the two curves might also show agreement again for higher values of foundation stiffness. For the current discussion only values of 0-24 for $\frac{\beta l^4}{16EI}$ are relevant, as we are only interested in the first mode (based on equation 2.16). If only these values are plotted (figure 4.12) the strong correlation is apparent. The slightly weaker correlation at each end of the curve is possibly due to the rounding which Timoshenko requires to keep his $m$ values as integers (they are either rounded up or down to the nearest whole number since it is not possible to have partial modes). At the beginning and the end of this range, the m values have the greatest amount of rounding error, which is not present in the code.

A final test of these two programs is to recreate figure 2.3. Using the two m-files and performing analysis as described in this section, critical load pairs were determined, overlaid on figure 2.3, and plot 4.13 was created.
### Table 4.4: $P_e$ iterations and Corresponding Zeros

<table>
<thead>
<tr>
<th>$P_e$</th>
<th>Zero</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.9000</td>
<td>0.0129</td>
</tr>
<tr>
<td>0.9100</td>
<td>0.0117</td>
</tr>
<tr>
<td>0.9200</td>
<td>0.0107</td>
</tr>
<tr>
<td>0.9300</td>
<td>0.0093</td>
</tr>
<tr>
<td>0.9400</td>
<td>0.0073</td>
</tr>
<tr>
<td>0.9500</td>
<td>0.0056</td>
</tr>
<tr>
<td>0.9600</td>
<td>0.0042</td>
</tr>
<tr>
<td>0.9700</td>
<td>0.0030</td>
</tr>
<tr>
<td>0.9800</td>
<td>0.0019</td>
</tr>
<tr>
<td>0.9900</td>
<td>4.1680</td>
</tr>
<tr>
<td>1.0000</td>
<td>4.1597</td>
</tr>
<tr>
<td>1.0100</td>
<td>4.1514</td>
</tr>
<tr>
<td>1.0200</td>
<td>4.1432</td>
</tr>
<tr>
<td>1.0300</td>
<td>4.1349</td>
</tr>
<tr>
<td>1.0400</td>
<td>4.1266</td>
</tr>
<tr>
<td>1.0500</td>
<td>4.1184</td>
</tr>
</tbody>
</table>

Figure 4.11: Stiffness Correlation to Theory
Table 4.5: $P_e$ iterations and Corresponding Zeros

<table>
<thead>
<tr>
<th>$P_e$</th>
<th>Zero</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.9800</td>
<td>0.0019</td>
</tr>
<tr>
<td>0.9810</td>
<td>0.0018</td>
</tr>
<tr>
<td>0.9820</td>
<td>0.0017</td>
</tr>
<tr>
<td>0.9830</td>
<td>0.0016</td>
</tr>
<tr>
<td>0.9840</td>
<td>0.0015</td>
</tr>
<tr>
<td>0.9850</td>
<td>0.0014</td>
</tr>
<tr>
<td>0.9860</td>
<td>0.0013</td>
</tr>
<tr>
<td>0.9870</td>
<td>0.0012</td>
</tr>
<tr>
<td>0.9880</td>
<td>0.0011</td>
</tr>
<tr>
<td>0.9890</td>
<td>0.0010</td>
</tr>
<tr>
<td>0.9900</td>
<td>0.0009</td>
</tr>
<tr>
<td>0.9910</td>
<td>0.0008</td>
</tr>
<tr>
<td>0.9920</td>
<td>0.0007</td>
</tr>
<tr>
<td>0.9930</td>
<td>0.0006</td>
</tr>
<tr>
<td>0.9940</td>
<td>0.0005</td>
</tr>
<tr>
<td>0.9950</td>
<td>0.0004</td>
</tr>
<tr>
<td>0.9960</td>
<td>0.0003</td>
</tr>
<tr>
<td>0.9970</td>
<td>0.0003</td>
</tr>
<tr>
<td>0.9980</td>
<td>0.0002</td>
</tr>
<tr>
<td>0.9990</td>
<td>4.1605</td>
</tr>
<tr>
<td>1.0000</td>
<td>4.1597</td>
</tr>
</tbody>
</table>
Table 4.6: Stiffness Correlation to Theory

<table>
<thead>
<tr>
<th>$\beta l_{16 E7}$</th>
<th>$P_{cr}$ (eq. 2.17)</th>
<th>Col. A*(16/$\pi^4$)</th>
<th>New Data</th>
<th>$(\frac{Col. B}{Col. D})$</th>
<th>Col. D*24.35</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0.033</td>
<td>35.264</td>
<td>0.804</td>
</tr>
<tr>
<td>1</td>
<td>1.164</td>
<td>0.164</td>
<td>0.058</td>
<td>25.704</td>
<td>1.412</td>
</tr>
<tr>
<td>5</td>
<td>1.821</td>
<td>0.821</td>
<td>0.074</td>
<td>24.611</td>
<td>1.802</td>
</tr>
<tr>
<td>10</td>
<td>2.644</td>
<td>1.643</td>
<td>0.105</td>
<td>25.180</td>
<td>2.557</td>
</tr>
<tr>
<td>15</td>
<td>3.468</td>
<td>2.464</td>
<td>0.129</td>
<td>26.882</td>
<td>3.141</td>
</tr>
<tr>
<td>20</td>
<td>4.287</td>
<td>3.285</td>
<td>0.149</td>
<td>28.769</td>
<td>3.628</td>
</tr>
<tr>
<td>30</td>
<td>5.236</td>
<td>4.928</td>
<td>0.182</td>
<td>28.772</td>
<td>4.432</td>
</tr>
<tr>
<td>40</td>
<td>5.642</td>
<td>6.570</td>
<td>0.211</td>
<td>26.740</td>
<td>5.138</td>
</tr>
<tr>
<td>50</td>
<td>6.067</td>
<td>8.213</td>
<td>0.253</td>
<td>23.979</td>
<td>6.161</td>
</tr>
<tr>
<td>75</td>
<td>7.073</td>
<td>12.319</td>
<td>0.288</td>
<td>24.560</td>
<td>7.013</td>
</tr>
<tr>
<td>100</td>
<td>8.117</td>
<td>16.426</td>
<td>0.333</td>
<td>24.375</td>
<td>8.109</td>
</tr>
<tr>
<td>200</td>
<td>12.226</td>
<td>32.851</td>
<td>0.471</td>
<td>25.957</td>
<td>11.469</td>
</tr>
<tr>
<td>300</td>
<td>14.457</td>
<td>49.277</td>
<td>0.577</td>
<td>25.056</td>
<td>14.050</td>
</tr>
<tr>
<td>500</td>
<td>18.108</td>
<td>82.128</td>
<td>0.744</td>
<td>24.338</td>
<td>18.116</td>
</tr>
<tr>
<td>700</td>
<td>21.836</td>
<td>114.979</td>
<td>0.881</td>
<td>24.785</td>
<td>21.452</td>
</tr>
<tr>
<td>1000</td>
<td>26.298</td>
<td>164.256</td>
<td>1.052</td>
<td>24.999</td>
<td>25.616</td>
</tr>
<tr>
<td>1500</td>
<td>31.210</td>
<td>246.384</td>
<td>1.185</td>
<td>26.338</td>
<td>28.855</td>
</tr>
<tr>
<td>2000</td>
<td>36.731</td>
<td>328.511</td>
<td>1.245</td>
<td>29.503</td>
<td>30.316</td>
</tr>
<tr>
<td>3000</td>
<td>45.043</td>
<td>492.767</td>
<td>1.364</td>
<td>33.023</td>
<td>33.213</td>
</tr>
<tr>
<td>4000</td>
<td>51.020</td>
<td>657.023</td>
<td>1.479</td>
<td>34.497</td>
<td>36.014</td>
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<tr>
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<td>821.297</td>
<td>1.592</td>
<td>36.050</td>
<td>38.765</td>
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<tr>
<td>8000</td>
<td>73.051</td>
<td>1314.046</td>
<td>1.912</td>
<td>38.207</td>
<td>46.557</td>
</tr>
<tr>
<td>10000</td>
<td>82.645</td>
<td>1642.557</td>
<td>2.112</td>
<td>39.131</td>
<td>51.427</td>
</tr>
</tbody>
</table>
Figure 4.12: Stiffness Correlation to Theory, \( \frac{\beta h^4}{16EI} \leq 24 \)

Figure 4.13: Interaction Stability Curves Comparison
Chapter 5

Results

In this section the complete mathematical model was solved to determine the relative effect of $\alpha$ on the critical load pairs.

When equation 4.43 was plotted, as can be seen in figure 5.1, the shape of the curves still resemble that of Rao’s plot, with loads above approximately $P_b = 0.7$ causing a shift from divergence to flutter instability. Figure 5.2 shows that for increasing $\alpha$ values, the divergence curves shift to the right, increasing possible critical Euler loads to many times the value possible with only the application of Beck load to the column. The flutter critical curve did not move with increased lateral support, but each coalescence pair occurred at a higher frequency $a\omega^2$.

The nonzero $\alpha$ curves did not all initially have the consistent shape and distribution visible in plot 5.2, however. Consider figure 5.3, and notice the solution points present in the two nonzero $\alpha$ curves shown. In each of these cases, some of the solution points deviate from the smooth curve of the rest of the data. If, however, the second zero of equation 4.43 is found for each of these points and the corresponding critical Euler point inserted in place of the first, the trend is preserved (figure 5.2). In order to explain this, Timoshenko (page 96) is referenced: for a flexible elastic bar, by increasing $\beta$, a condition arises where the critical load is actually smaller for $m = 2$ than for $m = 1$ [25]. In other words, some critical load values of the bar at a higher mode shape will occur at lower values than the critical loads of a lower mode shape. Also, as lateral support is increased in subsequent curves, there should be less occurrences of this phenomenon. The first curve ($\alpha = 1$) has 5 deviation
Critical Euler Load
1.5
Divergence
Flutter
1.25
1
Critical Beck Load (α)
0.75
0.5
0.25
0
-0.25
-0.5
-0.75
(α)
0 0.5 1 1.5 2 2.5 3 3.5 4 4.5

Figure 5.1: Critical Pairs with $\alpha = 0$

Critical Euler Load \( \frac{4P_e l^2}{\pi^2 EI} \)

Critical Beck Load \( \frac{4P_e l^2}{\pi^2 EI} \)

Divergence
Flutter
\( \alpha = 1 \)
\( \alpha = 5 \)
\( \alpha = 20 \)
\( \alpha = 50 \)
\( \alpha = 100 \)

Figure 5.2: Critical Pairs with $\alpha = 0, 1, 5, 20, 50, 100$
points in the region shown and the rest of the $\alpha$ curves had 3,2,1 and 0, respectively, which is also consistent with this observation.

From the preceding plots it is apparent that lateral support has the ability to affect the stability curves much more than increases in Beck or Euler load alone. In figure 5.4, zone A, we see the increase in Euler load possible without lateral support. In this case, if critical Beck load is increased from zero to approximately 0.5, a critical Euler load increase of approximately 2 is seen. If $\alpha$ is nonzero, however, much larger increases in critical Euler load for the same increase in critical Beck load can theoretically be realized. For example, observe zone B, which shows the difference between the critical Euler load with $\alpha=0$ and $\alpha=100$ (Beck load being increased to 0.5 in both cases). Here, increasing the critical Beck load would show an increase in critical Euler load of approximately 6 times compared to the case with $\alpha=0$, Beck load=0, which is more than 3 times the increase seen with no $\alpha$ and just non-conservative force application.
Figure 5.4: Critical Pairs with $\alpha = 0, 1, 5, 20, 50, 100$
Chapter 6

Conclusion and Future Work

To conclude, the present thesis has examined previous work done in the area of column stability, summarizing the views of scholars such as Leonard Euler and Max Beck, along with newer contemporaries such as Celep, Rao and Singh, etc. A new transcendental equation incorporating Beck and Euler forces, as well as foundation stiffness, was derived and numerical techniques were used to evaluate it. The findings show that, relatively speaking, the foundation stiffness of a column has more effect on its theoretical maximum critical Euler load than the addition of Beck loading to said column, which the author believes is a novel result.

When considering this paper’s results as applied to microneedles, a thin needle with high lateral support could pierce the tough stratum corneum of the human skin, possibly causing less pain than a larger needle of the same material. After piercing the stratum corneum, the subsequent layers (epidermis and dermis) are much softer and would not pose a threat to the structural integrity of the needle. If a sufficient support mechanism is in place, the relative size of microneedles could be smaller for a given material. The applied lateral support also need not be physical in nature - metallic needles could take advantage of magnetic lines stabilizing them, SMA materials might be kept rigid through the application of electric current, etc. Alternatively, research should be done in the areas of physical structural support to determine the best method for adding foundation stiffness to microneedles on this scale. Distributed piezo actuators have been used by Rao and Singh to apply follower forces to a cantilever beam [22]; Zehetner and Irshik also used piezos to
quell the vibrations of such a beam [29] - one of these setups might be modified to instead provide lateral support.

Figure 6.1: FEA Model and SEM Picture of Mosquito Fasicle

Future work in this area could involve the application of a sinusoidal input force to the column, to see how it is affected by Beck and Euler forces or foundation stiffness. A force of this type may aid in inserting said microneedle into the skin. Realistic values of $\beta$ for different materials should also be investigated to determine the maximum $\alpha$ possible. Finite element modeling could also be helpful in future work, as the author has explored this area and done some preliminary work including the modeling of a fascicle as seen in figure 6.1. The difficulty herein lies in the lack of data available for the structural properties of mosquito biological materials or human skin, although some work has recently been done in this area as well [24].
Bibliography


[18] Akio OKI, Madoka TAKAI, Hiroki OGAWA, Yuzuru TAKAMURA, Takayuki FUKASAWA, Jun KIKUCHI, Yoshitaka ITO, Takanori ICHIKI, and Yasuhiro HORIIKE.


Appendicies
Appendix A

Comparison of equation 4.43 with Celep

Comparing Celep’s terms with ours we have

\[
\begin{align*}
\lambda^2 &= a\omega^2 \\
\chi &= k^2 \\
\frac{\lambda_b}{\lambda_e + \lambda_b} &= \alpha \\
(1 - \alpha)\chi &= \lambda_e
\end{align*}
\]  

(A.1)

Starting with our final transcendental form

\[
(\lambda_1^4 + \lambda_2^4) + \lambda_e (\lambda_1^2 - \lambda_2^2) + \lambda_1\lambda_2 (-\lambda_1^2 + \lambda_2^2 - 2\lambda_e) (\sinh \lambda_1 \sin \lambda_2) \\
+ \left[2\lambda_1^2\lambda_2^2 + \lambda_e (-\lambda_1^2 + \lambda_2^2)\right] (\cosh \lambda_1 \cos \lambda_2) = 0 \quad (A.2)
\]

and taking the constant terms we have

\[k^4 + 2(a\omega^2) + \lambda_e(-k^2)\]

Now taking the constant terms from Celep, we can see equivalence with equation A

\[\chi^2 + 2\lambda^2 - (1 - \alpha)\chi(\chi) = 2\lambda^2 + \alpha\chi^2\]

Considering the sin sinh terms,

\[\sqrt{a\omega^2}(k^2 - 2\lambda_e)\]
corresponds to

\[ \lambda[\chi - 2\chi(1 - \alpha)] = \lambda[\chi - 2\chi + 2\chi\alpha] = \lambda[\chi(2\alpha - 1)] \]

And finally, the cos cosh terms

\[ 2\lambda^2\lambda_1^2 + \lambda_\epsilon(\lambda_2^2 - \lambda_1^2) \]

correspond to

\[ 2\lambda^2 + (1 - \alpha)\chi(\chi) = 2\lambda^2 + \chi^2(1 - \alpha) \]
% Adjust 1) Pe 2) Pb 3) Pb step & upper limit
% then run
clc;
clear all;
Pe = (3.5)/(4); % (Euler load)
Pb = (.75); % starting (Beck load)
Pb_step = (0.001);
Pb_upper = (.76);
alpha = (0)/pi^4; % foundation stiffness (beta*L^4/EI)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
interval_step=.01; % this controls accuracy of interpolation
% .1--> .001 is the useful range
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
n=0; % counter
x = 0:.01:5; % x-variable (aw^2) (match up to line 33)
y=zeros(n+1,size(x,2)); % initialize sol'n matrix
warning on verbose % gives more info about warnings
warning off MATLAB:legend:IgnoringExtraEntries; % suppress error
diff=1; % initialize loop variable
figure; % starts new figure
grid on;
hold on;
step=1;
while Pb<=Pb_upper
x = 0:.01:5; % x-variable (aw^2)
%(make sure this matches line 20)
b = (Pe + Pb)*pi^2; \quad \%b = k^2 = (Pe + Pb)

n = n + 1;

z = x*pi^4; \quad \%z = \pi^4*(aw^2)

r1 = sqrt(-b/2 + sqrt((b^2/4) + (z-alpha))); \quad \%r1 = \lambda_1

r2 = sqrt(b/2 + sqrt((b^2/4) + (z-alpha))); \quad \%r2 = \lambda_2

r3 = Pe*pi^2; \quad \%r3 = \lambda_E*pi^2

A = (r1.^4 + r2.^4) + r3.*(r1.^2 - r2.^2);

B = r1.*r2.*(-r1.^2 + r2.^2 - 2*r3).*sinh(r1).*sin(r2);

y = A + B + (2*r1.^2.*r2.^2 + r3.*(-r1.^2 + r2.^2)).*cosh(r1).*cos(r2);

\%y = Solution

plot(x,y),title('Zero Beck and Euler Force'),xlabel('aw^2')...
ylabel('Solution to Transcendental Equation'),...
legend('1','2','3','4','5','6','7','8','9','10');

\%FROM HERE DOWN IS THE INTERPOLATION
\%for the exact zero values

x1 = x;
y1 = y;

\%Generating linearly spaced x data
xmin = min(x1);
xmax = max(x1);
xlin = xmin:interval_step:xmax;

\%Interpolate the data
fn1 = @(x)interp1(x1, y1, x, 'linear');
ytemp = zeros(size(xlin));

for (i=1:length(xlin))
ytemp(i) = fzero(fn1, xlin(i));
end
xval = unique(ytemp);

\%Return the values that are not NAN
val = find(isnan(xval) == 0);
zero_values = xval(val);

\%get rid of any zero points
entries = find(zero_values);
zero_values = zero_values(entries);

\%get rid of dupe entries
zeroi = unique(single(zero_values))
diff = abs(zero1(1) - zero1(2))

Pb_table(step) = Pb;
diff_table(step) = diff;
step = step + 1;

%%%%%%%End of Interpolation%%%%%%%
Appendix C

Column_Diverge_Rao.m

To use this program:
% adjust: 1) Pb 2) Pe 3) Pe step & top limit
% 4) run and look at graph for first zero point approaching x=0
% 5) once the zero values are getting close, examine the output
% in the command window for the exact zero values to zoom in on exact Pe

clc;
clear all;
Pb = ( 0.01 );
Pe = (0.327)/4; % (Euler load) starting value
Pe_step=(.0001)/4;
Pe_upper=(.328)/4;

alpha = (20)/(pi^4); % foundation stiffness (beta*L^4/EI)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
interval_step=.01; % this controls accuracy of interpolation
% .1--> .001 is the useful range
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
n=0;
while Pe<=Pe_upper

b = (Pe + Pb)*pi^2; % b = k^2 = [(Pe + Pb)L^2/EI]*pi^2
n=n+1;

end
\[ z = x \pi^4; \quad \%z = ((aw^2)mL^4/EI)\pi^4 \]
\[ r_1 = \sqrt{-b/2 + \sqrt{(b^2/4) + (z-\alpha)}}; \quad \%r_1 = \lambda_1 \]
\[ r_2 = \sqrt{b/2 + \sqrt{(b^2/4) + (z-\alpha)}}; \quad \%r_2 = \lambda_2 \]
\[ r_3 = Pe\pi^2; \quad \%r_3 = \lambda_E\pi^2 \]
\[ A = (r_1^4 + r_2^4) + r_3*(r_1^2 - r_2^2); \]
\[ B = r_1*r_2*(-r_1^2 + r_2^2 - 2*r_3)*\sinh(r_1)*\sin(r_2); \]
\[ y = A + B + (2*r_1^2*r_2^2 + r_3*(-r_1^2 + r_2^2)) \cdot \cosh(r_1) \cdot \cos(r_2); \quad \%y = \text{Solution} \]

\begin{verbatim}
plot(x,y),title('Pure Euler Loading (no stiffness)'),xlabel('aw^2')...
ylabel('Solution to Transcendental Equation');...
legend('1','2','3','4','5','6','7','8','9','10');
\end{verbatim}

\begin{verbatim}
\%FROM HERE DOWN IS THE INTERPOLATION
\%for the exact zero values
\%
\% x1=x;
\% y1=y;
\% \Generating linearly spaced x data
\% xmin=min(x1);
\% xmax=max(x1);
\% xlin=xmin:interval_step:xmax;
\%
\% \Interpolate the data
\% fn1 = @(x)interp1(x1,y1,x,'linear');
\% ytemp=zeros(size(xlin));
\%
\% for(i=1:length(xlin))
\% ytemp(i)=fzero(fn1,xlin(i));
\% end
\% xval=unique(ytemp);
\%
\% display('Pe=');Pe*4
\%
\% Return the values that are not NAN
\% val=find(isnan(xval)==0);
\% zero_values = (xval(val))
\%
\% get rid of any zero points
\% entries=find(zero_values);
\% zero_values = zero_values(entries);
\%
\% Pe_table(step)=Pe*4;
\% zero_table(step)=zero_values(1);
\%
\% step=step+1; \%
\end{verbatim}
Pe = Pe + (Pe_step); % increments Pe
end

% clc;
% table(:,1)=(Pe_table.);table(:,2)=(zero_table.);
% disp('Table of Pe (left column) and associated zeros (right column):');
% table
Appendix D

*ColumnFlutter_corrected.m*

%Adjust 1) Pe 2) Pb 3) Pb step & upper limit
%then run

clc;
clear all;

Pe = (10); % (Euler load)
Pb = (6.45); % starting (Beck load)
Pb_step = (.01);
Pb_upper = (6.48);

alpha = (20); % foundation stiffness (beta*L^4/EI)

interval_step=.01; % this controls accuracy of interpolation 
% .1--> .001 is the useful range

n=0; % counter
x = 0:.01:300; % x-variable (aw^2) (match up to line 33)
y=zeros(n+1,size(x,2)); % initialize soln matrix
warning on verbose % gives more info about warnings
warning off MATLAB:legend:IgnoringExtraEntries; % suppress error
diff=1; % initialize loop variable

figure; % starts new figure
grid on;
hold on;
step=1;

while Pb<=Pb_upper
    x = 0:.01:300; % x-variable (aw^2)
    % (make sure this matches line 20)
\[ b = (P_{e} + P_{b}); \] \[ \text{\%}b = k^2 = (P_{e} + P_{b}) \]

\[ n = n + 1; \]

\[ z = x; \] \[ \text{\%}z = (aw^2) \]

\[ r_1 = \sqrt{(-b/2 + \sqrt{(b^2/4) + (z-alpha)})}; \] \[ \text{\%}r_1 = \lambda_1 \]

\[ r_2 = \sqrt{b/2 + \sqrt{(b^2/4) + (z-alpha)}}; \] \[ \text{\%}r_2 = \lambda_2 \]

\[ r_3 = P_{e}; \] \[ \text{\%}r_3 = \lambda_E \]

\[ A = (r_1^4 + r_2^4) + r_3(r_1^2 - r_2^2); \]
\[ B = r_1 \cdot r_2 \cdot (-r_1^2 + r_2^2 - 2r_3) \cdot \sinh(r_1) \cdot \sin(r_2); \]

\[ y = A + B + (2r_1^2 \cdot r_2^2 + r_3(-r_1^2 + r_2^2)) \cdot \cosh(r_1) \cdot \cos(r_2); \] \[ \text{\%}y = \text{Solution} \]

\[ \text{plot}(x,y), \text{title}('Zero Beck and Euler Force'), \text{xlabel}('aw^2')... \text{ylabel}('Solution to Transcendental Equation'),... \text{legend}('1', '2', '3', '4', '5', '6', '7', '8', '9', '10'); \]

\[ \% \text{FROM HERE DOWN IS THE INTERPOLATION} \]
\[ \% \text{for the exact zero values} \]

\[ % \]
\[ % \]
\[ % \]
\[ % \]
\[ % \text{Generating linearly spaced x data} \]
\[ % \]
\[ % \text{xmin} = \text{min}(x1); \]
\[ % \]
\[ % \text{xmax} = \text{max}(x1); \]
\[ % \]
\[ % \text{xlin} = \text{xmin}:\text{interval}\_\text{step}:\text{xmax}; \]
\[ % \]
\[ % \text{Interpolate the data} \]
\[ % \text{fn1} = @(x) \text{interp1}(x1, y1, x, 'linear'); \]
\[ % \text{ytemp} = \text{zeros} \text{(size}(xlin)); \]
\[ % \]
\[ % \text{for}(i = 1: \text{length}(xlin)) \]
\[ % \text{ytemp}(i) = \text{fzero} \text{(fn1, xlin}(i)); \]
\[ % \text{end} \]
\[ % \text{xval} = \text{unique}(\text{ytemp}); \]
\[ % \]
\[ % \text{Return the values that are not N\text{AN}} \]
\[ % \text{val} = \text{find}(\text{isnan}(\text{xval}) == 0); \]
\[ % \text{zero\_values} = \text{xval}(\text{val}); \]
\[ % \]
\[ % \text{get rid of any zero points} \]
\[ % \text{entries} = \text{find}(\text{zero\_values}); \]
\[ % \text{zero\_values} = \text{zero\_values}(\text{entries}); \]
\[ % \]
\[ % \]
\[ % \text{get rid of dupe entries} \]
\[ % \text{zero1} = \text{unique}(\text{single}(\text{zero\_values})); \]
\[ % \]
\[ % \]
\[ \text{Pb} \]
%% diff=abs(zero1(1)-zero1(2))
%%
%% Pb_table(step)=Pb;
%% diff_table(step)=diff;
%% step=step+1;
%%
%%%%%%%End of Interpolation%%%%%%%
To use this program:
1) Adjust \( \text{Pb} \) 2) \( \text{Pe} \) step & top limit
3) Run and look at graph for first zero point approaching \( x=0 \)
4) Once the zero values are getting close, examine the output
in the command window for the exact zero values to zoom in on exact \( \text{Pe} \)

```matlab
clc;
clear all;
Pb = ( 5 );        \%(Beck load)
Pe = (0);         \%(Euler load) starting value
Pe_step=(1);
Pe_upper=(16);

alpha = (100);    \% foundation stiffness \((\beta L^4/\text{EI})\)

interval_step=.01; \% this controls accuracy of interpolation
\% .1--->.001 is the useful range

n=0;                \% counter
x = 0:1e-3:.5;      \% x-variable \((aw^2)\)
y=zeros(n+1,size(x,2)); \% initialize sol'n matrix
figure;grid on;hold on;
step=1;             \% variable for \( \text{Pe} \)_table
warning off MATLAB:legend:IgnoringExtraEntries; \% suppress error

while \( \text{Pe}<\text{Pe}\_upper \)
    b = (Pe + Pb); \% \( b = k^2 = (\text{Pe} + \text{Pb}) \)
    n=n+1;
```
\[ z = x; \quad \text{\%}z = (aw^2) \]
\[ r1 = \sqrt{-b/2 + \sqrt{(b^2/4) + (z-\alpha)}}; \quad \%r1 = \lambda_1 \]
\[ r2 = \sqrt{b/2 + \sqrt{(b^2/4) + (z-\alpha)}}; \quad \%r2 = \lambda_2 \]
\[ r3 = Pe; \quad \%r3 = \lambda_E \]
\[ A = (r1.^4 + r2.^4) + r3.*(r1.^2 - r2.^2); \]
\[ B = r1.*r2.*(-r1.^2 + r2.^2 - 2*r3).*\sinh(r1).*\sin(r2); \]
\[ y = A + B + (2*r1.^2.*r2.^2 + r3.*(-r1.^2 + r2.^2)).*\cosh(r1).*\cos(r2); \]
\[ \%y = \text{Solution} \]

\texttt{plot(x,y),title('Pure Euler Loading (no stiffness)'),xlabel('aw^2')...}
\texttt{,ylabel('Solution to Transcendental Equation')...}
\texttt{legend('1','2','3','4','5','6','7','8','9','10');}

\texttt{\%FROM HERE DOWN IS THE INTERPOLATION}
\texttt{\%for the exact zero values}
\texttt{\%}
\texttt{\% x1=x;}
\texttt{\% y1=y;}
\texttt{\% Generating linearly spaced x data}
\texttt{\% xmin=min(x1);}
\texttt{\% xmax=max(x1);}
\texttt{\% xlin=xmin:interval_step:xmax;}
\texttt{\%}
\texttt{\% Interpolate the data}
\texttt{\% fn1= @(x)interp1(x1,y1,x,'linear');}
\texttt{\% ytemp=zeros(size(xlin));}
\texttt{\%}
\texttt{\% for(i=1:length(xlin))}
\texttt{\% ytemp(i)=fzero(fn1,xlin(i));}
\texttt{\% end}
\texttt{\% xval=unique(ytemp);}
\texttt{\%}
\texttt{\% display('Pe=');Pe*4}
\texttt{\%}
\texttt{\% Return the values that are not NAN}
\texttt{\% val=find(isnan(xval)==0);}
\texttt{\% zero_values = (xval(val));}
\texttt{\%}
\texttt{\% get rid of any zero points}
\texttt{\% entries=find(zero_values);}
\texttt{\% zero_values = zero_values(entries);}
\texttt{\%}
\texttt{\%}
\texttt{\% Pe_table(step)=Pe*4;}
\texttt{\% zero_table(step)=zero_values(1);}
\texttt{\%}
\texttt{\% step=step+1; \quad \%increments var. for tables}
Pe = Pe + (Pe_step);                  % increments Pe
end

% clc;
% table(:,1)=(Pe_table.');table(:,2)=(zero_table.');
% disp('Table of Pe (left column) and associated zeros (right column):');
% table