

ABSTRACT

HUDNUTT, BETHANY S. Teaching Functions with Dynamic Graphing Tools: A Study of Lesson Plans. (Under the direction of Dr. Hollylynne Lee).

Graphing technologies are replete in high school mathematics courses concerned with the teaching and learning of functions. Much research has been devoted to the examination of student use of such technology but tends not to differentiate between the technology itself and the lesson activities designed to implement that technology. Relatively little research, in comparison, has focused on how teachers choose to use such technology for teaching about functions and related concepts. This study investigates teachers' choices on how to implement two dynamic graphing java applets into their lessons about functions.

The investigation primarily uses two frameworks in analyzing lesson activities: Vinner's (1983) *concept images* and the *APOS* (Action, Process, Object, Schema) theory of mathematical knowledge acquisition, initially conceived by Dubinsky and Harel (1992) and expanded upon by Asiala et al (1996).

Using these two frameworks as a guide for analysis, teacher intentions for using the software are examined within their lesson planning process. Analysis of the lessons shows how teachers intend to develop a function's mathematical structure using the various representations of functions. Findings show a consistency between teachers' expressed learning goals with how they chose to integrate the software, a reliance on the historical definition of function to develop concept images, and helping students generalize how transformations occur through changing parameters but not necessarily why changing parameters transform the function in such a manner.

TEACHING FUNCTIONS WITH DYNAMIC GRAPHING TOOLS:

A STUDY OF LESSON PLANS

by

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DEDICATION

To my ever loving husband, David Hudnutt. I could not have accomplished this without you. You complement my whole being in mind, in body, and in spirit.

To my mother who read every word of this document and cared enough to spend many hours providing her gift of words.

To my family and friends for your thoughtful and caring support through this arduous yet rewarding process.

My mind is full. And so is my heart because I know you all cared so much, allowing me my space when I needed it or your time on a moment's notice because I was unable to plan ahead. I am looking forward to a renewed focus, spending time and energy with my many loved ones who I have missed so much these past years.

BIOGRAPHY

Bethany Snyder Hudnutt was born as Bethany Lynn Snyder, to the parents of Patricia Peppercorn Groves and James Floyd Snyder on January 3rd, 1972. Bethany graduated high school in 1990 from Upper Arlington, OH and proceeded to her undergraduate career at Bethany College. Bethany gained her baccalaureate degree from Ohio University in March 1995 in mathematics education.

Since obtaining her undergraduate degree, Bethany has had a variety of professional experiences relating to mathematics education. She taught applied mathematics at Tri-County Joint Vocational School in Nelsonville, OH until she and her significant other, David Hudnutt, moved to Durham, NC in the summer of 1997. She taught an additional year at South Granville High School in Creedmoor, NC. Beginning September 1998, Bethany worked for Measurement Incorporated as a lead item writer developing standardized tests.

In the summer of 2000 while on jury duty for Durham County, she met the executive director of Shodor, a non-profit company dedicated to the integration of computational science in education. That fall, Bethany took a position with the company working on materials development for Shodor's online mathematics education courseware, *Interactivate*, and continues to work as *Interactivate's* project manager.

Bethany continued her schooling to obtain a computer programming certificate through North Carolina State University's distance education program, DELTA. As a Shodor liaison to NCSU's MEGA network, Bethany met her advisor Dr. Hollylynn Lee. Realizing the high caliber of faculty in the mathematics education department and similar research interests, Bethany chose to attend NCSU for her Master's of Science degree.

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TABLE OF CONTENTS

LIST OF TABLES.....	viii
LIST OF FIGURES.....	ix
CHAPTER 1 - INTRODUCTION.....	1
Software to Promote Understanding of Functions.....	4
Description of the <i>Flyer</i> Graphing Activities	5
Problem Statement	8
CHAPTER 2 - LITERATURE REVIEW.....	10
Function Definitions	10
Graphing Technologies in the Classroom.....	11
Multiple Representations and the Learning of Functions	16
Action, Process, Object, Schema (APOS) Framework.....	19
Teaching Functions.....	24
Theoretical Framework Synopsis	26
Research Question	28
CHAPTER 3 - METHODOLOGY.....	29
Participants and Context of the Study.....	29
Data Sources and Methods of Collection.....	30
<i>Initial Interview</i>	30
<i>Lesson Plan Artifacts</i>	31
<i>Exit Interview</i>	32
Data Analysis Methodology	33
CHAPTER 4 - RESULTS.....	35

Assertions.....	39
Assertion #1	40
Assertion #2	45
Assertion #3	47
Assertion #4	49
Assertion #5	52
Assertion #6	53
Assertion #7	55
CHAPTER 5 - CONCLUSIONS	57
Summary of Study within the Framework.....	57
Findings.....	59
<i>Learning Goals</i>	59
<i>Function Definitions</i>	60
<i>Use of Function Representations</i>	61
Implication and Recommendations.....	63
<i>For Teaching</i>	63
<i>For Research</i>	64
<i>For Software and Supporting Materials</i>	65
Footnotes.....	68
REFERENCES	69
APPENDICIES.....	74
Appendix A - Form for Participant Identification and Selection.....	75
Appendix B - Initial Interview Questions.....	77

Appendix C - Lesson Plan Template.....78

Appendix D - Exit Interview Questions.....80

Appendix E - North Carolina Standard Course of Study Objectives by Lesson
Plan.....81

LIST OF TABLES

Table 1: Function Analysis Framework (Moschkovich, Schoenfeld, and Arcavi, 1993).	17
Table 2: Data Collection Context	32
Table 3: Lesson on Inverse Functions	36
Table 4: Lesson on Sine Curve Transformations and Data Fitting.....	36
Table 5: Lesson on Least Squares Line	37
Table 6: Lesson on Power Function Transformations	38
Table 7: Lesson on Parabola Transformations.....	38
Table 8: Lesson on Transformations across Function Families.....	39
Table 9: Lessons aligned by NCSCoS	41
Table 10: Summary learning goals about functions by course and teacher compared to implementation of software for the respective lessons.	43

LIST OF FIGURES

Figure 1. <i>Function Flyer</i> Interface.....	6
Figure 2. <i>Data Flyer</i> Interface	7
Figure 3. Other controls for <i>Data Flyer</i>	8
Figure 4: Function Framework Synthesis.....	27
Figure 5: Sample Row from Inverses Lesson Worksheet.....	46

CHAPTER 1

INTRODUCTION

Graphing technology potentially influences students' understanding of mathematical functions. This study concerns the impact of graphing technology on teachers' planning and implementation to meet learning goals for student understanding of mathematical functions. The formation of a well-defined function concept should be a central concern to the teaching of mathematics because functions are fundamental to most mathematical pursuits. For example, the study of advanced pure mathematics ultimately falls into two categories: sets and functions. In applied mathematics, functions are commonly used to develop mathematical models of numerous aspects in the world around us to examine trends and make predictions. Functional models also pervade all branches of science: classical sciences such as physics and chemistry, social sciences such as economics and sociology, natural sciences such as biology and geology, and engineering sciences such as computer science and architecture. Further, Piaget and his colleagues demonstrated children as young as four years old have a qualitative intuitive sense for functional relationships (1968/1977).

State and national curricula reflect the importance of understanding functions, usually including functions as a major organizing strand within the objectives. The *Principles and Standards* developed and adopted by the National Council of Teachers of Mathematics (NCTM) specify that "Instructional programs from prekindergarten through grade 12 should enable all students to understand patterns, relations, and functions standards," (NCTM, 2000, p. 37). In the *Standards Grades 9-12 Expectations*, the NCTM

specifies that “High school algebra also should provide students with insights into mathematical abstraction and structure.” Specifically in grades 9-12 student should--

- Understand relations and functions and select, convert flexibly among and use various representations for them;
- Understand and perform transformations such as arithmetically combining, composing and inverting commonly used functions...;
- Understand and compare the properties of classes of functions, including exponential polynomial, rational, logarithmic, and periodic functions. (p. 296)

Much of the literature regarding the teaching and learning of functions refers to the learner’s concept image of function, (Vinner, 1983; Vinner and Dreyfus, 1989; Tall, 1989; Slavit 1997; Sfard, 1994; Thompson, 1994). Vinner and Dreyfus found that a learner may know the formal definition of function yet not fully be able to apply it. This is the case, in large part, because the correct application depends upon the learner’s concept image of function, rather than the definition itself.

When deciding whether an example is or is not a function, a learner may apply a mental image believed to be a generic representative of a mathematical object, (Vinner and Dreyfus, 1989). Prior to fully integrating new knowledge, a learner often *compartmentalizes* new information and is unaware of potential inconsistencies contained within his/her cognitive schemes.

Hence, from an instructional point of view, before a teacher can expect learners to use and apply functions accurately, the teacher must assist in accurately developing their concept images to encompass the definition. Simply providing definitions is insufficient without assisting the learners in developing an intuition for the mathematical object.

Historically, the study of functions has primarily focused on symbolic notation. This has been due to a lack of ability to graph or compute tabular values quickly and

accurately. The NCTM *Principles and Standards* (2000) stress multi-representational approaches and the important role technology plays in teaching about functions.

Graphing without the use of computing technology, often referred to as “graphing by hand,” requires that a person creating a graph to already have an understanding of the behavior of the co-domain in relation to its domain. This is in order to accurately identify and subsequently graph changes in localized behavior such as maxima and minima or curvature.

Technology has the potential to fundamentally alter the order in which a learner develops a concept image for functions. Graphing software potentially inverts the historical sequencing of teaching functional representation or, at least, allows for the concurrent development of multiple representations in the mind of the learner. Software can create the graphical representation of a function which allows the user to investigate globalized functional behavior without any prior understanding of localized behavior. Indeed, teachers can now use technology-generated graphs as a pedagogical tool to help students investigate global functional behavior *in order to* develop a sense of localized behavior.

In their analysis of research on the teaching and learning of functions, Leinhardt, Zaslavsky, and Stein (1990) note that, “more than perhaps any other early mathematics topic, technology dramatically affects the teaching and learning of functions and graphs,” (p. 7). Teachers can utilize technology to have students make observations and conjectures within a variety of function representations such as equations, graphs, and tables. Students can then begin to make connections among the representations in order to develop a concept image without first having an in depth knowledge of function. With

the use of technology, teachers can expose students at a much earlier stage in their cognitive development to the function concept. This, in turn, allows students to explore the connections among representations enabling the learning of functions to become investigative in nature.

The NCTM supports this view of using technology to enhance student learning. As stated in one of the seven principles in the *Principles and Standards*, “Calculators and computers are reshaping the mathematical landscape, and school mathematics should reflect those changes. Students can learn more mathematics more deeply with the appropriate and responsible use of technology. They can make and test conjectures. They can work at higher levels of generalization or abstraction,” (NCTM, 2000, p. 25).

Software to Promote Understanding of Functions

Shodor, a company which promotes the integration of computational science in education, developed two graphing activities, *Data Flyer* and *Function Flyer* (Interactivate, 2007). These activities are available as java applets on Shodor’s *Interactivate* website. The intent of these graphing applications is to foster a global understanding of functions through dynamic linking among graphical and symbolic representations as well as output of tabular values. Shodor designed these applets to promote a learner’s ability to generalize behavior within a function family such as linear, quadratic, logarithmic, etc. These applets “allow the manipulation of the constants and coefficients in any function thereby encouraging the user to explore the effects on the graph of the function by changing those numbers,” (Interactivate, 2007).

By experimenting with changing parameters, the design also promotes an even broader generalization of function transformations amongst *all* function types. The user

can recognize patterns of shifts and stretches that exist across all function types (see Footnote 1).

Description of the *Flyer* Graphing Activities

The *Flyer* activities are web-based java applets belonging to Shodor's larger web-based project, *Interactivate* (2007). *Interactivate* contains a collection of over one hundred interactive activities supporting the teaching and learning of mathematics in the K-12 curriculum. The project also offers numerous support materials such as help files, lesson plans, worksheets, and discussions on mathematical concepts presented in the activities.

The *Flyer* activities offer the dynamic, visual aspects designed in the same spirit the NCTM suggests in its *Principles and Standards* (2000). From the *Technology Principle* (2000):

Using technological tools, students can reason about more-general issues, such as parameter changes, and they can model and solve complex problems that were heretofore inaccessible to them (pg 26).

As well,

Dynamic geometry software can allow experimentation with families of geometric objects, with an explicit focus on geometric transformations. Similarly, graphing utilities facilitate the exploration of characteristics of classes of functions (pg 26).

The *Function Flyer* java applet interface, shown in Figure 1, utilizes color coded slider bars to control parameters from a function entered by the user. The numeric values in the parameters change simultaneously with the graph, allowing the user to visualize the connection between the constants and coefficients in the function's symbolic and graphical form. The user may also choose to open the "Function Data" window to view tabular values produced by the function, although these values are not instantaneously

updated with the use of the sliders. Because of computational limitations, if the “Function Data” window is open, the user is prohibited from moving the sliders in order to prevent the tabular values from being updated dynamically.

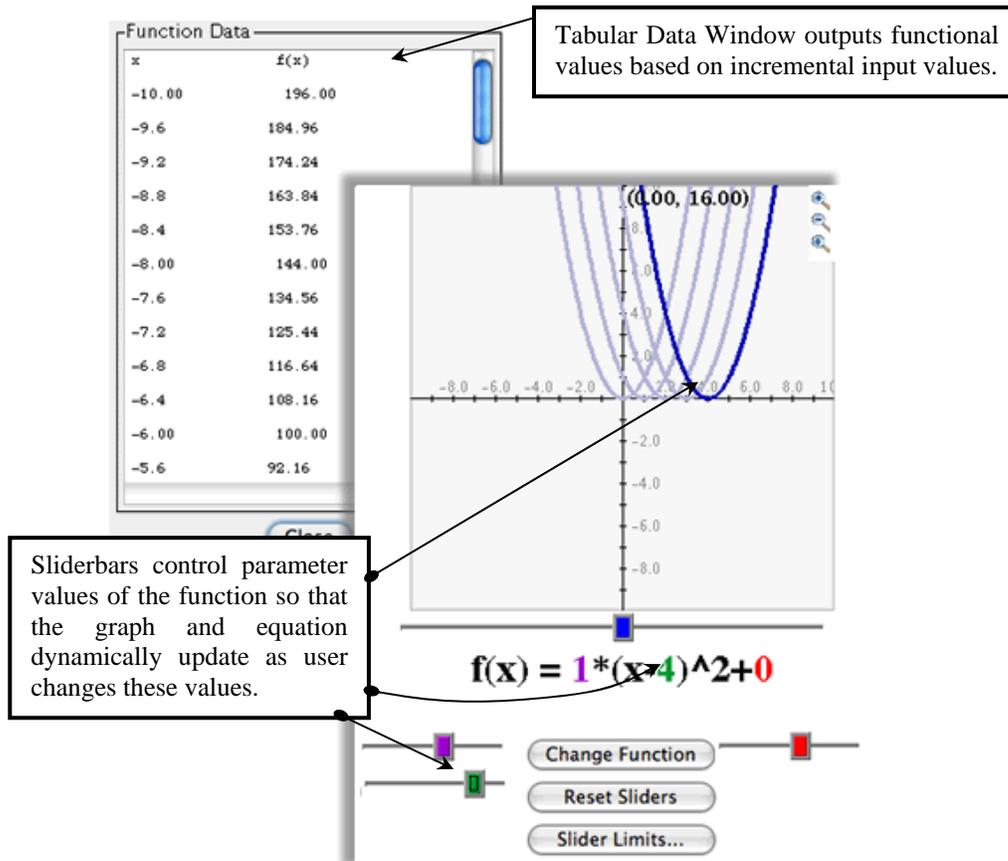


Figure 1. Function Flyer Interface

The *Data Flyer* interface, shown in Figure 2, is an expanded version of the *Function Flyer* interface that also allows the user to input and plot data on the same graph as the function, with the potential to manually fit a function to a data set. *Data Flyer* calculates and displays the individual squared deviations in a separate window and always displays the sum of the squares of the deviations in the main window. The user can then turn on the “Show Squares” option, which geometrically represents the

individual deviations by drawing squares on the graph corresponding to the “squared” vertical distance between the function values and the data. If the “Show Squares” feature is turned on, the squares dynamically update as the user changes the parameters of the function with the sliders.

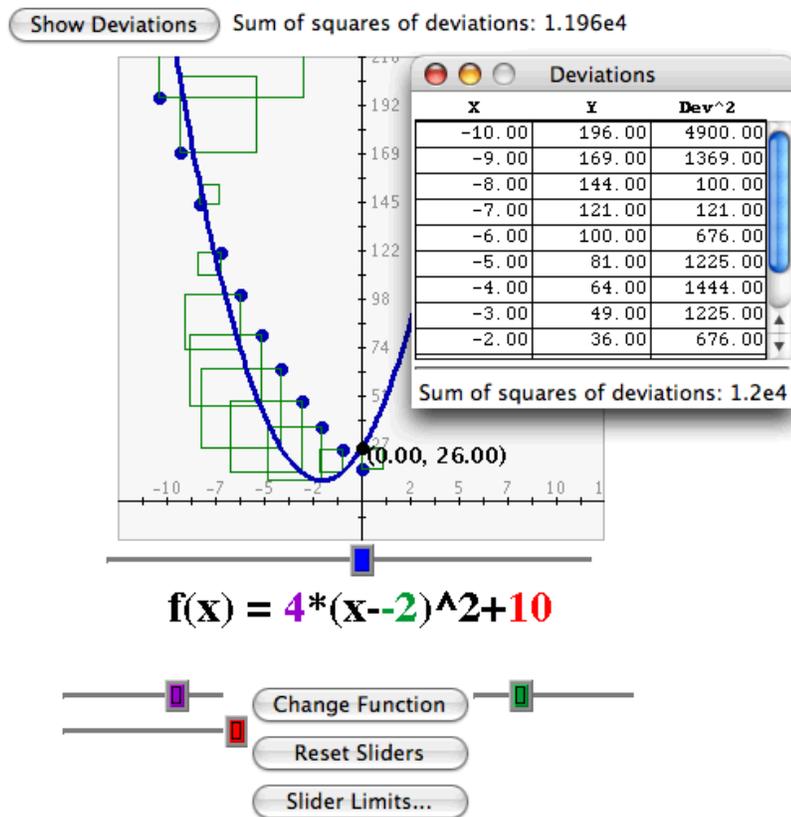


Figure 2. Data Flyer Interface

The interface also gives the user control of a number of other features in the activity. The controls for *Data Flyer*, shown in Figure 3, allow the user to manipulate the x and y min and max values; to change the max and min values controlled by the slider along with the sliders’ step size; to display vertical asymptotes; and to add grid lines to the graph. The “Show Trace” feature, shown in Figure 1, lightly marks previous

functions swept out from changing parameters with the sliders. The *Function Flyer* interface also contains all of these controls with the exception of the ones pertaining to input and output of data and subsequent features.

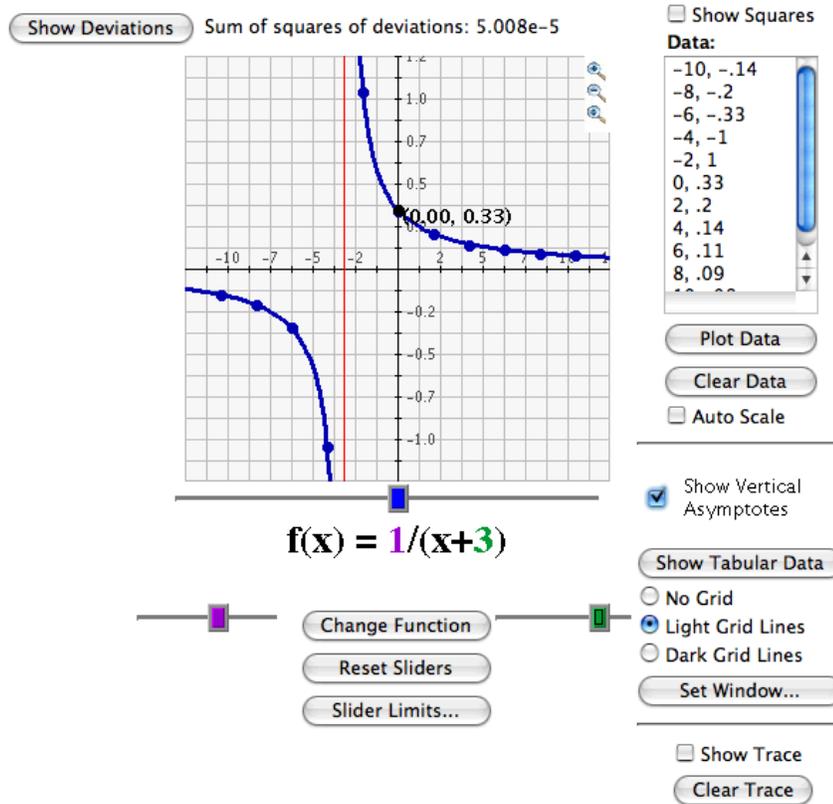


Figure 3. Other controls for *Data Flyer*

Problem Statement

Without goal directed action, open-ended graphing tools do little to facilitate student understanding of functions. Graphing tools can do little to facilitate a student's understanding of functions unless a teacher provides goal directed explorations. The

teacher can design and execute a lesson to leverage the power of the tools which focus student attention towards a specific learning goal.

As will be discussed in Chapter 2, a plethora of studies exists about the *effects* of technology integration into mathematics classroom instruction. Few studies investigate *how* teachers choose to implement particular kinds of technology into their lessons and how those choices influence student learning. This particular study qualitatively examines several teachers' implementations of the *Function Flyer* and *Data Flyer* graphing technologies into the lesson planning process.

Primarily, this study examines how teachers choose to utilize these specific java applets in their lessons with the purpose of gaining insights for future development of materials within the *Interactivate* site. Specifically, the research aims to (a) inform future development of support materials provided in *Interactivate* such as lesson plans, worksheets / labs, discussions; and (b) influence interface changes in the *Flyer* activities.

Secondly, and more generally, this study intends to provide insights on the manner in which teachers utilize graphing technologies to facilitate students' concept images when engaging in lesson planning and execution. Moreover, the research considers the potential influence of technology and teaching as bidirectional. It further explores how teachers leverage the properties of the technology in order to influence students' concept images of function and also examines how the technology influences teachers in their lesson development and implementation process.

CHAPTER 2

LITERATURE REVIEW

Graphing technology, as with all open-ended explorations and manipulatives, does little to improve conceptual understanding unless directed by well-defined goals. The teacher bears the primary responsibility for setting the didactic environment in order that students may uncover the patterns inherent to those manipulatives.

This study examines how teachers set the instructional environment for students to learn functions using a particular set of graphing tools. In order to frame this analysis, relevant literature is presented on the research and theory of how students develop the function concept. Potential effects of using graphing technologies for teaching and learning about functions are also explored.

Function Definitions

Historically, the definition of function was limited to the input and corresponding output of a two-variable equation (Malik, 1980). A function transforms numerical values taken on by the independent variable (the input) and through structured computations on that value, assigns the transformed value to a dependent variable (the output).

Towards the beginning of the 20th century, function was redefined in a more general, axiomatic sense. This modern conception, known as the Dirichlet-Bourbaki definition, defines mathematical functions as a correspondence between two non-empty sets that assigns to every element in the domain exactly one element in the co-domain, (Malik, 1980). Though the Dirichlet-Bourbaki conception of function is taught as the

formal definition in high school mathematics, most instruction focuses on functions described by a single mathematical equation, (Vinner & Dreyfus, 1989).

Because instructional focus has primarily remained on functions represented by equations, computing technologies are a natural extension for working with functions. Soon after Casio introduced the first graphing calculator into the market in 1986 (Japan Corporate News Network, 2007), the instructional climate for the teaching and learning of functions within the United States began to change substantially.

A number of graphing technologies now proliferates the mathematical education technology market and mathematics classrooms. *Texas Instruments, Inc.* offers a wide range of graphing calculators for use from middle school through the undergraduate curriculum. *Vernier Software and Technology, Inc.* produces a wide variety of probes for data collection to be used in conjunction with the calculators to create graphs of real data collected by devices such as motion detectors, light intensity meters, thermometers, and heart rate monitors. Graphing and function based software include spreadsheet applications, interactive java applets, *Geometer's Sketchpad* (Jackiw, 2001), *Fathom* (Finzer, 2001), *SimCalc* (Mathematics Education Research Group, 2007), *Function Probe* (Confrey, 1991), *Green Globes* (Dugdale & Kibbey, 1997), and *Function Sketcher* (Yerushalmy & Shternberg, 1998).

Graphing Technologies in the Classroom

Technology and mathematics go hand in hand. Ever since the introduction of computing technologies in the classroom, mathematics educators recognized their potential for teaching their subject. Computers can take the tedium out of computation

when the computation is not the point of the lesson, thus helping students to see emergent patterns without being bogged down by peripheral detail.

A number of studies have examined the use of graphing technology for learning about functions. Doerr and Zangor made in-depth observations and an analysis of a pre-calculus class that extensively used the graphing calculator. They found that this technology was instrumental in facilitating analytical thinking. The teacher designed the instructional tasks involving a variety of functions where students were required to describe patterns, represent data, and make and justify generalizations.

In regards to the ease in which functions can be graphed and manipulated with software, Dugdale (1993) finds that these tools have, “raised the possibility of visual representations of functions playing a more important role in mathematical reasoning, investigation, and argument. Relationships among functions can be readily observed, conjectures can be made and tested, and reasoning can be refined through graphical investigation,” (pg. 115).

This easy manipulation of a function’s various representations can help students construct their own internal, flexible images of function. Dynamic software can help students construct such images. Yerushalmy and Chazan (1990) found that through a dynamic geometry environment students were able to reason more flexibly about geometric concepts than their counterparts who learned with static diagrams. Similarly, Moschkovich, Schoenfeld, and Arcavi (1993) explored student learning of functions with the dynamic software, GRAPHER, and found that such an environment “allows students to operate on equations and graphs as objects... in ways not possible before the existence of such technologies,” (pg. 98). In this research study, they also found that it is not just

the dynamics on the screen that makes the difference in student learning. Students must integrate what they see on the computer screen into their own conceptual structures for any learning to take place.

Vinner and Dreyfus (1983) identify a learner's internal representation of a mathematical construct as a *concept image*. Tall (1989) examines the potential of well-designed software to improve a learner's concept image of function by allowing the learner to explore the complex structures of functions. The software allows students to experience higher level cognitive structures than they would be able to without such software.

Tall (1989) asserts that there is a danger in teaching mathematics by building from the simple to the complex. When presented with simplified versions of complex mathematical concepts, students tend to build their concept images from these contexts. "These deeply ingrained cognitive structures can cause serious cognitive conflict and act as obstacles to learning," (Tall, 1989, pg 37). He suggests students can more readily explore complex concepts through the use of technology, without first needing to build from the simple to the complex.

Dugdale (1993) concurs with Tall. Due to her own research on the use of technology to support student thinking with functions, as well as reviewing others' research, Dugdale states, "Such tools have facilitated the movement away from a focus on calculating values and plotting points toward a more global emphasis on the behavior of entire functions, and even families of functions," (pg 114). Hence students can study the more complex global aspects of function prior to or in parallel with studying functions as input / output machines.

Because mathematical ideas are often introduced to students in simplified contexts, students form inflexible concept images that rely on those simplifications. For instance, in a traditional algebra curriculum, single-variable linear equations are usually introduced prior to two-variable equations. Students are expected to solve single-variable equations to find the solution which makes it true. Two-variable linear equations are presented after much energy has been spent on solving a variety of the single-variable equations. Graphing two-variable equations comes even later in the curriculum sequence. Graphs are often developed from graphing individual coordinate pairs produced from an input / output table. The linear model becomes the learner's concept image for functions.

Indeed, through their research, Schwarz and Hershkowitz (1999) demonstrated students' propensity for linear models when it was inappropriate for the task at hand because of their linear concept image of function. Even though the students in the control group knew the correct definition of function, they were unable to apply it. In an attempt to foster a more robust concept image, students in the experimental group used software that dynamically linked equations, tabular values, and graphs of functions. The instructional model focused on inquiry within small groups, allowing students to choose which function representations and the linkage of those representations when solving problems. Students were also required to write and share reports to compare their solutions to other classmates. Those in the experimental group demonstrated improved use of appropriate functions beyond the linear model.

In Tall's (1986) research, he examined understanding of tangency with regards to functions and students' concept images. Tall found that students' concept image of

“tangent” related to a line touching the circle at a single point. Students taking calculus relied on this image when identifying a tangent line to a function at a given point and regularly concluded that if a tangent line intersected the function in more than one point the tangent could not exist. Using specially designed software, students in an experimental group graphically explored tangency as related to function, with instruction focused on teacher / student discourse. In this group, Tall found marked student improvement in concept imaging when compared to the control group.

However, when computers are used for simulation of mathematical constructs, there are still opportunities for misinterpretation. Goldenberg (1988) identified student difficulties with graphing technologies due to issues of scale. Students unfamiliar with notions of window sizing, scaling on the axes, and global behaviors experienced difficulties in using the technology to understand graphs. Moschkovich, Schoenfeld, and Arcavi (1993) found students interpreted the pixilation of a line represented on a graph as an actual property of the line.

Students must make sense of whatever they see on the computer screen and internalize that understanding with their already existing constructs of and internalize with their already existing constructs. Tall (1989) identifies three types of *insights* used to make sense of what is seen and experienced through the use of educational software. *External insight* occurs when the user does not know how the software was programmed or the algorithms it uses yet can use prior knowledge to check for validity of the output. When the user does have a sense of the algorithms the computer uses, such as a Riemannian sum for approximating integrals, Tall defines that as *analogue insight*. Finally, *specific insight* occurs when the user understands the programming language

used to create the software. Tall suggests that both teachers and students must have external insight for using software for learning and that it is helpful, though not necessary, for teachers to have a notion pertaining to analogue insight.

Multiple Representations and the Learning of Functions

In the study of functions in traditional high-school algebra, primary focus is given to two-variable equations with a given output for a particular input. Both Rider (2004) and Eisenberg and Dreyfus (1994) showed student preferences for this symbolic form of function when solving problems even when this form is not a convenient way to find a solution. Because of the widespread availability of graphing and computing software, mathematics educators are now able to leverage the use of a variety of functional *representations* for teaching, the most common of which include tables, equations, and graphs.

Much attention in the literature has been devoted to what is commonly known as the use of *multiple representations* in the teaching and learning of functions. (Heibert & Carpenter, 1992; Moschkovich, Schoenfeld, and Arcavi, 1993; DeJong et al, 1998; Porzio, 1999; Rider, 2004). This research shows improved student ability to use and apply different representations when solving function related problems when the instructor uses multiple representations of functions concurrently within the lessons. Moschkovich, Schoenfeld, and Arcavi (1993) developed a framework in which to analyze students' understanding of function. Their framework examined students' reasoning about functions from two perspectives, resulting in a two dimensional matrix shown in Table 1 below. The first perspective analyzed students' ability to move flexibly among graphical, tabular, and symbolic representations and being able to call upon other representations

where a particular representation did not provide sufficient information in solving a problem. The second perspective, which will be subsequently discussed at greater length, involves students' ability to view a function as a process of inputs and outputs and to view a function holistically as an object.

Table 1: Function Analysis Framework (Moschkovich, Schoenfeld, & Arcavi, 1993)

	Equation	Graphical	Tabular
Process			
Object			

Relying on the framework from Moschkovich et al (1993), Knuth (2000) examined students' ability to associate points on a line with the linear function itself. Knuth found that the students who relied almost exclusively on a single representation struggled more to provide proper reasoning about problems related to linear functions than those who used multiple representations.

In her research, Rider (2004) showed improved student problem solving and improved reasoning about functions when using a multi-representational approach. Rider compared a control group of students who were taught using a traditional algebra curriculum to students in an experimental group using a multi-representational approach. On the post-test, students in the experimental group clearly outperformed those in the control, whereas the pre-test showed students began the course at comparable ability levels.

For students to be competent, though, in translation between representations, they must internalize these constructs and understand that the representations are

representative of a *single* mathematical object. Schwarz and Dreyfus (1995) assert that compartmentalization arises in the translation among function representations because students do not comprehend these representations as different forms of a single object. Rather, students will tend to view these forms each as different mathematical objects in and of themselves as opposed to the same object in different forms. Thus, students only superficially connect these representations and do not find contradiction if the underlying structure of the forms do not align.

As well, the use of external representations of functions without carefully designed lessons to ground the representations in a context familiar to the learner does little to improve comprehension of the function concept. In an in-depth analysis in 1994, Schoenfeld, Smith and Arcavi found that a college student who was well-versed in connections among representations remained yet quite disconnected from other concepts.

Kaput (1989) and Thompson (1994) also stress caution about relying too heavily on the use of multiple representations for teaching functions without connection to student experience. Thompson (*ibid*, pg 23) states, “I agree with Kaput that it may be wrongheaded to focus on graphs, expressions, or tables as representations of function. We should instead focus on them as representations of something, that, from the students’ perspective, is *representable*, such as aspects of a specific situation...” and furthermore “[t]he situation being represented must be paramount in students’ awareness, for if they do not see something remaining the same as they move among tables, graphs, and expressions, then it increases the probability that they will see each as a ‘topic’ to be

learned in isolation of others.” Kaput continues to echo these sentiments in later years,

Just because representations are linked, if there is no connection to other knowledge in the learner’s schema / experience the linked representations are just as meaningless. Students may understand cause and effect relationships within the representations yet still not have a mental image of what any of the representations, linked or not, mean (pg 7, Kaput, 1998).

Van deWalle and Lovin (2006) also agree. In addition to what Kaput names as the “big three” (1998) Van deWalle and Lovin find two additional function representations: contextual and verbal. The context grounds the learner within his / her cognitive structure or personal experience. A verbal description allows the learner another route in which to understand the symbolism, relating the independent and dependent variables. Van deWalle and Lovin (2006) assert students should be able to see a connection among all five of these representations for any given function.

In lesson planning, the teacher makes premeditative choices in developing the context and then, during lesson execution, makes spontaneous choices to direct student attention to specific aspects of the situation being represented. The teacher bears responsibility to “weave a *representational context*” [original italics] (Ball, 2002, pg. 4) and act as arbiter in choosing how students interact with unfamiliar concepts. Throughout the lesson, the learning domain must remain relatively familiar to the learner, enabling him or her to extend and make sense of new concepts in relation to previously internalized cognitive schema, (Lampert, 1989).

Action, Process, Object, Schema (APOS) Framework

As noted earlier, functions, like many mathematical concepts, can be understood as both a process as well as a mathematical object. As a process, a function takes an input value, transforms that value using a specified rule, and from the transformation,

assigns it to a new value designated as the output. As an object, a function is the collection of all possible unique input values paired with all of their subsequent output values. A learner with a robust conception of function will be able to call upon both of these views of function as needed.

Based on the work of Dubinsky and Harel (1992), Asiala et al (1996) developed the *APOS* framework describing how a learner constructs mathematical knowledge. The acronym, APOS, is derived from the single-word descriptions given to each of four progressions; *action*, *process*, *object*, *schema*, describing this mathematical construct and integration into cognitive structure.

Action. During the *action* stage, the learner applies one or more external actions or rules to utilize the mathematical construct. In an example applied to the learning of linear functions, a student may know how to find the slope of the function by looking at the equation of a line, solving for the dependent variable, and writing down the value of the coefficient multiplied by the independent variable. In the action stage, the learner exhibits what Skemp (1978) refers to as relational understanding where he / she does not have a sound notion of why the actions work but can apply those actions in order to find a correct solution. If the problem situation requires multiple steps, the learner moves forward a single step at a time unaware of the steps in their entirety, i.e. unaware of the overall *process*.

Process. In order to move from an *action* conception into a *process* conception, the learner must actively and consciously reflect upon the actions performed in order to garner meaning behind those actions. Utilizing process conception, the learner has internalized the actions required to operate on, or with, the mathematical object though is

yet unaware of the construct as an object. At this stage, the learner is aware of all the steps required to for the construct but does not need to consciously think about the steps in order to perform them. The learner has immediate access to any step without actually having to perform each action.

A process view of function allows for the reversibility of actions. For example, in learning about the concept of slope pertaining to linear functions, the student would be able provide a variety of linear functions from a given slope. The learner views slope as a property of linear functions in general and is not concretely bound to a specific function in order to identify the slope.

However, the student with a process view cannot yet treat the linear function as a holistic object. In the case of slope, the student cannot envision the consequence of varying slope and how that variation dynamically transforms all of the outputs from the function's domain.

Object. When operating with an *object* view, a learner encapsulates the process into a unified entity. The mathematical idea has now been grounded in the mind of the learner as an object on which he or she can operate. The learner can now anticipate the effects of a transformation on the object without the need of stepping through the process of the transformation.

To continue with the example of slope and linear functions, the learner now anticipates the effect of changing the slope on the function. This includes the effects with regards to the function's structure so encompasses all representations. Given tabular values, symbolic form, or a graph of the function the learner can predict and visualize this transformation when operating within an object view. These various representations are

all viewed as different aspects of the same mathematical structure. Thus the learner understands that a variation in the slope on a graph, for example, creates changes within the other representations of the function because all of the representations are simply manifestations of a single object.

It is important to note, however, that the object can be transformed back into a process when necessary. When using a mathematical object, it often becomes necessary to deconstruct the encapsulated object and transform it back into a process in order to use the process. For example, a problem situation may require identification of a specific output for a given input of a function. If the learner recognizes and is able to switch back into a process mode, the learner has achieved a process/object duality (Sfard & Linchevski, 1994).

Sfard & Linchevski (1994) describe the capacity to flexibly move between process and object as *versatile* and *adaptable* thinking:

In certain circumstances a person may display his or her ability to see an expression as a process, in another context he or she may view it as the product of this process, and in still another situation as a function. One would say, therefore, that the versatility of his or her outlook is quite impressive. This, however, does not necessarily mean that the person will always be able to adapt the perspective to the task at hand, (p. 99).

Thus the thinking process of an individual who could view a construct as process or object given a particular situation would be characterized as versatile. Whereas the ability to apply the particular properties of the construct as process or object depending on the situation would be characterized as adaptable.

Schema. A *schema* encompasses all of an individual's *actions*, *processes*, and *objects* associated with a mathematical concept. The schema is a web of interconnected

ideas within ones cognitive structure. With regards to the slope example, the concept is now subsumed into the learner's overall linear function schema.

Dubinsky and McDonald (2001) compare a schema to concept image, "because... all mathematical entities can be represented in terms of actions, processes, objects, and schemas, the idea of schema is very similar to the concept image..." (pg. 3). However, they also differentiate schema from concept image because a schema must be, "coherent in the sense that it gives, explicitly or implicitly, means of determining which phenomena are in the scope of the schema and which are not," (pg. 3).

It is the nature of mathematics that structures are combined and compressed to form new objects. In this manner, once an individual reflects and acts upon these schemas, the schemas themselves become objects.

Objects can be transformed as higher level actions, leading to new processes, objects, and schemas. Hence we have a mechanism which may be envisioned as a spiraling of action, process, and object, within expanding schemas (Sfard & Linchevski, p. 99).

So, though these four conceptions, *Action, Process, Object, Schema*, are hierarchical to a degree, they are not discrete, (Dubinsky & McDonald, 2001). One view precedes the other, but there remains an oscillation between two views as the learner incorporates more complicated pieces of a concept into the schema obtaining the spiraling effect.

Furthermore, if a learner compartmentalizes objects which are actually parts of the same structure, it also raises the possibility the learner can understand an object in one form yet only have a process view of another. In working to avoid this compartmentalization, Schwarz and Dreyfus (1995) distinguish between a concrete representation of an object, which they call a *representative*, and the setting in which the

object is represented, which they name as the *representation*. They are careful to make this distinction because, similar to the *Action* of APOS they also recognize that,

concept acquisition is intimately linked to actions. The lack of such actions in certain setting implies that properties may not be seen as invariant through several settings. Such properties may, for the learner, become features of formal objects (graphs, formulae or tables) rather than properties of the concept itself. This is why the above compartmentalization of students' knowledge occurs, (pg. 264).

A number of researchers have applied the APOS framework to analyze student thinking about functions and related concepts. Dubinsky & McDonald (2001) cite over fifty articles in an annotated bibliography of research which use the APOS framework. As mentioned earlier in this literature review, Moschkovich and colleagues (1993), as well as Knuth (2000) examined student conceptualizations in conjunction with a multi-representational approach. Laige and Gaisman (2006) apply the framework to understanding transformations of functions.

Teaching Functions

To be effective, the teacher must not only have a well-mapped and accurate function schema, but must also grasp the roots of students' schemas, however well or malformed. Additionally, the teacher must understand a myriad of pathways that may shape the concept within the minds of his or her students.

As noted earlier, the nature of mathematics is that of a continuing abstraction and compression of ideas into objects which are integrated into a larger network of schemas.

Teachers must work to unpack these mathematical objects for their students. Ball and Bass (2000) refer to this as the ability to “decompress” the content.

...most personal knowledge of subject matter knowledge, which is desirably and usefully compressed, can be ironically inadequate for teaching. ... Because teachers must be able to work with content for students in its growing, not finished state, they must be able to do something perverse: work backward from mature and compressed understanding of the content to unpack its constituent elements, (pg. 98)

Hill, Rowan, and Ball, (2005) have given a good deal of attention in their research to this type of specialized teacher knowledge, what Schulman (1986) identified as pedagogical content knowledge. Teachers must be aware of their own internal constructs in order to communicate those ideas to their students. Teachers must also tease out their students’ cognitive structures in order to build upon them. In doing so, teachers deconstruct compacted mathematical objects and communicate about schemas to relay this interconnected web of related mathematical ideas to their students.

With regards to teaching functions using graphing technologies, Simmt (1997) examined the impact of the graphing calculator on how teachers design and implement lessons with it. As noted earlier, many educators and researchers herald the potential for graphing technology to fundamentally change how mathematics is taught. However, the bulk of the research on functions with graphing technologies examines how students learn with it rather than examining how teachers choose to teach with it.

Simmt found that teachers used the graphing calculators as an extension of their philosophies of teaching and beliefs about mathematics. She concluded that teachers use technology in parallel with their already existing paradigms, and that for technology to have any real impact, these philosophies also must change to incorporate the power of the tools.

Theoretical Framework Synopsis

This study examines teachers' intentions for promoting student thinking about functions via their lesson planning process and reflections on those lessons. How teachers intend to facilitate student thinking will be considered through APOS and concept images. Both APOS and concept images provide the framework for how learners internally conceptualize a mathematical construct. However, because student thinking is not observed in this research, the framework also incorporates an observational component. This component is used to examine the pedagogical context the teachers set through their learning goals, function definitions, and function representations and linkages among them.

With regards to the conceptualization of the function concept, Vinner and Dreyfus (1989) theorized that students call upon concept images when applying their knowledge to a mathematical task. Asiala and colleagues (1996) theorized that, when learning about and applying the function concept, a person operates in one of several modes: action, process, or object. When operating within the action mode the learner has yet to form any true understanding of the concept.

To flexibly use functions in mathematics, the learner must possess the ability to work with functions within both a process and an object mode. Hierarchically, a learner, typically acquiring a process view prior to understanding function as an object, tends to oscillate between these views as the concept becomes more complex.

A given mathematical task may lend itself to working with a function as a process or as an object. Within a given context, a person who operates on a function as either a process or object is characterized as a versatile thinker. A person who ably moves

between a function as process or object, depending on the appropriateness within a given context, is characterized as an adaptable thinker. Therefore the student who, given a mathematical task involving functions, actively chooses either a process or object mode, also must have an adaptable concept image to the given task. Figure 4 synthesizes these conceptual processes.

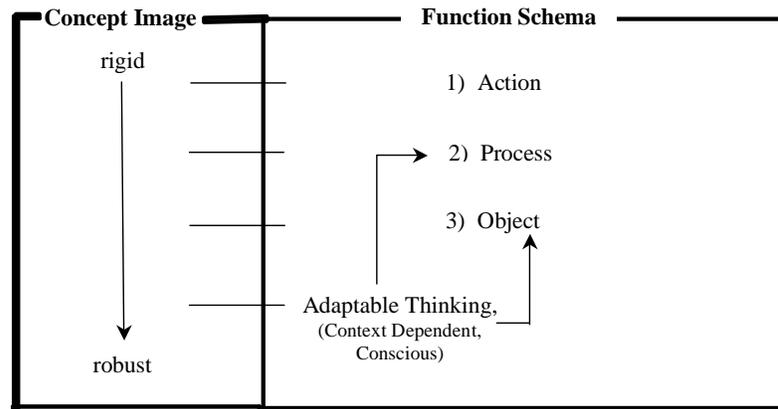


Figure 4: Function Framework Synthesis

With regards to the pedagogical context, the function framework described above can be observed through teachers' chosen learning goals, use of function definitions, and use of function representations. A teacher's choice of a particular lesson's learning goals will delimit possible student activities for that lesson. In turn, the learning goals in which a teacher chooses indicate what that teacher deems important for student learning. Thus, studying teachers' chosen learning goals provides insight regarding their intentions for the development of the function concept.

Examining which aspects of the function definition a teacher chooses to emphasize may be an indicator of the development of their students' concept images.

The more robust a learner's *concept image* of a mathematical construct, the more accurately that image will map to the definition. The robust concept image will be constrained by a definition's limits yet be flexible enough to include all of the construct's permissible properties. Subsequently, an examination of how a teacher intends to use the definition of function yields information on how that teacher may work to develop students' concept images.

Lastly, observing how a teacher chooses to incorporate function representations and links among those representations will help discern how that teacher develops students' understanding of function within the *APOS* framework. The utilization of function representations within the context of teachers' lessons demonstrates their intentions of how they encourage students to think about functions and how they want students to operate with and on them. For students to understand a function as an *object*, they need to view the representations of functions as different views of the same construct. For students' thinking to be *adaptable*, they need to readily move from *process* to *object* and vice versa, requiring fluid translation among representations.

Research Question

The research framework discussed is used as a lens to examine teacher intentions of developing students' concept images towards an adaptable understanding of function as process and object. Specifically, this study investigates teachers' planning and reflections to answer the following research question:

- What is the nature of how teachers who instruct courses beyond Algebra I intend to use the *Flyer* tools for the development of their students' understanding of functions?

CHAPTER 3

METHODOLOGY

To answer the research questions, three North Carolina mathematics teachers, currently teaching courses involving the study of functions beyond Algebra I, agreed to participate in the study. The researcher collected and qualitatively analyzed the data. This chapter describes participant selection and the context in which the data were collected. The instruments for, and methods of, data collection, and the methods of analysis are subsequently described in light of the research questions.

Participants and Context of the Study

Participants in the study included three high school teachers who participated in the Advanced Functions and Modeling Workshop developed and offered by faculty from the North Carolina School of Science and Mathematics (NCSSM) in summer of 2006. NCSSM developed and began offering this workshop in the summer of 2004 to help prepare teachers for the Advanced Functions and Modeling course, introduced into the North Carolina curriculum beginning fall of 2004. One of the concurrent sessions in the week-long workshop highlighted Shodor's *Flyer* tools contained in *Interactivate*. Hence, all participants had some training with the use of these tools.

The researcher discussed the study with teachers in the workshop and requested volunteers (see Appendix A). Of those that volunteered, four were purposely chosen as participants in the study based on their teaching schedule for the Fall. Of the four teachers chosen as participants, only three completed all requirements of the study.

Data Sources and Methods of Collection

The main data sources of interest include artifacts in the form of two lessons that were developed, implemented and then reflected upon by each participant. Lessons were chosen as the central data source because lessons provide for an unobtrusive method of gathering data on how teachers choose to implement the *Flyer* tools viewed through the theoretical framework. The context of data collection therefore took place in a natural, realistic setting for the teachers.

To provide additional context and clarification on the lessons, two other sources of subsidiary data were collected in the form of an initial, in-person interview between the researcher and participant and a phone based exit interview. The two supplementary data sources provided *triangulation* (Merriam, 1998; Stake, 1995), a qualitative research method in which data is collected and analyzed concurrently to assist in checking for consistency and redundancy in patterns that the researcher notices throughout the data. A description of the data sources is presented, followed by an explanation of how the data were analyzed in relation to each other.

Initial Interview

The initial interview was designed by the researcher to query the participants' beliefs regarding the teaching of functions, the teaching of mathematics with technology, and the teaching of functions with technology. The interviews provided background information with regards to each participant's goals of instruction when teaching about functions in order to examine how those beliefs may play a part in the design and implementation of the lesson. The interviews also provided a context of available technology at each school.

The researcher traveled to each participant's school to conduct the interview during October 2006. Each interview was digitally video recorded and later transcribed. The in-person interview also helped to establish rapport and trust between the researcher and participants. The interviews took place in thirty to forty-five minutes periods. Appendix B contains the interview questions.

Lesson Plan Artifacts

In order to compare lesson elements among participants, a lesson plan template was provided. The standardized lesson template provided consistency across lessons for the analysis. Elements of the template included: goals of the lesson; objectives; *North Carolina Standard Course of Study* objectives; prerequisite student knowledge; preliminary preparation; materials; lesson description; anticipation of student thinking; assessment. As part of the template, teachers were asked to respond to reflection questions after implementing the lesson. See Appendix C for the lesson template.

The participants were given electronic copies of the template prior to the initial interview and hard copies at the time of the interview. The researcher reviewed the template with each participant at that time to explain the template and clarify expectations of what each element from the template should contain.

During the study, each teacher developed two lesson plans focused on a function concept from a course beyond Algebra I. The courses included Algebra II, Advanced Functions and Modeling, and Trigonometry. The Advanced Functions and Modeling course is designed as a fourth year of mathematics for students in high school beyond Algebra II who are not yet mathematically prepared to take Pre-Calculus. The course focuses on the use of mathematics, particularly functions and data, for modeling real-

world phenomena. The participants had the option of choosing the instructional mode of tool use within the lesson; either as a demonstration tool using a single computer and projection device or in a computer lab. Table 2 presents the context for which data was collected by participant. Pseudonyms are used to protect the privacy of the participants.

Table 2: Data Collection Context

Participant Name	Number of Completed Years Teaching	<i>Course</i> , Lesson 1 Topic	<i>Course</i> , Lesson 2 Topic
Max	3	<i>AFM</i> , Power function transformations	<i>AFM</i> , Line of best fit
Carl	4	<i>Algebra 2</i> , Parabola transformations	<i>Algebra 2</i> , Synthesis / review of transformations
Wendy	3	<i>Algebra 2</i> , Function inverses	<i>Trigonometry</i> , Fitting data with sine function

Exit Interview

The exit interviews probed for reflection in order to further examine participants' reactions on the use of the *Flyer* tools within the context of the lesson implementation beyond the written reflections. The exit interview also sought clarifications of the lessons. These interviews were conducted over the phone during February 2007, after participants had written, implemented, and reflected on both lessons. Prior to the exit interview, the researcher reviewed the lessons and initial interviews in detail and developed individualized questions and written memos for later analysis.

The exit interview consisted of both prepared items asked of all participants and items specific to each lesson developed during the review of the previous data sources. The researcher took written notes as participants provided responses. Although the exit interviews were not recorded, the researcher verbally read the participants' responses back to the participant to ensure accuracy. Each interview lasted approximately thirty to forty-five minutes. See Appendix D for the exit interview questions.

Data Analysis Methodology

After examining a number of possible qualitative analysis design methods, the researcher chose to use *Interpretive Analysis* as described by Erickson (1986). Interpretive analysis emphasizes the analysis of *meaning perspectives* of those persons on whom data is collected in regards to both formal and informal modes of social organization (Erickson, 1986). A person's meaning perspective is characterized by "the creation of meaningful interpretation of the physical and behavioral objects that surround them in the environment," (p. 126).

Within the context of this study, the analysis focuses on the formal modes of how the *Flyer* tools are situated within the lesson design in light of the research questions. The analysis then examines the informal mode, via the participant reflections, in order to understand the teachers' interpretation of the physical object, i.e. the *Flyer* tools, to facilitate student understandings of function. The lessons and written reflections serve as the basis of the analysis with the initial and post interviews serving as a wider context and resource for what Erickson calls "sense-making," (1986). Similar to triangulation, sense-making connects and integrates the various sources of data into a unified whole.

To avoid confusion, it is important to emphasize that the variable of interest is the teachers' meaning perspectives of the tool use for meeting learning goals, not the students' perspectives. Thus, the analysis examines the data to identify key aspects of how the teacher interprets the *Flyer* tools usefulness for teaching about functions and chooses to use the *Flyer* tools within the lesson design and implementation.

The interpretive methodology does not use a coding scheme in order to analyze the data, as qualitative methods often suggest. Rather, hypotheses are formed and revised in an ongoing, iterative manner as the data sources are examined numerous times. Observations and patterns are noted *as the data are collected* and these observations are continually reviewed, as well as the raw data sources, throughout the collection process. The researcher is charged to seek and present both *confirming and disconfirming* evidence in order to continually refine or possibly reject the initial hypothesis (Erickson, 1986). Once these hypotheses are refined and accepted, they take the form of *assertions* for presentation in the analysis.

CHAPTER 4

RESULTS

This chapter provides a detailed analysis of the data. As noted in the previous chapter, the data are analyzed through the use of assertions. Each assertion is stated and then discussed in light of the research questions and literature base. A total of seven assertions were identified. To provide further context for the assertions, a restatement of the research question is provided.

- What is the nature of how teachers who instruct courses beyond Algebra I intend to use the *Flyer* tools for the development of their students' understanding of functions?

Each of the three teachers in the study completed two lessons. A synopsis of each of the six lessons appears in Table 3 through Table 8.

Table 3: Lesson on Inverse Functions

<i>Teacher</i>	Wendy
<i>Course</i>	Algebra II
<i>Learning goal</i>	Discover graphs of inverses for <i>constant, linear, quadratic, cubic,</i> and <i>exponential</i> functions.
<i>Tool</i>	<i>Data Flyer</i>
<i>Mode</i>	Computer lab
<i>Context of tool use</i>	Tool is used to graph functions and then graph inverted ordered pairs so students can visualize the shape of the inverse graph.
<i>Lesson activities overview</i>	<p>Students were provided with a worksheet with five equations, one example from each function family listed above. For each equation students were to</p> <ul style="list-style-type: none"> • graph the function using <i>Data Flyer</i> • identify ordered pairs from the function and record the ordered pairs in a table on the worksheet • reverse the x and y values to create an inverse table • graph those ordered pairs using <i>Data Flyer</i> • record the shape of the graph • develop the equation for the new graph
<i>Self-reported Results</i>	Wendy resorted to direct instruction because students had difficulty grasping how to use the technology in conjunction with the worksheet provided.

Table 4: Lesson on Sine Curve Transformations and Data Fitting

<i>Teacher</i>	Wendy
<i>Course</i>	Trigonometry
<i>Learning goal</i>	Reinforce generalizations of parameter changes on the graph of the <i>sine</i> function.
<i>Tool</i>	<i>Data Flyer</i>
<i>Mode</i>	Computer lab

Table 4 (continued)

<i>Context of tool use</i>	Tool is used to graph data with a sinusoidal appearance and model the data with the <i>sine</i> function.
<i>Lesson activities overview</i>	<ul style="list-style-type: none"> • Students were provided with the general form of the sine function required for input into <i>Data Flyer</i> so that sliders are produced for each parameter. • Students were provided with the data to input into <i>Data Flyer</i> • Wendy instructed students on how to use the “Slider Limits” window to adjust step sizes and bounds for the sliders. • Students were expected to find what they thought appeared to be the best match between the function and the data.
<i>Self-reported results</i>	Students successfully transformed the <i>sine</i> function to match the data though many were initially bothered that the data did not match the function exactly.

Table 5: Lesson on Least Squares Line

<i>Teacher</i>	Max
<i>Course</i>	Advanced Functions and Modeling
<i>Learning goal</i>	Develop an intuition for creating a least squares line from a data set.
<i>Tool</i>	<i>Data Flyer</i>
<i>Mode</i>	Single computer & projector
<i>Context of tool use</i>	Tool was used to try and fit a line using the sliders to student collected data.
<i>Lesson activities overview</i>	<ul style="list-style-type: none"> • Whole class review of regression terms through providing results of an investigation to students and discussing those results. • Students measured their arm lengths and heights. Max input the data into <i>Data Flyer</i> as it was collected. • After the data were graphed, Max engaged the students in a whole class discussion on what function best fit the data. After students decided on linear, discussion ensued on how to move the sliders to best fit the data to develop a line of best fit. Interpretation of the slope and intercept was also discussed. • Using their graphing calculators, students used the data to calculate the least squares line and compare it to the line they believed was a best fit from <i>Data Flyer</i>.
<i>Self-reported results</i>	The students were able to discuss slope and intercept in terms of the graph. Max thought students had a better grasp of minimizing residuals through changing the slope and intercept.

Table 6: Lesson on Power Function Transformations

Teacher	Max
Course	Advanced Functions and Modeling
Learning goal	Generalize effects of parameter changes power functions graphs.
Tool	<i>Function Flyer</i>
Mode	Computer lab
Context of tool use	Tool was used to analyze and generalize how the shape of a function's graph changed as numeric parameters changed.
Lesson activities overview	<ul style="list-style-type: none"> Reviewed homework problems that involved examination of similar graphs and subsequent equations in order to form generalizations about transformations. Students graphed $f(x)=x^3$ and then experimented by using the sliders changing the exponent, sketch resulting graphs in their notes and generalize results. Students explored on their own but were also asked to investigate specific functions, graphing and identifying the equations from verbal descriptions.
Self-reported results	Students explored a number of functions at their own discretion and were engaged during the lesson.

Table 7: Lesson on Parabola Transformations

Teacher	Carl
Course	Algebra II
Learning goal	Generalize effects of parameter changes on the graph of parabolic functions.
Tool	<i>Function Flyer</i>
Mode	Computer lab
Context of tool use	Analyze and generalize how the shape of a parabola's graph changed as the parameters changed.
Lesson activities overview	<p>Students were provided with a worksheet with six examples per equation of the forms: $\pm ax^2$, $\pm(x+h)^2$, $\pm x^2+k$ respectively.</p> <ul style="list-style-type: none"> For each example students were to sketch the graph; provide the value of a; direction of movement of the parabola from the x axis; determine if the vertex was the "highest" or "lowest" point in the graph; the axis of symmetry, and determine if the graph was "narrower" or "wider" compared to the graph of x^2. After responding to these items per equation type, students were asked a series of questions in order to generalize about how each parameter affected the graph.
Self-reported Implementation results	Carl thought this activity could have been done with a graphing calculator but <i>Function Flyer</i> provided instantaneous feedback which in turn helped the students make the generalizations easier.

Table 8: Lesson on Transformations across Function Families

Teacher	Carl
Course	Algebra II
Learning goal	Generalization of transformations across families of functions: <i>linear, quadratic, square root, exponential, absolute value.</i>
Tool	<i>Function Flyer</i>
Mode	Single computer & projector
Context of tool use	Review in a whole class discussion on function transformations of the functions.
Lesson activities overview	<ul style="list-style-type: none"> • Carl graphed each of the functions and asked the class to predict how the graph would change by changing parameters. Carl also asked students to make generalizations of parameter changes across function families. • Next, Carl mapped axes on the floor of the classroom. As a class, students were charged with the creation of a graph of a specified function. Using the axes drawn on the floor students, arm in arm, created the graph.
Self-reported results	Carl thought students improved their ability to generalize about transformations. The software allowed for discourse which in turn provided him insights into students' thinking.

Assertions

For the reader's reference and summary of findings, the following list contains the seven assertions identified through the data analysis. Each assertion is then subsequently discussed in further detail.

Assertion #1 *Teachers used the Flyer tools in their individual lessons in a manner consistent with both their own personal overall learning goals for functions within the course and in line with curriculum expectations.*

Assertion #2 *Teachers believed learning goals of the lessons were met when students understood the purpose of using the Flyer tools within the context of the lesson.*

Assertion #3 *The concept image of a function that teachers intended to develop through these lessons aligns with the historical definition relating two variables by an equation.*

Assertion #4 *Lesson designs that focused on the transformation of functions concept promoted a process conception of those transformations.*

Assertion #5 *Teachers used the Flyer tools in a manner such that a) permitted an inversion of the Action / Process / Object sequencing encouraging students to manipulate function as object externally via the software yet b) did not require students to internalize function as an object conception.*

Assertion #6 *Lessons facilitated neither versatile nor adaptable thinking in regards to understanding of function as process and object.*

Assertion #7 *Teachers did not always utilize features in the Flyer tools that directly related to their learning goals.*

Each assertion is presented individually followed by a discussion justifying and expanding its content. Evidence is drawn from each of the three teacher's interviews, lessons, and reflections.

Assertion #1

Teachers used the Flyer tools in their individual lessons in a manner consistent with both their own personal overall learning goals for functions within the course and in line with curriculum expectations.

Teachers designed lessons seemingly consistent with their own teaching goals as well as expectations put forth by the North Carolina Department of Public Instruction for the study of functions. The North Carolina Department of Public Instruction has curriculum documents for courses offered in the public schools of North Carolina. These documents are collectively known as the *North Carolina Standard Course of Study* (NCSCoS).

Teachers discussed teaching goals on the study of functions for students in general as well as specific to the course for which they developed the lessons during the initial interview. As part of the lesson plan template, teachers were also asked to identify their objectives and the objectives that would be addressed from the NCSCoS. In reviewing these data sources, there is consistency among the use of the *Flyer* tools within

the lesson, the required North Carolina curricula objectives, and their personal teaching goals.

Regarding the NCSCoS, Wendy and Max completely aligned their lesson activities to meet these objectives and Carl addressed supporting knowledge needed to meet the objectives. Table 9 displays each lesson with the stated NCSCoS objectives and a description of how the lesson met the objective with the use of the *Flyer* tools.

Table 9: Lessons aligned by NCSCoS

<i>Lesson / Teacher</i>	<i>NCSCoS Objective</i>	<i>How lesson used Flyer tools to meet objectives</i>
<i>Inverses / Wendy</i>	2.01 Use the composition and inverse of functions to model and solve problems; justify results.	Students were required to use <i>Data Flyer</i> to investigate inverses of functions and solve problems relating to generalizing about how to find a function's inverse.
<i>Sine / Wendy</i>	1.01 Create and use calculator-generated models of linear, polynomial, exponential, trigonometric, power, and logarithmic functions of bivariate data to solve problems. 2.04 Use trigonometric (sine, cosine) functions to model and solve problems; justify results. b. Create and identify transformations with respect to period, amplitude, and vertical and horizontal shifts.	Students were required to model bivariate data using <i>Data Flyer</i> to transform the sine function and find an estimated fit of the function to the data.

Table 9 (continued)

<p><i>Least Squares / Max</i></p>	<p>1.01 Create and use calculator-generated models of linear, polynomial, exponential, trigonometric, power, and logarithmic functions of bivariate data to solve problems.</p> <p>a. Interpret the constants, coefficients, and bases in the context of the data.</p> <p>b. Check models for goodness-of-fit; use the most appropriate model to draw conclusions and make predictions.</p>	<p>Students were required to use <i>Data Flyer</i> by creating a linear model to fit bivariate data collected in class and interpret slope and intercept of their linear model. They also used goodness of fit by comparing the fit from the calculator generated model and the model developed using <i>Data Flyer</i>.</p>
<p><i>Power / Max</i></p>	<p>2.03 Use power functions to model and solve problems; justify results.</p> <p>a. Solve using tables, graphs, and algebraic properties.</p> <p>b. Interpret the constants, coefficients, and bases in the context of the problem.</p>	<p>With <i>Function Flyer</i>, students were required to investigate power functions in graphical and algebraic forms and generalize about the behavior of constants and coefficients through the use of the slider bars.</p>
<p><i>Parabola/ Carl (see Footnote 2)</i></p>	<p>2.01 Write the equations in standard form of circles and parabolas; graph.</p> <p>3.05 Use quadratic equations and inequalities to solve problems.</p> <p>3.06 Find and interpret the maximum and minimum values and the intercepts of a quadratic function.</p> <p>The secondary objectives are:</p> <p>3.01 Describe graphically, algebraically and verbally real-world phenomena as functions; identify the independent and dependent variables.</p> <p>3.03 Graph relations and functions and find the zeros of functions.</p>	<p>With <i>Function Flyer</i> students were required to investigate a parabola written in standard form through manipulating the values of a, h, and k. Students solved problems related to generalizing about the parabola in this form by interpreting max / min and intercepts.</p>

Table 9 (continued)

<p><i>Function Families/ Carl</i> (see Footnote 2)</p>	<p>3.01 Describe graphically, algebraically and verbally real-world phenomena as functions; identify the independent and dependent variables. The secondary objectives are: 3.03 Graph relations and functions and find the zeros of functions. 3.15 Write and graph exponential functions of the form $f(x) = a b^x$.</p>	<p>Students were required to verbally describe and generalize about a variety of functions and their transformations through the use of the <i>Function Flyer</i> tool in graphical and algebraic forms including exponential functions.</p>
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In comparing the data sources by teacher, there is also a consistency between what the teacher believes students should learn about functions in the course and the lessons they developed. During the initial interview teachers were asked, “What should students understand about functions by the time they finish this course?” In comparing the data from the initial interviews and then examining the implementation of the software within the lessons, teachers consistently used the *Flyer* tools in a way that matched with their teaching goals. Table 10 summarizes teacher responses to the question by course, along with a description of how the software was used in that lesson.

Table 10: Summary learning goals about functions by course and teacher compared to implementation of software for the respective lessons.

<p>Teacher Course</p>	<p>Key aspects students should know about functions after this course</p>	<p>Lesson: Description of software use</p>
<p>Wendy <i>Alg II</i></p>	<ul style="list-style-type: none"> • “...correlations between those different representations but a lot of times we don’t have time to focus on that... We’re really pushed [by the <i>End of Course</i> exam].” (see Footnote 3) • “...how to translate any type of function. At least translate, and if they can stretch, that would be even better!” 	<p><i>Inverses:</i> Discover several functions’ inverse graphs and equations through attempted translations of original graphs and examining tables of values using <i>Data Flyer</i>.</p>

Table 10 (continued)

<p>Wendy <i>Alg II</i></p>	<ul style="list-style-type: none"> • “I like students to be able to see the relationship between the function notation - the equation, the table and the graph and how the three interact with each other or given one you can come up with the others.” • “From real world data how to use functions to model the data.” 	<p><i>Sine:</i> Fitting a sine function to a data set though graphing data and sine function concurrently in <i>Data Flyer</i> and using sliders to transform the function.</p>
<p>Max <i>AFM</i></p>	<ul style="list-style-type: none"> • “With the [AFM] kids, they’ve already taken and know all the algebra, now let’s actually do something with it.” • “...you know we can’t collect all the data in the world so they tell us if we collect a good sample then we can use a function to help us answer questions.” • “It’s kinda hard to do the stretches and shrinks and stuff like that. But it is basically what I would want them to get out of that.” 	<p><i>Least Squares:</i> Collecting data. Graphing and fitting a line to the data with <i>Data Flyer</i> using sliders the to transform and fit the function to the data.</p> <p><i>Power functions:</i> Generalizing about transformations of power functions by graphing and using sliders in <i>Function Flyer</i> to change exponents and other parameters.</p>
<p>Carl <i>Alg II</i></p>	<ul style="list-style-type: none"> • “Be able to look in those tool kits [i.e. function types]... just to be able to see the basic shape and then definitely know how to use the calculator to graph them and use the table in the graphing calculator to find points.” • “If I say graph this parabola, then I say at 4 - what is y going to be? You should be able to look at the graph and see that.” • “The table I don’t focus on too much other than a means to find the graph.” 	<p><i>Parabola:</i> Generalizing about transformations of a parabola by graphing specific functions with <i>Function Flyer</i> specified from a worksheet. The worksheet could be done using a graphing calculator.</p> <p><i>Function families:</i> Generalizing about transformations for the toolkit functions. Whole class discussion with computer / projector using <i>Function Flyer</i>.</p>

Assertion #2

Teachers believed learning goals of the lessons were met when students understood the purpose of using the Flyer tools within the context of the lesson.

If a learning activity incorporates technology, students will assume they are supposed to learn or observe something from the use of that technology. Within five of the six lessons, teachers clearly believed they incorporated the *Flyer* tools into the lesson activities such that students readily understood the purpose of the tool and what they were supposed to learn from its use.

In both written reflections and in the exit interview regarding these lessons, teachers expressed that the *Flyer* tools helped students to meet the learning goals. For example, Max wrote in his reflection on the *Power* (Table 6) lesson, “It worked great!... I feel they grasped the idea of movement and when we discussed translations later in the lesson, students would refer back to the *Function Flyer*.” In regards to the *Sine* (Table 4) lesson, Wendy states, “Using the *Data Flyer* was a great way to manually manipulate the amplitude, phase shift, and vertical and horizontal shift. The dynamic nature of the sliders helped students adjust quickly and see how the adjustments in numbers affected the graph.” Comments such as these are prolific in both the written reflections and exit interviews. In the reflection data, teachers emphasized the power of the sliders for helping students recognize these kinds of patterns.

Wendy, however, did not feel learning goals were met through the use of the tool within the *Inverses* (Table 3) lesson. In the written reflection data for this lesson, Wendy expresses such thoughts as

- “Adjusting the lesson may make a better use of the technology.”

- “Students struggled to get all inverses of the functions.”
- “[I]t could have been presented without the sliders if need be.”
- “I would do these [functions] one at a time as we discussed new functions.”

In review of the data regarding this lesson, it appears that the lesson design did not integrate the features of the technology well. Students were overwhelmed at how they were supposed to use the tool within the context of the lesson and what they were supposed to learn through using it. In the exit interview Wendy said she “...did not expect the inverses lesson to be completely smooth but did not expect the students would be satisfied with no answer at all.” Wendy resorted to “direct instruction” in order to get students to complete the activity.

Wendy notes that she did not expressly teach the students how to use the interface. The lesson activity required students to follow a lengthy procedure in order to reach the conclusions she was expecting students to identify. A sample row from the worksheet that Wendy expected students to complete is in Figure below. The worksheet contained five rows altogether with a different type of function for each row. Function types, one per row, included a constant, a linear, a quadratic, a cubic, and an exponential.

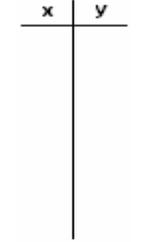
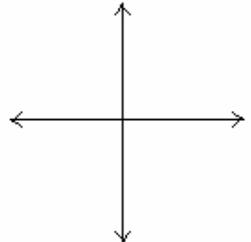
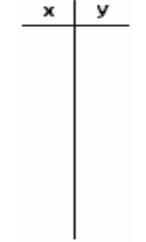
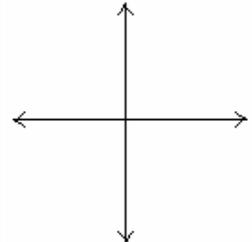
Function	Table	Graph	Inverse Table	Inverse Graph	Inverse Equation
Cubic $y=x^3$					

Figure 5: Sample Row from Inverses Lesson Worksheet

Thus, failure in meeting learning goals seem to be that in conjunction with a large number of procedural steps, there was a lack of clarity on how the tool should be used to observe inverses and a lack of instruction on how to use the software interface. Students became confused and gave up on the lesson activity.

Assertion #3

The concept image of a function that teachers intended to develop through these lessons aligns with the historical definition relating two variables by an equation.

A robust concept image allows a person to think flexibly about a mathematical concept and perform mental transformations on it within the constraints of the definition. All lessons provided evidence that teachers intended to use the *Flyer* tools to promote students' concept images of function. All six lessons use the software in a manner that promotes a visual connection between the analytic equations of the functions and their graphs.

Four of the lessons focus on the visual aspects of function transformations specifically. These four include the *Sine*(Table 4), *Power* (Table 6), *Function Families* (Table 8), and *Parabola* (Table 7) lessons. In these lessons, students are required to either make generalizations between changes in the constants and coefficients from the equation of a function and subsequent changes in its graph or use the transformations to solve a problem. The two remaining lessons, *Least Squares* (Table 5) and *Inverses* (Table 3) , also attempted to use the visual, dynamic power of the software to facilitate visual imagery of those respective topics.

However, all lessons exclusively focused on concept image development of an historical definition of function. Recall that the classical conception of function is a relationship between two variables described by an analytic equation (Malik, 1980). This

result is not surprising in light of the interface design of the software as it makes use, almost exclusively, of equations to create the graphs of functions. As well, on inspection of the supporting data sources, teachers are almost wholly focused on the historical definition regardless of the software.

The initial interview data also provides evidence that emphasis on understanding the modern definition is not a curricular priority for teachers. When asked about the key aspects of which functions teachers wanted students to learn by the time they graduated from high school, all teachers did mention that students should know the definition. However, when they expanded upon the response, all teachers continued to refer to understanding functions as equations. Teacher comments also expressed a desire that students should be able to visualize those equations in their graphical form to understand the global behavior of functions in relation to their domain. Responses included:

- Max: “I want them to be able to look at a graph and identify that it is a function and what function would fit the best. And could they get the best fit function without using a calculator so by hand reflections and stretches and shrinking of the functions to see, the parent functions you know where they start now what do you have to do to that to get it to fit this data...”
- Wendy: “In AFM I do a lot of that going between the function, the table, and the graph so we switch between those 3 connections, how to graph functions, some basic standard functions like constant, linear, quadratic, those types. I guess exponential, absolute value... the basic ones that are covered either in Algebra 2 or AFM.”

- Carl: “Basically they should know that whole tool kit of functions. Those graphs.... and should know if I see this, this is the basic general shape with pretty much any type of function.” As well, “I would hope that most of them would be able to look in those tool kits and say exponential it should be that same basic curve, a quadratic should be a parabola - sideways if its y that is squared, up and down if it is the x. ... and just to be able to see the basic shape...”

Wendy was the only teacher to briefly reference a more general definition, “In Algebra 2, we probably start off with the definition. We’re looking for each x is only paired with one y is maybe what I tell them. And then we may go to the graphing calculator and use the vertical line test.” However, references to the historical conception of function dominate the discussion on what teachers expect students to understand about functions. As well, several lessons which had plausible opportunities for discussion of a modern view of function, *Inverses* (Table 3), *Sine* (Table 4), and *Least Squares* (Table 5) were not explored. Teachers’ reflections on these lessons did not reveal a belief that they might need or want to emphasize anything besides the historical definition.

Assertion #4

Lesson designs that focused on the transformation of functions concept promoted a process conception of those transformations.

The lessons promoted student understanding of function transformations as a process. A process understanding of function transformations can be characterized by the ability to describe changes that take place as a result of the transformation and an ability to identify the base function. Students with a process conception will lack the ability to apply transformations among different representations (Lage & Gaisman, 2006).

Lessons in which the stated learning goal was to help students understand one or more transformations of a function include: *Function Families* (Table 8), *Parabolas* (Table 7), *Power Functions* (Table 6), and *Sine* (Table 4). Each of these lessons had similar designs in which students were asked to graph a function and then change the function's parameters in order to either fit data or simply generalize about the movement of its graph. Students were not required to examine or explain why the various changes to a parameter have certain effects on a graph with respect to the equation or tabular values.

For example, in the *Parabolas* (Table 7) lesson, the students were provided a worksheet containing a table of quadratic equations and each row listed a quadratic with a different set of parameters such as $f(x)=ax^2$, $f(x)=a(x-h)^2$, etc. Students were asked to use *Function Flyer* to graph each equation, sketch the general shape of the equation, and then describe how changing the parameters using the sliders affected the graph. Asking students to generalize in this manner without an underlying understanding of the relationships among the various representations is indicative of a process conception of function transformations.

Within the *Least Squares* (Table 5), students also had to draw conclusions about how changing an equation affects the graph but this was not the central learning goal. Students had to use the transformation of a linear function as a means to an end, fitting the function to data collected in class. Although five of the six lessons required students to transform functions, none of the lessons encouraged students to think about why a change in one representation would change the other representation as it did.

A process understanding of function transformations can also be related to Skemp's (1978) instrumental understanding. Skemp defines *relational understanding* as “knowing both what to do and why,” (pg. 9) whereas *instrumental understanding* is “‘rules without reasons’, without realizing that... the possession of such a rule and ability to use it was what [teachers and students] meant by ‘understanding’,” (pg. 9). Lesson activities encouraged students to generalize about the cause and effect relationship between changing an equation and its graph but did not ask students to analyze or explain these connections. Having students examine these relationships and generalize them (e.g., increasing the parameter c of $f(x)=ax^2+c$ causes the graph to shift vertically) but not asking them to understand why the relationship exists indicates instrumental understanding.

For example, in Carl's *Function Families* lesson he states as the goal of the lesson, “To help the students understand the differences between different functions.” In his assessment of what students learned during the lesson, he replied, “the students have a fairly good knowledge of functions and how changing the different numbers [coefficients, exponents, constants, etc.] will change the graph.” In the exit interview when asked about whether or not the students understood why changing those numbers affected the shape of the graph, he said that a connection was not made in that lesson or in previous ones. Note how this example can be applied both to a process conception of transformations as well as instrumental understanding.

Assertion #5

Teachers used the Flyer tools in a manner such that a) permitted an inversion of the Action / Process / Object sequencing encouraging students to manipulate function as object externally via the software yet b) did not require students to internalize function as an object conception.

As noted in the literature review presented in chapter two, Tall (1989) emphasizes that technology enables students to work with higher level mathematical concepts and thus reduces the need to present complex mathematics in simplified terms. It appears teachers use the *Flyer* tools in a similar fashion. Within the lesson, teachers utilize the software in such a manner that allows students to operate on “functions as if they were objects” through the dynamic nature of the tools, even though conceptually students do not understand the mathematical formalities of the system.

When operating with an object conception of function, the learner treats the function as a single entity and has the ability to operate on that entity. For the mathematical concept to actually be an object though, the operations must occur within the mind of the learner. With use of the *Flyer* tools, the operation on the “function as object” occurs externally. However, to facilitate an internalization of “function as object” the lesson design should have students explicitly address the relationship between the changes in the equation and changes in the graph. None of the lessons attempted to make such a connection with regards to the function concept. The *Least Squares* (Table 5) lesson did seem to attempt to promote an object conception of the least squares concept, however, but not of function.

As elaborated in Assertion #4, each of the lessons on transformations as well as the *Inverses* (Table 3) lesson did not make attempts to help students understand the

connection between the changing parameters in the equation and the patterns of animation resulting from those changes or vice versa. All lesson activities asked students to notice patterns but did not ask students to dig deeper and inquire why such patterns appeared as they did.

Programmatically within the *Flyer* tools, the computer operates on individual domain values according to the parameter changes, calculates corresponding outputs, graphs the coordinate pairs, and then connects those points. This functionality hides from the user the underlying process the computer performs in order to create the transformation. The software allows for an inversion of this action/process/object sequencing externally so that students can explore the complexities of transformations without needing to first understand the details.

The danger, however, occurs when this inversion is not explicitly addressed. Because it appears students can externally operate on “functions as if they were objects” through the use of the software, it may be easy to bypass development of these cause and effect relationships entirely between the changing numeric parameters and the changing graphs. Even though students can generalize about the patterns they see because of the tool, the students still need to be guided to internalizing why these patterns occur in the manner they observe.

Assertion #6

Lessons facilitated neither versatile nor adaptable thinking in regards to understanding a function as a process and an object.

Versatile thinking, in regards to functions, is characterized by Sfard & Linchevski (1994) as the ability to think of a function as a process *or* as an object, though the learner does not volitionally choose the process or object orientation. For the versatile thinker,

the context of the problem determines the process or object mode. Adaptable thinking, in turn, is characterized as the ability to think of a function as both process *and* object. The adaptable thinker can consciously choose which mode of operation is more appropriate for solving the problem within a context. Versatile thinking is reactive, whereas adaptable thinking is proactive.

Due to a learner's ability to compartmentalize, a learner can treat a function as an object yet not operate with an object understanding of subordinate concepts, such as transformations. For instance, a learner could explain and justify from an object perspective why a particular function behaves as it does across its domain yet not fully grasp why a transformation, such as multiplying the input values, stretches the shape of a graph.

Assertion #4 showed how teachers encouraged process thinking about function transformations within their lessons. As discussed in Assertion #4, teachers used the software in the lesson design to get students to operate on a function as if it were a single entity to be manipulated. Operating on functions in such a manner is indicative of an attempt at getting students to think of functions from an object conception, even if students do not internalize why the transformations occur as they do. The teachers are trying to leverage the software to teach students about a function's structural aspects, i.e. function as object.

There is little evidence that teachers encouraged students to think about functions as a process. Process thinking of functions is characterized by conceptualizing functions as input / output machines where the learner focuses on individual outputs of the function rule and thinks of the function as a collection of those values. Five of the lessons focused

themselves entirely on the holistic representations of functions and one, *Inverses* (Table 3), focused mostly on holistic representations.

The *Inverses* lesson, conceivably, could be viewed as encouraging versatile thinking because students did have to move back and forth between using functions as process and as object. The lesson did have students use functions to find individual input / output pairs. However, the purpose of this activity was a means to an end, not a focal point. Thus, *Inverses* may have implicitly encouraged students to switch between process and object views. This is difficult to assess, however, without direct evidence from the classroom implementation.

Working with functions as a process is an essential component for versatile or adaptable thinking, yet was not present in the lessons. Because the lessons did not focus on function as process, none of the lessons directly encouraged versatile or adaptable thinking.

Assertion #7

Teachers did not always utilize features in the tools that could help them meet their learning goals for a particular aspect of the function concept.

Two lessons, *Least Squares* (Table 5) and *Inverses* (Table 3), did not use powerful features in *Data Flyer* directly related to the lesson activities. *Data Flyer* has output features for displaying numeric values in a table and geometric representations of “Squares of the Deviations” from a fitted function to the actual data points. Both of these features could have been incorporated in the respective lessons to support students in achieving the teachers’ stated learning goals of these lessons, but were not.

The *Least Squares* (Table 5) lesson was designed to help students develop a concept image of the regression line calculation through minimizing deviations. The

teacher, Max, chose to use *Data Flyer* for collecting and graphing linear data, then used the sliders to transform and “eyeball” a linear function that modeled the data. The *Data Flyer* tool contains several features to examine deviations: an output in the main window of the value “Sum of the Squares of the Deviations”; a “Show Deviations” pop-up window accessed by a labeled button that displays the individual deviations, and a “Show Squares,” also accessed from a labeled button which, when pressed, geometrically displays the squares of the deviations on the graph. As explained in Chapter 1, these features are either displayed in the main window or are accessed from a button in the main window labeled as such. These features are directly related to Max’s learning goals for the lesson providing other visual and numeric representations of the concept. Max expressed disappointment at not having taken advantage when asked about them during the exit interview.

In the *Inverses* (Table 3) lesson, Wendy asked students to list tabular values from each function in the worksheet. During the exit interview she states, “I had to ‘take over’ for the sake of time.” Both *Flyer* tools contain a “Show Tabular Data” pop-up window that lists discrete values of the graphed function. Calculating the input and output values from the function consumed class time and was not a focal point of the lesson. Students easily could have used this window to get individual data points and save time during the lesson. Figure 1 of Chapter 1 displays the “Tabular Data” window.

After reviewing the lesson plans the researcher recognized these features were not used and subsequently asked the teachers in the final interview why they chose not to use those features. In both cases, teachers stated they were unaware the features existed.

CHAPTER 5

CONCLUSIONS

This study sought to examine how teachers choose to implement the dynamic graphing *Flyer* tools to meet learning goals for student understanding of functions. The theoretical framework for analyzing the data is based on the theory of *concept images* developed by Vinner (1983) and also on comprehension and integration of mathematical constructs as put forth in *APOS* (Asiala et al, 1996). This research serves to inform practitioners, professional development providers, teacher educators, and software developers concerned with advancing student conception of functions as mathematical objects and their development of robust concept images. This chapter concludes the study with an overview of the study and framework, a discussion of the analysis in light of the research questions, and implications with recommendations for teaching, materials development, and future research.

Summary of Study

To address the research question, lesson plans from three teachers were qualitatively examined, together with their written reflections and records of their initial and exit interviews. Each participant developed and taught two lessons incorporating one of two dynamic graphing applets: *Function Flyer* or *Data Flyer*. Four North Carolina teachers, whose courses focused on functions beyond those taught in Algebra I, were selected from a professional development workshop in which the *Flyer* tools were featured, assuring familiarity with these tools. Assertions, as described by Erickson

(1986), were used for the analysis of this data. Seven assertions were identified, examined and discussed as a result.

The *APOS* theory (*Action, Process, Object, Schema*) of mathematical knowledge acquisition was used in conjunction with concept images as the framework for analysis of teachers' intentions for facilitating students' learning of functions. Because the study analyzed the pedagogical context in which learning occurred, and did not observe students directly, the framework uses three observable factors in which to judge how teachers promoted these internal constructs: learning goals, function definitions, and the use of function representations.

APOS theorizes that a learner holds an action conception of function when applying rote algorithms to work with functions. Upon reflection of these algorithms and internalization, the learner operates with a process conception. If the learner condenses the process, forming a connected mathematical structure, then the function becomes an object on which the learner can operate. All of the related actions, processes, and objects create an overall schema of that concept for the learner.

A learner has an *adaptable* conception of function (Sfard & Linchevski, 1994) if he/she can call upon a process view or object view at will, as appropriate within the context of the problem. An *adaptable* concept of function exists when a process view or object view is called upon at will, as appropriate within the context of the problem.

With regards to the concept image, a learner calls upon a mental image when using a mathematical construct such as a function. The more accurate the concept image, the more flexible the image will be, yet still be constrained by the structure of that mathematical object. A flexible concept image allows the learner to visualize a function

as both process and object when appropriate for the task at hand. A concept image is similar to a schema though a schema is, by definition, a coherent, connected, web of mathematical ideas whereas a concept image need not be.

Findings

Research question: What is the nature of how teachers who instruct courses beyond Algebra I intend to use the Flyer tools for the development of their students' understanding of functions?

To answer this question, the three observable elements of the pedagogical context are used to organize the response. Within the pedagogical context of this study, the researcher was able to observe teachers' learning goals, definition of a function used in their lessons, and use of and attention to representations of a function.

Learning Goals

As described in Assertion #1, teachers intended to implement the Flyer tools in their lessons so they met both their own learning goals for their students and the goals set forth by the North Carolina curricula. For each lesson, teachers identified the North Carolina Standard Course of Study (NCSCoS) curricular objectives which aligned with their lessons. In examination of the lessons, the NCSCoS objectives fit well within the teachers expected outcomes for student learning. The North Carolina curricular objectives specify particular functions, including what teachers refer to as the "toolkit" functions: linear, polynomial, logarithmic, exponential, and absolute value (see Appendix E for a listing of course objectives used within the lessons).

Even though teachers did not take full advantage of all the features that could have helped them meet their learning goals as shown in Assertion #7, teachers were

pleased with the use of the tools. Assertion #2 presents evidence that teachers were content within their lesson designs and with the outcomes, particularly when students understood the purpose of using the tool.

Teacher beliefs about what students should know with regards to functions remained consistent throughout the study. No discrepancies were identified between the lesson implementations and consequent participant reflections. Teachers written reflections and the exit interviews demonstrated a belief that the integration of the Flyer tools within their lessons helped them meet learning goals effectively. The thoughts expressed in the initial interviews, the lesson goals, the use of the tools to meet those goals, and their reflections all align. The reflections did not reveal new insights for the participants on how they might teach functions and related concepts more effectively or new insights into how students learn.

Teachers used the Flyer tools within the lessons to have students operate on functions in a manner they believed would reach their teaching goals more quickly. Consequently, they did not require their students to analyze why processes, such as function transformations, behave as they do. Essentially, teachers did not innovate with the use of the tool to alter teaching goals with functions. Rather, they used the tools to solidify students' comprehension of defined goals that they had been teaching previously.

Function Definitions

Assertion #3 showed how teachers utilized the Flyer tools in all of the lessons to help students develop concept images of functions' behaviors (e.g. absolute value, linear, exponential) as basic shapes with changing features. They used the Flyer tools to show how features of the curve changed depending on which parameters from the equation

changed. When teachers discussed what students should understand about functions responses referred to a desire that students understand a function as something that relates one variable to another, i.e. an historical definition (Malik, 1980), as well as knowledge of one or more specific “tool kit” functions and their behavior.

In all data sources, teachers emphasized the historical conception of function in which the function can be represented by an equation relating two variables in conjunction with its graphical image on the coordinate plane. The Flyer tools do lend themselves to exploring function from the historical perspective because they rely on equations to be inputted in order to generate a graphical representation.

Use of Function Representations

Assertion #4, #5, and #6 all provide evidence of how teachers chose to use the Flyer tools to familiarize students with a variety of functions behaviors using different representations and linking of those representations. With regards to a function’s “behavior,” teachers referred to how a function behaves across its range of values with respect to its domain, suggesting visual imagery such as "graph", "shape" or "curve" to explain its global properties. Teachers wanted students to relate two representations of function, the equation with its graph or vice versa.

Four of the six lessons - *Sine*, *Power*, *Function Families* and *Parabolas* - leveraged the tools to help develop a mental image of how changing parameters in the equation form of a function affected the graph of one or more functions. As discussed in Assertion #4, these lessons facilitated a process conception of function transformations, though generally lacked a process view of function as input-output that could potentially support a deeper understanding of the transformations. Students were expected to know

how parameters from the equation affected the graph but were not asked to explore why the changes would have such an effect. This process view of transformations is similar to an instrumental understanding (Skemp, 1978) of linking function representations.

Interestingly, none of the lessons provided evidence that teachers used the Flyer tools to develop a process conception of the overall function concept. Instead, as Assertion #5 demonstrates, the lessons showed that teachers intended to have students use the software to operate on functions as if they were objects. The design of the software allows for such external operations. Yet none of the lessons incorporated activities which encouraged students to internalize these operations.

For students to internalize their observations through using the Flyer tools, they would need to reflect on why changing the numerical parameters effects the graphs in the manner they observed. Using a process view of function as an input-output system could be useful in addressing the underlying reasons for these effects. However, this kind of activity was not present in the lessons. Thus, the lessons did not attempt to facilitate the creation of functions as process and objects within the minds of the students.

Furthermore, as Assertion #6 indicates, lessons did not encourage versatile or adaptable thinking of the overall function concept. Lesson activities encouraged process thinking of function transformations but not of the function concept in general. For a process conception of the overall function concept, lesson activities would needed to have had students move between process and object in one or more representations.

Teachers believed that the incorporation of software into their lessons improved understanding of functions. Students more readily recognized relationships among function representations, particularly in graphs and in equations. The tools apparently

helped the teachers to be more efficient, as they felt they were able to help more students recognize patterns. However, although students may have recognized the patterns, teachers did not reveal a belief that students understood why those patterns existed.

Implications and Recommendations

For Teaching

As Ball (2002) stresses, “it is the teacher who bears the primary responsibility to weave the representational context.” As the key element in implementing the curriculum, a teacher needs to learn how to weave that context to promote adaptable thinking with flexible concept images. This skill, identified as pedagogical content knowledge, is specific to the teaching profession.

In this study, the teachers used the *Flyer* tools in a manner consistent with what they already teach. The data show that these teachers are teaching the curriculum required by the NCSCoS. Although the NCSCoS specifies students should be able to work within and translate among a variety of functional representations, it does not necessarily require this should be accomplished through structural understanding. Consequently, as noted in chapter two from the Schwarz and Dreyfus (1995) study, students without the underlying object / structural understanding of functions, students will tend to compartmentalize these representations into objects themselves. Students do not find contradiction when the structures of these separate objects do not align because they are not understood as different aspects of the same object.

Findings in this study closely align with Simmt’s results (1997). Simmt found teachers integrated graphing calculators into their lessons in a manner consistent with their own beliefs. For teachers to utilize powerful tools such as the *Flyers* for teaching

functions in innovative ways that facilitate adaptable thinking, the curriculum would need to change as will their own teaching goals in a manner that emphasizes adaptable thinking.

For Research

Formative research would be helpful for improvements in interface design with the teachers as the primary participants in this research. Because the teachers are the ones responsible for integrating the technology into their lessons, the technology should be intuitive for their use in their classrooms. Software interfaces in this environment have two distinct sets of users. The software interface must be usable by students. However, it is the teachers that must determine how to best use the interface in order to meet learning goals. Research therefore should also focus on the teachers as the target users for design principles.

The research presented in the literature review reveals the power of technology in the development of robust concept images and in facilitating both process and object views of functions. Most of those studies focus directly on the students' use of the software when implemented through activities designed by the researchers themselves. Formative research would be useful to study how teachers currently lesson plan and implement those plans with graphing technologies. These results should inform new studies that result in innovative suggestions for lesson plans and implementations. These lesson implementations can then be studied as well. The research would then directly benefit teacher participants as they become empowered through learning how to incorporate flexible concept images and how to adapt thinking about structural connections among function representations.

Additionally, research should focus on how to best support teachers' integration of graphing technologies in their lessons to empower student learning in their classrooms. For teachers to use and integrate powerful graphing tools such as *Function Flyer* and *Data Flyer*, they rely on their pedagogical content knowledge in judging the tools as effective and for decisions on how to implement them in facilitating learning goals. Further research should examine how to help teachers develop this type of knowledge so they can identify and use quality tools more easily and efficiently.

For Software and Supporting Materials

With regards to implications for the interface design of software, this study identifies no specific conclusions for improvement. From the data, it is apparent that teachers were unaware of functionality in the software that feasibly would have helped students reach learning goals for that lesson. Neither the data from the lessons nor the interviews with the teachers suggest how the interfaces might be improved so teachers would notice or be encouraged to explore the functionality of the tools.

However, Shodor's *Interactivate* can be leveraged to incorporate *content* of a caliber which supports adaptable thinking of functions in both its online materials and professional development or workshops are conducted by Shodor staff. It is apparent from the analysis that providing graphing technologies that could be used to support an object conception of function is not enough. As discussed in Assertion #1, teachers tend to integrate the technology in a manner consistent with their own learning goals and rely on a curriculum that does not explicitly encourage adaptable thinking.

Interactivate contains more than the java applets. Currently, there are over ninety lesson plans and other supporting materials to assist teachers in the integration of these

tools into their classrooms. Lesson plans which exemplify the process / object duality that characterizes adaptable thinking should be implemented as part of these materials.

As part of the lesson plan template used in the study, teachers were asked if they wanted to have the lessons they developed published within *Interactivate*. For those that agreed, Shodor will work with the teachers to revise these lessons in a manner that supports adaptable thinking and robust concept imaging. Additionally, this study will inform future materials yet to be developed for *Interactivate*.

Interactivate currently averages more than one million page accesses per month on its website (Shodor, 2007). Developing materials for *Interactivate* that integrate the framework has potentially far reaching effects. The materials are used directly by teachers in their classrooms. Feasibly, these materials could also serve as models for teachers and educators as they adapt and create their own lessons after using the content found in *Interactivate*.

With regards to professional development and training, Shodor conducts a variety of national and local workshops serving to facilitate the integration of computational science resources in the teaching and learning of mathematics and science at a variety of levels. Workshops and professional development participants include undergraduate faculty and teachers, teacher and faculty leaders responsible for training others in their respective institutions, as well as workshops for middle school and high school students. Shodor also provides internships for undergraduate students. As part of these internships, the undergraduates participate as instructors for the middle and high school level workshops under the supervision of Shodor staff.

This study will inform Shodor as the provider of these educational experiences. It is apparent that the tools, in and of themselves, do not support an object oriented conception of mathematical concepts. The instructors still bear the responsibility of designing lesson activities which help students make the internal connections of what they see and experience externally through the use of these tools. As a result of this study, Shodor can now incorporate this pedagogy into the professional development and workshop instruction, thus reaching a gamut of persons involved in the teaching and learning of mathematics and science education.

Footnotes

¹The author is the project manager of Interactivate at the Shodor Foundation and influences the development of the interfaces, as well as develops curriculum materials contained on the site.

²The NCSCoS was revised for the 2005-2006 school year. In the Parabola and Function Families lessons, Carl lists objectives from the previous Standard Course of Study.

³The North Carolina Department of Public Instruction requires all public schools to administer standardized tests, called End of Course exams, for a number of high school mathematics courses including Algebra II. The Advanced Functions and Modeling and Trigonometry courses do not have End of Course exams.

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APPENDICES

Appendix A - Form for Participant Identification and Selection

Technology and Graphing lesson study
Bethany Hudnutt

Benefits of participation:

- A well-designed lesson intergrating graphing technologies for use in your classroom with time to reflect on the lesson as well.
- 1 technology CEU
- Providing professional expertise to the Shodor Education Foundation so we may better suit your needs with our graphing activities

The purpose of this study is to examine how teachers utilize graphing technologies when planning and implementing lessons about modeling real-world phenomena with functions. As a participant in this study, you will be asked to complete the following:

1. Complete an initial brief survey describing your beliefs about mathematics and teaching mathematics, and the role of technology in teaching mathematics. Approximate time is 30 minutes
2. Participate in an individual orientation and interview for approximately 90 minutes;
3. Plan a lesson for a specified topic in the *Advanced Functions and Modeling* course that utilizes either the graphing calculator or a computer based java applet, *Data Flyer*, developed by the Shodor Education Foundation, Inc. This lesson and accompanying student material will be submitted electronically to the researcher. Approximate time will vary by individual during the planning phase (12-17 hours).
4. Teach the specified lesson in your *Advanced Functions and Modeling* course.
5. Participate in a follow-up interview in order to ask for your reflections on the lesson and implementation. Approximately 60 minutes.

We anticipate a total of fifteen to twenty hours of your time will be required for the survey, interviews, communications, and lesson design.

All interviews will be audio and video taped, though kept strictly confidential. Your name will be removed from the survey and lesson plan materials. Results will be shared in both written papers and conference presentations.

By participating in the research you will have the opportunity to design and reflect on a technology based lesson for your *Advanced Functions and Modeling* course. You will also receive a letter addressed to your district credit recommending one Continuing Education Unit (CEU) for participation. The knowledge gained from the study will benefit the research base of mathematics education in providing insight into teacher lesson planning with technology, specifically for teaching functions for modeling real-world phenomena. There will be no risk associated with your participation in the project.

Please check one of the following:

I am not interested in participating in the study at this time.

I may be interested in participating but have concerns or questions. (Please list these below)

I am interested in participating. Please provide:

name

address

city / state / zip

school

district

contact phone

email

number of years teaching described by grade level / topic

number of years teaching AFM

Note that expressing interest does not necessarily mean you will be selected for participation.

Appendix B - Initial Interview Questions

Demographics

Name:

Number of years teaching:

Number of years teaching mathematics:

Number of years teaching a course involving functions:

Number of years teaching this course:

Textbook:

Function Concept and Pedagogy

1. What are the key aspects students should understand about functions? *(If interviewee tries to describe tasks try to elicit the general goals from those tasks)*
 - a. What are your goals *in this course* for your students' in understanding functions
2. Do you think it is important for students to understand the function concept? Why?

General Technology Pedagogy

3. What do you believe is the purpose of technology in teaching and learning mathematics?

Technology Pedagogy with Functions

4. Describe your experiences teaching about functions with various graphing technologies.
 - a. Describe your experiences teaching with Data Flyer.
5. How does your access to technology limit or expand the way you choose to use technology for the teaching of functions?
6. Is there any technology you would choose to use in your teaching of functions if you had access to but currently do not?
7. What are the benefits and drawbacks of using technology for student understanding about functions?

Appendix C - Lesson Plan Template

Lesson Plan Guidelines

Your lesson plan should contain the following elements:

Goal of the lesson: What knowledge do you expect the student to have gained through your lesson?

Objectives of the lesson: What specific outcomes do you expect the student to achieve through the lesson?

NCSCoS Objectives: Which SCoS objectives do you intend to address with this lesson? Which SCoS objectives will be used within the lesson even if they are not the main focus?

Prerequisite knowledge: To be successful with the lesson what must the student already know? Please include both technology and content prerequisites.

Preliminary preparation: As the instructor teaching the lesson what must you do to prepare?

Materials: What materials will students and teacher need during the lesson?

Lesson description: Please include *detailed procedures* of how you anticipate the lesson to progress. Please describe classroom configuration throughout the lesson, specifically your role and students' roles. Include any of the following that apply:

- oral directions
- questions to prompt student thinking
- exact wording of examples, tasks, or problems used in the lesson
- how the technology will be used in the lesson by you or students
- worksheets / written directions
- follow-up activities / tasks

Anticipate student thinking: Describe students' possible misconceptions, likely approaches to tasks / problems, use of technology, etc. How do you plan to address these issues in your lesson?

Assessment: What types of informal and formal assessment would you use either during this lesson or in the future to determine the students achieved the goal(s) of your lesson? Please provide at least one example of a formal assessment item.

____ Check here if you are interested in having your lesson published in *Interactivate*. You will receive credit on the page containing the lesson if you would like.

Reflection: After you have taught the lesson please respond to the following prompts:

1. Describe your assessment about what students understand regarding the concepts addressed in the lesson.
2. Discuss the effectiveness of the activities/tasks used in the lesson for developing students' understanding of the function concept.
3. Discuss the effectiveness of the technology used in the lesson for developing students' understanding of the function concept.
4. If you were to teach this lesson again, what do you think were the strengths of the lesson and thus would keep the same?
5. If you were to teach this lesson again, what would you change, and why?
6. Did you have any insights on the teaching of functions through developing and teaching this lesson? Please explain.

Appendix D - Exit Interview Questions

Pedagogy

1. What was your overall impression of the lesson implementation?
 - a. What do you think went well?
 - b. Did anything in the lesson happen that was unexpected?
 - c. Would you teach this lesson again?
 - d. Would you make modifications? If so, what modifications would you make?

2. Please describe the process you went through to develop the lesson. What did you decide first and how did the plan evolve from there?
 - a. When you were planning this lesson what other resources did you use?

Technology and Functions

Question 2 applies to the teacher and question 3 applies to the student.

3. Did the technology enable you to teach in a different manner than if the technology was unavailable? Please explain with specific examples.

 4. Did the technology enable your students to learn in a different manner than if the technology was unavailable? Please explain with specific examples.

 5. Describe how the use of the Flyer activities may have changed the way you teach the function concepts in your lessons.

 6. Are there instances where you observed that the student interaction with the technology hindered their understanding of the lesson objectives?
 - a. Do you have suggestions for changes in the Data / Function Flyer activity that would help to improve student understanding? How?

 7. Describe aspects of the function concept you were expecting the student would understand better as a result of this lesson.
 - a. How do you think these aspects were affected by the use of the Flyer activities?
- Get specific examples of student interaction / learning insofar as possible.
 - Read through lessons carefully to identify possible ambiguities and ask for clarification.
 - Ask follow-up questions from the reflections.

Appendix E - North Carolina Standard Course of Study Objectives by Lesson Plan

Algebra II, Parabolas Lesson

- 2.01** Write the equations in standard form of circles and parabolas; graph.
- 3.05** Use quadratic equations and inequalities to solve problems.
- 3.06** Find and interpret the maximum and minimum values and the intercepts of a quadratic function.

The secondary objectives are:

- 3.01** Describe graphically, algebraically and verbally real-world phenomena as functions; identify the independent and dependent variables.
- 3.03** Graph relations and functions and find the zeros of functions.

Algebra II, Function Families Lesson

- 3.01** Describe graphically, algebraically and verbally real-world phenomena as functions; identify the independent and dependent variables.

The secondary objectives are:

- 3.03** Graph relations and functions and find the zeros of functions.
- 3.15** Write and graph exponential functions of the form $f(x) = a b^x$.

Algebra II, Inverses Lesson

- 2.01** Use the composition and inverse of functions to model and solve problems; justify results.

Trigonometry, Sine Lesson

- 1.01** Create and use calculator-generated models of linear, polynomial, exponential, trigonometric, power, and logarithmic functions of bivariate data to solve problems.
- 2.04** Use trigonometric (sine, cosine) functions to model and solve problems; justify results.
 - b. Create and identify transformations with respect to period, amplitude, and vertical and horizontal shifts.

Advanced Functions and Modeling, Power Functions Lesson

- 2.03** Use power functions to model and solve problems; justify results.
 - a. Solve using tables, graphs, and algebraic properties.
 - b. Interpret the constants, coefficients, and bases in the context of the problem.

Advanced Functions and Modeling, Least Squares Lesson

1.01 Create and use calculator-generated models of linear, polynomial, exponential, trigonometric, power, and logarithmic functions of bivariate data to solve problems.

- a. Interpret the constants, coefficients, and bases in the context of the data.
- b. Check models for goodness-of-fit; use the most appropriate model to draw conclusions and make predictions.