ABSTRACT

PUSEY, ELEANOR LOUISE. The van Hiele Model of Reasoning in Geometry: A Literature Review. (Under the direction of Dr. Hollylynne Stohl)

The objective of this paper has been to examine the van Hiele model of learning used to describe how students reason in geometry, the research that many others have conducted with regards to it, other related theories of learning, and the implications of van Hiele's theory for curricula, teacher education, and classroom practice.

Initially, the levels of the van Hiele model are described in detail for the reader. The research on the van Hiele model is highlighted with respect to four different areas: 1) appropriate ways to assess students' levels of geometric reasoning and the results of those assessments, 2) assessment of preservice and inservice teachers' levels of reasoning, 3) instructional interventions used with students based on the van Hiele model, 4) interventions with both preservice and inservice teachers to promote awareness of the theory and improved knowledge of geometry content. A brief description of the developmental theory of Piaget and the SOLO Taxonomy of Biggs and Collis is given as a means of comparison to the van Hiele model. A synthesis of the advantages and disadvantages of each of the three theories is outlined. Finally, implications of the research on the van Hiele model are given with respect to curriculum, teacher education and teaching of geometry in addition to my own personal recommendations supported by this research and NCTM's standards.

THE VAN HIELE MODEL OF REASONING IN GEOMETRY:

A LITERATURE REVIEW

By

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A thesis submitted to the Graduate Faculty of
North Carolina State University
in partial fulfillment of the
requirements for the Degree of
Master of Science

MATHEMATICS EDUCATION

Raleigh

2003

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BIOGRAPHY

I was born and raised in the great state of Virginia in a little town called Blacksburg (home of the Virginia Tech Hokies!). I am one of four children and have a twin brother, Brian, who still resides in Blacksburg, an older brother, Boyd, who lives in Johnson City, Tennessee and an older sister, Elizabeth, who lives in Virginia Beach with her husband and two children. My parents, Bob and Betty Pusey, were both professors at the college level and still reside in Blacksburg.

Growing up, some of my hobbies included playing basketball and taking piano lessons. At a young age, I had always dreamed of becoming a teacher one day. I attended Radford University for my undergraduate degree in mathematics and worked with the women's' basketball team as a team manager. I graduated in May of 1992 excited at the prospect of teaching high school math full time and coaching girls' basketball.

For whatever reason, I was unable to find employment at that time in Virginia and ended up applying for a teaching position in Wake County, North Carolina. Within a few months, I had received several offers and began my teaching career in November of 1992 at Wake Forest-Rolesville High School. After a few years, there I was hired as the junior varsity girls basketball coach and stayed there coaching and teaching until my decision to attend graduate school as a full-time student in the fall of 2001.

I taught a number of courses there including Pre-Algebra, Algebra I, Geometry, Algebra II, Algebra III and Trigonometry, and an SAT Math prep course. I particularly enjoyed using some curricula from the University of Hawaii's (UH) Curriculum and

Research Development Group called *Process Algebra* and *Geometry: A Moving Experience*. I became especially excited about geometry after attending a summer institute under Dr. Gary Martin from UH who was responsible for the research and development of the geometry curriculum. It was his influence that caused major changes in my teaching, inspired me to go to graduate school, and sparked my interest in writing my thesis on the van Hiele model.

I have been in Raleigh now for 10 years. I am part of an incredible church that has taught me so much about life and people, but most importantly how to have a personal relationship with God. I enjoy being involved with the singles' ministry of the church and value the many friendships I have made that have helped to make North Carolina my "home away from home".

In my spare time, I enjoy working part-time at the RBC Center for Carolina Hurricanes hockey and North Carolina State basketball games. I also enjoy returning to Blacksburg to visit with family and watch the Hokies play football. I also have a small Boston terrier named "Miss Scarlett" (very spoiled and thus, appropriately named) who keeps me busy, but entertained most of the time.

I have thoroughly enjoyed my graduate school experience. I look forward to returning to teaching full-time upon completion of my master's program.

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CHAPTER 1

INTRODUCTION

Other nations are consistently outperforming children in the United States (US) in mathematics achievement. According to the Third International Mathematics and Science Study (TIMSS), US students scored above the international average in overall mathematics achievement in 1999, however, they ranked 19th out of 38 participating countries (Mullis et al., 2000). Within the five specific content areas (algebra, fractions and number sense, data analysis and probability, measurement, and geometry), the US received their lowest rankings in geometry and measurement (27th and 23rd respectively) compared to the other 38 countries. Trends for the US on the geometry portion of the achievement test indicated an average of 50% correct in 1995 and 52% in 1999, below the international average of 65% correct in 1995 and 62% correct in 1999. This data suggests that the needs of American students are not being met in the geometry classroom.

The National Council of Teachers of Mathematics' (NCTM) *Curriculum and Evaluation Standards* (1989) and *Principles and Standards for School Mathematics* (2000) introduced high expectations for all students in school mathematics for grades K-12, including geometry. NCTM recommended an increased emphasis on geometry throughout a student's school experience to encourage a progression in reasoning ability over time, as proposed by a model of geometric reasoning developed by Pierre van Hiele. The van Hiele model describes levels of thinking that students progress through in reasoning about geometry that should ultimately prepare them for formal proof-writing. According to the National Assessment for Educational Progress (NAEP), in 1992 the majority of students at grades 4, 8, and 12 appeared to demonstrate holistic reasoning (the lowest level described by van Hiele) in geometry (Strutchens & Blume, 1997).

The poor achievement of students in geometry would indicate the challenge which exists in the teaching of geometry at all grade levels. In lieu of this challenge, a reasonable question to contemplate is the amount of exposure teachers have had to geometry as a subject as well as the amount of training they have received in teaching geometry. The National Assessment for Educational Progress (NAEP) collects this kind of data in the form of a survey to answer these kinds of questions. In terms of exposure to geometry as a topic, in 1992, 84% of grade 4 and 95% of grade 8 teachers reported that they had received little or no exposure to geometry beyond high school (Nation's Report Card, NAEP Data Tool, http://nces.ed.gov/nationsreportcard/naepdata/). In 1996, the situation was no better with 85% of grade 4 teachers (86% in 2000) and 94% of grade 8 teachers (95% in 2000) indicated they had received little or no exposure to geometry. Another area of interest would be the degree of course work that has taken place in a teacher's undergraduate program concerning the teaching of geometry for elementary or middle grades. In 1990, 67% of teachers at grade 4 and 61% at grade 8 reported they had not had any course on the teaching of geometry at the undergraduate or graduate level. Fortunately, if preservice teachers lack necessary preparation in the teaching of geometry, there is always the opportunity to participate in inservice professional development. Unfortunately, in 1992, 71% of grade 4 teachers and 76% of grade 8 teachers reported they had not had any experiences with inservice geometry training. In 1996 and 2000, 64% of grade 4 teachers gave no response in reporting the level of exposure received in professional development workshops or seminars in geometry (for grade 8 teachers, 78% in 1996 and 69% in 2000). The results in these three areas suggest there is reason for concern in mathematics education for the preparation of teachers to teach geometry.

Many teachers devote a substantial number of years teaching high school geometry courses and attending various inservice programs to help them learn to teach geometry better. The rationale and inspiration behind this thesis stems from my personal classroom experience and the experience of so many other teachers. The following vignette was written to capture the essence of these real experiences within the story of a fictitious teacher named Mrs. Trump.

Vignette: Mrs. Trump's Experiences Teaching Geometry

> *Mrs. Trump was a young zealot ready to begin her third year as a high school mathematics teacher. Things were beginning to "click" for her professionally and she was more excited than ever for the new school year and the challenge of a new class of young adults. After a few years of teaching, her principal and department chair decided she was ready to take on the challenge of teaching geometry. The other geometry teachers in the department had to be wondering, "How long will this one last?" They had spoken often of their struggles to teach geometry as well as the high failure rates (about one-third) of students.*

With a gleam in her eye and the same textbook Mrs. Trump had used when she took high school geometry in 1985, she embarked on a mission to teach reasoning and proof-writing to a new batch of tenth graders. She was feeling some intimidation and insecurity with the curriculum because she had not had much training in her undergraduate courses. There were vague memories from her mathematics methods course of a few references to some of the content covered in a high school geometry course; however, realistically she was relying on her minimal recall of her own high school experience and ultimately knew she was going to be "learning" it along with her students.

Mrs. Trump "survived" the first year but unfortunately over one-third of her students did not pass the class. She began her second year more determined than ever to do better and try some innovative lessons she had learned about at professional conferences. She was enthusiastic about using some of these new ideas, but found they did little to help. These isolated one-shot lesson ideas may have provided some interesting activities for students, but overall the curriculum was still discontinuous and seemed too hard for the students.

As a last ditch effort before requesting to be assigned a different course load that did not include geometry, Mrs. Trump decided to attend a summer institute on the teaching of geometry. She hoped that maybe a workshop of more substance could explain her struggles and offer longer *lasting solutions to her problems in the classroom. The focus of the course was geometry content, research related to student's learning of geometry, as well as effective teaching practices in geometry. Part of the foci of the course material was reading and discussing how students progress through levels in learning and reasoning about geometry. This theory of learning was called the van Hiele model and seemed to make perfect sense of why she and her students were having difficulty.*

Not surprisingly, Mrs. Trump's next several years of teaching geometry showed a "transformation" in her and her students' level of success. This time, she built the course content up from the lower thinking levels, in the hopes that she could foster the development of students' thinking at progressively higher levels over time. For example, writing formal proofs was the ending point of the course instead of the beginning point. Students were expected to make informal arguments throughout the course and had to justify their claims to one another as a community of learners in class discussion. Students became responsible for constructing their own understanding as Mrs. Trump facilitated this process. Mrs. Trump's excitement with her new-found success and students' enthusiasm for learning geometry prompted her to conduct several professional development seminars for other teachers.

A Closer Look at Mrs. Trump's Experiences

The source of Mrs. Trump's transformation and the events leading up to it are the rationale and inspiration for this thesis. Her story is one that describes a common struggle (teaching of geometry), points to the source of such struggles (students' reasoning abilities) and offers some solutions to address these struggles. Ultimately, the intent of this thesis is to help other educators gain a theoretically-grounded perspective on research on students' reasoning in geometry so they may possibly experience similar success stories in the teaching and learning of geometry with other students.

The first challenge facing Mrs. Trump was the "stigma" attached to teaching geometry for the high failure rates of students, the stories from teachers who are frustrated with teaching high school geometry, and a host of bad memories and experiences of those who have taken it. The van Hiele model (to be described in detail in Chapter 2) can help educators understand why this course is problematic for students. Fuys (1985) notes:

> according to the van Hiele's, the levels can be used to explain why students have difficulties in geometry. They believed that high school geometry involves thinking at a relatively high level and that many students have not had sufficient experiences in thinking at prerequisite lower levels (p. 448).

Students often attribute their difficulty in understanding geometry to having a hard time memorizing theorems and writing proofs (Senk, 1989). There is certainly more to geometry than proofs; however, the responsibility for changing such attitudes in students lies with the classroom teacher. A good place to start is with Fuys' recommendation to give students experiences at these lower levels before asking them to do formal proofs.

The second challenge for Mrs. Trump was the obvious lack of preparation in her undergraduate program for teaching a course like geometry. There was no education to equip her for the challenges that lie ahead. Unfortunately, her preservice coursework did not focus on current research or theories of learning. One can only wonder what kind of difference it could have made for her students and countless others who are unaware that such theories exist. According to Jaime and Gutierrez (1995),

> the van Hiele model of mathematical reasoning has become a proved descriptor of the progress of students' reasoning in geometry and is a valid framework for the design of teaching sequences in school geometry as acknowledged by the NCTM's *Curriculum and Evaluation Standards for School Mathematics* (1989) and by the *Addenda* Series books devoted to geometry (p. 592).

Statements like this one would indicate the importance of studying such models in mathematics teacher education courses.

Although Mrs. Trump was able to improve student's performance in her geometry class, her efforts to try new ideas and activities did not offer long-term help or discover the source of students' problems in geometry. This particular source was most likely a language barrier between the teacher and students. This is a main point of emphasis in the van Hiele model of learning. Senk (1989) says, "as a predictor, the van Hiele model states that two persons reasoning at different levels may not understand each other...a student who has attained only level n will not understand the thinking of level $n+1$ or higher" (p. 310). Awareness of this communication gap allows teachers to prepare for this potential barrier in the learning of geometry.

The victory of Mrs. Trump's story is that her and her students' trouble was something she could control when armed with appropriate knowledge. Teachers need to be aware of theories of learning to improve classroom instruction. Swafford and Jones (1997) say, "the common belief is that the more a teacher knows about a subject and the way students learn, the more effective that individual will be in nurturing mathematical understanding" (p. 467). The implications from the van Hiele model offer strategies for teaching that help students develop and improve their levels of reasoning. Once this kind of support is in place for teachers, they may be better able to overcome some of the challenge in teaching geometry.

Another important element of Mrs. Trump's classroom was the change from a teacher-focused setting to a student-centered classroom. Students benefit from learning in settings where they are able to communicate the mathematics they are thinking about and are forced to give justification to such reasoning. Teachers in such settings have the flexibility to be facilitators who can ask questions that promote and assess student learning. Clements and Battista (1992) say a benefit of this model is "students must decide what they believe to be mathematical truths" (p. 442) which can be particularly significant in the study of geometry to give support for writing proofs.

A final lesson of Mrs. Trump's story is the importance and impact of continuing this cycle of teaching and learning by passing her understanding and classroom experiences with using the van Hiele model as a guide on to other geometry teachers. Jacobson and Lehrer (2000) suggested that teachers should be exposed to current research on student reasoning and share the results of using these research results to guide their own instructional decisions. They say, "in the two classes in which teachers were more knowledgeable about students' thinking about space and geometry, not only did students learn more than did their counterparts, but this difference in learning was maintained over time" (p. 86). Inservice teachers with more experience implementing theory in their practice can help make changes in mathematics education through collaboratively planning instruction with colleagues and leading professional development workshops.

Research Focus

The purpose of this paper is to describe the van Hiele model in more detail, present research related to the van Hiele model, synthesize the results of such research, compare the van Hiele model to other theoretical models, and discuss classroom implications. Since van Hiele's theory was proposed, a variety of research has been done to answer a number of questions related to the theory (Clements & Battista, 1992). The research related to the van Hiele model has followed several different lines of work. One avenue of research has focused on testing the van Hiele theory itself and some of its assumptions. For instance, some have tried to find evidence that levels of reasoning exist and form a hierarchy. Another question posed by research is whether students reason at the same level across topics within geometry. A question of great interest in research has been van Hiele's assumption that the levels are discrete. Finally, researchers have tried to identify the most basic level of reasoning exhibited by students. They have considered whether a level exists that is even lower than those that van Hiele describes.

A second avenue of research has attempted to find appropriate ways to assess the levels and discuss implications of these assessments. For instance, some assessment techniques have included multiple choice tests, open-ended response tests, and clinical interviews. Indicators have been developed for each of the levels to create interview protocols. Assessments have been conducted with students of middle and upper grades as well as preservice and inservice teachers. Areas of interest with student assessment have been how well van Hiele levels can predict success in geometry and proof-writing as well as readiness for taking geometry. Areas of interest with teacher assessment have been identifying where teachers stand in terms of the levels as well as their possible misconceptions.

A third avenue of research with the van Hiele theory has looked at the effects of interventions with students and teachers based on the model. With students, the research has sought to determine if instruction based on van Hiele's recommendations is effective in fostering improved reasoning. With teachers, the research has sought to determine if interventions to improve content knowledge and awareness of the theory are effective in helping teachers and students.

Chapter 2 will provide details about the results in these three avenues of research regarding the van Hiele model. Since there are other models that have been proposed, Chapter 3 will briefly discuss Piaget's developmental theory of learning and Biggs and Collis' SOLO Taxonomy as a basis for comparison with the van Hiele model. This will include a synthesis of the benefits and drawbacks associated with each of the three models. Finally, Chapter 4 provides implications of the research results on the van Hiele model for curriculum development, pedagogy and teacher education.

CHAPTER 2

RESEARCH ON THE VAN HIELE MODEL

An abundance of studies have shown that students are performing horribly in the geometry classroom (e.g., Strutchens & Blume, 1997; Mullis et al., 2000). It has become clear that most students do not understand the meaning of a proof, much less how to go about writing one, even after completing a year-long course in geometry (Senk, 1989). These difficulties prompted a lot of research by educators in the Soviet Union from 1930-1950 (e.g., Psyhkalo, 1968 and Wirszup, 1976 as cited in Clements & Battista, 1992), whose goal was to try and identify the source of this problem. Little progress came out of these initial efforts of the Soviets. However, Jean Piaget and a couple, Pierre van Hiele and Dina van Hiele-Geldof concurrently and separately set out to create theoretical frameworks that could describe geometric thinking. It was not really until the theories of these individuals became public that their impact began to be felt in the geometry classroom. The Soviets then re-focused their research efforts more on these two theories and subsequently began seeing some significant changes occurring in the classroom (Fuys, 1985).

According to Pegg and Davey (1998), Piaget's framework has not produced much change in the classroom while the van Hiele's work has (possible reasons for this will be addressed later in Chapter 3). Jaime and Gutierrez (1995) noted "the van Hiele model of mathematical reasoning has become a proved descriptor of the progress of students' reasoning in geometry and is a valid framework for the design of teaching sequences in school geometry" (p. 592). The chapter is focused on research based on the van Hiele model. The work of Piaget, as well as Biggs and Collis, will be discussed in chapter three as a comparison to the van Hiele model.

The van Hieles were Dutch educators experiencing challenges in their own classroom that inspired their focus for their doctoral dissertations. Their goals were to try and categorize student thinking in geometry by levels, hereafter referred to as the van Hiele model. Each of them took a different angle in their research. Dina van Hiele-Geldof strove to explain from a teaching perspective how to help children make progress with the levels, and described five teaching phases within each level. Her husband, Pierre van Hiele, was responsible for coming up with the model and describing these levels in more detail (Lawrie & Pegg, 1997). The focus of this chapter will be on the five levels of thinking used in geometric reasoning as described by Pierre van Hiele and subsequent research with this model. The teaching phases proposed by Dina van Hiele-Geldof will be discussed later in this chapter.

Description of the van Hiele Model

According to van Hiele's (1986) model, there are five levels of geometric thought that are sequential and hierarchical. At the first level, students view figures holistically by their appearance. When asked to explain why a particular quadrilateral is a square, a typical response would be, "because it looks like one." Students might associate a square with other familiar things that are shaped like a square. At the second level, a figure is no longer viewed as a whole, but rather by its features. It may be identified with a "laundry list" of properties, yet no relationships exist amongst these properties or to other closely related classes of figures. The third level is where students see figures related to each other because of their properties. Further, a student could begin to see how one figure could be characterized by several different names if they share the same properties (i.e., a square is a rectangle, but a rectangle is not necessarily a square). The fourth level is where deductive reasoning and writing proof takes on meaning and value. Students are able to define things and limit them with a minimum amount of information (vs. the "laundry list" associated with the second level). The fifth and final level is described by students being able to go between geometries (Euclidean vs. non-Euclidean) and think outside of one axiomatic system. The van Hieles also described periods which explain the kind of growth needed for a student to reach the next level. Within these periods, they identify a distinction between symbol character and signal character.

The first period bridges the gap between the first two levels. Its purpose is to bring students from seeing things holistically to objects with specific components. In this period, the figures themselves are the symbol character. The goal is for students to begin to learn the properties of these symbols and consequently to focus less on the figure itself. This could be tested if by taking the figure away and providing the list of properties instead, the students could still identify the figure. At this stage, the symbol character has become a signal character.

The second period connects the second and third level. In this period, the properties are the symbol character and students are encouraged to relate properties. This can be accomplished by investigating groups of figures (i.e., squares and rectangles) and seeing their common properties. This leads to the notion that order exists and that one figure “is” another if it shares all of those properties, even if it has additional properties, too (i.e., a square is a special type of rectangle). When the properties (symbol character) of a rectangle elicit a student’s anticipation of another figure (a square), the image of this symbol character has become a signal character.

The third period closes the gap between the third and fourth level. Here is where students build the notion of axioms and theorems and the idea of one thing implying another. Not much is mentioned about this period or any that follows because van Hiele was of the opinion that most students did not pass the third level (Pegg & Davey, 1998).

Another point of significance in the theory according to van Hiele was that the levels were discrete and being able to move from one level to the next was not gradual but a “jump.” According to Lawrie and Pegg (1997), a “crisis of thinking” was necessary prior to a student reaching a new level. In addition, the discontinuity of the levels presents communication problems in a classroom because terms take on different meaning for students functioning at different levels. (This phenomenon certainly could have contributed to the frustrations experienced by Mrs. Trump and other geometry teachers.)

These five levels are not necessarily associated with a certain age range. "According to the van Hiele model, an important characteristic of mathematical reasoning is that growth in age does not necessarily imply growth in a student's level of reasoning. Instruction plays a central role in a student's progression throughout the levels" (Jaime & Gutierrez, 1995, p. 592). Also, for students to gain true understanding, it is not feasible to skip a level. Unfortunately, it is reported that teachers may be guilty of "level reduction" when students have to resort to rote memorization instead of functioning at the appropriate higher level (Clements & Battista, 1992). Fuys, Geddes, and Tischler (1988) describe this phenomenon with students when "some tried to recall (rather than think out) what their teacher had told them…thus, when geometry was taught, it appeared to be mainly at a recall or knowledge level" (p. 155).

At the third level, van Hiele asserts that students will not be successful constructing proofs unless the steps involved are minimal. He claims that any success with proof for a student below the third level would indicate that a student had memorized a proof rather than originally producing a proof. Essentially, he predicts that the ability to construct formal proofs on a regular basis would only be feasible for students operating at the top two levels (Senk, 1989).

The levels have undergone some changes since their inception (Pegg & Davey, 1998). The levels as proposed by van Hiele were originally numbered from zero to four. Since then researchers saw a need to categorize thinking that was not even at the introductory level. Van Hiele did not include a level "below 0" and claimed that all students are at least at level 0 (Senk, 1989). The problem was that his subjects were secondary students and others since have tried to use the model with elementary students as well. Rather than label these students as below level 0, van Hiele and a number of other researchers have adopted a new system that renumbered the levels one to five to accommodate for this. Therefore, a student not at the first level, previously below 0, was now assigned level 0 (pre-recognition). Both numbering schemes have been used throughout the research on the van Hiele model and will be specified accordingly for each study.

Research On and With the van Hiele Model of Geometric Reasoning

Much research has been done to question and validate van Hiele's theory. Studies on the van Hiele (VH) theory have attempted to answer several critical questions since its initial introduction to American educators by Izaak Wirszup in 1976 (Clements & Battista, 1991). Some researchers' objectives have been to identify these levels of reasoning and create instruments to assess the VH levels. Researchers have tried to determine whether the VH model is accurate in describing geometric reasoning and if a student reasons consistently across topics within geometry. Research has also attempted to validate if the VH levels are discrete and form a hierarchy. Some have questioned the existence of a pre-recognition level more basic than the visualization level. Others have done studies in the classroom and looked at how VH levels are related to student achievement and whether they are able to predict success in geometry. Still others have done studies with mathematics teachers, both pre-service and in-service, and assessed their VH levels. Efforts have been made to inform teachers of the theory as well as suggest strategies for how it might impact classroom instruction. Finally, van Hiele-based units of instruction have been created and implemented to establish the value in using such types of intervention.

Identifying and Assessing Students' Levels of Reasoning

One of the first major studies done with the VH model was by Usiskin (1982, as cited in Fuys, 1985) at the University of Chicago. Usiskin developed a multiple-choice test to measure a student's VH level of reasoning. This test has been widely used by others, yet has fallen under some criticisms as well. One advantage of Usiskin's test is the ease in administration and grading. Usiskin wanted to find out if these tests could at all predict student achievement in geometry. He tested 2900 10th graders and looked for a correlation between their VH level and their geometry grades. The results showed that there was a moderately strong relationship (r=.64). He consistently found that most tenth graders were not ready for high school geometry. They were generally at levels 0 or 1 (based on the original 0-4 numbering scheme) and had little experience or recall of geometry before their high school course.

Crowley (1990) questioned the validity and reliability of the VH results from Usiskin's (1982, as cited in Fuys, 1985) study. Usiskin and Senk's (1990) reply to the criticism of Crowley suggested that their purpose in using the test was to determine if the VH theory was accurate in describing students' thinking and useful for predicting success in geometry. However, Usiskin and Senk claim some have used it in other contexts of doing research for which it was not originally designed. Wilson (1990) reanalyzed the data from Usiskin's (1982, as cited in Fuys, 1985) study using the Rasch model, a probabilistic model. Wilson claimed that the test items were written to fit subjects into the 5 levels and thus Usiskin and Senk were assuming that which they hoped to validate with their research (i.e., whether the levels were appropriate for describing students' thinking). Usiskin and Senk's (1990) response to this criticism was that if they wanted to test the descriptive power of the theory they had to construct test questions that matched up with the theory proposed by the van Hieles and could show the degree to which levels existed in student reasoning. For this reason, Usiskin and Senk suggest that content validity be the top criteria in critiquing their VH test.

The research of Burger and Shaughnessy (1986) sought to answer three questions regarding the VH model. First, they wanted to know if these levels were reasonable for classifying students' thinking in geometry. Second, they were looking for specific indicators in students' reasoning that might be aligned with each of the levels (Table 1). Finally, they were interested in developing an interview template, instead of a written test, to see if certain levels were dominant in students' thinking while working on a particular task. The subjects varied in age from primary grade to college-level. Burger and Shaughnessy felt that their results did support the notion that the levels were reasonable in assigning students' levels of thinking and they were able to assign certain behaviors to each level. Also, they were successful in creating an interview protocol that allowed them to compare student responses on the same problems.

Table 1: Level Indicators for first four levels (based on the original 0-4 numbering scheme) for Interview Protocol (Burger & Shaughnessy, 1986, pp.43-45)

Level 0	o Use of imprecise properties (qualities) to compare drawings and to identify, characterize, and sort shapes. o References to visual prototypes to characterize shapes. o Inclusion of irrelevant attributes when identifying and describing shapes, such as orientation of the figure on the page. o Inability to conceive of an infinite variety of types of shapes. o Inconsistent sortings; that is, sortings by properties not shared by the sorted shapes. o Inability to use properties as necessary conditions to determine a shape; for example, guessing the shape in the mystery shape task after far too few clues, as if the clues triggered a visual image.

Table 1 (continued)

1	○ Comparing shapes explicitly by means of properties of their components. ○ Prohibiting class inclusions among general types of shapes, such as quadrilaterals. ○ Sorting by single attributes, such as properties of sides, while neglecting angles, symmetry, and so forth. ○ Application of a litany of necessary properties instead of determining sufficient properties when identifying shapes, explaining identifications, and deciding on a mystery shape. ○ Descriptions of types of shapes by explicit use of their properties, rather than by type names, even if known. For example, instead of rectangle, the shape would be referred to as a four-sided figure with all right angles. ○ Explicit rejection of textbook definitions of shapes in favor of personal characterizations.
2	○ Formation of complete definitions of types of shapes. ○ Ability to modify definitions and immediately accept and use definitions of new concepts. ○ Ability to accept equivalent forms of definitions. ○ Acceptance of logical partial ordering among types of shapes, including class inclusions. ○ Ability to sort shapes according to a variety of mathematically precise attributes. ○ Explicit use of "if, then" statements. ○ Ability to form correct informal deductive arguments, implicitly using such logical forms as the chain rule (if p implies q and q implies r, then p implies r) and the law of detachment (modus ponens). ○ Confusion between the roles of axiom and theorem.
3	○ Clarification of ambiguous questions and rephrasing of problem tasks into precise language. ○ Frequent conjecturing and attempts to verify conjectures deductively. ○ Reliance on proof as the final authority in deciding the truth of a mathematical proposition. ○ Understanding of the roles of the components in a mathematical discourse, such as axioms, definitions, theorems, proof. ○ Implicit acceptance of the postulates of Euclidean geometry.

Burger and Shaughnessy's (1986) study included an interview of a college-level mathematics major. Although this was not the original audience targeted for their study, they realized the students in high school who had taken geometry were not at the formal deduction level. Thus, in trying to develop an interview script that would be suitable for subjects at the highest levels, they had to go to a university. One task asked the subject to identify and define the quadrilaterals pictured on a page by labeling them accordingly and justifying their "labels." The subject was also asked to explain how he would tell someone else to find a particular shape (such as a rectangle). In his response, he came up with meanings for the different figures separately and then checked back to see that class inclusions supported his definitions. Another task had him identify a mystery shape based on clues revealed one at a time until they deemed that enough information was known to "uncover" the mystery shape. Again, the subject formally deduced from the clues to decide on the mystery shape. A third task, completed only by him and the secondary school students, was to explain and give examples of an axiom, theorem, or postulate. The response of the college student indicated an understanding that theorems had to be proven, but that postulates were so elementary that it was not necessary to prove them. Based on the different responses to these tasks, the reviewers assigned a predominant VH level. The assigned levels varied according to the task for many of the school-age subjects in this study; however, the college mathematics major was consistently assigned level 3 (based on original 0-4 numbering scheme) thinking.

The van Hiele's claimed that the levels are discrete; yet, Burger and Shaughnessy (1986) contend that the levels seem more continuous because they found it difficult to choose between levels and often identified students oscillating between levels, even on the same question. They suggested this kind of behavior made sense coming from students transitioning between levels, particularly between levels 1 and 2; however, they suggested that a larger sample size of college-level students should be interviewed to see transitioning students between levels 2 and 3. This claim of non-discreteness to the levels has also been supported by other researchers (e.g., Fuys, Geddes, and Tischler, 1988; Usiskin, 1982 as cited in Fuys, 1985; Clements, Battista, and Sarama, 2001).

Burger and Shaughnessy's (1986) research has supported and refuted some of the other research done on the VH model. Burger and Shaughnessy and Usiskin's (1982, as cited in Fuys, 1985) research both pointed to the existence of some students who were unidentifiable, or in transition between levels. This would go against van Hiele's theory that the levels are discrete. Burger and Shaughnessy also supported the results of Mayberry (1983) and Usiskin (1982, as cited in Fuys, 1985) who reported that a very limited number of children were found to be functioning at the formal deduction level. Burger and Shaughnessy's research also supported that of Mayberry's (1983) in concluding that a student's level may vary across topics and tasks in geometry. Finally, their research upheld observations proposed by Fuys (1985) and Mayberry (1983) that the levels are hierarchical in nature.

Gutierrez, Jaime and Fortuny (1991) presented an alternate way of assessing VH levels to especially handle those "in-between" students. Their method of evaluating tried to measure the acquisition of levels quantitatively and qualitatively. If a student has no awareness or fails to recognize a need for a level, they are classified as having "no acquisition" and fall in a range of 0-15 out of 100. Students with "low acquisition" fall in the range of 15-40 since they begin to try working at that level, yet fail and resort to a lower level. At the "intermediate acquisition" of a level (40-60), students are comfortable working at a level, but when they have difficulty they regress to the lower level. This degree is easily recognized if a student habitually goes back and forth between two levels. “High acquisition” of a level (60-85) would mean a students’ thinking matches the descriptors of that level, but is prone to making mistakes. Finally, a student has “complete acquisition” of a level (85-100) when they are able to fully understand and operate at that level with no difficulty.

Gutierrez, Jaime, and Fortuny (1991) administered their newly devised Spatial Geometry test to a group of 41 primary school teachers and nine 8th grade students and used their newly devised acquisition of levels scale to analyze their results. They noted:

> The way of evaluating thinking levels explained in this article allows for the possibility that a student can develop two consecutive levels of reasoning at the same time, although what usually happens is that the acquisition of the lower level is more complete than the acquisition of the upper level (Gutierrez et al., 1991, p.250).

Years later, Gutierrez and Jaime (1998) presented a test they designed for a longitudinal study with primary and secondary students looking at the VH levels; again, they used the techniques described above in analyzing the results of this study.

Gutierrez, Jaime, Burger and Shaughnessy (1991) teamed up to compare their two different approaches for assigning VH levels to students, as well as their testing instruments. Gutierrez and Jaime used open-ended questions in a paper and pencil format and assigned levels using a continuum measuring degree of acquisition. Burger and Shaughnessy used clinical interviews and assigned a dominant level of thinking (or in some cases two). From both original samples, a sub-sample of six students was chosen that would well-represent the spectrum of thinking. Then each pair of researchers scored the other pair's sub-sample with their original scoring technique. One obvious comparison was that the degree of acquisition (4x1 matrix of percents for each level) seemed to give a much clearer picture of a student's thinking than a single dominant level (or levels); however, this again goes against van Hiele's theory that the levels are discrete. Also, there seemed to be a high level of consensus between the two different means of assessment. There was a slightly better fit with Burger and Shaughnessy's students, probably due to the interview setting which allowed for clarification of unclear responses.

Fuys, Geddes, and Tischler (1988) conducted an interview study with students in the sixth and ninth grades, looking to assign them an appropriate VH level. Subjects were assigned an "entry level" for topics covered prior to and in the 6^{th} grade curriculum and then a "potential level" based on evident changes in their thinking as the interview progressed. The van Hieles claimed that students' thinking level was based on their prior learning experiences. Thus, the entry level was a static assessment and might not give a true picture of a student's ability to think. In the course of the interview, if the interviewer came across a topic unfamiliar to the student, this required some prompting and instruction during the interview. In this case, a student's level of thinking was still identified, but considered a "potential level" since they were being exposed to a topic for the first time.

The results of the 6^{th} grade interviews yielded mostly level zero and some level one (based on original 0-4 numbering scheme) for entry level. This could be due to a lack of exposure to geometry in school, or lack of experiences other than those typical of level 0. On the other hand, the assessment of potential levels identified some students still at level 0 (3 of 16), more at level 1 (5 of 16), and the rest transitioning to level 2. The ninth grade subjects broke down similarly into three groups, with fewer students entering at level 0 (2 of 16), several entering between levels 0 and 1 (7 of 16) and the rest entering at level 1. Likewise, some students had potential levels at level 0, the "transitioning" group at level 1, as well as some showing progress toward level 2, and the rest at level 2.

Senk (1989) also conducted a study in conjunction with Usiskin's (1982, as cited in Fuys, 1985) study at the University of Chicago looking at VH levels as a possible predictor of success in proof-writing. She did this through a series of tests, looking for correlation amongst them. In the fall, 241 students enrolled in high school geometry were given the VH test (Usiskin's 1982 multiple-choice test) and an "entering geometry" achievement test; in the spring, they administered a proof test and standardized geometry test as well as a re-test for VH level. The proof test consisted of 6 items; two were short-answer items and four were proofs. The proofs were scored on a 4 point scale with a 4 being completely correct. A student was considered to have mastered proof if they scored a 3 or higher on at least three of the four proofs.

It was reported that the higher the fall VH level, the more likely a student would master proof in the spring (r=.5 in the fall vs. r=.6 in the spring). Also, the higher the spring VH level, the more likelihood of success and less likelihood of failure on the spring proof test (Table 2). Thus, there was definitely a positive correlation between VH levels (regardless of time of year) with proof score (r=.5 in the fall vs. r=.57 in the spring), as well as with standardized geometry tests (r=.57 in the fall vs. r=.63 in the spring). Analysis of variance was done with mean proof-writing achievement scores at the different van Hiele levels. Statistical significance ($p < 0.01$) was evident with students whose VH level was 2 or 3 (based on a 1-5 numbering scheme) scoring higher on the proof exam than students at level 0 or 1. Senk (1989) claimed that her results indicate the importance of a student's entry level and how it plays a big part in a student's success in high school geometry. She points out the implications of this finding are the need for a strong geometry curriculum prior to a formal high school course, with plenty of experiences in the lower levels of the VH model.

Table 2: Percent of Students Mastering and Failing Proof Writing vs. Fall/Spring VH Level (Senk, 1989, p. 316)

	Fall		**Spring**	
Van Hiele level	*% Mastering*	*% Failing*	*% Mastering*	*% Failing*
0	9	41	4	50
1	30	21	13	38
2	56	8	22	27
3	100	0	57	3
4	0	0	85	8
5	-	-	100	0

Mason (1997) conducted research with 120 academically gifted students in grades 6-8, prior to them having a formal course in geometry. All students completed a multiple-choice VH test (Usiskin, 1982, as cited in Fuys, 1985) and about half of them also participated in interviews based on the protocol used by Mayberry (1983). The results showed a statistically significant difference ($p < 0.0001$) between the VH levels exhibited by these talented students and Senk's (1989) high school study. This means that even though Mason's study involved a younger group of students, overall they exhibited higher VH levels than those in Senk's study that were beginning high school geometry. Mason's research also confirmed the claims that the levels form a hierarchy, are not dependent on a student's age, and may vary across geometry content objectives.

Identifying and Assessing Teachers' Levels of Reasoning

There has clearly been a lot of research done with students to investigate ways to measure VH levels and validate or refute aspects of van Hiele's theory. Another area of focus in research of the VH model has been with preservice and inservice teachers. Some studies have looked at teachers' levels of geometric reasoning. Additionally, some have sought to identify misconceptions held by preservice and inservice teachers.

Mayberry (1983) was one of the first to study teachers VH levels of geometric reasoning, where she interviewed 19 preservice elementary teachers. Questions were written at each level, covering 7 major topics prevalent in the elementary curriculum. The analysis of teachers' responses indicated a lack of readiness for a formal deductive geometry course (13 of them had already had high school geometry). Mayberry's other objective was to evaluate whether the questions posed to the participants formed a hierarchy that would correspond to the VH levels. This assertion was confirmed using the Guttman scalogram analysis and has been supported by other researchers (e.g., Fuys, Geddes, and Tischler, 1988; Mason, 1997; Clements and Battista, 1992).

Lawrie and Pegg (1997) raised some concerns with Mayberry's (1983) testing instrument and test results. They sought to conduct another study with 60 pre-service primary teachers, using Mayberry's interview protocol to create an equivalent written test, and field-tested it to ensure its reliability. Lawrie and Pegg made every effort to evaluate these tests in accordance with Mayberry in order to allow for a comparison between results. As with Mayberry's study, they found the majority of preservice teachers either at level 0 or level 1 (based on original 0-4 numbering scheme). It is significant to note that both samples tested had similar exposure to geometry in the past. The question of Mayberry's results came up when Lawrie and Pegg found 19 response pattern errors. These errors occurred when they found that some teachers were incorrectly assigned VH levels (by the researchers who coded their responses) and thus did not support the hierarchy associated with the VH levels. In investigating further, they also found that concepts tested using Mayberry's protocol as a guide were treated unequally across levels and the number of questions were not equitably distributed and balanced across levels. They claimed that fixing these errors with the testing instrument then resulted in only 6 pattern errors.

Mason and Schell (1988) also studied the levels of reasoning and misconceptions present in preservice elementary and secondary teachers as well as inservice secondary teachers. Subjects were questioned using Mayberry's (1983) interview protocol. The secondary inservice teachers tended to score highest of the three groups in terms of VH level (at the top two levels). The results of the secondary preservice teachers showed that over 75 % were at level 3 or higher (based on original 0-4 numbering scheme). At the same time, some misconceptions were noted with this group; for example, some did not indicate that parallel lines had to lie in the same plane. Others applied theorems of isosceles triangles to equilateral triangles; however, their previous definition of isosceles triangles stated that exactly two sides had to be congruent. For the preservice elementary teachers, 38% were operating below level 3 and 8% not even at level 0. Some of them also shared the incorrect definition of isosceles triangle. Another area of deficiency was distinguishing between sufficient and necessary conditions. Some indicated that rectangles were squares. Some also had trouble noting congruence of corresponding angles with similar figures.

In addition to identifying reasoning levels of students, research has shown interest in identifying reasoning level of teachers. In particular, some preservice teachers have had been identified as reasoning below the formal deductive level. Also, they exhibited some of the same misconceptions in geometry as noted by students. However, the purpose of assessing levels of reasoning is to identify where deficiencies exist, so that instruction can be catered to the needs of students (and teachers) to strengthen those problem areas.

Instructional Intervention with Students

Dina van Hiele-Geldof's (1984, as cited in Fuys, Geddes, & Tischler, 1988) contributions to the VH theory included five instructional phases that were designed for teachers to help bring students from one level to the next. These phases were information, guided orientation, explicitation, free orientation, and integration. The information phase involves the student becoming familiar with the topic of interest; this is possible through the introduction of examples and non-examples. The guided orientation phase is where students might perform teacher-specified tasks intended to notice possible relationships. The explicitation phase is where students exhibit awareness of such relationships and attempt to communicate about them. After students verbalize their understanding in their language, the teacher would make explicit the standard vocabulary and mathematical language used to describe that same context. The free orientation phase is where the students' task becomes more challenging as they investigate on their own and see if relationships would transfer to other cases (established properties of one shape being true of a new shape). The integration phase includes students' synthesis of established relationships across cases. It is expected that at the completion of this phase, the student will have now reached the next level of reasoning in the VH model.

Some researchers have tried to test out the VH theory by creating instructional units based on the VH levels and phases of instruction. Their goal was to see if these van Hiele-based units were effective in improving students' levels of reasoning.

Parsons, Stack and Breen (1998) did a study with 11 eighth-graders to see if computer-assisted instruction (CAI) could improve students' VH levels, specifically to level 2 (informal deduction, based on a 0-4 numbering scheme). The students took the VH geometry test (Usiskin, 1982, as cited in Fuys, 1985) before and after computer instruction. This sample began with about 18 % below 0, 45% at level 0, and 36% at level 1. A one-tailed t test showed a significant increase in VH levels ($p < 0.001$) from the pre-test to post-test. The students also took 2 other pre/post tests: one on standard geometry content and the other on vocabulary. A two-tailed t test showed no significant difference ($p < 0.05$) from pre-test to post- test for either of these tests. Parsons, Stack and Breen claimed that CAI did impact VH levels and helped students get to level 2. However, with such a limited sample size (n=11), the results of this study are extremely tenuous and unreliable.

Clements, Battista, and Sarama (2001) designed a research-based curriculum using the Logo programming language to look at how elementary students learn geometric concepts. Their project also sought to assess student learning in this microworld setting, and characterize how Logo might facilitate students' learning. A number of studies have already attempted to evaluate Logo as a technological tool, portraying a host of both good and bad results. "In sum, studies showing the most positive effects involve carefully planned sequences of Logo activities" (Clements et al., 2001, p. 9). Clements, Battista, and Sarama developed a curriculum, Logo Geometry (LG), with the theories of Piaget and van Hiele as the underlying models to inform curricular and assessment decisions.

The LG curriculum was composed of three major strands. The first strand was *paths* to introduce the idea of movement both with and without use of the computer. The second strand was *shapes* which built upon these paths to create figures with special names (i.e., rectangle, square, etc.). This strand also brought up the notion of angle and angle measure in constructing these shapes. Finally, the third strand was *motions* where students investigate transformations and the notions of congruence and symmetry.

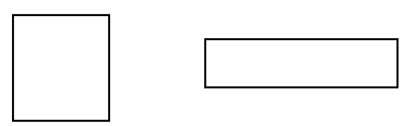
One group of students participated in the LG curriculum (experimental group) while another group did not (control group). Overall, on total achievement tests, the

"Logo geometry students scored significantly higher than control students, making double the gains of the control groups" (Clements et al., 2001, p. 90). Of particular note to the VH model, students in the LG classes showed higher gains in describing properties of shapes (level 2 thinking on a 1-5 numbering scheme) than students in the control group. This supports a premise of the LG curriculum that having students engage in construction of more complex paths (shapes) helps them to transition between level one thinking (visual) to level two thinking (analysis).

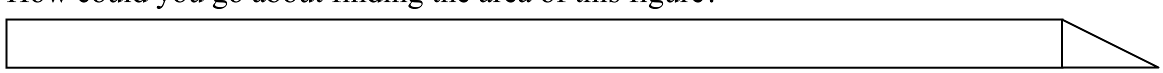
Another point of interest was that the experimental LG group outperformed the control group on angle tasks assessing their understanding of degrees of turn. Students completed a triad sorting task designed to assess VH level by asking them to look at three polygons and decide which two were most alike and why. The results of this task yielded gains in VH levels from pre- to post-test of 32%, 20%, and 13% in subsequent years. The motions strand showed evidence of some gains as well. LG students did better than those in the control group in identifying lines of symmetry for a given figure, and justifying why pairs of figures were congruent. Finally, the experimental group showed better understanding of slides, flips, and turns than the control group based on a number of motions sorting activities.

Mistretta (2000) did a field study testing a sample instructional unit that would help foster improving students' VH levels in 23 8th graders. Before the unit was taught, the students' VH level was assessed using a pre-test. The pre-test results indicated that 22% of the students were at level 0 (based on original 0-4 numbering scheme), 30% were at level 1, 13% were at level 2, and 35% could not be assigned (meaning they could not correctly answer 60% of the questions of a particular reasoning level). She also interviewed the subjects to gather more evidence and concluded that the students had difficulty in applying VH levels 1 and 2, as shown on the pre-test. Figure 1 highlights some of these sample questions asked in the students' interviews. They were confused about area concepts, commonly applying perimeter notions instead. They also had trouble with class inclusion and sufficient conditions for quadrilaterals. Based on these issues, she planned a unit that would encourage students to use higher-level thinking. Mistretta incorporated use of the VH teaching phases, stressed conceptual understanding (why) versus procedural understanding (how), and led student-centered discussions. After teaching the unit for a month, she tested the students' VH levels again. The post-test results showed that 4% of the students were at level 0, 26% were at level 1, and 70% of the students were now at level 2.

Identify these two geometric shapes.
How are they alike?
How are they different?
Are all squares rectangles? Why or why not?
Are all rectangles squares? Why or why not?



How could you go about finding the area of this figure?



Is there a relationship between the area of a parallelogram and the area of a triangle?

Figure 1. Sample Interview Questions assessing VH levels (Mistretta, 2000, p. 372)

Instructional Interventions with Teachers

Some research has been aimed at trying to prepare teachers to assess VH levels of their students and of various tasks in textbooks. Other research has focused on professional development courses to improve teachers' content knowledge of geometry and educate them on current learning theories, like the VH model. Researchers sought to find out whether the institutes improved the teachers VH levels and how knowledge of the VH model impacted their teaching decisions.

Fuys, Geddes, and Tischler (1988) interviewed 8 pre-service and 5 in-service teachers to see if the teachers could learn how to assign VH levels to characterize a student's thinking as well as in textbook materials. The teachers got a brief introduction of the VH model and then completed some of the same instructional tasks as the 6^{th} and 9^{th} grade subjects, as reported earlier in this chapter. Then, the teachers watched video clips of students completing those same tasks and were guided in matching student responses to the VH descriptors. They also examined some pages from a textbook and assigned VH levels to the explanations used as well as thinking required by students to successfully complete its exercises. Finally, they had a follow-up session where they repeated this assessment of videotapes and written materials. The results of the teachers assigning levels to the students' work on the videotapes indicated 87% agreement with the project staff; the results of the textbook evaluation showed 78% agreement on the exposition and 84% agreement on the exercises. Fuys, Geddes, and Tischler claimed that it was possible for teachers to learn how to assess VH levels with student responses as well as instructional material.

Swafford and Jones (1997) created a teacher institute for in-service teachers designed to improve content knowledge and awareness of research on student learning in geometry. The institute included 49 middle grades teachers and was conducted for 4 weeks. The researchers wanted to use the institute to investigate how their primary goals of helping classroom teachers with content and theory would ultimately impact student instruction. The teachers were given pre and post tests of Usiskin's (1982, as cited in Swafford and Jones, 1997) multiple-choice VH test to assess changes in the teachers' VH levels and an achievement test to assess improved content knowledge. In addition, follow-up observations and interviews were conducted with a sub-sample of the participants during the next school year to report the teachers' perceptions and changes in instruction. The results showed that this course was effective in improving the teachers' VH levels and understanding of content as well as impacting teachers to change their instruction based on the VH theory. The teachers came away feeling more confident with geometry, open to taking risks and encouraging the same in their students, wanting to take more time teaching a geometry unit, and convinced that students needed more opportunities with lower level experiences of the VH model. "This study demonstrated that an intervention program that enhances teachers' knowledge of geometry and their knowledge of research-based findings on student cognition in geometry can influence instructional practice" (Swafford & Jones, 1997, p. 480).

The research of Jacobson and Lehrer (2000) also suggests that teachers be exposed to current research on student reasoning. They conducted professional development with four elementary teachers on student understanding of arithmetic; two of the teachers also attended seminars addressing children's ideas of geometry and space. Class discussion of students' conversations and the teacher's questioning were videotaped from both sets of classes. Students also took an eight-item test on transformational geometry at the end of the unit and again a month later to assess retention of learning. They report, "in the two classes in which teachers were more knowledgeable about students' thinking about space and geometry, not only did students learn more than did their counterparts, but this difference in learning was maintained over time" (p. 86). This would certainly seem to give support to keeping teachers abreast of learning theories like the VH model, which have made a difference for classroom teachers and their students.

Sharp (2001) examined the effect of a professional development course offered to 11 K-7 inservice teachers through distance education. She wanted to give teachers the chance to experience a constructivist setting as learners because she assumed that many teachers had not had these kinds of experiences in their own schooling. The goal of her course was to help them improve their content knowledge of geometry and model what she considered to be an effective teaching practice. An additional element of the course included exposure to the van Hiele model of learning and helping teachers recognize the various levels in their own thinking as well as the thinking of their students.

Throughout the course, teachers participated in a variety of activities during class as well as on their own. In particular, the teachers had the chance to investigate geometry through the use of manipulatives. Teachers were responsible for reading assignments that were related to geometry content (various textbooks) and research on student learning in geometry. Class sessions consisted of discussions in which teachers recognized each others' level of VH reasoning and shared classroom experiences of analyzing their own students' reasoning. Finally, the teachers planned and carried out

VH-based lessons noting the level of reasoning demonstrated by their students. These lessons were followed up with additional questions for students that would be appropriate based on the previous level ascertained.

The results were self-reported by the participating teachers with the following themes emerging. All participants felt they had grown in their understanding of geometry. Seven of the eleven admitted they had better attitudes about geometry in general as well as increased enthusiasm for teaching it. All but one perceived students' learning of geometry differently based on knowledge of the VH model. Sharp (2001) attributed these changes to the course emphasis on teachers analyzing VH levels as well as the experiences which improved their content knowledge of geometry.

Summary

In summary, research on the van Hiele model can be divided into three main categories. First, the initial efforts were aimed toward determining if the VH levels are reasonable in describing geometric reasoning and through what means these levels could be assessed. Second, the success of these efforts made it possible to assess the VH levels of students in grades 6-12 as well as teachers in training and those currently in the field. Finally, researchers have sought to test the effectiveness of the VH model by using interventions with both students and teachers.

There has been support to indicate that the VH levels are an appropriate way to characterize geometric reasoning, although research has raised doubts of the discrete nature of these levels. Different forms of assessment have included a multiple choice test, an open-ended test, and clinical interviews. Assessment of students has indicated many are not prepared for success in high school geometry, and thus proof-writing, because of their lack of background with lower levels prior to high school. Assessment of teachers at the elementary level has shown evidence of them reasoning at low levels; this suggests that the gap in students' understanding could be partially due to the lack of confidence and content knowledge of teachers in the early grades. On a positive note, interventions using van Hiele-based instruction have proven effective in improving students' thinking levels. Interventions in professional development have also shown evidence that helping teachers with content knowledge and awareness of learning theories like the VH model has helped them and their students.

The next chapter will strive to compare and the contrast the van Hiele theory of learning with other prominent theories in education. A synthesis of benefits and drawbacks to each theory will be highlighted as well. The results of Chapters 2 and 3 will inform the implications suggested in Chapter 4.

CHAPTER 3

COMPARISON OF VAN HIELE WITH OTHER THEORIES

In order to fully appreciate the van Hiele model of levels of geometric thought, it is necessary to consider other models that have been posited and used in research. Two models of interest are a developmental theory by Jean Piaget and the SOLO (Structure of the Observed Learning Outcomes) Taxonomy created by John Biggs and Kevin Collis. These models have similarities as well as differences when compared to each other and the van Hiele model. Prior to van Hiele, Jean Piaget similarly set out to describe the nature of children's thinking in geometry. Two of Piaget's most famous books that outline his theory include *The Child's Conception of Space* (Piaget & Inhelder, 1967) and *The Child's Conception of Geometry* (Piaget, Inhelder, & Szeminska, 1960). Lehrer, Jenkins and Osana (1998) say "the pioneering efforts of Piaget and van Hiele remain the most extensive sources of information about school-age children's initial conceptions about space and corresponding trajectories of change" (p. 137). Since the van Hiele model, John Biggs and Kevin Collis proposed a structure called the SOLO Taxonomy to describe the responses of students qualitatively across a variety of subjects, not just mathematics. Biggs and Collis share this proposed organization in *Evaluating the Quality of Learning: the SOLO Taxonomy* (1982). To contrast, Bloom's Taxonomy (1956, as cited in Biggs & Collis, 1982) presents a hierarchy of questioning, whereas the SOLO Taxonomy presents criteria to evaluate the responses to such questions. The purpose of this chapter will be to briefly describe Piaget's theory and the SOLO

Taxonomy and draw comparisons and contrasts to the van Hiele theory. Ultimately though, the goal will be to describe how these three theories have and can impact the teaching of geometry.

Piaget

Jean Piaget was a genetic epistemologist whose goal was to describe the developmental nature of children's thinking in a variety of domains, one of which is space and geometry. The overarching organization of his theories about development in a variety of domains was structured using stages of cognitive development that were typically associated with certain ages. These stages of development are *sensorimotor* (infancy), *preoperational* (early childhood through preschool), *concrete operational* (childhood to adolescence) and *formal operational* (early adulthood) (as cited in Collis, Romberg, & Jurdak, 1986). Piaget assumed that these stages of cognitive growth were inevitable based on a person's mental structures developing and were not linked with or necessarily influenced by instruction (as cited in Lehrer et al., 1998).

There were two major themes in the work of Piaget. One major component of Piaget's theory was the topological primacy thesis (Clements & Battista, 1992). This describes a logical sequence in which children organize and construct ideas about geometry beginning with topological relations, projective relations, and then finally Euclidean relations. Piaget says "Topological relations are concerned only with proximities within figures or configurations; projective space involves a coordination of points of view from which objects are observed; Euclidean space involves the coordination of the objects themselves" (1960, p.405). For example, a child might transform a figure topologically through stretching or compressing; thus, all of its original properties do not remain invariant (a concave kite becomes a triangle). Projective relations involve the notion of things looking different based on the viewpoint of the observer (how a kite would look from the front vs. the side). If a child reasoned that two triangles were congruent because their angles and sides were congruent, their construction of ideas would be based on Euclidean relations. However, according to Clements and Battista (1992), this notion of Piaget's has not been completely supported by research. Rather, they claim that all three of these ideas develop together in an integrated way. Another major tenet of Piaget's theory is associated with constructivism; that is, a child's representation of space is developed through their own activity and interaction within the environment.

Piaget's theory has contributed to the field of education by giving description to children's thinking. In terms of teaching geometry, there has not been evidence overall to suggest that Piaget's theory has really been effective (Clements & Battista, 1992). Again, Piaget claims that students' development in their thinking is of a physiological nature that just happens over time as they grow older. Thus, it is not something that teaching can affect or improve. Piaget would say that a child's growth is already dictated and not reversible through planned instructional techniques or activities. Pegg and Davey (1998) concur with Clements and Battista (1992), questioning whether Piaget's work has really changed classroom instruction; they also share others' doubts about Piaget's ideas of topological primacy.

SOLO Taxonomy

Biggs and Collis' (1982) SOLO Taxonomy concentrates on student responses as opposed to levels of thinking or stages of development. In describing the taxonomy, they say that it is "the only instrument available for assessing quality retrospectively in an objective and systematic way that is also easily understandable by both teacher and student" (p. xi). They were interested in describing pre-existing criteria that could be used for qualitative evaluation of students' responses in any subject area, not just mathematics.

Similar to Piaget's work, Biggs and Collis proposed five modes of functioning (also linked to certain age ranges) where learning takes place: sensorimotor, ikonic, concrete symbolic, formal, and postformal. Then they suggest five levels of response within a particular mode: prestructural, unistructural, multistructural, relational, and extended abstract. A prestructural response indicates a student is actually operating at the preceding mode. Unistructural, multistructural and relational responses indicate the expected mode. Extended abstract responses indicate a student operating in the succeeding mode. This general model is also summarized in Table 3. So essentially, the goal of the taxonomy is to take observable behaviors and classify them by mode of thinking and level of response.

Biggs and Collis (1982) outline the response levels in the following way. A student giving a prestructural response would either not respond or give information irrelevant or unrelated to the question. A unistructural response would indicate a student gave one piece of information relevant to the question; a multistructural response would elicit several pieces of information relevant to the question. A relational response would indicate relationships among these pieces of information and fit them together as a coherent whole. Finally, an extended abstract response is evident when a student is "able to derive a general principle from the integrated data that can be applied to new situations" (Olive, 1991, p.92).

Table 3: Modes and Levels in the SOLO Taxonomy (Biggs & Collis, 1991, p. 65)

MODE		Structural Level (SOLO)
Next	5	*Extended abstract*: The learner now generalizes the structure to take in new and more abstract features, representing a new and higher mode of operation.
Target	4	*Relational:* The learner now integrates the parts with each other, so that the whole has a coherent structure and meaning.
	3	*Multistructural:* The learner picks up more and more relevant or correct features but does not integrate them.
	2	*Unistructural:* The learner focuses on the relevant domain, and picks up one aspect work with.
Previous	1	*Prestructural:* The task is engaged, but the learner is distracted or misled by an irrelevant aspect belonging to a previous stage or mode

Biggs and Collis (1982) intended for the SOLO Taxonomy to be of particular interest to the educator. They hoped for teachers to use it in evaluating their instructional decisions. Such decisions might include teacher intentions, remediation, analysis of curriculum, and instructional tasks. The other obvious way this structure can benefit education is as a tool to evaluate students' answers in other ways besides the traditional quantitative method (number correct).

Comparison of the van Hiele Model to Piaget's Theory

An area of interest for researchers has been trying to compare and contrast these three theories. One of the more obvious differences between van Hiele and Piaget's theories is that one describes *levels* of thinking and the other describes *stages* of development. The distinction between stage and level has been clarified by Glasersfeld and Kelley (1982). They classify a stage such that it "designates a stretch of time that is characterized by a qualitative change that differentiates it from adjacent periods and constitutes one step in a progression" (p. 157). For instance, Piaget asserts that infancy is the time frame when children operate in the *sensorimotor* stage, characterized by children's first interaction with the physical environment (developing motor skills). On the other hand, they claim that a level is not defined in terms of time; instead level "implies a specific degree or height of some measurable feature or performance" (p.157). For example, van Hiele's levels are described according to the ways a student is reasoning about a figure. A particular level is possible at any age and can change at any time, indicated by a student reasoning from a different source (according to how a figure looks to what properties it has).

Clements and Battista (1992) drew some comparisons between the ideas of van Hiele and Piaget. They claim that both promote students' ownership in building understanding; also, "both theorists believe that a critical instructional dilemma is teaching about objects that are not yet objects of reflection for students" (p. 437). This goes back to the idea of level mismatch; students will struggle to make sense of ideas being taught at higher levels/stages than they have reached. Moreover, they suggest that in both cases the theorists support the conflicts that arise for students in trying to think at opposing levels and are an essential and healthy part of the learning process. Additionally, they suggest that van Hiele and Piaget do not support the notion that clear explanations are what define good teaching. Finally, Clements and Battista say that both theorists avoid two perspectives on teaching. One perspective is to perceive a lower level as inferior once a higher level is achieved; the other is attempting to force students quickly through succeeding levels once a students' current level has been established. In other words, neither Piaget nor van Hiele view their theories as an avenue to speed up development.

Some distinctions between the two theories also seem apparent. Pandiscio and Orton (1998) claim one difference is the two theorists' stance on students' movement among levels or stages. They say that Piaget would suggest this is dependent on activity, whereas van Hiele would suggest it is dependent on language. Another important distinction that Pandiscio and Orton make is that the purposes of the two theories are quite different. They suggest van Hiele was trying to help teachers to improve instruction by describing levels of thinking for students. Alternatively, Piaget was interested more simply in describing the progression of thinking and when it could be expected to occur. In other words, the van Hiele model is a theory that can inform instruction and Piaget's model is a theory of development.

Clements and Battista (1992) also make a distinction in the way Piaget and van Hiele believe students develop in their thinking about reasoning and proof. They indicate that van Hiele would say this growth is dependent on increasing understanding of geometric knowledge and relationships. They say, "according to Piaget, however, certain logical operations develop in students independent of the content to which they are applied" (p. 441). This suggests that van Hiele would say a student was ready to prove something if his understanding of the content is at an appropriate level (formal deduction). Thus, instruction is a controllable factor that can prepare them for this kind of reasoning. Yet, Piaget would argue that understanding content is unrelated to a child's readiness for formal argumentation. These kinds of opposing claims in Piaget and van Hiele's model have encouraged some to conduct more recent research.

Lehrer, Jenkins, and Osana (1998) conducted a longitudinal study on how children reason about space and geometry. In the course of the study, they brought into question the theories of Piaget and van Hiele. One task gave students a triad of figures (two and three-dimensional) and asked them to choose which two were most alike. The researchers intended for this to elicit multiple levels of VH reasoning for students. The results indicate that the majority of children based their choices on the appearance of a figure. Yet, it was clear that the difference in what they were paying attention to and describing varied a great deal. They claim that "these distinctions appear to defy description by a single, visual level of development" (Lehrer et al., 1998, p. 142). They did note instances as well where children classified two figures in the triad as "alike" according to attributes (indicating the descriptive level of reasoning). And in fact, the classic response noted by students included multiple nonconsecutive levels of reasoning in their justification. Due to this result, they claim that van Hiele's assumption of discrete levels is questionable. This is consistent with the result of Clements, Battista, and Sarama (2001) who also found evidence that students' thinking is not neatly described in discrete levels. Lehrer, Jenkins, and Osana also wanted to test the predictive capability of the VH levels over time. For instance, van Hiele would assert that over time students would think less visually and more descriptively. In assessing this piece of his model, they were unable to confirm van Hiele's hypothesis that this general pattern of growth exists. In conclusion, Lehrer, Jenkins and Osana assert that the detail in which children think about space is greater than what van Hiele explains is his theory.

With respect to Piaget's theory, Lehrer, Jenkins and Osana (1998) said "children's thinking was much more sensitive to context than to grade; age-related trends suggested that, if anything, experiences in school and in the world did little to change children's conceptions of shape…" (p. 145). This would seem to go against Piaget's theory because he suggests development is tied to a child's age. On the other hand, the results of this study did give support to Piaget's topological primacy theory. They based this on the prevalence of students "morphing"-strategies of pushing and pulling a figure's sides or corners to make other figures which they suggest is a topological type of transformation.

Comparison of van Hiele Model to SOLO Taxonomy

There exist similarities between the SOLO Taxonomy and the Van Hiele model of learning. Obviously, both models give teachers a way to assess students' reasoning. Additionally, the goal of both models was to empower teachers with a resource to guide assessment as well as instruction. As mentioned previously, this was not an objective that Piaget was interested in with his theory of development. Also, in both models the extremes of the levels indicate similar responses. Van Hiele's formal deduction is evident at the SOLO higher end as opposed to very limited reasoning at the lower end.

Jurdak (1989) suggests a comparison of the theories of Biggs and Collis and van Hiele. He claims a distinction is "classifying learning outcomes by looking at their structure rather than classifying individuals by looking at indicators of some cognitive abilities" (p.156). He used the "Identifying and Defining Task" from Burger and Shaughnessy's (1986) study to try and make a correspondence between the two models. He claims that if a correspondence exists, it would alleviate the need associated with the VH model to devise indicators for different tasks. Jurdak felt that he was reasonably able to match up the two sets with the exception of two levels. These exceptions were the SOLO Taxonomy's prestructural level (indicates no response or an irrelevant answer) and the VH model's level of rigor (comparing axiomatic systems). The existence of this highest VH level has also been questioned or noted difficult to assess in the research of the VH model (Usiskin, 1982, as cited in Wilson, 1990).

On the other hand, one of the distinct differences lies in the fact that the SOLO Taxonomy is not subject specific; yet, van Hiele's motivation to devise a theory grew out of his own frustration in attempting to teach geometry. The SOLO Taxonomy makes provisions for classifying students in transition. Biggs and Collis even assigned numbers to these transitional levels: 1A (prestructural to unistructural), 2A (unistructural to multistructural), 3A (multistructural to relational), and 4A (relational to extended abstract). On the other hand, van Hiele did not assume that these "in-between" levels even existed. In fact, this has been a great source of controversy of Van Hiele's theory.

Van Hiele claims the levels are of a discrete nature and that there is a "jump" from one level to the next. Yet, Burger and Shaughnessy (1986) contend that the levels seem more continuous because they found it difficult in assessment to choose between levels and often identified students oscillating between levels, even on the same question. They suggested this kind of behavior made sense coming from students transitioning between levels, particularly between levels 1 and 2; however, a larger sample size of college-level students were suggested to see transitioning students between levels 2 and 3. This claim has also been supported by other researchers as well (e.g., Usiskin, 1982 as cited in Fuys, 1985; Fuys, Geddes, and Tischler, 1988; Gutierrez, Jaime, and Fortuny, 1991). Since van Hiele's assumption of discrete levels has been such a controversy and not been evidenced by the research, it would seem reasonable to suggest an alternative to this part of his theory.

Clements, Battista, and Sarama (2001) took the strengths of both Piaget's theory and the VH model and used a synthesis of the two as a basis for their research on the Logo Geometry project. They propose a different approach that does not assume students think only one way at a time, but instead can be reasoning at multiple levels at the same time.

Advantages and Disadvantages

To summarize, there are points of strength and weakness to all of the theories discussed in this chapter. Piaget's idea about students constructing their own knowledge and making sense of their environment through active manipulation continues to be stressed as essential in mathematics learning. Also, Pandiscio and Orton (1998) say Piaget's developmental theory is more general than van Hiele's and can be applied more extensively. Furthermore, they say Piaget's ideas can give us guidance in understanding why a child may have difficulty understanding a geometric concept. The corresponding drawback is that his theory would not offer help in alleviating the problem whereas van Hiele's model would. Thus, one of Piaget's major weaknesses has been the lack of impact felt in the classroom.

The SOLO Taxonomy has strength in its generality as well. It is not specific to geometry or even mathematics so it does not require revamping for different content objectives (as VH level indicators would). By the same token, it does not appear to be anything more than a qualitative rubric for evaluating student responses. Although this is valuable for educators, it does not give implications for classroom practice (like Piaget's theory). Classifying a student by response level and mode of functioning gives more precision in assessing reasoning. The SOLO Taxonomy does allow for the possibility of evaluating students who might exhibit transitional thinking unlike van Hiele's original model.

In evaluating the van Hiele model as a theory, it has strengths that set it apart from the SOLO Taxonomy and Piaget's theory. First, it is much easier to understand from the perspective of a classroom teacher. I would attribute some of this to the fact that the van Hieles were both teachers and their theory grew out of this context; thus, teachers can relate to and identify with its tenets. A second strength of the VH model is that it was written specifically for geometry as opposed to the SOLO Taxonomy which applies across a variety of subject areas and Piaget's theory which applies to several areas of mathematics. Finally, from the perspective of a classroom teacher, van Hiele's theory offers the most hope of meeting the challenge of students' varying levels of reasoning within one geometry class. Van Hiele's greatest contribution with his theory is that differences in reasoning level are under the teacher's control and can be facilitated with appropriate instruction.

On the other hand, there are some drawbacks associated with the VH model. First and most importantly, many would agree that its greatest disadvantage is the lack of substantiation for the levels being discrete. As mentioned already, research has not upheld this notion and has actually made a more plausible case for discontinuity in the levels existing. This has been emphasized by the difficulty of assigning a level to students who do not seem to fit a particular level or are in between levels. Secondly, the research has given reason to suggest that the levels differ across content objectives which makes the assessment of levels that much more cumbersome. One of the issues of difficulty with the VH model has been finding a way to assess the levels appropriately. This consequence suggests that assessment of levels would need to be done with each different topic. Finally, there has not been support to suggest that textbooks are aligning with the VH model, as will be further discussed in Chapter 4. Therefore, if a teacher wanted to use this model as a framework for building units of instruction, they would have to figure out how to do this on their own. This would require much more time for planning and make the textbook somewhat useless.

The implications of these theories for teaching geometry will be discussed in the final chapter. In particular, results of curriculum evaluations based on the ideas of van Hiele will be shared. I will also use a synthesis of all the literature and research done with the van Hiele model to provide suggestions for teacher education and curriculum and instruction in geometry in K-12.

CHAPTER 4

VAN HIELE MODEL INFLUENCES ON PRACTICE

There have been efforts to offer recommendations and suggestions for teaching due to the research that has validated the existence of levels proposed by van Hiele. Some of these recommendations apply to developing appropriate curricula and textbook materials in association with the VH levels. In addition, there are opportunities to better prepare preservice teachers for the challenges of teaching geometry in light of the VH model. Others have suggested ideas for the classroom that inservice teachers can implement to meet the challenge of a diverse group of thinkers. Although anecdotal in nature, I will also share ideas I have tried that were successful in improving the teaching of geometry in my own classroom. This chapter concludes with additional implications for teaching and teacher education based on the research done throughout this literature review.

Curriculum

Fuys, Geddes, and Tischler (1988) analyzed the student text and teacher's edition of three textbook series (K-8) to see how they compared to the sequencing of the VH levels. Van Hiele (as cited in Fuys et al., 1988) himself said "the Brooklyn College Project investigation made clear that in the geometry materials in grades K-8 textbooks the van Hiele levels are mixed up--not sequenced--and because of this the higher levels are rarely reached" (p. iii). The general pattern noted in the study was the minimal attention to level two (on a 0-4 numbering scheme) and that which was present did not show up until grades 7-8. Lessons and tasks that could promote level one thinking were present in all three textbook series beginning at grade 3; however, Fuys, Geddes, and Tischler (1988) claimed that students could manage to be successful with the tasks in the textbook with only level zero thinking up until grade 8. This is seemingly a problem since research (Senk, 1989) has advocated that students encounter the most success with geometry in high school if they are entering with reasoning at least at level two.

Fuys, Geddes, and Tischler (1988) also made some recommendations for textbook authors and curriculum developers to align their materials more closely with the VH model. First, they suggest including notes in the teacher's edition that indicate the VH levels correlated with the text and ideas teachers could use to encourage students to reason at higher VH levels where appropriate. This might also be a suitable place to include strategies for incorporating the VH teaching phases, as described by Dina van Hiele-Geldof (as cited in Fuys et al., 1988). Secondly, they suggested that more of the tasks, lessons, and tests in the textbooks require and promote level one thinking since this seemed to be lacking. These include tasks which support students looking for characteristics and classes of shapes. Also, they propose asking questions to help students work on their communication in geometry by incorporating vocabulary in the exercises and formulating explanations that require mathematical language. One possible barrier mentioned was the strict requirements of reading levels for textbook authors. To deal with this issue, it was suggested that attention be given to using grade-appropriate reading level but not at the expense of reducing the level of thinking. Fuys, Geddes, and

Tischler (1988) recommend that teachers not depend on the text and supplement the level gaps present by using probing activities, providing additional examples, and using the textbook more as follow-up. Finally, they recommend future research should take a look at some of the curricula developed by the Soviets (Pyshkalo, 1981, as cited in Fuys et al., 1988) and the kinds of successes Soviets are experiencing from revising their study of geometry according to the VH model.

A group of researchers (Whitman et al., 1997) compared American and Japanese curricula relative to the VH model. Table 4 gives a comparison of Japan and the US by grade level with the percentage of lessons at various levels. The difference in the two countries is glaring. Japanese curricula are much more appropriately aligned to support the recommendations that students have level one and two experiences in the elementary and middle grades. Another distinction noted between the two series was that US texts tended to oscillate between levels zero and one from grade to grade and within a grade level. Alternatively, levels were developed in order in each section of the Japanese textbook.

Another related question this group of researchers (Whitman et al., 1997) sought to answer was how Japanese students performed on VH tests compared to the American students in grades 4, 7, 9, and 11. The tests were divided up to assess each of the levels from 0-3 (based on a 0-4 numbering scheme); the fourth grade test did not include questions to assess level 3. They reported that the results indicate a 2-year gap between the two groups (favoring the Japanese); presumably, this could partially be attributed to the difference in VH levels across the two curriculums.

Table 4: Percentage of Lessons at Maximum Level 0,1,2 (according to a 0-4 numbering scheme) in US Texts Compared to Japanese Texts (Whitman et al., 1997, p. 221)

	Japan			**US**		
Grade	Level 0	Level 1	Level 2	Level 0	Level 1	Level 2
1	100	0	0	100	0	0
2	5	95	9	100	0	0
3	0	80	20	93	7	0
4	0	82	18	83	17	0
5	0	47	53	76	13	0
6	0	92	8	70	29	1

In summary, there is concern over geometry curriculum and textbook development aligning with the VH levels to support student's progression of geometric reasoning. Teacher's editions and supplementary materials need to include recommendations and guidelines that help teachers to provide instruction that meets students at the lower levels and brings them to the higher levels. The Soviet and Japanese curriculum are examples of curricula that have already moved in the direction of corresponding to the VH levels in a way that helps students develop their reasoning more appropriately. Awareness of these concerns with geometry curriculum need to be included in preservice teacher programs to facilitate their transition to the classroom and meet the growing needs of students to improve their reasoning ability. "The validity of the van Hiele theory not only has strong implications for curriculum development, but it should greatly influence how prospective teachers are trained in the teaching of school geometry" (Whitman et al., 1997, p. 217).

Suggestions for Teacher Education

Bischoff, Hatch, and Watford (1999) advocate an emphasis in three areas to help better prepare preservice teachers: 1) strong content knowledge, 2) awareness of learning theories, and 3) effective teaching strategies. They assert that it is critically important for teachers to "own" the material they are trying to teach in the classroom. This is not to suggest that teachers must be "experts," but that lack of confidence greatly inhibits one's ability to provide effective instruction. Also, they claimed, based on their research of preservice teachers' planning, a teacher's definition of a successful lesson plan was not related to student understanding, but rather covering all the objectives. Bischoff, Hatch, and Watford advocate the practice of reflection and suggest that non-reflection could hinder a teacher's opportunity for growth in pedagogical understanding. They believe teacher education programs should require preservice teachers to engage in self-reflection and teacher educators should give lots of guidance and input to help them improve their reflections. Finally, they advocated that preservice teacher education should include strategies for teaching and learning according to specific areas of content.

Bush (1986) attempted to do descriptive research with five preservice teachers enrolled in mathematics methods' courses. The teachers were involved in various teaching assignments during their program of study. Bush interviewed them before a teaching episode to determine what sources were underlying their decisions in creating lessons; then he followed up after the lesson to keep track of what sources prompted their decisions during the lesson. The primary source for justifying instructional decisions noted by the preservice teachers was material learned in their methods courses. Other sources for their instructional decisions were textbooks, previous teachers in school, and their cooperating teacher. Bush was concerned with the amount of "authority" the preservice teachers placed on the textbook. He suggested that there is a strong need for preservice programs to prepare teachers in effective ways of utilizing textbooks to guide their lesson plans. Additionally, Bush felt that preservice teachers should be aware of the possible limitations inherent with using a textbook.

Graeber (1999) tried to identify the "big ideas" of mathematics education programs that should be emphasized for mathematics teacher education programs. Concerning pedagogy, she said preservice teachers need to know what makes certain topics difficult for students and why. This would include being aware of preconceived notions students might have as well as misconceptions. She mentioned that theories like the van Hiele model would shed light on this. She also noted specifically in geometry the importance of continually assessing student understanding on concepts. She claims "if preservice teachers enter the classroom without valuing student understanding, they are not apt to assess understanding or use knowledge of students' current understanding to make instructional decisions" (p. 193). Graeber suggested that one strategy is to have preservice teachers read articles that highlight documented misconceptions and then have opportunities to tutor or interview students to focus on students' possible misconceptions.

Another big idea Graeber (1999) mentioned was sharing instructional strategies that have shown to be effective and promote retention of learning. One of these is how students gain understanding through discussion and communication of their ideas. The preservice teachers need these same kinds of experiences as learners of mathematics to appreciate the benefit of such contexts for their students.

Fuys, Geddes, and Tischler (1988) conducted interviews with preservice and inservice teachers to see if they were capable of assigning students a VH level. Implications for teacher education and professional development were shared by the subjects of the study. The teachers felt that their previous learning experiences could be characterized by rote memorization. However, they said their experience in learning about VH levels and characterizing thinking associated with the levels was valuable and should be available for all teachers. The teachers really appreciated the activities that promoted higher levels of thinking and were hopeful about implementing the same techniques and manipulative materials in their own classroom.

Fuys, Geddes and Tischler (1988) also shared their own ideas about teacher education based on their study with the sixth and ninth grade students. They noted that the sixth graders had not experienced much geometry in school. They felt teachers had to be responsible for teaching geometry in the curriculum and that accountability should be monitored by principals, mathematics supervisors, and district curriculum specialists. Fuys, Geddes, and Tischler (1988) gave some specific guidelines for teachers at specific grade ranges. They encouraged teachers of grades K-4 to give students opportunities to explore properties and use more sophisticated language in communicating ideas.

Typically, grades K-4 stress activities associated with level zero (on a 0-4 numbering scheme); instead, they should be introducing and supporting level one tasks. Likewise, teachers in grades 5-9 should be incorporating some level one activities, but moving toward mostly level two activities. This would imply teachers asking probing questions that ask for explanation and evaluation of other students' responses. They suggest these ideas could be implemented through thoughtful study and discussion of the van Hiele model in preservice and inservice teacher programs.

In general, several themes for teacher education seem apparent that should be emphasized in preservice teacher programs. Preservice teachers need a strong understanding of content knowledge in geometry, awareness of students' misconceptions and learning theories that explain the source of these misconceptions, and teaching strategies to handle these misconceptions. Some specific examples included careful use of textbooks for lesson plans, assessment of understanding from class discussion, and activities that engage and support the lower levels of the VH model.

Suggestions for Teaching

Some researchers and teachers have made suggestions for teaching based on their research or knowledge of the VH levels. A common theme that emanated was using the VH levels as a basis to frame lessons and assessment of students' reasoning.

Recall the study by Burger and Shaughnessy (1986), in which they identified some recurring themes and misconceptions about geometry in student interviews. They noted that students' meaning of terminology covers a wide spectrum of possibilities that teachers might not anticipate. For instance, some students might define a triangle to include anything with three sides (including curvature); on the other hand, some might only characterize something as a triangle if all three of its sides are congruent. Another misconception evident from their interviews was in classifying properties of figures. Students did not have a natural intuition to limit these properties, but instead perceived all of them as necessary. This would also impact the precision of students' language in constructing definitions.

Shaughnessy and Burger (1985) have ideas about the implications of these themes for the classroom. First, they suggested high school students study geometry informally in the first semester (and a great number may need even longer than this). They should still be building arguments and justifying their reasoning in this portion of the course, but should be less formal. Secondly, they propose teachers write lesson plans or units in such a way to purposely support students' progress through the VH levels. Finally, they note that students need to see geometry sooner than high school and in much greater detail than is usually given at the elementary and middle grades. Even if students are getting these experiences early on, they still insist on the importance of a half-year of informal geometry in high school. They cite the Soviet Union's example in this area where they have rewritten curriculum in grades 1-3 to correspond to VH level one. Additionally, Soviet students continue to see geometry for the next seven years (up until where we in the US typically teach the secondary course).

Hoffer (1981) suggests a more focused effort in teaching five basic skills in geometry: visual, verbal, drawing, logical and applied. He says visual skills are important in building stronger spatial sense and these need time to develop through a variety of activities. He claims verbal skills are very important because geometry is the course where language is emphasized more than any other mathematics course. Students must learn definitions, recall properties, and apply postulates and theorems. They tend to not be ready for this because of their inexact language; nonetheless, it is necessary that they be able to work through these difficulties. Hoffer says drawing is a very practical skill that extends even into our adult life. There is a need for drawing to make sense of a lot of the axioms and theorems, so geometry is a very appropriate place in the curriculum to develop this skill. He expressed the importance of logical skills being developed as opposed to memorization of proofs. This can be done by evaluating another student's argument or deciding if a figure has enough information to reach a desired conclusion. An example of a logical task designed to correspond to each of the VH levels can be found in Figure 2. Lastly, he stressed the development of applied skills that connect geometry to other areas of study and the world.

Overall, Hoffer (1981) opposed the notion that so much emphasis in geometry is on proof and that "students get through it" by memorizing and miss the importance of learning how to reason and why. Instead, teachers should encourage students to justify their ideas informally early on to lay the groundwork for a more formal writing of proofs later on.

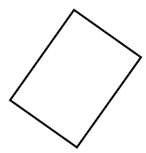
Van Hiele Level	LOGICAL SKILL
I Recognition	If a rectangle is turned as shown, is the new figure also a rectangle? 
II Analysis	Is the area of rectangle determined by its perimeter? If two rectangles have equal perimeters, are their areas equal also?
III Ordering	Which are true and which are false? 1. Each rectangle is a square. 2. Each square is a rectangle. 3. If the diagonals of a parallelogram are congruent, the figure is a rectangle.
IV Deduction	Prove or disprove: If the diagonals of a quadrilateral are congruent, the figure is a rectangle.
V Rigor	Since rectangles do not exist in non-Euclidean geometry, how are the areas of figures determined?

Figure 2. Sample Problems for Logical Skill by VH Level (Hoffer, 1981, p. 16)

Recall Mason's (1997) study to assess readiness for geometry in academically gifted students (discussed in Chapter 2). Based on the results of this study, Mason and Delano (1997) recommended that a test assessing VH levels (as opposed to progress in algebra) could better identify those students prepared to take a formal course in geometry. Mason (1997) also recommended that these younger students gain experiences initially in geometry that are less formal. This is possible through the early use of manipulatives and dynamic geometry software, instead of rigorous proof-writing.

Others have also used the VH levels to guide assessment. Carroll (1998) suggested the use of three different open-ended activities for middle school and scoring rubrics that are aligned with the VH levels. He claimed these tasks are good because they promote higher thinking levels, can tell a teacher more than the typical short answer response, may help in identifying students' misconceptions, and elicit various means of addressing these issues. For example, one task asked students whether they agree with another student, "Sheila", who says she can draw a triangle with two right angles. An example of responses to the triangle task and scoring rubric can be found in Table 5. Another task involved estimating the measure of an angle.

Table 5: Rubric and Sample Responses to Sheila's Triangle (Carroll, 1998, p. 400)

Level	DESCRIPTION AND SAMPLE OF STUDENTS' RESPONSES
0	No response or off task; geometric language is not used: "I disagree with Sheila because you can't do it."
1	Incorrect response, but some reasoning is attempted: "Yes, because all triangle have a right angle and a left angle." "Yes, you can make one at the top and one at the bottom." Partially correct response, but reasoning is weak: "No, because all triangles have right angles.
2	Correct response, but reasoning is not complete: "No, because you can only put 1 right angle in a triangle."
3	Correct response and good reasoning. Explanation goes beyond level 2 but relies on concrete and visual understanding rather than on abstract knowledge of properties: "Because if you put 2 right angles together, you already have 3 sides, and the sides are not closed." "No, because if you draw 2 right angles and try to connect them, you get a square or a rectangle. Two right angles is already 3 sides."
4	Exemplary response. Student used knowledge of triangles and angles: "Triangles have 3 angles and 180^0. If there are 2 right angles, then it would equal 180^0. But that is only 2 angles." "How could you possibly have 2 right angles equaling 180^0 when you only have $\frac{2}{3}$ of a triangle done?" "You would have 2 parallel sides."

Mistretta (2000) experienced some success in helping eighth graders progress in their thinking levels by using a van Hiele-based instructional unit. Some of the activities included were: 1) illustrating parallel and perpendicular lines with concept cards, 2) using graph paper or other strategies to find irregular-shaped areas, and 3) creating three-dimensional figures and analyzing their attributes. Her recommendations were for teachers to include more level 0 and level 1 (based on original 0-4 numbering scheme) experiences for middle school students. She said, "Since the geometry taught in high school requires thinking level 2, elementary and junior high schools need to prepare students to function at this level by providing a strong foundation in levels 0 and 1. Lessons should build upon each other, rather than merely offer isolated facts" (Mistretta, 2000, p. 380).

Jaime and Gutierrez (1995) suggest specific activities for teaching a unit on isometries for grades 3-12. Activities are divided up according to the VH levels and teaching phases. The teaching objectives they outlined for each of the VH levels are given in Table 6. They recommend that teachers should be able to recognize these different levels of student thinking. Also, they feel an important key to helping improve their understanding is building on students' previous knowledge. For instance, one of the last activities requires writing a formal proof (level three); the teachers allowed the students to give less formal explanations and then used these to lead students through the formal technique of proof-writing.

Table 6: Characteristics and Teaching Objectives of the van Hiele Levels in Isometries (Jaime & Gutierrez, 1995, p. 593)

VH Level	Characteristics of Thinking	Teaching Objectives
0	Students have a global nonmathematical view of isometries	-To recognize each isometry as a movement and to perform isometries using appropriate manipulatives -To recognize the invariance of shape and size under an isometry
1	The elements and properties that define each isometry are considered by students.	-To explicitly use the elements that characterize each isometry -To begin to use the definition of each isometry -To discover new properties of isometries from experimentation, to state and use them -To make products of isometries by moving cutouts -To discover that compositions of translations commute, as do compositions of rotations around the same center -To begin to use mathematical notation and vocabulary for the different isometries
2	Students can discover and use properties of, and relations among, isometries; can understand general mathematical reasoning; and can make informal arguments as proofs.	-To understand and use the intersection of perpendicular bisectors to determine the center of a rotation -To complete the analysis of the product of two isometries started in level 1 -To understand and use the infinite number of ways of decomposing rotations or translations into a product of two symmetries -To justify and use properties of isometries -To understand the formal definition of each isometry -To understand simple formal proofs that are shown and explained by the teacher
3	Students can reason without any concrete support.	-To complete the knowledge in previous levels by deducing and proving more complex properties of isometries -To acquire a global view of the set of plane isometries and its algebraic group structure -To do formal proofs, in particular the theorem of classification of isometries: *Every isometry is one of these four: a translation, rotation, symmetry, or glide reflection.*

Van Hiele-based tasks and units have proven successful in helping students grow in their geometric reasoning. These kinds of activities should be open-ended and build on students' prior knowledge. In general, geometry should begin less formal with decreased emphasis on proof initially, and encourage students to create informal arguments. Previous research (e.g., Senk, 1989; Usiskin, 1982 as cited in Fuys, 1985; Mason, 1997; Fuys et al, 1988) support these ideas that lower VH level experiences are lacking and would improve students' success in the more formal high school geometry course.

Recommendations

Spending the last nine months critically analyzing research with the van Hiele theory and reflecting on my own experiences as a preservice teacher, classroom teacher, and graduate student has allowed me to conceptualize my own beliefs about suggestions for these three experience levels in regards to understanding and utilizing the van Hiele theory effectively. Teacher education programs can better prepare preservice teachers to meet the needs of developing students' geometric reasoning. I have formulated new perspectives, as well as practical ideas for teaching geometry, which I feel could benefit many current teachers of geometry and mathematics. Some of these align closely with suggestions already offered in the research of this paper. Finally, I offer suggestions for how graduate programs could better prepare their students to become "teacher leaders" in the field.

Teacher Education

I believe one of the biggest obstacles faced in education today is breaking away from the "old" way of teaching that the majority of teachers experienced themselves. It seems that mathematics teachers have perpetuated a cycle of "teaching like we were taught" that probably contributes to the increased gap in student achievement. This relates back to teachers encouraging rote memorization or "reduction of level" when the teacher's level of reasoning is higher than the student (Clements and Battista, 1992). As a beginning teacher, I taught geometry in a very traditional way for two years, similar to Mrs. Trump, and could not understand why students struggled with learning geometry despite my "best" efforts. I often resorted to blaming my failure on students being forced into a high school geometry course that weren't ready or capable of learning what I was teaching. My own sense-making of the VH levels helped me understand that my own students' difficulties were based on some of the barriers that van Hiele proposes as opposed to students' lack of intelligence or my "unclear" explanation of the subject matter. These barriers are real and until we decide to give merit to understanding how and why students think the way they do, I believe we will continue to be frustrated in our efforts to educate them.

I think a solution is that teachers must reverse their typical attitude and approach of "it worked for me." This reversal must begin at least in preservice education. Bush (1986) noted that preservice teachers' strongest source of making instructional decisions came from their methods courses. Teacher education courses should not be mere discussions of theoretical, effective pedagogy, but should model this type of effective pedagogy. Preservice teachers need to engage as learners in these kinds of settings that may be radically different than what they might have experienced thus far in school. I envision these settings should look like a community of learners in a classroom that is rich in discourse where mathematics is something that has meaning because of personal investment and responsibility for learning. A major change in attitude about how effective teaching "looks" will only happen if we can get preservice teachers to understand and adopt new perspectives.

Practically speaking, I think the typical methods courses required for preservice teachers should begin much sooner. There is so much valuable input that teachers need about how to teach specific content areas of mathematics and an obstacle is the lack of time allotted for this in the undergraduate program of study. For instance, I propose for courses to be created for each of the five content standards suggested by NCTM's (2000) *Principles and Standards of School Mathematics* (PSSM). These include number and operation, algebra, measurement, geometry, and data analysis and probability. The objectives within each course would be threefold.

First of all, these methods courses should help teachers strengthen their personal content knowledge of the subject at a conceptual level. Recall the research of Mayberry (1983) and Mason and Schell (1988) identified low reasoning levels in preservice elementary teachers. Secondly, the course should include reading about and analyzing different theories of learning associated with that standard as well as common misconceptions. Jacobson and Lehrer (2000) attributed greater success of students to teachers who had awareness of theories like this. There should be firsthand experiences with students for witnessing and critically assessing such theories at work. Recall that Fuys, Geddes, and Tischler (1988) found teachers were successful in assessing student VH levels, given training and practice with this type of assessment. Third, the course should highlight effective pedagogy that has been validated in light of those theories. This should include discussion throughout the course, where appropriate, of NCTM's (2000) process standards: problem solving, reasoning and proof, communication, connections, and representation. The Mathematical Association of America (2001) advocates some similar ideas in its report, *The Mathematical Education of Teachers* (MET). Moreover, this document suggests that the sequence of mathematics courses required for preservice secondary teachers should be revamped to support the changes going on in education. These changes include increased use of class discussion, technology, and exploratory activities as well as the change in high school curricula.

One of the obstacles to implementing a plan like this for methods course would be again the amount of time it would take out of an undergraduate plan of study for preservice teachers. In an ideal world, I think it would be optimal if all five courses were required regardless of grade level (elementary, middle, or high). The NCTM (2000) expects these topics to be integrated at each of the grade bands so all preservice teachers should be prepared within each of these content areas. Teachers' lack of confidence with teaching geometry may cause them to skip this topic or "glaze over it" in the earlier grades. The overwhelming message of a lot of the research on the VH model is that students need a lot more experience with geometry at a younger age to build up the lower levels of reasoning.

Classroom Practice

Many teachers fear teaching geometry for two possible reasons: 1) their personal lack of confidence in their content knowledge and 2) known difficulty students have with this subject. If these are issues for non-geometry teachers of mathematics, I propose that we make professional development courses available that have the same focus as noted with the preservice teacher method courses (course content, research on learning, and research on teaching). Recall the research of Swafford and Jones (1997) which showed evidence that teacher institutes were effective for improving teachers' content knowledge. These courses could be taught through partnerships established between local universities and a school district. Again, this can make it possible to educate teachers already in the field and hope to bridge the gap between research and practice.

As a practicing teacher, I experienced some success in teaching geometry when I participated in a workshop which introduced me to the van Hiele model. It is from that experience that I made some drastic changes in the way I taught geometry (and mathematics in general) and eventually inspired me to attend graduate school. Research has shown that educating teachers about learning theories is helpful in changing views of teaching to better support students (e.g., Fuys et al., 1988; Swafford and Jones, 1997). I share some of these strategies that were effective for me as a teacher, but more importantly helped my students to experience success.

First, teachers should avoid making some assumptions about students' attitudes regarding geometry. It is unfair to assume students do not like the course, are unable to learn its content, or are incapable of reasoning logically. The thing that became clearest to me in making sense of van Hiele's model was that I had been talking "over their head" the whole time and it was my fault they could not get it; I might as well have been speaking French to them! In turn, I had a false sense of security about what they really understood because of the language barrier that van Hiele describes. The first step in making a change is admitting this mismatch of language exists. The existence of this mismatch of levels is validated by both van Hiele and Piaget

On a practical level, I claim communication of mathematics is crucial whether in a whole-class discussion or small-group setting. The classroom should be a place where students can share their ideas (correct and incorrect) and be encouraged to conjecture. Teachers should take on the role of learners as well and not be seen as the "authority" in the classroom. Recall Sharp's (2001) research which reported teachers improved understanding of geometry from being able to have this kind of discussion with each other. More effective and meaningful dialogue can occur if students are allowed to be an "authority" and have their questions directed at one another as opposed to being directed at the teacher. With this "freedom" for students, a challenge for teachers is to not interject ideas or re-direct if the discussion seems off course with the teacher's objective (like the "family circus" path of discourse). This is not to suggest encouraging off-task conversations; however, if students are never allowed to get off track with their incorrect notions, it is hard to know those incorrect ideas exist and be able to correct them. Additionally, unwillingness on the teacher's part to let students become "authorities" will hinder them in reaching depth in their discussions because students will anticipate the teacher's interruption.

This course should be driven by a curriculum that builds from the lower VH levels up. Fuys, Geddes, and Tischler (1988) encouraged these types of experiences for students. This keeps students from getting frustrated early on in the course and affords them the opportunity to not be left behind by their peers functioning at higher levels. Textbooks should not be relied upon to direct the course; Bischoff, Hatch, and Watford (1999) support this notion in their suggestions for preservice teachers, but it would still apply to inservice teachers. Rather, the teacher should be responsible for directing the pace and sequence through a progression of reasoning levels. For instance, students should be allowed to construct and write their own definitions. They can update, clarify and minimize these as the course progresses. The task of writing definitions with only necessary information is considered a level two task (informal deduction) for which many students will likely be unprepared. Furthermore, as they grow in their reasoning it will become evident in the language they use to write definitions.

Another general recommendation based on the VH model is to avoid jumping into proof early on in the course. The research has been hard-pressed to identify many students functioning at the level of formal deductive reasoning (e.g., Senk, 1989; Mayberry, 1983; Usiskin, 1982 as cited in Senk, 1989) except for college-level mathematics majors (Burger & Shaughnessy, 1986) even after completing a course in geometry. Formalized proof should be more of an ending point than a beginning point of a high school geometry course. However, students can construct informal arguments starting from the first day by having to explain and justify their thinking in written and verbal contexts. Teachers can have tendencies to define success in geometry as being able to write two-column proofs. This seems narrow-minded and overly focused on the product as opposed to the process. The role of geometry in the high school curriculum should be to help students think logically and build thoughtful arguments. It is likely that more children will want to justify and validate their theories if teachers stop dictating the form in which students must present their argument.

Use of technological tools like computers and software can be an effective avenue to support the VH model. These tools can offer experiences at lower levels, yet promote thinking at higher levels. For instance, dynamic geometry software can allow students to make predictions about properties (lower level) and test them as well as discover invariant properties (higher level). Piaget's (1967) notion that students' representation of space develops through manipulating their environment would support the use of children working in this kind of dynamic geometry microworld. Battista (1997) developed a series of lessons for students to use pre-made "Shape Makers" templates in the Geometer's Sketchpad (Jackiw, 2001) to facilitate the development of higher levels of geometric reasoning. These activities are but one example of how technology can be a powerful medium for facilitating conceptual understanding of mathematics using different representations and generally promotes students' enthusiasm and interest in learning geometry.

Graduate Teacher Education

At the graduate level, I propose more training to prepare masters and doctoral students to become leaders in the educational field. The wealth of knowledge and meaning that I have gained in my graduate work is indescribable. I highly recommend that teachers return to graduate school after having several years of teaching experience because it gives you a different “lens” to look through. As a graduate student who is months away from returning to the teaching profession, I have a wealth of ideas about mathematics education to take back to the classroom. I have some ideas about how to make changes in my own classroom but feel less sure about how to effectively take what I have learned and help other teachers in the profession.

Some appropriate avenues to influence other mathematics educators would be doing staff development or making presentations at mathematics conferences at the local, regional, and national level. Graduate work should encourage master teachers to make presentations and give specific guidelines on how to plan and conduct workshops for inservice teachers. The impact of staff development for teachers has been supported in some of the research previously described (e.g., Swafford and Jones, 1999; Fuys et al, 1988; Sharp, 2001; Jacobson and Lehrer, 2000). An environment for this kind of preparation could happen at the university level with graduate students teaching preservice teachers or conducting inservice with teachers in a local district. I personally plan to use my knowledge gained in graduate school to help bridge the gap between research and practice.

Final Thoughts

I often have to remind myself that the most important aspect of teaching is the children. I strongly encourage teachers to avoid traps in education of “covering the material" and "teaching to the test." When I left the classroom to attend graduate school, the emphasis was on teacher accountability and high stakes testing. I found I was guilty of placing more emphasis on teaching all the objectives of the curriculum than providing quality education for all students. Geometry is a course that leaves many children behind because they have not had much exposure to it prior to high school or the few experiences they have had did not require thinking above the visual level. Thus, students encounter the secondary course unprepared for the stated goals and objectives for high school geometry. I implore us as a profession to not ignore the evidence and research that has sought to explain why these difficulties arise (like the van Hiele model). Instead, I propose we use this data to direct our pedagogical decisions and thereby give support to children in the learning of geometry. Perhaps the next generation of teachers will have a different story than that shared by Mrs. Trump.

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