MANTRI, PARAG. Deployment Dynamics of Space Tether Systems. (Under the direction of Dr. Andre Mazzoleni.)

The purpose of this research is to model and understand the deployment dynamics of space systems with long and short tethers. This research is divided into two parts; in the first part, a model for short and medium length tether systems is developed and simulated by solving equations of motion. A detailed parametric study is conducted after identifying important parameters affecting the deployment and studying the effect of each parameter on the deployment performance. Certain tools are developed to assist mission planners in predicting the deployment performance of a space tether system for a given set of parameters. The second part of the research is motivated by Space Elevator (SE) dynamics. SE is a futuristic and highly challenging technology based on the idea of connecting Earth and Space by an approximately 100,000 km long tether. The tether used for the SE would be deployed from Geostationary Orbit (GEO). With this motivation, the short tether analysis from the previous section is extended to the analysis of long tethers. A model for long tether deployment is developed and governing equations of motions are formulated. Critical parameters are identified and problems involved in SE deployment are investigated. Tether mass is initially included in the model, but it is found that that the mass of the tether has very little effect on the overall qualitative dynamics of the system. Hence, for further analysis, a massless tether model is adopted. Upon simulating the system, it is found that long tethers can be highly unstable during deployment and can crash onto the Earth. However, a considerable fraction of the tether can be deployed successfully without any external control mechanism before the instability manifests itself. Hence, alternate SE designs with shorter tether deployment requirements may be a possibility.
DEPLOYMENT DYNAMICS OF SPACE TETHER SYSTEMS

by

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A dissertation submitted to the Graduate Faculty of North Carolina State University in partial fulfillment of the requirements for the degree of Doctor of Philosophy

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Dedication

I dedicate this thesis to my Parents and in memory of my late grandparents.
Biography

Parag Mantri was born on September 25th, 1979 in the historical city of Ujjain, MP, India. After finishing his undergraduate degree in Mechanical Engineering from Osmania University, Hyderabad, India in 2001, he came to the USA to pursue a Master’s degree in Mechanical Engineering at Tufts University in Boston, MA. He started his Ph.D in Aerospace Engineering at North Carolina State University in the Fall of 2004. Parag’s parents (Subhash Mantri and Pushpa Mantri) live in Hyderabad, AP, India.
About three years ago, my life would have taken a completely different path if my adviser, Dr. Andre Mazzoleni, would not have given me an opportunity to pursue Ph.D. at NCSU. Today I am able to achieve this goal only because of his support in my academic as well as personal life and most importantly his attitude and dedication toward teaching and successful research. I thank him for all this and for everything I have learned from him. I would also like to thank my committee members, Dr. Larry Silverberg, Dr. Gregory Buckner and Dr. James Selgrade for serving on my committee and guiding me through my thesis. I am grateful to Dr. Jintai Chung for some of the very critical feedback on my research. Also, thanks to my research lab members, David, Alex, Scott and Brian, for their constant feedback on my work. My internship at LiftPort, under the guidance of Michael Laine and Tom Nugent, has immensely contributed to my research and I am very thankful to them for giving me this opportunity. Other people whom I would to thank are Dr. Gary Howell for his technical support on high performance computing at NCSU, MATLAB support team for their prompt responses to my queries, thousands of $\LaTeX$ users around the world who have posted unlimited information, tips and packages on internet which helped me in smoothly documenting this thesis and my presentation. On personal side, my parents, my sister and my brother-in-law have very important contribution of all the emotional support I needed during my research. And on the lighter side, I thank my DELL Inspiron 5100 laptop for taking intense load of running almost every software I have used during my research.
# Table of Contents

List of Tables vi

List of Figures vii

List of Symbols viii

List of Abbreviations xi

1 Introduction 1

1.1 Background ........................................ 2

1.1.1 Short and Medium Length Tethers ....................... 2

1.1.2 Long Tethers .................................. 3

1.2 Motivation for Studying Tether Deployment Dynamics .......... 4

1.2.1 Short and Medium Length Tethers ....................... 4

1.2.2 Long Tethers .................................. 5

2 Short and Medium Length Tether Analysis 7

2.1 Model ............................................. 7

2.2 Equations of Motion: Derivation .......................... 10

2.2.1 Forces ...................................... 10

2.2.2 Acceleration ................................ 12

2.3 Analysis .......................................... 15

2.3.1 Operating Ranges of Dimensional Parameters ............. 15

2.3.2 Effect of Each Parameter on Deployment ................. 17

2.3.3 Nondimensionalization ................................ 27

2.3.4 Grids and Log-plots ................................ 30

2.4 Applications ....................................... 36

2.4.1 Mission Design from Given Parameters .................. 36

2.4.2 Mission Parameter Determination ....................... 42
3 The Space Elevator 44
3.1 Background ................................................. 44
3.2 System Geometry ........................................... 45
3.3 Material Requirements ..................................... 47
3.4 Other Space Elevator Ideas ................................. 48
   3.4.1 Planetary space elevators ............................ 48
   3.4.2 Alternate Configurations .............................. 49
4 Long Tether Analysis 51
4.1 Deployment of Space Elevator and other long tethered systems .... 51
4.2 Model ....................................................... 52
4.3 Equations of Motion: Derivation ............................ 53
4.4 Massless tether model ..................................... 53
   4.4.1 Forces .................................................. 57
   4.4.2 Equations with nondimensionalized mass .......... 61
4.5 Massive tether model ...................................... 63
4.6 Massless vs Massive Tether Model ......................... 72
4.7 Massless Tether Analysis ................................ 87
4.8 Deployment as a Function of Initial Alignment ............. 94
4.9 Deployment from Various Altitudes ......................... 96
4.10 Summary of Analysis ..................................... 99
5 Conclusions 100
5.1 Short and Medium Length Tethers ........................ 100
5.2 Long Tether Analysis ...................................... 101
5.3 Concluding Remarks ....................................... 101

List of References 103
## List of Tables

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Deployment results as a function of increasing or decreasing individual</td>
<td>28</td>
</tr>
<tr>
<td></td>
<td>dimensional parameters</td>
<td></td>
</tr>
<tr>
<td>3.1</td>
<td>Taper ratio comparison</td>
<td>47</td>
</tr>
</tbody>
</table>
List of Figures

2.1 Deployment Steps ............................................. 9
2.2 Representation of $\theta$ and $\phi$ .................................. 9
2.3 Deployment as a function of IPs at $[T_0 = 100 \text{ mN}, \bar{m} = 10 \text{ kg}, \dot{L}_0 = 5 \text{ m/s}]$ ........................................... 22
2.4 Deployment as a function of IPs at $[T_0 = 100 \text{ mN}, \bar{m} = 15 \text{ kg}, \dot{L}_0 = 5 \text{ m/s}]$ ........................................... 22
2.5 Deployment as a function of IPs at $[T_0 = 100 \text{ mN}, \bar{m} = 17.5 \text{ kg}, \dot{L}_0 = 5 \text{ m/s}]$ ........................................... 23
2.6 Deployment as a function of IPs at $[T_0 = 100 \text{ mN}, \bar{m} = 17.5 \text{ kg}, \dot{L}_0 = 4 \text{ m/s}]$ ........................................... 23
2.7 Deployment as a function of IPs at $[T_0 = 75 \text{ mN}, \bar{m} = 17.5 \text{ kg}, \dot{L}_0 = 4 \text{ m/s}]$ ........................................... 24
2.8 Deployment as a function of IC at $[T_0 = 50 \text{ mN}, \bar{m} = 17.5 \text{ kg}, \dot{L}_0 = 4 \text{ m/s}]$ ........................................... 24
2.9 Deployment as a function of IPs at $[T_0 = 50 \text{ mN}, \bar{m} = 17.5 \text{ kg}, \dot{L}_0 = 3 \text{ m/s}]$ ........................................... 25
2.10 Deployment as a function of IPs at $[T_0 = 50 \text{ mN}, \bar{m} = 17.5 \text{ kg}, \dot{L}_0 = 4 \text{ m/s}]$ ........................................... 25
2.11 Deployment as a function of IPs at $[T_0 = 50 \text{ mN}, \bar{m} = 17.5 \text{ kg}, \dot{L}_0 = 5 \text{ m/s}]$ ........................................... 26
2.12 AD Grid for Selected $(\bar{L}_0', \bar{T}_0)$ values ...................... 32
2.13 PFD Grid for Selected $(\bar{L}_0', \bar{T}_0)$ values ...................... 33
2.14 AD Log Plot for Various Values of $(\bar{L}_0', \bar{T}_0)$ .................. 34
2.15 PFD Log Plot for Various Values of $(\bar{L}_0', \bar{T}_0)$ .................. 35
2.16 Zoom in on AD grid for $(\bar{L}_0 = 1.01, \bar{T}_0 = 5.02)$ ............ 37
2.17 Zoom in on PFD grid for $(\bar{L}_0 = 1.01, \bar{T}_0 = 5.02)$ ............ 37
2.18 AD log plot for $(\bar{L}_0 = 2.02, \bar{T}_0 = 2.87)$ .................. 39
2.19 PFD log plot for $(\bar{L}_0 = 2.02, \bar{T}_0 = 2.87)$ .................. 39
2.20 Deployment curves for $(\bar{L}_0 = 1.01, \bar{T}_0 = 5.02)$ ............ 40
2.21 Deployment curves for \( \bar{L}_0 = 2.02, \bar{T}_0 = 2.87 \) ........................................ 40
2.22 Deployment curves for \( \bar{L}_0 = 1.4, \bar{T}_0 = 2.02 \) ........................................ 41
2.23 AD Subgrid for Parameter Identification. \( \bar{L}_0 \) arranged by columns and \( \bar{T}_0 \) arranged by rows .................................................. 43

3.1 SE System Geometry ......................................................... 46
3.2 Lunar Space Elevator ......................................................... 48
3.3 LEO Space Elevator .......................................................... 50

4.1 System Definition .............................................................. 53
4.2 Deployment stages of Bead Model ........................................... 64
4.3 \( \theta \) behavior for parameters \( \bar{m} = -0.8, \bar{L}_0 = 1 \text{ m/s} \) ......................... 75
4.4 \( \theta \) behavior for parameters \( \bar{m} = -0.8, \bar{L}_0 = 10 \text{ m/s} \) ..................... 76
4.5 \( \theta \) behavior for parameters \( \bar{m} = -0.8, \bar{L}_0 = 10 \text{ m/s} \) ..................... 77
4.6 \( \theta \) behavior for parameters \( \bar{m} = 0, \bar{L}_0 = 1 \text{ m/s} \) ............................ 80
4.7 \( \theta \) behavior for parameters \( \bar{m} = 0.8, \bar{L}_0 = 1 \text{ m/s} \) ......................... 81
4.8 \( \theta \) behavior for parameters \( \bar{m} = 0.8, \bar{L}_0 = 10 \text{ m/s} \) ..................... 82
4.9 \( \theta \) behavior for parameters \( \bar{m} = 0.8, \bar{L}_0 = 10 \text{ m/s} \) ..................... 83
4.10 \( \theta \) behavior for parameters \( \bar{m} = 0.8, \bar{L}_0 = 10 \text{ m/s} \) ..................... 84
4.11 \( \theta \) behavior for parameters \( \bar{m} = 0.8, \bar{L}_0 = 10 \text{ m/s} \) ..................... 85
4.12 \( \theta \) behavior for parameters \( \bar{m} = 0.8, \bar{L}_0 = 10 \text{ m/s} \) ..................... 86
4.13 \( \theta \) behavior for parameters \( \bar{m} = 0.8, \bar{L}_0 = 10 \text{ m/s} \) ..................... 87
4.14 \( \theta \) behavior for parameters \( \bar{m} = 0.8, \bar{L}_0 = 10 \text{ m/s} \) ..................... 88
4.15 Deployment dynamics at various speed for \( \bar{m} = -0.8 \) .......................... 88
4.16 Deployment dynamics at various speed for \( \bar{m} = -0.4 \) .......................... 88
4.17 Deployment dynamics at various speed for \( \bar{m} = 0 \) ............................... 89
4.18 Deployment dynamics at various speed for \( \bar{m} = 0.4 \) ............................. 89
4.19 Deployment dynamics at various speed for \( \bar{m} = 0.8 \) ............................. 90
4.20 Dynamics of \( R \) with respect to \( e \) and \( L \) at \( \bar{L}_0 = 1 \text{ m/s} \) .................. 91
4.21 Dynamics of \( R \) with respect to \( e \) and \( L \) at \( \bar{L}_0 = 5 \text{ m/s} \) .................. 92
4.22 Dynamics of \( R \) with respect to \( e \) and \( L \) at \( \bar{L}_0 = 10 \text{ m/s} \) ............... 92
4.23 Dynamics of \( R \) with respect to \( e \) and \( L \) at \( \bar{L}_0 = 25 \text{ m/s} \) ................. 93
4.24 \( R \) and \( \theta \) behavior. Case: \( \bar{L}_0 = 25 \text{ m/s}, \bar{m} = -0.4 \) ...................... 94
4.25 \( R \) and \( \theta \) behavior. Case: \( \bar{L}_0 = 25 \text{ m/s}, \bar{m} = 0 \) ......................... 95
4.26 \( R \) and \( \theta \) behavior. Case: \( \bar{L}_0 = 25 \text{ m/s}, \bar{m} = 0.4 \) ....................... 95
4.27 \( R \) and \( \theta \) behavior. Case: \( \bar{L}_0 = 1 \text{ m/s}, \bar{m} = -0.8 \) ....................... 97
4.28 \( R \) and \( \theta \) behavior. Case: \( \bar{L}_0 = 10 \text{ m/s}, \bar{m} = 0 \) ......................... 97
4.29 \( R \) and \( \theta \) behavior. Case: \( \bar{L}_0 = 25 \text{ m/s}, \bar{m} = 0.8 \) ....................... 98
List of Symbols

\(<E, e_1, e_2, e_3>\) Inertial axes

\(<O, o_1, o_2, o_3>\) Orbital axes

\(<S, s_1, s_2, s_3>\) Satellite axes

e Orbital eccentricity

\(f_a\) Thruster force on end-mass A

\(f_b\) Thruster force on end-mass B

\(h\) Orbital altitude

\(G\) Universal gravitational constant = \(6.6742 \times 10^{-11} \, \text{m}^3 \, \text{s}^{-2} \, \text{kg}^{-1}\)

g Gravitational acceleration at the surface of the Earth = \(9.8 \, \text{m/s}^2\)

\(M_e\) Mass of the Earth = \(5.98 \times 10^{24} \, \text{kg}\)

\(k_a\)

\[\frac{-GM_e}{|\mathbf{r}_{A/E}|^3} = \frac{-GM_e}{(R^2 + L^2_a + 2RL_a \cos \theta)^{\frac{3}{2}}}\]

\(k_b\)

\[\frac{-GM_e}{|\mathbf{r}_{B/E}|^3} = \frac{-GM_e}{(R^2 + L^2_b - 2RL_b \cos \theta)^{\frac{3}{2}}}\]
\[ k_j \quad \frac{-GM_e}{|\vec{F}_{j/E}|^3} = \frac{-GM_e}{(R^2 + L_j^2 - 2RL_j \cos \theta)^2} \]

\[ K_1 \quad k_a + k_b \]

\[ K_2 \quad k_a - k_b \]

\[ L_f \quad \text{Total length of the tether} \]

\[ L \quad \text{Deployed length of the tether} \]

\[ L_a \quad \text{Tether length from center of mass to end-mass A} \]

\[ L_b \quad \text{Tether length from center of mass to end-mass B} \]

\[ L_j \quad \text{Tether length from center of mass to bead j} \]

\[ m_a, m_b \quad \text{Two end-masses of the system} \]

\[ \tilde{m}_a \quad \frac{m_a}{m} \]

\[ \tilde{m}_b \quad \frac{m_b}{m} \]

\[ \tilde{m} \quad \text{Effective mass} = \frac{m_a m_b}{m_a + m_b} \]

\[ \tilde{m} \quad \text{Mass ratio} = \tilde{m}_a - \tilde{m}_b \]

\[ R \quad \text{Radius of the Earth} = 6378 \, \text{km} \]

\[ R_e \quad \text{Radius of the Earth} = 6378 \, \text{km} \]

\[ R_0 \quad \text{Orbital radius} = R_e + h \]
\( \vec{r}_{A/E} \) = Radius vector from point E to point A

\( \vec{r}_{B/E} \) = Radius vector from point E to point B

\( \vec{r}_{j/E} \) = Radius vector from \( j^{th} \) point to point B

\( T \) = Tether tension

\( T_0 \) = Tether tension

\( \tilde{L} \) = Nondimensional tether length = \( L/L_f \)

\( \tilde{t} \) = Nondimensional time = \( t\Omega \)

\( \tilde{T}_0 \) = Nondimensional tether tension = \( T_0/(\bar{m}L_f\Omega^2) \)

\( \tilde{L}_0' \) = Nondimensional initial separation velocity = \( \dot{L}(0)/(\Omega L_f) \)

\( \theta \) = In-plane angle between the tether axis and the local vertical

\( \phi \) = Out-of-plane angle between the tether axis and the local vertical

\( \nu \) = True Anomaly

\( \Omega \) = Orbital angular velocity = \( \sqrt{\frac{GM_e}{(R_e+h)^3}} \)

Superscripts:

\( \prime \) = First derivative with respect to nondimensional time

\( \prime' \) = Second derivative with respect to nondimensional time
List of Abbreviations

AD  Average Deployment
GEO Geostationary Orbit
LEO Low Earth Orbit
MEO Middle Earth Orbit
HEO High Earth Orbit
PFD Percentage of Full Deployment
SE Space Elevator
sf Safety Factor
Chapter 1

Introduction

Space tethers are long flexible members which connect two or more end-bodies in the space environment. The tether tension force acting on the end-bodies influences the dynamics of the entire system. Because of the variety of interesting dynamics they exhibit, space tethers have many scientific and research applications [1, 2, 3]. While some space tether applications are rudimentary, for example space tethers are used to ensure that astronauts and equipment do not separate from a vehicle during space-walks, some of them have extremely sophisticated applications such as electrodynamic propulsion, artificial gravity generation, momentum exchange tethers and space elevators. Some of these applications will be discussed in detail in later sections. Depending upon the application, tethered space systems can use very short tethers or extremely long ones. Since the dynamics of tether systems varies considerably depending on the tether length, in our study we divide tether systems into two categories; ‘short and medium length tether systems’ and ‘long tether systems’. Models for each kind are separately developed, analyzed and discussed.
1.1 Background

1.1.1 Short and Medium Length Tethers

We categorize tether systems with tether lengths up to a few thousands of meters as short or medium length tethers. Most of the tethered satellite systems in present day applications fall under this category. Among these applications, one of the most prominent ones is electrodynamic propulsion. Electrodynamic tethers can reduce the need for conventional energy sources such as chemical propellants and stored electrical power [4, 2]. As a conductive tether travels with high velocity through the Earth’s magnetic field, a Lorentz force can be imparted to the tether by forcing current through the tether; this force can be used for orbital transfer [5]. In addition to providing orbital propulsion, electrodynamic tethers can be used to de-orbit LEO (Low Earth Orbit) satellites that have fulfilled their missions [6, 4]; such satellites would be designed to deploy an electrodynamic tether to initiate the de-orbiting process. After the satellite is deorbited in a controlled manner, it no longer poses a collision threat to other satellites in LEO [7, 8]. Nonelectrodynamic tethers also have very useful and advanced applications and have been the subject of extensive research. A literature review by Kumar [9] provides an in-depth analysis of most of the prominent research on nonelectrodynamic tethers to date. Beyond simply enhancing the capabilities of satellite systems, tethers have also attracted considerable interest among researchers as data-gathering and space access tools; a number of these applications have been proposed [10, 11, 12, 13]. One such tethered system concept under study uses tethered satellites to employ a momentum transfer technique with the goal of transferring payloads from low orbit to a higher orbit [14, 15]. Tethered satellites can also be used to produce an artificial gravity effect in orbit [16]. The artificial gravity effect produced by a rotating body is currently under study because significant bone and muscle loss can occur in humans exposed to a microgravity environment for long periods of time [17]. The Tethered Artificial Gravity (TAG) satellite mission was designed to study the operation and dynamics of an artificial-gravity-generating tethered satellite system. The TAG mission profile involves boosting the TAG system to LEO in a packaged configuration using a conventional rocket, deploying the tether,
and then causing the system to spin about its center of mass in order to produce an artificial gravitational effect on objects located within the end-bodies. Space Tethers have also been proposed and investigated as possible tools for landing, take-off and return from celestial bodies [18]. A detailed list of other applications of space tethers and tethered satellites can be found in the Tethers in Space Handbook [19].

1.1.2 Long Tethers

We have classified space systems with more than 100 km of length as long tethers. Though the basic physics governing long tether dynamics is similar to short and medium length tether dynamics, in-depth analysis of these systems needs special treatment, especially regarding stability issues [20, 21]. One of the many challenges involved in analyzing long tethers is the inclusion of tether mass in the model, which is generally ignored for short and medium length tether analysis. Developing a continuous model is very complicated [22] and difficult to solve. As an alternative to a continuous model, several finite element models have been developed to incorporate tether mass [23]. A lumped mass model [22, 24, 25] or a bead model [26, 27] is generally adapted for such purposes. Each model varies in technical details based on the assumptions and needs of the problem. We have chosen to adopt a bead model to analyze massive long tethers, and this is presented in detail in Chapter 4. One far-term application of such long tethers is a ‘Space Elevator’ [14, 28, 29]. A Space Elevator (SE) consists of a cable which is deployed from a satellite in geosynchronous orbit. The purpose of a space elevator is to provide support for crawlers which move from the surface of the earth to space; space elevators present an economical alternative to chemical rockets for delivering payloads to space. A detailed background and introduction on space elevators is presented in Chapter 3.
1.2 Motivation for Studying Tether Deployment Dynamics

1.2.1 Short and Medium Length Tethers

In order to ensure the success of future missions requiring tether deployment, it is necessary to understand the dynamics of these systems thoroughly. This has inspired many researchers to study the dynamics and control of space tether systems. [13, 30, 31, 32]. A survey conducted by Misra and Modi in 1986 [33] and Kumar in 2006 [9], analyze a large portion of the research efforts on dynamics and controls of tethered satellites. Other than normal two body tethered systems, the dynamics of multi-body tethered systems is also investigated by many researchers [34, 35, 36, 37]. Stability of these systems is also a critical issue and has been the subject of much research [38, 39, 40]. Researchers have investigated stability in three dimensions and have proposed necessary conditions for keeping the system stable [41, 42, 43, 44]. Many tethered satellite missions, including the ones discussed above, have mission profiles that involve a packaged body being launched into orbit and deployed. These missions generally require that the tether deploys completely in order for the mission to be successful. Lack of full deployment can endanger the craft, any crew members that are present, and result in behavior outside the operational envelope for most systems; such behavior is to be avoided. For these reasons, it is necessary to study the deployment dynamics of tethered systems and understand the parameters affecting deployment performance. Previous work on deployment dynamics of tethered satellite systems include derivations of equations of motion and analyses of various deployment control laws [11, 45, 31, 46, 47] and computer simulation of these systems [48]. Most of the controls laws are based on the ones proposed by Rupp [49]. In spite of this extensive research on dynamics, control and several applications of tethered satellites and their deployment, a comprehensive parametric study of deployment dynamics which identifies all the parameters affecting the deployment and their contribution to the overall dynamics of the systems is missing. In this study, we will examine the general tethered satellite equations of motion in order to identify
the factors that affect tether deployment and we will perform a parametric analysis of
the parameters present in the equations of motion in order to determine the effect of
each identified parameter on the final deployed length achieved. In the course of this
study, we will identify trends and tendencies of the system deployment with respect to
the identified system parameters and develop tools capable of qualitatively and quan-
titatively predicting the final deployed length achieved for a given set of parameters.
These tools will be of assistance to mission planners in that they will clearly delineate
the combinations of parameters which will result in full tethered system deployment.
The goal of this study is to numerically examine the dynamics of a deploying tethered
satellite system in an uncontrolled state. Such a system is analogous to that
employed by the SEDS missions flown in the early 1990s [50, 51]. Attitude dynamics
of deploying systems have been studied by Grassi et al. and Mazzoleni et al. [52, 53],
who both examined the effect of initial alignment on end-body attitude stability and
the importance of initial alignment to overall deployment performance. We expand
on these works by varying five different parameters which affect the dynamics of a
 tethered satellite system. We determine the effect of each of these parameters in-
dividually and then, through nondimensionalization, show that the dynamics of the
tethered satellite system are determined through equations of motion governed by two
nondimensional parameters. We then determine the behavior of the system under a
variety of different combinations of these two parameters.

1.2.2 Long Tethers

Similar to short tethers, long tethers have many potential applications to space ad-
vancement, especially in the area of space transportation. Though there is no rigid
classification based on length for short and long tethers, for this analysis long tethers
are of the order of 100s of kilometers. This classification for long tethers requires
the avoidance of certain simplifying assumptions which are made during the analysis
of short and medium length tethers. These assumptions will be discussed in detail
in further chapters. Of the several applications suggested for long tethers, one of
the most intriguing and challenging application is the ‘Space Elevator’ (SE). A de-
tailed overview on SE is presented in Chapter 3. For this application, a hundred
thousand km long tether is proposed. Clearly, a detailed and extensive analysis of the dynamics of deployment for this immensely long tether is required before it is actually deployed. Though the SE has become a topic of recent research interest, its deployment dynamics has not been extensively studied. Hence, motivated by the desire to explore further the deployment dynamics of the Space Elevator, a model for its deployment is created, simulated and analyzed in detail.
Chapter 2

Short and Medium Length Tether Analysis

2.1 Model

The system used as a prototypical tethered satellite in this analysis of short tethers is similar to that of the TAG satellite mission discussed in the previous section and described in detail by Mazzoleni and Hoffman [17]. A simplified model of the system consists of two point masses connected by a massless tether. The center of mass of the system is assumed to move in a circular orbit about the Earth; the result of this assumption is that the orbital height $h$ and the orbital angular frequency $\Omega$ are both constant [54]. The tether is considered to be a rigid, inextensible rod which is capable of non-elastic changes in length through a spooling-out process. It has been shown previously that the rigid-rod model (also known as a planar dumbbell model) can be a good approximation to the behavior of a real tethered system [55]. We also assume that the spool inertia is negligible, the length of the tether is small compared to the distance between the center of the earth and the center of mass of the tethered satellite system, and that gravity and the tether tension force are the only forces which act on the end-bodies of the tethered satellite system. For the purposes of this study, we initially assume that the total mass of the satellite (the sum of the two end-body masses) is assumed to be $70 \text{ kg}$ and the mass of the tether is assumed to be
negligible. We also assume that the deployer friction remains constant throughout the deployment process and that the tension force traveling through the tether is equal to the deployer friction. The assumption that the deployer friction (and thus the tether tension) remains constant throughout the spooling process represents a simplified approximation of actual deployer friction profiles and was made in order to keep the analysis as general as possible. While most deployer friction profiles vary with parameters such as the deployment velocity, simulations performed assuming constant deployer friction can represent worst-case scenarios or can give insight into deployment characteristics under average deployer friction conditions. The initial velocity that separates the two end-bodies to begin the spooling-out process is assumed to be imparted by a spring mounted in one of the two end-bodies [53]. The deployment is unforced beyond the action of the spring and hence not controlled by any external system. The motion of the satellite is defined so that at any point in time, the in-plane angle, $\theta$, and the out-of-plane angle, $\phi$, fall in the range $-90^\circ$ to $90^\circ$. These angles are defined with respect to the local vertical ($\theta = 0$) and the orbital plane ($\phi = 0$). Under this definition, $\theta=\phi=0$ is the position where the tether axis is aligned to the local vertical and is completely in the orbital plane. As the tether spools out and increases in length, the satellite rotates in and out of the orbital plane. The spool-out process eventually stops after deploying to some length and the remainder of the mission can be initiated. For this analysis, the dynamics occurring after the spool-out process is complete is not considered, i.e., this analysis is concerned only with the deployment process. Sketches of the stages involved in deployment and the angles $\theta$ and $\phi$ are shown in Figs. 2.1 and 2.2, respectively.
Figure 2.1: Deployment Steps; Adapted from the *Tethers In Space Handbook* [19]

Figure 2.2: Representation of $\theta$ and $\phi$
2.2 Equations of Motion: Derivation

The equations of motion for a tethered satellite consisting of two point masses in a circular orbit are derived below. Representation of the in-plane and out-of-plane angles $\theta$ and $\phi$ are shown in Fig. 2.2.

We begin by writing Newton’s Second Law in the form

$$\frac{\mathbf{F}_A}{m_a} - \frac{\mathbf{F}_B}{m_b} = \mathbf{a}_{A/E} - \mathbf{a}_{B/E} = \mathbf{a}_{A/B} \quad (2.1)$$

where $\mathbf{F}_A$ and $\mathbf{F}_B$ are the net forces acting on end bodies A and B, respectively, $m_a$ and $m_b$ are the masses of end bodies A and B, respectively, and $\mathbf{a}_{A/E}$, $\mathbf{a}_{B/E}$ and $\mathbf{a}_{A/B}$ are the acceleration vectors from the center of the Earth to end-body A, from center of the Earth to end-body B, and from end-body B to end-body A, respectively.

2.2.1 Forces

We assume that each end-body is only acted upon by a gravitational force and tether tension force. Hence the left hand side of Equation (2.1) is expressed:

$$\frac{\mathbf{F}_A}{m_a} - \frac{\mathbf{F}_B}{m_b} = \left( -GM_e \frac{\mathbf{r}_{A/E}}{|\mathbf{r}_{A/E}|^3} + \frac{T}{m_a} \frac{\mathbf{r}_{C/A}}{|\mathbf{r}_{C/A}|} \right) - \left( -GM_e \frac{\mathbf{r}_{B/E}}{|\mathbf{r}_{B/E}|^3} + \frac{T}{m_b} \frac{\mathbf{r}_{C/B}}{|\mathbf{r}_{C/B}|} \right)$$

$$\Rightarrow \frac{\mathbf{F}_A}{m_a} - \frac{\mathbf{F}_B}{m_b} = \left( -GM_e \frac{\mathbf{r}_{A/E}}{|\mathbf{r}_{A/E}|^3} + \frac{T}{m_a} \frac{\mathbf{r}_{C/A}}{|\mathbf{r}_{C/A}|} \right) - \left( -GM_e \frac{\mathbf{r}_{B/E}}{|\mathbf{r}_{B/E}|^3} + \frac{T}{m_b} \frac{\mathbf{r}_{C/B}}{|\mathbf{r}_{C/B}|} \right) \quad (2.2)$$

where $\mathbf{r}_{A/E}$ and $\mathbf{r}_{B/E}$ are the position vectors from the center of the Earth, E to the end-masses A and B, respectively, and $\mathbf{r}_{C/A}$ and $\mathbf{r}_{C/B}$ are the position vectors to the center of mass, C, of the tether system from the end-masses A and B, respectively. $G$ is the universal gravitational constant ($6.673 \times 10^{-11} \text{ m}^3\text{kg}^{-1}\text{s}^{-2}$).

Vectors $\mathbf{r}_{A/E}$ and $\mathbf{r}_{B/E}$ can be expressed in the satellite reference frame, S, as:
\[ \{\vec{r}_{A/E}\}_S = \{\vec{r}_{A/C}\}_S + \{\vec{r}_{C/E}\}_S = \begin{bmatrix} L_a + R \cos \theta \cos \phi \\ -R \sin \theta \\ -R \cos \theta \sin \phi \end{bmatrix} \]

\[ \Rightarrow |\vec{r}_{A/E}| = \left( R^2 + L_a^2 + 2RL_a \cos \theta \cos \phi \right)^{\frac{1}{2}} = R \left( 1 + \frac{L_a^2}{R^2} + \frac{2L_a}{R} \cos \theta \cos \phi \right)^{\frac{1}{2}} \]

where \( R \) is the orbital radius and \( L_a \) is the length of tether from end-mass A to the center of mass of the system, C, and \( \theta \) and \( \phi \) are the in-plane and out-of-plane angle respectively, made by the tether axis with local vertical. If the total tether length is small compared to the orbital radius, the term \( L_a^2/R^2 \) is negligible compared to 1 and can be assumed to be approximately equal to zero. Furthermore, using the binomial expansion to compute the reciprocal of the cube of the magnitude, we can write

\[ |\vec{r}_{A/E}|^{-3} \approx \frac{1}{R^3} \left( 1 - 3 \frac{L_a}{R} \cos \theta \cos \phi \right) \]

Similarly,

\[ \{\vec{r}_{B/E}\}_S = \begin{bmatrix} -L_b + R \cos \theta \cos \phi \\ -R \sin \theta \\ -R \cos \theta \sin \phi \end{bmatrix}, \quad |\vec{r}_{B/E}|^{-3} \approx \frac{1}{R^3} \left( 1 + 3 \frac{L_b}{R} \cos \theta \cos \phi \right) \]

where \( L_b \) is the length of tether from end-mass B to the center of mass of the system, C.

Note that

\[ \left\{ \frac{\vec{r}_{C/A}}{|\vec{r}_{C/A}|} \right\}_S = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad \text{and} \quad \left\{ \frac{\vec{r}_{C/B}}{|\vec{r}_{C/B}|} \right\}_S = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \]
Substituting all the terms in the right hand side of Equation (2.2) with the expressions derived above, and simplifying, yields

\[
\left\{ \frac{\vec{F}_A}{m_a} - \frac{\vec{F}_B}{m_b} \right\}_S = \begin{bmatrix}
-\Omega^2 L + 3\Omega^2 L \cos^2 \theta \cos^2 \phi - \frac{T}{\bar{m}} \\
-3\Omega^2 L \cos \theta \sin \theta \cos \phi \\
-3\Omega^2 L \cos^2 \theta \cos \phi \sin \phi 
\end{bmatrix}
\]

(2.3)

where \( \Omega = \sqrt{\frac{GM_e}{R^3}} \) is the orbital angular velocity, \( L = L_a + L_b \) is the total tether length deployed and \( \bar{m} = \frac{m_am_b}{m_a+m_b} \) is the effective mass.

### 2.2.2 Acceleration

The relative angular velocity and angular acceleration between the inertial E frame and the satellite S frame can be expressed in the coordinates of the S reference frame as (note that the angular acceleration is obtained by differentiating the angular velocity):

\[
\left\{ E \vec{\omega}^S \right\}_S = \begin{bmatrix}
(\dot{\theta} + \Omega) \sin \phi \\
-\dot{\phi} \\
(\dot{\theta} + \Omega) \cos \phi
\end{bmatrix}, \quad \left\{ E \vec{\alpha}^S \right\}_S = \begin{bmatrix}
\ddot{\theta} \cos \phi + (\dot{\theta} + \Omega) \dot{\phi} \cos \phi \\
-\ddot{\phi} \\
\ddot{\theta} \cos \phi - (\dot{\theta} + \Omega) \dot{\phi} \sin \phi
\end{bmatrix}
\]

Note that, the position vector, \( \vec{r}_{A/B} \), and its subsequent derivatives differentiated in the S reference frame, all can be expressed in the coordinates of S reference frame as

\[
\left\{ \vec{r}_{A/B} \right\}_S = \begin{bmatrix}
L \\
0 \\
0
\end{bmatrix}, \quad \left\{ \dot{\vec{r}}_{A/B} \right\}_S = \begin{bmatrix}
\dot{L} \\
0 \\
0
\end{bmatrix}, \quad \left\{ \ddot{\vec{r}}_{A/B} \right\}_S = \begin{bmatrix}
\ddot{L} \\
0 \\
0
\end{bmatrix}
\]

Note that the notation, \( E \dot{\eta} \) represents \( E \frac{d}{dt} \eta \) and \( E \ddot{\eta} \) represents \( E \frac{d^2}{dt^2} \eta \) and so forth.
The inertial acceleration, i.e., the derivative of the velocity in the E frame, can be found using the relation

\[
E\dddot{\vec{r}}_{A/B} = S\dddot{\vec{r}}_{A/B} + E\vec{\alpha}^S \times \vec{r}_{A/B} + 2E\vec{\omega}^S \times S\dot{\vec{r}}_{A/B} + E\vec{\omega}^S \times \left(E\vec{\omega}^S \times \vec{r}_{A/B}\right)
\]  

Equation (2.4)

where \(E\dddot{\vec{r}}_{A/B}\) is the second derivative of the position vector from B to A, differentiated in the E reference frame with respect to time. Expressing this vector in the S reference frame by substituting each term in the Equation (2.4) with the corresponding vectors in the S reference frame, we obtain:

\[
\left\{E\dddot{\vec{r}}_{A/B}\right\}^S = \begin{bmatrix} \dddot{L} \\ 0 \end{bmatrix} + \begin{bmatrix} \theta \cos \phi + (\dot{\theta} + \Omega)\dot{\phi} \cos \phi \\ -\dddot{\phi} \end{bmatrix} \times \begin{bmatrix} L \\ 0 \end{bmatrix} + 2 \begin{bmatrix} (\dot{\theta} + \Omega) \sin \phi \\ -\ddot{\phi} \\ (\dot{\theta} + \Omega) \cos \phi \end{bmatrix} \times \begin{bmatrix} \dot{L} \\ 0 \end{bmatrix} + \begin{bmatrix} (\dot{\theta} + \Omega) \sin \phi \\ -\ddot{\phi} \\ (\dot{\theta} + \Omega) \cos \phi \end{bmatrix} \times \left(\begin{bmatrix} (\dot{\theta} + \Omega) \sin \phi \\ -\ddot{\phi} \\ (\dot{\theta} + \Omega) \cos \phi \end{bmatrix} \times \begin{bmatrix} L \\ 0 \end{bmatrix}\right)
\]

after simplifying,

\[
\left\{E\dddot{\vec{r}}_{A/B}\right\}^S = \begin{bmatrix} \dddot{L} - \ddot{\phi}^2 L - \left(\dot{\theta} + \Omega\right)^2 \cos^2 \phi L \\ \dot{\theta} \cos \phi L - 2 \left(\dot{\theta} + \Omega\right) \dot{\phi} \sin \phi + 2 \left(\dot{\theta} + \Omega\right) \dot{\phi} L \cos \phi \\ \ddot{\phi} L + 2 \dddot{\phi} L + \left(\dot{\theta} + \Omega\right)^2 \cos \phi \sin \phi L \end{bmatrix}
\]  

Equation (2.5)
Using equation (2.1), equations (2.3) and (2.5) can be set equal to each other to obtain:

\[
\begin{bmatrix}
\ddot{L} - \dot{\phi}^2 L - (\dot{\theta} + \Omega)^2 \cos^2 \phi L \\
\dot{\theta} \cos \phi L - 2 (\dot{\theta} + \Omega) \dot{\phi} \sin \phi + \\
2 (\dot{\theta} + \Omega) \dot{L} \cos \phi \\
\dot{\phi} L + 2 \ddot{\phi} L + (\dot{\theta} + \Omega)^2 \cos \phi \sin \phi L
\end{bmatrix}
= \begin{bmatrix}
-\Omega^2 L + 3\Omega^2 L \cos^2 \theta \cos^2 \phi - \\
\frac{T}{m} \\
-3\Omega^2 L \cos \theta \sin \theta \cos \phi \\
-3\Omega^2 L \cos^2 \theta \cos \phi \sin \phi
\end{bmatrix}
\]

The final differential equations of motion can be now presented as:

\begin{align*}
\ddot{L} &= L \left[ (\dot{\theta} + \Omega)^2 \cos^2 \phi + \dot{\phi}^2 + 3\Omega^2 \cos^2 \theta \cos^2 \phi - \Omega^2 \right] - \frac{T}{m} \quad (2.6a) \\
\ddot{\theta} &= 2 (\dot{\theta} + \Omega) \dot{\phi} \tan \phi - 2 L^{-1}(\dot{\theta} + \Omega) - 3\Omega^2 \cos \theta \sin \theta \quad (2.6b) \\
\ddot{\phi} &= -2 \frac{\dot{L}}{L} \dot{\phi} - \left[ (\dot{\theta} + \Omega)^2 + 3\Omega^2 \cos^2 \theta \right] \cos \phi \sin \phi \quad (2.6c)
\end{align*}

The quantities \(L\), \(\theta\), and \(\phi\) are three time-dependent variables which are determined under appropriate initial conditions. For the purposes of simulation, we assume that the system starts from rest with respect to the orbital reference frame \((\dot{\theta}(0) = \dot{\phi}(0) = 0)\). Also, since the tether is completely wound in one of the end-bodies, the initial tether length prior to deployment, \(L(0)\), is also zero. However, to avoid mathematical error during numerical simulation (due to the term \(\dot{L}/L\) in equation (2.6c)), we approximate \(L(0)\) to be a very small number; in practice, we make the approximation \(L(0) \approx 10^{-5}\). The angular alignment of the system with respect to the orbital plane before the start of deployment is specified by \(\theta(0)\) and \(\phi(0)\) and is
referred to as the Initial Position or “IP” of the system. Since one of the objectives of this study is to understand the deployment performance as a function of the system IP, we use combinations of \( \theta(0) \) and \( \phi(0) \) spanning from \(-90^\circ\) to \(90^\circ\) in steps of \(5^\circ\), i.e. a total of 1369 IPs for each simulation. In order to solve these equations, we need to specify the tether tension, \( T_0 \), the initial separation velocity, \( \dot{L}_0 \), the system effective mass, \( \bar{m} \), the total tether length, \( L_f \), and the orbital angular velocity of center of mass, \( \Omega \). These quantities are the five parameters which completely describe the dynamics of the tethered system for a given IP and they must specified for every mission. Note that the initial separation velocity (the separation velocity at instant \( t = 0 \)) does not appear explicitly in the equations of motion but must be specified as an initial condition when solving these equations. When designing a mission, it is likely that the orbital altitude of the tethered satellite will be of more importance than the orbital angular velocity. However, note that the orbital angular velocity is a function only of constant parameters and the orbital altitude \( \left( \Omega^2 = GM_e / (R_e + h)^3 \right) \) where \( R_e = 6378 \text{ km} \) is the radius of the Earth and \( h \) is the orbital altitude. Therefore, we replace \( \Omega \) by orbital altitude, \( h \), in the list of parameters which affect tethered satellite motion. The five parameters discussed are collectively referred to as System Parameters. Once these System Parameters are specified, the system dynamics can be simulated. Each simulation is repeated over the entire range of IPs (1369 values). Each case is simulated until either the separation velocity reaches zero or until the deployed length reaches the total tether length. (i.e \( \dot{L} = 0 \) or \( L = L_f \)). Since, for the deployer systems considered for this study, static friction is assumed to be significantly higher than the kinetic friction, we assume that there will not be any deployment after the deployment speed reaches zero, justifying the condition \( \dot{L} = 0 \) for simulation termination.
2.3 Analysis

2.3.1 Operating Ranges of Dimensional Parameters

Among the system parameters, $T_0$, $\bar{m}$ and $\dot{L}_0$ are functions of the system design and hence are collectively referred to as design parameters. The remaining two parameters, $h$ and $L_f$, are requirements for each mission irrespective of deployment performance that can be achieved, and hence they will be collectively referred to as mission parameters. Under these definitions, a mission with $T_0 = 100 \, mN$, $\bar{m} = 15 \, kg$ and $\dot{L}_0 = 5 \, m/s$, $h= 1000 \, km$ and $L_f= 5 \, km$ would be denoted as a mission with design parameters $[T_0 = 100 \, mN, \, \bar{m} = 5 \, kg, \, \dot{L}_0 = 5 \, m/s]$ and mission parameters $[h = 100 \, km, \, L_f = 5 \, km]$. Note that the tension is expressed in $mN$, separation velocity in $m/s$, and orbital altitude and tether length in $km$. From the design perspective, we have more liberty to change design parameters than the mission parameters; therefore the following section will detail deployment performance as a function of the design parameters. As for the mission parameters, we work in the range 100 $m$ to 10,000 $m$ for total tether length $L_f$ and 300 $km$ to 1200 $km$ for orbital altitude $h$. These ranges are selected on the basis of typical values of these parameters for similar missions. The operating ranges of design parameters for this analysis is determined from information detailed by Mazzoleni [56]:

**Tether Tension/Deployer Friction** $T_0$: When the tether is being spooled out, the friction between the tether and the deployer accounts for the tether tension. As long as the tether is being deployed, the tether tension is equal to the deployer friction. However, after tether deployment is complete, there is no deployer friction but it is possible that the tethered system is rotating with respect to the orbital reference frame. The centripetal acceleration which allows the end-bodies to rotate around each other without moving away from each other is provided by a tension traveling through the tether; i.e., after deployment the tether tension can be nonzero despite the fact that the deployer friction force is zero [3, 20]. However, this study concentrates only on the tether deployment process and thus the tether tension force is assumed to be equal to the tether deployer friction force for all simulations. We also assume that
the tether tension remains constant throughout the deployment process. This is not always true especially in systems with high ejection velocity. However, as is discussed in the next paragraph, for the purposes of this study we do not consider very high ejection velocities. Experiments on proposed tether deployers have shown that tether tension due to deployer friction is of the order 50 mN at room temperature [53]. However, since the effects of extreme temperatures and other conditions in orbit are not known, we use a safety factor of 4 and assume that the deployer friction can take values as high as 200 mN. Hence, the operating range of tether tension, $T_0$, is $[50 \ mN, 200 \ mN]$.

**Initial Separation Velocity $\dot{L}_0$:** The initial separation velocity is imparted by the springs mounted on the deployer, and hence is directly dependent on the spring constant and degree of compression of the springs. Based on a previous study performed by Mazzoleni et al. [53], the operating range for this parameter is selected to be $[3 \ m/s, 6 \ m/s]$. Because of space limitations on the deployer, the maximum spring size and spring compression are both limited; therefore, the maximum deployment velocity is limited. We assume that this maximum velocity is 6 m/s.

**Effective Mass $\bar{m}$:** The effective mass is defined as $\bar{m} = m_a m_b/(m_a + m_b)$. The dynamics of a tethered satellite are more closely related to the mass distribution between the tethered satellite end-bodies than to the total mass of the tethered satellite. For this reason, the total end-body mass of 70 kg is constant throughout the analysis and the distribution of this mass on either end is varied. The value of $\bar{m}$ is at a maximum when the masses are equally distributed and is at a minimum when the difference between the mass of the two end bodies is greatest, i.e. when one of the end-bodies has the minimum allowed mass. To accommodate the hardware and instruments, the mass of each end-body must have a lower bound; that lower bound is assumed to be 12 kg for this study. Therefore, the range of $\bar{m}$ for a 70 kg total mass can be shown to be $[10 \ kg, 17.5 \ kg]$.
2.3.2 Effect of Each Parameter on Deployment

Having defined the operating range of each of the parameters, we can now determine the solutions to the equations of motion. For each set of system parameters, the total deployed length is plotted as a function of its IP on two and three dimensional plots as shown in Figs. 2.3 – 2.11. We also compute the average of all the final deployed lengths achieved at each IP, (referred to as Average Deployment (AD)) and the percentage of the IPs resulting in full deployment, (referred to as Percentage of Full Deployment (PFD)). Note that the term “full deployment” or “complete deployment” are used to indicate situations when for a given IP, the final deployed length is at least equal to the total tether length \( L = L_f \). The term “level of deployment” is used to represent the percentage of total tether length to which the tether is deployed.

The AD of a tethered system measures the probability of a successful deployment (and thus a successful mission) and hence it is of interest because there conceivably are missions that can be completed with a high degree of success even if a tethered satellite system achieves a high (but not necessarily full) level of deployment. An AD of 100% indicates that all the IPs have deployed completely whereas an AD of 0% indicates that none of the IPs have deployed at all. Similarly, 100% PFD indicates all the IPs have deployed completely (effectively same as 100% AD) and 0% PFD indicates that none of the IPs have reached complete deployment (Note that this is not equivalent to 0% AD).

Both the 2-D and 3-D plots are generated from the same data. However they may not necessarily give redundant information. The 3-D plots are generated from calculations of the deployment achieved at each IP and hence the level of deployment for each of the simulated IPs can be determined. However, it can sometimes be difficult to read the deployment level from the 3-D plots. Therefore, to present the data in a more readable form, the range of deployment, i.e. 0% to 100% of total tether length deployed, is divided into six intervals and each interval is assigned to a shade of gray from black to white. A grid is generated on the \( \theta-\phi \) plane such that each element of the grid corresponds to a unique IP. Then, each point on the plane is colored with a shade of gray based on the deployment performance of the IPs. The shades of gray act as third axis and present the deployment performance on a 2-D
plot instead of a 3-D plot. While these plots do not give precise deployment achieved at a given IP, they are more effective than 3-D plots in determining the deployment of any given IP within 25%.

We now examine the effect of each of the design parameters, \([T_0, \bar{m}, \dot{L}_0]\), on deployment performance. For this entire analysis, we select mission parameters \([h = 1200 \text{ km}, \ L_f = 2.85 \text{ km}]\). Each of the design parameters is varied one at a time within their operating range. Note that the numerical method used to generate all simulations discussed is the MATLAB ode45 method; this method is a Dormand-Prince pair which uses a fourth-order Runge-Kutta scheme with an adaptive time step. The scheme is fourth-order accurate with respect to the time step.

**Effect of Effective Mass \(\bar{m}\) (Figs. 2.3 to 2.5)**

For Figs. 2.3 – 2.5, \(T_0\) is 100 \text{ mN} and \(\dot{L}_0\) is 5 \text{ m/s}. We vary \(\bar{m}\) from its minimum value of 10 \(\text{ kg}\) to its maximum value of 17.5 \(\text{ kg}\). We start with \(\bar{m} = 10 \text{ kg}\), the results of which are presented in Fig. 2.3; the design parameters for this case are \([T_0 = 100 \text{ mN}, \ \bar{m} = 10 \text{ kg}, \ \dot{L}_0 = 5 \text{ m/s}]\). Upon inspection of both Fig. 2.3(a) and Fig. 2.3(b), we see that the deployment is fairly uniform for all of the IPs except for a small portion at the center of each plot which shows a slightly higher level of deployment. The AD is below 50% and PFD is 0%. In Fig. 2.3(b), we see that the small portion in the center of the plot (indicating a higher level of deployment), is in the [50, 75)% range and the rest of the plot has a uniform shade representing [25, 50)% AD. The absence of a white region in this plot shows that none of the IPs result in full deployment; thus the PFD is 0%. In Fig. 2.4, the effective mass is raised to 15 \(\text{ kg}\) and we see a prominent increase in the AD of the system to about 66%. However, less than 1% of IPs result in full deployment as indicated by the PFD of about 0.5%. This small fraction of IPs resulting in complete deployment can be seen in Fig. 2.4(b) as a small white region at the center of the plot. This white region is surrounded by another small region of \([75, 100)%\) deployment and the rest of the plot has a shade representing \([50, 75)%\) AD. Finally, for Fig. 2.5, we increase \(\bar{m}\) to its maximum value of 17.5 \(\text{ kg}\). We can see that the AD is slightly above 75% and the PFD also increases considerably to about 16%. Fig. 2.5(b) has same three shades of gray as in Fig. 2.4(b), but the white region
representing the full deployment in Fig. 2.5(b) is bigger than the corresponding white region in Fig. 2.4(b). Clearly, the level of deployment has increased as we progress from Figs. 2.3 to 2.5 which indicates that the deployment favors higher effective mass; therefore, to achieve full deployment, the end-body masses should be distributed as uniformly as possible.

**Effect of Tether Tension, $T_0$ (Figs. 2.6 to 2.7)**

For Figs. 2.6 – 2.7, $\bar{m}$ is 17.5 $kg$ and $\dot{L}_0$ is 4 $m/s$. We vary $T_0$ from 100 $mN$ to its minimum value of 50 $mN$. The design parameters for Fig. 2.6(a) are $[T_0 = 100 \text{ mN}, \bar{m} = 17.5 \text{ kg}, \dot{L}_0 = 4 \text{ m/s}]$. We can see that the AD is slightly below 50% while the PFD is 0%. In Fig. 2.6(b) we see a region of [50, 75)% deployment level develop in the center of the plot surrounded by a region of [25, 50)% in the rest of the plot. There is still no white region anywhere on the plot and the calculated PFD is 0%. For Fig. 2.8, the tether tension is reduced to 75 $mN$. We see that the AD increases to about 68% and about 18% of IPs result in full deployment. This fraction of PFD can be seen in Fig. 2.8(b) as a white region at the center of the plot. This white region is surrounded by a region of [75, 100)% of full deployment and the rest of the plot has a shade representing [50, 75)% deployment. Finally, in Fig. 2.7, where $T_0$ is at its minimum value of 50 $mN$, we calculate that the AD has increased to about 88% and the PFD has increased to about 48%. Again, Fig. 2.7(b) has the same three shades of gray as in Fig. 2.8(b), but the region of full deployment and region of [75, 100)% deployment have both spread out widely, touching the top edge of the plot and re-emerging from the bottom. These plots indicate that with decreasing deployer friction the total level of deployment reached by a tethered satellite system increases. This conclusion is consistent with the fact that friction opposes motion in general and, specifically, deployer friction opposes tether deployment.

**Effect of Initial Separation Velocity $\dot{L}_0$ (Figs. 2.9 to 2.11)**

For Figs. 2.9 – 2.11, $\bar{m}$ has a value of 17.5 $kg$ (the maximum value of $\bar{m}$ considered in this study) and $T_0$ has a value of 50 $mN$ (the minimum value of $T_0$ considered in this
study). From the previous two sections, we can conclude that these two parameters are optimized for full deployment. In order to determine the effect of the initial separation velocity on AD and PFD, we vary $\dot{L}_0$ from $3 \text{ m/s}$ to $5 \text{ m/s}$. We can see that for this configuration of $[T_0 = 50 \text{ mN}, \bar{m} = 17.5 \text{ kg}, \dot{L}_0 = 3 \text{ m/s}]$ (illustrated by Fig. 2.9), the AD is about 64% while the PFD is about 28%. Fig. 2.9(b) shows that the region of high deployment on the two dimensional shade plot spreads outward from the center of the plot; this is consistent with previous two dimensional shaded plots. Note that the shaded plots seem to be periodic in nature with respect to the in-plane angle $\theta$; i.e., if the top of a shaded plot is folded down to meet the bottom of the shaded plot, the shaded regions match. We see the corresponding behavior in three-dimensional representation in Fig. 2.9(a). For Fig. 2.10, we increase the initial separation velocity to $4 \text{ m/s}$. We calculate that the AD has increased to about 88% and about 48% of IPs result in full deployment. Note that this configuration is identical to one studied in the previous subsection for Fig. 2.7. Finally for Fig. 2.11, we increase the separation velocity to $5 \text{ m/s}$. We can see that the AD is now 100% which indicates that PFD is also 100%. This is the ideal deployment scenario because each simulated IP results in full deployment. In Fig. 2.11(b) we see that the entire plot is white which indicates complete deployment of all of the simulated IPs. Note that the deployment has increased with increases in $\dot{L}_0$ and this is consistent with the intuitive expectation that higher initial separation velocities should result in a higher probability of deployment. We have shown that full tethered satellite deployment can be achieved when the initial separation velocity is $5 \text{ m/s}$. Recall that in the definition of the operating ranges for each variable, we indicated that the operating range of the initial separation velocity is $[3 \text{ m/s}, 6 \text{ m/s}]$. Because complete deployment can be achieved at an initial separation velocity of $5 \text{ m/s}$, we assume that full deployment is achieved for any initial separation velocity greater than $5 \text{ m/s}$; this indicates that the effective operating range for which useful data may be gathered is $[3 \text{ m/s}, 5 \text{ m/s}]$. Therefore, simulations using larger separation velocities than $5 \text{ m/s}$ are not presented.
Figure 2.3: Deployment as a function of IPs at $T_0 = 100 \, mN$, $\bar{m} = 10 \, kg$, $\dot{L}_0 = 5 \, m/s$

(a) 3-D representation with IPs on horizontal plane and deployed length on vertical axis
(b) 2-D representation with $\phi$ on horizontal axis and $\theta$ on vertical axis

Figure 2.4: Deployment as a function of IPs at $T_0 = 100 \, mN$, $\bar{m} = 15 \, kg$, $\dot{L}_0 = 5 \, m/s$

(a) 3-D representation with IPs on horizontal plane and deployed length on vertical axis
(b) 2-D representation with $\phi$ on horizontal axis and $\theta$ on vertical axis
(a) 3-D representation with IPs on horizontal plane and deployed length on vertical axis

(b) 2-D representation with $\phi$ on horizontal axis and $\theta$ on vertical axis

Figure 2.5: Deployment as a function of IPs at $[T_0 = 100\, mN, \bar{m} = 17.5\, kg, \dot{L}_0 = 5\, m/s]$

(a) 3-D representation with IPs on horizontal plane and deployed length on vertical axis

(b) 2-D representation with $\phi$ on horizontal axis and $\theta$ on vertical axis

Figure 2.6: Deployment as a function of IPs at $[T_0 = 100\, mN, \bar{m} = 17.5\, kg, \dot{L}_0 = 4\, m/s]$
(a) 3-D representation with IPs on horizontal plane and deployed length on vertical axis

(b) 2-D representation with $\phi$ on horizontal axis and $\theta$ on vertical axis

Figure 2.7: Deployment as a function of IPs at $[T_0 = 75 \text{ mN}, \bar{m} = 17.5 \text{ kg}, \dot{L}_0 = 4 \text{ m/s}]$

---

(a) 3-D representation with IPs on horizontal plane and deployed length on vertical axis

(b) 2-D representation with $\phi$ on horizontal axis and $\theta$ on vertical axis

Figure 2.8: Deployment as a function of IC at $[T_0 = 50 \text{ mN}, \bar{m} = 17.5 \text{ kg}, \dot{L}_0 = 4 \text{ m/s}]$
(a) 3-D representation with IPs on horizontal plane and deployed length on vertical axis

(b) 2-D representation with $\phi$ on horizontal axis and $\theta$ on vertical axis

Figure 2.9: Deployment as a function of IPs at $[T_0 = 50 \text{ mN}, \bar{m} = 17.5 \text{ kg}, \dot{L}_0 = 3 \text{ m/s}]$

(a) 3-D representation with IPs on horizontal plane and deployed length on vertical axis

(b) 2-D representation with $\phi$ on horizontal axis and $\theta$ on vertical axis

Figure 2.10: Deployment as a function of IPs at $[T_0 = 50 \text{ mN}, \bar{m} = 17.5 \text{ kg}, \dot{L}_0 = 4 \text{ m/s}]$
Figure 2.11: Deployment as a function of IPs at $[T_0 = 50 \text{ mN}, \ m = 17.5 \text{ kg, } \dot{L}_0 = 5 \text{ m/s}]$
A general observation concerning the previously described simulations is that for IPs closer to $\theta = 0$ and $\phi = 0$, i.e. the IPs located at the center of the two dimensional shaded plot, a high level of deployment is reached for a wider range of tethered satellite configurations than simulations involving IPs near the edge of the two dimensional plots. In other words, if the design parameters do not result in complete deployment for all simulated IPs, the IPs which are closer to $(\theta(0), \phi(0)) = (0, 0)$ tend to deploy to a higher level than those far from the point $(\theta(0), \phi(0)) = (0, 0)$. This suggests that prior to deployment, the system should be initially aligned close to the local vertical and in the orbital plane. Note that the region of high deployment tends to spread symmetrically about the line $\phi = 0$ as the design parameters result in higher levels of deployment. However, the region of high deployment does not spread symmetrically about the line $\theta = 0$. In other words, the deployment plots are symmetrical about the $\phi(0)$ axis but not about the $\theta(0)$ axis. Note also that the regions of high deployment are larger on the half of the IP plane which contains positive $\theta(0)$ values; this indicates that the deployment occurs more readily for positive initial in-plane angles than for negative initial in-plane angles. This asymmetry is most likely due to the interaction between the gravity gradient torque and the orbital angular velocity of the system; this hypothesis is a subject for further mathematical investigation. However, upon inspection of the equations of motion (Eqs.(2.6a) - (2.6c)), it can be seen that the equations are symmetric with respect to $\phi$ but not with respect to $\theta$, which is consistent with the observations regarding the deployment plots.

2.3.3 Nondimensionalization

While the above analysis gives an overview of how each of the design parameters affects the deployment achieved by a given tethered satellite system, it has a few limitations:

- Because each design parameter was varied only twice, we know the general trend of the deployment with respect to a given parameter, but we do not know at what point the level of deployment transitions from one range of deployment to another.
Table 2.1: Deployment results as a function of increasing or decreasing individual dimensional parameters

<table>
<thead>
<tr>
<th>$T_0$ $mN$</th>
<th>$\bar{m}$ $kg$</th>
<th>$\dot{L}_0$ $m/s$</th>
<th>AD</th>
<th>PFD</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>10</td>
<td>5.0</td>
<td>43.32</td>
<td>0</td>
</tr>
<tr>
<td>100</td>
<td>15</td>
<td>5.0</td>
<td>66.25</td>
<td>0.51</td>
</tr>
<tr>
<td>100</td>
<td>17.5</td>
<td>5.0</td>
<td>76.98</td>
<td>16.58</td>
</tr>
<tr>
<td>100</td>
<td>17.5</td>
<td>4</td>
<td>49.03</td>
<td>0</td>
</tr>
<tr>
<td>75</td>
<td>17.5</td>
<td>4</td>
<td>68.65</td>
<td>18.41</td>
</tr>
<tr>
<td>50</td>
<td>17.5</td>
<td>4</td>
<td>88.43</td>
<td>48.14</td>
</tr>
<tr>
<td>50</td>
<td>17.5</td>
<td>3</td>
<td>64.07</td>
<td>27.98</td>
</tr>
<tr>
<td>50</td>
<td>17.5</td>
<td>4</td>
<td>88.43</td>
<td>48.14</td>
</tr>
<tr>
<td>50</td>
<td>17.5</td>
<td>5</td>
<td>100</td>
<td>100</td>
</tr>
</tbody>
</table>

- Because only one parameter at a time was varied, we do not know the effect of varying more than one (or even all) of the parameters at a time.

- The analysis only examined the effect of the design parameters. The effect of varying mission parameters, $[h, L_f]$, on the deployment performance was not evaluated.

In order to completely analyze the system, we must study the effect of all the five system parameters, $[T_0, \bar{m}, \dot{L}_0, h, L_f]$, and use multiple values of each parameter within their respective operating ranges and vary them simultaneously in different combinations. In order to reduce the complexity of this analysis, we nondimensionalize the equations of motion and thus reduce to two the number of parameters affecting the deployment.

We nondimensionalize the equations of motion by first introducing nondimensional basic units, nondimensional time, $\tilde{t} = t\Omega$, and nondimensional length, $\tilde{L} = L/L_f$. Then, using these nondimensional variables, we rewrite the equations of motion as:
\[
L'' = L \left[ \left( \theta' + 1 \right)^2 \cos^2 \phi + \phi'^2 + 3 \cos^2 \theta \cos^2 \phi - 1 \right] - \tilde{T}_0 
\]
(2.7)

\[
\theta'' = 2 \left( \theta' + 1 \right) \phi' \tan \phi - 2 \frac{\tilde{L}'}{L} \left( \theta' + 1 \right) - 3 \cos \theta \sin \theta 
\]
(2.8)

\[
\phi'' = -2 \frac{\tilde{L}'}{L} \phi' - \left[ \left( \theta' + 1 \right)^2 + 3 \cos^2 \phi \right] \cos \phi \sin \phi 
\]
(2.9)

Note that all nondimensional variables are denoted by a top-placed tilde. Since \( \theta \) and \( \phi \) are nondimensional by definition, we drop the use of tildes on \( \theta \) and \( \phi \). Also note that the superscript \( \prime \) denotes the derivative of a given variable with respect to nondimensional time \( \tilde{t} \) and not with respect to dimensional time.

In the above nondimensional equations of motion, the following two nondimensional parameters are introduced:

\[
\tilde{T}_0 = \frac{T_0}{\bar{m} L_f \Omega^2} \quad \text{and} \quad \tilde{L}_0' = \frac{\dot{L}_0'}{\Omega L_f} \quad (2.10)
\]

Note that the definitions of these two nondimensional parameters include all of the design parameters, \([T_0, \bar{m}, \dot{L}_0]\) and mission parameters, \([h, L_f]\) (\(h\) is included implicitly in \(\Omega\)). In other words, variations of each of the five dimensional system parameters can be effectively reduced to variations of the two nondimensional parameters. Note that we no longer differentiate between design and mission parameters. Now we investigate the operating range of the nondimensional parameters based on the operating range of the corresponding dimensional system parameters. Based on the definitions of these two nondimensional parameters, \(\tilde{T}_0\) and \(\tilde{L}_0'\) in Eq. (2.10), and using the ranges of each of the dimensional parameters discussed earlier, we can derive the range of \(\tilde{T}_0\) as \([0.21343, 218.31]\) and \(\tilde{L}_0'\) as \([0.25929, 62.87]\).

Even though the maximum value for \(\tilde{L}_0'\) is calculated as 62.87, for values of \(\tilde{L}_0'\) greater than 22, we have AD = PFD = 100% for any given value of \(\tilde{T}_0\) within the range calculated above. Therefore, values of \(\tilde{L}_0'\) beyond 22 are not investigated.
For convenience, the range of $\tilde{T}_0$ is taken to be $[0.1, 220]$ and the range of $\tilde{L}_0'$ is taken to be $[0.2, 22]$. Note that the the analysis set forth in the remainder of this study of short tethers is valid for any combination of the dimensional parameters which result in nondimensional parameters within the previously stated ranges; i.e. it is no longer necessary to restrict the dimensional parameters to their previously described ranges as long as the nondimensional parameters are restricted to their respective ranges. For example, the analysis is valid for a value of $T_0$ less than $50 \, mN$ or greater than $200 \, mN$ as long as the values of $\bar{m}$, $\Omega$, and $L_f$ are such that $\tilde{T}_0$ falls in the range $[0.1, 220]$. The following nondimensional analysis is insensitive to different combinations of dimensional parameters which yield the same set of nondimensional parameters.

### 2.3.4 Grids and Log-plots

To investigate the deployment behavior of tethered satellite systems in terms of nondimensional parameters over the relatively wide operating range of the nondimensional parameters, the equations of motion are simulated and analyzed for 30 values of $\tilde{T}_0$ and 30 values of $\tilde{L}_0'$. The nondimensional equations of motion (Eq. (2.7)-(2.9)) are therefore simulated over the usual range of IPs for 900 different combinations of $\tilde{T}_0$ and $\tilde{L}_0'$. We tabulate the AD and PFD for each combination of nondimensional parameters over the usual range of IPs and show the results in Fig. 2.12 and Fig. 2.13 respectively. Each entry in the grid corresponds to a different combination of $(\tilde{L}_0', \tilde{T}_0)$ and is assigned a shade of gray which indicates the level of either AD or PFD reached at each combination of nondimensional variables. The allocation of the shade are shown in the shade-bar on the right side of the grid. Note that the shade-bars for AD and PFD grids are similar but not identical. This is to highlight the differences between important levels of PFD and AD. For example, 0% PFD indicates that none of the IPs result in full deployment for a given combination of nondimensional parameters; it is important to know when this happens and hence a shade is dedicated only for that phenomena. On the other hand, 0% AD indicates that none of the IPs deploy at all, which is very unlikely. So a shade is assigned for a low range of AD (rather than just for 0%). However, since 100% AD and 100% PFD
represent the same phenomenon (all the IPs deploying completely), the white region is identical in both of the grids.

Now we analyze the deployment behavior in terms of the nondimensional parameters in the context of these grids. From the AD grid in Fig. 2.12, we can see that the AD increases both with increases in $\tilde{L}_0'$ and decreases in $\tilde{T}_0$. Since $\tilde{L}_0'$ depends on the initial separation velocity $\dot{L}_0$, and $\tilde{T}_0$ depends on the tether friction $T_0$, the above analysis is consistent with the analysis performed using dimensional parameters. We see that at $\tilde{L}_0' = 22$, complete deployment for all the IPs is achieved, i.e. $AD = 100\%$ for all the values of $\tilde{T}_0$. Because the level of deployment reached increases with increases in $\tilde{L}_0'$ and the level of deployment over the range of IPs cannot produce an AD higher than 100%, we can assume that the level of deployment will be 100% for all simulations in which $\tilde{L}_0' > 22$ regardless of the value of $\tilde{T}_0$. Therefore, there is no need to perform simulations for values of $\tilde{L}_0'$ above 22. Note that in Fig. 2.12, the transition between high and low levels of AD generally occurs diagonally throughout the grid starting from the bottom left corner and ending at top right corner. The top left triangle of the grid is mostly black which represents low levels of deployment; the bottom right triangle is mostly white which represents full deployment. This is consistent with the intuitive assertion that deployment favors low $\tilde{T}_0$ and high $\tilde{L}_0'$. For any given point along the transition region shown on Fig. 2.12, increasing $\tilde{L}_0'$ results in an increase in AD; however, increasing $\tilde{T}_0$ results in a decrease in AD. These tendencies create the diagonal nature of the transition region. The same diagonal nature is seen in Fig. 2.13 with the exception that the transition seems to deviate from a diagonal in the upper right of the figure. In order to further explore the nature of the transition region, the data represented by Fig. 2.12 and Fig. 2.13 were plotted on logarithmic scales in Fig. 2.14 and Fig. 2.15. We have selected 101 points each on the $\tilde{L}_0'$ and $\tilde{T}_0$ scales, thus $101 \times 101 = 10201$ combinations of nondimensional variables were simulated and the resulting AD and PFD are shown on a log-log scale. In these log plots, the shade scales are identical to those in Figs. 2.12 and 2.13 but the grid lines are removed for clarity. More detail can be seen in Figs. 2.14 and 2.15 than in Figs. 2.12 and 2.13, especially in the transition region. Also, since we have used an exponential scale, any point in the operating range of $\tilde{L}_0'$ and $\tilde{T}_0$ can be located
on these plots and the corresponding AD and PFD can be estimated. A case study featuring the use of these plots as a design tool will be described in the next section.

Figure 2.12: AD Grid for Selected \( (\bar{L}_0', \bar{T}_0) \) values
Figure 2.13: PFD Grid for Selected ($\tilde{L}_0$, $\tilde{T}_0$) values
Figure 2.14: AD Log Plot for Various Values of \((\tilde{L}_0, \tilde{T}_0)\)
Figure 2.15: PFD Log Plot for Various Values of \((\tilde{L}_0, \tilde{T}_0)\)
2.4 Applications

We now demonstrate two applications of the methods previously described. These two applications illustrate how the principles discussed until now in this analysis can be used in two different ways to determine the best set of parameters for optimum tethered satellite system deployment. For convenience, the two nondimensional parameters are written in coordinate form when they appear together to represent the entire system configuration. For example, \( \tilde{L}'_0 = 2 \) and \( \tilde{T}_0 = 3 \) are written as \( (\tilde{L}'_0 = 2, \tilde{T}_0 = 3) \).

2.4.1 Mission Design from Given Parameters

We will show how the tools previously developed can be used to adjust tethered satellite mission parameters in order to achieve full tethered satellite deployment. Such an analysis could be performed mid-way through the mission design process in order to determine adjustments which need to be made to tethered satellite parameter values in order to achieve full deployment. In order to estimate the amount of deployment that will take place with the given parameters, we calculate the values of \( \tilde{L}'_0 \) and \( \tilde{T}_0 \) and use Figs. 2.14 and 2.15 to estimate the expected level of average deployment and the percentage of IPs reaching full deployment. Based on the location of the point \( (\tilde{L}'_0, \tilde{T}_0) \) relative to the transition region, we determine the most appropriate change in the design parameters that will result in full tethered satellite deployment. Let us assume a 70 kg tethered satellite mission similar to the TAG mission previously discussed with mission parameters \([h = 1000 km, L_f = 2.4 km]\). Let us also assume that the two halves of the tethered satellite separate at a velocity of 3 m/s and that the internal deployment mechanism friction (and thus the tether tension) is 185 mN. We also assume that the end-body masses are distributed so that the effective mass is 10 kg. The design parameters corresponding to this physical configuration are \([T_0 = 185 mN, \bar{m} = 10 kg, \dot{L}_0 = 3 m/s]\). From Eq. 2.10 this configuration gives the nondimensional variable values \((\tilde{L}'_0 = 1.01, \tilde{T}_0 = 5.02)\). It would be difficult to estimate the AD for such a configuration using only Fig. 2.12 because the spacing between the grid points of Fig. 2.12 is too large. We can narrow down our region of
interest by zooming in on a subgrid around the point of interest as shown in Fig. 2.16

<table>
<thead>
<tr>
<th>$\tilde{T}_0$</th>
<th>$\tilde{L}_0'$</th>
<th>1</th>
<th>1.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 2.16: Zoom in on AD grid for $(\tilde{L}_0' = 1.01, \tilde{T}_0 = 5.02)$

<table>
<thead>
<tr>
<th>$\tilde{T}_0$</th>
<th>$\tilde{L}_0'$</th>
<th>1</th>
<th>1.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 2.17: Zoom in on PFD grid for $(\tilde{L}_0' = 1.01, \tilde{T}_0 = 5.02)$
We see that this subgrid has only the bottom two shades of the shade-bar, together indicating [0 25)% of deployment at the point of interest. Now we examine the same subgrid on a plot showing PFD as a function of $\tilde{L}_0'$ and $\tilde{T}_0'$; this subgrid is shown in Fig. 2.17. The entire grid is black and hence we can be certain that, at this point of interest, the PFD is 0%.

We can see that AD is below 25% and PFD is 0%. To achieve better performance, we need to move diagonally down the grid and toward right. This can be done by increasing $\tilde{L}_0'$ and decreasing $\tilde{T}_0'$. In order to increase $\tilde{L}_0'$, let us assume that the separation mechanism employs a stiff spring resulting in the highest considered initial separation velocity: $6 \text{ m/s}$. The new configuration yields ($\tilde{L}_0' = 2.02$, $\tilde{T}_0 = 5.02$), which moves the point of interest toward the right of the grid. Now in order to move down the grid, we lower the value of $\tilde{T}_0'$. We accomplish this by increasing the effective mass through equal redistribution of the end-body masses, giving the maximum value of $\bar{m}$, as calculated earlier to be 17.5 kg. The nondimensional parameters are now ($\tilde{L}_0' = 2.02$, $\tilde{T}_0 = 2.87$). With these two steps, we effectively moved the point of interest diagonally down the grid toward the right. Now instead of using the grid plots, we zoom in on the more refined tool, i.e., the log plots. We can locate the point ($\tilde{L}_0' = 2.02$, $\tilde{T}_0 = 2.87$) on the both plots as shown in the Figs. 2.18 and 2.19. We notice that this configuration is located on the point representing an AD in [50, 75)% range and PFD in the (0 25)% range. This can be verified by the deployment versus IP curves shown in the Fig. 2.21. The AD is still not at its maximum value, even at optimum values for $\bar{m}$ and $\tilde{L}_0'$. We now attempt to raise the AD by assuming a reduction in $T_0$, i.e., by reducing $\tilde{T}_0'$ and hence, moving further down the AD grid. Assume a decrease in $T_0$ (i.e., a decrease in the internal friction of the deployment mechanism) to 90 mN. The nondimensional design parameters of this configuration are ($\tilde{L}_0' = 2.02$, $\tilde{T}_0 = 1.40$). From the initial grids (Figs. 2.12 and 2.13) it is clear that for this configuration, AD and PFD are 100%. The deployment curves for this configuration can be seen in Fig. 2.22. This application demonstrates how the PFD and AD grids can be used to determine changes necessary to a tethered satellite physical configuration in order to achieve an AD of 100%.
Figure 2.18: AD log plot for \((\tilde{L}_0' = 2.02, \tilde{T}_0 = 2.87)\)

Figure 2.19: PFD log plot for \((\tilde{L}_0' = 2.02, \tilde{T}_0 = 2.87)\)
Figure 2.20: Deployment curves for \((\tilde{L}_0 = 1.01, \tilde{T}_0 = 5.02)\)

(a) 3-D representation with IPs on horizontal plane and deployed length on vertical axis  
(b) 2-D representation with \(\phi\) on horizontal axis and \(\theta\) on vertical axis

Figure 2.21: Deployment curves for \((\tilde{L}_0 = 2.02, \tilde{T}_0 = 2.87)\)

(a) 3-D representation with IPs on horizontal plane and deployed length on vertical axis  
(b) 2-D representation with \(\phi\) on horizontal axis and \(\theta\) on vertical axis
(a) 3-D representation with IPs on horizontal plane and deployed length on vertical axis

(b) 2-D representation with $\phi$ on horizontal axis and $\theta$ on vertical axis

Figure 2.22: Deployment curves for ($\tilde{L}_0' = 1.4$, $\tilde{T}_0 = 2.02$)
2.4.2 Mission Parameter Determination

We will now demonstrate how mission parameters can be actively selected (rather than adjusted) in order to obtain the desired deployment behavior. This differs from the previous design method in that we will demonstrate how mission parameters can be selected to achieve full deployment with a limited amount of knowledge about the system rather than demonstrating how mission parameters can be adjusted to achieve full tethered satellite deployment. Assume that the given design criteria call for a 70 kg tethered satellite mission operating at an orbital height \( h = 1200 \text{ km} \) with a final deployed tether length \( L_f = 3 \text{ km} \). We now attempt to determine a set of parameters that will result in full deployment. From the given information, and the definition of \( \tilde{T}_0 \) and \( \tilde{L}_0' \) as given in Eq.(2.10), we can immediately calculate

\[
\tilde{T}_0 = \frac{T_0}{0.0027482} \quad \text{and} \quad \tilde{L}_0' = \frac{\dot{L}(0)}{2.8714} \tag{2.11}
\]

From the operating range of dimensional parameters, discussed earlier in the chapter, we can also calculate the range of \( T_0/\bar{m} \) as \([0.0028571 \text{ mN/kg}, 0.02 \text{ mN/kg}]\) and we know that range of \( \dot{L}_0 \) is \([3 \text{ m/s}, 6 \text{ m/s}]\). From this we can compute the range of nondimensional parameters as \( \tilde{T}_0 \) as \([1.039, 7.27]\) and that of \( \tilde{L}_0' \) as \([1.048, 2.0896]\)

Now, with the minimum and maximum value of both the nondimensional parameters, we can narrow the AD grid as shown in Fig. 2.23. The final parameters selected for the tethered satellite system must give nondimensional parameter values which lie in the subgrid described by Fig. 2.23. In this subgrid, only two points give 100% PFD and AD, \( (\tilde{L}_0' = 2, \tilde{T}_0 = 1.1) \) and \( (\tilde{L}_0' = 2, \tilde{T}_0 = 1.4) \). It is generally difficult to reduce the \( T_0 \) because of its dependence on the internal friction of the deployment mechanism; therefore, we choose a point which allows for higher \( T_0 \), i.e., higher \( \tilde{T}_0 \). We choose the point \( (\tilde{L}_0' = 2, \tilde{T}_0 = 1.4) \) as nondimensional parameters and choose dimensional parameters which correspond to these nondimensional parameters. Substituting the nondimensional parameters into Eqs. 2.11, we have \( \dot{L}_0 = 5.75 \text{ m/s} \) and \( T_0/\bar{m} = 0.0038 \text{ mN/kg} \). If we equally distribute the mass, i.e., set \( \bar{m} = 17.5 \text{ kg} \), then \( T_0 = 67.3 \text{ mN} \). In summary, the dimensional design parameters for this system
Figure 2.23: AD Subgrid for Parameter Identification. $\tilde{L}_0$ arranged by columns and $\tilde{T}_0$ arranged by rows

are $[T_0 = 67.3 \, mN, \, \bar{m} = 17.5 \, kg, \, \dot{L}_0 = 5.75 \, m/s]$ and the mission parameters are $[h = 1200 \, km, \, L_f = 3 \, km]$ and the tethered satellite system exhibiting these parameters will deploy fully for all initial angular displacements. The study presented in this chapter is adapted from the work done by Mantri et al [57].
Chapter 3

The Space Elevator

As mentioned in Chapter 1, our primary motivation for studying long tethers is to analyze the deployment dynamics of a space elevator. Before we go into detailed long tether analysis, we present some background information related to Space Elevators.

3.1 Background

The use of tethers for space travel has been proposed for a long time in different forms [58, 59]. One of the most revolutionary ideas is the Space Elevator(or the Orbital Tower). While there are several designs proposed for the Space Elevator (SE), one of the popular and more researched designs involves a 100,000 km tether connecting Earth and Space balanced at Geostationary Orbit (GEO). Some of the other alternate designs will be discussed briefly at the end of this Chapter. The idea of the Space Elevator was first presented by Russian scientist Konstantin Tsiolkovsky (who is also known for developing basic rocket equation) in 1895 [60]. Inspired by the Eiffel tower, Tsiolkovsky imagined a tall tower reaching up to space. This idea was further investigated by researchers such as Artustontov [61], Pearson [29] and Isaacs et al. [62]. A classic science fiction novel by Arthur Clarke [63] (who is also famous for popularizing the concept of the geostationary satellite) and a semi-technical paper in the following year by the same author [64] also popularized the SE concept. However, at the time of the Clarke paper, the high strength material requirements could not
be addressed, as no known material at that time had strength promising enough to build a Space Elevator. Only when fullerenenes (e.g. carbon nanotubes), which showed a theoretical strength of 130 Gpa, were first discovered in 1991 [65], did SE research regain its popularity. In 2000, David Smitherman prepared a report on the Space Elevator for NASA [66] based on the findings of a workshop on space elevators in 1999. This report, after arguing the pro and cons of the space elevator concept, ends by recommending that a space elevator be built for use in the latter part of the 21st century. However, a study by Dr. Bradley Edwards, under a grant from the NASA Institute of Advanced Concepts [67], also investigated the concept in detail, and Edwards proposed constructing a space elevator within the first half of this century. A brief overview of his findings can be found in [28] and [14]. Another important paper on the space elevator is written by Chobotov [68]. In that paper, a detailed study on orbital transfer using the space elevator for interplanetary missions is presented.

3.2 System Geometry

The basic heuristics of using a long tether to serve as a space elevator were first presented in a paper by Pearson [29]. In that paper, it was pointed out that if the tether is long enough and extended far enough above GEO, it can remain completely in tension instead of compression due to gravity gradient forces (given that the material is strong enough to stand under its own weight). Another critical aspect of SE, also analyzed in detail in the Pearson paper is tether taper. To minimize the tether mass and to maintain constant stress throughout its cross section, the tether can be tapered such that its cross sectional area is maximum at GEO and decreases with exponential taper toward either end. The cross section of the tower at any point is sufficient to support the weight of all the material below it. It should be noted that the height of a tower with uniform cross section would be the same as a tapered tower. As an alternative to building a 144,000 km tower with no counter weight, it has been proposed by Edwards that a 100,000 km tether can be deployed provided a large enough counter weight is utilized. This tether, (consisting of many separate
tethers in parallel), would serve as a Space Elevator. The tether tension of such a system is maximum at GEO, decreasing to a minimum at either end. In the absence of a ballast mass or counter-weight, the tether tension at either end is theoretically zero. To introduce a finite tether tension, $T_0$, at either end, a ballast mass can be introduced as shown in Figure 3.1

![Figure 3.1: SE System Geometry](Adapted from [69])

The exponential tether taper ratio, $TR$, as derived by Pearson is

$$TR = \frac{A_s}{A_0} = e^{0.776r_0}$$

(3.1)

Since the SE is deployed from GEO, the most convenient and dynamically stable anchor location for the SE would be along the equator [70]. According to Pearson, this would also be geographically stable as the equator experiences low average wind-speed and almost no hurricanes.
3.3 Material Requirements

The material requirements for the space elevator can be best explained in terms of the characteristic length, $L_c$, of a material; this is defined as the ratio of yield stress to specific weight. Clearly, a higher $L_c$ value represents a stronger material and a lower mass density and hence is strongly desired in the construction of a space elevator [29]. Characteristic length can also be interpreted as the height up to which a cylindrical tower can sustain its own weight in a uniform 1-g gravity field. Constructing an SE is equivalent to constructing a 4900 km high tower in 1-g [29]. This means that a material with $L_c$ of at least 4900 km is required for constructing a SE. However, as mentioned earlier, this strength requirement can be reduced by tapering the tether. A constant stress tapered tether can reach an infinite length which means that, theoretically, any material could be used for constructing a SE by making its diameter at GEO large enough. However, for most known materials, this required diameter is beyond any practical possibility. For example, the taper ratio, $TR$, which is defined as the cross sectional area at GEO divided by the cross sectional area at the anchor, required for a SE made of steel, is of the order $10^{50}$, and that for a graphite SE is about 100 [67].

Table 3.1 shows a significant difference in required taper ratio for spectra, graphite and fullerene.

<table>
<thead>
<tr>
<th></th>
<th>TR(sf=1)</th>
<th>TR(sf=2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spectra</td>
<td>$2 \times 10^6$</td>
<td>$4.5 \times 10^{12}$</td>
</tr>
<tr>
<td>Graphite</td>
<td>100</td>
<td>9000</td>
</tr>
<tr>
<td>Fullerene</td>
<td>1.7</td>
<td>2.8</td>
</tr>
</tbody>
</table>

A Space elevator would grant tremendously easy access to space to the outer planets. This is possible because of the drastic reduction in energy requirements. A payload sliding up the SE to GEO from Earth would need only a potential energy component. The kinetic energy of the orbital velocity is already imparted by the SE as the payload ascends [29].
3.4 Other Space Elevator Ideas

3.4.1 Planetary space elevators

Synchronous space elevators are well suited for planets with weak gravity and fast rotation rates, such as Mars. The tether payload mass ratio, $r_m$, required on Mars would be about 42, and hence graphite whiskers could be used [58]. Pearson [71, 72] also proposed a space elevator to the Earth-Moon Lagrange points L1 and L2. Although the height of such a system would be significantly greater than a standard Earth SE (twice as large for the SE to L1 and four times as large for the SE to L2), the strength requirement would be much lower, and the mass and taper ratio could be achieved with present day materials. In addition to Pearson, other researchers have also analyzed the lunar space elevator and have discussed its potential applications [73, 74]. Such a space elevator would help in building a permanent communication link from Earth to the far side of the Moon, and thus would facilitate access to large regions of the Earth-Moon system. A schematic of the lunar Space Elevator as presented by Pearson is shown in Figure 3.2

![Figure 3.2: Schematics of Lunar Space Elevator. [Adapted from [71]]](image-url)
3.4.2 Alternate Configurations

**Tall Tower SE**

According to the Smitherman report [66], the tall tower SE design is the most efficient and technically feasible construction method. In this design, proposed by Landis [75], the SE would be anchored to a 300 km to 1500 km tall tower instead of the ground. This greatly reduces the strength required and hence the required mass ratio (200, which is possible using graphite whiskers). The tower would be in compression, and as a pressurized structure it would have improved stability. In Landis’ design analysis, it is argued that the construction of such long towers is not limited by construction or material technology available today and hence, could be built in the present day.

**LEO Space Elevator**

In this design, a relatively short tether of length 2000 – 4000 km would be used with the center of gravity of the system in a Low Earth Orbit. Since the orbital velocity of this system is not synchronized with that of Earth, it cannot be anchored to the Earth. However, this can be used as an advantage as it allows the system to be placed in an inclined orbit with respect to plane of ecliptic. This would be greatly beneficial for interplanetary travel [66]. A schematic of a LEO Space Elevator is shown in Figure 3.3.
Figure 3.3: Schematics of LEO Space Elevator [Adapted from [66]]
Chapter 4

Long Tether Analysis

We now analyze long tethered systems with the motivation of understanding the deployment dynamics of an initial space elevator ribbon.

4.1 Deployment of Space Elevator and other long tethered systems

The deployment of a space elevator as proposed by Edwards involves a single tether that would be deployed from GEO and then reinforced incrementally from the ground once anchored [67]. This is a promising strategy, but the orbital stability of the system before anchoring needs to be investigated, as will be discussed below.

Very long tether systems have positive orbital energy (i.e. enough energy to escape orbit) [21] and are orbitally unstable [76]. Studies on very-long tether dumbbells in MEO (Middle Earth Orbit) with positive orbital energy indicate that the orbit can be bound but libration must be carefully controlled.

Once anchored, the stability problem is likely to disappear but further analysis is required. The above discussion on energy indicates that the deployment can be unstable. Hence, it is important to develop a model which will show this instability and thus help in better understanding SE deployment dynamics. Such a model will also help in pinpointing specific problems and consequently in exploring alternate design
possibilities. Though there exists some literature concerning the deployment dynamics of a space elevator, there is no detailed parametric study resulting from a set of governing equations that has been published. To fill this gap, we first start with developing the equations of motion for space elevator cable deployment. Acknowledging the difficulty in developing a massive model, and also being motivated to understand the effect of tether mass in deployment dynamics, it is worthwhile to develop two models separately; one with a massive tether and other with a massless tether. Both models are developed only for planar motion. We first start with deriving equations of motion which assume the tether to be massless.

4.2 Model

The space elevator deployment scenario we are considering in this paper consists of deploying a long tether toward and away from the Earth from a location initially in orbit several thousand kilometers above the Earth. We first develop a model which treats the tether as massless and has all of the mass of the system contained in the two end-bodies, and then we develop a model which accounts for the mass of the tether. A comparison of results from the massless and massive models is then conducted, and these results show that good qualitative results can be obtained using the massless model, so we then present a detailed parametric study of the deployment dynamics using the massless model.

The orbital radius, $R$, in-plane angle, $\theta$, tether length, $L$, and true-anomaly, $\nu$, are the variables used as a basis for this set of equations. The variables and frames of reference used are defined and shown in Figure 4.1. Point C represents the center of mass of the system.
4.3 Equations of Motion: Derivation

4.4 Massless tether model

By Newton’s second law of motion, we know that the total force acting on the system is equal to the product of its mass and the acceleration of the center of mass of the system. This can be stated as:

\[
\frac{1}{m} \left\{ \bar{F}_A + \bar{F}_B \right\} = \vec{a}_{C/E}
\]  

(4.1)

where \( m \) is the total end mass, \( m_a + m_b \), \( \vec{a}_{C/E} \) and \( \vec{a}_{A/B} \) are the acceleration vectors
from the center of the Earth, E to the center of mass of the tether system, C and from the end-mass B to the end-mass A respectively.

Applying Newton’s Second Law of Motion to the individual end-masses, we get:

\[
\vec{F}_A = m_a \vec{a}_{A/E} \tag{4.2}
\]

\[
\vec{F}_B = m_b \vec{a}_{B/E} \tag{4.3}
\]

The above equations can be rewritten as:

\[
\frac{\vec{F}_A}{m_a} = \vec{a}_{A/E} \tag{4.4}
\]

\[
\frac{\vec{F}_B}{m_b} = \vec{a}_{B/E} \tag{4.5}
\]

where \(\vec{F}_A\) and \(\vec{F}_B\) are the net forces acting on the end-bodies A and B, respectively, \(m_a\) and \(m_b\) are the respective end-masses and \(\vec{a}_{A/E}\) and \(\vec{a}_{B/E}\) are the acceleration vectors directed from the center of the Earth, E, to the end masses, A and B, respectively.

Subtracting Equations (4.4) and (4.5), we get:

\[
\frac{\vec{F}_A}{m_a} - \frac{\vec{F}_B}{m_b} = \vec{a}_{A/B} \tag{4.6}
\]

**Acceleration**

Now we individually derive the acceleration and force components of the deploying system. We start with deriving the acceleration of C with respect to center of the Earth, E, in the inertial reference frame \(< E1, E2, E3 >\) as follows:

\[
E_\omega \vec{r}_{C/E} = O_\omega \vec{r}_{C/E} + E \alpha \vec{r}_{C/E} + 2E \vec{\omega} \times O \vec{r}_{C/E} + E \vec{\omega} \times (E \vec{\omega} \times \vec{r}_{C/E}) \tag{4.7}
\]
where $\dot{E}_{\vec{r}_{C/E}}$ and $\dot{O}_{\vec{r}_{C/E}}$ are the second derivatives of position vectors from point E to point C, differentiated with respect to time in the E and the O reference frames, respectively. For the planar motion considered in this derivation, the angular velocity, $E_{\vec{\omega}^O}$, and the angular acceleration, $E_{\vec{\alpha}^O}$, of the O frame with respect to the E frame can be simply expressed in coordinates of the O reference frame in terms of the true anomaly, $\nu$, as:

$$\{E_{\vec{\omega}^O}\}_O = \begin{bmatrix} 0 \\ 0 \\ \dot{\nu} \end{bmatrix} \text{ and } \{E_{\vec{\alpha}^O}\}_O = \begin{bmatrix} 0 \\ 0 \\ \ddot{\nu} \end{bmatrix}$$

The position vector $\vec{r}_{C/E}$ and its subsequent derivatives differentiated with respect to time in the O reference frame, can be expressed in the coordinates of the O reference frame as:

$$\{\vec{r}_{C/E}\}_O = \begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix}, \quad \{O_{\vec{r}_{C/E}}\}_O = \begin{bmatrix} \dot{R} \\ 0 \\ 0 \end{bmatrix}, \quad \{O_{\ddot{\vec{r}}_{C/E}}\}_O = \begin{bmatrix} \ddot{R} \\ 0 \\ 0 \end{bmatrix}$$

where R is the orbital radius from the center of the Earth, E, to the center of mass of the tether system, C. Substituting each term on the right hand side of Equation (4.7)
with the corresponding vector in the O reference frame, we obtain

\[
\begin{align*}
\left\{ \frac{E}{\vec{r}_{C/E}} \right\}_O &= \left[ \begin{array}{c} \ddot{\hat{R}} \\ 0 \\ 0 \\ \dot{\vec{R}} \end{array} \right] + \left( \begin{array}{c} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} \hat{R} \\ 0 \end{bmatrix} \end{array} \right) + \left( \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} \dot{\hat{R}} \\ \dot{\vec{R}} \end{bmatrix} \right) + \left( \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ \dot{\vec{R}} \end{bmatrix} \right) + \left( \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} \dot{\vec{R}} \\ 0 \end{bmatrix} \right) \\
&= \left[ \begin{array}{c} \ddot{\hat{R}} - \dot{\vec{R}}^2 R \\ \ddot{\vec{R}} + 2 \dot{\vec{R}} \dot{\vec{R}} \\ 0 \end{array} \right] (4.8)
\end{align*}
\]

Similarly, we derive the inertial acceleration of A with respect to B as:

\[
\frac{E}{\vec{r}_{A/B}} = S \ddot{\vec{r}}_{A/B} + \frac{E}{\vec{r}_{A/B}} S \times \frac{E}{\vec{r}_{A/B}} + 2 \frac{E}{\vec{r}_{A/B}} S \times \frac{E}{\vec{r}_{A/B}} S + \frac{E}{\vec{r}_{A/B}} \left( \frac{E}{\vec{r}_{A/B}} S \times \frac{E}{\vec{r}_{A/B}} S \right) (4.9)
\]

Here the angular velocity, \( \frac{E}{\vec{r}} \), and the angular acceleration, \( \frac{E}{\vec{r}} S \), of the satellite frame, S, with respect to the inertial reference frame, E, can be expressed as:

\[
\begin{align*}
\frac{E}{\vec{r}} S &= \begin{bmatrix} 0 \\ 0 \\ \theta + \dot{\theta} \end{bmatrix} \\
\frac{E}{\vec{r}} S &= \begin{bmatrix} 0 \\ 0 \\ \theta + \dot{\theta} \end{bmatrix}
\end{align*}
\]

where \( \theta \) is the planar angle made by tether axis with the local vertical. The position vector \( \vec{r}_{A/B} \) and its subsequent time derivatives differentiated in the S reference frame,
can be expressed in the S frame as

\[
\begin{bmatrix}
\mathbf{r}_{A/B}^{S} \\
\mathbf{\dot{r}}_{A/B}^{S} \\
\mathbf{\ddot{r}}_{A/B}^{S}
\end{bmatrix} =
\begin{bmatrix}
L \\
0 \\
0
\end{bmatrix},
\begin{bmatrix}
\mathbf{\dot{r}}_{A/B}^{S} \\
\mathbf{\ddot{r}}_{A/B}^{S}
\end{bmatrix} =
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

where \( L \) is the total tether length. Substituting each term on the right hand side of Equation (4.9) with the corresponding vector in the S reference frame, we obtain:

\[
\Rightarrow \begin{bmatrix}
\mathbf{\ddot{r}}_{A/B}^{S}
\end{bmatrix} =
\begin{bmatrix}
\ddot{L} - L \left( \dot{\theta} + \dot{\nu} \right)^2 \\
L \left( \dddot{\theta} + \dddot{\nu} \right) + 2\dot{L} \left( \dot{\theta} + \dot{\nu} \right) \\
0
\end{bmatrix}
\]

\[ (4.10) \]

### 4.4.1 Forces

Once we have the required acceleration components, we derive the force components. The forces acting on each end-body are: the gravitational force, the tether tension and the external thruster force. With this information, the net force acting on the end-body A is given by:

\[
\begin{aligned}
\mathbf{F}_A &= \mathbf{F}_{GA} + \mathbf{F}_{TA} + \mathbf{F}_{CA} \\
&= - \frac{GM_e m_a}{|\mathbf{r}_{A/E}|^3} \mathbf{r}_{A/E} + T \frac{\mathbf{r}_{B/A}}{|\mathbf{r}_{B/A}|} + f_a \frac{\mathbf{r}_{A/B}}{|\mathbf{r}_{A/B}|}
\end{aligned}
\]

\[ (4.11) \]
where \( \vec{F}_{GA} \), \( \vec{F}_{TA} \) and \( \vec{F}_{CA} \) are the gravitational, tension and control forces, respectively, acting on the end-body A.

This net force \( \vec{F}_A \) can be expressed in the coordinates of the S reference frame as:

\[
\{ \vec{F}_A \}_S = k_a m_a \left( \begin{bmatrix} L_a \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} R \cos \theta \\ -R \sin \theta \\ 0 \end{bmatrix} \right) + T \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + f_a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
\]

\[
= \begin{bmatrix} k_a m_a (L_a + R \cos \theta) - (T - f_a) \\ -k_a m_a R \sin \theta \\ 0 \end{bmatrix}
\]

\[
\Rightarrow \frac{1}{m_a} \{ \vec{F}_A \}_S = \begin{bmatrix} k_a (L_a + R \cos \theta) - \frac{1}{m_a} (T - f_a) \\ -k_a R \sin \theta \\ 0 \end{bmatrix}
\]

Similarly, \( \vec{F}_B \), the net force acting on end body B, expressed in coordinates of the S reference frame can be computed as:

\[
\{ \vec{F}_B \}_S = \begin{bmatrix} k_b m_b (-L_b + R \cos \theta) + (T - f_b) \\ -k_b m_b R \sin \theta \\ 0 \end{bmatrix}
\]

\[
\Rightarrow \frac{1}{m_b} \{ \vec{F}_B \}_S = \begin{bmatrix} k_b (-L_b + R \cos \theta) + \frac{1}{m_b} (T - f_b) \\ -k_b R \sin \theta \\ 0 \end{bmatrix}
\]

Note that as defined in the nomenclature section, \( m_a \) and \( m_b \) are the masses of end-bodies, A and B respectively, T is the tether tension, \( f_a \) and \( f_b \) are thruster forces.
acting on end-bodies, A and B, respectively, and

\[ k_a = \frac{-GM_e}{|\vec{r}_{A/E}|^3} = \frac{-GM_e}{(R^2 + L^2 + 2RL\cos\theta)^{\frac{3}{2}}} \]

and

\[ k_b = \frac{-GM_e}{|\vec{r}_{B/E}|^3} = \frac{-GM_e}{(R^2 + L^2 - 2RL\cos\theta)^{\frac{3}{2}}} \]

Adding Equation (4.12) and Equation (4.13), we obtain:

\[
\{ \vec{F}_A + \vec{F}_B \}_S = \begin{bmatrix}
\bar{m}L(k_a - k_b) + f_2 + R(k_a m_a + k_b m_b)\cos\theta \\
-R(k_a m_a + k_b m_b)\sin\theta \\
0
\end{bmatrix}
\]

(4.14)

where \( \bar{m} = \frac{m_a m_b}{m_a + m_b} \) is the effective mass and \( f_2 \) is defined as \( f_a - f_b \). Expressing this in the coordinates of the O reference frame yields:

\[
\{ \vec{F}_a + \vec{F}_b \}_O = \begin{bmatrix}
\cos\theta (\bar{m}L(k_a - k_b) + f_2) + R(k_a m_a + k_b m_b) \\
(\bar{m}L(k_a - k_b) + f_2)\sin\theta \\
0
\end{bmatrix}
\]

(4.15)

Note that any given vector \( \vec{r} \), expressed in the S frame, can be converted from the S frame to the O frame by the transformation:

\[
\{ \vec{r} \}_O = ^O[C]^S \{ \vec{r} \}_S
\]

where \(^O[C]^S\) is the direction cosine matrix for the transformation from the S frame to the O frame. In the planar case, the direction cosine matrix is simply the rotational matrix from the S frame to the O frame and is given by:

\[
^O[C]^S = R_\theta = \begin{bmatrix}
\cos\theta & -\sin\theta & 0 \\
\sin\theta & \cos\theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Now, subtracting Equation (4.12) and Equation (4.13), and expressing the result in
the S reference frame, we obtain:

\[
\frac{1}{m_a} \{\vec{F}_a\}_S - \frac{1}{m_b} \{\vec{F}_b\}_S = \begin{bmatrix}
  k_a L_a + k_b L_b + R (k_a - k_b) \cos \theta - \frac{T}{m} + \left( \frac{f_a}{m_a} + \frac{f_b}{m_b} \right) \\
  -R (k_a - k_b) \sin \theta \\
  0
\end{bmatrix}
\]  

(4.16)

Finally, equating Equation (4.8) and Equation (4.15) as per Equation (4.1) and equating Equation 4.10 and Equation (4.16) as per Equation (4.6), we get equations of motion as follows:

\[
\ddot{R} = \frac{1}{m} \left[ \cos \theta (\dot{m} L (k_a - k_b) + f_2) + R (k_a m_a + k_b m_b) \right] + \nu^2 R 
\]  

(4.17a)

\[
\dot{\nu} = \frac{1}{m R} (\dot{m} L (k_a - k_b) + f_2) \sin \theta - \frac{\dot{R}}{R} \dot{\nu} 
\]  

(4.17b)

\[
\ddot{\theta} = -\frac{R (k_a - k_b) \sin \theta}{L} - \frac{\dot{L}}{L} (\dot{\theta} + \dot{\nu}) - \ddot{\nu} 
\]  

(4.17c)

\[
\ddot{L} = (k_a L_a + k_b L_b) + R (k_a - k_b) \cos \theta - \frac{T}{m} + \left( \frac{f_a}{m_a} + \frac{f_b}{m_b} \right) + \left( \dot{\theta} + \dot{\nu} \right)^2 L 
\]  

(4.17d)

Though L is also one of the variables in these equations, if we assume constant deployment, then we eliminate the need for Equation (4.17d), unless we need to determine the tether tension, T. Note that only Equation (4.17d) contains the tether tension , and that equations Equation (4.17a), Equation (4.17b) and Equation (4.17c) are independent of tether tension. We also assume equal and opposite thruster forces which eliminates the only thruster term (f_2) in the above equations. This also assists us in partially nondimensionalizing these equations as is presented in next section. So, with the assumption of a constant deployment rate and equal and opposite thruster
forces, and re-arranging some of the mass terms, we can rewrite Equation set (4.17) as:

\[ \ddot{R} = \left[ \cos \theta \left( \frac{m_a m_b}{m} L (k_a - k_b) \right) + \dot{R} \left( k_a m_a + k_b m_b \right) \right] + \dot{\nu}^2 R \]  

\[ (4.18a) \]

\[ \ddot{\nu} = \frac{1}{R} \left( \frac{m_a m_b}{m} L (k_a - k_b) \right) \sin \theta - 2 \frac{\dot{R}}{R} \dot{\nu} \]

\[ (4.18b) \]

\[ \ddot{\theta} = - \frac{R (k_a - k_b) \sin \theta}{L} - 2 \frac{\dot{L}}{L} \left( \dot{\theta} + \dot{\nu} \right) - \dot{\nu} \]

\[ (4.18c) \]

### 4.4.2 Equations with nondimensionalized mass

The above equations need \( m_a \) and \( m_b \) to be specified independently of each other. However, to generalize the analysis over different combination of end-masses, we can nondimensionalize the mass terms in the above equations. For this purpose, we first introduce nondimensional masses, \( \bar{m}_a = \frac{m_a}{m} \) and \( \bar{m}_b = \frac{m_b}{m} \). Now we define a nondimensional mass parameter \( \tilde{m} = \bar{m}_a - \bar{m}_b \). So we have:

\[ \bar{m}_a + \bar{m}_b = 1 \]  

\[ (4.19) \]

\[ \bar{m}_a - \bar{m}_b = \tilde{m} \]  

\[ (4.20) \]

From the above equations we can easily derive,

\[ \bar{m}_a = \frac{1 + \tilde{m}}{2} \]  

\[ (4.21) \]

\[ \bar{m}_b = \frac{1 - \tilde{m}}{2} \]  

\[ (4.22) \]
Note that since \( L_a \) and \( L_b \) are the distances from the center of mass of the tether system to end-bodies A and B, respectively, we know that

\[
m_a L_a = m_b L_b = \bar{m} L
\]  
(4.23)

where \( L = L_a + L_b \) is the total tether length and \( \bar{m} \) is effective mass \( \frac{m_a m_b}{m_a + m_b} \). So now we can write,

\[
m_a L_a = \frac{m_a m_b}{m} L
\]  
(4.25)

\[
\Rightarrow L_a = \bar{m}_b L = \frac{1 - \bar{m}}{2} L
\]  
(4.26)

Similarly,

\[
m_b L_b = \frac{m_a m_b}{m} L
\]  
(4.27)

\[
\Rightarrow L_b = \bar{m}_a L = \frac{1 + \bar{m}}{2} L
\]  
(4.28)

With the above definitions and nondimensionalization, our final equations of motion can be presented as

\[
\ddot{R} = \frac{(1 - \bar{m}^2)}{4} L K_2 \cos \theta + R \left( \frac{K_1 + \bar{m} K_2}{2} \right) + \dot{\nu}^2 R
\]  
(4.29a)

\[
\ddot{\nu} = \frac{(1 - \bar{m}^2)}{4R} L K_2 \sin \theta - 2 \frac{\dot{R}}{R} \dot{\nu}
\]  
(4.29b)

\[
\ddot{\theta} = - \frac{R K_2 \sin \theta}{L} - 2 \frac{\dot{L}}{L} \left( \dot{\theta} + \dot{\nu} \right) - \dot{\nu}
\]  
(4.29c)

where \( K_1 \) and \( K_2 \) are defined as \( k_a + k_b \) and \( k_a - k_b \), respectively. We can now simulate the SE system deployment. It can be seen from the equations that the two parameters of interest for this analysis are the non-dimensional mass, \( \bar{m} \) and initial separation velocity, \( \dot{L}_0 \). Once these two parameters are specified, the above equations can be
solved for a tether deploying from GEO.

Before we present results from massless deployment, we derive a model with a massive tether. This will enable us to compare results from both models.

4.5 Massive tether model

We now derive the equations of motion for the system which accounts for the mass of the tether. For this derivation, we will use the so-called ‘bead-model’ [26, 27]. Since the tether is massive, we have that as the tether deploys, the tether connecting the two end-masses should increase in mass while the end-masses from which the tether is spooled should decrease in mass. The deployment process is divided into several stages, with the number of stages depending on the number of beads used in the model. At the beginning of every stage, a bead emerges from each of the end masses, and collectively both of the beads represent the tether mass between them, hence, the tether between beads can be modeled as massless. The deployment process begins with some tether already deployed and a single bead present at the center of that tether. So, if the model uses S stages, the total number of beads will be 2S-1. The distance between any two adjacent beads will be d (a constant value), and the portion of tether between each of the end masses and the bead nearest to it will increase as the tether is deploying. The deployment schematic of the bead model is presented in Figure 4.2.

While deriving these equations, we make use of the fact that we will be using a thruster/deployer controlled system to prescribe the deployment rate and hence we do not have an equation relating tether length and tether tension. We derive the equation at an arbitrary stage, s, with n = 2s-1 beads. In a given stage, the tether can be divided into sections, with each section having a massless tether in-between, with two massive bodies (end-masses, A and B, or beads) at either end.

Forces

The forces are summed over all the sections to derive the equations of motion. As mentioned earlier, since we do not need to use the equation involving tether tension
to simulate the system dynamics, we calculate only the net external force acting on the system. With equal and opposite thruster forces, we have only gravitational pull acting on each end-body, given by \[ F_G = \frac{GM_e M_i}{|\vec{r}_{i/E}|^3} \vec{r}_{A/E}, \] in the earth centered initial reference frame. So the total external force acting on the system is:
\[ \vec{F} = \sum_{j=a,b,1}^{n} (k_j m_j) \vec{r}_{j/E} \]
\[ = \sum_{j=a,b,1}^{n} (k_j m_j) \left( \vec{r}_{j/C} + \vec{r}_{C/E} \right) \]
\[ = \left( \sum_{j=a,b,1}^{n} (k_j m_j) \right) \vec{r}_{C/E} + \sum_{j=a,b,1}^{n} (k_j m_j \vec{r}_{j/C}) \]

where, as defined in the nomenclature section, \( \vec{r}_{j/C} \) is the position vector from the center of mass of the system to any given bead or end-mass and \( \vec{r}_{C/E} \) is the position vector from point C to the center of the Earth, E, and \( m_j \) is the mass of any given bead or end-body, and \( k_j \) is given by:

\[ k_j = \frac{-GM_e}{R^2 + L_j^2 - 2RL_j \cos \theta} \]

Note that the shorthand notation \( \sum_{j=a,b,1}^{n} (\lambda_j) \) represents \( \lambda_a + \sum_{i=1}^{n} (\lambda_j) + \lambda_b \). Expressing the force in the coordinates of the \( O \) frame yields:

\[ \left\{ \vec{F} \right\}_O = \left( \sum_{j=a,b,1}^{n} (k_j m_j) \right) \begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix} + \sum_{j=a,b,1}^{n} (k_j m_j \begin{bmatrix} L_j \cos \theta \\ L_j \sin \theta \\ 0 \end{bmatrix}) \]

\[ = \begin{bmatrix} \sum_{j=a,b,1}^{n} (k_j m_j) R + \sum_{j=a,b,1}^{n} (k_j m_j L_j \cos \theta) \\ \sum_{j=a,b,1}^{n} (k_j m_j L_j \sin \theta) \\ 0 \end{bmatrix} \]

(4.30)

where R is the orbital radius from the center of the Earth, E, to the center of
mass of the system, \( C \), \( L_j \) is the tether length from any given bead or end-body to the point \( C \), and \( \theta \) is the planar angle made by the tether axis with the local vertical.

**Torque**

When computing the torque about the center of mass of the system, all the thruster forces disappear because they act along the line of action of the center of mass. Hence the net external torque acting about the center of mass of the system, \( \vec{\tau}_{cm} \) can be formulated as:

\[
\vec{\tau}_{cm} = \sum_{j=a,b,1}^{n} (k_j m_j \vec{r}_{j/E} \times \vec{r}_{j/C}) \\
= \sum_{j=a,b,1}^{n} (k_j m_j (\vec{r}_{j/C} + \vec{r}_{C/E})) \times \vec{r}_{j/C} \\
= - \left( \sum_{j=a,b,1}^{n} (k_j m_j \vec{r}_{j/C}) \right) \times \vec{r}_{C/E}
\]

Expressing this torque in the coordinates of the \( O \) reference frame, we have:

\[
\{ \vec{\tau}_{cm} \}_O = - \left( \sum_{i=1}^{n} \left( k_i m_i \begin{bmatrix} L_j \cos \theta \\ L_j \sin \theta \\ 0 \end{bmatrix} \right) \right) \times \begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix}
\]

\[
\Rightarrow \{ \vec{\tau}_{cm} \}_O = R \sin \theta \left[ \begin{array}{c} 0 \\ 0 \\ \sum_{j=a,b,1}^{n} (k_j m_j L_j) \end{array} \right]
\]

**Acceleration**

Now that we have the forces and torques acting on the system, we compute the acceleration and angular momentum. The vector joining the center of mass of the
system, C, to the center of the Earth, E, can be expressed in the O frame as
\[
\{\vec{r}_{C/E}\}_O = \begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix}
\] (4.32)

Taking a second derivative of this in the E reference frame and expressing it in coordinates of the O reference frame, we obtain:
\[
\left\{ \dot{E} \ddot{\vec{r}}_{C/E} \right\}_O = \left\{ \dot{O} \ddot{\vec{r}}_{C/E} + E \vec{\alpha}^O \times \vec{r}_{C/E} + 2E \vec{\omega}^O \times \vec{r}_{C/E} + E \vec{\omega}^O \times \left( E \vec{\omega}^O \times \vec{r}_{C/E} \right) \right\}_O
\] (4.33)

where for the planar motion considered in the derivation, the angular velocity \( E \vec{\omega}^O \) and the angular acceleration, \( E \vec{\alpha}^O \) of the O frame with respect to the E frame can be simply expressed in coordinates of the O reference frame in terms of the true anomaly, \( \nu \), as
\[
\{ E \vec{\omega}^O \}_O = \begin{bmatrix} 0 \\ 0 \\ \dot{\nu} \end{bmatrix}
\]
and
\[
\{ E \vec{\alpha}^O \}_O = \begin{bmatrix} 0 \\ 0 \\ \ddot{\nu} \end{bmatrix}
\]

The position vector \( \vec{r}_{C/E} \) and its subsequent time derivatives differentiated in the O reference frame, can be expressed in coordinates of the O frame as
\[
\left\{ \dot{\vec{r}}_{C/E} \right\}_O = \begin{bmatrix} \dot{R} \\ 0 \\ 0 \end{bmatrix}, \quad \left\{ \ddot{\vec{r}}_{C/E} \right\}_O = \begin{bmatrix} \ddot{R} \\ 0 \\ 0 \end{bmatrix}, \quad \left\{ \vec{r}_{C/E} \right\}_O = \begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix}
\]
Substituting each term on the right hand side in Equation (4.33) with the corresponding vector, all expressed in the O frame, we obtain

$$\{E \ddot{\vec{r}}_{C/E}\}_O = \begin{bmatrix} \dddot{R} \\ 0 \\ 0 \end{bmatrix} + \begin{pmatrix} 0 \\ \dot{\nu} \end{pmatrix} \times \begin{bmatrix} 0 \\ 0 \end{bmatrix} + 2 \begin{pmatrix} 0 \\ \dot{\nu} \end{pmatrix} \times \begin{bmatrix} \dddot{R} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\nu} \end{bmatrix} \times \begin{pmatrix} 0 \\ 0 \end{pmatrix} \times \begin{bmatrix} \dddot{R} \\ 0 \end{bmatrix}$$

$$\Rightarrow \{E \ddot{\vec{r}}_{C/E}\}_O = \begin{bmatrix} \dddot{R} - \dot{\nu}^2 R \\ \ddot{\nu} R + 2 \dot{\nu} \dddot{R} \\ 0 \end{bmatrix}$$

(4.34)
Angular Momentum

The angular momentum of the system about its center of mass is given by

$$E \vec{h}_{cm} = \sum_{j=a,b,1}^{n} (\vec{r}_{j/C} \times m_j \vec{E}_{\vec{r}_{j/E}})$$

$$= \sum_{j=a,b,1}^{n} (\vec{r}_{j/C} \times m_j (\vec{E}_{\vec{r}_{j/C}} + \vec{E}_{\vec{r}_{C/E}}))$$

$$= \sum_{j=a,b,1}^{n} (\vec{r}_{j/C} \times m_j \vec{E}_{\vec{r}_{j/C}}) + \sum_{j=a,b,1}^{n} (\vec{r}_{j/C} \times m_j \vec{E}_{\vec{r}_{C/E}})$$

$$= \sum_{j=a,b,1}^{n} (\vec{r}_{j/C} \times m_j \vec{E}_{\vec{r}_{j/C}}) - \vec{E}_{\vec{r}_{C/E}} \times \sum_{j=a,b,1}^{n} (m_j \vec{r}_{j/C})$$

$$= \sum_{j=a,b,1}^{n} (\vec{r}_{j/C} \times m_j \vec{E}_{\vec{r}_{j/C}}) \quad \text{(since } \sum_{j=a,b,1}^{n} (m_j \vec{r}_{j/C}) = 0 \text{)} \quad (4.35)$$

Now, the time derivative of the angular momentum can be found by differentiating the above vector in the inertial reference frame, E, as:

$$\frac{d}{dt} (E \vec{h}_{cm}) = \sum_{j=a,b,1}^{n} (m_j (\vec{E}_{\vec{r}_{j/C}} \times \vec{E}_{\vec{r}_{j/C}} + \vec{E}_{\vec{r}_{j/C}} \times \vec{E}_{\vec{r}_{j/C}}))$$

$$= \sum_{j=a,b,1}^{n} (m_j \vec{r}_{j/C} \times \vec{E}_{\vec{r}_{j/C}}) \quad (4.36)$$

$$= \sum_{j=a,b,1}^{n} (m_j \vec{r}_{j/C} \times \vec{E}_{\vec{r}_{j/C}}) \quad (4.37)$$

$$= \sum_{j=a,b,1}^{n} (m_j \vec{r}_{j/C} \times \vec{E}_{\vec{r}_{j/C}}) \quad (4.38)$$
Note that:

\[
\left\{ \mathbf{\ddot{r}}_{j/C} \right\}_S = 0 \quad \text{(Constant Deployment)}
\]

\[
\left\{ \mathbf{E} \mathbf{\ddot{r}}_{j/C} \right\}_S = \left\{ \mathbf{E} \mathbf{\ddot{r}}_{j/C} \right\}_S + \mathbf{E} \mathbf{\alpha}^S \times \mathbf{r}_{j/C} + 2 \mathbf{E} \mathbf{\ddot{\omega}}^S \times \mathbf{r}_{C/P} + \mathbf{E} \mathbf{\bar{\omega}}^S \times \left( \mathbf{E} \mathbf{\bar{\omega}}^S \times \mathbf{r}_{j/C} \right)
\]

\[
= \begin{bmatrix}
0 \\
0 \\
(\dot{\theta} + \dot{\nu})
\end{bmatrix} \times \begin{bmatrix}
L_j \\
0 \\
0
\end{bmatrix} + 2 \begin{bmatrix}
0 \\
(\dot{\theta} + \dot{\nu}) \\
0
\end{bmatrix} \times \begin{bmatrix}
L_j \\
0 \\
0
\end{bmatrix} +
\]

\[
\begin{bmatrix}
0 \\
0 \\
(\dot{\theta} + \dot{\nu})
\end{bmatrix} \times \left( \begin{bmatrix}
0 \\
0 \\
(\dot{\theta} + \dot{\nu})
\end{bmatrix} \times \begin{bmatrix}
L_j \\
0 \\
0
\end{bmatrix} \right)
\]

\[
= \begin{bmatrix}
- (\dot{\theta} + \dot{\nu})^2 L_j \\
(\dot{\theta} + \dot{\nu}) L_j + 2 (\dot{\theta} + \dot{\nu}) \dot{L}_j \\
0
\end{bmatrix}
\]

Expressing the derivative of angular momentum, \( \frac{E}{dt} \left( \mathbf{E} \mathbf{\bar{h}}_{cm} \right) \), in the S frame, yields:

\[
\frac{E}{dt} \left( \mathbf{E} \mathbf{\bar{h}}_{cm} \right)_S = \sum_{j=a,b,1}^n \left( m_j \begin{bmatrix}
L_j \\
0 \\
0
\end{bmatrix} \times \begin{bmatrix}
- (\dot{\theta} + \dot{\nu})^2 L_j \\
(\dot{\theta} + \dot{\nu}) L_j + 2 (\dot{\theta} + \dot{\nu}) \dot{L}_j \\
0
\end{bmatrix} \right)
\]

\[
\Rightarrow \frac{E}{dt} \left( \mathbf{E} \mathbf{\bar{h}}_{cm} \right)_S = \begin{bmatrix}
0 \\
0 \\
\sum_{j=a,b,1}^n \left( (\dot{\theta} + \dot{\nu}) m_j L_j^2 + 2 (\dot{\theta} + \dot{\nu}) m_j \dot{L}_j L_j \right)
\end{bmatrix}
\]

where the notation \( \frac{E}{dt} \) means we are taking time derivative in the E reference frame.
and the superscript ‘E’ in $E\vec{h}$ means that the velocities in the angular momentum expression are all computed with respect to the E reference frame. Note that since we have only planar motion, the above derivative of the angular momentum could be expressed identically in any frame.

Now we combine all the equations we have derived so far using the Newton-Euler approach which can be expressed as

$$\vec{F} = m \vec{F}_{cm/E}$$  \hspace{1cm} (4.40)

$$\vec{r}_{cm} = E \frac{d}{dt} (E\vec{h}_{cm})$$  \hspace{1cm} (4.41)

Substituting Equations (4.30) and (4.34) in Equation (4.40), we obtain:

$$\begin{bmatrix}
\sum_{j=a,b,1}^{n} (k_j m_j) R + \sum_{j=a,b,1}^{n} (k_j m_j L_j) \cos \theta \\
\sum_{j=a,b,1}^{n} (k_j m_j L_j) \sin \theta \\
0
\end{bmatrix} = m \begin{bmatrix}
\ddot{R} - \dot{\nu}^2 R \\
\dot{\nu} R + 2 \dot{\nu} \dot{R} \\
0
\end{bmatrix}$$  \hspace{1cm} (4.42)

and substituting Equations (4.31) and (4.39) in Equation (4.41) on, we obtain

$$\begin{bmatrix}
\sum_{j=a,b,1}^{n} (k_j m_j L_j) \\
(\dot{\theta} + \dot{\nu}) \sum_{j=a,b,1}^{n} (m_j L_j^2) + 2 (\dot{\theta} + \dot{\nu}) \sum_{j=a,b,1}^{n} (m_j \dot{L}_j L_j)
\end{bmatrix} = m \begin{bmatrix}
0 \\
0
\end{bmatrix}$$  \hspace{1cm} (4.43)

Combining Equations (4.42) and (4.43), the equations of motion for any given stage in the bead model, with constant deployment rate, and equal and opposite thruster
forces, can be expressed as:

$$\ddot{R} = \frac{1}{m} \left[ \sum_{j=a,b,1}^n (k_j m_j) R + \sum_{j=a,b,1}^n (k_j m_j L_j) \cos \theta \right] + \dot{v}^2 R$$  

(4.44a)

$$\dot{\nu} = \frac{1}{R} \left[ \frac{1}{m} \sum_{j=a,b,1}^n (k_j m_j L_j) \sin \theta - 2 \dot{\nu} \dot{R} \right]$$  

(4.44b)

$$\ddot{\theta} = -\frac{1}{n} \sum_{j=a,b,1}^n \left( \frac{R \sin \theta}{m_j L_j^2} \sum_{j=a,b,1}^n (k_j m_j L_j) + \left( \dot{\theta} + \dot{v} \right) \sum_{j=a,b,1}^n \left( 2 m_j L_j L_j \right) \right) - \ddot{\nu}$$  

(4.44c)

### 4.6 Massless vs Massive Tether Model

The equations for the massive and massless tether models can now be simulated and compared to estimate the effect of exclusion of tether mass in the massless model. We will be adopting a constant tether deployment model for this analysis, so Equation set (4.29) for the massless tether analysis and Equation set (4.44) for the massive tether analysis will be used. The massless tether equations require only two parameters to be specified: the nondimensionalized mass distribution parameter, $\tilde{m}$ and the dimensional constant separation velocity, $\dot{L}_0$. However, in the case of the massive tether model, one additional parameter needs to be specified. This parameter is bead mass, or the mass-density of the tether material. For the analysis presented in this paper, this is specified from [28]. Once we fix the tether material density and cross section, we can simulate these equations by specifying the end-masses and separation rate. However, the massive tether equations do not have a nondimensionalized mass term as we have in the massless model. In other words, in the massive tether
analysis, $m_a$ and $m_b$ need to be specified separately. Since we simulate the massless tether equations with a certain specified value of $\bar{m}$, we must choose $m_a$ and $m_b$ for the massive tether equations such that they yield a value of $\bar{m}$ which is the same as that used in the massless tether model. $\dot{L}_0$ will be the same for both of the analyses, and tether mass density and cross sectional area will be assumed constant along the length of the tether for the massive tether analysis.

Several cases were analyzed to make a comparison between the dynamics observed for the massless tether analysis and for the massive tether analysis; we present the results of these cases below. In the course of studying the deployment process, we found that the dynamics of variables $R$ and $\theta$ cover most of the interesting dynamics of the system, so most of the plots in this analysis show the behavior of $R$ and $\theta$ as a function of the total amount of tether deployed (recall that we are deploying the tether at a prescribed constant rate). Furthermore, it was also found that once the deployment starts, the system does not remain in its original orbit at GEO but falls rapidly to lower orbits. In order to concentrate on the stable dynamics of the system, we arbitrarily define ‘stability’ as $e \leq 0.01$ and tether length deployed until that point is recorded and is plotted against $R$ and $\theta$.

Several cases with different combinations of $\bar{m}$ and $\dot{L}_0$ were simulated. We present some of these cases below. Figure 4.3 and Figure 4.4 show $R$ vs. $L$ and $\theta$ vs. $L$ plots, respectively. These plots are simulated with parameters $\bar{m} = -0.8$ and $\dot{L}_0 = 1 \text{ m/s}$. We see that both the plots are qualitatively, and to good extent, quantitatively, similar to each other. Similar observations are drawn for Figures 4.5 through 4.14 where the $R$ and $\theta$ behavior for several combinations of $\bar{m}$ and $\dot{L}_0$ is qualitatively similar for both the massive and the massless tether analyses. We can see from all the $\theta$ vs. $L$ plots that the system initially tilts about $-60^\circ$ from the local vertical before it stabilizes along the local vertical. This is because of the Coriolis forces acting on the system which tend to rotate the tether perpendicular to the local vertical. However, as the tether deploys and tether length increases, gravity gradient forces become significant and they align the system back with local vertical. The $R$ vs. $L$ plots show that the as soon as the deployment begins, the center of mass of the system falls rapidly toward the Earth. Similar trends can be observed in all of the other cases which were
simulated to compare both of the models. Based on this comparison study, we assume that qualitatively valid results regarding general trends in deployment dynamics for long tether systems can be obtained by using the massless tether model. The massless tether model has the advantage of allowing us to nondimensionalize the system and conduct parametric studies based on just two parameters, $\tilde{m}$ and $\tilde{L}_0$, and hence, in the next section, we present a detailed analysis of the dynamics of the deployment of a Space Elevator system based on the massless tether model.
Figure 4.3: R behavior for parameters $\tilde{m} = -0.8, \dot{L}_0 = 1 \text{ m/s}$
Figure 4.4: $\theta$ behavior for parameters $\tilde{m}=-0.8, \tilde{L}_0=1 \text{ m/s}$
Figure 4.5: R behavior for parameters $\tilde{m}=-0.8, \dot{L}_0=10 \text{ m/s}$
Figure 4.6: $\theta$ behavior for parameters $\tilde{m}=-0.8, \dot{L}_0=10 \text{ m/s}$
Figure 4.7: R behavior for parameters $\tilde{m}=0, \dot{L}_0=1 \text{ m/s}$
Figure 4.8: $\theta$ behavior for parameters $\tilde{m}=0, \tilde{L}_0=1 \, m/s$
Figure 4.9: $R$ behavior for parameters $\tilde{m}=0, \dot{L}_0=10 \text{ m/s}$
Figure 4.10: $\theta$ behavior for parameters $\bar{m}=0, \bar{L}_0=10 \text{ m/s}$
Figure 4.11: R behavior for parameters $\tilde{m}=0.8, \tilde{L}_0=1 \text{ m/s}$
Figure 4.12: $\theta$ behavior for parameters $\bar{m}=0.8, \bar{L}_0=1\ m/s$
Figure 4.13: R behavior for parameters $\bar{m}=0.8, \dot{L}_0=10 \text{ m/s}$
Figure 4.14: $\theta$ behavior for parameters $\tilde{m}=0.8, \dot{L}_0=10 \text{ m/s}$
4.7 Massless Tether Analysis

From the plots presented in the previous section, we have seen that some of the important behaviors to be observed are the decreasing orbital radius as the tether deploys and the initial in-plane oscillation of the system with respect to local vertical. In this section, we first look at some additional cases to verify this trend. For plots presented in this section, we have defined the ‘stability’ criteria as $e=0.1$. This means that the deployment in all of the plots is shown until the eccentricity of the system’s orbit reaches 0.1.

Figures 4.15 to 4.19 show $R$ vs. $L$ and $\theta$ vs. $L$ plots for various mass ratios, $\tilde{m}$, from -0.8 to 0.8. Each figure has plots for various deployment speeds, $\dot{L}_0$, from 1 m/s to 25 m/s. The $R$ and $\theta$ behavior seen in the previous section is confirmed by these plots. Another interesting observation from these plots is that for any given $\tilde{m}$, the dynamics of $R$ is almost identical for all the deployment speeds. This suggests that for the range of speeds considered in this analysis, the dynamics of the orbital radius, $R$, as a function of the deployed length, $L$, is independent of the deployment speed, $\dot{L}_0$. In other words, at any given altitude, during the fall of the center of mass of the system from GEO toward the Earth, the total deployed tether length would be same for any deployment speed. So, a higher deployment speed can result in faster deployment, but not better deployment. One of the most important observations from this analysis is that for none of the cases studied, does the system reach complete deployment; i.e. the system leaves the GEO orbit before deploying even half of the actual desired tether length (100,000 km). This shows the instability present in the system during the deployment and the need for further research and development in the area of devising deployment schemes for SE systems.
Figure 4.15: Deployment dynamics at various speed for $\bar{m} = -0.8$

Figure 4.16: Deployment dynamics at various speed for $\bar{m} = -0.4$
Figure 4.17: Deployment dynamics at various speed for $\bar{m} = 0$

Figure 4.18: Deployment dynamics at various speed for $\bar{m} = 0.4$
Figure 4.19: Deployment dynamics at various speed for $\tilde{m} = 0.8$
Figures 4.20 to Figure 4.23 show the dynamics of the orbital radius, R, with respect to deployed length, L, and with respect to eccentricity, e, for various deployment speeds, \( \dot{L}_0 \), from 1 m/s to 25 m/s and each figure is plotted for several values of mass distribution ratios, \( \tilde{m} \), from -0.8 to 0.8. As we see from these plots, the deployment is highest for the most negative value of \( \tilde{m} \) (\( m_b > m_a \)) and gradually decreases to lowest for the most positive value of \( \tilde{m} \) (\( m_a > m_b \)). In other words, deployment favors initial alignment such that the heavier mass is pointed toward the Earth. We also see from the eccentricity plots that the case with \( \tilde{m} = -0.8 \) is more stable than other cases in that at any given altitude, the orbital eccentricity for the system with \( \tilde{m} = -0.8 \) is lowest, and for the system with \( \tilde{m} = 0.8 \), it is highest. With these observations, we conclude that deployment favors negative mass distribution (\( m_b > m_a \)). This could be a critical design criteria during the deployment of the SE, because this would mean that the counterweight at the bottom of the ribbon should be heavier than the ballast mass (Section 3.2) at the top.

Figure 4.20: Dynamics of R with respect to e and L at \( \dot{L}_0 = 1 \) m/s
Figure 4.21: Dynamics of $R$ with respect to $e$ and $L$ at $\dot{L}_0 = 5 \text{ m/s}$

Figure 4.22: Dynamics of $R$ with respect to $e$ and $L$ at $\dot{L}_0 = 10 \text{ m/s}$
Figure 4.23: Dynamics of R with respect to e and L at $\dot{L}_0 = 25 \text{ m/s}$
4.8 Deployment as a Function of Initial Alignment

In the above analysis, the deployment begins when the tether axis is initially inclined to the local vertical, i.e. when the initial condition for $\theta$ is zero. In this analysis, we study the deployment behavior for various initial values of $\theta$. Figures 4.24 to 4.26 show deployment dynamics in terms of $R$ and $\theta$ for several combinations of $\tilde{m}$ and $\dot{L}_0$. Each figure is plotted for several initial alignments, i.e. for several values of $\theta_0$ ($\theta$ at $t=0$).

![R vs L and θ vs L plots](image)

Figure 4.24: $R$ and $\theta$ behavior. Case: $\dot{L}_0 = 25 \text{ m/s}$, $\tilde{m} = -0.4$
Figure 4.25: $R$ and $\theta$ behavior. Case: $\dot{L}_0 = 25 \, m/s$, $\bar{m} = 0$

Figure 4.26: $R$ and $\theta$ behavior. Case: $\dot{L}_0 = 25 \, m/s$, $\bar{m} = 0.4$
From the above figures it is clear that when $\theta_0 < 0$, the axis of the system tends to balance itself about an angle of 180° relative to the local vertical instead of about 0°, as is the case when $\theta_0 \geq 0$. When $\tilde{m}$ is negative (Figure 4.24), the deployment performance is better for $\theta_0 \geq 0$; when $\tilde{m}$ is positive (Figure 4.26), the deployment performance is better for $\theta_0 \leq 0$; and when $\tilde{m} = 0$ (Figure 4.25), the deployment performance is identical for all chosen values of $\theta$. Hence, $\theta_0 = 0$ would always assure better deployment irrespective of the mass ratio.

4.9 Deployment from Various Altitudes

In the previous section, we observed that the orbital altitude decreases considerably as the system deploys, so in this section we examine cases where the deployment starts from a higher orbit above GEO. The purpose of this analysis is to examine the effect of altitude on the dynamics of the system and determine if changing the altitude of deployment could result in a more successful deployment. Figures 4.27, 4.28 and 4.29 show R and $\theta$ behavior as a function of different altitudes for various sets of $\tilde{m}$ and $\dot{L}_0$. From these figures it is clear that the overall qualitative dynamics of the system remains unchanged as a function of altitude, and this opens up the possibility of deploying the system from above GEO so that at the end of deployment (or when $e$ reaches 0.1), the center of mass of the system is closer to the desired GEO altitude.
Figure 4.27: R and $\theta$ behavior. Case: $\dot{L}_0 = 1\, m/s$, $\dot{m} = -0.8$

Figure 4.28: R and $\theta$ behavior. Case: $\dot{L}_0 = 10\, m/s$, $\dot{m} = 0$
Figure 4.29: R and $\theta$ behavior. Case: $\dot{L}_0 = 25 \text{ m/s}$, $\bar{m} = 0.8$
4.10 Summary of Analysis

In this chapter, deployment models with both massless and massive tethers were developed, simulated and compared. It was found that the qualitative behavior of the deployment dynamics remains unchanged even if the tether mass is excluded from the analysis (Figures 4.3 through 4.14). With this conclusion, a massless tether model was further investigated. For a given initial alignment of the system with the local vertical, two parameters which need to be specified in order to simulate the system are nondimensional mass, \( \tilde{m} \), and deployment speed, \( \dot{L}_0 \). The system is simulated for several combinations of \( \tilde{m} \) and \( \dot{L}_0 \) and the resulting R vs. L and \( \theta \) vs. L behavior is presented. For some cases, R vs. e plots are also presented. Primarily, it was found that though the gravity gradient forces align the system with the local vertical, the system is unstable in that the system tends to fall from the GEO orbit eventually crashing on the Earth. For the range of speed considered in this analysis (1 m/s to 25 m/s), it was found that the deployment performance is independent of the deployment speed (Figures 4.15 through 4.19). It was also found that cases with negative \( \tilde{m} \) deploy more and are relatively more stable than other cases (Figures 4.20 through 4.23). When \( \theta_0 \geq 0 \), the system aligns itself back to the local vertical. However, when \( \theta_0 \leq 0 \), the system aligns at an angle of 180° to the local vertical, i.e. the system turns upside down (Figures 4.24 through 4.26). Finally, the system was simulated such that the deployment begins at altitude higher than GEO and it was found that the dynamics of the system remains more or less unchanged for any initial altitude (Figures 4.27 through 4.29).
Chapter 5

Conclusions

5.1 Short and Medium Length Tethers

The first part of this thesis presents a detailed study of the factors affecting the deployment dynamics of short and medium length tethered systems. Five dimensional parameters which affect the percentage of the total length to which a tether deploys are identified. These parameters include the initial separation velocity, the tether tension force, the orbital height, the effective mass (which measures the distribution of the tethered satellite end-body masses) and the final desired tether length. Through numerical simulations, we determined that the final length a tethered satellite reaches under uncontrolled deployment increases with increasing initial separation velocity, increasing effective mass, and decreasing tether tension (which is equal to deployer friction). Furthermore, we note that the final deployment length increases as the angle between the tether and the orbital plane approaches zero; i.e. the final deployment length is larger if the tether deployment begins in the orbital plane than if the tether deployment begins outside of the orbital plane. We also note that the maximum final tether length with respect to the in-plane initial angle occurs at a point close to, but not equal to, an in-plane angle of zero. By nondimensionalizing the equations of motion, we define two nondimensional variables which completely describe the tether deployment dynamics. By numerically integrating the nondimensional equations of motion, we show that the final tether deployment length increases with decreasing
nondimensional tension and increases with increasing nondimensional initial velocity. We also show the utility of the nondimensional parameters from a mission design standpoint and provide case studies which demonstrate the use of the nondimensional parameters as tools useful in determining the final deployment length of a tethered satellite without solving the equations of motion.

5.2 Long Tether Analysis

Important parameters for long tethers which should be controlled for successful deployment are identified and analyzed. Instability of long tethers during deployment is demonstrated. The current deployment model for Space Elevators commonly described in the literature includes deploying the tether up and down until one end reaches the Earth. However, the analysis in this thesis shows that the system falls from the GEO orbit and tends to crash on the Earth before it deploys completely. Thus, deficiencies in currently envisioned deployment schemes for Space Elevators have been demonstrated, illustrating the need for developing a new scheme for SE deployment.

5.3 Concluding Remarks

This thesis on space tether dynamics has involved non-planar short tether space systems and planar long tether space systems. Independent models were developed for both systems and they were analyzed systematically by developing governing equations of motion and performing parametric studies on each of them. Though the basic physics and the approach toward deriving equations of motion for both kinds of tether systems is similar, there are significant differences in the dynamics of each system. While we have nondimensionalized the equations of motions for short tether systems and have generalized the results over a wide range of parameters, the long tether system equations can not be completely nondimensionalized. The Tether mass which was ignored for short and medium tether analysis, was initially accounted for by developing a ‘bead-model’ for massive tether deployment, however it was found
that the qualitative dynamics of the system does not change appreciably by excluding tether mass from the equations and hence rest of the analysis for long tether systems was carried out with a massless tether model. Though the dynamics of short tether systems have been extensively researched, not as much work on long tether dynamics has been published, especially with respect to space elevator applications. The analysis in this thesis shows that Space Elevator deployment dynamics, as currently envisioned by SE advocates, is fundamentally unstable, and demonstrates that alternate deployment schemes need to be developed and analyzed.
List of References


