

## **Abstract**

CAO, YINGFANG. Adaptability and Comparison of the Wavelet-based with Traditional Equivalent Linearization Method and Potential Application for Damage Detection. (Under the direction of Mohammad N. Noori.)

The main objective of this research work is to compare a wavelet transform based Equivalent linearization technique, presented herein with the traditional Equivalent linearization method. In addition, the application of this wavelet-base method to the damage detection of a single Degree of Freedom system has also been presented.

In many practical applications the system of concern is nonlinear. In general, and especially in random vibration analysis, it is difficult to obtain a closed form solution for dynamic response of a nonlinear system. Therefore, it is necessary to replace the nonlinear system with an equivalent linear system. Method of equivalent linearization has been extensively used in these engineering applications. The wavelet analysis allows us to capture temporal variations in the energy and frequency content. In combining the wavelet analysis technique and the well-known Equivalent linearization method, a nonlinear system can be approximated as a time dependent linear system. By comparing the information of the time-varying natural frequency coefficient between the healthy system and the damaged system, the introduced damage can be identified at a particular time.

The first part of this document consists of a detailed description of some of the most common types of time-frequency methods that have been developed or used for vibration and frequency analysis over the last several decades. The fundamentals of the traditional

Equivalent Linearization Method have been introduced to demonstrate the general procedure to linearize a system. An important linearization method, utilizing Wavelet based transformation is then presented.

In this study, the benefits of this proposed method have been verified by its application in a typical nonlinear system. Different wavelets and different excitations are used in performing this analysis. To verify the accuracy of the wavelet based method, the traditional Equivalent Linearization results of displacement have been compared with those obtained by this method. In order to check the feasibility of the methodology, the relationship of the restoring force vs. displacement has been compared between the original nonlinear system and the equivalent linear system. For further application in the structural health monitoring, this method has been applied to some simple cases to see the effect of damage detection. To simplify the entire procedure of derivation, some formulation work introduced by Basu, Gupta [1997] and Roberts and Spanos [1990] have been followed.

Results from this study demonstrate that this approach is more accurate for non-stationary excitation compared with the traditional Equivalent Linearization method. It is also suitable for damage detection even when the introduced damage is fairly small.

**ADAPTABILITY AND COMPARISON OF THE WAVELET-BASED WITH  
TRADITIONAL EQUIVALENT LINEARIZATION METHOD AND POTENTIAL  
APPLICATION FOR DAMAGE DETECTION**

by  
**Yingfang Cao**

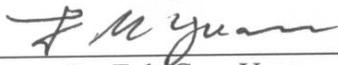
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## Biography

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# **Chapter 1 : Introduction**

## ***1.1 Time-Frequency Analysis***

Time-frequency analysis plays a central role in signal analysis. Time-frequency methods are simply the extensions of the well-known Fourier Transform (FT) in which most signals of practical interest can be decomposed into a sum of sinusoidal components with different frequencies. As a tool for applications, Fourier Transform is used in virtually all areas of science and engineering, especially in its discrete form since computational power has increased dramatically. Fourier Transform demonstrates that every signal has a spectrum, while the Fourier inversion theorem implies that a function and its Fourier transform are really equivalent in the sense that one determines the other. The invention of the computer algorithm known as the Fast Fourier Transform (FFT) in the mid-1960s opened up the possibility of an even wider use of the FT in computational physics and many other areas of Physics and Mathematics. Nowadays, computers being used in computational physics are frequently evaluated principally on their capacity to perform the FFT algorithm, and many special computers or add-on cards are available to perform the FFT algorithm at ultra-high speed.

Even after the introduction of the FFT algorithm, the Fourier Transform still faces some limitations. In many applications, the system being studied generates quickly changing (impulsive) sounds or vibrations. The Fourier Transform cannot provide temporal localization necessary for proper engineering analysis. This characteristic makes it only suitable for the analysis of stationary signals where the frequency content of the signal being analyzed does not evolve with time. This limitation encouraged the search for

a new tool, which can be used to give information about signals simultaneously in the time domain and the frequency domain.

One of the first time-frequency representations developed is the Wigner-Ville transform. In 1932, Wigner derived a distribution over the phase space in quantum mechanics [Wigner, 1932]. Some 15 years later, Ville, searching for an "instantaneous spectrum" - influenced by the work of Gabor - introduced the same transform in signal analysis [Ville, 1948]. Unfortunately the non-linearity of the Wigner distribution causes many interference phenomena, which makes it less attractive for many practical purposes [Cohen, 1995].

A different approach to obtain a local time-frequency analysis (suggested by various scientists, among them Ville), is to cut the signal first into slices, followed by doing a Fourier analysis on these slices. But the functions obtained by this crude segmentation are not periodic, which will be reflected in large Fourier coefficients at high frequencies, since the Fourier transform will interpret this jump at the boundaries as a discontinuity or an abrupt variation of the signal. To avoid these artifacts, the concept of windowing has been introduced. By segmenting the signal into small sections and analyzing each section separately there is a better chance to procure meaningful analysis. If the signal has sharp transitions, windowing the input data such that the sections converge to zero at the endpoints helps to localize the signal in time. Adjacent sequences of windowed Fourier analyses join together to produce a time ordered view of the frequency content of a signal.

While the windowed Fourier Transform compromise between the time and frequency information can be useful, the drawback is that once the particular size of the time

window is chosen, that window is same for all frequencies. Many signals require a more flexible approach—varying the window size to determine more accurately either time or frequency.

A revolutionary change from the concept of time-frequency analysis came with the introduction of Wavelet Transform (WT). Wavelet theory was initially proposed by J. Morlet, a geophysicist, and A. Crossman, a theoretical physicist [1984], together with their fellow Frenchman, Y. Meyer, their ‘French school’ developed the mathematical foundations of wavelets. Due to its mathematical complexity, the wavelet analysis remained purely theoretical and had very limited practical applications at that time. It was the work of Daubechies [1988,1990,1992], and Mallat [1989], which established the necessary links between the rigorous mathematical requirements and the wide applications in signal processing.

Wavelet analysis allows the use of long time intervals where more precise low-frequency information can be obtained and shorter regions where high-frequency information can be gained. Instead of breaking down the signal into harmonics, the Wavelet Transform break down the signal into a series of local basis functions called wavelets. These basis functions are nothing but variable length window functions, where the length depends on a variable scale parameter. This scale parameter then could be related to a center frequency of the window and thus allowing for the construction of a time-frequency map of the signal. Compared with sine waves, wavelets tend to be irregular and asymmetric. Signals with sharp changes might be better analyzed with an irregular wavelet than with a smooth sinusoid. With its capability of revealing the aspects of data that other signal analysis techniques might miss, Wavelet Transform have gained

great acceptance in several fields, especially in the applications of image processing which take great advantage of the variable window property of this new transform. This same property will be the one, which will pose the Wavelet transform as a useful tool for structural health monitoring applications.

## ***1.2 Equivalent Linearization Method***

In many practical applications, due to the high intensity nature and often complex nature of environmental loads such as earthquakes, wind loads, and sea waves, the systems subjected to these loadings may experience excessive stress or displacements that results in elastic or even hysteretic behavior. This is particularly the case under high intensity random excitation. Under these conditions, it is difficult to obtain the closed form solution for dynamic response of a nonlinear system. Therefore, for higher accuracy, one must resort to the analysis of nonlinear systems under non-stationary forces.

Of the various possible approaches that are available, over last several decades, the method of equivalent linearization has been proved to be the most useful approximation technique and has been extensively used in engineering applications. Basically, the method is the statistical extension of Krylov and Bogoliubov's [1963] linearization technique and is often referred to as the 'describing function method' in the electrical engineering literature. An adaptation of the approach to deal with stochastic problems was originally developed by Booton [1953] and used as a tool in control engineering. Subsequent developments of this method in this field have been described by Sawaragi [1962], Kazakov [1965], Gelb and Van Der Velde [1968], Atherton [1975] and Sinitsyn [1976].

Independently, the method, (now variously known as ‘statistical linearization’ or ‘equivalent linearization’) was proposed by Caughey [1963] as a means of solving nonlinear stochastic problems in structural dynamics. With Caughey’s work, the linear part of the nonlinear equation was uncoupled using the modal matrix, and then linearization technique was applied to the resulting single-degree-of-freedom equations separately. This is quite effective when applicable, but the conditions imposed on the instantaneous correlation matrix of the excitation limited its application except for some special cases. In Foster’s (1968) more general approach, numerical methods must be used in order to obtain the equivalent linear stiffness and damping matrices. By this way, the procedure of approximating closed form solutions would require an excessive amount of algebraic computation. Later on, Iwan and Yang (1971 and 1972) were able to formulate the approach such that the linear stiffness and damping matrices may be determined analytically. Their approach is based on the concept of decomposing the nonlinear forces of the actual system into the sum of simpler approximate nonlinear forces, each of which depends solely on the relative displacement and velocity between discrete masses of the system. In some cases, this method will yield closed form solutions. However, given a system of nonlinear equations, accuracy of the solution depends heavily on the choice of these approximate forces. Also, when the masses are distributed, the identification of the forces might be quite complex.

This approach is well suited to mass-spring type systems. Numerous applications of the technique for studying the response of non-linear oscillators to random excitation have been described in the literatures [Roberts, 1981b, and Spanos, 1981a]. Although from the perspective of treatable physical problems, the equivalent linearization method

has proved a useful analytical tool across a very wide spread of engineering applications [Roberts and Spanos, 1990], generally, in this technique, the excitations are assumed to be stationary and gaussian. Thus, in some cases, such as systems excited by ground motion, it is necessary to approximate the nonlinear system to be a time-varying system. Some theoretical developments in this area have been illustrated by Spanos [1978,1980 and 1981], Iwan and Mason [1980], Ahmadi [1980], and Noori et. al. [1986].

In recent years, it has been shown that Wavelet analysis provides a convenient way for the response analysis of systems subjected to non-stationary ground motion (Basu, Gupta, [1997]). Specifically, the wavelet transformation has been used to obtain both frequency and temporal information of system response caused by non-stationary excitation. Based on this capability, the wavelet analysis method can be combined with traditional equivalent linearization method to obtain a time-varying equivalent system [Basu and Gupta, 1999]. This recently developed wavelet-based equivalent linearization method has proved to be more accurate when dealing with some nonlinear SDOF structures [Cao and Noori, 2002].

### ***1.3 Structural Health Monitoring***

Nearly all in-service structures require some form of maintenance for monitoring their integrity and health condition to prolong their lifespan or to prevent catastrophic failure. The interest in the ability to monitor a structure and detect damage at the earliest possible stage is pervasive throughout the civil, mechanical and aerospace engineering communities. This is now commonly referred to as ‘Structural Health Monitoring and Damage Detection’.

Structural health monitoring can be defined as the diagnostic monitoring of the integrity or condition of a structure. The intent is to detect and locate damage or degradation in structural components and to provide this information quickly and in a form easily understood by the operators or occupants of the structure. The damage may result from fatigue, large earthquake, strong winds, and explosion or vehicle impact. Early detection of damage or structural degradation prior to local failure can prevent "runaway" catastrophic failure of the system. In engineering applications, damage is understood intuitively as an imperfection or impairment of the function and working condition of a structure or machine. Damage can be described in many ways depending on the structure and its function. Hence damage detection has many definitions based on what type of damage is being measured [Staszewski, 1998 and Sone, 1995]. Since the health monitoring was defined as use of in-situ, nondestructive sensing and analysis of system characteristic, including structural response, for the purpose of detecting changes, the term, structural health monitoring and damage identification are usually interchangeable.

Research over the past several decades within the structural engineering and other related engineering fields have resulted in advancements and technologies that assist structural engineers in their attempts to ensure the safety and reliability of structures over their life spans. Two major reasons responsible for this growth of the health monitoring of structural systems have been: the advances in sensors, data acquisition, data communications, and real-time data analysis, allowing for the implementation of highly reliable and accurate monitoring and diagnostics hardware with very advanced features [Housner et al., 1997], and the growing interest in certain markets, such as military,

aerospace, and civil infrastructures, which see the application of health monitoring systems as means of assuring the correct and safe performance of these engineered systems, as well as protecting their investments [Chang, 1999].

The structural health monitoring can be performed on both global and local basis. Global based health monitoring focuses on monitoring and verifying the performance of the entire system. By monitoring the output, such as vibration displacement, velocity or acceleration, of the system during its operation, the occurrence of the damage can be detected. On the other hand, local based health monitoring is interested in monitoring certain key elements of the total system. The applications of local based health monitoring usually rely on the use of localized non-destructive evaluation techniques, such as acoustic or ultrasonic methods, magnetic fields methods, radiography, eddy-current methods, or thermal methods [Doherty, 1993]. All of these experimental techniques require that the vicinity of the damage is known a priori and that the portion of the structure being inspected is readily accessible [Doebling, 1998]. Subject to these limitations, the need for additional global damage detection methods has led to the development of the methods that examine changes in the vibration characteristics of the structure.

Damage detection, as determined by changes in the dynamic properties or responses of structures, is a subject that has received considerable attention in the literature. The basic idea is that modal parameters (frequencies, mode shapes, etc.) are functions of the physical properties will cause changes in the modal properties [Doebling et al. 1996].

Damage effects on a structure can be classified [Doebling et al. 1998] into linear or non-linear. A linear damage scenario is one in which the linearly elastic behavior of the

structure is preserved after the occurrence of the damage. The changes of modal properties are resulted from the changes in geometry or material properties of the structure, but the behavior of the structure can still be modeled using linear equations of motions. On the other hand non-linear effects are the ones in which the system does no longer exhibit a linearly elastic behavior after the damage has been introduced. Loosing of fastening elements, where the separation of mating parts induce non-linear responses in the system, can be cited as these non-linear effects. Another example of nonlinear damage is the formation of a fatigue crack that subsequently opens and closes under the normal operating vibration environment.

As mentioned previously, the field of structural health monitoring is very broad. Among those different issues, an important one is to use signal processing techniques, which includes such method as Fourier analysis, time-frequency analysis and wavelet analysis. There exist two major and complementary components that form the basis for structural health monitoring: hardware or the experimental tools, and methodologies or analytical/computational tools. Previous analytical and computational studies on damage detection have been tailored toward identifying variations of modal parameters, such as natural frequencies, modal shapes, and modal damping ratios obtained from the vibration signals [Doebbling, et al 1996, 1998]. Some of these methods allowed for assessing changes in these parameters, which then were related to some structural damage, but they could not provide information of the exact time of the occurrence of the damage neither detecting the location of a fault, which may be an important part in the total picture of what structural health monitoring implies. A time frequency based method for examining

the changes of the natural frequency to detect damage will be introduced in this thesis work.

Methods such as the windowed Fourier Transforms and Wigner Distribution were among the first time-frequency methods to be implemented. In recent years, the application of the Wavelet Transform provided a new tool for time frequency analysis, which has been effectively used in health monitoring and damage detection of various structures.

## **1.4 Scopes**

As introduced earlier, for some nonlinear problems, such as the dynamic behavior of buildings to wind loading and earthquake, the vibration of vehicles traveling over rough ground and the excitation of aircraft and missile by atmospheric and jet noise, the classical linear theory is not applicable. With various possible nonlinear analysis approaches, the method of equivalent linearization has been proved more effective. However, most studies have assumed the response to be harmonic in nature leading to an equivalent time invariant linear system. This may work well for steady-state response with the frequency of response centered in a narrow band. Therefore, if the response energy fluctuates temporally, a time-varying system may approximate the response better [Mason, 1979]. Wavelet analysis provides a convenient way for the response analysis of system subjected to non-stationary ground motion. Based on the capability to obtain both temporal and frequency information of system response caused by non-stationary excitation, wavelet analysis has been combined with the traditional equivalent linearization method to obtain a time-varying equivalent system.

A pioneering work for developing the wavelet-based equivalent linearization method was proposed by Basu and Gupta [1999]. In this work, based on a newly presented technique of using wavelet analysis to solve the response of a single-degree-of-freedom system subjected to transient excitation, a time-varying equivalent system can be obtained. In this study, a SDOF system with nonlinear stiffness was considered.

To verify the effectiveness of this new method, it is necessary to compare the results obtained using this way with other traditional methods. Therefore, the main goal of the present work is to compare the system responses gained from different approaches. The quantification of the error between nonlinear system and linearized system will physically verify the applicability of this method.

The structural health monitoring and damage detection utilizing wavelet analysis have been widely studied. By utilizing an orthogonal wavelet decomposition of the vibration signal analysis, Masuda [1995] developed a way to determine the instant at which damage occurred. Hou and Noori [1999] extended this orthogonal wavelet decomposition method to evaluate damages in multi-degree of freedom systems. Alonso and Noori [2001] built up a milestone of wavelet analysis application on structural health monitoring. It also has been demonstrated [Alonso and Noori, 2001] that there is a limitation of this wavelet decomposition for non-deterministic excitation such as Gaussian white noise.

In using the newly developed wavelet-based equivalent linearization method, the damage introduced to the nonlinear system under non-stationary excitation can be determined. The effect of the damage detection will be explored for the cases where

system is subject to random loads and harmonic loads. The information provided by the frequency changes contains instant message and the extent of the damage.

Finally, suggestions for future work will be discussed.

## **Chapter 2 : Fourier Analysis**

Signal analysts already have an impressive arsenal of tools. Perhaps, the most well known of these is Fourier analysis, which break down a signal into constitute sinusoids of different frequencies. This chapter is devoted to an overview of this method. Instead of presenting detailed information of Fourier analysis, this overview will only discuss some basic concepts and important properties that might be useful in the understanding of Wavelet analysis, a extension of the Fourier analysis, which will be widely used in this research work. Fourier analysis basically involves the decomposition of the signals in terms of sinusoidal components. With such decomposition, a signal is said to be represented in the frequency domain. For the class of periodic signals, the decomposition is called a Fourier series. For the class of finite energy signals, the decomposition is called the Fourier transform.

### ***2.1 Fourier series***

To know Fourier analysis, the first element should be known is Fourier series. The theory of Fourier series lies in the idea that most signals, and all engineering signals, can be decomposed into a sum of sinusoid waves. J. Fourier, a French mathematician, used such trigonometric series expansions in describing the phenomenon of heat conduction and temperature distribution through bodies. Although his work was motivated by the problem of heat conduction, the mathematical techniques that he developed during the early part of the nineteenth century now find application in a variety of problems encompassing many different fields, including vibrations in mechanical systems, electromagnetics, and system theory [J. Proakis and D. Manolakis, 1992].

The Fourier series of an infinite signal  $f(t)$  with period  $T_p$  can be represented as

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \quad (2.1)$$

Where,  $j = \sqrt{-1}$ ,  $\omega_0 = 2\pi/T_p$  is the fundamental frequency of the sinusoids, and  $c_k$  are the Fourier coefficients, which can be obtained from

$$c_k = \frac{1}{T_p} \int_{T_p} f(t) e^{-jk\omega_0 t} dt \quad (2.2)$$

An important issue that arises in representing a periodical signal  $f(t)$  by the Fourier series is whether or not the series converges to  $f(t)$  for every value of  $t$ . The so-called Dirichlet conditions will guarantee that the signal  $f(t)$  will be equal to a sum of harmonic series. The Dirichlet conditions are:

1. The signal  $f(t)$  has a finite number of discontinuities in any period.
2. The signal  $f(t)$  contains a finite number of maxima and minima during any period.
3. The signal  $f(t)$  is absolutely integrable in any period, that is

$$\int_{T_p} |f(t)| dt < \infty \quad (2.3)$$

Generally, if a signal is periodical and satisfies the Dirichlet conditions, it can be accurately represented by Fourier series.

## 2.2 Fourier Transform

### 2.2.1 Definition

The Fourier Transform is a generalization of the Fourier series. Strictly speaking it applies to continuous and aperiodic functions, but the use of the impulse function allows the use of discrete signals. The Fourier transform can be defined as:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt \quad (2.4)$$

It has been shown that  $F(\omega)$  is a function of the continuous variable  $\omega$ . It does not depend on  $T_p$  or  $\omega_0$ .

The inverse Fourier transform can be defined as:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{j\omega t} d\omega \quad (2.5)$$

The set of conditions that guarantee the existence of the Fourier transform is the Dirichlet conditions, which may be expressed as:

1. The signal  $f(t)$  has a finite number of finite discontinuities.
2. The signal  $f(t)$  contains a finite number of maxima and minima.
3. The signal  $f(t)$  is absolutely integrable, that is

$$\int_{-\infty}^{\infty} |f(t)| dt < \infty \quad (2.6)$$

The Dirichlet conditions are sufficient but not necessary for the existence of the Fourier transform. In any case, nearly all finite energy signals have a Fourier transform. It is apparent that the essential difference between the Fourier series and the Fourier

transform is that the spectrum in the latter is continuous and hence the synthesis of an aperiodic signal from its spectrum is accomplished by integration instead of summation.

## 2.2.2 Properties of the Fourier Transform

There are a variety of properties associated with the Fourier Transform and the inverse Fourier transform. The following are some of the most relevant for time-frequency analysis.

1. Linearity: The Fourier and the inverse Fourier transforms are linear operations.

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} (af_1(t) + bf_2(t))e^{-j\omega t} dt = a \int_{-\infty}^{\infty} (f_1(t)e^{-j\omega t}) dt + b \int_{-\infty}^{\infty} (f_2(t)e^{-j\omega t}) dt \\ &= aF_1(\omega) + bF_2(\omega) \end{aligned} \quad (2.7)$$

2. Shift: If a signal  $f(t)$  is shifted by  $\tau$ , the Fourier transform can be:

$$F_{\tau}(\omega) = \int_{-\infty}^{\infty} (f(t - \tau)e^{-j\omega t}) dt = \int_{-\infty}^{\infty} f(u)e^{-j\omega(u - \tau)} du = e^{-j\omega\tau} F(\omega) \quad (2.8)$$

Following this property, for a time shifting signal  $f(at)$ , the Fourier transform will be

$$F_a(\omega) = \int_{-\infty}^{\infty} f(at)e^{-j\omega t} dt = \frac{1}{a} \int_{-\infty}^{\infty} f(t)e^{-j\omega(t/a)} dt = \frac{1}{a} F\left(\frac{\omega}{a}\right) \quad (2.9)$$

Where  $a$  is a non-zero constant. If  $a$  is greater than 1, the spectrum will be expanded. If  $a$  is less than 1, the spectrum will be contracted.

Based on the process used for time shifting, the frequency shift can be illustrated as:

$$f_o(t) = f(t)e^{j\omega_o t} \quad (2.10)$$

Where frequency shifting is defined as:  $F_o(\omega - \omega_o)$ .

With the form of  $F_a(\omega) = F(a\omega)$ , the inverse Fourier transform will be:

$$f_a(t) = \frac{1}{a} f\left(\frac{t}{a}\right) \quad (2.11)$$

Where  $a$  is a non-zero constant.

### 3. Moments:

The  $n$ th order moment of a signal  $f(t)$  can be defined as:

$$M_n = \int_{-\infty}^{\infty} t^n f(t) dt \quad (2.12)$$

### 4. Convolution:

Convolution of function  $f$  and  $g$  is the new function  $f \otimes g$  such as

$$f \otimes g = \int_{-\infty}^{\infty} f(\tau) g(t - \tau) d\tau \quad (2.13)$$

By taking the Fourier transform of the convolution, it can be proved that in the frequency domain the convolution in time of two functions is equivalent to the multiplication of the Fourier transform of the respective functions [Brigham, 1988], which can be expressed as:

$$F(\omega) = F_f(\omega) F_g(\omega) \quad (2.14)$$

Where,  $F_f(\omega)$  is the Fourier transform of  $f$  and  $F_g(\omega)$  is the Fourier transform of  $g$ .

### 5. Parseval's Theorem:

The energy,  $E$ , in a signal can be measured either in the spatial domain or the frequency domain. For a signal with finite energy, the total energy calculated in the time domain is equal to the total energy measured in the frequency domain:

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega \quad (\text{For continuous space}) \quad (2.15)$$

$$\sum_{k=-\infty}^{\infty} |f(k)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(\omega)|^2 d\omega \quad (\text{For discrete space}) \quad (2.16)$$

This "signal energy" is not to be confused with the physical energy in the phenomenon that produced the signal. If, for example, the value  $f(k)$  represents a photon count, then the physical energy is proportional to the amplitude of  $f(k)$  and not the square of the amplitude. This is generally the case in video imaging.

### **2.3 Discrete Fourier Transform**

In practical applications, computers cannot handle a continuous signal. They have to have discrete samples. Therefore, a discrete form of the Fourier Transform (DFT) has been applied to approximate the spectrum of the signal. The Fourier Transform of a finite energy discrete time signal  $f(n)$  is defined as:

$$F(\omega) = \sum_{n=-\infty}^{\infty} f(n)e^{-j\omega n} \quad (2.17)$$

Physically,  $F(\omega)$  represents the frequency content of the signal  $f(n)$ . In other words,  $F(\omega)$  is the decomposition of  $f(n)$  into its frequency components.

The equation (2.17) can be written as:

$$F_N(\omega) = \sum_{n=-N}^N f(n)e^{-j\omega n} \quad (2.18)$$

if for every  $\omega$ ,  $F_N(\omega) \rightarrow F(\omega)$  as  $N \rightarrow \infty$ .

Practically, by sampling the signal  $f(t)$  in  $N$  number of points equally spaced, the signal can be represented as:

$$f(t) \Rightarrow f(kh) \quad , \quad k = 0, 1, 2, \dots, N-1 \quad (2.19)$$

Where  $h$  is the sampling period, and the frequency can be defined by it:

$$\omega_n = \frac{2n\pi}{Nh}, \quad n = 0, 1, 2, \dots, N-1 \quad (2.20)$$

From the equation (2.17), it is noticed that the integral has been replaced by a summation. As  $N$  (the number of samples) increases the result goes to infinity, so to stop this happening a factor of  $h$  has been multiplied. The Fourier transform in its discrete form will be:

$$F(\omega_n) = h \sum_{k=0}^{N-1} f(kh) e^{-j2\pi kn/N} \quad (2.21)$$

It should be noticed that for improperly sampled signal, the spectral estimate might be distorted due to a so-called characteristic: leakage. To reduce it, the Nyquist criterion has to be applied when sampling the signal. It requires the sampling rate to be at least twice of the highest frequency component presented in the signal.

## **2.4 Fast Fourier Transform**

The Fast Fourier Transform is simply a method of laying out the computation, which is much faster for large values of  $N$ , where  $N$  is the number of samples in the sequence.

The idea behind the FFT is the dividing and conquering approach, to break up the original  $N$  point sample into two  $(N / 2)$  sequences. This is because a series of smaller problems is easier to solve than one large one. The DFT requires  $(N-1)^2$  complex multiplications and  $N*(N-1)$  complex additions as opposed to the FFT's approach of breaking it down into a series of 2 point samples which only require 1 multiplication and 2 additions and the recombination of the points which is minimal. In using this method, it is possible to calculate the DFT more efficiently, which reduces the number of multiplications from  $N^2$  to  $O(N \log_2 N)$ .

## ***2.5 Application of the Fourier Transform***

The use the Fourier transform has many applications, in fact any field of physical science that utilizes sinusoidal signals in its theory, such as engineering, physics, applied mathematics, and chemistry, will make use of Fourier theory and transforms. As you can see it would be near impossible to give examples of all the areas where the FT is involved, so the applications listed are related to engineering and signal processing.

1. Communications: The field of communications over a vast range of applications from high level network management down to sending individual bits down a channel. The Fourier transform is usually associated with low level aspects of communications.
2. Seismic: Seismic research has always been a common user for the Discrete Fourier Transform (and the FFT). In fact if you look at the history of the FFT you will see that the original use for the FFT was to distinguish between natural seismic events and nuclear test explosions. The FFT is able to tell the difference between the two because natural seismic events have a different frequency response (spectrum) to seismic events caused by nuclear explosions.

3. Filtering: Most filters are designed to filter out some frequency components of a signal be it low pass, high pass or band pass/stop. The Fourier Transform provides a valuable tool in order to take signals across into the frequency domain to view their characteristics before and after the filtering.
4. Acoustics: In Acoustics the signal is the fluctuations in pressure, which exist in the path of a sound wave. The Fourier transform of this signal is effectively breaking down the sound into pure tones as recognized by musicians as a note of definite pitch. The Inverse Fourier transform is also used in acoustics by combining tones of different frequencies to produce a desired resulting sound.

## **2.6 Summary**

The Fourier transform is an important tool in practical applications. Especially the DFT and the inverse DFT (IDFT), as computation tools, play a very important role in many engineering and digital signal processing applications, such as frequency analysis (spectrum analysis) of signals, power spectrum estimations, etc. The importance of the DFT and IDFT in such practical applications is due to the large extent on the existence of computationally efficient algorithm, which is known as the Fast Fourier Transform, for computing the DFT and IDFT.

## **Chapter 3 : Introduction of Time-Frequency Analysis**

### ***3.1 Introduction***

Although essential to many fields of study, the Fourier Transform is faced with a severe limitation since it is unable to provide information about how the spectral content of the signal changes with time [Newland 1993b]. If the signal properties do not change much over time, that is, if it is the stationary signal, this drawback isn't very important. However, most interesting signals contain numerous non-stationary or transitory characteristics: drafts, trends, abrupt changes, and beginnings and ends of the events. These characteristics are often the most important part of the signals, and Fourier Transform is not suited to detect them. Therefore, the need to describe the changes in frequencies of a particular signal and the limitation of the Fourier Transform of providing this type of information induced the development of new signal analysis tools. These new signal analysis tools and the field in which they were developed are known today as time-frequency analysis, which sole purpose is to provide detail information on how the spectral content of a signal varies with time. Every time-frequency signal analysis tool provides a time-frequency representation in the form of a time-frequency map, which is a two dimensional plot which illustrate the variation of the frequencies with time.

### ***3.2 Uncertainty Principles***

Although the verbal description of the goals of time-frequency analysis sounds perfectly reasonable, the corresponding mathematical model does not always match the intuition and presents some counterintuitive obstacles in the form of the uncertainty principle. This is a fundamental obstruction to the ideal time-frequency analysis.

One of the most common approaches to achieve time-frequency representation is by the implementation of windowing functions. The window functions consist of functions defined in certain interval, called the support, and whose value outside that interval is zero. These window functions can be defined in both, time and frequency domains, so their support could have units of time or frequency respectively. When this window is centered around a particular value of time or frequency and then multiplied to the function we are interested to analyze, the result is a new function with the characteristics of the original function and the shape of the window. The result outside the support of the window will be zero. This process introduces, the sense of locality, since the characteristics of the resulting function from this multiplication will be unique only around the value this window was placed.

For a window function  $\psi(t)$  defined on the space  $\psi \in L^2(\mathfrak{R})$ , its Fourier transform is defined as:

$$\hat{\psi}(\omega) = \int_{\mathfrak{R}} \psi(t) e^{-i\omega t} dt \quad (3.1)$$

In mathematics, uncertainty principles in the narrow sense are inequalities that involve both  $\psi$  and  $\hat{\psi}$ . There are many uncertainty principles. Heisenberg's Uncertainty Principle is the most classical one.

The center of the window can be defined as:

$$t^* = \frac{\int_{-\infty}^{\infty} t |\psi(t)|^2 dt}{\int_{-\infty}^{\infty} |\psi(t)|^2 dt} \quad (3.2)$$

Now suppose the wave is confined to the region in space with a width whose value is twice that of its room-mean-square (RMS):

$$\Delta \psi = \sqrt{\frac{\int_{-\infty}^{\infty} (t - t^*) |\psi(t)|^2 dt}{\int_{-\infty}^{\infty} |\psi(t)|^2 dt}} \quad (3.3)$$

With Fourier transform  $\hat{\psi}$ , the frequency center is defined as:

$$\omega^* = \frac{\int_{-\infty}^{\infty} \omega |\hat{\psi}(\omega)|^2 d\omega}{\int_{-\infty}^{\infty} |\hat{\psi}(\omega)|^2 d\omega} \quad (3.4)$$

Then its RMS radius will be

$$\Delta \hat{\psi} = \sqrt{\frac{\int_{-\infty}^{\infty} (\omega - \omega^*) |\hat{\psi}(\omega)|^2 d\omega}{\int_{-\infty}^{\infty} |\hat{\psi}(\omega)|^2 d\omega}} \quad (3.5)$$

It should be noticed that the window width is different from the actual support or duration of the window. Here, the width of the frequency window can be obtained as  $2\Delta \hat{\psi}$ .

What Heisenberg discovered is that a window function confined to a very small region must be made up of a lot of different frequency lengths. In other words, if the uncertainty in the position of the particle is small, the uncertainty in the momentum is large. And similarly, a particle whose window width is made up of only a few frequency lengths will be spread out over a large region. It is a common practice to refer to the width of a specific window as the resolution of that window in that particular domain. Thus, the resolution of a time-frequency window will be the product of the time resolution times the frequency resolution for that particular window.

The mathematical expression of this uncertainty principle will be:

$$\Delta \psi \cdot \Delta \hat{\psi} \geq \frac{1}{2} \quad (3.6)$$

This inequality then specifies that the value for the product of the time and frequency radius of the window cannot be smaller than  $\frac{1}{2}$ . It also shows that if the radius of the window in the time domain is reduced in order to obtain more local information of the signal, how the radius of the window in the frequency domain will increase in order to satisfy uncertainty principle. Therefore, making it impossible to improve the time localization by reducing the time window width, without affecting adversely the frequency localization by instantly increasing the frequency width to satisfy Uncertainty Principle.

### **3.3 Short Time Fourier Transform (STFT)**

In order to obtain information about local frequency spectrum, Dennis Gabor [1946] adapted Fourier transform to analyze only small section of the signal at a time-this is the so-called Short Time Fourier Transform (STFT), maps a signal into a two-dimensional function of time and frequency. The STFT represents a sort of compromise between a time and frequency based views of a signal. It provides some information about both when and at what frequencies a signal event occurs.

#### **3.3.1 Definition**

Given a signal  $f(t)$ , multiplying it by a finite time window function  $\psi(t - \tau)$ , which is translated by  $\tau$ , the information of  $f(t)$  in the neighborhood of  $t = \tau$  will be obtained. By applying Fourier transform, the spectral information of the signal in that instant will be gained. If this process is repeated by translating the window along the whole time

length of the signal, a complete description of the spectral signal varying with time can be obtained. The STFT of a signal  $f(t)$  with respect to  $\psi$  can be expressed as:

$$S_f(\omega, \tau) = \int_{-\infty}^{\infty} f(t)\psi(t-\tau)e^{-j\omega t} dt \quad (3.7)$$

Where  $\omega$  is the frequency at which the window will be located in the frequency domain.

To illustrate the windowing effects more clearly in the frequency domain, in using Parseval's identity, the equation (3.7) can be written as:

$$S_f(\xi, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\hat{\psi}^*(\omega-\xi)e^{j\omega\tau} d\omega \quad (3.8)$$

Where  $\hat{\psi}^*(\omega-\xi)$  is the conjugate of the Fourier transformed window function. This expression of the STFT allows illustrating the windowing effect that occurs in the frequency domain. Notice that irrespective of how it is expressed, the result of the STFT of a one-dimensional signal will be a complex value function of two real parameters: time  $\tau$  and frequency  $\omega$  in the two-dimensional time-frequency space.

Similar with the Inverse Fourier transform, the signal  $f(t)$  can be reconstructed by applying the inversion formula for the STFT:

$$f(t) = \frac{1}{2\pi\|\psi(t)\|^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_f(\omega, \tau)\psi(t-\tau)e^{j\omega t} d\omega d\tau \quad (3.9)$$

Where  $\|\psi(t)\|^2$  is the 2-norm of  $\psi$  defined as:

$$\|\psi(t)\|^2 = \left[ \int_{-\infty}^{\infty} |\psi(t)|^2 dt \right] \quad (3.10)$$

### 3.3.2 Remarks

1. If  $\psi$  is compactly supported with its support centered at the origin, then  $S_f$  is the Fourier transform a segment of  $f$  centered in a neighborhood of  $\tau$ . As  $\tau$  varies, the window slides along the  $t$  axis to different positions. For this reason the STFT is often called the “Sliding Window Fourier Transform”. With some reservations,  $S_f(\omega, \tau)$  can be thought of as a measure for the amplitude of the frequency band near  $\omega$  at time  $\tau$ .
2. In signal analysis, at least in dimension  $d = 1$ ,  $\mathfrak{R}^2$  is called the time-frequency plane, and in physics, it is called phase space.
3. The STFT is linear in  $f$  and conjugate linear in  $\psi$ . Usually the window  $\psi$  is kept fixed, and  $S_f$  is considered as a linear mapping from functions on  $\mathfrak{R}$  to functions on  $\mathfrak{R}^2$ . Clearly the function  $S_f$  and the properties of the mapping  $f \rightarrow S_f$  depend crucially on the choice of the window function  $\psi$ .

### 3.3.3 Disadvantages

The problem with STFT can best be demonstrated due to Heisenberg Uncertainty Principle demonstrated last section. This principle originally applied to the momentum and location of moving particles, can be applied to time-frequency information of a signal. According to the principle, one cannot know the exact time-frequency representation of a signal, i.e., one cannot know what spectral components exist at what instances of times. What one can know are the time intervals in which certain bands of frequencies exist, which is a resolution problem.

The problem with the STFT has something to do with the width of the window function that is used. To be technically correct, this width of the window function is known as the support of the window. If the window function is narrow, then it is known as compactly supported. This terminology is more often used in the wavelet world.

If a window of infinite length is used, we will get the FT, which gives perfect frequency resolution, but no time information. Furthermore, in order to obtain the time domain information, in STFT, a short enough window will be used, in which the signal is stationary. The narrower the window, the better the time resolution, and better the assumption of stationarity, but poorer the frequency resolution:

Narrow window → good time resolution, poor frequency resolution.

Wide window → good frequency resolution, poor time resolution.

To use STFT, it is unavoidable to face the resolution problem. In order to obtain good resolution in both time and frequency domain, a good choice of window function will be crucial. The Wavelet transform (WT) can solve the dilemma of resolution to a certain extent, as will be introduced in next chapter.

### ***3.4 Wigner-Ville Distribution (WVD)***

#### **3.4.1 Definition**

Wigner-Ville Distribution (WVD) is another popular tool for time-frequency analysis. It can also be thought of as a short-time Fourier transform (STFT) where the windowing function is a time-scaled, time-reversed copy of the original signal. It is the Fourier

transform of the signal's autocorrelation function with respect to the delay variable. It generally has much better resolution than the STFT.

The Wigner-Ville distribution was introduced Wigner [1932]. A linear signal space  $\chi$  is a collection of signals  $f(t)$  such that any linear combination  $c_1 f_1(t) + c_2 f_2(t)$  of two elements  $f_1(t) \in \chi$  and  $f_2(t) \in \chi$  is again an element of  $\chi$  [franks, 1969, Naylor and Sell, 1982]. The WVD of a signal  $f(t)$  is defined as:

$$WVD_f(t, \omega) = \int_{\tau} f\left(t + \frac{\tau}{2}\right) f^*\left(t - \frac{\tau}{2}\right) e^{-i\omega\tau} d\tau \quad (3.11)$$

The WVD can also be represented in the frequency domain as

$$WVD_f(t, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F\left(\omega + \frac{\xi}{2}\right) F^*\left(\omega + \frac{\xi}{2}\right) e^{i\xi t} d\xi \quad (3.12)$$

Where  $1/2\pi$  is a normalization factor,  $F(\omega)$  is the Fourier Transform of  $f(t)$ . As a result of either Equation (3.1) or Equation (3.2), the WVD will be a time-frequency joint representation of  $f(t)$  in the time-frequency space.

The inverse WVD can be obtained from the inverse Fourier Transform of Eqn. (3.1). Introducing the variables  $t_1 = t + \tau/2$  and  $t_2 = t - \tau/2$ , the inverse Fourier Transform of the Wigner-Ville distribution of Eqn. (3.1) yields

$$f(t_1) f^*(t_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} WVD_f\left(\frac{t_1 + t_2}{2}, \omega\right) e^{i(t_1 - t_2)\omega} d\omega \quad (3.13)$$

Finally, the signal  $f(t)$  can be recovered from the inverse WVD by letting  $t_1 = t$  and  $t_2 = 0$ , then Eqn. (3.3) can be written as:

$$f(t) f^*(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} WVD_f\left(\frac{t}{2}, \omega\right) e^{i\omega t} d\omega \quad (3.14)$$

Where  $f^*(0)$  is constant. In other words, the signal  $f(t)$  is reconstructed from the inverse Fourier Transform of the Wigner-Ville distribution  $WVD_f(t/2, \omega)$ , dilated in the time domain.

Wigner-Ville distribution has many useful properties in the time and frequency domains: time shift and frequency modulation properties, relation between Wigner-Ville distribution and instantaneous frequency or group delay, and its uniqueness condition. Another important property of Wigner-Ville distribution is related to the smoothing. It's that general time-frequency analysis can be obtained by smoothing the Wigner-Ville distribution with a two dimensional smoothing window. Smoothing the Wigner-Ville distribution corresponds to multiplying a two-dimensional window to the ambiguity function. The window is often called as the kernel. There can be many kernels, which lead to the same number of distributions, depending on the targets or objectives to be concerned.

### 3.4.2 Properties

In this section, some useful properties of the WVD are stated.

1. Real-valued: The WVD of a space is real-valued function which, however, is not guaranteed to be everywhere nonnegative.
2. Time-frequency integral: The integral of WVD over the entire time-frequency plane equals the dimension of  $\chi$ ,

$$\int_t \int_{\omega} WVD_f(t, \omega) dt d\omega = N_{\chi} \quad (3.15)$$

3. Norm: The squared norm of  $WVD_f(t, \omega)$  equals the dimension of  $\chi$ ,

$$\|WVD_f\|^2 = \int_t \int_\omega [WVD_f(t, \omega)]^2 dt d\omega = N_\chi \quad (3.16)$$

4. Finite support: If all signals  $f(t) \in \chi$  are time-limited to an interval  $[t_1, t_2]$ , then  $WVD_f(t, \omega)$  is zero for all  $t$  outside  $[t_1, t_2]$ .

### 3.4.3 Disadvantages

The Wigner-Ville distribution has two problems: Cross-terms and Aliasing. These problems have been largely dealt with, although people are still on the lookout for better smoothing functions and alias-free distributions.

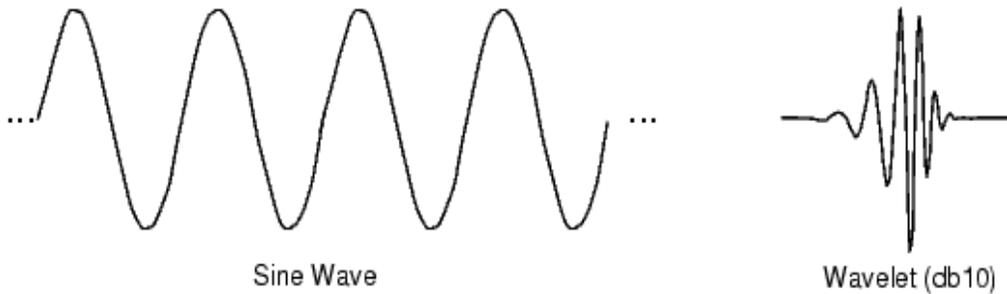
## **Chapter 4 : Wavelet Transform (WT)**

Compared with other time-frequency analysis techniques, especially the Short Time Fourier Transform, wavelet analysis represents the next logical space: a windowing technique with variable sized regions. Wavelet analysis allows use of long time intervals where more precise low frequency needed and shorter regions where high frequency information needed. The original impetus for wavelets came from the analysis of earthquake records [Goupillaud, 1984], but wavelet analysis has found important applications in engineering and signal processing and is now a significant tool in signal analysis generally.

### ***4.1 What are Wavelets***

A Wavelet is a waveform of effectively limited duration that has an average value of zero. Wavelet analysis consists of breaking up a signal into shifted and scaled versions of the original (mother) wavelets. Some of the wavelet forms will be introduced briefly in this chapter. More detailed introduction can be found in the MATLAB wavelet toolbox.

Here illustrated in Figure 4.1 is a comparison between sine wave and one of the wavelet forms. As the basis function of Fourier analysis, sine wave does not have limited duration and it is smooth and predictable. On the other hand, wavelets tend to be irregular and asymmetric. Fourier analysis breaks up a signal into sine waves of various frequencies. Similarly, Wavelet analysis breaks up a signal into mother wavelet with different length and position. From this point of view, signals with sharp changes might be better analyzed with an irregular wavelet than with a smooth sinusoid.

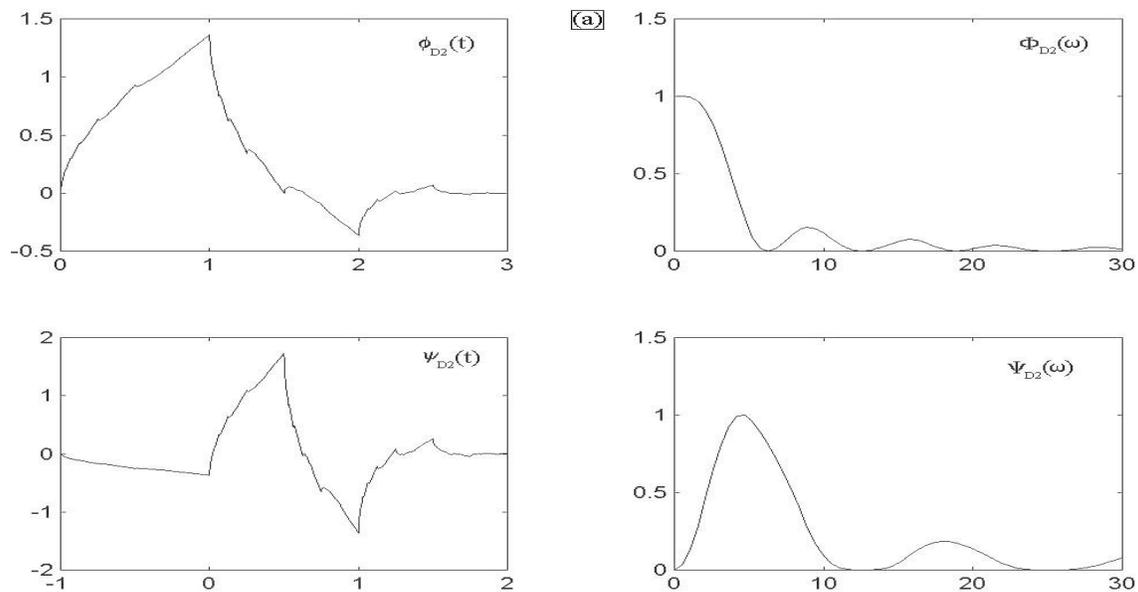


**Figure 4.1 Comparison of the Sine Wave**

As one of the major differences between the Wavelet Transform and Fourier Transform, the wavelet basis function will be explicitly defined here. The idea behind the wavelet theory is to define a set of general properties, which the basis function needs to satisfy in order to be used. This allows for infinite possibilities of basis functions to be used for the Wavelet Transform, which range from analytical to numerical, orthonormal or non-orthonormal, etc. The discussion about the properties used to define the wavelet basis will be presented later in this chapter.

Several families of wavelets that have proven to be especially useful are introduced here. Notice how these wavelets functions are different in both time and frequency domain. This fact makes it essential to properly specify, based on what wavelet basis function the transform was performed, in order to make correct sense of the results. This wavelet basis is called mother wavelet. The mother wavelet is chosen to serve as a prototype for all windows in the process. All the windows that are used are the dilated (or compressed) and shifted versions of the mother wavelet. There are a number of functions that are used for this purpose. The Daubechies wavelet, the Morlet wavelet and the Mexican hat function are some candidates of them.

- Ingrid Daubechies, one of the brightest stars in the world of wavelet research, invented what are called compactly supported orthonormal wavelets – thus making discrete wavelet analysis practicable. The name of the Daubechies family wavelets is written  $dbN$ , where  $N$  is the order, and db is the “surname” of the wavelet. The family includes the Haar wavelet, which is written as  $db1$ . Figure 4.2 illustrates some of the scaling and wavelet functions along with their respective Fourier transform for the Daubechies family.



**Figure 4.2 Daubechies wavelets**

- Mexican hat wavelet:

The analytical expression of the Mexican hat wavelet is

$$\psi(x) = \left( \frac{2}{\sqrt{3}} \pi^{-1/4} \right) (1 - x^2) e^{-x^2/2} \quad (4.1)$$

This function is proportional to the second derivative function of the Gaussian probability density function. The effective support of the wavelet is  $[-5 \ 5]$ .

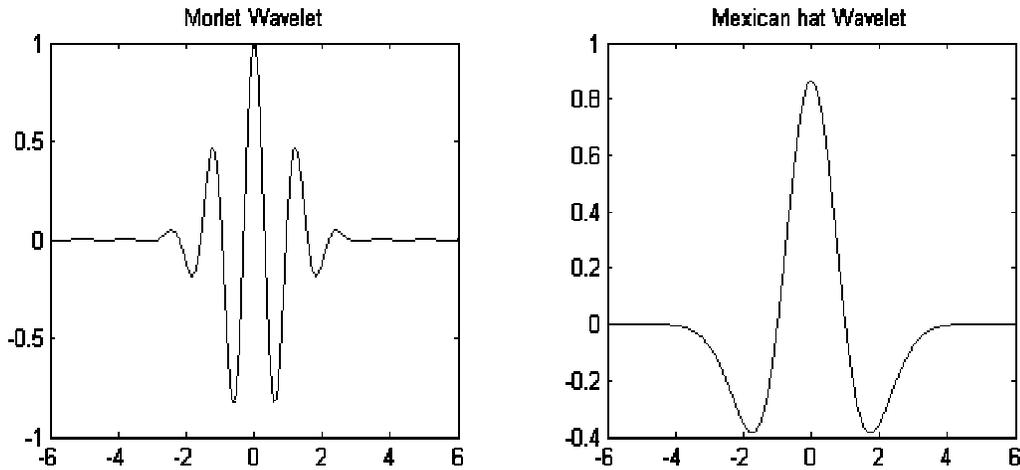
### 3. Morlet wavelet

Morlet wavelet has  $[-4, 4]$  as effective support. The analytical equation of it is defined as:

$$\psi(x) = e^{-x^2/2} e^{j\omega_0 t} \quad (4.2)$$

Where  $\omega_0 = 1.75\pi$ , is selected in order to satisfy one of the wavelet function conditions [Grossmann, et al, 1989]. Figure 4.3 shows the basic form of the both Morlet and Mexican hat wavelets. Introducing the scale and translation parameters of the Wavelet Transform to equation 3.2 yields

$$\psi\left(\frac{t-\tau}{a}\right) = \frac{1}{\sqrt{a}} e^{-\left(\frac{t-\tau}{2a}\right)^2} e^{j\frac{\omega_0}{a}(t-\tau)} \quad (4.3)$$



**Figure 4.3 Morlet and Mexican hat Wavelet forms**

## 4.2 Continuous Wavelet Transform (CWT)

The Continuous Wavelet Transform was developed as an alternative approach to the short time Fourier Transform to overcome the resolution problem. The wavelet analysis

is done in a similar way to the STFT analysis, in the sense that the signal is multiplied with a function, (or the wavelet), similar to the window function in the STFT, and the transform is computed separately for different segments of the time-domain signal. However, there are two main differences between the STFT and the CWT:

1. The Fourier transforms of the windowed signals are not taken, and therefore single peak will be seen corresponding to a sinusoid, i.e., negative frequencies are not computed.
2. The width of the window is changed as the transform is computed for every single spectral component, which is probably the most significant characteristic of the wavelet transform.

#### 4.2.1 Definition

For a signal  $f(t)$ , the continuous Wavelet Transform is defined as the sum over all time space of the signal multiplied by scaled, shifted version of the wavelet function  $\psi$ :

$$Wf(a,b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} f(t) \psi^* \left( \frac{t-b}{a} \right) dt \quad (4.4)$$

Where	$\psi(t)$ :	Mother wavelet (* indicates the conjugate)
	$f(t)$ :	Signal being analyzed
	a:	Dilation parameters (a>0)
	b:	Translation parameters (b∈R)

Herein, the variable  $a$  scales the wavelet basis function such that the wavelet function convolutes with  $f(t)$  in different temporal windows, and the variable  $b$  acts as a shift operator by centering the window at several locations. Interpreting in frequency domain, the variation in  $a$  gives basis function with different frequency contents. The possibility

whether those frequencies are present at a particular stretch of the signal is analyzed by centering the basis function with the help of variable  $b$ . The  $1/\sqrt{a}$  factor is used with the purpose of energy normalization so that, all the scaled wavelets have the same energy [Rao and Bopardikar, 1998]. The choice of the Mother wavelet is restricted by one of the most important characteristics-admissibility condition:

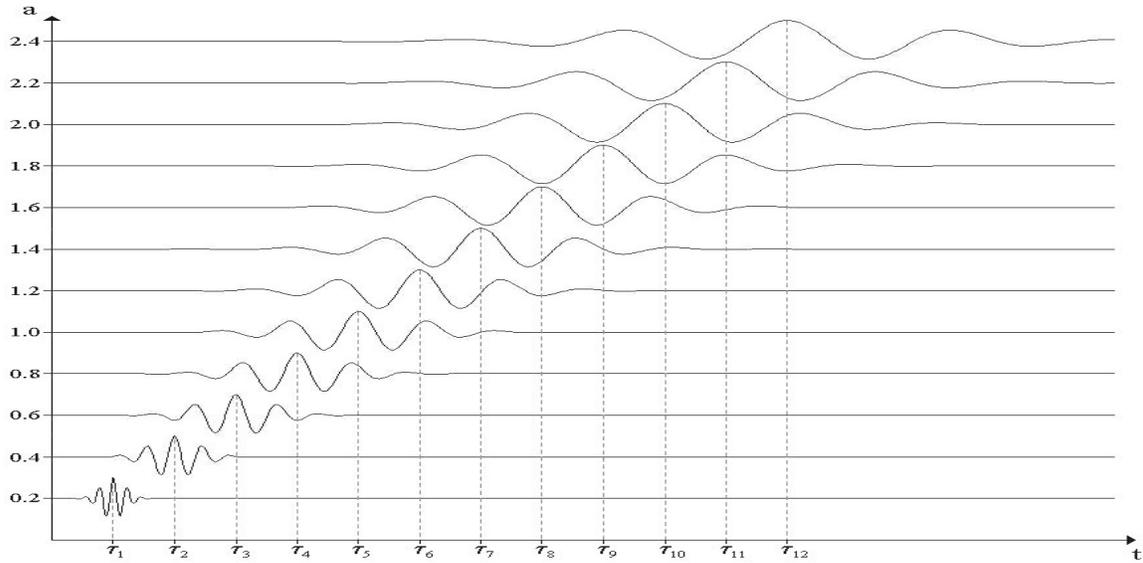
$$C_{\psi} = \int_{-\infty}^{+\infty} \frac{|\Psi(\omega)|^2}{|\omega|} d\omega < \infty \quad (4.5)$$

Where

$$\Psi(\omega) = \int_{-\infty}^{+\infty} \psi(t) \exp(-i\omega t) d\omega \quad (4.6)$$

is the Fourier transform of  $\psi(t)$ .

The results of the CWT are Wavelet coefficients  $C$ , which are a function of scale and position. Multiplying each coefficient by the appropriately scaled and shifted wavelet can yield the constituent wavelets of the original signals. Here the scale factor, which is commonly denoted by the letter  $a$ , is introduced. Scaling a wavelet simply means to stretch it. That means the smaller the scale factor, the more compressed the wavelet has been. It is also clearly shown from the Figure 4.4 that the scale is related to the frequency of the signal.



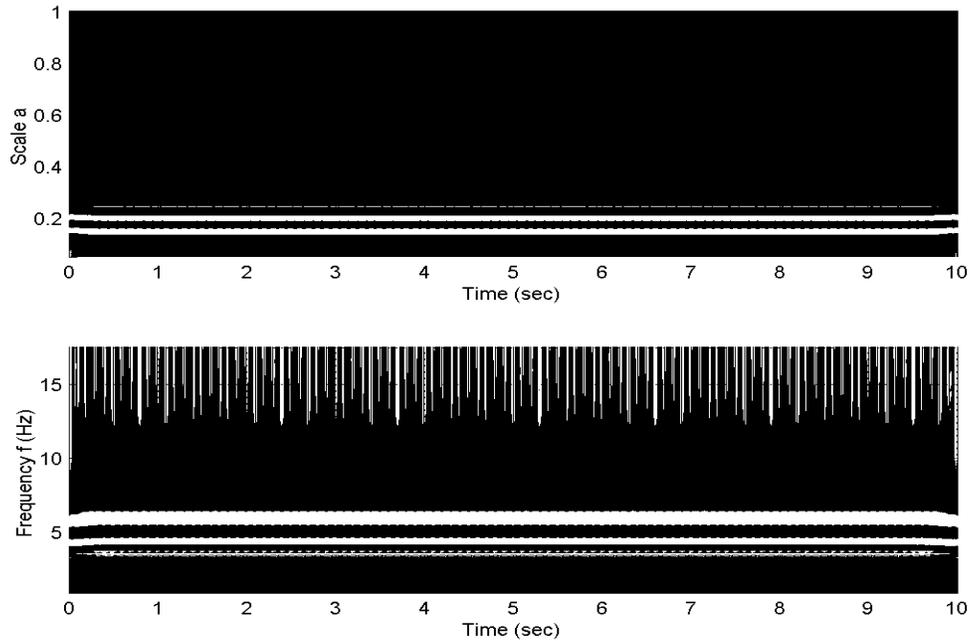
**Figure 4.4 Scaling and shifting of a wavelet**

The use of the scale parameter represents the first difference between the Wavelet Transform and other types of transforms previously discussed. The result of the Wavelet Transform of a function yields a two-dimensional time-scale representation by scaling and translating the window function. Another way to view the Wavelet Transform is as a decomposition of a function into a linear combination of the wavelets, where the coefficients  $Wf(a, b)$  are the inner products between the function and the wavelet basis used. By Parseval's identity, the Wavelet transform can be represented in the frequency domain, which can be written as:

$$Wf(a, b) = \frac{\sqrt{a}}{2\pi} \int_{-\infty}^{\infty} F(\omega) \Psi^*(a\omega) e^{j\omega b} d\omega \quad (4.7)$$

Where,  $F(\omega)$  is the Fourier transform of  $f(t)$  and  $\Psi^*(a\omega)$  is the complex conjugate of Fourier transform of the mother wavelet. Therefore, the Wavelet Transform can be seen as a bank of filters with different scales. Here applying the Morlet wavelet to show (Figure 3.6) how varying the scale parameter generates a set of filters in the frequency

domain. It can be seen as the scale value decreases the filter reduces its amplitude and increases its bandwidth. This behavior is due to the fact that the energy of the filter is constant.



**Figure 4.5 Scale effect on frequency domain**

Where the relationship between the scales and frequency is

$$f = \frac{\omega_0}{2\pi a} \quad (4.8)$$

Thus, the correspondence of wavelet scales and frequency can be revealed as:

- 1 Low scale  $a \Rightarrow$  Compressed wavelet  $\Rightarrow$  Rapidly changing details  $\Rightarrow$  High frequency  $\omega$ .
- 2 High scale  $a \Rightarrow$  Stretched wavelet  $\Rightarrow$  Slowly changing, coarse features  $\Rightarrow$  Low frequency  $\omega$ .

It is important to notice that the time-scale analysis instead of the time-frequency is strength of this technique.

#### 4.2.2 Properties of CWT

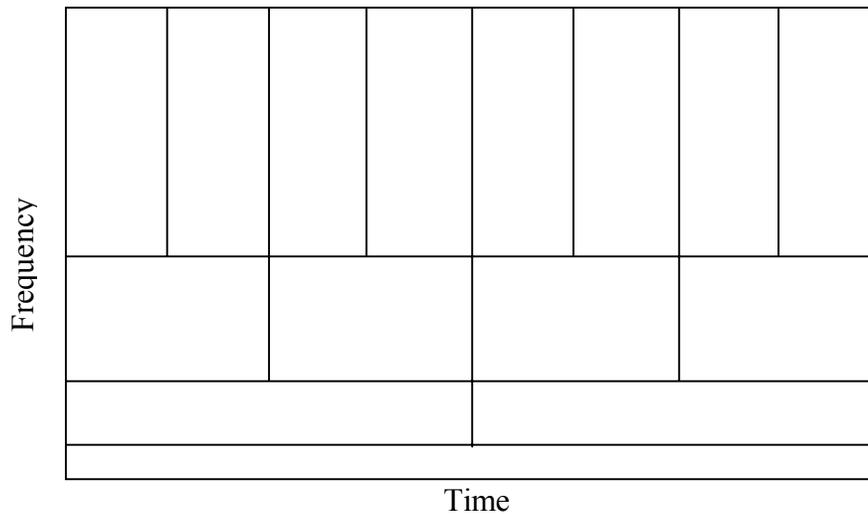
**Linearity:** One of the most useful properties of the Wavelet transform is the fact that it is a linear operation similar to other analysis tools. That means, for  $f(t) = f_1(t) + f_2(t)$ , the Wavelet transform will be:

$$Wf(a,b) = Wf_1(a,b) + Wf_2(a,b) \quad (4.9)$$

This property allows for the evaluation of signals with multiple frequency components by the Wavelet Transform, without the cross terms effects seen for the WVD.

**Resolution property:** Another important property is the resolution property. Unlike the STFT which has a constant resolution at all times and frequencies, the WT has a good time and poor frequency resolution at high frequencies, and good frequency and poor time resolution at low frequencies.

The illustration in Figure 4.5 is commonly used to explain how time and frequency resolutions should be interpreted. Every box in Figure 4.5 corresponds to a value of the wavelet transform in the time-frequency plane. Note that boxes have a certain non-zero area, which implies that the value of a particular point in the time-frequency plane cannot be known. All the points in the time-frequency plane that falls into a box are represented by one value of the WT.



**Figure 4.6 Time and Frequency resolution of the Wavelet transform**

First thing to notice from Figure 4.6 is that although the widths and heights of the boxes change, the area remains constant. That is each box represents an equal portion of the time-frequency plane, but giving different proportions to time and frequency. Note that at low frequencies, the height of the boxes are shorter (which corresponds to better frequency resolutions, since there is less ambiguity regarding the value of the exact frequency), but their widths are longer (which correspond to poor time resolution, since there is more ambiguity regarding the value of the exact time). At higher frequencies the width of the boxes decreases, i.e., the time resolution gets better, and the heights of the boxes increase, i.e., the frequency resolution gets poorer.

It is worthwhile to mention how the partition looks like in the case of STFT. Recall that in STFT the time and frequency resolutions are determined by the width of the analysis window, which is selected once for the entire analysis, i.e., both time and frequency resolutions are constant. Therefore the time-frequency plane consists of squares in the STFT case.

Regardless of the dimensions of the boxes, the areas of all boxes, both in STFT and WT, are the same and determined by Heisenberg's inequality. As a summary, the area of a box is fixed for each window function (STFT) or mother wavelet (CWT), whereas different windows or mother wavelets can result in different areas. However, all areas are lower bounded by  $1/4 \pi$ . That is, we cannot reduce the areas of the boxes as much as we want due to the Heisenberg's uncertainty principle. On the other hand, for a given mother wavelet the dimensions of the boxes can be changed, while keeping the area the same. This is exactly what wavelet transform does.

The mathematical way to look the uncertainty property of the CWT is by recalling the basic idea introduced in section 3.2. The equation used to define the time localization of the wavelet will be:

$$t_0 = \frac{\int_{-\infty}^{\infty} t |\psi(t)|^2 dt}{\int_{-\infty}^{\infty} |\psi(t)|^2 dt} \quad (4.10)$$

For the center of the frequency:

$$\omega_0 = \frac{\int_{-\infty}^{\infty} \omega |\Psi(\omega)|^2 d\omega}{\int_{-\infty}^{\infty} |\Psi(\omega)|^2 d\omega} \quad (4.11)$$

Where

$$\Psi(\omega) = \int_{-\infty}^{\infty} \psi(t) e^{-j\omega t} dt \quad (4.12)$$

is the Fourier transform of the mother wavelet  $\psi(t)$ .

Then the RMS radius of the time-frequency window can be written as:

$$\Delta t_{\psi} = \sqrt{\frac{\int_{-\infty}^{\infty} (t - t_0) |\psi(t)|^2 dt}{\int_{-\infty}^{\infty} |\psi(t)|^2 dt}} \quad (4.13)$$

and

$$\Delta \omega_{\Psi} = \sqrt{\frac{\int_{-\infty}^{\infty} (\omega - \omega_0) |\Psi(\omega)|^2 d\omega}{\int_{-\infty}^{\infty} |\Psi(\omega)|^2 d\omega}} \quad (4.14)$$

By applying the time scaling property of the Fourier transform, the RMS radius of the window function for different scale values in the time domain are found as:

$$\Delta t_{\psi(a)} = |a| \Delta t_{\psi} \quad (4.15)$$

The RMS radius in the frequency domain will be:

$$\Delta \omega_{\Psi(a)} = \frac{\Delta \omega_{\Psi}}{|a|} \quad (4.16)$$

From these equations, the same idea illustrated by Figure 3.7 has been shown that the time window's length will increase for increasing values of scale  $a$  while the frequency window's length will reduce. On the other hand if the scale values reduce, the time window will shrink but the frequency one will expand.

For a scale value of  $a=1$  the resolution of the time frequency window is given as:

$$[t_o + \tau - \Delta t, t_o + \tau + \Delta t] \times [\omega_o - \Delta \omega, \omega_o + \Delta \omega] \quad (4.17)$$

Where  $\Delta t$  and  $\Delta \omega$  are the RMS radius of the time and frequency calculated for the mother wavelet. Now consider the case in which the scale value is half the original.

Therefore, the time-frequency window will have a different resolution. Applying the Eqn. (4.15) and Eqn. (4.16) the new time-frequency window will be

$$[t_o + \tau - \Delta t/2, t_o + \tau + \Delta t/2] \times [\omega_o - 2\Delta\omega, \omega_o + 2\Delta\omega] \quad (4.18)$$

On the other hand, if the scale value is twice the original value the new time-frequency window resolution will be written as

$$[t_o + \tau - 2\Delta t, t_o + \tau + 2\Delta t] \times [\omega_o - \Delta\omega/2, \omega_o + \Delta\omega/2] \quad (4.19)$$

Although the time-frequency in the Wavelet Transform is a function of the scale value, the area will remain constant for every scale value:

$$\Delta t_{\psi(a)} \Delta \omega_{\psi(a)} = |a| \Delta t \frac{\Delta \omega}{|a|} = \Delta t \Delta \omega = c \geq \frac{1}{2} \quad (4.20)$$

Where  $c$  is a constant [Rao and Bopardikar, 1998].

**Reconstruction of the signal:** The analyzed signal  $f(t)$  can be constructed from its Wavelet transform  $Wf(a, b)$ . In order to achieve perfect reconstruction, no information should be lost during the Wavelet Transform. This requirement introduces the need for the resolution of identity, which can be defined as:

$$\int_{-\infty}^{\infty} \frac{da}{a^2} \int_{-\infty}^{\infty} d\tau \langle f_1(t), \psi_{a,\tau} \rangle \langle \psi_{a,\tau}, f_2 \rangle = c_h \langle f_1, f_2 \rangle \quad (4.21)$$

Where  $\langle f_1, f_2 \rangle$  denotes the inner product of  $f_1$  and  $f_2$  etc. The equation can be further simplified to:

$$\frac{C_\psi}{2\pi} \int_{-\infty}^{\infty} F_1(\omega) F_2^*(\omega) d\omega = C_\psi \int_{-\infty}^{\infty} f_1(t) f_2(t) dt \quad (4.22)$$

Where  $C_\psi$  is defined in the equation 4.5.

The significance of the resolution of identity is simply to state that the energy in the time-scale space, defined in the left hand side of Eqn.4.21 must be equal to the energy of the signal in the time domain defined by the right hand side of Eqn.4.21. In fact if  $f_1(t) = f_2(t)$ , the resolution of identity can be rewritten as:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Wf(a,b)|^2 db \frac{da}{a^2} = C_\psi \int_{-\infty}^{\infty} |f(t)|^2 dt \quad (4.23)$$

Equation 4.23 describe the energy conservation relation for the Wavelet transform. As demonstrated in the definition of the Wavelet transform,  $C_\psi$  satisfies the admissibility condition. This admissibility condition forces the Fourier transform of the wavelet function to have a zero component at the zero frequency, so that

$$\Psi(0) = 0 \quad (4.24)$$

This means that in the Fourier domain the wavelet is a band-pass filter. Also, in the time domain the wavelet must be oscillatory with zero mean value as:

$$\int_{-\infty}^{\infty} \psi(t) dt = 0 \quad (4.25)$$

Then the expression for the inverse Wavelet Transform can be obtained from the resolution of identity:

$$f(t) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Wf(a,b) \frac{1}{\sqrt{a}} \psi\left(\frac{t-b}{a}\right) db \frac{da}{a^2} \quad (4.26)$$

Based on all these properties discussed, it can be seen that any square integrable function, which satisfies the admissibility condition, can be used as a wavelet basis function. At the same time, the use of this type of function allows for the proper recovery of the function from its Wavelet transforms by means of the Inverse Wavelet Transform (IWT) defined by Eqn. 2.16.

In addition, a regularity condition is imposed on the wavelet functions in order to achieve that the wavelet transform decrease quickly with decreasing values of the scale value  $a$ . This requires that the wavelet function  $\psi(t)$  and its Fourier Transform  $\Psi(\omega)$  exhibit some smoothness and localization in both time and frequency domains. It can be found that the speed of convergence to zero of the Wavelet Transform of smooth functions, as the scales values decreases, is determined by the first nonzero moment of the basic wavelet  $\psi(t)$ . Therefore, a wavelet of order  $n$  will be one that satisfies:

$$M_p = \int_{-\infty}^{\infty} t^p \psi(t) dt, \quad p = 0, 1, 2, \dots, n-1 \quad (4.27)$$

This condition defined by Eqn. 4.27 is equivalent to have the derivatives of the Fourier transform of the wavelet basis  $\psi(t)$  equal to zero up to order  $n$  at  $\omega = 0$ . Equation 4.27 can be represented in the frequency domain as

$$\Psi^{(p)}(0) = 0, \quad p = 0, 1, 2, \dots, n-1 \quad (4.28)$$

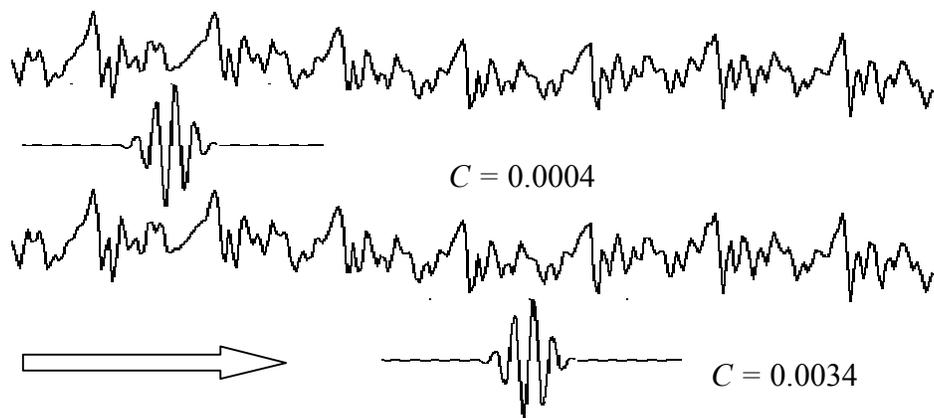
All of these are basic properties that the wavelet function should satisfy. Obviously, they have not been discussed in great detail in here, since such a detail discussion is beyond the scope of this work. The interested reader should see Daubechies [1992] for an in

depth and detail discussion of the properties and characteristics of the wavelet basis functions.

### 4.2.3 Computation of CWT

Let  $f(t)$  be the signal to be analyzed. The mother wavelet is chosen to serve as a prototype for all windows in the process. All the windows that are used are the dilated (or compressed) and shifted versions of the mother wavelet. There are five steps of an easy recipe for creating a CWT:

1. Take a wavelet and compare it to a section at the start of the original signal  $f(t)$ .
2. Calculate the value of Wavelet coefficient  $C$ , which represents how closely correlated the wavelet is with this section of the signal. The higher  $C$  is, the more the similarity. More precisely, if the signal energy and the wavelet energy are equal to one,  $C$  may be interpreted as a correlation coefficient.
3. Shift the wavelet to the right and repeat step 1 and 2 until the whole signal is covered.



**Figure 4.7 Process of the Wavelet transform**

4. Stretch the wavelet and repeat steps 1 through 3.
5. Repeat steps 1 through 4 for all scales.

As a result, for every scale and for every time (interval), one point of the time-scale plane is computed. The computations at one scale construct the rows of the time-scale plane, and the computations at different scales construct the columns of the time-scale plane.

Although it is relatively simple and easy to compute the CWT, there exist a limited number of functions for which the Wavelet transform can be calculated in the closed-form and they are usually very simple. Therefore, for most other functions, the Wavelet Transform needs to be calculated numerically, requiring discretization of the function and the transform.

Before presenting the actual discussion on the implementation, it is important to point out another difference between the Wavelet Transform and other transformations. Normally the process of implementing a transform numerically will be called discrete, as for the case of the discrete Fourier Transform or the discrete STFT. In the case of the Wavelet Transform, even though it is calculated in a discrete manner it is still called a Continuous Wavelet Transform (CWT). The reason is that the transform is calculated for every single scale value, and there exists one special form of discretizing the scale domain and only that process takes the name of the Discrete Wavelet Transform (DWT). The discussion of the DWT is presented later on in this chapter.

By discretizing the integral in the definition using expressions:

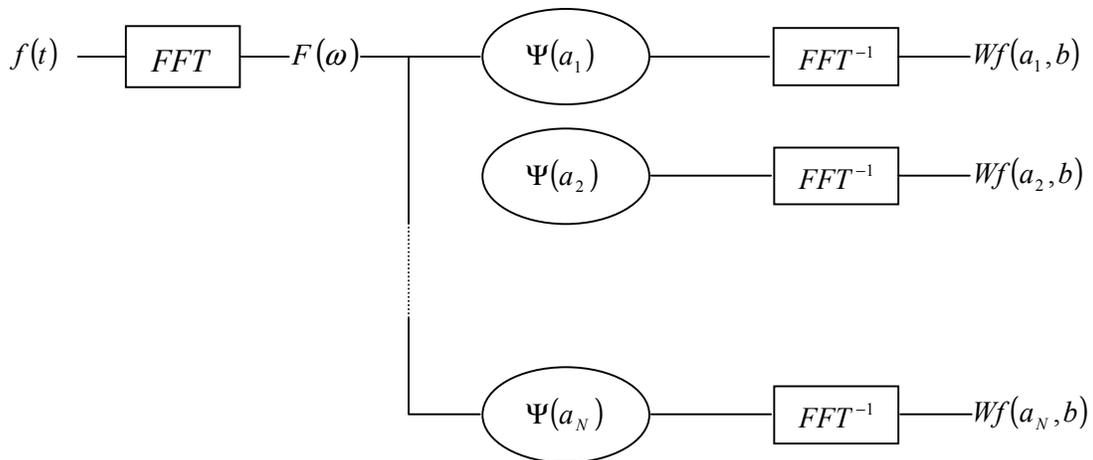
$$\begin{aligned}
 a_n &= n\Delta a & , & & n = 1,2,3,\dots,M \\
 b_k &= k\Delta t & , & & k = 0,1,2,3,\dots,N-1 \\
 t &= h\Delta t & , & & n = 0,1,2,3,\dots,N-1
 \end{aligned}
 \tag{4.29}$$

The discrete version of the CWT can be expressed as:

$$Wf(k, n) = \sum_{k=0}^{N-1} f(h\Delta t) \frac{1}{\sqrt{n\Delta a}} \psi^* \left( \frac{(h-k)\Delta t}{n\Delta a} \right) \Delta t \quad (4.30)$$

Where  $M$  is the number of scale values desired,  $N$  is the number of discrete time points of the function,  $\Delta a$  is the scale step, and  $\Delta t$  is the time step. Notice that the scale value will never be zero, since a zero value of  $a$  will make the value of the wavelet basis function go to infinity, which is not desirable.

The Wavelet Transform can also be calculated from a frequency domain approach by making use of Eqn. 4.7. Similar to STFT, the first step is to take the Fourier Transform of the signal to be analyzed. Then the Fourier Transform of the first scaled wavelet function is calculated. Finally, the inverse Fourier Transform of the product between the Fourier Transforms of the signal and the scaled wavelet yields the coefficients of the CWT for that scale value. Repeating this process for all the scale values chosen for the analysis produces the CWT of the signal. Figure 4.9 will be an illustration of the procedure behind the calculation of the CWT from a frequency domain perspective.



**Figure 4.8 Schematic of the frequency-based algorithm of CWT**

### 4.3 Discrete Wavelet Transform (DWT)

Although the discretized Continuous Wavelet Transform enables the computation of the continuous wavelet transform by computers, it is not a true discrete transform. As a matter of fact, the wavelet series is simply a sampled version of the CWT, and the information it provides is highly redundant as far as the reconstruction of the signal is concerned. This redundancy, on the other hand, requires a significant amount of computation time and resources. The discrete wavelet transform (DWT), on the other hand, provides sufficient information both for analysis and synthesis of the original signal, with a significant reduction in the computation time. The DWT is considerably easier to implement when compared to the CWT. The basic concepts of the DWT will be introduced in this section along with its properties and the algorithms used to compute it.

#### 4.3.1 Definition

Because of the high redundancy in continuous Wavelet transform, it is possible to use a hyperbolic grid to discretize the transformation parameters. Normally, the dyadic scales and positions are used. Then we obtain the analysis form – Discrete Wavelet Transform (DWT).

By using the dyadic values:

$$a = a_0^j, \quad b = a_0^j k b_0 \quad (4.31)$$

Where  $j, k \in Z$  are positive integers and  $a_0 > 1, b_0 > 0$ . In this manner, large basis functions are shifted in large steps, while small basis functions are shifted in small steps. In order for sampling of the time-scale plane to be sufficiently fine,  $a_0$  has to be chosen

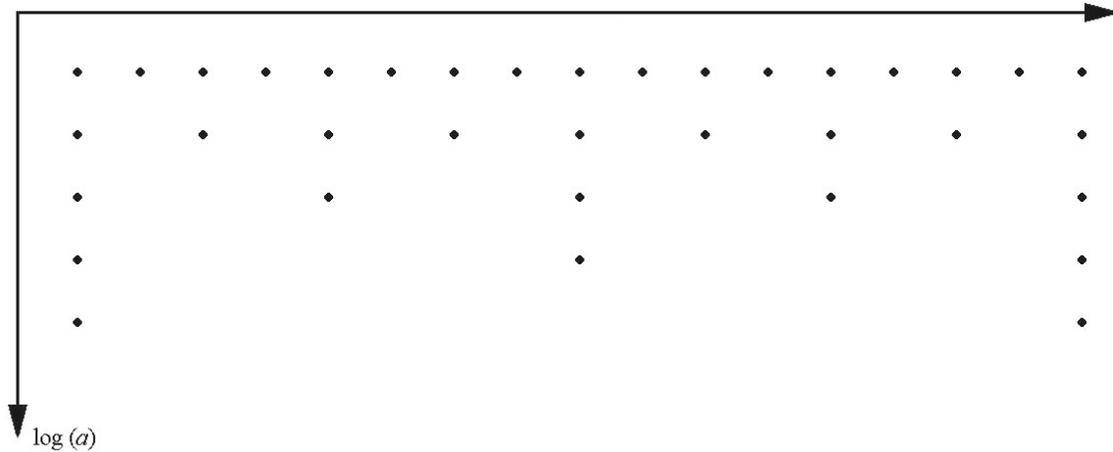
close to 1 and  $b_0$  has to be chosen close to 0. Then the desired discretized wavelet can be written as:

$$\psi_{j,k}(t) = a_0^{-j/2} \psi(a_0^{-j}t - kb_0) \quad (4.32)$$

In using discrete mother wavelets, the Wavelet Transform will be calculated at discrete time and scale values, which will correspond to the sampling points in the time-scale space. For the time axis, since the sampling is proportional to the defined discrete scale factor, the sampling interval will be  $b_0 a_0^j$ . That is why for small scale values the time sampling step will be small and will increase as the scale value increases. On the other hand if  $a_0$  is constant the sampling points in the scale axis will be logarithmic, since  $\log(a) = j \log(a_0)$ . It is common in practical problems to use  $a_0 = 2$ , which correspond to a dyadic sampling in the frequency. Then the discretized wavelet will be rewritten as:

$$\psi_{j,k}(t) = 2^{-j/2} \psi(2^{-j}t - kb_0) \quad (4.33)$$

This means that if  $b_0 = 1$ , when the scale factor increases the time-sampling doubles. This is known as dyadic translation. Figure (4.9) illustrates how the sampling points are distributed in the time-scale space for a dyadic case. Notice that the sampling along the time axis remains constant for each scale value. Also notice the effect of doubling in the time sampling rate as the scale value increases.



**Figure 4.9 Sampling of time-scale plane**

If the discrete wavelet basis function in Eqn. 4.33 is selected to perform the Wavelet transform using a dyadic scale, the DWT of a function  $f(t)$  can be expressed as:

$$W f(j, k) = 2^{-j/2} \sum_h f(t) \psi_{j,k}^*(t) \quad (4.34)$$

The result of the DWT defined by Eqn. 4.34 will be a series of wavelet coefficients at different levels. This decomposition is often referred to as wavelet series decomposition, since the function  $f(t)$  can be reconstructed as:

$$f(t) = A \sum_j \sum_k W f(j, k) \psi_{j,k}(t) \quad (4.35)$$

Where  $A$  is a constant and independent of  $f(t)$ . The wavelet series decomposition is the equivalent to the CWT, in the same way that the Fourier series to the Fourier Transform. In fact their objective is the same, i.e., trying to decompose a function into a set of basis functions. Although, similar in purpose, the wavelet series decomposition differentiates drastically with respect to the Fourier series. The inherent variable window capability of the DWT allows the wavelet decomposition to obtain more detail information regarding fast and drastic changing phenomena in the function.

The implementation of the DWT reduces considerably the over sampling and redundancy obtained from the CWT. In fact, redundancy can be eliminated if the wavelet basis functions are chosen so that they are orthonormal with each other. Another advantage of the DWT over the CWT is that its implementation is much faster thanks to a tree-structured algorithm developed in the Multi-resolution analysis framework [Mallat, 1989]. One limitation is that in order to be able to take advantage of all these useful characteristics of the DWT, severe constraints are required for the discrete wavelet basis functions. This reduces considerably the number of wavelet basis functions that could be used in the DWT.

Daubechies [1990], proved that the necessary conditions in order to achieve a stable reconstruction of a function  $f(t)$  from its wavelet coefficients  $Wf(j,k)$  was that the energy of the function  $\|f(t)\|^2$ , must lie between two positive bounds:

$$A\|f(t)\|^2 \leq \sum_{j,k} |\langle f(t), \psi_{j,k}(t) \rangle|^2 \leq B\|f(t)\|^2 \quad (4.36)$$

Where  $A > 0$ ,  $B < \infty$ , and both are independent of  $f(t)$ . If the relation defined in Eqn.4.34 is satisfied the family of functions obtain by dilation and translation of the wavelet basis function is referred as a frame, where  $A$  and  $B$  are the frame bounds [Chui, 1992]. Satisfying this requirement ensures that no information is lost from the reconstruction of the function from its wavelet coefficients obtained from the DWT.

It has also been shown [Daubechies, 1990] that the accuracy of the reconstruction is highly dependent on the frame bounds. For example, the closer the bounds  $A$  and  $B$  are, the more accurate the reconstruction will be. The ideal scenario is when  $A=B$ , then the frame is tight, meaning that the discrete wavelets are orthonormal basis. This means that the discrete wavelets are orthogonal to their own translations and dilations. That means

$$\int \psi_{j,k}(t) \psi_{m,n}^*(t) dt = \begin{cases} 1 & \text{if } k = m \text{ and } j = n \\ 0 & \text{otherwise} \end{cases} \quad (4.37)$$

Which can be achieved by proper selection of the basis wavelet function. Even when  $A \neq B$ , reconstruction from the DWT can be achieved by using two different wavelet basis functions, one for decomposition and another for synthesis.

### 4.3.2 Multiresolution Analysis

The foundations of the DWT go back to 1976 when Croiser, Esteban, and Galand devised a technique to decompose discrete time signals. Crochiere, Weber, and Flanagan did a similar work on coding of speech signals in the same year. They named their analysis scheme as subband coding. In 1983, Burt defined a technique very similar to subband coding and named it pyramidal coding which is also known as multiresolution analysis. Later in 1989, Vetterli and Le Gall made some improvements to the subband coding scheme, removing the existing redundancy in the pyramidal coding scheme. Subband coding is explained below. A detailed coverage of the discrete wavelet transform and theory of multiresolution analysis can be found in a number of articles and books that are available on this topic.

The basic idea of DWT is similar with CWT, which is a correlation between a wavelet at different scales and the signal with the scale being used as a measure of similarity. The continuous wavelet transform was computed by changing the scale of the analysis window, shifting the window in time, multiplying by the signal, and integrating over all times. In the discrete case, filters of different cutoff frequencies are used to analyze the signal at different scales. The signal is passed through a series of high pass

filters to analyze the high frequencies, and it is passed through a series of low pass filters to analyze the low frequencies.

The resolution of the signal, which is a measure of the amount of detail information in the signal, is changed by the filtering operations, and the scale is changed by upsampling and downsampling (subsampling) operations [Polikar, 1996]. Subsampling a signal corresponds to reducing the sampling rate, or removing some of the samples of the signal. For example, subsampling by two refers to dropping every other sample of the signal. Subsampling by a factor  $n$  reduces the number of samples in the signal  $n$  times. Upsampling a signal corresponds to increasing the sampling rate of a signal by adding new samples to the signal. For example, upsampling by two refers to adding a new sample, usually a zero or an interpolated value, between every two samples of the signal. Upsampling a signal by a factor of  $n$  increases the number of samples in the signal by a factor of  $n$ .

In the multiresolution analysis framework, the orthogonal Wavelet Transform is based on the scaling function  $\phi(t)$ . The scaling function, different from the wavelet, does not satisfy the admissible condition. Therefore, the mean value of the basic scaling function is not equal to zero and is usually normalized as

$$\int \phi(t)dt = 1 \quad (4.38)$$

The correlation between the scaling function and the function  $f(t)$  produces the averaged approximation  $x_1(n)$  the function equivalent to the low-pass filter in the subband coding. Since the theory in the subband coding requires the filter to be discrete, the scaling function need to be defined in a discrete manner. Therefore, the scaling function  $\phi(t)$  can be shifted by discrete translation factors to construct an orthonormal basis at the same

resolution level. Also, the scaling function is dilated by dyadic scale factors so that the discrete version of the scaling function can be written as

$$\phi_{j,k}(t) = 2^{-j/2} \phi(2^{-j}t - k) \quad (4.39)$$

The high-pass filtering aspect of the subband coding scheme correspond to the wavelet basis function in the Wavelet Transform. The discrete orthonormal wavelet basis functions are generated from the scaling function. Also they are orthogonal to the scaling function within the same scale. Then the wavelet basis functions with dyadic scales and translation can be written as

$$\psi_{j,k}(t) = 2^{-j/2} \psi(2^{-j}t - k) \quad (4.40)$$

In the multiresolution analysis framework, the orthonormal Wavelet Transform is the decomposition of a function into approximations at lower and lower resolutions with less detail information by the projections of the function onto the orthonormal scaling function bases. Although, similar to the multiresolution subband coding in the sense that they are discrete algorithms, the multiresolution Wavelet Transform leads to a series expansion that decomposes the continuous function into a series of continuous wavelet functions.

The procedure starts with passing this signal (sequence) through a half band digital lowpass filter with impulse response  $h[n]$ . Filtering a signal corresponds to the mathematical operation of convolution of the signal with the impulse response of the filter. The convolution operation in discrete time is defined as follows:

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] \quad (4.41)$$

A half band lowpass filter removes all frequencies that are above half of the highest frequency in the signal. After passing the signal through a half band lowpass filter, half of the samples can be eliminated according to the Nyquist's rule, since the signal now has a highest frequency of  $\pi/2$  radians instead of  $\pi$  radians. Simply discarding every other sample will subsample the signal by two, and the signal will then have half the number of points. The scale of the signal is now doubled. Note that the lowpass filtering removes the high frequency information, but leaves the scale unchanged. Only the subsampling process changes the scale. Resolution, on the other hand, is related to the amount of information in the signal, and therefore, it is affected by the filtering operations. Half band lowpass filtering removes half of the frequencies, which can be interpreted as losing half of the information. Therefore, the resolution is halved after the filtering operation. Note, however, the subsampling operation after filtering does not affect the resolution, since removing half of the spectral components from the signal makes half the number of samples redundant anyway. Half the samples can be discarded without any loss of information. In summary, the lowpass filtering halves the resolution, but leaves the scale unchanged. The signal is then subsampled by 2 since half of the number of samples are redundant. This doubles the scale. This procedure can mathematically be expressed as:

$$a[n] = \sum_{k=-\infty}^{\infty} x[2n - k] h[k] \quad (4.42)$$

The DWT is actually computed as follows: The DWT analyzes the signal at different frequency bands with different resolutions by decomposing the signal into coarse approximation and detail information. DWT employs two sets of functions, called scaling functions and wavelet functions, which are associated with low pass and highpass filters,

respectively. The decomposition of the signal into different frequency bands is simply obtained by successive highpass and lowpass filtering of the time domain signal. The original signal  $x[n]$  is first passed through a halfband highpass filter  $g[n]$  and a lowpass filter  $h[n]$ . After the filtering, half of the samples can be eliminated according to the Nyquist's rule, since the signal now has a highest frequency of  $\pi/2$  radians instead of  $\pi$ . The signal can therefore be subsampled by 2, simply by discarding every other sample. In order to obtain the detail information lost by the lowpass filtering, let the highpass filter  $g[n] = h[-n]$ . Detailed low-resolution signal can be obtained as:

$$d[n] = \sum_n x[n] g[2k - n] \quad (4.43)$$

### 4.3.3 Wavelet Reconstruction

The process of discrete wavelet transform can be called decomposition or analysis. The other part of this technique is how those components can be assembled back into the original signal without loss of information. This process is called reconstruction or synthesis. The mathematical manipulation that effect synthesis is called the inverse discrete wavelet transform (IDWT).

One important property of the discrete wavelet transform is the relationship between the impulse responses of the highpass and lowpass filters. The highpass and lowpass filters are not independent of each other, and they are related by:

$$g[L - 1 - n] = (-1)^n h[n] \quad (4.44)$$

Where  $g[n]$  is the highpass,  $h[n]$  is the lowpass filter, and  $L$  is the filter length (in number of points). Note that the two filters are odd index alternated reversed versions of each

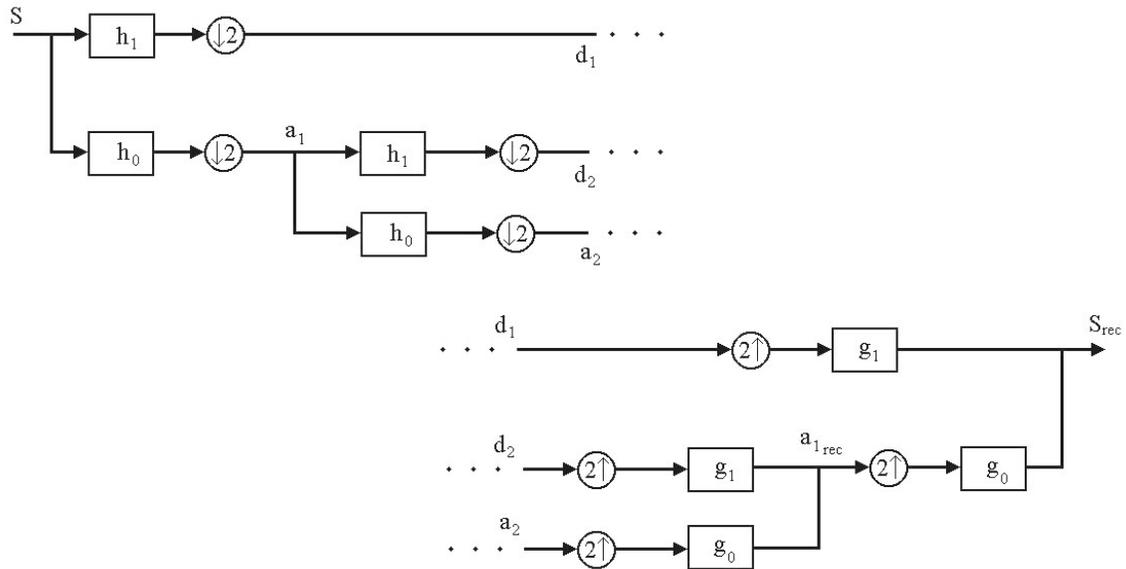
other. Lowpass to highpass conversion is provided by the  $(-1)^n$  term. Filters satisfying this condition are commonly used in signal processing, and they are known as the Quadrature Mirror Filters (QMF). The two filtering and subsampling operations can be expressed by

$$\begin{aligned} a[n] &= \sum_n x[n]h[2k-n] \\ d[n] &= \sum_n x[n]g[2k-n] \end{aligned} \quad (4.45)$$

The reconstruction in this case is very easy since halfband filters form orthonormal bases. The above procedure is followed in reverse order for the reconstruction. The signals at every level are upsampled by two, passed through the synthesis filters  $g'[n]$ , and  $h'[n]$  (highpass and lowpass, respectively), and then added. The interesting point here is that the analysis and synthesis filters are identical to each other, except for a time reversal. Therefore, the reconstruction formula becomes (for each layer):

$$x[n] = \sum_{-\infty}^{\infty} (a[n]h[2k-n] + d[n]g[2k-n]) \quad (4.46)$$

In Figure 4.10, the algorithm of decomposition and reconstruction is shown.



**Figure 4.10 Tree algorithms for wavelet decomposition and reconstruction**

If the filters are not ideal halfband, then perfect reconstruction cannot be achieved. Although it is not possible to realize ideal filters, under certain conditions it is possible to find filters that provide perfect reconstruction. The most famous ones are the ones developed by Ingrid Daubechies, and they are known as Daubechies' wavelets.

The Daubechies wavelet bases are a family of orthonormal, compactly supported scaling and wavelet functions that have the maximum regularity for a given length of the support of the quadrature mirror filter (QMF). Where a QMF is a type of filter in which its low-pass and high-pass filters are mirror images of each other. There exists a close link between the orthonormal Wavelet Transform and the two-band perfect reconstruction quadrature mirror filter (PRQMF) bank, which is a QMF that allows the perfect reconstruction of the signal from its low-pass and high-pass components. A complete and detail discussion of this can be found in Daubechies [1988] and will not be discussed here since it is beyond the scope of this work.

## **Chapter 5 : General Equivalent Linearization Method**

### **5.1 Introduction**

In many practical applications the system of concern is nonlinear. In general, and especially in random vibration analysis, it is difficult to obtain the closed form solution for dynamic response of a nonlinear system. There are various approaches which have been developed over the years to treat the nonlinear problems. Most commonly used methods include Perturbation, Fokker-Plank-Kolmogorov (F-P-K), Monte Carlo simulation, and stochastic equivalent linearization techniques. In this chapter, stochastic equivalent linearization method will be discussed, starting with the overview of these techniques. Further detailed development can be found in Caughey [1971], Iwan [1974], Roberts [1981,1984,1987], Spanos [1976, 1978, 1981], Roberts and Spanos [1990], Baber [1980a], Noori [1984] and Dobson [1997].

#### **5.1.1 Perturbation**

This approach has been known in deterministic vibration theory for many years [Stoker, 1950] and was generalized to the case of stochastic excitation by Crandall [1963]. The basic idea is to expand the solution to the non-linear set of equations in terms of a small scaling parameter, which characterizes the magnitude of the non-linear terms. The first term of the expansion is the linear response when all the nonlinear terms in the system are removed. The following terms express the information of the nonlinearity. In general, the computation in using this technique is lengthier as the order of the scaling parameter increases. In practice, results are obtained only to the first order of the parameter. Perturbation techniques are limited to weak geometric nonlinearities.

### **5.1.2 Markov method**

The work of physicists on Brownian motion has disclosed that the response of dynamic systems to wide-band random excitation can, in many cases, be accurately modeled in terms of multi-dimensional Markov processes. The state transition probability function for such a process is governed by a linear partial differential equation, known as the Fokker-Planck-Kolmogoro (FPK) equation, or forward diffusion equation [Arnold, 1973, Soong, 1973]. The FPK equation, which governs the diffusion of probability mass in the state space, is analogous to the diffusion equations which govern the diffusion of heat, or mass in thermal fluid mechanics problem. It is possible to relate the drift and diffusion coefficients in the FPK equation directly to the parameters in the dynamic equations of motion of the system under consideration. The drawback of this method is it can only apply to some limited systems.

### **5.1.3 Monte Carlo Simulation**

Numerical methods that are known as Monte Carlo methods can be loosely described as statistical simulation methods, where statistical simulation is defined in quite general terms to be any method that utilizes sequences of random numbers to perform the simulation. Monte Carlo methods have been used for centuries, but only in the past several decades has the technique gained the status of a full-fledged numerical method capable of addressing the most complex applications. The theoretical foundation of Monte Carlo simulation is associated with the fact that the stochastic differential equation governing the motion of the system can be interpreted as an infinite set of deterministic differential equations [Soong, 1973]. For each member of this set, the input is a sample function of the excitation process, and the output is the corresponding sample function of

the response process. However, this technique tends to be computer intensive, with many problems taking minutes or hours to solve on a high-speed computer. For this reason, Monte Carlo simulation is avoided when closed form or other simple solutions exist for a problem.

#### **5.1.4 Stochastic Equivalent Linearization**

The method of the stochastic equivalent linearization is based on the idea that a nonlinear system may be replaced by a linear system by minimizing the mean square error of the two systems. The most commonly applied and convenient procedure uses Atalik and Utuk's [1974] suggestion to estimate the linearization coefficients in context with Wen's introduction of an analytical expression for the restoring force [Schueller, 2001]. This method has seen the broadest application because of their ability to accurately capture the response statistics over a wide range of response levels while maintaining relatively light computational burden. The method will be briefly discussed in the following sections. For More detailed introductions, refer Roberts and Spanos [1990].

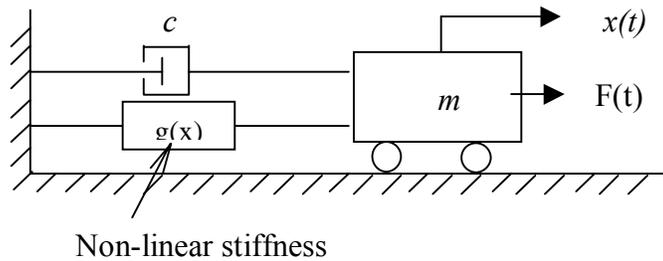
#### **5.2 Stochastic Equivalent Linearization Method**

The equivalent linearization method has been studied widely. Basically, it is the statistical extension of Krylov and Gogoliubov's [1943]. The original work in the development of an equivalent linearization method, reported in the literature, is by Booton [1953]. Caughey [1960] utilized a linearization technique for the response analysis of systems with a bilinear hysteresis restoring force. Fundamental theory of this linearization procedure can be found in Caughey [1963], Iwan [1973]. In the work

presented in this paper, the formulation of equivalent linearization by Roberts and Spanos [1990] is followed. From the perspective of treatable physical problems, the statistical linearization method has proved a useful analytical tool across a very wide spread of engineering applications [Spanos, 1981a]. Examples include the analysis of the sliding motion of a structure on a randomly moving surface [Constantinou and Tadjbakhsh, 1984, Noguchi, 1985], the response of offshore structures to wave loads [Spanos and Agarwal, 1984], the sloshing of liquids in tanks subjected to earthquake excitation [Sakata, 1984] and the effect of nonlinear soil-structures interaction on the dynamic response of buildings [Chu, 1985].

### 5.2.1 System model

To illustrate the procedure of equivalent linearization theory, let us consider the following oscillator with a nonlinear restoring force component.



**Figure 5.1 SDOF mass-spring-damper system**

The ordinary differential equation of the motion can be written as:

$$m\ddot{x}(t) + c\dot{x}(t) + g(x) = F(t) \quad (5.1)$$

Where  $m$  is the mass,  $c$  is the viscous damping coefficient,  $F(t)$  is the external excitation signal with zero mean and  $x(t)$  is the displacement response of the system.

Dividing the equation by  $m$ , the equation of motion can be rewritten as:

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + h(x) = f(t) \quad (5.2)$$

Where  $\zeta$  is the critical damping factor, and  $\omega_n$  is the undamped natural frequency, for the linear system.

We can always find a way to decompose the nonlinear restoring force to one linear component plus a nonlinear component

$$h(x) = \omega_n^2(x + H(x)\lambda) \quad (5.3)$$

Where  $\lambda$  is the nonlinear factor to control the type and degree of nonlinearity in the system.

### 5.2.2 Implementation of the Stochastic Equivalent Linearization

The idea of linearization is replacing the equation (5.2) by a linear system:

$$\ddot{x}(t) + 2\zeta_{eq}\omega_{eq}\dot{x}(t) + \omega_{eq}^2x(t) = f(t) \quad (5.4)$$

Where

$$\zeta_{eq} = \frac{\omega_n}{\omega_{eq}}\zeta \quad (5.5)$$

is the damping ratio of equivalent linearized system and  $\omega_{eq}$  is the natural frequency of the equivalent linearized system.

To find an expression for  $\omega_{eq}$ , it is necessary to minimize the expected value of the difference between equations (5.2) and (5.4) in a least square sense. Now the difference is simply the difference between the nonlinear stiffness and linear stiffness terms, which is

$$e = h(x) - \omega_{eq}^2 x \quad (5.6)$$

The value of  $\omega_{eq}$  can be obtained by minimizing the expectation of the square error  $E\{e^2\}$ :

$$\frac{d}{d\omega_{eq}^2} E\{e^2\} = 0 \quad (5.7)$$

Substituting the equation (5.6) into Eqn. (5.7), performing the necessary differentiation, the expression of  $\omega_{eq}$  can be obtained as:

$$\omega_{eq}^2 = \omega_n^2 \left( 1 + \lambda \frac{E\{xH(x)\}}{\sigma_x^2} \right) \quad (5.8)$$

Where  $\sigma_x$  is the standard deviation of  $x(t)$ . This equation shows how the nonlinear component of the stiffness element affects the value of  $\omega_{eq}$ .

In equation (5.8), the exact evaluation of  $E\{xH(x)\}/\sigma_x^2$  requires knowledge of the first-order density function of the response process  $x(t)$ . Let the process  $x(t)$  assume to be Gaussian, then the standard deviation can be found from equation (5.4). For convenience, the Duffing oscillator has been used to illustrate this procedure here, with which the nonlinear restoring force is written as:

$$h(x) = \omega_n^2 (x + x^3 \lambda) \quad (5.9)$$

Where the nonlinear factor  $\lambda$  controls the type and degree of nonlinearity in the system. A higher value of  $\lambda$  indicates a stronger nonlinearity. A positive value of  $\lambda$  represents a

hardening system while a negative value represents a softening system behavior. In this case, the coefficient  $\omega_{eq}$  can be expressed

$$\omega_{eq}^2 = \omega_n^2 \left( 1 + \lambda \frac{E\{x^4\}}{\sigma_x^2} \right) \quad (5.10)$$

Taking the density function of  $x(t)$  to be of Gaussian form

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left(-\frac{x^2}{2\sigma_x^2}\right) \quad (5.11)$$

Using the definition of the expectation operator, it is found from equation (5.8) that:

$$\omega_{eq}^2 = \omega_n^2 (1 + 3\lambda A) \quad (5.12)$$

Where in this case

$$A = E[x^4] = \int_{-\infty}^{\infty} x^4 f_X(x) dx \quad (5.13)$$

By performing integration, the equation (5.12) can be expressed using Gamma function:

$$A = \frac{2}{\sqrt{\pi}} \sigma_x^2 \Gamma\left(\frac{3}{2}\right) \quad (5.14)$$

Utilizing a well-known frequency domain input-output formula, the spectrum of  $x(t)$  can be determined by:

$$S_x(\omega) = |\alpha(\omega)|^2 S_f(\omega) \quad (5.15)$$

Where  $S_x(\omega)$  and  $S_f(\omega)$  are the spectral density matrix for  $x(t)$  and  $f(t)$  respectively.

Here the appropriate frequency response function  $\alpha(\omega)$  is given as:

$$\alpha(\omega) = \frac{1}{(\omega_{eq}^2 - \omega^2 + 2j\zeta_{eq}\omega_{eq}\omega)} \quad (5.16)$$

Then

$$\sigma_x^2 = \int_{-\infty}^{\infty} S_y(\omega) d\omega \quad (5.17)$$

Combining the equation (5.11) and (5.16), two algebraic relationships are obtained for the two unknown  $\omega_{eq}$  and  $\sigma_x$ . By iterative process, finally, the desired equivalent coefficient can be obtained. The cyclic procedure can be devised as follow:

1. Assign an initial value to the equivalent coefficient  $\omega_{eq}$
2. Use equation (5.14) (5.15) and (5.16) to get  $\sigma_x$
3. Solve equation (5.11) and (5.13) for the new  $\omega_{eq}$
4. Repeat steps 2 and 3 until results from cycle to cycle are similar.

## **Chapter 6 : Wavelet-based Stochastic Linearization**

### **6.1 Introduction**

As introduced in the Chapter 5, methods most commonly used to predict random response of nonlinear systems include Fokker-Planck-Kolmogorov (FPK), numerical simulation and stochastic equivalent linearization techniques. All these approaches have limitations. Although the use of the Equivalent linearization method to analyze nonlinear systems has been quite common in engineering applications, it is still limited in cases where the response is harmonic in nature. Thus, if the response energy fluctuates temporally, a time-varying system may approximate the response better [Mason, 1979]. Further, the assumed form of response should also account for the temporal variation of the frequency content. Excitations, due to the earthquake ground motions, are often characterized by temporal variations in the energy and frequency composition. To overcome this drawback, the approach of equivalent linearization should be modified accordingly. Since it has been shown that Wavelet analysis provides a convenient way for the response analysis of systems subjected to non-stationary ground motion [Basu and Gupta, 1997]. Specifically, the wavelet transformation has been used to obtain both frequency and temporal information of system response caused by non-stationary excitation. Based on this capability, to develop a wavelet-based method might be a good alternative to traditional equivalent linearization method.

The basic idea of this wavelet-based equivalent linearization method was previously suggested by Basu and Gupta [1999]. By performing the Wavelet transform for the equations of motion, the transitional property of excitation can be obtained. Then following the traditional equivalent linearization technique to minimize the square error,

a time varying linear system can be obtained. In next section, the procedure will be demonstrated, for more detailed information, see Basu and Gupta [1997, 1998, 1999 and 2000].

## 6.2 Input-Output relationship for a linear system

### 6.2.1 Review of Wavelet Theory

From the introduction in Chapter 4, the Wavelet transform for a function  $f(t)$ , which may represent a ground motion or the response of a dynamical system to this motion, can be written as:

$$W_{\psi} f(a, b) = \langle f, \psi_{a,b} \rangle = \frac{1}{|a|^{1/2}} \int_{-\infty}^{\infty} f(t) \psi^* \left( \frac{t-b}{a} \right) dt \quad (6.1)$$

And the reconstruction of the  $f(t)$  will be:

$$f(t) = \frac{1}{2\pi C_{\psi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{a} W_{\psi} f(a, b) \psi \left( \frac{t-b}{a} \right) da db \quad (6.2)$$

Where,  $C_{\psi}$  is constant and  $\psi^*$  is the conjugate of the mother wavelets.

For numerical calculation, in using:

$$a_j = 2^j, \quad b = (j-1)\Delta b \quad (6.3)$$

Such that:

$$\Delta b_j = [(b_{j+1} - b_j) + (b_j - b_{j-1})] / 2 = \Delta b \quad (6.4)$$

And

$$\Delta a_j = [(a_{j+1} - a_j) + (a_j - a_{j-1})] / 2 = \frac{3}{4} a_j \quad (6.5)$$

Then the discretized form of Eqn. (6.2) can be expressed as:

$$f(t) = \sum_j \sum_k \frac{K\Delta b}{a_j} W_\psi f(a, b) \psi_{a,b}(t) \quad (6.6)$$

With

$$K = \frac{3}{8\pi C_\psi} \quad (6.7)$$

To further analyze the time-frequency characteristics of the  $f(t)$ , the expression for the inner product of  $f$  [Daubechies, 1992] will be used here:

$$\langle f, f \rangle = \int_{-\infty}^{\infty} f^2(t) dt = \frac{1}{2\pi C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{[W_\psi f(a, b)]^2}{a^2} da db \quad (6.8)$$

The instantaneous mean square response will be obtained by discretizing the equation (6.8) [Basu and Gupta, 1996]:

$$f^2|_{t=b_k} = K \sum_j \frac{1}{a_j} [W_\psi f(a, b)]^2 \quad (6.9)$$

## 6.2.2 Input-output relation

In this part, the input-output differential relationship of a linear SDOF dynamic system in the wavelet domain has been introduced. The procedure is originally derived by Basu and Gupta [1997]. It has been used to obtain the relation of input and output wavelet coefficient of the system. Considering the a linear SDOF system with equation of motion

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2x(t) = f(t) \quad (6.10)$$

Where  $\zeta$  is the critical damping factor, and  $\omega_n$  is the undamped natural frequency. On performing the Wavelet transform of both sides of the equation (6.10), there is

$$W_\psi \ddot{x}(a, b) + 2\zeta\omega_n W_\psi \dot{x}(a, b) + \omega_n^2 W_\psi x(a, b) = W_\psi f(a, b) \quad (6.11)$$

Where  $W_\psi \ddot{x}, W_\psi \dot{x}, W_\psi x$  and  $W_\psi f$  denote the wavelet transform of  $\ddot{x}, \dot{x}, x$  and  $f$  respectively. Performing the integration by part on  $W_\psi \ddot{x}$ , according the characteristic of fast decaying for wavelet basis, the following expression can be obtained:

$$W_\psi \ddot{x}(a, b) = \frac{1}{a^2} W_{\ddot{\psi}} x(a, b) \quad (6.12)$$

Where the term in the right hand side is the wavelet transform with the wavelet basis as  $\ddot{\psi}$ . The second order partial differential equation of  $W_\psi x(a, b)$  with respect  $b$  can be shown as:

$$\frac{\partial^2}{\partial b^2} W_\psi x(a, b) = \frac{\partial^2}{\partial b^2} \int_{-\infty}^{\infty} x(t) \psi\left(\frac{t-b}{a}\right) dt \quad (6.13)$$

Exchanging the differential operator with the integral operator on the right hand side, the equation (6.13) can be rewritten as:

$$\frac{\partial^2}{\partial b^2} W_\psi x(a, b) = \frac{1}{a^2} W_{\ddot{\psi}} x(a, b) \quad (6.14)$$

Then it is clearly shown that

$$W_\psi \ddot{x}(a, b) = \frac{\partial^2}{\partial b^2} W_\psi x(a, b) \quad (6.15)$$

Similarly

$$W_{\psi}\dot{x}(a, b) = \frac{\partial}{\partial b} W_{\psi}x(a, b) \quad (6.16)$$

The equation (6.11) can be expressed as:

$$\frac{\partial^2}{\partial b^2} W_{\psi}x(a, b) + 2\zeta\omega_n \frac{\partial}{\partial b} W_{\psi}x(a, b) + \omega_n^2 W_{\psi}x(a, b) = W_{\psi}f(a, b) \quad (6.17)$$

From this equation, the wavelet coefficient of the output can be interpreted according to the wavelet coefficient input. Considering the ground motion processes, which are broadband and non-stationary, it is possible to decompose them into sets of narrow-banded processes through their wavelet coefficient. Since the parameter  $b$  contains the temporal information, the information of the response output in time domain can be obtained.

### 6.3 Wavelet-Based Equivalent Linearization

Considering the nonlinear SDOF oscillator discussed in the Chapter 5, by using the input-output relation introduced previously, performing the wavelet transform on both sides of the equation (5.2) and (5.4), the following form can be obtained:

$$\frac{\partial^2}{\partial b^2} Wx(a, b) + 2\zeta\omega_n \frac{\partial}{\partial b} Wx(a, b) + Wh(a, b) = Wf(a, b) \quad (6.18)$$

$$\frac{\partial^2}{\partial b^2} Wx(a, b) + 2\zeta\omega_n \frac{\partial}{\partial b} Wx(a, b) + \omega_{eq}^2 Wx(a, b) = Wf(a, b) \quad (6.19)$$

Following the basic procedure of the traditional Equivalent linearization introduced in the last chapter, firstly to find the difference between the equations (6.18) and (6.19)

$$e^2 = \left( \omega_{eq}^2 Wx(a, b) - Wh(a, b) \right)^2 \quad (6.20)$$

Practically, the wavelet coefficients are evaluated numerically, in this case, parameter  $a$  and  $b$  will be discretized as described in section 6.2.1. By summing the discretized error over all  $j$  values and multiplying by a factor  $1/a_j$ , which represents the error in the instantaneous energy of the response at a instant time  $b_k$ , then minimizing the square error with respect to  $\omega_{eq}^2$ , the instantaneous equivalent natural frequency will be obtained as

$$\omega_{eq}^2(k) = \frac{\sum_j W_{j,k} x W_{j,k} h}{\sum_j W_{j,k}^2 x} \quad (6.21)$$

It is obviously shown that  $\omega_{eq}$  is a time varying equivalent frequency.

## 6.4 Implementation to Duffing Oscillator

For the case of the Duffing oscillator, the wavelet transformation of the nonlinear part can be expressed as:

$$Wh(a, b) = \omega_n^2 Wx(a, b) + \lambda Wx^3(a, b) \quad (6.22)$$

From the equation (6.9), there is

$$x^2(t) = K \sum_j \frac{1}{a_j} [Wx(a_j, t)]^2 \quad (6.23)$$

Plugging this equation into Eqn. (6.22), this equation will be rewritten as:

$$Wh(a, b) = \omega_n^2 Wx(a, b) + \frac{\lambda K}{\sqrt{a}} \int_{-\infty}^{\infty} \sum_j \frac{1}{a_j} W^2 x(a_j, t) x(t) \psi(t) dt \quad (6.24)$$

On exchanging the integral and summation operator of the equation (6.24), applying the fast decay property of the wavelet basis on either side, discretizing the time factor with  $t = b_k$ , the discretized form of the equation (6.24) will be:

$$W_{j,k} h = \omega_n^2 W_{j,k} x + \lambda K \omega_n^2 W_{j,k} x \sum_j \frac{W_{j,k}^2 x}{a_j} \quad (6.25)$$

Substituting the Eqn. (6.25) into the Eqn (6.21), the equivalent natural frequency can be obtained for Duffing oscillator:

$$\omega_{eq}^2(k) = \omega_n^2 \left( 1 + \lambda \sum_j \frac{K}{a_j} W_{j,k}^2 x \right) \quad (6.26)$$

Similar with traditional Equivalent linearization method, combining the equation (6.19) and (6.26), two algebraic relationships are obtained for the two unknown  $\omega_{eq}$  and  $W_{j,k} x$ . By iterative process, finally, the desired equivalent coefficient can be obtained. The cyclic procedure can be devised as follow:

1. Assign an initial value to the equivalent coefficient  $\omega_{eq}$
2. Use equation (6.19) to get  $W_{j,k} x$
3. Solve equation (6.26) for the new  $\omega_{eq}$
4. Repeat steps 2 and 3 until results from cycle to cycle are similar.

With this wavelet-based equivalent linearization approach, it is shown that by performing the wavelet transform, the property of excitation signal in time domain has been built into the equivalent coefficient  $\omega_{eq}$ . Therefore, it is not necessary to know the

distribution of the excitation signal and thus, the technique leads to a simpler way as compared with the standard equivalent linearization.

## Chapter 7 : Numerical Solutions

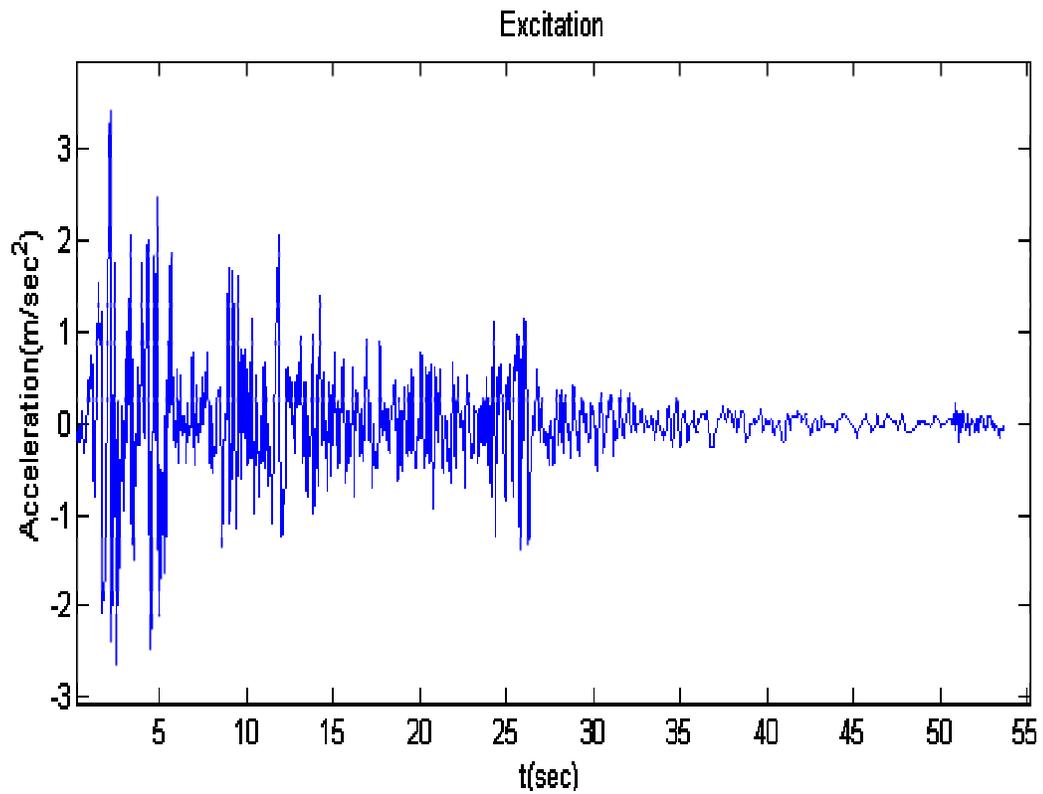
The main objective of this research work was to compare the wavelet-based Equivalent linearization method with the widely used traditional linearization technique and to verify the feasibility of this method. The displacement response of a Duffing Oscillator under different excitations is compared with numerical solution of the nonlinear system response. To check the effect of the different mother wavelet, two of the most often used mother wavelets Morlet wavelet and Mexican hat wavelet have been applied. Combing the basic idea of structural health monitoring, this method can be applied to some cases for damage detection.

### ***7.1 Comparison of the Wavelet-based and Traditional EQL***

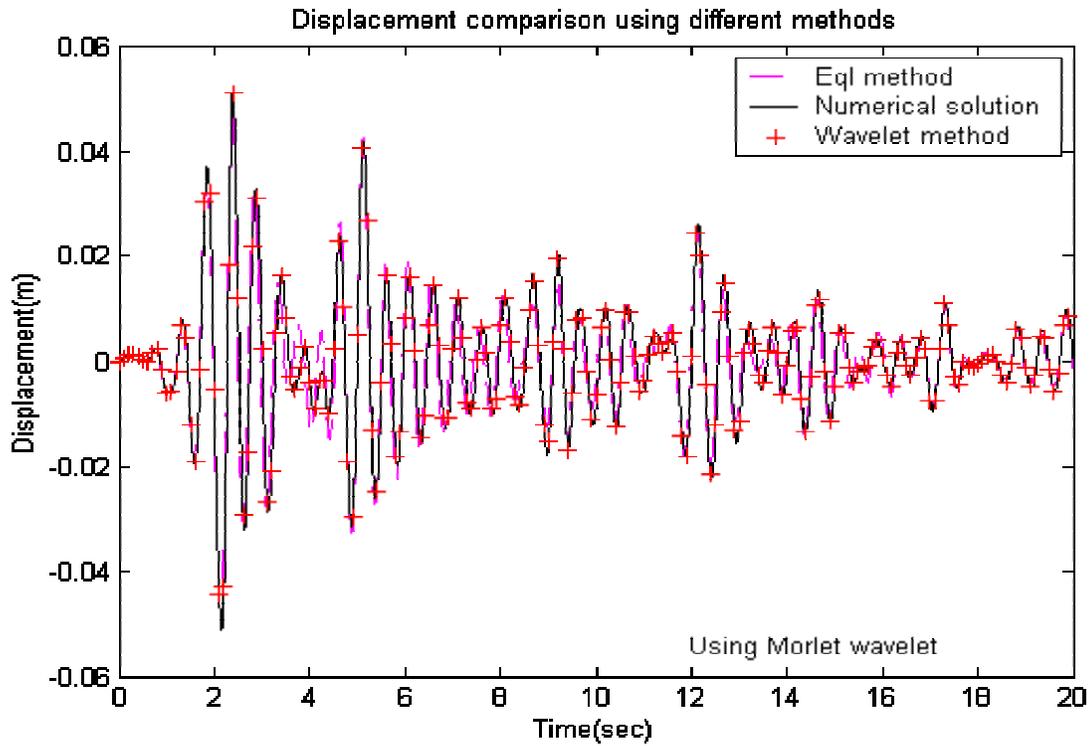
In the application presented below, we set the system to be a unit mass system, with nonlinear factor  $\lambda=0.5$ . The linear natural period  $T_p=0.5$  and let  $c = 0.1\omega_n$ .

- The first case is when the system is excited by an El Centro earthquake signal (Figure 7.1). With the wavelet-based method, using the procedure demonstrated above, initializing  $\omega_{eq} = 1$ , after 4 steps of iteration a set of equivalent natural frequency can be obtained that are varying with time. In Figure 7.2, what is shown are the displacement response of the nonlinear system and its corresponding linear system obtained using both this wavelet based approach and the result based on traditional linearization method. The fourth order Runge-Kutta method is used to get the numerical solution of nonlinear system. For convenience, the response of the first 20 seconds is illustrated here. From the figure, it is shown clearly that the response obtained

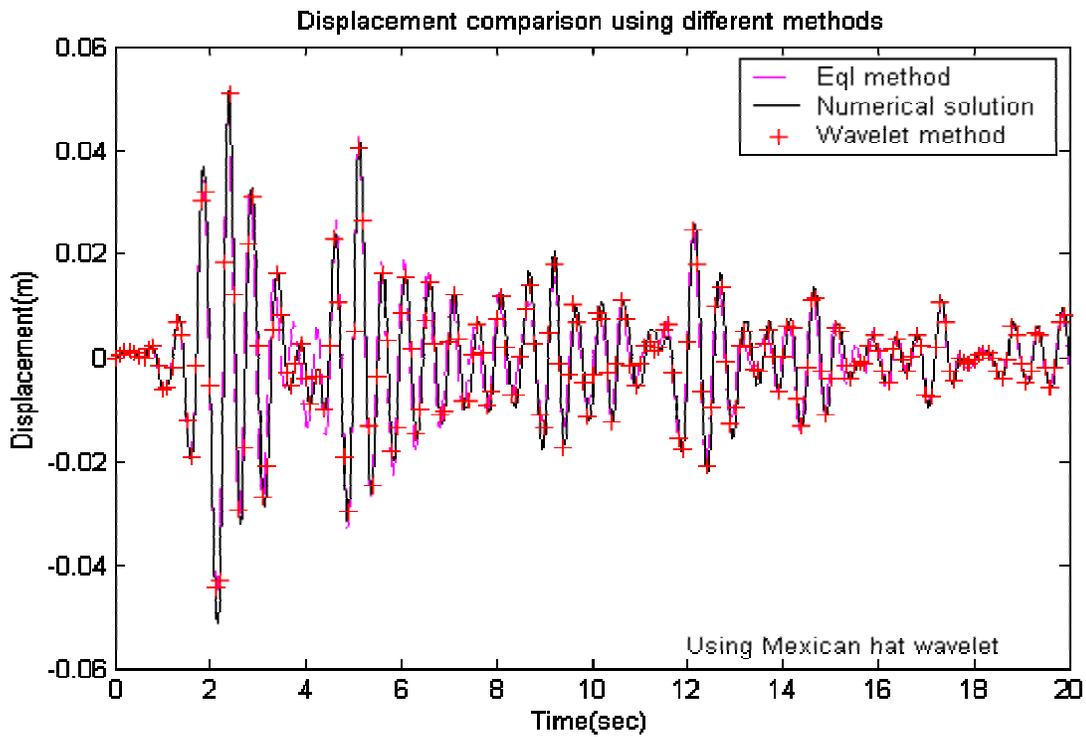
by both wavelet-based method and the traditional methods are matching well with the numerical solution. However, there is no obvious difference with different wavelets in this case.



**Figure 7.1 Excitation 1- El Centro earthquake acceleration**

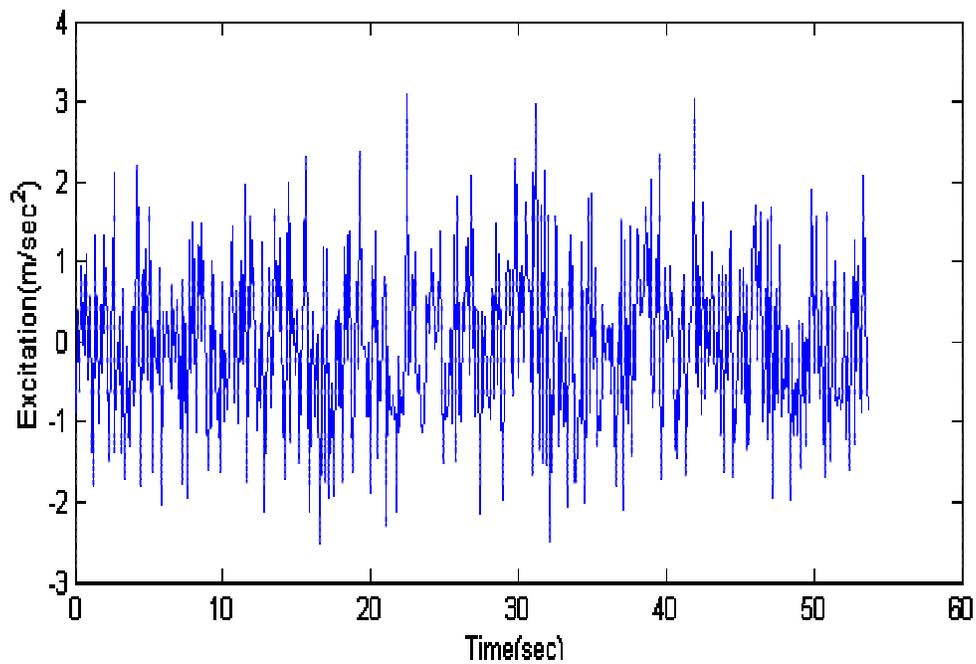


**Figure 7.2 Using Mexican hat wavelet**

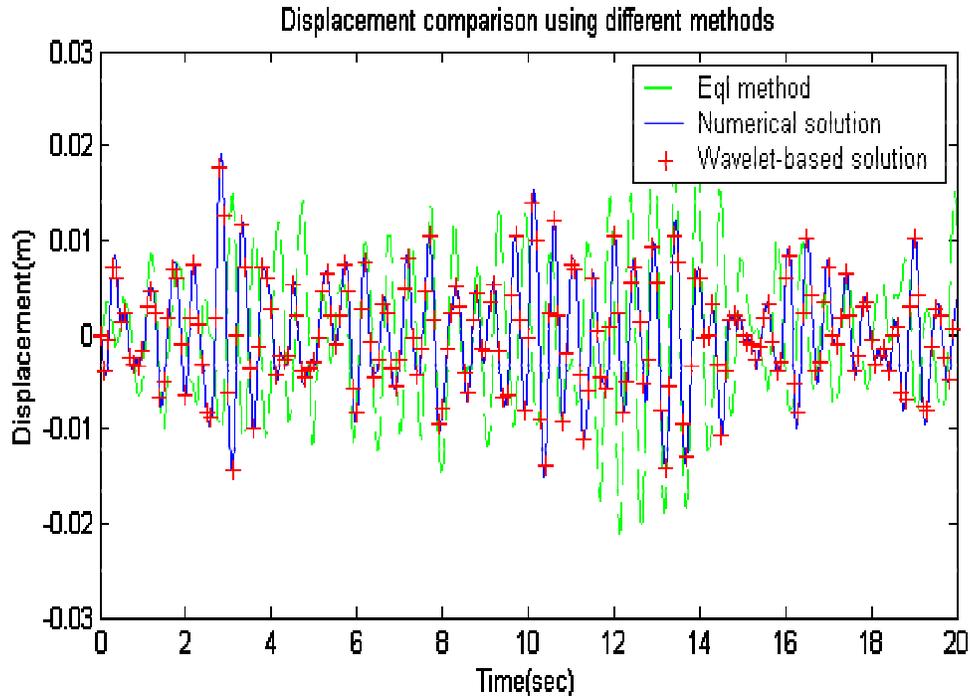


**Figure 7.3 Using Morlet wavelet**

- To further verify the procedure, a White noise excitation is used (Figure7.4). Following the same procedure, with wavelet-based method, from the Figure 7.5, it is obvious that a good match is occurred. But with the general equivalent procedure, there is relatively large discrepancy. During the simulation, it is also found that it is harder to achieve the required convergence unless we actually use a value close to the fundamental frequency of the system, i.e.,  $\omega_{eq} = \omega_n$ . For this case, it is obvious that the wavelet-based method has better performance.



**Figure 7.4 Excitation 2-White noise**



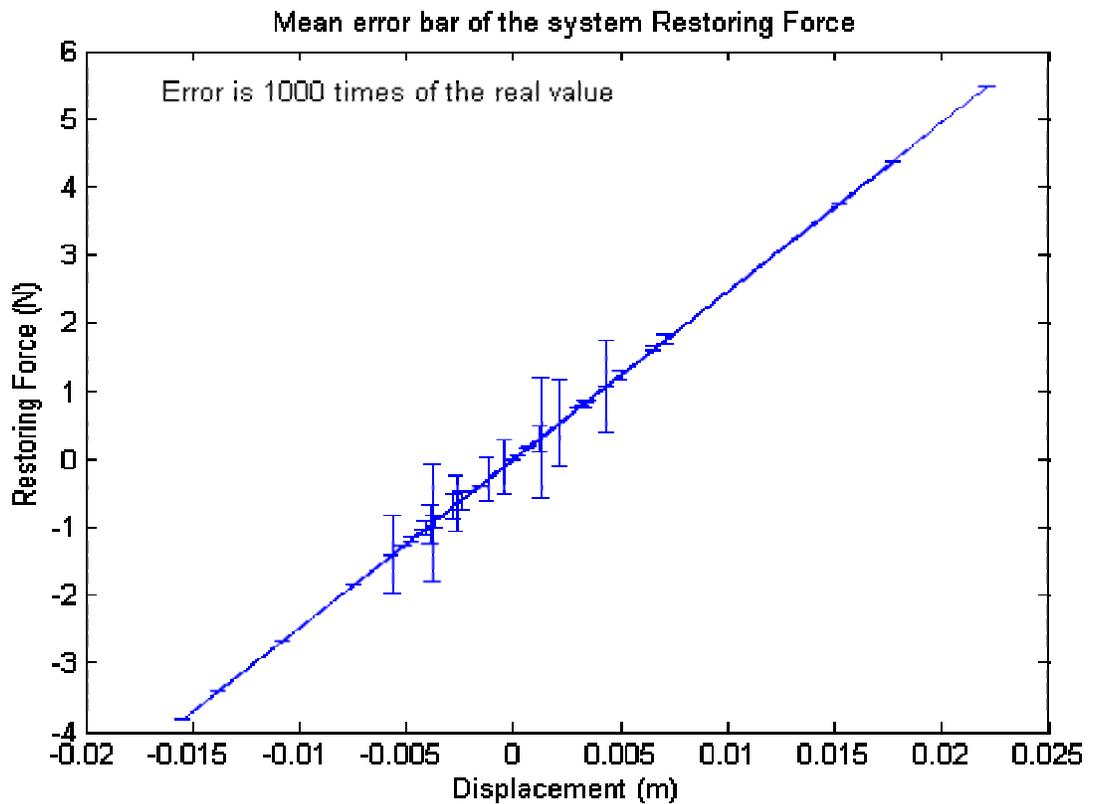
**Figure 7.5 Comparison of different method under white noise excitation**

## **7.2 Effects of the Nonlinearity**

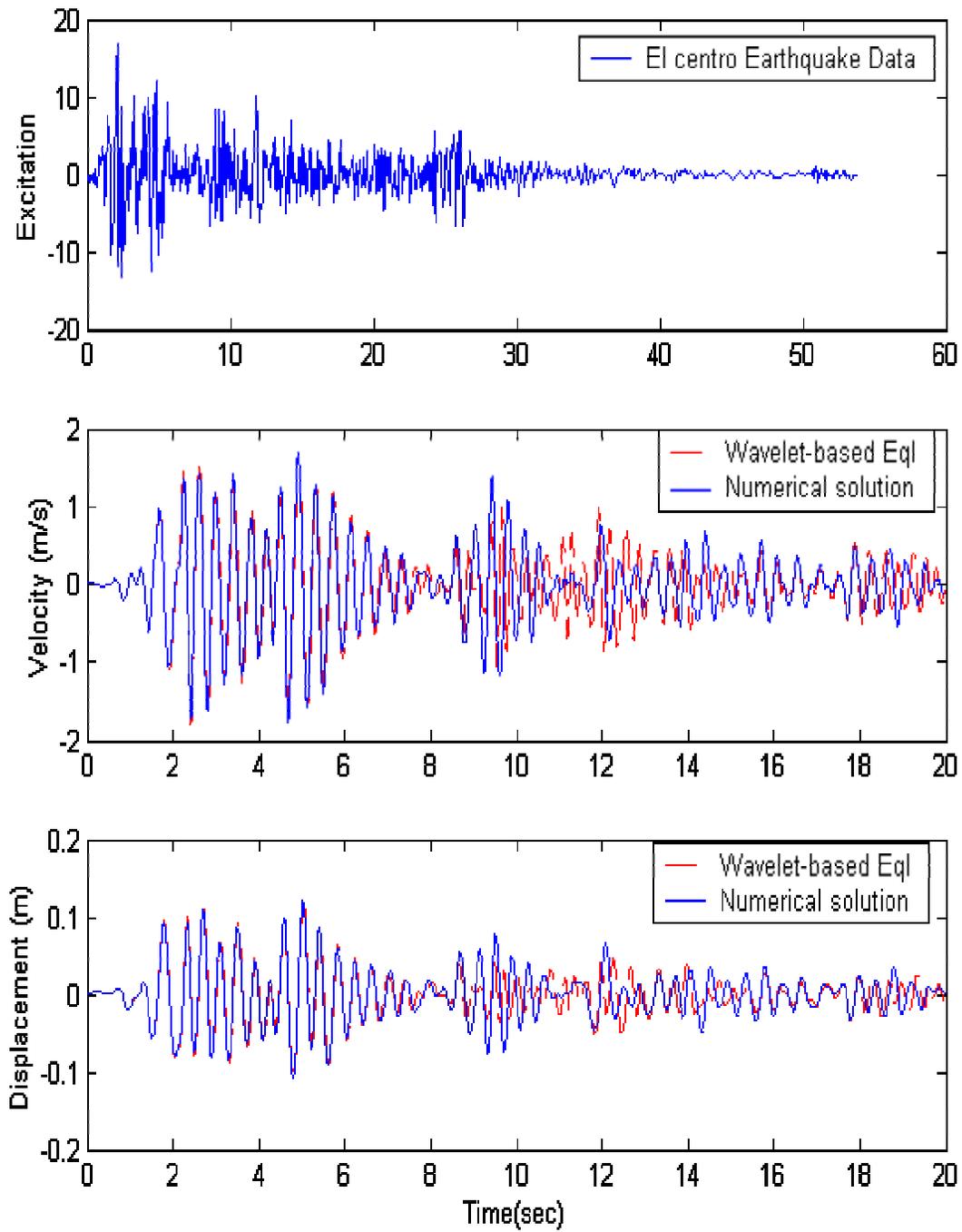
Results listed in the previous cases are for the system with small-nonlinearity. In order to see the effects of the nonlinearity on this wavelet-based technique, the restoring force vs. displacement relationship between the nonlinear system and the equivalent linear system has been compared. To quantify the difference between the original nonlinear system and the equivalent linear system, the relation of restoring force, the nonlinear part of the system and the displacement has been studied. By numerical curve fitting, the approximation of the linearized restoring force can be obtained and compared with nonlinear restoring force vs. displacement.

As illustrated in Figure 7.6, for the case with small nonlinearity introduced in the previous section, the restoring force is almost linear, and the difference between two

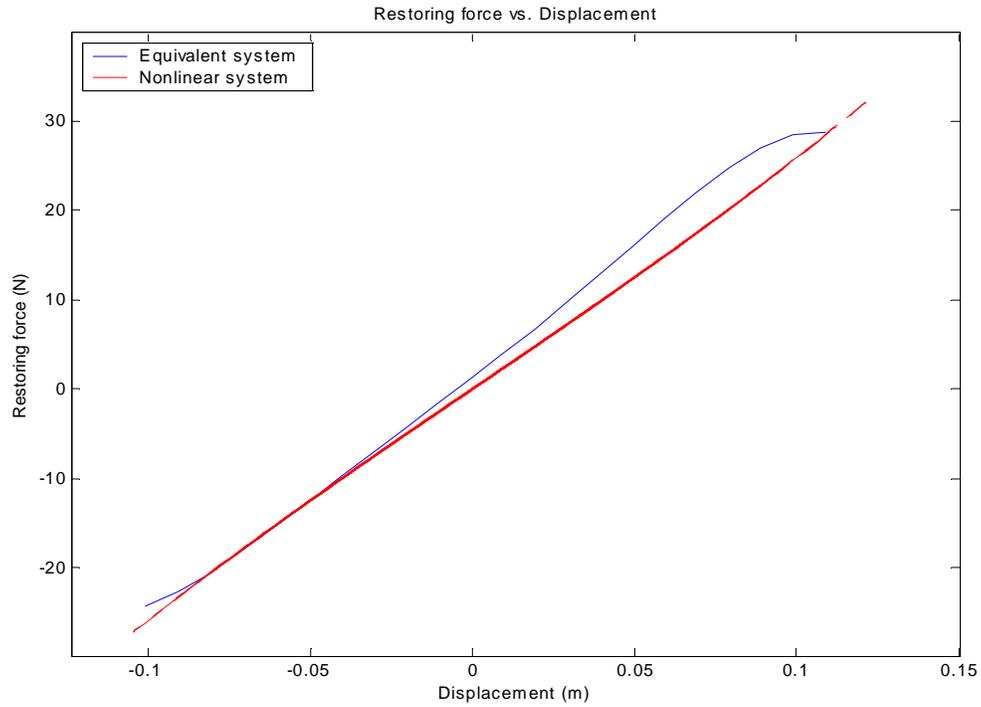
systems is hard to see. On using numerical technique to plot the error bar of the restoring force, it is shown that the error is less than 0.7 %. To increase the nonlinearity of the system, the nonlinear factor has been increase to  $\lambda=3$ , the amplitudes of the excitation have been increased 5 times. The responses are obtained as in Figure 7.7. It is clearly shown that there is some obvious deviation of the response with the increase of the nonlinearity. As illustrated in the Figure 7.8, the relative error of restoring force has been increased to 10% for the case of  $\lambda=5$ .



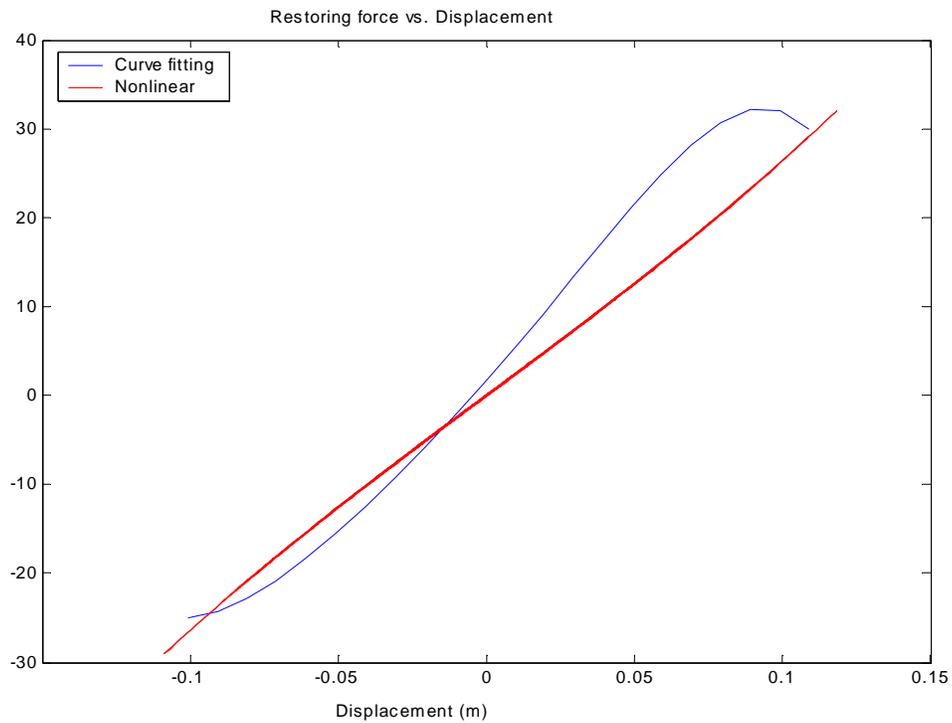
**Figure 7.6 Restoring force vs. Displacement with small nonlinearity**



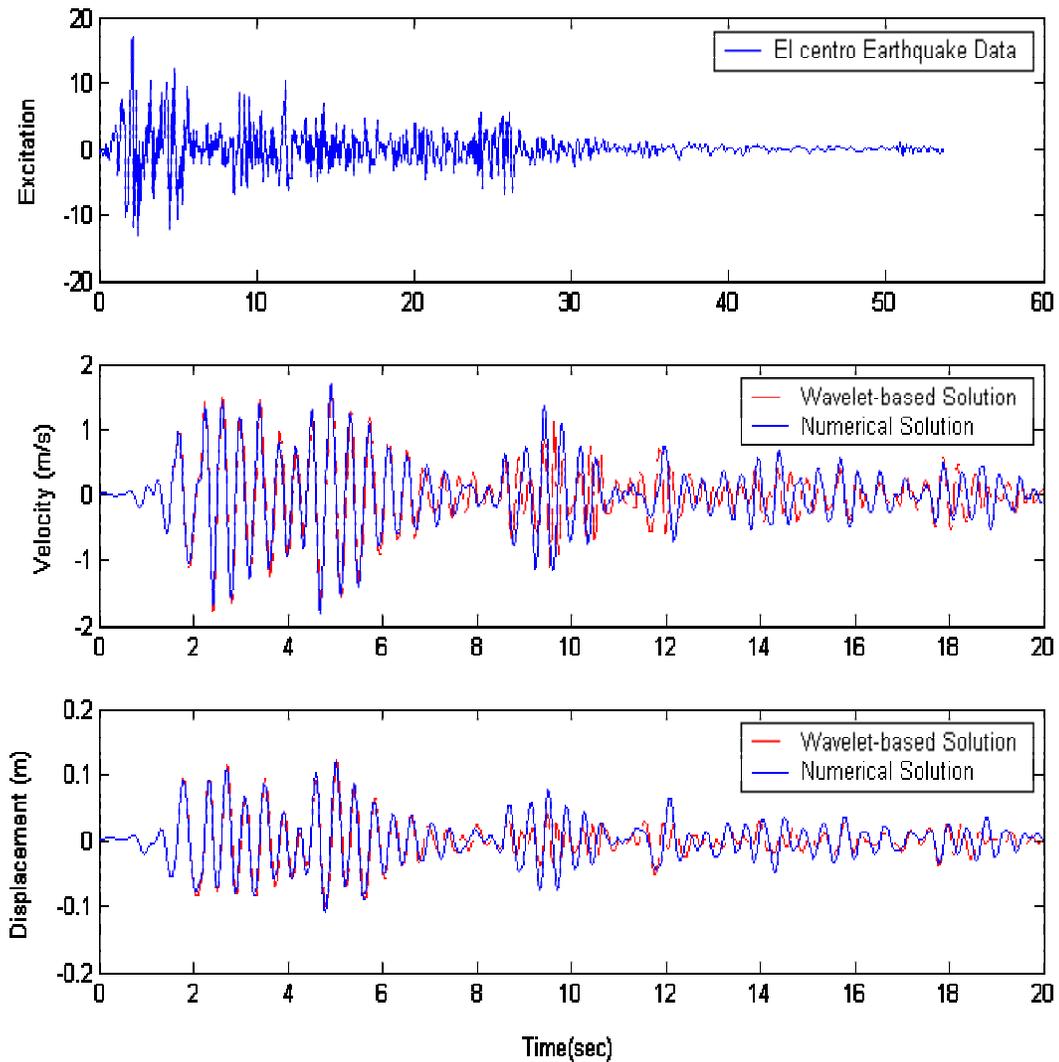
**Figure 7.7 Response results with nonlinear factor  $\lambda=3$**



**Figure 7.8 Restoring Force vs. Displacement for case for  $\lambda=3$**



**Figure 7.9 Restoring Force vs. Displacement for case for  $\lambda=5$**

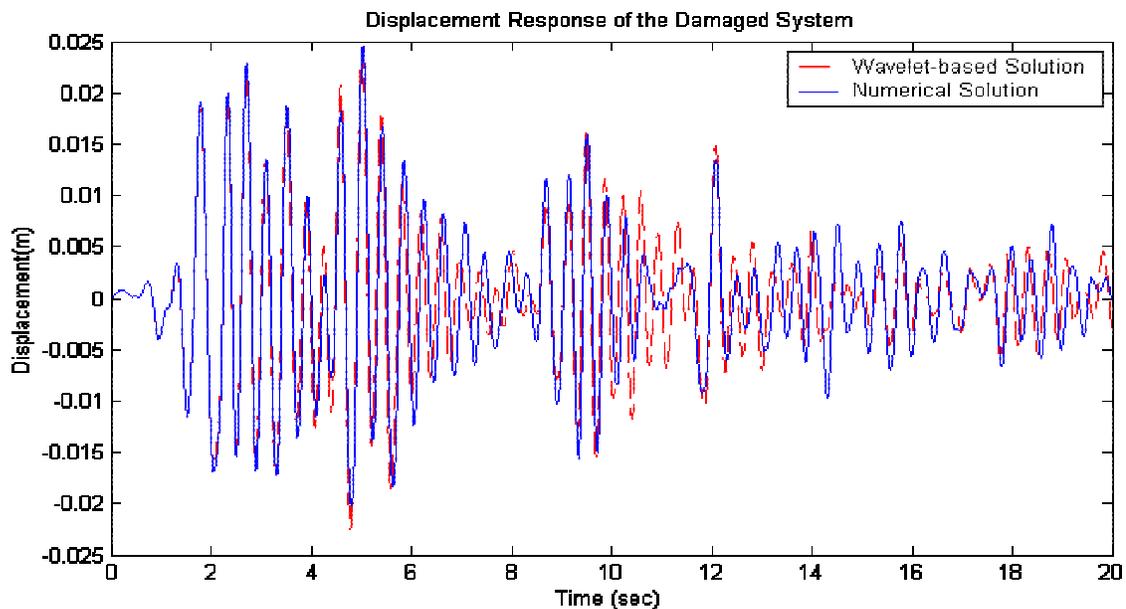


**Figure 7.10 Restoring Force vs. Displacement for case for  $\lambda=5$**

### **7.3 Damage Detection**

The interest in the ability to monitor a structure and detect damage at the earliest possible stage is pervasive throughout the civil, mechanical and aerospace engineering communities. Several works can be found in the literature in which the Wavelet analysis is used for damage detection purpose. In the work of Masuda [1993], the wavelet analysis

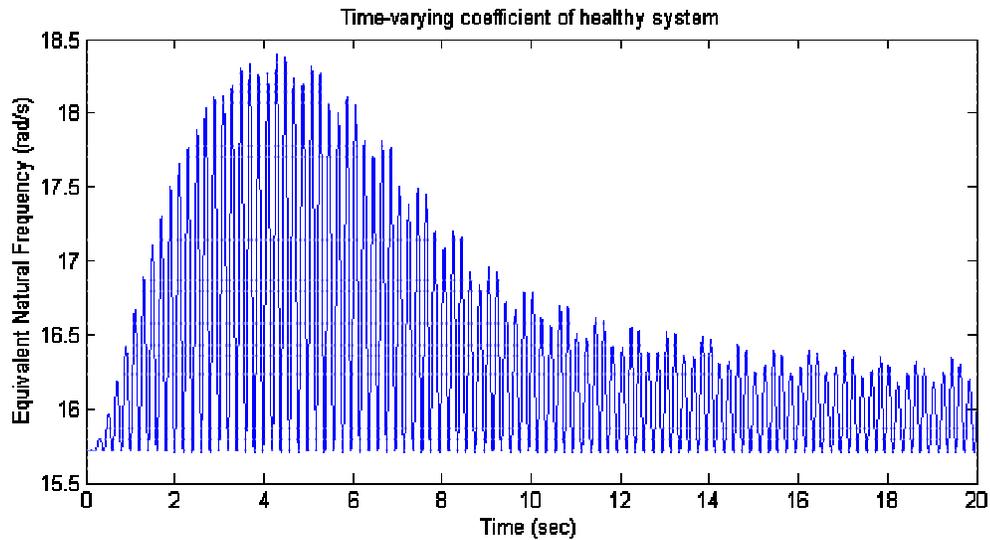
can be utilized to detect fatigue signals which are modeled as impulses superimposed to the input data. It was shown that the moments when these impulses were applied can be clearly observed by spikes in the wavelet representations of the response signal [Al-Khalidy and Noori, 1997]. Similar idea has been applied in the work of Hou and Noori [2000], Noori and Alonso [2000,2001]. Inspired by these works, some trials on the effect of damage detection using the wavelet-based Equivalent linearization have been carried out in this research work. For the system discussed previously, by changing the nonlinear factor from  $\lambda=4$  to  $\lambda=5$  at a particular time  $t=10\text{sec}$ , the system response can be obtained as in Figure 7.11.



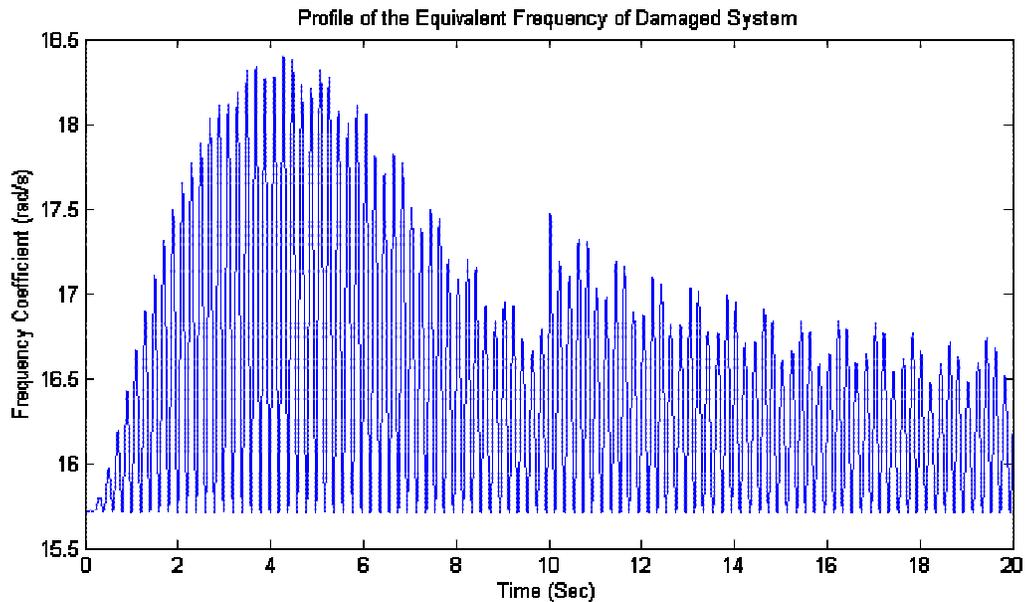
**Figure 7.11 Displacement response of the damaged system**

From the response of the system, it is not possible to get the information of the damage for both nonlinear system and the equivalent linear system. However, the property of the Wavelet transform enables us to obtain temporal information of the signal. In this case, the equivalent natural frequency obtained using this wavelet-based

technique can provide the information desired. Figure 7.12 is the profile of the equivalent time varying natural frequency for the healthy system (there is system parameters changed). Comparing the profile of the equivalent time varying natural frequency for the damaged system that illustrated in Figure 7.13, with the healthy one, clearly, it has been shown that the system has been changed at time  $t = 10$  sec .



**Figure 7.12 Equivalent time-varying natural frequency of the health system**



**Figure 7.13 Equivalent time-varying natural frequency of the damaged system**

In using this way, the extent of the damage can be as small as 1%, the comparison the profiles is able to provide the information of the system change and when change happened.

## **Chapter 8 : Conclusions and Recommendations**

### **8.1 Conclusions**

In the research work presented in this thesis, the feasibility of utilizing the Wavelet Analysis as a signal analysis tool has been studied, the recently emerged wavelet-based equivalent linearization method has been compared with the traditional Equivalent linearization method and the possible application of this method on the structural health monitoring has been introduced. To explicitly demonstrate the whole procedure, a SDOF Duffing oscillator modal has been used in this research work.

#### **8.1.1 Wavelet-based Equivalent Linearization Method**

For the case of a Duffing Oscillator, the system response is shown to match better by applying wavelet-based method than the traditional linearization results. This is the attribution of the Wavelet transform with which the time information of the system can be learned. This technique is also easier to apply the wavelet-based method since it is not necessary to have the probability distribution of the excitation and/or make an assumption about the statistical nature of the response.

One of the limitations of this particular approach lies on its case dependence. Moreover, in order to implement this technique, the knowledge of the original system is required. Besides, there is no general form of the equivalent natural frequency coefficient. That means for every particular system, the equivalent coefficient should be derived according to equation of the motion for that system. To consider the application on the practical problems, another disadvantage of this technique should be mentioned that the computation time to obtain the response, using Wavelet based method, is much

larger than that of the traditional approach. This is mainly caused by the iterative procedure as introduced in chapter 6. In each step of the iteration, there is 4<sup>th</sup> order Runge-Kutta process involved to get the wavelet coefficient. This limitation will limit the real-time application of the approach.

Since the procedure to derive the equivalent frequency has the assumption that the mother wavelet function used in the Wavelet transform has to be at least twice differentiable, the mother wavelet should be chosen carefully. In order to check the effect of the different mother wavelet, two different wavelet forms: Morlet wavelet and Mexican hat wavelet have been applied. For these two wavelets, it didn't show obvious difference on the accuracy of the results.

### **8.1.2 Nonlinearity effects**

For nonlinear problems, much of the research effort has been directed towards developing suitable methods to approximate the nonlinear systems. However, there is no way to find a perfect method to deal with all kinds of problems. For instance, with a small nonlinearity problem, it has been much easier to find an applicable technique to solve it. Thus for a newly developed technique, it will be necessary to study its applicability on different cases.

In this research work, based on the same SDOF mass-spring-damper system with different nonlinear factor, the response has been studied. To quantify the effect of the nonlinearity on this approach, the restoring force vs. displacement relations for both nonlinear system and the equivalent linear system have been compared. For small nonlinearity, especially when the excitation force is relatively small, the effect is fairly

small, the restoring force can be closely approximated, and the responses for both systems are matching well. If big nonlinearity is introduced, the error of the restoring force between the nonlinear system and equivalent linear system will be much larger. The response obtained using this method still can be used to approximate the numerical one.

### **8.1.3 Damage Detection**

The analysis categories of damage detection include changes in modal frequencies, changes in measured flexibility coefficients [Doebbling etc., 1996]. On applying this wavelet-based method, changes of the equivalent natural frequency were tractable. It has been shown that for an abrupt change introduced into the system, it is possible to detect the change by tracking the time-varying equivalent natural frequency. However, due to the limitations introduced in previous section, the application of this method on the structural health monitoring need to be further studied.

## **8.2 Recommendations**

Based on the results obtained there are some areas, which are worthwhile to investigate in the future.

- ❖ It will be worthwhile to investigate some other types of nonlinear systems to see the feasibility of this approach on different cases. Especially for some multi-degree-freedom systems, if a relatively general form of the equivalent linear system can be obtained, it will be a valuable contribution.
- ❖ Another possible extension is to try different wavelet forms in this approach. There are numerous families of wavelet basis with different properties. It might be worthwhile to check the effect of different mother wavelets on this approach.

- ❖ And it will be also useful to find out the applicability of different wavelets on different problems.
- ❖ Another important extension and very promising application is the use of this approach to detect damages under random excitation environment.

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