

ABSTRACT

LAMBERTUS, AMANDA JANE. Students' Understanding of the Function Concept: Concept Images and Concept Definitions. (Under the direction of Karen Norwood.)

Misconceptions about the function concept can occur across a variety of representations. These misconceptions may be compartmentalized allowing students to answer questions incorrectly by evoking certain parts of the concept image. The concept image is the mental pictures that students construct for each mathematical concept. The concept definition is a formal definition of a mathematical concept.

This study is exploratory nature. The researcher is trying to compare students' understanding of the function concept by examining their concept images and concept definition when they are introduced to function concept through a formal definition versus an informal approach.

The participants were traditional college students enrolled in Intermediate Algebra at a large university in the southeast region of the United States. The students completed a questionnaire that asked them to identify functions and non-functions, mentally construct functions from verbal statements, and provide a definition for the function concept.

There is evidence that students do not make connections between their concept images and concept definitions in one class. There are some students that can provide a concept definition even when their classroom instruction does not specifically state one.

**Students' Understanding of the Function Concept:
Concept Images and Concept Definitions**

by
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DEDICATION

For my students.

BIOGRAPHY

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CHAPTER 1

INTRODUCTION TO THE PROBLEM

The concept of function is fundamental to the learning and understanding of mathematics (Bowman, 1997; Dreyfus & Eisenberg, 1982; Eisenberg, 1991; Gagatsis, Elia, Panaoura, Gravvani, & Spyrou, 2006.). The function concept provides a basis for high school mathematics courses as well as college courses. In order to help improve students' knowledge of functions, the National Council of Teachers of Mathematics (NCTM) emphasizes: students as early as grades 3-5 need to begin looking at representing and analyzing functions using words, tables, and graphs. Teachers should emphasize the importance of using and interpreting several representations, while working with functions, throughout the student's mathematical education (NCTM, 2000). This emphasis should help students develop a repertoire of many different types of functions and their respective representations as they progress through their middle school and high school mathematics courses. They should be able to manipulate and interpret a variety of functions using several different representations, graphical, tabular, verbal, and symbolic (Eisenberg, 1991; Moschkovich, Schoenfeld, & Arcavi, (1993); NCTM, 2000). Representations in this study are considered the tools for interpreting and depicting functions.

Unfortunately, research indicates that college and high school students do not have a well developed understanding of the function concept (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; DeMarois, 1997; Doyle, 1986; Mousoulides & Gagatsis, 2004; Vinner, 1983; Vinner & Dreyfus, 1989). Students possess a variety of misconceptions and beliefs that range from continuity issues to conflicts stemming from the representations of functions (DeMarois, 1997; Doyle, 1986; Dreyfus & Eisenberg, 1982; Vinner, 1983).

Vinner (1983, 1991) used the constructs of concept images and concept definitions to analyze students' understandings and misconceptions of the function concept. A *concept definition* is a verbal description of a mathematical concept that accurately describes the concept in a non-circular way; whereas, a *concept image* is the mental picture that is associated with the concept name in a student's mind (Vinner, 1983). The use of these two constructs allows researchers to look at distinctions between the formal definition of a concept and the students' images or the way that students think about a concept and the association it has with the formal definition (Lloyd & Wilson, 1998).

Vinner and Dreyfus (1989) found that students compartmentalize their concept images and concept definitions; meaning students do not always connect a formal definition to their mental images. The implication is students can possess different and possibly conflicting views of what constitutes a function and not be concerned with the fact that these notions are in conflict. Another option is the students simply do not realize the conflict exists and therefore are ignorant of it. This compartmentalization can lead to misconceptions between the concept definition and one's concept image, misconceptions in the concept image, and a misunderstanding of the formal definition of a concept.

Students' evoked concept images often focus on a single image or piece of information about the function concept that allows the student to answer a particular mathematics question without consulting the concept definition. Consequently, this partial use of the concept image prevents the development of conceptual understanding (Vinner & Dreyfus, 1989). If students' concept images and definitions are not compatible, in subsequent mathematics classes students will experience difficulties.

Misconceptions

A student's misconceptions about the function concept may occur for several reasons. For example, students may not fully understand the formal definition of function. Lack of understanding a definition may lead to conflicts between students' images and their concept definition (DeMarois, 1996; Slavit, 1997; Vinner, 1983; Vinner, 1991; Vinner & Dreyfus, 1989). Early in their high school mathematics careers, students are typically introduced to a formal definition of function: *correspondence between two non-empty sets that assigns to every element in the first set (the domain) exactly one element in the second set (the co-domain)*, (Vinner & Dreyfus, 1989, p. 357). This definition is not unique. A survey of mathematics textbook revealed many different definitions for the function concept. No matter what definitions of function students are presented in Algebra courses, the students are not forced to use the formal definition in a rigorous manner. The definition is noted, and the teacher moves on in the instruction to help the students create and build concept images. As a result students will tend to rely on their concept images for function when answering questions and solving problems.

Typically, students do not consult the formal definition of a concept when presented with an unfamiliar function (Doyle, 1986; Slavit, 1997; Vinner, 1983; Vinner 1991). This has led researchers to investigate what alternative strategies are viable for teachers to use when introducing the concept of function. The fact that the students do not consult their concept definitions is not unusual. According to Thompson (1994), most people in lay situations rely on their mental imagery to understand words. "They operate from the basis of imagery, not from the basis of conventional constraints adopted by a community" (Thompson, 1994, p. 23). Is it reasonable to expect students to have a better conceptual

understanding or at least a more well-defined concept image for function after completing an Algebra course?

Problem Statement

Mathematical learning may not be able to take place at the higher levels without the aide of a teacher or textbook. However, given specific teaching strategies, it may be difficult to determine how much of the learning was due to the learning processes or influenced by the teacher (Sfard, 1991). How does the type of instruction influence the students' ability to form connections between concept images and concept definitions or does it influence the students' understanding at all? The purpose of this study is to examine the nature of college students' understandings of function when the definition is presented formally versus informally.

Research Question

Does classroom instruction affect the development and adjustment of the concept image and definition for the function concept among students in an Intermediate Algebra college course?

CHAPTER 2

LITERATURE REVIEW

Background

This chapter discusses the roles of reform, technology, and representations in students' understanding and the development of the function concept in the high school and college algebra curriculums. It will identify students' misconceptions in three different representations (algebraic, graphical, tabular, and verbal) and the effects that these misconceptions have on student understanding. Due to the fact that representations play an important role in the NCTM Standards, they will also be discussed. Finally, the chapter will suggest changes to the curriculum for teaching the function concept based on research and the object/process conceptions for understanding mathematical concepts.

Reform

The reform movement concerned with teaching the function concept in algebra and pre-algebra classes has sparked debate. In the past, teaching of the function concept focused on symbol manipulation skills, drill, and memorization, without linking the concept to any other representations such as graphs or tables (Brenner et al., 1997; Cates, 2002). The ideas behind this type of curriculum, where students are expected to master procedures for solving problems, are that they will be able to apply these procedures in the context of new problems (Chazan & Yerushalmy, 2003). Current reform movements addressing the teaching of function emphasize problem solving, the use of multiple representations, connections between multiple representations, and development of concepts enhanced through the use of technology (O'Callaghan, 1998; Patterson, 2002).

Role of Technology

With calculators and computers widely available in the schools for student use, technology is also a large factor in reform movements. Using graphing calculators or computers, students can explore the different representations of functions with relative ease (Fey, 1989; O'Callaghan, 1998). Having technology in the classroom, allows the teacher to re-vitalize the methods in which they teach mathematical concepts, and produce more meaningful lessons that would help the student create meaning and learn for understanding (Fey, 1989; Kaput, 1992). The visual and exploratory nature of the graphing calculator provide learning experiences which allow students to form a deeper understanding of the content they are learning (Cates, 2002; Erbas, Ledford, Polly, & Orrill, 2004; Fey, 1989). The algebra curriculum needs to be adjusted in order to account for the influx of technology and the fact that most high school students do not develop the expected levels of proficiency and understanding in algebra courses (Fey, 1989).

Technology in the classroom can also allow educators to address concerns about students' difficulties in translating between different representations (Kaput, 1987). The graphing calculators allow students to see at least two representations at one time and to view a third one with the push of a button. The focus of the algebra courses should be redirected to help students interpret these representations (Fey, 1989; Kaput, 1987). NCTM (2000) states that with the use of computers and graphing calculators we can change the nature of what students can do in the classroom with the various representations. The technology will enable them to view the differences between conventional (static) representations produced with pencil and paper in which each new representation is create from scratch and the fluent nature of the representations meaning they can be altered and created with ease, displayed on

the calculator.

Concept Definition/Concept Image

Despite NCTM's (2000) recommendations for students to begin exploring functions in the third grade, a surprising number of students lack a conceptual understanding of the function concept when entering college level courses (DeMarois, 1997; Doyle, 1986; Mousoulides & Gagatsis, 2004; Vinner & Dreyfus, 1989). This does not mean that students do not have some understanding about functions. Indeed, they have constructed concept images and memorized formal definitions, but they may not have formed a connection between the two "cells" (Vinner, 1991). In Vinner's (1991) description of cells, he states that the concept image cell remains empty until there is some meaning associated with the concept name. As soon as meaning (correct or not) is attached, the image cell begins to "fill". The concept definition cell is "filled" when the student learns a new concept definition. It is important to note that the definition does not have to be accurate, if the student believes that it is. However, there may not be a complete transfer of the definition to the "cell" if the student does not understand the meaning behind a formal definition. Students may only transfer part of the definition or an incorrect interpretation of the definition. Vinner (1991) believes there is supposed to be "interaction between the concept image cell and the concept definition cell, although the two cells can be formed independently" (p. 70). Yet, until meaning between the definition and the images is created there is little if any interaction between the two cells.

The independent formation of a concept image and concept definition may be a result of students memorizing a formal definition without connecting meaning to it (Edwards & Ward, 2004; Vinner 1991). However, once the students have created a concept image the

definition may become dispensable with students no longer relying on it to formulate answers (Vinner, 1991). When formal definitions are introduced early in a mathematics curriculum, with not much emphasis placed on them, the definitions can be easily pushed aside in favor of more accessible images. Hitt's (1998) investigation of in-service teachers found that the teachers do not rely on their concept definitions when determining if a relation is a function. It is not surprising then that students do not consult their concept definitions. Both the students and teachers rely on their concept images to identify functions until the students have more experiences in mathematics which force them to access and rely on formal definitions (Gagatsis et al, 2006; Hitt, 1998; Tall & Bakar, 1991).

In several research studies, students were asked to provide definitions for the term function (DeMarois 1997; Gagatsis, et al, 2006; Tall & Bakar 1991; Vinner & Dreyfus 1989; Williams 1998). The definitions students provided included complete and appropriate definition for functions and a partial definition with necessary parts missing, as well as some students being unable to provide a definition. However, most of the students could provide a more advanced example of a function than the definition they provided and preferred to do so rather than state a definition (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Gagatsis, et al, 2006). Concept images are a result of the students' experiences, examples, and familiarity with a particular mathematical concept (Tall, 1992; Tall & Bakar, 1991; Vinner & Dreyfus 1989). This image however, often does not agree with the concept definition (Gagatsis et al, 2006; Keller & Hirsch, 1998; Tall, 1992; Vinner, 1983). Therefore, it is imperative that students form an accurate and adaptable concept image. Familiarity of a mathematical concept refers to different functions in which the students may be acquainted with through instruction or their own experiences. Students, in mathematics courses, high school through

college, construct concept images through a variety of methods. These methods include, but are not limited to, exploration of examples and non-examples, working problems, classroom instruction, and applying the definition.

Since, students do not consult their concept definition when confronted with new situations, it is important that students' concept images are as accurate as possible. Students should also be willing to adapt and change their concept images as they progress through more advanced mathematics. Providing the student opportunities to create, modify and adjust "definitions" as necessary when they encounter new and different scenarios allows them to learn to adjust their concept images. Their concept images will continue to grow and when they encounter a concept definition the students will be able to mesh it together with the images that they have, or apply that definition to a new problem and add the result to their concept image.

Students' Misconceptions about Functions

According to Vinner (1983), students may have expectations of functions that may not logically relate to the definition (p. 302). Expectations on the students' part are factors in their misconceptions of functions. These misconceptions range across the different representations that the students encounter.

Verbal. One common misconception in the verbal representations are that functions exist only if mathematicians give names to them - meaning that all quadratics, polynomials, and trigonometry ratios are functions because those groups of mathematical objects have specific names (Vinner, 1983; Williams, 1998). It also implies that students may believe that circles, ellipses, and vertical lines are also functions because they have names and equations that can be easily written and identified by the students.

This type of misconception can be linked to the familiarity issues of teaching functions in the high school. Typically, teachers present students with these specific types of functions on a regular basis as part of the classroom instruction. The students become comfortable with them and understand that quadratics and polynomials are functions without calling on the concept definition. However, when a student encounters an unfamiliar equation, for example a split domain equation, they do not consult their concept definitions to determine if the resulting curve is a function. The student may simply state that it is not a function, because they do not recognize it and the function does not have a specific mathematical name.

Students are not the only population to have these verbal misconceptions about functions. Even and Tirosh (1995) found teachers often have the same difficulties as students when working with familiar curves and their verbal representations. In their study, Brian, a teacher could demonstrate that he understood the correspondence requirement for functions. He demonstrated this knowledge while working with arbitrary functions such as: “let $g(x)$ equal x , if x is a rational number and zero, if x is an irrational number” (p. 7). However, he also believed that “familiar graphs such as circles and ellipses are function” (p. 7) even though they do not fulfill the correspondence requirement. Brian seemed unaware of this conflict in his concept image and concept definition. Not until he was presented with a situation, in which the definition and the image did not correspond, did Brian confront his misconceptions. However, instead of adjusting his concept image to match the definition, he decided that the vertical line test did not work on all functions and that he would only use it with his students when discussing linear functions.

Tall and Bakar (1992) give an excellent example of how concept images and concept

definitions are at odds with a person's everyday experiences. Normally, a concept would be developed by looking at examples that one encounters at the outset, and focuses on the features of those examples. Eventually, the person will encounter an example that does not fit with the features they have allocated to the concept and adjustments need to be made to the image. For example: "That is a bird. ... A bird *flies* ... it had *wings*, ... and *feathers* ... and a *beak* ... and *lays eggs*" (Tall and Bakar, 1992, p. 40). Eventually, one will encounter a new example that needs to be tested against these features. "Is a Chicken a bird?" (p. 40). The person will have to adjust their features to include that some birds do not fly. Here we can see that the person is willing to adjust their criteria for what constitutes a bird. The same types of opportunities need to be available in the classroom for students that are developing or adding to their concept images. They develop an organized and complex concept image without having to memorize a difficult formal definition for a concept.

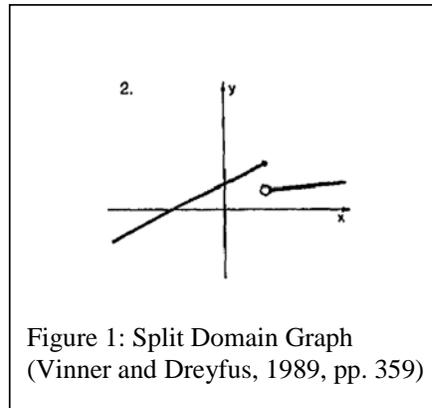
Algebraic. Algebraic misconceptions come in the following forms of student beliefs: functions are equations; functions are a process that numbers go through to get a number in return, a function is given by a rule, and if a variable is missing from the equation, it is not a function (Tall & Bakar, 1991; Vinner 1983; Vinner & Dreyfus, 1989; Williams, 1998). These ideas are a direct result of the students using pieces of their evoked concept images to answer questions, rather than the formal definition. Students that are comfortable working with functions represented as expression or equations have a challenging time thinking about functions that can be constructed arbitrarily (Jones, 2006). For example, "Does there exist a function whose values for integral numbers are non-integral?" (Vinner & Dreyfus, 1989, p 359). Not only is this function statement arbitrary, it can not be written using algebraic symbols. It has to be written in a verbal form. Therefore, it requires students to use their

definition of the function concept to determine if such a function exists. If the concept definition is incomplete or non-existent, the student will not be able to determine if the statement represents a function.

Sfard (1992) found that students believed that all functions can be expressed in a regular manner relating x and y , and that all functions can be expressed by computational formulas. In other words, if we provided a student with a value for the x term, they can find the exact value for the y term. According to Vinner (1991), a student should not formulate an answer without first consulting the formal definition, which should be the student's concept definition. But, the students do not use these formal definitions introduced in high school algebra courses or introductory algebra courses at the college level as they classify functions. Consequently, the students rarely understand these formal definitions (Gagatsis et. al. 2006; Tall & Bakar, 1991). Instead, the students rely solely on their concept images. These images become distorted when elements of the function concept are not introduced nor reinforced; hence, these element are forgotten (Vinner, 1983).

Graphical. In addition to verbal and algebraic misconceptions, students can also apply their concept image distortions to graphical representations of functions. Students often believe that functions are continuous (Jones, 2006). This is not surprising since the first functions that most students encounter in high school textbooks are graphical representations of continuous curves. When students are forced to work with split domain graphs (Figure 1) in the coordinate plane for the first time, they may not categorize these as functions, because they are unfamiliar and they fail to meet the expectations of the students, i.e. that functions are curves that are continuous. It is interesting to note that this same idea of continuity is used to reject functions. For example, a common justification for why a

discrete function or a set of ordered pairs do not represent a function is that “functions are continuous graphs”.



The students also come to rely on the vertical line test, possibly a result of teachers’ reliance on the vertical line test for determining if curves are functions during classroom instruction. The vertical line test, states that a curve or discrete relation is a function if and only if any given vertical line passes through the curve or relation once at a given point. This implies that visually, students and teachers can quickly categorize functions.

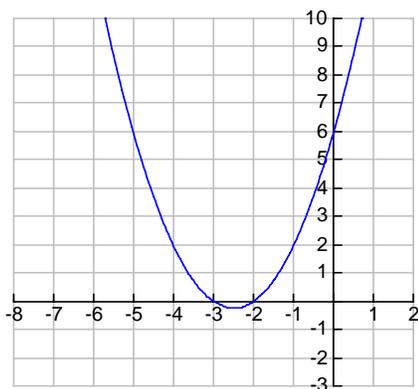
“Strange” or unfamiliar graphs cause the students to pause. They do not want to accept these new types of graphs as functions, because it does not fit with their images of what a function should look like: i.e. the graph of a function should be a reasonable, curved line in the coordinate system, the graphs of a function should not be curtailed but rather, go on forever (Jones, 2006; Tall & Bakar, 1991; Vinner, 1983; Williams, 1998). Some researchers believe that this type of misconception is a result of the students’ exposure to only a few “familiar” types of functions, dictated by the high school mathematics curriculum (Keller & Hirsch, 1998; Lloyd & Wilson 1998; Tall & Bakar, 1991).

Some students do not connect the graph of a function to its symbolic representation. They see graphs as separate entities than the equations. If the students are not explicitly

asked to graph a function, they may see the graph as “extra baggage” not essential to the problem (Eisenberg and Dreyfus, 1994).

Representations

According to NCTM (2000), in “the middle grades, students should be able to understand the relationships among tables, graphs, and symbols and to judge the advantages and disadvantages of each way of representing relationships for particular purposes” (p. 37). However, research has shown that students who have an idea on how to apply the function concept in one representation have difficulties in applying the same concepts in a different representation (Eisenberg, 1991). In fact, students may not make the connection that an equation, table, and graph all communicate the same information in different forms (Figure 2). The difficulty that students have with making connections between the representations leads to a student who can claim a circle is function because it has a mathematical name in the verbal or symbolic representations and then can look at the graph of a circle and claim that it is not a function because the graph does not extend toward infinity or pass the vertical line test.



$$f(x) = x^2 + 5x + 6$$

x	$f(x) = x^2 + 5x + 6$
-8	30
-7	20
-6	12
-5	6
-4	2
-3	0
-2	0
-1	2
0	6
1	12
2	20
3	30

Figure 2: Three representations of a function

Some students do not view the symbolic and graphical representations as descriptions of the same object (Eisenberg, 1991). When students or teachers draw or write down a function, they should realize that the picture or the symbols are only one possible representation for the abstract concept of function, which in itself can not be experienced by the five senses (Sfard, 1991). These “different representations often illuminate different aspects of a complex concept or relationship” (NCTM, 2000), while simultaneously conveying the same information. Rather, students see the different representations as separate entities. This poses a problem for the students’ cognitive development and understanding of the function concept. Students allowed to view the two representations as separate pieces will continue to compartmentalize their concept images and concept definition. Moschkovich, Schoenfeld, & Arcavi (1993) discuss the need for students to come to terms with the different perspectives regarding the functions themselves, while making the connections between the representations. The two perspectives they discuss are the “process perspective” and “object perspective” (p. 71). Using a process perspective, the student is relating the x and y values, however, this relation may be represented as an equation expression, table or graph. With an object perspective, the student views the function and its representations as entities.

Further, Mousoulides & Gagatsis (2004) claim most students correctly solve linear functions algebraically. But, the number of correct solutions drops considerably when the students are presented with quadratic functions and graphical representations. They argue that students who can effectively use graphical representations efficiently make more connections and relations within the function concept.

No matter what the representation may be, the functions can be viewed differently

when considered as separate concepts (Bowman, 1993; Eisenberg & Dreyfus, 1994).

Janvier's Star model provides an aid in helping teachers decide where the gaps or holes are in instructional methods of the function concept. Janvier's star suggests that a definition with a single representation can not encompass all the meaning of a notion; that students need to be able to translate through the different representations to gain a richer knowledge (Janvier, 1987). It also provides a way to link the different ideas and representations of functions (Bowman, 1993; Janvier, 1987). Janvier (1987) states that when "a function is envisaged as a *variable*, the role of the domain is often played down if not disregarded" (p. 68). However, on the Cartesian plane the curve is a "natural illustration" of the variable and its domain. The same functions can also be represented in a variety of different settings and representations (Dreyfus & Eisenberg, 1982). Examples of these types of settings and representations can be viewed in the Dreyfus & Eisenberg (1982) study (Appendix A).

The multiple representations can yield a deeper and more flexible understanding of the function concept (Keller & Hirsch, 1998). The different representations all develop and display different aspects of a function. Depending on the problem context, setting, and question, a different representation may be more helpful than another. Consider the following example, students working with functions in "break even problems" graphically they can locate a break even point faster. But, interpretation of the meaning of that break even point and the interpretation of the different functions is still something that the students need to be taught. In this particular case, a graphical representation provided easier access to the information than a table or symbolic representation may have provided (Lloyd & Wilson, 1998).

Curriculum Suggestions

Due to the fact that students' concept images are often their only reference for classifying functions (Tall & Bakar, 1991; Vinner, 1991), it is important that teachers provide students with experience that will help them create accurate concept images and definitions. Eisenberg (1982) stated that teachers should "provide the students with a good exposition and appropriately structured exercise to reveal various aspects of the notions, and students will understand, internalize and master the notion" (p. 140). In order to provide these different exercises, the teacher can introduce the function concept in a variety of representations, contexts, activities, and examples (Eisenberg, 1982; Gagatsis et al, 2006; Tall & Bakar, 1991). Each of these methods has its own disadvantages as well as advantages. For example, teaching functions using only examples and non-examples may lead students to erroneous impressions of the general concept; if the only examples are continuous functions that extend in both directions, and then the student may think that is the only type of graph for a function. Therefore, it is important for educators to use a combination of the methods.

Another teaching technique that challenges the students' concept image and forces them to use a definition is to present the students with tasks that cannot be solved correctly using only a concept image (Vinner, 1991). To help the teacher understand the students' concept images, Williams (1998) suggests the use of advanced organizers. By having the students construct concept maps using advanced organizers; teachers can gain insight into what images the students hold. Concepts maps can also help the teacher create dialogues to address the different notions and misconceptions held by the students. Advanced organizers will help both the student and the teacher evaluate the concept image, keeping in mind that

there is no possible way for the entire concept image to be placed in an advanced organizer.

Textbooks often start the unit on functions by having the students construct tables of ordered pairs, plot these ordered pairs on the Cartesian Plane, and then connect the points to form the corresponding lines. This is a poor technique since students may not be familiar with the concept of functions; therefore, the students raise barriers against the meaning behind these constructions because they look strange. On the other hand, graphical representations and coordinate ordered pairs are familiar to the students and provide a method for introducing the function as a symbolic representation for a line. This would provide meaning for the students about the function concept and help the students develop the skills necessary to move between the different representations (French, 2002).

Mousoulides and Gagatsis (2004) claim a pure algebraic approach to functions gives students a local image of the function concept, “on the contrary, a geometric approach gives students a global approach of the concept of function” (p. 386). The geometric approach, in this case, is referring to using a graphing approach to solve problems involving functions. By having students use different representations, they can construct mental images of the functions and build a broader concept image. Using representations “has the potential [for] making the process of learning algebra more meaningful and effective” (Friedlander & Tabach, 2001).

Working with the different representations in different problem contexts allows the students to see the advantages and disadvantages of each, what situations would promote efficiency for certain representations and accommodate the individual thinking styles of each student (Friedlander & Tabach, 2001). Further, technology can provide the means with which different representations can be employed efficiently in the classroom.

In many classrooms, the textbook is the sole resource that teachers use. They develop lessons, retrieve exercises, and use the testing strategies from these books (Jones, 2006). Also, teachers often will use the definitions for concepts provided by the textbooks. Using these definitions without questions can cause issues for the students later, as every textbook publishes its own version of the definition. For example, the following are definitions for function found in high school and college texts.

1. “A function is a relation in which no two different ordered pairs have the same first coordinate.” (Bello, 1998, p. 124)
2. “A variable y is a function of a variable x if each value of x determines a unique value for y .” (Kime & Clark, 2001, p. 16).
3. “A function is a relation that pairs each element in a domain D with exactly one element in a range R .” (Hall, 1994, p. 502).
4. “A function is a correspondence between a first set, called the *domain*, and a second set, called the *range*, such that each member of the domain corresponds to *exactly one* member of the range.” (Bittinger & Ellenbogen, 1998, p. 75).
5. “A function is a correspondence that matches each input value with exactly one value of the output variable.” (Hall & Mercer, 2007, p. 133).

From examining these different definitions of the function concept, we can see that teachers who rely solely on textbooks and their definitions may miss some of the ideas behind the concept. For example, Bello (1998) has chosen to define a function in terms of ordered pairs. A student first learning this difficult concept may think that all functions have to be represented as ordered pairs. However, Sfard (1991) suggests that defining mathematical concepts as abstract objects is not the only option, and that it might be helpful to the students to have a more concrete definition that they can build on as they encounter new ideas connected to the function concept.

Object & Process Conceptions

To understand the function concept means to go beyond the manipulation of symbols and formulas and using the vertical line test. Eisenberg and Dreyfus (1994) state “a central

aspect to function sense is the use of more than one representation for the same mathematical situation” (p. 46). Students should be able to “make sense of a situation by constructing a mental process that transforms (mental) objects” (Breidenbach, et al, 1992, p. 247).

Sfard (1991) refers to student understanding of the function concept from structural and operational points of view. In an operational conception, students are thinking of transformations as process, algorithms, and actions. In the structural conception, the student can think of functions as static objects which they can manipulate (Sfard, 1991). These two conceptions are not distinct entities. For the same mathematical concept, such as functions, these conceptions are often blended together to help the student create meaning. Students often begin thinking of functions in an operational manner. For example, given $f(x) = x^2 + 5x + 6$, they want to “do something”. That something ranges from graphing the function to finding the roots often without any instruction. Students do not see $f(x) = x^2 + 5x + 6$ as an object. However, graphing a function may provide a link from the operational to structural conceptions in that the graph represents all “the infinitely many components of the function” (Sfard, 1991, p. 6) represented as a continuous smooth line. The students can see the function as a whole on the plane.

Visualization is an important aspect of students’ understanding of the function concept. It makes abstract ideas more accessible and the key to solving many problems concerning functions (Eisenberg & Dreyfus, 1994; Sfard, 1991). However, visualization does have disadvantages. Many students that prefer to visualize are not able to move beyond the concreteness of the image (Eisenberg & Dreyfus, 1994). A good concept image can support the structural conception, because they can be manipulated similar to real objects and the present a holistic construction of the mathematical concept (Sfard, 1991). However, this

structural conception does not occur in students without the help of teachers, it “is a lengthy, often painfully difficult process” (Sfard, 1991, p. 16).

For a student to possess an operational view of the function concept, they would still be working in a stage where computation was necessary. They would be relying on algorithms to help them manipulate the functions, Breidenbach et al (1992) calls these algorithms and manipulations *actions* (p. 249). However, if the entire manipulation or transformation is internal, Breidenbach and his colleagues (1992) refer to the action as being *interiorized* and thus it becomes a *process* (p. 249).

In their study, Breidenbach et al (1992) asked pre-service teachers, “What is a function?” followed by “Give an example.” (p. 252). They then classified the participants’ answers into four categories, prefunction, action, process, and unknown. A typical prefunction response was “I do not know”, “a mathematical equations with variables”, “a mathematical statement that describes something”, or “a social gathering”. An action response was one in which the students emphasized substituting numbers for input and output, but did not state that one starts with a value and does something to get a new value, the procedure was tied to an equation, or the input was restricted. A process response had the input, some sort of transformation, and output, but the statement was general (see Breidenbach et al, 1992, for specific examples). Of the sixty-two responses, 64% were at the prefunction or action level, and 21% were unknown.

Researchers grouped the examples provided by the participants into eight different categories;

- I. Something that could not really be interpreted as function
- II. Omitted
- III. An equation in two or more variables
- IV. A graph

- V. $F(x)$ = some algebraic or trigonometric expression
- VI. Y = some algebraic or trigonometric expression
- VII. An algebraic or trigonometric expression alone
- VIII. Some attempt to describe a process (p. 253)

What is interesting about their findings is that 64 % of the participants gave examples in either category V. or VI. (p. 253). Another interesting fact is that prior to the experiment, the pre-service teachers which were questioned, only 3.7% gave a *process* example of function and none of them gave an example of a function in graphical form (p. 254). If the concept definitions are dispensable when students and teachers form concept images, it would not be uncommon that they could provide examples that are more complex than the definitions they provided. This is true with other concepts not related to mathematics as well. For example, if one was to ask a group of people what they thought the definition of a tree would be, there would be a variety of answers. However, those people would be able to provide many examples of trees, many of them may be more complex and encompass many of the attributes of trees than the definition they supplied. In fact, one may not even know the definition for the concept of trees, in order to provide examples. Incidentally, Dictionary.com defines a tree as “a plant having a permanently woody main stem or trunk, ordinarily growing to a considerable height, and usually developing branches at some distance from the ground”. One can see here that a formal definition can be very broad to encompass all the possible outcomes. However, definitions provided by people may be more specific if the person is thinking of certain families of trees or the definition could become cumbersome if a person tries to include all the things that they know about trees. The same is true for mathematical examples as well. Therefore it is important to have definitions that meet the requirements of the course and the needs of the students.

It should not be surprising, therefore that some of the misconceptions students have are enriched by their teachers' misconceptions of the function concept. These teachers do not have a well developed understanding of the subject matter. In a study by Even and Tirosh (1995), they found that teachers are able to provide definitions for functions that include the univalence requirement and they are able to classify functions and non-functions. However, further investigation into the teachers' subject matter knowledge revealed that their knowledge of functions was shallow. (The univalence property of functions is the correspondence of one element in the domain to exactly one element in the range.)

Breidenbach et al.'s (1992) process understanding is still at Sfard's (1991) operational conception of function. It is not until the process is encapsulated that it becomes an object. Encapsulation occurs when an action transforms a process into an object (Breidenbach et al., 1992). However, this operational conception, while unconditionally necessary and adequate for problem solving, it can not be easily processed. It is stored "in unstructured, sequential cognitive schemata"... "it must be processed in a piecemeal, cumbersome manner" (Sfard, 1991, p. 26). This operational conception may lead to insufficient understanding of the concept.

Students possess a variety of misconceptions and inaccurate concept images about the function concept. These misconceptions include thinking: that functions are continuous, that functions are rules, formulas, or equations, that functions need to have specific names, and that functions should be familiar. Students also have difficulty in connecting the graphical, tabular, and symbolic representations of functions. These representations are viewed as separate entities and do not relate to one another. Further, once a student has constructed a concept image, they no longer refer to the concept definition.

CHAPTER 3

THEORETICAL FRAMEWORK

There is a wide range of misconceptions that students may possess when it comes to understanding the nature of functions. As a result, the framework chosen to analyze the nature of the students' concept images had to be versatile enough to allow for all the types of misconceptions and still look at the relationship between the concept images and concept definitions. Therefore, the objective of this chapter is to discuss Vinner and Dreyfus's (1989) study which informed the framework for the study described in this paper. This chapter will also describe the adjustments made to the categories described by Vinner and Dreyfus (1989) and how the framework was applied to this study.

Framework Background

Vinner and Dreyfus's (1989) work focused on the concept images of 271 college students and 36 junior high school teachers. They compared the participants' images to their concept definitions for the function concept. Their study explored the following questions:

1. *What are the common definitions of the function concept given by college students before they start their calculus course?*
2. *What are the main images of the function concept that these students use in identification and construction tasks?*
3. *Are there statistically significant differences between groups of students with different majors in the way they conceive functions?*
4. *How frequently do students compartmentalize their formal definition of function and their image of the function concept? p. 357*

The researchers separated the students into groups determined by the level of mathematics courses their chosen degrees would require. The teachers were participating in an in-service training program. The participants were asked to fill out a questionnaire concerning functions. The questionnaire contained seven questions; the first four were identification

problems, the fifth and sixth were construction problems and the seventh question asked the students to provide their definition for the concept function. The identification problems were designed to elicit participants' images of function, i.e. whether or not they could identify a function. The construction problems were probing the participants' definition of function and their ability to apply it to a particular situation. Finally, Vinner, Dreyfus and a research assistant, analyzed the answers.

The researchers categorized students' definitions into six categories – a refinement of Vinner's (1983) previous work and they categorized students' concept images as revealed through their work on problems 1-6 into four categories. The six categories the researchers used to describe students' concept definitions included: “*correspondence*”, “*dependence relation*”, “*rule*”, “*operation*”, “*formula*”, and “*representation*” (Vinner, 1983, Vinner and Dreyfus, 1989, p. 359-360).

Correspondence refers to a function in the formal definition (Dirichlet-Bourbaki definition), meaning that a function is expressed as a correspondence where every element of the domain is paired with exactly one element of the range. *Dependence relations* are definitions in which the students stated that one variable would be dependent on the other. Student responses did not state what this dependence might be or how it affected the different variables. *Rule* suggests that the students expect there to be regularity between the variables, not an arbitrary correspondence. The category *operation*, states, “a function is an operation or manipulation” (pg 360). The student acts on a number (input) by a given set of algebraic and arithmetic operations then the output is then considered the image. *Formula* is similar to *operation*; however, the student states that the function is a formula, expression or equation. The student did not discuss correspondence between variables or set. The final category is

representation. Vinner and Dreyfus (1989) state that the “function identified, in a possibly, meaningless way, with one of its graphical or symbolic representations” (pg 360). The analysis showed that students giving some version of the correspondence category responses increased with the level of the mathematics courses that the students were taking. Also, the higher the level of mathematics required by the students’ the more correct answers and explanations they provided.

The four categories for concept images (in order of importance for considering what objects might be functions according to Vinner & Dreyfus (1989)) included: “*one-valuedness*”, “*discontinuity*”, “*split domain*”, “*exceptional point*” (pg 361). One-valuedness focused on assigning correspondence between exactly one value to every element of the domain. The students with this type of response primarily used a version of the formal definition. Discontinuity centered on graphs that had gaps, holes, or jumps in function. Some students would use these gaps as a reason for a graph to not be a function, while others used discontinuity as a reason for accepting a graph as a function. Split domains created graphs that may not have smooth flowing curves. Rather, the domain of the graph was split into different parts called sub-domains which may have different rules of correspondence. Again, the split domain reasoning was used for both rejecting and accepting graphs as functions. Finally, the exceptional point category focused on the graph having a point in which the given correspondence did not hold. For example, in the graph $y = \frac{1}{x}$, zero would be an exceptional point.

Implementation of Framework

The purpose of the current study is to determine if there is a difference in students' understanding of the function concept by examining students' concept images and concept definitions after the students are introduced to the function concept using different instructional strategies, rather than to compare their concept images and definitions based on the number of mathematics classes they have taken. Findings from Vinner and Dreyfus' (1989) research served as a framework for describing students' understandings of function.

The framework of Vinner and Dreyfus (1989) served as a basis for categorizing the students' concept images and concept definitions since both Vinner and Dreyfus's study and this study collected data on college students' understanding of the function concept. The students in this study were mostly incoming freshman.

Vinner and Dreyfus (1989) used this framework in an attempt to evaluate students' knowledge in their ability to answer questions about functions. Further, the researcher used the framework in the current study to look at students' understanding about the concept definition for the function concept and their ability to identify functions in different representations.

CHAPTER 4

COMPONENTS AND METHODS

The components of the study and the methods for conducting the study will be described will be discussed in this chapter. The study is exploratory in nature meaning that it is looking for evidence of a difference in students' understanding of the function concept using different classroom instruction methods for introducing functions. Therefore some time will be spent describing the different students and the setting of the study. Further, the data collection process and analysis will be introduced and discussed.

Context of Study

The study was situated in an Intermediate Algebra course at a large southern urban university. Students enrolled in the course do not earn mathematics credit towards their degree, and thus it is considered a remedial course. The course focused on using technology, specifically a graphing calculator, and multiple representations to teach algebraic concepts. The students enrolled in the course must have competed at least Algebra I, Algebra II and Geometry at the high school level or in the community college. The students opting to take the course either did not take the college math placement test, placed into the course, or chose to take the course as a refresher returning to college after several years in the work place. Credit for the course is only given if the student passes the class with a C, 70% or better. Although the course does not satisfy the one of the mathematics requirement for general education requirements, the grade is calculated in the students' grade point average.

The course required that the students be able to use a graphing calculator efficiently. Throughout the course, the instructors teach the students how to use the technology. A course design was to have the students use the graphing calculator on all of the quizzes,

exams and homework assignments. The students were expected to be fluent in the use of the technology and move between the three representations of functions with confidence.

Furthermore, the students were expected to look at the three representations of a function and understand their meanings, and be able to answer questions about them. By using the graphing calculator, students were able to move through the different representations of functions with the fluency expected and focus on the interpretation of the representations.

The researcher chose to use participants from four classes taught by herself and another instructor for a variety of reasons. First, having four classes would permit greater participation in the study. Second, it initially balanced the number of students in the treatment and control groups. Third, the instructor of the remaining sections not used in the study was not as experienced in teaching the course. This was her first year semester teaching the Intermediate Algebra courses.

The researcher has considerable more teaching experience than the instructor of the control group. The researcher spent four years teaching algebra, geometry, and technical mathematics in public high schools, one year teaching undergraduate mathematics at Indiana University, and three years teaching the Intermediate Algebra courses, for a total of nine sections. The control group instructor has no experience teaching in the public school system. However, she had taught the Intermediate Algebra for a year prior to the study. For this reason, the researcher taught the treatment classes, and the other instructor taught the control group.

Both instructors have taken the same graduate level courses for teaching mathematics at the master's level. These classes have provided many opportunities for the instructors to gain confidence in teaching with technology. Therefore, both of them were comfortable

using the graphing calculators as part of the instruction as well as comfortable demonstrating how to use the graphing calculators. In the planning stages of the study, the two instructors talked about the misconceptions of students' understanding of the function concept, and possible strategies for teaching a control and treatment group.

Participants

Forty six of the 62 students enrolled in four sections of Intermediate Algebra in Fall 2006, initially participated in the study. Two sections were taught by the researcher and served as the treatment group ($n=22$), in which students were introduced to functions without providing a formal definition of function. In the two section serving as the control group ($n=24$), taught by an experienced Intermediate Algebra course instructor, students were introduced to the function concept with the use of a formal definition. However, not all of the volunteers were kept in the study.

Three of the four classes used in the study had similar demographics. Meaning, the average student was a traditional incoming freshman. The students all had similar mathematics background in high school, and were almost split fifty-fifty by gender. These classes met during the morning hours, Monday through Friday for fifty minutes each day. The evening class had a different make-up of students as consistent with previous night classes. Many of the students in this class were older and many had previous college mathematics courses. Most of these students were returning to school for second degrees of degrees beyond Associates. They were enrolled in the course for a refresher to mathematics. While the evening class provided a different type of student, it was still valuable to look at their concept images and concept definitions.

Several participants were removed from the sample because of their experiences and

age. The night class traditionally provides a different variety of students in the Intermediate Algebra courses. Since many of these students are returning for further degrees, they should not be compared to traditional students as a whole, unless one is comparing the two groups understanding of function. This however was not in the scope of this particular study.

Instruction Sequence

The instructor teaching the treatment group was the researcher. She used an alternative method to the formal definition for introducing the function concept (Appendix B). This included allowing the students to create their own classroom definitions for the function concepts, as well as presenting a variety of functions and non-functions for them to apply their definition to. The students in the treatment classes created their own definition of function based on their prior experiences, concept images, examples and non-examples provided by the instructor, and classroom discussion. Initially, the definition started out vague. However, it was gradually expanded to include all the necessary parts.

The control group instructor taught the function concept by introducing a formal definition. This definition was taken from the textbook: “A *function* is when exactly one element in the range corresponds to each element in the domain” (S. Wilson, personal communication, October 3, 2006). See her lesson outline in Appendix C. The students used this definition and the vertical line test while looking at several examples and non-examples to determine whether or not they were functions.

The instruction of the treatment classes and the control classes was similar in the fact that both instructors used many examples and non-examples to demonstrate the differences between functions and non-functions as well as a definition. The definitions and examples varied depending on the section. For treatment classes, the students used their constructed

definitions, and the control classes used the formal definition from the textbook. The course focused on using multiple representations, therefore these representations were used in all classes to present functions in alternative ways to the symbolic notation, in particular, graphs and tables. The students were all assessed using the same quizzes, tests, and homework assignments. The students in all the classes were also provided the same review assignments to help them prepare for quizzes and exams. The three morning classes followed the same schedule throughout the semester. The night class followed a faster paced schedule, since they met three days a week instead of five.

The biggest difference in the classes was the teaching style of the two instructors. The instructor of the control groups presents the material in an organized and methodical manner. She uses a variety of graphs, tables, drawings and other visual aides to supplement the lecture and to ensure student understanding. While the instructor of the treatment classes also uses a wide range of representations, the class discussed the different concepts and produced definitions based on the discourse. The instructor of the treatment group was more flexible in the order of the material covered and often allowed the students to dictate the direction of the class.

The participants volunteered to complete the survey. The decision to participate in the study had no effect on the students' grades for the course. All students read, signed, and were given a copy of a consent form as required by the IRB, the Institutional Review Board, at the participating university (Appendix D). Since the data was originally collected as part of an academic class, the Principle Investigator listed on the IRB is the instructor of that class.

Data Collection Instrument

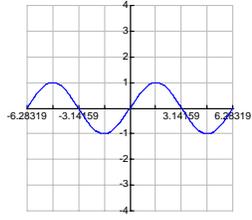
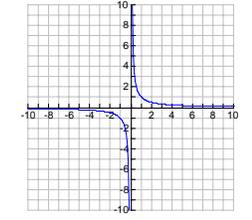
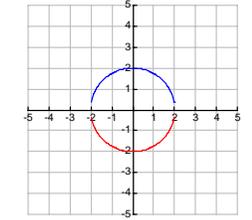
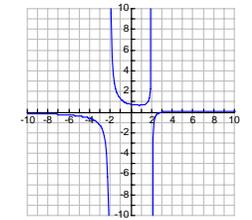
The instrument used to help determine the students' concept images and concept definitions and therefore their understanding of the function concept was a questionnaire. The questionnaire surveyed the students' knowledge of identifying functions in three different representations, their ability to use a definition of function to construct functions from a verbal representation, and the students' definition for what a function is. This questionnaire was similar to the questionnaires used by Vinner and Dreyfus (1989), as well as Hitt (1998).

The first part of the survey asked the students to identify functions and non-functions in three different representations; graphical, tabular, symbolic. The purpose of these questions was two-fold. First, to gain understanding of the students' evoked concept images, and to determine in which representation the students were able to classify functions and non-functions more accurately. The second part, asked the students to interpret verbal expressions and determine if these expressions represented functions. Finally, the last question asked the students to provide a definition for the function concept. The first two parts of the questionnaire to designed to help the researcher determine possible categories for students' concept images. The last section was designed to determine the students' concept definition.

At the end of the unit on rational functions, the respective instructor of each class asked the students to fill out this questionnaire (Appendix E). The questionnaire was constructed based on examples taken from Vinner and Dreyfus (1989), and Hitt (1998). The first three questions asked specifically if the students could identify functions in three different representations: graphical, tabular, and symbolic. Table 1 (below) provides further

explanations on why each of the questions was chosen.

Table 1: Justifications of Survey Questions

Representation	Question	Justification for Using
Graphical	 <p data-bbox="553 596 911 632">Sine Function: $f(x) = \sin(x)$</p>	<p data-bbox="1019 344 1382 558">This was a nice continuous graph with a mathematical name and is easily recognized by students with some trigonometry background.</p>
	 <p data-bbox="553 890 911 926">Rational Function: $f(x) = 1/x$</p>	<p data-bbox="1019 638 1382 852">This graph represents a discontinuous function, and was representative of the functions graphed by the students in their rational function chapter.</p>
	 <p data-bbox="553 1184 781 1220">Circle: $x^2 + y^2 = 4$</p>	<p data-bbox="1019 932 1349 1041">To test the misconception that a named curve is a function.</p>
	 <p data-bbox="553 1478 967 1556">Rational Function: $f(x) = \frac{x-3}{x^2-4}$</p>	<p data-bbox="1019 1226 1373 1335">This graph was chosen to address the misconceptions involving continuity.</p> <p data-bbox="1019 1373 1373 1768">The students in these particular classes should have recognized both rational curves as functions without being told specifically what the domain in each of the pictures was. This would have been a result of the instructional strategies in the course.</p>

Tabular	<table border="1"> <thead> <tr> <th>x</th> <th>y1(x)</th> </tr> </thead> <tbody> <tr><td>-5</td><td>56</td></tr> <tr><td>-4</td><td>42</td></tr> <tr><td>-3</td><td>30</td></tr> <tr><td>-2</td><td>20</td></tr> <tr><td>-1</td><td>12</td></tr> <tr><td>0</td><td>6</td></tr> <tr><td>1</td><td>2</td></tr> <tr><td>2</td><td>0</td></tr> <tr><td>3</td><td>0</td></tr> <tr><td>4</td><td>2</td></tr> <tr><td>5</td><td>6</td></tr> <tr><td>6</td><td>12</td></tr> <tr><td>7</td><td>20</td></tr> <tr><td>8</td><td>30</td></tr> <tr><td>9</td><td>42</td></tr> </tbody> </table>	x	y1(x)	-5	56	-4	42	-3	30	-2	20	-1	12	0	6	1	2	2	0	3	0	4	2	5	6	6	12	7	20	8	30	9	42	<p>This table represents some of the ordered pairs for a particular parabola. The interesting piece is the two zeros in the center of the table. Having studied how to use the different representations for solving quadratic functions, the researcher wanted to see how changing the question from find the roots to is it a functions would affect the students interpretation of the table.</p>
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1	11																																	

Symbolic	$x = y^2$	The first four questions were used to find out if students would be able to recognize square root, linear, and quadratic functions as well as curves that are not functions. For the first problem, the students in the study would have thought x was still the independent variable. This has to do with the fact that the students used graphing calculators throughout the course.
	$y = \sqrt{x-4}$	
	$y = 2x + 5$	
	$y = 3x^2 - 4x + 12$	
	$y = \begin{cases} 1 & \text{if } x \text{ is even} \\ 0 & \text{if } x \text{ is odd} \end{cases}$	This question was used to test their concept definition. Most of the students would not have seen functions of this sort.

The next three questions asked the students to use their concept images and definitions to construct a function from a verbal representation. Since the students did not answer these construction questions, they were removed from the analysis. The final question asked the students to provide their definition for a function. The questionnaire asked students to provide justification for all of their answers in the form of a few words or a sentence.

The instructor of each of the classes gave the questionnaires during a regular classroom session and the students completed them during that time.

Analysis

Initially, all questionnaires collected were analyzed, including those where the student(s) replied with question marks or “I don’t know”. Later, the researcher decided to remove those questionnaires from the study in which the students did not take the questions

seriously and the questionnaires in which every answer was “I don’t know”. The researcher did not feel that the students who responded with justifications such as “No, really!” would improve the knowledge base of research on students’ understanding of the function concept. Also, students whose answers were simply that they did not know, may not have had enough time to complete the questionnaire, did not take the questionnaire seriously, or simply did not know the answers to the questions. In any case these students were left in the study and included in the analysis. Further, there is a gap between being able to answer a mathematical question and being able to communicate why that particular answer was chosen. Therefore, questionnaires with answers and not justifications were also kept in the analysis.

The questionnaire asked the students to identify functions and non-functions, as well as, to provide justification for the decisions. The identification of functions and non-functions was based on correct yes or no responses. For each representation, the students were given four graphs and tables and five equations, in which to classify as functions or not. By looking at the responses in each of the two groups, the researcher was able to find a percentage of correct answers in each of the representations for both the treatment group and the control group. The questionnaire was scored as either having correct answers or incorrect answers for each of the representations during this part of the analysis. There was not an analysis for comparing total scores on the questionnaire for the treatment and the control groups.

The justifications from the questionnaire were used to evaluate the students’ evoked concept images. It was not important whether or not the student correctly classified the functions and non-functions for this part. It was the justification for the decision that they made that was important in helping to evaluate the evoked concept images. For example, if a

student incorrectly labeled the fourth table as a function, the researcher was interested in knowing why they thought that was a function. Based on the justification, they could be “placed” into a category representing their concept image. The justifications were not scored per se. Rather the researcher grouped the justifications into respective types and these types were compared to the categories adopted from Vinner and Dreyfus’s (1989) framework. The definitions provided by the students were also grouped according to the categories from Vinner and Dreyfus (1989).

The questionnaire responses were analyzed using both Excel and Fathom. Fathom is a statistical software program that will create graphs, calculate statistical measures, and perform tests on data. The data can be entered by hand or imported from other spreadsheets such as Excel. The software has some unique features that allow the user to click on pieces of a specific histogram and see where those students fell in the other graphs or on in the table. Excel was used to help organize the data and look at relationships students have between their concept images and concept definition.

The researcher looked at the answer to each of the questions and then the justifications on the questionnaire given to the participants. The justifications for the answers were used to place students in a particular category for describing their concept images and concept definitions. For example, in the fourth graph on the questionnaire, Figure 3, a student stated that this was not a function, because it was not continuous. This student would fall in the category “discontinuity”.

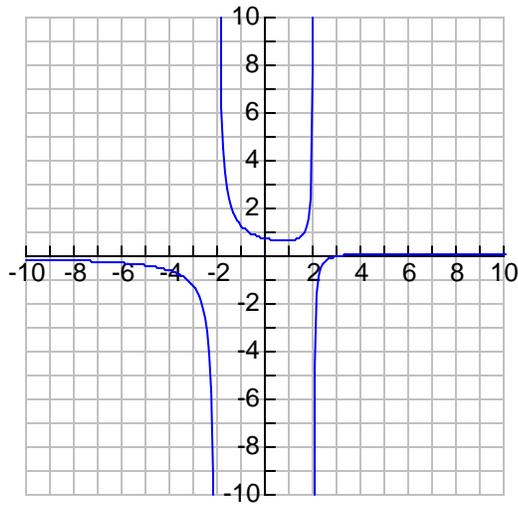


Figure 3: Continuity Linked to Functions

If we look at one of the definitions: “ $f(x)$, what is the equation that will solve or x .” Here, one can see that the student does not have a clear understanding of the definition of a function. The researcher would categorize this definition as a “formula”, because the student has demonstrated that they desire a rule that will allow them to solve for x .

The questionnaire provided the researcher the opportunity to formally investigate the students’ understanding of function after several weeks of instruction. The students provided their justifications and definitions for the function concept. The students’ responses provide a brief snapshot of their concept images, and understanding. It is by no means an entire summary of what they believe to be true about functions.

One should note, however, that the researcher does not believe that the categories are all inclusive, or that a single category may describe the entire concept image of a student. As a teacher, we may never “see” the entire concept image of a student for the function concept. First, concept images are a dynamic representation of the students’ beliefs. Second, the

students only evoke parts of the image when considering problem solving approaches and solutions to problems.

After classifying students' understanding of the function concept, the researcher looked to see if there was a difference in the students' understandings based on the instruction of the class. To help the researcher find differences in the students' understandings, we are going to look at the various justifications and how consistent the students were in applying their definition throughout the questionnaire.

CHAPTER 5

ANALYSIS

The results and the findings of the study are presented in this chapter. They describe the results for the two different groups as well as a comparison between the two groups. The chapter will start with an overall analysis of the students' accuracy on the questionnaire. It will then describe the findings after looking closer at comparing the two groups. The justifications which the students used will be examined and described. Finally, the researcher will look at comparing the students' concept images and concept definitions.

The treatment class spent a period discussing in small groups and answering the question "what is a function?" From their work, conversations, and whole class discussion the following definition was reached: "a function is a relation that provides an association between the domain of a set and the range, and it passes the vertical line test and the x-values do not repeat." While this definition does not encompass the mathematical terms found in textbook definitions, it does a reasonable job of providing a basis for which the students can determine if a relation is a function. The students' definition is very concrete; it uses a testing strategy for part of the definition. However, the vertical line test is useful for the graphical representation. This means that to employ this definition the students in the treatment courses are going to have to use multiple representations for determining if a relation is a function. For example, if given $y = \frac{(x-2)}{3(x+4)(x-7)}$ they are going to have to graph the equation to see if it is a function. However, this technique will only work if the students are able to produce a graph. In the case of the construction problems, where they were asked to determine if a function exists *that maps integer numbers to non-integer and*

non-integer numbers to integer numbers. None of the students tried to produce a graphical interpretation of this problem, they either gave an answer without a justification or the abstained from answering the question.

Initially, the researcher checked the questionnaires for correct answers. However, some of the participants were removed; one of the treatment classes, specifically the night class, had a large percentage of non-traditional students enrolled. The researcher decided to eliminate these participants also from the sample, as several of these students already had degrees, and one was a high school mathematics teacher.

Overall Analysis of Accuracy of Responses

The researcher first scored the questionnaires for accuracy and computed the percent of questions answered correctly in each of the representations. This analysis involved only right or wrong answers, not justifications. Also, if the students only answered three of the four questions involving graphs their percentage correct was calculated from the number of questions answered not the total number of questions. This decision was made due to the fact that the questionnaires were given during a class period and some students may not have felt had accurate time to complete the survey as thoroughly as they would have liked. Table 2 shows the percentage of correct answers based on the three representations and separated by control group and the treatment group. The percentage of correct answers were computed by looking at each of the participants questionnaires and checking for correct answers and calculating percentages out of 4 for the graphs and tables, and out of 5 for the equations. Table 3 shows the percentages and number of students for each of the questions in the survey and the break down of their scores. The researcher analyzed questionnaire for 22 members of the control group and 15 members of the treatment group.

Table 2

Percentage of Correct Answers

Type of Question	Treatment Group (N=15)	Control Group (N=22)
Graphs	70.31%	82.95%
Tables	65.63%	80.68%
Equations	57.81%	50.90%
Totals	64.58%	69.93%

Table 3
Break Down by Question of Correct Answers

			Treatment Group (N=15)		Control Group (N=22)	
1. Identify function from graph * (#) Numbers in this form are responses of “not sure”, “I don’t know”, or “?”	trigonometric function	Correct	11	73%	21	95%
		Incorrect	4	27%	1	5%
		Left Blank	-	-	-	-
	rational function with one vertical asymptote	Correct	11	73%	16	73%
		Incorrect	4	27%	6	27%
		Left Blank	-	-	-	-
	circle	Correct	13	87%	21	95%
		Incorrect	2	13%	1	5%
		Left Blank	-	-	-	-
	rational function with two vertical asymptotes	Correct	10	67%	15	68%
		Incorrect	5	33%	6	27%
		Left Blank	-	-	1	5%
2. Identify function from table	Quadratic Function	Correct	8	53%	17	77%
		Incorrect	7	47%	4	18%
		Left Blank	-	-	- (1)	5%
	Rational Function	Correct	12	80%	17	77%
		Incorrect	2	13%	5	23%
		Left Blank	1	7%	-	-
	Horizontal Line	Correct	11	73%	17	77%
		Incorrect	4	27%	5	23%
		Left Blank	-	-	-	-
	Vertical Line	Correct	11	73%	20	91%
		Incorrect	4	27%	2	9%
		Left Blank	-	-	-	-
3. Identify function from an equation	$x = y^2$	Correct	5	33%	6	27%
		Incorrect	8	53%	12	55%
		Left Blank	1 (1)	13%	2 (2)	18%
	$y = \sqrt{x - 4}$	Correct	7	47%	17	77%
		Incorrect	6	40%	3	14%
		Left Blank	1 (1)	13%	1 (1)	9%
	$y = 2x + 5$	Correct	13	87%	17	77%
		Incorrect	1	7%	3	14%
		Left Blank	1	7%	2	9%
	$y = 3x^2 - 4x + 12$	Correct	10	67%	12	55%
		Incorrect	2	13%	5	23%
		Left Blank	2 (1)	20%	3 (2)	23%
	$y = \begin{cases} 1 & \text{if } x \text{ is even} \\ 0 & \text{if } x \text{ is odd} \end{cases}$	Correct	2	13%	4	18%
		Incorrect	7	47%	6	27%
		Left Blank	5 (1)	40%	7 (5)	55%

From the tables and looking at the percentages resulting from this study, there does not appear to be any significant difference in college algebra students' ability to determine if relations are functions in the three representations, when the concept is introduced using different instructional approaches. The classes with the alternative instruction method (treatment group) did comparably well when compared to the control group when classifying functions in their different representations. In fact, the treatment group did slightly better than the control group with three of questions using the symbolic representation. However, by examining the separate questions and considering the questions that were left blank; the two groups appear to have similar notions of the function concept in terms of being able to identify functions and non-functions.

At an initial glance, the treatment group provided more justification for their answers and a wider range of definitions. The researcher categorized the justifications by simply looking to see if the students provided an explanation for their answer and what category this justification would have fallen into. Also, the treatment group provided a wider variety of types of definitions in the way that they worded the justifications and the different categories according to Vinner and Dreyfus (1989).

Digging Deeper in Comparing the Two Groups

Only comparing the treatment and control group with percent correct in each question does not portray the potential variability that may exist within a group and among the students. Using the Fathom Software to analyze the data, the researcher created several graphs and tables. The graph in Figure 4 shows a comparison between the students in the treatment group and the control group for the total number of questions they each identified correctly as functions and non-functions (maximum correct of 13—4 graphical, 4 tabular,

and 5 symbolic). The vertical lines on the histogram represent the mean (red) and median (blue). By looking at the graph, one can see that the means for each group are close together. The treatment group mean is 8.27 correct responses and the mean for the control group is 9.09 correct answers.

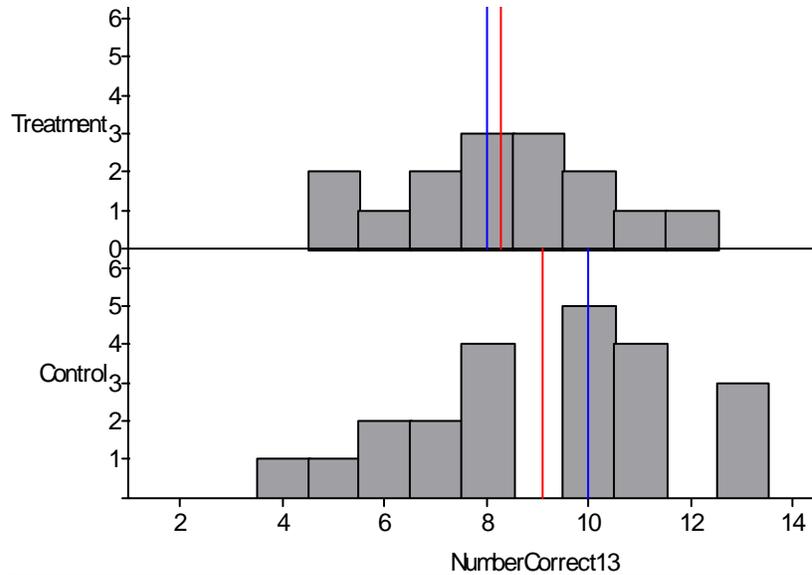


Figure 4: The Total Number of Correct Answers, The red line is the mean, and the blue is the median.

There is an obvious split in the control group’s number of correct responses. This split could indicate that in the control group the students either had a reasonably accurate concept image of functions or they did not have a good grasp on the concept. The treatment group’s responses are more evenly spread across the range of the data set. The number of students with 9 or more correct answers in the treatment group is 7 or about 47%. The number of student with 9 or more correct answers in the control group is 12 or about 55%.

The graph suggests that the control group preformed slightly better than the treatment group on the questionnaire. The students with 9 or more correct answers demonstrated some

knowledge of function in all three representations used on the questionnaire. However, the split in the control group's responses suggests that the control group students understanding may have been limited to specific representations for some of the students. This will be discussed further in a comparison of the concept images and concept definition.

It is interesting to note that all of the students in both groups, who got nine or more of the questions correct, also got all four graphing questions correct. This accounts for all but two of the fours in the following figure (Figure 5). In these histograms, the red line indicates the mean values of correct answers for the two groups. The other two students who also answered the graphical question correctly, one in the treatment group and one in the control group, answered 8 of the 13 survey questions on representations accurately.

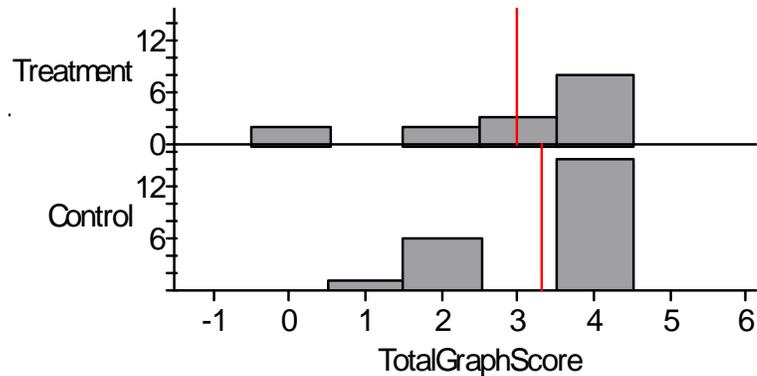


Figure 5: Graph Scores

By examining the bar graphs in Figure 5, for the students' number of correct answers on the graphical questions, we can see that the control group is again split in to two separate groups. Those that correctly identified all the graphs and those that missed two or more. Seventy percent of the treatment group and 68% of the control group had correctly identified all four graphs as functions and non functions. For both groups the two graphs involving rational functions created the most conflicts in the students' concept images. For example, students stated that graphs of rational functions could not be functions because "the line is

broken” or “it is not continuous”.

In the treatment group, of the two students who did not answer any of the graphical questions correctly, one of them had an inaccurate concept image. The justification that this particular student used was “if the x-axis is crossed more than once, it [the curve] is not a function.” The other student used the vertical line test as their justification, but it may be that this student did not understand what the purpose of the vertical line test is or did not understand the nature of vertical asymptotes.

The histograms in figure 6 show all of the participants’ number of correct answers for the entire questionnaire and each of the representations. The red color highlights individual participants who correctly identified function in 9 or more questions and where they fell in each of the histograms. Using these diagrams, the researcher compared the number of total correct answers to the number of correct answers in each of the representations. The students who answered a total of nine or more questions on the survey are highlighted in the other three graphs, representing the graphical, tabular, and symbolic questions.

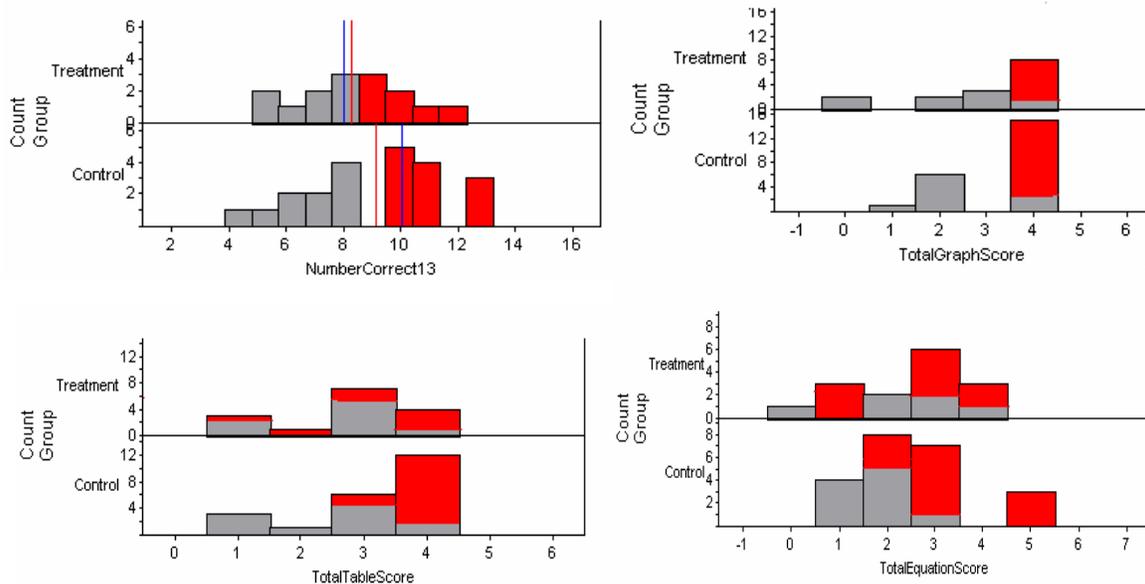


Figure 6: Comparison of Total Correct Answers Across Representations

Of the students that answered nine or more questions correctly, they identified all four of the graphs as representing either functions or non-functions (as seen in the graph in the top right corner). In the control group, these students were able to identify at least three of the tables as representing functions or not. Most of the students in this group also correctly identified three or more of the symbolic representations as functions or non-functions. By examining the distribution of the red color in the control group, it seems that a group of these students could apply their concept images to the three different representations with some consistency. However, the other members of the group seem to have demonstrated a lack of accuracy when applying their concept images.

The treatment group does not clearly show a line of demarcation among its members. Their number of correct answers has a greater variability throughout the representations, with the exception of the graphs.

Examining Students' Justifications for Identifying Functions

Treatment Group

Although 15 students were included in the prior analyses, only 12 of these students provided justifications for their responses. Thus, in this section, only 12 students from the treatment group were included.

Graphical Representation. Students used a variety of reasons to support their decision of whether or not a graph represented a function. The graphs were counted as four different problems for each of the students, so there were a total of 48 graphs in the treatment group determined by 12 students to be functions or not. The students used the vertical line test in 31 of the 48 cases for determining if the graph was a function. This was by far the most popular justification for the treatment group, and in every case, except for one student, the students could accurately determine if the curve represented a function. These students would be difficult to categorize in Vinner and Dreyfus's (1989) categories, because implementation of the vertical line test does not mean that the students understand it is a test for correspondence, or if they think of it as an easy classification method.

One student used continuity as a justification. In the case of the trigonometric function, he determined the curve to be a function; he also correctly determined the circle to not be a function on the bases of the vertical line test. However, for the two rational functions, he incorrectly determined that they were not functions by using a continuity argument. This person would have a concept image of *discontinuity* according to Vinner and Dreyfus (1989). This is also one of the misconceptions found by Tall and Bakar (1991) and Vinner (1983).

Another student had conflicting answers for the two rational functions. In the first

rational function, he correctly applied the vertical line test and concluded that the curve was a function. However, in the second rational function graph, he stated that it could not be a function because of the existence of vertical asymptotes. It is clear here that this student has some confusion between the vertical line test and a vertical asymptote. This student may be demonstrating the *exceptional point* category (Vinner and Dreyfus, 1989).

Tabular Representation. The student had a much wider variety of justifications for this representation than the graphical. For the 48 tables, the most popular justification was again the vertical line test. However, it was used only 23 times, and five of the tables had sketches of graphs constructed next to them. The second most widely used justification was the “x’s don’t repeat”. This justification was used seven times. The seven times that the students used this justification yielded correct interpretations of the tables; this was not true for this representation with the vertical line test. It is difficult to tell if the students are using the uniqueness of x in tables in a correspondence manner or as testing strategy without further assessment. It is clear though, that some of the students preferred the graphical representation enough to sketch graphs of the ordered pairs, rather than determine if the relation is a function from the table.

Symbolic Representation. This representation presented the most difficulty for the students. Many of the students did not provide justifications. Although, the research saw many of the students graph these equations on their graphing calculators and presumably used the vertical line test. Most of the students did not answer the fifth question under this representation, Figure 7.

$$y = \begin{cases} 1 & \text{if } x \text{ is even} \\ 0 & \text{if } x \text{ is odd} \end{cases}$$

Figure 7: Symbolic Representation of a Function

This type of function is novel to the class and was never addressed as part of the course.

Also, the first question switched the order of the variables, i.e. $x = y^2$. This may have caused some confusion amongst the students. Regardless, it was included in the analysis.

Control Group

Of the 22 students included in the prior analysis, only 17 provided justifications for why they chose their answers. Therefore, the following section only includes those 17 students.

Graphical Representation. The control group preformed better than the treatment group. Their justifications were using the vertical line test, continuity, “no shared x’s”, and “circles can not be functions”. The vertical line test was the most popular justification. It was used a total of 58 times on 68 graphs which were in question. Of all these questions, the vertical line test was used appropriately and achieved the correct classification except in one instance. The one student, who used the vertical line test incorrectly, did so on the rational functions. She believed that the curves passed the vertical line test, but sense they had an exceptional point, “broken”, she stated that the curves were not functions (Vinner and Dreyfus, 1989).

Tabular Representation. The control group did not use the vertical line test at all for classifying the tabular questions. Instead, a majority, 46 of 72, justifications were worded similar to “all x-values are different”. Once again, this does not indicate that the students understand a correspondence definition or are even thinking about the justification in that manner. They may just be using a testing strategy similar to the vertical line test which

works for tables. None the less, the use of the justification provided a correct answer in cases in which it was used.

There are some instances of students exhibiting an *exceptional point* image. A few of the students say that the second table can not be a function because it has an undefined value for y at x equals zero. There is also a case of a students using continuity to claim the second table is not a function. One student used the justification “ y repeats” or “ y does not repeat”, for determining if a relation is a function. It seems that this particular student was confused on the definition of function, but we can not know for sure, without further questioning.

Symbolic Representation. For questions b, c, and d in this section, it appears that the students graphed these on their calculators and used the vertical line test. The researcher can not say for certain that this is what happened, as she was not the one to distribute the questionnaire. But several of the students used the vertical line test as a justification. When this method was employed, the students answered the question correctly.

Comparison Across Groups

In both groups, we can see that the students used the vertical line test the most often to determine if curves are functions. When, it comes to the tables, the two groups differ on their methods for determining if the table represents a function. The symbolic representation was the most difficult for both groups, in deciding whether the equation represented a function and in providing a justification. The fact that many of the students in both groups did not address the construction problems made it impossible to make any generalizations on this part. In the future, an interview technique would be a good way for bringing their thinking to light on these kinds of problems. The few that did attempt them did not provide any justifications for their conclusions.

Analysis of Students' Concept Definitions

Many of the students provided definitions that did not easily fit into one of the categories proposed by Vinner and Dreyfus (1989). These definitions included: “A function is a curved line that when it passes through its points can not be touched by a vertical line more than once.” or “a function passes the vertical line test”. Normally, these types of definitions would fall into the representation category or a correspondence category, but we can not know for sure how the students are using the vertical line test without conducting an interview. Therefore, the researcher assumed there might be a different meaning here that can not be assigned to one of Vinner and Dreyfus’s (1989) categories for function definitions. Vinner and Dreyfus’s (1989) categories were modified to include another category; the definition of function which includes using a testing strategy. The following table, Table 4, shows the distribution of the types of definitions given by the students in the two different groups.

Table 4: Percentage of Students in each of the Concept Definition Categories

Definition Category	Treatment Group (N=15) n (%)	Control Group (N=22) n (%)	Both Groups (N=37) n (%)
Corresponding	1 (6.67%)	2 (10%)	3 (8.11%)
Dependence Relation	0 (0%)	1 (4.50%)	1 (2.70%)
Rule	0 (0%)	0 (0%)	0 (0%)
Operation	0 (0%)	0 (0%)	0 (0%)
Formula	1 (6.67%)	0 (0%)	1 (2.70%)
Representation	0 (0%)	0 (0%)	0 (0%)
Testing Strategy	5 (33.33%)	5 (22.70%)	10 (27.01%)
Do Not Know/No Response	6 (40.00%)	12 (54.5%)	18, (48.65%)
Inaccurate Definition	2 (13.33%)	2 (9.10%)	4 (10.81%)

Notice, in the treatment group 6.67% (1 student out of 15) of the students provided a “correspondence” definition despite the fact that they were not presented a formal definition

during classroom instruction. In the control group, 9.10 % or 2 of the 22 students provided a correspondence definition even though their classes were provided the formal definition as part of their classroom instruction. Also, there are a larger percentage, over half, of students in the control group that stated that they did not know a definition for the function concept, left the questions blank, or provided a wrong definition, than the students in the treatment group.

The histogram in Figure 8 displays the total number of correct answers on the x-axis and the different definition categories on the y-axis for all of the participants. The red lines represent the average number of correct responses in each category. The students that did not provide a definition on the questionnaire have a lower average of correct answers than any other category.

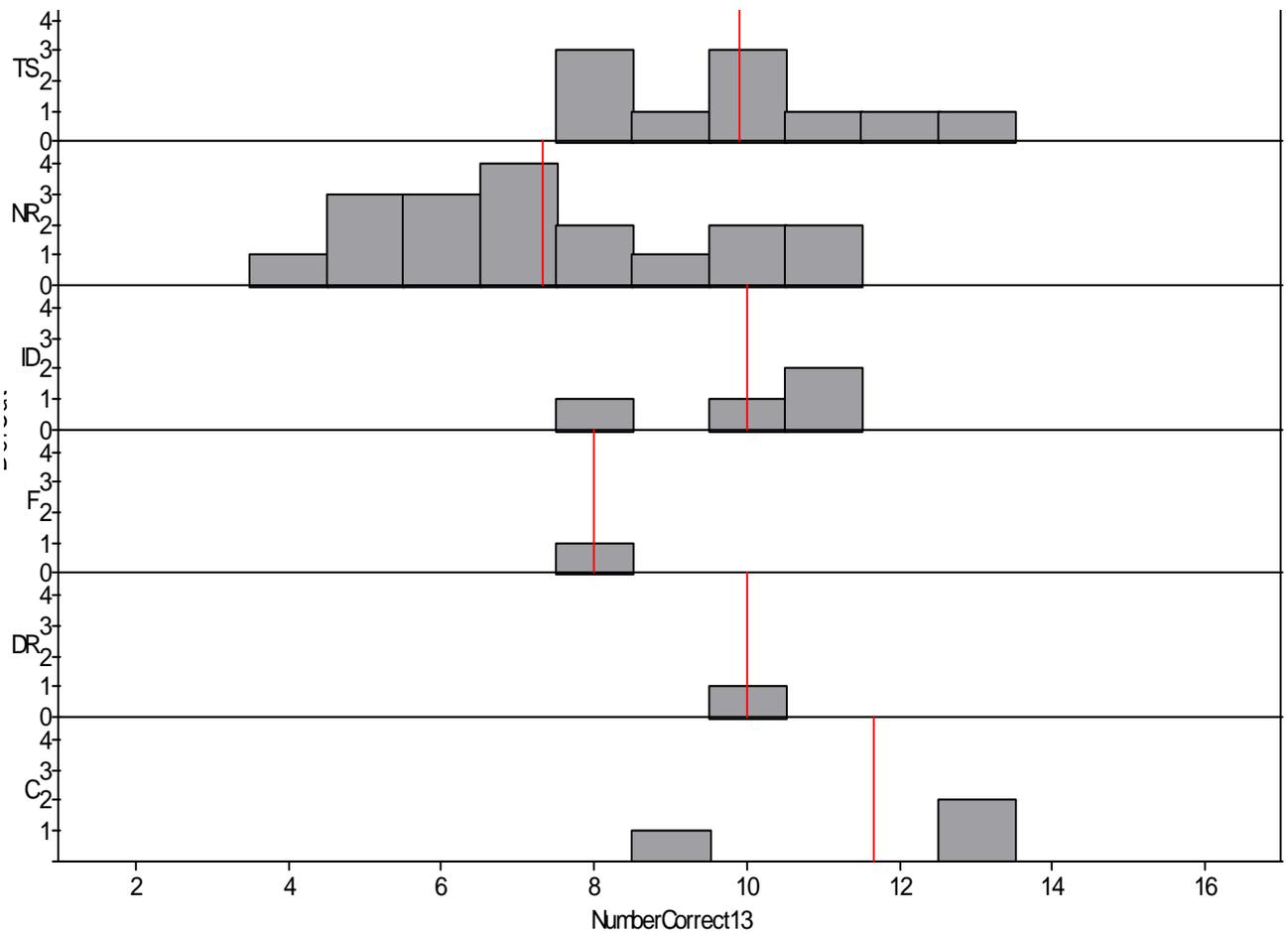


Figure 8: Number Correct and Definition Type

As far as looking for misconceptions in the students' understanding of function, there does not appear to be any differences in the two groups. All of the students who participated in the study had most likely already constructed some level of a concept image for the function concept, since prior mathematics courses were a prerequisite. One of the most noticeable misconceptions dealt with continuity of curves. In particular, students with this notion of continuity being necessary did not believe that graphs with vertical asymptotes were functions; despite the fact, they had just finished a unit on rational functions. Furthermore, the class examined a variety of functions in the graphical representation that

included both continuous curves and discontinuous curves.

For these classes, this implies that regardless of the students' concept definition or understanding of the function concept, they can apply testing strategies in the graphical and tabular representations. However, in the symbolic representation these testing strategies do not work as fluently unless the students are able to change the representation. The student either has to change the representation and understand the connections between them, or has to be familiar with the symbolic notation enough to know if it represents a function or not.

Comparing Students' Concept Image and Definition

It is important to examine the relationship between students' concept images and definitions, because these two "cells" need to be connected in order for the students to have complete understanding of a mathematical concept such as functions. If the students; do not form connections between their images and the definition, they may not be aware of contradicting beliefs within their images. The evoked concept image can provide enough information for students to answer some questions, but an understanding of the definition will be needed to answer all questions about a mathematical concept accurately.

For comparing the students concept image to their definitions, the researcher created eight categories: 1) has concept image they can use to answer some questions but gave no concept definition; 2) concept image used to answer questions has some inconsistencies with the definition given; 3) gives correct concept definition but does not seem to have a concept image useful for answering any of the questions; 4) concept image and concept definition appear to correspond; 5) not enough evidence for classifying a concept image; 6) gave inaccurate definition, but have some concept images; 7) evoked concept images has both correct pieces and incorrect pieces, and 8) image and definition support each other (definition

is not a concept definition. For the fifth category, there were students that answered many of the questions correctly and provided a definition for the function concept, but they did not justify any of their answers on the questionnaire. Therefore, there is little or no evidence of what their evoked concept images may look like.

Table 5 shows examples for each of the categories from the control group and the frequency of participants that fell into each category, except for *gives correct concept definition but does not seem to have a concept image useful for answering any of the questions*; it is unlikely that students at this level of mathematics would fall into this category. Table 6 shows examples for each of the categories provided from the treatment group.

Table 5

Connections between Concept Images and Definitions Control Group (N=22)

Categories	Student Examples	Percent of Students
1) Has concept image they can use to answer some questions but gave no concept definition	9002 – image involves continuity, and some testing strategies, no definition 15002 – used testing strategies as justifications, no definition was provided 6001 – student has a concept image that they can apply with accuracy 11 of 13 correct, but gave no definition 3001, 2002, 5002	27.27%
2) Concept image used to answer questions has some inconsistencies with the definition given	5001 – provided a dependence relation definition, and could apply this to most of the questions, conflict with undefined terms.	4.55%
3) Gives correct concept definition but does not seem to have a concept image useful for answering any of the questions		0
4) Concept image and concept definition correspond	6002 -- correctly answered all 13 questions, provided a concept definition 7002 -- correctly answered all 13 questions, provided a concept definition	9.09%
5) Not enough evidence for classifying a concept image	1001 – did not provide any justifications nor a definition 14002 – no justifications provided, 8 of 13 correct, Testing strategy definition provided 11002 – no justifications provided, 5 of 13 correct. No definition given 12002 – provided two justifications “X does not repeat”, no definition provided	18.18%
6) Gave inaccurate definition have some concept image	2001 – correctly answered 11 of 13 questions, did not provide justifications except for VLT on one graph, and the last table. 4002	9.09%
7) Evoked concept image has both correct pieces and incorrect pieces	7001 – correctly applies the VLT, but focuses on repeating or non repeating y values in the tables, no definition given 4001 – correctly applies the VLT but there are problems with graphs that have “breaks” 3002	13.64%
8) Image and Definition support each other, (not a concept definition)	13002 – using testing strategies as justifications, and gives a testing strategy type definition 10002 – using testing strategies as justifications, and gives a testing strategy type definition 1002, 8002	18.18%

Table 6

Connections between Concept Images and Definitions Treatment Group (N=15)

Categories	Student Examples	Percent of Students
1) Has concept image they can use to answer some questions but gave no concept definition	8003 – consistently used the VLT to identify functions in the graphical representation, no definition provided 6011 – consistently used the VLT to identify functions in the graphical representation only, not definition given 3003 – used the VLT test for the graphs and tables, did not answer the symbolic questions, no definition given	20%
2) Concept image used to answer questions has some inconsistencies with the definition given	2003 – used testing strategies for answering question, but definition was a formula 5011—could apply the testing strategies to the tables and graphs, had trouble with the equations, provided testing strategy definition	13.33%
3) Gives correct concept definition but does not seem to have a concept image useful for answering any of the questions		0%
4) Concept image and concept definition correspond	5003 – correctly answered questions from every category, had a concept definition, incorrect answers may be a result of not being able to communicate effectively	6.67%
5) Not enough evidence for classifying a concept image	4003 – no justifications, and no definition provided 13011 – justifications do not make sense, and provided and inaccurate definition	13.33%
6) Gave inaccurate definition have some concept image	1003 – could accurately apply the VLT, used discontinuous as justifications for things not being functions, inaccurate definition provided	6.67%
7) Evoked concept image has both correct pieces and incorrect pieces	9003 – could correctly identify function and non function by table, not by graphs, solved some of the equations for x, no definition given 10003 -- could correctly identify function and non function by table, not by graphs, no definition given	13.33%
8) Image and Definition support each other, (not a concept definition)	1011 –successfully applied this definition to the graphical and tabular representations, was not successful with the symbolic representations, testing strategy definition given 7011 – effectively used a testing strategy to answer 10 of 13 questions correctly, a testing strategy definition was given 4011, 3011	26.67%

The tables tell us that most of the student have some sort of concept image about the function concept that they can evoke to answer questions that ask them to identify functions and non-functions. The tables also show that most of the students are not connecting these images to concept definitions. Only 33% of the students in the treatment group and 27% of the students in the control group had formed any connections between their concept images and definitions.

CHAPTER 6

DISCUSSION

This chapter will discuss the implications for Mathematics Educations, as well as questions for further research. Due to the exploratory nature of the study, more questions arose from the findings than were answered; providing researchers with the opportunity to explore further.

Implications for Mathematics Education

Using a formal definition for introducing mathematical objects and concepts may not always be the best educational approach in secondary and early college level mathematics. While these formal definitions may be unavoidable in higher level college mathematics classes were alternate approaches are not feasible. For secondary and elementary level mathematics, there are a variety of alternative methods, such as, using examples and non-examples to help students build definitions that they can use and make sense to them, using a variety of different representations to interpret and represent the function concept, arrow diagrams, tables, algebraic symbolic notation, graphical representations, etc. This implies teachers do not have to follow the textbook and use formal definitions as the only method for presenting a concept. They can approach the function topic using a myriad of techniques.

Educators should be aware of the students' concept images before teaching a concept. Knowing the students' concept images would provide a starting point for a lesson. It is possible that the students do not have an image for a particular concept if it is the first time they have encountered it. However, the more mathematics class they take the more likely they will have some previous notion of the concepts. For example, calculus students would have encountered functions in Algebra I and Algebra II, but they may not have a clear

understanding of what a function is. In specific tasks, only part of the students' concept image or concept definition may be evoked. Therefore, teachers can not judge a student's understanding of any mathematical concept on the basis of one observed behavior (Vinner, 1983).

As an instructor for the course, the researcher noticed that several students used continuity as part of their justifications. This may be an indication that students are unwilling to change their previous notions about functions. Instead, they may be adding new knowledge to their concept images without replacing or removing incorrect images. An example of the students' apparent unwillingness to alter their concept images and remove faulty information was noticed when students presented justifications and definitions stating that functions needed to be "continuous". Continuity of graphs was beyond the scope of the class and whether or not a curve was continuous was not discussed in class.

The teacher should address the students' concept images if there are distorted images. As found in this study and others (Vinner, 1983), students will hold on to their concept images despite conflicting information. Instruction that specifically addresses the inaccurate concept images may help the students be more willing to let go of their previous images, and replace them with accurate and non-conflicting images. It is clear that giving the students a definition and some examples is not satisfactory for the students to form advantageous concept images (Vinner, 1983). By encouraging students to address their own misconceptions, teachers help students learn that they can change their concept images and definitions as they move through a course or the mathematics curriculum. The need to change concept images and concept definitions is not limited to the function concept. For example, students may take several years of mathematics believing that the square root of a

negative number does not exist. Eventually, the students will have to adjust their concept image for roots when they are introduced to complex numbers.

Questions for Further Research

In this study, it is difficult to tell if the instructional variation created different concept image structures or concept definition in the students from the two groups. To gain a clearer picture, one would need to observe both instructors on multiple days to look for variations in the teaching styles and methods. The influence of the instructor was not considered in this study. The instructor of the treatment groups had eight years of experience teaching, including four years in the public school systems. The instructor of the control group has one year of teaching experience at the college level. The difference in experience levels would influence the classroom instruction.

The students previously had formed concept images and possibly “concept definitions” about functions, from prior experiences and mathematics classes. In some cases, the students were unwilling to reconstruct their concept images regardless of the new information. For example, several students were deciding if a graph was a function, based on continuity, but continuity was not a topic discussed in the course. It is interesting to ask, “What would happen if the function concept was first introduced with out a formal definition in a high school or middle school Algebra I class?” We could then follow the students through their higher level mathematics and study their misconceptions and concept images. Would the researcher get different results?

Another topic for research is to look at how definitions are used in the mathematics classroom where proofs are not explicitly a part of the curriculum. How can we recreate the algebra classroom to use the definitions in a more concise and mathematical way in order to

provide students the opportunity to use and learn these formal definitions?

Now that technology, at least the graphing calculators, is in most classrooms, it would be note worthy to start looking at the misconceptions that technology has fostered. Do students loose understanding when they compare functions with shifts rather than drawing the function themselves? Are the misconceptions about the function concept changing or are they staying the same even with the use of technology? Does the way technology is implemented in the classroom affect or change the students' misconceptions? How the findings relate to those of other researchers?

The students in this study held similar misconceptions about the function concept that students in previous studies did (Brenner, et al., 1997; DeMarois, 1996; Dreyfus and Eisenberg, 1982; Eisenberg, and Dreyfus, 1994; Gagatsis et al., 2006; Tall and Bakar, 1991; Vinner and Dreyfus, 1989). In fact, there has been little change in the types of misconceptions about functions held by students in the last 20 years. Students are still not gaining a good understanding of the function concept. However, pre-service teachers may also hold some of the same misconceptions (Breidenbach et al., 1992)

The students in this study preferred graphical representations over the symbolic representations. They worked more comfortably with this particular representation in answering and justifying if the curve represented was a function. This may be a result of the fact that the instructors of the course privileged graphical representations more than the others and that the course had a focus on using multiple representations and fluency between them. In the classroom and on other assignments, such as homework, the students also tend toward this same representation to help them interpret problems and even to solve them. Mousoulides & Gagatsis (2004) found that students with a strong graphical approach to

functions were more effective in solving complex geometric problems than their counterparts which relied on the algebraic representations. Since, the students in this study preferred the graphical representation; it would have been interesting to look at their problem solving strategies as well.

Students in this study were better at determining if a relation was function than providing an accurate definition. Gagatsis et al. (2006) had similar findings. In their study, they found that 60.6% of students could provide a mathematically correct definition for function, but 74.6% could provide an example (p. 140). These findings suggest that students do not need to have complete understanding of a concept in order to work at an operational conception (Sfard, 1991). This ability to produce examples without understanding a formal definition also reinforces Vinner and Dreyfus's research (1989).

Hitt's (1998) research focused on in service teachers and their understanding of the function concept. He found that teachers do not use their concept definitions when answering questions about functions. He asked thirty teachers to determine if the graph in Figure 9 was a function. While 29 of the teachers correctly stated that the graph did not represent a function, their reasoning varied:

- Two teachers used a definition of ordered pairs
- Ten wrote that there was more than one image in certain points
- Six teachers explicitly used a vertical line cutting the curve in more than one point
- Eleven teachers said that it was not a function without giving a reason (Hitt, 1998, p. 127).

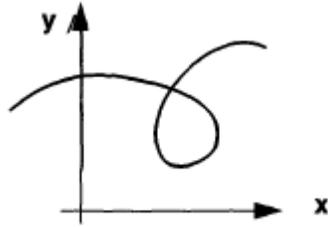


Figure 9: Concept Images of Functions (Hitt, 1998, p. 126)

Notice that the teachers are using evoked concept images to answer and justify their answers, and only two of the teachers used a definition of function as a justification. It is not unexpected then, that teachers present a formal definition in the classroom, but they encourage students to use concept images as a strategy for answering questions. While, teachers in Hitt's (1998) study evoked correct concept images for this question that was not the case throughout his entire study. For example, when the teachers were presented with conic sections with principal axis on the x -axis (see Figure 10), teachers abandoned their previous definitions for functions in favor of an algebraic expression (Hitt, 1998). Since, students frequently make the same connections between an algebraic expression and a function, misconceptions about functions teachers hold should be addressed so they do not pass them on to their students.

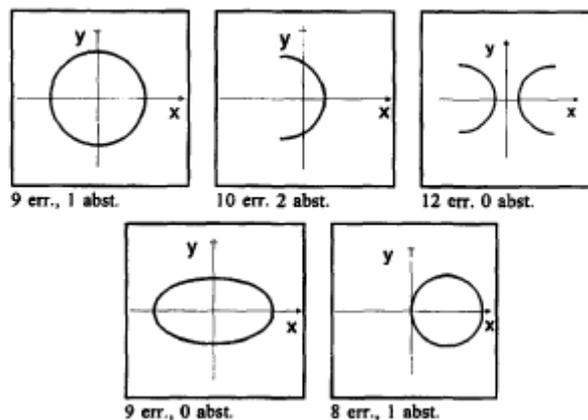


Figure 10: Errors with Conic Sections (Hitt, 1998, p. 126)

All of the students in this study appeared to favor the graphical and tabular representations over the algebraic. There are several possible reasons for this: the students probably were not confident in their algebraic manipulation skills. This would not be uncommon given that they chose to enroll in an Intermediate Algebra course. Further the complexity of the construction problems may have presented difficulties. The students enrolled in these classes are often not comfortable working with word problems which they have to interpret (Keller and Hirsch, 1998). Further influence for students' comfort level with graphical representation in particular is that both instructors privileged graphical representations more than the others. Keller and Hirsch (1998) found that students have preferences for specific representations based on the situation in which the problem is presented. For mathematical based problems students preferred to use equations, while in contextual problems students preferred tables and graphs to equations. This study found that the students worked better with graphical representations of functions for determining if a relation was a function, in purely mathematical settings. This preference is linked to the strategies that the students used to determine functions, namely the vertical line test.

It is worth noting that the instructor of the treatment courses taught the class, she was aware of those misconceptions and worked to help the students overcome them. In the end, after six chapters on different types of functions, the students still possessed many of the misconceptions that they had at the beginning. The students just added images and meaning to their concept image to encompass what they learned in the course. Further, the research conducted in this study indicates that misconceptions of the function concept are not being addressed in the mathematics curriculum. The Vinner and Dreyfus (1989) study is still a good model and framework for categorizing students' understanding seventeen years later. This indicates that regardless of the reform movements discussed in mathematics education circles, they are slow in implementation.

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APPENDICES

Appendix A

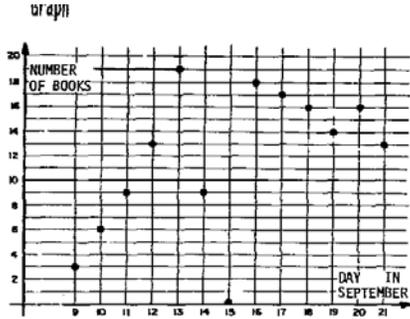
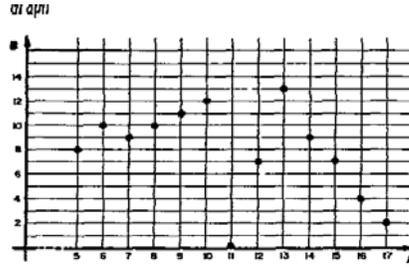
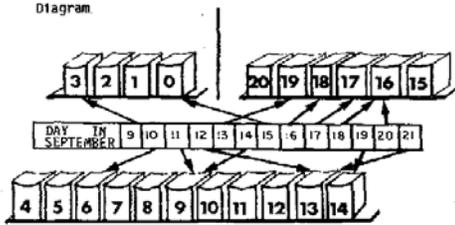


Image Question

How many books did the pupils take on loan from the library on September 14?

- a. 19
- b. 9
- c. 14
- d. impossible to determine from the data given.



Extremum Question

What is the smallest number in set B which is associated with one of the numbers between 10 and 14 in set A?

- a. 7
- b. 10
- c. 2
- d. another number.

Growth Question

The number of books taken from the library between September 16 and September 19

- a. increased each day
- b. decreased each day
- c. increased each day but one
- d. decreased each day but one
- e. none of the above answers a, b, c, d is correct.

Table

A	5	6	7	8	9	10	11	12	13	14	15	16	17
B	8	10	9	10	11	12	0	7	13	9	7	4	2

Preimage Question

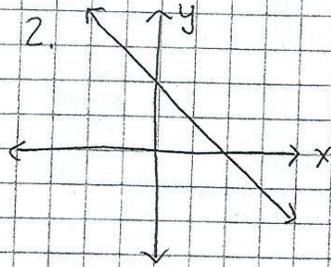
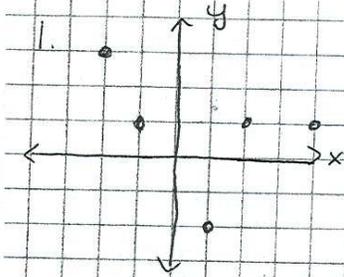
The list of numbers in set A to which there corresponds the number 7 in set B is:

- a. 12
- b. 2
- c. 12, 13, 14, 15
- d. 12, 15.

Dreyfus & Eisenberg, 1982, p. 368

Appendix B

For each of the following graphs Name:
 a) Is it a function
 b) What is the domain (interval notation)
 c) What is the range (interval notation)



3.

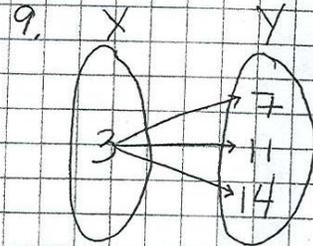
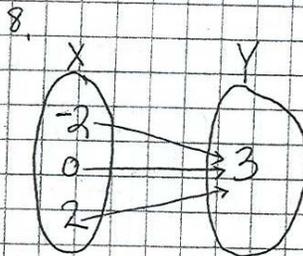
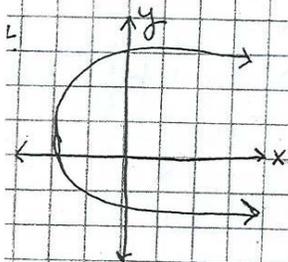
x	y
3	1
2	2
3	3
2	4
5	5

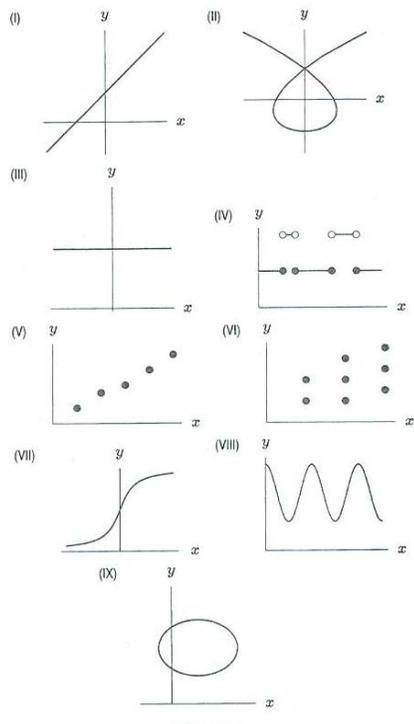
4.

x	y
-10	70
-8	54
-6	40
-4	18
-2	10

5. $3x + 4y = 12$

6. $y = 3x^2 - 7x + 11$





2.4 - RELATIONS AND FUNCTIONS

OBJECTIVES:

1. GIVEN A GRAPH, YOU WILL BE ABLE TO DETERMINE ITS DOMAIN & RANGE
2. GIVEN A TABLE OR A GRAPH, YOU WILL BE ABLE TO DETERMINE WHETHER OR NOT IT IS A FUNCTION.

• A RELATION IS A CORRESPONDENCE BETWEEN THE ELEMENTS OF ONE SET (THE DOMAIN) AND THE ELEMENTS OF ANOTHER SET (THE RANGE).

• DOMAIN IS ^{THE SET OF} ALL LEGAL INPUTS (X-COORDINATES)

• RANGE IS THE SET OF MEANINGFUL OUTPUTS (Y-COORDINATES)

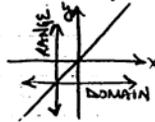
• WHAT ARE LEGAL INPUTS?

• ANYTHING THAT WON'T GIVE YOU A GOTO/QUIT ERROR

• GO TO $y =$

IN $y_1 = x$ → GRAPH (THIS IS A SNAPSHOT)

→ TABLE



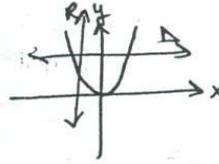
ALL X-VALUES ARE LEGAL INPUTS

DOMAIN: $(-\infty, \infty)$

ALL Y-VALUES ARE MEANINGFUL OUTPUTS

RANGE: $(-\infty, \infty)$

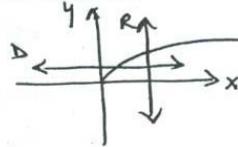
• In $y_1 = x^2 \rightarrow$ GRAPH
 \rightarrow TABLE



ALL X-VALUES ARE LEGAL INPUTS
 DOMAIN: $(-\infty, \infty)$

Y-VALUES MEANINGFUL FROM \emptyset TO ∞
 RANGE: $[0, \infty)$

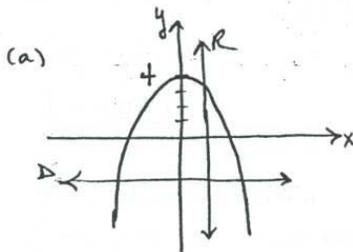
• In $y_1 = \sqrt{x} \rightarrow$ GRAPH
 \rightarrow TABLE



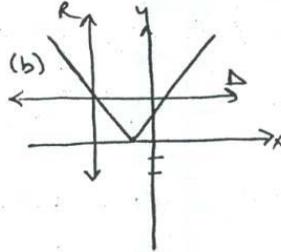
X-VALUES LEGAL FROM \emptyset TO ∞
 DOMAIN: $[0, \infty)$

Y-VALUES MEANINGFUL FROM \emptyset TO ∞
 RANGE: $[0, \infty)$

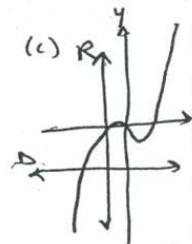
• EX: FIND THE DOMAIN & RANGE OF THE FOLLOWING:



DOMAIN: $(-\infty, \infty)$
 RANGE: $(-\infty, 4]$



DOMAIN: $(-\infty, \infty)$
 RANGE: $[0, \infty)$



DOMAIN: $(-\infty, \infty)$
 RANGE: $(-\infty, \infty)$

* DOESN'T MATTER THE NUMBER OF SQUIGGLS.

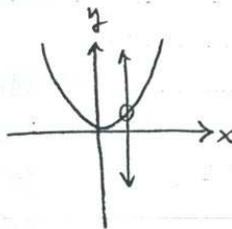
• A FUNCTION is when EXACTLY ONE ELEMENT IN THE RANGE CORRESPONDS TO EACH ELEMENT IN THE DOMAIN.

→ A FUNCTION IS A RELATION THAT PAIRS EACH INPUT WITH EXACTLY ONE OUTPUT.

→ X BELIEVES IN MONOGAMY ;)

Ex: $y = x^2$

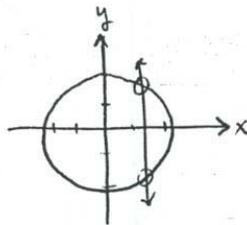
x	y
-2	4
-1	1
0	0
1	1
2	4



• DOES EACH X PAIR WITH EXACTLY 1 y?
→ YES, SO THIS IS A FUNCTION

Ex: $y = \pm\sqrt{4-x^2}$

x	y
-2	0
-1	$\pm\sqrt{3}$
0	± 2
1	$\pm\sqrt{3}$
2	0



• DOES EACH X PAIR WITH EXACTLY 1 y?
→ NO, SO THIS IS NOT A FUNCTION.

• Also ask:

Ex: Do the following tables represent functions?

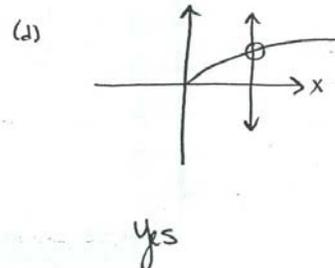
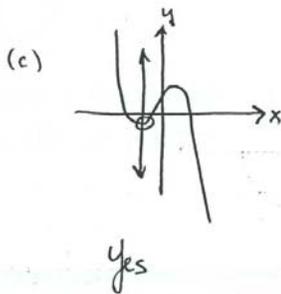
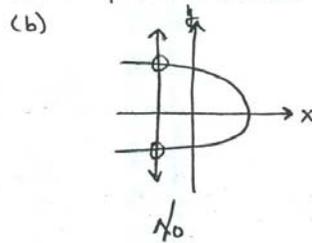
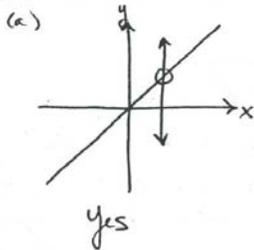
(a)		(b)	
x	y	x	y
-4	9	-4	1
-3	7	-3	2
-2	3	-2	3
-1	7	-1	4
0	9	-3	5

→ ANOTHER METHOD FOR DETERMINING WHETHER A GRAPH IS A FUNCTION IS THE VERTICAL LINE TEST.

• IF A GRAPH INTERSECTS ANY VERTICAL LINE MORE THAN ONCE, THE GRAPH DOES NOT REPRESENT A FUNCTION.

→ THIS TEST WORKS b/c IN ORDER FOR A VERTICAL LINE TO INTERSECT A GRAPH MORE THAN ONCE THERE MUST BE AT LEAST ONE X-VALUE THAT CORRESPONDS TO TWO OR MORE Y-VALUES.

Ex: DETERMINE WHETHER THE FOLLOWING GRAPHS REPRESENT FUNCTIONS.



Appendix D

Exploring students' advanced mathematical thinking while solving mathematics problems

Dr. Karen F. Hollebrands, Principal Investigator

You are invited to participate in a research project. The purpose of this project is to learn how students solve mathematics problems. You will contribute to this research by meeting with the researcher for about an hour to solve some mathematics problems and talk about your thoughts on the problems you are working. This session will allow the researchers to gain a deeper understanding of your mathematical reasoning on such tasks. This interview will be videotaped and these tapes will be destroyed after analysis.

There will be no risk associated with your participation in the research study. Your grade in your mathematics class will not be affected by your decision to participate in the study. The knowledge the researcher gains from your experiences will add to the knowledge base in mathematics education, especially with regard to how students solve mathematics problems. Information derived from this session will be kept strictly confidential, with your name removed from the work. It will be stored securely in a locked file and will be made available only to the researchers unless you specifically give permission in writing to do so. No reference will be made to your name either in oral or written reports and transcripts that could link you individually to the study.

You are free to withdraw from the study at any time. If you have any questions at any time, you may contact Dr. Karen Hollebrands at 513-0505. Her address is 326K Poe Hall, NC State University. If you feel your rights as a participant in research have been violated during the course of this project, you may contact Dr. Matthew Zingraff, Chairperson of the NCSU Human Subjects Committee, 8101, NCSU Campus.

CONSENT

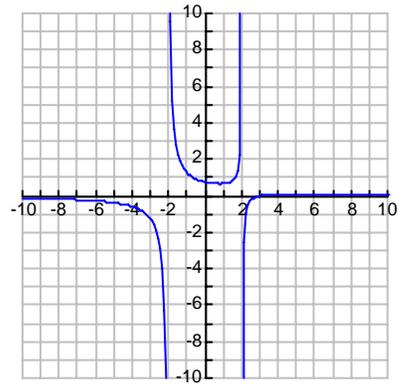
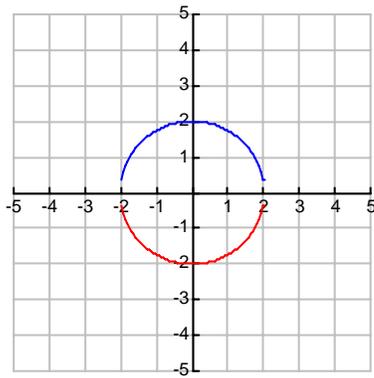
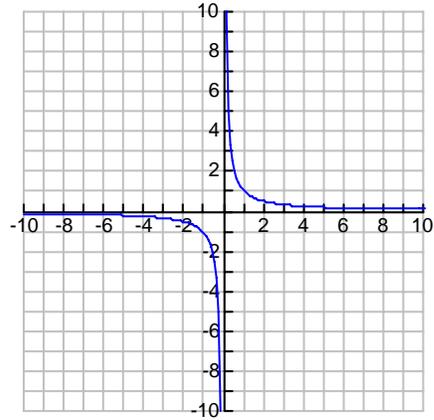
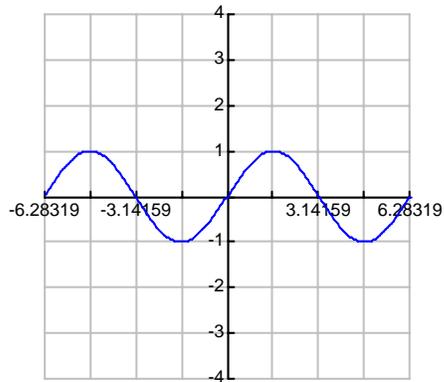
I have read and understood the above information. I have received a copy of this form. I agree to participate in this study.

Participant's signature _____ Date _____

Investigator's signature _____ Date _____

Appendix E

1. Do the following graphs represent functions? Justify your answer.



2. Do the following tables represent functions? Justify your answers.

x	y1(x)
-5	56
-4	42
-3	30
-2	20
-1	12
0	6
1	2
2	0
3	0
4	2
5	6
6	12
7	20
8	30
9	42

x	y1(x)
-5	1.8
-4	1.75
-3	1.66667
-2	1.5
-1	1
0	undef
1	3
2	2.5
3	2.33333
4	2.25
5	2.2
6	2.16667
7	2.14286
8	2.125
9	2.11111

x	y
-4	7
-3	7
-2	7
-1	7
0	7
1	7
2	7
3	7
4	7
5	7
6	7

x	y
1	1
1	2
1	3
1	4
1	5
1	6
1	7
1	8
1	9
1	10
1	11

3. Determine if the following equations represent functions, Justify your answers.

a. $x = y^2$

b. $y = \sqrt{x-4}$

c. $y = 2x + 5$

d. $y = 3x^2 - 4x + 12$

e. $y = \begin{cases} 1 & \text{if } x \text{ is even} \\ 0 & \text{if } x \text{ is odd} \end{cases}$

4. Is there a function that maps every number different from zero to its square and maps 0 to 1? Explain

5. Is there a function that maps integer numbers to non-integer and non-integer numbers to integer numbers? Explain

6. Is there a function that maps the temperature in degrees Fahrenheit to the temperature in degrees Celsius? Explain

7. What is the definition for a function?