ABSTRACT

MOJICA, GEMMA FOUST. Middle School Teachers’ Conceptions of Empirical and Theoretical Probability: A Study of Pedagogy. (Under the direction of Dr. Hollylynne Stohl Lee).

The purpose of this study is to address the insufficiency of research in teachers’ understanding of probability. This present study specifically investigates middle school teachers’ conceptions of empirical and theoretical probability using teachers’ pedagogical decisions as a lens for their understanding. This study also explores how teachers’ utilize models and representations, examples, and approaches of empirical and theoretical probability.

The participants in the study are inservice classroom teachers in North Carolina public schools who are involved in a 5 year NSF funded professional development project. Four participants from the project were chosen as case studies. The design of this qualitative study utilizes Pretest and Posttest assessment instruments, a videotaped lesson of a probability teaching episode involving empirical and theoretical probability, and a written reflection of the videotaped lesson by each participant.

The conceptual framework for the study considers two theoretical perspectives as a basis for understanding middle school teachers’ conceptions of empirical and theoretical probability: 1) Lee, Rider, and Tarr’s (2005) bi-directional model relating empirical data and theoretical probability, and 2) Even and Kvatinisky’s (2002) analytical framework for teacher knowledge and understanding about empirical and theoretical probability.

Findings from the study indicate that participants attempt to make connections between empirical and theoretical probability while teaching probability lessons; however,
they do not have a robust understanding of the relationship between empirical and theoretical probability. Although participants rely heavily on a theoretical estimation as the *true* probability, some have difficulty calculating the likelihood of uncertain events using a theoretical approach. Some participants are unable to move flexibly between empirical data and a theoretical model, ignoring sample size, variation, and independence. Most participants’ use of representations and basic repertoire of examples are limited. Participants use approaches of probability both appropriately and inappropriately.
MIDDLE SCHOOL TEACHERS’ CONCEPTIONS OF EMPIRICAL AND 
THEORETICAL PROBABILITY: A STUDY OF PEDAGOGY

by

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Dr. Karen Hollebrands
DEDICATION

To Chris.
BIOGRAPHY

Gemma Foust Mojica was born at Camp Lejeune, NC, on December 19, 1975. Upon graduation from White Oak High School in Jacksonville, NC, she enrolled at East Carolina University in Greenville, NC in 1994. At East Carolina, she was a North Carolina Teaching Fellow and active member of the Gamma Chapter of the North Carolina Council of Teachers of Mathematics. She was also actively involved in Pi Mu Epsilon, Omicron Delta Kappa, and the University Honors program. In 1998, she received a Bachelor of Science in Mathematics and became licensed to teach mathematics in grades 8-12 in North Carolina public schools.

Gemma married Paul Christopher Mojica in June of 1998 and moved to Wake County where she began her career as a mathematics educator teaching middle school students. She taught mathematics at Zebulon GT Magnet Middle School for 5 years and taught for 1 year at Moore Square Magnet Middle School in downtown Raleigh. While at Zebulon Middle, Gemma taught Mathcounts as an elective and coached the school’s Mathcounts competition team.

In 2002, Gemma became a Lead Teacher in the North Carolina Middle Math Project. As a result of participation in this 5 year project, she began pursuit of her Master’s Degree in Mathematics Education and National Board Certification. In 2004, she decided to leave the classroom and become a full time graduate student at North Carolina State University while working as a Research Assistant in the Center for Research in Mathematics and Science Education. She received National Board Certification in Early Adolescence in Mathematics in 2005. After receiving a Master of Science, Gemma plans to pursue a Ph.D. in Mathematics Education at North Carolina State University.
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CHAPTER 1
INTRODUCTION

Noss and Hoyles (1996) suggest that mathematics plays a role in our everyday lives despite the fact that we are unaware of its powerful orchestration over our existence. Mathematics is omnipresent but may remain hidden to the casual observer. While the average person may be unaware of the numerous ways in which applications of mathematics affect his or her life on a daily basis, Pratt (2005) suggests that probability is different from other areas of mathematics in that it explicitly permeates life on a day-to-day basis. Probability influences the way we interpret the world around us, and shapes many of our decisions including those of critical events in our lives (Batanero, Godino, & Roa, 2004; Pereira & Swift, 1981; Vere-Jones, 1995). People regularly make judgments about chance in the context of weather, sports, games, and other events. Probability is used to make decisions in business, economics, government, medicine and research that affect how ordinary people live their lives. Will it rain today? Should I buy a lottery ticket? How likely is it that I will get stuck in rush hour traffic? These are all examples of ordinary situations in which people are required to make predictions about uncertain events using probabilistic reasoning.

Perhaps it is for this reason among others that probability has become more prominent in the K-12 mathematics curriculum in the United States within the last fifteen years (National Council of Teachers of Mathematics [NCTM], 1989, 2000). Other countries, like Australia and England, have also adopted probability as part of their mainstream mathematics curriculum (Ahlgren & Garfield, 1991; Greer & Mukhopadhyay, 2005; Pratt, 2005; Shaughnessy, 1992; Watson, 2001). In fact, many countries have recognized statistics and probability as an important area of the curriculum long before its rise to prominence in
the United States. Many advocates of teaching probability in primary and secondary schools point to the usefulness of probability in every day life. (Batanero et al., 2004; Pereira & Swift, 1981; Pratt, 2005; Shaughnessy, 1981, 1992; Vere-Jones, 1995). These authors, as well as others, recognize the importance of being an informed citizen and the critical role that school plays in helping students develop their probabilistic reasoning. Students live in an increasingly sophisticated world. An understanding of probability can help students navigate through this world and become more savvy consumers.

Other arguments that justify the teaching of probability in schools include its relationship with other disciplines as well as other areas of mathematics. Vere-Jones (1995) points to importance of statistical reasoning in the fields of industry and business, while Pereira and Swift (1981) identifies the importance of having knowledge of statistics and probability to understand biology and the social sciences. Number, measurement, estimation, statistics, data analysis, and problem solving are all areas of mathematics that can be linked to probability (NCTM, 1989). Probability should also be included in the school curriculum because it will be necessary knowledge for students as they encounter higher level mathematics (Pereira & Swift, 1981). An understanding of probability can also help younger students develop concepts such as ratios, fractions, percents, and decimals (NCTM, 1989).

Despite the utility of probability in understanding other disciplines, its connection to other areas of mathematics, and its usefulness in everyday life, many individuals have difficulty understanding ideas and concepts related to probability theory (Batanero, Henry, & Parzysz, 2005). Research revealing people’s misunderstandings and misconceptions of probability are prevalent throughout literature (Shaughnessy, 1992). Shaughnessy (1981) suggests that “when students enter college, most of them are (a) unfamiliar with probability
and (b) subject to misconceptions of probability that are deeply entrenched and therefore hard to overcome” (p. 95). These misconceptions are not held only by pre-college students, but adults as well (Kahneman & Tversky, 1972; Konold, 1991, 1995; Tversky & Kahneman, 1971, 1973).

Difficulties in understanding probability have major implications for our schools and the teaching and learning of this subject. Since probability has become a part of the school curriculum in many middle and secondary schools, teachers are faced with the task of teaching content related to probability, even though they are often inexperienced and unprepared to teach such content (Greer & Mukhopadhyay, 2005; Shaughnessy, 1992; Stohl, 2005). Stohl (2005) suggests that some preservice and inservice teachers may be operating with nonnormative beliefs about probability. It seems an impossibility that teachers can convey normative ideas about probability to their students if they have a weak understanding of probabilistic concepts themselves. After all, as Jones and Thornton (2005) point out “the critical person in any learning environment is the teacher” (p. 82). The teacher is ultimately responsible for choosing which aspects of the curriculum to teach in his or her classroom and the methods and experiences that will be used to convey content to students.

One important issue related to the teaching and learning of probability in the classroom is the approach that teachers use to interpret uncertainty. Teachers’ perspectives of probability will both directly and indirectly impact the development of probabilistic reasoning in their students. When dealing with issues of uncertainty, either an objective or subjective approach can be taken. When uncertain events are repeatable and assumed to be identical, an objective approach estimates probability using a classical Laplacean perspective or frequentist view. The classical view is an a priori approach which estimates the
probability of an event as the ratio of favorable outcomes to the number of elements in the sample space. A frequentist perspective is an *a posteriori* approach that involves a simulation or experiment. After an experiment is repeated enough times, the probability is the value at which the relative frequencies stabilize. The different approaches of probability are discussed in more detail at the beginning of Chapter 2.

Stohl (2005) indicates that many teachers prefer to use a classical, or theoretical, approach to probability. This classical view of probability subjugated the secondary mathematics curriculum before the 1970’s and was later complemented with an axiomatic approach (Batanero et al., 2005). Only recently has an empirical approach to probability been initiated in the classroom (Jones, 2005; Lee, Rider, & Tarr, 2005). Batanero et al. (2005) indicated that “there is a growing interest in an empirical introduction of the notion of probability as a limit of the stabilized frequency” (p. 31).

The NCTM (1989) recommends that students in grades 5 – 8 should have learning experiences that encompass both an empirical and theoretical approach to probability. The exploration of probability in the middle school should lead students to:

- model situations by devising and carrying out experiments or simulations to determine probabilities;
- appreciate the power of using a probability model by comparing empirical results with mathematical expectations; [and]
- make predictions that are based on empirical or theoretical probabilities (p. 109).

Shaughnessy (1981) also encourages classroom teachers to provide activity-based learning experiences that consider both an empirical and theoretical approach to probability. He
suggests that students are more likely to overcome certain judgmental heuristics (e.g. representativeness and availability) if probability is introduced by experimentation.

Shaughnessy advises that students should

work in small groups, make guesses for the likelihood of events, perform an experiment, organize and analyze the data, and make estimates for the likelihood of events based on their data. More advanced students can perhaps attempt to build a theoretical model for the experiment and then compare theoretical predictions to empirical outcomes (p. 96).

Statement of the Problem

Within the last couple of decades, very little research focusing on the teaching and the learning of probability has emerged. In 1992, Shaughnessy acknowledged there is a critical need for more research in this field, specifically in the area of teachers’ conceptions of stochastics. Research has fallen short in fulfilling this request (Watson, 2001). Stohl (2005) indicates that there is still an insufficient knowledge base of teacher’s content knowledge and pedagogical content knowledge in relation to probability. As a result, some teacher education programs and professional development projects have not adequately prepared teachers for the challenges of teaching a curriculum that includes probability (Stohl, 2005). Some of these teachers have had little or no experience with probability as a learner; however, they are expected to teach an area of mathematics that they themselves may not fully understand.

Since it is known that both students and adults have misconceptions concerning probability, insight into teachers’ understanding could help inform teacher education programs. If the knowledge base of teachers’ conceptions of probability were broader, it could be used in the design of teacher education programs and professional development.
projects. More effective preparation of both preservice and inservice teachers could ultimately lead to teachers’ ability to provide better quality instruction for students regarding probability.

The purpose of this study is to address the insufficiency of research in teachers’ understanding of probability. This present study specifically investigates middle school teachers’ conceptions of empirical and theoretical probability using teachers’ pedagogical decisions as a lens for their understanding. This study also explores how teachers’ utilize models and representations, examples, and approaches of empirical and theoretical probability. The specific research questions investigated in this study will be stated at the end of Chapter 2.
CHAPTER 2

REVIEW OF LITERATURE

As indicated in Chapter 1, as a response to curricular changes throughout the United States, probability and statistics have become a part of the mainstream curriculum in many schools. The rationale for including probability in the school curriculum is based on but not limited to its utility in real life situations, its relationship to critical thinking, and its connection to other areas of mathematics and disciplines. In order for teachers to communicate normative ideas about stochastics to their students, teachers must have an understanding of probabilistic and statistical concepts, like empirical and theoretical probability. Teachers must also possess pedagogical knowledge about the teaching and learning of stochastics in order to foster probabilistic reasoning within the students that they teach. In this chapter, literature related to these ideas will be examined.

The literature review begins by describing the different approaches that can be taken in interpreting probability and is followed by literature related to the law of large numbers. Research related to student and adult misconceptions of probability is reviewed next. Afterwards, literature concerning teacher subject matter content knowledge and pedagogical content knowledge is examined. The last section of the literature review looks at research related to teacher’s beliefs and content knowledge of probability. Since much of this research shaped the conceptual framework for this study, particularly research related to approaches to probability and literature related to teacher knowledge, the conceptual framework will be presented after the last section of the literature review. The specific research questions that were investigated in the present study will be listed at the end of this chapter.
Approaches to Probability

There are two approaches that can be taken when dealing with probability, the *objective* approach or the *subjective* approach (Batanero et al., 2005; Borovcnik, Bentz, & Kapadia, 1991; Lee et al., 2005; Stohl, 2005). An objective perspective estimates uncertainty using a *classical* or *frequentist* view. This viewpoint can only be used for events that are repeatable and assumed to be identical. The classical Laplacean perspective of probability embodies an *a priori* approach which means probability can be calculated before any trials occur. In using a classical approach, the probability of an event is estimated by finding the ratio of favorable outcomes to the number of elements in the sample space. The classical perspective is representative of the physical intricacies and symmetry of the probability of an event (Borovcnik et al., 1991; Lee et al., 2005; Stohl, 2005). Students usually have experiences in primary and middle schools that consider equiprobable events (e.g. rolling dice, spinning spinners, and flipping coins). For example, students are often asked to roll a fair six-sided die and consider the probability of obtaining a “three.” The theoretical estimate of 1/6 takes into account the symmetry of the die.

In some cases, like rolling a fair six-sided die, equiprobability seems obvious, but other cases, especially those involving humans or nature, may be more complicated for students (Batanero et al., 2005). To muddy the waters even more, an experiment may also uncover different symmetries (Borovcnik et al., 1991). Even the simple case of rolling a six-sided die is not free from complications. The equiprobability of rolling a die may be evident to some learners, but this is not always obvious to young students (Lee et al., 2005; Watson & Moritz, 2003).
If students were asked to conduct an experiment by repeatedly rolling a fair six-sided die and obtain the proportion of “threes” that were rolled, they would be estimating the probability of rolling a “three” using a frequentist approach. A frequentist approach approximates probability *a posteriori*, after the trials in an experiment have been conducted. When an experiment is repeated enough times, the probability is the value at which the relative frequencies stabilize. Thus, different experiments may result in different empirical estimates of the probability of the same event. It may also be difficult for students to determine the number of trials needed to estimate the probability of an event (Batanero et al., 2005; Borovcnik et al. 1991). In a student’s mind, it may be unclear as to what constitutes “enough times” for the frequencies to stabilize.

A subjective approach views probability as a degree of belief or confidence of the individual assigning the probability. This perspective of probability is not based on objective measurable properties. Hence, individuals may assign different probabilities to the same event depending on these immeasurable properties. Differences in estimation can even arise when different individuals observe the same empirical results (Lee et al., 2005).

Many teachers harbor the belief that mathematics is about applying algorithms that result in right or wrong solutions, thus favoring a classical approach to probability in their classrooms (Stohl, 2005). Teachers embracing a deterministic view of probability “assume that the purpose of teaching and learning about probability is to use procedures to calculate theoretical probabilities in the absence of considering the real world application of these probabilities” (p. 347). Stohl indicates a consequence of solely viewing probability from this perspective is that these teachers rely on counting techniques, ask students to calculate probabilities that result in a single theoretical answer, and deny students the opportunity to
consider realistic approximations. Stohl states, “Only in an approach to teaching that embraces both a classical and frequentist approach for estimating probability can students develop appropriate probability intuitions” (p. 348).

Steinbring (1991) also supports the argument that empirical and theoretical probability should be developed concurrently in the classroom. Steinbring states

If taken absolutely, the frequentist and the classical approach to define the concept of probability result in circularities in their epistemological characterization. Their educational derivations in the shape of developing the concept by abstraction or developing the concept operatively are also trained by one-sidedness and show that the respective opposite aspect is required in order to understand the concept completely. Teaching stochastics cannot simply be based on explicit definitions of probability and chance which are provided so comprehensively that they contain, in principle, all the mathematical consequences. Rather, the concept of probability and the concept of chance must be developed while being mutually related to each other (p. 143).

He relates the object (empirical probability), the sign (theoretical probability), and the concept of chance and probability as a triangle (see Figure 1). Jones and Thornton (2005) state that “it is the relationships between these vertices that not only defines the meaning of chance but also provides the pedagogical orientation for organizing classroom learning” (p. 78). Steinbring indicates that viewing probability as a relational triangle of object, sign, and concept can help teachers avoid a perspective that overemphasizes either mathematical
calculations or experiments. A teacher’s understanding of both approaches, therefore, is critical to developing normative reasoning in students.

![Steinbring’s triangle of relations of object, concept, and sign.](image)

**Figure 1.** Steinbring’s triangle of relations of object, concept, and sign.

The Law of Large Numbers

Only recently has the frequentist approach, based on the *law of large numbers*, been introduced into the classroom (Jones, 2005; Lee et al., 2005). Since the *law of large numbers* relates empirical probabilities to theoretical probabilities, it is important that teachers have an understanding of this principle when conducting experiments and interpreting empirical probabilities in light of data.

Individuals can misinterpretation the *law of large numbers* (Sedlmeier & Gigerenzer, 1997; Stohl, 2005). People often confuse the *empirical law of large numbers* with the mathematical theorem of the *law of large numbers*. Stohl states that “one source of many misconceptions (e.g. gambler’s fallacy, law of small numbers) may be due to an incorrect interpretation of this law as implying that empirical probabilities *limit* to the theoretical probability” (p. 348). As the number of trials increases in an experiment, the “probability of a large difference between the empirical probability and the theoretical probability limits to zero” (p.348). Simply stated, the empirical *law of large numbers* tells us that larger samples
usually estimate theoretical probability better than smaller samples. Larger samples usually result in more accurate estimates of the population parameter than do smaller samples. This principle does not imply that the empirical probability will always come close to the theoretical probability when the number of trials is large. When teachers have misconceptions about the law of large numbers, they may “misguide students into expecting a necessary convergence of empirical probabilities with a large number of trials” (Stohl, 2005, p. 348).

Kahneman and Tversky’s work in the early ‘70’s ascertained that naïve subjects ignored sample size. Tversky and Kahneman (1971) also found that even trained psychologists held erroneous intuitions about chance and sample size. Professional psychologists at the meetings of the Mathematical Psychology Group and of the American Psychological Association responded to a questionnaire concerning research decisions.

Tversky and Kahneman posited some “people’s intuitions about random sampling appear to satisfy the law of small numbers, which asserts that the law of large numbers applies to small numbers as well” (p. 106). The believer in the law of small numbers often does not recognize sampling variation. Thus, this fallacy will remain intact in the mind of the believer. This study shows that despite training with formal logic and probability theory, erroneous intuitions can not always be eliminated, even in adults.

Research on Student and Adult Misconceptions about Probability

Most of the research that has been conducted in mathematics education regarding probability and statistics has been carried out by cognitive psychologists or mathematics and statistics educators (Shaughnessy, 1981). Researchers from both traditions have found that students and adults have misconceptions about probability (Kahneman & Tversky, 1972,
1973; Konold, 1995; Tversky & Kahneman, 1971). While some literature indicates that novices in probability and statistics have difficulty understanding concepts of stochastics, there is evidence to suggest that even trained psychologists can fall prey to these misconceptions (Tversky & Kahneman, 1971). Middle school mathematics teachers, whether they have been trained or untrained in the teaching of probability, may have misconceptions.

**Representativeness and Availability**

In the early 1970’s, psychologists Tversky and Kahneman found that people use specific judgment heuristics, *representativeness* and *availability*, when reasoning about uncertainty (Kahneman & Tversky, 1972; Tversky & Kahneman, 1973). Their work consisted of a series of ten studies which included 1500 subjects. The subjects were comprised of several groups of college level and pre-college level students. Two groups of college level students participated, one from the University of Oregon and the other from Stanford University. College-preparatory students from different high schools in Israel also participated; these students were between the ages of 15 and 18.

Kahneman and Tversky (1972, 1973) found these novices did not use principles of probability theory when evaluating the probability of uncertain events. They also found that they could predict the type of judgments that people would use and that these judgment heuristics, *representativeness* and *availability*, were systematic and difficult to eliminate. The dependence on heuristics biases subjective probabilities.

People that reason using a *representative* heuristic make judgments based on how similar an event is to the parent population or the similarity between the process by which the event is generated (Kahneman & Tversky, 1972). The following question and participant response illustrates how people reason using a *representative* heuristic:
All families of six children in a city were surveyed. In 72 families the exact order of births of boys and girls was GBGBBG. What is your estimate of the number of families surveyed in which the exact order of births was BGBBBB?

Although the two sequences are equally likely under the assumption that the probability of a boy birth is the same as a girl birth, 82% of the participants in the study erroneously reasoned that the sequence BGBBBB was less likely than the sequence GBGBBG. To those using representativeness, the number of boy births and girl births should be nearly equal. If the proportion of boys to girls is considered in the sequence BGBBBB, it is less representative of the population’s proportion of the same number of boy births to that of girl births. Participants in the study also thought that the sequence BBBGGG was less likely than GBBGBG because BBBGGG appears to be less random.

Tversky and Kahneman (1973) also found that people make decisions about uncertain events by employing an availability heuristic. When people use this approach to estimate frequency or probability they make decisions based on how easy it is to construct occurrences of the event. An individual’s perception of the frequency of an event also characterizes the availability heuristic. For example, if you have been given a traffic citation for speeding, you might predict a higher estimate for the number of drivers who have received citations for speeding. This estimation of the frequency of drivers who have been cited for speeding is based on one’s own personal experience of the event and could be biased. Tversky and Kahneman (1973) asked subjects in their study to consider the following problem:
In the drawing below, there are ten stations along a route between Start and Finish. Consider a bus that travels, stopping at exactly \( r \) stations along the route.

START        FINISH

What is the number of different patterns of \( r \) stops that the bus can make?

There was an overall underestimation of the number of patterns that could be created. The only exception was that people found it easier to find different patterns for two stops. Even though there are as many patterns for two stops as there are for eight stops, the most available case to the subjects in the study was two stops. Tversky and Kahneman’s analysis of this phenomenon suggests that patterns of two stops are easier to notice right away, and these patterns are easier to distinguish and visualize.

**Outcome Approach**

In an interview study with college aged students, Konold (1991, 1995) found that when individuals make decisions about uncertainty, they attempt to predict a single trial of an experiment, rather than treating the outcome as part of the experiment as a whole. This is called the *outcome approach*. Individuals using an *outcome approach* reduce reasoning about uncertainty to decisions of yes or no. For example, subjects were asked to interpret the weather forecast “70% chance of rain.” Individuals using the *outcome approach* interpreted the prediction to mean “It’s going to rain today.” They were considering one specific event instead of considering the forecast over a long period of time. *Outcome* oriented individuals tend to use the values 0%, 50%, and 100% as points of reference. Values close to 0% would be interpreted to mean “No, it will not rain” while values close to 100% would be interpreted to mean “Yes, it will rain today.”
Like Kahneman and Tversky, Konold (1995) found that students’ beliefs about probability are persistent. He also found that students could have contradictory beliefs about a specific situation. While conducting computer simulations to predict the probability of various outcomes of an event, undergraduates switched among different perspectives about a distinct situation. The following questions were given to students:

Coin Problem: Part 1. Which of the following sequences is most likely to result from flipping a fair coin 5 times?

a. HHHTT
b. THHTH
c. THTTT
d. HTHTH
e. All four sequences are equally likely.

Coin Problem: Part 2. Listed below are the same sequences of H’s and T’s as listed above. Which of the sequences is least likely to result from flipping a fair coin 5 times?

a. HHHTT
b. THHTH
c. THTTT
d. HTHTH
e. All four sequences are equally unlikely.

Around 70% of the subjects responded correctly in Part 1, all sequences are equally as likely. Written justifications by subjects did not confirm that students really understood the underlying principles of why the sequences were equally as likely. Therefore, Parts 1 and 2
were administered to a second group of twenty subjects during clinical interviews. Konold found that students used the *outcome approach* in Part 1 and the *representativeness* heuristic in Part 2. In Part 1, students thought they were being asked to predict which sequence would occur. In Part 2, students believed that the sequence THTTT is less likely to occur because it has only one H. Konold reasoned that when students were unable to employ the *outcome approach* they instead fell back on the *representativeness* heuristic.

Research on Teacher Knowledge

Regardless of the subject being taught, the knowledge base of the teacher plays a role in student learning (Shulman, 1987). In order to comprehend issues related to the teaching and learning of probability, it is important to have an understanding of both teacher’s content knowledge of probability and teacher’s knowledge of how to teach probabilistic concepts (Stohl, 2005). Therefore, it is necessary to first explore teacher knowledge in general in order to have a broader lens for interpreting a teacher’s knowledge specifically related to probability. Research regarding teacher knowledge will first be explored followed by a review of literature relating to teacher’s knowledge of probability.

*General Teacher Knowledge*

According to Shulman, (1987) the teacher is the “primary source of student understanding of subject matter” (p. 9). Teaching starts with the teacher’s understanding of knowledge and the teacher’s conceptions of how this knowledge is to be taught to students. (Shulman, 1987). In the classroom, the teacher is ultimately responsible for designing and providing opportunities for students to ascertain new knowledge. This task is not easy.

A teacher makes a multitude of decisions and engages in mathematical reasoning prior to and during instruction in order to create learning experiences for students; the nature
of these decisions have an impact on what can be learned by students (Ball, 2000; Fennema & Franke, 1992; Wilson, Shulman, & Richert, 1987). For example, a teacher must decide which parts of the curriculum to highlight in his or her classroom and which types of tools and activities will be presented to students. During instruction, the teacher must also make many decisions that affect the outcome of learning in his or her classroom. For example, the teacher must determine appropriate questioning techniques and suitable interactions with his or her students. All pre-active and interactive decisions made by teachers are dependent on the knowledge base of the teacher.

In the 1980’s, Shulman and his colleagues studied the knowledge base of teachers. Shuman (1986, 1987) specifically studied the transition of student teachers to first year teachers in order to identify how new teachers learn to teach. This research focused on the development of secondary teachers in California in the subject areas of English, biology, mathematics, and social sciences. Shulman and his colleagues also studied veteran teachers to compare their practices to the novice teachers. Shulman identified that there are three kinds of content knowledge that teachers possess: subject matter content knowledge, pedagogical content knowledge, and curricular knowledge. Shulman (1986) emphasizes the importance of subject matter content knowledge by stating that

Teachers must not only be capable of defining for students the accepted truths in a domain. They must also be able to explain why a particular proposition is deemed warranted, why it is worth knowing, and how it relates to other propositions, both within the discipline and without, both in theory and in practice (p.9).
Pedagogical content knowledge is the knowledge that allows teachers to communicate conceptions of their discipline to their students. Shulman (1986) defines pedagogical content knowledge as

useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations – in a word, the ways of representing and formulating the subject that make it comprehensible to others (p. 4).

According to Shulman, the teacher must have an understanding of the different types of representations, founded in research and in the practice of teaching, that are needed to help students develop an understanding of the content. Teachers must also be able to identify what makes the learning of topics easy or difficult for students.

The third category of content knowledge that teachers possess is curricular knowledge. This type of knowledge guides teachers in their ability to make decisions concerning instructional materials such as textbooks, technology, and manipulatives. This type of knowledge is not limited to the selection of instructional materials and tools. It also includes an understanding of how to sequence these materials within a curriculum. All three types of knowledge are important when examining teachers’ conception of probability.

Wilson et al. (1987) argue that although it is necessary for teachers to have an understanding of the subject matter that they teach, subject matter content knowledge alone is insufficient to foster understanding in students. They claim that teachers must also possess pedagogical content knowledge, the knowledge that allows teachers to communicate conceptions of their discipline to their students. Although both types of knowledge are equally important to teachers, they are often treated as separate domains.
In accordance with Shulman, Ball (2000) confirms the dual importance of the understanding of theoretical knowledge about subject matter and the development of pedagogical methods. Ball argues that understanding content knowledge is crucial to creating worthwhile learning opportunities for students in mathematics. Ball states that “No matter how committed one is to caring for students, to taking students’ ideas seriously, to helping students develop robust understandings, none of these tasks of teaching is possible without making use in context of mathematical understanding and insight” (p. 243). Ball also claims that knowing how to use this knowledge of mathematics coupled with the understanding of mathematics itself is the heart of teaching all students. She also asserts that issues regarding teacher understanding of subject matter and their ability to use this understanding in teaching should be viewed from the perspective of the practice of teaching.

Teacher Beliefs and Content Knowledge of Probability

Literature related to teachers’ conceptual understanding of probability and their knowledge for teaching probability is far less extensive than research regarding students’ understanding of probability (Stohl, 2005). Studies in the United States and New Zealand have found that inservice and preservice teachers do not have a strong understanding of probability (Begg & Edwards, 1999; Carnell, 1997; Carter & Capraro, 2005; Haller, 1997; Lui, 2005). These studies indicate that inservice and preservice teachers hold the same misconceptions as students. Some of these misconceptions persist even after instruction or participation in professional development (Carter & Capraro, 2005; Haller, 1997). Teachers’ conceptions of probability are often situational, changing from one context to the next (Lui, 2005).
Studies by Begg and Edwards (1999) and Carter and Capraro (2005) have similar findings. To assess teachers’ statistical background and knowledge, Begg and Edwards studied 22 inservice New Zealand elementary school teachers and 12 preservice teachers who were enrolled in a college of education. Carter and Capraro (2005) studied 108 preservice teachers’ misconceptions of probability. Participants were enrolled in an introductory statistics class that did not explicitly cover topics of probability.

In clinical interviews, Begg and Edwards found that the teachers had a weak understanding of the law of large numbers. Likewise, in Carter and Carpraro’s study, participants responding to an on-line assessment instrument exhibited a weak understanding of the law of large numbers. Participants in both studies also reasoned using the representativeness heuristic. In the Begg and Edwards study, there was no evidence to indicate that teachers believed order or patterns are associated with random events. For example, even though 24 of the participants indicated that they had played the lotto and 9 indicated that they understood how to play the lotto, the sequence 1, 2, 3, 4, 5, 6 and the sequence 5, 10, 15, 20, 25, 30 were judged less random than the sequence 2, 13, 19, 27, 30, 38. Participants in the Carter and Capraro study erroneously disregarded sample size, believing that if the proportions of favorable events were equal in two samples of different sizes, the probabilities were equally as likely. For example, some participants reasoned that the chance of getting 3 tails out of 5 flips of a coin is equally as likely as getting 3000 tails out of 5000 flips.

The two studies produced mixed results in terms of teachers’ understanding of the concept of equally likely. Most of the teachers in the Begg and Edwards study had a good understanding or were familiar with the concept of equally likely, but only half understood
randomness. In contrast, Carter and Capraro found evidence to suggest that preservice teachers have a weak conceptual understanding of the concept of equally likely. Participants assumed equality of events in inappropriate situations.

Carter and Capraro also found that preservice K-8 teachers had difficulty with conditional probability. Carnell (1997) found similar results when conducting a study to investigate preservice middle school teachers’ conceptions of conditional probability. This study was grounded in Falk’s analysis of major difficulties. Carnell studied 13 preservice teachers’ reasoning related to difficulties in the following: defining a conditioning event; the temporal order of the conditioning event and the target event, and confusing conditionality and causality. Carnell found that the preservice teachers in her study held misconceptions about conditional probability. All of the participants had difficulty defining the conditioning event and using it appropriately. Regarding the misconception of confusing conditionality with causality, a majority of the participants showed evidence of having this misconception, as well.

In regards to the misconception related to the temporal order of the conditioning and target events, evidence suggested that participants ignored the conditioning event. In particular, Carnell found that the preservice middle school teachers had did not utilize information about the conditioning event if the target event occurred before the conditioning event in real time. These teachers did not have a strong understanding of independence and dependence. Begg and Edwards also found that less than half of the teachers in their study had an understanding of independence.

Another important finding of Carnell included the misapplication of procedures and participants’ readiness to make problems involving conditional probability simpler than they
It was not uncommon for the preservice teachers to consider a sample space that was incomplete. Consider the following problem from Carnell’s study:

A woman is expecting fraternal twins. A chromosomal test is performed on random cells out of one of the amnion. The results show it is a boy. Assuming that boys and girls are equally likely to be conceived, what is the probability that the woman is expecting two boys?

If \( b \) represents the probability of conceiving a boy and \( g \) the probability of conceiving a girl, the sample space, \( S \), of the event of expecting twin boys could be described as \( S = \{ \frac{b_1}{b_2}, \frac{g_1}{g_2}, \frac{b_2}{b_1}, \frac{g_2}{g_1}, \frac{g}{b} \} \). However, the participants gave incomplete definitions of the sample space. The sample space was defined as one of the following: \{boy/boy, boy/girl\}, \{boy/boy, boy/girl, girl/girl\}, \{boy/boy, boy/girl, girl/boy, girl/girl\}, or \{boy/girl\}.

In a 2001 study, Watson gained insight into teachers’ subject matter content knowledge and pedagogical content knowledge by developing an instrument to profile teacher achievement in probability and statistics. This instrument was also designed to assess the needs of Australian teachers in terms of a new mathematics curriculum. Watson’s study included 43 Australian public and private school mathematics teachers of grades 1 – 6 and grades 7 – 12. She found a need to improve teachers’ content knowledge of probability as well as their pedagogical content knowledge.

Watson found several critical issues relating to the teaching of stochastics. Watson indicated that there was not evidence to support coherent teacher planning with respect to chance and data at the primary level. Primary teachers needed experience with probability in equally likely outcomes, basic probability outcomes, odds, and sampling. Teachers were generally more familiar with the concept of average than the concept of sample.
When questioned about topics that they would include in preparing a unit on chance and data, primary teachers chose topics such as surveys, graphs, chance in general, and probability contexts that involved rolling dice. The most common topic mentioned by secondary teachers was probability. More than half of these teachers chose topics that they considered to be their favorite. Almost 30% of the teachers indicated that the most enjoyable topics to teach were graphing, normal distributions, surveys and data collection, analyzing and interpreting data, and different facets of probability. These teachers indicated that these topics were enjoyable for their students as well. Watson (2001) stated that “these data suggest that many teachers are either comfortable describing the teaching of topics they enjoy or come to enjoy the topics they are more familiar with teaching” (p. 317).

Studies investigating teachers’ conceptual knowledge of probability have also been fruitful in unveiling teachers’ attitudes towards the use of technology and manipulatives for teaching probability. These studies have produced mixed results. Begg and Edwards (1999) found that technologies such as calculators and computers were not widely used by teachers. When technology was used, it was used for tasks like computing percentages or for word processing. In contrast, Watson revealed that teachers are using concrete materials and technology in their instruction. 70% to 81% of the teachers used calculators or computers, concrete materials, materials with chance outcomes, and sources of data to teach topics that involved chance or data.

By examining the results of studies that have previously been discussed in this section of the literature review, some insight into teachers’ beliefs about their ability to teach probability have also been uncovered. Begg and Edwards (1999) indicated that teachers felt less confident about teaching ideas related to probability than those related to statistics, and
their initial attitudes in relation to statistics were negative in orientation. In terms of teacher confidence, Watson (2001) found significant differences in the confidence level of high school teachers compared to the other teachers. High school teachers felt more confident teaching the following topics: *equally likely outcomes, average, basic probability calculations, median, graphical representation*, and *sampling*. Watson posited that this may have been influenced by the differences in teachers’ backgrounds; high school teachers had stronger mathematical backgrounds than the other teachers in the study. Haller (1997) saw an increase in confidence in subject matter competency as a result of probability instruction, particularly in the area of *sample space* while studying 35 inservice middle school teachers who were involved in a NSF funded summer institute for professional development. Participants in the study also acknowledged that it is difficult to teach probability.

Studies regarding teacher beliefs and understanding of probability indicate that there is some evidence that teachers favor deterministic and theoretical approaches to probability. In an NSF funded professional development project involving 8 inservice secondary teachers, Lui (2005) found that participants fostered a deterministic view of probability. They believed that problems involving uncertainty have one *right* answer. Participants found it difficult to entertain multiple solutions or interpretations of an uncertain event. They expressed a need to have their own students come to a consensus when dealing with probabilistic situations. Further, participants stated that they would avoid situations in the classroom where multiple interpretations of probability were evident. Carter and Capraro suggest that probabilistic misconceptions may persist because contemporary education “forces deterministic cognitive strategies rather than supporting the development of probability concepts in an indeterministic environment” (p. 110). There was evidence that some secondary teachers
taught probability using a theoretical approach, but these teachers did not balance their curriculum with learning opportunities that included experiments or simulations (Watson, 2001).

Literature regarding the implementation of probability lessons by teachers is scarce (Stohl, 2005). Steinbring (1991) studied teaching episodes focusing on the concept of chance in German classrooms. In these episodes, students conducted some type of chance experiment, described the outcome with a stochastical model, and tried to explain the difference between the empirical results and their theoretical predictions. Steinbring found that the concept of “chance emerged as a universal object for explaining the connections between the actual outcome of an experiment and theoretical prediction” (p. 518); students explained the differences in terms of “magic” or “luck.” Steinbring states that reexamining empirical conditions and assumptions of a theoretical model is important in the development of probabilistic reasoning. The teacher’s role in this analysis is pivotal in terms of questioning and fostering social interaction in the classroom.

Haller (1997) also studied classroom teachers’ instructional practices of probability. Haller selected 4 cases from 35 participants who were involved in a professional development project during the summer. Each of the 4 participants in her case study were inservice middle school teachers in the United States. Based on Pretests from the summer institute, teachers initially exhibited a weak understanding of probability. Some participants specifically showed weaknesses in their knowledge of two-stage events, multi-stage events, and conditional probability. Through classroom observations, interviews, and other assessment instruments, Haller found that these teachers held misconceptions during the implementation of their probability lessons. They also failed to link multiple representations
of probability (i.e. fractions, decimals, percents) while teaching probability. These teachers also relied on their textbooks for instruction.

Conceptual Framework for Studying Teachers’ Conceptions of Empirical and Theoretical Probability

The conceptual framework for this study considers two theoretical perspectives as a basis for understanding middle school teachers’ conceptions of empirical and theoretical probability. One of the theoretical perspectives considered is the analytical framework of Lee et al. (2005) that models the bi-directional relationship between empirical data and a theoretical model of probability. The other perspective considered is Kvatinsky and Even’s (2002) analytical framework for teacher knowledge and understanding about empirical and theoretical probability.

Bi-directional Model of the Relationship between Empirical and Theoretical Probability

The first of the theoretical perspectives considered is an analytical framework that models the bi-directional relationship between empirical data and a theoretical model of probability (Lee et al., 2005). This model (see Figure 2) was adapted from Stohl and Tarr’s (2002) original model that describes students’ probabilistic reasoning. In this model, student thinking can either originate with an empirical or theoretical approach to probability. Based on prior experiences and previous knowledge, students create an initial mental image of what they expect to observe. Their original image can then be shaped by considering a different perspective of probability. Students compare their original image to a hypothesis about the probability of an event based on how well their image and hypothesis match in light of empirical evidence.
Student reasoning may enter the model from the theoretical side. Students may begin considering the probability of an event with a mental image that is grounded in a classical Laplacean view. If students participate in an experiment or consider empirical data that has been collected, they then compare their original image with the empirical results. Observations made during the experiment or analysis of the empirical data may lead students to question their original assumptions or hypothesis; however, their observations may actually match their original image. As a result, students may decide to collect and evaluate more data to test the match between their original theoretical image and results from repeated trials of an experiment. Lee et al. identify three influences that affect students’ reasoning about empirical data in comparison to their initial hypothesized theoretical probability: sample size, variability and independence.

Students’ thinking about the probability of an event may enter the model from the empirical side. Sometimes students may have had no prior experience with the event being considered. There are some events that cannot be described by a classical model of probability. Students would construct an initial image and form a hypothesis about the theoretical probability based on their evaluation of the empirical data and its relative frequencies. Students use their image to decide to collect more empirical data. If students determine that more empirical data needs to be collected, they then make decisions regarding how more empirical data should be collected.
The second of the theoretical perspectives considered in this study is based on Kvatinsky and Even’s (2002) analytical framework for teacher knowledge and understanding about empirical and theoretical probability. This framework was based on Even’s (1990) earlier work on subject matter knowledge for teaching mathematics. Even postulates that teachers are more adept at helping students attain a meaningful understanding of mathematics if they possess strong mathematical knowledge. Even claims that there are many aspects of teacher’s subject matter that contributes to a teacher’s understanding of a specific mathematical concept. Even identified seven such aspects: 1) essential features, 2) the strength of the concept, 3) different representations, 4) alternate ways of approaching, 5) basic repertoire, 6) knowledge and understanding of a concept, and 7) knowledge about mathematics.

Applying Even’s framework to the concept of probability, Kvatinsky and Even identified seven aspects of knowledge that contribute to teachers’ understanding of empirical and theoretical probability. Kvatinsky and Even distinguish between the three categories of

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*Framework for Teacher Knowledge and Understanding about Probability*

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![Bi-directional model of probabilistic reasoning.](image)

Figure 2. Bi-directional model of probabilistic reasoning.
teacher content knowledge as identified by Shulman (1986, 1987). The Kvatinsky and Even framework is grounded in

(a) examination of the role and importance of probability in mathematics, as well as in other disciplines and real-world situations,
(b) the role of probability in mathematics school curriculum,
(c) research and theoretical work on learning, knowledge and understanding of mathematical topics and concepts in general and probability in particular, and
(d) research and theoretical work on teachers’ subject matter knowledge and its role in teaching (p. 1).

One component of this framework is concerned with the essential features of probability. These essential features distinguish probability from other areas of mathematics. Kvatinsky and Even claim that there are two essential features of probability that make it different from other domains of mathematics. Kvatinsky and Even define probability as being a mathematical way to deal with uncertainty. They assert that probabilistic thinking is fundamentally different from the deterministic thinking that can be used to reason about other areas of mathematics. The second essential feature of probability is that it is characterized by two different approaches to probability: the objective approach and the subjective approach. These approaches are based on two different interpretations of its meaning.

A second aspect of this framework deals with the strength of probability. Even (1990) remarks that teachers should be aware of the unique and powerful characteristics of the concept they are teaching. Kvatinsky and Even (2002) point to probability’s utility in almost
every field of life. It is applied to everyday situations and many areas that affect people’s lives.

A third aspect of Kvatinsky and Even’s framework describes the different representations and models that teachers of probability need to understand. Teachers need to have an understanding of representations such as tables, Venn diagrams, tree diagrams, formulas, and pipe diagrams. Teachers should be able to translate between different representations and models and link them as well.

The fourth component of this framework, alternate ways of approaching, is closely related to the first and third. According to Kvatinsky and Even, teachers need to be aware of the different approaches to probability (e.g. classical, frequentist, and subjective). It is critical that they know when it is appropriate to use each approach and that the best approach is chosen for the context of the problem.

The fifth aspect of Kvatinsky and Even’s framework is basic repertoire. A basic repertoire of probability should include “powerful examples that illustrate important ideas, principles, properties, theorems, etc.” specific to probability (p. 3). Teachers should also have an understanding of mathematical terms and sub-topics related to the field of probability.

A sixth component of this analytical framework is knowledge and understanding of probability. Kvatinksy and Even acknowledge the terms knowledge and understanding have multiple meanings and different forms of these knowledge exist (e.g. conceptual, procedural, instrumental, relational, formal, etc.). They claim that knowledge and understanding do exist in different forms in relation to mathematics. They also believe that combining and integrating these forms of knowledge empowers an individual, especially when considering probability. Many intuitions about probability often lead people astray. Kvatinsky and Even
claim that “integrating other types of knowledge could serve as a control” for intuitive knowledge (p. 5).

The final aspect of this framework is related to knowledge about mathematics. According to Kvatinsky and Even, there is a relationship between general knowledge of mathematics and knowledge specific to probability. The nature of mathematical knowledge sometimes supports probability knowledge, while at other times withholds it. For example, inductive and deductive reasoning are used throughout mathematics in general. While inductive reasoning supports making conjectures, proof is also required. Being involved in experimentation when considering the probability of an event motivates individuals to form conjectures and test hypotheses. This is an example of how mathematical knowledge supports the knowledge of probability. An example of mathematical knowledge that withholds knowledge of probability is the concept of a limit. Individuals often misinterpret the law of large numbers when evaluating empirical data by inferring that the empirical probability should converge to the theoretical probability.

**Conceptual Framework**

The conceptual framework of this study was designed to consider four aspects of teacher subject matter knowledge of empirical and theoretical probability. These four aspects of the Kvatinsky and Even framework were chosen because of the nature of the data collected in the study and those aspects that would be most prevalent in the data. The following components of teachers’ knowledge of empirical and theoretical probability were chosen to be used in the conceptual framework for this study: representations and models, alternate ways of approaching, basic repertoire, and essential features.
Each of the four aspects of teacher subject matter knowledge represents important components in assessing a teacher’s understanding of empirical and theoretical probability. Teachers’ understanding plays an important role in the decisions that teachers make both before and during instruction. Each of these components, therefore, influences teachers’ pedagogical decisions in the classroom. These aspects all influence the pedagogical choices that teachers’ make in portraying the relationship between empirical and theoretical probability.

The conceptual framework allows one to identify aspects of teacher subject matter knowledge of empirical and theoretical probability. Once an aspect is identified, the conceptual framework considers how this aspect influences the teacher’s understanding of the relationship between empirical and theoretical probability by using the bi-directional model. The bi-directional model can explain how the teacher negotiates his or her original image or hypothesis in light if the classroom episode that is being analyzed. This may shed light on the teacher’s pedagogical understanding and decisions made during instruction.

Within each aspect of the conceptual framework, there are specific guiding questions that can be used to assess a teacher’s understanding of a specific component of empirical and theoretical probability. For example, consider the aspect of representations and models. The following questions can be useful in examining a teacher’s understanding of probability based upon his or her use of representations and models:

- What representations and/or models does the teacher use in relation to empirical and theoretical probability?
- Does the teacher link multiple representations and/or models of empirical and theoretical probability?
- How does the teacher use representations and/or models to illustrate the relationship between empirical and theoretical probability?

Each aspect of the conceptual framework contains key questions that can be useful in assessing teacher’s understanding of each component. These guiding questions used in the conceptual framework can be found in Figure 3.
Figure 3. Conceptual framework.
Research Questions

Shulman (1986, 1987) was the first to identify categories of teacher content knowledge (e.g. subject matter content knowledge, pedagogical content knowledge, and curricular knowledge). This study investigates middle school teachers’ content knowledge of empirical and theoretical probability. The study also explores how teachers’ understanding unfolds in the classroom during instruction and how teachers’ utilize models and representations, examples, and approaches of empirical and theoretical probability. By using the analytical frameworks of Lee et al. (2005) and Kvatinsky and Even (2002), each theoretical perspective provides a lens for addressing both questions of the study. The Lee et al. bi-directional model provides a lens for analyzing teachers’ reasoning about the relationship between empirical and theoretical probability, while the Kvatinsky and Even framework provides a lens for examining teachers’ knowledge of empirical and theoretical probability.

The specific research questions are as follows:

1. What do middle school teachers’ pedagogical decisions during a probability lesson imply about their understanding of empirical and theoretical probability?

2. How do middle school teachers’ utilize models and representations, examples, and approaches of empirical and theoretical probability?
CHAPTER 3

METHODOLOGY

The researcher chose to do a case study of 4 different middle school mathematics teachers who are all involved in a professional development project in North Carolina. The purpose of this chapter is to describe the context of the study, the methods used in the data collection and analysis, and the participants involved in the study.

Context for the Study

Participants in the current study are involved in a state-wide professional development project for middle school mathematics teachers funded by the National Science Foundation. The project is a collaboration between 9 university-based professional development centers. The major goals of the project are two-fold: to improve mathematics education in middle school, and to retain and support teachers in their professional development by providing teachers with the opportunity for academic growth and financial compensation. Teachers in the project were expected to enroll in Master’s courses at one of the 9 universities and apply for National Board Certification in Early Adolescence Mathematics.

The project spans five years involving over 100 teachers. Two cohorts of teachers were chosen to participate at different stages of the project. The first cohort consists of 18 teachers, 2 from each of the 9 university-based centers. The second cohort consists of approximately 130 teachers, between 10 and 30 were chosen from each university-based center. Teachers were selected to participate in the project based on years of teaching experience, the highest degree that he or she held, his or her teaching license (e.g. subject, grade level), and whether he or she was willing to pursue National Board Certification.
The project created 3 graduate-level courses for teachers, focusing on the content areas of data analysis and probability, geometry and measurement, and number and algebra. The courses were designed such that teachers in the project could use this coursework and other initiatives of the project in the pursuit of National Board Certification in Early Adolescence Mathematics and as a foundation in obtaining a Master’s degree in mathematics education. All participants took the 3 graduate courses.

*The Data Analysis and Probability Course*

Twenty-nine teachers in the second cohort from 3 universities in the same region of the state enrolled in the Data Analysis and Probability course in 2003. The teachers met for one week in June for 6.25 hours of instruction for each of four days, and for 3.5 hours on the fifth day. During the summer, teachers met in a central location in the state with an instructor. The instructor is an assistant professor in a mathematics education department at a large southeastern university in the United States. Five teachers from the first cohort assisted the university instructor as co-teachers.

Instruction in the fall of 2003 began in October and ended in December. The class met once a week for 3 hours at a time through distance education. In the fall, teachers met at their respective universities. All 3 university centers were linked by teleconferencing. Each center was able to see the other 2 centers and communicate in real-time. Five teachers from the first cohort again served as facilitators and sometimes as co-teachers during each class.

The Data Analysis and Probability course was designed to increase both teachers’ content knowledge and pedagogical knowledge of probability and statistics. Teachers were exposed to standards-based curricula during the course. Standards-based tasks and lessons were modeled to show participants how to implement these lessons and tasks with middle
school students. Teachers were actively involved in participating in these activities to increase their own content knowledge and better understand their students’ perspectives and conceptions. The courses attempted to heighten participants’ curricular knowledge by exposing teachers to technology, manipulatives, and multiple representations. Teachers had experiences using manipulatives such as spinners, dice, and other concrete tools. They used software such as Excel, Fathom, Probability Explorer, and the Geometer’s Sketchpad.

Teachers also used graphics calculators as well as an applet designed for the calculator to do simulations (e.g. Probability Simulation). At the beginning of the course, teachers were given Fathom, Probability Explorer, Geometer’s Sketchpad, and a TI-83 Plus overhead graphics calculator which they kept after completing the course.

Throughout the course, concepts related to both empirical and theoretical probability were explored. Emphasis was placed on increasing teachers’ understanding of subject matter knowledge related to both perspectives of probability. Many of the lessons in the course highlighted the relationship between the two perspectives. One of the major pedagogical objectives of the Data Analysis and Probability course was increasing teachers’ understanding of how to conduct experiments and simulations to estimate probabilities. Teachers were engaged in experiments and simulations that used concrete manipulatives such as spinners, as well as technology such as the Probability Simulation applet on the TI-83 graphics calculator and computer software like Probability Explorer. A detailed list of the course objectives can be found in Appendix A.

Sources of Data

Three sources of data were collected for this study: videotaped lessons of probability teaching episodes, participants’ written reflections of the videotaped lessons, and Pretest and
Posttest assessments. One source of data was the videotape from the Analyzing Teaching Project. During the course, teachers were required to complete and submit an Analyzing Teaching Project as part of their final grade for the course. The Analyzing Teaching Project counted for 30% of the teachers’ course grades. A complete description of the Analyzing Teaching Project can be found in Appendix B. Teachers were asked to view, reflect, and analyze one videotape of their classroom teaching where a concept or technique in data analysis or probability was central to the lesson. They were given the choice of utilizing either manipulatives or some form of technology in this lesson. Teachers were directed to submit a 15 minute, uninterrupted, unedited segment of the videotaped lesson in VHS format. They were also asked to write a 2 – 3 page reflection about their analysis and answer specific guiding questions for the videotaped lesson. Teachers were asked to choose only 15 minutes of their lesson for submission because that is what is required for the Portfolio Entries for National Board Certification. Many teachers submitted videotapes and written reflections that were considerably longer than the requirements for the project.

The guiding questions for reflection were similar to questions participants would encounter in Portfolio Entries of the National Boards. Teachers were given the description of the project on the first day of instruction in June 2003 and could implement and videotape the lesson between then and the due date in December of 2003.

Pretest and Posttest instruments were used to assess participants’ prior content knowledge of different concepts related to probability and statistics, as well as participants’ knowledge after completion of the course. Both instruments were created by the mathematics educator who created and instructed the course. A Pretest was administered to all 29 teachers.
After completion of the course, the same teachers were given a Posttest as a final exam. For the purpose of this study, only 5 questions from each assessment instrument were analyzed.

Since the research methods chosen for this study are qualitative in nature, the researcher used a *triangulation* protocol. According to Stake (1995), “for data source triangulation, we look to see if the phenomenon or case remain the same at other times, in other spaces, or as persons interact differently” (p. 112). Therefore, the researcher chose to analyze written reflections from the Analyzing Teaching Project along with the videotape data and the assessment instruments. The researcher was looking for evidence that supported or disconfirmed analysis of the videos, participant reflections, or assessment instruments.

**Methods and Analysis of Data**

The 3 sources of data were analyzed separately in 5 phases (see Figure 4). In order to avoid bias based on teachers’ knowledge from the assessment instruments and teachers’ own reflections, the videotaped teaching episodes were analyzed first.

![Figure 4. Phases of data analysis.](image-url)
Phase 1: Describing Classroom Teaching Episodes

Powell, Francisco, and Maher (2003) developed a model for analyzing videotape data based on nearly two decades of research. Their model consists of seven interactive phases which are nonlinear. The phases of data analysis of videorecording are as follows:

- Viewing attentively the video data
- Describing the video data
- Identifying critical events
- Transcribing
- Coding
- Constructing storyline
- Composing narrative.

This model for video analysis was used to analyze the videotaped lessons in this current study during Phase 1 of the analysis. Some of the phases of the Powell et al. model were combined into one phase for the video analysis of this study.

Videotaped lessons of all 29 teachers in the central region of the state were collected. The videotapes were sorted into two categories: videos of data analysis or statistics lessons and videotapes of probability lessons. Of the 29 teachers participating in the project in the central region, 10 of the teachers chose to submit a lesson on probability while 19 submitted a lesson on statistics or data analysis. In the first phase of analysis, each of the 10 videotapes of probability lessons was viewed. Initial viewing of the videotapes was conducted so that the researcher could become familiar with the data.

Then, videotapes were viewed several times so that the teaching episode could be described. According to Powell et al. (2003), the purpose of viewing video data at the
descriptive phase should be to create a record so that an outside observer would be able to have an objective idea of what occurred in the video. In this phase, transcripts of the lessons were created. All 10 videotapes of probability lessons were transcribed. The transcripts included all sounds that could be heard. Utterances and noises made by both the students and teachers were recorded. Any movements or actions that were visible were also included in the transcript. Representations created by the teacher, either on the board or overhead projector, were recorded. All of the information that was recorded in the transcript was descriptive; descriptions were not intended to be interpretative or inferential.

Next, all of the videotapes were converted from analog VHS tapes to digital Real Media files using the software Studio (version 9). Using the software Transtool, a multimedia presentation of each of the 10 lessons was created by the researcher such that the Real Media files and each transcript could be viewed simultaneously in Real Media.

After viewing each of the 10 multimedia presentations, initial descriptions of the probability classroom episodes’ objectives, as identified by the researcher were made. The multimedia presentations were viewed again for an overall impression of each classroom episode as a whole and marginal notes were taken on the transcripts. The researcher looked for overall themes amongst the classroom episodes. Some of the themes that the researcher observed were teachers using simulations and data collection, lessons focusing on empirical versus theoretical probability, and probability and ratio lessons.

*Case study selection.* Once themes were identified, a protocol of questions was created to gauge first impressions of the classroom episodes. Four teachers’ classroom episodes were chosen as case studies based on their themes. Each of the four teachers in this study was chosen based on *criterion sampling*. Miles and Huberman (1998) describe the
purpose for this type of sampling as “all cases that meet some type of criterion” (Creswell, 1998, p. 119). The criterion in this case was that the overarching concept of the probability lesson was related to *empirical and theoretical probability*. Stake (1995) suggests that the first criterion in selecting cases for case study “should be to maximize what we can learn” (p. 4). This was also a consideration in choosing cases. Cases were selected because the researcher felt the most knowledge could be gained from studying each of the cases as opposed to others.

All 4 teachers in this study signed an informed consent form; however, each of the participants has been given a pseudo name. All case study participants are licensed middle school teachers who teach mathematics in North Carolina public schools. Two of the participants in the study teach at the same school, and 3 teach in rural schools. Frank, a teacher with 5 years of experience, is a Caucasian male. Helen is a Caucasian female with 6 years of teaching experience. Kathy is a Caucasian female with 8 years of teaching experience, and Pam is an African-American female who has 15 years of teaching experience. The table below provides some additional information about the participants.

Table 1. Participant information.

<table>
<thead>
<tr>
<th>Teacher</th>
<th>Years of Teaching Experience</th>
<th>License</th>
<th>Grade Level Taught</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frank</td>
<td>5</td>
<td>Math 6-8</td>
<td>7th Grade</td>
</tr>
<tr>
<td>Helen</td>
<td>6</td>
<td>Math 6-8</td>
<td>8th Grade</td>
</tr>
<tr>
<td>Kathy</td>
<td>8</td>
<td>K-6</td>
<td>6th Grade</td>
</tr>
<tr>
<td>Pam</td>
<td>15</td>
<td>K-6</td>
<td>6th Grade</td>
</tr>
</tbody>
</table>
**Phase 2: Coding Video Data**

After each case was chosen, Phase 2 of the analysis began. Both the digital media file of the video and transcript were used as data. Each video was watched many times in digital format using Real Media. The transcripts were also read and reread during this phase of analysis. Critical events were identified relating to the teacher’s subject matter content knowledge and pedagogical content knowledge.

Before the videos were coded, each video was segmented into sections. The criterion for segmentation was different for each case. In Kathy’s teaching episode, for example, her lesson was characterized by using a science analogy. Her video was partitioned into the following segments:

- Helen creates analogy to science experiment
- Students share ideas about science experiments
- Helen and students formulate image of empirical probability
- Helen creates analogy to science theory
- Students share ideas about theories in science
- Helen and students formulate image of theoretical probability and how it is different from empirical probability, and
- Students find the theoretical probability of drawing a blue, green, red, and yellow chip from an envelope (with replacement).

The segmentation not only helped in identifying critical events, but allowed the researcher to focus on the teacher’s and students’ reasoning before and after the critical events.

Each of the 4 classroom episodes chosen as case studies was then coded. Once videos were coded, the initial codes did little to answer the research questions in terms of the
conceptual framework. New codes were then created, and all the videos were recoded. The final set of codes identifies key aspects of the conceptual framework of this study. Table 2 lists the codes used in the analysis.

Table 2: Codes used in data analysis of videos.

<table>
<thead>
<tr>
<th>Code</th>
<th>What Code Identifies</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rep</td>
<td>Representations of empirical and theoretical probability</td>
</tr>
<tr>
<td>Mod</td>
<td>Models of empirical and theoretical probability</td>
</tr>
<tr>
<td>App</td>
<td>Ways of approaching empirical and theoretical probability</td>
</tr>
<tr>
<td>Ex</td>
<td>Examples used during instruction regarding empirical and theoretical probability</td>
</tr>
<tr>
<td>Unc</td>
<td>Presence of uncertainty or uncertain events</td>
</tr>
<tr>
<td>Det</td>
<td>Deterministic reasoning</td>
</tr>
<tr>
<td>Non Det</td>
<td>Nondeterministic reasoning</td>
</tr>
<tr>
<td>Int</td>
<td>Institutions about empirical and theoretical probability</td>
</tr>
</tbody>
</table>

A table was created to quantify the number of times each code was observed in order to observe patterns in the data. The researcher also looked at the different combinations of codes. Finding every combination and counting its occurrence was tedious and time consuming; therefore, the researcher focused on looking for patterns of the most common combinations.

Phase 3: Analysis of Teacher Reflections

During the third phase of analysis, each of the participants’ written reflections of the videotaped lesson were read and analyzed. Participants were specifically instructed to write their reflections after they had viewed their videotapes using the guiding questions from the Analyzing Teaching Project. The purpose of the reflection was to engage participants in thinking about their own practice and its relationship to student understanding. After the
researcher analyzed participant reflections, the researcher compared each participant’s perceptions to the observations from the videotaped lesson.

*Phase 4: Analysis of Assessment Instruments*

In Phase 4 of analysis, the Pretests and Posttests of each participant were analyzed for accuracy by a graduate student who was not the researcher. Each instrument was given a score with a maximum of 40 points using a rubric. After the videotapes were analyzed, the researcher examined the rubric used to grade the Pretest and Posttest. The researcher used the rubric to give each participant a score based on a 40 point scale. These scores were compared to the previous scores for both test instruments of each participant.

The researcher then examined participant responses more carefully on both the Pretest and Posttest. Comparisons of each participant’s Pretest and Posttest results were made. Only five questions from both instruments were used in the data analysis. Since the instruments measured participants’ knowledge of both probability and statistics, some of the questions were not pertinent for this study. The five questions that were selected all relate to teacher’s subject matter knowledge or pedagogical content knowledge of empirical or theoretical probability.

*Phase 5: Creating Narrative*

In the fifth and final phase of analysis, a storyline and narrative were written. Merriam (1998) suggests that the end result of qualitative research should be richly descriptive. The storylines and narratives were based on evidence of participant knowledge in the Pretest and Posttest, the videotape analysis, and participants’ written reflections.
CHAPTER 4

RESULTS

Chapter 4 presents the findings of the data analysis. Results are reported case by case. As indicated in previous chapters, three sources of data were analyzed for this study: videos of probability lessons, teachers’ reflections of their teaching episode, and Pretest and Posttest responses. Each case begins with a description of the classroom context and task, followed by the results of each participant’s teaching episode. Afterwards, each participant’s reflections are described, along with a comparison of the teacher’s perceptions and those of the researcher. Finally, findings from examining the participant’s Pretest and Posttest are presented.

As previously indicated at the end of Chapter 2, this study seeks to answer the following questions:

1. What do middle school teachers’ pedagogical decisions during a probability lesson imply about their understanding of empirical and theoretical probability?

2. How do middle school teachers’ utilize models and representations, examples, and approaches of empirical and theoretical probability?

The Case of Frank

Frank is the case of a teacher who approaches probability from a theoretical perspective. He believes using a theoretical approach will result in computing an accurate probability. Frank prefers to use algorithms and emphasizes procedures, like simplifying fractions, even when these procedures have little value in promoting probabilistic reasoning.
There is some evidence to indicate Frank is the case of a teacher who might be transitioning to reasoning that moves back and forth between empirical data and theoretical probability.

*Frank’s Classroom Context and Task*

In this teaching episode, Frank introduces the concepts of empirical and theoretical probability through whole class discourse. He begins the teaching episode by defining theoretical and empirical probability. Frank then shows students how to find the number of outcomes when rolling a pair of dice and three dice. Next, he instructs students to conduct three brief experiments with two six-sided dice, making comparisons between the empirical and theoretical probabilities.

Students conduct the three experiments individually at their desks. Data collection takes approximately two to four minutes per experiment. In the first experiment, students roll their dice until they get double 1’s. They are only allowed to roll a maximum of 36 times, but they must stop once double 1’s are obtained. Students record the number of rolls it took them to get double 1’s, if this event occurs. In the second experiment, students roll their dice until they get any doubles (i.e. 1-1, 2-2, 3-3, etc.). Again, Frank limits their rolls to a maximum of 36 trials. They must stop once they obtain doubles which means $n \leq 36$. Students record the number of rolls it took them to get doubles, if this event occurs. In the third and final experiment, students roll their dice a maximum of 36 times. Frank again asks students to stop rolling once they obtain doubles and record which roll doubles occurs, if it does indeed occur. The video ends with students collecting this data.

*Frank’s Teaching Episode*

Frank does not use a wide variety of examples to illustrate ideas about uncertainty. All of Frank’s examples involve situations which are equiprobable (i.e. rolling fair dice,
tossing a fair coin). Although it could be inferred that his basic repertoire is limited, Frank chooses rolling a die as a basic example and builds on this by asking students to consider more complicated situations like rolling a pair of dice and then three dice. There may also be another explanation for Frank’s pedagogical choices. He only chooses situations that he assumes his students may be familiar with or have experienced. For example, Frank begins the lesson by asking students if they have ever rolled doubles before. This is an experience that many middle school students have previously encountered by playing games or in other mathematics classes. All of Frank’s students indicated they have had this experience prior to this lesson.

The language that Frank uses to introduce empirical and theoretical probability is very informal. It is hypothesized that the purpose of this informality is an attempt to try and relate ideas about probability to experiences that students have already had. Frank suggests that theoretical probability “means thinking.” Frank goes on to say it is having an “idea, but you haven’t actually done anything yet.” Frank relates empirical probability to theoretical by stating empirical probability is “actually doing the experiment;” it’s “recording results.”

Although analogies can be useful in introducing concepts, they need to be linked to terminology and normative ideas about probability. Frank’s analogy only relates the two perspectives on a superficial level. While it is true that theoretical probability is an approximation, it is based on symmetry and physical complexities; it is much more than merely making a prediction. Empirical probability encompasses more than recording “results.” You can also find “results” from examining the theoretical probability as well.

Frank’s analogy implies theoretical probability is merely a guess. Predictions of probability can be based on subjective and frequentist approaches, also. If students make
predictions using criteria other than symmetry, their predictions are not necessarily based on a theoretical perspective. The conjecture may be made using a subjective approach. This may indicate that Frank does not recognize a subjective view of probability or he assumes his students predictions are based on a theoretical perspective.

In Frank’s approach to probability, he leads students to believe that theoretical probability is a better indicator of uncertainty than using empirical data. Frank seems to believe, empirical probability is not as reliable as theoretical. Franks states many times that the theoretical probability of an event is what the probability is supposed to be. This may give students the impression that the theoretical probability is the true probability.

Frank focuses on highlighting what he perceives as the difference between empirical and theoretical approaches, giving students the impression that the results will always be different. Frank’s idea about these two approaches is that results obtained empirically probably won’t be close to theoretical values. He does not attend to issues such as sample size, and how the sample size of an experiment relates to the closeness of empirical and theoretical probability.

Frank’s image that empirical and theoretical probabilities are always different is the basis for all of the experiments in which students engage. Frank has the idea that he can show students that the two approaches yield different results by not allowing students to fulfill the theoretical probability. According to Frank, the task (n ≤ 36) design will illustrate to students that there will be a mismatch between empirical and theoretical probability, although the reason for the possible mismatch is not addressed.

Frank precedes the experiments by constructing the theoretical probability of rolling two dice and obtaining double 1’s (1/36). In the first experiment, Frank wants to show
students that they will not obtain an empirical probability of 1/36. Let’s say, for example, a student in Frank’s class rolled double 1’s on the third roll. The student would stop rolling. His empirical probability would actually be 1/3 which is certainly not close to 1/36. Frank believes this illustrates to students that empirical and theoretical probabilities will approximate different values. He explains the difference by reminding students he did not allow them to fulfill the theoretical probability. If he allowed the hypothetical student above to roll another 33 times, it may be that the student would not obtain any other double 1’s. In this case the student’s experiment would result in an empirical probability of 1/36. Frank assumes none of his students obtain double 1’s on the 36th roll.

In the last two experiments, Frank uses different reasoning than in the first. He still wants to illustrate to his students that the empirical probability of obtaining any doubles will be different than the theoretical probability of 6/36, which he refers to as 1/6. In these instances, he wants students to see a difference. Students again stop rolling once they obtain doubles. Frank records the number of rolls that it took for students to obtain doubles in a chart with the intervals 1–5, 6-10, and 11-15. All students roll doubles by their tenth roll. Frank indicates that students should roll doubles once in every 6 rolls as opposed to thinking of the expectation that students would obtain 6 doubles out of 36. One of the issues with Frank’s representation of the empirical results is that the students all have different trial sizes, \( \{n \mid 1 \leq n \leq 10\} \), yet Frank combines their data to compare it to the theoretical probability.

Frank uses multiple representations to convey ideas about empirical and theoretical probability. When the class is considering the outcome of rolling a die, Frank begins a tree diagram to illustrate the sample space. He then asked students to consider rolling 2 dice.
Instead of finishing the tree diagram, he partially completes the diagram and encourages students to use an algorithm to obtain the number of all possible outcomes (see Figure 5).

Figure 5. Frank’s tree diagram.

Frank then asks students to consider rolling 3 dice. Again, he does not complete the tree diagram and suggests students use a rule to find all possible outcomes. His students are unable to reason about the number of possible outcomes of 3 dice and the rule. Frank does allude to why the algorithm works, but it is evident that students do not have a conceptual understanding of how these two representations are linked.
Frank was very quick to use the multiplication rule for finding all the possible outcomes. To make a link between the tree diagram and the algorithm, the researcher believes Frank should have developed the diagram completely. It appeared that Frank was the only person on the video constructing the diagram; he missed an opportunity to get students actively involved by giving them a chance to construct the tree diagram. In rushing students to a rule that they clearly did not understand, Frank missed out on an opportunity to link the multiplication rule to the patterns in the diagram.

Frank uses different symbolic representations to express empirical and theoretical probability. For example, when finding the theoretical probability of obtaining double 1’s when rolling a pair of die, Frank uses the following representations: 1 out of 36, 1:36, 1/36, 1 to 36.

*Frank’s Reflections and Perceptions*

Frank felt that it was not difficult for him to select a 15 minute video segment from his 90 minute lesson because he felt this segment is the best example of his interaction and discourse with students. He stated that students were “thinking and answering the how and why questions” in this segment. In his own words, Franks states “students used logical reasoning and problem solving skills to connect the concepts of rolling two dice to rolling three.”

From the researcher’s perspective, this is not what was observed on the videotape. Students seemed confused about finding all possible outcomes in regards to rolling two and three dice. Below is an excerpt from the transcript of this particular teaching episode:
Frank: There’s six numbers on each die. Sure. Ok. Make sense? Nod your head “yes.” It’s very easy. It gets confusing. I know, but that’s ok. What about three dice? How many possible outcomes for three dice?

[Some student responses are inaudible.]

Several Students: 18.

Student: 196.

Several Students: 18.

Frank: 18. 196, that’s really close. Think about it. How did I get my two dice?

Student: 198.

Frank: 6 times 6. So, for three dice, Caitlin?

Student: 316.

Frank: 36 times 6. Would did you get, Pablo?

Student: 216.

Frank: Thank you, sir.

Frank goes on to say, “That all makes sense.” However, many students indicate that it does not. Earlier in the lesson, students thought there should be 12 outcomes for rolling two dice. Some students shout “18” in reference to three dice. They are using additive reasoning. Other students appear to be guessing. Based on Frank’s tone, students alter their response. Pablo correctly states “216,” but Frank had just said “36 times 6” in the previous line. Pablo does not justify his answer. Thus, it is unclear if his response was based on Frank’s previous comment, “36 times 6,” or his own reasoning. The speed in which this dialogue takes place also refutes claims that Frank walks students through “slowly, to make sure that the misconceptions and misunderstandings … are addressed and redirected appropriately.”
In his reflection, Frank pointed out that he did not have access to technologies such as *Probability Explorer* and graphics calculators. His choice of tools for conducting experiments was limited to dice. Frank believes that students might have made better connections and engaged in more meaningful dialogue if they had used more than one manipulative. He also felt that he could have exposed students to more variations of probabilistic situations. While Frank could not control his access to technology or other manipulatives, he did have autonomy over the kinds of examples he selected for instruction. During the teaching episode, Frank used seven examples involving uncertainty; tossing dice was the context for every example.

Frank’s reflection gives some indications of his ideas about empirical and theoretical probability. He wants students to “understand the fundamental difference” between theoretical and empirical probability as a goal of his lesson. This is the same belief observed in the videotaped lesson. He feels disheartened that at the end of the entire lesson half of the class did not understand this “difference.” One explanation students give for this “difference” is theoretical probability is “what should happen” while empirical probability is “what did happen.” When the videotape is viewed, this is the language that Frank uses to define the perspectives. He also stated that other students think theoretical probability is the “probability if all outcomes were fair and equal” and empirical probability was the “probability if all outcomes were not equal.”

Frank’s Pretest and Posttest Results

The questions assessed in the Pretest and Posttest, along with Frank’s responses, can be found in Appendix C. In question #1 on the Pretest, Frank identifies one topic, *combinations and permutations*, as an important topic for middle school students to learn.
about probability. He reasons that problems involving combinations and permutations make students think mathematically. While he acknowledges that this topic is difficult for middle school students, he believes that computations involving this topic are “one of the easiest computations to perform.” As in the videotaped lesson, Frank’s response indicates he values using algorithms to solve problems involving probability.

On Frank’s Posttest, he lists three topics that he feels are important for the learning of probability: dependent vs. independent events, outcomes, and more or less likely. Frank uses the “Hospital Activity”, which is actually a question from both assessment instruments, as rationale to support the importance of these topics. His response indicates that his students have misconceptions concerning the law of large numbers. He believes students need broader experiences with probability, especially those that challenge students’ deterministic views of probability. Frank’s Posttest response seems to shift the selection of topics from those that illicit a computational response to those that develop nondeterministic reasoning in students.

Frank’s Pretest indicates that the word sample made him think of “example,” “sample problem,” “sample population,” and “free.” The context in which he describes the meaning of sample is in terms of getting free samples in the mail or in stores. He indicates that “sample population” applies “to surveys and statistical information.” It is unclear what the term “sample population” means to him. Frank indicates that he would introduce sample in the context of consumers receiving free samples from companies or stores according to his response to question #2.

In his Posttest response, Frank thinks of “a survey.” He states that “the word sample implies a smaller group selected from a larger group (population) that is representative.” The context that Frank gives in his Posttest response is less broad than his first response but gives
more insight into his understanding of *sample* in a statistical context. His second response clearly indicates that he sees a *sample* as a subset of a population. This is the context that would be used to introduce *sample* to his students. Frank describes using a specific survey that his students complete each month. He would consider the sampling method and the idea of a sample being representative. On both assessment instruments, Frank suggests introducing *sample* in a real world context that his students could relate to.

On question #3 of the Pretest, Frank draws a diagram that illustrates having 10 batteries, 8 good and 2 defective. He literally draws a picture of 10 batteries, but he does not indicate what the sample space is with his diagram. On his Posttest, his diagram is much like a tree diagram without the lines. His Posttest representation correctly illustrates that the sample space is \{DD, DG, GD, GG\} where D represents drawing a defective battery and G a good battery. When asked to calculate probabilities, he correctly calculates 2 out of 4 with his Posttest diagram whereas he correctly calculated only 1 using his Pretest diagram.

On question #4 in the Pretest, Frank indicated that he believed the sequence HTTHHTHHTT, which he labeled as 5 heads and 5 tails, was more likely to occur than HHHHTTHHTT, which he labeled as 7 heads and 3 tails, even though these sequences are equally as likely to occur. Perhaps, Frank is relying on the *representativeness heuristic*. According to his intuition, the first sequence represents obtaining an equivalent number of heads and tails. However, on the Posttest, Frank correctly reasons that the sequences are equally likely and each toss has the same probability of obtaining heads or tails.

Frank’s responses to question #5 indicate he has a better understanding of the *law of large numbers*. By the end of the Data Analysis and Probability course, he is considering sample size in terms of empirical data and the theoretical probability. On the Pretest, he
reasons that it is equally likely to obtain 4 female births and 1 male birth as it is to obtain 20 female births and 5 male births, given it is equally likely for a female or male birth to occur. On the Posttest, Frank states that 4 out of 5 babies being born female is more likely than 20 out of 25 babies being born female since “unlikely events are more likely to happen in a smaller sample.”

It should also be noted that on the Pretest he indicates the empirical probability shows it is “more likely to have a girl”, but the theoretical probability “states that it is equally likely to occur.” This statement emphasizes Frank’s reliance on a theoretical view. On the Posttest, Frank used the term sample space when he meant sample. These two ideas are important in having a strong conceptual understanding of empirical and theoretical probability.

The Case of Helen

Helen is the case of a middle school mathematics teacher who has a weak understanding of probability. She is unable to convey ideas about relating empirical and theoretical probability to students. Helen makes mistakes calculating both the empirical and theoretical probability during her lesson. She has a weak understanding of sample space, the law of large numbers, and concepts related to sampling, like sample size and sample selection.

Helen’s Classroom Context and Task

In Helen’s teaching episode, students explore whether or not the “Additive Game” is fair or unfair. Students play the game in pairs and record their results. Each student rolls a pair of fair six-sided dice 18 times. Player A wins a point if the sum is odd, and player B wins a point if the sum is even. The player with the most points wins the game. Student pairs toss their dice a total of 36 times, alternating rolls between each player. The theoretical
probability of the “Additive Game” is the same as comparing the theoretical probability of rolling an even sum (18/36) to the probability of rolling an odd sum (18/36).

Before the video of the probability lesson begins, Helen has asked students to make a prediction on the fairness of the game. It takes students approximately seven minutes to collect and record their results from the game. After data collection is complete, Helen leads a whole class discussion that focuses on whether the game is fair or unfair based on the empirical data while making comparisons to the theoretical probability.

*Helen’s Teaching Episode*

In this videotaped lesson, Helen creates a scenario in which students compare the empirical and theoretical probability of rolling an even sum to rolling an odd sum. Helen and her students model this mathematical idea by playing a game with hands-on manipulatives: dice. The model creates a real life context which enables students to draw on their previous experiences.

While the only model being used in the lesson is the “Additive Game” itself, Helen uses a couple of symbolic representations in relation to empirical and theoretical probability: fractions and percents. Students use the same representations, as well. Helen emphasizes that the fractional representations should be reduced. In representations of probability, the denominator in the ratio takes on a special meaning. From a theoretical view, it represents all the possible outcomes of the event, the sample space; while from an empirical perspective, it represents the number of trials in an experiment. An emphasis on simplification does not take advantage of the significance of the denominator. Simplifying fractional representations does not always support the development of probabilistic reasoning.
In general, Helen has a weak understanding of sample space; this will be discussed later in the analysis. Her conceptions of sample space may be linked to her ideas about how ratios should be used to represent probability. More specifically, since she can not identify the sample space in this particular lesson, she may not make the connection between the number of all possible outcomes and the denominator of the theoretical probability or the number of trials conducted by students and the denominator.

The bi-directional model can be used to describe Helen’s reasoning throughout her teaching episode. Initially, students are asked to make a prediction as to whether the “Additive Game” is fair or unfair. It is unknown how students form their hypotheses, but it is clear that Helen’s prediction is based on her theoretical image of the probability of rolling an even sum and odd sum. Helen’s intuition tells her the game is unfair, while most students predict the game is fair. Even though students collect and report their empirical data in a whole class discussion, it seems that Helen has already decided what the outcomes of the experiment should be before they actually take place. Helen’s belief in her theoretical image of the probability is so strong that it has a serious impact on her pedagogical decisions and how empirical data is handled.

Before results regarding how this empirical data is handled are revealed, it is imperative to have an understanding of Helen’s image of the theoretical probability of rolling an even sum versus an odd sum. As mentioned earlier, Helen does not have a strong knowledge base of sample space. She never explicitly uses the terminology sample space in the lesson. Her conception of the sample space may relate to her ideas of independence. Helen sees rolling a 1-2 as the same as a 2-1. In her sample space there are 21 possible outcomes when rolling two dice, instead of 36 (see Figures 6 and 7). As a result, Helen’s
original image is that the game is unfair. She believes that the theoretical probability of obtaining an even sum and odd sum is 12/21 and 9/21 respectively. Throughout the teaching episode, Helen is never sure what constitutes the sample space. She struggles with whether the rolls (i.e. possible pairs) or the sums of the rolls constitute the sample space. Perhaps if Helen had used two different colored dice (blue and yellow), it may have helped her see rolling a 1 on blue and 2 on yellow is different than a 2 on blue and 1 on yellow.

Figure 6. Helen’s conception of sample space.

Figure 7. Sample space of sums of rolling two dice.
When Helen considers the empirical data, it is evident that she is confounded about how to analyze data generated from the simulation. In fact, Helen uses two different approaches in her analysis of the empirical data during the probability lesson. Using a winning approach, Helen first concludes the game is unfair because more student pairs obtained odd sums than even sums. Next, using a volunteer sampling method, Helen determines the game is unfair based on only 6 student pairs obtaining more even sums than odds. The latter conclusion matches Helen’s original theoretical image.

Helen does not address the issue that her two approaches are contradictory. Although both indicate the game is unfair, Helen’s first method results in an empirical probability of more odd sums while her second produces an empirical probability of more even sums. Further, there is never an explanation for why the empirical probabilities and theoretical probability do not match. Helen also does not attend to issues dealing with variability, sample size, and independence relating to the empirical data that was collected.

Instead of calculating the total number of evens and odds each pair of students obtain, Helen initially uses a winning mentality to tally whether the pair got more evens or more odds. She represents this by putting one tally mark under the word evens if the pair had more evens than odds and vice versa for odds. If students obtained the same frequencies of even and odd sums, Helen put a tally mark under evens and odds.

After Helen tallies the class results, she indicates that they have 11 odds to 6 evens. She miscalculates the empirical probabilities, expressing them as 11/36 for odds and 6/36 for evens. This has an impact on her students’ probabilistic reasoning. Almost all of the students predicted the game was fair before they did the experiment. After Helen compares odds to evens, almost everyone alters their thinking and declares the game unfair.
This *winning mentality* may suggest that Helen does not have a good understanding of the *law of large numbers*. She does not look at the long run frequencies of all odds and evens student pairs obtained. Neither does she compare the classes’ frequencies to the total number of trials. Perhaps, students would also have come to a different conclusion of fairness if they had examined the empirical results in terms of long run frequencies.

From the perspective of the researcher, Helen’s *winning approach* to computing empirical probability impeded the possible development of probabilistic reasoning in her students. If Helen had calculated the frequency of even and odd sums and compared these frequencies to the total number of trials, she could have addressed issues about the variability of the empirical probabilities of each pair. Helen could also have begun to develop ideas about how the sample size of an experiment affects empirical results and its relationship to the theoretical estimate of the probability. Also, calculating the empirical probability as a class would have dealt with ideas of independence.

In Helen’s second approach to analyzing the empirical data in this lesson, Helen asks 6 of the 17 pairs of students to report the total number of even and odd sums they obtained in their simulation. Helen asks students to report the empirical probabilities in simplest form and records the results on the overhead. Helen chooses the pairs using a *volunteer sampling* method, but she never justifies why she only looks at the empirical data of 6 pairs. Of the 6 pairs, 4 of these pairs had more even sums than odd sums. Students had already reported whether they had more even or odd sums.

Helen’s theoretical image is that there should be more even sums. Is she specifically choosing a sample from the class so the empirical probability will match her image? Her treatment of data indicates she either has a poor understanding of sampling or she is trying to
select data to match her image. Helen missed an opportunity to engage students in a
discussion about sampling. One of her students comments that she didn’t collect data from all
the pairs in the class. Helen ignores his comment. This could have been an opportunity to
justify her method and discuss sampling methods as well as sample size.

Based on the results of the 6 student pairs that were recorded on the overhead, the
empirical probability of obtaining an even sum is $\frac{116}{216}$ and an odd sum is $\frac{100}{216}$. These
probabilities were calculated by the researcher; Helen never pooled the data of the 6 student
pairs. The probabilities are actually very similar. Instead of pooling the data, she writes each
pairs’ empirical probability as a fraction in simplest form. If Helen had not emphasized the
simplification of fractions, the representations might have motivated students to consider all
the rolls together.

After Helen concludes that the empirical results of the experiment implies it is more
likely to obtain an even sum based on the 6 student pairs’ results, a student reconciled that it
made sense that you should get more even sums. The student reasons that the sum of an even
and an even number always results in an even, as does the sum of an odd number and an odd
number. Further, he goes on to explain that the only way to obtain an odd number is by
summing an even and odd number. While this is true, Helen missed an opportunity to point
out that the deterministic reasoning used in mathematics in general can lead you astray when
reasoning about probability.

It is conjectured that Helen selected this lesson from the Connected Mathematics
text in her classroom. Helen’s implementation is almost identical to the directions given to
students in the student textbook in Investigation 2 (p. 22) for the “Additive Game.” If Helen
is implementing the lesson based on the “Additive Game,” from the Connected Mathematics Project, she clearly did not read the discussion of the lesson in the teacher’s edition discussing the sample space. What does this imply about Helen’s preparation? Perhaps, she felt confident in her ability to do the mathematics and thought it was unnecessary to prepare ahead of time.

Helen’s Reflections and Perceptions

An analysis of Helen’s reflection of her teaching episode reveals that her main objective was 1) to determine if the “Additive Game” was fair to both players and 2) to examine how the theoretical probability relates to the empirical. Helen felt that the goals of the lesson were adequately met based on evidence from the videotape. From the researcher’s perspective, the goals were not adequately met.

The class and Helen did come to a consensus on the fairness of the game; however, there was never an explicit attempt to reconcile why the two approaches (i.e. empirical and theoretical) yielded different probabilities. On numerous instances in her reflection, Helen emphasizes the “difference” between the two approaches. Helen’s conviction in her theoretical image was so strong that she ignores empirical evidence and attempts to make the empirical data match her theoretical image. As a result, she did not attend to developing reasoning that may explain the mismatch between the two.

Helen admits that “probability is not a strong suit” for her. She makes a reference to having the tendency to “skip over probability when time runs short at the end of the year” and she spends little time on this strand of the curriculum. From the researcher’s point of view, the analysis does expose some of Helen’s misconceptions dealing with the uncertainty of the “Additive Game.” Helen miscalculates both the empirical and theoretical probabilities.
She does not have a complete understanding of all the possible outcomes of rolling a pair of six-sided dice. Helen’s conception is rolling a 1-2 is the same as rolling a 2-1; she would not include both outcomes. Thus, her sample space contains less than 36 outcomes. Helen’s conception of the sample space favors obtaining an even sum, theoretically. She is not viewing the individual die tosses as being independent of one another when tossed simultaneously. Perhaps, if Helen had used representations, like tree diagrams or charts, when reasoning about sample space, her image may have been normative.

As further justification for her theoretical image, she cites a student’s explanation that the sum of two even numbers and two odds both result in an even number. She goes on to state that the only way to get an odd sum is to roll an even and an odd number. While this is true, this type of deterministic reasoning leads her down the wrong path within the context of this problem. This also shows that her own intuitions play an important roll in her pedagogical decisions.

As a result of Helen’s misconceptions about sample space, her students’ probabilistic reasoning was impacted. Helen writes that as a result of concluding that the “Additive Game” was unfair, her students intuitively believed that other games involving obtaining specific sums or multiples were also unfair. She indicated that students would conclude the games were unfair based on their theoretical model before they had finished conducting their empirical trials. Helen emphasized that students would not make conclusions based on their empirical results. This may indicate Helen’s belief that theoretical probability is the correct indicator of uncertainty has been internalized by her students.
Helen’s Pretest and Posttest Results

Helen’s Pretest and Posttest responses can be found in Appendix D. Although Helen identifies different topics of importance on the assessments in question #1, the difference between probability and odds on the Pretest and independent and dependent events on the Posttest, her rationale on both assessments is similar. Helen’s conception of her students’ understanding is her underlying justification for the importance of these topics. Helen asserts that these topics are both areas of probability in which students have difficulties or misconceptions.

In response to questions #2, Helen only uses one context on both the Pretest and Posttest. Her description of a sample on the Pretest is vague, “a small selection representing something larger,” while her Posttest response contains more details. She describes a sample as a “snapshot of a whole population.” Helen makes reference to using a proportional sampling method, and the idea that a sample should be representative of the population.

On the Pretest Helen indicates that she would use two different contexts to introduce sample to her students. She would use the context of a sample being a small amount of a product, and then broaden sample in terms of data and probability; however, Helen does not describe how she would do this. In contrast, on the Posttest, Helen is more detailed. She describes introducing sample by surveying teens to find out about the types of music they listen to. She formulates specific questions she would ask her students and mentions choosing a sampling method to conduct this survey. Helen’s second response, as opposed to her first, is actually an idea she could implement in her classroom.

Regarding question #3, both of Helen’s diagrams do not depict the sample space. Both drawings represent 10 batteries, 8 good and 2 defective, but not possible outcomes of
drawing two batteries. This supports the evidence from the videotape that Helen has a weak understanding of sample space. She was unable to calculate the probabilities of simple and compound events on the Pretest, but calculates two of the four on the Posttest.

With respect to question #4, Helen responds that both sequences are as likely to occur; however, her justification on the Pretest to the Posttest changes. Although she answers the question correctly on the Pretest, her reasoning does not accurately explain why the sequences have the same likelihood of occurring. Nonetheless, her response gives insight into her conceptions about the relationship between empirical and theoretical probability. The bi-directional model can describe how Helen relates empirical and theoretical probability in the context of this problem. Helen’s original image is that she expects “50% heads and 50% tails.” After examining the data given in the problem, Helen notes that the empirical probability does not match her original theoretical image. She concludes that the sample size, 10 flips, is not large enough “to see a 50/50 split.” On the Posttest, Helen reasons that each coin flip is independent of the other and is as equally likely to land on heads as it is on tails.

Helen used proportional reasoning on both assessment instruments to conclude that 4 out 5 female births was equally as likely as 20 out of 25 female births in question #5. This supports findings from the teaching episode which indicate she does not have a deep conceptual understanding of the law of large numbers. On the Pretest, Helen explained that 20/25 is the same as 4/5. She used similar reasoning on the Posttest, instead of comparing female births to total births, Helen compares female births to male births in the hospitals, concluding that 20/5 is equivalent to 4 to 1.
The Case of Kathy

Kathy is the case of a teacher who teaches empirical and theoretical probability through analogy. Since Kathy teaches both mathematics and science, she draws on students’ previous experience in science class to develop probabilistic reasoning. She believes approaching the concepts of empirical and theoretical probability through analogy makes the mathematics easier for her students. It is evident that Kathy has an understanding of these concepts herself; however, she filters her ideas, stripping away the probabilistic content in an attempt to make the mathematics less challenging.

Kathy’s Classroom Context and Task

In this teaching episode, Kathy introduces empirical and theoretical probability through whole class discussion. Most of the discourse focuses on constructing definitions of empirical and theoretical probability using an analogy between science and mathematics. In the last 5 minutes of the video, Kathy asks students to find the theoretical probability of drawing specific colored chips from an envelope. Each student had been given an envelope containing 2 yellow chips, 1 red chip, 1 blue chip, and 1 green chip. Later, students will randomly draw one chip at a time, with replacement, in order to compare the empirical and theoretical probabilities.

Kathy’s Teaching Episode

In Kathy’s school district, it is fairly common for middle school teachers to teach both mathematics and science. Even when this is not the case, there is often an emphasis in middle grades classrooms to integrate ideas from different subjects. It appears Kathy adopts this philosophy and wants to draw on students existing knowledge about science to develop probabilistic reasoning.
To introduce the concepts of empirical and theoretical probability, Kathy adopts the approach of creating an analogy. Capitalizing on her students’ prior experiences in science, Kathy compares science experiments and theories to probability in order to specifically develop ideas about empirical and theoretical perspectives. It is evident that Kathy is also attempting to relate the two approaches in this teaching episode; however, the use of the analogy in conjunction with informal language may actually impede the development of normative probabilistic reasoning in Kathy’s students.

Kathy spends quite a bit of time questioning students about their ideas related to science experiments and theories. It is plausible to believe that Kathy’s purpose is to relate probability to students’ real life experiences. As a result, the language used by Kathy and her students is very informal. Instead of clearly identifying what empirical and theoretical probability are, the informal language in the teaching episode muddies the waters. This may also be impacted by Kathy’s need to make probability easier for her students. She states several times that she is “trying to make it [probability] real easy.”

While it is good pedagogical practice to build on students’ prior knowledge and personal experiences, the science analogy in this case may lead to nonnormative views of probability. Using the science experiment as a reference, students described an experiment as a test for proof of a hypothesis, a method for finding out how something works, and a validation of truth.

Students refer to a theory in science as something that is not true yet or something that could be proved. Students believe theory is based on facts. Kathy refers to theoretical probability as “what you think could get but haven’t done anything to prove yet.” The relationship between empirical and theoretical seems to be summarized in the following way:
theoretical probability is a hypothesis, a guess, or what you think could happen, and empirical probability is a tool used to prove your hypothesis or guess by doing something (see Figure 8). While a guess or hypothesis could be based on a theoretical image, it could also be based upon a subjective view. Students do not always make predictions based on the symmetries of the sample space and physical complexities of an uncertain situation. Kathy’s view of probability may not include a subjective approach. It may also be the case that she assumes all students make predictions bases on a theoretical perspective.
Figure 8. Kathy’s analogy of a science experiment.
The idea that an experiment is used to prove something may indicate the existence of a right answer. Kathy may unknowingly give students the impression that the right or true answer can be determined in all probabilistic situations, which may promote deterministic reasoning. Often times in the video, Kathy and her students accept the theoretical probability as the true probability, even after collecting empirical data. For example, when referring to a previously conducted experiment, Kathy and her students express the probability of obtaining heads when tossing a fair coin as \( \frac{1}{2} \). It is unlikely all students in the class obtained heads and tails exactly 50% of the time.

Using an analogy in and of itself does not impede probabilistic reasoning. Teachers often create analogies and use metaphors to develop mathematical thinking in students and to represent new ideas. It is often the case that these analogies are used to make more abstract ideas seem easier to students. However, analogies need to be paired with normative ideas of stochastics.

Kathy missed an opportunity to link terminology and normative ideas about probability to the analogy that she created. Accepting experimentation as a test to validate the truth of theoretical probability does not embody all the important ideas related to empirical probability. When Kathy adopted this language from her students’ ideas, she did not attend to the idea that empirical probability may lead to the rejection of a hypothesis. Empirical data will not always support theory, and theoretical probability is only an estimate of the true probability. Theoretical probability is not a predetermined true value that predicts the likelihood of an event.

There is evidence to support the hypothesis that Kathy does believe that there is a right answer when dealing with probabilistic situations. While using the science experiment
analogy to create a definition for empirical probability, Kathy asks students, “what would you expect to happen if you flipped a coin let’s say 50 times?” A student replies that you would get 25 heads and 25 tails. This shows that both Kathy and her student have an image that begins with the theoretical probability. Kathy suggests that they would have to actually flip the coin “in order to figure out if that was actually true.” This suggests that empirical data should confirm or disconfirm the hypothesis. Kathy’s language throughout the lesson, however, insinuates the data should confirm a hypothesis.

Based on the videotape analysis, Kathy’s basic repertoire of examples is limited. Almost every example provided by Kathy deals with probabilistic events that are theoretically equiprobable. Furthermore, Kathy uses flipping a fair coin as the context for every example, except one. When dealing with the uncertainty of tossing a coin, Kathy favors a theoretical approach to approximating the outcome of obtaining a heads or tails. In every case, she states the probability is $\frac{1}{2}$.

There is one occasion in which Kathy relates the theoretical probability of obtaining a heads to empirical probability. In this instance, Kathy refers to an experiment the students did the day before (i.e. flipping a fair coin). Kathy asks the students to remember this experiment and wants to know the probability of obtaining heads. A student replies 50%. Kathy acknowledges this is correct. Collecting empirical data did not alter Kathy’s original theoretical image. It seems that it did not alter her students either. This may be tied to the idea that there is one true answer to the likelihood of obtaining heads.

Kathy uses two representations for both empirical and theoretical probabilities: fractions and percents. Students use the same representations, as well. The informal use of language and lack of attention to terminology supports these representations. Kathy and her
students discuss probability using the language of ratios. They refer to the numerator as “how many you have,” “how many you can or did get,” and “how many pieces or what the size of the piece was.” The denominator is referred to as the “whole” or “whole piece.” This kind of language supports proportional reasoning. Kathy makes only one allusion to sample space in terms of possible outcomes. She states that the probability of obtaining heads from a coin toss is \( \frac{1}{2} \), the number you want over the number you have for possible outcomes. Kathy does not link the representations of ratios to probabilistic terminology.

At the end of the video, the students calculate the theoretical probability for an experiment that they will conduct. Kathy uses manipulatives, colored chips and an envelope to model some situation. It is unclear what students are modeling. They are simply told they have an envelope that contains colored chips. There is no context or task to justify doing the experiment. Kathy designs her lesson such that she and her students consider the theoretical probability of an event before they conduct the experiment.

*Kathy’s Reflections and Perceptions*

In her reflection, Kathy explains that she set up an experiment for her students such that the theoretical probability of every event is identical. Recall students are given an envelope which contains 2 yellow, 1 red, 1 blue, and 1 green chip. The theoretical probability of drawing a yellow chip would be \( \frac{2}{5} \), while the theoretical probability of drawing a red, blue, and green would be \( \frac{1}{5} \). The probabilities are not quite equal, although very close. Kathy does favor presenting students with equiprobable situations. All of the verbal examples that Kathy used in her video dealt with flipping a fair coin.

In situations dealing with uncertainty, Kathy’s reflection suggests that she favors a theoretical approach to probability. In her reflection, she acknowledges that when she asked
students what the probability of flipping a coin and obtaining heads is that the correct response is \( \frac{1}{2} \). The videotape supports this assertion; her response to dealing with uncertainty is always that the likelihood of an event is to be determined by the theoretical probability. Although this was not seen on the videotape, Kathy states that the class has a discussion about how empirical and theoretical probabilities could be different or the same. She also acknowledges that her students had difficulty explaining why empirical results were different from person to person. She states that some students explain the difference as chance.

Kathy did not plan on focusing part of her lesson on ratios and simplifying fractions. She believes that extending the lesson to include these ideas promoted student reasoning. While it is important to have an understanding of ratios, Kathy’s diversion from her original course is a distraction from the researcher’s point of view. It may have promoted student thinking, but it did nothing to enhance probabilistic reasoning. Kathy put an emphasis on simplifying fractions and rewriting equivalent fractions with different denominators. For example, after students calculated the theoretical probability of choosing each colored chip from their envelope, Kathy asked students to calculate the theoretical probability if there were 10 chips instead of 5. The discussion focused on the mechanics of rewriting the fractions instead of on the sample space. A fractional representation is important to her. Throughout Kathy’s reflection, she refers to probabilities in the form of a ratio.

*Kathy’s Pretest and Posttest Results*

Kathy’s responses can be found in Appendix E in their entirety. In question #1 on the Pretest, Kathy listed two topics which she viewed as important to the learning of probability for middle school students: *types of events that can occur* and *making predictions from the probability that an event might occur*. Kathy’s justification for her first topic is that
probabilities depend on the dependence or independence of events. She feels students should learn about making predictions because analyzing the likelihood of events will help them plan for the future. In her initial response, there is no link to how she would approach making predictions about probabilistic events.

Kathy identifies three topics on her Posttest: *probabilities of dependent and independent events*, *difference between empirical and theoretical probability*, and *combinations and permutations*. She holds on to belief that dependence and independence should be part of the curriculum, as well as the idea that students should make predictions and generalizations. Kathy’s Posttest response indicates that she believes probability should be approached from both an empirical and theoretical perspective. In her mind, there is a “difference” or there should be a “difference” in estimating probability from these two viewpoints. This supports findings from the teaching episode. Kathy also believes that students should do experiments and relate the results to theoretical probability.

Kathy uses two different contexts to describe *sample* in question #2 of her Pretest. Kathy used language that is commonly used when referring to ratios, “a smaller piece of a whole.” She also describes *sample* in terms of a population, a section of a population that could be used to draw conclusions or make generalizations about the population. She uses the same contexts on her Posttest, making references to a “small part of a larger group or object” and “a piece of something that represents the whole object.” Kathy also uses the context of a sample being “part of an entire population.” Using the context of ratios is significant because it relates to specific types of representations, like fractions and percents, which may be used to represent probabilities.
On both assessment instruments, she responds that she would introduce \textit{sample} by using the context of receiving a trial size or free sample of sausage or cheese from a store, like a grocery store. On her Posttest, Kathy also indicated that she would introduce \textit{sample} in another context, as well. She would discuss how data is collected for polls or television ratings. Both of these contexts are real life scenarios that her students are probably familiar with. Adding a discussion about data collection would give her students a context of \textit{sample} other than as a ratio.

There is a distinct difference in the representations that Kathy drew in response to question #3 on the assessment instruments. On the Pretest, her diagram represents 10 batteries, 8 circles represent good batteries while 2 filled in circles represent defective batteries. The diagram does not address the sample space. However, on Kathy’s Posttest, her diagram, a tree diagram, is an accurate representation of the sample space \{DD, DG, GD, GG\}. On both the Pretest and Posttest, she is only able to calculate the probability of drawing a good battery on the 1\textsuperscript{st} draw. This may indicate that she is able to calculate simple probabilities but has difficulty with compound events.

On both the Pretest and Posttest, Kathy responds that the sequences are equally likely in question #4. On the Pretest, she reasons that each flip is an independent event and each event has the same probability. Since there are an equal number of coin flips, the sequences are equally likely. On the Posttest she reasons in terms of each sequence being part of the set of all sequences.

Kathy’s response to question #5 on the assessment instruments would indicate that she has a stronger understanding of the \textit{law of large numbers}. On the Pretest, she uses proportional reasoning to conclude that the events are equally likely. She shows that the
ratios and percents are equivalent using the language of comparing the part to the whole. However, on the Posttest, Kathy reasons that 4 female births out of 5 births is more likely to occur because “the more babies born, the more chances there are of having a male, so the \( P(\text{female}) \) is reduced.”

Another significant difference in her Posttest response is that she calculates the theoretical probability of female births in both samples and compares the theoretical probabilities. Kathy finds the theoretical probability of 20 female births occurring is much smaller. Although it is unclear if her original image is based on the empirical data provided in the problem or the theoretical probability that she calculated, it is evident that she is reasoning using both sides of the bi-directional model, relating empirical and theoretical probability. She is also taking sample size into account. However, her response may also indicate her reliance on a theoretical perspective to demonstrate which event is most likely.

The Case of Pam

Pam is the case of a teacher who is unable to successfully achieve the learning goals of a standards-based lesson. Pam chooses to implement a task from an NCTM resource that was designed to engage students and promote probabilistic reasoning. However, Pam’s weak understanding of empirical and theoretical probability, coupled with her lack of confidence in her pedagogical decisions, inhibited the execution of the lesson as intended in the curriculum materials. She is a veteran teacher who may be trying a different pedagogical approach to teaching, but she is unable to use these strategies effectively in a probability lesson.

*Pam’s Classroom Context and Task*

During Pam’s teaching episode, students engage in a task and conduct an experiment, with replacement, to simulate obtaining a gumball from a gumball machine. The “Gumball
The "Machine" task is from NCTM's *Navigating through Probability in Grades 6 – 8* (Bright, Frierson, Tarr, & Thomas, 2003). Pam distributes a copy of the student activity sheet that is provided by *Navigating through Probability in Grades 6 – 8* to students. Each pair has been given 4 blue chips, 3 yellow chips, 2 green chips, and 1 red chip. During the simulation, pairs put all 10 chips into a paper cup, mix the chips in the cup, and draw a chip without looking, collecting a total of 10 trials. Pam leads a whole class discussion focusing on comparing predictions made before the simulation and the results of the experiment.

Before Pam’s teaching episode begins, she asks students to predict which color gumball would be drawn if they were to draw one gumball from the machine. Pam then asks students to select the true statement from the following three statements:

1. Blue is likely to be drawn.
2. Blue is likely not to be drawn.
3. Blue is equally likely to be drawn or not to be drawn.

Finally, before the students conduct the experiment, they are asked to predict the frequency of each color (blue, yellow, green, and red) if they were to choose 10 gumballs from the machine with replacement.

**Pam’s Teaching Episode**

As indicated earlier, Pam chose a task from an NCTM *Navigations* book (Bright et al., 2003) for her lesson. This book not only provides tasks and student activity sheets but also gives pedagogical strategies for teaching probabilistic concepts. Each task is preceded by a discussion section that explains the mathematics involved in the activity, important ideas related to the concept, and research that pertains to the teaching and learning of a particular probabilistic concept.
The discussion section related to the “Gumball Activity” specifically addresses a student misconception: it is more likely to draw blue gumballs than any other color. Students with this misconception may be reasoning with the outcome approach. Instead of comparing the likelihood of obtaining a blue gumball to its complement (i.e. getting a gumball that is not blue), students compare the likelihood of drawing blue to each disjoint event (i.e. drawing yellow, drawing green, and drawing red). Approaching this task by developing an empirical and theoretical approach simultaneously can help students address this misconception.

After analysis of the videotape, it can be concluded that Pam’s students are reasoning using the outcome approach. Almost every student has the misconception that theoretically you should draw more blues than any other color. At the end of the video, at least half of the class still reasons that blue is likely to be drawn.

It appears that Pam tries to implement the suggestions from the NCTM Navigations discussion section (Bright et al., 2003), even asking questions that had been scripted in the discussion. However, Pam appears to have difficulty promoting discourse while engaged in the lesson.

Pam does attempt to approach the problem from an empirical and theoretical perspective. The bi-directional model can explain the reasoning of Pam and her students; it is also useful in identifying where the breakdown in this well selected lesson occurs. Both Pam and her students form a hypothesis and enter the bi-directional model from the theoretical side. Students then conduct a simulation of 10 trials.

Based on the discussion of the empirical results, students are aware that their empirical data does not match their predictions. However they are unable to explain the
mismatch between the empirical and theoretical probabilities. Examining empirical data did not motivate students to question their original hypothesis.

Pam was unable to craft questions and lead students in a discussion about the mismatch between their empirical and theoretical results. Instead, the discourse focuses on numerical differences between predictions and empirical data. For example, students made comments like this, “I was 1, um, 2 off on blue, 2 chips off on yellow, 1 chip off on green, and I was right on the money on red.” On the occasions that students did attempt to reason about the mismatch, it was explained by chance or randomness. One student questioned the randomness of the experiment, suggesting the chips weren’t mixed well.

Pam not only has a difficult time creating questions and leading a discussion that promotes probabilistic reasoning, she also makes comments that reinforce student misconceptions. A student reports that her and her partner got 5 blue, 2 yellow, 1 green, and 1 red gumball. Pam responds by laughing and saying, “Well, you should have.” Does this mean that Pam believes that this result is close to the theoretical probabilities for each color when drawing 10 gumballs, or does she believe you should get more blues? Her comment without further explanation may lead students to believe you should get more blue gumballs.

Although Pam was unable to shift the focus of her lesson to meet her learning goals, Pam had at least 5 good opportunities to move the conversation to a level that promoted the development of probabilistic reasoning. Consider the following 5 empirical results from students’ data collection:

- 3 blue, 3 yellow, 3 green, and 1 red;
- 4 blue, 6 yellow;
- 1 blue, 6 yellow, 2 green, 1 red;
2 blue, 4 yellow, 3 green, and 1 red; and
1 blue, 6 yellow, 2 green, and 1 red.

In each of the cases above, students obtained fewer blues than any of the other colors. In a couple of these cases, students only obtained 1 blue gumball. Pam could have asked students to reason about why these results were different from their predicted values. Focusing on any of these empirical results might have challenged students to reevaluate their original hypothesis and search for an explanation for the mismatch.

There is evidence to suggest that Pam’s approach to probability did help at least one student address his misconception. Ben predicted that he would draw more blue gumballs than any other color. His empirical data persuaded him to question his original image when he didn’t draw more blue gumballs. By collecting empirical data, Ben reasoned you have a better chance of drawing one of the other three colors than you do blue. After Ben makes this assertion, Pam does not address whether his new hypothesis is correct or incorrect. She does not use Ben’s reasoning to foster her learning goals for her students. Ben’s comments could have been a turning point in the lesson, but Pam did not respond to his commentary.

**Pam’s Reflections and Perceptions**

In Pam’s reflection, she spends a considerable amount of time describing the off task behavior of a particular student, Bob, and the behavior of students in close proximity to Bob. From the researcher’s perspective, the behavior goes unnoticed while watching the video. The class as a whole is engaged and well-behaved during the duration of the probability lesson. Pam concludes the off task behavior is a result of “the manipulative aspect coupled with the student-to-student interaction.” She also indicates the importance of being skilled in class management in her reflection. Perhaps Pam believes that using manipulatives and
cooperative learning leads to unruly behavior in students which to her is an indicator of poor classroom management. A classroom that does not support the use of manipulatives and cooperative learning may not be conducive to developing ideas of empirical probability. Pam’s responses seem to indicate she is trying a new pedagogical approach to teaching probability.

According to Pam, there were three learning goals for the “Gumball Machine” task. The task should enable students to

- develop an understanding of the concept of independent events,
- develop an understanding that events associated with small probabilities can and do occur, and
- realize that the distribution of data from small samples often does not reflect the parent distribution.

Pam indicates that the learning goals of the lesson could have been better achieved if the quality of teacher-to-student questioning had been better. Analysis of the videotape by the researcher corroborates Pam’s hypothesis. At the end of the teaching episode, at least half of Pam’s students still believe it is more likely to draw blue than any other color. None believed that blue was not likely to be drawn. The lesson began with a theoretical image, however the collection of data did not motivate students to question their original hypothesis. Pam did not attend to topics such as variability, independence, or sample size in trying to relate the two perspectives. She does not explicitly reason about why the mismatch in predictions and empirical probability occur.

Pam writes that the focus of the lesson centered on whether or not the prediction made at the beginning matched closely the actual outcomes of the draw. Pam’s inability to
structure and motivate questioning to promote her learning goals could be a direct result of Pam’s lack of confidence. She indicates that she is not comfortable with the lesson and teaching probability, statistics, and data analysis in general. She acknowledges numerous times in her reflection that she is not as knowledgeable about the subject of probability as she should be. Having weak subject matter knowledge, low confidence, and possibly inexperienence in using manipulatives and cooperative learning may have all contributed to the stagnation in Pam’s lesson. Even though students were not reasoning about the probabilistic situation in the video, Pam validates the importance of students exploring, conjecturing, and reasoning in her reflection.

Pam’s Pretest and Posttest Results

Pam’s Pretest and Posttest responses can be found in Appendix F. Pam identifies events and chance as topics that are important for middle school students to learn in regards to probability. Her rationale did not support the importance of these topics in question #1. Instead, she states “students can be aware of how likely an event is,” and that they “can determine the chance of an event taking place.” This indicates that she is aware of two topics, but she does not know why they should be included in her curriculum.

On the Posttest, Pam identifies theoretical and empirical probability as important topics for middle school students. In contrast to Pam’s Pretest response, her rationale is cohesive, and it supports her statement. Pam reasons that middle school students make predictions on a daily basis, and testing these conjectures through experimentation is informative. This suggests that Pam believes an empirical approach should be used with middle school students.
Pam’s response to question #2 on the Pretest shows that she thought of a *sample* as a smaller part of a larger whole. She also believes that the process of selecting a *sample* should be random. Pam’s response on the Posttest was very similar. She describes a *sample* in terms of partitioning, but this time she also uses contexts such as food, fragrance, in addition to describing the whole as a population. Her response to introducing *sample* to students is also similar on both assessment instruments. She suggests conducting an experiment in both instances to answer a question of interests to students. Her Posttest response contains a little more detail than the Pretest.

Pam did not attempt to answer question #3 on the Pretest. Her response on the Posttest would indicate that she does not have an understanding of the sample space or how to represent the sample space with a diagram. Pam’s representation consisted of the numerical values “10, 9, 8.” She drew lines underneath each number. The researcher’s interpretation of this response is that perhaps 10 indicates the number of batteries total, 9 indicates the number of batteries after the 1st draw, and 8 is the number of batteries after the 2nd draw. Or, perhaps, Pam is trying to use a counting technique. She was unable to calculate any of the probabilities correctly in question #3.

Pam indicates that she believes the sequence HTTHHTHHTT is more likely to occur than the sequence HHHHHTTHHT on question #4 of the Pretest. It seems as though Pam is using the *representativeness heuristic*. She indicates that “a fair coin is just as likely to land on heads as it is to land on tails,” but her response does not indicate that she sees each toss as a separate event having the same probability. However, on the Posttest, Pam correctly asserts that the sequences are equally likely. Interestingly, her reasoning on both assessment
instruments is almost identical. Even though Pam chooses the correct answer on the Posttest, there is not enough evidence to indicate she understands why it is accurate.

On both assessment instruments, Pam shows a weak understanding of the law of large numbers. Regarding question #5, Pam believes that it is as likely to obtain 4 female births and 1 male birth as it is to obtain 20 female births and 5 male births, given the likelihood that a female and male birth is the same. Since more babies are born female, she reasons that choice A and choice B are likely to occur because they indicate more female births. On the Posttest, Pam reasons that the ratios are equal; therefore, the likelihood is the same. She does not use nondeterministic reasoning or a theoretical approach to this problem.
CHAPTER 5
SUMMARY, CONCLUSIONS, IMPLICATIONS, AND RECOMMENDATIONS

The purpose of this study is to use middle school teachers’ pedagogical decisions as a lens to investigate their understanding of empirical and theoretical probability. Several elements constitute the rationale for this study: 1) probabilistic reasoning is useful in making decisions in every day life (Batanero et al., 2004; Pereira & Swift, 1981; Pratt, 2005; Shaughnessy, 1981, 1992; Vere-Jones, 1995) and is linked to other disciplines (Pereira & Swift, 1981), as well as other areas of mathematics (NCTM, 1989); 2) Steinbring (1991) and Stohl (2005) indicate empirical and theoretical probability should be developed concurrently in the classroom; 3) the NCTM (1989) recommends students in grades 5 – 8 should have learning experiences that encompass both an empirical and theoretical approach to probability; and 4) there is insufficient knowledge of teacher’s content knowledge and pedagogical content knowledge in relation to probability (Stohl, 2005; Watson, 2001).

Chapter 5 is divided into three sections: summary and conclusions, implications, and recommendations. A summary of the study is presented first, followed by conclusions of the research questions. The next section identifies the study’s implications for teacher education. Finally, the chapter concludes with recommendations for future research.

Summary and Conclusions

This study utilized case study research. The 4 participants were selected from 29 teachers who were involved in a 5 year NSF funded professional development project. At the time of data collection, participants were enrolled in a graduate level mathematics education course, focusing on data analysis and probability. Videotapes of teachers’ implementations of a lesson focusing on empirical and theoretical probability were analyzed in this study;
teachers had the option of choosing to use manipulatives or technology in the lesson. Due to the qualitative nature of the study, triangulation protocol was used. Teachers wrote reflections responding to specific questions after viewing the videotape of their probability lessons. Pretest and Posttests were administered to participants prior to and after completion of the Data Analysis and Probability course. Teacher reflections and Pretest and Posttest instruments were used to confirm or disconfirm findings from the videotape analysis. The data was used to answer the following research questions:

1. What do middle school teachers’ pedagogical decisions during a probability lesson imply about their understanding of empirical and theoretical probability?

2. How do middle school teachers’ utilize models and representations, examples, and approaches of empirical and theoretical probability?

Research Question 1

Participants’ pedagogical decisions indicate they have a strong reliance on theoretical probability. Frank and Kathy both indicate that there is one right answer when dealing with uncertain situations. The correct answer, in their minds, is the theoretical probability. Both participants do not see the theoretical probability as an estimate but an indicator of truth. Helen shares this view of theoretical probability, ignoring empirical data that contradicts her own theoretical image. This may be due in part to an emphasis on deterministic thinking. Frank, Helen, and Kathy all focused on procedures like simplifying fractions. Frank also emphasized the use of algorithms.

Even though participants rely heavily on theoretical probability, they may not all have a complete understanding of using a theoretical approach. Helen’s theoretical image of
rolling two dice is inaccurate. Only Frank explicitly addressed the sample space and used a representation (e.g. tree diagram) to partially develop the sample space. Although Frank initiated the idea, he did not complete a tree diagram.

While participants indicated they were teaching lessons about empirical and theoretical probability, they all downplayed the value of using empirical data to approximate the likelihood of uncertain events. The participants in the study attempted to make connections between empirical and theoretical probability; however, the relationship participants developed did not promote a robust understanding of the relationship between the two perspectives.

Participants’ conceptions of the relationship between empirical and theoretical probability is that the two approaches will always result in different estimations of the likelihood of an uncertain event. Even though participants emphasized this relationship in their probability lessons, none of the participants explicitly dealt with why there was a mismatch between the two approximations when a difference did occur. Students sometimes concluded it was due to chance.

Participants understanding of the relationship between empirical and theoretical probability did not include the idea that the probabilities may match, or be close in value. Participants chose very small trial sizes for their experiments. Frank chose trials that were less than or equal to 36, while Helen chose 36 trials. Pam had students conduct 10 trials followed by 10 more. None of the participants pooled the class data to obtain a larger number of trials. Thus, participants did not attend to ideas like sample size and how sample size relates the two approaches.
Participants’ pedagogical decisions may be based on their understanding of the law of large numbers. The Pretest data suggests all participants had a weak understanding of the law of large numbers when they began the Data Analysis and Probability course. On the Pretest assessment, all 4 of the participants missed the question assessing their knowledge of the law of large numbers. Helen and Pam both missed the question again on the Posttest. Frank and Kathy both showed a better understanding of the law of large numbers on their Posttest assessments.

The teaching and learning of probability is very complex. When a new element is added to the mix (e.g. conducting an experiment or simulation), teachers may be unable to handle this additional complexity. Participants in the study seemed confounded by situations in which they had to deal with empirical data. One participant even miscalculated the empirical probabilities during her lesson. It seemed difficult for participants to make decisions and craft questions during their teaching episodes when dealing with data. Most have a weak understanding of sampling. They chose not to attend to sample size, variability of data, or independence.

Research Question 2

Participants in this study all chose to model situations involving uncertainty using manipulatives. Frank and Helen both use dice, while Kathy and Pam use colored chips. The use of manipulatives and technology were both emphasized in the Data Analysis and Probability course to promote probabilistic reasoning. Participants could choose either tool in their videotaped lesson. None of the participants chose to use technology. Frank is the only participant in the study who expresses a rationale for his pedagogical decision to use manipulates over technology. Frank communicates his preference for using technology, like
the software *Probability Explorer*. However, he does not have access to computers or graphics calculators.

Most of the participants in the study are not using multiple representations in their lessons involving empirical and theoretical probability. The most popular representations used by participants are ratios and percents; Frank is the only teacher that uses representations other than ratios and percents. Three of the teachers in the study emphasize the “reduction” of fractions when expressing empirical and theoretical probability. In fractions that represent probability, the numerators and denominators of the ratio have special meanings. An emphasis on “reduction” can sometimes devalue this meaning in a probabilistic context.

Overall, participants did not appropriately link their representations to the models they designed. Frank was the only participant to link the denominator of the theoretical probability of rolling dice to the sample space. Helen inappropriately linked the number of overall wins of evens and odds (e.g. 6 groups obtained more evens and 11 groups obtained more odds) to the number of trials in an experiment (e.g. 36). When Kathy linked her representations of uncertainty, she used fractional language in lieu of language related to probability. The denominator of the theoretical probability was referred to as “the whole piece” instead of the sample space or all possible outcomes. The numerator was referred to as “how many pieces or what the size of the piece was.”

Participants’ basic repertoire of examples is limited. When approaching uncertainty, participants usually provide one or two contexts for verbal examples which almost always have equally likely outcomes, like tossing a fair coin or rolling a fair die. In fact, Frank uses tossing a coin for the context of every verbal example in his videotaped lesson and rolling a
die or dice for the other examples. Helen uses tossing fair dice in her only example. Kathy also uses tossing a coin almost exclusively. She provides one example dealing with uncertainty that is not equiprobable (e.g. chips); however, the likelihood of obtaining a yellow, red, blue, or green chip is almost equally likely. It should be noted that Kathy refers to this example as equiprobable in her reflection.

Participants use approaches of empirical and theoretical probability both appropriately and inappropriately. In the scenarios participants created for students during their teaching episodes, both empirical and theoretical models could be used to predict uncertain events. Participants did not create tasks for students to consider where one approach would be more appropriate.

Frank and Helen used empirical approaches inappropriately. The design of Frank’s experiments in which students were not allowed to fulfill the theoretical probability was inappropriate. Helen improperly compared the number of wins to the number of trials to obtain empirical probabilities, and she later used a biased sampling method to obtain a contradictory result, a result which matched her own theoretical image. The analogies created by participants were never connected to normative ideas about empirical and theoretical probability. The attempt by Frank and Kathy to use analogy and informal language to make mathematics easier for students does not promote probabilistic reasoning.

Implications

Literature indicates that teachers do not have a strong understanding of probability (Begg & Edwards, 1999; Carnell, 1997; Carter & Capraro, 2005; Haller, 1997; Lui, 2005). As in studies by Carter and Caparo (2005) and Haller (1997), participants in this study held misconceptions even after instruction in a graduate mathematics education course focusing
on data analysis and probability. As found in studies by Begg and Edwards (1999) and Carter and Capraro (2005), participants do not have a strong understanding of the law of large numbers and reason using judgment heuristics. Like participants in the Begg and Edwards study, at least one participant has difficulty with the concept of independence. One participant made mistakes related to content during her probability lesson as did participants in Haller’s (1997) study. Finally, two participants have a weak understanding of sample space, as did participants in Carnell’s (1997) study. As in Lui’s (2005) study, participants’ reasoning could be characterized as deterministic, and they also failed to link representations.

This study adds additional knowledge to the existing literature of teachers’ conceptions of probability, particularly in the area of teachers’ conceptions of empirical and theoretical probability. Participants do feel that empirical and theoretical probability are important topics in the middle school mathematics curriculum. These participants are teaching probability lessons that approach the likelihood of uncertain events using both approaches. Every participant but one linked experiments and simulation to a real world context.

Even though participants are attempting to link empirical and theoretical probability, this current research specifically reveals participants do not have a robust understanding of the relationship between empirical and theoretical probability. Participants’ conception of the relationship is the two approaches will always result in different estimations of the likelihood of an uncertain event. Although they rely heavily on a theoretical estimation as the true probability, some have difficulty calculating the likelihood of uncertain events using a theoretical approach. Participants are unable to move flexibly within the bi-directional model, ignoring sample size, variation, and independence. There is evidence to suggest at
least two participants may be beginning to conceptualize how empirical data can inform theoretical probability. These participants have a better understanding of the law of large numbers and the relationship between the two approaches. Participants’ use of representations and basic repertoire of examples is limited. All of these findings contribute to new knowledge about middle school teachers’ understanding of probability.

Based on the findings of this current study, teacher education programs need to include courses that focus on developing the content and pedagogical content knowledge of probability simultaneously. Participants in this study had some misconceptions relating to their subject matter knowledge of probability. This study shows participants need a stronger conceptual understanding of ideas related to statistical inference and probability such as, sampling, sample space, independence, and the law of large numbers. Participants were faced with making challenging pedagogical decisions during their lessons. Their content knowledge and pedagogical decisions are almost always interrelated. Therefore, both content and pedagogical knowledge need to be fostered concurrently.

Prospective and practicing middle school teachers need to be required to take a course that approaches probability from a content and pedagogical perspective. A probability course for teachers should provide preservice and inservice teachers with experiences emphasizing the relationship between empirical and theoretical probability. One course was not enough to build a strong conceptual knowledge of the relationship between empirical and theoretical probability for all of the participants in this study. Perhaps, teacher education programs should include concepts related to empirical and theoretical probability in other courses as well, or develop a series of courses addressing different issues relating to the teaching and learning of probability.
All middle school teachers do not have the means or desire to enroll in courses at universities or colleges. There is a need to provide these inservice middle school teachers with professional development opportunities to enhance their conceptual and pedagogical understanding of empirical and theoretical probability. Probability has become an area of emphasis in many schools in the United States. In many states, important ideas related to the teaching and learning of probability have been placed at the beginning of the middle school curriculum. For example, in the state where this study took place all topics relating to probability have been recently placed in the 6th grade curriculum, including empirical and theoretical probability. Almost all topics relating to probability were previously in the 8th grade standard course of study. In North Carolina, some middle school mathematics teachers hold an elementary license (K-6). Most of these teachers have had little experience with probability in their teacher education programs, but they are now required to teach such topics. These teachers need professional development opportunities to build a better conceptual understanding of empirical and theoretical probability.

Professional development should also emphasize how to use standards-based curricula in professional development activities. Helen and Pam both were unable to implement standards-based probability lessons effectively. Access to standards-based curricula alone does not guarantee that a task or lesson will be implemented as intended by the curricular developers of standards-based materials.

Many times, teachers participate in professional development and are unable to implement new pedagogical approaches, like the use of technology, as in Frank’s case. Teacher education programs must provide support for these teachers once they return to their schools in order to create any significant change. In some cases, teachers have limited or no
access to technology. Perhaps, partnering with teachers to write grants might help them obtain the tools they need to have a real impact on student learning.

Teachers also need to be supported in other ways to sustain new pedagogical approaches. Follow-up’s by professional development providers may need to be conducted once teachers return to their schools to determine what additional support teachers may need. Teachers may actually need university faculty or other specialized staff to make visits to the teacher’s classroom. Teachers could also be supported by meeting with other teachers who are also trying to institute new pedagogical approaches.

Recommendations

Since this study was based on case study research, the generalizability of the findings are limited. Future research needs to be conducted to further investigate middle teachers’ conceptual understanding of empirical and theoretical probability. It would be important to learn about how this conceptual understanding affects students’ probabilistic reasoning. Since all the participants in this study were part of a professional development project, the following questions would be important to answer:

- What kind of support do teachers need to implement pedagogical change in their classrooms after their involvement in professional development?
- What are the effects of a course specifically designed to develop the relationship between empirical and theoretical probability in terms of teachers’ understanding and pedagogical approaches?
- What would teachers’ probability lessons look like over a period of 3 years?
REFERENCES


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APPENDICES
## Data Analysis and Probability for Middle School Summer Course Objectives

<table>
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<tr>
<th>Lesson</th>
<th>Objectives</th>
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| **Lesson 1 Summer 2003** | Mathematical Objectives:  
  • Teachers will use raw data and different techniques to analyze and display the data in order to answer questions about the data set.  
  • Teachers will review the NCTM Standards and NCDPI Objectives for Data Analysis Standards in grades 6-8.  
  Pedagogical Objectives:  
  • Teachers will discuss how different representations can be used by students to display data, and the benefits or drawbacks of using each type of representation.  
  • Teachers will discuss how the NCTM and NCDPI standards and objectives provide a framework for their curriculum. |
| **Lesson 2 Summer 2003** | Mathematical Objectives:  
  • Teachers will learn how to do various mathematical operations, statistics, and create graphs with GCs, Excel, and Fathom.  
  • Teachers will learn how to design, simulate and analyze data from experiments using Probability Explorer.  
  Pedagogical Objective:  
  • Teachers will think about the benefits and drawbacks of using various technologies for learning probability and statistics. |
| **Lesson 3 Summer 2003** | Mathematical Objectives:  
  • Given a bag of marbles with a known total but unknown contents, teachers will use the Probability Explorer tools to simulate and analyze data and predict contents of the bag.  
  • Teachers will analyze data from sampling with replacement from the bag of marbles.  
  • Given a lake with 2 different types of fish and an unknown total, teachers will use the Probability Explorer tools to simulate and analyze data to estimate the likelihood of catching each type of fish in a lake.  
  • Teachers will explore several mystery lakes with ratios for 3 fish and discuss connections between the ratio, probability, pie graph, bar graph, and percents for each ratio (1:2:2, 1:2:3, 2:6:8, 1:1:2, and 2:2:6). |
| **Lesson 4 Summer 2003** | Mathematical Objectives:  
  • Teachers will understand factors that make probability problems more cognitively demanding, such as knowing N, sample size, possible outcomes, etc. |
| Lesson 5 Summer 2003 | Mathematical Objective:  
• Teachers will understand how to find different measures of central tendency, spread, and extreme values.  
Pedagogical Objectives:  
• Teachers will discuss how different computation techniques may be used by students to calculate and display measures of central tendency, and spread. Teachers will also analyze and discuss the merits and demerits each type of representation.  
• Teachers will discuss and reflect on the connections of this discussion to the NCTM and NCDPI standards and objectives and the relevance to their curriculum. |
| Lesson 6 Summer 2003 | Mathematical Objectives:  
• Develop an understanding for estimation in an unusual manner using a practical example.  
• Sharpen our ability to make intelligent guesses particularly with respect to population sizes.  
• Use proportion and randomization in order to develop reasonable estimates for unknown quantities. |
| Lesson 7 Summer 2003 | Mathematical Objective:  
• Teachers will think about the role of probabilistic reasoning involved for a middle school student to solve the Schoolopoly task.  
Pedagogical Objectives:  
• Teachers will analyze students’ work on the Schoolopoly Task in 3 different forms: video of computer work, poster presentations, and test items.  
• Based on analysis teachers will discuss what each student understands about concepts such as fairness, empirical probability, theoretical probability, variability, sample size, and making inferences. |
## Data Analysis and Probability for Middle School Fall Course Objectives

<table>
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<tr>
<th>Lesson</th>
<th>Objectives</th>
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| Lesson 8 Fall 2003 | **Mathematical Objective:**  
  - Students will transition from univariate data analysis to bivariate data analysis.                                                                                                                  |
| Lesson 9 Fall 2003 | **Mathematical Objectives:**  
  - Students will gain a deeper conceptual understanding of the computation of the standard deviations for univariate data, and residuals, squared error, and linear regression for bivariate data.  
  - Students will gain a deeper conceptual understanding of the computation of the correlation coefficient and the effects of outliers on this computation. |
| Lesson 10 Fall 2003 | **Pedagogical Objectives:**  
  - Teachers will discuss the Mathematical Task Framework and come to understand the level of cognitive demand required by different types of tasks.  
  - Teachers will discuss how use of tools can influence the level of a task as well as students’ conceptual understanding. |
| Lesson 11 Fall 2003 | **Mathematical Objectives:**  
  - Teachers will understand the purpose of sampling in light of the entire population.  
  - Teachers will understand different types of sampling methods and the advantages and disadvantages of each. Teachers will have an understanding of the term *bias*.  
  - Teachers will gain an understanding of the effect of sample size when conducting simulations.  
  - Pedagogical Objective:  
    - Teachers will learn to use various methods to conduct simulations of an experiment (e.g. spinners, random number tables, graphic calculator, Excel, Probability Explorer). |
| Lesson 12 Fall 2003 | **Mathematical Objective:**  
  - Learn about binomial situations.                                                                                                                  |
| Lesson 13 Fall 2003 | **Mathematical Objectives:**  
  - Teachers will understand how to use the binomial distribution to calculate probabilities.  
  - Teachers will understand the normal distribution and use it to solve problems.                                                                     |
Appendix B – Analyzing Teaching Project

Data Analysis and Probability for Middle Grades
(from the course syllabus)

Analyzing Teaching Project
You will view, reflect on, and analyze one video of your classroom teaching where a concept or technique in data analysis or probability is central to the lesson. One video-taped lesson should be with a probability concept or with a data analysis concept. You will write a 2-3-page reflection about your analysis and answer specific guiding questions for the video-taped lesson. See information below

ANALYZING TEACHING
To help you think critically about issues of teaching and learning probability and statistics, it is important for you to implement ideas from this course in your classroom. Analyzing this implementation will allow you to have a “window” into your own teaching and learning issues of your students retrospectively rather than “on-the-fly” as you make decisions during lessons. This process will also help you in your preparation for National Board Certification. You will need to tape one class session. You will either tape one classroom lesson when you are teaching a probability concept or one that is focused on data analysis. The video should incorporate some type of technology tool or manipulative. You will pick a 15 minute segment (continuous, unedited) to focus on for your analysis and reflection. The structure of this assignment is parallel to what is expected by National Boards! You will choose either Video 1 or Video 2 to submit as your Analyzing Teaching Project.

Video 1: Whole Class Mathematical Discourse (Option 1)
You will provide a videotape of 15 minutes from a lesson that demonstrates how you use a classroom discussion and targeted questioning to develop student understanding about an important probability or data analysis idea. You will provide evidence of your ability to engage students in mathematical discourse as the whole class investigates, explores, or discovers important concepts, procedures, or reasoning processes within a stimulating and inclusive environment that promotes student development of mathematical power. You will need to use technology or manipulatives as part of this class discussion.

The questions that you will answer regarding your video in your 2-3 page reflection are:
1. How does what we see in the videotape fit into the lesson as a whole?
2. How well were the learning goals for the lesson achieved? What is the specific evidence for your answer?
3. How did your design and execution of this lesson affect the achievement of your learning goals?
4. How do you ensure fairness, equity, and access to learning for all students in your class?
5. What interactions on the videotape show students learning to reason and think mathematically and to communicate that reasoning and thinking?
6. Does your analysis of this lesson suggest that your learning goals for these students were best achieved through a whole class discussion?
7. How did your use of manipulative materials and/or technology affect the students’ learning experience?
8. How does what you noticed in your classroom compare with what you learned in the course about students’ reasoning with probability or data analysis?

**Video 2: Small Group Mathematical Collaborations (Option 2)**

You will provide a videotape of 15 minutes from a lesson that demonstrates how you interact with students working in small groups in order to promote mathematical discourse and to develop student understanding about an important idea in probability or data analysis. You are required to show how you use manipulative materials or appropriate technology to provide access to or deepen mathematical understanding. You will also show how you model questioning strategies and mathematical thinking and reasoning processes to promote interactions between you and the students, as well as among the students in small groups.

The questions that you will answer regarding your video in your 2-3 page reflection are:

1. How does what we see in the videotape fit into the lesson as a whole?
2. How well were the learning goals for the lesson achieved? What is the specific evidence for your answer?
3. How did your design and execution of this lesson affect the achievement of your learning goals?
4. How do you ensure fairness, equity, and access to learning for all students in your class?
5. What interactions on the videotape show students learning to reason and think mathematically and to communicate that reasoning and thinking?
6. Does your analysis of this lesson suggest that your learning goals for these students were best achieved through small group collaborations?
7. How did your use of manipulative materials and/or technology affect the students’ learning experience?
8. How does what you noticed in your classroom compare with what you learned in the course about students’ reasoning with probability or data analysis?

**Before you videotape, you must have permission forms from every person who will be viewed in your videos.**

*Some notes about the NTBS Process:* Videos must be submitted on previously blank VHS videotapes. Each entry must be **15-minutes** of class time that is continuous and unedited.
## Pretest Question #1
What topics do you consider the most important for middle school students to learn about probability? Why?

### Frank’s Response:
**Topic:** Permutations & Combinations

**Rationale:** P’s and C’s, as I call them, tend to be one of the most difficult objective for MS students to master. Yet, it is one of the easiest computations to perform. P’s and C’s are problems that make the students think mathematically. I have yet to meet a student who wants to “hand write” all the combinations for the Subway menu. They would much rather think about them. Not only does this promote the learning of P’s and C’s but also enhances their mental math capabilities.

## Posttest Question #1
What topics do you consider the most important for middle school students to learn about probability? Why?

### Frank’s Response:
**Topic:** Dependent vs. Independent events; Outcomes (FCP); More or less likely

**Rationale:** After completing the “Hospital” activity, I realized that the understanding my students had of probability was limited to only what they had done in my class. They also failed to see the difference between the “babies” question. This led me to believe that the proportionality of probability is not enough. The students must understand when an event is more likely depending on the whole situation.
**Pretest Question #2**  
Part I. What do you think of when you hear the word “sample”? Consider its meaning in different contexts.

**Frank’s Response:**  
When I hear the word sample, I first think “example” or “sample” problem. I also think of “sample population” and how this applies to surveys and statistical info. Sample also brings to mind “Free” (which a lot of students pay attention to). I ask the students to think about free samples they either get in the mail or in stores.

Part II. Briefly describe how you might introduce “sample” to your students.

**Frank’s Response:**  
I ask them to describe these “samples.” Most often their descriptions include the words “small,” “test/trial size,” and “mini.” I ask them to predict why companies and stores give out samples, and a lot of them answer with “it’s easier to handle, mail, give out” etc … This leads them to understand that samples in statistics are merely the “trial size” of what we are really looking at.

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**Posttest Question #2**  
Part I. What do you think of when you hear the word “sample”? Consider its meaning in different contexts.

**Frank’s Response:**  
When I hear the word sample, I immediately think of a survey. The word “sample” implies a smaller group selected from a larger group (population) that is representative (i.e. a well rounded selection) of the entire population.

Part II. Briefly describe how you might introduce “sample” to your students.

**Frank’s Response:**  
I introduce the concept of “sample” and likewise, “population” to my students. I would begin by having them think about our team’s “Safe School Survey” that they must complete at the end of every month. I would then ask them if they think that the entire school would respond as they did. I would then ask them to extend the “population” to all of NC, or even the US.
Pretest Question #3
A box contains 10 batteries, 2 of which are defective and the rest in good condition. Suppose you randomly select two of these batteries without replacement. Describe the sample space with a diagram.

Consider the events:
X={a good battery is drawn in the 1st drawing}
Y={a good battery is drawn in the 2nd drawing}

Find P(X), P(Y), P(X and Y), and P(X or Y).

Frank’s Response:

P(X) = \frac{4}{5}
P(Y) = \frac{7}{9}
P(X \text{ and } Y) = \frac{36}{45} + \frac{35}{45} = \frac{4}{5} + \frac{7}{9} = \frac{71}{91}
P(X) = \frac{4}{5} \cdot \frac{7}{9} = \frac{28}{45}

Posttest Question #3
A box contains 10 batteries, 2 of which are defective and the rest in good condition. Suppose you randomly select two of these batteries without replacement. Describe the sample space with a diagram.

Consider the events:
X={a good battery is drawn in the 1st drawing}
Y={a good battery is drawn in the 2nd drawing}

Find P(X), P(Y), P(X and Y), and P(X or Y).

Frank’s Response:

P(X) = \frac{8}{10}
P(Y) = \frac{7}{9}
P(X \text{ and } Y) = \frac{8}{10} \cdot \frac{7}{9} = \frac{56}{90} = \frac{28}{45}
P(X \text{ or } Y) = \text{no response}
<table>
<thead>
<tr>
<th>Pretest Question #4</th>
<th>Posttest Question #4</th>
</tr>
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<tbody>
<tr>
<td>Suppose James and Joleene flipped the same fair coin 10 times and got the following two sequences of outcomes.</td>
<td>Suppose Kimmie and Karl flipped the same fair coin 10 times and got the following two sequences of outcomes.</td>
</tr>
<tr>
<td>James: HHHHHTTTHT</td>
<td>Kimmie: HHHHHTHHHT</td>
</tr>
<tr>
<td>Joleene: HTTHHTHTHTT</td>
<td>Karl: HTTHHTHTHTT</td>
</tr>
<tr>
<td>Are either of these ordered sequences more likely to occur? Why or why not?</td>
<td>Are either of these ordered sequences more likely to occur? Why or why not?</td>
</tr>
</tbody>
</table>

**Frank’s Response:**
I think that Jolene’s sequence is more likely since the coin has an equally likely chance to be either H’s or T’s.

**Frank’s Response:**
I believe that each sequence is as likely to occur as the other. The fact is that each times a coin is tossed, the chances are equal that the coin is heads or tails. It’s only when the probability of the exact sequence is examined, that we find Kimmie’s sequence more likely.
Pretest Question #5

Suppose that two hospitals kept track of the number of babies born on a given day. Hospital A is in a big city while Hospital B is in a small town.

Hospital A.
20 Female Babies and 5 Male Babies

Hospital B.
4 Female Babies and 1 Male Baby

Assuming that, for any given birth, it is equally likely for a boy or girl to be born, which do you think is more likely to occur?

a) 20 out of 25 of the babies born in Hospital A are female,

b) 4 out of 5 of the babies born in Hospital B are female,

c) Events a) and b) are equally likely to occur.

Explain your reasoning.

Frank’s Response: Even though the empirical probability (choices a and b from above pictures) says that it is more likely to have a girl, the theoretical probability states that it is “equally likely to occur” leaving answer Choice C

Posttest Question #5

Suppose that two hospitals kept track of the number of babies born on a given day. Hospital A is in a big city while Hospital B is in a small town.

Hospital A.
20 Female Babies and 5 Male Babies

Hospital B.
4 Female Babies and 1 Male Baby

Assuming that, for any given birth, it is equally likely for a boy or girl to be born, which do you think is more likely to occur?

a) 20 out of 25 of the babies born in Hospital A are female,

b) 4 out of 5 of the babies born in Hospital B are female,

c) Events a) and b) are equally likely to occur.

Explain your reasoning.

Frank’s Response: I feel that Choice B is more likely to happen. 4 out of 5 times has a not so likely probability, making it an unlikely event. Unlikely events are more likely to happen in a smaller sample space. I originally thought that because they had the same proportionality (80%) that they were equally likely. Now I know.
### Appendix D – Helen’s Pretest and Posttest Responses

<table>
<thead>
<tr>
<th>Pretest Question #1</th>
<th>Posttest Question #1</th>
</tr>
</thead>
<tbody>
<tr>
<td>What topics do you consider the most important for middle school students to learn about probability? Why?</td>
<td>What topics do you consider the most important for middle school students to learn about probability? Why?</td>
</tr>
<tr>
<td><strong>Helen’s Response:</strong> Topic: Difference between probability &amp; odds</td>
<td><strong>Helen’s Response:</strong> Topic: Independent and dependent events</td>
</tr>
<tr>
<td><strong>Rationale:</strong> Students that I teach always seem to think they are the same thing</td>
<td><strong>Rationale:</strong> I think this because my students always seem to have trouble dealing with outcomes when there is or isn’t replacement</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Pretest Question #2</th>
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<td>Part I. What do you think of when you hear the word “sample”? Consider its meaning in different contexts.</td>
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</tr>
<tr>
<td><strong>Helen’s Response:</strong> A small selection representing something larger</td>
<td><strong>Helen’s Response:</strong> “Sample” is a snapshot of a whole population. You don’t include everyone, but you include as many types of people that you can in proportion to their representation of population.</td>
</tr>
<tr>
<td>Ex: sample size</td>
<td>Part II. Briefly describe how you might introduce “sample” to your students.</td>
</tr>
<tr>
<td><strong>Helen’s Response:</strong> I would use sample size products – have them tell me what they think sample is – broaden to sample in terms of data/prob.</td>
<td><strong>Helen’s Response:</strong> If I were surveying on types of music listened to by teens, would I ask every teen in America? Why wouldn’t I? And from this, I would lead in to how to choose a sample based on the demographics, white/black, girls/boys, geographical regions, etc.</td>
</tr>
</tbody>
</table>
### Pretest Question #3
A box contains 10 batteries, 2 of which are defective and the rest in good condition. Suppose you randomly select two of these batteries without replacement. Describe the sample space with a diagram.

Consider the events:
- \(X\) = {a good battery is drawn in the 1\(^{st}\) drawing}
- \(Y\) = {a good battery is drawn in the 2\(^{nd}\) drawing}

Find \(P(X)\), \(P(Y)\), \(P(X \text{ and } Y)\), and \(P(X \text{ or } Y)\).

**Helen’s Response:**

\[
P(X) = \frac{8}{10} \text{ or } \frac{4}{5}
\]

\[
P(Y) = \frac{7}{9}
\]

\[
P(X \text{ and } Y) = \frac{8}{10} \cdot \frac{4}{5} = \frac{4 \cdot 7}{5 \cdot 9} = \frac{28}{45}
\]

\[
P(X) = \frac{8}{10} \text{ or } \frac{4}{5}
\]

### Posttest Question #3
A box contains 10 batteries, 2 of which are defective and the rest in good condition. Suppose you randomly select two of these batteries without replacement. Describe the sample space with a diagram.

Consider the events:
- \(X\) = {a good battery is drawn in the 1\(^{st}\) drawing}
- \(Y\) = {a good battery is drawn in the 2\(^{nd}\) drawing}

Find \(P(X)\), \(P(Y)\), \(P(X \text{ and } Y)\), and \(P(X \text{ or } Y)\).

**Helen’s Response:**

\[
P(X) = \frac{8}{10} \cdot \frac{7}{9} = \frac{56}{90} = \frac{28}{45}
\]

\[
P(Y) = \frac{2}{10} \cdot \frac{1}{9} = \frac{2}{90} = \frac{1}{45}
\]

\[
P(X \text{ and } Y) = \frac{8}{10} \cdot \frac{1}{9} = \frac{8}{90} = \frac{4}{45}
\]

\[
P(X \text{ or } Y) = 100\%
\]

Do you mean getting only \(X\)’s on both draws and only \(Y\)’s on both draws? If not then

\[
P(X) = \frac{8}{10} = \frac{4}{5}
\]

\[
P(Y) = \frac{2}{10} = \frac{1}{5}
\]
<table>
<thead>
<tr>
<th><strong>Pretest Question #4</strong></th>
<th><strong>Posttest Question #4</strong></th>
</tr>
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<tr>
<td>Suppose James and Joleene flipped the same fair coin 10 times and got the following two sequences of outcomes.</td>
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</tr>
<tr>
<td><strong>Helen’s Response:</strong></td>
<td><strong>Helen’s Response:</strong></td>
</tr>
<tr>
<td>No, b/c although in theory we expect 50% H &amp; 50% T empirical probability isn’t the same. 10 times is not enough tosses to see a 50/50 split.</td>
<td>No. each has an equally likely chance of happening b/c each of the 10 slots could either be H or T equally.</td>
</tr>
</tbody>
</table>
**Pretest Question #5**

Suppose that two hospitals kept track of the number of babies born on a given day. Hospital A is in a big city while Hospital B is in a small town.

<table>
<thead>
<tr>
<th>Hospital A.</th>
<th>Hospital B.</th>
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<tbody>
<tr>
<td>20 Female Babies and 5 Male Babies</td>
<td>4 Female Babies and 1 Male Baby</td>
</tr>
</tbody>
</table>

Assuming that, for any given birth, it is equally likely for a boy or girl to be born, which do you think is more likely to occur?

a) 20 out of 25 of the babies born in Hospital A are female,

b) 4 out of 5 of the babies born in Hospital B are female,

c) Events a) and b) are equally likely to occur.

Explain your reasoning.

**Helen’s Response:**

c) \( \frac{20}{25} \) is the same as \( \frac{4}{5} \)

**Posttest Question #5**

Suppose that two hospitals kept track of the number of babies born on a given day. Hospital A is in a big city while Hospital B is in a small town.

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Assuming that, for any given birth, it is equally likely for a boy or girl to be born, which do you think is more likely to occur?

a) 20 out of 25 of the babies born in Hospital A are female,

b) 4 out of 5 of the babies born in Hospital B are female,

c) Events a) and b) are equally likely to occur.

Explain your reasoning.

**Helen’s Response:**

c) b/c \( \frac{20}{25} \) is the equivalent of four to 1
Appendix E – Kathy’s Pretest and Posttest Responses

<table>
<thead>
<tr>
<th>Pretest Question #1</th>
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</tr>
</thead>
<tbody>
<tr>
<td>What topics do you consider the most important for middle school students to learn about probability? Why?</td>
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</tr>
</tbody>
</table>

**Kathy’s Response:**

**Topic:** Types of events that can occur; Making predictions from the probability that an event might occur

**Rationale:** The probability of an event depend on whether they are dependent or independent.

Students can be taught how to plan for the future if they can analyze the likelihood of an event.

**Kathy’s Response:**

**Topic:**
6th grade – probability of events, both dependent and independent;
7th grade – differentiate between empirical and theoretical probability;
8th grade – combinations and permutations

**Rationale:** Given a basic understanding of probability and the numbers assigned to it. They should then be able to compare and make generalizations and predictions. Also, to be able to explain why an event did or did not occur as expected. They should also be able to tell the number of ways an event could occur to realize the numbers they get from experimentation and relate it to P(events).
Pretest Question #2
Part I. What do you think of when you hear the word “sample”? Consider its meaning in different contexts.

Kathy’s Response:
I think of a smaller piece of a whole. Sometimes it is a test of sausages in the market, or a little package of detergent to try. It could also be to take a section of something, like a population, and use it to draw generalizations or conclusions about the entire population.

Part II. Briefly describe how you might introduce “sample” to your students.

Kathy’s Response:
I would start by discussing the trial size or sample table section in grocery stores or Wal-Mart. I might even pass out “samples” of something, like a cookie or candy bar.

Posttest Question #2
Part I. What do you think of when you hear the word “sample”? Consider its meaning in different contexts.

Kathy’s Response:
1) A small part of a larger group or object.
2) To test or try.
3) A piece of something that represents the whole object.
4) Part of an entire population.

Part II. Briefly describe how you might introduce “sample” to your students.

Kathy’s Response:
I would discuss a trip to the store – maybe Target, and relate their “sample size” section. Also, when they go to the grocery store, to have them consider times they got a free piece of sausages, cheese, etc. We would discuss other ways you could have a sample of something, and talk about polls or TV rating and how data is gathered for that.
Pretest Question #3
A box contains 10 batteries, 2 of which are defective and the rest in good condition. Suppose you randomly select two of these batteries without replacement. Describe the sample space with a diagram.

Consider the events:
X = {a good battery is drawn in the 1st drawing}
Y = {a good battery is drawn in the 2nd drawing}

Find P(X), P(Y), P(X and Y), and P(X or Y).

Kathy’s Response:

\[
P(X) = \frac{8}{10} \\
P(Y) = \frac{8}{10} \cdot \frac{7}{9} = \frac{63}{90} = \frac{21}{30} \\
P(X \text{ and } Y) = \frac{21}{30} \\
P(X) = \text{don’t know}
\]
### Pretest Question #4
Suppose James and Joleene flipped the same fair coin 10 times and got the following two sequences of outcomes.

<table>
<thead>
<tr>
<th>James</th>
<th>HHHHHTTHHTT</th>
</tr>
</thead>
<tbody>
<tr>
<td>Joleene</td>
<td>HTHHTHHTTT</td>
</tr>
</tbody>
</table>

Are either of these ordered sequences more likely to occur? Why or why not?

**Kathy’s Response:**
No, each flip is an independent event, so out of 10 flips the probability is \( \left( \frac{1}{2} \right)^{10} \) for both or \[ \frac{1}{2} \]
however many possible combinations there are with 10 flips.

### Posttest Question #4
Suppose Kimmie and Karl flipped the same fair coin 10 times and got the following two sequences of outcomes.

<table>
<thead>
<tr>
<th>Kimmie</th>
<th>HHHHHTHHHTT</th>
</tr>
</thead>
<tbody>
<tr>
<td>Karl</td>
<td>HTHHTHHTTT</td>
</tr>
</tbody>
</table>

Are either of these ordered sequences more likely to occur? Why or why not?

**Kathy’s Response:**
No, they are equally likely. There is only one chance in all of the combinations that that exact sequence would occur and each member of the set has an equal chance of occurring.
### Pretest Question #5
Suppose that two hospitals kept track of the number of babies born on a given day. Hospital A is in a big city while Hospital B is in a small town.

<table>
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<th>Hospital A.</th>
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Assuming that, for any given birth, it is equally likely for a boy or girl to be born, which do you think is more likely to occur?

- a) 20 out of 25 of the babies born in Hospital A are female,
- b) 4 out of 5 of the babies born in Hospital B are female,
- c) Events a) and b) are equally likely to occur.

Explain your reasoning.

**Kathy’s Response:**

c) I made a ratio from the percents. The total number of babies in A is 25. For females, 20/25 = 80%. For males, 5/25 = 20%. For B, 5 babies were born: 4/5 female = 80%. 1/5 male = 20%.

\[
\frac{4}{5} = \frac{20}{25} = \frac{80}{100} \quad \text{and} \quad \frac{1}{5} = \frac{5}{25} = \frac{20}{100}
\]

### Posttest Question #5
Suppose that two hospitals kept track of the number of babies born on a given day. Hospital A is in a big city while Hospital B is in a small town.

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- a) 20 out of 25 of the babies born in Hospital A are female,
- b) 4 out of 5 of the babies born in Hospital B are female,
- c) Events a) and b) are equally likely to occur.

Explain your reasoning.

**Kathy’s Response:**

- Hospital B is in a small town and the baby population is much smaller than the baby population at hospital A.
- B: \( 5 \cdot \binom{4}{4}(1 - .5)^{(5-4)} = .156 \)
- A: \( 25 \cdot \binom{20}{5}(1 - .5)^5 = .002 \)

Hospital B has a higher probability than hospital A. The more babies born, the more chances there are of having a male, so the P(female) is reduced.
### Appendix F – Pam’s Pretest and Posttest Responses

<table>
<thead>
<tr>
<th><strong>Pretest Question #1</strong></th>
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</tbody>
</table>

**Pam’s Response:**  
**Topic:**  
Events  
Chance  

**Rationale:** Students can be aware of how likely an event is to happen. Students can determine the chance of an event taking place.

<table>
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<tr>
<th><strong>Pretest Question #2</strong></th>
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<td>Part I. What do you think of when you hear the word “sample”? Consider its meaning in different contexts.</td>
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**Pam’s Response:**  
I think about some smaller part of something larger. I think about randomly selecting parts of a larger whole.

**Part II. Briefly describe how you might introduce “sample” to your students.**  
**Pam’s Response:**  
The sixth grade students are planning a trip to an amusement park. They have two choices. Which park would the students prefer to attend? Ask a number of students from each classroom to get a sampling of the entire grade.

**Posttest Question #1**  
Topic:  
Theoretical and empirical  

**Rationale:** Because making predictions about things are an every day occurrence and constructing experiments to test them out are informative and interesting

**Pretest Question #2**  
Part I. What do you think of when you hear the word “sample”? Consider its meaning in different contexts.  

**Pam’s Response:**  
The word sample brings to mind a partition of: most anything – food, fragrance, and in statistics, population.

**Part II. Briefly describe how you might introduce “sample” to your students.**  
**Pam’s Response:**  
I might prepare a nontypical food and offer them a sample or discuss a survey of our school students body by choosing randomly 4 or 5 students from each of the three grade levels.
Pretest Question #3
A box contains 10 batteries, 2 of which are defective and the rest in good condition. Suppose you randomly select two of these batteries without replacement. Describe the sample space with a diagram.

Consider the events:
X={a good battery is drawn in the 1st drawing}
Y={a good battery is drawn in the 2nd drawing}

Find P(X), P(Y), P(X and Y), and P(X or Y).

Pam’s Response:
No response

Posttest Question #3
A box contains 10 batteries, 2 of which are defective and the rest in good condition. Suppose you randomly select two of these batteries without replacement. Describe the sample space with a diagram.

Consider the events:
X={a good battery is drawn in the 1st drawing}
Y={a good battery is drawn in the 2nd drawing}

Find P(X), P(Y), P(X and Y), and P(X or Y).

Pam’s Response:

\[
\begin{array}{c}
10 \\
9 \\
8
\end{array}
\]

P(X) = \frac{1}{8}
P(Y) = \frac{1}{7}
P(X \text{ and } Y) = \frac{2}{6} \text{ or } \frac{1}{3}
P(X) = \frac{2}{4} \text{ or } \frac{1}{2}
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<td>Are either of these ordered sequences more likely to occur? Why or why not?</td>
<td>Are either of these ordered sequences more likely to occur? Why or why not?</td>
</tr>
<tr>
<td>Pam’s Response: The sequence of Jolene is more likely to occur because a fair coin is just as likely to land on heads as it is on tails.</td>
<td>Pam’s Response: Neither sequence is more likely to occur because either event is equally likely to happen and results will happen randomly.</td>
</tr>
</tbody>
</table>
Pretest Question #5
Suppose that two hospitals kept track of the number of babies born on a given day. Hospital A is in a big city while Hospital B is in a small town.

Hospital A.                                                       Hospital B.
20 Female Babies and 5 Male Babies              4 Female Babies and 1 Male Baby

Assuming that, for any given birth, it is equally likely for a boy or girl to be born, which do you think is more likely to occur?

a) 20 out of 25 of the babies born in Hospital A are female,
b) 4 out of 5 of the babies born in Hospital B are female,
c) Events a) and b) are equally likely to occur.

Explain your reasoning.

Pam’s Response:
c) More female babies are born than males in general. Therefore, in either A or B the event is likely to occur.

Posttest Question #5
Suppose that two hospitals kept track of the number of babies born on a given day. Hospital A is in a big city while Hospital B is in a small town.

Hospital A.                                                       Hospital B.
20 Female Babies and 5 Male Babies              4 Female Babies and 1 Male Baby

Assuming that, for any given birth, it is equally likely for a boy or girl to be born, which do you think is more likely to occur?

a) 20 out of 25 of the babies born in Hospital A are female,
b) 4 out of 5 of the babies born in Hospital B are female,
c) Events a) and b) are equally likely to occur.

Explain your reasoning.

Pam’s Response:
c) 20: 5 and 4:1 are equal ratios