ABSTRACT

Ayyar, Sajidul Rahman A. Single and Multiple Server Queues with Vacations: Analysis and Algorithms. (Under the direction of Professor Xiuli Chao).

In this thesis we are concerned with the analysis and algorithm development of multiple server queueing systems with finite buffer and vacations.

In chapter 2, we analyze a G/M(n)/1/K queueing system where the server applies an N policy and takes multiple exponential vacations when the system is empty. This includes G/M/n/K queues with vacation and other multiple server models. Using the method of supplementary variables to derive the system equations, we develop a recursive algorithm for numerically computing the stationary queue length distribution of the system. The only input requirement is the Laplace-Stieltjes transform of the arrival distribution. In chapter 3, we extend the above results to the case where the server takes multiple state-dependent exponential vacations.

In chapter 4, we study a M(n)/G/1/K queueing system where the server applies an N policy and takes multiple arbitrary vacations when the system is empty. We provide a recursive algorithm using the supplementary variable technique to numerically compute the stationary queue length distribution of the system. The only input requirements are the Laplace-Stieltjes transforms of the service time distribution and the vacation time distribution.

The results of this research can be applied to the design and optimization of computer and communication systems.
Single and Multiple Server Queues with Vacations: Analysis and Algorithms

by

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To

*Mummy and Puppa*
Biography

Sajidul Rahman Ayyar was born on March 11th, 1979 in Calicut, India. He completed his undergraduate studies in Industrial Engineering from Indian Institute of Technology at Kharagpur, India. The studies culminated with a Bachelor of Technology Degree in June 2000.

In 2000, he joined North Carolina State University to pursue his graduate studies in Industrial Engineering. During his studies he worked as an industrial engineering intern with Abbott Laboratories, Chicago, IL.
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Chapter 1

Introduction

1.1 Background

A queue is formed when service requests arrive at a service facility and are forced to wait while the server is busy working on other requests. Mathematically, a queueing system consists of three components: the source of service requests, the queue and a server (or several servers). Our study of queueing systems is motivated by its use in the study of communication systems and computer networks. The various computers, routers and switches in such a network may be modeled as individual queues. The phenomenon of queueing also arises in operations research and industrial engineering as the facilities studied in these areas may also be modeled as either individual queues or queueing networks.

Queueing systems with vacations are characterized by the fact that the idle time of the server may be utilized for other secondary jobs. During this period of time the system is unavailable to further arrivals to the system. For example, in a digital system, the processor is multiplexed among a number of jobs, and hence is not available all the time to handle one job type. If we take any job type as a reference point, the processor is alternately busy handling that type of work and absent doing work elsewhere. Some of the different kinds of vacation models are as follows:

- Single vacation: The server takes exactly one vacation immediately when the system
becomes empty. If it finds an empty system upon returning from the vacation, it becomes idle until a customer arrives. The customer is served as soon as it arrives.

- Multiple vacation: The server takes a vacation each time the system becomes empty. If the server returns from a vacation to find a non-empty system, it starts service immediately and continues until the system becomes empty (exhaustive service). If the server returns from a vacation to find an empty system, it begins another vacation immediately, and continues until it finds one or more customers upon returning from a vacation.

- Single threshold: The server starts exhaustive service when it finds the system size above a particular threshold.

- Vacations with threshold: The server continues to take multiple vacations after the system is empty till it finds the system size above a particular threshold.

Queueing systems with vacations have found wide applications in the modeling and analysis of communication networks and various other engineering systems in which a single server attends to more than one workstations. Modeling such systems as single server queues with vacations allows one to analyze each workstation in relative isolation since the time the server attends to the other workstations can be modeled as a vacation.

1.2 Problem Definition

Consider a $G/M/1$ queue with finite capacity in which the service times are exponentially distributed with the service rates being $\mu_n$ when there are $n$ customers in the system ($n = 1, 2, \ldots, K$). The model described above is referred to as a $G/M/1/K$ queue with state-dependent services, or simply a $G/M(n)/1/K$ queue according to Kijima and Makimoto [19]. Similarly defined a $M(n)/G/1/K$ queue is a $M/G/1/K$ with state-dependent arrivals, where $\lambda_n$ is the arrival rate when there are $n$ customers in the system ($n = 0, 1, \ldots, K - 1$).

We use the supplementary variable technique to study the operating characteristics of a removable server in $G/M(n)/1/K$ and $M(n)/G/1/K$ queueing systems with exhaustive service and multiple vacations. The removable server follows an $N$ policy and takes a
sequence of vacations when the system is empty till the arrivals reach a pre-decided number. The $N$ policy means the server is turned on when $N$ or more customers are present, and off only when the system is empty. After the server is turned off, it does not operate until at least $N$ customers are present in the system.

The $G/M(n)/1/K$ queueing system has the following characteristics. The inter-arrival times follow a general probability distribution $A$. It is assumed that the system can hold up to $K$ customers including the one being served at any point of time. The service times follow exponential distribution with state-dependent rates, i.e., the service rate is $\mu_n$ when there are $n$ customers in the system. When the system is empty, the server takes a sequence of vacations till there are $N$ customers in the system. The vacation times follow exponential distribution with state-dependent rates.

The $M(n)/G/1/K$ queueing system has the following characteristics. The arrival process is Poisson with state-dependent rates, i.e., the arrival rate is $\lambda_n$ when there are $n$ customers in the system. It is assumed that the system can hold up to $K$ customers including the one at service at any point of time. The service times follow a general probability distribution $B$. When the system is empty, the server takes a sequence of vacations till there are $N$ customers in the system. The vacation times are assumed to follow an arbitrary distribution $V$.

1.3 Literature Review

When the input is a non-Poisson process, modeling and mathematical analysis is difficult. Chatterjee and Mukherjee [3] used the embedded Markov Chain to examine the $GI/M/1$ queueing system with exponential vacations. Tian, et al. [30] proposed a similar analysis for the ordinary $GI/M/1$ queueing system with exponential server vacations. Karasemen and Gupta [16] used the embedded Markov chain to study the ordinary $GI/M/1/K$ queueing system with server vacations. The provided heuristic methods to compute the blocking probability and provided an explicit solution for special arrival processes. Laxmi and Gupta [22] used both the embedded Markov chain and the supplementary variable technique to study the ordinary $G/M/1/K$ queueing system with batch service.
The supplementary variable technique was introduced by Cox [7], and has been widely applied to $M/G/1$ queueing systems by Keilson and Kooharian [17], Takacs [27], Cohen [5], Hokstad [15] and others. This approach can also be used to study the finite source model as applied to the machine repair problem. Based on this technique, Gupta and Rao [10, 11] provided a recursive method to develop the steady state probability distribution of the number of failed machines in the $M/G/1$ machine repair problem with no spares and the cold-standby $M/G/1$ machine repair problem, respectively.

The $N$ policy without vacations was initially studied by Yadin and Naor [35]. Heyman [13] was the first to study the $N$ policy $M/G/1$ queueing system. The extensions of this model can be referred to Bell [1, 2], Hersh and Brosh [12], Kimura [20], Tijms [31], Teghem [29], Gakis et al. [9] and others. Wang and Huang [32, 33] derived the explicit solutions for the $M/E_k/1$ queueing system with $N$ policy and performed a sensitivity analysis. Recently Wang and Ke [34] developed a recursive method to compute the steady-state probability distributions of the number of customers for the $N$ policy $M/G/1$ queueing system.

Queueing systems with server vacations were first studied by Levy and Yechiali [24]. By taking vacations, idle time can be utilized for other purposes by the server. A thorough analysis of queueing systems with vacations is found in Doshi [8] and Takagi [28]. $N$ policy queueing systems with multiple vacations were first studied by Lee and Srinivasan [23] and Kella [18]. They respectively dealt with the batch arrival $M/G/1$ and the single unit arrival $M/G/1$ queueing systems, examined system performances and obtained the optimal threshold under a stationary cost function.

Kijima and Makimoto [19] studied the stationary queue length distributions of the $M(n)/G/1/K$ and $GI/M(n)/1/K$ queues via the Neuts’ method (see Neuts [25]). They also obtained recursion formulas to compute the stationary queue length distribution when the state-dependent rates are all different. The coefficients in these recursion formulas are not difficult to compute. However, the recursion formulas have not been explicitly derived for cases where some of the state-dependent rates are equal because they are extremely complicated. The authors have mentioned that these can be obtained in principle by taking
the limits of the coefficients in the previous recursion formulas. Yang [36] developed an iterative algorithm for numerically computing the stationary queue length distributions of $M(n)/G/1/K$ and $GI/M(n)/1/K$ when some of the state-dependent rates are the same.

There have been several proposals in the literature for computing the stationary queue length distributions of $M/G/1/K$ queues which are special cases of $M(n)/G/1/K$ queues (Chaudhry, Gupta and Agarwal [4], among others). Most of the studies are based on the embedded Markov chain method. For $M(n)/G/1/K$ queues, due to state-dependent arrivals, it is difficult to obtain the one-step transition matrix in the embedded chain for efficient and practical computations (Courtis and Georges [6]). Other important special cases of $M(n)/G/1/K$ and $G/M(n)/1/K$ queues are machine interference models (Stecke and Aronson [26], Gupta and Rao [10, 11]) and $GI/M/C/N$ queues (Hokstad [14], Kleinrock [21]).

1.4 Organization of Thesis

In Chapter 2, we study the $G/M(n)/1/K$ queueing system where the server applies an $N$ policy and takes multiple exponential vacations when the system is empty. In Chapter 3, we study the above system where the server takes multiple state-dependent exponential vacations. In Chapter 4, we study the $M(n)/G/1/K$ queueing system where the server applies an $N$ policy and takes multiple arbitrary vacations when the system is empty.

Each chapter is organized as follows. The queueing model is described briefly following which the notations used are summarized. We first analyze the system using the method of supplementary variables and derive the corresponding system equations. We then develop a recursive algorithm for numerically computing the stationary queue length distribution of the system. We also illustrate how to calculate performance measures such as blocking probability, the expected waiting time, average queue length, etc. We then illustrate the algorithm by presenting a few simple examples.
Chapter 2

G/M(n)/1/K queue with exponential vacations

2.1 Model Description

In this system, we assume that the inter arrival times are independently and identically distributed (i.i.d.) with general distribution $A(x)(x \geq 0)$ and probability density function (p.d.f.) $a(x)(x \geq 0)$ and mean interarrival time $E(A)$. Arriving units at the server form a single waiting line and are served in the order of their arrivals i.e., the service discipline is FIFO. The service times for successive customers are independently and exponentially distributed with parameter $\mu_n$ where $n$ is the number of customers in the system. As soon as the system becomes empty the server is turned off and takes a vacation of random length with $\Pr[V \leq t] = 1 - e^{-\mu t}$. The server takes repeated vacations $V$ until there are at least $N(N \geq 1)$ customers to be served on return from a vacation. Once service is started, the server will continue service until the queue is exhausted.

We first derive the system equations for the $N$ policy $G/M(n)/1/K$ queues with multiple exponential vacations using the method of supplementary variables. We also illustrate how to calculate performance measures such as blocking probability, the expected
waiting time, average queue length, etc. Next we develop a recursive algorithm for numerically computing the stationary queue length distribution in the \( N \) policy \( G/M(n)/1/K \) queues with multiple exponential vacations. The complexity of the algorithm is \( O(n^3) \). Then we illustrate the algorithm by presenting a few simple examples.

**Notations and probabilities**

The two-dimensional state-space of the system is defined as
\[ \{(i,n) : (i = 0 \text{ and } n = 0,1,\ldots,K) \text{ or } (i = 1 \text{ and } n = 1,2,\ldots,K)\} \]

The following notations are used in this chapter

- \( N \) – threshold level
- \( K \) – system capacity \((N \leq K)\)
- \( A(x) \) – c.d.f. of interarrival time \( A \)
- \( a(x) \) – p.d.f. of interarrival time \( A \)
- \( \tilde{A}(s) \) – Laplace-Stieltjes transform of interarrival time \( A \)
- \( \tilde{A}^{(l)}(s) \) – \( l \)th order derivative of \( \tilde{A}(s) \) with respect to \( s \)
- \( P_{0,n}(t) \) – probability of \( n \) customers in the system at time \( t \) when the server is on vacations, where \( n = 0,1,\ldots,K \)
- \( P_{1,n}(t) \) – probability of \( n \) customers in the system at time \( t \) when the server is working, where \( n = 0,1,\ldots,K \)
- \( P_{0,n} \) – steady-state probability of \( n \) customers in the system when the server is on vacations, where \( n = 0,1,\ldots,K \)
- \( P_{1,n} \) – steady-state probability of \( n \) customers in the system when the server is working, where \( n = 1,2,\ldots,K \)
- \( \bar{P}_{i,n}(s) \) – Laplace-Stieltjes transform of \( P_{i,n}(x) \) where \( i = 0,1 \)
- \( \bar{P}_{i,n}^{(l)}(s) \) – \( l \)th order derivative of \( \bar{P}_{i,n}(s) \) with respect to \( s \)
- \( E(A) \) – mean interarrival time
- \( v \) – vacation rate of the server
- \( \mu_n \) – service rate of the system in state \( (1,n) \), where \( n = 1,2,\ldots,K \)
- \( P_{0,n}^- \) – probability of \( n \) customers in the system immediately prior to an arrival when the server is idle
- \( P_{1,n}^- \) – probability of \( n \) customers in the system immediately prior to an arrival when the server is busy
- \( P_B \) – probability an arriving customer is blocked because the system is full
- \( L \) – average number of customers in the system
- \( L_q \) – average number of customers in the queue
2.2 Formulation

We first establish the mathematical equations to govern the system by employing the remaining interarrival time as the supplementary variable. Next, we develop a recursive method to derive the steady-state probability distributions of the number of customers in the system.

The state of the system at time $t$ is given by $[Q(t), U(t)]$, where

- $Q(t)$ – number of customers in the system at time $t$
- $U(t)$ – remaining interarrival time for the customers who is arriving

Let us define

\[
P_{0,n}(x,t)dx = \text{Prob}\{Q(t) = n, x < U(t) \leq x + dx, \text{ server is on vacations}\},
\]
\[
x \geq 0, n = 0, 1, \ldots, K
\]

\[
P_{1,n}(x,t)dx = \text{Prob}\{Q(t) = n, x < U(t) \leq x + dx, \text{ server is busy}\},
\]
\[
x \geq 0, n = 1, 2, \ldots, K
\]

Relating the state of the system at time $t$ and $t + dt$, we set up the following partial differential equations

\[
- \frac{\partial P_{0,0}(x,t)}{\partial x} + \frac{\partial P_{0,0}(x,t)}{\partial t} = \mu_1 P_{1,1}(x,t) \tag{2.1}
\]
\[
- \frac{\partial P_{0,n}(x,t)}{\partial x} + \frac{\partial P_{0,n}(x,t)}{\partial t} = a(x)P_{0,n-1}(0,t), \quad 1 \leq n \leq N - 1 \tag{2.2}
\]
\[
- \frac{\partial P_{0,n}(x,t)}{\partial x} + \frac{\partial P_{0,n}(x,t)}{\partial t} = a(x)P_{0,n-1}(0,t) - vP_{0,n}(x,t), \quad N \leq n \leq K - 1 \tag{2.3}
\]
\[
- \frac{\partial P_{0,K}(x,t)}{\partial x} + \frac{\partial P_{0,K}(x,t)}{\partial t} = a(x)[P_{0,K-1}(0,t) + P_{0,K}(0,t)] - vP_{0,K}(x,t) \tag{2.4}
\]
\[
- \frac{\partial P_{1,1}(x,t)}{\partial x} + \frac{\partial P_{1,1}(x,t)}{\partial t} = \mu_2 P_{1,2}(x,t) - \mu_1 P_{1,1}(x,t) \tag{2.5}
\]
From (2.1)-(2.8), steady-state equations are obtained as follows:

\[-\frac{\partial P_{1,n}(x,t)}{\partial x} + \frac{\partial P_{1,n}(x,t)}{\partial t} = a(x)P_{1,n-1}(0,t) + \frac{\mu_{n+1}P_{1,n+1}(x,t)}{\mu_nP_{1,n}(x,t)}, \quad 2 \leq n \leq N - 1 \]  

(2.6)

\[-\frac{\partial P_{1,n}(x,t)}{\partial x} + \frac{\partial P_{1,n}(x,t)}{\partial t} = vP_{0,n}(x,t) + a(x)P_{1,n-1}(0,t) + \frac{\mu_{n+1}P_{1,n+1}(x,t)}{-\mu_nP_{1,n}(x,t)}, \quad N \leq n \leq K - 1 \]  

(2.7)

\[-\frac{\partial P_{1,K}(x,t)}{\partial x} + \frac{\partial P_{1,K}(x,t)}{\partial t} = vP_{0,K}(x,t) + a(x)[P_{1,K-1}(0,t) + P_{1,K}(0,t)] - \frac{\mu_KP_{1,K}(x,t)}{-\mu_KP_{1,K}(x,t)} \]  

(2.8)

In steady state, we define,

\[ P_{0,n} = \lim_{t \to \infty} P_{0,n}(t), \quad n = 0, 1, \ldots, N, \ldots, K \]

\[ P_{0,n}(x) = \lim_{t \to \infty} P_{0,n}(x,t), \quad n = 0, 1, \ldots, N, \ldots, K \]

\[ P_{1,n} = \lim_{t \to \infty} P_{1,n}(t), \quad n = 1, 2, \ldots, N, \ldots, K \]

\[ P_{1,n}(x) = \lim_{t \to \infty} P_{1,n}(x,t), \quad n = 1, 2, \ldots, N, \ldots, K \]

From (2.1)-(2.8), steady-state equations are obtained as follows:

\[-\frac{dP_{0,0}(x)}{dx} = \mu_1P_{1,1}(x) \]  

(2.9)

\[-\frac{dP_{0,n}(x)}{dx} = a(x)P_{0,n-1}(x), \quad 1 \leq n \leq N - 1 \]  

(2.10)

\[-\frac{dP_{0,n}(x)}{dx} = a(x)P_{0,n-1}(x) - vP_{0,n}(x), \quad N \leq n \leq K - 1 \]  

(2.11)

\[-\frac{dP_{0,K}(x)}{dx} = a(x)[P_{0,K-1}(0) + P_{0,K}(0)] - vP_{0,K}(x) \]  

(2.12)

\[-\frac{dP_{1,1}(x)}{dx} = \mu_2P_{1,2}(x) - \mu_1P_{1,1}(x) \]  

(2.13)

\[-\frac{dP_{1,n}(x)}{dx} = a(x)P_{1,n-1}(x) + \frac{\mu_{n+1}P_{1,n+1}(x)}{-\mu_nP_{1,n}(x)}, \quad 2 \leq n \leq N - 1 \]  

(2.14)

\[-\frac{dP_{1,n}(x)}{dx} = vP_{0,n}(x) + a(x)P_{1,n-1}(x) + \frac{\mu_{n+1}P_{1,n+1}(x)}{-\mu_nP_{1,n}(x)}, \quad N \leq n \leq K - 1 \]  

(2.15)

\[-\frac{dP_{1,K}(x)}{dx} = vP_{0,K}(x) + a(x)[P_{1,K-1}(0) + P_{1,K}(0)] - \frac{\mu_KP_{1,K}(x)}{-\mu_KP_{1,K}(x)} \]  

(2.16)
2.3 Analysis

We define the following Laplace-Stieltjes transforms,

\[
\begin{align*}
\tilde{A}(s) &= \int_0^\infty e^{-sx} a(x) \, dx \\
\tilde{P}_{i,n}(s) &= \int_0^\infty e^{-sx} P_{i,n}(x) \, dx, \quad i = 0, 1 \\
\tilde{P}_{i,n}(0) &= \int_0^\infty P_{i,n}(x) \, dx, \quad i = 0, 1
\end{align*}
\]

\[
\int_0^\infty e^{-sx} \frac{\partial}{\partial x} P_{i,n}(x) \, dx = s\tilde{P}_{i,n}(s) - \tilde{P}_{i,n}(0)
\]

Taking Laplace-Stieltjes transforms on both sides of (2.9)-(2.16), we obtain the following

\[
\begin{align*}
-s\tilde{P}_{0,0}(s) &= \mu_1 \tilde{P}_{1,1}(s) - \tilde{P}_{0,0}(0) \quad (2.17) \\
-s\tilde{P}_{0,n}(s) &= \tilde{A}(s)P_{0,n-1}(0) - \tilde{P}_{0,n}(0), \quad 1 \leq n \leq N - 1 \quad (2.18) \\
(v-s)\tilde{P}_{0,n}(s) &= \tilde{A}(s)P_{0,n-1}(0) - \tilde{P}_{0,n}(0), \quad N \leq n \leq K - 1 \quad (2.19) \\
(v-s)\tilde{P}_{0,K}(s) &= \tilde{A}(s)[P_{0,K-1}(0) + P_{0,K}(0)] - \tilde{P}_{0,K}(0) \quad (2.20)
\end{align*}
\]

\[
\begin{align*}
(\mu_1 - s)\tilde{P}_{1,1}(s) &= \mu_2 \tilde{P}_{1,2}(s) - P_{1,1}(0) \quad (2.21) \\
(\mu_n - s)\tilde{P}_{1,n}(s) &= \mu_{n+1} \tilde{P}_{1,n+1}(s) + \tilde{A}(s)P_{1,n-1}(0) - \tilde{P}_{1,n}(0), \quad 1 \leq n \leq N - 1 \quad (2.22) \\
(\mu_n - s)\tilde{P}_{1,n}(s) &= v\tilde{P}_{0,n}(s) + \mu_{n+1} \tilde{P}_{1,n+1}(s) + \tilde{A}(s)P_{1,n-1}(0) - \tilde{P}_{1,n}(0), \quad N \leq n \leq K - 1 \quad (2.23) \\
(\mu_K - s)\tilde{P}_{1,K}(s) &= v\tilde{P}_{0,K}(s) + \tilde{A}(s)[P_{1,K-1}(0) + P_{1,K}(0)] - \tilde{P}_{1,K}(0) \quad (2.24)
\end{align*}
\]

Substituting \(s=0\) in (2.17), we get

\[
P_{0,0}(0) = \mu_1 \tilde{P}_{1,1}(0) = \mu_1 P_{1,1}
\]

Substituting \(s=0\) in (2.18), we get

\[
P_{0,n}(0) = P_{0,n-1}(0), \quad 1 \leq n \leq N - 1 \quad (2.25)
\]
Differentiating (2.18) w.r.t $s$,

$$
\tilde{P}_{0,n}(0) = -\tilde{A}^{(1)}(0)P_{0,n-1}(0), \quad 1 \leq n \leq N - 1
$$

$$
\Rightarrow P_{0,n} = E(A)P_{0,n-1}(0), \quad 1 \leq n \leq N - 1 \quad (2.26)
$$

Substituting $s = v$ in (2.19) and (2.20),

$$
P_{0,n}(0) = \tilde{A}(v)P_{0,n-1}(0), \quad N \leq n \leq K - 1
$$

$$
P_{0,N}(0) = \tilde{A}(v)P_{0,N-1}(0)
= \tilde{A}(v)P_{0,0}(0)
\Rightarrow P_{0,n}(0) = [\tilde{A}(v)]^{n-N+1}P_{0,0}(0), \quad N \leq n \leq K - 1 \quad (2.27)
$$

$$
(1 - \tilde{A}(v))P_{0,K}(0) = [\tilde{A}(v)]^{K-N+1}P_{0,0}(0)
$$

$$
P_{0,K}(0) = \frac{[\tilde{A}(v)]^{K-N+1}P_{0,0}(0)}{1 - \tilde{A}(v)} \quad (2.28)
$$

Substituting $s = 0$ in (2.19) and (2.20),

$$
v\tilde{P}_{0,n}(0) = \tilde{A}(0)P_{0,n-1}(0) - P_{0,n}(0), \quad N \leq n \leq K - 1
$$

$$
\Rightarrow \tilde{P}_{0,n}(0) = \frac{P_{0,n-1}(0) - P_{0,n}(0)}{v}, \quad N \leq n \leq K - 1
\Rightarrow P_{0,n} = \frac{1}{v}[1 - \tilde{A}(v)][\tilde{A}(v)]^{n-N}P_{0,0}(0), \quad N \leq n \leq K - 1 \quad (2.29)
$$

$$
v\tilde{P}_{0,K}(0) = P_{0,K-1}(0)
\Rightarrow P_{0,K} = \frac{\tilde{A}(v)^{K-N}P_{0,0}(0)}{v} \quad (2.30)
$$

Adding equations (2.17) to (2.24) and simplifying we obtain,

$$
\sum_{n=0}^{K} \tilde{P}_{0,n}(s) + \sum_{n=1}^{K} \tilde{P}_{1,n}(s) = \frac{1 - \tilde{A}(s)}{s} \left[ \sum_{n=0}^{K} P_{0,n}(0) + \sum_{n=1}^{K} P_{1,n}(0) \right]
$$

Taking $\lim_{s \to 0}$ and using L’Hospital’s rule, we get

$$
\sum_{n=0}^{K} \tilde{P}_{0,n}(0) + \sum_{n=1}^{K} \tilde{P}_{1,n}(0) = E(A) \left[ \sum_{n=0}^{K} P_{0,n}(0) + \sum_{n=1}^{K} P_{1,n}(0) \right]
$$
\[
\sum_{n=0}^{K} P_{0,n} + \sum_{n=1}^{K} P_{1,n} = E(A) \left[ \sum_{n=0}^{K} P_{0,n}(0) + \sum_{n=1}^{K} P_{1,n}(0) \right] \quad (2.31)
\]

\[
\sum_{n=0}^{K} P_{0,n}(0) + \sum_{n=1}^{K} P_{1,n}(0) = \frac{1}{E(A)} \quad (2.32)
\]

Setting \( s = \mu_n \) in (2.22),(2.23) and (2.24) we get,

\[
P_{1,n}(0) = \mu_{n+1} \tilde{P}_{1,n+1}(\mu_n) + \tilde{A}(\mu_n) P_{1,n-1}(0)
\]

\[
P_{1,n}(0) = v \tilde{P}_{0,n}(\mu_n) + \mu_{n+1} \tilde{P}_{1,n+1}(\mu_n) + \tilde{A}(\mu_n) P_{1,n-1}(0)
\]

\[
P_{1,K}(0)[1 - \tilde{A}(\mu_K)] = v \tilde{P}_{0,K}(\mu_K) + \tilde{A}(\mu_K) P_{1,K-1}(0)
\]

\[
\Rightarrow P_{1,K-1}(0) = \frac{1 - \tilde{A}(\mu_K)}{\tilde{A}(\mu_K)} P_{1,K}(0) - \frac{v \tilde{P}_{0,K}(\mu_K)}{\tilde{A}(\mu_K)}
\]

\[
P_{1,n-1}(0) = \frac{P_{1,n}(0)}{\tilde{A}(\mu_n)} - \frac{\mu_{n+1} \tilde{P}_{1,n+1}(\mu_n)}{\tilde{A}(\mu_n)} - \frac{v \tilde{P}_{0,n}(\mu_n)}{\tilde{A}(\mu_n)}
\]

\[
P_{1,n-1}(0) = \frac{P_{1,n}(0)}{\tilde{A}(\mu_n)} - \frac{\mu_{n+1} \tilde{P}_{1,n+1}(\mu_n)}{\tilde{A}(\mu_n)}
\]

where, \( \tilde{P}_{0,n}(\mu_n) \) are given by the following,

For \( w \neq v \)

\[
\tilde{P}_{0,K}(w) = \frac{1}{1 - \frac{\tilde{A}(w) - \tilde{A}(v)}{v - w}} \left[ \sum_{i=N}^{K-1} \tilde{A}(v) \right] P_{0,n}(0)
\]

\[
\tilde{P}_{0,n}(w) = \left[ \frac{\tilde{A}(w) - \tilde{A}(v)}{v - w} \right] \left[ \sum_{i=N}^{n-1} \tilde{A}(v) \right] P_{0,0}(0) \quad N \leq n \leq K - 1
\]

For \( w = v \)

\[
\tilde{P}_{0,K}(w) = -\tilde{A}(w)[P_{0,K-1}(0) + P_{0,K}(0)]
\]

\[
\tilde{P}_{0,n}(w) = -\tilde{A}(w)P_{0,n-1}(0) \quad N \leq n \leq K - 1
\]

and, \( \tilde{P}_{1,n}(\mu_{n-1}) \) are given by the following,

For \( w \neq \mu_K \)

\[
\tilde{P}_{1,K}(w) = \frac{v \tilde{P}_{0,K}(w) + \tilde{A}(w)[P_{1,K-1}(0) + P_{1,K}(0)] - P_{1,K}(0)}{\mu_K - w}
\]

(4.40)
For $w = \mu_K$
\[
\tilde{P}_{1,K}(w) = -\left(v\tilde{P}_{0,K}^{(1)}(w) + \tilde{A}^{(1)}(w)[P_{1,K-1}(0) + P_{1,K}(0)]\right)
\] (2.41)

For $w \neq \mu_n$
\[
\tilde{P}_{1,n}(w) = \frac{v\tilde{P}_{0,n}(w) + \mu_{n+1}\tilde{P}_{1,n+1}(w) + \tilde{A}(w)P_{1,n-1}(0) - P_{1,n}(0)}{\mu_n - w}
\] $N \leq n \leq K - 1$ (2.42)
\[
\tilde{P}_{1,n}(w) = \frac{\mu_{n+1}\tilde{P}_{1,n+1}(w) + \tilde{A}(w)P_{1,n-1}(0) - P_{1,n}(0)}{\mu_n - w}
\] $2 \leq n \leq N - 1$ (2.43)

For $w = \mu_n$
\[
\tilde{P}_{1,n}(w) = -\left(v\tilde{P}_{0,n}^{(1)}(w) + \mu_{n+1}\tilde{P}_{1,n+1}^{(1)}(w) + \tilde{A}^{(1)}(w)P_{1,n-1}(0)\right)
\] $N \leq n \leq K - 1$ (2.44)
\[
\tilde{P}_{1,n}(w) = -\left(\mu_{n+1}\tilde{P}_{1,n+1}^{(1)}(w) + \tilde{A}^{(1)}(w)P_{1,n-1}(0)\right)
\] $2 \leq n \leq N - 1$ (2.45)

Differentiating (2.19) and (2.20) $l$ times with respect to $s$, and setting $s = w$ we get the following expressions for $\tilde{P}_{0,n}^{(l)}(w)$

For $w \neq v$ and $n = K$
\[
\tilde{P}_{0,K}^{(1)}(w) = \frac{A^{(1)}(w)[P_{0,K-1}(0) + P_{0,K}(0)] + \tilde{P}_{0,K}(w)}{v - w}
\] (2.46)
\[
\tilde{P}_{0,K}^{(l)}(w) = \frac{A^{(l)}(w)[P_{0,K-1}(0) + P_{0,K}(0)] + l\tilde{P}_{0,K}^{(l-1)}(w)}{v - w}
\] $l \neq 1$ (2.47)

For $w \neq v$ and $N \leq n \leq K - 1$
\[
\tilde{P}_{0,n}^{(1)}(w) = \frac{A^{(1)}(w)P_{0,n-1}(0) + \tilde{P}_{0,n}(w)}{v - w}
\] (2.48)
\[
\tilde{P}_{0,n}^{(l)}(w) = \frac{A^{(l)}(w)P_{0,n-1}(0) + l\tilde{P}_{0,n}^{(l-1)}(w)}{v - w}
\] $l \neq 1$ (2.49)

For $w = v$
\[
\tilde{P}_{0,K}^{(l)}(w) = -\frac{A^{(l+1)}(w)[P_{0,K-1}(0) + P_{0,K}(0)]}{l + 1}
\] (2.50)
\[
\tilde{P}_{0,n}^{(l)}(w) = -\frac{A^{(l+1)}(w)P_{0,n-1}(0)}{l + 1}
\] $N \leq n \leq K - 1$ (2.51)
Differentiating (2.22), (2.23) and (2.24) \( l \) times with respect to \( s \), and setting \( s = w \) we get the following expressions for \( \tilde{P}^{(l)}_{1,n}(w) \)

For \( w \neq \mu_K \)

\[
\begin{align*}
\tilde{P}^{(1)}_{1,K}(w) &= \frac{v\tilde{P}^{(1)}_{0,K}(w) + \tilde{A}^{(l)}(w)[P_{1,K-1}(0) + P_{1,K}(0)]}{\mu_K - w} + \tilde{P}_{1,K}(w) \\
\tilde{P}^{(l)}_{1,K}(w) &= \frac{v\tilde{P}^{(l)}_{0,K}(w) + \tilde{A}^{(l)}(w)[P_{1,K-1}(0) + P_{1,K}(0)] + l\tilde{P}^{(l-1)}_{1,K}(w)}{\mu_K - w}, \ l \neq 1 
\end{align*}
\] (2.52)

For \( w = \mu_K \)

\[
\tilde{P}^{(l)}_{1,K}(w) = -\frac{v\tilde{P}^{(l+1)}_{0,K}(w) + \tilde{A}^{(l+1)}(w)[P_{1,K-1}(0) + P_{1,K}(0)]}{l+1} 
\] (2.54)

For \( w \neq \mu_n \) and \( N \leq n \leq K - 1 \)

\[
\begin{align*}
\tilde{P}^{(1)}_{1,n}(w) &= \frac{v\tilde{P}^{(1)}_{0,n}(w) + \tilde{A}^{(1)}(w)P_{1,n-1}(0) + \mu_{n+1}\tilde{P}^{(1)}_{1,n+1}(w) + \tilde{P}_{1,n}(w)}{\mu_n - w} \\
\tilde{P}^{(l)}_{1,n}(w) &= \frac{v\tilde{P}^{(l)}_{0,n}(w) + \tilde{A}^{(l)}(w)P_{1,n-1}(0) + \mu_{n+1}\tilde{P}^{(l)}_{1,n+1}(w) + l\tilde{P}^{(l-1)}_{1,n}(w)}{\mu_n - w}, \ l \neq 1 
\end{align*}
\] (2.55)

For \( w \neq \mu_n \) and \( 2 \leq n \leq N - 1 \)

\[
\begin{align*}
\tilde{P}^{(1)}_{1,n}(w) &= \frac{\tilde{A}^{(1)}(w)P_{1,n-1}(0) + \mu_{n+1}\tilde{P}^{(1)}_{1,n+1}(w) + \tilde{P}_{1,n}(w)}{\mu_n - w} \\
\tilde{P}^{(l)}_{1,n}(w) &= \frac{\tilde{A}^{(l)}(w)P_{1,n-1}(0) + \mu_{n+1}\tilde{P}^{(l)}_{1,n+1}(w) + l\tilde{P}^{(l-1)}_{1,n}(w)}{\mu_n - w}, \ l \neq 1 
\end{align*}
\] (2.57)

For \( w = \mu_n \)

\[
\begin{align*}
\tilde{P}^{(l)}_{1,n}(w) &= -\frac{v\tilde{P}^{(l+1)}_{0,n}(w) + \tilde{A}^{(l+1)}(w)P_{1,n-1}(0) + \mu_{n+1}\tilde{P}^{(l+1)}_{1,n+1}(w)}{l+1} \quad N \leq n \leq K - 1 \\
\tilde{P}^{(l)}_{1,n}(w) &= -\frac{\tilde{A}^{(l+1)}(w)P_{1,n-1}(0) + \mu_{n+1}\tilde{P}^{(l+1)}_{1,n+1}(w)}{l+1}, \ 2 \leq n \leq N - 1 
\end{align*}
\] (2.59)
Setting $s = \mu_1$ in (2.21) we get,

$$P_{1,1}(0) = \mu_2 \bar{P}_{1,2}(\mu_1) \quad (2.61)$$

Substituting $s = 0$ in (2.17), (2.22) and (2.23)

$$P_{1,n} = \frac{P_{1,n-1}(0) + P_{0,n-1}(0)}{\mu_n} \quad 2 \leq n \leq K \quad (2.62)$$

Let $P^{-}_{0,n}$ and $P^{-}_{1,n}$ denote the probabilities of $n$ customers in the queue immediately prior to an arrival when the server is idle and busy, respectively. We observe that,

$$P^{-}_{i,n} = \frac{P_{i,n}(0)}{\sum_{n=0}^{K} P_{0,n}(0) + \sum_{n=1}^{K} P_{1,n}(0)}, \quad i = 0, 1$$

Using the result from (2.32) we get,

$$P^{-}_{0,n} = E(A)P_{0,n}(0), \quad 0 \leq n \leq K \quad (2.63)$$

$$P^{-}_{1,n} = E(A)P_{1,n}(0), \quad 1 \leq n \leq K \quad (2.64)$$

An important performance measure for a finite queueing system is the blocking probability. The blocking probability is defined as the probability that an arriving customer is blocked because the system is full. The blocking probability $P_B$ can be determined as follows

$$P_B = P^{-}_{0,K} + P^{-}_{1,K}$$

$$\Rightarrow P_B = E(A)(P_{0,K}(0) + P_{1,K}(0)) \quad (2.65)$$

Let $\lambda'$ denote the effective arrival rate. Since $1 - P_B$ is the probability that an arriving customer is accepted, we get

$$\lambda' = \frac{1 - P_B}{E(A)} \quad (2.66)$$

The expected number of customers in the system $L$ and expected number of customers in
queue $L_q$ at an arbitrary time is given by

$$L = \sum_{n=0}^{K} n P_{0,n} + \sum_{n=1}^{K} n P_{1,n}$$

$$L_q = \sum_{n=0}^{K} n P_{0,n} + \sum_{n=1}^{K} (n-1) P_{1,n}$$

Using Little’s law, the expected waiting time in the system $W$ and expected waiting time in the queue $W_q$ are obtained as

$$W = \frac{L}{\lambda'}$$

$$W_q = \frac{L_q}{\lambda'}$$

### 2.4 Algorithm

Algorithm for calculating $P_{1,n}(0), P_{0,n}(0), P_{1,n}, P_{0,n}$ is as follows

Step 1: For $n = 0, 1, \ldots, K$, calculate $P_{0,n}(0)$ in terms of $P_{0,0}(0)$ as follows

$$P_{0,0}(0) = 1 \quad 0 \leq n \leq N - 1$$

$$P_{0,n}(0) = \widetilde{A}(v) * P_{0,n-1}(0) \quad N \leq n \leq K - 1$$

$$P_{0,K}(0) = \widetilde{A}(v) * P_{0,K-1}(0)/(1 - \widetilde{A}(v))$$

Step 2: For $n = 1, 2, \ldots, K$, calculate $P_{0,n}$ in terms of $P_{0,0}(0)$ as follows

$$P_{0,n} = E(A) \quad 1 \leq n \leq N - 1$$

$$P_{0,n}(0) = (P_{0,n-1}(0) - P_{0,n}(0))/v \quad N \leq n \leq K - 1$$

$$P_{0,K}(0) = P_{0,K-1}(0)/v$$

Step 3: For $n = N, N+1, \ldots, K$, calculate $\widetilde{P}_{0,n}(\mu_j)$ for all $j = 1, 2, \ldots, n$ in terms of $P_{0,0}(0)$ as follows
If \( v = \mu_j \)

\[
\tilde{P}_{0,K}(\mu_j) = -\tilde{A}^{(1)}(\mu_j) \cdot (P_{0,K-1}(0) + P_{0,K}(0)) \\
\tilde{P}_{0,n}(\mu_j) = -\tilde{A}^{(1)}(\mu_j) \cdot P_{0,n-1}(0), \quad N \leq n \leq K - 1
\]

Else

\[
\tilde{P}_{0,K}(\mu_j) = (\tilde{A}(\mu_j) \cdot (P_{0,K-1}(0) + P_{0,K}(0)) - P_{0,K}(0))/(v - \mu_j) \\
\tilde{P}_{0,n}(\mu_j) = (\tilde{A}(\mu_j) \cdot P_{0,n-1}(0) - P_{0,n}(0))/(v - \mu_j), \quad N \leq n \leq K - 1
\]

End If

Step 4: For \( n = N, N + 1, \ldots, K \), calculate \( \tilde{P}^{(l)}_{0,n}(\mu_j) \) for all \( j = 1, 2, \ldots, n \) and \( l = 1, 2, \ldots, n-1 \) in terms of \( P_{0,0}(0) \) as follows

If \( v = \mu_j \)

\[
\tilde{P}^{(l)}_{0,K}(\mu_j) = -\frac{\tilde{A}^{(l+1)}(\mu_j) \cdot (P_{0,K-1}(0) + P_{0,K}(0))}{l + 1} \\
\tilde{P}^{(l)}_{0,n}(\mu_j) = -\frac{\tilde{A}^{(l+1)}(\mu_j) \cdot P_{0,n-1}(0)}{l + 1}, \quad N \leq n \leq K - 1
\]

Else If \( l = 1 \)

\[
\tilde{P}^{(1)}_{0,K}(\mu_j) = \frac{\tilde{A}^{(1)}(\mu_j) \cdot (P_{0,K-1}(0) + P_{0,K}(0)) + \tilde{P}_{0,K}(\mu_j)}{v - \mu_j} \\
\tilde{P}^{(1)}_{0,n}(\mu_j) = \frac{\tilde{A}^{(1)}(\mu_j) \cdot P_{0,n-1}(0) + \tilde{P}_{0,n}(\mu_j)}{v - \mu_j}, \quad N \leq n \leq K - 1
\]

Else

\[
\tilde{P}^{(l)}_{0,K}(\mu_j) = \frac{\tilde{A}^{(l)}(\mu_j) \cdot (P_{0,K-1}(0) + P_{0,K}(0)) + l \cdot \tilde{P}^{(l-1)}_{0,K}(\mu_j)}{v - \mu_j} \\
\tilde{P}^{(l)}_{0,n}(\mu_j) = \frac{\tilde{A}^{(l)}(\mu_j) \cdot P_{0,n-1}(0) + l \cdot \tilde{P}^{(l-1)}_{0,n}(\mu_j)}{v - \mu_j}, \quad N \leq n \leq K - 1
\]

End If

Step 5: For \( n = K, K - 1, \ldots, 1 \), let us express \( P_{1,0}(0) \), \( \tilde{P}_{1,n}(\mu_j) \) and \( \tilde{P}^{(l)}_{1,n}(\mu_j) \) in terms
of $P_{0,0}(0)$ and $P_{1,K}(0)$ as follows

\[
\begin{align*}
P_{1,n}(0) &= c_n P_{0,0}(0) + d_n P_{1,K}(0) \\
\tilde{P}_{1,n} \left( \mu_j \right) &= e_{n,j} P_{0,0}(0) + f_{n,j} P_{1,K}(0) \\
\tilde{P}_{1,n}^{(l)} \left( \mu_j \right) &= e_{n,j}^{(l)} P_{0,0}(0) + f_{n,j}^{(l)} P_{1,K}(0)
\end{align*}
\]

(i) Calculate $c_n$ and $d_n$ as follows

If $n = K$

\[
\begin{align*}
c_K &= 0 \\
d_K &= 1 \\
c_{K-1} &= -\frac{v \tilde{P}_0, K(\mu_K)}{A(\mu_K)} \\
d_{K-1} &= 1 - \frac{\tilde{A}(\mu_K)}{A(\mu_K)}
\end{align*}
\]

Else If $N \leq n \leq K - 1$

\[
\begin{align*}
c_{n-1} &= \frac{c_n - \mu_{n+1} e_{n+1,n} - v \tilde{P}_{0,n}(\mu_n)}{A(\mu_n)} \\
d_{n-1} &= \frac{d_n - \mu_{n+1} f_{n+1,n}}{A(\mu_n)}
\end{align*}
\]

Else $2 \leq n \leq N - 1$

\[
\begin{align*}
c_{n-1} &= \frac{c_n - \mu_{n+1} e_{n+1,n}}{A(\mu_n)} \\
d_{n-1} &= \frac{d_n - \mu_{n+1} f_{n+1,n}}{A(\mu_n)}
\end{align*}
\]

End If

(ii) For $j = n - 1, n - 2, \ldots, 1$,

(a) Calculate $e_{n,j}$ and $f_{n,j}$ as follows

If $n = K$
If $\mu_K = \mu_j$

$$e_{K,j} = -vP_{0,K}(\mu_j) - \bar{A}^{(1)}(\mu_j)[c_{K-1} + c_K]$$
$$f_{K,j} = -\bar{A}^{(1)}(\mu_j)[d_{K-1} + d_K]$$

Else

$$e_{K,j} = \frac{vP_{0,K}(\mu_j) + \bar{A}(\mu_j)[c_{K-1} + c_K] - c_K}{\mu_K - \mu_j}$$
$$f_{K,j} = \frac{\bar{A}(\mu_j)[d_{K-1} + d_K] - d_K}{\mu_K - \mu_j}$$

End If

Else If $N \leq n \leq K - 1$

If $\mu_n = \mu_j$

$$e_{n,j} = -(vP_{n,n}(\mu_j) + \mu_{n+1}e_{n+1,j}^{(1)} + \bar{A}^{(1)}(\mu_j)c_{n-1})$$
$$f_{n,j} = -(\mu_{n+1}f_{n+1,j}^{(1)} + \bar{A}^{(1)}(\mu_j)d_{n-1})$$

Else

$$e_{n,j} = \frac{v* P_{n,n}(\mu_j) + \mu_{n+1}e_{n+1,j} + \bar{A}(\mu_j)c_{n-1} - c_n}{\mu_n - \mu_j}$$
$$f_{n,j} = \frac{\mu_{n+1}f_{n+1,j} + \bar{A}(\mu_j)d_{n-1} - d_n}{\mu_n - \mu_j}$$

End If

End If

Else If $2 \leq n \leq N - 1$

If $\mu_n = \mu_j$

$$e_{n,j} = -(\mu_{n+1}e_{n+1,j}^{(1)} + \bar{A}^{(1)}(\mu_j)c_{n-1})$$
$$f_{n,j} = -(\mu_{n+1}f_{n+1,j}^{(1)} + \bar{A}^{(1)}(\mu_j)d_{n-1})$$

Else

$$e_{n,j} = \frac{\mu_{n+1}e_{n+1,j} + \bar{A}(\mu_j)c_{n-1} - c_n}{\mu_n - \mu_j}$$

End If
\[ f_{n,j} = \frac{\mu_{n+1}f_{n+1,j} + \tilde{A}(\mu_j)d_{n-1} - d_n}{\mu_n - \mu_j} \]

End If

End If

(b) Calculate \( e_{n,j}^{(l)} \) and \( f_{n,j}^{(l)} \) for \( l = 1, 2, \ldots, n - 2 \) as follows

If \( n = K \)

If \( \mu_K = \mu_j \)

\[ e_{K,j}^{(l)} = -\frac{v\hat{D}_{0,K}^{(l+1)}(\mu_j) + \tilde{A}^{(l+1)}(\mu_j)[c_{K-1} + c_K]}{l + 1} \]

\[ f_{K,j}^{(l)} = -\frac{\tilde{A}^{(l+1)}(\mu_j)[d_{K-1} + d_K]}{l + 1} \]

Else

\[ e_{K,j}^{(1)} = \frac{v\hat{D}_{0,K}^{(1)}(\mu_j) + \tilde{A}^{(1)}(\mu_j)[c_{K-1} + c_K] + e_{K,j}}{\mu_K - \mu_j} \]

\[ f_{K,j}^{(1)} = \frac{\tilde{A}^{(1)}(\mu_j)[d_{K-1} + d_K] + f_{K,j}}{\mu_K - \mu_j} \]

\[ e_{K,j}^{(l)} = \frac{\tilde{A}^{(l)}(\mu_j)[c_{K-1} + c_K] + l * e_{K,j}^{(l-1)} + v\hat{D}_{0,K}^{(l)}(\mu_j)}{\mu_K - \mu_j}, \quad l \neq 1 \]

\[ f_{K,j}^{(l)} = \frac{\tilde{A}^{(l)}(\mu_j)[d_{K-1} + d_K] + l * f_{K,j}^{(l-1)}}{\mu_K - \mu_j}, \quad l \neq 1 \]

End If

Else If \( N \leq n \leq K - 1 \)

If \( \mu_n = \mu_j \)

\[ e_{n,j}^{(l)} = -\frac{v\hat{D}_{0,n}^{(l+1)}(\mu_j) + \tilde{A}^{(l+1)}(\mu_j)c_{n-1} + \mu_{n+1}e_{n+1,j}^{(l+1)}}{l + 1} \]

\[ f_{n,j}^{(l)} = -\frac{\tilde{A}^{(l+1)}(\mu_j)d_{n-1} + \mu_{n+1}f_{n+1,j}^{(l+1)}}{l + 1} \]
Else

\[ e_{n,j}^{(1)} = \frac{v \tilde{P}_{0,n}^{(1)}(\mu_j) + \tilde{A}^{(1)}(\mu_j)c_{n-1} + \mu_{n+1} e_{n+1,j}^{(1)} + e_{n,j}}{\mu_n - \mu_j} \]

\[ f_{n,j}^{(1)} = \frac{\tilde{A}^{(1)}(\mu_j)d_{n-1} + \mu_{n+1} f_{n+1,j}^{(1)} + f_{n,j}}{\mu_n - \mu_j} \]

\[ e_{n,j}^{(l)} = \frac{\tilde{A}^{(l)}(\mu_j)c_{n-1} + \mu_{n+1} e_{n+1,j}^{(l)} + l * e_{n,j}^{(l-1)} + v \tilde{P}_{0,n}^{(l)}(\mu_j)}{\mu_n - \mu_j}, \quad l \neq 1 \]

\[ f_{n,j}^{(l)} = \frac{\tilde{A}^{(l)}(\mu_j)d_{n-1} + \mu_{n+1} f_{n+1,j}^{(l)} + l * f_{n,j}^{(l-1)}}{\mu_n - \mu_j}, \quad l \neq 1 \]

End If

Else If 2 ≤ n ≤ N − 1

If \( \mu_n = \mu_j \)

\[ e_{n,j}^{(l)} = \frac{-\tilde{A}^{(l+1)}(\mu_j)c_{n-1} + \mu_{n+1} e_{n+1,j}^{(l+1)}}{l + 1} \]

\[ f_{n,j}^{(l)} = \frac{-\tilde{A}^{(l+1)}(\mu_j)d_{n-1} + \mu_{n+1} f_{n+1,j}^{(l+1)}}{l + 1} \]

Else

\[ e_{n,j}^{(1)} = \frac{\tilde{A}^{(1)}(\mu_j)c_{n-1} + \mu_{n+1} e_{n+1,j}^{(1)} + e_{n,j}}{\mu_n - \mu_j} \]

\[ f_{n,j}^{(1)} = \frac{\tilde{A}^{(1)}(\mu_j)d_{n-1} + \mu_{n+1} f_{n+1,j}^{(1)} + f_{n,j}}{\mu_n - \mu_j} \]

\[ e_{n,j}^{(l)} = \frac{\tilde{A}^{(l)}(\mu_j)c_{n-1} + \mu_{n+1} e_{n+1,j}^{(l)} + l * e_{n,j}^{(l-1)}}{\mu_n - \mu_j}, \quad l \neq 1 \]

\[ f_{n,j}^{(l)} = \frac{\tilde{A}^{(l)}(\mu_j)d_{n-1} + \mu_{n+1} f_{n+1,j}^{(l)} + l * f_{n,j}^{(l-1)}}{\mu_n - \mu_j}, \quad l \neq 1 \]

End If

End If

Step 6: Calculate \( P'_{1,1}(0) \) in terms of \( P_{0,0}(0) \) and \( P_{1,K}(0) \) as follows

\[ c'_1 = \mu_2 e_{2,1} \]

\[ d'_1 = \mu_2 f_{2,1} \]
Step 7: Equating $P_{1,1}(0)$ from Step 4 and $P'_{1,1}(0)$ from Step 6, obtain $P_{1,K}(0)$ in terms of $P_{0,0}(0)$ as follows

$$P_{1,K}(0) = r \cdot P_{0,0}(0),$$

where $r$ is given by

$$r = \frac{c_1 - c_1'}{d_1 - d_1'}$$

Step 8: For $n = 1, 2, \ldots, K$, calculate $P_{1,n}(0)$ and $P_{1,n}$ in terms of $P_{0,0}(0)$ as follows

$$P_{1,n}(0) = c_n + r \cdot d_n$$

$$P_{1,1} = P_{0,0}(0) / \mu_1$$

$$P_{1,n} = (P_{1,n-1}(0) + P_{0,n-1}(0)) / \mu_n, \quad 2 \leq n \leq K$$

Step 9: Determine $P_{0,0}$ in terms of $P_{0,0}(0)$ as follows

$$P_{0,0} = E(A) \cdot \left[ \sum_{n=0}^{K} P_{0,n}(0) + \sum_{n=1}^{K} P_{1,n}(0) \right] - \left[ \sum_{n=1}^{K} P_{0,n} + \sum_{n=1}^{K} P_{1,n} \right]$$

Step 10: For $n = 0, 1, \ldots, K$, The numerical values of $P_{0,n}$ and $P_{1,n}$, $P'_{0,n}$ and $P'_{1,n}$ respectively are computed as follows

$$P'_{0,n} = \frac{P_{0,n}}{\sum_{n=0}^{K} P_{0,n} + \sum_{n=1}^{K} P_{1,n}}$$

$$P'_{1,n} = \frac{P_{1,n}}{\sum_{n=0}^{K} P_{0,n} + \sum_{n=1}^{K} P_{1,n}}$$

We find that the number of iterations in the algorithm is in the order of $K^3$ where $K$ is the maximum capacity of the system.

**Remark:** It should be noted that if $N = 1$ the algorithm has to be modified slightly for the calculation of $P_{1,1}(0)$ and $P_{1,1}$. 
2.5 Numerical Examples

We consider three instances of $G/M(n)/1/K$ queueing system with $K = 10$ and mean interarrival time $1/8$. The arrival processes considered are Poisson, 3-stage Erlang and hyperexponential respectively. The values of $\mu_n$ are given in Table 2.1 and the vacation rate is 12. $N$ is varied from 1 to 10 and the blocking probability $P_B$, the average number of customers in the system $L$ and the waiting time in queue $W_q$ are calculated for each instance. Each of these measures is plotted against $N$ for the different arrival distributions (See Figures 2.1, 2.2 and 2.3).

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\mu_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.0</td>
</tr>
<tr>
<td>2</td>
<td>5.5</td>
</tr>
<tr>
<td>3</td>
<td>6.0</td>
</tr>
<tr>
<td>4</td>
<td>6.5</td>
</tr>
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<td>8.5</td>
</tr>
<tr>
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<td>8.0</td>
</tr>
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<td>8</td>
<td>8.5</td>
</tr>
<tr>
<td>9</td>
<td>9.0</td>
</tr>
<tr>
<td>10</td>
<td>9.5</td>
</tr>
</tbody>
</table>

For Poisson arrival with mean interarrival time $1/8$, the interarrival time distribution and the Laplace-Stieltjes transform are given as follows

\[ a(x) = 8e^{-8x} \]
\[ A(x) = 1 - e^{-8x} \]
\[ \hat{A}(s) = \frac{8}{8 + s} \]

For 3-stage Erlang arrival process ($E_3$) with mean interarrival time $1/8$, the interarrival time distribution and the Laplace-Stieltjes transform are given as follows

\[ a(x) = 24e^{-24x} \frac{(24x)^2}{2!} \]
\[ A(x) = 1 - \sum_{r=0}^{2} e^{-24x} \frac{(24x)^r}{r!} \]
\[
\bar{A}(s) = \left( \frac{24}{24 + s} \right)^3
\]

For hyperexponential arrival process with \( q_1 = 0.4, q_2 = 0.6 \) and \( \lambda_1 = 16, \lambda_2 = 6 \) (mean interarrival time = 1/8), the interarrival time distribution and the Laplace-Stieltjes transform are given as follows

\[
a(x) = 0.4 \cdot 16e^{-16x} + 0.6 \cdot 6e^{-6x} \\
a(x) = 0.4(1 - e^{-16x}) + 0.6(1 - e^{-6x}) \\
\bar{A}(s) = 0.4 \left( \frac{16}{16 + s} \right) + 0.6 \left( \frac{6}{6 + s} \right)
\]

For deterministic arrival with interarrival time = 1/8, the interarrival time distribution and the Laplace-Stieltjes transform are given as follows

\[
\bar{A}(s) = e^{-s/8}
\]

In figure 2.1, we see that \( L \) is higher for the interarrival time distribution with the lower variance. It is also seen that \( L \) increases with increasing \( N \). Also from figure 2.2, \( W_q \) is lower for the interarrival time distribution with the higher variance when \( N \) is 1. As \( N \) increases \( W_q \) increases, with the increase in \( W_q \) more for the interarrival time distribution with the higher variance. Once \( N \) crosses 6, \( W_q \) is found to be higher for the interarrival time distribution with the higher variance. And from figure 2.3 the blocking probability \( P_B \) is lower for the interarrival time distribution with the lower variance. Also \( P_B \) increases with increasing \( N \).

Now, we try to see how the vacation rate \( v \) influences the system performance measures. We consider a \( G/M(n)/1/K \) queueing system with \( K = 10, N = 3 \) and mean inter arrival time 1/8. \( v \) is varied from 2 to 14 and the blocking probability \( P_B \), the average number of customers in the system \( L \) and the waiting time in queue \( W_q \) are calculated for each instance. Each of these measures is plotted against \( v \) for the different arrival distributions (See Figures 2.4, 2.5 and 2.6).

In figure 2.4, we see that \( L \) is higher for the interarrival time distribution with the lower variance. It is also seen that \( L \) decreases with increasing \( v \). Also from figure 2.5, \( W_q \) is higher for the interarrival time distribution with the higher variance when \( v \) is 2. As \( v \)
increases $W_q$ decreases, with the decrease in $W_q$ more for the interarrival time distribution with the higher variance. Once $v$ crosses 8, $W_q$ is found to be lower for the interarrival time distribution with the higher variance. And from figure 2.6 the blocking probability $P_B$ is lower for the interarrival time distribution with the lower variance. Also $P_B$ decreases with increasing $v$. 
Figure 2.1: Effect of $N$ on $L$ for interarrival time distributions $M$, $E_3$, $H_2$ and $D$

Figure 2.2: Effect of $N$ on $W_q$ for interarrival time distributions $M$, $E_3$, $H_2$ and $D$
Figure 2.3: Effect of $N$ on $P_B$ for interarrival time distributions $M$, $E_3$, $H_2$ and $D$

Figure 2.4: Effect of $v$ on $L$ for interarrival time distributions $M$, $E_3$, $H_2$, $D$ and $N=3$
Figure 2.5: Effect of $v$ on $W_q$ for interarrival time distributions $M$, $E_3$, $H_2$, $D$ and $N=3$

Figure 2.6: Effect of $v$ on $P_B$ for interarrival time distributions $M$, $E_3$, $H_2$, $D$ and $N=3$
Chapter 3

G/M(n)/1/K queue with state-dependent vacations

3.1 Model Description

In this system, we assume that the inter arrival times are independently and identically distributed (i.i.d.) with general distribution $A(x)(x \geq 0)$ and probability density function (p.d.f.) $a(x)(x \geq 0)$ and mean interarrival time $E(A)$. Arriving units at the server form a single waiting line and are served in the order of their arrivals i.e., the service discipline is FIFO. The service times for successive customers are independently and exponentially distributed with parameter $\mu_n$ where $n$ is the number of customers in the system. As soon as the system becomes empty the server is turned off and takes a vacation of random length with $\Pr[V \leq t] = 1 - e^{-\nu_n t}$. The server takes repeated vacations $V$ until there are at least $N (N \geq 1)$ customers to be served on return from a vacation. Once service is started, the server will continue service until the queue is exhausted.

We first derive the system equations for the $N$ policy $G/M(n)/1/K$ queues with multiple state-dependent exponential vacations using the method of supplementary variables. We also illustrate how to calculate performance measures such as blocking probability,
the expected waiting time, average queue length, etc. Next we develop a recursive algorithm for numerically computing the stationary queue length distribution in the $N$ policy $G/M(n)/1/K$ queues with multiple state-dependent exponential vacations. The complexity of the algorithm is $O(n^3)$. Then we illustrate the algorithm by presenting a few simple examples.

**Notations and probabilities**

The two-dimensional state-space of the system is defined as
$\{(i,n) : (i = 0 \text{ and } n = 0,1,\ldots,K) \text{ or } (i = 1 \text{ and } n = 1,2,\ldots,K)\}$

The following notations are used in this chapter:

- $N$ - threshold level
- $K$ - system capacity ($N \leq K$)
- $A(x)$ - c.d.f. of interarrival time $A$
- $a(x)$ - p.d.f. of interarrival time $A$
- $\tilde{A}(s)$ - Laplace-Stieltjes transform of interarrival time $A$
- $\tilde{A}^{(l)}(s)$ - $l$th order derivative of $\tilde{A}(s)$ with respect to $s$
- $P_{0,n}(t)$ - probability of $n$ customers in the system at time $t$ when the server is on vacations, where $n = 0,1,\ldots,K$
- $P_{1,n}(t)$ - probability of $n$ customers in the system at time $t$ when the server is working, where $n = 1,2,\ldots,K$
- $P_{0,n}$ - steady-state probability of $n$ customers in the system when the server is on vacations, where $n = 0,1,\ldots,K$
- $P_{1,n}$ - steady-state probability of $n$ customers in the system when the server is working, where $n = 1,2,\ldots,K$
- $\tilde{P}_{i,n}(s)$ - Laplace-Stieltjes transform of $P_{i,n}(x)$ where $i = 0,1$
- $\tilde{P}_{i,n}^{(l)}(s)$ - $l$th order derivative of $\tilde{P}_{i,n}(s)$ with respect to $s$
- $E(A)$ - mean interarrival time
- $v_n$ - vacation rate of the system in state $(0,n)$, where $n = 0,1,\ldots,K$
- $\mu_n$ - service rate of the system in state $(1,n)$, where $n = 1,2,\ldots,K$
- $P_{0,n}^-$ - probability of $n$ customers in the system immediately prior to an arrival when the server is idle
- $P_{1,n}^-$ - probability of $n$ customers in the system immediately prior to an arrival when the server is busy
- $P_B$ - probability an arriving customer is blocked because the system is full
- $L$ - average number of customers in the system
- $L_q$ - average number of customers in the queue
3.2 Formulation

We first establish the mathematical equations to govern the system by employing the remaining interarrival time as the supplementary variable. Next, we develop a recursive method to derive the steady-state probability distributions of the number of customers in the system.

The state of the system at time $t$ is given by $[Q(t), U(t)]$, where

- $Q(t)$ – number of customers in the system at time $t$
- $U(t)$ – remaining interarrival time for the customers who is arriving

Let us define

- $P^{0,0}(x,t)dx = \text{Prob}\{Q(t) = n, x < U(t) \leq x + dx, \text{server is on vacations}\}$, $x \geq 0, n = 0, 1, \ldots, K$
- $P^{1,0}(x,t)dx = \text{Prob}\{Q(t) = n, x < U(t) \leq x + dx, \text{server is busy}\}$, $x \geq 0, n = 1, 2, \ldots, K$

Relating the state of the system at time $t$ and $t + dt$, we set up the following partial differential equations

$$\begin{align*}
- \frac{\partial P^{0,0}(x,t)}{\partial x} + \frac{\partial P^{0,0}(x,t)}{\partial t} &= \mu_1 P^{1,1}(x,t) \quad (3.1) \\
- \frac{\partial P^{0,n}(x,t)}{\partial x} + \frac{\partial P^{0,n}(x,t)}{\partial t} &= a(x)P^{0,n-1}(0,t), \quad 1 \leq n \leq N - 1 \quad (3.2) \\
- \frac{\partial P^{0,n}(x,t)}{\partial x} + \frac{\partial P^{0,n}(x,t)}{\partial t} &= a(x)P^{0,n-1}(0,t) - v_n P^{0,n}(x,t), \quad N \leq n \leq K - 1 \quad (3.3) \\
- \frac{\partial P^{0,K}(x,t)}{\partial x} + \frac{\partial P^{0,K}(x,t)}{\partial t} &= a(x)[P^{0,K-1}(0,t) + P^{0,K}(0,t)] - v_K P^{0,K}(x,t) \quad (3.4) \\
- \frac{\partial P^{1,1}(x,t)}{\partial x} + \frac{\partial P^{1,1}(x,t)}{\partial t} &= \mu_2 P^{2,1}(x,t) - \mu_1 P^{1,1}(x,t) \quad (3.5)
\end{align*}$$
From (3.1) (3.8), steady-state equations are obtained as follows

\[-\frac{\partial P_{1,n}(x,t)}{\partial x} + \frac{\partial P_{1,n}(x,t)}{\partial t} = a(x)P_{1,n-1}(0,t) + \mu_{n+1}P_{1,n+1}(x,t) - \mu_nP_{1,n}(x,t),\]

\[2 \leq n \leq N - 1 \quad (3.6)\]

\[-\frac{\partial P_{1,n}(x,t)}{\partial x} + \frac{\partial P_{1,n}(x,t)}{\partial t} = v_nP_{0,n}(x,t) + a(x)P_{1,n-1}(0,t) + \mu_{n+1}P_{1,n+1}(x,t) - \mu_nP_{1,n}(x,t), \quad N \leq n \leq K - 1 \quad (3.7)\]

\[-\frac{\partial P_{1,K}(x,t)}{\partial x} + \frac{\partial P_{1,K}(x,t)}{\partial t} = v_KP_{0,K}(x,t) + a(x)[P_{1,K-1}(0,t) + P_{1,K}(0,t)] - \mu_KP_{1,K}(x,t) \quad (3.8)\]

In steady state, we define,

\[P_{0,n} = \lim_{t \to \infty} P_{0,n}(t), \quad n = 0, 1, \ldots, N, \ldots, K\]

\[P_{0,n}(x) = \lim_{t \to \infty} P_{0,n}(x,t), \quad n = 0, 1, \ldots, N, \ldots, K\]

\[P_{1,n} = \lim_{t \to \infty} P_{1,n}(t), \quad n = 1, 2, \ldots, N, \ldots, K\]

\[P_{1,n}(x) = \lim_{t \to \infty} P_{1,n}(x,t), \quad n = 1, 2, \ldots, N, \ldots, K\]

From (3.1)–(3.8), steady-state equations are obtained as follows

\[-\frac{dP_{0,0}(x)}{dx} = \mu_1P_{1,1}(x) \quad (3.9)\]

\[-\frac{dP_{0,n}(x)}{dx} = a(x)P_{0,n-1}(0), \quad 1 \leq n \leq N - 1 \quad (3.10)\]

\[-\frac{dP_{0,n}(x)}{dx} = a(x)P_{0,n-1}(0) - v_nP_{0,n}(x), \quad N \leq n \leq K - 1 \quad (3.11)\]

\[-\frac{dP_{0,K}(x)}{dx} = a(x)[P_{0,K-1}(0) + P_{0,K}(0)] - v_KP_{0,K}(x) \quad (3.12)\]

\[-\frac{dP_{1,1}(x)}{dx} = \mu_2P_{1,2}(x) - \mu_1P_{1,1}(x) \quad (3.13)\]

\[-\frac{dP_{1,n}(x)}{dx} = a(x)P_{1,n-1}(0) + \mu_{n+1}P_{1,n+1}(x) - \mu_nP_{1,n}(x), \quad 2 \leq n \leq N - 1 \quad (3.14)\]

\[-\frac{dP_{1,n}(x)}{dx} = v_nP_{0,n}(x) + a(x)P_{1,n-1}(0) + \mu_{n+1}P_{1,n+1}(x) - \mu_nP_{1,n}(x), \quad N \leq n \leq K - 1 \quad (3.15)\]

\[-\frac{dP_{1,K}(x)}{dx} = v_KP_{0,K}(x) + a(x)[P_{1,K-1}(0) + P_{1,K}(0)] - \mu_KP_{1,K}(x) \quad (3.16)\]
3.3 Analysis

We define the following Laplace-Stieltjes transforms,

\[
\tilde{A}(s) = \int_0^\infty e^{-sx}a(x)dx
\]

\[
\tilde{P}_{i,n}(s) = \int_0^\infty e^{-sx}P_{i,n}(x)dx, \quad i = 0, 1
\]

\[
P_{i,n} = \tilde{P}_{i,n}(0) = \int_0^\infty P_{i,n}(x)dx, \quad i = 0, 1
\]

\[
\int_0^\infty e^{-sx} \frac{\partial}{\partial x} P_{i,n}(x)dx = s\tilde{P}_{i,n}(s) - \tilde{P}_{i,n}(0)
\]

Taking Laplace-Stieltjes transforms on both sides of (3.9)–(3.16), we obtain the following

\[
-s\tilde{P}_{0,0}(s) = \mu_1 \tilde{P}_{1,1}(s) - P_{0,0}(0)
\]

(3.17)

\[
-s\tilde{P}_{0,n}(s) = \tilde{A}(s)P_{0,n-1}(0) - P_{0,n}(0), \quad 1 \leq n \leq N - 1
\]

(3.18)

\[
(v_n - s)\tilde{P}_{0,n}(s) = \tilde{A}(s)P_{0,n-1}(0) - P_{0,n}(0), \quad N \leq n \leq K - 1
\]

(3.19)

\[
(v_K - s)\tilde{P}_{0,K}(s) = \tilde{A}(s)[P_{0,K-1}(0) + P_{0,K}(0)] - P_{0,K}(0)
\]

(3.20)

\[
(\mu_1 - s)\tilde{P}_{1,1}(s) = \mu_2 \tilde{P}_{1,2}(s) - P_{1,1}(0)
\]

(3.21)

\[
(\mu_n - s)\tilde{P}_{1,n}(s) = \mu_{n+1} \tilde{P}_{1,n+1}(s) + \tilde{A}(s)P_{1,n-1}(0) - P_{1,n}(0), \quad 1 \leq n \leq N - 1
\]

(3.22)

\[
(\mu_n - s)\tilde{P}_{1,n}(s) = v_n \tilde{P}_{0,n}(s) + \mu_{n+1} \tilde{P}_{1,n+1}(s) + \tilde{A}(s)P_{1,n-1}(0) - P_{1,n}(0), \quad N \leq n \leq K - 1
\]

(3.23)

\[
(\mu_K - s)\tilde{P}_{1,K}(s) = v_K \tilde{P}_{0,K}(s) + \tilde{A}(s)[P_{1,K-1}(0) + P_{1,K}(0)] - P_{1,K}(0)
\]

(3.24)

Substituting \(s=0\) in (3.17), we get

\[
P_{0,0}(0) = \mu_1 \tilde{P}_{1,1}(0)
\]

\[
= \mu_1 P_{1,1}
\]

Substituting \(s=0\) in (3.18), we get

\[
P_{0,n}(0) = P_{0,n-1}(0), \quad 1 \leq n \leq N - 1
\]

(3.25)
Differentiating (3.18) w.r.t $s$,

\[ \tilde{P}_{0,n}(0) = -\tilde{A}^{(1)}(0)P_{0,n-1}(0), \quad 1 \leq n \leq N - 1 \]

\[ \Rightarrow P_{0,n} = E(A)P_{0,n-1}(0), \quad 1 \leq n \leq N - 1 \quad (3.26) \]

Substituting $s = v_n$ in (3.19) and (3.20),

\[ P_{0,n}(0) = \tilde{A}(v_n)P_{0,n-1}(0), \quad N \leq n \leq K - 1 \]
\[ P_{0,N}(0) = \tilde{A}(v_N)P_{0,N-1}(0) = \tilde{A}(v_0)P_{0,0}(0) \]

\[ \Rightarrow P_{0,n}(0) = \prod_{i=N}^{n} \tilde{A}(v_i)P_{0,0}(0), \quad N \leq n \leq K - 1 \quad (3.27) \]

\[ (1 - \tilde{A}(v_K))P_{0,K}(0) = \prod_{i=N}^{K} \tilde{A}(v_i)P_{0,0}(0) \]
\[ P_{0,K}(0) = \frac{\prod_{i=N}^{K} \tilde{A}(v_i)P_{0,0}(0)}{1 - \tilde{A}(v_K)} \quad (3.28) \]

Substituting $s=0$ (3.19) and (3.20),

\[ v_n\tilde{P}_{0,n}(0) = \tilde{A}(0)P_{0,n-1}(0) - P_{0,n}(0), \quad N \leq n \leq K - 1 \]

\[ \Rightarrow \tilde{P}_{0,n}(0) = \frac{P_{0,n-1}(0) - P_{0,n}(0)}{v_n}, \quad N \leq n \leq K - 1 \]

\[ \Rightarrow P_{0,n} = \frac{1}{v_n}[1 - \tilde{A}(v_n)]\prod_{i=N}^{n-1} \tilde{A}(v_i)P_{0,0}(0), \quad N \leq n \leq K - 1 \quad (3.29) \]

\[ v_K\tilde{P}_{0,K}(0) = P_{0,K-1}(0) \]
\[ \Rightarrow P_{0,K} = \frac{[\prod_{i=N}^{K-1} \tilde{A}(v_i)]P_{0,0}(0)}{v_K} \quad (3.30) \]

Adding equations (3.17) to (3.24) and simplifying we obtain,

\[ \sum_{n=0}^{K} \tilde{P}_{0,n}(s) + \sum_{n=1}^{K} \tilde{P}_{1,n}(s) = \frac{1 - \tilde{A}(s)}{s} \left[ \sum_{n=0}^{K} P_{0,n}(0) + \sum_{n=1}^{K} P_{1,n}(0) \right] \]
Taking \( \lim_{s \to 0} \) and using L'Hôpital's rule, we get
\[
\sum_{n=0}^{K} \bar{P}_{0,n}(0) + \sum_{n=1}^{K} \bar{P}_{1,n}(0) = E(A) \left[ \sum_{n=0}^{K} P_{0,n}(0) + \sum_{n=1}^{K} P_{1,n}(0) \right] \\
\Rightarrow \sum_{n=0}^{K} P_{0,n}(0) + \sum_{n=1}^{K} P_{1,n}(0) = E(A) \left[ \sum_{n=0}^{K} P_{0,n}(0) + \sum_{n=1}^{K} P_{1,n}(0) \right] \quad (3.31) \\
\Rightarrow \sum_{n=0}^{K} P_{0,n}(0) + \sum_{n=1}^{K} P_{1,n}(0) = 1/E(A) \quad (3.32)
\]

Setting \( s = \mu_n \) in (3.22), (3.23) and (3.24) we get,
\[
P_{1,n}(0) = \mu_{n+1} \bar{P}_{1,n+1}(\mu_n) + \bar{A}(\mu_n) P_{1,n-1}(0) \\
P_{1,n}(0) = v_n \bar{P}_{0,n}(\mu_n) + \mu_{n+1} \bar{P}_{1,n+1}(\mu_n) + \bar{A}(\mu_n) P_{1,n-1}(0)
\]
\[
P_{1,K}(0)[1 - \bar{A}(\mu_K)] = v_K \bar{P}_{0,K}(\mu_K) + \bar{A}(\mu_K) P_{1,K-1}(0) \\
\Rightarrow P_{1,K-1}(0) = \frac{1 - \bar{A}(\mu_K)}{\bar{A}(\mu_K)} P_{1,K-1}(0) - \frac{v_K \bar{P}_{0,K}(\mu_K)}{\bar{A}(\mu_K)} \\
P_{1,n-1}(0) = \frac{P_{1,n}(0)}{\bar{A}(\mu_n)} - \frac{\mu_{n+1} \bar{P}_{1,n+1}(\mu_n)}{\bar{A}(\mu_n)} - \frac{v_n \bar{P}_{0,n}(\mu_n)}{\bar{A}(\mu_n)} \\
P_{1,n-1}(0) = \frac{P_{1,n}(0)}{\bar{A}(\mu_n)} - \frac{\mu_{n+1} \bar{P}_{1,n+1}(\mu_n)}{\bar{A}(\mu_n)}
\]

where, \( \bar{P}_{0,n}(\mu_n) \) are given by the following,

For \( w \neq v_n \)
\[
\bar{P}_{0,K}(w) = \frac{1}{v_K - w} \left( \frac{\bar{A}(w) - \bar{A}(v_K)}{1 - \bar{A}(v_K)} \right) \left[ \prod_{i=N}^{K-1} \bar{A}(v_i) \right] P_{0,0}(0) \quad (3.36)
\]
\[
\bar{P}_{0,n}(w) = \left[ \frac{\bar{A}(w) - \bar{A}(v_n)}{v_n - w} \right] \left[ \prod_{i=N}^{n-1} \bar{A}(v_i) \right] P_{0,0}(0) \quad N \leq n \leq K - 1 \quad (3.37)
\]

For \( w = v_n \)
\[
\bar{P}_{0,K}(w) = -\bar{A}(w)[P_{0,K-1}(0) + P_{0,K}(0)] \quad (3.38)
\]
\[
\bar{P}_{0,n}(w) = -\bar{A}(w)P_{0,n-1}(0) \quad N \leq n \leq K - 1 \quad (3.39)
\]
and, $\bar{P}_{1,n}(\mu_{n-1})$ are given by the following,

For $w \neq \mu_K$

$$\bar{P}_{1,K}(w) = \frac{v_K \bar{P}_{0,K}(w) + \bar{A}(w)[P_{1,K-1}(0) + P_{1,K}(0)] - P_{1,K}(0)}{\mu_K - w}$$  \hspace{1cm} (3.40)

For $w = \mu_K$

$$\bar{P}_{1,K}(w) = -\left(v_K \bar{P}_{0,K}^{(1)}(w) + \bar{A}^{(1)}(w)[P_{1,K-1}(0) + P_{1,K}(0)]\right)$$  \hspace{1cm} (3.41)

For $w \neq \mu_n$

$$\bar{P}_{1,n}(w) = \frac{v_n \bar{P}_{0,n}(w) + \mu_{n+1} \bar{P}_{1,n+1}(w) + \bar{A}(w)P_{1,n-1}(0) - P_{1,n}(0)}{\mu_n - w} \hspace{1cm} (3.42)$$

$$\bar{P}_{1,n}(w) = \frac{\mu_{n+1} \bar{P}_{1,n+1}(w) + \bar{A}(w)P_{1,n-1}(0) - P_{1,n}(0)}{\mu_n - w} \hspace{1cm} (3.43) \hspace{1cm} 2 \leq n \leq N - 1$$

For $w = \mu_n$

$$\bar{P}_{1,n}(w) = -\left(v_n \bar{P}_{0,n}^{(1)}(w) + \mu_{n+1} \bar{P}_{1,n+1}^{(1)}(w) + \bar{A}^{(1)}(w)P_{1,n-1}(0)\right) \hspace{1cm} (3.44)$$

$$\bar{P}_{1,n}(w) = -\left(\mu_{n+1} \bar{P}_{1,n+1}^{(1)}(w) + \bar{A}^{(1)}(w)P_{1,n-1}(0)\right) \hspace{1cm} (3.45) \hspace{1cm} 2 \leq n \leq N - 1$$

Differentiating (3.19) and (3.20) $l$ times with respect to $s$, and setting $s = w$ we get the following expressions for $\bar{P}_{0,n}^{(l)}(w)$

For $w \neq v_K$ and $n = K$

$$\bar{P}_{0,K}^{(1)}(w) = \frac{\bar{A}^{(1)}(w)[P_{0,K-1}(0) + P_{0,K}(0)] + \bar{P}_{0,K}(w)}{v_K - w}$$  \hspace{1cm} (3.46)

$$\bar{P}_{0,K}^{(l)}(w) = \frac{\bar{A}^{(l)}(w)[P_{0,K-1}(0) + P_{0,K}(0)] + l\bar{P}_{0,K}^{(l-1)}(w)}{v_K - w}, \hspace{1cm} l \neq 1$$  \hspace{1cm} (3.47)
For \( w \neq v_n \) and \( N \leq n \leq K - 1 \)

\[
\tilde{P}^{(1)}_{0,n}(w) = \frac{\tilde{A}^{(1)}(w)P_{0,n-1}(0) + \tilde{P}_{0,n}(w)}{v_n - w} \quad (3.48)
\]

\[
\tilde{P}^{(l)}_{0,n}(w) = \frac{\tilde{A}^{(l)}(w)P_{0,n-1}(0) + l\tilde{P}^{(l-1)}_{0,n}(w)}{v_n - w}, \quad l \neq 1 \quad (3.49)
\]

For \( w = v_n \)

\[
\tilde{P}^{(l)}_{0,K}(w) = -\frac{\tilde{A}^{(l+1)}(w)[P_{0,K-1}(0) + P_{0,K}(0)]}{l + 1} \quad (3.50)
\]

\[
\tilde{P}^{(l)}_{0,n}(w) = -\frac{\tilde{A}^{(l+1)}(w)P_{0,n-1}(0)}{l + 1}, \quad N \leq n \leq K - 1 \quad (3.51)
\]

Differentiating (3.22), (3.23) and (3.24) \( l \) times with respect to \( s \), and setting \( s = w \) we get the following expressions for \( \tilde{P}^{(l)}_{1,n}(w) \)

For \( w \neq \mu_K \)

\[
\tilde{P}^{(1)}_{1,K}(w) = \frac{v_K\tilde{P}^{(1)}_{0,K}(w) + \tilde{A}^{(1)}(w)[P_{1,K-1}(0) + P_{1,K}(0)] + \tilde{P}_{1,K}(w)}{\mu_K - w} \quad (3.52)
\]

\[
\tilde{P}^{(l)}_{1,K}(w) = \frac{v_K\tilde{P}^{(l)}_{0,K}(w) + \tilde{A}^{(l)}(w)[P_{1,K-1}(0) + P_{1,K}(0)] + l\tilde{P}^{(l-1)}_{1,K}(w)}{\mu_K - w}, \quad l \neq 1 \quad (3.53)
\]

For \( w = \mu_K \)

\[
\tilde{P}^{(l)}_{1,K}(w) = -\frac{v_K\tilde{P}^{(l+1)}_{0,K}(w) + \tilde{A}^{(l+1)}(w)[P_{1,K-1}(0) + P_{1,K}(0)]}{l + 1} \quad (3.54)
\]

For \( w \neq \mu_n \) and \( N \leq n \leq K - 1 \)

\[
\tilde{P}^{(1)}_{1,n}(w) = \frac{v_n\tilde{P}^{(1)}_{0,n}(w) + \tilde{A}^{(1)}(w)P_{1,n-1}(0) + \mu_n+1\tilde{P}^{(1)}_{1,n+1}(w) + \tilde{P}_{1,n}(w)}{\mu_n - w} \quad (3.55)
\]

\[
\tilde{P}^{(l)}_{1,n}(w) = \frac{v_n\tilde{P}^{(l)}_{0,n}(w) + \tilde{A}^{(l)}(w)P_{1,n-1}(0) + \mu_n+1\tilde{P}^{(l)}_{1,n+1}(w) + l\tilde{P}^{(l-1)}_{1,n}(w)}{\mu_n - w}, \quad l \neq 1 \quad (3.56)
\]

For \( w \neq \mu_n \) and \( 2 \leq n \leq N - 1 \)

\[
\tilde{P}^{(1)}_{1,n}(w) = \frac{\tilde{A}^{(1)}(w)P_{1,n-1}(0) + \mu_n+1\tilde{P}^{(1)}_{1,n+1}(w) + \tilde{P}_{1,n}(w)}{\mu_n - w} \quad (3.57)
\]

\[
\tilde{P}^{(l)}_{1,n}(w) = \frac{\tilde{A}^{(l)}(w)P_{1,n-1}(0) + \mu_n+1\tilde{P}^{(l)}_{1,n+1}(w) + l\tilde{P}^{(l-1)}_{1,n}(w)}{\mu_n - w}, \quad l \neq 1 \quad (3.58)
\]
For $w = \mu_n$

$$P_{1,n}(w) = -\frac{v_n P_{0,n}(w) + \tilde{A}(n)P_{1,n}(0) + \mu_n+1 P_{1,n+1}(w)}{l+1} \quad N \leq n \leq K - 1 \quad (3.59)$$

$$P_{2,n}(w) = -\frac{\tilde{A}(n)P_{1,n}(0) + \mu_n+1 P_{1,n+1}(w)}{l+1} \quad 2 \leq n \leq N - 1 \quad (3.60)$$

Setting $s = \mu_1$ in (3.21) we get,

$$P_{1,1}(0) = \mu_2 \tilde{P}_{1,2}(\mu_1) \quad (3.61)$$

Substituting $s = 0$ in (3.17), (3.22) and (3.23)

$$P_{1,n} = \frac{P_{1,n-1}(0) + P_{0,n-1}(0)}{\mu_n} \quad 2 \leq n \leq K \quad (3.62)$$

Let $P_{0,n}^-$ and $P_{1,n}^-$ denote the probabilities of $n$ customers in the queue immediately prior to an arrival when the server is idle and busy, respectively. We observe that,

$$P_{i,n}^- = \frac{P_{i,n}(0)}{\sum_{n=0}^{N} P_{0,n}(0) + \sum_{n=1}^{N} P_{1,n}(0)}, \quad i = 0, 1$$

Using the result from (3.32) we get,

$$P_{0,n}^- = E(A)P_{0,n}(0), \quad 0 \leq n \leq K \quad (3.63)$$

$$P_{1,n}^- = E(A)P_{1,n}(0), \quad 1 \leq n \leq K \quad (3.64)$$

An important performance measure for a finite queueing system is the blocking probability. The blocking probability is defined as the probability that an arriving customer is blocked because the system is full. The blocking probability $P_B$ can be determined as follows

$$P_{B} = P_{0,K}^- + P_{1,K}^-$$

$$\Rightarrow P_{B} = E(A)(P_{0,K}(0) + P_{1,K}(0)) \quad (3.65)$$
Let $\lambda'$ denote the effective arrival rate. Since $1 - P_B$ is the probability that an arriving customer is accepted, we get

$$\lambda' = \frac{1 - P_B}{E(A)}$$  \hspace{1cm} (3.66)

The expected number of customers in the system $L$ and expected number of customers in queue $L_q$ at an arbitrary time is given by

$$L = \sum_{n=0}^{K} n P_{0,n} + \sum_{n=1}^{K} n P_{1,n}$$

$$L_q = \sum_{n=0}^{K} n P_{0,n} + \sum_{n=1}^{K} (n-1) P_{1,n}$$

Using Little’s law, the expected waiting time in the system $W$ and expected waiting time in the queue $W_q$ are obtained as

$$W = \frac{L}{\lambda'}$$

$$W_q = \frac{L_q}{\lambda'}$$

### 3.4 Algorithm

Algorithm for calculating $P_{1,n}(0), P_{0,n}(0), P_{1,n}, P_{0,n}$ is as follows

Step 1: For $n = 0, 1, \ldots, K$, calculate $P_{0,n}(0)$ in terms of $P_{0,0}(0)$ as follows

$$P_{0,n}(0) = 1 \quad 0 \leq n \leq N - 1$$

$$P_{0,n}(0) = \tilde{A}(v_n) * P_{0,n-1}(0) \quad N \leq n \leq K - 1$$

$$P_{0,K}(0) = \tilde{A}(v_K) * P_{0,K-1}(0)/(1 - \tilde{A}(v_K))$$

Step 2: For $n = 1, 2, \ldots, K$, calculate $P_{0,n}$ in terms of $P_{0,0}(0)$ as follows

$$P_{0,n} = E(A) \quad 1 \leq n \leq N - 1$$

$$P_{0,n}(0) = (P_{0,n-1}(0) - P_{0,n}(0))/v_n \quad N \leq n \leq K - 1$$

$$P_{0,K}(0) = P_{0,K-1}(0)/v_K$$
Step 3: For \( n = N, N+1, \ldots, K \), calculate \( P_{0,n}(\mu_j) \) for all \( j = 1, 2, \ldots, n \) in terms of \( P_{0,0}(0) \) as follows

If \( v_n = \mu_j \)
\[
\begin{align*}
\tilde{P}_{0,K}(\mu_j) &= -\tilde{A}^{(1)}(\mu_j) * (P_{0,K-1}(0) + P_{0,K}(0)) \\
\tilde{P}_{0,n}(\mu_j) &= -\tilde{A}^{(1)}(\mu_j) * P_{0,n-1}(0), \quad N \leq n \leq K - 1
\end{align*}
\]
Else
\[
\begin{align*}
\tilde{P}_{0,K}(\mu_j) &= (\tilde{A}(\mu_j) * (P_{0,K-1}(0) + P_{0,K}(0)) - P_{0,K}(0))/v_K - \mu_j \\
\tilde{P}_{0,n}(\mu_j) &= (\tilde{A}(\mu_j) * P_{0,n-1}(0) - P_{0,n}(0))/(v_n - \mu_j), \quad N \leq n \leq K - 1
\end{align*}
\]
End If

Step 4: For \( n = N, N+1, \ldots, K \), calculate \( \tilde{P}_{0,n}(\mu_j) \) for all \( j = 1, 2, \ldots, n \) and \( l = 1, 2, \ldots, n-1 \) in terms of \( P_{0,0}(0) \) as follows

If \( v_n = \mu_j \)
\[
\begin{align*}
\tilde{P}_{0,K}^{(l)}(\mu_j) &= -\tilde{A}^{(l+1)}(\mu_j) * (P_{0,K-1}(0) + P_{0,K}(0)) \\
\tilde{P}_{0,n}^{(l)}(\mu_j) &= -\tilde{A}^{(l+1)}(\mu_j) * P_{0,n-1}(0) \\
\end{align*}
\]
Else If \( l = 1 \)
\[
\begin{align*}
\tilde{P}_{0,K}^{(1)}(\mu_j) &= \tilde{A}(\mu_j) * (P_{0,K-1}(0) + P_{0,K}(0)) + \tilde{P}_{0,K}(\mu_j)/v_K - \mu_j \\
\tilde{P}_{0,n}^{(1)}(\mu_j) &= \tilde{A}(\mu_j) * P_{0,n-1}(0) + \tilde{P}_{0,n}(\mu_j)/v_n - \mu_j, \quad N \leq n \leq K - 1
\end{align*}
\]
Else
\[
\begin{align*}
\tilde{P}_{0,K}^{(l)}(\mu_j) &= \tilde{A}(\mu_j) * (P_{0,K-1}(0) + P_{0,K}(0)) + l * \tilde{P}_{0,K}^{(l-1)}(\mu_j)/v_K - \mu_j \\
\tilde{P}_{0,n}^{(l)}(\mu_j) &= \tilde{A}(\mu_j) * P_{0,n-1}(0) + l * \tilde{P}_{0,n}^{(l-1)}(\mu_j)/v_n - \mu_j, \quad N \leq n \leq K - 1
\end{align*}
\]
End If

Step 5: For \( n = K, K - 1, \ldots, 1 \), let us express \( P_{1,n}(0) \), \( \tilde{P}_{1,n}(\mu_j) \) and \( \tilde{P}_{1,n}^{(l)}(\mu_j) \) in terms of \( P_{0,0}(0) \) and \( P_{1,K}(0) \) as follows

\[
P_{1,n}(0) = c_n P_{0,0}(0) + d_n P_{1,K}(0)
\]

\[
\tilde{P}_{1,n}(\mu_j) = e_{n,j} P_{0,0}(0) + f_{n,j} P_{1,K}(0)
\]

\[
\tilde{P}_{1,n}^{(l)}(\mu_j) = e_{n,j} P_{0,0}(0) + f_{n,j} P_{1,K}(0)
\]

(i) Calculate \( c_n \) and \( d_n \) as follows

If \( n = K \)

\[
c_K = 0
\]

\[
d_K = 1
\]

\[
c_{K-1} = \frac{v_K \tilde{P}_{0,K}(\mu_K)}{A(\mu_K)}
\]

\[
d_{K-1} = \frac{1 - \tilde{A}(\mu_K)}{A(\mu_K)}
\]

Else If \( N \leq n \leq K - 1 \)

\[
c_{n-1} = \frac{c_n - \mu_{n+1} e_{n+1,n} - v_n \tilde{P}_{0,n}(\mu_n)}{A(\mu_n)}
\]

\[
d_{n-1} = \frac{d_n - \mu_{n+1} f_{n+1,n}}{A(\mu_n)}
\]

Else \( 2 \leq n \leq N - 1 \)

\[
c_{n-1} = \frac{c_n - \mu_{n+1} e_{n+1,n}}{A(\mu_n)}
\]

\[
d_{n-1} = \frac{d_n - \mu_{n+1} f_{n+1,n}}{A(\mu_n)}
\]

End If

(ii) For \( j = n - 1, n - 2, \ldots, 1, \)
(a) Calculate $e_{n,j}$ and $f_{n,j}$ as follows

If $n = K$

If $\mu_K = \mu_j$

\[
\begin{align*}
e_{K,j} & = -v_K \tilde{P}_{0,K}^{(1)}(\mu_j) - A^{(1)}(\mu_j)[c_{K-1} + c_K] \\
f_{K,j} & = -A^{(1)}(\mu_j)[d_{K-1} + d_K]
\end{align*}
\]

Else

\[
\begin{align*}
e_{K,j} & = \frac{v_K \tilde{P}_{0,K}(\mu_j) + A(\mu_j)[c_{K-1} + c_K] - c_K}{\mu_K - \mu_j} \\
f_{K,j} & = \frac{A(\mu_j)[d_{K-1} + d_K] - d_K}{\mu_K - \mu_j}
\end{align*}
\]

End If

Else If $N \leq n \leq K - 1$

If $\mu_n = \mu_j$

\[
\begin{align*}
e_{n,j} & = -(v_n \tilde{P}_{0,n}^{(1)}(\mu_j) + \mu_{n+1}e_{n+1,j}^{(1)} + \tilde{A}^{(1)}(\mu_j)c_{n-1}) \\
f_{n,j} & = -(\mu_{n+1}f_{n+1,j}^{(1)} + \tilde{A}^{(1)}(\mu_j)d_{n-1})
\end{align*}
\]

Else

\[
\begin{align*}
e_{n,j} & = \frac{v_n \cdot \tilde{P}_{0,n}(\mu_j) + \mu_{n+1}e_{n+1,j} + \tilde{A}(\mu_j)c_{n-1} - c_n}{\mu_n - \mu_j} \\
f_{n,j} & = \frac{\mu_{n+1}f_{n+1,j} + \tilde{A}(\mu_j)d_{n-1} - d_n}{\mu_n - \mu_j}
\end{align*}
\]

End If

Else If $2 \leq n \leq N - 1$

If $\mu_n = \mu_j$

\[
\begin{align*}
e_{n,j} & = -(\mu_{n+1}e_{n+1,j}^{(1)} + \tilde{A}^{(1)}(\mu_j)c_{n-1}) \\
f_{n,j} & = -(\mu_{n+1}f_{n+1,j}^{(1)} + \tilde{A}^{(1)}(\mu_j)d_{n-1})
\end{align*}
\]
Else
\[ e_{n,j} = \frac{\mu_{n+1} e_{n+1,j} + \tilde{A}(\mu_j) c_{n-1} - e_n}{\mu_n - \mu_j} \]
\[ f_{n,j} = \frac{\mu_{n+1} f_{n+1,j} + \tilde{A}(\mu_j) d_{n-1} - d_n}{\mu_n - \mu_j} \]
End If

End If

(b) Calculate \( e_{n,j}^{(l)} \) and \( f_{n,j}^{(l)} \) for \( l = 1, 2, \ldots, n - 2 \) as follows

If \( n = K \)

\( e_{K,j}^{(l)} = - \frac{v_K \tilde{P}_0^{(l+1)}(\mu_j) + \bar{A}^{(l)}(\mu_j)[c_{K-1} + c_K]}{l + 1} \)
\[ f_{K,j}^{(l)} = - \frac{\bar{A}^{(l)}(\mu_j)[d_{K-1} + d_K]}{l + 1} \]

Else
\[ e_{K,j}^{(1)} = \frac{v_K \tilde{P}_0^{(1)}(\mu_j) + \bar{A}^{(1)}(\mu_j)[c_{K-1} + c_K] + e_{K,j}}{\mu_K - \mu_j} \]
\[ f_{K,j}^{(1)} = \frac{\bar{A}^{(1)}(\mu_j)[d_{K-1} + d_K] + f_{K,j}}{\mu_K - \mu_j} \]
\[ e_{K,j}^{(l)} = \frac{\bar{A}^{(l)}(\mu_j) [c_{K-1} + c_K] + l * e_{K,j}^{(l-1)} + v_K \tilde{P}_0^{(l)}(\mu_j)}{\mu_K - \mu_j}, \quad l \neq 1 \]
\[ f_{K,j}^{(l)} = \frac{\bar{A}^{(l)}(\mu_j)[d_{K-1} + d_K] + l * f_{K,j}^{(l-1)}}{\mu_K - \mu_j}, \quad l \neq 1 \]
End If

Else If \( N \leq n \leq K - 1 \)

If \( \mu_n = \mu_j \)
\[ e_{n,j}^{(l)} = - \frac{v_n \tilde{P}_0^{(l+1)}(\mu_j) + \bar{A}^{(l+1)}(\mu_j) c_{n-1} + \mu_n^{(l+1)} e_{n+1,j}^{(l+1)}}{l + 1} \]
\[ f_{n,j}^{(l)} = - \frac{\overline{A}^{(l+1)}(\mu_j) d_{n-1} + \mu_{n+1} f_{n+1,j}^{(l+1)}}{l + 1} \]

Else

\[ e_{n,j}^{(1)} = e_{n+1,j} \frac{\bar{P}_{0,n}^{(1)}(\mu_j) + \overline{A}^{(1)}(\mu_j) c_{n-1} + \mu_{n+1} e_{n+1,j}^{(1)} + e_{n,j}}{\mu_n - \mu_j} \]

\[ f_{n,j}^{(1)} = \overline{A}^{(1)}(\mu_j) d_{n-1} + \mu_{n+1} f_{n+1,j}^{(1)} + f_{n,j} \]

\[ e_{n,j}^{(l)} = \frac{\overline{A}^{(l)}(\mu_j) c_{n-1} + \mu_{n+1} e_{n+1,j}^{(l)} + l * e_{n,j}^{(l-1)} + v_n \tilde{P}^{(l)}_{0,n}(\mu_j)}{\mu_n - \mu_j}, \ l \neq 1 \]

\[ f_{n,j}^{(l)} = \overline{A}^{(l)}(\mu_j) d_{n-1} + \mu_{n+1} f_{n+1,j}^{(l)} + l * f_{n,j}^{(l-1)}, \ l \neq 1 \]

End If

Else If \( 2 \leq n \leq N - 1 \)

If \( \mu_n = \mu_j \)

\[ e_{n,j}^{(l)} = - \frac{\overline{A}^{(l+1)}(\mu_j) c_{n-1} + \mu_{n+1} e_{n+1,j}^{(l+1)}}{l + 1} \]

\[ f_{n,j}^{(l)} = - \frac{\overline{A}^{(l+1)}(\mu_j) d_{n-1} + \mu_{n+1} f_{n+1,j}^{(l+1)}}{l + 1} \]

Else

\[ e_{n,j}^{(1)} = \frac{\overline{A}^{(1)}(\mu_j) c_{n-1} + \mu_{n+1} e_{n+1,j}^{(1)} + e_{n,j}}{\mu_n - \mu_j} \]

\[ f_{n,j}^{(1)} = \overline{A}^{(1)}(\mu_j) d_{n-1} + \mu_{n+1} f_{n+1,j}^{(1)} + f_{n,j} \]

\[ e_{n,j}^{(l)} = \frac{\overline{A}^{(l)}(\mu_j) c_{n-1} + \mu_{n+1} e_{n+1,j}^{(l)} + l * e_{n,j}^{(l-1)}}{\mu_n - \mu_j}, \ l \neq 1 \]

\[ f_{n,j}^{(l)} = \overline{A}^{(l)}(\mu_j) d_{n-1} + \mu_{n+1} f_{n+1,j}^{(l)} + l * f_{n,j}^{(l-1)}, \ l \neq 1 \]

End If

End If
Step 6: Calculate $P_{1,1}^\prime(0)$ in terms of $P_{0,0}(0)$ and $P_{1,K}(0)$ as follows
\[ c_1' = \mu_2 e_2,1 \]
\[ d_1' = \mu_2 f_2,1 \]

Step 7: Equating $P_{1,1}(0)$ from Step 4 and $P_{1,1}^\prime(0)$ from Step 6, obtain $P_{1,K}(0)$ in terms of $P_{0,0}(0)$ as follows
\[ P_{1,K}(0) = r \ast P_{0,0}(0), \text{ where } r \text{ is given by} \]
\[ r = \frac{c_1 - c_1}{d_1 - d_1'} \]

Step 8: For $n = 1, 2, \ldots, K$, calculate $P_{1,n}(0)$ and $P_{1,n}$ in terms of $P_{0,0}(0)$ as follows
\[ P_{1,n}(0) = c_n + r \ast d_n \]
\[ P_{1,1} = \frac{P_{0,0}(0)}{\mu_1} \]
\[ P_{1,n} = \frac{(P_{1,n-1}(0) + P_{0,n-1}(0))/\mu_n, \quad 2 \leq n \leq K}{P_{0,0}(0)} \]

Step 9: Determine $P_{0,0}$ in terms of $P_{0,0}(0)$ as follows
\[ P_{0,0} = E(A) \ast \left[ \sum_{n=0}^{K} P_{0,n}(0) + \sum_{n=1}^{K} P_{1,n}(0) \right] - \left[ \sum_{n=1}^{K} P_{0,n} + \sum_{n=1}^{K} P_{1,n} \right] \]

Step 10: For $n = 0, 1, \ldots, K$, The numerical values of $P_{0,n}$ and $P_{1,n}$, $P_{0,n}$ and $P_{1,n}$ respectively are computed as follows
\[ P_{0,n}' = \frac{P_{0,n}}{\sum_{n=0}^{K} P_{0,n} + \sum_{n=1}^{K} P_{1,n}} \]
\[ P_{1,n}' = \frac{P_{1,n}}{\sum_{n=0}^{K} P_{0,n} + \sum_{n=1}^{K} P_{1,n}} \]

We find that the number of iterations in the algorithm is in the order of $K^3$ where $K$ is the maximum capacity of the system.

Remark: It should be noted that if $N = 1$ the algorithm has to be modified slightly for the calculation of $P_{1,1}(0)$ and $P_{1,1}$.
3.5 Numerical Examples

We consider three instances of $G/M(n)/1/K$ queueing system with $K = 10$ and mean interarrival time $1/8$. The arrival processes considered are Poisson, 3-stage Erlang and hyperexponential respectively. The values of $\mu_n$ and $v_n$ are given in Table 3.1. $N$ is varied from 1 to 10 and the blocking probability $P_B$, the average number of customers in the system $L$ and the waiting time in queue $W_q$ are calculated for each instance. Each of these measures is plotted against $N$ for the different arrival distributions (See Figures 3.1, 3.2 and 3.3).

Table 3.1: Values of $\mu_n$ and $v_n$ for $n = 1, 2, \ldots, 10$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\mu_n$</th>
<th>$v_n$ when $N \geq n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.0</td>
<td>3.0 when $N \geq 1$</td>
</tr>
<tr>
<td>2</td>
<td>5.5</td>
<td>4.0 when $N \geq 2$</td>
</tr>
<tr>
<td>3</td>
<td>6.0</td>
<td>5.0 when $N \geq 3$</td>
</tr>
<tr>
<td>4</td>
<td>6.5</td>
<td>6.0 when $N \geq 4$</td>
</tr>
<tr>
<td>5</td>
<td>7.0</td>
<td>7.0 when $N \geq 5$</td>
</tr>
<tr>
<td>6</td>
<td>8.5</td>
<td>8.0 when $N \geq 6$</td>
</tr>
<tr>
<td>7</td>
<td>8.0</td>
<td>9.0 when $N \geq 7$</td>
</tr>
<tr>
<td>8</td>
<td>8.5</td>
<td>10.0 when $N \geq 8$</td>
</tr>
<tr>
<td>9</td>
<td>9.0</td>
<td>11.0 when $N \geq 9$</td>
</tr>
<tr>
<td>10</td>
<td>9.5</td>
<td>12.0 when $N = 10$</td>
</tr>
</tbody>
</table>

For Poisson arrival with mean interarrival time $1/8$, the interarrival time distribution and the Laplace-Stieltjes transform are given as follows

$$a(x) = 8e^{-8x}$$
$$A(x) = 1 - e^{-8x}$$
$$\overline{A}(s) = \frac{8}{8 + s}$$

For 3-stage Erlang arrival process ($E_3$) with mean interarrival time $1/8$, the interarrival time distribution and the Laplace-Stieltjes transform are given as follows

$$a(x) = 24e^{-24x} \frac{(24x)^2}{2!}$$
$$A(x) = 1 - \sum_{r=0}^{2} e^{-24x} \frac{(24x)^r}{r!}$$
\[ \tilde{A}(s) = \left( \frac{24}{24 + s} \right)^3 \]

For hyperexponential arrival process with \( q_1 = 0.4, q_2 = 0.6 \) and \( \lambda_1 = 16, \lambda_2 = 6 \) (mean interarrival time = 1/8), the interarrival time distribution and the Laplace-Stieltjes transform are given as follows

\[
\begin{align*}
\alpha(x) &= 0.4 \cdot 16e^{-16x} + 0.6 \cdot 6e^{-6x} \\
A(x) &= 0.4(1 - e^{-16x}) + 0.6(1 - e^{-6x}) \\
\tilde{A}(s) &= 0.4 \left( \frac{16}{16 + s} \right) + 0.6 \left( \frac{6}{6 + s} \right)
\end{align*}
\]

For deterministic arrival with interarrival time = 1/8, the interarrival time distribution and the Laplace-Stieltjes transform are given as follows

\[ \tilde{A}(s) = e^{-s/8} \]

In figure 3.1, we see that \( L \) is higher for the interarrival time distribution with the lower variance. It is also seen that \( L \) increases with increasing \( N \). Also from figure 3.2, \( W_q \) is lower for the interarrival time distribution with the higher variance when \( N \) is 1. As \( N \) increases \( W_q \) increases, with the increase in \( W_q \) more for the interarrival time distribution with the higher variance. Once \( N \) crosses 6, \( W_q \) is found to be higher for the interarrival time distribution with the higher variance. And from figure 3.3 the blocking probability \( P_B \) is lower for the interarrival time distribution with the lower variance. Also \( P_B \) increases with increasing \( N \).
Figure 3.1: Effect of $N$ on $L$ for interarrival time distributions $M$, $E_3$, $H_2$ and $D$

Figure 3.2: Effect of $N$ on $W_q$ for interarrival time distributions $M$, $E_3$, $H_2$ and $D$
Figure 3.3: Effect of $N$ on $P_B$ for interarrival time distributions $M$, $E_3$, $H_2$ and $D$
Chapter 4

**M(n)/G/1/K queue with arbitrary vacations**

**4.1 Model Description**

In this system, the arrival process is a Poisson process with various rates $\lambda_n$ where $n$ is the number of customers in the system. Arriving units at the server form a single waiting line and are served in the order of their arrivals i.e., the service discipline is FIFO. We assume that the service times are independently and identically distributed (i.i.d.) with general distribution $B(x)(x \geq 0)$ and probability density function (p.d.f.) $b(x)(x \geq 0)$ and mean service time $E(B)$. As soon as the system becomes empty the server is turned off and takes a vacation of random length with general distribution $V(x)(x \geq 0)$ and probability density function (p.d.f.) $V(x)(x \geq 0)$ and mean vacation time $E(V)$. The server takes repeated vacations $V$ until there are at least $N(N \geq 1)$ customers to be served on return from a vacation. Once service is started, the server will continue service until the queue is exhausted.

We first derive the system equations for the $N$ policy $M(n)/G/1/K$ queues with vacations using the method of supplementary variables. We also illustrate how to calculate
performance measures such as blocking probability, expected waiting time, average queue length, etc. Next we develop a recursive algorithm for numerically computing the stationary queue length distribution in the $N$ policy $M(n)/G/1/K$ queues with vacations. The complexity of the algorithm is $O(n^3)$. Then we illustrate the algorithm by presenting a few simple examples.

**Notations and probabilities**

The two-dimensional state-space of the system is defined as

$$\{(i,n) : (i = 0 \text{ and } n = 0,1,\ldots,K) \text{ or } (i = 1 \text{ and } n = 1,2,\ldots,K)\}$$

The following notations are used in this chapter

- $N$ – threshold level
- $K$ – system capacity ($N \leq K$)
- $\lambda_n$ – arrival rate of the system in state $[i,n]$, where $n = 0,1,\ldots,K$
- $B(x)$ – c.d.f. of service time $B$
- $b(x)$ – p.d.f. of service time $B$
- $\tilde{B}(s)$ – Laplace-Stieltjes transform of service time $B$
- $\tilde{B}^{(l)}(s)$ – $l$th order derivative of $\tilde{B}(s)$ with respect to $s$
- $V(x)$ – c.d.f. of vacation time $V$
- $v(x)$ – p.d.f. of vacation time $V$
- $\tilde{V}(s)$ – Laplace-Stieltjes transform of vacation time $V$
- $\tilde{V}^{(l)}(s)$ – $l$th order derivative of $\tilde{V}(s)$ with respect to $s$
- $P_{0,n}(t)$ – probability of $n$ customers in the system at time $t$ when the server is on vacations, where $n = 0,1,\ldots,K$
- $P_{1,n}(t)$ – probability of $n$ customers in the system at time $t$ when the server is working, where $n = 1,2,\ldots,K$
- $P_{0,n}$ – steady-state probability of $n$ customers in the system when the server is on vacations, where $n = 0,1,\ldots,K$
- $P_{1,n}$ – steady-state probability of $n$ customers in the system when the server is working, where $n = 1,2,\ldots,K$
- $\tilde{P}_{0,n}(s)$ – Laplace-Stieltjes transform of $P_{0,n}(x)$ where $i = 0,1$
- $\tilde{P}_{1,n}^{(l)}(s)$ – $l$th order derivative of $\tilde{P}_{1,n}(s)$ with respect to $s$
- $E(B)$ – mean service time
- $E(V)$ – mean vacation time
- $P_B$ – probability an arriving customer is blocked because the system is full
- $L$ – average number of customers in the system
- $L_q$ – average number of customers in the queue
4.2 Formulation

We first establish the mathematical equations to govern the system by employing the remaining service time (vacation time) as the supplementary variable. Next, we develop a recursive method to derive the steady-state probability distributions of the number of customers in the system.

The state of the system at time \( t \) is given by \([Q(t), U(t)]\), where

\[
\begin{align*}
Q(t) & \quad \text{number of customers in the system at time } t \\
U(t) & \quad \text{remaining service time for the customers who is in service} \\
U(t) & \quad \text{remaining vacation time when server is on vacation}
\end{align*}
\]

Let us define

\[
\begin{align*}
P_{0,n}(x,t)dx & = \text{Prob}\{Q(t) = n, x < U(t) \leq x + dx, \text{server is on vacations}\}, \quad x \geq 0, n = 0,1,\ldots,K \\
P_{1,n}(x,t)dx & = \text{Prob}\{Q(t) = n, x < U(t) \leq x + dx, \text{server is busy}\}, \quad x \geq 0, n = 1,2,\ldots,K
\end{align*}
\]

Relating the state of the system at time \( t \) and \( t + dt \), we set up the following partial differential equations

\[
\begin{align*}
- \frac{\partial P_{0,0}(x,t)}{\partial x} + \frac{\partial P_{0,0}(x,t)}{\partial t} & = v(x)P_{1,1}(0,t) - \lambda_0 P_{0,0}(x,t) + v(x)P_{0,0}(0,t) \\
- \frac{\partial P_{0,n}(x,t)}{\partial x} + \frac{\partial P_{0,n}(x,t)}{\partial t} & = \lambda_{n-1} P_{0,n-1}(x,t) - \lambda_n P_{0,n}(x,t) + v(x)P_{0,n}(0,t), \quad 1 \leq n \leq N - 1 \\
- \frac{\partial P_{n,n}(x,t)}{\partial x} + \frac{\partial P_{n,n}(x,t)}{\partial t} & = \lambda_{n-1} P_{0,n-1}(x,t) - \lambda_n P_{n,n}(x,t), \quad N \leq n \leq K - 1 \\
- \frac{\partial P_{0,K}(x,t)}{\partial x} + \frac{\partial P_{0,K}(x,t)}{\partial t} & = \lambda_{K-1} P_{0,K-1}(x,t)
\end{align*}
\]

where \( \lambda_i \) are the effective arrival rates into the system.
In steady state, we define,

\[
\begin{align*}
    P_{0,n} &= \lim_{t \to \infty} P_{0,n}(t), \quad n = 0, 1, \ldots, N, \ldots, K \\
    P_{0,n}(x) &= \lim_{t \to \infty} P_{0,n}(x,t), \quad n = 0, 1, \ldots, N, \ldots, K \\
    P_{1,n} &= \lim_{t \to \infty} P_{1,n}(t), \quad n = 1, 2, \ldots, N, \ldots, K \\
    P_{1,n}(x) &= \lim_{t \to \infty} P_{1,n}(x,t), \quad n = 1, 2, \ldots, N, \ldots, K
\end{align*}
\]

From (4.1)-(4.8), steady-state equations are obtained as follows

\[
\begin{align*}
    -\frac{\partial P_{1,1}(x,t)}{\partial x} + \frac{\partial P_{1,1}(x,t)}{\partial t} &= b(x)P_{1,2}(0,t) - \lambda_1 P_{1,1}(x,t) \quad (4.5) \\
    -\frac{\partial P_{1,n}(x,t)}{\partial x} + \frac{\partial P_{1,n}(x,t)}{\partial t} &= b(x)P_{1,n+1}(0,t) + \lambda_{n-1} P_{1,n-1}(x,t) - \lambda_n P_{1,n}(x,t), \quad 2 \leq n \leq N - 1 \quad (4.6) \\
    -\frac{\partial P_{1,n}(x,t)}{\partial x} + \frac{\partial P_{1,n}(x,t)}{\partial t} &= b(x)P_{0,n}(0,t) + b(x)P_{1,n+1}(0,t) + \lambda_{n-1} P_{1,n-1}(x,t) \\
    -\frac{\partial P_{1,K}(x,t)}{\partial x} + \frac{\partial P_{1,K}(x,t)}{\partial t} &= b(x)P_{0,K}(0,t) + \lambda_{K-1} P_{1,K-1}(x,t) \quad (4.8)
\end{align*}
\]
4.3 Analysis

We define the following Laplace-Stieltjes transforms,

\[
\begin{align*}
\tilde{B}(s) &= \int_0^\infty e^{-sx} b(x) \, dx \\
\tilde{V}(s) &= \int_0^\infty e^{-sx} v(x) \, dx \\
\tilde{P}_{i,n}(s) &= \int_0^\infty e^{-sx} P_{i,n}(x) \, dx, \quad i = 0, 1 \\
P_{i,n} &= \tilde{P}_{i,n}(0) = \int_0^\infty P_{i,n}(x) \, dx, \quad i = 0, 1 \\
\int_0^\infty e^{-sx} \frac{\partial}{\partial x} P_{i,n}(x) \, dx &= s\tilde{P}_{i,n}(s) - P_{i,n}(0)
\end{align*}
\]

Taking Laplace-Stieltjes transforms on both sides of (4.9)-(4.16), we obtain the following

\[
\begin{align*}
(\lambda_0 - s)\tilde{P}_{0,0}(s) &= \tilde{V}(s)P_{1,1}(0) - P_{0,0}(0) + \tilde{V}(s)P_{0,0}(0) \quad (4.17) \\
(\lambda_n - s)\tilde{P}_{0,n}(s) &= \lambda_{n-1} \tilde{P}_{0,n-1}(s) - P_{0,n}(0) + \tilde{V}(s)P_{0,n}(0), \quad 1 \leq n \leq N - 1 \quad (4.18) \\
(\lambda_n - s)\tilde{P}_{0,n}(s) &= \lambda_{n-1} \tilde{P}_{0,n-1}(s) - P_{0,n}(0), \quad N \leq n \leq K - 1 \quad (4.19) \\
-s\tilde{P}_{0,K}(s) &= \lambda_{K-1} \tilde{P}_{0,K-1}(s) - P_{0,K}(0) \quad (4.20)
\end{align*}
\]

\[
\begin{align*}
(\lambda_1 - s)\tilde{P}_{1,1}(s) &= \tilde{B}(s)P_{1,1}(0) - P_{1,1}(0) \quad (4.21) \\
(\lambda_n - s)\tilde{P}_{1,n}(s) &= \tilde{B}(s)P_{1,n+1}(0) + \lambda_{n-1} \tilde{P}_{1,n-1}(s) - P_{1,n}(0), \quad 2 \leq n \leq N - 1 \quad (4.22) \\
(\lambda_n - s)\tilde{P}_{1,n}(s) &= \tilde{B}(s)P_{0,n}(0) + \tilde{B}(s)P_{1,n+1}(0) + \lambda_{n-1} \tilde{P}_{1,n-1}(s) - P_{1,n}(0), \quad N \leq n \leq K - 1 \quad (4.23) \\
-s\tilde{P}_{1,K}(s) &= \tilde{B}(s)P_{0,K}(0) + \lambda_{K-1} \tilde{P}_{1,K-1}(s) - P_{1,K}(0) \quad (4.24)
\end{align*}
\]

Substituting \( s = \lambda_n \) in (4.17),(4.18)and(4.19) and \( s = 0 \) in (4.20) we get,

\[
P_{0,0}(0) = \frac{\tilde{V}(\lambda_0)P_{1,1}(0)}{1 - \tilde{V}(\lambda_0)} \quad (4.25)
\]
\[
P_{0,n}(0) = \frac{\lambda_{n-1} P_{0,n-1}(\lambda_n)}{1 - V(\lambda_n)}, \quad 1 \leq n \leq N - 1 \tag{4.26}
\]
\[
P_{0,n}(0) = \lambda_{n-1} P_{0,n-1}(\lambda_n), \quad N \leq n \leq K - 1 \tag{4.27}
\]
\[
P_{0,K}(0) = \lambda_{K-1} P_{0,K-1} \tag{4.28}
\]

where, \( \tilde{P}_{0,n}(\lambda_{n+1}) \) are given by the following,

For \( w \neq \lambda_0 \)
\[
\tilde{P}_{0,0}(w) = \frac{V(w)(P_{1,1}(0) + P_{0,0}(0)) - P_{0,0}(0)}{(\lambda_0 - w)} \tag{4.29}
\]

For \( w = \lambda_0 \)
\[
\tilde{P}_{0,0}(w) = -V^{(1)}(w)(P_{1,1}(0) + P_{0,0}(0)) \tag{4.30}
\]

For \( w \neq \lambda_n \)
\[
\tilde{P}_{0,n}(w) = \frac{\lambda_{n-1} \tilde{P}_{0,n-1}(w) - P_{0,n}(0) + \tilde{V}(w)P_{0,n}(0)}{(\lambda_n - w)}, \quad 1 \leq n \leq N - 1 \tag{4.31}
\]
\[
\tilde{P}_{0,n}(w) = \frac{\lambda_{n-1} \tilde{P}_{0,n-1}(w) - P_{0,n}(0)}{(\lambda_n - w)}, \quad N \leq n \leq K - 1 \tag{4.32}
\]

For \( w = \lambda_n \)
\[
\tilde{P}_{0,n}(w) = - \left( \lambda_{n-1} \tilde{P}_{0,n-1}^{(1)}(w) + \tilde{V}^{(1)}(w)P_{0,n}(0) \right), \quad 1 \leq n \leq N - 1 \tag{4.33}
\]
\[
\tilde{P}_{0,n}(w) = -\lambda_{n-1} \tilde{P}_{0,n-1}^{(1)}(w), \quad N \leq n \leq K - 1 \tag{4.34}
\]

Substituting \( s = 0 \) in (4.17),(4.18)and(4.19) we get,

\[
\lambda_0 P_{0,0} = P_{1,1}(0) \tag{4.35}
\]
\[
\lambda_n P_{0,n} = \lambda_{n-1} P_{0,n-1}, \quad 1 \leq n \leq N - 1 \tag{4.36}
\]
\[
\lambda_n P_{0,n} = \lambda_{n-1} P_{0,n-1} - P_{0,n}(0), \quad N \leq n \leq K - 1
\]
\[
P_{0,n} = \frac{P_{1,1}(0)}{\lambda_n}, \quad 0 \leq n \leq N - 1 \tag{4.35}
\]
\[
P_{0,n} = \frac{P_{1,1}(0) - \sum_{i=N}^{n} P_{0,i}(0)}{\lambda_n}, \quad N \leq n \leq K - 1 \tag{4.36}
\]
Adding equations (4.17)-(4.20), we get,

\[-s \sum_{i=0}^{K} P_{0,n}(s) = \tilde{V}(s)P_{1,1}(0) + \sum_{i=0}^{N-1} \tilde{V}(s)P_{0,n}(0) - \sum_{i=0}^{K} P_{0,n}(0) \quad (4.37)\]

Differentiating (4.37) w.r.t \(s\) and setting \(s = 0\), we get,

\[- \sum_{i=0}^{K} P_{0,n} = \tilde{V}^{(1)}(0)P_{1,1}(0) + \sum_{i=0}^{N-1} \tilde{V}^{(1)}(0)P_{0,n}(0) \Rightarrow P_{0,K} = -\tilde{V}^{(1)}(0)P_{1,1}(0) - \sum_{i=0}^{N-1} \tilde{V}^{(1)}(0)P_{0,n}(0) - \sum_{i=0}^{K-1} P_{0,n} \]
\[\Rightarrow P_{0,K} = E(V)P_{1,1}(0) + \sum_{i=0}^{N-1} E(V)P_{0,n}(0) - \sum_{i=0}^{K-1} P_{0,n} \quad (4.38)\]

Substituting \(s = \lambda_n\) in (4.21),(4.22)and(4.23) and \(s = 0\) in (4.24) we get,

\[P_{1,1}(0) = \tilde{B}(\lambda_1)P_{1,2}(0)\]
\[P_{1,n}(0) = \tilde{B}(\lambda_n)P_{1,n+1}(0) + \lambda_{n-1}\tilde{P}_{1,n-1}(\lambda_n), \quad 1 \leq n \leq N - 1\]
\[P_{1,n}(0) = \tilde{B}(\lambda_n)P_{0,n}(0) + \tilde{B}(\lambda_n)P_{1,n+1}(0) + \lambda_{n-1}\tilde{P}_{1,n-1}(\lambda_n), \quad N \leq n \leq K - 1\]
\[P_{1,K}(0) = P_{0,K}(0) + \lambda_{K-1}P_{1,K-1}\]
\[P_{1,2}(0) = \frac{P_{1,1}(0)}{B(\lambda_1)} \quad (4.39)\]
\[P_{1,n+1}(0) = \frac{P_{1,n}(0) - \lambda_{n-1}\tilde{P}_{1,n-1}(\lambda_n)}{B(\lambda_n)}, \quad 1 \leq n \leq N - 1 \quad (4.40)\]
\[P_{1,n+1}(0) = \frac{P_{1,n}(0) - \lambda_{n-1}\tilde{P}_{1,n-1}(\lambda_n) - \tilde{B}(\lambda_n)P_{0,n}(0)}{B(\lambda_n)}, \quad N \leq n \leq K - 1 \quad (4.41)\]

where \(\tilde{P}_{1,n}(\lambda_{n+1})\) are given by the following,

For \(w \neq \lambda_1\)

\[\tilde{P}_{1,1}(w) = \frac{\tilde{B}(w)P_{1,2}(0) - P_{1,1}(0)}{(\lambda_1 - w)} \quad (4.42)\]
For $w = \lambda_1$

$$\bar{P}_{1,1}(w) = -\tilde{B}^{(1)}(w)P_{1,2}(0)$$  \hspace{1cm} (4.43)

For $w \neq \lambda_n$

$$\bar{P}_{1,n}(w) = \frac{\tilde{B}(w)P_{1,n+1}(0) + \lambda_{n-1}\bar{P}_{1,n-1}(w) - P_{1,n}(0)}{(\lambda_n - w)}, \hspace{0.5cm} 2 \leq n \leq N - 1$$  \hspace{1cm} (4.44)

$$\tilde{P}_{1,n}(w) = \frac{\tilde{B}(w)(P_{1,n+1}(0) + P_{0,n}(0)) + \lambda_{n-1}\bar{P}_{1,n-1}(w) - P_{1,n}(0)}{(\lambda_n - w)}$$

$$N \leq n \leq K - 1$$  \hspace{1cm} (4.45)

For $w = \lambda_n$

$$\bar{P}_{1,n}(w) = -\left(\tilde{B}^{(1)}(w)P_{1,n+1}(0) + \lambda_{n-1}\bar{P}_{1,n-1}(w)\right), \hspace{0.5cm} 2 \leq n \leq N - 1$$  \hspace{1cm} (4.46)

$$\tilde{P}_{1,n}(w) = -\left(\tilde{B}^{(1)}(w)(P_{1,n+1}(0) + P_{0,n}(0)) + \lambda_{n-1}\bar{P}_{1,n-1}(w)\right),$$

$$N \leq n \leq K - 1$$  \hspace{1cm} (4.47)

Setting $s = 0$ in (4.21),(4.22)and(4.23) we get,

$$\lambda_1P_{1,1} = P_{1,2}(0) - P_{1,1}(0)$$

$$\lambda_nP_{1,n} = \lambda_{n-1}P_{1,n-1} + P_{1,n+1}(0) - P_{1,n}(0)$$

$$\lambda_nP_{1,n} = \lambda_{n-1}P_{1,n-1} + P_{0,n}(0) + P_{1,n+1}(0) - P_{1,n}(0)$$

$$P_{1,n} = \frac{P_{1,n+1}(0) - P_{1,1}(0)}{\lambda_n}, \hspace{0.5cm} 1 \leq n \leq N - 1$$  \hspace{1cm} (4.48)

$$P_{1,n} = \frac{P_{1,n+1}(0) - P_{1,1}(0) + \sum_{i=N}^{n} P_{0,i}(0)}{\lambda_n}, \hspace{0.5cm} N \leq n \leq K - 1$$  \hspace{1cm} (4.49)

Adding equations (4.21)-(4.24), we get,

$$-s \sum_{i=1}^{K} \bar{P}_{1,n}(s) = \tilde{B}(s) \sum_{i=2}^{K} P_{1,n}(0) + \sum_{i=N}^{K} \tilde{B}(s)P_{0,n}(0) - \sum_{i=1}^{K} P_{1,n}(0)$$  \hspace{1cm} (4.50)

Differentiating (4.50) w.r.t $s$ and setting $s = 0$, we get,

$$- \sum_{i=1}^{K} P_{1,n} = \sum_{i=2}^{K} \tilde{B}^{(1)}(0)P_{1,n}(0) + \sum_{i=N}^{K} \tilde{B}^{(1)}(0)P_{0,n}(0)$$
\[ \Rightarrow P_{1,K} = - \sum_{i=2}^{K} \tilde{B}^{(1)}(0)P_{1,n}(0) - \sum_{i=N}^{K} \tilde{B}^{(1)}(0)P_{0,n}(0) - \sum_{i=1}^{K-1} P_{1,n} \]

\[ \Rightarrow P_{1,K} = \sum_{i=2}^{K} E(B)P_{1,n}(0) + \sum_{i=N}^{K} E(B)P_{0,n}(0) - \sum_{i=1}^{K-1} P_{1,n} \quad (4.51) \]

Differentiating (4.17), (4.18) and (4.19) \( l \) times with respect to \( s \), and setting \( s = w \) we get the following expressions for \( \tilde{P}_{0,n}^{(l)}(w) \)

For \( w \neq \lambda_0 \)

\[ \tilde{P}_{0,0}^{(1)}(w) = \frac{\tilde{V}^{(1)}(w)(P_{1,1}(0) + P_{0,0}(0)) + \tilde{P}_{0,0}(w)}{(\lambda_0 - w)} \quad (4.52) \]

\[ \tilde{P}_{0,0}^{(l)}(w) = \frac{\tilde{V}^{(l)}(w)(P_{1,1}(0) + P_{0,0}(0)) + l\tilde{P}_{0,0}^{(l-1)}(w)}{(\lambda_0 - w)}, \quad l \neq 1 \quad (4.53) \]

For \( w = \lambda_0 \)

\[ \tilde{P}_{0,0}^{(l)}(w) = -\frac{\tilde{V}^{(l+1)}(w)(P_{1,1}(0) + P_{0,0}(0))}{l + 1} \quad (4.54) \]

For \( w \neq \lambda_n \) and \( 1 \leq n \leq N - 1 \),

\[ \tilde{P}_{0,n}^{(1)}(w) = \frac{\lambda_{n-1}\tilde{P}_{0,n-1}^{(1)}(w) + \tilde{V}^{(1)}(w)P_{0,n}(0) + \tilde{P}_{0,n}(w)}{(\lambda_n - w)} \quad (4.55) \]

\[ \tilde{P}_{0,n}^{(l)}(w) = \frac{\lambda_{n-1}\tilde{P}_{0,n-1}^{(l)}(w) + \tilde{V}^{(l)}(w)P_{0,n}(0) + l\tilde{P}_{0,n}^{(l-1)}(w)}{(\lambda_n - w)}, \quad l \neq 1 \quad (4.56) \]

For \( w \neq \lambda_n \) and \( N \leq n \leq K - 1 \),

\[ \tilde{P}_{0,n}^{(1)}(w) = \frac{\lambda_{n-1}\tilde{P}_{0,n-1}^{(1)}(w) + \tilde{P}_{0,n}(w)}{(\lambda_n - w)} \quad (4.57) \]

\[ \tilde{P}_{0,n}^{(l)}(w) = \frac{\lambda_{n-1}\tilde{P}_{0,n-1}^{(l)}(w) + l\tilde{P}_{0,n}^{(l-1)}(w)}{(\lambda_n - w)}, \quad l \neq 1 \quad (4.58) \]

For \( w = \lambda_n \)

\[ \tilde{P}_{0,n}^{(l)}(w) = -\frac{(\lambda_{n-1}\tilde{P}_{0,n-1}^{(l+1)}(w) + \tilde{V}^{(l+1)}(w)P_{0,n}(0))}{l + 1}, \quad 1 \leq n \leq N - 1 \quad (4.59) \]

\[ \tilde{P}_{0,n}^{(l)}(w) = -\frac{\lambda_{n-1}\tilde{P}_{0,n-1}^{(l+1)}(w)}{l + 1}, \quad N \leq n \leq K - 1 \quad (4.60) \]

Differentiating (4.21), (4.22) and (4.23) \( l \) times with respect to \( s \), and setting \( s = w \) we get
the following expressions for $\tilde{P}_{1,n}^{(l)}(w)$

For $w \neq \lambda_1$

\[
\tilde{P}_{1,1}^{(1)}(w) = \frac{\tilde{B}^{(1)}(w)P_{1,2}(0) + \tilde{P}_{1,1}(w)}{(\lambda_1 - w)} \quad (4.61)
\]
\[
\tilde{P}_{1,1}^{(l)}(w) = \frac{\tilde{B}^{(l)}(w)P_{1,2}(0) + l\tilde{P}_{1,1}^{(l-1)}(w)}{(\lambda_1 - w)}, \quad l \neq 1 \quad (4.62)
\]

For $w = \lambda_1$

\[
\tilde{P}_{1,1}^{(l)}(w) = -\frac{\tilde{B}^{(l+1)}(w)P_{1,2}(0)}{l+1} \quad (4.63)
\]

For $w \neq \lambda_n$ and $2 \leq n \leq N - 1$,

\[
\tilde{P}_{1,n}^{(1)}(w) = \frac{\tilde{B}^{(1)}(w)P_{1,n+1}(0) + \lambda_{n-1}\tilde{P}_{1,n-1}^{(1)}(w) + \tilde{P}_{1,n}(w)}{(\lambda_n - w)} \quad (4.64)
\]
\[
\tilde{P}_{1,n}^{(l)}(w) = \frac{\tilde{B}^{(l)}(w)P_{1,n+1}(0) + \lambda_{n-1}\tilde{P}_{1,n-1}^{(l)}(w) + l\tilde{P}_{1,n}^{(l-1)}(w)}{(\lambda_n - w)} \quad (4.65)
\]

For $w \neq \lambda_n$ and $N \leq n \leq K - 1$,

\[
\tilde{P}_{1,n}^{(1)}(w) = \frac{\tilde{B}^{(1)}(w)(P_{1,n+1}(0) + P_{0,n}(0)) + \lambda_{n-1}\tilde{P}_{1,n-1}^{(1)}(w) + \tilde{P}_{1,n}(w)}{(\lambda_n - w)} \quad (4.66)
\]
\[
\tilde{P}_{1,n}^{(l)}(w) = \frac{\tilde{B}^{(l)}(w)(P_{1,n+1}(0) + P_{0,n}(0)) + \lambda_{n-1}\tilde{P}_{1,n-1}^{(l)}(w) + l\tilde{P}_{1,n}^{(l-1)}(w)}{(\lambda_n - w)}, \quad l \neq 1 \quad (4.67)
\]

For $w = \lambda_n$

\[
\tilde{P}_{1,n}^{(l)}(w) = -\frac{\tilde{B}^{(l+1)}(w)P_{1,n+1}(0) + \lambda_{n-1}\tilde{P}_{1,n-1}^{(l+1)}(w)}{l+1}, \quad 1 \leq n \leq N - 1 \quad (4.68)
\]
\[
\tilde{P}_{1,n}^{(l)}(w) = -\frac{\tilde{B}^{(l+1)}(w)(P_{1,n+1}(0) + P_{0,n}(0)) + \lambda_{n-1}\tilde{P}_{1,n-1}^{(l+1)}(w)}{l+1}, \quad N \leq n \leq K - 1 \quad (4.69)
\]

An important performance measure for a finite queueing system is the blocking probability. The blocking probability is defined as the probability that an arriving customer is blocked because the system is full. The blocking probability $P_B$ can be determined as follows

\[
P_B = P_{0,K} + P_{1,K} \quad (4.70)
\]
The expected number of customers in the system $L$ and expected number of customers in queue $L_q$ at an arbitrary time is given by

$$L = \sum_{n=0}^{K} nP_{0,n} + \sum_{n=1}^{K}nP_{1,n}$$

$$L_q = \sum_{n=0}^{K} nP_{0,n} + \sum_{n=1}^{K} (n - 1)P_{1,n}$$

Using Little’s law, the expected waiting time in the system $W$ and expected waiting time in the queue $W_q$ are obtained as

$$W = \frac{L}{\lambda'}$$

$$W_q = \frac{L_q}{\lambda'}$$

where, the effective arrival rate $\lambda'$ is given by

$$\lambda' = \sum_{n=0}^{K-1} \lambda_nP_{0,n} + \sum_{n=1}^{K-1} \lambda_nP_{1,n}$$

### 4.4 Algorithm

Algorithm for calculating $P_{0,n}(0), P_{1,n}(0), P_{0,n}, P_{1,n}$ is as follows

Step 1: Calculate $P_{1,1}(0)$ in terms of $P_{0,0}(0)$ as follows

$$P_{1,1}(0) = [1 - \bar{V}(\lambda_0)]/\bar{V}(\lambda_0)$$

Step 2: For $n = 0, 1, \ldots, K - 1$,

(i) Calculate $P_{0,n}(0)$ in terms of $P_{0,0}(0)$ as follows

If $n = 0$

$$P_{0,n}(0) = 1$$
Else If \( n < N \)
\[
P_{0,n}(0) = \frac{\lambda_{n-1} \ast \bar{P}_{0,n-1}(\lambda_n)}{1 - \bar{V}(\lambda_n)}
\]
Else If \( N \leq n \leq K - 1 \)
\[
P_{0,n}(0) = \lambda_{n-1} \ast \bar{P}_{0,n-1}(\lambda_n)
\]
End If

(ii) For \( j = n + 1, n + 2, \ldots, K \),

(a) Calculate \( \bar{P}_{0,n}(\lambda_j) \) in terms of \( P_{0,0}(0) \) as follows

If \( n = 0 \)

If \( \lambda_0 = \lambda_j \)
\[
\bar{P}_{0,0}(\lambda_j) = -\bar{V}^{(1)}(\lambda_j) \ast (P_{1,1}(0) + P_{0,0}(0))
\]
Else
\[
\bar{P}_{0,0}(\lambda_j) = \frac{\bar{V}(\lambda_j) \ast (P_{1,1}(0) + P_{0,0}(0)) - P_{0,0}(0)}{(\lambda_0 - \lambda_j)}
\]
End If

Else If \( 1 \leq n \leq N - 1 \)

If \( \lambda_n = \lambda_j \)
\[
\bar{P}_{0,n}(\lambda_j) = -\left( \lambda_{n-1} \ast \bar{P}_{0,n-1}^{(1)}(\lambda_j) + \bar{V}^{(1)}(\lambda_j) \ast P_{0,n}(0) \right)
\]
Else
\[
\bar{P}_{0,n}(\lambda_j) = \frac{\lambda_{n-1} \ast \bar{P}_{0,n-1}(\lambda_j) - P_{0,n}(0) + \bar{V}(\lambda_j) \ast P_{0,n}(0)}{(\lambda_n - \lambda_j)}
\]
End If
Else If $N \leq n \leq K - 1$

If $\lambda_n = \lambda_j$

\[
\tilde{P}_{0,n}(\lambda_j) = -\lambda_{n-1} * \tilde{P}_{0,n-1}^{(1)}(\lambda_j)
\]

Else

\[
\tilde{P}_{0,n}(\lambda_j) = \frac{\lambda_{n-1} * \tilde{P}_{0,n-1}(\lambda_j) - P_{0,n}(0)}{(\lambda_n - \lambda_j)}
\]

End If

End If

(b) Calculate $\tilde{P}_{0,n}^{(l)}(\lambda_j)$ for $l = 1, 2, \ldots, K - n$, in terms of $P_{0,0}(0)$ as follows

If $n = 0$

If $\lambda_0 = \lambda_j$

\[
\tilde{P}_{0,0}^{(l)}(\lambda_j) = -\frac{\tilde{V}^{(l+1)}(\lambda_j) * (P_{1,1}(0) + P_{0,0}(0))}{l + 1}
\]

Else

\[
\tilde{P}_{0,0}^{(1)}(\lambda_j) = \frac{\tilde{V}^{(1)}(\lambda_j) * (P_{1,1}(0) + P_{0,0}(0)) + \tilde{P}_{0,0}(\lambda_j)}{(\lambda_0 - \lambda_j)}
\]

\[
\tilde{P}_{0,0}^{(l)}(\lambda_j) = \frac{\tilde{V}^{(l)}(\lambda_j) * (P_{1,1}(0) + P_{0,0}(0)) + l * \tilde{P}_{0,0}^{(l-1)}(\lambda_j)}{(\lambda_0 - \lambda_j)}, \quad l \neq 1
\]

End If

Else If $1 \leq n \leq N - 1$

If $\lambda_n = \lambda_j$

\[
\tilde{P}_{0,n}^{(l)}(\lambda_j) = -\frac{\left(\lambda_{n-1} * \tilde{P}_{0,n-1}^{(l+1)}(\lambda_j) + \tilde{V}^{(l+1)}(\lambda_j) * P_{0,n}(0)\right)}{l + 1}
\]
Else

\[ \tilde{P}_{0,n}^{(1)}(\lambda_j) = \frac{\lambda_{n-1} \ast \tilde{P}_{0,n-1}^{(1)}(\lambda_j) + \tilde{V}^{(1)}(\lambda_j) \ast P_{0,n}(0) + \tilde{P}_{0,n}(\lambda_j)}{(\lambda_n - \lambda_j)} \]

\[ \tilde{P}_{0,n}^{(l)}(\lambda_j) = \frac{\lambda_{n-1} \ast \tilde{P}_{0,n-1}^{(l)}(\lambda_j) + \tilde{V}^{(l)}(\lambda_j) \ast P_{0,n}(0) + l \ast \tilde{P}_{0,n}^{(l-1)}(\lambda_j)}{(\lambda_n - \lambda_j)} \], \ l \neq 1

End If

Else If \( N \leq n \leq K - 1 \)

If \( \lambda_n = \lambda_j \)

\[ \tilde{P}_{0,n}^{(l)}(\lambda_j) = -\frac{\lambda_{n-1} \ast \tilde{P}_{0,n-1}^{(l+1)}(\lambda_j)}{l + 1} \]

Else

\[ \tilde{P}_{0,n}^{(1)}(\lambda_j) = \frac{\lambda_{n-1} \ast \tilde{P}_{0,n-1}^{(1)}(\lambda_j) + \tilde{P}_{0,n}(\lambda_j)}{(\lambda_n - \lambda_j)} \]

\[ \tilde{P}_{0,n}^{(l)}(\lambda_j) = \frac{\lambda_{n-1} \ast \tilde{P}_{0,n-1}^{(l)}(\lambda_j) + l \ast \tilde{P}_{0,n}^{(l-1)}(\lambda_j)}{(\lambda_n - \lambda_j)} \], \ l \neq 1

End If

End If

Step 3: For \( n = 0, 1, \ldots, K - 1 \), calculate \( P_{0,n} \) in terms of \( P_{0,0}(0) \) as follows

\[ P_{0,0} = P_{1,1}(0)/\lambda_0 \]

\[ P_{0,n} = \lambda_{n-1} \ast P_{0,n-1}/\lambda_n, \quad 1 \leq n \leq N - 1 \]

\[ P_{0,n} = (\lambda_{n-1} \ast P_{0,n-1} - P_{0,n}(0))/\lambda_n, \quad N \leq n \leq K - 1 \]

Step 4: Calculate \( P_{0,K} \) and \( P_{0,K}(0) \) in terms of \( P_{0,0}(0) \) as follows

\[ P_{0,K}(0) = \lambda_{K-1} \ast P_{0,K-1} \]

\[ P_{0,K} = E(V) \ast P_{1,1}(0) + E(V) \ast \sum_{i=0}^{N-1} P_{0,n}(0) - \sum_{i=0}^{K-1} P_{0,n} \]
Step 5: For \( n = 1, 2, \ldots, K - 1, \)

(i) Calculate \( P_{1,n}(0) \) in terms of \( P_{0,0}(0) \) as follows

If \( n = 1 \)

\[
P_{1,n+1}(0) = P_{1,1}(0) / \bar{B}(\lambda_1)
\]

Else If \( n < N \)

\[
P_{1,n+1}(0) = \frac{P_{1,n}(0) - \lambda_{n-1} \cdot \bar{P}_{1,n-1}(\lambda_n)}{\bar{B}(\lambda_n)}
\]

Else If \( N \leq n \leq K - 1 \)

\[
P_{1,n+1}(0) = \frac{P_{1,n}(0) - \lambda_{n-1} \cdot \bar{P}_{1,n-1}(\lambda_n) - \bar{B}(\lambda_n) \cdot P_{0,n}(0)}{\bar{B}(\lambda_n)}
\]

End If

(ii) For \( j = n + 1, n + 2, \ldots, K, \)

(a) Calculate \( \bar{P}_{1,n}(\lambda_j) \) in terms of \( P_{0,0}(0) \) as follows

If \( n = 1 \)

If \( \lambda_1 = \lambda_j \)

\[
\bar{P}_{1,1}(\lambda_j) = -\bar{B}^{(1)}(\lambda_j) \cdot P_{1,2}(0)
\]

Else

\[
\bar{P}_{1,1}(\lambda_j) = \frac{\bar{B}(\lambda_j) \cdot P_{1,2}(0) - P_{1,1}(0)}{(\lambda_1 - \lambda_j)}
\]

End If

Else If \( 2 \leq n \leq N - 1 \)
If $\lambda_n = \lambda_j$

$$\tilde{P}_{1,n}(\lambda_j) = -\left(\tilde{B}^{(1)}(\lambda_j) * P_{1,n+1}(0) + \lambda_{n-1} * \tilde{P}_{1,n-1}^{(1)}(\lambda_j)\right)$$

Else

$$\tilde{P}_{1,n}(\lambda_j) = \frac{\tilde{B}(\lambda_j) * P_{1,n+1}(0) + \lambda_{n-1} * \tilde{P}_{1,n-1}(\lambda_j) - P_{1,n}(0)}{(\lambda_n - \lambda_j)}$$

End If

Else If $N \leq n \leq K - 1$

If $\lambda_n = \lambda_j$

$$\tilde{P}_{1,n}(\lambda_j) = -\left(\tilde{B}^{(1)}(\lambda_j) * (P_{1,n+1}(0) + P_{0,n}(0)) + \lambda_{n-1} * \tilde{P}_{1,n-1}^{(1)}(\lambda_j)\right)$$

Else

$$\tilde{P}_{1,n}(\lambda_j) = \frac{\tilde{B}(\lambda_j) * (P_{1,n+1}(0) + P_{0,n}(0)) + \lambda_{n-1} * \tilde{P}_{1,n-1}(\lambda_j) - P_{1,n}(0)}{(\lambda_n - \lambda_j)}$$

End If

End If

(b) Calculate $\tilde{P}_{1,n}^{(l)}(\lambda_j)$ for $l = 1, 2, \ldots, K - n$, in terms of $P_{0,0}(0)$ as follows

If $n = 1$

If $\lambda_1 = \lambda_j$

$$\tilde{P}_{1,1}^{(l)}(\lambda_j) = -\frac{\tilde{B}^{(l+1)}(\lambda_j) * P_{1,2}(0)}{l + 1}$$

Else

$$\tilde{P}_{1,1}^{(l)}(\lambda_j) = \frac{\tilde{B}^{(l)}(\lambda_j) * P_{1,2}(0) + \tilde{P}_{1,1}(\lambda_j)}{(\lambda_1 - \lambda_j)}$$

$$\tilde{P}_{1,1}^{(l)}(\lambda_j) = \frac{\tilde{B}^{(l)}(\lambda_j) * P_{1,2}(0) + l * \tilde{P}_{1,1}^{(l-1)}(\lambda_j)}{(\lambda_1 - \lambda_j)}, \quad l \neq 1$$
End If

Else If 2 ≤ n ≤ N – 1

If \( \lambda_n = \lambda_j \)

\[
\tilde{P}_{1,n}^{(l)}(\lambda_j) = -\left[ \frac{\tilde{B}^{(l+1)}(\lambda_j) * P_{1,n+1}(0) + \lambda_{n-1} * \tilde{P}_{1,n-1}^{(l+1)}(\lambda_j)}{l+1} \right]
\]

Else

\[
\tilde{P}_{1,n}^{(l)}(\lambda_j) = \frac{\tilde{B}^{(l)}(\lambda_j) * P_{1,n+1}(0) + \lambda_{n-1} * \tilde{P}_{1,n-1}^{(l)}(\lambda_j) + \tilde{P}_{1,n}(\lambda_j)}{(\lambda_n - \lambda_j)}
\]

End If

End If

Else If N ≤ n ≤ K – 1

If \( \lambda_n = \lambda_j \)

\[
\tilde{P}_{1,n}^{(l)}(\lambda_j) = -\left[ \frac{\tilde{B}^{(l+1)}(\lambda_j) * (P_{1,n+1}(0) + P_{0,n}(0)) + \lambda_{n-1} \tilde{P}_{1,n-1}^{(l+1)}(\lambda_j)}{l+1} \right]
\]

Else

\[
\tilde{P}_{1,n}^{(l)}(\lambda_j) = \frac{\tilde{B}^{(l)}(\lambda_j) * (P_{1,n+1}(0) + P_{0,n}(0)) + \lambda_{n-1} \tilde{P}_{1,n-1}^{(l)}(\lambda_j)}{(\lambda_n - \lambda_j)}
\]

\[
+ \frac{l * \tilde{P}_{1,n}^{(l-1)}(\lambda_j)}{(\lambda_n - \lambda_j)}, \quad l \neq 1
\]

End If

End If

Step 6: For \( n = 1, 2, \ldots, K - 1 \), calculate \( P_{1,n} \) in terms of \( P_{0,0}(0) \) as follows
\[ P_{1,1} = \frac{P_{1,2}(0) - P_{1,1}(0)}{\lambda_1} \]

\[ P_{1,n} = \frac{\lambda_{n-1} * P_{1,n-1} + P_{1,n+1}(0) - P_{1,n}(0)}{\lambda_n}, \quad 2 \leq n \leq N - 1 \]

\[ P_{1,n} = \frac{\lambda_{n-1} * P_{1,n-1} + P_{0,n}(0) + P_{1,n+1}(0) - P_{1,n}(0)}{\lambda_n}, \quad N \leq n \leq K - 1 \]

Step 7: Calculate \( P_{1,K} \) in terms of \( P_{0,0}(0) \) as follows

\[ P_{1,K} = E(B) * \sum_{i=2}^{K} P_{1,n}(0) + E(B) * \sum_{i=N}^{K} P_{0,n}(0) - \sum_{i=1}^{K-1} P_{1,n} \]

Step 8: For \( n = 0, 1, \ldots, K \), The numerical values of \( P_{0,n} \) and \( P_{1,n} \), \( P'_{0,n} \) and \( P'_{1,n} \) respectively are obtained as follows

\[ P'_{0,n} = \frac{P_{0,n}}{\sum_{n=0}^{K} P_{0,n} + \sum_{n=1}^{K} P_{1,n}} \]

\[ P'_{1,n} = \frac{P_{1,n}}{\sum_{n=0}^{K} P_{0,n} + \sum_{n=1}^{K} P_{1,n}} \]

We find that the number of iterations in the algorithm is in the order of \( K^3 \) where \( K \) is the maximum capacity of the system.

**Remark:** It should be noted that if \( N = 1 \) the algorithm has to be modified slightly for the calculation of \( P_{1,1}(0) \) and \( P_{1,1} \).

### 4.5 Numerical Examples

We consider a \( M(n)/G/1/K \) queueing system with \( K = 10 \), mean service time 1/8 and mean vacation time 1/10. The values of \( \lambda_n \) are \( \lambda_0 = 5.0, \lambda_1 = 5.5, \lambda_2 = 6.0, \lambda_3 = 6.5, \lambda_4 = 7.0, \lambda_5 = 7.5, \lambda_6 = 8.0, \lambda_7 = 8.5, \lambda_8 = 9.0 \) and \( \lambda_9 = 9.5 \). \( N \) is varied from 1 to 10 and the blocking probability \( P_B \), the average number of customers in the system \( L \) and the waiting time in queue \( W_q \) are calculated for each instance.
First we consider different service time distributions with exponential vacation times. The service time distributions studied are exponential, 3-stage Erlang and hyperexponential respectively. For each of these distributions, $L$, $W_q$ and $P_B$ are plotted against $N$ respectively (See Figures 4.1, 4.2 and 4.3).

For exponential service with mean service time $1/8$, the service time distribution and the Laplace-Stieltjes transform are given as follows

$$b(x) = 8e^{-8x}$$
$$B(x) = 1 - e^{-8x}$$
$$\tilde{B}(s) = \frac{8}{8 + s}$$

For 3-stage Erlang service ($E_3$) with mean service time $1/8$, the service time distribution and the Laplace-Stieltjes transform are given as follows

$$b(x) = \frac{24}{24 + s} e^{-24x} \left(e^{-(24x)^2/24} - 1\right)$$
$$B(x) = 1 - \sum_{r=0}^{2} e^{-24x} \left(e^{-(24x)^r/24} - 1\right)$$
$$\tilde{B}(s) = \left(\frac{24}{24 + s}\right)^3$$

For hyperexponential service with $q_1 = 0.4$, $q_2 = 0.6$ and $\lambda_1 = 16$, $\lambda_2 = 6$ (mean service time = $1/8$), the service time distribution and the Laplace-Stieltjes transform are given as follows

$$b(x) = 0.4 \cdot 16e^{-16x} + 0.6 \cdot 6e^{-6x}$$
$$b(x) = 0.4(1 - e^{-16x}) + 0.6(1 - e^{-6x})$$
$$\tilde{B}(s) = 0.4 \left(\frac{16}{16 + s}\right) + 0.6 \left(\frac{6}{6 + s}\right)$$

For deterministic service with service time $= 1/8$, the service time distribution and the Laplace-Stieltjes transform are given as follows

$$\tilde{A}(s) = e^{-s/8}$$

In figure 4.1, we see that $L$ is higher for the service time distribution with the lower variance. As $N$ increases $L$ increases, with the increase in $L$ more for the service time
distribution with the lower variance. Also from figure 4.2, $W_q$ is higher for the service time distribution with the higher variance. As $N$ increases $W_q$ increases, with the increase in $W_q$ more for the service time distribution with the higher variance. And from figure 4.3 the blocking probability $P_B$ is lower for the service time distribution with the lower variance. Also $P_B$ increases with increasing $N$.

Then we consider exponential service with different vacation time distributions. The vacation time distributions considered are exponential, 3-stage Erlang and hyperexponential respectively. For each of these distributions, $L$, $W_q$ and $P_B$ are plotted against $N$ respectively (See Figures 4.4, 4.5 and 4.6).

For exponential vacation with mean vacation time $1/10$, the vacation time distribution and the Laplace-Stieltjes transform are given as follows

$$b(x) = 10e^{-10x}$$
$$B(x) = 1 - e^{-10x}$$
$$\tilde{V}(s) = \frac{10}{10 + s}$$

For 3-stage Erlang vacation ($E_3$) with mean vacation time $1/10$, the vacation time distribution and the Laplace-Stieltjes transform are given as follows

$$v(x) = 30e^{-30x}\frac{(30x)^2}{2!}$$
$$V(x) = 1 - 2 \sum_{r=0}^{2} e^{-30x}\frac{(30x)^r}{r!}$$
$$\tilde{V}(s) = \left(\frac{30}{30 + s}\right)^3$$

For hyperexponential vacation with $q_1 = 0.4$, $q_2 = 0.6$ and $\lambda_1 = 16$, $\lambda_2 = 8$ (mean vacation time = 1/10), the vacation time distribution and the Laplace-Stieltjes transform are given as follows

$$v(x) = 0.4 \times 16e^{-16x} + 0.6 \times 8e^{-8x}$$
$$V(x) = 0.4(1 - e^{-16x}) + 0.6(1 - e^{-8x})$$
$$\tilde{V}(s) = 0.4\left(\frac{16}{16 + s}\right) + 0.6\left(\frac{8}{8 + s}\right)$$
For deterministic vacation with vacation time = 1/10, the vacation time distribution and the Laplace-Stieltjes transform are given as follows

\[ \tilde{V}(s) = e^{-s/10} \]

In figure 4.4, we see that \( L \) is higher for the vacation time distribution with the higher variance. As \( N \) increases \( L \) increases, with the increase in \( L \) more for the vacation time distribution with the lower variance. Also from figure 4.5, \( W_q \) is higher for the vacation time distribution with the higher variance. As \( N \) increases \( W_q \) increases, with the increase in \( W_q \) more for the vacation time distribution with the lower variance. And from figure 4.6 the blocking probability \( P_B \) is lower for the vacation time distribution with the lower variance. Also \( P_B \) increases with increasing \( N \).
Figure 4.1: Effect of $N$ on $L$ for service time distributions $M$, $E_3$, $H_2$ and $D$.

Figure 4.2: Effect of $N$ on $W_q$ for service time distributions $M$, $E_3$, $H_2$ and $D$. 
Figure 4.3: Effect of $N$ on $P_B$ for service time distributions $M$, $E_3$, $H_2$ and $D$

Figure 4.4: Effect of $N$ on $L$ for vacation time distributions $M$, $E_3$, $H_2$ and $D
Figure 4.5: Effect of $N$ on $W_q$ for vacation time distributions $M$, $E_3$, $H_2$ and $D$

Figure 4.6: Effect of $N$ on $P_B$ for vacation time distributions $M$, $E_3$, $H_2$ and $D$
Chapter 5

Conclusions and Future Research

5.1 Conclusions

In this thesis, the problem of finding the stationary queue length distribution of multiple server queueing systems with finite buffer and vacations was addressed. Using the method of supplementary variables to analyze the system, we developed a recursive algorithm for numerically computing the stationary queue length distribution of the following $N$ policy queueing systems with vacations:

- $G/M(n)/1/K$ queues with multiple exponential vacations
- $G/M(n)/1/K$ queues with multiple exponential vacations that are state-dependent
- $M(n)/G/1/K$ queues with multiple arbitrary vacations

The complexity of the algorithm was found to be $O(n^3)$ for all the three cases. The algorithm is powerful and easy to implement. Using the steady state results, we also calculated system performance measures such as blocking probability, expected queue length, expected waiting time, etc.
5.2 Future Research

A natural extension of this work would be to consider priority queues with server vacations. As a first step, queueing systems with multiple customer classes and server vacations and no priorities can be considered. Another possible extension is considering systems where the server is unreliable.

Other possible extensions include studying queues where the service discipline is different from FIFO (First In First Out). It is also possible to consider queues with non-exhaustive service (e.g., gated service).
Bibliography


