ABSTRACT

BASCIANO, CHRISTOPHER ANTHONY. Computational Analysis and Simulation of the Non-Linear Arterial Wall Dynamics with Application to Abdominal Aortic Aneurysms. (Under the direction of Clement Kleinstreuer.)

The arterial wall is a complex fiber-reinforced composite structure. Its bio-mechanical response is a function of anisotropic material properties, nonlinear behavior, local stimuli, and the related time-dependent decay or remodeling. The purpose of this study is to identify the complexities of the arterial wall and investigate the necessary techniques that will adequately model the wall’s mechanical response. Specific detail will be given to the pathological case of abdominal aortic aneurysms.

After a detailed review of the nonlinear theory needed to adequately model the arterial wall, new models for the healthy abdominal aorta and abdominal aortic aneurysm that achieve a new degree of biomechanical realism will be developed and critically compared to the current “state-of-the-art” models. A comparative study of the two prominent abdominal aortic aneurysm models is then conducted via a finite element analysis.

The new models of the healthy and aneurysmatic abdominal aortic wall report physiologically realistic loading of the healthy and aneurysmatic arterial wall’s microstructure. Additionally, the macroscopic stress and strain of both models adequately matches experimental data and previous phenomenological models. The comparative study revealed a stark contrast in each model’s stress distribution and radial deformation. Therefore, the work completed within this study provides valuable information to accurately model the abdominal aortic aneurysm wall. Potential applications include enhanced analyses of abdominal aortic aneurysm tissue that have increased clinical applicability and the optimization of endovascular treatment through more realistic computational simulations.
Christopher Anthony Basciano was born on May 10, 1983 in Teaneck, NJ. He is the son of Frank and Judy Basciano and is third generation Italian-American. The author’s family moved to Spring Hill, FL shortly after he was born. While in Florida, the author’s younger brother and sister were born and the author went through grade, middle, and high-school, eventually graduating from Hernando Christian Academy in May 2001.

He then attended Mercer University in Macon, GA where he met his fiancé, Amanda Chappell, and in May 2005, earned a Bachelor of Science in Engineering with a specialization in Biomedical Engineering and a minor in Business Administration. While at Mercer, Christopher had many wonderful experiences. One of them was being mentored by Dr. Sinjae Hyun, who sparked his interest in computational research within the field of biomedical engineering. In August 2005, the author continued to pursue biomedical computational research and began work on this thesis under the direction of Dr. Clement Kleinstreuer within The Department of Mechanical and Aerospace Engineering at North Carolina State University.

Christopher will begin the next chapter in his life when he marries Miss Amanda Chappell in June 2007.
ACKNOWLEDGMENTS

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The support and encouragement from my parents has inspired me to strive for excellence in all things. I am truly in awe of the remarkable examples of faith and love they have poured into my life. My fiancé Amanda has stood by my endeavors even at the sacrifice of her own aspirations. I owe her great amounts of personal gratitude. Finally, I wish to acknowledge the lordship of Jesus Christ who has entrusted me with abilities I could never imagine.
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NOTATION AND SYMBOLS

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<td>AAA</td>
<td>Abdominal Aortic Aneurysm</td>
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<td>HAA</td>
<td>Healthy Abdominal Aorta</td>
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<td>FSI</td>
<td>Fluid-Structure Interaction</td>
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<td>SEF</td>
<td>Strain-Energy Function</td>
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<td>WSS</td>
<td>Wall Shear Stress</td>
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<td>SMC</td>
<td>Smooth Muscle Cell(s)</td>
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<td>PK</td>
<td>Piola-Kirchoff</td>
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Symbols

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<tr>
<td>$\psi$</td>
<td>Per unit volume Helmholtz free energy / Strain-Energy function</td>
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<td>$F_{ij}$</td>
<td>Deformation Gradient</td>
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<td>$C_{ij}$</td>
<td>Right Cauchy-Green stretch tensor</td>
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<tr>
<td>$I_c, II_c, III_c, IV_c,$</td>
<td>Invariants of right Cauchy-Green stretch tensor</td>
</tr>
<tr>
<td>$V_c, VI_c, VII_c, VIII_c$</td>
<td>Invariants of right Cauchy-Green stretch tensor</td>
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<td>$\sigma_{ij}$</td>
<td>Cauchy stress tensor</td>
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<td>$T_{ij}$</td>
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<tr>
<td>$S_{ij}$</td>
<td>Second Piola-Kirchoff stress tensor</td>
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<td>$p$</td>
<td>Incompressible Lagrangian multiplier</td>
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<td>$\delta_{ij}$</td>
<td>Kronecker delta</td>
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1. BIOLOGICAL AND MECHANICAL PROPERTIES
OF THE ARTERIAL WALL

1.1 Introduction

One of the greatest challenges of computational hemodynamic simulations is to accurately incorporate the mechanical properties of the arterial wall. The arterial wall is itself a complex system that contains its own biomechanical processes and is a highly active piece of multi-layered tissue. It has been shown that some of the complexities of the arterial wall include: shape changes in response to different biomechanical stimuli, anisotropic mechanical properties, and multi-layer composition of viscoelastic/anisotropic material.

Incorporating these complex effects in computational simulations is vital for accurate fluid structure interaction (FSI) analyses of blood vessels. The mechanical properties of the wall determine the deformation the wall will experience which, in turn, influences the local hemodynamics. Multiple investigations have been conducted that incorporate some of the arterial wall properties. These studies have reported significant differences between investigations that used simple rather than complex characteristics of the arterial wall. Thus, in order to achieve a new degree of realism needed for accurate FSI simulations of stented aneurysms, aforementioned complexities of the arterial wall must be included.
1.2 Cellular Composition of the Healthy Arterial Wall

The cellular composition of the arterial wall provides the fundamental basis for the wall’s mechanical behavior and characterization as a fiber-reinforced composite. Three primary molecules play pivotal roles in the microstructure of the arterial wall: collagen, elastin, and smooth muscle cells.

1.2.1 Collagen

Collagen is a protein that has a triple-helix polypeptide structure. The three polypeptide chains that compose the triple-helix are given the term α-chains and are considered to be the building blocks of the collagen molecule. The α-chains contain the repetitive sequence [Gly-X-Y]ₙ, where Gly is glycine, and X and Y are varying amino acids (usually proline). Over 20 different types of collagen fibers have been identified, each having a unique composition of collagen α-chains and amino acids (Haarer and Dee, 2006). Types I and III are predominantly found within the arterial wall and are also targets of proteolytic enzymes for pathological conditions such as atherosclerosis and aneurysms.

Strands of collagen form nanofibrils that are 4 – 8 nm in diameter, yet the collagen triple-helix molecules do not contact each other in the axial direction, but are separated by a gap of approximately 40 nm. Instead of direct axial contact, the collagen molecules are cross-linked in a repeating, staggered pattern that has a length of 65 – 67 nm between the cross-links of the collagen molecules (Humphrey, 2002; Haarer and Dee, 2006). Collections of nanofibrils are then organized into fibrils, which have a cross-sectional diameter of 10 – 500 nm. The fibrils then in turn compose collagen fibers which have a diameter of 1 – 500 μm and are distributed throughout the desired tissue. Humphrey (2002) stated that the diameter of the
collagen fibers is dependent on the stress and strain that the surrounding tissue experiences during physiological conditions. The collagen structure and organization is illustrated in Fig. 1.1.

Collagen fibers exhibit a nonlinear stress-strain curve. This is caused by the molecular structure of the collagen fibers. Collagen fibers have a crimped/undulated helical shape that remains crimped until a threshold amount of strain straightens the collagen fibers, and applies stress to the cross links between the collagen molecules. Upon full extension of the cross links, the collagen fiber has undergone full extension and will begin to have an exponential increase in stress with relatively little if any strain/deformation. Figure 1.2 illustrates the

<table>
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<td>4 - 8 nm</td>
</tr>
<tr>
<td>fibril</td>
<td>10 - 500 nm</td>
</tr>
<tr>
<td>fiber</td>
<td>1 - 500 ?m</td>
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Fig 1.1: Structure and Organization of Collagen within biological tissue
(vertically yellow lines are the cross-links between molecules)
From: http://courses.cm.utexas.edu/jrobertus/ch339k/overheads-1/ch6_collagen.jpg

Fig 1.2: Diagram of collagen fibers and surrounding tissue

α-chains
nanofibril
fibril
Section of collagen molecule
Heads of collagen molecules
Cross-striations 65-67 nm
surrounding tissue
250 nm
1 - 500 ?m
structural forms of the collagen fibers in conjunction with its nonlinear stress-strain relationship (Haarer and Dee, 2006).

![Stress-strain curve of collagen fibers](image)

**Fig. 1.2: Stress-strain curve of collagen fibers**  
*From: Haarer and Dee, 2006*

Collagen has a relatively high failure strength and is used to reinforce the arterial wall (preventing overstretch and rupture) (Sonesson et al. 1999; Humphrey, 2002; Haarer and Dee, 2006). The ultimate tensile strength of collagen fibers ranges from 50 – 100 MPa, yet has very little compliance and exemplifies a brittle failure (Haarer and Dee, 2006; Humphrey, 2002; Fung, 1993). Although the tensile strength has a fundamental affect on the mechanical response of collagen-reinforced tissue, the orientation of the collagen fibers play a greater role in the overall mechanical response of the tissue. The primary method of reporting collagen fiber distribution is by measuring the angle between the direction of the fiber and a principle axis of the tissue that holds the fibers. Holzapfel et al. (2000, 2006) reported the orientation of the collagen fibers as a mean angle from the circumferential principle direction. However, the experimental work of Holzapfel et al (2002) and Sacks et
al. (1998) have shown that collagen fibers are oriented throughout the cardiovascular system in a multiple angular directions with respect to the circumferential principle direction (cf. Fig. 1.3).

The mean angles displayed in Fig 1.3 indicate agreement with the trend recorded by Rhodin (1980), i.e., collagen fibers in the human aortic media orient themselves close to the circumferential direction. However, Holzapfel (2006) reported mean collagen distributions in a different set of human aortic tissue and found significantly different values. The mean angle in the media was ±37.8° and the mean angle in the adventitia was ±58.9°.

Furthermore, the innermost layer of the aorta (the intima) was reported to have a mean angular distribution of ±18.8°. Thus, it appears that variability of collagen distribution...
results in a significant inter-subject variability, making the exact angular distribution of the collagen fibers patient specific.

1.2.2 Elastin

Elastin is an insoluble, tightly cross-linked polymer containing covalent bonds. Its structure is quite random in unstretched conditions, yet straightens and forms a more ordered structure upon tensile loading. The cross links of elastin are hydrophilic while regions in between the cross links are hydrophobic. Because the hydrophilic regions exist only at the cross links, most of the elastin molecule is hydrophobic. However, despite its hydrophobic tendencies, elastin has a high degree of hydration (Haarer and Dee, 2006). Figure 1.4 illustrates the structure, cross links, and behavior of elastin under tensile loading.

![Fig. 1.4: Structure and behavior of elastin fiber under tensile load (red lines are the cross links between molecules)](http://137.222.110.150/calnet/cellbio/image/connective%20tissue%20matrix-elastin.jpg)

The amino acid building blocks of the elastin molecule are somewhat similar to collagen. In the order of high-to-low concentrations, the amino acids are: glycine, alanine, valine, and proline. It is important to note that the amount of glycine found within a molecule is correlated to the amount of elasticity/mechanical flexibility of the polypeptide chains. With glycine being the prominent amino acid of the elastin molecule, the elastin molecule is
expected to have a high flexibility and large amounts of elastic deflection (Haaer and Dee, 2006).

The mechanical behavior of elastin is considered “the most linearly elastic natural protein material” (Haaer and Dee, 2006). The resulting tensile elastic modulus is 0.6 MPa and can undergo uni-axial extensions of up to 150% prior to breaking (characteristic of high glycine content); whereas collagen fibers can only undergo less than 10% extension before breaking. Sonesson et al. (1999) reported that collagen is actually 1000 times stiffer than elastin. Based on these material properties it is evident that elastin’s role in the arterial wall is to ensure the wall maintains elastic deformation rather than preventing overstretch and rupture (Haaer and Dee, 2006; Humphrey, 2002). Although the combined mechanical behavior of elastin and collagen is non-linear, a linear representation of the differences in the two fiber’s stiffness and their combined mechanical response is illustrated in Fig. 1.5.

1.2.3 Smooth muscle cells and ground substance

Smooth muscle cells are involuntary muscular tissue that regulates the diameter and distensibility (dilation due to internal pressure) of arteries. The cells tend to be oriented in a
near circumferential configuration and have an approximate length of 100 μm and an average diameter of about 5 μm. Each cell has a fusiform morphology and is adjoined to other cells in an interlocking fashion (Humprey and McCulloch, 2003). Figure 1.6 illustrates the morphology and interaction of the smooth muscle cells.

![Collection of smooth muscle cells illustrating interlocking orientation](image)

**Fig. 1.6**: Collection of smooth muscle cells illustrating interlocking orientation (blue lines are reinforcing membrane around the cells)

*From: Marieb (1998)*

In elastic arteries (arteries closer to the heart), the smooth muscle cells (SMCs) are organized into concentric layers 5 – 15 μm thick that are separated by elastin sheets with an approximate thickness of 3 μm. Some elastic arteries have as many as 40 to 70 concentric layers of smooth muscle tissue. Muscular arteries (those further away from the heart) usually have a single collection of SMCs not separated by any elastin, yet the SMCs still orient themselves in multiple concentric layers. A maximum of 25 to 35 concentric layers of have been recorded for larger muscular arteries (Humphrey and McColluch, 2003).

Smooth muscle reacts much slower than the traditional striated, skeletal muscle. In the arterial wall, the smooth muscle may not begin to contract until 5 – 100 ms after the initial stimulus. The cells then take on the order of 1 – 10 s to reach the maximum contraction stage. However, the smooth muscle cells can maintain a prolonged lower level contraction much longer than striated, skeletal muscle and only require 0.25 – 5% of the energy required
for a comparable skeletal muscle contraction. Thus, healthy SMCs are well suited to perform their roles of maintaining a homeostatic balance of stress and deformation across the arterial wall (Humphrey and McColluch, 2003).

The ground substance/matrix of the arterial wall surrounds the collagen fibers, elastin fibers/sheets, and the SMCs. It is a collection of elastic components such as fibronectin and laminin, which provide the needed support for cells to remain anchored to each other. The ground matrix also contains integrins, which provide biochemical pathways that allow the other components of the arterial wall to communicate in other methods than the gap junctions that are inherent of the cross linked design of collagen and elastin (Humphrey, 2002).

1.3 Biological Composition of the Healthy Arterial Wall

Most healthy, arterial walls have three distinct layers, where each layer has a different distribution of materials and fibers (see Fig. 1.7). The two fibers that are of utmost importance to the structure of the arterial wall are collagen and elastin. Over twenty different types of collagen have been identified, yet the main load bearing collagen fibers present in the arterial wall are types I and III (Haarer and Dee, 2006; Humphrey, 2002). It is these collagen fibers which tether the vessel to surrounding tissue and prevent excessive dilation of the vessel. The elastin fibers, however, respond to

Fig. 1.7: Composition of arterial wall
pressure and can expand or shrink in volume to maintain a steady flow of blood through the vessel (Vito and Dixon, 2003).

The outermost layer is called the tunic adventitia and primarily contains collagen arranged in a loosely woven pattern. Small nerve fibers, lymphatic vessels, and even smaller blood vessels (in large arteries only) are located within this layer. Within the middle layer, the tunic media, significant amounts of sheets of elastin and smooth muscle tissue are located (Marieb, 1998). Types I, III, and IV collagen fibers are found to be tightly woven, almost embedded, in the smooth muscle cells and the elastin sheets (Vito and Dixon, 2003). It is this layer that normally is the dominant layer of the arterial wall; thus, bearing most of the mechanical load that the artery encounters. Furthermore, the artery’s ability to dilate or constrict is governed by the elastin sheets and smooth muscle cells within this layer. It is no surprise that this is the layer most affected by cardiovascular diseases such as atherosclerosis and arteriosclerosis. The layer that comes in contact with blood flow (under normal conditions) is the innermost layer, the tunic intima. It is this layer that contains the endothelium, a lining of simple squamous epithelial cells that align themselves to favor a mean shear stress of about 1.5 Pa (Mariab, 1998; Barner, 2002; Rachev, 2000a).

An important note is that the distribution of fibers and tissue within each arterial layer varies throughout the body. An example of this is how arteries more proximal to the heart are more elastic while arteries more distal to the heart are more muscular (Holzapfel, 2000b; Humphrey, 2002). Silver et al. (2003) conducted a study on the porcine aorta, carotid artery, and vena cava and reported the weight fractions of the primary mechanical components in the wall of each blood vessel (cf. Table 1.1).
### Table 1.1: Composition of different blood vessel walls

<table>
<thead>
<tr>
<th>Vessel</th>
<th>Collagen wet (dry) percentage</th>
<th>Elastin wet (dry) percentage</th>
<th>Smooth Muscle wet (dry) percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Abdominal Aorta</strong></td>
<td>13.7% (40%)</td>
<td>9.0% (27%)</td>
<td>11.35% (33%)</td>
</tr>
<tr>
<td><strong>Carotid Artery</strong></td>
<td>12.2% (40%)</td>
<td>9.30% (30%)</td>
<td>8.94% (30%)</td>
</tr>
</tbody>
</table>

From: Silver et al. (2003)

### 1.4 Mechanical Properties of the Arterial Wall

Some of the complexities of the arterial wall become even more evident when analyzing the wall’s mechanical properties. As mentioned, the arterial wall is considered to be a multi-layered anisotropic, composite material due to the different materials located within each tissue layer and the wavy, helical orientation of the collagen and elastin fibers (Holzapfel et al., 2000b; Li, 2004; Taber and Humphrey, 2001; Humphrey, 2002). Because each layer of the wall has a different composition discussed in the previous section, the mechanical properties of each layer are significantly different (Holzapfel et al., 2000b). An illustration of these fibers is shown in Fig. 1.8.

![Fig. 1.8: Illustration of composite structure of arterial wall](From: Holzapfel et al., 2000)
Direct contributions of the very thin tunic intima on the solid mechanical properties of the wall are very small and often negligible for young healthy arteries. Conversely, as the arteries age, this intima layer thickens and stiffens. This stiffening and thickening becomes even more pronounced upon the onset of atherosclerosis, which significantly changes the mechanical properties of each layer composing the arterial wall.

The tunic media consists of the tissue that primarily determines the mechanical properties of the entire composite structure of the healthy arterial wall (Holzapfel et al., 2000b; Humphrey, 2002; Vito and Dixon, 2003). As shown in Fig. 1.8, elastic lamina located within the tunic media divide the section into distinct fiber-reinforced layers (Rhodin, 1980). In elastic arteries (those closer to the heart, ie, aorta and iliac) such layers are difficult to distinguish from the elastic lamina itself, and can be considered as one prominent layer. Furthermore, each fibril of collagen and elastin, the elastic laminae, and smooth muscle cells form close junctions that collectively compose a continuous, fibrous helix. This helix is primarily circumferentially oriented, yielding an innate ability of the tunic media to resist both longitudinal and circumferential loads (Holzapfel et al., 2000b; Humphrey, 2002).

The outermost layer of the arterial wall, the tunic adventitia, has large amounts of variation of thickness based on the physiologic function and anatomic location of the artery (cf. Fig. 1.7). Furthermore, the large amounts of collagen within the adventitia are also arranged in helical patterns that have a primary contribution to the strength and stability of the arterial wall. It is interesting to note that the adventitia provides primary reinforcement against rupture and overstretch. The adventitia is thus somewhat limp when unloaded/relaxed; yet, at high levels of pressure, the adventitia becomes much more rigid as
the collagen fibers reach their straightened lengths (Holzapfel et al., 2000). Thus, it has been shown that this layer plays a pivotal role in the rupture of various aneurysms throughout the body (Vorp and Vande Geest, 2005). A photo illustrating the relaxed/unstrained tunic adventitia and the tunic media is shown in Fig. 1.9.

![Fig. 1.9: Picture showing the stiff tunic media and the limp tunic adventitia Source: Holzapfel et al. (2000) with permission](image)

1.5 Mechanical Response of the Arterial Wall

A collective analysis of the three arterial layers reveals that a healthy arterial wall forms a highly deformable and nearly incompressible, composite structure that exhibits a non-linear stress-strain response. Additionally, at higher loads the stress-strain response reveals exponential stiffening due to the collagen fibers of the various layers. Fung (1993) referred to the arterial walls as a pseudo-elastic material that exhibits hysteresis loops between the loading and unloading of the arterial tissue (negating the definition of a true elastic material), yet after a period of complete loading and unloading cycles, the arterial tissue yields a repeatable stress-and-strain relationship. Holzapfel et al. (2000) clearly illustrated this phenomenon when uni-axially testing strips of the tunic media under passive conditions. The
results are shown in Fig. 1.10 and exhibit a pre-conditioned response, where the material exhibits a nearly repeatable cyclic behavior after a certain number of load cycles.

![Nonlinear stress-strain behavior of arteries](Source: Holzapfel et al. (2000) with permission)

Small hysterisis loops exist between the loading and unloading of the arterial wall, noting that the energy lost due to heat is relatively small. Point one (I) in the figure corresponds to the yield point of the arterial tissue, where elastoplastic (permanent) structural deformation begins to occur (significant changes in the arterial wall’s mechanical behavior occur after this point in the stress-strain diagram). A state of stress-softening (lowered stress response) exists between points two (II) and three (III), where at point III the stress-strain curve reveals a perfect elastic/viscoelastic response. To combine the different regions into an approximate mechanical response curve, Holzapfel et al. (2000) constructed an engineering response curve that neglects preconditioned responses of the artery. It should be noted that although most of the stress in Fig. 1.10 is outside the normal physiologic range, such high stress on the arterial wall is encountered in many vascular therapies and invasive procedures i.e. angioplasty, stenting, etc. (Halzapfel et al., 2000). Furthermore, the stress on the arteries
would be higher than the normal physiologic region when encountering cardiovascular diseases, such as hypertension.

Although Fig. 1.10 illustrates the non-linear stress-strain behavior of an artery, it has been widely shown that the arterial tissue is anisotropic and requires biaxial analyses to accurately quantify the mechanical behavior of the arterial wall (Holzapfel et al., 2000; Humphrey, 2002; Fung, 1993; Sacks and Sun, 2003). The two primary anisotropic directions of arteries are the circumferential and the longitudinal directions; where the circumferential direction can be depicted by a line drawn tangent the circular cross section of artery and the longitudinal direction is depicted as a line that is parallel to the length of the artery (cf. Fig. 1.11). It is sometimes difficult to interpret the physical meaning of the circumferential stress. A good description of this stress can be made if an imaginary slit is cut along the length of the cylinder of Fig 1.11 in the longitudinal direction. When an internal pressure (along the inner wall of the cylinder) is applied to the (now sliced) cylinder, the unconnected circular edges of the cylinder would want to move apart. The circumferential stress is analogous to the stress needed to keep circular edges of the sliced cylinder flush together when an internal pressure is applied to the interior of the cylinder. Because this type of stress is predominantly encountered in circular and spherical structures, the circumferential stress is often referred to as the hoop stress. Thus, the two terms refer to the same concept of stress and have equivalent numerical values.
Figure 1.12 contains a stress-strain curve from a biaxial test, illustrating the anisotropic response of the healthy abdominal aorta (HAA). The stress and strain in the circumferential direction is depicted as $S_{TT}$ and $E_{TT}$, respectively; while stress and strain in the longitudinal direction is depicted as $S_{LL}$ and $E_{LL}$, respectively.

The different curves in the two graphs of Fig. 1.12 represent different ratios between circumferential and axial tensions, for the different loading ratios as given in the figure legends as $\text{Load}_{TT} : \text{Load}_{LL}$. It is important to notice the different responses to the different loading ratios. The circumferential stresses appear to have a slightly steeper increase in stress at higher strains. A possible explanation of this could be due to the orientation of the collagen fibers and their higher resistance to circumferential stretching than to axial stretching. It is very important to capture these different mechanical responses in a realistic model of the arterial wall as presented by Vande Geest et al. (2006a).
Another important response of the arterial wall is the changing of the wall’s mechanical properties with respect to time. Lanne et al. (1992) conducted a study on the effects of age on the mechanical response of the abdominal aorta. It was found that in healthy, male subjects the mean diameter and the stiffness of the abdominal aorta increased with respect to time. The diameter and strain data from Lanne et al. (1992) clearly reveal a change in the aorta’s mechanical response with the progression of time (cf. Fig. 1.13).

All data was recorded in-vivo through a real-time, non-invasive echo-tracking analysis.

Strain and mean diameter were calculated from the Eqs. (1.1) and (1.2).

\[
\text{strain} = \frac{D_{\text{systolic}} - D_{\text{diastolic}}}{D_{\text{diastolic}}} \tag{1.1}
\]

\[
\text{mean diam.} = \frac{D_{\text{systolic}} + D_{\text{diastolic}}}{2} \tag{1.2}
\]

The terms \(D_{\text{systolic}}\) and \(D_{\text{diastolic}}\) refer to the artery’s diameter at systolic and diastolic pressures, respectively. To enhance the general trend of the strain and diameter data, exponential and logarithmic trend-lines have been fit the respective curves. The corresponding R\(^2\) values are
also displayed on the graphs. It is important to note that the strain graph illustrates a relatively consistent decrease in strain. While the diameter plot shows a steep increase during developmental stages, yet begins to level off as time progresses.

Vande Geest et al. (2004) also illustrated the stiffening of the abdominal aorta by conducting biaxial analyses on healthy abdominal aortas of different ages. Their results illustrated a significant decrease of deformation when patients were older than 30 yrs old. A plot illustrating their results is shown in Fig. 1.14.

![Areal Strain of Abdominal Aorta](image)

Fig. 1.14: Areal strain of the abdominal aorta vs. age
From: Vande Geest et al. (2004)

The source of the discrepancy between the plots of Lanne et al. (1992) and Vande Geest et al. (2004) is their experimental methods. Lanne et al. (1992) calculated strain from the in vivo diameter of the abdominal aorta under systolic and diastolic pressure while the Vande Geest et al. (2004) conducted biaxial tensile experiments on excised abdominal aortic tissue. Furthermore, the areal strain is a combination of the deformations in both the circumferential and longitudinal directions, whereas the strain of Lanne et al. (1992) is the circumferential strain.
It has thus been shown that the mechanical response of the arterial wall is anisotropic and dependent on the direction/orientation of the load applied to the wall. Additionally, the mechanical response of the arterial wall changes with the passing of time and involves a vast amount of biological processes. Thus, studies that depend on the mechanical response of the artery will need to incorporate the afore-mentioned biological and mechanical properties of the arterial wall.

1.6 The Process of Arterial Remodeling and Adaptation

In addition to the complex architecture and mechanical behavior of the arterial wall, the arterial tissue has an innate ability to adapt and modify its morphological structure and material properties in response to acutely and chronically altered loading conditions (Fung, 1993; Humphrey, 2002). The thickening of the arterial wall in response to hypertension is an example of the adaptive capabilities of the arterial wall, but arterial adaptation is also seen in healthy arterial conditions. Dienne et al. (2001) reported that the arterial wall of healthy men, without any cardiovascular disease and regularly participate in endurance exercise, exhibited a thickening behavior in response to the internal conditions during exercise. Additionally, the smooth muscle cells within the arterial walls exhibit an active response that contracts under periods of elevated pressure and dilates/expands under lowered lumen pressure (Marieb, 1998).

Scientists have thus defined all adaptive behavior of the arterial wall with the terms arterial remodeling or adaptation. However, such terms describe a very broad range of adaptive behaviors ranging from the thickening of the wall without affecting the area of the
vessel lumen to the increased production of collagen fibers to reinforce regions of the wall undergoing increased levels of deformation. Known cells that play a pivotal role in this adaptive process include the endothelial cells, smooth muscle cells (SMC), fibroblasts, collagen and elastin fibers, and the extracellular matrix that contains numerous sensors for biochemical remodeling signals. The common mechanism that triggers all forms of remodeling is the human body’s natural desire to maintain homeostatic conditions. Arterial walls and cells have an instinctive target for basal stress levels and the processes of arterial remodeling are aimed to restore/maintain these basal stress levels (Glagov 1994, Alexander and Dzau 2000, Jackson et al. 2002).

One of the primary methods for stress compensation, that does not involve any macroscopic morphological changes to the arterial structure, is the degradation and synthesis of different components of the arterial wall. The wall itself is living tissue and thus exhibits a homeostatic balance between new, healthy components and old, worn components. Matrix metalloproteinases (MMPs) are the primary enzymes that break down components of the vascular wall, while tissue inhibitors of MMPs (TIMP) are the enzymes that regulate the arterial wall’s response to secreted MMPs. More than twenty MMPs exist, and the entire population of MMPs has been divided into 4 categories: collagenases, gelatinases, stromelysins, and membrane-type MMPs (Loftus and Thompson, 2002). Recalling that the matrix surrounding the smooth muscle cells of the arterial wall is predominantly composed of elastin and collagen, the MMP enzymes target the degradation of these two fibers.

A careful balance between MMP degradation and TIMP inhibition must be established otherwise the collagen and elastin of the arterial wall might have a significant loss in tensile
strength and become too compliant or the arterial wall may become too rigid and raise stress levels above the tensile strength of newly constructed collagen fibers (Lehoux and Tedgui, 2005). Fig. 1.14 provides a visual illustration of the careful balance between MMP and TIMP activity.

![Fig.1.15: Scale of MMP and TIMP activity](From: Lehoux and Tedgui 2005)

The flow of blood through the arteries is a common stimulus of previously discussed metabolic and anabolic activity of the arterial wall as well as significant changes in the morphology of the arterial wall. Rachev (2000a) computationally analyzed the response of arteries to altered blood flow and found that significant adaptations occurred in both the lumen & wall radius and arterial wall thickness. In his study he also noted that the arteries respond differently to an increase and a decrease in blood flow.

An increase in blood flow triggers a two step process that compensates the increased stress levels due to more blood flowing through the artery. The first step is an acute response that is primarily focused on the lumen diameter of the artery; while the second phase is a chronic response that involves significant reconstruction of the arterial wall. In healthy
arteries, each phase has a feedback response that will terminate the adaptation process when the stress levels of different portions of the arterial wall have been returned to the desired basal level (Rachev, 2000a).

A decrease in blood flow results in a lower shear stress exerted on the endothelial cells of the tunic intima. It is important to note that the arterial remodeling in response to decreased blood flow does not occur within the same timescale as the artery’s response to increased blood flow; however, a multi-stage response is still present. The first compensatory action of the arterial wall is to raise the stress along the walls by decreasing the lumen diameter. A persistent decrease in blood flow primarily affects the endothelium of the tunic intima. The remodeling process causes the intima to increase in thickness and accumulate smooth muscle cells from the tunic media. In some arteries, the basal levels of stress can never be reached; thus, the body’s adaptive response continues to increase the intima’s thickness, creating a permanent hindrance to blood flow known as a stenosis (Rachev, 2000a).

1.6.1 Acute remodeling and adaptation response

The vast majority of the loading on the arterial wall is due to the pulsating flow of blood through the arterial lumen (Nichols and O’Rourke 1990). Changes or disturbances in the flow can trigger the release of certain hormones that dilate or contract the lumen of the artery. One of the fundamental stimuli for all acute adaptation is the shear stress along the endothelial wall of the tunic intima, given the term wall shear stress (WSS). It has been shown that the arteries tend to favor a mean WSS of about 1.5 Pa (Rachev, 2000a; Langille, 1993; Langille, 1995; Brownlee and Langille, 1991).
The WSS is directly proportional to the viscosity of the fluid and the local velocity gradient (see Eq. (1a) and (1b)). Based on Stokes’ hypothesis for incompressible Newtonian fluids the wall shear stress (WSS) can be written as:

\[
\vec{\tau} = \eta \left[ \nabla \vec{v} + (\nabla \vec{v})^T \right]
\]  

(1.2a)

where \( \vec{\tau} \) is the stress tensor, \( \eta = \eta(\mu, \nabla \vec{v}, Ht) \) is the apparent non-Newtonian blood viscosity, \( \vec{v} \) is the velocity vector, \( \nabla \) is the del operator, and \( (A)^T \) is the transpose of the tensor A. For Poiseuille flow of a Newtonian fluid the wall shear stress can be written as:

\[
WSS = \tau_{wall} = \mu \frac{du}{dr}\bigg|_{wall} = \frac{\mu \bar{u}}{R} = \frac{\frac{1}{2} \frac{\Delta P}{L}}{2}
\]  

(1.2b)

where \( \mu \) is the fluid viscosity, \( u(r) \) is the axial velocity, where \( r \) is the normal/radial coordinate, \( \bar{u} \) is the average velocity, \( R \) is the tube radius, and \( \frac{\Delta P}{L} \) is the constant pressure gradient. WSS can vary greatly in areas of disturbed flow, making it a possible link to many vascular diseases. The acute arterial response tends to be a local phenomenon, which contributes to the localized lesions of different cardiovascular diseases (Buchanan et al., 2003; Kleinstreuer et al., 2001).

Koller et al. (1993) revealed that an increase in blood viscosity, while maintaining consistent physiologic flow and pressure, resulted in an increase in vasodilation (lumen expansion due to relaxing of smooth muscle cells). Since viscosity is directly related to the WSS, Koller also revealed the effects of increased shear stress on the endothelium when flow and pressure are maintained at a constant level (Barner, 2002). Vallance and Chan (2001) and Moncada (1991) report that WSS induced vasodilation can occur via multiple
biochemical pathways that begin with signal transduction within the epithelial cells of the tunic intima. The primary molecule that carries the signal for vasodilation is Nitrous Oxide. The secretion of Nitrous Oxide (NO) is a local response and can be triggered in small arterioles while the large arteries remain unaffected. NO can also have a global affect since high concentrations of NO have been shown to affect both large and small arteries (Sausbier et al, 2000).

NO is synthesized at various locations within the body, yet within the epithelial cells a specific synthase enzyme titled epithelial nitrous synthase (eNOS) produces NO and the byproduct L-citrulline from O2 and L-arginine. The resulting NO then diffuses out of the epithelial cell membrane and into the smooth muscle cells (SMCs) of the tunic media. Within the SMCs, the NO triggers guanylate cyclase (GC) to synthesize cyclic guanosine 3’,5-monophosphat (cGMP), the messenger that directly stimulates vascular tone and platelet function. Figure 1.16 provides a visual illustration of WSS induced endothelial NO production and resulting SMC vasodilation.

![Fig. 1.16: Mechanism for vasodilation from WSS stimulus](image-url)
NO has a very short half life that is less than 4 sec in biological solutions (Vallance and Chan, 2001). To accommodate the molecule’s short existence, the body’s response to NO is almost immediate, which explains the popular use of nitroglycerine for expanding arteries in cases of congested heart failure and myocardial infarction.

Regardless of the body’s reaction time, eNOS can synthesize NO in multiple methods. Ca$^{2+}$, K$^+$, Cl$^-$ ion channel activation of NO release merely takes a few seconds after the endothelial cell is exposed to WSS. The prominent ion that drives the channels is Ca$^{2+}$ that increases in concentration during pulsatile flow conditions and can trigger intracellular depolarizations that open the various ion channels (Vallance and Chan, 2001; Marieb, 1998). Another method of NO synthesis is the phosphorylation of intracellular proteins, which after hours of endothelial WSS exposure, can increase the intracellular concentration of eNOS by up to 6-fold. The last prominent method of endothelial NO secretion is the gene transcription of eNOS in response to changes in the WSS on the endothelial wall. Gene transcription is a more permanent method of NO secretion, and is more similar to a chronic response rather than an acute response.

1.6.2 Chronic remodeling and adaptation response

The chronic response is a long-term, adaptive process that can includes cell growth or apoptosis, extracellular matrix expansion or contraction, and activation or inhibition of specific proteolytic enzymes or glycosidases (Alexander and Dzau, 2000). It is therefore evident that the chronic response of the arterial wall very extensive and yields alterations in composition, internal structure, and external morphology (Lehoux and Tedgui, 2005).
Many chronic adaptations of the arterial wall are a mere continuation of an acute response. The stimulation for the chronic response is a persistent condition that alters the stress on the arterial wall (Rachev, 2000a). Masuda et al. (1989) reported a compensatory thickening of the arterial wall under a consistent increase in blood flow. The results were subsequently supported by the research of Glagov et al., (1993), which also revealed a thickening response of the total arterial wall associated with a persistent increase of blood flow.

Some pathological conditions also elicit a chronic adaptation response. During sustained hypertension, the arterial wall must develop a more permanent compensation than a constant release of NO to dilate the arterial lumen. The elevated pressure exerted on the arterial walls increases the accumulation of types I, III, and IV collagen fibers to counteract the increased circumferential stress placed on the arterial wall (Intengan et al., 1999). Additionally, Rachev et al. (1995) showed that arteries adapt to increases of blood pressure through the growth of vascular tissue. His results revealed that the arterial wall thickness increased monotonically until a new steady state had been established. Additionally, the residual stress of the artery itself changed with respect to the increased pressure and revealed a non-monotonic increase of the opening angle of the artery (Rachev et al., 1995; cf Fig. 1.17).
Another pathological condition that involves chronic arterial remodeling and adaptation is atherosclerosis. One of the primary concerns of atherosclerosis is that the lesions and plaque cause intimal thickening, and begin to hinder blood flow (Kleinstreuer, 2006). Directly stimulated from the change in hemodynamic forces, the arteries begin to take compensatory action and attempt to restores levels of stress in the wall. The results of such attempts usually can be classified in two broad categories of expansive/positive or constrictive/negative remodeling. Expansive remodeling is characterized by an increase in wall and plaque volume that does not significantly decrease the lumen area of the blood vessel while constrictive remodeling is used to identify an increase in wall and plaque volume that decreases the lumen area of the blood vessel (Gyongyosi et al., 2004; Burke et al., 2002; Pasterkamp et al., 2002).

Interesting studies have shown that although the expansive remodeling tends to not hinder the flow of blood and create a stenosis, the plaque formed during expansive remodeling is more susceptible to unstable atherosclerotic conditions. Some of the primary

Fig. 1.17: Changes in: (a) Arterial wall thickness and (b) Arterial opening angle over time
From: Rachev et al. (1995)
observations of expansive atherosclerotic remodeling were that the plaque contained more inflammatory cells (lymphocytes, macrophages, etc), a greater number of thrombi, and a greater risk of being detached from the arterial wall and passed through the cardiovascular system (Pasterkamp et al., 2002).

The endovascular treatment of stenting and the grafting of anastomoses can also produce chronically elevated levels of stress within the arterial wall. The biological response to vascular stenting is fueled by the arterial remodeling process. Moore and Berry (2002) reported that an inflammation, proliferation, and remodeling phase occur in response to stenting. Where the remodeling phase is characterized primarily by collagen deposition and restructuring in the adventitia and media while the inflammation and proliferation target the smooth muscle cells of the arterial wall.

Rachev et al. (2000b) computationally investigated the arterial remodeling in response to the increased stress levels at the junctions of the stent or graft and the normal artery. The results from their research concluded that localized, elevated levels of stress at the stent/graft junction with the artery caused a thickening of the arterial wall at those locations until the stress levels stabilized over time. Unfortunately, the thickening of the arterial often does not stabilize the local stress until a majority of the vessel’s lumen is again occluded at the edges of the stent. Thus, the arterial wall may never achieve the desired basal level of stress at junctions between the stent and the artery, until another stenosis is formed. The term restenosis has been given to the occlusion of a stented artery at the regions of contact between the stent edges and the arterial wall (Frank et al. 2002). Berry et al. (2002) proposed a stent design that matched the stent and arterial wall compliance at the junction between the
stent ends and arterial wall. In the study, it was shown that the stress in the junction near the stent edges and arterial wall was not as elevated as compared to traditional stent designs. Furthermore, the new design was investigated in a porcine medical trial where the compliance matching design triggered a significantly less inflammatory response than traditional stent design (Berry et al, 2002).

1.7 Arterial Wall Mechanics of Aneurysms

One of the many irreversible, pathological conditions that affect the arteries is the aneurysm. An aneurysm is a local dilation of the blood vessel to where the lumen of the vessel expands to at least 1.5 times the original diameter. The dilations often have complex morphologies that range from a fusiform or a saccular shape to a more tortuous geometry that is highly irregular. Further geometric complexities exist such as variable wall thickness and presence of intraluminal thrombus (ILT) that lines portions of the aneurysm wall. Aneurysms most frequently occur in the abdominal aorta (almost 90%), but numerous cases have reported aneurysms in the thoracic aorta, the cerebrum, and the coronaries (cf. Fig. 1.18).
One of the prominent dangers of an aneurysm is its tendency to rupture which results in massive internal bleeding and a local collapse of the surrounding blood vessels. Within the US over two million people are diagnosed each year to have an aneurysm and over 18,000 patients are annually killed by aneurysms. Furthermore, aneurysm rupture is the 13th most common cause of death in the Western World (Choke et al., 2005). Numerous investigations have been completed that analyze the pathology and mechanism of aneurysm rupture. The most common agreement is that the process is multifaceted with several factors/parameters that influence the severity of the aneurysm. Some of the common parameters agreed on by multiple researchers are the morphological geometry of the aneurysm, the biological/biochemical balance of normal vascular remodeling, the three dimensional morphology of the aneurysm, and the mechanical/material properties of the aneurysm wall (Choke et al., 2005; Kleinstreuer and Li, 2006; Vorp and Vande Geest, 2005; Lindholt et al. 2001).
1.7.1 Composition of the aneurysm wall

The foundational components of the normal arterial wall are significantly altered in the aneurysm wall. Nichols and O’Rourke (1990) reported that the normal collagen to elastin ratio of the arterial wall is around 1.58; however, He and Roach (1994) presented data that revealed the collagen to elastin ratio is much higher in aneurysms (cf. Table 1.2).

<table>
<thead>
<tr>
<th></th>
<th>Normal aorta</th>
<th>Aneurysm</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Elastin</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average</td>
<td>22.7</td>
<td>2.4</td>
</tr>
<tr>
<td>Maximum</td>
<td>32.5</td>
<td>6.7</td>
</tr>
<tr>
<td>Minimum</td>
<td>16.1</td>
<td>0.2</td>
</tr>
<tr>
<td><strong>Muscle</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average</td>
<td>22.6</td>
<td>2.2</td>
</tr>
<tr>
<td>Maximum</td>
<td>33.6</td>
<td>6.4</td>
</tr>
<tr>
<td>Minimum</td>
<td>15.5</td>
<td>0.4</td>
</tr>
<tr>
<td><strong>Collagen and ground substances</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average</td>
<td>54.8</td>
<td>95.5</td>
</tr>
<tr>
<td>Maximum</td>
<td>63</td>
<td>98</td>
</tr>
<tr>
<td>Minimum</td>
<td>48</td>
<td>91.4</td>
</tr>
</tbody>
</table>

Table 1.2: Composition of normal aorta and aneurysm wall
From: He and Roach, 1994

Similar analyses have also reported the drastic decrease in elastin and smooth muscle within the abdominal aortic aneurysm (AAA) wall (Gandhi et. al 1994). Baxter et al (1994) reported that during the formation of AAAs the elastin concentration of the arterial wall decreases while the collagen concentration of the arterial wall increases, supporting the results of He and Roach (1994). While several investigations confirm the decrease of elastin and smooth muscle within the aneurysm wall, analyses have yielded contradictory results concerning the origin of the increase in collagen concentration within the aneurysm wall. White et al. (1993) reported that the increase in collagen concentration was due to increased
collagen synthesis and deposition within the arterial wall. However, Menashi et al. (1987) claimed that the increased collagen concentration was caused by the massive decrease in elastin concentration without increased collagen synthesis. Carmo et al. (2002) investigated the cross links of collagen and elastin in both the healthy abdominal aorta and the abdominal aortic aneurysm (AAA). Their study revealed that elastin cross links decrease by about 90%, while the collagen cross links increased by almost 350%. However, the study also revealed that the total content of both elastin and collagen fibers decreased by 47% and 91% respectively. By combining these trends with histological stains Carmo et al. (2002) also discovered a prevalence of aged type-I collagen fibers within the aneurysm wall, suggesting a complication with the synthesis of type-I collagen fibers within the aneurysm wall.

1.7.2 Imbalance of arterial remodeling within the aneurysm wall

A contributor to the variation in both collagen and elastin of AAA-walls is an imbalance in the biological/biochemical enzymes and proteinases that degrade or synthesize the elastin and collagen fibers within the arterial wall. Kowalewski et al. (2006) claimed that the AAA wall represents a severe case of unstable arterial remodeling and proteoglycan metabolism of both elastin and collagen fibers. One should recall from section 1.6 that normal arterial remodeling involves the matrix metalloproteinase (MMP) enzymes that break down both collagen and elastin fibers while tissue inhibitor MMP (TIMP) enzymes encourage collagen and elastin synthesis. The primary MMP enzymes that affect collagen (types I and III) and elastin have been listed in Table 1.3.
Table 1.3: MMPs and their degradation targets
From: Choke et al. (2005) and Alexander (2004)

<table>
<thead>
<tr>
<th>MMP</th>
<th>Collagen I</th>
<th>Collagen III</th>
<th>Elastin</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Collagenases</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MMP-1</td>
<td>x</td>
<td>x</td>
<td>x*</td>
</tr>
<tr>
<td>MMP-8</td>
<td>x</td>
<td>x</td>
<td></td>
</tr>
<tr>
<td>MMP-13</td>
<td>x</td>
<td>x</td>
<td></td>
</tr>
<tr>
<td>MMP-18</td>
<td>x</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Gelatinases</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MMP-2</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>MMP-9</td>
<td></td>
<td></td>
<td>x</td>
</tr>
<tr>
<td><strong>Stromelysins</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MMP-3</td>
<td>x*</td>
<td>x</td>
<td>x*</td>
</tr>
<tr>
<td>MMP-10</td>
<td>x*</td>
<td></td>
<td>x</td>
</tr>
<tr>
<td><strong>Membrane MMPs</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MMP-14</td>
<td>x</td>
<td>x</td>
<td>x*</td>
</tr>
<tr>
<td>MMP-15</td>
<td>x*</td>
<td>x*</td>
<td>x*</td>
</tr>
<tr>
<td>MMP-16</td>
<td>x*</td>
<td>x*</td>
<td>x*</td>
</tr>
</tbody>
</table>

The enzymes in Table 1.3 affect the degradation of collagen I, collagen III, and elastin through direct interaction (x) or through indirect stimulation of another MMP that has direction interaction with the target (x*). Several studies have shown that MMP-2 and MMP-9 are the primary MMPs found within aortic aneurysms (Alexander, 2004). Crowther et al. (2000) has shown that smooth muscle cells (SMCs) taken from the aneurysm wall produce three times more MMP-2 than SMCs of age matched, healthy arteries. Peterson et al. (2002) reported a correlation between MMP-2 and the max diameter of an aneurysm, and suggested that MMP-2 could be closely linked with the initial formation/bulging of aneurysms.
Conversely, MMP-9 is often found in larger aneurysms and is attributed to later stages of elastin degradation and the inflammation of the tunic adventitia. Some investigations of MMP-9 have reported a decreased elastin degradation and suppression of aneurysm formation by disrupting the MMP-9 enzyme (Alexander, 2004). Studies involving collagenase (the first group of MMPs in table 3) have shown that the enzyme targets immature and damaged collagen fibers rather than mature fibers. Furthermore, multiple investigators have shown that MMP-1 (a collagenase) is present in aortic aneurysms and that the aneurysm cells do not respond to TIMP deactivation (Alexander, 2004). Thus, a disruption in a feedback loop of the biological system is evident, and in turn also supports the results of Carmo et al.’s (2002) research that aged collagen fibers are more prominent in the aneurysm wall. Despite the support of the two studies, the exact mechanism and biochemical pathways that govern aneurysm formation and severity remains unsolved. Currently, several studies have also produced conflicting results that illustrate the need for further investigation into the enzymes involved with the degradation of elastin and collagen within aneurysms (Loftus and Thompson, 2002).

1.7.3 Mechanical properties of the aneurysm wall

In addition to an imbalance in arterial remodeling, aneurysms also have significantly different mechanical properties than healthy arteries. Lanne et al. (1992) reported an increase in the pressure strain elastic modulus ($E_p$) of AAAs when compared to age-matched, healthy abdominal aortas. The equation for $E_p$ is given in Eq (1.3) where $P$ is the internal pressure of the artery, $D$ is the diameter, and the subscripts $sys$ and $dia$ refer to the systolic and diastolic conditions respectively.
Vorp and Van de Geest (2005) indicated that numerous studies reported an increase in arterial stiffness in AAAs when compared to healthy abdominal aortas. Wilson et al. (1999) stated that aneurysms with a larger diameter exhibited a lower compliance or increased stiffness. The limitation of both $E_p$ and arterial stiffness is that the calculations do not generate enough information to construct a model of the aneurysm wall that relates the values of stress and strain. Furthermore, most of the studies used to calculate both $E_p$ and arterial stiffness have been investigated noninvasively and cannot provide details such as ultimate strength which is needed to further quantify the properties of AAAs.

Initial ex vivo studies tested AAA tissue through uni-axial tensile tests, and found that AAAs exhibited a decreased ultimate tensile strength when compared to healthy abdominal aortas (Raghavan et al., 1996; Vorp and Van de Geest, 2005). Raghavan et al (1996, 2002) conducted experiments that revealed the stress-strain curve of the aneurysm moves considerably to the left with a large decrease in ultimate strength while exhibiting an increase in wall stiffness (cf. Fig. 1.19). Vorp et al. (1996) reported a 50% decrease in tensile strength in the AAA wall when compared to abdominal aortas without aneurysms. Further analyses by Di Martino et al. (2004) revealed that the wall strength of ruptured AAAs is significantly less than the strength of electively repaired AAAs (cf. Table 1.4).

$$E_p = K \frac{P_{sys} - P_{dia}}{(D_{sys} - D_{dia})/D_{dia}}$$

(1.3)
Thubrikar et al. (2001a) studied the mechanical properties of different spatial locations within AAAs. His research revealed that tissue from the anterior, posterior, and lateral regions of the aneurysm bulge had significantly different biomechanical properties. The anterior region had the lowest strength and the wall stiffness in circumferential direction was consistently greater than the stiffness in the longitudinal direction, illustrating the anisotropic behavior of AAAs (Thubrikar et al., 2001a). These results were supported by Vallabhaneni et al. (2004) who illustrated the heterogenic mechanical characteristics of the arterial wall and a large variation of mechanical properties between each AAA specimen.

Despite the severity of AAAs and their rising commonality, very few experimental studies have been performed to analyze the anisotropic mechanical response of the collective layers of the arterial wall. To date, Vande Geest et al. (2006a) is the only reported investigation of such a study. The stress strain plots of their study are reported in Fig. 1.20.

![Fig. 1.19: Ultimate strength of aneurysm wall and healthy aortic wall](From: Raghavan, 2002)

![Table 1.4: Wall strength of ruptured and repaired AAAs](From: Di Martino et al. (2004))

<table>
<thead>
<tr>
<th>Aneurysm Type</th>
<th>Wall Strength [N/cm²]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ruptured</td>
<td>54.2±5.6</td>
</tr>
<tr>
<td>Repaired</td>
<td>82.3±9.0</td>
</tr>
</tbody>
</table>

Thubrikar et al. (2001a) analyzed the mechanical properties of different spatial locations within AAAs. His research revealed that tissue from the anterior, posterior, and lateral regions of the aneurysm bulge had significantly different biomechanical properties. The anterior region had the lowest strength and the wall stiffness in circumferential direction was consistently greater than the stiffness in the longitudinal direction, illustrating the anisotropic behavior of AAAs (Thubrikar et al., 2001a). These results were supported by Vallabhaneni et al. (2004) who illustrated the heterogenic mechanical characteristics of the arterial wall and a large variation of mechanical properties between each AAA specimen.

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The different curves in the two graphs of Fig. 1.19 represent different ratios between circumferential and axial tensions for different loading ratios as given in the figure legend as Load_TT : Load_LL. It is important to compare these trends with the normal bi-axial properties of the normal aorta in Fig. 1.12. Through such a comparison, it is evident that the maximal strains encountered by AAA walls are significantly less than healthy abdominal aortic (HAA) walls; while the maximum stress of the AAA walls is significantly larger than HAA wall stress. Thus, the trend of increased arterial stiffness within AAA tissue is clearly evident in the data reported by Vande Geest et al. (2006a).

1.7.4 The effects of intraluminal thrombus on the aneurysm wall

Intraluminal thrombi (ILT) are present in approximately 75% of all AAAs and have been shown to influence the local stress and the local strength of the aneurysm walls (Vorp et al., 2001). An ILT itself is a heterogeneous, isotropic structure composed of fibrin intertwined with platelets, blood cells, blood proteins, and cellular debris (Wang et al., 2001; Vande
Geest, 2006b; Harter, 1982). Furthermore, the ILT can be a load-bearing material adjacent to the aneurysm wall. An illustration of ILT orientation and structure is located in Fig. 1.21.

The photos clearly illustrate significant differences in the microstructure of each ILT layer. An important observation is that the ILT closest to the blood flow (the luminal layer) contains more fibrin molecules arranged in a netlike pattern. Conversely the layer closest to the arterial wall (the abluminal layer) has little if any orientation of its fibers. Thus, a general trend is observed that ILT closest to the arterial lumen tends to be more load bearing and has a more definitive solid structure. Vande Geest et al. (2006b) noted the more solid, load-bearing abilities of the luminal layer when performing biaxial analyses of the ILT.

The load bearing nature of the complete ILT structure has been shown through computational investigations to decrease the local stress levels by up to 30% and create a
completely different stress distribution along the AAA wall in regions where ILT is present (Di Martino et al., 1998; Wang et al., 2002). Although such a decrease in wall stress may appear to be advantageous, Vorp et al. (2001) illustrated that regions of the aneurysm wall adjacent to thick layers of ILT had significantly lower wall strength than regions of the aneurysm wall adjacent to thinner or no ILT. Furthermore, Kazi et al. (2003) revealed that AAA walls adjacent to ILT had less smooth muscle cells and an overall lower thickness than AAA walls not adjacent to ILT. It is highly possible that such behavior of the AAA wall adjacent to an ILT is related to a decreased amount of elastin fibers in AAA walls adjacent to ILT (Vorp and Vande Geest, 2005). The ILT also serves as a site of inflammatory response within the aneurysm bulge. Vorp et al. (1996) and Kazi et al. (2003) both have reported that ILT contains macrophages that could possibly release proteolytic enzymes (MMPs) under stress conditions found within AAAs. Kazi et al. (2003) also reported that AAA walls adjacent to ILT contained a greater amount of macrophages and other inflammatory cells when compared to AAAs without ILTs. Thus, in areas of a thick ILT the decrease in stress acting on the aneurysm wall is accompanied by multiple negative conditions such as wall weakening, increased degradation of the AAA wall, and an increase in AAA wall inflammation.

1.8 Appropriate Experimental Conditions When Testing an Artery

To accurately collect data on the material properties several conditions must be satisfied. For example, Holzapfel et al. (2000) discussed some important factors that must be considered when testing arterial segments for mechanical behavior. The primary concern is
to simulate an *in vivo* environment as closely as possible when testing the arterial segments. Some *in vivo* environment contributors include: temperature, pH, length (arteries often decrease in length when excised, and must be stretched back to *in vivo* length upon testing), osmotic pressure, CO₂ and O₂ partial pressures, and ionic concentrations. The authors proposed that arteries should be tested in “appropriate oxygenated, temperature controlled, salt solutions as fresh as possible” (Holzapfel et al., 2000; Humphrey, 1995). Additionally, Holzapfel et al. (2000) et al. stated that to assess the true anisotropic behavior of the arterial tissue, testing beyond uni-axial extension tests must be completed. (Holzapfel et al., 2000; Vaishnav et al., 1983). Some examples of additional experimental tests include biaxial tension, torsion studies, and shear analysis.
2. RESEARCH OBJECTIVES AND JUSTIFICATION

The motivation for this study is to enhance the realism of computational simulations of abdominal aortic aneurysms (AAAs) and stented AAAs. Groundbreaking research has already been accomplished using simple properties of the aneurysm wall (Li, 2005; Thubrikar et al., 2001b; DiMartino et al., 2001). However, Vande Geest (2004, 2006a) conducted experimental analyses of the healthy and aneurysmatic abdominal aortic wall and found significant amounts of material complexity and anisotropy as outlined in Sections 1.4, 1.5, and 1.7. Investigations regarding the impact of these complexities on computational analyses of healthy and aneurysm tissue have not been conducted. Thus, it is unknown how the material complexities such as anisotropy will affect computational investigations of the aneurysm wall and ultimately realistic fluid-structure interactions (FSIs) of AAAs as well as stented AAAs.

2.1 Establishment of a Theoretical Framework for Nonlinear Arterial Models

The first objective of this study is to establish a theoretical foundation for modeling nonlinear elastic materials such as the arterial wall, lung parenchyma, and the alveolar region (see Ch.4). Additional topics included are the concepts of anisotropy, near incompressibility, and the numerical stability of nonlinear models. This objective is important because future investigations, well beyond the scope of this individual study, will base their analyses on the basic theory presented within this report.
2.2 Development of New Models for Healthy and Aneurysmatic Abdominal Aortic Walls

The theoretical foundation will immediately be implemented to derive a new model for the healthy and aneurysmatic abdominal aortic wall (see Ch.5). Novel features of this model are enhanced realism of physiological tissue, numerical stability, and streamlined implementation within the commercial finite element software ANSYS v.11. Following, the new model’s development, comparisons will be made to existing models of the AAA wall, using known analytical solutions of nonlinear materials.

2.3 Comparative Study of Two Isotropic Abdominal Aneurysm Wall Models

Following the derivation of a new model for the AAA wall, the two most prominent AAA models will be compared through a finite element analysis (see Ch.6). Currently the two models are an isotropic linear elastic model and a nonlinear isotropic hyperelastic model. Until now, a direct comparison between the two models has not been published. This initial analysis will set the stage for future comparisons between the newly developed model and existing models.

2.4 Significance of Research Objectives

After the objectives of this study have been met, a deeper understanding of the different models used to describe nonlinear materials within the elastic regime, will allow for more realistic FSI simulations, including alveolar tissue within the lung airways or the abdominal aneurysm wall. In summary, the current work will greatly enhance the clinical relevance of
computational biomedical research, based on the theoretical framework and fundamental model comparisons provided.
3. CONTINUUM THEORY AND CONSTITUTIVE RELATIONS OF LARGE, FINITE STRAINS APPLIED TO THE ARTERIAL WALL

3.1 Introduction

Continuum mechanics seeks to calculate the deformed shape of an object at a specific time given an initial state of the object, the material that composes the object, and the loads acting on the object (Batra, 2006). It is based on the continuum assumption, which states that the material or object analyzed is made of a continuous medium with no gaps or breaks in the material. It would thus appear that the continuum assumption and underlying continuum mechanics theory would not apply to most biological tissues such as arteries, since it has been well established that the tissue is heterogeneous with gaps in its molecular structure (making the artery discontinuous). However, since the characteristic dimensions of most tissues are considerably larger than the dimensions of its molecular components (i.e., the diameter of the artery is much greater than the diameter of a collagen fiber), it is possible to make the continuum assumption regarding arterial tissue and utilize continuum mechanics theory to quantify the arterial tissue’s mechanical response (Humphrey, 2002).

Relations that are equally essential to continuum mechanics are the constitutive equations. In the majority of continuum mechanics problems, the stress equilibrium equations coupled with the strain displacement relations do not provide enough equations to match the number of independent unknowns in the grouped equations. The constitutive
equations provide a relationship between the independent stress and strain components and are then coupled with the equilibrium and strain displacement equations. After this coupling, the system of equations has the same number of equations as independent unknowns. Thus, it is the constitutive equations that describe the material properties of the continuum object and complete the system of equations utilized to solve continuum mechanics problems (Hollister, 2006a-c).

3.2 Characteristics of Constitutive Equations

A constitutive equation appears generally as a phenomenological model that describes the effects of the mechanical conditions and material properties. Clearly a constitutive equation that accurately describes all forms of biological tissue under all mechanical conditions does not exist. Each type of tissue usually behaves quite differently, and tissues such as the arterial wall have been shown to possess different mechanical properties based on its location within the human body (Silver, 2003; Fung, 1993). As discussed in Ch. 1, arteries infected with pathological diseases, such as atherosclerosis and aneurysms, have vastly different mechanical properties than healthy, normal arteries (Humphrey, 2002). Also, arteries with abdominal aortic aneurysms (AAAs) have been shown to have significant variability in the local mechanical properties of each region of the aneurysm wall (Thubrikar et al., 2001a, Vallabhaneni et al., 2004). Thus, to accurately use constitutive equations to model arterial behavior, the constitutive equation can only be valid for the specified conditions of the current situation being modeled.
The constitutive equation itself consists of unknown constants, measures of strain or deformation, and sometimes the rate of deformation. Benchmark experimental sets of stress vs. strain data are thus required to select the constants within the constitutive equation so that the equation matches the experimental data. Due to a constitutive equation’s dependency on mechanical conditions, it is imperative to match the experimental loading with the loading that will be entered into the model during numerical analyses. In other words, although the constitutive model is used to computationally investigate the behavior of an object, its components require experimental data from mechanical conditions similar to those that the model will encounter during simulation. The development of constitutive models is an iterative procedure that includes obtaining experimental data, significant theoretical groundwork, and frequent critical analyses. The procedure will be discussed in greater detail in Ch. 4. Throughout Ch.2, the focus will be the theoretical framework used to create constitutive correlations for the arterial wall.

3.3 Foundations for Constitutive Equations of Large Deformations

Large deformations are typically defined as strain greater than 3 to 5 %, and most soft, biological tissue tends to exhibit deformations within this classification. Furthermore, simple engineering equations of strain and the small-strain tensor do not contain the needed parameters to accurately describe large deformations. Thus, more advanced equations with additional parameters are needed to quantify the behavior of solid bodies undergoing large strain.
The following sections contain intense mathematical operations and theory that upon first glance appear unrelated to modeling the arterial wall. However, these operations and underlying theory lay the foundation for new measures of strain, stress, and deformation that are essential to creating elastic nonlinear models of the arterial wall. The most important terms of the following theory for arterial wall modeling are listed in Table 3.1.

<table>
<thead>
<tr>
<th>Table 3.1: Important terms and symbols for nonlinear arterial wall models</th>
</tr>
</thead>
<tbody>
<tr>
<td>Per unit volume Helmholtz free energy</td>
</tr>
<tr>
<td>Deformation Gradient</td>
</tr>
<tr>
<td>Right Cauchy-Green stretch tensor</td>
</tr>
<tr>
<td>Invariants of right Cauchy-Green stretch tensor</td>
</tr>
<tr>
<td>Cauchy stress tensor</td>
</tr>
<tr>
<td>First Piola-Kirchoff stress tensor</td>
</tr>
<tr>
<td>Second Piola-Kirchoff stress tensor</td>
</tr>
</tbody>
</table>

Utilizing the previous terms, nonlinear models of the arterial wall within the elastic regime can be constructed. The following sections will discuss in great detail the theoretical framework of arterial wall modeling in both healthy and aneurysm cases using the underlying theory of hyperelasticity.

### 3.3.1 Mathematical Preliminaries

Tensors are the fundamental components in the description of continuum mechanics. Thus, a solid knowledge of certain aspects of tensor/matrix mathematics is crucial to the application of continuum mechanics to arterial tissue. All symmetric tensors have a principal coordinate system that when the symmetric tensor is expressed in terms of the
principal coordinate system, the tensor contains only diagonal elements. The principal values and corresponding directions of these values are the eigenvalues and eigenvectors of the symmetric tensor, respectively. Thus, if $\Gamma_{ij}$ is any symmetric tensor (not the deformation gradient), $\mu$ are the corresponding eigenvalues, and $\delta_{ij}$ is the Kronecker delta (which is the identity matrix of dimension $i$ and $j$), the principal values of the symmetric matrix $\Gamma_{ij}$ satisfy the following condition:

$$\det(\Gamma_{ij} - \mu \delta_{ij}) = 0 \quad (3.1)$$

To aid in solving the eigenvalues three scalars called principal invariants will be defined. The numerical value of each invariant is not dependent on the coordinate system that $\Gamma_{ij}$ is based on. Thus, for any given symmetric tensor, each invariant will yield the same scalar value regardless of the coordinate system. However, the principal invariants are not constants, because if the symmetric tensor expresses a quantity over a specified area, it is possible that the invariants could have different values at different points within the area. Their change in value would not be caused by the coordinate system though, rather it would change based on the different values within the symmetric tensor. The equations for the three invariants are:

$$I_\Gamma = \Gamma_{ij} \delta_{ij} = \Gamma_{ij} = \text{tr}(\Gamma) \quad (3.2)$$

$$II_\Gamma = \frac{1}{2} \left( \Gamma_{ij} \Gamma_{ji} - \Gamma_{ij} \Gamma_{ji} \right) = \frac{1}{2} \left[ \left( \text{tr}(\Gamma) \right)^2 - \text{tr}(\Gamma^2) \right] \quad (3.3)$$

$$III_\Gamma = \det \Gamma_{ij} = \text{J}_\Gamma \quad (3.4)$$

Using the three invariants in Eqs. 3.2–3.4, it is possible to expand Eq. (3.1) to the cubic function:
\[-\mu^3 + I_\Gamma \mu^2 + II_\Gamma \mu + III_\Gamma = 0\]  \hspace{1cm} (3.5)

The three roots of Eq (3.5) are the eigenvalues/principal values of the coordinate system. To find the corresponding directions of these values, each eigenvalue must be individually inserted into the following equation and solved for the vector \(\vec{u}\).

\[
\left(\Gamma_{ij} - \mu \delta_{ij}\right) \cdot \vec{u}_i = 0
\]  \hspace{1cm} (3.6)

Thus, the principal coordinates for a 3x3 symmetric tensor will be the tensor’s three eigenvalues (principal values) each having its own corresponding eigenvector (principal basis/direction).

The Cayley-Hamilton theorem yields important relations in the field of continuum mechanics and states that the symmetric tensor \((\Gamma_{ij})\) can be substituted for \(\mu\) in Eq. (3.5). Thus, it is possible to calculate \((\Gamma_{ij})^3\) from \(\Gamma_{ij}\), \((\Gamma_{ij})^2\), \(\delta_{ij}\), and the principal invariants of \(\Gamma_{ij}\).

\[-\left(\Gamma_{ij}\right)^3 + I_\Gamma \left(\Gamma_{ij}\right)^2 - II_\Gamma \left(\Gamma_{ij}\right) - III_\Gamma \delta_{ij} = 0\]  \hspace{1cm} (3.7)

If Eq. (3.7) is multiplied by the inverse of \(\Gamma_{ij}\), the result will yield an equation that relates the inverse of the symmetric tensor \(\Gamma_{ij}^{-1}\) to the same variables that Eq (3.7) is dependent on.

\[
\left(\Gamma_{ij}\right)^2 - I_\Gamma \Gamma_{ij} + II_\Gamma \delta_{ij} = III_\Gamma \left(\Gamma_{ij}\right)^{-1}
\]  \hspace{1cm} (3.8)

Further decomposition of the equation can be conducted by taking the first invariant of all the tensors in Eq. (3.8). Such an operation, results in a useful quantity that relates the first invariant of \((\Gamma_{ij})^{-1}\) to the second and third invariants of \((\Gamma_{ij})\).

\[I_{\Gamma^{-1}} = \frac{II_\Gamma}{III_\Gamma}\]  \hspace{1cm} (3.9)
In conclusion, the three invariant equations can be differentiated with respect to the original symmetric tensor. The derivatives of the principal invariants will be vital for developing continuum relations of specific materials, ie, the arterial walls.

\[
\frac{\partial \Gamma_\gamma}{\partial \Gamma_{ij}} = \delta_{ij} \tag{3.10}
\]

\[
\frac{\partial \Gamma_\gamma}{\partial \Gamma_{ij}} = I_\gamma \delta_{ij} - \Gamma_{ij}^T \tag{3.11}
\]

and

\[
\frac{\partial \Gamma_\gamma}{\partial \Gamma_{ij}} = III_\gamma \Gamma_{ij}^{-T} \tag{3.12}
\]

### 3.3.2 Deformation Gradient Tensor

After a certain load has been applied to an unloaded solid, some form of deformation occurs (cf. Fig. 3.1)

![Fig. 3.1: Unloaded and deformed configurations with displacement vector \( \vec{u} \)](image)

Assuming that the spatial domain is a real \( \mathbb{R}^3 \) space (subscripts have indices 1, 2, and 3 unless otherwise noted), the equations can be expressed using Einstein’s index notation.
The deformed vector $\vec{y}$ can then be related to the unloaded/original vector $\vec{x}$ by a displacement vector $\vec{u}$ (cf. Eq. (3.13)).

$$y_i = x_i + u_i \quad (3.13)$$

Additional vectors exist that describe the material orientation within both the unloaded (reference) and the deformed configuration. The term $dx$ will be used to express the orientation in the unloaded configuration while $dy$ will be used to express the orientation in the deformed orientation. A mapping between the two orientations can be achieved by differentiating via the chain rule.

$$dy_i = \frac{\partial y_i}{\partial x_j} dx_j \quad (3.14)$$

$$F_{ij} = \frac{\partial y_i}{\partial x_j} \quad (3.15)$$

$$dy_i = F_{ij} dx_j \quad (3.16)$$

This mapping ($F_{ij}$) is defined as the deformation gradient, and is the fundamental measure of the object’s deformation. The two free indices indicate it is a second order tensor with nine terms, yet $F_{ij}$ is not symmetric. Within the nine terms, the deformation gradient describes rigid body motion (rotation and translation), deformations (extension, compressions, and shear), or some combination of them both (Humphrey, 2002). The deformation gradient tensor can thus be decomposed into a rotation and stretch component. It is possible to place either the rotation ($R_{ij}$) or the stretch ($U_{kj}$ or $V_{ik}$) component as the first term in the decomposition. The $U_{kj}$ is considered the right stretch tensor while $V_{ik}$ is considered the left stretch tensor due to their location in the corresponding equations. The principal values of
both the right and left stretch tensors are the principal stretches (current length divided by the reference length) of the solid body. Furthermore, the rotation component has a significant characteristic that when the transpose of the rotation tensor is multiplied by the original rotation tensor, the product is the identity matrix (cf. Eqs. (3.17 – 3.19)).

\[
F_{ij} = R_{ik}U_{kj} \quad (3.17)
\]

\[
F_{ij} = V_{ik}R_{kj} \quad (3.18)
\]

\[
R_{ik}R_{jk} = RR^T = \delta_{ij} \quad (3.19)
\]

The deformation gradient can also be derived from Eq. (3.13) in conjunction with the Kronecker delta \( \delta_{ij} \) which is identical to an identity matrix of the appropriate dimension based on the indices \( i \) and \( j \).

\[
\frac{\partial y_i}{\partial x_j} = \frac{\partial x_i}{\partial x_j} + \frac{\partial u_i}{\partial x_j} \quad (3.20)
\]

\[
\frac{\partial x_i}{\partial x_j} = \delta_{ij} + \frac{\partial u_i}{\partial x_j} \quad (3.21)
\]

\[
F_{ij} = \delta_{ij} + \frac{\partial u_i}{\partial x_j} \quad (3.22)
\]

A vital tool in using the deformation gradient is to utilize its ability to map lines, areas, and volumes between the unloaded and the deformed configurations of the solid. A line can be mapped by using the deformation gradient and a unit vector \( \tilde{M} \) that expresses the line in the original orientation. Such a mapping will yield the square of the stretch ratio (\( \Lambda \)), which is the square of the deformed length over the original length (cf. Eq. (3.23)). An area can be mapped by using the inverse of the deformation gradient, the third invariant, and normal
vectors \( n_j \) and \( N_i \) for the deformed and unloaded configurations respectively. To map a volume between the two configurations is much simpler since it only requires the third invariant of the deformation gradient (\( J_F \)) in addition to the original and deformed volumes (cf. Eqs. (3.24 – 3.26))

\[
\Lambda^2 = \left( \frac{dy}{dx} \right)^2 = M_i F_{ij} M_j \tag{3.23}
\]

\[
F_{ij}^{-1} = \frac{\partial x_i}{\partial y_j} \tag{3.24}
\]

\[
n_j dA_y = J_F N_i F_{ij}^{-1} dA_x \tag{3.25}
\]

\[
dV_y = J_F dV_x \tag{3.26}
\]

3.3.3 Deformation gradient of multiple, successive deformations

Due to the periodic, pulsatile behavior of blood flow, arteries tend to experience multiple deformations in succession to the previous deformation. An example of such deformation is illustrated in Fig. 2.2.

![Multiple, successive deformations](image)

Fig. 3.2: Multiple, successive deformations with displacement vectors \( \vec{u}_1 \) and \( \vec{u}_2 \)
Infinitesimal small vectors in each configuration will be depicted by $dx$, $dy$, and $dz$ in the original, 1$^{\text{st}}$ deformation, and 2$^{\text{nd}}$ deformation respectively. In order to construct equations that relate the different configurations, the displacement vectors must be a function of either the 1) original configuration or 2) the previous deformation as noted in Fig. 3.2. The deformation gradients of the two deformed states are formed in the same manner as previously mentioned in Eqs. (3.14 – 3.16) (cf. Eqs. (3.27 - 3.32)).

$$dy_i = \frac{\partial y_i}{\partial x_j} dx_j$$ \hspace{1cm} (3.27)

$$F_{ij}^{(1)} = \frac{\partial y_i}{\partial x_j}$$ \hspace{1cm} (3.28)

$$dy_i = F_{ij}^{(1)} dx_j$$ \hspace{1cm} (3.29)

$$dz_i = \frac{\partial z_i}{\partial y_j} dy_j$$ \hspace{1cm} (3.30)

$$F_{ij}^{(2)} = \frac{\partial z_i}{\partial y_j}$$ \hspace{1cm} (3.31)

$$dz_i = F_{ij}^{(2)} dy_j$$ \hspace{1cm} (3.32)

The next step is to create an overall deformation gradient that will relate the last deformation to the original configuration.

$$dz_i = \frac{\partial z_i}{\partial x_j} dx_j = F_{ij} dx_j$$ \hspace{1cm} (3.33)

$$F_{ij} = F_{ik}^{(2)} F_{kj}^{(1)}$$ \hspace{1cm} (3.34)

Therefore, the cumulative mapping from the original configuration to the 2$^{\text{nd}}$ (or last) deformation is a mere application of the chain rule.
\[ dz_i = \frac{\partial z_i}{\partial y_j} \frac{\partial y_j}{\partial x_k} dx_k \]  

Equation (3.34) has a profound significance, since it describes how the last deformation (after a series of successive deformations have occurred) can be related back to the original/unloaded configuration of the structure.

### 3.3.4 The right and left Cauchy-Green stretch tensors

Although the deformation gradient is the fundamental measure of change in a solid body’s morphology due to applied loads, it is not ideally suited for elastic analysis. The primary limitations of \( F_{ij} \) are that it is not symmetric and it includes both rigid body motion and stretching of the solid body. A symmetric tensor greatly simplifies mathematical analyses and since rigid body motion does not deform the structure of the solid body it is often desirable to isolate the stretch portion of the deformation gradient. To remedy the situation, tensors that contain only the deformation gradient’s stretch components can be derived. (cf. Eqs. (3.36 – 3.37)).

\[
C_{ij} = F_{ik}F_{kj} = F^T F \tag{3.36}
\]

\[
B_{ij} = F_{ik}F_{jk} = FF^T \tag{3.37}
\]

\( C_{ij} \) of Eq. (2.36) and \( B_{ij} \) of Eq. (3.37) are the right and left Cauchy-Green stretch tensors, respectively. Where the right Cauchy-Green stretch tensor relates all stretches to reference configuration and the left Cauchy-Green stretch tensor relates all deformations to the loaded/deformed configuration. Through the polar decomposition of \( F_{ij} \) outlined in Eq. (3.17) and (3.18), both Cauchy-Green stretch tensors can be constructed using the right and left stretch tensor \( U_{ij} \) and \( V_{ij} \) respectively.
(cf. Eqs. (3.38) and (3.39)).

\[
C_{ij} = F^T F = R_{ik} U_{li} R_{jm} U_{mj} = R^T U^T RU \tag{3.38}
\]

\[
B_{ij} = F F^T = V_{ik} R_{jm} V_{mk} = VRV^T R^T \tag{3.39}
\]

Since Eq. (3.19) states that the rotation tensor \((R_{ij})\) times its transpose yields the identity matrix, it is thus possible to completely eliminate any rotation terms in both the right and left Cauchy-Green stretch tensors.

\[
C_{ij} = R^T U R = U_{li} \delta_{mj} U_{mj} = U_{li} U_{lj} \tag{3.40}
\]

\[
B_{ij} = VRV^T R = V_{ij} \delta_{jm} V_{jm} = V_{ij} V_{jl} \tag{3.41}
\]

Furthermore, both the right and left Cauchy-Green stretch tensor are a symmetric, positive definite tensor, which ensures that the tensor will have three real eigenvalues. Such a constraint is very important because the square roots of these eigenvalues are the principal stretches of the solid body.

### 3.3.5 The Green-Lagrange and Almansi-Eulerian strain tensors

It is often desirable to quantify the solid body’s deformation as zero when no stretching occurs. However, both the right and left stretch tensors yield the identity matrix under loading conditions that yield no stretching of the solid body. Thus, the quantity titled strain is derived to be a new measure of deformation that has a value of zero when no stretching of the solid body occurs. To begin the derivation, it is prudent to return to the foundation of all deformation, the deformation gradient \(F_{ij}\).

\[
E_{ij} = \frac{1}{2} \left( F_{ik} F_{kj} - \delta_{ij} \right) \tag{3.42}
\]
\[ E_{ij}^* = \frac{1}{2} \left( \delta_{ij} - F_{ik}^{-1} F_{jk}^{-1} \right) \]  

(3.43)

In Eqs. (3.42) and (3.43), \( E_{ij} \) represents the Green-Lagrange strain while \( E_{ij}^* \) represents the Almansi-Eulerian strain. The important difference between the two tensors is that the Lagrange strain refers to the reference/unloaded configuration while the Eulerian strain refers to the deformed configuration. Reinforcing this concept, Eqs. (3.38) and (3.39) can be used to express both strain tensors in terms of the right and left stretch tensors.

\[ E_{ij} = \frac{1}{2} \left( C_{ij} - \delta_{ij} \right) \]  

(3.44)

\[ E_{ij}^* = \frac{1}{2} \left( \delta_{ij} - B_{ij}^{-1} \right) \]  

(3.45)

Equation (3.44) will be very useful in future calculations because it reveals that the Green-Lagrange strain tensor is symmetric, since it is dependent on two symmetric tensors.

Furthermore, it is possible to expand Eqs. (3.42) and (3.43) in terms of the displacement vector \( u_i \).

\[ E_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right) \]  

(3.46)

\[ E_{ij}^* = \frac{1}{2} \left( \frac{\partial u_i}{\partial y_j} + \frac{\partial u_j}{\partial y_i} - \frac{\partial u_k}{\partial y_i} \frac{\partial u_k}{\partial y_j} \right) \]  

(3.47)

The infinitesimal (linearized) strain tensor \( \epsilon_{ij} \) only contains the first two terms in Eq. (3.46):

\[ \epsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \]  

(3.48)
The lack of the third term in the infinitesimal strain tensor implicates that the infinitesimal strain tensor is linearly dependent on the deformation gradient $F_{ij}$. Thus it can only be used as a meaningful measure of strain when both rigid body motion and deformation are small. Thus, the Lagrange and Eulerian strain tensors fully capture all effects of large deformations and also contain a quadratic term, making any calculation involving these tensors nonlinear.

### 3.3.6 Cauchy stress tensor

The Cauchy stress tensor ($\sigma_{ij}$) always refers to the deformed state/configuration of the tissue. It assumes that an infinitesimal volume has been shrunken down and the stresses on the resulting element have the following configuration.

\[
\sigma_{ij} = \begin{bmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_{21} & \sigma_{22} & \sigma_{23} \\
\sigma_{31} & \sigma_{32} & \sigma_{33}
\end{bmatrix}
\]

Fig 3.3: Visual representation of Cauchy stress tensor with its matrix form

Source: www.eml.ou.edu/equation/SOLIDS/STRESS/STRESS.HTM

The primary limitation when using the Cauchy stress tensor to analyze large deformations is that the area of the deformed configuration is often unknown. Thus, although the Cauchy stress is a common method of reported force/area stress, a transformation of this tensor should be used when analyzing large deformations. It will be important to understand and keep track of all the following transformations. Thus, it is helpful to focus on the deformation gradient ($F_{ij}$) in each equation. Whenever the original form of the deformation
gradient is used in an equation, the result is mapped forward to the deformed configuration. Conversely, whenever the inverse of the deformation gradient is present in an equation, the result is mapped backwards to the original/unloaded configuration.

3.3.7 First Piola-Kirchoff stress tensor

In order to remedy the difficulty of not knowing the area of the deformed state, the Cauchy stress tensor will be mapped to the original/unloaded configuration of the solid model. The constraint of the new equation is to ensure that the total force each component of the Cauchy stress tensor generates will be conserved in the new stress tensor. Thus to begin the transformation, the calculation of the total force of each Cauchy stress component must be completed:

\[ dP_i^y = \sigma_{ij} n_j dA_y \] (3.49)

Where \( dP_i^y \) is the total force, \( \sigma_{ij} \) is the Cauchy stress tensor, \( n_j \) is a vector normal to the surface of the deformed configuration, and \( dA_y \) is the surface area of the deformed configuration appropriate face. In order to conserve \( dP_i^y \), a new equation is formed based on a new stress tensor \( T_{ij} \), a new normal vector \( N_j \), and \( dA_x \) the surface area of the original configuration’s face.

\[ dP_i^y = T_{ij} N_j dA_x \] (3.50)

The new stress tensor \( T_{ij} \) is the 1st Piola-Kirchoff stress tensor and can be derived through the use of the third invariant of the deformation gradient \( J_F \), the mapping in Eq. (3.25), and Eqs. (3.49) and (3.50):

\[ T_{ij} = \sigma_{ik} J_F F_{jk}^{-1} = \sigma J_F F^{-T} \] (3.51)
Additionally, the transformation from $T_{ij}$ to $\sigma_{ij}$ is:

$$\sigma_{ij} = \frac{F_{jk} T_{ik}}{J_F} = J_F^{-1} F^T = J_F^{-1} F F^T \quad (3.52)$$

Although the 1st Piola-Kirchoff stress relates the actual force on the deformed configuration to the original configuration, it has two primary limitations. The first is that it cannot be used with the Green-Lagrange strain tensor because their product is not equivalent to the product of the Cauchy stress tensor and the small deformation tensor. The second is that the 1st Piola-Kirchoff tensor is not symmetric, which makes it more difficult to include in numerical analyses. Thus, a second transformation of the Cauchy stress tensor should be constructed.

### 3.3.8 Second Piola-Kirchoff stress tensor

To derive a new stress tensor, a different force value must be conserved than the total force of the Cauchy stress tensor on the deformed configuration (as was used to derive the 1st Piola-Kirchoff tensor). Thus, the total force in the deformed configuration $P_{ij}^y$ is mapped to the original configuration $dP_{ij}^x$ using the inverse of the deformation gradient tensor $F_{ij}^{-1}$.

This new force is a fictitious force that cannot be measured and only calculated, while the force used in the 1st PK and the Cauchy stress tensor is a real applied force to the structure.

$$dP_{ij}^x = F_{ij}^{-1} dP_{ij}^y \quad (3.53)$$

Eq. (3.49) can now be substituted into Eq. (3.53) in order to map the force from the Cauchy stress tensor to the original configuration.

$$dP_{ij}^x = F_{ij}^{-1} \sigma_{ik} n_k dA_y \quad (3.54)$$

A new stress tensor can now be constructed by conserving the mapped force of Eq. (3.54):
\[ dP_i^x = S_{ij} N_j dA_x \]  
(3.55)

where \( S_{ij} \) is the new stress tensor, \( N_j \) is a vector normal to the face of the original configuration, and \( dA_x \) is the surface area of the original configuration’s face. \( S_{ij} \) is the 2nd Piola-Kirchoff stress tensor and it can be derived by relating the third invariant of the deformation gradient \( F_J \), the mapping outlined in Eq. (3.25), and Eqs.(3.54) and (3.55).

\[
S_{ij} = F^{-1}_{ik} \sigma_{kl} J_F F^{-1}_{jl} = F^{-1} \sigma J_F F^{-T}
\]  
(3.56)

\[
S_{ij} = F^{-1}_{ik} T_{kj} = F^{-1} T = S^T
\]  
(3.57)

The transformation from \( S_{ij} \) to \( \sigma_{ij} \) is:

\[
\sigma_{ij} = \frac{F_{ik} S_{km} F_{jm}}{J_F} = J_F^{-1} F S F^{-T}
\]  
(3.58)

The advantage of the 2nd Piola-Kirchoff (PK) stress tensor is primarily threefold: 1) it is a symmetric, making it easier to use in numerical analyses, 2) it is calculated in the original configuration rather than the deformed configuration, and 3) when multiplied by the Green-Lagrange strain it produces the same strain energy as the Cauchy stress tensor and the small strain tensor (cf. Eq. (3.59)).

\[ S_{ij} E_{ij} = \sigma_{ij} E_{ij} \]  
(3.59)

It is advantageous to define a quantity called material stiffness that is expressed in terms of the 2nd PK stress and the Green-Lagrange strain.

\[ K_{lmkj} = \frac{\partial S_{lm}}{\partial E_{kj}} \]  
(3.60)
The physical meaning behind this quantity is the change in stress related to a respective change in strain. Thus, a large value for the material stiffness would reflect a large increase in stress correlated with a small increase in strain. An important note is that the matrix $K_{lmkj}$ is a fourth order tensor that corresponds to the linear stiffness matrix.

Since the small strain tensor is not applicable in large deformation problems, the previously discussed characteristics of the 2$^{nd}$ PK stress tensor make it the primarily stress used to solve large deformation problems. After the 2$^{nd}$ PK stress has been calculated, the Cauchy stress is then derived from the 2$^{nd}$ PK stress using Eq. (3.58). To fully grasp the technique it is important to understand the physical meaning of each stress tensor. The primary difference between the stress tensors is whether the force and area are mapped in the original or deformed configuration. The Cauchy stress tensor has its force (the real, applied force) and area mapped in the deformed configuration while the 1$^{st}$ PK stress has its force (the real, applied force) mapped in the deformed configuration and its area in the original configuration. Conversely, the 2$^{nd}$ PK stress has its force (a fictitious force) and its area mapped to the original configuration. An illustration of the different mappings is located in Fig 3.4.
### 3.3.9 The strain-energy function

When an elastic solid body undergoes any form of deformation, mechanical/strain energy is stored in a portion of the body’s potential energy. A fundamental understanding of this mechanical energy needs to begin with the rate-based form of the conservation of energy (1\textsuperscript{st} Law of Thermodynamics). It is expressed in Eq. (3.61)

\[
\frac{Du}{Dt} = T_{ij} \frac{DF_{ij}}{Dt} - \nabla_i \dot{Q}_i + \dot{g} 
\]

Fig. 3.4: The mappings of the Cauchy, 1\textsuperscript{st} PK, and 2\textsuperscript{nd} PK stress tensors
in a referential, per unit volume form where \( \frac{Du}{Dt} \) represents the material time derivative of the internal energy, \( T_\theta \) represents the 1st PK stress tensor, \( \frac{DF_{ij}}{Dt} \) represents the material time derivative of the deformation gradient, \( \nabla_i \) is the del operator in the reference configuration, \( \dot{Q}_i \) is the time dependent Piola-Kirchoff heat flux in the reference configuration, and \( \dot{g} \) is a heat source/generation term per unit time and per unit volume in the reference configurations. Through this equation, the local form of the balance of energy in the material description is expressed. In summary, the material’s internal energy with respect to time is related to the rate of internal mechanical work \( T_\theta : \frac{DF_{ij}}{Dt} \) and the net rate of heat added to the system \( -\nabla_i \dot{Q}_i + \dot{g} \).

To isolate the energy that is stored due to deformation, the internal energy must be expanded in terms of the Helmholtz potential/free-energy and the entropy combined with temperature:

\[
u = \hat{\psi} + \eta \xi
\]

\[
\frac{Du}{Dt} = \frac{D\hat{\psi}}{Dt} + \eta \frac{D\xi}{Dt} + \xi \frac{D\eta}{Dt}
\]

(3.62)

where \( \hat{\psi} \) represents the Helmholtz potential, \( \eta \) is the entropy, and \( \xi \) is the absolute temperature (all per unit volume of the reference configuration). Equation (3.62) can than be coupled with the Clausius-Planck inequality (the 2nd Law of Thermodynamics) to derive an
expression that relates the Helmholtz potential and the rate of internal mechanical work (Holzapfel, 2000).

\[ T_{ij} : \frac{DF_{ij}}{Dt} - \frac{D\hat{\psi}}{Dt} \geq 0 \]  

(3.63)

Eq. (3.63) can be further simplified if the material being studied is considered to be isothermal / purely mechanical.

\[ T_{ij} : \frac{DF_{ij}}{Dt} \geq 0 \]  

(3.64)

In perfectly elastic materials, Truesdell and Knoll (2004) state that the deformed material produces no entropy (all energy is recoverable) and thus Eq. (3.64) should be equivalent to zero.

\[ T_{ij} : \frac{dF_{ij}}{dt} - \frac{D\hat{\psi}}{Dt} = 0 \]  

(3.65)

Although no material truly satisfies the condition of perfect elasticity, the energy lost is often deemed negligible for many materials. If \( \hat{\psi} \) is only a function of the deformation \( \{ \psi = \hat{\psi}(F_{ij}) \} \), its material time derivative has the following form.

\[ \frac{D\hat{\psi}}{Dt} = \frac{\partial \hat{\psi}}{\partial F_{ij}} : \frac{DF_{ij}}{Dt} \]  

(3.66)

Substituting Eq. (3.66) into Eq. (3.65) yields:

\[ \left( T_{ij} - \frac{\partial \hat{\psi}}{\partial F_{ij}} \right) : \frac{DF_{ij}}{Dt} = 0 \]  

(3.67)

The \( DF_{ij}/Dt \) term in Eq. (3.67) is not always zero. Thus, in order for Eq. (3.67) to hold for every possible value of \( F_{ij} \), the term in parenthesis must be equal to zero, which results in an
expression that relates the 1st PK Stress tensor and the Helmholtz potential / strain energy function.

\[ T_{ij} - \frac{\partial \hat{\psi}}{\partial F_{ij}} = 0 \]  
(3.68)

\[ T_{ij} = \frac{\partial \hat{\psi}}{\partial F_{ij}} \]  
(3.69)

Equation (3.69) is a powerful equation that enables purely elastic materials to be characterized by their Helmholtz potential of elastic deformation. Since the Helmholtz potential is dependent on deformation alone, the free energy can be characterized as the energy that is stored for a given amount of deformation/strain. Thus, the Helmholtz potential can be referred to as a per-unit-volume, strain-energy-function.

The strain energy function (SEF) has several fundamental characteristics that remain constant regardless of any material properties. First and most important is that the SEF should be differentiable with respect to all components of the deformation gradient. Thus, \( \frac{\partial \hat{\psi}}{\partial F_{ij}} \) should exist for all values of \( F_{ij} \). The second characteristic is that the SEF should be polyconvex, such that ellipticity of the SEF is maintained. Ball (1977) pioneered the constraint of polyconvexity and illustrated the constraint’s utility through calculus of variations theory. A function is polyconvex if and only if the following condition is satisfied:

\[ W(F) = P\left(F, \det[F]F^{-1}, \det[F]\right) \]  
(3.70)

Where \( P \) is a convex function, \( F \) is any invertible matrix and \( \det[\bullet] \) is the determinant of \( \bullet \). Polyconvexity of the SEF ensures an unambiguous global solution exists within the domain of possible solutions by implying the concept of quasiconvexity introduced by Morrey.
(1952). Additionally, quasiconvexity implies the global solution also reports physically meaningful mechanical behavior through the concept of rank 1 convexity (Ball, 1977; Schroder and Neff, 2003; Balzani et al., 2006).

\[ \text{polyconvexity} \Rightarrow \text{quasiconvexity} \Rightarrow \text{rank-one convexity} \]

The converse implications of the previous conditions are not true. Further results of polyconvexity include numerical stability, real wave speeds of elastic deformations, the system of equations forms a well-posed problem, and the acoustic tensor that corresponds to the stored energy is elliptic for all deformations (Schroder and Neff, 2003; Blazani et al., 2006). Thus, it is evident the condition of polyconvexity has a profound affect on the behavior of the SEF. Closely tied to the condition of polyconvexity is the convexity of the SEF energy contours for arbitrary deformations. The convex shape of the contours ensures an increase in strain energy when there is an increase in deformation (Holzapfel et al., 2000). Since the Helmholtz potential and the object’s density are always positive, the SEF must also be non-negative for all possible deformations.

\[ \psi(F_{ij}) \geq 0 \]  \hspace{1cm} (3.71)

Conversely, if no deformation occurs than the deformation gradient will be equivalent to the identity matrix, and the SEF must be equivalent to zero. Such a condition establishes a referential/stress free configuration.

\[ \psi(F_{ij} = \delta_{ij}) = 0 \]  \hspace{1cm} (3.72)

The SEF is also independent of any translation or rotation, making it dependent only on the deformation of the material. Thus, if an initial deformation has a deformation gradient of \( F_{ij} \) and the prescribed deformation is followed by a rotation or translation, which changes the
overall deformation gradient to $R_{ij}$, the strain energy density functions of both deformation
gradients should be equal.

$$\psi(F_{ij}) = \psi(R_{ij}) \tag{3.73}$$

Further extrapolation of the SEF being independent of rigid body motion states that it is
possible to equally express the SEF as a function of other deformation tensors. Therefore, if
a deformation is described by the Green-Lagrange strain tensor ($E_{ij}$), the right Cauchy
deformation tensor ($C_{ij}$), or the deformation gradient ($F_{ij}$), the SEFs based on the three
deformation tensors must contain equivalent amounts of energy.

$$\psi(F_{ij}) = \psi(C_{ij}) = \psi(E_{ij}) \tag{3.74}$$

The final characteristic of the SEF ensures that the material cannot be infinitely expanded or
crushed to a volume of zero (neither create nor destroy matter). To enforce these conditions
it is necessary to impose limits on the SEF that are not physically possible when such
conditions occur.

$$\psi(F_{ij}) \rightarrow +\infty \quad \text{as} \quad \det F_{ij} \rightarrow +\infty \tag{3.75}$$

$$\psi(F_{ij}) \rightarrow +\infty \quad \text{as} \quad F_{ij} \rightarrow 0^+ \tag{3.76}$$

Equations (3.75) and (3.76) require an infinite amount of strain energy to compress the given
amount of material to a vanishing volume or expand the amount to an infinite volume
(Holzapfel, 2000). A summary of the fundamental characteristics of the SEF can be found in
Table 3.2.
### Table 3.2: Characteristics of an ideal strain energy function (SEF)

<table>
<thead>
<tr>
<th>Condition</th>
<th>Property</th>
</tr>
</thead>
<tbody>
<tr>
<td>SEF must be differentiable with respect to all components of the deformation gradient</td>
<td>( \frac{\partial \hat{\psi}}{\partial F_{ij}} ) must exist for all ( F_{ij} )</td>
</tr>
<tr>
<td>SEF should be polyconvex and exhibit convex energy contours for arbitrary deformations</td>
<td>( \hat{\psi}(F) = P(F, \det[F]F^{-1}, \det[F]) )</td>
</tr>
<tr>
<td>SEF should exhibit convex energy contours for arbitrary deformations</td>
<td>Must be tested for each SEF</td>
</tr>
<tr>
<td>SEF must be positive</td>
<td>( \hat{\psi} \geq 0 )</td>
</tr>
<tr>
<td>SEF only reports energy of deformation, not rigid body motion</td>
<td>( \hat{\psi}(F_{ij}) = \hat{\psi}(U_{ij}) )</td>
</tr>
<tr>
<td>( U_{ij} ) is the symmetric right stretch tensor</td>
<td>( \hat{\psi}(F_{ij} = \delta_{ij}) = 0 )</td>
</tr>
<tr>
<td>SEF equals zero under conditions where no deformation occurs</td>
<td>( \hat{\psi}(F_{ij}) = \hat{\psi}(C_{ij}) = \hat{\psi}(E_{ij}) )</td>
</tr>
<tr>
<td>SEF should calculate the same amount of energy whether it is a function of the deformation gradient, right Cauchy-Green stretch tensor, or the Green-Lagrange strain tensor</td>
<td>( \hat{\psi}(F_{ij}) \rightarrow +\infty ) as ( \det F_{ij} \rightarrow +\infty )</td>
</tr>
<tr>
<td>Material can neither be created nor destroyed</td>
<td>( \hat{\psi}(F_{ij}) \rightarrow +\infty ) as ( \det F_{ij} \rightarrow 0^+ )</td>
</tr>
</tbody>
</table>

#### 3.3.10 Stress tensors and the strain energy function

Materials that exhibit large finite strains and are characterized through the resulting strain energy are called hyperelastic materials. In order to fully characterize the mechanical behavior of hyperelastic materials, constitutive equations between stress and a form of displacement must be derived from the previous equations of non-linear continuum mechanics. Equation (3.69) is a powerful formula that relates the 1st PK stress to the SEF and a measure of deformation. Thus, each stress tensor can be related to the SEF and the deformation gradient by Eqs. (3.69), (3.52), and (3.57).
\[ T_{ij} = \frac{\partial \hat{\psi}}{\partial F_{ij}} \quad (3.77) \]

\[ S_{ij} = F_{ik}^{-1} T_{kj} = F_{ik}^{-1} \frac{\partial \hat{\psi}}{\partial F_{ij}} = F_{ik}^{-1} \frac{\partial \hat{\psi}}{\partial F} \quad (3.78) \]

\[ \sigma_{ij} = \frac{T_{ak} F_{jk}}{J_F} = \frac{1}{J_F} \frac{\partial \hat{\psi}}{\partial F_{ik}} F_{jk} = J_F^{-1} \frac{\partial \hat{\psi}}{\partial F} F^T \quad (3.79) \]

Equations (3.76) to (3.78) enable most of the mechanical properties of a hyperelastic material to be expressed in a single, scalar equation: the SEF. However, the desired constitutive relation must relate stress components to their corresponding displacement/strain components independent of rigid body motion and the components of the deformation gradient contain information regarding rigid body motion and displacement. It is thus desirable to express Eqs. (3.76 – 3.78) in a form of displacement that excludes information about rigid body motion. Using the transformations listed in Eqs. (3.79) and (3.80) plus the expression in Eq. (3.73), Eqs. (3.76 – 3.78) can be expressed in terms of the symmetric, right Cauchy stretch tensor.

\[ J_F^{-1} \frac{\partial \hat{\psi}}{\partial F_{ik}} F_{jk} = J_F^{-1} \frac{\partial \hat{\psi}}{\partial F} F^T = J_F^{-1} F \left( \frac{\partial \hat{\psi}}{\partial F_{ij}} \right)^T \quad (3.80) \]

\[ \left( \frac{\partial \hat{\psi}(F_{ij})}{\partial F_{ij}} \right)^T = 2 \frac{\partial \hat{\psi}(C_{ik})}{C_{ik}} F_{jk} \Rightarrow \left( \frac{\partial \hat{\psi}(F^T)}{\partial F} \right)^T = 2 \frac{\partial \hat{\psi}(C)}{C} F^T \quad (3.81) \]

\[ \sigma_{ij} = \frac{1}{J} F_{im} \frac{\partial \hat{\psi}}{\partial C_{mn}} F_{jn} = \frac{1}{J} F \frac{\partial \hat{\psi}}{\partial C} F^T \quad (3.82) \]

\[ T_{ij} = 2 F_{ik} \frac{\partial \hat{\psi}}{\partial C_{kj}} = 2 F \frac{\partial \hat{\psi}}{\partial C} \quad (3.83) \]
\[ S_{ij} = 2 \frac{\partial \hat{\psi}}{\partial C_{ij}} = \frac{\partial \hat{\psi}}{\partial E_{ij}} \quad (3.84) \]

Equations (3.82 – 3.84) provide the needed relations between stress and displacement that excludes information of rigid body motion. Furthermore, the expressions listed in Eqs. (3.82 – 3.84) fully characterize hyperelastic materials when coupled with the equations of equilibrium, compatibility, and the equation for the SEF.

3.4 Modeling the Arterial Wall

The non-linear behavior of the arterial wall and the large, finite deformation that the wall undergoes require that non-linear structural theory be used to construct viable models of the arterial wall’s mechanical behavior. Thus, the classical engineering constitutive expression, Hooke’s Law, does not apply for the arterial wall (Fung, 1993; Humphrey, 2002; Holzapfel et al., 2000). Instead the constitutive relation between stress and strain/displacement must be obtained through nonlinear methods such as the Helmholtz potential as presented in the previous section on non-linear continuum mechanics of isothermal solids.

3.4.1 Phenomenological vs. Structural Models

Two primary categories of arterial wall models exist: phenomenological models and the more detailed, structural models. Phenomenological models focus on the macroscopic behavior of the tissue. They only report the end-results of loading the tissue, and do not consider the underlying microstructure of the arterial wall. Thus, such models have no physical meaning behind any constants within the equations, yet can still be used to closely match experimental data and obtain the overall mechanical response of the tissue. The most
well-known phenomenological model was developed by Fung (titled the Fung-elastic model); its general form will be presented in Ch. 4. A structural model of the arterial wall models the same macroscopic mechanical response; however, it incorporates the underlying microstructure of the arterial wall. More specifically, a structural model has specific terms that incorporate the reinforcing behavior of the collagen fibers within the arterial wall, while the phenomenological model will only capture the end result of the collagen fiber reinforcement. One of the most recognized structural models of the arterial wall was developed by Holzapfel; its general form will be discussed alongside the Fung elastic model in Ch. 4 (Holzapfel et al., 2000; Sacks, 2000).

3.4.2 Modeling the Primary Characteristics of the Arterial Wall

The properties of the arterial wall have been discussed throughout Ch. 1. A summary of the arterial wall’s defining characteristics are given in Table 3.3.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Dynamic adaptation and remodeling</td>
<td>Nearly incompressible and isothermal</td>
</tr>
<tr>
<td>Nonlinear stress vs. strain curve</td>
<td>Collagen and elastin fiber reinforced</td>
</tr>
<tr>
<td>Viscoelastic behavior with hysteresis</td>
<td>Anisotropic material properties</td>
</tr>
<tr>
<td>Residual stress and strain</td>
<td>Heterogeneous composition</td>
</tr>
</tbody>
</table>

Arteries are known for their change in mechanical and material properties due to metabolic activities, arterial disease, and the passing of time. The difficulty in combining such phenomena with analyses of arterial hemodynamics is that the arterial remodeling and adaptation often occurs on a different time scale than the hemodynamics of the arteries. Thus, incorporating both phenomena into a single analysis on the same timescale is currently
not a viable option when studying arterial diseases such abdominal aortic aneurysms (AAAs). However, a few options do exist to incorporate the remodeling (or deterioration) of the arterial wall in hemodynamic analyses: (i) focus on the remodeling of the arteries and simulate mean (steady) blood flow through the arteries, (ii) construct different material models that quantify different stages in the arterial remodeling/adaptation process, while analyzing each material model with the same pulsatile, hemodynamic model, or (iii) conduct analyses of the arterial remodeling and hemodynamics as if both phenomena occur at the same timescale.

The investigator must select one of the preceding analysis options with respect to the system that requires the most accuracy. In the current modeling effort the passive arterial response was desired to have the greatest realism under pulsatile conditions. Thus, method (ii) where the various stages of the remodeling and adaptation process are included internally within the model parameters was utilized to model the arterial wall. In future efforts, the active response of the artery may be included as an attempt to further improve the model’s biomechanical realism.

Another defining characteristic of arterial tissue is its nonlinear stress vs. strain curve with a presence of hysteresis between the loading and unloading of arteries. The hysteresis region reveals that all of the energy stored within the strain energy is not fully recovered when the arterial tissue is unloaded (Humphrey, 2002; Holzapfel et al., 2000; Fung, 1993). By definition an elastic material has the same loading and unloading curves, time-independent behavior, and the material returns to its original configuration when the external loading is removed. Thus, due to the hysteresis of arterial tissue, the arterial wall cannot be
considered a perfectly elastic material, and tends to be classified as a viscoelastic material with creep (change in deformation under constant loads) and stress relaxation (change in stress under constant loads). However, Fung (1993) made a remarkable simplification to the viscoelastic classification of the artery: an artery exhibits a pseudo-elastic behavior. The pseudo-elastic classification states that the tissue forms repeatable hysteresis curves after a number of cycles have been completed. Experimentalists often refer to the tissue as preconditioned when it exhibits this repeatable response. Due to the consistent behavior of the material, it is possible to use two elastic models (one for the loading of the tissue, another for the unloading of the tissue) to quantify the two curves that make up the hysteresis loop. Thus, an elastic model can be used to quantify a portion of the hysteresis loops of arterial tissue. Moreover, if the amount of hysteresis is small, a single elastic model can be used to model the material’s behavior with reasonable accuracy.

In conjunction with the nonlinear stress vs strain curve, the arterial wall also exhibits a high degree of anisotropy: meaning that its material properties change based on location within the geometry and the direction that the material is loaded (cf. Sections 1.2 and 1.3). The primary source of the anisotropy is the collagen fibers embedded within the arterial wall. Structural models must include the directions and orientations of the collagen fibers embedded within the wall in order to accurately base the Helmholtz potential on the microstructure of the artery. However, the previous equations of hyperelastic materials are not suited to handle material anisotropy. Thus, new equations based on sound theory must be introduced.
In order to quantify the most realistic mechanical response of arterial tissue, multiple constitutive equations are needed to model the mechanical response of the arterial wall that is dependent on the physical location within a reference frame and the different regions of the analyzed material (ie, the neck, posterior, and anterior regions of an AAA). Another contributor to the anisotropy of the wall is the heterogeneous structure of the arterial wall. As discussed in Ch. 1, the arterial wall has multiple layers, each having unique mechanical properties (Von Maltzahn et al., 1984; Yu et al., 1993; and Xie et al., 1995). Thus, to achieve the greatest amount of realism, separate constitutive models for each mechanically unique layer should be constructed.

The final characteristics of the arterial tissue are perhaps some of the most mathematically important other than the classification of pseudo-elasticity. Arterial tissue exhibits a near incompressible behavior, and follows the conservation of volume when the dynamic biological response is assumed to be at a quasi-static case. Since the volumes of the unloaded and reference configurations are equivalent under this assumption, the third invariant of the deformation tensor is equivalent to unity and the density in reference and deformed configuration are also equal (cf. Eq. (3.26) and (3.65)).

\[ III_f = J_f = \frac{\rho_r}{\rho} = 1 \] (3.85)

Since both the right Cauchy-Green stretch and the Green-Lagrange strain tensors depend on the deformation gradient, setting the third invariant of the deformation gradient to unity also affects the stretch and strain tensors. Eq (3.38) and the commutative property of the determinant operation force the determinant of the Cauchy-Green stretch tensor to also be equal to unity if the third invariant of the deformation gradient has a value of one.
\[ III_C = J_C = \det C_{ij} = 1 \quad \text{if} \quad J_F = 1 \quad (3.66) \]

Eq (3.66) also affects the principal stretches of the material because invariants by definition are independent of the coordinate system used to express the matrix. Thus, if \( C_{ij} \) is expressed in terms of its principal stretches (meaning the matrix only has diagonal elements equal to its eigenvalues squared), the product of three principle stretches is also equal to unity.

\[ \lambda_1 \lambda_2 \lambda_3 = 1 \quad (3.87) \]

Material incompressibility also requires the SEF to be split into deviatoric/isochoric and volumetric components. The deviatoric component stores energy of volume preserving deformations, while the volumetric portion stores energy from deformations that change the material’s volume. (Note that the third invariant appears only in the volumetric component.)

\[ \dot{\psi} = \dot{\psi}_{\text{dev}}(I_C, II_C, IV_C - VIII_C) + \dot{\psi}_{\text{vol}}(III_C) \quad (3.88) \]

In addition to the volumetric split, a deviatoric operator must be included when calculating the 2\(^{nd}\) Piola-Kirchoff stress.

\[
S_{ij} = S_{ij}^{\text{dev}} + S_{ij}^{\text{vol}} = \text{Dev}(\tilde{S}_{ij}) + S_{ij}^{\text{vol}} \\
= \text{Dev}\left(2 \frac{\partial \dot{\psi}_{\text{dev}}}{\partial C_{ij}} + 2 \frac{\partial \dot{\psi}_{\text{vol}}}{\partial C_{ij}}\right) \quad (3.89)
\]

\[ \text{Dev}(\bullet) = (\bullet) - \frac{1}{3} \left( C_{ij} : (\bullet) \right) C_{ij}^{-1} \]

The volumetric stress term has multiple forms and varies with each SEF. It is often reported through a Lagrangian multiplier (\( \rho \)), which is dependent on position and possibly time. The preferred method of determining the numerical value of \( \rho \) is through the boundary conditions of current analysis.

\[
S_{ij} = \text{Dev}(\tilde{S}_{ij}) - \rho C_{ij}^{-1} \quad (3.90)
\]
The purpose of the Lagrangian multiplier is to report a stress in the object when there is no deformation due to the constraint of incompressibility. A situation where this is common is under conditions of hydrostatic pressure. The pressure does not induce a deformation of the object due to conservation of volume, yet the object still experiences stress in all directions. However, the Lagrange multiplier $p$ is equivalent to the hydrostatic pressure only when the first invariant of the extra stress/deformation dependent term is zero (Humphrey, 2002). The remaining stress tensors are calculated by transforming the deviatoric operator and the Lagrangian multiplier into the appropriate form or by transforming the final 2nd Piola-Kirchoff stress into the other two stress tensors through Eqs. (3.57) and (3.58). Many commercial finite-element programs compute stress using the Cauchy stress. Thus, the transformation for the Cauchy stress is listed below.

$$\sigma_{ij} = \sigma_{ij}^{\text{dev}} + \sigma_{ij}^{\text{vol}} = \text{dev}(\bar{\sigma}_{ij}) + \sigma_{ij}^{\text{vol}}$$

$$= \text{dev}\left(2F_{im} \frac{\partial \bar{\psi}_{\text{dev}}}{\partial C_{mk}} F_{jk}\right) + 2F_{im} \frac{\partial \bar{\psi}_{\text{vol}}}{\partial C_{mk}} F_{jk}$$

$$\text{dev}(\bullet) = (\bullet) - \frac{1}{3} (\delta_{ij} : (\bullet))\delta_{ij}$$

$$\sigma_{ij} = \text{dev}(\bar{\sigma}_{ij}) - p\delta_{ij}$$ (3.92)

Equations (3.88 – 3.90) represent the most general nonlinear constitutive relations that are based upon the Helmholtz potential/SEF (Holzapfel, 2000). Often, the term containing the Lagrangian multiplier is referred to as the reaction stress and the deformation dependent term is referred to as the extra stress.

It is also possible to enforce incompressibility without incorporating the deviatoric & volumetric split or a Lagrange multiplier ($p$). By including the result of Eq. (3.82) in stress-
strain relation, the need for a Lagrange multiplier is eliminated since the reactive stress is incorporated internally within the stress-strain relation. In this configuration, the strain energy will then depend on the principle strains only. An advantage of this formulation is that the strain energy function will be much simpler; however, the principle components of stress and strain only coincide under certain symmetries, which are not present in every constitutive model of the arterial wall. Thus, the deviatoric & volumetric split with a Lagrange multiplier is a more general method of enforcing incompressibility (Humphrey, 2002).

3.4.3 Modeling the characteristics of the abdominal aortic aneurysm wall

As outlined in the first chapter, the abdominal aortic aneurysm also exhibits a nonlinear response and thus requires nonlinear structural theory to construct viable models of the wall’s mechanical behavior. Some of the fundamental characteristics of the aneurysm wall are the same as the characteristics of the healthy arterial wall: anisotropic behavior, nearly incompressible, nonlinear response, etc. However, the aneurysm wall has multiple characteristics that are different than the healthy arterial wall, and are summarized in Table 3.4.

<table>
<thead>
<tr>
<th>Increased rigidity of wall</th>
<th>Presence of intra-luminal thrombus</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hyperactivity of wall degrading enzymes</td>
<td>Decrease of elastin within the wall</td>
</tr>
<tr>
<td>Variable wall strength / local weakening of wall</td>
<td>Decrease of smooth muscle cells within the wall</td>
</tr>
<tr>
<td>Greater amount of anisotropic behavior</td>
<td>Decrease of collagen cross links within the wall</td>
</tr>
</tbody>
</table>

Table 3.4: Mechanical characteristics of the aneurysm wall that differ from the healthy arterial wall
Many of the items in Table 3.4 can be modeled by constructing a new strain energy function that will yield the appropriate stress and deformation patterns caused by the changes in the arterial tissue. Whereas, some of the items require additional steps to accurately model the system of the aneurysm wall. The ILT requires a separate nonlinear material to be created in contact with the arterial wall, further increasing the nonlinearity of the resulting stress and deformation analyses. Whereas, the variable wall strength of the wall must be incorporated as a spatially varying parameter of the aneurysm wall that can be compared to the resulting anisotropic stress distribution. Such significant differences between the healthy abdominal aortic wall and the AAA wall require experimental testing of the aneurysm wall to construct accurate nonlinear models of the aneurysm wall.

3.4.4 Experimental data needed for new constitutive models of the arterial wall

The characteristics mentioned above make the arterial wall a difficult mechanical structure to model. Furthermore, all constitutive models should be based on experimentation, which tests the material of interest over a specified range of conditions (cf. Sections 1.8 and 3.2). Thus, experimental setup and design are vital to the development of new models.

Each characteristic of the arterial wall mentioned above affects the type or form of experiment needed to quantify the behavior of arterial tissue. The fundamental experimental conditions are governed by the physiological conditions that the constitutive model desires to predict. Thus, the experimental conditions used to derive constitutive models for angioplasty are not going to be the same experimental conditions used to derive models for normal, physiologic conditions, hypertension, or aneurysm conditions. Furthermore, the greater
range of phenomena that the constitutive equation must contain, the more complex the resulting equation will be.

In general, the only data that phenomenological models require is the macroscopic behavior of the arterial wall under the desired loading conditions. Often this results in a form of stress vs. strain data for uniaxial or multiaxial tension tests. However, structural models require two primary sets of data. The first is the macroscopic stress vs. strain data for tension tests, but they also require information about the histology of the tissue. In the arterial wall, the cells of interest are the collagen molecules that reinforce the wall, and their distribution throughout the tissue. Holzapfel et al. (2002) wrote an algorithm that would locate the directions of the collagen fibers from a scanned image of histological tissue. Conversely, direct methods for measuring collagen fiber distribution from the tissue also exist. Sacks et al. (1997) developed a novel technique that passes a Helium-Neon laser through a tissue specimen and records the spatial intensity distribution of scattered light, which can be used to obtain the angular distribution of the of the collagen fibers (Sacks and Sun, 2003). Furthermore, it is possible to obtain fiber directions through histological staining. Holzapfel et al. (2006) had histological samples of the human aorta stained and the angular directions of the fibers measured by a skilled histopathologist. New methods of measuring arterial behavior continue to be investigated. However, the primary focus of measuring the mechanical behavior of the arterial wall remains consistent with each new approach.

Currently, experimental data for the abdominal aorta is rare for both healthy and aneurysm tissue. One of the primary limitations is the location of the artery and the efforts needed to extract the appropriate tissue. The group of test samples must be from patients of
comparative age and the same gender for accurate comparison, making the pool of tests
samples even smaller. Thus, a lack of abdominal aortic tissue that can be used for
experimental testing is common throughout the medical field. To date, only a handful of
constitutive models for the abdominal aortic aneurysm wall have been developed. To the
knowledge of the author, not one of the current models is a structural model, but rather all are
phenomenological. Thus, an important piece of data needed to fully characterize the
aneurysm wall is the collagen fiber distribution within the wall and the fibers’ regional
location within the aneurysm. Another important piece of information is the spatially
varying mechanical strength of the aneurysm wall. Thus, future experiments should
investigate different regions of the aneurysm and test if each region has different mechanical
response that requires a separate SEF.
4. SPECIFIC STRAIN-ENERGY FUNCTIONS FOR NONLINEAR MATERIALS AND THE ARTERIAL WALL

4.1 Introduction

Numerous attempts have been made to accurately model the human arterial wall’s mechanical response to different loads. Quantifying the behavior of the arterial wall remains one of the prominent areas of biomechanical research. In the following section an overview of multiple mechanical models and some preliminary results will be discussed.

4.2 Fundamental Isotropic Strain Energy Functions

Prior to investigating the complex models of the arterial wall, some of the basic isotropic models used to describe non-linear elasticity are briefly discussed. The most fundamental non-linear elastic model is the neo-Hookean relation. Although the equation is simple, it has the ability to accurately describe rubber behavior up to stretches of about 30% under isothermal conditions. The general form of the neo-Hookean SEF is given as:

$$\psi = c_1(I_C - 3)$$

(4.1)

where $c_1$ and $I_C$ are half of the initial shear modulus and the first invariant of the right Cauchy-Green stretch tensor, respectively.

Another fundamental non-linear isotropic model is the Mooney-Rivlin model and is one of the benchmark models for describing isotropic rubber-like materials. It has been shown to
describe the non-linear behavior of elastomers (highly elastic materials, with large
deformations) over a large range of strains with more accuracy than the neo-Hookean model
(Humphrey, 2002). The model has even been used to describe the mechanics/behavior of
arterial tissue such as cardiac (heart) tissue (Jernigan, 2006). Multiple forms of the Mooney-
Rivlin model exist, ranging from a two-parameter model to a nine-parameter model. A three-
parameter model can be written as:

$$\psi = c_1 (I_c - 3) + c_2 (I_c - 3)^2 + c_3 (I_c - 3)^3$$  \hspace{1cm} (4.2)

where $c_1 > 0$, $c_2 > 0$, and $c_3 < 0$ are material parameters (constants) whose signs are inferred
from Green-Lagrange strain inequalities.

Another very useful, isotropic model that is capable of modeling very large deformations
while maintaining numerical stability is the three-parameter Yeoh model:

$$\hat{\psi} = c_1 (I_c - 3) + c_2 (I_c - 3)^2 + c_3 (I_c - 3)^3$$  \hspace{1cm} (4.3)

Each of the three coefficients must be greater than zero in order for the model to maintain
polyconvexity and exhibit convex energy contours. The model still remains very popular for
its ease of implementation and robust ability to model a wide array of deformations including
stress stiffening at higher degrees of deformation. In general, the Yeoh model has a higher
amount of numerical stability due its polyconvexity. Thus, unpublished investigations by the
author suggest the Yeoh model to be a better overall model than the Mooney-Rivlin models.
The Yeoh is more robust, and can model highly nonlinear behavior over a very large range of
experimental deformations.
4.3 General Strain-Energy Functions for Fiber Reinforced Composite Materials

As previously mentioned, the arterial wall can be considered as a fiber-reinforced composite structure. However, the classical SEF potentials must be modified to include the strain energy of the reinforcing collagen fibers. The fibers are assumed to be continuously distributed throughout the material so that the continuum theory may be used to model the material. Furthermore, the material potential is strongly dependent on the direction or alignment of the collagen fibers. Thus, the fiber-reinforced materials are referred to as anisotropic.

A hyperelastic material may be characterized by the primary three invariants. However, following Spencer (1984) and Holzapfel (2000), additional invariants are required to characterize an anisotropic hyperelastic material.

\[
IV_C = a_i C_{ij} a_j = \lambda^2 \tag{4.4}
\]

\[
V_C = a_i C_{ij}^2 a_j \tag{4.5}
\]

The vector \( a_i \) is the preferred direction of the reinforcing fiber(s) while the matrix \( C_{ij} \) is the right Cauchy-Green stretch tensor. It is important to note that the fourth invariant is equivalent to the square of the stretch ratio in direction of the fiber \( (a_i) \).

4.3.1 Materials with one set of reinforcing fibers

A fiber-reinforced composite with one preferred direction of fibers is referred to as a transversely isotropic material. The implication of this classification and arrangement of fiber reinforcements is that the material has an isotropic behavior in directions orthogonal to the preferred direction of the reinforcing fibers. Connective biological tissues such as tendons fall under this classification.
The resulting continuum equations for an incompressible transversely isotropic material are:

\[
\dot{\psi}_{\text{dev}} = \dot{\psi}_{\text{dev}} \left[ I_C (C_{g}) ; II_C (C_{g}) ; IV_C (C_{g}, a_i) ; V_C (C_{g}, a_i) \right] \tag{4.6}
\]

\[
S_{ij} = 2 \left[ \frac{\partial \dot{\psi}}{\partial I_C} + I_C \frac{\partial \dot{\psi}}{\partial II_C} \right] \delta_{ij} - \frac{\partial \dot{\psi}}{\partial II_C} C_{ij} + 2 \left[ \frac{\partial \dot{\psi}}{\partial IV_C} a_i \otimes a_j + \frac{\partial \dot{\psi}}{\partial V_C} \left( a_i \otimes C_{g} a_i + a_j \otimes C_{g} a_i \right) \right] - pC_{g}^{-1} \tag{4.7}
\]

where the invariants \( I_C, II_C, III_C, IV_C, \) and \( V_C \) are the first, second, third, fourth and fifth invariants of the right Cauchy stretch tensor respectively; \( a_i \) is the preferred direction of the reinforcing fibers; \( C_{g} \) is the right Cauchy-Green stretch tensor, \( S_{ij} \) is the second PK stress tensor, and \( \otimes \) represents the dyadic product that forms a second-order tensor (in matrix form) from two vectors.

4.3.2 Materials with two sets of reinforcing fibers

A significant number of materials exhibit fiber reinforcements in two preferred directions. Figure 4.1 illustrates two sets of fibers embedded within a ground matrix \((\vec{a} & \vec{g})\). Their direction is characterized by an angle between the preferred fiber directions (note: this angle is not always 90°). Some popular materials containing two sets of reinforcing fibers are glass, alumina, silica.
carbide, and graphite (Holzapfel, 2000). According to Spencer (1984) and Holzapfel (2000) additional invariants are required to characterize the two directions of reinforcing fibers:

\[
VI_C = g_i C_{g_j} g_j = \lambda^2
\]

\[
VII_C = g_j C_{g_i} g_i^2
\]

\[
VIII_C = (a_i g_i) a_j C_{g_i} g_i
\]

where \(g_i, a_i, \) and \(C_{ij}\) are the preferred direction of the second fiber, the preferred direction of the first fiber, and the right Cauchy stretch tensor respectively. An additional invariant exists, but does not contain any information regarding the deformation or strain energy of the material and is thus omitted from the governing continuum equations of the material.

Utilizing the additional invariants, the governing equations for an incompressible fiber reinforced material with two sets of reinforcing fibers are:

\[
\hat{\psi}_{\text{dev}} = \hat{\psi}_{\text{dev}} \begin{bmatrix} I_C(C_{g_j}); II_C(C_{g_j}); \\ IV_C(C_{g_i}, a_j); V_C(C_{g_j}, a_i); \\ VI_C(C_{g_i}, g_j); VII_C(C_{g_j}, g_i); \\ VIII_C(C_{g_i}, a_i, g_i) \end{bmatrix}
\]

\[
S_{ij} = 2 \left( \frac{\partial \hat{\psi}}{\partial I_C} + I_C \frac{\partial \hat{\psi}}{\partial II_C} \right) \delta_{ij} - \frac{\partial \hat{\psi}}{\partial II_C} C_{ij} + 2 \left( \frac{\partial \hat{\psi}}{\partial IV_C} a_i \otimes a_j + \frac{\partial \hat{\psi}}{\partial V_C} \left( a_j \otimes C_{g_j} a_j + a_i C_{g_j} \otimes a_i \right) \right) + 2 \left( \frac{\partial \hat{\psi}}{\partial VI_C} g_i \otimes g_j + \frac{\partial \hat{\psi}}{\partial VII_C} \left( g_j \otimes C_{g_j} g_j + g_i C_{g_j} \otimes g_i \right) \right) + \frac{\partial \hat{\psi}}{\partial VIII_C} (a_i g_i) (a_i \otimes g_j + g_i \otimes a_j) - p C_{ij}^{-1}
\]
Where $I_C, II_C, III_C, IV_C, V_C, VI_C, VII_C$, and $VIII_C$ are the first, second, third, fourth, fifth, sixth, seventh, and eighth invariants of the right Cauchy stretch tensor respectively; $a_i$ is the preferred direction of the first reinforcing fiber; $g_i$ is the preferred direction of the second reinforcing fiber; $C_{ij}$ is the right Cauchy-Green stretch tensor, $S_{ij}$ is the second PK stress tensor, and $\otimes$ represents the dyadic product that forms a second order tensor (in matrix form) from two vectors.

An important subclass of materials with two preferred fiber directions is orthotropic materials where the two families of fibers have orthogonal preferred directions. Thus, the dot product between the two fibers is zero, setting $VIII_C = 0$. Eqs. (4.11) and (4.12) become:

$$\dot{\psi}_{\text{dev}} = \dot{\psi}_{\text{dev}} \begin{bmatrix} I_C (C_{ij}); II_C (C_{ij}); IV_C (C_{ij}, a_i); V_C (C_{ij}, a_i); VI_C (C_{ij}, g_i); VII_C (C_{ij}, g_i) \end{bmatrix}$$

$$S_{ij} = 2 \left[ \frac{\partial \dot{\psi}}{\partial C_{ij}} \delta_{ij} - \frac{\partial \dot{\psi}}{\partial C_{ij}} C_{ij} \right] + 2 \left[ \frac{\partial \dot{\psi}}{\partial V_C} a_i \otimes a_j + \frac{\partial \dot{\psi}}{\partial V_C} (a_i \otimes C_{ij} a_i + a_j C_{ij} \otimes a_i) \right] + 2 \left[ \frac{\partial \dot{\psi}}{\partial VII_C} g_i \otimes g_j + \frac{\partial \dot{\psi}}{\partial VII_C} (g_i \otimes C_{ij} g_i + g_j C_{ij} \otimes g_i) \right] - pC_{ij}^{-1} \quad (4.14)$$

Arteries have often been considered and experimentally supported to be an orthotropic material in the cylindrical reference frame (Fung, 1993; Holzapfel et al., 2000b, 2002, 2005; Humphrey, 2002; Vito and Dixon, 2003).
4.4 Strain-Energy Functions for the Passive Behavior of the Healthy Arterial Wall

As outlined in Ch. 1, the arterial wall is an active piece of tissue with multiple sources of dynamics and variations. However, many of the strain-energy describe the passive behavior of the wall and take a still-frame image of the artery’s active behavior. The prominent strain-energy function for the passive arterial wall was constructed by Fung et al. (1979) as a strain-energy equation for two-dimensional problems. Fung et al. (1983) revisited the model and presented a generalized three-dimensional model that assumed principle directions of the stress tensor which coincide with the radial, circumferential, and axial directions of the artery. These assumptions did not consider any shearing deformations that may occur throughout the physiologic response (Holzapfel et al., 2000). Numerous modifications and adaptations have been proposed to account for shear and more accurate three-dimensional analyses. In 1995 Humphrey presented a general Fung-type strain energy equation that is capable of three dimensional deformations, including shearing deformations. The most general form of the Fung-type equation is listed below:

\[
\psi = \frac{1}{2} c \left[ \exp(Q) - 1 \right]
\]

\[
Q = b_1 E_{\theta \theta}^2 + b_2 E_{ZZ}^2 + b_3 E_{RR}^2 + 2b_4 E_{\theta \theta} E_{ZZ} + 2b_5 E_{ZZ} E_{RR} + 2b_6 E_{RR} E_{\theta \theta} + b_7 E_{\theta \theta}^2 + b_8 E_{ZZ}^2 + b_9 E_{RR}^2
\]

(4.15)

where \( \psi \) is a pure isochoric (incompressible, deformation only) contribution to the Helmholtz free-energy equation, variable \( c \) is a material parameter, \( E_{ij} \) is the modified Green-Lagrange strain tensor in cylindrical polar coordinates, and \( b_1 - b_9 \) are non-dimensional material constants to be determined from experimental data. Although the model is very straightforward and functionally simple, the computational structure of the generalized Fung-
type equation does not exhibit numerical stability for all values of the material constants (Holzapfel et al., 2000; Holzapfel et al., 2004; Sun and Sacks, 2005). Furthermore, the model is purely phenomenological in nature where the material constants have no significant meaning other than curve-fitting parameters.

A simpler model that reports an isotropic strain potential of the arterial wall was proposed by Delfino et al. (1997). It consisted of two material parameters and the first invariant of the Cauchy-Green tensor. The equation is shown below:

\[
\psi = \frac{a}{b} \left\{ \exp \left[ \frac{b}{2} (I_c - 3) \right] - 1 \right\}
\]

Where \(a\) and \(b\) are the material parameters and \(I_c\) is the first invariant of the Cauchy-Green tensor. The primary limitation of this model is that it can never accurately model the anisotropic complexities of the collagen fibers throughout the arterial wall. However, numerically, the equation is stable through strict, local convexity, and its results show a good qualitative picture of mechanical response of the healthy arterial wall (Holzapfel et al., 2000).

One of the most accurate models currently available was developed by Holzapfel et al. (2000, 2004). It is a two-layer structural model capable of modeling the anisotropic mechanical response of the artery in all three dimensions, including shear and torsion loads. Furthermore, it splits the stress-energy equation into two different parts: an isotropic term and an anisotropic term. The isotropic term is a non-linear neo-Hookean model that captures the matrix surrounding the collagen fibers. Conversely, the anisotropic component models the collagen fibers. Additionally, the media and adventitia have different equations due to the heterogeneous structure of the arterial wall. It was previously thought that for healthy arteries that the innermost layer, the tunic intima, did not play a pivotal role in the
mechanical properties and was thus not included in the original model. However, Holzapfel et al. (2006) incorporated an intima layer with experimentally derived properties in the computational model and found significant stress patterns within the layer. To incorporate such effects, the original Holzapfel model would be modified to add another system of equations for the tunic intima. The Holzapfel model for one layer has the following equations:

\[
\psi = \psi_{iso} + \psi_{aniso} \tag{4.17}
\]

\[
\psi_{iso} = \frac{c}{2} (I_c - 3) \tag{4.18}
\]

\[
\psi_{aniso} = \frac{k_1}{k_2} \left\{ \frac{1}{2} \left( \exp \left[ k_2 \left( IV_c - 1 \right)^2 \right] - 1 \right) + \frac{1}{2} \left( \exp \left[ k_2 \left( IV_c - 1 \right)^2 \right] - 1 \right) \right\} \tag{4.19}
\]

Where \( c \) and \( k_1 > 0 \) are stress-like material parameters, \( k_2 > 0 \) is a dimensionless parameter, and \( I_i \) is an invariant of the right Cauchy stretch tensor. By selecting the appropriate \( k_1 \) and \( k_2 \) values, the model will include the collagen fibers’ contribution to the mechanical response of the arterial wall only at elevated intra-arterial pressures. An important note is that the constants of the model do not depend on any geometry, residual stress, or fiber angle of the tissue. Furthermore, strict, local convexity is satisfied for all possible constants given the specifications above. Overall, the Holzapfel model maintains good numerical stability and models a realistic structural response of the artery.

Multiple models exist that are modifications of the general Holzapfel model. A common modification to Holzapfel model is to include a Gaussian distribution of fiber directions/orientations in the arterial wall. Thus, instead of only two directions being
prescribed for the embedded fibers, a large range of fiber directions are included in the model with the amount of fibers aligned in a preferred direction being controlled by Gaussian distribution parameters (Gasser et al., 2006; Driessen et al., 2005; Sacks, 2003). However, many of the current models that incorporate a Gaussian distribution of collagen fiber directions are functions of the principle stretches rather than invariants of the right Cauchy-Green tensor. Zulliger et al. (2004) modified the Holzapfel model to account for the volume fractions of the arterial wall’s primary components.

4.5 Specific Constitutive Relations for the Passive Abdominal Aortic Aneurysm Wall

Only two constitutive equations have been developed for the AAA tissue. Furthermore, neither equation is a structural model that is based on the underlying microstructure of the tissue. The first equation was proposed by Raghavan and Vorp (2000) and is a hyperelastic equation based upon the invariants of the left Cauchy-Green stretch tensor. Data from uniaxial tensile tests was used to construct the model while the aneurysm was assumed to an isotropic, hyperelastic structure. Using the analytical solution for uniaxial tensile experiments of isotropic, hyperelastic materials, the general form of the strain energy function (Eq. (4.20)) was determined and the unknown coefficients were fit to the AAA tensile data.

\[
\psi = \alpha \left(I_B - 3\right) + \beta \left(I_B - 3\right)^2
\]

(4.20)

The terms \(\alpha\) and \(\beta\) are the unknown coefficients while \(I_B\) is the first invariant of the left Cauchy-Green tensor. It is important to note that the first three invariants of the right
Cauchy-Green and left Cauchy-Green tensor are equivalent. Thus, Eq. (4.20) can also be expressed as a function of the first invariant of the right Cauchy-Green stretch tensor.

Multiple assumptions and limitations exist for the previous model. Although the simplicity and the ease of its implementation in finite element code are attractive, the model cannot capture the anisotropic behavior of the aneurysm wall. Van de Geest et al. (2006a) was the first to conduct biaxial experimentation on AAA tissue and his results clearly illustrate the biaxial anisotropic behavior of both the healthy abdominal aorta and the AAA tissue. They also used their experimental data to construct a phenomenological anisotropic model of the AAA wall using a modified Fung-type SEF without any shear components.

\[ \tilde{\Psi}_{dev} = c_0 \left( e^{0.5 b_1 E_{aL}} + e^{0.5 b_2 E_{aL}} + e^{b_3 E_{aL}} - 3 \right) \]  

(4.21)

An advantage of this SEF is the separation of the exponential function, which reduces the amount of parameter covariance. Table 4.1 lists the average parameters for both the isotropic and anisotropic passive models of the AAA wall.

<table>
<thead>
<tr>
<th>Table 4.1: AAA model parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Model</strong></td>
</tr>
<tr>
<td>-------------------</td>
</tr>
<tr>
<td>AAA Isotropic</td>
</tr>
<tr>
<td>AAA Anisotropic</td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>

**4.6 Strain-Energy Functions of the Intraluminal Thrombus**

Experimental results from Wang et al. (2001) and Vande Geest et al. (2006b) illustrate that the intraluminal thrombus (ILT) of AAAs has a very different mechanical response than the AAA wall. Thus, a separate SEF is needed to accurately model the ILT structure. As mentioned before there are 3 primary layers for the ILT. The layers have slightly different
mechanical properties, yet the structure is still able to be modeled using a single homogeneous SEF (Di Martino and Vorp, 2003). Table 4.2 summarizes the mechanical response of the luminal and medial layers.

<table>
<thead>
<tr>
<th>Region</th>
<th>Orientation</th>
<th>Stiffness [kPa]</th>
<th>Strength [kPa]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Luminal Layer</td>
<td>Longitudinal</td>
<td>540±70</td>
<td>520±70</td>
</tr>
<tr>
<td></td>
<td>Circumferential</td>
<td>570±70</td>
<td>540±70</td>
</tr>
<tr>
<td>Medial Layer</td>
<td>Longitudinal</td>
<td>330±70</td>
<td>300±70</td>
</tr>
<tr>
<td></td>
<td>Circumferential</td>
<td>270±40</td>
<td>220±50</td>
</tr>
</tbody>
</table>

When compared with uniaxial and biaxial tensile data, the information in Table 4.2 can be used to develop an accurate model of the ILT structure using an isotropic hyperelastic function. Wang et al. (2001, 2002) developed the first SEF used to model the ILT structure. The general form is listed in Eq. (4.22).

\[ \psi = c_1 (I_2^c - 3) + c_2 (I_2^c - 3)^2 \] (4.22)

A significant drawback of the SEF in Eq. (4.22) is that it is not polyconvex. Thus, there is a risk for unstable mechanical behavior and inaccurate modeling. Vande Geest et al. (2006b) revisited the aforementioned shortcoming and performed equibiaxial experimentation on the luminal layer of the ILT, confirming the structure’s anisotropy and the lack of load bearing ability in the underlying layers. Their analysis yielded a new SEF for ILT that is both more accurate and polyconvex. The general form of the new SEF is listed in Eq. (4.23).

\[ \psi = c_1 (I_3^c - 3) + c_2 (I_3^c - 3)^2 \] (4.23)
Notice the only difference between the two models is the dependence on the first or the second invariant of the right Cauchy-Green stretch tensor. Table 4.3 lists the average parameters that best fit the experimental ILT data for both models.

<table>
<thead>
<tr>
<th>Model</th>
<th>C₁</th>
<th>C₂</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>2nd Invariant</td>
<td>2.80</td>
<td>2.86</td>
<td>Di Martino and Vorp (2003)</td>
</tr>
<tr>
<td>1st Invariant</td>
<td>7.98</td>
<td>8.71</td>
<td>Vande Geest et al. (2006b)</td>
</tr>
</tbody>
</table>

4.7 Model for the Spatially Varying Wall Strength of the Aneurysm Wall

A challenge for all investigations of AAA tissue is the estimation of the AAA wall strength. Both Thubrikar et al. (2001a) and Vallabhaneni et al. (2004) reported significant amounts of spatially varying wall strength. Currently there are multiple methods of non-invasively assessing the wall strength of the AAA wall (Kleinstreuer and Li, 2006; Vande Geest et al., 2006c). An initial investigation of a non-invasive equation used to determine rupture risk included factors such as a normalized diameter term, patient and family history, sex/gender, and ILT thickness (Vande Geest et al., 2006c). After several statistical analyses and mixing models, the authors arrived at the following equation to assess the spatially varying wall strength.

\[
Wall \text{ Strength} = 71.9 - 37.9\left(\sqrt{t_{ILT}} - 0.81\right) - 15.6(D_{\text{norm}} - 2.46) - 21.3\alpha + 19.3\beta
\]

The terms \(t_{ILT}\) and \(D_{\text{norm}}\) represent the ILT thickness and AAA normalized diameter respectively. While the terms \(\alpha\) and \(\beta\) represent terms that account for family history and
sex/gender respectively. The most significant limitations of this model is the lack of clinical
testing and small ranges of $t_{ILT}$ and $D_{norm}$ the model can be applied (0, 3.6) and (1.06, 3.9)
respectively.

4.8 Strain-Energy Functions That Incorporate Active Arterial Response

One of the challenges of incorporating the dynamics of arterial behavior is the large time-scale of normal tissue maintenance and adaptation. Endothelial cells of a mature arterial wall under normal loading conditions replace a maximum of a few percent of cells per day. Furthermore, the turnover of smooth muscle cells of a mature arterial wall occurs about 0.06% each day; while, the half life of collagen is on the order of weeks or months (Humphrey, 2002). However, in situations of local of global alterations in stimulation the arterial tissue can undergo significant modification in a short period of time. The $2^{nd}$ law of thermodynamics becomes one of the fundamental methods used for modeling such dynamics, since many of the common conservation laws (mass, volume, etc.) do not apply in situations where tissue is being created, destroyed, or some uneven combination of both processes.

In general, the SEFs that incorporate active response depend on more than just the material’s deformation. The time, history of deformation, and change of mass and/or volume are other terms that the SEF include. Furthermore, each of these terms is a function of biological parameters that dictate the change in the material’s response. Baek et al. (2006) created a model for the formation and changes within intercranial/cerebral aneurysms. The general form of their SEF is shown below:

$$\tilde{\psi} = \tilde{\psi}(\lambda_i, \dot{\lambda}_i, \dot{m})$$  (4.24)
where $\lambda_i$ is the deformation in each of the principal direction, $\dot{\lambda}_i$ time rate of change for the deformation in each principal direction, and $\dot{m}$ is the time rate of change in mass due to collagen creation and elimination.

Most commercial finite-element software packages may not have the capability to incorporate the complexities and additional parameters of SEFs that incorporate active tissue response. Thus, ease of implementation is a paramount limitation of most active models. The analyses conducted within the current investigation have focused primarily on the biomechanical response of the tissue as a result of the biochemistry and dynamic adaptation.

4.9 Basic Comparison Between Linear and Nonlinear Models

Using simple states of loading and deformation, it is possible to construct direct comparisons between the results of isotropic hyperelasticity and isotropic linear elasticity. Two loading states that enable a direct comparison are the uniaxial extension of a rectangular piece of material, and the uniform inflation of a hollow, thin-walled sphere (condition of equibiaxial plane stress).

The material models used are the Neo-Hookean, three-parameter Mooney Rivlin, three-parameter Yeoh, and the traditional linear Young’s modulus with accompanying Poisson ratio. The parameters for each model were set to ensure the initial shear modulus of each model was equivalent to 225 MPa after the approximate shear modulus for porcine coronary arteries at internal pressure of 16kPa (Lu et al., 2003). Equations for the shear modulus of isotropic hyperelastic and linear materials are:
\[ \mu = 2 \left( \frac{\partial \psi}{\partial I_C} + \frac{\partial \tilde{\psi}}{\partial I_C} \right) \]  \hspace{1cm} (4.25) \\
\[ \mu = \frac{Y}{2(1+v)} \]  \hspace{1cm} (4.26) \\

where \( Y \) and \( v \) are the Young/elastic modulus and the Poisson ratio respectively. After equating the initial shear moduli the parameters for each model are listed in Table 4.4.

<table>
<thead>
<tr>
<th>Model</th>
<th>Initial Shear Modulus [MPa]</th>
<th>( C_1^* )</th>
<th>( C_2^* )</th>
<th>( C_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear Elastic</td>
<td>0.250</td>
<td>0.750</td>
<td>0.49</td>
<td>n/a</td>
</tr>
<tr>
<td>Neo-Hookean</td>
<td>0.250</td>
<td>0.125</td>
<td>n/a</td>
<td>n/a</td>
</tr>
<tr>
<td>3-param Mooney-Rivlin</td>
<td>0.250</td>
<td>0.100</td>
<td>0.025</td>
<td>1.9778e-005</td>
</tr>
<tr>
<td>3-param Yeoh</td>
<td>0.250</td>
<td>0.125</td>
<td>6.2004e-005</td>
<td>6.2004e-005</td>
</tr>
</tbody>
</table>

*C1 and C2 correspond to the elastic modulus and Poisson ratio respectively for the linear elastic model

It is important to note that the nonlinear terms in the three-parameter Mooney-Rivlin and Yeoh model create a dynamic shear modulus that increases with increasing tension.

4.9.1 Uniaxial extension

The fundamental experimental test for all materials is the case of uniaxial extension. A diagram of uniaxial extension with a corresponding global coordinate system is shown in Fig. 4.2.
By assuming the material deforms equally in the 22 and 33 directions, the resulting deformation gradient and right Cauchy-Green stretch tensor have the following forms.

\[
F_{ij} = \begin{bmatrix}
\lambda & 0 & 0 \\
0 & \lambda^{-\frac{1}{2}} & 0 \\
0 & 0 & \lambda^{-\frac{1}{2}}
\end{bmatrix}
\Rightarrow
C_{ij} = \begin{bmatrix}
\lambda^2 & 0 & 0 \\
0 & \lambda^{-1} & 0 \\
0 & 0 & \lambda^{-1}
\end{bmatrix}
\] (4.27)

Thus, a single stretch ratio (\(\lambda\)) can be specified and the principle invariants and resulting stress distribution can be calculated for each state value of \(\lambda\). Additionally, the incompressible Lagrangian multiplier is solved by enforcing the boundary conditions of zero stress in the 33 and 22 directions. The deviatoric and volumetric split need only to be performed in situations of anisotropic hyperelasticity, and thus is not needed in this situation.

After the previous deformations have been specified, the corresponding stress values can be solved through the SEF’s whose parameters are defined in Table 3.2. Since the material is isotropic and a single value of displacement is used to characterize the deformation, a single value of stress corresponds to the deformation and the results can be illustrated in the following graphs.
Figure 4.3 clearly illustrates the nonlinear behavior of all hyperelastic models in comparison to the linear elastic model based on engineering strain and plane stress. Furthermore, it is evident that the Yeoh model provides the most apparent nonlinear trends due to the exponential increase based on the first invariant.
Figure 4.4 illustrates the dynamic values of the shear modulus for the Yeoh model under the specified conditions of uniaxial displacement. Note that all curves have the same initial value of 0.250 MPa, yet as the tensile deformation increases, the shear modulus also increases. The physical meaning of this trend is that the Yeoh material is more resistant to shear at higher tensile loads in uniaxial tension while the other material models retain the same resistance to shear throughout the deformation. Such knowledge is of utmost importance to ensure that the Yeoh SEF is the optimal choice to model a specific material’s behavior.

4.9.2 Uniform inflation of a sphere

The other state of deformation is the uniform inflation of a hollow, thin-walled sphere. A diagram illustrating the resulting deformation is shown in Fig. 3.5.
The uniform inflation causes an equibiaxial state of plane stress and thus the deformation gradient and right Cauchy-Green stretch tensor have the following forms.

\[
F_y = \begin{bmatrix}
\lambda & 0 & 0 \\
0 & \lambda & 0 \\
0 & 0 & \lambda^{-2}
\end{bmatrix} \Rightarrow C_y = \begin{bmatrix}
\lambda^2 & 0 & 0 \\
0 & \lambda^2 & 0 \\
0 & 0 & \lambda^{-4}
\end{bmatrix}
\]  \hspace{1cm} (4.26)

Similar to the case of uniaxial extension, a single stretch ratio (\(\lambda\)) can be specified, which enables the resulting stress to be calculated. The same material models used in the uniaxial case are implemented within this loading state. Thus, a single state of displacement can be used to calculate the corresponding stress value and the trends are shown in the following graphs.

Fig 4.5: Uniform inflation of a hollow, thin-walled sphere
Once again, the nonlinear behavior of the hyperelastic models is clearly visible in Fig. 3.6. In this set of data, the Mooney-Rivlin model illustrates the greatest amount of nonlinear behavior. This trend is due to its dependence on the second principal invariant. The 2nd invariant is much larger than the first, and subsequently the square of \( I_2 \) is also much larger than the cube of \( I_1 \). An interesting note is that the neo-Hookean model does not exhibit considerable difference from the traditional linear elastic models. Such a trend is supported by previous investigations, which state that the neo-Hookean model has limited applications where it can accurately model the material’s nonlinear response.

\[ \text{Cauchy Stress [MPa]} \]
\[ \text{Stretch Ratio } \lambda \]

Fig. 4.6: Cauchy stress resulting from uniform inflation of hollow, thin-walled sphere
The curves within Fig. 4.7 reveal that the Yeoh model has the greatest increase in its shear modulus when compared to the other material models. In fact, the shear modulus for the Yeoh model is over 400% larger than the original shear modulus at the maximum stretch ratio.

The previous results provide a clear distinction between the traditional linear, engineering stress and strain response and the response of multiple hyperelastic functions. It is also apparent that for certain models, the shear modulus is not constant and will thus affect the overall mechanical behavior of the system. Thus, care must be taken when selecting the SEF used to model the nonlinear material, and shear stress studies of nonlinear material can play a pivotal role in the determination of the optimal material model.

Fig. 4.7: Shear moduli of uniform inflation of hollow, thin-walled sphere
5. DEVELOPMENT OF A NEW STRUCTURAL ANISOTROPIC MODEL FOR THE ABDOMINAL AORTIC ANEURYSM WALL

5.1 Introduction

As previously mentioned in Ch.3, the only models that have been developed for AAA tissue are phenomenological in nature. Furthermore, the effects of anisotropic wall models on the resulting stress distribution of a AAA geometry remain uninvestigated. To address this need a structural anisotropic hyperelastic model for the AAA wall has been developed utilizing approximated experimental data from biaxial tensile experiments of human AAA tissue.

The development of constitutive models is a gradual, multi-step process. Humphrey (2002) listed a five-step outline as the general procedure required to create a constitutive correlation:

1. Describe the general characteristics of the desired material under the appropriate environmental conditions through diagrams and lines (ie, stress/strain graphs)
2. Establish an accurate theoretical framework
3. Identify specific functional form of the relation
4. Calculate “best-fit” values of the associated material parameters
5. Evaluate predictive capability of the final relation

The steps mentioned above frequently overlap and require the investigator to often revisit previous steps and procedures. An illustration of the procedure is given in Fig. 5.1.
The experimental data and theoretical framework have already been discussed in detail in Chs. 1 and 3. Thus, the next stage of the modeling procedure is the development of the model’s functional form.

5.2 Functional Form of New Structural Anisotropic Model

As this initial study focused on the effects of a structural anisotropic model on computational analyses of aneurysms, a fundamental characteristic of the new model was a streamlined implementation within the commercial finite-element software ANSYS (v.11). Currently ANSYS has only one anisotropic function that derives its anisotropy from hyperelastic theory with embedded fibers. Thus, the general functional form of the new model was derived from the ANSYS anisotropic hyperelastic function shown in Eq. (5.1).
\[ \dot{\psi}_{\text{vol}} = \frac{1}{d} (III_C - 1)^2 \]

\[ \dot{\psi}_{\text{dev}} = \sum_{i=1}^{3} a_i (I_{C} - 3)^i + \sum_{j=1}^{3} b_j (II_{C} - 3)^j \]

\[ + \sum_{k=2}^{6} c_k (IV_{C} - 1)^k + \sum_{j=2}^{6} d_j (V_{C} - 1)^j \]

\[ + \sum_{m=2}^{6} e_m (VI_{C} - 1)^m + \sum_{n=2}^{6} f_n (VII_{C} - 1)^n \]

\[ + \sum_{\alpha=2}^{6} g_\alpha \left( VIII_{C} - (\hat{\alpha} \cdot \hat{\mathbf{g}})^2 \right) \]

(5.1)

The roman numerals \( I_C \) – \( VIII_C \) represent the eight invariants of the right Cauchy-Green stretch tensor for fiber reinforced materials with the fibers principal directions being \( \hat{\alpha} \) & \( \hat{\mathbf{g}} \); while, the letters \( a_i \) – \( \hat{g}_\alpha \) represent constant coefficients that are determined by fitting the model to experimental data.

Several limitations exist with the functional form listed in Eq. (5.1). One of the most obvious is the excessive amount of constants that are needed to fully define the model behavior. An efficient SEF should accurately model the material behavior with the least amount of coefficients possible. Furthermore, Merodio and Ogden (2006) illustrate that the inclusion of the eighth invariant induces a significant amount of instability and affects the ellipticity of the SEF. Another significant drawback is that the SEF in Eq. (5.1) is not polyconvex. The lack of polyconvexity may result in numerical instability and inaccurate modeling regardless of the selected coefficients. Thus, prior to fitting the SEF in Eq. (5.1) to experimental data, the SEF must be modified to ensure polyconvexity \textit{a priori}. Combining the work of Hartmann and Neff (2003) with Balzani et al. (2006) specific terms of the SEF in Eq. (5.1) can be collected to form a new SEF that satisfies the requirements of polyconvexity.
\[
\dot{\psi}_{\text{vol}} = \frac{1}{d}(III_c - 1)^2
\]
\[
\dot{\psi}_{\text{dev}} = \sum_{i=1}^{3} a_i (I_c - 3)^i + \sum_{k=2}^{6} c_k (IV_c - 1)^k + \sum_{m=2}^{6} e_m (VI_c - 1)^m
\]

where: \( a_i \geq 0 \), \( c_k \geq 0 \), \( e_m \geq 0 \) (5.2)

Following Holzapfel et al (2000), it is possible to split the deviatoric portion of the SEF in Eq. (5.2) into isotropic and anisotropic components.

\[
\dot{\psi}_{\text{dev, iso}} = \sum_{i=1}^{3} a_i (I_c - 3)^i
\]
\[
\dot{\psi}_{\text{dev, aniso}} = \sum_{k=2}^{6} c_k (IV_c - 1)^k + \sum_{m=2}^{6} e_m (VI_c - 1)^m
\]

After the modification of polyconvexity to the original SEF in Eq. (5.1), the resulting SEF does not contain the instabilities of the eighth invariant and also requires less than half of the coefficients listed in Eq. (5.1). Thus, the general form of the SEF listed in Eqs. (5.2) and (5.3) is ready to be fit to the experimental data.

### 5.3 Fitting the Model to Experimental Data

The experimental data was approximated from published biaxial tensile data of human AAA tissue (cf. Fig. 1.19) by Vande Geest et al. (2006a). An illustration of a biaxial tensile experiment is located in Fig. 5.2.
For more information on biaxial tension the interested reader is referred to the detailed procedure and theory outlined by Sacks (2000). The experiments were conducted at different ratios (1:1, 0.75:1, 1:0.75) between the two loading directions (11 and 22), such that equibiaxial tension was not the only data acquired. Thus, each loading ratio corresponded to a single set of data (3 sets total) that contained stress and deformation information for both the 11 and 22 directions. The principal directions for the two sets of embedded collagen fibers were set to ±43.4° from the 11 direction respectively. The fiber directions were obtained from collagen fiber directions in the healthy abdominal aorta (Balzani et al., 2006). Currently, there has not been any experimental assessment of the collagen fiber direction within AAA tissue, which makes the assumption that the collagen fibers do not change orientation plausible in this initial investigation.

5.3.1 Curve fitting procedure

A custom MATLAB constrained nonlinear optimization routine was written to collectively fit the SEF in Eqs. (5.2) and (5.3) to all 3 sets of the experimental data simultaneously. The concept of constrained nonlinear optimization was used to ensure the model coefficients would not contain negative values, ensuring the requirements of polyconvexity. A weighted sum of least squares between the stress calculated from the specified model and the experimental stress values was minimized in the optimization routine.
\[ err_m = \left[ \sum_{i=1}^{n} \left[ K_{11} \left( S_{11e} - S_{11a} \right)_i^2 + K_{22} \left( S_{22e} - S_{22a} \right)_i^2 \right] \right]_m \]

\[ K_{11} = \frac{1}{\max(S_{11e})} \quad K_{22} = \frac{1}{\max(S_{22e})} \]  \hspace{1cm} (5.4)

\[ Err = err_1 + err_2 + \cdots + err_m \]

\( S_{11e} \) and \( S_{11a} \) are the experimental and analytical 2\textsuperscript{nd} Piola-Kirchhoff stress in the 11 direction respectively and \( S_{22e} \) and \( S_{22a} \) are the experimental and analytical 2\textsuperscript{nd} Piola-Kirchhoff stress in the 22 direction respectively. The \( Err \) term is a collective error term that combines the normalized error from all data sets into a single sum. To further assess the fit of the model to the experimental data, an r-squared value was calculated from the square of Pearson’s correlation coefficient:

\[ r^2 = \left[ \frac{\sum_{i=1}^{n} \{ (S_i - \bar{S})_e \cdot (S_i - \bar{S})_a \}^2}{\sum_{i=1}^{n} (S_i - \bar{S})_e^2 \cdot \sum_{i=1}^{n} (S_i - \bar{S})_a^2} \right] \] \hspace{1cm} (5.5)

where \( S \) is the 2\textsuperscript{nd} Piola-Kirchhoff stress, \( \bar{S} \) is the mean 2\textsuperscript{nd} Piola-Kirchhoff stress for the appropriate data set, \((\bullet)_a\) denotes to analytical/model values, and \((\bullet)_e\) denotes experimental values. Sufficient matches to the experimental data yield \( r^2 \)-values between 0.9 and 1.0, but it is preferred that \( r \) should fall in the range of \( 0.95 \leq r \leq 1.00 \) (Humphrey, 2002). An important consequence of the biaxial data is that there will be two separate correlation coefficients for each set of data that can be combined into a single coefficient.

\[ r^2 = \frac{r_1 + r_2}{2} \] \hspace{1cm} (5.6)
The $r^2$ value for each set can then be averaged together to calculate a mean $r^2$ value that can represent all sets of data.

5.3.2 Modifications to the SEF model

After a significant portion of curve fitting attempts, a portion of the model’s coefficients were less than 1e-006. Thus, in an attempt to limit the amount of needless coefficients and to provide equal load across both sets of fibers, the general form of the SEF was simplified to form listed in Eq. (5.7).

$$
\psi_{dev, iso} = \alpha (I_C - 3)^2 \\
\psi_{dev, aniso} = \beta (V^C - 1)^6 + \beta (V^C - 1)^6
$$  (5.7)

The resulting model contains only two unknown coefficients $\alpha$ and $\beta$ while maintaining polyconvexity for all values of $\alpha$ and $\beta$ greater than zero. Thus, the original 31 coefficient model has been simplified to only a two coefficient model.

5.3.3 Fitting the simplified model to experimental data

Using the smoothed sets of experimental data from biaxial tensile data of AAA tissue (Vande Geest, 2006), the simplified model was fit to all three sets of data simultaneously using the custom MATLAB nonlinear optimization routine. Table 5.1 contains the model’s coefficients determined from the curve fitting analysis.

<table>
<thead>
<tr>
<th>Table 5.1: AAA model coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$ [kPa]</td>
</tr>
<tr>
<td>7.433</td>
</tr>
</tbody>
</table>
The resulting $\text{Err}$ was less than 3.000 and the mean $r^2$ value was 0.9987 with a minimum of 0.9980. Figure 5.3 provides a visual illustration of the model’s fit to the experimental data.

The plots in figure 5.3 reveal a good correlation between the experimental data and the new AAA model. At higher stretches, the error begins to increase, but the stress remains reasonably close to the experimental values. Furthermore, at low stretch ratios, all the models appear to have similar (if not exactly the same) material response. Such behavior is acceptable and physiologically realistic, since the collagen fibers are the source of anisotropy and are inactive at low values of stretch.
The new model outlined in Eq. (5.7) was extended to the case of healthy abdominal aortic (HAA) tissue. Experimental biaxial tensile data (cf Fig. 1.12) of healthy abdominal aortas within the same age range of the previous AAA data was approximated from Vande Geest et al. (2004). Using the same custom MATLAB optimization routine, the general form of the model was fit to the experimental data. Table 5.2 contains the model’s coefficients for healthy abdominal aortic tissue.

<table>
<thead>
<tr>
<th>Table 5.2: HAA model coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>𝛼 [kPa]</td>
</tr>
<tr>
<td>77.228</td>
</tr>
</tbody>
</table>

The resulting $Err$ was less than 1.5 and the mean $r^2$ value was 0.9976 with a minimum of 0.9970. Figure 5.4 provides a visual of the model’s fit to the experimental data.
A noticeable difference between the curves in Figs. 5.3 – 5.4 is the greater amount of deformation and enhanced elastic behavior of the HAA tissue in Figure 5.4. Thus, the new model based on the microstructure of collagen fibers within an isotropic medium matches the experimental data for both HAA and AAA tissue with different sets of material coefficients.

**5.4 Characteristics of the New AAA Model**

Prior to implementation, the new model must be analyzed for numerical stability, predictive capabilities, and compared against existing models of AAA tissue. Too often this
step of further analysis is neglected, creating the possibility for immense errors in subsequent investigations where the model may be incorrectly applied. Thus, the information gained from such analyses will determine when the model is best suited for implementation within computational investigations of AAA tissue and assess the model’s overall ability to mimic the behavior of AAA tissue.

5.4.1 Numerical stability and behavior

Although the model maintains polyconvexity *a priori*, the energy contours of the model should still be tested to ensure they form a monotonically increasing relationship with increasing deformation. Holzapfel et al (2000) illustrated for several models that convex energy contours of the strain energy function imply a monotonic increase of strain energy. Thus, the energy contours of the new model with the best-fit coefficients listed in Table 5.1 were computed for arbitrary, equibiaxial Lagrange strains of (-0.4) – 1.

![Fig 5.5: Convex energy contours of SEF in [kPa]](image-url)
The contours within Fig 5.5 are arranged in a convex pattern and also reveal a monotonic increase in energy with an increase in strain/deformation. Thus, the new model fit to AAA tissue satisfies the fundamental requirements of numerical stability outlined in Ch.3.

One of the advantages of the new model is its basis on the artery’s microstructure which divides the model into isotropic and anisotropic components. A comparison between of the stress within each component will yield fundamental knowledge regarding the load-bearing nature of AAA and HAA microstructure. Thus, a loading ratio was constructed based on the isotropic and anisotropic components of the total stress and is listed in Eq. (5.8).

\[
\chi_{iso} = \frac{\sigma_{iso}}{\sigma} \quad \chi_{aniso} = \frac{\sigma_{aniso}}{\sigma}
\]

(5.8)

where: \( \sigma = \sigma_{iso} + \sigma_{aniso} \)

\( \sigma > 0 \)

where \( \sigma_{iso} \) is the isotropic stress of the model, \( \sigma_{aniso} \) is the anisotropic stress of the model, \( \sigma \) is the total stress of the model. Figure 5.7 illustrates the different values of \( \chi \) for the AAA and HAA models at various stretch ratios of equibiaxial extension. The ratios do not have any significant differences based on the 11 or 22 biaxial directions, which enables the general trends to be represented by a single graph.
It is very apparent that the anisotropic component of AAA tissue begins bearing most of the load at very low deformations, while the anisotropic component of HAA tissue does not bear a majority of the load until the upper range of deformations. The stretch ratio where 95% of the load is included within the anisotropic component of the AAA model is 1.035 whereas the corresponding stretch ratio of the HAA model is 1.26.

The previous loading ratios imply that the new model incorporates changes in the tissue histology. As outlined in Ch.1, elastin, its cross links, and smooth muscle cells are vastly diminished within AAA tissue (He and Roach, 1994; Carmo et al., 2002). The characteristics of these cells (cf. Section 1.2.2), it would be classified as the primary load-bearing structures within the isotropic component of the new structural model. Thus, when the new AAA model predicts a vast decrease in the loading of the isotropic stress component of AAA tissue, the model is accurately reporting the trend of diminished elastin and smooth muscle within the aneurysm wall. The result is a rapid increase of anisotropic stress, which the

Fig 5.6: Change in loading ratio vs. increase in deformation
model interprets as the embedded fibers bearing the vast majority of the mechanical load. Such a trend is also supported by analyses of the AAA wall histology, which have shown increased cross links of collagen, a decrease of new collagen fiber synthesis, while a simultaneous increase of overall percentage of collagen within the AAA wall (cf. Section 1.7.1).

5.4.2 Predictive capabilities and comparison with existing models

As mentioned in Ch 2., a SEF is only valid over the range of experimental data that the model has sufficiently matched. Prior to extrapolating the SEF to situations outside of the experimental data, the predictability of the model must be assessed. Humphrey (2002) stated that to evaluate a model’s predictive abilities, the model should fit another set of experimental data that was not used to determine the model’s parameters. It is preferred that the new set of data is outside the range of the original data. To date, only one set of experimental biaxial tensile data for AAA tissue and HAA tissue has been recorded, and this data was used to generate the model parameters. Thus, it is impossible to completely ascertain the model’s predictive abilities outside the range of experimental data.

However, *ex-vivo* uniaxial tensile data may be used to create a simulated state of biaxial tension through the conditions of incompressibility outlined in Holzapfel (2006). *Ex-vivo* uniaxial tensile analyses of AAA tissue have been performed by Raghavan et al. (1996) and Thubrikar et al. (2001a). The experimental results of Raghavan et al. (1996) implied that AAA tissue is more rigid in the axial direction than the circumferential direction (a stark contradiction to the biaxial Vande Geest (2006a) results). Conversely, the experimental data of Thubrikar et al. (2001a) revealed a greater circumferential stiffness, a similar trend to the
Vande Geest (2006) data and also contained data outside the range of the biaxial tensile experiment. Thus, the average trends of Thubrikar’s et al’s (2001a) experimental data were used to simulate a state of biaxial tension for an incompressible material utilizing similar techniques as Holzapfel (2006). The stress corresponding to the experimental deformation was calculated for the two existing AAA models (modified Fung and Isotropic) and the new polyconvex anisotropic AAA model, where the resulting stress-deformation curve of each model is compared to the experimental data.

Fig 5.7: Comparison between AAA models and experimental data from Thubrikar et. al. (2001a)

Despite the vast difference between the models’ and experimental values an ultimate determination regarding the models’ predictability cannot be assessed by the previous information alone. More experimental data is needed to completely assess the models of AAA tissue. Currently, true biaxial data is the closest resemblance to *in-vivo* loading of arterial tissue in an *ex-vivo* environment. The experimental data of Thubrikar et al. (2001a) is true uniaxial tensile data, which is based on very different loading conditions than a biaxial tensile experiment. Thus, to truly analyze the effectiveness of an AAA model, additional
biaxial tensile experiments should be performed with measurements of collagen fiber orientation. The differences between the three AAA models should still be critically compared with one another. Each model was thus placed under a general state of equibiaxial strain. The resulting stress-deformation curves of each model for the 11 direction are shown in Fig. 5.8.

Figure 5.8 reveals significant differences between the three models at the different stages of deformation. Under stretch ratios less than 1.10, the isotropic AAA model reports significantly higher stress values than both the Fung and the new polyconvex models of AAA.
tissue. However, at stretch ratios greater than 1.125 both the Fung and the new polyconvex model begin to have an exponential increase in stress that soon far exceeds the values predicted by the isotropic model. At stretch ratios less than 1.09, the Fung and the new polyconvex model are almost identical, which is an important trend since the experimental data used to construct each model had stretch ratios less than 1.09. Thus, since both models yield relatively the same response for arbitrary deformations within the range of the experimental data, the new structural model captures the same phenomena as the modified-Fung model over the range of experimental data. Despite their near exact response over the experimental range of data, the two models begin to quickly diverge at stretch ratios greater than 1.1. The new polyconvex model reports a greater deformation and lower stress while the Fung model conversely reports a state of lessened deformation and increased stress. At the max stretch ratio of 1.5 the corresponding stress of all three models is listed in Table 5.3.

<table>
<thead>
<tr>
<th>AAA Model</th>
<th>Stretch</th>
<th>Stress [kPa]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Isotropic</td>
<td>1.50</td>
<td>1.197e+4</td>
</tr>
<tr>
<td>New polyconvex</td>
<td>1.50</td>
<td>2.459e+6</td>
</tr>
<tr>
<td>Fung-type</td>
<td>1.50</td>
<td>6.600e+70</td>
</tr>
</tbody>
</table>

The critical comparison between all the AAA models revealed large differences both within and outside the range of experimental data used to construct the anisotropic models. Thus, to determine which model reports has the greatest range of accurate biomechanical prediction; additional experiments must be conducted to determine the *in-vivo* biomechanical response outside of the current *ex-vivo* data.
5.5 Summary of New AAA Model

A new model for AAA and HAA tissue has been constructed from realistic \textit{ex-vivo} experimental data and the fundamental microstructure of each tissue. The new model predicts realistic trends of the loading of each tissue’s microstructure, something the existing models of both tissues can not accomplish. Furthermore, the new model captures the same phenomenological response as the modified-Fung model for AAA tissue within the range of experimental data. Therefore, the new AAA and HAA model incorporates a new degree of realism that was previously unavailable in the existing AAA and HAA models. Additionally, the model has been designed with a high degree of numerical stability and streamlined implementation into the commercial finite element software ANSYS. A detailed breakdown of the new AAA and HAA model is shown below.

\[
\hat{\psi} = \hat{\psi}_{\text{deviatoric}} + \frac{1}{\eta} (III_C - 1)^2
\]

\[
\hat{\psi}_{\text{deviatoric}} = \hat{\psi}_{\text{dev,iso}} + \hat{\psi}_{\text{dev,aniso}}
\]

NOTE: the differences between the AAA and HAA models are encapsulated within the constant coefficients \(\alpha\) and \(\beta\), which have the values specified in Tables 5.1 and 5.2 for AAA and HAA tissue respectively and are listed within the derivation below.
\[ \hat{\psi}_{\text{dev, iso}} = \alpha \left( I_C - 3 \right)^2 \]

Constant coefficient  Incorporation of isotropic deformation from elastin and smooth muscle cells

\[ \hat{\psi}_{\text{dev, aniso}} = \beta \left( (IV_C - 1)^6 + (VI_C - 1)^6 \right) \]

Constant coefficient  Incorporation of anisotropy deformation from embedded collagen fibers

\[ \hat{\psi} = \alpha \left( I_C - 3 \right)^2 + \beta \left( (IV_C - 1)^6 + (VI_C - 1)^6 \right) + \frac{1}{\eta} \left( III_C - 1 \right)^2 \]

Isotropic component  Anisotropic component  Incompressible component

Isotropic Strain energy  Anisotropic Strain energy  Volumetric Strain energy

The stress and strain are calculated from the previous model by the following equations:

\[ S_{ij} = 2 \frac{\partial \hat{\psi}_{\text{dev}}}{\partial C_{ij}} - pC_{ij}^{-1} \]

\[ = 2 \left( \frac{\partial \hat{\psi}_{\text{dev}}}{\partial I_C} \frac{\partial I_C}{\partial C_{ij}} + \frac{\partial \hat{\psi}_{\text{dev}}}{\partial IV_C} \frac{\partial IV_C}{\partial C_{ij}} + \frac{\partial \hat{\psi}_{\text{dev}}}{\partial VI_C} \frac{\partial VI_C}{\partial C_{ij}} \right) - pC_{ij}^{-1} \]

Isotropic 2\(^{\text{nd}}\) PK Stress  Anisotropic 2\(^{\text{nd}}\) PK Stress  Volumetric 2\(^{\text{nd}}\) PK Stress
In conclusion, the new model contains both isochoric and volumetric strain energy while exhibiting polyconvexity, numerical stability, and a new degree of biomechanical realism. Some limitations of the model are the assumptions that the active behavior of the arterial wall is in a quasi-static state and the lack of experimental data for further validation. Despite these limitations, it is the first model of AAA and HAA tissue that is based on the tissue microstructure and predicts realistic trends of microstructure loading in healthy and pathological cases. Thus, the model is ready for implementation within a Finite-Element analysis.

\[
S_{ij} = 2\alpha(I - 3)\delta_{ij} + 6\beta(I - 1)\left(a_i \otimes a_j\right) + 6\beta(VI - 1)\left(g_i \otimes g_j\right) - pC_{ij}^{-1}
\]

<table>
<thead>
<tr>
<th></th>
<th>AAA</th>
<th>HAA</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\alpha) [kPa]</td>
<td>7.433</td>
<td>77.228</td>
</tr>
<tr>
<td>(\beta) [kPa]</td>
<td>63.597e+003</td>
<td>2.804e+003</td>
</tr>
</tbody>
</table>

The previous equations fully describe the coupled stress and deformation of the new AAA and HAA models. Figures 5.3 and 5.4 contain representative stress vs. deformation curves of the AAA and HAA model respectively.

Either stress or deformation must be specified to calculate its corresponding counterpart.

Numerical techniques such as the Newton-Raphson algorithm are then used to calculate roots of the resulting system of nonlinear equations.

In conclusion, the new model contains both isochoric and volumetric strain energy while exhibiting polyconvexity, numerical stability, and a new degree of biomechanical realism.
5.6 Application of Non-Linear Modeling to Tracheobronchial Airways

The inner tracheobronchial airways and the alveolar region exhibit significant amounts of deformation during the inspiration and exhalation of the lung parenchyma. Such a large amount of deformation would have a profound affect on particle depositions within these highly elastic regions (Dailey and Ghadiali, 2007). However, no models of the inner tracheobronchial airways and alveolar region exist that are based on the region’s local mechanical properties. In general, the lung is modeled with a global perspective that accounts for the change in the lung’s volume during respiration. Such global models do not provide enough detail to formulate a model of the inner tracheobronchial airways and alveolar region.

Since the lung parenchyma itself can be modeled using nonlinear continuum mechanics, models of the alveolar region and inner tracheobronchial airways may also be constructed using the previously discussed methods used to construct a new model of the HAA and AAA (Holzapfel, 2005; Fung, 1993). Currently, no experimental data exists that characterizes the local mechanical response of the inner tracheobronchial airways and the alveolar region. Thus, the previous modeling procedure used for HAA and AAA tissue will be used to initiate research of the local mechanical response of the inner tracheobronchial airways and alveolar region. The affects of the region’s local mechanical response on particle deposition throughout the region can then be analyzed, achieving a new degree of realism that is currently unavailable in computational models.
6. COMPARATIVE STUDY OF TWO ABDOMINAL AORTIC ANEURYSM MODELS FOR FINITE ELEMENT ANALYSES

6.1 Introduction

Numerous computational investigations of AAA tissue continue to be conducted as method to optimize clinical treatment of the irreversible pathological condition. Recent improvements to these investigations include patient specific geometries, incorporation of isotropic nonlinear response, and experimental biaxial analyses of AAA tissue. However, a critical comparison between the various wall models used to describe the AAA wall has yet to be analyzed. To provide this needed comparison, a comparative study attempting to isolate the effects of the AAA model on the resulting stress distribution and deformation has been initiated.

6.2 Finite-element methodology

The preferred method for most computational simulations of solid deformations is the Galerkin finite element method. All of the following studies were conducted using the commercial finite element software ANSYS v.11 from Canonsburg, PA.

6.2.1 Idealized AAA geometry

An idealized aneurysm geometry was created from the published clinical data of Hirsch et al. (2005) and Espinosa et al. (2005) so that each AAA wall model can be tested over the
same domain. The geometric characteristics were selected to contain a significant amount of physiological realism and minimize conditions that would adversely affect an objective comparison between the multiple models. Some of the primary idealizations of the geometry were the assumption of constant wall thickness, a single homogeneous layer of the aneurysm wall, and the removal of the aorto-iliac bifurcation at the proximal end of the aneurysm sac/bulge. Figure 6.1 contains a visual illustration of the idealized AAA geometry.

The geometry contained no planes of symmetry to ensure that a full three-dimensional study was conducted on all AAA wall models. Furthermore, the axial direction of the geometry corresponded the z-coordinates whereas the x and y coordinates characterize the cross section of the geometry.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inlet radius</td>
<td>1.063 [cm]</td>
</tr>
<tr>
<td>Outlet radius</td>
<td>0.865 [cm]</td>
</tr>
<tr>
<td>Max diameter</td>
<td>5.25 [cm]</td>
</tr>
<tr>
<td>Wall thickness</td>
<td>2.00 [mm]</td>
</tr>
<tr>
<td>Total length</td>
<td>10 [cm]</td>
</tr>
</tbody>
</table>

Fig. 6.1: Idealized aneurysm geometry and parameters
6.2.2 Numerical conditions and assumptions

The geometry was constrained in the axial directions at the proximal and distal ends of the structure to simulate vascular tethering at these locations. A uniform static gauge pressure of 120 mmHg was applied to the inner surfaces of the aneurysm geometry while a uniform, static gauge pressure of 0 mmHg was applied to the exterior surfaces. Residual stress of the aneurysm was neglected for all material models while each model was assumed to be nearly incompressible.

A mesh independence study was conducted to ensure that the computational solutions were independent of mesh refinement. The max Von Mises stress of four different meshes was compared, where each mesh had a different number of elements. Table 6.1 contains the overall results from the mesh independence analysis.

<table>
<thead>
<tr>
<th>Mesh No.</th>
<th>Elements</th>
<th>Max Von Mises Stress [kPa]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>25,089</td>
<td>214.383</td>
</tr>
<tr>
<td>2</td>
<td>31,761</td>
<td>214.437</td>
</tr>
<tr>
<td>3</td>
<td>36,880</td>
<td>214.718</td>
</tr>
<tr>
<td>4</td>
<td>41,191</td>
<td>214.516</td>
</tr>
</tbody>
</table>

Mesh two was selected as the optimal mesh based on a combination of overall element distribution, acceptable solution time, and negligible change in accuracy in future mesh refinements.
6.3 Validation Study

Prior to initializing the comparative investigation, the nonlinear capabilities of ANSYS must be validated by comparing sample finite element computations to known solutions of the same problem. The analytical solution of isotropic hyperelastic materials undergoing inflation of a thick-walled cylinder with a constant length was thus compared to the ANSYS finite element solutions for the same problems. Summaries of the analytical solution are presented within the following text. Refer to Appendix I for a detailed derivation of the analytical solution according to the theory of nonlinear continuum mechanics.

The analytical solution for the inflation of a tube with a constant length has its origins when Mooney and Rivlin conducted some of their initial tensile experiments of rubberlike material (Humphrey, 2002). Figure 6.2 clearly illustrates the inflation of a cylinder with a constant length and the resulting deformation.

![Diagram of inflation of a thick-walled cylinder with a constant length](image)

**Fig. 6.2:** Inflation of a thick-walled cylinder with a constant length

The corresponding deformation gradient for the solution has the form listed in Eq. (6.1)
\[ \bar{F} = \begin{bmatrix}
\frac{dr}{dR} & 0 & 0 \\
0 & \frac{r}{R} & 0 \\
0 & 0 & 1
\end{bmatrix} \quad (6.1) \]

Where \( r \) is the radii in the deformed orientation and \( R \) is the corresponding radii in the undeformed orientation. Coupling the deformation gradient in Eq. (6.1) with the conditions of incompressibility, zero pressure on the exterior surface of the cylinder, conditions of static equilibrium, and a known inflation pressure an analytical solution for the problem can be solved for a specific SEF. For this problem, a two-parameter Mooney-Rivlin SEF was used, which is the same equation listed in Eq. (4.2) without the squared term containing the second invariant. The parameters for the Mooney-Rivlin model are listed in Table 6.2.

<table>
<thead>
<tr>
<th>Table 6.2: Mooney-Rivlin parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_1 )</td>
</tr>
<tr>
<td>( C_2 )</td>
</tr>
</tbody>
</table>

Three different pressures were used to inflate a thick-walled cylinder of dimensions that matched average values of the HAA. The three pressures were 80, 100, and 120 mmHg, simulating average values for human diastolic, mean arterial, and systolic pressure. The finite element computations and analytical values for the Von Mises stress and circumferential stretch ratio \((r/R)\) through the thickness of the wall were compared for each inflation pressure. It was found that ANSYS nonlinear simulations of hyperelastic material run much smoother when the number of substeps is specified rather than using the default
automatic time-stepping feature. Figure 6.3 illustrates the comparison between analytical and finite element computations.

The max percent difference between a single finite element value and its corresponding analytical solution was less than 3.7%. Since, the percent difference is well within the acceptable range of error (<10%) the results of the subsequent comparative analysis have a significant amount of credibility.

6.4 Results of Initial Finite Element Simulations

The initial simulations feature a direct comparison between two popular isotropic models of the AAA wall. Model one is a linear elastic model with a specified Young’s modulus and Poisson ratio, whereas model two is the isotropic hyperelastic two-parameter Yeoh model. Table 6.3 contains their material parameters.

Fig. 6.3: Validation of finite element simulations with analytical solutions
Table 6.3: AAA wall model parameters

<table>
<thead>
<tr>
<th>Model</th>
<th>$C_1^*$</th>
<th>$C_2^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear Elastic (Thubrikar et al., 2001b)</td>
<td>4.66 [MPa]</td>
<td>0.45</td>
</tr>
<tr>
<td>2-param Yeoh (Raghavan, 2000)</td>
<td>0.174 [MPa]</td>
<td>1.881 [MPa]</td>
</tr>
</tbody>
</table>

*C$_1$ and C$_2$ correspond to the Young’s modulus and Poisson ratio, respectively for the linear elastic model.

To date, an objective comparison of each model’s affects on a single geometry’s resulting stress and deformation has yet to be completed. Figures 6.4 – 6.6 illustrate the different Von Mises stress distributions along the exterior and interior surfaces of the aneurysm for each AAA wall model; whereas, Figure 6.7 illustrates the spatially varying radial deformation of each model.
Fig 6.4: Von Mises stress distributions along exterior surface of AAA geometry for different AAA wall models

Isotropic Linear Elastic – view 1

Isotropic Hyperelastic – view 1

Von Mises Stress [Pa]

2.15E+05
2.01E+05
1.87E+05
1.73E+05
1.59E+05
1.45E+05
1.31E+05
1.18E+05
1.04E+05
8.96E+04
7.57E+04
6.18E+04
4.79E+04
3.39E+04
2.00E+04

Isotropic Linear Elastic – view 2

Isotropic Hyperelastic – view 2

Fig 6.4: Von Mises stress distributions along exterior surface of AAA geometry for different AAA wall models
Fig 6.5: Von Mises stress distributions along the x-plane interior surface of the AAA geometry for different AAA wall models
Fig 6.6: Von Mises stress distributions along the y-plane interior surface of the AAA geometry for different AAA wall models.

- **Isotropic Linear Elastic** – \( y \geq 0 \)
- **Isotropic Hyperelastic** – \( y \geq 0 \)
- **Isotropic Linear Elastic** – \( y \leq 0 \)
- **Isotropic Hyperelastic** – \( y \leq 0 \)

Von Mises Stress [Pa]:
- Red: 2.15E+05
- Orange: 2.01E+05
- Green: 1.87E+05
- Light Green: 1.73E+05
- Light Blue: 1.59E+05
- Blue: 1.45E+05
- Light Green: 1.31E+05
- Green: 1.18E+05
- Orange: 1.04E+05
- Orange: 8.96E+04
- Light Green: 7.57E+04
- Light Blue: 6.18E+04
- Blue: 4.79E+04
- Light Blue: 3.39E+04
- Light Green: 2.00E+04
Fig 6.7: Radial displacement along surface of AAA geometry for different AAA wall models
6.7 Discussions and Clinical Relevance

Figures 6.4 – 6.7 reveal significant differences between the AAA wall linear and nonlinear models. Thus, although the arteries may appear to have a linear stress-strain relationship at internal pressures greater than 80 mmHg, nonlinear models that have little change in slope during this region of internal pressure generate a very different stress distribution and radial expansion than the linear models. The source of these trends is from each model’s formulation, the hyperelastic model is designed to allow large amounts of deformation without an excessive increase of stress, whereas the linear model cannot mimic such behavior.

It is well-known that arteries undergo radial expansions that are classified as large strains. However, to the best knowledge of the author, the numerical data of a large-scale clinical study of in-vivo AAA radial expansions remains to be published. It is thus currently imprudent to develop ultimate conclusions regarding the accuracy of the linear and nonlinear calculations of radial deflection. Well-posed hypotheses of aneurysm behavior based on ex-vivo biaxial studies are currently the most realistic descriptions of in-vivo AAA pressurization.

When compared to the experimental data of Vande Geest (2006a), the initial realization is that a significant portion of the computational circumferential stretch ratios are outside the range of experimental values. Referring back to Section 5.4.2, each model behaves very differently outside the range of experimental data, making these regions incapable of being compared to the experimental data. However, both models have select regions of the idealized AAA geometry that have circumferential stretches within the range of experimental
data. The Von Mises stress and circumferential stretch was measured at three specific points of the idealized AAA geometry for both the linear and the isotropic hyperelastic model. The computational circumferential stretch ratios at each point were then used to obtain corresponding experimental Von Misses stress values from Vande Geest et al. (2006) equibiaxial tensile data. Figure 6.8 compares the experimental and computational results.

![Comparison between experimental and computational results](image)

Fig. 6.8: Comparison between experimental and computational results

The trends in Fig 6.8 reveal vast differences between the experimental and the computational results. Such trends have very high clinical relevance since engineers are attempting to use stress distributions within AAA geometries as predictors of aneurysm rupture and severity. However, many engineers have understated the importance of accurate constitutive models to describe the AAA tissue. The discrepancies between the linear and the nonlinear isotropic models, both not matching experimental data clearly illustrate the possible errors generated with the current models. Thus, there is a need for experimental/clinical validations of AAA constitutive models prior to utilizing the maximum stress criteria as the prevailing method for
rupture risk assessment. Furthermore, clinical data sets are needed for rigorous engineering testing and validation of FSI simulations.
In conclusion, all three research objectives (see Ch.2) of the previous study have been met. A strong theoretical framework has been developed, a new structural model containing a new degree of realism has been constructed, and the two most prominent models of the AAA wall have been critically compared to experimental data. The results from the previous study should have a significant impact on future analyses of AAA tissue in both the experimental and computational domains. Whereas, new experiments should provide quantitative data, physical insight, and computer model validation for computational investigations to analyze the impact of the new AAA-wall model. Clinicians will thus gain a greater understanding of computational biomedical engineering analyses and their use for optimal medical management.

7.1 Conclusions

The summary of all the knowledge assimilated from this study pertained to the modeling efforts of AAA tissue. Experiments have clearly shown the AAA wall exhibits a significant amount of anisotropy and rigidity within its stress-strain data. The new model fitted to the experimental data exhibits polyconvexity and monotonically increasing energy contours for all possible deformations, and the model reports the microstructure loading of AAA and HAA tissue with a new degree of realism. A comparison between experimental data and the
current prominent AAA models revealed large discrepancies that suggest the current models are incapable of accurately modeling AAA tissue.

The clinical relevance of this study is significant, because most medical practitioners are unfamiliar with the methods engineers use to model AAA tissue. Furthermore, as engineers continue to promote the use of wall stress as a method of rupture risk assessment, the present study reveals that the calculated wall stress has a strong dependence on the constitutive model used for AAA tissue behavior. A wall stress analysis of AAA tissue can therefore be conducted with sufficient realism in terms of geometry, loading, and multi-physics coupling, yet still report unrealistic wall stress data based on the constitutive model used for the AAA wall.

### 7.2 Future Work

The results from this study uncover a wide range of future research projects. First is to compare the new anisotropic model of AAA tissue with the current isotropic models in a finite element analysis of AAAs. The resulting study would further validate the new model and document the affects of anisotropy in finite element analysis of AAA tissue. Following this subsequent validation, the new model and current isotropic models can be inserted in simulations that incorporate fluid-structure interactions between pulsatile blood flow and the AAA wall. The results from these studies can then be used to enhance the effectiveness of current endovascular therapies such as stent-graft implants. The knowledge from this study can also be applied to additional nonlinear soft tissues. Currently an area that has yet to be
thoroughly investigated is the bronchial airways and alveolar region of the human lung (cf. Section 5.6).

Further computational results would be to investigate the effects of a heterogeneous wall with residual stress. Currently, all models of AAA tissue assume a homogeneous wall, yet histological staining and biological investigations have clearly shown three heterogeneous layers of the arterial wall (cf. Ch. 1). However, experimentation has not been conducted on the different layers of AAA tissue. Additional computational work can also include dividing the AAA geometry into different regions and using a separate constitutive equation for each region. Such an approach would be similar to the experimental methods of Thubrikar et al. (2001a). Another assumption of current models is quasi-steady state behavior. By extending the underlying theory to include the dynamic behavior of the arterial wall, further realism could be achieved.

Future experimental work of AAA tissue should include the quantification of collagen fiber distribution throughout the AAA geometry. The in-vivo and ex-vivo outer radii of AAA morphologies should be recorded for the validation and calibration of AAA constitutive models. Additional biaxial tensile experiments should also be conducted over a stricter range of samples (gender, age, and anatomic location within the AAA). Such results will provide much needed data for future constitutive equations and experimentally assess the significance of specific characteristics of AAA tissue not included in the initial biaxial study of Vand Geest (2006a).
It is the desire of the author that this study will prove to be an aid for future analyses of AAA tissue. The theoretical framework of the study was designed to be a detailed reference for future efforts in nonlinear modeling, while clinical relevance was analyzed as part of each research objective. Hopefully the study will spur new research that will yield enhanced treatment of multiple pathological conditions, such as abdominal aortic aneurysms.
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APPENDICES
APPENDIX I: Analytical solutions of nonlinear continuum mechanics

The following pages discuss the detailed derivations for the analytical solutions of biaxial extension and the static inflation of a thick-walled cylinder with constant length. Below is a listed of important variables and symbols applicable in this Appendix only.

\[ \bar{F} = \text{deformation gradient} \]
\[ \bar{C} = \text{right Cauchy – Green stretch tensor} \]
\[ \bar{B} = \text{left Cauchy – Green stretch tensor} \]
\[ \sigma = \text{Cauchy stress tensor} \]
\[ T = \text{1st Piola – Kirchoff stress tensor} \]
\[ P_1 = \text{tensile load in 11 direction} \]
\[ P_2 = \text{tensile load in 22 direction} \]
\[ I_c = \text{first in variant of the right Cauchy – Green stretch tensor} \]
\[ II_c = \text{second in variant of the right Cauchy – Green stretch tensor} \]
\[ III_c = \text{third in variant of the right Cauchy – Green stretch tensor} \]
\[ I_B = \text{first in variant of the left Cauchy – Green stretch tensor} \]
\[ II_B = \text{second in variant of the left Cauchy – Green stretch tensor} \]
\[ III_B = \text{third in variant of the left Cauchy – Green stretch tensor} \]
\[ p = \text{Lagrange multiplier} \]
\[ \bar{I} = \text{identity matrix} \]
\[ \hat{\psi} = \text{Strain energy function (per unit volume) for the deviatoric/volume preserving loading} \]

\[ \lambda_r = \text{radial stretch ratio} \]
\[ \lambda_\theta = \text{circumferential stretch ratio} \]
\[ \lambda_z = \text{axial stretch ratio} \]

\[ R = \text{radii from the reference configuration} \]
\[ \Theta = \text{theta from the reference configuration} \]
\[ Z = \text{axial length from the reference configuration} \]
\[ r = \text{radii from the deformed configuration} \]
\[ \theta = \text{theta from the deformed configuration} \]
\[ z = \text{axial length from the deformed configuration} \]
\[ R_i = \text{reference inner radius} \]
\[ R_o = \text{reference outer radius} \]
\[ r_i = \text{deformed inner radius} \]
\[ r_o = \text{deformed outer radius} \]
\[ P_i = \text{inner pressure of hollow cylinder} \]
**In-plane biaxial extension of a thin sheet of nonlinear, isotropic material**

The analytical solution for biaxial tensile loading of nonlinear materials is one of the most useful analytical solutions in nonlinear elasticity. A primary use of the solution is to couple the analytical solution with experimental biaxial tensile data and derive constitutive relations that define nonlinear material behavior.

Humphrey (2002) and Sacks (2000) provide the foundational theoretical groundwork that fully describes the system of biaxial loading. One of the fundamental simplifications this system exhibits is the condition of plane stress, which makes all stress parallel to the 33 plane equivalent to zero. Another fundamental condition is that at the center of the material there exists a sub-domain that exhibits homogeneous deformation. It is this region of homogeneous deformation that makes data acquisition of biaxial tensile experiments possible. The region of homogeneous deformation is illustrated by the four dots in the figures depicting the unloaded and loaded orientations of the material. A final condition of the biaxial tension is negligible shear strain, which reduces the complexity of the equations used to characterize the deformation of the material.

Using the condition of negligible shear, a point in the original / unloaded configuration \((X_1, X_2, X_3)\) can be mapped to coordinates in the deformed configuration \((x_1, x_2, x_3)\) through the expressions listed below.

\[
x_1 = \lambda_1 X_1 ; \quad x_2 = \lambda_2 X_2 ; \quad x_3 = \lambda_3 X_3
\]
The $\lambda$ term is assumed to remain constant at each stable deformed configuration and represents the stretch of the material in the corresponding direction. From the previous equations the deformation gradient can be derived.

$$F = \begin{bmatrix}
\frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\
\frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\
\frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3}
\end{bmatrix} = \begin{bmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{bmatrix}$$

Since the deformation gradient has now been derived, the right and left Cauchy-Green stretch tensors can now be derived. The condition of negligible shear has made the deformation gradient a diagonal matrix, which results in the right and left Cauchy-Green tensors being equivalent.

$$\overline{C} = F^T F = \begin{bmatrix}
(\lambda_1)^2 & 0 & 0 \\
0 & (\lambda_2)^2 & 0 \\
0 & 0 & (\lambda_3)^2
\end{bmatrix}$$

$$\overline{B} = FF^T = \overline{C}$$

The first three invariants of the two tensors are always equal (Humphrey, 2002; Holzapfel, 2000). Such a concept will be vital to deriving an equation that relates the stress with a corresponding deformation. The first three invariants of the right Cauchy-Green stretch tensor are listed below.

$$I_1 = (\lambda_1)^2 + (\lambda_2)^2 + (\lambda_3)^2$$

$$II_1 = (\lambda_1)^2(\lambda_2)^2 + (\lambda_2)^2(\lambda_3)^2 + (\lambda_3)^2(\lambda_1)^2$$

$$III_1 = (\lambda_1)^2(\lambda_2)^2(\lambda_3)^2 \quad \Rightarrow \quad \lambda_3 = \frac{1}{\lambda_1\lambda_2}$$
The third invariant is equal to unity when the material is considered to be incompressible, a common characteristic of most biological soft tissue (Fung, 1993; Holzapfel et al., 2000; Humphrey, 2002). The remaining analytical solution will be derived under the condition of material incompressibility, which results in a strain energy function that needs to be defined only by the first two invariants rather than all three.

By definition, a strain energy function (SEF) yields an equation for the stress distribution when the derivative of the SEF is taken with respect to the corresponding strain. The chain rule than be utilized to derive an expression relating stress and the right Cauchy-Green stretch tensor.

\[ \sigma = -p I + 2F \frac{\partial \psi}{\partial C} F^T \]

Through dividing the total strain energy function into two components, one that is deviatoric / volume preserving and the other terms which is dependent on volume change, it is possible to begin the derivation of the equation relating stress and deformation.

\[ W = \psi(I_c, II_c) - \tilde{p}(III_c - 1) \]
\[ \sigma = \frac{2}{III_c} F \frac{\partial W}{\partial C} F^T \]

The derivative(s) of W with respect to the right Cauchy-Green stretch tensor must now be derived.

\[ \frac{\partial W}{\partial C} = \frac{\partial \psi(I_c, II_c)}{\partial C} - \frac{\partial \tilde{p}(III_c - 1)}{\partial C} \]
\[ = \frac{\partial \psi(I_c, II_c)}{\partial I_c} \frac{\partial I_c}{\partial C} + \frac{\partial \psi(I_c, II_c)}{\partial II_c} \frac{\partial II_c}{\partial C} - \frac{\partial \tilde{p}(III_c - 1)}{\partial III_c} \frac{\partial III_c}{\partial C} \]
\[ = \frac{\partial \psi(I_c, II_c)}{\partial I_c} I + \frac{\partial \psi(I_c, II_c)}{\partial II_c} (I_c I - C^{-T}) - \tilde{p} \cdot III_c \cdot C^{-T} \]
Inserting the previous equation into the first equation for Cauchy stress yields the following expression.

\[
\sigma = 2F \left[ \frac{\partial \tilde{\psi}}{\partial I_C} \bar{I} + \frac{\partial \tilde{\psi}}{\partial II_C} \left( I_C \bar{I} - \bar{C} \right) - \bar{p} \cdot III_C \cdot \bar{C}^{-\text{T}} \right] F^T
\]

Using the knowledge that the material is incompressible, and that by definition the right Cauchy-Green stretch tensor is symmetric, the previous equation can be simplified into the form below.

\[
\sigma = 2F \left[ \frac{\partial \tilde{\psi}}{\partial I_C} \bar{I} + \frac{\partial \tilde{\psi}}{\partial II_C} \left( I_C \bar{I} - \bar{C} \right) - \bar{p} \bar{C}^{-1} \right] F^T
\]

\[
= 2 \frac{\partial \tilde{\psi}}{\partial I_C} F F^T + 2 \frac{\partial \tilde{\psi}}{\partial II_C} \left( I_C F F^T - F C F^T \right) - 2 \bar{p} F C^{-1} F^T
\]

\[
F F^T = F F^T = \bar{B}
\]

\[
F C F^T = \left( F F^T \right)^2 = \bar{B}^2
\]

\[
F C^{-1} F^T = \left( F F^{-1} \right) F^{-T} F^T = \bar{I}
\]

\[
\sigma = 2 \frac{\partial \tilde{\psi}}{\partial I_C} \bar{B} + 2 \frac{\partial \tilde{\psi}}{\partial II_C} \left( I_C \bar{B} - \bar{B}^2 \right) - 2 \bar{p} \bar{I}
\]

The equation can be further simplified through the following relation of the Caley-Hamilton Theorem and the fact that the invariants of the right and left Cauchy-Green Stretch tensor are equal (Humphrey, 2002).

\[
I_C = I_B = \bar{B} + II_B \bar{B}^{-1} - III_B \bar{B}^{-2} = \bar{B} + II_B \bar{B}^{-1} - \bar{B}^2
\]

Coupling the preceding steps into a simplified equation yields the solution below:

\[
\sigma = 2 \frac{\partial \tilde{\psi}}{\partial I_C} \bar{B} - 2 \frac{\partial \tilde{\psi}}{\partial II_C} \bar{B}^{-1} - p \bar{I}
\]

where: \[ p = 2 \left( \bar{p} - \frac{\partial \tilde{\psi}}{\partial II_C} I_B \right) \]
After the previous step, the equation presented is much easier to utilize when deriving an analytical solution for isotropic material. An important note is that although both $\tilde{p}$ and $p$ are Lagrange multipliers, their numerical values will differ due to the formulation of $p$. However, when using the previous equation to construct an analytical solution, the value of $\tilde{p}$ does not affect the solution, since the multiplier $p$ contains all the needed reaction stress.

After several manipulations of the previous equations involving the material’s isotropy, (see Appendix I for a detailed account of the procedure), the following equations are derived. If the material were anisotropic (arteries, aneurysm tissue, tendons, etc.) the remaining equations do not apply to the material.

\[
\sigma = 2 \frac{\partial \psi}{\partial I_C} B - 2 \frac{\partial \psi}{\partial II_C} B^{-1} - p\bar{I}
\]

\[
\sigma_{11} = -p\bar{I} + 2 \frac{\partial \psi}{\partial I_C} (\lambda_1)^2 - 2 \frac{\partial \psi}{\partial II_C} (\lambda_1)^2
\]

\[
\sigma_{22} = -p\bar{I} + 2 \frac{\partial \psi}{\partial I_C} (\lambda_2)^2 - 2 \frac{\partial \psi}{\partial II_C} (\lambda_2)^2
\]

\[
\sigma_{33} = -p\bar{I} + 2 \frac{\partial \psi}{\partial I_C} (\lambda_3)^2 - 2 \frac{\partial \psi}{\partial II_C} (\lambda_3)^2
\]

The Lagrange multiplier $p$ in the previous equations must be determined from the equations of local, mechanical equilibrium under static conditions with no body forces and the boundary condition of there being zero stress for all stress parallel to the 33 plane.

\[
local, mechanical equilibrium \ (static \ conditions, \ no \ body \ forces) \quad \frac{\partial \sigma_y}{\partial x_i} = 0
\]

\[
\frac{\partial}{\partial x_i} \left( \frac{\partial \psi}{\partial I_C} \right) \frac{\partial \psi}{\partial x_i} \left( \frac{\partial \psi}{\partial II_C} \right) = 0
\]

\[
\frac{\partial p}{\partial x_i} = 0 \quad \Rightarrow \quad Lagrange \ multiplier \ independent \ of \ position
\]
After solving for the Lagrange multiplier, it is possible to finalize the equations that relate stress and deformation.

\[
\sigma_{11} = -2\left[\frac{\partial\psi}{\partial I_C}(\lambda_3)^2 - \frac{\partial\psi}{\partial II_C}(\lambda_3)^2\right] + 2\frac{\partial\psi}{\partial I_C}(\lambda_1)^2 - 2\frac{\partial\psi}{\partial II_C}(\lambda_1)^2
\]

\[
\sigma_{22} = -2\left[\frac{\partial\psi}{\partial I_C}(\lambda_3)^2 - \frac{\partial\psi}{\partial II_C}(\lambda_3)^2\right] + 2\frac{\partial\psi}{\partial I_C}(\lambda_2)^2 - 2\frac{\partial\psi}{\partial II_C}(\lambda_2)^2
\]

\[
\vdots
\]

\[
\sigma_{11} = 2\left[\lambda_1^2 - (\lambda_3)^2\right]\left[\frac{\partial\psi}{\partial I_C} + (\lambda_2)^2\frac{\partial\psi}{\partial II_C}\right]
\]

\[
\sigma_{22} = 2\left[\lambda_2^2 - (\lambda_3)^2\right]\left[\frac{\partial\psi}{\partial I_C} + (\lambda_1)^2\frac{\partial\psi}{\partial II_C}\right]
\]

Using the equation for \(\lambda_3\) that was derived from incompressibility, the final equations that relate Cauchy stress and deformation of the material are shown below.

\[
\sigma_{11} = 2\left[\lambda_1^2 - (\lambda_1\lambda_2)^2\right]\left[\frac{\partial\psi}{\partial I_C} + (\lambda_2)^2\frac{\partial\psi}{\partial II_C}\right]
\]

\[
\sigma_{22} = 2\left[\lambda_2^2 - (\lambda_1\lambda_2)^2\right]\left[\frac{\partial\psi}{\partial I_C} + (\lambda_1)^2\frac{\partial\psi}{\partial II_C}\right]
\]

In many situations, it is desired to relate the applied force/load on the material and the resulting deformation. The first Piola-Kirchoff stress tensor is defined as the applied load divided by the original/undeformed area. Thus, by utilizing a relation between the Cauchy
stress and the first Piola-Kirchoff stress, it is possible to derive an expression that relates the applied load to the material’s original/undeformed dimensions.

\[ \sigma = T F^T \]

\[ \sigma_{11} = T_{11} \lambda_1 = \lambda_1 \frac{P_1}{L_2 H} \]

\[ \sigma_{22} = T_{22} \lambda_2 = \lambda_2 \frac{P_2}{L_1 H} \]

Substituting the previous equations into the final equations that include the Cauchy stress yields the following two equations that relate the applied loads and the resulting deformation.

\[ P_1 = 2 \frac{L_2 H}{\lambda_1} \left[ (\lambda_1)^2 - (\lambda_1 \lambda_2)^2 \right] \left[ \frac{\partial \hat{\psi}}{\partial l_c} + (\lambda_2)^2 \frac{\partial \hat{\psi}}{\partial H_c} \right] \]

\[ P_2 = 2 \frac{L_1 H}{\lambda_2} \left[ (\lambda_2)^2 - (\lambda_1 \lambda_2)^2 \right] \left[ \frac{\partial \hat{\psi}}{\partial l_c} + (\lambda_1)^2 \frac{\partial \hat{\psi}}{\partial H_c} \right] \]

The variables that remain unknown are the derivatives of the SEF. Rivlin and Saunders (1951) conducted numerous experimental procedures of biaxial loading of isotropic nonlinear material and used the previous analytical solution to construct a novel SEF (Humphrey, 2002; Rivlin and Saunders, 1951). Another use is to supply a known SEF and determine the analytical stress distribution along with the material deformation. Therefore, the analytical solution listed above provides near endless amounts of possible information regarding materials that exhibit structural nonlinearity.
Inflation of a thick-walled hollow cylinder of nonlinear, isotropic material

Another useful analytical solution is the inflation of a thick-walled hollow cylinder composed of nonlinear, isotropic cylinder, while it maintains a constant length. In the field of biomedical engineering, many investigations of arterial wall dynamics begin with this analytical solution (Holzapfel et al., 2000). Furthermore, the numerical stability of known strain energy functions (SEF) can be tested on this analytical solution. Although the solution has multiple uses, one can not derive a constitutive relation that fully characterizes a material’s nonlinear behavior when coupling this solution with experimental data.

Under extension, torsion, and extension of a tube; a point in the referential coordinates \((R, \Theta, Z)\) is mapped to the deformed coordinates \((r, \theta, z)\) with the following expressions.

\[
\begin{align*}
    r &= r(R) \\
    \theta &= \Theta + \gamma Z \\
    z &= \Lambda Z
\end{align*}
\]

The function that maps the deformed radial coordinates will be derived using the conditions of material incompressibility at a later point in the solution. Even though the function is unknown, it is possible to construct the deformation gradient tensor in cylindrical coordinates.

\[
\bar{F} = \begin{bmatrix}
\frac{\partial r}{\partial R} & 1 & \frac{\partial r}{\partial \Theta} & \frac{\partial r}{\partial Z} \\
\frac{\partial \theta}{\partial R} & \frac{r}{R} & \frac{\partial \theta}{\partial \Theta} & \frac{\partial \theta}{\partial Z} \\
\frac{\partial z}{\partial R} & 1 & \frac{\partial z}{\partial \Theta} & \frac{\partial z}{\partial Z}
\end{bmatrix}
= \begin{bmatrix}
\frac{\partial r}{\partial R} & 0 & 0 \\
0 & \frac{r}{R} & \gamma r \\
0 & 0 & \Lambda
\end{bmatrix}
\]
If the cylinder is restricted to only a change in radius and both torsion and the elongation of the cylinder are negligible, the deformation reduces to the form below, which provides easy derivation of the right Cauchy-Green stretch tensor.

\[
\overline{F} = \begin{bmatrix}
\frac{dr}{dR} & 0 & 0 \\
0 & \frac{r}{R} & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
\overline{C} = \overline{F}^T \overline{F} = \begin{bmatrix}
\left(\frac{dr}{dR}\right)^2 & 0 & 0 \\
0 & \left(\frac{r}{R}\right)^2 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Since the right Cauchy-Green stretch tensor is a diagonal matrix, the principle stretch ratios are the square root of each of the diagonal elements of the matrix. Thus, the circumferential stretch ratios are represented by the following equations, which are inserted into the equations for the invariants of the right Cauchy-Green stretch tensor.

\[
\lambda_r = \frac{dr}{dR} \quad I_C = \left(\lambda_r\right)^2 + \left(\lambda_\theta\right)^2 + \left(\lambda_z\right)^2
\]

\[
\lambda_\theta = \frac{r}{R} \quad \Rightarrow \quad II_C = \left(\lambda_r\right)^2 \left(\lambda_\theta\right)^2 + \left(\lambda_\theta\right)^2 \left(\lambda_z\right)^2 + \left(\lambda_z\right)^2 \left(\lambda_r\right)^2
\]

\[
\lambda_z = 1 \quad \Rightarrow \quad III_C = \left(\lambda_r\right)^2 \left(\lambda_\theta\right)^2 \left(\lambda_z\right)^2
\]

The condition of material incompressibility sets the third invariant to be equivalent to unity, providing a method to determine the unknown mapping function of the deformed radial coordinates.
\[ III_C = 1 \quad \Rightarrow \quad \frac{dr}{dR} = 1 \]
\[ r dr = R dR \]
\[ \int_{r_i}^{r_o} r dr = \int_{r_i}^{R} R dR \]
\[ r_o^2 - r_i^2 = (R_o^2 - R_i^2) \]

For general \( r \) and \( R \) values:
\[ r = \sqrt{R^2 - R_i^2 + r_i^2} \]

After deriving the equation that will map the reference radial values to the deformed radial values, it is possible to derive an expression that will relate the Cauchy stress with cylindrical stretch ratios by utilizing the technique outlined in Appendix I.

\[ \sigma = 2 F F^T \frac{\partial \hat{\psi}}{\partial C} - p I \]
\[ \sigma = 2 \frac{\partial \hat{\psi}}{\partial I_C} B - 2 \frac{\partial \hat{\psi}}{\partial II_C} B^{-1} - p I \]
\[ \sigma_{rr} = -p + 2 \frac{\partial \hat{\psi}}{\partial I_C} (\lambda_r)^2 - 2 \frac{\partial \hat{\psi}}{\partial II_C} (\lambda_r)^2 \]
\[ \sigma_{\theta\theta} = -p + 2 \frac{\partial \hat{\psi}}{\partial I_C} (\lambda_\theta)^2 - 2 \frac{\partial \hat{\psi}}{\partial II_C} (\lambda_\theta)^2 \]
\[ \sigma_{zz} = -p + 2 \frac{\partial \hat{\psi}}{\partial I_C} (\lambda_z)^2 - 2 \frac{\partial \hat{\psi}}{\partial II_C} (\lambda_z)^2 \]

In order to determine the Lagrange multiplier \( p \), the equations for local, mechanical equilibrium in cylindrical coordinates under static conditions and no body loads must be utilized.
radial equilibrium:
\[ \frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \quad \Rightarrow \quad \frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \]

circumferential equilibrium:
\[ \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} = 0 \quad \Rightarrow \quad -\frac{\partial p}{\partial \theta} = 0 \]

axial equilibrium:
\[ \frac{\partial \sigma_{zz}}{\partial z} = 0 \quad \Rightarrow \quad -\frac{\partial p}{\partial z} = 0 \]

The equations of local mechanical equilibrium signify that the Lagrange multiplier is a function of the deformed radii only and is independent of the circumferential and axial directions. Using the boundary condition \( \sigma(r = r_i) = -P_i \), the resulting equation from radial equilibrium can be manipulated to derive a function for the Lagrange multiplier.

\[ \frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \]
\[ \int_{r_i}^{r} \frac{d\sigma_{rr}}{dr} dr = \int_{r_i}^{r} \frac{\sigma_{\theta\theta} - \sigma_{rr}}{r} dr \]
\[ \sigma(r) + P_i = \int_{r_i}^{r} \frac{\sigma_{\theta\theta} - \sigma_{rr}}{r} dr \]

The next step in the derivation is to substitute the general equations of the Cauchy stress into the equation and solve for the Lagrange multiplier.
\[ \sigma(r) + P_i = \int_{r_i}^{r_f} \frac{\sigma_{\theta\theta} - \sigma_{rr}}{r} \, dr \]

\[ - p(r) + 2 \frac{\partial \hat{\psi}}{\partial \lambda_C} \left( \lambda_r \right)^2 - 2 \frac{\partial \hat{\psi}}{\partial \lambda_C} \left( \lambda_r \right)^2 + P_i = \int_{r_i}^{r_f} \frac{\sigma_{\theta\theta} - \sigma_{rr}}{r} \, dr \]

\[ p(r) = P_i + 2 \frac{\partial \hat{\psi}}{\partial \lambda_C} \left( \lambda_r \right)^2 - 2 \frac{\partial \hat{\psi}}{\partial \lambda_C} \left( \lambda_r \right)^2 - \int_{r_i}^{r_f} \frac{\sigma_{\theta\theta} - \sigma_{rr}}{r} \, dr \]

The difficulty with the previous expression is the integration, which is dependent on the deformed radial values that are unknown prior to solving the problem. Transforming the integration to be dependent on the undeformed radial values would eliminate this situation.

The function of \( r \) that was derived from the condition of incompressibility is essential to completing the aforementioned transformation.

\[ \int_{r_i}^{r_f} \frac{2 \left( \frac{\partial \hat{\psi}}{\partial \lambda_C} \left( \lambda_r \right) \right)^2}{r} \left( \frac{dr}{d\lambda} \right)^2 \, dr = \int_{r_i}^{r_f} \frac{\partial \hat{\psi}}{\partial \lambda_C} \left( \frac{dr}{d\lambda} \right)^2 - \frac{R^2}{r^2} \, dr \]

\[ r = \sqrt{R^2 - R_i^2 + r_i^2} \implies R = \sqrt{r^2 - r_i^2 + R_i^2} \]

\[ \frac{dr}{dR} = \frac{R}{\sqrt{R^2 - R_i^2 + r_i^2}} \]

Using the integration method of substitution, the previous integral is transformed into one that is dependent on values of the undeformed radii.
\[
\int_{r_i}^{R} \frac{2}{r} \left\{ \hat{\psi}_i \left( \lambda_{\theta}^2 - \lambda_{r}^2 \right) + \hat{\psi}_i \left( \lambda_{r}^{-2} - \lambda_{\theta}^{-2} \right) \right\} dr = \int_{r_i}^{R} \frac{2R}{\alpha} \left\{ \hat{\psi}_i \left[ \frac{\alpha}{R^2} - \frac{R^2}{\alpha} \right] + \hat{\psi}_i \left[ \frac{\alpha}{R^2} - \frac{R^2}{\alpha} \right] \right\} dR
\]

where: \( \hat{\psi}_i = \frac{\partial \hat{\psi}}{\partial \xi_i} \) & \( \hat{\psi}_i = \frac{\partial \hat{\psi}}{\partial \xi_i} \)

\( \alpha = R^2 - R_i^2 + r_i^2 \)

The resulting \( p(R) \) Lagrange multiplier has the form:

\[
p(R) = P_i + 2 \hat{\psi}_i \frac{R^2}{\alpha} - 2 \hat{\psi}_i \frac{R^2}{\alpha} - \int_{r_i}^{R} \frac{2R}{\alpha} \left\{ \hat{\psi}_i \left[ \frac{\alpha}{R^2} - \frac{R^2}{\alpha} \right] + \hat{\psi}_i \left[ \frac{\alpha}{R^2} - \frac{R^2}{\alpha} \right] \right\} dR
\]

Now that a general form of \( p(R) \) has been derived the three nonzero Cauchy stress equations can be further analyzed.

\[
\sigma_{rr} = -p(R) + 2 \left[ \hat{\psi}_i \frac{R^2}{\alpha} - \hat{\psi}_i \frac{R^2}{\alpha} \right]
\]

\[
\sigma_{\theta\theta} = -p(R) + 2 \left[ \hat{\psi}_i \frac{R^2}{\alpha} - \hat{\psi}_i \frac{R^2}{\alpha} \right]
\]

\[
\sigma_{zz} = -p(R) + 2 \hat{\psi}_i - 2 \hat{\psi}_i
\]

However, even after the transformation, one deformed radii value is needed (\( r_i \)) to solve the stress distribution throughout the thickness of the wall and the resulting deformation. The boundary condition of zero pressure being applied to the exterior surface of the cylinder is the key to solving for the needed variable of \( r_i \). If the bounds of integration on \( p(R) \) are changed to \( R_i \) and \( R_o \), the corresponding radial Cauchy stress at \( R = R_o \) is zero. A relation between the applied pressure and integral of \( p(R) \) can be derived where the only variable is \( r_i \).

\[
P_i = \int_{r_i}^{R} \frac{2R}{\alpha} \left\{ \hat{\psi}_i \left[ \frac{\alpha}{R^2} - \frac{R^2}{\alpha} \right] + \hat{\psi}_i \left[ \frac{\alpha}{R^2} - \frac{R^2}{\alpha} \right] \right\} dR
\]
Using the previous equation to find $r_i$, all of the variables needed to fully characterize the stress distribution and the resulting deformation are identified. With $r_i$ known, all deformed radii values can be calculated from the corresponding undeformed / reference radii values. Subsequently, the stress distribution can than be calculated for each deformed and undeformed radii pair.

Although this solution cannot be used to fully characterize a new strain energy function equation based on experimental data, many uses still exist for the solution. Validation of computational simulations is one common use, and testing new strain energy functions for numerical stability is another. Thus, the significance of the previous solution should not be undermined due to its inability to derive new strain energy functions from experimental data.
APPENDIX II: Sample format for publication of present novel study

Introduction

The introduction would begin with a condensed discussion of Chs. 1 – 4. Current, “state-of-the-art” developments would be introduced along with their underlying theory.

Formulation of New Model

The modeling procedure would be discussed, highlighting the combination of published data and the reasons for its novel approach. Following the specified discussion, the functional form of the new model would be introduced and discussed.

Results

Present the basic comparisons between the new model and current AAA models. Additionally, reveal the comparative study conducted on the two isotropic AAA models and their discrepancy with the experimental data.

Conclusions

Discuss clinical relevance of present study. Introduce future work and the desired direction of the research. Additionally, specify needs of future experimental data.
APPENDIX III: Written permission to use figures from previous publication

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