ABSTRACT

WILSON, PETER HOLT. Teachers’ Uses of a Learning Trajectory for Equipartitioning. (Under the direction of Dr. Jere Confrey).

Recent work by some researchers has focused on synthesizing what is known about students' thinking of particular concepts. These syntheses elaborate core progressions called learning trajectories that articulate how thinking matures from informal ideas to increasingly complex understandings. Though useful at the level of curriculum, assessment, and standards development, it remains to be shown that learning trajectories can be incorporated into teachers’ practice and become a tool to understand students’ thinking, for planning instructional activities, for interacting with students during instruction, and for assessing students’ understandings. Further, the impact of such incorporation in the classroom on students’ learning is unknown.

This design study investigated K-2 teachers’ uses of a learning trajectory for equipartitioning in instruction. Thirty-three teachers participated in 20 hours of professional development focused on a learning trajectory for equipartitioning and key instructional practices including clinical interviewing, task selection and adaptation, analysis of student work, and classroom interactions. A subset of the participants was observed teaching a lesson on equipartitioning and gains in students' learning were measured with pilot items for a diagnostic assessment system. Findings from the study indicate that the introduction of the learning trajectory assisted teachers to varying degrees in building more precise and adequate models of students’ thinking, identifying specifically what students need to learn next, deepening their own understandings of equipartitioning, and facilitating coherent instruction.

These results suggest that learning trajectories can act as a tool for coordinating (1) a student's behaviors and verbalizations with cognition, (2) various models of thinking among groups of students, and (3) these models with instructional practices.

Teachers' Uses of a Learning Trajectory for Equipartitioning

by
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APPROVED BY:

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DEDICATION

For my students, teachers, and colleagues – who help me to make sense of my world

BIOGRAPHY

Peter Holt Wilson, II was born on June 30, 1979 in Raleigh, NC. Holt grew up in rural eastern North Carolina and graduated from Smithfield-Selma High School in Smithfield, NC in 1997. He attended East Carolina University in Greenville, NC and graduated summa cum laude in May 2000 with Bachelor of Science degree in mathematics. Upon graduation, Holt returned to Smithfield-Selma High School where he taught Algebra, Discrete Mathematics, and Statistics courses. Concurrently, he pursued a Master of Arts in Education at East Carolina University, graduating in May 2003. In 2004, he was certified by the National Board of Professional Teaching Standards in Adolescent and Young Adult Mathematics and continued his graduate studies at East Carolina University, receiving a Curriculum and Supervision license in 2004.

Holt returned to graduate school full time at North Carolina State University in 2005 to pursue a doctoral degree in mathematics education. He worked as a graduate research assistant on various projects throughout his studies, including "Preparing to Teach Mathematics with Technology: An Integrated Approach," a year-long teaching experiment on early algebraic reasoning, and most recently the "Diagnostic e-Learning Trajectories Approach" project. From this work, Holt co-authored papers and presented at national and international conferences and submitted publications to journals.

After receiving his degree, Holt plans to search for a university faculty position and pursue research related to teacher development and instructional guidance for teachers.

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"For of him, and through him, and to him, are all things: to whom be glory forever."

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CHAPTER 1

Current discourse in American education policy continues to focus on reform. The expectations of high-stakes accountability models (Elmore, 2002), the changing focus of curriculum toward 21st century skills (Partnership for 21st Century Skills, 2007), and the climate of the No Child Left Behind (NCLB) legislation places new demands on teachers. Among these demands is the need for teachers to assume new roles of understanding students' thinking and how to use these ways of thinking to guide their instruction.

Over the last two decades, systemic initiatives have attempted to address reform. State accountability models and later NCLB sought reform through new policies. Concurrently, professional organizations such as the National Council of Teachers of Mathematics (NCTM) outlined and revised standards in efforts to guide the discussion. Curriculum development projects were funded to support teachers in the reform effort. Yet, the cumulative effect of these has resulted in only modest gains in students' learning. For instance, Corcoran, Fritz, and Mosher (2009) point out that the result of policy mandates has often led to the lowering of achievement levels for standardized tests. Groups such as Mathematically Correct have challenged the standards movement, resulting in disagreements and contributed to a mitigating of the efforts of those standards. A lack of robust outcome measures to support the reform-oriented curricula has left the field unable to provide evidence acceptable to the greater community of their efficacy (NRC, 2004).

Teachers stand at the center of this debate, negotiating the demands of policy, the suggested direction of standards, and the goals of curriculum. Though expertise in isolated cases may assist them in focusing on and understanding students' thinking, many teachers lack tools for diagnosing these understandings in a coherent way. Even if such expertise led to conclusions about what students may not understand, few structured guidelines based on research exist to inform teachers' instructional decisions. In *Knowing What Students Know* (NRC, 2001), the authors present a framework called the assessment triangle (figure 1). In this framework, observations are made of students' behaviors and verbalizations and interpretations are made about students' cognition. Thus, a possible new approach to reform may be providing teachers with tools for better interpretations of their observations of students and better inferences concerning their students' cognition. Moreover, these tools should provide instructional guidance based on those interpretations.

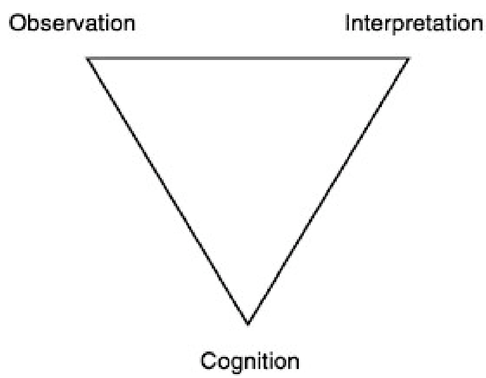


Figure 1. Assessment Triangle (NRC, 2001).

Much is known about the factors associated with student learning, such as how students learn (NRC, 1999) and students' thinking within specific domains (e.g. Confrey & Scarano, 1999; Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996). However, this knowledge is rarely enacted in the context of real classrooms on a large scale. Thus, the barrier between education research and practice plagues reform efforts. In response, some researchers are turning to research syntheses as a way of summarizing what is known about domain-specific knowledge and then articulating learning trajectories that describe the ways that students' thinking about particular concepts mature from informal understandings to more sophisticated conceptions over time (Corcoran et al., 2009). They posit these trajectories may provide teachers with a framework for observing, interpreting, and responding to students' ideas.

Rational Number Reasoning

One particular area of importance in mathematics education is rational number reasoning (RNR). RNR is essential in students' preparation for more advanced mathematics and science courses (Lesh, Post, & Behr, 1988). Reasoning about rational numbers is a great predictor of students' success in algebraic reasoning, often identified as a gatekeeper to both higher and technical education (Lamon, 2007). Professional organizations such as NCTM reflect this importance by the emphasis on developing rational number concepts in elementary and middles grades standards (NCTM, 2006, 2000). Yet, researchers have long indicated that developing RNR is a monumental challenge in children's development of mathematical knowledge (Lamon, 2007). Some document students' difficulties with rational number reasoning, such as the conservation of operations and the belief that multiplication results in a larger quantity (Greer, 1987). Others have developed curricular approaches to better assist students in developing RNR (e.g. Moss & Case, 1999; Cramer, Post, & del Mas, 2002).

Yet despite these initiatives, students' difficulties persist. Though much of the elementary and middles grades curriculum is comprised of RNR, only 32% of students in

Grade 8 were proficient in mathematics (NAEP, 2008). In the words of Lamon (2007), of all school curricular topics, rational numbers

> arguably hold the distinction of being the most protracted in terms of development, the most difficult to teach, the most mathematically complex, the most cognitively challenging, the most essential to success in higher mathematics and science, and one of the most compelling research sites (p. 629).

Assisting students in coming to understand rational numbers remains a significant challenge for mathematics educators.

Research Focus

Under the direction of Drs. Confrey and Maloney at North Carolina State University, the DELTA (Diagnostic e-Learning Trajectory Approach) group is synthesizing the research in rational number reasoning from a database of 662 articles (Confrey, Maloney, Nguyen, Wilson, & Mojica, 2008). Based on these syntheses, they are articulating learning trajectories for constitutive parts of RNR. Originally, the synthesis was intended to inform the creation of a diagnostic assessment system for RNR to provide instructional feedback for teachers in grades K – 8. As the work progressed, other implications of the syntheses and trajectories emerged. In addition to influencing assessments, the learning trajectories research was extended to informing the revision of the North Carolina State Mathematics Standards (Confrey, 2009), to considering the ways in which learning trajectories may interact with existing curricula (Wilson, in preparation), and investigating the ways that learning trajectories may support prospective teacher education (Mojica, 2009).

The research reported here focuses on the ways in which learning trajectories may be used by practicing teachers in advance of the forthcoming diagnostic assessment system.

Specifically, it investigates the ways that a learning trajectory can inform the teachers' use of the three vertices of the assessment triangle, the observations, interpretations, and inferences concerning cognition that teachers make in the domain of RNR. It aims to characterize the ways that trajectories may affect the instructional practices of teachers within classrooms. Lastly, the study seeks to describe relationships between teachers' uses of a learning trajectory and their students' learning.

Outline of Dissertation

The dissertation is organized into seven chapters. Chapter One provides an introduction to the problem area of the study and a description of the research's focus. Chapter Two provides a review of the research literature pertaining to the problem area and identifies salient points from this review. It focuses on the nascent domain of learning trajectories research, rational number reasoning and equipartitioning, and the professional development literature. It concludes with a presentation of the research questions. Chapter Three presents the theoretical perspectives in which the study is grounded and the conceptual framework guiding the research. It outlines and justifies the methodology and finally describes the participants, context, sources of data, and methods of analysis used. Chapters Four, Five, and Six present the ongoing and retrospective analyses of the data. Finally, Chapter Seven discusses the findings from the study in response to the research questions, as well as the limitations, implications, and future research related to the study.

CHAPTER 2

Given the description of the research problem presented in Chapter One, this chapter reviews the literature concerning the different areas of education research that surround the context of the study. First, I present the literature on learning trajectories and their current uses. Next, I address research studies on a specific area of RNR and present a learning trajectory for that area. After summarizing some of the literature concerning professional development, I underscore key aspects of the review and present the purpose and research questions of this study.

Learning Trajectories

Currently, education researchers and policy makers are exploring how core progressions of how students develop understandings of concepts over time. In their report *Learning Progressions in Science*, Corcoran, Mosher, and Rogat (2009) present learning progressions as a tool for improving student learning and meeting the goals of education reform. Learning progressions, also known as learning trajectories, are likely paths by which students move from informal to more complex understandings over time. They are related to the hypothetical learning trajectory offered by Simon (1995). For him, a hypothetical learning trajectory refers

> to the teacher's prediction as to the path by which learning might proceed. It is hypothetical because the actual learning trajectory is not knowable in advance. It characterizes an expected tendency. Individual students' learning proceeds along idiosyncratic, although often similar, paths. This assumes that an individual's learning has some regularity to it, that the classroom community constrains mathematical activity often in predictable ways, and that many of the students in the same class can benefit from the same mathematical task (p. 135).

Essentially, learning trajectories and progressions refer to the same idea. In mathematics education, they are often referred to as trajectories due to the idea's early beginnings in Simon's hypothetical learning trajectory. Thus, this dissertation uses the term learning trajectory unless in reference to literature from outside of the field that uses learning progressions.

Learning Trajectories Defined

Current uses of the terms, learning trajectories develop Simon's idea in a more global manner. For example, Clements and Sarama (2004) conceptualize learning trajectories as

> descriptions of children's thinking and learning in a specific mathematical domain and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain (p. 83).

In his work, Battista (2004) describes their developmental progression using the concept of levels of sophistication, levels through which students progress from informal, pre-instructional reasoning to different *cognitive plateaus*, ending in formal mathematical concepts that are the focus of instruction. He uses the term *cognitive terrai*n to indicate the plateau's multidimensionality and its inclusion of key processes and ideas needed to support them. These levels include descriptions of the plateaus, conceptualizations and reasoning, cognitive obstacles, and mental processes needed at each level and to advance to the next. Because of diverse background experiences and different mental processes, his model asserts that although individual students' progressions through the terrain may differ, research indicates that many pass through the same major landmarks. He refers to these well-traveled routes as *cognitive itineraries*. Similar to Battista, Smith, Wiser, Anderson, and Krajcik (2006) define learning progressions as “a sequence of successively more complex ways of thinking about an idea that might reasonably follow one another in a student’s learning” (p. 5) and note that progressions are dependent on instruction, not maturation, and emphasize that there is no single correct pathway of a concept’s development.

In their recent report summarizing the domain of learning progressions, Corcoran et al. (2009) define a learning progression as “a hypothesized description of the successively more sophisticated ways student thinking about an important domain of knowledge or practice develops as children learn about and investigate that domain over an appropriate span of time” (p. 37). These progressions are “based on research of how students’ learning actually progresses – as opposed to selecting sequences of topics and learning experiences based only on logical analysis of current disciplinary knowledge and on personal experiences in teaching” (p. 8). Characteristics of progressions include a basis in education and cognitive psychology research, a focus on foundational and generative disciplinary knowledge and practices, and an internal conceptual coherence. They can be “empirically tested …are not developmentally inevitable… [and] are crucially dependent on the instructional practices provided for the students” (p. 38).

Similar to the aforementioned definitions, the DELTA research group has used its own definition of learning trajectory in framing their research. They define learning trajectories as:

> A researcher-conjectured, empirically-supported description of the ordered network of constructs a student encounters through instruction (i.e. activities, tasks, tools, forms of interaction and methods of evaluation), in order to move from informal

> ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time (Confrey, Maloney, Nguyen, Wilson, Mojica, & Myers, in preparation).

Consistent with the characteristics from the Corocoran et al. report, the definition encompasses the theoretically- and evidence-based nature of the construct and accentuates the essential role instruction plays in the refinement of students' conceptions.

Potential Benefits of Learning Trajectories

One conclusion of the Corcoran et al. report (2009) was that though there were potentials of learning progressions, most work involving them still in preliminary stages of development and validation. These potential benefits include "more focused standards, better designed curricula, better assessments, and ultimately more effective instruction and improved student learning" (p.17). Confrey (in preparation b) offers an explanation as to how these potentials might be realized. Drawing from interdisciplinary work, she views learning trajectories as boundary objects, carrying meaning and common understanding among policy makers, curriculum and assessment designers, researchers, teacher educators, and teachers. By communicating essential understandings about the development of students' thinking over time, learning trajectories may ensure more coherence among these systemic forces affecting students' learning.

For instance, whereas standards typically decompose targeted performances or understandings into logical chunks for incremental mastery, learning trajectories are developed based on evidence of how students actually develop those understandings over time and across grade levels, thus providing support for articulating standards. One example of this is the work of Catley, Lehrer, and Reiser (2005). In preparing a learning progression for evolution, they claim that standards should instead articulate the knowledge, skills and forms of activity that students need to gain knowledge rather than a description of ideas students should learn. By expressing standards as learning performances, cognitive processes that support the standard are clarified and the practices that support those understandings are emphasized. In another example, Confrey's (2009) learning trajectories research informed the revision of the North Carolina Mathematics Standard Course of Study by clarifying how mathematical ideas develop across grades, supporting a broader view of curricular coherence for teachers.

Learning trajectories may provide frameworks by which curricula may be written. Rather than sequencing topics according to individual authors' expertise or a logical development of a domain, curricula can be organized around trajectories. According their report, Corcoran et al. (2009) claim learning progressions differ from curriculum and standards in two fundamental ways. First, they are based on disciplinary analysis *and* research findings of how students learn a particular idea, as opposed to a "conventional wisdom or consensus processes" (p. 23). Second, they are validated by evidence rather than by "the authority of experts, professional bodies, and government agencies" (p. 23). For example, Clements, Wilson, and Sarama (2004) report on a curriculum and software development project based on a learning trajectory for composing and decomposing shapes. Initially, they conjectured a trajectory based on a review and synthesis of the research literature and observations. After creating and piloting activities designed to guide students through proposed trajectory, they made revisions based on input from other researchers, the outcomes of the pilot, and cognitive psychology research.

In addition, learning trajectories can inform the development of assessments that seek to characterize how students thinking develops over time (e.g. Battista, 2004; Clements, Sarama, & Wilson, 2004; Confrey, Maloney, Nguyen, Wilson, & Mojica, 2008). Rather than report levels of achievement of a student, a new purpose of assessment may indicate increasing levels of proficiency as students move towards the learning target (Corcoran et al, 2009). One example is Battista's (2004) Cognition Based Assessment system. The assessment system provides descriptions of core mathematical ideas and reasoning processes, research-based descriptions of the cognitive constructions students must make in developing the understanding of an idea, and coherent sets of assessment items that enable teachers to locate students' positions within a range of conceptual development. Battista claims that the system can support instructional improvement by providing teachers with knowledge of the content and how students' come to understand it, identifying what students are not learning, and in communicating instructional foci for teachers.

Lastly, the Corcoran et al. report (2009) claims that learning progressions, because they can help teachers understand how student thinking develops over time, can lead to improved instruction. They state that learning progressions can:

> provide teachers with a conceptual structure that will inform and support their ability to respond appropriately to evidence of their students' differing stages of progress by adapting their instruction to what each student needs in order to stay on track and make progress toward the ultimate learning goals (p. 19).

Learning trajectories may support the pedagogical content knowledge that is needed to guide instruction by providing "detailed support for teachers' choices about what to do when they see evidence of how students are progressing" (p.23). The report comments on the potential to improve the professional development of practicing teachers by providing "an informed framework for teacher to gain better understanding about how students' ideas develop" (p. 53). However, studies of teachers' uses of learning trajectories in practice are notably absent in the literature with one exception. Bardsley (2006) reports on a case study of 14 pre-kindergarten teachers implementing a research-based curriculum, *Building Blocks* (Clements & Sarama, 2007). The curriculum, based on learning trajectories described by the developers, was accompanied by a document of developmental progressions. She found that teachers' uses of this document varied according to their reason for participating in the professional development. Teachers characterized as wanting to know *more math* activities tended focus on moving students through the levels. In contrast, teachers who participated in the professional development in order to learn *better math* used the trajectories in order to guide instructional decisions. But in her study, the developmental progressions document was related to the *Building Blocks* curriculum and was used in the professional development as a means of supporting its implementation. Though Bardsley's study provides some insights into the ways that teachers may use learning trajectories in practice, very little is known about the ways that they may use a trajectory that is not embedded within curriculum.

To realize these potentials for standards, assessment, curriculum, and teacher education, the Corcoran et al. report (2009) makes a series of recommendations to researchers, developers, policymakers, and other education professionals. One of these recommendations is for researchers to create existence proofs of the value and significance of learning progressions. They state, “an effort should be made to collect evidence that using learning progressions to inform curriculum instruction, assessment design, professional development, and /or education policy results in meaningful gains in student achievement” (p. 50).

Equipartitioning

As discussed in Chapter One, learning to reason about rational numbers is a significant challenge for students. Researchers have documented many of the difficulties students encounter and misconceptions they must overcome to formulate an understanding of rational number. These include the nonconservation of operations of multiplication and division when numbers within a given context change from whole numbers to rational numbers (Greer, 1987), misconceptions surrounding decimals such as *longer is larger* (Stacey & Steinle, 1998), that belief that *multiplication makes bigger and division makes smaller* (Fischbien, Deri, Nello, & Marino, 1985), and an additive misconception where the product of factors is believed to be their sum (Empson & Turner, 2006). Researchers have also noted that teachers hold many of these same misconceptions (Graeber, Tirosh, & Glover, 1989). Additionally, they note that many teachers’ understandings of rational number is procedurally based (Ball, 2000; Tirosh, 2000; Zazkis & Campbell, 1996). Many have difficulties with ideas relating to equivalence of fractions, the concept of units, and qualitative compensation (Cramer & Lesh, 1998).

These difficulties may be the result of the multiple interpretations of rational number. Kieren and Southwall (1979) proposed five distinct constructs of rational number: part-whole, ratio, quotient, measure, and operator. The Rational Number Project (Behr, Post, Silver, & Mierkiewicz, 1980), a long-standing and multi-university research project, used and elaborated these interpretations, creating complex subconstructs to provide a sound theoretical foundation of the domain (Behr & Harel, 1990).

One outcome of the DELTA research group's synthesis on rational number reasoning is a conceptual mapping of the domain of rational numbers (see figure 2). This mapping simplifies Kieren and Southwall's (1979) five subconstructs into three: rational number as a ratio, as an operator, and as a number (Confrey, 2008). In the mapping, Kieren and Southwall's part-whole and ratio construct are subsumed into rational number as ratio; quotient and operator are combined for rational number as operator; and fractions and measurement are considered rational number as number. It map depicts the development of the different constructs from early to more complex understandings and connections between the different interpretations. The map positions an early understanding of fair sharing as the foundation of all three constructs. Fair sharing (shown in the shaded boxes) is central for the development of an operator interpretation and is supportive of the development of both rational number as fraction and as ratio.

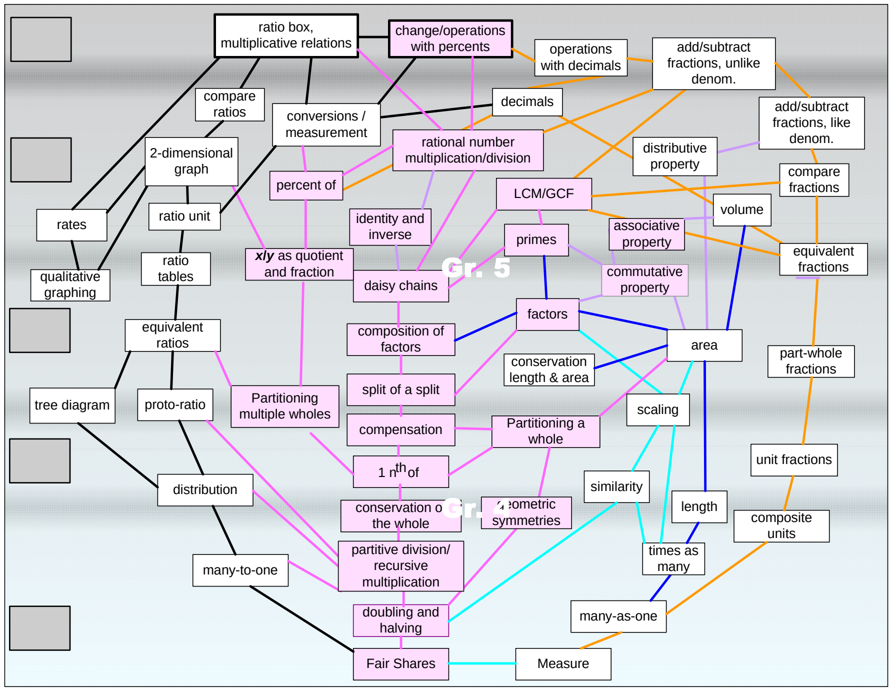


Figure 2. DELTA's conceptual mapping of RNR (Confrey, 2008).

The centrality of fair sharing within the mapping is a result of the DELTA research group's synthesis and is influenced by Confrey's splitting conjecture (1988, 1999). Across the rational number literature, the DELTA research group identified a substantial collection of studies with a primary focus on partitioning, i.e. students' actions of making equal-sized groups or parts. Because the studies identified involved creating equal-sized groups or parts rather than decompositions of any size, they distinguished the operation as *equipartitioning* and define it as "cognitive behaviors that have the goal of producing equal-sized groups (from collections) or equal-sized parts (from continuous wholes) as "fair shares" for each of a set of individuals" (Confrey et al., in preparation a).

The nature of the studies permitted the introduction of a case structure to describe the literature. One group of studies (Case A) involved students creating equal-sized groups of a collection of discrete objects. A second group (Case B) involved creating equal-sized parts of a single, continuous whole. Remaining studies involved sharing a collection of continuous wholes with the outcome yielding what could be thought of as a proper fraction (Case C) or an improper fraction (Case D). In the next section, I review some of the studies from each case to provide background for the context of the learning trajectory for equipartitioning. For a full summary of the cases including different strategies students use, related mathematical reasoning, and emergent properties, see Confrey et al. (in preparation a).

The Cases

Studies in Case A involved equipartitioning of discrete collections, such as crackers, sweets, or toys into multiple groups in the context of sharing fairly (Cowan & Biddle, 1989; Davis, 1991; Davis & Hunting, 1990; Pepper & Hunting, 1998; Squire & Bryant, 2002). When given a collection to share, many young children deal or "distributive count" as a means of creating equal-sized groups, indicating one-to-one correspondence as a sufficient means of justifying the creation of equivalent groups for the children (Cowan & Biddle, 1998; Davis & Pepper, 1991; Miller, 1984). Some second graders expressed that counting was needed to establish the equivalence (Davis, 1991; Davis & Pitkethly, 1990), perhaps as a result of its heavy emphasis in primary schooling. A study of younger children demonstrated that development in counting and in dealing were independent (Pepper & Hunting, 1998).

In Case B, children are asked to equipartition a single continuous whole into a given number of parts. In *The Child's Conception of Geometry*, Piaget, Inhelder, and Szeminska (1960) reported on a study of children ages four through six who subdivided a circular slab of modeling clay in order to analyze the genesis of part-whole relationships and their quantification as fractions. They described general stages of children's' abilities to subdivide continuous wholes, beginning with general fragmentations and proceeding to successful dichotomous dividing, through trichotomous dividing, and ultimately to an anticipatory scheme for dividing into five and six parts. In this analysis, they described seven conditions for progress in understanding fractions. Three of these criteria are crucial for correctly equipartitioning a single whole: 1) creating the correct number of pieces, 2) making all the pieces equal in area, and 3) exhausting the area to be equipartitioned (Piaget et al., 1960)[1]. For the remainder of the dissertation, I refer to these as the *three equipartitioning criteria.* In another group of studies, Pothier and Sawada explore students' equipartitioning of different geometric shapes (Pothier & Sawada, 1990, 1983; Pothier, 1989). From their work, they offered a theoretical framework to describe the development of equipartitioning skills. At the initial level, students have an understanding of "half" of rectangular or circular regions that is social and non-mathematical, such as referring to any fractional part of a whole as "half" regardless of the relationship between the part to the whole. At the next level called "algorithmic halving," students are doubling the number of equipartitions by repeatedly halving. At the third level, students can use their understanding of halving to create an even number of equipartitions by attending to the shape and size of pieces they create. This is

[1] Piaget et al. (1960) identified seven characteristics indicative of understanding fractions (pp. 309-311), three of which are essential to equipartitioning:
1. there is a divisible whole;
2. there are a determinate number of parts (often in correspondence with the number of persons sharing);
3. subdivision is exhaustive and produces no remainder;
4. a fixed relationship between the number of parts into which a continuous whole is to be divided and the number of parts is one more than the number of intersections;
5. all the parts are equal;
6. there is a nesting system, and not just juxtaposition; the fractions are parts of the original whole, and can themselves act as wholes and be subdivided; and
7. the sum of the fractional parts equals the whole.

extended to odd numbers in the fourth level. Students can equipartition into any number of parts based on factors in the last level called composition of factors. Important to this progression is the recognition of and connections between the number theoretic and geometric properties of equipartitioning tasks. They state:

> To trace children's partitioning capabilities as precursors of rational number acquisition, the relevant mathematical constructs, contrary to expectation, are not those of rational number. Rather, we propose that the key theoretical constructs come primarily from number theory and secondarily from geometry. Our description of the emergence and differentiation of the partitioning process hinged critically on the use of bipolar number-theoretic concept: odd/even, prime/composite, factor/multiple. We synthesize the bipolar constructs into a final level of partitioning capability suggested by the fundamental theorem of arithmetic. We made refinements in each level of capability by using concepts from motion geometry: translation, reflection, rotation, similarity, congruence, and symmetry (p. 309-310).

Cases C and D contain studies of equipartitioning collections of continuous wholes. Many of these studies center on students' development of fraction understandings and of division (Charles & Nason, 2000; Empson et al., 2006; Lamon, 1996; Toluk & Middleton, 2003). Lamon (1996) identified various strategies students use to equipartition collections of continuous wholes such as pizzas, including: first equipartitioning pieces, distributing those pieces, and then equipartitioning again if needed (a distribution strategy); equipartitioning each whole and allocating a piece from each whole (a mark-all strategy); and allocating wholes initially and then equipartitioning remaining wholes (a preserves pieces strategy). Another strategy that used ratio reasoning was identified in an investigation of how equipartitioning transforms into fraction knowledge by Empson et al. (2006). In analyzing and consolidating the different strategies documented in the literature, Charles and Nason (2000) offered a taxonomy which classified the strategies by increasing sophistication from repeated cutting and distribution to the realization that any fair sharing situation can be viewed as a division construct they called the "partitive fraction quotient construct." Toluk and Middleton (2003) investigated a trajectory of moving from a part-whole understanding of fractions, through equipartitioning, and ultimately to fraction as division, finding that students need instructional experiences to make connections among these different interpretations.

Learning Trajectory for Equipartitioning

Based on the synthesis of the research literature for each of the cases and clinical interviews with 37 children from Kindergarten through Grade 6, the DELTA research group created a learning trajectory for equipartitioning (Confrey et al, in preparation a; Confrey et al., 2008). The trajectory begins with the fair sharing of collections of items or a single, continuous whole between two people. After experiences with sharing a collection of discrete objects among larger numbers of people, students encounter sharing a single, continuous whole among four and eight people, then among three, other even numbers of people, and finally among other odd numbers. Later, students work with sharing collections of continuous wholes resulting in proper and improper fractions. Within each type of equipartitioning, there is a sequence of values used as tasks draw upon number theoretic qualities and geometry. Through these experiences, students develop an understanding of the fundamental mathematical properties of compensation, equivalence, and composition.

In each of these new situations, children use a variety of strategies, develop different mathematical reasoning practices, and encounter emerging mathematical properties. For instance when sharing collections, students may deal one at a time or with composite units and justify their solutions with strategies such as constructing arrays or measuring. When sharing continuous wholes, students may rely on geometric and number theoretic properties to create equal-sized parts and use stacking, measuring, or folding to justify their solutions. As students move through the trajectory, they learn about “splits of splits” or composition of factors (Pothier & Sawada, 1983), experiencing the multiplicative nature of equipartitioning. They use the compensatory principle to anticipate changes in equipartitioning resulting from creating more or fewer groups or parts, first at a qualitative level and later by factor-based changes. Later, they encounter ratio equivalence, formulate ideas of transitivity, and ultimately develop notions of the distributive property.

As previously mentioned, a learning trajectory may be viewed as a boundary object (Confrey, 2009) and can thus be represented in multiple ways. In the case of the DELTA research group’s diagnostic assessment work, their assessment partners use the idea of a construct map (Wilson, 2005) to describe the nature of a concept to be measured. Though understandings can be complex, construct maps assume two extremes such as low to high or informal to formal with qualitative levels defined between these two extremes. In terms of a model for measurement, a construct map is represented as a progress variable (Wilson & Sloane, 2000). Progress variables map learning within a specific content area as a progression towards increasingly sophisticated understanding. They assume that learning is conceptualized, not in terms of acquiring more knowledge and skills, but as growth toward higher levels of competence and deeper understandings.

A progress variable can be viewed as a representation of the learning trajectory for equipartitioning (Confrey et al., 2008). An early version of the DELTA research group’s progress variable for equipartitioning is shown in table 1 and table 2. Table 1 shows a global view of the progress variable based on the case structure from the synthesis. Level 1 combines cases A and B for equipartitioning to form two groups or parts. Level 2 involves equipartitioning collections of discrete objects to form more than two groups (Case A). In Levels 3 through 6, equipartitioning of a single continuous whole progresses from creating 4, 8, or 16 parts (powers of 2), through creating 3 parts and even parts, and finally through creating an odd number of parts (Case B). Levels 7 (Case C) and 8 (Case D) involve equipartitioning collections of single wholes, though this ordering was uncertain in this version of the progress variable.

Table 1. Equipartitioning Progress Variable

1.8 m objects shared among p people, $m > p$
1.7 m objects shared among p people, $p > m$
1.6 Splitting a continuous whole object into odd # of parts ($n > 3$)
1.5 Splitting a continuous whole object among $2n$ people, $n > 2$, and $2n \neq 2i$
1.4 Splitting continuous whole objects into three parts
1.3 Splitting continuous whole objects into $2n$ shares, with $n > 1$
1.2 Dealing discrete items among $p = 3 - 5$ people with no remainder; mn objects, $n = 3, 4$, or 5
1.1 Partitioning using 2-split (continuous and discrete quantities)

Within each of these levels as children progress from one extreme of the learning trajectory to the other, their behaviors and verbalizations increase in sophistication. Students first solve the task, solve the task in multiple ways, justify their solutions, mathematically name their solution, learn to reverse the process, and, at the highest level of proficiency, demonstrate understanding of the fundamental mathematical properties. This within-level framework is depicted in table 2.

Table 2. Within-level framework for equipartitioning progress variable.

Properties	Composition, Compensation, Transitivity, Equivalence
Reversibility	“If we put everyone’s share back together, what would it be?”
Naming	“What would you call a share?”
Justification	“How do you know this is a fair share?”
Multiple Methods	“Is there another way to share?”
Methods	“How could you share?”

Thus, based on the rational number synthesis and the field’s recent work in learning trajectories, the DELTA group expressed a learning trajectory for equipartitioning as this progress variable by which students refine their informal understandings of fair sharing to the rational number constructs of ratio, operator, and number described by Confrey (2008a).

Professional Development

Reform efforts requires practices different than many practicing teachers use (Ball & Cohen, 1999; Loucks-Horsley, Hewson, Love, & Stiles, 1998) and demands that teachers be more informed of the ways in which their students may be thinking. Yet, many professional development opportunities do not provide the support teachers find valuable in making these changes. Ball and Cohen (1999) characterize many professional developments as “intellectually superficial, disconnected from deep issues of curriculum and learning, fragmented, and non-cumulative” (p. 3 - 4).

The National Staff Development Council outlines a set of standards for professional development experiences based on research and a consensus of the staff development community to assist in designing and selecting programs that are focused on teacher learning and change (NSDC, 2001). They call for staff development focused on improving learning for all students through organizing educators into learning communities, providing leadership in the continuous improvement of instruction, and providing resources to support teachers in implementing changes in practice. These programs should use student data and multiple sources of information to inform decisions concerning instructional changes, prepare teachers to make decisions that are based on research about students' learning, be designed to employ learning strategies consistent with what is known about human learning and change, and support collaboration among educators around students' learning. The programs should assist teachers in understanding all students, deepen knowledge of content and research-informed instructional strategies and a variety of classroom assessments, and provide skills in involving other stakeholders in the process of improved student learning.

Researchers in professional development have identified a number of common characteristics of effective programs. These include programs that are focused on student learning, grounded in practice, sustained, ongoing, focused on specific curriculum content, foster collegial relationships among participants, and promote collective participation (Darling-Hammond et al., 2009; Desimone, 2009; Heck, Banilower, Weiss, & Rosberg, 2008; Yoon, Duncan, Lee, Scarloss, & Shapley, 2007). Specifically within the context of mathematics education, Loucks-Horsley (1995) identified principles of effective professional development for mathematics and science education. Such programs: (1) are driven by a clear, well-defined image of effective classroom learning and teaching; (2) provide teachers with opportunities to develop knowledge and skills and broaden their teaching approaches so they can create better learning opportunities for their students; (3) use instructional methods to promote learning for adults which mirror the methods to be used with students; (4) build or strengthen the learning community of science and mathematics teachers; (5) prepare and support teachers to serve in leadership roles; (6) consciously provide links to other parts of the educational system; and (7) include continuous assessment.

Using these principles and a review of the research literature concerning mathematics and science professional development, Loucks-Horsley et al. (1998) offer a framework of professional development design for mathematics and science education (see figure 3). At the center of this frame is a planning sequence of goal setting, planning, doing, and reflecting. Influencing the first two stages are four inputs: knowledge and beliefs, context, critical issues, and strategies. Reflection at the end of the implementation sequence informs these four inputs for successive refinement of the professional development plan. Knowledge and beliefs inputs are described to be knowledge of learners and learning, knowledge of teachers and teaching, knowledge of the discipline of mathematics or science, knowledge of effective professional development principles, and knowledge of change and the change process. Important facets of context noted by the authors include the audience (teachers and students), the current state of practice, relevant policies, available resources, and organizational structure. Critical issues noted include equity and diversity, professional culture, leadership effective use of standards and frameworks, time for professional development, and its evaluation and assessment. They identify strategies used in professional development implementation to meet its goals.

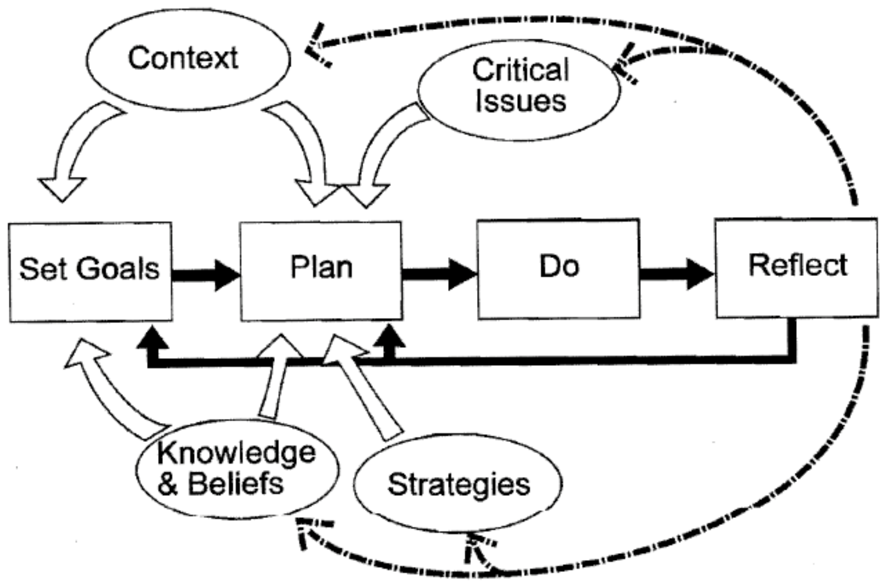


Figure 3. Loucks-Horsley et al. (1998) framework for designing mathematics and science educational professional development.

Several researchers report on the effectiveness of focusing on students' thinking within professional development. Cognitively-Guided Instruction (CGI) is a program that was developed to assist teachers in constructing conceptual understandings of the development of children's mathematical thinking in specific content domains (Carpenter, Fennemaa, & Franke, 1996). Through a series of studies, the researchers demonstrate teachers' development of students' mathematical thinking results in changes in instructional practices, teacher beliefs, and increased student achievement (Carpenter et al., 1998; Fennema et al, 1996). Early in their research, they found that teachers have intuitive ideas of students' thinking, but that these ideas lack coherence and thus rarely play a role in teachers' decision making (Carpenter et al., 1988). After professional development using CGI, a study comparing 20 participating teachers with 20 nonparticipating teachers in their instructional interactions with students, they found that CGI teachers showed a greater emphasis on conceptual understandings and processes with their students, expected multiple approaches from students, tried to makes sense of students' verbalizations more, and understood more about students' thinking than control group (Carpenter et al., 1989). Further, the students in CGI classrooms had levels of achievement significantly higher than those students in the comparative classrooms. In another CGI study, the researchers conducted a three-year longitudinal study of 21 teachers. At the conclusion of the study, 19 of the teachers' classrooms were significantly influenced by teachers' understandings of students' thinking. These teachers posed problems and questions to uncover students' thinking, and understood and expected a variety of solution strategies from their students. Additionally, 7 of those 19 "conceptualized instruction in terms of the thinking of the children in their classes… they not only applied their knowledge to asses students' thinking and plan instruction, but also regarded it as a framework for developing a deeper understanding of children's thinking in general" (Carpenter, Fennema, Franke, Levi, & Empson, 1999, p. 108). Thus, there is significant evidence in the literature that focusing on students' thinking within professional development can be an effective means to support changes in practice and student learning.

Curriculum Adaptation and Student Work Analysis

Loucks-Horsley et al. (1998) list several strategies for offering professional development. In particular, two are relevant to providing instructional guidance for teachers within the context of their practice in focusing on students' thinking. One of these is curriculum adaptation. The strategy provides teachers with opportunities to modify existing instructional materials according to new professional knowledge for the purpose of more fully meeting the needs of students. Another strategy for assisting teachers in instructional guidance uses artifacts of learning as an entry for an analysis of students' thinking. They suggest that through the process of adapting curriculum materials, teachers increase their content and pedagogical knowledge as a result of thinking deeply about curriculum goals and students' learning. In addition, because student learning is the ultimate goal of instruction, they claim careful consideration of real outcomes can provide opportunities for teachers to deepen their understanding of content and pedagogy. In describing possible solutions to the problems of professional development, Ball and Cohen (1999) echo these suggestions:

> Professional development could be improved by seeking ways to ground its "curriculum" in the tasks, questions, and problems of practice. One way to do this is to use the actual contexts of teachers' ongoing work: their efforts to design particular units of instruction ... Another way would be to collect concrete records and artifacts of teaching and learning that teachers could use as the curriculum for professional inquiries – for example, students' work, curriculum materials, videotapes of classroom teaching, and student assessments (p. 20).

Ben-Peretz (1990) points out that the adaptation of curriculum materials is a central practice of teachers and that no curriculum is used without such adaptations. Teachers make adjustments to instructional tasks as results of their own understandings of the lesson and the mathematics, their perceived abilities of students, and constraints of time and resources (Ball & Cohen, 1996). Ben-Peretz presents a concept of curriculum as "too much and too little"; curricular materials represent more than a teacher could possibly use in the context of teaching and, at the same time, are inadequate alone to address the demands of the diverse, particular situations of instruction. It is within the instances of being "too little" that curriculum adaptation occurs. With this conception, curriculum materials "are more than the embodiment of their developers' intentions, and offer teachers a wide array of curriculum potential depending on their purposes and the demands of their classroom situations" (p. xiv). She asserts that teachers need knowledge and expertise in identifying potentials within materials to make adaptations to respond to the specific needs of their students, classrooms, and contexts. In a case study of two elementary teachers using *Children's Math Worlds* (Fuson, 2000), Drake and Sherin (2006) report that teachers follow distinct patterns of curriculum adaptation. These changes are based on early recollections of themselves as mathematics learners, interactions with children in informal settings, and their current views of themselves of mathematics learners. When a teacher's curriculum in not aligned with standards, teachers must adapt instructional materials to meet those goals. Confrey & Maloney (in press a) suggest several ways in which adaptations can be made, one of which is the adoption of a different mathematical goal.

The other strategy identified by Loucks-Horsley et al. (1998) is a focus on artifacts of students' learning. Krebs (2005) reports on 20 middle grades teachers' examination of three pairs of students' work and related video recordings on a quadratic pattern task. When the teachers examined the students' work, she reports that they observed some of what they expected, but that the examination revealed unanticipated strategies and surprises about students' abilities. Additionally, the analysis raised questions for the teachers about the students' understandings and led teachers to consider how these observations would influence their instruction. When viewing a video recording of the students working on the task, the teachers gained more insights into the ways that the students were approaching the pattern task, raised additional questions about the students' understandings, and called into question some of their previously-held assumptions.

Several mathematics educators and researchers report on using video recordings to focus teachers on the mathematical thinking of students (Goldsmith & Seago, 2008; Horvath & Lehrer, 2000; Lampert & Ball, 1998; Maher, 2008). Video recasts teachers as *observers* rather than *actors*, permitting them to focus on particular pedagogical activities rather than attend to all situations that arise in classrooms requiring their attention. Furthermore, video recordings allow teachers to shift their focus from actions performed and decisions made by the teacher to the mathematical thinking of students (Sherin & van Es, 2005; van Es & Sherin, 2002).

Key Aspects from the Literature

From this review of the literature, I identified several key aspects that informed this study. First, learning trajectories have potential to provide instructional guidance to teachers. In addition, trajectories may assist teachers in grounding their knowledge of content and students and provide a more stable conception of the goals and sequences of instruction for practicing teacher development. However, little is known about how teachers will use learning trajectories within their practices or how this integration may relate to students' learning.

Second, equipartitioning is a significant operation needed to develop robust rational number reasoning. Through engaging in equipartitioning tasks over time, students can move from their informal understandings towards more complex conceptions over time, develop mathematical reasoning skills, and encounter mathematical properties. New synthesis research in rational number reasoning has led to the development of a learning trajectory for equipartitioning.

Finally, professional development that is focused on and grounded in students' thinking can be an effective way to affect instructional practices and move toward a realization of the goals of reform. Experiences with artifacts of students' thinking and video recordings of students engaging in tasks can support teachers in focusing on students' thinking. Further, providing opportunities for teachers to consider how insights into students' thinking can manifest in regular practice, such as through the use of existing curricular materials, can provide opportunities to deepen understanding of content and pedagogy.

Purpose and Research Questions

These key aspects shaped the purposes of the study and informed the development of the research questions. The primary purpose of the study was to explore how teachers used a learning trajectory for equipartitioning within the context of a professional development experience for K-2 teachers to make sense of students' thinking and to inform their practices of planning, assessing, and teaching. A secondary purpose was to investigate the relationships between the teachers' uses of the learning trajectory and their students' learning. Specifically, this study will investigate three research questions:

(1) In what ways and to what extent do teachers use a learning trajectory for equipartitioning to build models of students' thinking?
(2) In what ways and to what extent do teachers use a learning trajectory for equipartitioning to inform their adaptations of curricular instructional tasks, their interactions with students during instruction, and their assessment of students' understanding?
(3) What are the relationships between teachers' knowledge of equipartitioning and uses of a learning trajectory for equipartitioning and their students' learning?

CHAPTER 3

The previous chapter provided an overview of the research literature related to learning trajectories, equipartitioning, and professional development. In this chapter, I present the theoretical perspectives grounding the work of the DELTA research group and this study. I outline the conceptual framework driving and situating the study within the greater body of research. Next, I select and justify a methodology for the study and describe the study's specific design, sources of data, and methods of analysis.

Theoretical Perspectives

The work of the DELTA research group, their construct definitions, and the results of this study are grounded in and influenced by two theoretical perspectives, Constructivism and Socio-cultural Theory. This section provides an overview of each theory, highlighting tenets particularly relevant to the study. Then, I relate these theories to the DELTA research group's definition of learning trajectories and the purpose, focus, and design of the study.

Constructivism

Constructivism is a theory of learning asserting that knowledge is actively constructed by individuals to make sense of their world experiences. It rejects a "received view" of reality in favor of viewing knowledge as "apprehendable in multiple, intangible mental constructions dependent in form and content on the individual persons holding constructions" (Guba & Lincoln, 2000, p. 110). As a paradigm for mathematics education with roots in research in students' problem solving abilities, systematic misconceptions, and Piaget's genetic epistemology, it focuses on "the strengths and resources children [bring] to tasks… by making their active involvement and participation central" (Confrey & Kazak, 2006, p. 306). Key components of this theory include genetic epistemology, the "cycle of constructive activity" (p. 318), schemes and their evolution, and sequential stages of development.

In *Genetic Epistemology*, Piaget (1970) rejected the traditional view of epistemology because it focused on knowledge independent of the knower and neglected its constant evolution. His fundamental hypothesis for a new perspective on knowledge was that there was a correspondence between the logical organization of knowledge and the formation of psychological processes. That is, genetic epistemology explains knowledge based on the way it is formed and the way it evolves. He stated, "Knowledge results from continuous construction, since in each act of understanding, some degree of invention is involved; in development, the passage from one stage to the next is always characterized by the formation of new structure which did not exist before" (p. 77). This generative thinking, or "invention," is a result of individuals bringing their current understandings to bear on new situations. In this way, knowing and acting are mutually constitutive. As Piaget stated:

> Human knowledge is essentially active. To know is to assimilate reality into systems of transformations. To know is to transform reality in order to understand how a certain state is brought about… Knowing an object does not mean copying it – it means acting on it… knowing reality means constructing systems of transformations that correspond, more or less adequately, to reality…Knowledge is a system of transformations that become progressively adequate (1970, p. 15).

For Piaget, knowledge is abstracted from acting on objects. He referred to knowledge abstracted from a single action as simple abstraction, whereas knowledge derived from coordinating actions was called *reflective abstraction.* It is through reflection that reorganization of knowledge takes place. Of particular interest to Piaget were the mental operations that arose from reflective abstraction. These operations had four characteristics: they must result from actions that can be carried out internally and materially; they must be reversible; they consist of some invariant; and they exist within a system of other operations.

Mental structures called schemes develop from combinations of mental operations. Schemes are the result of equilibration, or the need for an internal consistency of the constructed operations. Equilibration is reached through assimilation or accommodation, where new objects are incorporated into existing schemes or existing schemes are adjusted to account for the new object, respectively. Von Glasersfeld (1984) recast equilibration as a search for viability. In the positivist tradition, *truth* was seen as a correspondence with objective reality, i.e. a *match* between a conception and reality. However, since an objective reality cannot be known or does not exist in constructivist thought, von Glasersfeld supplanted this match with an idea of *fit*, drawing on a Darwinian notion of survival as long as a concept has not proven unviable. He stated, "Just as the environment places constraints on the living organisms and eliminates all variants that in some way transfers the limits within which they are possible or 'viable', so the experiential world… constitutes the testing ground for our ideas" (p. 22).

In reviewing Constructivism from a perspective of mathematics education, Confrey and Kazak (2006) describe the process of refining schemes as the *cycle of constructive activity*. They describe that individuals abstract qualities of objects based upon actions which are the result of some perturbation experienced by the individual, similar to von Glasersfeld's notion of a concept proving unviable. These abstractions are coordinated through reflection and ultimately create patterns of thought, or schemes. Actions are purposeful activities on physical or mental objects; those that are internalized, reversible, and based on some invariance become operations. Through reflective abstraction, previously abstracted objects create new objects or are coordinated with existing schemes, become objects themselves, and are in turn subjected to operations. Thus, the cycle continues towards progressive adequacy and knowledge is taken as the result of this cycle.

In addition, Confrey and Kazak (2006) suggest one of Piaget's contributions to Constructivism is the idea of general stages of thought that occur sequentially. Piaget is criticized often for his developmental stages based on age (Sensorimotor, Pre-Operational, Concrete Operational, and Formal Operational). However, the essential element in his theory was not the age at which transformations in thinking occur but rather the *sequence* in which they occur. In fact, towards the end of his career, Piaget himself rejected the interpretation of clear demarcation of stages based on age (Confrey, 2008b).

Socio-cultural Theory

Socio-cultural Theory posits the primary origin of knowledge does not lie in the structure of the objective world or in the interaction between the individual and the world, but in the "social and material history of the culture of which the subject is a part" (Case, 1996). With foundations in the work of Vygotsky, an essential idea of the theory is one's simultaneous enculturation into and transformation of society. As knowledge develops through interactions with the culture, that same knowledge serves as a means of mastering the environment and changing the culture. Vygotsky (1981) suggested that, "Any function in the child's cultural development appears twice or on two planes. First, it appears on the social plane, and then on the psychological plane. First, it appears between people as an interpsychological category, and then within the child as an intrapsychological category" (p. 163). Thus, knowledge is imbued from society rather than constructed internally. This internalization occurs through activity which is mediated by: 1) signs, which are "artificial, or self-generated, stimuli" that "extend the operation of memory beyond the biological dimensions of the human nervous system" (Vygotsky, 1978, p. 39); and 2) tools, which are material or mental artifacts that are employed in goal-directed activities. Whereas tool use leads to change in the object on which it operates and is thus externally oriented, the use of signs "changes nothing in the object of a psychological operation. It is a means of internal activity aimed at mastering oneself; the sign is internally oriented" (p.55). For Vygotsky, mediation is not taken as an intermittent link between a stimulus and a response but as a change in the stimulus as a result of a response to it through the use of a tool (p. 14). It is through mastery and appropriation of cultural tools for mediation that higher psychological processes develop.

Although Piaget contended that development (physical maturation) precedes learning, Vygotsky (1978) suggests that learning actually advances development through the Zone of Proximal Development (ZPD):

> It is the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers" (p. 86).

Because Vygotsky saw learning as the internalization of culture, interaction with adults or "more capable peers" advances learning by supporting less mature ideas. Similarly, the development of scientific concepts precedes the development of spontaneous concepts

(Vygotsky, 1986). In the same way that interaction with another supports the maturation of ideas with the ZPD, formal ideas (scientific concepts) provide structure for the refinement of spontaneous ideas. He writes,

> Systematicity and consciousness do not come from outside, displacing the child's spontaneous concepts, but… on the contrary, they presuppose the existence of rich and relatively mature representations… Systematic reasoning, being initially acquired in the sphere of scientific concepts, later transfers its structural organization into spontaneous concepts, remodeling them 'from above' (p. 172).

Constructivist principles influenced the work of the DELTA research group and the study in several important ways. Primarily, equipartitioning developed by a child is viewed as a psychological operation that develops into a mathematical operation. As a scheme, the origins of equipartitioning rest with a child's experiences and are subject to perturbations, assimilations, and accommodations. Instructional experiences play a key role in its development, not only in creating problematic situations where children can act on and with their existing ideas, but also in providing opportunities for representation, articulation, and reflection. Over time, there are general, sequential stages through which children's thinking progresses from their informal ideas to an increasingly sophisticated understanding of equipartitioning. These philosophical commitments manifested in the study in two important ways other than its content. First, the professional development was designed with a view of teachers as learners. Just as instruction provides students with problematic situations in order to render current understandings unviable, the teachers were provided opportunities to act on and refine their existing understandings of equipartitioning and students. Second, this belief about learning was communicated throughout the professional development so that teachers could come to consider this perspective about their students' learning.

Likewise, the influences of Socio-cultural Theory can be seen in the role instruction plays in the learning trajectories definition. The profound influences of the social and cultural worlds of children on their learning are not ignored but rather influence instructional activities and the interactions between students and among students and teachers. Similarly, learning is mediated by the material and psychological tools that children use. Thus, the affordances and constraints of context, media, curricula, and forms of interaction should be considered in instructional design. Additionally, the scientific concept of equipartitioning provides structure for students' spontaneous concepts of creating equal-sized parts or groups. These factors influenced the study in the design of the professional development, from the creation of discourse community during and between the sessions to the different types of learning activities in which the teachers engaged. Moreover, this theory suggested that the interactions between students and teachers would be a potential site for investigation into the ways that teachers use the learning trajectory.

Conceptual Framework

Reform calls for a shift from instruction where the teacher is the sole source of information to instruction where students' ideas are regarded as the points of departure for teachers' designs of instructional activities. This shift requires teachers to be more informed of the ways in which their students may be thinking and how learning occurs. Further, it demands that teachers consider carefully the role of instruction in learning, such as the types of interactions, curricular choices, and methods of assessment they provide for students.

Thus, the conceptual framework driving this study entails a component of how teachers come to understanding students' thinking and also the influences of those understandings on their practices.

Models of Students' Thinking

The constructivist orientation of this study signifies that teachers cannot access students' thinking directly but must infer thought based on its externalizations, such as students' behaviors, verbalizations, and representations (von Glasersfeld, 1995). In this sense, students' work on instructional activities and accompanying words may be seen as evidence of their thinking, and teachers' assessment practices may be seen as making conjectures about students' thinking based on this evidence. Cobb and Steffe (1983) describe this work of teachers as building models of students' understandings of mathematical ideas. These models must be specific enough to yield information useful to the teacher yet parsimonious in order to be portable and applicable to other students.

In this conceptual framework, models of students' thinking are based on Black's (1962) discussion of theoretical models. For Black, theoretical models use a better-known secondary domain to understand less familiar domain of investigation. They relate objects, mechanisms, systems, and structures from the more familiar domain to make inferences and to build understandings of the less-known area. Thus, as teachers use their own understandings of mathematics to make sense of students' behaviors and verbalizations, they are creating a theoretical model of students' thinking that allows them to relate and infer students' understandings of mathematics.

Hollebrands, Wilson, & Lee (2008) identified four processes that prospective teachers used in creating models of a pair of middle school students' thinking working on an instructional task. One such process is *Describing*, where teachers identified portions of students' work and language that may provide evidence for later inferences of thinking. Another process is *Comparing*, where teachers implicitly or explicitly compared evidence of students' work on a task with their own work and thinking. Also, teachers engaged in the process of *Inferring*, where they used the students' work and language and in relation to their own knowledge of the task to build a model of students' cognition. A fourth process was *Restructuring*, where teachers' models of students' thinking informed their next pedagogical move.

While *Describing*, *Comparing*, and *Inferring* focus on students' thinking, *Restructuring* is a process that is focused on teachers' own thinking. Restructuring is similar to Confrey's (1998) description of the *voice – perspective dialectic.* While the close listening and articulating of students' mathematics with their *voice* assists teachers in understanding the actions of the student, the teacher's *perspective* influences what they hear and observe. In seeking to understand the student's voice, the teacher's perspective is changed. Thus Hollebrands et al.'s *Restructuring* can be thought of as teachers' changes in their perspectives as a result of students' voices.

Teachers' Instructional Practices

One purpose of this study is to understand the relationship between the ways that teachers use a learning trajectory and the models of students' thinking that they construct.

Yet, the Socio-cultural orientation of the study suggests that teachers' instructional practices affect the ways that students formulate their understandings of a particular concept. To structure this portion of the study, I draw on Confrey and LaChance's (2000) four components of instruction: curriculum, methods of instruction, the role of the teacher, and methods of assessment. These components are well-related to the purposes of this study, namely *planning* for instruction by the selection and design of instructional tasks (curriculum), *assessing* of students' thinking (methods of assessment), and *teaching*, particularly the interactions between the teachers and students (methods of instruction and the role of the teacher).

Thus, the model-building processes along with the components of instructional design are combined with the learning trajectories definition to frame this study. As a learning trajectory makes explicit students' behaviors, verbalizations, and related ways of thinking at different stages of conceptual development, this study explores the ways that teachers use it to model students' understandings of equipartitioning concepts. The ways that teachers enact this knowledge within instruction can be described as the different components of instructional design. As a purpose of the study is to understand better the role a learning trajectory may play in both understandings students' thinking and how those understandings manifest in instructional practices, the framework seeks to highlight the relationships between the learning trajectory, the teachers' models of students' thinking, and the teachers' instructional practices (see figure 4). The creation of the conceptual framework was informed by the research questions, a review of the research literature, and the theoretical perspectives of the study.

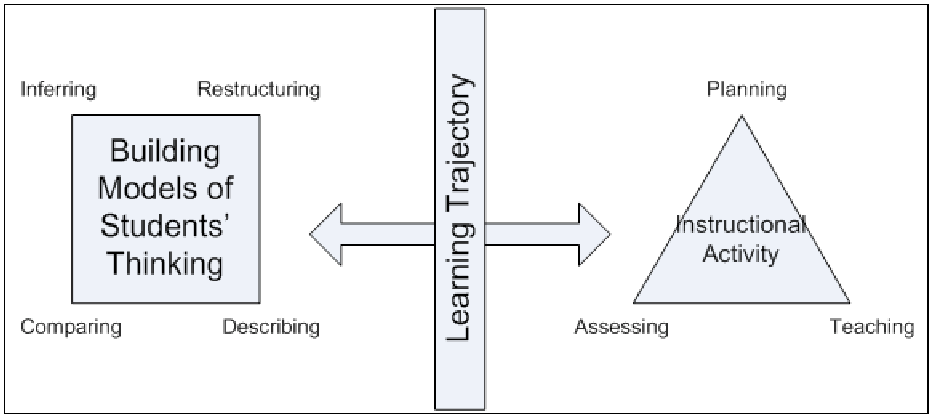


Figure 4. Conceptual framework for the study.

Methodology

The research questions, review of the literature, the theoretical perspectives, and the conceptual framework all influenced the selection of design studies as the methodology for the dissertation study. In *Scientific Research in Education*, Shavelson and Towne (2002) guided their discussion of methodology around three key questions for social science research: what is happening (description); is there a systemic reason it is happening (causation); and why and how is it happening (mechanism). They emphasize that is not the method employed in inquiry that renders the study scientific but the appropriate selection of method to address the research question. Since this study is a question of mechanism, i.e. it investigates the ways and the extent to which the learning trajectory construct can be utilized in teachers' practice, the selection of design studies methodology appropriately addresses the questions of process (Confrey, 2006). Additionally, in order to understand the ways that the learning trajectory affects teachers' practices in classrooms, an observational study focused on teachers' instruction was selected to address the second and third research questions.

In her chapter The Evolution of Design Studies as Methodology, Confrey (2006) traces the roots of design studies to both Constructivism and Socio-cultural Theory. Both theories contributed to the realization that research in classrooms is not "deterministic but complex and conditional" (p.134). She described how Piaget's clinical method and research in pedagogy from the Soviet Union grounded in Vygotsky's work evolved into teaching experiment methodology. Steffe and Thompson (2000) described the main purpose of the teaching experiment methodology is "for researchers to experience, firsthand, students' mathematical learning and reasoning" (p. 267). Confrey and Lachance (2000) introduced the notion that teaching experiments should be conjecture-driven in their description of "transformative teaching experiments." Design studies grew out of the teaching experiment tradition, taking classrooms as whole systems and having a goal of generating theory to provide "domain-specific guidance" for learning in context (Confrey, 2006).

The purpose of design studies is to provide "systematic and warranted knowledge about learning and to produce theories to guide instructional decision making" (Confrey, 2006). They "entail both 'engineering' particular forms of learning and systematically studying those forms of learning within the context defined by the means of supporting them" (Cobb, Confrey, diSessa, Lehrer, and Schauble, 2003, p. 9). They are used to develop theories and are ideal for addressing the complexities of education. Cobb et al. identify five features of design experiments. First, they develop an understanding of the processes of learning and the means to support it in a particular context. They are interventions and not only do they generate theories but test them as well. They are iterative in nature in the sense that outcomes inform the next intervention. Finally, the theories generated must be robust.

In preparation for a design experiment, conjectured starting points for the intervention are articulated (Cobb et al., 2003; Confrey & Lachance, 2000) and elements of the intervention are outlined (Cobb, 2000; Confrey & Lachance, 2000). During the implementation of the intervention, researchers engage in ongoing analyses in relation to initial conjectures and make revisions accordingly (Cobb et al., 2003; Confrey & Lachance, 2000). At the conclusion of the intervention, researchers conduct a retrospective or final analysis (Cobb et al., 2003; Cobb, 2000; Confrey & Lachance, 2000; Steffe & Thompson, 2000).

The Conjectures

In preparation for the study, I articulated five conjectures related to teachers' uses of a learning trajectory and the relationships to students' learning described by the research questions. They were informed by the review of the research literature, consultations with the DELTA research group, and my own experiences from professional practice. The conjectures driving the study are:

> *Conjecture One:* The use of a learning trajectory for equipartitioning will assist teachers in building models of students' thinking by highlighting facets of students' behaviors and language and relating those to a particular conception. Whereas initially teachers will not notice distinctions among students' activities and language, the learning trajectory will act as a lens for teachers, accentuating language and actions that are indications of thinking while filtering irrelevant work or words.

> *Conjecture Two:* The use of a learning trajectory for equipartitioning will inform the adaptation of instructional tasks by helping them adapt tasks through altering the mathematical goals and assist them in judging the relative difficulty of tasks.

> *Conjecture Three:* The use of a learning trajectory for equipartitioning will inform teachers' formative analysis of students' work on instructional tasks by helping them identify evidence of students' thinking, conjecture about what students may or may not know, and raise questions that inform teachers' next instructional steps.

> *Conjecture Four:* The use of a learning trajectory for equipartitioning will inform interactions with students during instruction by assisting teachers in identifying students' ideas that will support their mathematical goals, by influencing teachers' questions, and in sequencing students' work during whole group discussions.

> *Conjecture Five:* Teachers' knowledge and uses of a learning trajectory for equipartitioning will be positively associated with their students' learning.

Design of the Study

In this section, I provide a description of participants in the study, the context of the investigation, and an outline of the elements of the intervention.

Participants

Thirty-five primary grades teachers from two elementary schools participated in a professional development experience required by their district. All teachers were invited to participate in the study; 11 Kindergarten, 11 First Grade, and 11 Second Grade teachers volunteered. Of these 33 participating teachers, 88% received undergraduate training in elementary education and 24% held a master's degree in education. Their average number of undergraduate credit hours was 6.4 hours in mathematics (s = 4.1 hours) and 4.4 in mathematics methods courses (s = 2.1 hours). Six of the participating teachers were currently enrolled in a graduate program in elementary education and two of the teachers were certified by the National Board of Professional Teaching Standards. On average, the teachers had 12.0 years of experience (s = 7.2 year), 9.2 of which were with students in grades K-2 (s = 5.8 years). Finally, the teachers were participating in an average of 1.3 other professional development opportunities (s = 1.3 opportunities) during this study.

Context

The schools were from a rural district within one medium-sized city in the southeastern United States. I gained access to this district by approaching the district mathematics coordinator and offering to provide professional development relating anticipated revisions of the state mathematics standards, recent progress in the DELTA research group's work, and the district's adopted mathematics curriculum. In addition, low performance on high-stakes testing by these two schools contributed the district's desire for additional professional development.

Professional Development Outline

The Loucks-Horsely et al. framework (1998), its principles, and the conjectures of the study were used to design the professional development entitled Early Rational Number Reasoning Professional Development (ERNR PD). Consistent with this framework, the ERNR PD had three goals: 1) to assist teachers in understanding and using a learning trajectory for equipartitioning; 2) to equip teachers with strategies to help model and use students' thinking in their practice (clinical interviews, instructional task analysis and adaptation, formative assessment using artifacts of students' work, and principles for interactions with students); and 3) to advance students' learning of equipartitioning. The ERNR PD focused on developing strategies for modeling students' thinking and presented pedagogical strategies that support students in refining their own mathematical ideas within the context of the district's adopted curriculum, *Investigations in Data, Number, and Space, 2nd Edition* (TERC, 2007). In preparation, a review of *Investigations* to identify curricular units amenable to the goals of the study was conducted (Wilson, in preparation). Successful completion of the ERNR PD entitled teachers to two continuing education credits towards the renewal of their professional license.

The professional development met these goals through 20 hours of professional development comprised of 10 hours of face-to-face instruction where I served as instructor and 10 hours of classroom-based activities. The face-to-face instruction used three strategies identified by Loucks-Horsley et al. as effective for mathematics and science teacher professional development to introduce a learning trajectory for equipartitioning: curriculum adaptation, the examination of students' work and thinking, and the development of professional networks. This time was divided into five two-hour sessions and focused on the learning trajectory for equipartitioning and the following practices: clinical interviewing, instructional task analysis and adaptation, formative assessment using artifacts of students' work, and questioning principles for interactions with students. In a final session, teachers discussed the implementation of the practices and their uses of the learning trajectory in instruction following opportunities to teach units related to equipartitioning. Embedded between the fourth and fifth sessions was an observational study of the participating second grade teachers' classrooms. The first four sessions were conducted at each of the participating schools after the instructional day. The final session pooled teachers from both schools for one larger meeting.

The professional development used classroom-based activities that engaged teachers in the strategies studied in the ERNR PD with their own students and colleagues. Between the face-to-face sessions, teachers completed classroom-based assignments involving their students or their curriculum. These assignments included conducting clinical interviews and the adaptation of an instructional task from *Investigations*. After a brief training, teachers received two Flip-cams provided to each of the participating schools for teachers to collect these artifacts. In addition, school technology coordinators received training on the Flip-cams and the ERNR PD website in order to offer support to teachers. Artifacts collected with the Flip-cams, accompanying analyses, and reflections were posted in the secure online environment for other participants to analyze and to foster professional collaboration around students' thinking. Table 3 displays the structure of the professional development and the embedded observational study. Descriptions of the sessions and learning activities are provided in the next section. The decisions concerning these activities and the evolution of the ERNR PD are detailed in Chapter Four.

Table 3. Timeline and structure of the study.

Week 1	Session I	Teacher Pre-test Clinical interview training
Week 2	Classroom-based Activity One	DELTA Pirate Treasure Clinical Interview
Week 3	Session II	Learning Trajectory for Equipartitioning I: Equipartitioning collections of discrete wholes Problem-Centered Instruction (Confrey, in press)
Week 4	Classroom-based Activity Two	*Investigations* Task Adaptation
Week 5	Session III	Learning Trajectory for Equipartitioning II: Equipartitioning a continuous whole
Week 6	Classroom-based Activity Three	DELTA Pirate Birthday Clinical Interview
Week 7	Session IV	Learning Trajectory for Equipartitioning III: Equipartitioning collections of continuous wholes Teacher Post-test
Weeks 8 – 13	Observational Study	Student Pre-test Observations Observation Interviews
Week 14	Session V	Professional Networking

Session I. The first session was devoted to collecting base-line data and to the practice of clinical interviewing. After introductions and a description of the goals of the ERNR PD, the teachers completed the background survey and the pre-assessment. Using a module on clinical interviewing for teachers (Confrey & Maloney, in press) as a guide, I led the teachers through a brief history of Piaget's clinical method, including its rationale, associated difficulties, and procedures, including questioning techniques to elicit predictions, explanations, arguments, and justifications from the student. Next, the teachers were presented with a clinical interview of a child sorting a collection of geometric figures into groups that are "the same". In small groups, teachers analyzed the interview. The session ended with an explanation of the first classroom-based assignment.

In this session, the learning activity that was most relevant to the goals of the ERNR PD was the teachers' group analysis of the similarity clinical interview. Since the main goal of the session was for teachers to develop clinical interviewing skills, I used an activity from the Confrey and Maloney module (in press a). First, teachers were shown a collection of geometric figures and asked to consider how they would sort them into groups that were "the same." Next, the teachers viewed a video example of a clinical interview. In the video, a sixth grader named Keisha was given a collection of geometric figures on a table and asked to complete the sorting task. Teachers were given a transcript of the interview on which they took notes and marked lines that indicated where the interviewer asked questions to elicit predictions, explanations, arguments, and justifications. Immediately following, the teachers watched the interview again with a focus on what the student may know about similar geometric figures. After the second viewing, teachers met in grade-level groups to summarize what they believe the student knew and why they believed so.

Classroom-based activity one. For the first classroom-based activity, teachers selected a student and conducted the Pirate Treasure clinical interview. Using an interview protocol provided for them, the teachers gave a collection of 24 pieces of “pirate treasure” to the student and asked him or her to help two pirates share the treasure fairly. After a series of questions, the student was asked to share among four pirates and then among six pirates. The teachers digitally recorded the clinical interview using a Flip-cam. After reviewing the recording, they analyzed the interview by answering the questions “What does the student know about fair sharing? Why do you think that?” After completing the analysis, the teachers posted their analyses and recorded clips illustrating their claims to the website. Next, they reviewed a colleague’s post from a different grade level or school and provided additional analyses.

The activity was designed to provide further base-line information on the teachers’ abilities to create models of students’ thinking. The DELTA interview protocol was developed to elicit behaviors and verbalizations significant to equipartitioning. Through iterative development, testing, and revising, the protocol was piloted prior to the study and found to be effective for uncovering evidence of thinking about equipartitioning. The activity was placed in the ERNR PD before the teachers’ introduction to the learning trajectory for equipartitioning in order to provide an informal comparison between teachers’ analyses prior to and after exposure. The Pirate Treasure protocol is included in Appendix E.

Session II. This session included the introduction of learning trajectories in general, equipartitioning and its location within the revised state standards and *Investigations*, the learning trajectory for equipartitioning concerning discrete collections, Problem-Centered Instruction (Confrey & Maloney, in press b), and developing strategies for adapting instructional tasks. After a debriefing discussion on conducting the Pirate Treasure clinical interview, I introduced the learning trajectories construct to the teachers using the DELTA research group's definition. Next, I oriented the teachers to equipartitioning by presenting the DELTA research group's definition, making clear distinctions between breaking and splitting, partitive and quotitive division, and outlining three of Piaget's seven conditions for understanding fractions applicable to equipartitioning (see page 16). These were referred to throughout the ERNR PD as the *three equipartitioning criteria*. From this point, we discussed the proposed changes to the state standards and located equipartitioning in the current and revised document. In anticipation of the learning trajectory, the grade-level groups discussed their clinical interviews where they attempted to identify behaviors and verbalizations that were more or less sophisticated than others and the beginnings of mathematical properties. Using the equipartitioning progress variable as a representation of the learning trajectory, I led the teachers through the first two levels which concerned equipartitioning a discrete collection using video examples of different behaviors collected by the teachers from their classroom-based activity. After this exposition, teachers reflected individually on their clinical interview analyses noting behaviors and verbalizations that they noticed or overlooked and whether their conclusions about their interviewee's understanding of fair sharing had changed. The remainder of the session concerned Problem-Centered

Instruction (Confrey & Maloney, in press b), the location of equipartitioning in *Investigations*, strategies for adapting mathematical tasks with a focus on altering a task's mathematical goal to permit other goals, and an example of a task from Gr. 1 *Investigations* that I had adapted. The session concluded with an overview of the second classroom-based activity.

Two learning activities from this session directly affected the teachers' progress in meeting the goals of the ERNR PD, the introduction to the learning trajectory for equipartitioning discrete collections and the task adaptation discussion. To introduce teachers to learning trajectories and equipartitioning, I gave a brief overview of each. First, I presented the DELTA research group's definition of learning trajectories, emphasizing the constructivist and socio-cultural influences, the role of instruction, and the idea of refining informal ideas into complex understandings. Next, I elaborated on equipartitioning as a process of making equal-sized groups and related this to fair sharing, a familiar context for the teachers. This elaboration focused on the differences between equipartitioning and breaking, its independence from counting, and its basis for rational number reasoning. The distinction between equipartitioning of discrete collections as partitive division and quotitive division was emphasized. In addition, three of Piaget's conditions for fraction understanding were presented as three separate cognitive processes that one must successfully coordinate in order to equipartition.

Rather than directly instructing the teachers about the learning trajectory for equipartitioning, I asked the teachers to anticipate what they believed a learning trajectory for equipartitioning might contain in order for them to anticipate and then ultimately reflect on their understandings. In grade-level groups, the teachers were guided by a series of questions based on the progress variable levels. After this anticipation, I distributed a copy of the two progress variable levels concerning discrete collections (see Appendix F) and described the different levels using video clips recorded by the teachers from their clinical interviews. After completing this, the teachers completed a reflection of their clinical interview in light of the learning trajectory. Both the anticipation and reflection component as well as the use of their own interviews was meant to allow teachers to actively build and refine their own understandings of students and equipartitioning.

Based on the framework and the study conjectures, I believed that the learning trajectory would influence instructional practices. Accordingly, the second portion of Session II addressed instructional tasks. First, I introduced teachers to Problem-Centered Instruction. In this model, students are presented with a compelling problematic that creates a sense of tension for them. Students act on this problematic to resolve this tension, practice those new actions to achieve fluency, and reflect on how the problematic was resolved. Essential to this model of instruction are instructional tasks. We discussed characteristics of such tasks; they are engaging, create the need for a new idea, have multiple points of entry, and represent significant mathematical ideas. Next, we discussed different techniques to adapt instructional tasks, including reversing or generalizing exercises and solving tasks in multiple ways. Special attention was given to changing the mathematical goal of a task as it was most useful in assisting teachers in including equipartitioning within their chosen curriculum.

I identified different units from *Investigations* for all three grade-levels that could be adapted to include a goal of equipartitioning a collection of discrete objects. As an example of a task adaptation, I selected an activity from the *Investigations* Gr.1, Unit 8 called *Roll Tens*. In this activity, students are to roll a die and count out the corresponding number of snap cubes, make towers of 10 cubes, and record their work as numerical sums creating 10. Once the tower was 10 cubes high, student were to begin a new tower. The teachers correctly identified that the mathematical goals of the activity were counting, building fluency with combinations of 10, and writing equivalent expressions for sums of 10. To demonstrate an adaptation for the teachers, I changed the goal of the activity for students to equipartition a given number of cubes and to write equivalent expressions for those decompositions. As the problematic, students would to be given 12 snap cubes and told to make towers of the same height and record the corresponding sums. The teachers discussed which numbers would and would not be appropriate for the number to equipartition. I concluded the example with a video of a second grade student working on the task. The student in the video successfully created two towers of six cubes and four towers of three cubes but did not create three towers of four cubes or six towers of two cubes. We discussed the primacy of the 2-split from the learning trajectory and related this to the students' creation of two towers and then four towers.

Classroom-based activity two. For the second classroom-based activity, teachers selected and analyzed a task from a selected unit of *Investigations* using the Problem-Centered Instruction framework and determine its mathematical goal. Next, they used the learning trajectory to provide a rationale for an adaptation of the task that included a goal of equipartitioning. As part of the rationale, they included two possible approaches a student may use. Digital pictures of the original and adapted tasks accompanied by their rationales were to be posted on the website. Next, they reviewed a colleague's post from a different grade level or school and identify elements of the learning trajectory they observed or provide additional approaches the students may take.

The activity was designed to help teachers identify tasks within their curriculum that could be adapted for equipartitioning in response to its increased emphasis in the revised state standards and to investigate their uses of the learning trajectory in this effort. Informed by a review of *Investigations*, the units selected included tasks that concerned number decomposition in hopes that teachers would adapt them to include or privilege a splitting decomposition as well as an additive decomposition.

Session III. The session began with a debriefing discussion about the task adaptations and the differences between partitive and quotitive division. Next, using paper folding and a revised progress variable chart, teachers explored the second part of the learning trajectory with video exemplars. Using the research of Black, Harrison, Lee, Marshall, & Wiliam (2004), I presented information about formative assessment and provided teachers with a set of guiding questions to use when analyzing students' work. Grade-level groups were given two samples of students' equipartitioning of a continuous whole to analyze with the guiding questions and the revised progress variable chart. The session concluded with an overview of the final classroom-based activity.

Two learning activities were selected for closer analysis from Session III because of their close relation to the goals of the ERNR PD. The first was the continuation of the learning trajectory to include the equipartitioning of a continuous whole. The second was the formative assessment discussion and the students' work analysis activity. Because the teachers had not conducted interviews on equipartitioning of a continuous whole as they had for collections prior to Session II, I chose to actively involve the teachers in the exploration of the trajectory using paper-folding. Each group of teachers was given a stack of paper rectangles and circles. As I progressed through the levels of the progress variable representing the learning trajectory, teachers attempted to fold into the targeted number of equal-sized regions. I used these as examples of different methods in addition to video exemplars originating from work conducted by the DELTA research group.

The framework and study conjectures suggested that the teachers would use the learning trajectory in assessing students' work. Beginning with the Council of Chief State School Officers (2008) definition of formative assessment, I presented the teachers with research findings outlining the benefits of formative assessment practices. Principles of effective formative assessment outlined by Black, Harrison, Lee, Marshall, & Wiliam (2004) were discussed with an emphasis on questioning and providing descriptive feedback for the student. Next, I shared guidelines for analyzing students' work adapted from Krebs (2005). These guiding questions encouraged teachers to describe what they believe the student knows, what the student does not know, what questions were raised by the work sample, what descriptive feedback they would provide the student, and what they would do next as the teacher. I adapted these guidelines in order to encourage teachers to move from simple evaluations of the students' knowledge to a more careful consideration of which components of their thinking would prove productive. As an example, I showed the teacher a digital picture of the outcome of a Kindergartener sharing a rectangular cake between two people and used the guiding questions to analyze the sample.

The remainder of the session was devoted to grade-level groups considering sets of hypothetical work samples. For each sample, the teachers were to use the guiding questions to analyze the samples and make a conjecture about what the student understands about equipartitioning. Some of the samples were created based on clinical interviews conducted by the DELTA research team. Others were specifically designed to draw attention to a student's focus on one or two of the three equipartitioning criteria while ignoring the other or to provide teachers with opportunities to notice the mathematical properties of composition, compensation, and equivalence. Teachers discussed these in their grade-level groups and shared their findings in a larger group discussion. The complete activity is included in Appendix H.

Classroom-based activity three. For the final classroom-based activity, teachers worked in pairs to conduct the Pirate Birthday clinical interview with one of the students who had participated in the Pirate Treasure interview earlier in the ERNR PD. Using an interview protocol, the pairs of teachers gave the student a paper or play dough rectangular "birthday cakes" and asked him or her to pretend they we sharing the cake between two pirates. After a series of questions, the student was asked to share among four pirates and then among three and six pirates. As one teacher conducted the interview, the partner teacher took notes. After the student had completed all of the rectangular birthday cakes, the teachers exchanged roles and repeated the protocol with circular cakes. The clinical interviews were digitally recorded. After reviewing the recording, each teacher analyzed the portion of the interview that they had observed by answering the questions "What does the student know about fair sharing? Why do you think that?" After completing the analysis, the teachers posted their analyses and recorded clips illustrating their claims to the website. Next, they reviewed their partner's post and provide an additional analysis making comparisons between their portion and their partners. Because of district-level mandated assessments, the teachers did not complete the activity before Session IV. The Pirate Birthday interview protocol is included in Appendix I.

The activity was designed for teachers to conduct a clinical interview on equipartitioning a continuous whole. The interview protocol was developed by the DELTA research team to elicit behaviors and verbalizations significant to equipartitioning. Through iterative development, testing, and revising, the protocol was piloted prior to the study and found to be effective for uncovering evidence of thinking about equipartitioning. In contrast with the first classroom-based activity, teachers had had experiences with the learning trajectory when they conducted the interviews.

Session IV. The session began with a review the different interpretations of division. Teachers worked in grade-level groups to classify a collection of problems taken from Ma's (1999) work as partitive and quotitive division. Next, teachers considered a set of student responses to tasks of equipartitioning a collection of continuous wholes in a sorting activity. Using the revised progress variable chart, the items from the sorting activity as anticipations, and the DELTA research group's video exemplars, I presented teachers with the remaining portion of the learning trajectory. After a brief presentation about student discourse and guidelines for its support by teachers adapted from Kazemi and Hintz (2007), the session concluded with the post-assessment.

The sorting activity was created to meet the goals of the ERNR PD, to provide teachers with experiences with tasks and students' work from the upper levels of the learning trajectory for equipartitioning, and was informed by the study conjectures. The activity consisted of a set of tasks of the equipartitioning of a collection of continuous wholes with hypothetical solutions. These hypothetical student responses to the tasks were based on empirical work of the DELTA research team and documented strategies from the literature (for an elaborated description of these strategies, see Confrey et al., in preparation a). These included: (1) the reduction of the task to dealing a discrete collection and then equipartitioning the remaining continuous whole(s), such as sharing three cookies between two people by dealing one cookie to each person and splitting the remaining cookie (e.g. figure 5.a); (2) the reduction of a task to multiple instances of sharing one continuous whole, such as sharing five graham crackers between two people by splitting each in half (e.g. figure 5.b); and (3) the use of landmark fractions in sharing, such as two pizzas among three friends by using halves and then splitting the remaining half into three parts (e.g. figure 5.c). Teachers were asked to examine the tasks and responses and to sort them into two like groups and explain their sorting criterion. They repeated this several times varying the number of categories and recorded their criteria. For example, the teachers may have grouped the examples in figures 5.a and 5.b together in one sort because they both involved sharing between two people. In a different sort, they may have classified them differently, placing them in separate groups based on the difference in approaches. The activity is included in Appendix J.

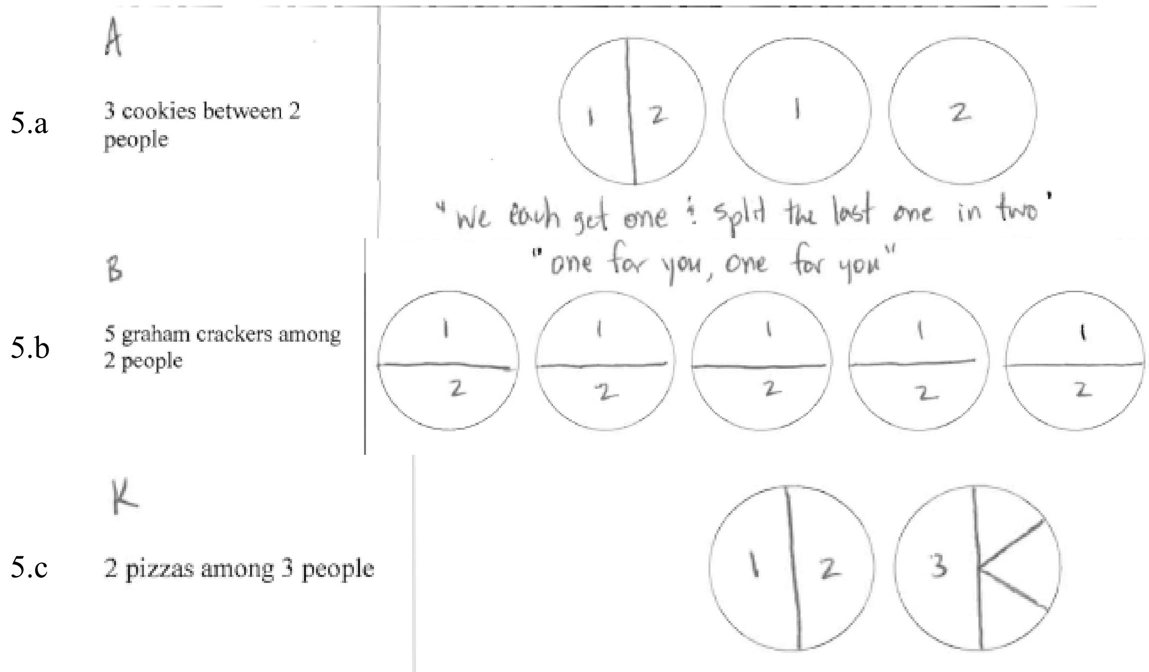


Figure 5. Sample tasks and solutions from the sorting activity.

The activity was designed for several purposes. Primarily, I wanted the teachers to consider the remainder of the learning trajectory by grounding it in a variety of methods that students use to approach these situations. Also, I believed that the set of tasks would provide teachers with the opportunity to evaluate the tasks for difficulty using the learning trajectory. Further, it provided teachers with an opportunity to continue to work on their analyses of students' work.

Session V. For this session, I designed four activities to assess how teachers used the learning trajectory to construct models of students' thinking, in assessing the relative difficult of instructional tasks, in analyzing students' work, and in interacting during instruction with students. Teachers were assigned to grade-level groups that blended the two participating schools. As groups, the teachers completed each of the three following tasks: the task-ranking activity, the student work analysis activity, and the classroom interactions activity.

Teachers were given 15 minutes for each activity. Next, they watched portions of a clinical interview conducted by the researcher. For each segment, the recording was paused, and teachers were asked to summarize individually the student's approach, what the student knew about fair sharing, and what they anticipated she would do on the next task. The session concluded with teachers completing a feedback questionnaire about their perceived uses of learning trajectories and its efficacy in practice.

To further test Conjecture Two concerning task adaptations, I created 11 equipartitioning tasks that spanned the entire learning trajectory. In addition, I included one quotitive division problem to see if teachers would identify it as different from the partitive problems. In groups, the teachers were to rank the tasks from least to most difficult and provide a rationale for their ranking. The task-ranking activity is included in Appendix M. To further test Conjecture Three concerning the analysis of students' work, I used two work samples from a clinical interview of a student sharing circular cakes among four and three people conducted by the DELTA research team. On each of the samples, the student had invented a notation for relating the one share among the total number of shares using an "=" sign. For example, to note one of three pieces, the student wrote "1 = 3." In groups, the teachers were to describe what they believed the student did and did not know about equipartitioning, what was unclear based on the sample, and what questions they would pose. Student work analysis activity is included in Appendix M. To further test Conjecture Four concerning classroom interactions, I created four hypothetical responses by groups of students to a task of folding a piece of paper in half three times in a lesson on equipartitioning. In groups, the teachers were to describe each approach and evaluate it as correct or incorrect. Next, they were to order the approaches from least to most sophisticated and provide a rationale. They were to suggest a sequence for sharing the ideas in a whole class discussion and list questions for each groups' approach to advance the class discussion. The classroom interactions activity is included in Appendix M. To further test Conjecture One concerning models of students' thinking, I selected four clips from a clinical interview using the Pirate Birthday protocol where the student shared a rectangular cake among three pirates, a rectangular cake among six pirates, a circular cake among three pirates, and a circular cake among six pirates. After each clip, the teachers summarized, in writing, the student's sharing and then anticipated what they would do for the next task. The interview analysis activity is included in Appendix M.

Methods of Data Collection and Analysis of Data

Data from the study included a teacher assessment of knowledge and uses of the learning trajectory for equipartitioning, a collection of student pilot items for equipartitioning, video and audio recordings of each session, classroom observations, and artifacts from the teachers' work with learning activities from the professional development. This section provides a description of collection and analysis for each of these sources of data.

Data Collection and Analysis of Teacher Assessment and Pilot Items Data

The teacher assessment was developed by the DELTA research team to assess respondents' content and pedagogical content knowledge of equipartitioning. Its construction was informed by the DELTA groups' synthesis on rational number reasoning (e.g. Empson & Turner, 2006; Lamon, 1996; Pothier, 1981; Pothier & Sawada, 1989) and their current research on equipartitioning. The assessment was piloted with four teachers who were either currently teaching elementary mathematics or had only recently left the classroom. This piloting process helped in the revision of the items and in anticipating the time needed to complete the assessment.

In scoring the assessments, rubrics were iteratively developed in order to assign responses to four different levels based on student responses from previous studies relating to equipartitioning and rational number reasoning (Empson & Turner, 2006; Lamon, 1996; Pothier, 1981; Pothier & Sawada, 1989) and the ongoing work of the DELTA research group, including responses from clinical interviews and theoretical work. Complete, incomplete, and incorrect explanations were identified within the responses. These were defined as those who are logical and complete, logical but incomplete or implicit, and described within the rubrics. In collaboration with another doctoral candidate who used the same assessment and rubrics in her investigation of how prospective teachers use learning trajectories in preparing for practice (Mojica, 2009), cycles of scoring and revision to the rubrics were conducted. During these cycles of revision, both sets of responses were scored concurrently. Responses that were not easily rated resulted in revisions to the rubrics. Mojica and I scored samples of items and compared results as we stabilized the rubrics. Once stable, the rubrics were used to assign scores to the tests. The teacher assessment and rubrics are included in Appendix A and B.

The student assessment was developed by the DELTA research team in consultation with their assessment partners, the Berkeley Evaluation and Assessment Research (BEAR) team, to assess respondents' knowledge of equipartitioning. The items were written to measure students' progress in the learning trajectory and served as pilot items from the group's larger diagnostic assessment work. They were informed by the DELTA group's synthesis and current research on equipartitioning.

Based on students' responses and classroom observations, the items related to equipartitioning a continuous whole were selected for scoring. Rubrics were developed to assign responses based on: 1) analysis and preliminary notes of the DELTA research group related to the specific items; 2) responses from previous studies relating to equipartitioning and rational number reasoning; and 3) the ongoing synthesis and research work of the DELTA research group. Cycles of scoring and revision to the rubrics were conducted, with responses not easily rated causing revisions to the rubrics. Once stabilized, the rubrics were used to assign scores to the responses. The student pilot items and rubrics are included in appendix C and D.

Composite and gain scores for both the teacher assessment and the student items were created. Exploratory data analysis techniques and Shapiro-Wilk tests were used to investigate the samples. These investigations along with the ordinal nature of the scores indicated that distribution-free methods of analysis were more appropriate than parametric methods due to the unlikely assumption that the samples were from a population that was normally distributed. Wilcoxon Sign Rank tests, Kruskal-Wallace tests, and Spearman's rho measures were used to investigate hypotheses of significant gains, differences in scores among teachers' students, and the relationship between teachers' knowledge and their students' gains, respectively.

Data Collection and Analysis of Video, Audio, and Classroom Observation Data

Each session of the ERNR PD and the classroom observations were videotaped. The video data was processed using Powell, Francisco, & Maher's (2003) model for videotape analysis. Critical events, defined to be instances in which teachers were engaged in activities related to the framework of the study, were identified and transcribed. Audio recordings of small group discussions surrounding the learning activities during the sessions and interviews with the teachers following classroom observations were audio recorded. These audio recordings were transcribed.

All transcribed video and audio data were initially coded using the study's conceptual framework for the ongoing analysis. Subsequently for the retrospective analysis, all data for each primary code was reviewed. Open coding and the constant comparison method (Glaser, 1992, 1967) suggested themes that were confirmed or refuted through further analysis and triangulation. For example, a portion of one participant's Pirate Birthday clinical interview analysis was initially coded as *inferring* because she was using the student's behaviors to infer what the student may have been thinking: "In observing Sam dividing the round cake between three pirates, it is obvious that she is only interested in the number of pieces. She was not concerned at all about each pirate having an equal size." Later, when all data coded as inferring were reviewed, open coding was used to identify patterns within each. For the above example, codes of *three criteria* and *discriminating* were assigned. A complete list of codes with definitions is included in Appendix N.

Data Collection and Analysis of Other Artifacts

Other artifacts from the ERNR PD sessions that served as data include written responses to various learning activities and work on the classroom-based activities. These responses were collected and coded initially with the conceptual framework. As with the transcriptions, secondary coding and constant comparison were used to identify patterns within these data. Additionally, two of the classroom-based activities involved teachers conducting clinical interviews with students. In these cases, I viewed the interview clips several times, creating a description of the students' behaviors and verbalizations and noting my analysis of the interview. These summaries served to support the coding and analysis of the teachers' interview analyses.

Validity and Reliability

Validity in the teacher instrument was ensured by two methods. First, the items were piloted with a sample of teachers similar to those who took the assessment. In terms of content validity, the items were based on items from the DELTA group's synthesis work. Additionally, Confrey and members of her research team brought expertise in the area of equipartitioning to the creation of the assessment. Validity for the student items was derived from the expertise of the DELTA and BEAR research groups. Interrater reliability of scoring was established with two separate mathematics education researchers independent of the research group to address the reliability of the both the teacher and student scoring. The reliability and validity of the other data sources were ensured through the collection of multiple sources of data, triangulation, and by searching for discrepant cases (Maxwell, 2005).

In this chapter, I provided the theoretical foundations for the study and elaborated the conceptual framework used to guide its development and ultimately to situate its findings within the greater research literature. The selection and justification of the methodology was presented, as well as a description of the study's design, data sources, and methods of analysis. Chapter Four will present findings from the ongoing analysis of the intervention.

CHAPTER FOUR

Design studies entail two levels of analysis: an ongoing analysis during the implementation of the intervention and a retrospective analysis (Cobb et al., 2003). Accordingly, I report the findings from the design study in two chapters. In the first part of Chapter Four, I detail the implementation of the professional development and accompanying analyses of the sessions and learning activities that affected the evolution of the conjectures. Chapter Five details findings from a retrospective analysis of the study. In Chapter Six, I report findings from the observational study embedded within the design study. Together, these chapters provide foundations for an explanation of the process by which teachers came to know and use a learning trajectory for equipartitioning, the means that supported that learning, and effects on classroom instruction discussed in Chapter Seven.

Examples included in Chapters Four, Five, and Six are selected to exemplify findings from the analysis. They are representative of the cadre of participating teachers, though some were more articulate and more clearly illustrated the essence of the ideas. Because of the number of teachers in the study, codes for each teacher were used (e.g. W13) rather than pseudonyms to identify each teacher.

Implementation of the ERNR PD

The design of the ERNR PD was influenced by the conjectures driving the study as discussed in Chapter Three. As the sessions were conducted, the teachers' work on the activities was reviewed in light of these conjectures. In many cases, these observations required an adjustment to the original plan for the sessions and ultimately the revision of the conjectures. The conjectures were based on a review of the literature, the expertise of the

DELTA research group, and my professional experiences. For reference, these conjectures are listed below:

> *Conjecture One:* The use of a learning trajectory for equipartitioning will assist teachers in building models of students' thinking by highlighting facets of students' behaviors and language and relating those to a particular conception. Whereas initially teachers will not notice distinctions among students' activities and language, the learning trajectory will act as a lens for teachers, accentuating language and actions that are indications of thinking while filtering irrelevant work or words.
> *Conjecture Two:* The use of a learning trajectory for equipartitioning will inform the adaptation of instructional tasks by helping them adapt tasks through altering the mathematical goals and assist them in judging the relative difficulty of tasks.
> *Conjecture Three:* The use of a learning trajectory for equipartitioning will inform teachers' formative analysis of students' work on instructional tasks by helping them identify evidence of students' thinking, conjecture about what students may or may not know, and raise questions that inform teachers' next instructional steps.
> *Conjecture Four:* The use of a learning trajectory for equipartitioning will inform interactions with students during instruction by assisting teachers in identifying students' ideas that will support their mathematical goals, by influencing teachers' questions, and in sequencing students' work during whole group discussions.
> *Conjecture Five:* Teachers' knowledge and uses of a learning trajectory for equipartitioning will be positively associated with their students' learning.

The following sections discuss chronologically an ongoing analysis affecting the implementation of the remainder of the ERNR PD and the study. For each session, I identify the learning activities and sources of data affecting the evolution of the design of the sessions. I analyze these data and draw conclusions linked to the conjectures of the study and describe how these affected the remainder of the ERNR PD.

Session I

Session I focused the practice of clinical interviewing. The similarity clinical interview activity described in Chapter Three was selected as most relevant to the goals of the ERNR PD. Thus, audio recordings served as data from this activity as well as the video recording of the complete session. After Session I, digital audio recordings and video recordings of the grade-level and large group discussions of the $n = 33$ participating teachers were viewed and used to determine the effectiveness of the activity in meeting the goals for the session and to reexamine the conjecture about teachers' models of students' thinking (Conjecture One) in light of the outcomes of the first session. Using the study's framework and methods of analysis detailed in Chapter Three, I noted several commonalities across the groups for the participants' descriptions of Keisha's actions and language, their interest in her grade, their evaluations of her understanding, and their own mathematical knowledge associated with similar geometric figures. Keisha's work on the task is discussed in Chapter Three.

In their descriptions, many of the groups noticed Keisha's nonmathematical language when describing shapes and the ways in which she made connections to her environment, such as when she related rectangles to stereos and doors or triangles to Christmas trees or ice cream cones. Some believed this was inventive and had positive reactions to Keisha's assignment of names and connections, such as calling obtuse triangles "kooky houses" and isosceles triangles "ramps". Other groups questioned rather the use of informal language and environmental comparisons were appropriate for students in upper elementary school. One teacher who believed that "kooky houses" were right triangles stated, "Well, I mean, she didn't know like what a right triangle was. I feel like she should have had that, she should have had that knowledge, don't you think? I mean… by that time?" Another group expressed this concern by stating, "she should have had that knowledge [appropriate vocabulary] by that time." Though some groups stated that relating the geometric figures to concrete examples was desirable, other groups insisted that Keisha should be "at a more abstract level" because of her age.

Also when describing the ways in which Keisha sorted the shapes, the teachers tended to focus only on a basic sorting and were concerned that Keisha did not initially create groups of four basic shapes: triangles, rectangles, squares, and circles. Three of the six groups mentioned that she "focused on angles" and one group noted that she rotated all of the shapes to the same orientation to make comparisons. However, none of the groups of teachers noticed that Keisha used the grain of the wooden table as a reference by which she differentiated acute, right, and obtuse triangles. Also, all but one group failed to notice that Keisha overlaid and tiled smaller rectangles on larger ones. This group commented that "she was able to manipulate the shapes to make other shapes, and had some understanding of how to do that." From this comment, it is unclear whether they were suggesting that Keisha was decomposing larger rectangles into smaller ones that could be iterated to compose the larger rectangle or if they believed that she was using this as "growing" argument for similarity.

Within the groups' discussions of language and actions, there were many comparisons of Keisha to a hypothetical student from the participants' grade. At the introduction of the activity, the teachers were told that Keisha was in sixth grade, but the majority of the groups asked about this again when they began their analysis. A comparison between what the teachers noticed in Keisha's interview and what they believed was appropriate knowledge given her age was a prevalent factor in their discussions. For instance, extending the analysis above about manipulating shapes, this group of teachers said:

> And, she showed through that that she had some fractional relationships, but for a sixth grader, they probably weren't as advanced as they should have been. It is interesting because she made that very elementary comparison of these are squares, these are rectangles, but she never brought them together and said a square is a special rectangle. That was kind of a kindergarten, first grade assessment of squares and rectangles.

Accompanying many of these comparisons were broad evaluations of Keisha's understanding about similarity. Some teachers made positive claims such as "I think she knows a lot about similarity because she can explain them [her groupings of shapes] in more than one way" or "she knows her shapes." Others, however, focused on deficits in what they believed Keisha should know. For example, one teacher commented, "We think that she really only knows basic things. She is not a real in-depth thinker. She compared the sizes and the shapes, but not much beyond that. Her vocabulary wasn't really there to be able to explain beyond that point."

Another factor that affected what the participating teachers noticed, and consequently their models of Keisha's understanding, was their own understandings of similar geometric figures. Though the teachers had explored the task, their understandings did not support a full exploration of similarity. Whereas all of the groups focused on attributes of shapes, including angles and number of sides, only one teacher out of all six groups raised the question of congruency versus similarity of the figures; this issue was not adopted by the group for further discussion. One particular group illustrated an underdeveloped notion of geometric similarity which affected their model of Keisha's understanding. A teacher from the group stated, "I think that maybe she was trying to make it harder than what it was. Maybe she thought it couldn't be something as simple as putting them into four groups." From this, I inferred that the teacher believed that a basic categorization of rectangles, squares, triangles, and circles sufficiently addressed the task of creating groups that were "the same," rather than further exploring the different similar figures within these groupings. Later in their discussion, they stated:

> W17: She kind of went wishy-washy, because she said they were different if they were different sizes, and then she [the interviewer] went back and she said that, 'uhm, and this one, they are all Christmas trees?' And then she [Keisha] was like, 'yeah, they are just smaller.' So sometimes she thought they were the same if they were different sizes and sometimes she didn't. So she is kind of inconsistent.
> W15: Yeah, she didn't have like a solid, 'this makes this this.' She kind of -
> W5: She didn't refer to size too much either as far as just bigger and smaller. Other than when she talked about the rectangles and making them -
> W1: It was almost like she wanted to make them the same, like even though they were rectangles she had to make them the same size to make them the same.

I believe that the essential idea Keisha described in the clinical interview is that, though similar geometric figures are "different" in terms of size, they are the "same" in that relationships between angles and sides are preserved. The fact that Keisha "went wishy-washy" indicated that she had some understanding of this key idea, yet the teachers in this group believed that Keisha made "it harder than it was."

From these initial observations, I concluded that participating teachers may have had some skill in listening to students' language but make evaluations rather than use these observations as a means of making sense of the students' understandings. Many of their comments focused on what Keisha did not know or could not do. Also, the participants noticed some actions taken by Keisha as she worked to share her understanding of geometric figures that were "the same", though two of Keisha's strategies remained virtually unnoticed.

Moreover, the teachers did not link these actions to potential underlying cognitive meanings, such as linking attention and comparison of angles to the use of the table as a standard of reference. In addition, the teachers tended to make evaluate Keisha's understandings based on both their knowledge of students at different grade levels and in terms of their own knowledge of similarity. These conclusions indicated that subsequent learning activities in the sessions should include a focus on ways to assist teachers in parsing their evaluations into what students may know and not know rather than broad judgments that focused predominately on what students did not know or did not do. Also, these activities should assist teachers in connecting their observations of students' actions and verbalizations to cognition.

Classroom-Based Activity One

The first classroom-based activity, teachers were to select a student and conduct the Pirate Treasure clinical interview. Posts on the website, including written analyses, illustrative video clips, and additional comments and analyses from other teachers were reviewed prior to Session II. Twenty-nine of the participating teachers completed the activity. Similar to the similarity interview activity, I focused this review on teachers' skill in conducting clinical interviews and in their construction of models of their interviewee's thinking using the framework as a guide. A secondary purpose was to select video clips exemplifying behaviors from the learning trajectory to be used in introducing the teachers to it during Session II.

In these data, I observed more examples of teachers noticing some behaviors and verbalizations while overlooking others. Some teachers noticed that students used dealing as a strategy for creating equal-sized groups. One teacher wrote, “I can tell by the way she separated the coins that she had fair-shared things before. I know this because she takes a group of coins and passes one to each pirate until her pile is gone.” Other teachers commented on students’ abilities to verbalize and name the outcomes of their activity, such as “she never did say what she could call a pirate’s share of the coins.” Some would comment on behaviors but were unaware of their importance in indicating the interviewee’s thinking. For instance, one student shared coins between two pirates by dealing simultaneously which may indicate that he was coordinating two actions at one time. However, the teacher only noted, “He counted by ones to organize and separate the treasure in two piles.” In another instance, a student created visual patterns to assist in justifying the results of equipartitioning. Rather than commenting that the student was using these patterns to recognize the equivalence of each share, the teacher did not comment on this, stating, “[the student] knew that when shared with four pirates, each pirate would be given six pieces of gold. Referred to the gold at one point that there were 3 pieces of gold on each side. Each pirate had two sides.” The two sides refer to one column of a 3 X 2 array that was created for each of the four shares of treasure. Still another teacher did not attend to her student’s use of arrays when sharing among four pirates. The student created two 6 X 2 arrays of coins when sharing between two pirates. When asked to share among four pirates, he created the same two arrays but left a larger space between rows three and four and commented that he “knew that 12 has six in it, and so I put six and then six [touching the smaller 3 X 2 arrays], and then six and six because I had two 12’s.” Yet, the teacher did not notice this composition of factors in her analysis, i.e. splitting each group of 12 coins in half rather than splitting the 24 coins into four groups. She wrote: “He was very much at ease during his entire interview. In fact some of the questions I was to ask him did not get asked because he would answer them as a tag to the previous questions. He did a very good job with the project.”

Another observation supporting my assertions from the Session I data was that teachers did not relate behaviors and verbalizations to what the student may have been thinking. In one of the participating teacher’s interviews, the student formed three lines of coins when sharing among three pirates containing seven, nine, and eight coins, respectively. The coins were not aligned with equal spaces between them, so the student may have been justifying the outcome using an early measurement strategy of attending to the alignment of endpoints. When she counted to justify, she realized that the second row had two coins more than the first, pulled them aside, counted the third row, and pulled one coin aside, creating three rows of seven coins with three remaining. When asked how many each pirate had, the student quickly pulled the remaining coins from the table, smiled, and said, “Seven.” The teacher provided the following analysis:

> It was interesting to see what Sam did with the three chips she wasn’t sure about. It seemed that she was looking at the groups as lines or columns. It looked like the middle column was shorter because they were closer together. I wonder if that is what gave her some difficulty in knowing where to put the three chips. She figured it out in the end. She seems to know what fair shares are.

From the student’s actions, I infer that she does not understand that fair sharing requires exhausting the original whole but does understand that the goal is to create equal-sized groups. From the analysis, the teacher noticed behaviors that indicate these issues, but did not connect them and ultimately declared that “she seems to know what fair shares are.”

Also in accordance with previous observations, some of the teachers made broad evaluations of the students' abilities to share fairly with little or no evidence mentioned. For instance, one teacher wrote, "This kindergarten student understands that fair sharing means to separate objects. She [doesn't] know how to share equally. She struggled to listen to directions and didn't answer any of my questions." Another teacher stated, "Nautica knew what fair share meant. She had a good understanding of doubles and 'groups of'. She has a good concept of number sense. She had no problem with the activity even when she shared the treasure with 3 and 4 pirates."

These observations supported the findings from Session I data. Participating teachers had some initial abilities to notice behaviors and verbalizations of students that may be related to their understanding of particular mathematical concept, such as Sam's teacher's observation about lining up the coins to judge the fairness of her sharing. However, they had some difficulties in relating these to cognitive behaviors. In the case of Sam, her alignment suggested that she was using an early measurement idea of comparing endpoints to justify equality, not a one-to-one correspondence. Moreover, they may have overlooked or ignored the importance of some actions that could provide a better window into students' understanding, such as Sam's initial acceptance of not using all of the coins from the collection in her sharing. Finally, the teachers tended to holistically evaluate the students' understanding of a particular concept rather than identify components of their thinking that may be correct or productive in formulating a correct conception.

Session II

Prior to providing Session II, I reconsidered the conjectures about models of students' thinking (Conjecture One) and how teachers may use a learning trajectory to adapt instructional tasks (Conjecture Two) in light of three issues that arose from the initial analyses of Session I and the first classroom-based activity. First, the teachers were noticing some behaviors and verbalizations important to equipartitioning but were missing others. They needed ways to link those actions and words to particular cognitive behaviors. Finally, for the anticipated benefits of learning trajectories to be realized, the teachers would need to move from making holistic evaluations of a student's understanding to parsing students work on equipartitioning tasks into component parts, some of which may or may not ultimately strengthen the students' conception of equipartitioning. Based on the conjectures, I believed that the introduction of the learning trajectory would address each of these issues by highlighting important behaviors and verbalizations, linking those with cognition, and equipping teachers with knowledge to dissect students' work for a clearer understanding of what is known and not known about equipartitioning.

Audio recordings of grade-level group discussions anticipating the learning trajectory, video recordings of the learning trajectory and task adaptation presentations, and individual written reflections of the Pirate Treasure interview of the n = 32 participating teachers attending Session II served as data for the initial analysis. Each source was reviewed using the framework and the conjectures relevant to this session.

Many of the observations from Session I were confirmed by teachers' discussion when anticipating the learning trajectory. Some teachers noticed key behaviors such as dealing and the use of arrays to structure the outcomes of the sharing but failed to make connections between these and the student's thinking. New in this discussion was one teacher's observation that sharing between two pirates and among four pirates seemed to be easier for students than sharing among three. She reported this example from her Pirate Treasure clinical interview to her grade-level group:

> S14: I thought that was interesting, like you said, what I said before when I gave him – my kid, with the two groups, he immediately used that higher counting strategy of starting to pull groups and counting by twos. But, then when the number got bigger, he reverted back to one at a time, when I got to the fours. But, then when I backed up to three – I mean, you watch it on the video, he was going 24 and he wasn't talking out loud but, his mouth was moving. He was trying to expand on his number facts of 24 to figure out which multiple worked.
>
> S12: It's harder for them to divide by three than it is by four - because four is still halves, and they get that it's half and then half again.

The teacher noticed how sharing between two seemed easier than sharing among four because the student dealt with composite units, which may have been a result of the student seeing how the composite unit of two would be first applicable for sharing by two and then not being sure of its use of four. When sharing among three, he may have been questioning whether it would have allowed him to end evenly. Also, the teacher deduced that the student was using number facts and that "dividing" among three was more difficult than by fours.

In addition, they continued to note different verbalizations made by students. In particular, they noticed different names for shares offered by the students. However, the teachers did not link these different names to the ways that students may have been thinking about the results of their sharing. For example, a discussion from a group of Kindergarten teachers focused on naming the results of equipartitioning a collection from the Pirate Treasure interviews:

S4: I don't think any of our kindergarteners could say, "a half," or "a third," or "a fourth," I know mine didn't.
All agreeing
S5: Not even the second grader that I interviewed came up with that.
S4: She said – I don't think she even came up with the right number, she kept saying "Twelve. This one's got twelve."
S1: Mm-hmm.
S4: "That's the same."
S6: Mine didn't –
S5: She said, "same," that's good.
S1: Mine didn't count them and she didn't get the right number. Well, she did count, but she counted something different, like –
S2: Mine was just excited it was treasure. Like, "it's treasure." Like, it doesn't matter, it's treasure.
…
S5: Mine kept saying – because every time you would divide in two and three, she'd come up with something. One was "jewelry," and the next was "money," and the next was something else that was like, a commercial thing.

Though teachers noted nonmathematical names, counts, and anticipated fractional names, they did not use these as evidence from which they inferred about students' understanding of equipartition. Other groups of teachers, however, did imply connections between students' actions and words and what it may indicate about their thinking. One group discussed a student who did not exhaust the whole collection:

S11: Kerr didn't have a real concise way of dividing it. It was kind of –
S10: What did he - did he know it?
S11: Well, he just had them and – at first, it looked like he was going to do two, and two, and two and then he kind of didn't. I think the first time, he ended up with thirteen in one pile and he didn't seem to know that– I don't know, it was kind of strange. But then, when he got to the threes, he divided into groups of seven and threw the three in the sea.
S10: So, after he did that, did he come back and put the three?
S11: No, he threw them in the sea.

S9: That was it? Okay, so they were gone.
S10: They stayed in the sea?
S11: Mm-hmm. They stayed in the sea.

Implicit in their conversation was that the student was satisfied with groups of the same number of coins even though he had not successfully coordinated the need to exhaust the original collection. These observations supported the conclusions from Session I that teachers noticed some behaviors and verbalizations but did not explicitly state how these related to underlying cognition. Though in a few cases they used these implicitly as evidence from which to infer about students' thinking, most teachers continued to describe actions and words of students and make broad judgments that students could or could not share fairly rather than link those behaviors and verbalizations to the three equipartitioning criteria.

During the large group discussion of the learning trajectory, a variety of methods, justifications, and naming strategies were presented with the progress variable representation[2] and discussed in relation to the research literature. From the video recordings of the large group presentation of the learning trajectory, I observed that the teachers noticed both commonalities and differences among their interviewees and those of other teachers. For instance, I used the video of Sam measuring and not exhausting the whole collection (described in the initial analysis of classroom-based activity one) as an example in the presentation. Kerr's teacher immediately commented that he did something similar and "threw them in the sea." Though most of the teachers' interviews did not feature this, "throw them in the sea" became an informal way for this group of teachers to refer to a students'

[2] This was an early and longer version of the progress variable chart than is currently being used by the DELTA research group.

failure to exhaust the whole. As the levels of the progress variable were presented, connections between behaviors and verbalizations and student cognition were made in the discussion. However, few teachers made notes of these connections on the progress variable chart.

A review of the teachers' written reflections about their interviews after they had studied the learning trajectory revealed three themes. First, most teachers had a deficit perspective (Valencia, 1997), that is, they focused on deficiencies in the students' activity and knowledge when it came to discussing what their students knew. Many highlighted the behaviors and words that the students did not say, specifically with respect to naming the results of the equipartitioning. In response to the question asking "what behaviors and words did you notice in your interviews that are a part of the learning trajectory? What words or behaviors did you not notice?" one teacher wrote, "She could not complete the task. She did not understand fair share and she did not use words half and equal. Jennifer was not able to split pirate's treasure correctly." Also, teachers listed many of the behaviors from the progress variable and then commented that their student did or did not use them. For instance in response to the same question, another teacher wrote,

> 1.1.1 She dealt one object at a time until all dealt. She used a general name, calling the share coins. She justified the groups by counting. She did not use a relational number. She knows how to create fair shares. She knows how to justify. She also made arrays as she was checking.

In this example, the teacher indicated the number level from the progress variable, further leading me to believe that teachers were not using the learning trajectory as a way to understand students thinking but as a checklist of behaviors or as a list of behaviors that students must exhibit. Rather than using the progress variable levels as descriptions of students' strategies and as a framework for interpreting those observations in relation to cognition, they simply identified different progress variable levels and commented on the ways the students met or failed to meet those levels. This use may have prevented them from recognizing the subtleties of their behaviors and verbalizations and possible interactions among strategies, justifications, and naming. Finally, many of the second grade teachers related the students' work from the interviews to their curriculum. At the time of the interviews, the second grade classrooms were studying a unit from *Investigations* that focused on building addition fluency using number facts. Several of the teachers commented that their student used his or her knowledge of doubles facts to share between two pirates and then among four. One teacher noticed this and related it to the property of composition; "She used the composition property because she split a split (used doubles facts)."

Though there was little dialogue among the teachers in the video recordings of the session, I noticed that the teachers were growing increasingly frustrated in coordinating the multiple pages of the progress variable chart handout and their handouts on Problem-Centered Instruction. The combination of the level of detail of the chart, the nonstandard way in which it is read, and the additional handout frustrated many of them to the point of simply disengaging in the presentation.

From these initial observations, I concluded that video exemplars were an effective way for teachers to understand parts of the learning trajectory, evidenced by the ways they referred to them during their discussions and analyses. Also, though useful as a reference, the progress variable chart was too cumbersome for teachers to use efficiently in its current form.

Furthermore, the chart did not explicitly relate behaviors and verbalizations to cognition and in some sense was promoting checklist mentality, which may have contributed to the teachers overall deficit perspective. Lastly, some teachers demonstrated that curricular experiences were affecting their analyses of students' thinking and may be interacting with their own formulation of knowledge of the learning trajectory, particularly the use of doubles facts.

Classroom-Based Activity Two

For the second classroom-based activity, teachers were to select, analyze, and adapt a task from a selected unit of *Investigations.* Posts on the website, including written analyses, digital pictures, and additional comments and analyses from other teachers of the $n = 24$ participating teachers who completed the activity were reviewed prior to Session III. Using the framework and the conjecture concerning teachers' adaptations (Conjecture Two), I focused this review on the teachers' use the learning trajectory in the task adaptation. I observed two related issues in this review: the teachers' use of the learning trajectory and how the trajectory interacts with *Investigations*.

First, some of the teachers viewed equipartitioning a discrete collection simply as the reverse of multiplication. Despite the emphasis on the differences between the partitive and quotitive interpretations of division, 10 of the 24 teachers who completed the activity chose tasks related to lessons from *Investigations* that either build multiplication through repeated addition or focused on efficient addition by grouping. For example, one teacher adapted a task that depicted four hands and asked for how many fingers were present to asking how many hands there would be if there were 20 fingers. This task is not an equipartitioning task because the size of the group is known (five fingers) and the goal is to find the number of groups of that size (four hands). Other teachers used similar problems with variations of eyes, legs, cat paws, and wheels on cars. Though eleven teachers did create equipartitioning problems including contexts such as sharing apples in bags or goldfish in bowls, three of the twenty-four teachers created tasks that were neither quotitive nor partitive in nature. One of them began with a problem of combining three red apples and five yellow apples represented by red and yellow counters. Her adaptation used those same counters and asked, "You have 8 apples. Six are red. How many are yellow?"

Also related to their use of the learning trajectory are the ways that they used the progress variable. Some of the teachers used the numbered levels of the progress variable much like curricular objectives, such as "I anticipate that students will use reversibility (1.1.6) and justification (1.1.4) as ways to approach this problem" or "I think they would use reversibility (1.1.6) to solve this problem." One teacher in particular called them objectives and listed them as a part of her rationale: "Adapted Task: Learning trajectory objectives: 1.1.6 Reversibility, 1.2.3 Multiple Methods, 1.2.4 Justification. I am hoping the students will use their background knowledge of how many fingers or toes a person has." These uses of the progress variable imply that teachers may have believed that attention to the levels would cause students to progress, rather than using the levels as descriptions of possible observations indicative of a particular way of thinking. That is, the teachers may have interpreted the progress variable chart as prescriptive rather than descriptive, its intended function.

Another group of observations concerned the interaction between *Investigations* and the learning trajectory. In this activity, five of the teachers created tasks with a secondary goal of determining if a number was even or odd. This was surprising, but a closer look at the *Investigations* unit revealed a possible explanation. The materials develop the notion of even numbers using an analogy of partners and teams. The number of a group of students is called "even" if two teams of the same number can be made without anyone being left out or if everyone in the group can find a partner. The teachers were using the idea of sharing between two with the idea of evenness in their tasks. Another observation was that some teachers may have used their understanding of equipartitioning and the learning trajectory to select numbers for the size of the collection to be shared. Fifteen of the twenty-four who completed the activity used a number that would allow for a composition of factors. The most popular choice was 12, followed by 16. However, seven of the twenty-four used numbers that are important in building an understanding of the base-ten number system (10, 20, and 100) which is a large focus in K-2 *Investigations*. The materials emphasize heavily across all grades additive combinations that make 10 and the development of place value understanding. In second grade, there is an emphasis on using groupings of twos, fives, and tens to simplify addition expressions. One teacher wrote in her rationale, "I adapted the task in several ways. First, I changed the number of balls that Emma found to equal 10. I used 10 because this is such an important base number in kindergarten and can be shared equally by two students."

Three conclusions drawn from this analysis were considered in the subsequent planning of the study and revisions of Conjectures One and Two. First, some of the teachers were unclear about what equipartitioning was and how it was different from division. Though the distinction between partitive and quotitive division had been made, some teachers still interpreted equipartitioning of a discrete collection as the inverse of multiplication. Second, teachers were using the progress variable chart in ways that were unintended. The use of the level numbers as objectives indicated that the teachers may be interpreting them as a collection of behaviors that students must accomplish before mastering the concept rather than a means of interpreting observations in relation to cognition. Lastly, the confluence of the learning trajectory for equipartitioning and the teacher's experiences with *Investigations* was significant. The curriculum's focus on additive combinations of 10 and place value development competed with an understanding of equipartitioning which affected their adaptations. Though this was somewhat anticipated, the degree to which the curriculum influenced the teachers' adaptations suggested they may need more assistance in understanding how the learning trajectory deviates from or parallels with *Investigations*.

Session III

In preparing for Session III, I revisited the conjecture concerning the analysis of students' work (Conjecture Three) and my plans for subsequent sessions in light of issues that arose from the initial analyses of Sessions I and II and both classroom-based activities. The teachers' uses of the progress variable chart as a checklist or as a list of objectives that students needed to meet were a concern. For the learning trajectory to assist teachers in the ways conjectured, the progress variable chart needed to become more descriptive of cognition for the teachers. Further, the means by which I had helped teachers connect behaviors and verbalizations to cognition needed to be strengthened. Teachers needed more explicit connections between what students were doing and saying and what those may indicate about their thinking.

Video recordings of the learning trajectory and formative assessment discussions, audio recordings of grade-level group discussions, and groups' written notes of the work samples of the n = 29 participating teachers in attendance served as data for the initial analysis of Session III. My preliminary analysis regarding the outcomes of the session was guided by the framework of the study and the conjecture about students' work analysis (Conjecture Three).

During the previous sessions, teachers were familiar with the video clips as they had collected them and viewed them on the blog. Also between Sessions II and III, I altered the format of the progress variable chart in efforts to assist teachers in linking behaviors, verbalizations, and cognition (see appendix G for a sample of the altered progress variable chart). Therefore, I viewed the video recordings of the learning trajectory portion of the session with special attention to the teachers' engagement. During the sessions, many of the teachers were continually folding paper circles and rectangles, first in efforts to complete the task themselves, but then subsequently to anticipate of how students may approach the task or to mimic the outcomes depicted in the video exemplars. This exercise was useful for having the teachers to confront their own misunderstandings about the multiplicative nature of paper-folding. For example, when discussing six-splits on a circle, one teacher folded a circular piece of paper in half three times and did not realize that she had actually created eight parts instead of six. Another teacher in the group demonstrated how she had folded a circular piece of paper to create six parts by folding in half and then folding the resulting semicircle into thirds. The first teacher questioned, "I just did it like this [demonstrating]. Why can't you just do it that way?" As she was demonstrating her method, she unfolded and counted to see that she had, in fact, created eight parts rather than six. At the other school, teachers at the first grade table worked at finding many different ways that students might approach the same problem. After folding a circular piece of paper into six equal-sized parts, they produced an example with five parallel folds as well as an example where the paper was folded in half and then thirds similar to how one would fold to create 6 equal-sized parts on a rectangle. The variety of examples supported a rich discussion about differences in shape and connections to strategies for creating two, four, and three equal-sized parts for circles and rectangles. During the discussions that followed each video example, I observed many of the teachers actively made notes on the revised progress variable chart. They recorded descriptions of the students' behaviors or verbalizations and then, as a group, we discussed what those might indicate about the students' thinking. As opposed to the previous session, the teachers remained actively engaged throughout the activity.

An analysis of the student work activity provided evidence of progress in many of the ERNR PD goals. Specifically, teachers began to make hypotheses about the students' thinking processes based on their work, formulate questions to pose to students related to those hypotheses, and plan subsequent tasks in response to the students' work. Also, there was evidence that the teachers were becoming more sensitive towards behaviors from the learning trajectory and more instances of how *Investigations* was interacting with the teachers' analyses. Finally, this analysis provided the chance to observe if teachers noticed the mathematical properties associated with the learning trajectory. To assist in reporting my analysis, table 4 contains the different work samples to which the teachers were referring in their dialogue.

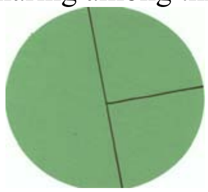
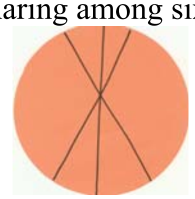
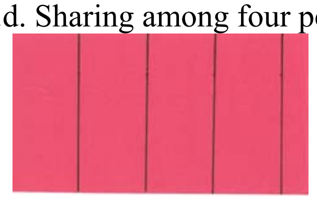
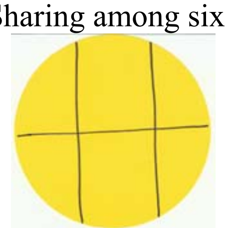
	Sample	Intended Features
K	Figure 6.a. Sharing among three people 	Making cuts with a focus on creating three pieces rather than creating equal-sized pieces
K	Figure 6.b. Sharing among four people 	Making four horizontal cuts resulting in five pieces and then orthogonally making three cuts to create four pieces and allocating five pieces piece potentially revealing equivalence and composition
1	Figure 6.c. Sharing among six people 	Making cuts with a focus on creating six pieces rather than creating equal-sized pieces piece potentially revealing and understanding of composition
1	Figure 6.d. Sharing among four people 	Making four horizontal cuts while focusing on creating equal-sized pieces and then ignoring the last piece
2	Figure 6.e. Sharing among six people 	Making cuts with a focus on creating six pieces rather than creating equal-sized pieces piece potentially revealing and understanding of composition
2	Figure 6.f. Sharing among four people 	Making one vertical cut to create two equal-sized pieces then treating each piece as a new piece and making one vertical and one horizontal cut to create two equal-sized pieces out of each piece potentially revealing transitivity

Figure 6. Student work samples for the analysis activity.

The teachers made inferences about thinking based on work sample related to the *three equipartitioning criteria*. Moreover, they made less indiscriminant claims that the student "understands" or "does not understand" equipartitioning in favor of highlighting students' foci and areas of neglect. When the first grade teachers at one school examined the artifact in figure 6.c, they noticed that four of the parts were of equal size with a small remainder. From this, they concluded that the student may have been focusing on creating equal-sized pieces and perhaps did not know to exhaust the whole, though they had not incorporated the word "exhaust" into their vocabulary yet:

S10: Student was asked to divide a rectangle into 4 equal parts
S9: There's an extra part there
S11: He drew the 4 lines, he wasn't thinking about -
S10: The extra piece there. Alright, so what does the student understand about mathematics of the problem?
S9: The other parts are equal except that there's that little skinny extra piece. They look pretty equal.
S8: I think he got to that last line and realized it wasn't - after he did 3 lines, he realized there was a great big other part to make it equal and had some left over.
S10: That was the, uh -what do you call it when they throw it away- I can't think of the word - when they try to get rid of that extra part?
S9: Like with the pirate's treasure, they tried to throw it in the sea. I cannot remember the word, either.
S10: So what does he understand about the mathematics? It looks like he understands that the pieces should be the same.
All: Uh huh.

At the other school, the first grade teachers made similar observations about the work sample (figure 6.c) but noticed that the student may have been working through the difficulty of making one less cut that the desired number of pieces and wrestled with what this might mean in terms of the student's understanding of equipartitioning.

W12: The four parts that are up there are equal amounts but there's a little left over there. So maybe they didn't understand the difference between making the three lines to make four parts and just made that extra line there. But knew that they really should be equal amounts but -

W8: Maybe they don't know what to do with that extra piece there. Or maybe they just didn't - cut it in half at the right spot. Maybe they don't understand halves. They would have to cut it in half and then half again.

W12: Right – I just see that all the time. I do that sometimes, you know, when I am making lines and things or cutting things, I'll make the wrong number of cuts. They knew that they needed to make the cuts, but -

W8: Well, she understands – well, she's got 5 there, maybe she doesn't have good number sense or maybe she just made the wrong cut there since there is 5 pieces.

W10: There is supposed to be four, though.

W8: Yeah, I know, so do they really understand 4 since there are 5 pieces?

All: I think the problem was –

W8: I think they went 1, 2, 3, 4 – made 4 cuts. They understand 4, but they just have left-overs. Is that what you are saying?

W12: No. I thought that they understood that they had to make cuts they just didn't know how many cuts to have four pieces.

W8: I got you

W11: They thought four lines would give them four pieces

W12: Uh huh. So I think he does not understand that four cuts makes 5 pieces

From these and other examples, I concluded that the revised progress variable chart and the adapted guidelines for analyzing students' work may have assisted teachers in relating evidence of behaviors to possible cognition as well as moving from holistic to more nuanced treatments of students' work. This may also be related to the differences in equipartitioning a collection of discrete objects and a single continuous whole.

Also related to the teachers' use of the learning trajectory was their reference to the video exemplars. In their discussions, some of the teachers made references to the recordings that were used to illustrate the trajectory for both collections and for a continuous whole. For example, the Kindergarten teachers from one school examined the artifact in figure 6.b. As they discussed why there were more than four pieces, they recalled a video example of Cameron, a child trying to share a rectangular cake among four people. After Cameron made four parallel cuts and realized that this created five pieces of cake, he tried again and made three cuts yielding the correct number of pieces. This video was shown to illustrate how a student's focus on creating equal-sized pieces while neglecting the number of pieces can lead students to discover this relationship. They stated:

> S2: Well the four lines of across. I'm guessing he didn't want to have 5 shares.
>
> S5: The same thing the boy in the video did where he kept cutting it in 4s or he kept making an extra cut, drew four lines instead of three.

The teachers had noticed the four vertical parallel line and the three parallel horizontal lines and concluded that the student had learned from his of her mistake. It was unclear from their discussion whether the teachers believed that the student had simply reused the same cake and was disregarding the four vertical cuts or if the student intended for each person to get five pieces. In another instance, the first grade teachers referred to two different video exemplars. One video was of Cameron which was collected by the DELTA research team. The other video was of Kerr, which was collected by one of the participating teachers and discussed in the initial analysis of Session II. Recall that Kerr shared 24 coins among three pirates by failing to exhaust the whole, giving each pirate seven coins and then throwing the remaining coins in the sea. They stated:

> W8: She might do like that one little boy did, just cut it and cut it and…
>
> W12: Those are going in the sea!

From these references, I concluded that the teachers were using the videos as a way of understanding the different levels of the progress variable, as a means of connecting behaviors with cognition by having examples illustrating different levels of coordinating the *three equipartitioning criteria*, and relating the learning trajectory to another student's work.

Another observation from this activity was that some teachers tended to formulate their questions about the student's understanding based on the *three equipartitioning criteria*. Related to this observation is that some tended to tailor their next activity for the student based on their model of what the students did or did not understand about equipartitioning. For instance, in the work samples shown in figure 6.a and 6.c which contained pieces of unequal size such, the teachers suggested asking the student, "which piece do you want?" or having the student cut out the pieces and stack them to see that they were of unequal size. When examining 6.c, one group of first grade teachers commented:

> S9: I just question why if the student understands that some pieces are way larger than other pieces. I'll bet if you ask the student which piece would want they'd pick out the biggest piece.
>
> S11: Well, you can ask him to cut the pieces apart and see if they matched to show even and uneven.

In the instances where there were more than the target number of pieces (figure 6.b), teachers suggested asking students about who gets which pieces or to have the students cut and allocate the pieces. One teacher said, "What kind of descriptive feedback would help the student? Maybe count the pieces and how many pieces do you have? How would count the 20 pieces…[and ask him] which part of the cake does each person get?" Finally, in the sample where the teachers believed the student was not exhausting the whole (figure 6.d), the teachers suggested asking about the fate of the left-over piece. One group of teachers stated:

S9: It seems like he wants to throw that piece away. He's ignoring that piece.
S10: He knows that it doesn't belong there, that that one was too big - so he wants it to be equal.
S11: Well, you could ask, "do you have four pieces?" He's going to say, 'No, you have five', and then -
S9: And then you need to ask -
S11: What are you going to do about that last piece?

Aside from relating the next learning activity to their model of students' thinking, the teachers suggested changing shapes, the number of people sharing, or reattempting as their next steps. On teacher questioned, "Could he do it with a different shape if it wasn't a circle? Could he do it with a different shape? Can he… does he know why it's not even? Can you show a picture of another circle that's not cut equally and ask him if that's equal? And see what he would say." In another example, teachers discussing 6.e stated:

S12: Start over. Yeah, me too. I'd have him start over.
S14: Maybe even, do you think if they try it first with another number, like if they tried it with four pieces and then you ask them to look at the one they've done before - well, maybe not. But I was trying to think, to try to get them to form some kind of relationship between the two to see a non-example, to get them to see the number and equivalence and to see that they have made a non-example.

Already seen are examples of teachers' sensitivity to the behaviors that may have resulted in the artifacts. Another example of this awareness was when one of the groups of teachers analyzed the artifact shown in figure 6.b. They were curious about the order of the cuts and if the student had planned the sharing strategy or simply discovered it in the process of enacting a different plan. They discussed how the student's reasoning may differ based on which cuts preceded the others.

S2: I wonder what happened first?
S4: My first question, did he do this one or -
S6: But then even if he did it that way, why didn't he stop?
S5: Either way that he did it, yeah, either way that he did it he would have had 4 equal shares if he would have stopped.
S6: No, no. Not this one. Actually, not this one, 'cause look - He was thinking 1, 2, 3, 4 cuts.
S2: Yeah, he has four cuts then three cuts. So -
S1: Anybody who's seen a birthday cake is not going to stop, you know…they are going to try to make pieces.
S6: They think – birthday cake. Pieces.
S2: It looks like 'cause the line is going down are very straight, it looks like possibly the student could have halved and halved it, and then like you said the birthday cake concept – did the four lines across, creating five rows.

The analysis of the student work activity provided further support for my observations from previous sessions of the influential role of the teachers' curriculum. For instance, at the time of this session, the second grade teachers were using a unit from *Investigations* in their classrooms. One mathematical emphasis of this unit was building a notion of area measurement. The materials included activities of covering various shapes with pattern blocks and square tiles to meet this goal. In the session, one group of second grade teachers considering the artifact shown in figure 6.f thought aloud about other experiences of the student, stating "I wonder if they could do things like fill in shapes with pattern blocks or something." At the other school, the second grade teachers discussed:

W13: Well, you could literally cut it out and then have them stack them, you know when he was talking about what the children would do
W17: But you cannot do that with this – cut it out - or you could give them those kind of
W15: What do you call those? One inch - ?

W17: Square tiles -
W15: And measure them.

The teachers were connecting different justification strategies described in the learning trajectory with approaches that coincided with other curricular foci.

In another case, one teacher made a connection between *Investigation's* development of even and odd numbers and the work sample show in figure 6.e.

S16: And they're thinking about the number not the size. It's definitely not… you can tell that person put some thought into that…. but it is definitely not equal.
S12: They have developed that… whenever they have to share something, it has to be in half… so they automatically go to that. And they are probably like, oh wait. So they understand, they understand the three on the two split, but they don't understand the size.

These instances provided more support of my growing belief that the interactions, both in terms of productive and conflicting ideas, between the learning trajectory and the teachers' curriculum were more substantial than originally anticipated.

Finally, the analysis of the teachers' discussions during this activity suggested that the teachers noticed some of the mathematical properties described in the progress variable and discussed in the ERNR PD but not to the degree that I had hoped. For example, though three of the six artifacts included a use of composition (one for each grade-level group), none of the teachers mentioned this property. Though the sample shown in figure 6.b provided teachers with a chance to demonstrate an awareness of equivalence, one group did not mention the property. The other group was unclear about how the student would allocate the pieces and did not focus on the idea that five of the smaller 20 pieces were the same as one of the larger four pieces. However, both second grade groups discussed implicitly the transitive property as they examined the artifact shown in figure 6.f. One group discussed:

S16: The back one says a student was asked to share a pan of brownies fairly among 4 people. They didn't choose the obvious. What do you think that the student understands about the mathematics of the problem?
S14: That half is a half no matter how you half it.
S16: I still think that they go for that half line first… and then thought from there.
S12: I wonder which cut they made first.
All: [agreeing]
S16: But what made them do it that way? I mean, that's odd to me. He obviously knew there were four.
S17: They're fair.
S12: It is.
S16: The lines were drawn… [pause]
All: It's fair. It's fair.
S14: Unless he or she stumbled on it.
…
S16: He understands half of a half, because that is probably what he did.
S14: But see, my question is, was it accidental and happened upon, or does he just, or he or she, whatever, understand halves well enough to know that it doesn't matter either way.
S17: That half of a half is a quarter
S16: But I'd also like to see them do it because…
S17: I think the order in which they are cut…
S16: I mean did they cut it and then go, uh oh, I've got to do these. Or did he take his time because those lines are pretty good.
S17: I think the order in which the cuts were made will tell you a lot.

Here, some of the teachers seemed to understand the transitive property and how it was related to this example, stating "half is a half no matter how you half it." Others may not have been as certain as they paused and then all agreed simultaneously. The group at the other school wrestled with the same example:

W17: And I wonder if he is thinking that these two – is he thinking that this piece is equal to this piece?
W13: I think he is.

W17: Because see, it doesn't look like, I mean if you look at it, you know kids don't think that this right here is equal to that because this is shorter -

W15: So you could ask him if we cover these with one inch little cubes, do you think that's going to -

W17: Will the same number of cubes cover this portion as this portion and see what he says.

W15: And see what he says, if he would say yes or no – if he said no, well then he doesn't quite understand.

In this example, the teachers clearly knew that the pieces were equal-sized. They appealed to their curriculum's emphasis on area measurement using one-inch tiles as a means to convince students rather than developing the transitive property or appealing to decomposition and re-composition.

The analysis of Session III taken with my observations from the previous sessions and classroom-based activities suggested that the learning trajectory was influencing the models of students' thinking that teachers created. Teachers' were becoming increasingly sensitive to behaviors described in the learning trajectory and indications of those behaviors. They were more precisely connecting those behaviors to cognition to create models. Moreover, they were posing questions related to those models and were planning their teaching moves based on them. Also, these observations supported the growing realization of the effects of the teachers' curriculum on their uses of the learning trajectory. Finally, there were some instances where misunderstandings within the teachers' mathematical knowledge were engaged by the activities designed to provide opportunities to use a learning trajectory for equipartitioning, such as understanding that noncongruent pieces could be the result of equipartitioning.

Session IV

In preparing for Session IV, I revisited the conjecture concerning interactions with students (Conjecture Four) and my plans for subsequent sessions in light of issues that arose from the initial analyses of Sessions I, II, and III and the first two classroom-based activities. Previous observations suggested that many of the teachers viewed equipartitioning as simply the inverse of multiplication and were not differentiating partitive and quotitive contexts. Also, the previous presentations of the learning trajectory had been supported by teachers anticipating and reflecting on how the trajectory might look and actively engaging in considering a variety of students work. The conjectures and my observations thus far led me to believe that teachers should be able to use the learning trajectory and their ability to construct models of students' thinking to examine and sort different responses to problems of equipartitioning collections of continuous wholes. Lastly, discussions from the analysis of student work activity indicated that teachers formulated questions for students related to their models. Conjecture Four suggested the learning trajectory would support teachers' identification of student ideas to build a more robust understanding through instruction as well as serve as a tool with which teachers may judge relative difficulty of equipartitioning tasks.

The primary source of data was the sorting activity. Groups of teachers were given a collection of tasks for sharing multiple wholes coupled with hypothesized student responses to them and told to create groups of tasks they believed to be similar and to articulate their sorting criterion. They repeated this activity several times. Video recordings of the sorting activity, related audio recordings of grade-level group discussions, and groups' written notes of the n = 24 participating teachers in attendance served as data for the initial analysis of Session IV. The complete activity is located in appendix J, and an explanation of it is included in Chapter 3. This preliminary analysis was guided by the framework. I noted two tendencies in their discussions: groupings based on the mathematical demands of a task and groupings based on perceived strategies used by students.

Several of the grade-level groups used several different mathematical characteristics of the task as a means of grouping. One of these characteristics was the number of splits required by the task. For example, a first grade group used a criterion based on the number of people involved in the problem:

> S10: Alright, four graham crackers among three people. Ok. Two pizzas among three people. And we have three people, three people. Four people. Five people.
> S7: Now how are we sorting?
> S10: By the amount of people that we're dividing it by.
> S10: So four divided by three, 2 divided by three, 10 divided by 3
> …
> S11: We just did it by the number of people.

Another group of Kindergarten teachers created groupings based on whether the number of people involved in the task was even or odd.

> W1: Oh, oh, oh - I have another way, I have another way. What about the number of people, put 'em in even, like when the number is even or odd.
> W4: Even-Odd. So you want me to have the even and odd?
> W1: Yeah. The number of people, even and odd.
> W4: Where's odd pile?
> All: [sorting tasks] Even, even. Even, odd.

In these two examples, the groups of teachers were sorting the tasks based on the number of equipartitions demanded by the task, either as even and odd or as the number itself.

Two other groups used the same criterion, but added a sense of difficulty for students within their classification scheme. For instance, one group of teachers sorted according to tasks that "used halves" and those that did not, recalling that two-splits are easier for students than others:

W17: Something that is in halves and then fourths in one group and everything else into something else?
[teachers sorting]
W17: So we've got eighths and thirds. This is a fifth. Fifths, thirds and eighths - Alright, so group one is – [teachers naming tasks for their groupings]
R: Explain this sort to me.
W15: Ok. Which one? The one we just did?
R: This one here. This is an interesting sort. Tell me about it.
W15: Basically, this one would just be easier to start out with halves and fourths when you're teaching and then you move on, to like the thirds, the fifths…how to split those.
R: Why, why would you start off with those over there?
W15: They're just easier, easier splits.
R: How do you know?
W15: 'Cause it's harder to do these.
W17: These occur more naturally.
W15: Yeah.
W17: Split something in half. I mean you're always cutting burgers and sandwiches in half. Kids are use to halves and fourths.

The other group classified as easy, medium, and hard, a judgment made based on the number of splits required to complete the task:

S16: You could do it as those that are divided in half, in fourths…
S17: In others.
S14: [writing] thirds - so this would be, let's divide them into easy, medium, and hard.
S12: Well, anything that's on a half is going to be easy.
S16: It is going to be easy, so this, this, this -
S17: This would definitely be more difficult. That would be more difficult. This would be easy.
…
S16: Yeah, because this one has, this one -

S12: - is strictly halves. Yes.
S15: They could manage this.
S17: So, it's like easy, medium, and hard.
S12: So we are doing it by level of difficulty?
S14: Level of difficulty - or level of sophistication.
S12: See, this would even go in here. Even if they had to share like…
…
S14: So the last one would be G, L, D. And the more sophisticated would be A, H, J, & I. And the most sophisticated would be E, C, D, A and F.

In both of these cases, the teachers were using the number of splits required by the tasks to create groupings that were ranked by the perceived level of difficulty for students.

Finally, one group considered classifying the tasks based on the outcome in terms of creating a quantity greater or less than a whole. After creating two piles, one teacher from a second grade group stated, "the way we did it was more items than the number of people and fewer items than the number of people." Their sorting had separated all tasks that resulted in shares greater than one whole from the tasks yielding a share less than one whole. In all of these examples, whether by attending to the number of splits required by the tasks or the tasks' outcomes, these groups of teachers use using mathematical characteristics to create classes of problems. In some instances, they ranked these according to difficulty.

Alternatively, teachers also sorted tasks based on their perceptions of the ways that students had approached the task. For instance, one group used the students' strategy of reducing the task to sharing a collection of wholes and then equipartitioning the remainder as a way of identifying similar items.

S2: These two together and everyone else would be in its own category, right?
S5: Yeah
S6: They get wholes and they split what is left
S2: Yeah.

S5: Yeah – they don't split from the beginning.

These teachers grouped items where the students' seemed to have dealt wholes initially and then equipartition the remainder. Another way teachers used their perceptions of students' strategies was through identifying items where students' used landmark fractions. Each of the six groups either commented on or used this as a means of classifying the items, such as this discussion from a group of first grade teachers:

R: Have you done the sort? Okay, tell me about your first sort.
W10: We sorted them as these, you know, these are all the fractions are the same. These all have six parts.
R: Okay so all the wholes are created, are partitioned in the same number of parts.
W12: And these are more mixed, like a half plus a third and two fourths plus two fifths which makes it a more complicated problem to figure out.

From the initial analysis of the sorting activity, I made three observations that influenced my ongoing revisions to Conjectures One through Four and plans for the final session. First, the teachers were using a sense of the increasing difficulty of splits from the learning trajectory in their examination of the tasks, such as the group that created a category with repeated two-splits from others. They used these analyses to discuss the difficulty of tasks compared to others and even referred to these as levels of sophistication, the language used when describing the learning trajectory with the progress variable chart. Though they had not learned about the portion of the learning trajectory for equipartitioning multiple continuous wholes that were not evenly divisible by the number of people sharing, their understanding of the lower levels influenced their initial considerations of the problems.

Secondly, the mathematical content knowledge needed to understand the tasks and hypothetical students' work was significantly greater than that which was needed for previous sessions. As many of the teachers were long-time primary grades teachers and had not taught concepts demanded by these types of tasks, the teachers were uncomfortable with the mathematics of the problem or were unaware of their mistakes. Teachers used "mixed numbers" and "improper fractions" interchangeably, ignoring the cognitive indications related to this distinction. For instance, a mixed number interpretation suggests a reduction of a task to dealing and then considering the remainder, while an improper fraction result may suggest that students considered the task as multiple instances of equipartitioning a single continuous whole and counting resulting parts. Only one group noticed that the use of landmark strategies to solve the tasks resulted in more complex computations because different units.

Finally, there was more evidence of the effect of the teachers' curriculum on the PD tasks. One teacher, when discussing a method of preserving wholes when sharing three cookies among two people stated, "that messes up my odd and even stuff. Because I'm like, you can't fair share three cookies. You know… but now they are like, you can break it in half. Or when they go to do this they'll be like you can't break it in half, you can't break it in half – you know? It is going to mess up my even and odd stuff." Presumably, this teacher uses the idea of fair sharing between 2 as a way of classifying numbers as even or odd that is developed in *Investigations* through making teams. Once students encounter this problem where an odd number of items can be fairly shared, the teacher recognized that this definition would need to be modified.

In this chapter, I described the implementation of the professional development and the ongoing analyses that affected the evolution of the conjectures. In Chapter Five, I describe outcome measures used in the retrospective analysis.

CHAPTER FIVE

The preceding chapter described an ongoing analysis of the ERNR PD and the influences of the analysis on evolving conjectures of the study. This chapter details a retrospective analysis of the data collected. First, I examine data directly related to two of the ERNR PD goals to assess its effectiveness. Next, I describe the primary and secondary sources of data from the design study used in the analysis. I analyze these data to understand more fully the results of the professional development and to assess its effectiveness in meeting the other goals of the professional development.

Effectiveness of the Professional Development

For the final analysis, I began by determining if two of the goals of the ERNR PD were met, the understanding and use of the learning trajectory by teachers and the advancement of students' learning of equipartitioning. These goals are most directly addressed by the teacher assessment and the student assessment. Inasmuch, the following sections describe the analysis and conclusions of each of these measures.

Teacher Gain Scores

Pretest and posttest items for the $n = 33$ participating teachers were scored using the rubrics included in Appendix B. A mathematics educator unassociated with the research group was trained on the rubrics and independently scored a randomly selected subset of the items (50% of the responses). These scores were compared to assess consistency among raters with 79% agreement. I calculated a kappa coefficient, a measure of agreement corrected for chance, as $\kappa = 0.726$. Conventionally, kappa coefficients greater than 0.7 are taken to indicate acceptable agreement between raters, therefore I concluded that the scoring was reliable.

Participants had 45 minutes to complete both the pretest and posttest. In both cases, teachers reported that they needed more time to complete all items. As such, items left blank were scored as "blank," indicating either a lack of time to respond to the item or a respondent's inability to address the demands of the question. Because the pretest and posttest was in an early stage of development, it is unclear if the teachers had enough time to complete the tasks. Therefore, equating blank scores to incorrect answers was not warranted. Thus, items scored as blank were disregarded in all statistical analyses. Scores on the items from the pretest to posttest were transformed into composite scores and thus were neither adversely nor favorably affected by non-responses.

Items 1 - 12 on the assessment measure knowledge of equipartitioning while items 13-18 measure pedagogical knowledge related to equipartitioning. These two sets of items are referred to as content items (1-12) and pedagogy items (13-18). Two subscores were created based on responses in the items in both sets, a disaggregation that is theoretically supported (c.f. Hill, Ball, & Schilling, 2008; Shulman, 1986). In addition, a subscore recording of the total number of items scored as "blank" was created. The maximum content subscore possible is 36 points and the maximum pedagogy subscore is 18 points. Descriptive statistics for the pretest and posttest subscores are displayed in table 4.

Table 4. Descriptive statistics for pretest and posttest subscores.

	Pretest		Posttest	
	Median	IQR	Median	IQR
Content	19	7	23	5.5
Pedagogy	2	4.5	8	5.5
Blank	4	5	1	2.5

n = 33

An examination of the pretest and posttest distributions suggested favorable changes in performances on the assessment after the PD. There appeared to be gains in both the content and pedagogy dimensions and a decrease in the number of items left blank. To further explore this, I created gain scores for each of the three dimensions. Before statistical inference procedures could be conducted, I investigated the sample distributions of each of the gain scores to see if the standard assumption of normality was warranted. Initially, box plots and normal quantile plots of the content and blank gain subscores suggested normality (see figure 7 and 9). In both instances, the box plots were roughly symmetric with centered density, and the normal quantile plots were roughly linear and lacked systematic deviations from linearity. When pedagogy gain scores were subjected to the same procedure however, the box plot of the distribution of the sample suggested skewness, and the normal quantile plot revealed systematic deviations from linearity (see figure 8). Shapiro-Wilk tests were conducted to determine if the samples could be assumed to be taken from a normal population. These tests assume that the sample is taken from a normal population with an alternative hypothesis that the sample is taken from a non-normal population. In the cases of the content and blank gains, the test failed to reject the hypothesis of non-normality, indicating that both content and blank gain subscores could be assumed to be from normal populations ($W = 0.9719$, $p = 0.5339$; $W = 0.9600$, $p = 0.2590$, respectively). However, the test for the pedagogy gains indicated that the assumption of a normal distribution of the population from which these scores were taken was not supported (W = 0.9320, p = 0.0398). Thus, while the use of parametric inference procedures for both content and blank gains was warranted, it was inappropriate to apply these procedures to the pedagogy subscore gains. For consistency, nonparametric methods were employed across all three composite subscores. The benefits of these procedures include a lack of assumptions about the underlying distributions of samples, and are appropriate for ordinal data such as rubric scores, whereas parametric methods assume the use of interval or continuous data (Hollander & Wolfe, 1999). Nonparametric methods are statistically less powerful than parametric methods.

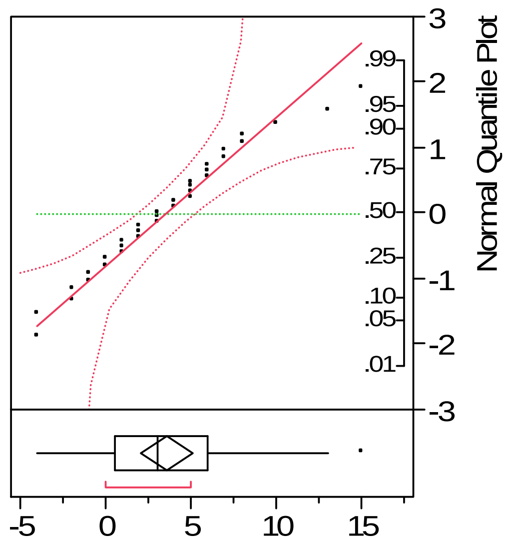


Figure 7. Sample distribution of content gains.

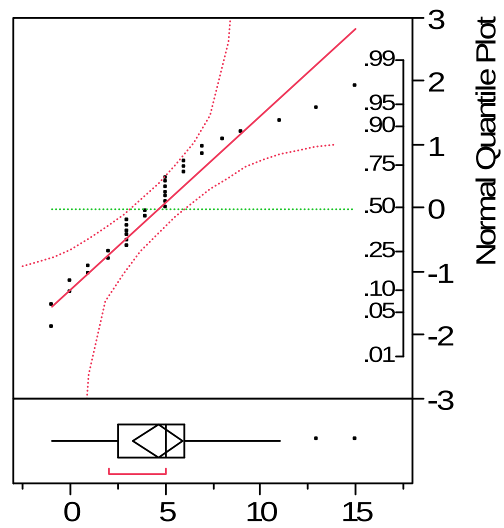


Figure 8. Sample distribution of pedagogy gains.

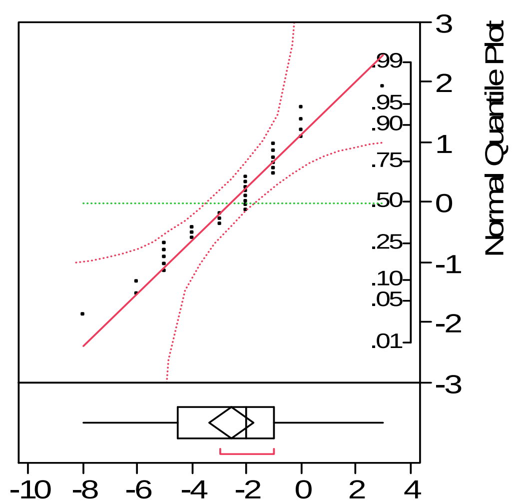


Figure 9. Sample distribution of gains in blanks.

Descriptive statistics were calculated for gains in the content, pedagogy, and blank items. These suggested that there were positive gains in both content and pedagogy subscores and negative gains in the number of items scored as blank. To assess if these empirical findings were statistically significant, I conducted Wilcoxon Sign Rank tests on each of the gain subscores. This test assumes that the samples are simple random samples but does not make assumptions about the underlying distribution. The null hypothesis for the test is that the median of the distribution of gain subscores is zero. For each of the content, pedagogy, and blank gains, this hypothesis was rejected in favor of a one-sided alternative. This indicates that both the content and pedagogy scores on the posttest were statistically significantly higher and that there were significantly less blanks on the posttest. Descriptive statistics, test statistics, and p-values are reported in table 5.

Table 5. Gains in content, pedagogy, and blank scores.

	Median	IQR	Wilcoxon Test Statistic	p-value
Content gains (36 points possible)	3 points	5 points	S = 195.00	$p < 0.0001$
Pedagogy gains (18 points possible)	5 points	3 points	S = 243.00	$p < 0.0001$
Blank gains (18 items possible)	-2 item	3.5 items	S = -201.00	$p < 0.0001$

n = 33

Though theoretically supported, I investigated the distinction between content and pedagogy items empirically. Using the content and pedagogy gain scores, I constructed a scatter plot of pedagogy gains versus content gain (see figure 10) which indicated no association between the two scores. Since the assumption of normal population for pedagogy scores was not supported, a measure of association based on rank, Spearman's rho, was calculated to determine a relationship. A nonparametric test of association using rho was conducted which assumes no monotonic relationship between the content and pedagogy gain scores. The test failed to reject this hypothesis (ρ = 0.0410, p = 0.8208), suggesting that there is no relationship between content and pedagogy gains.

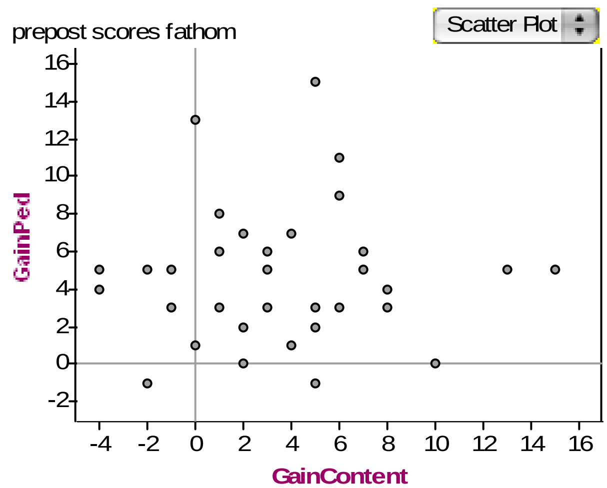


Figure 10. Association between content and pedagogy gains.

Following this, I investigated the top quartiles and outliers in the content and pedagogy samples and the lowest quartile of the blanks sample. Examinations of box plots of the content gains revealed several participants with large gains. To investigate the degree to which this might be attributed to instructional effects, I controlled for pretest content subscores by analyzing gains versus pretest scores graphically (figure 11). From this, I confirmed my expectation that participants with low content pretest subscores had the highest gains. Next, I conducted a similar investigation of the pedagogy gains. Again, when controlling for pretest content scores, participants with the highest gains had the lowest pedagogy pretest scores (figure 12). Lastly, an investigation of the gains of blank scores was conducted in a similar fashion (figure 13). Participants in the lower quartile corresponded to those who had fewer blank responses on the posttest than the pretest. The graphs used in these analyses are included as figures 11, 12, and 13.

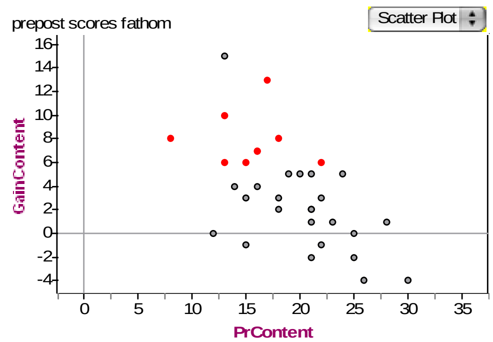


Figure 11. Content gains versus content pretest scores.

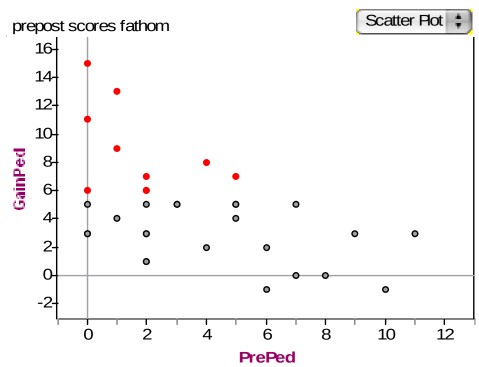


Figure 12. Pedagogy gains versus pedagogy pretest scores.

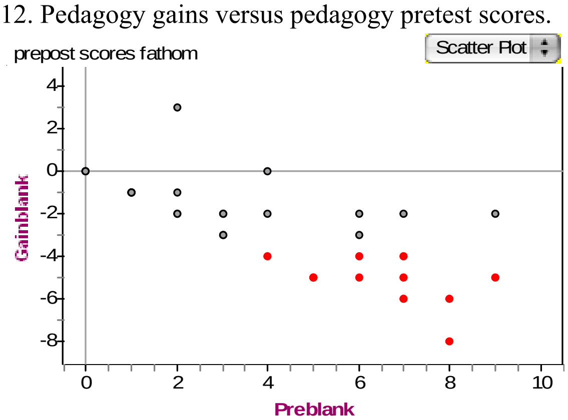


Figure 13. Gains in blanks versus pretest blanks.

Descriptive Investigation of Content Items

The above analysis suggests that the ERNR PD as a whole was effective at increasing teachers' knowledge of equipartitioning and uses of the learning trajectory in instruction. To understand better the nature of this increase, I conducted an item by item analysis of the content items. Table 6 displays the median scores for each of the content items from the pretest, posttest, and gains.

Table 6. Median scores on content items.

Item	Pretest	Posttest	Gain
Q1	1	1	0
Q2	2	3	1
Q3	3	3	0
Q4	3	3	0
Q5	2	2	0
Q6	2	2	0
Q7	2	2	0
Q8	1	1	0
Q9	3	3	0
Q10	2	3	1
Q11	0	2	2
Q12	2	2	0

$n = 33$

Three of the items had positive gain scores. Item 2 concerned a generalization of equipartitioning as division. Items 10 and 11 involved equipartitioning two collections of continuous wholes and comparing the results. Item 10 required the use of the compensation property. Item 11 involved numbers that would distract respondents who held an additive misconception. Other items had no change from pretest to posttest. For Items 3 and 9, the median score on both was 3, the highest level attainable on the rubrics. These items involved sharing collections and reversibility. Items 5, 6, and 7 showed no change in median scores, but each had a score of 2 from the rubric, indicating correct conceptions but incomplete reasoning. These items concerned equipartitioning irregular wholes (e.g. darts, eight-pointed stars, etc.) and the equivalence of noncongruent equipartitions. However, items 1 and 8 involve multiple ways of naming shares and generating multiple methods of equipartitioning a continuous whole. On these items, the median score of 1 remain unchanged from pretest to posttest. This indicates that the teachers failed to gain flexibility in naming shares in a variety of ways (e.g. as a count, in relation to the whole) or using factors to find multiple ways of composing splits to generate a desired outcome. This analysis indicates that though the ERNR PD made some progress with the teachers in deepening their understanding of the mathematics of equipartitioning in a few areas, the majority of the concepts measured by the items remain unchanged.

Descriptive Investigation of Pedagogy Items

Similar to the content item analysis, I investigated the difference in the median scores of the pedagogy items. Table 7 shows the median scores per pedagogy item from the pretest, the posttest, and the difference in gains.

Table 7. Median scores on pedagogy items.

Item	Pretest	Posttest	Gain
Q13	0.5	1	0.5
Q14	0	1	1
Q15	1	2	1
Q16	1	1	0
Q17	1	2	1
Q18	1	1.5	0.5

$n = 33$

All but one of the pedagogy items saw an increase in median score. Both items 15 and 17 addressed assessing the difficulty of equipartitioning tasks. Item 18 assessed the ability to model of a student's thinking about equipartitioning based on a work sample. Whereas the impact on content knowledge was limited, the ERNR PD positively affected almost all of the concepts measured by the pedagogy items.

Three additional data sets were collected from two of the items that are not directly represented in the gain scores. Item 13 on both forms of the test asked respondents to list multiple strategies that students may use from a variety of levels of sophistication when splitting a continuous whole. Item 14 on both forms of the test asked respondents to list multiple correct and incorrect strategies that students may use to split a continuous whole. The items were parallel in the sense that they both had teachers generate strategies but were not equivalent across the two contexts of the forms. Since more strategies exist for the rectangle than the circle, gains in these three areas could not be calculated accurately. These differences are more clearly explained in table 8.

Table 8. Descriptions of items 13 and 14.

Form A	Form B
13 – Number of strategies for a 4-split on a rectangle	13 – Number of strategies for a 6-split on a circle
14 – Number of correct and incorrect strategies for a 6-split on a circle	14 – Number of correct and incorrect strategies for a 4-split on a rectangle

However, by taking the two items on the pretest as a pair, I created a measure of the number of strategies teachers can generate for both items by summing the strategies for item 13 with the number of correct and incorrect solutions on item 14. Thus, a gain in the number of strategies from pretest to posttest could be investigated. Since the items were not parallel, an item by item investigation similar to the one above for items 15, 17, and 18 was inappropriate. Therefore, an investigation similar to the other composite gain scores was conducted. An examination of the sample distributions with a box plot and a normal quantile plots (see figure 14) suggested normality, and Shapiro-Wilk test confirmed that the sample could be assumed to have been taken from a normal population ($W = 0.9657$, $p = 0.3706$). Since these data are interval data, parametric methods are appropriate. A matched pairs t-test indicated that there was significant gains in the number of strategies teachers generated from pretest to posttest ($t = 5.5211$, $p < 0.0001$). For consistency, I conducted a Wilcoxon Sign Rank test and found significance, as well ($S = 165.5$, $p < 0.0001$).

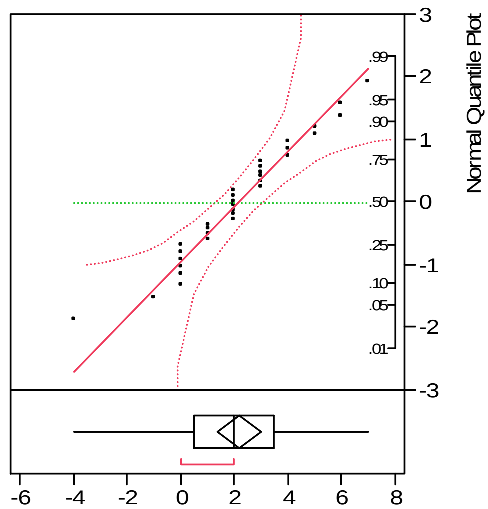


Figure 14. Sample distribution of strategy gains.

From this analysis, I concluded with reservation that the first goal of the ERNR PD pertaining to increasing teachers' knowledge and use of the learning trajectory for equipartitioning was met. The significant content subscore gains suggested that the teachers' understanding of equipartitioning improved marginally from the beginning to the end of the sessions. However, significant gains in the pedagogy subscore suggested that teachers' knowledge of students and equipartitioning improved as measured by items that targeted teachers' uses of the trajectory in instruction.

Student Gain Scores

Pretest and posttest items for the $n = 113$ students agreeing to participate in the study were scored using the rubrics included in Appendix D. A mathematics educator unassociated with the research group was trained on the rubrics and independently scored a randomly selected subset of the items (roughly 30% of the responses). These scores were compared to assess consistency among raters with 79% agreement. I calculated a kappa coefficient, a measure of agreement corrected for chance, as $\kappa = 0.703$. Conventionally, kappa coefficients greater than 0.7 are taken to indicate acceptable agreement between raters, therefore I concluded that the scoring was reliable. Composite scores of students who completed all pretest and posttest items were created. Descriptive statistics for the pretest and posttest are displayed in table 9. The maximum score allowable by the rubric is 18 points.

Table 9. Descriptive statistics for student pretest and posttest scores.

	Pretest		Posttest	
	Median	IQR	Median	IQR
All (n = 95)	6	4	7	4
S12 (n = 7)	4	1.5	5	4
S13 (n = 8)	5.5	1.5	6	1.5
S14 (n = 2)	8.5	2.5	6.5	2.5
S16 (n = 14)	8.5	3.75	8	4.75
S17 (n = 5)	4	4	6	1
W13 (n = 10)	4	2.25	8	4
W15 (n = 6)	8.5	2.5	8	3.75
W16 (n = 14)	5	4.5	7	1.75
W17 (n = 15)	7	4	8	5.5
W18 (n = 14)	6	2	6.5	2

To explore further, gain scores were calculated. The distribution of student gain scores was investigated to determine if parametric methods were appropriate (see figure 15). A Shapiro-Wilk test rejected the null hypothesis that the sample is from a normally distributed population ($W = 0.9670$, $p = 0.0177$). Further, the ordinal nature of the gain scores supported the use of distribution-free methods.

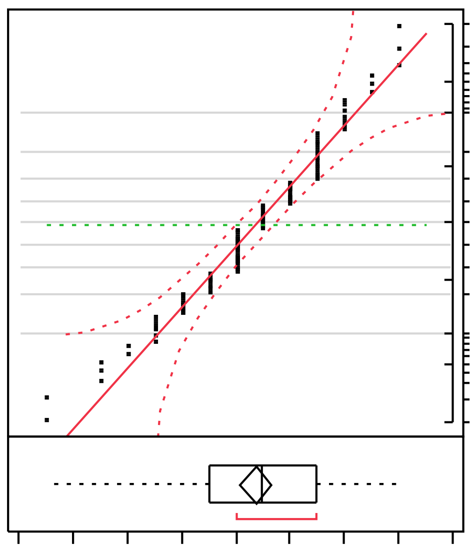


Figure 15. Sample distribution of gain scores.

Descriptive statistics for the student gain scores were calculated and suggested an overall increase in scores on the posttest and are included in table 10. A Wilcoxon Sign Rank Test determined that there was a significant increase in student scores from pretest to posttest ($S = 486.00$, $p = 0.0053$). From this, I concluded that there was a small positive gain in scores for all participating students.

Table 10. Distribution of student gain scores by teacher.

Class	Median	IQR
All (n = 95)	1	4
S12 (n = 7)	2	4.5
S13 (n = 8)	0	1.75
S14 (n = 2)	-2	0
S16 (n = 14)	0	3.25
S17 (n = 5)	1	3
W13 (n = 10)	3	3.75
W15 (n = 6)	0	4.5
W16 (n = 14)	0.5	3
W17 (n = 15)	1	5
W18 (n = 14)	0.5	1.75

Conjecture Five suggested that the students' gains would be positively related to their teachers' knowledge and uses of the learning trajectory. Accordingly, I investigated whether students of different teachers performed differently. I examined the distribution of gain scores disaggregated by teacher graphically (figure 16.). A Kruskal-Wallis Test using a χ^2 approximation determined that class membership was not significantly related to performance gains (χ^2 = 15.1456, p = 0.0870, df = 9). This suggests that although students performed better after the teachers' lessons on equipartitioning, the teacher from whom they received the instruction did not influence those gains significantly. S14's class had only two students consent to participate in the study who completed all items on the pretest and posttest. Due to the small sample size of this class, I explored how these findings would change if these were removed. The exclusion of this class did not change the findings in relation to the aggregate scores above. However, the p-value of the revised Kruskal-Wallis test indicated even less association with gains and class membership (χ^2 = 12.7281, p = 0.1216, df = 8).

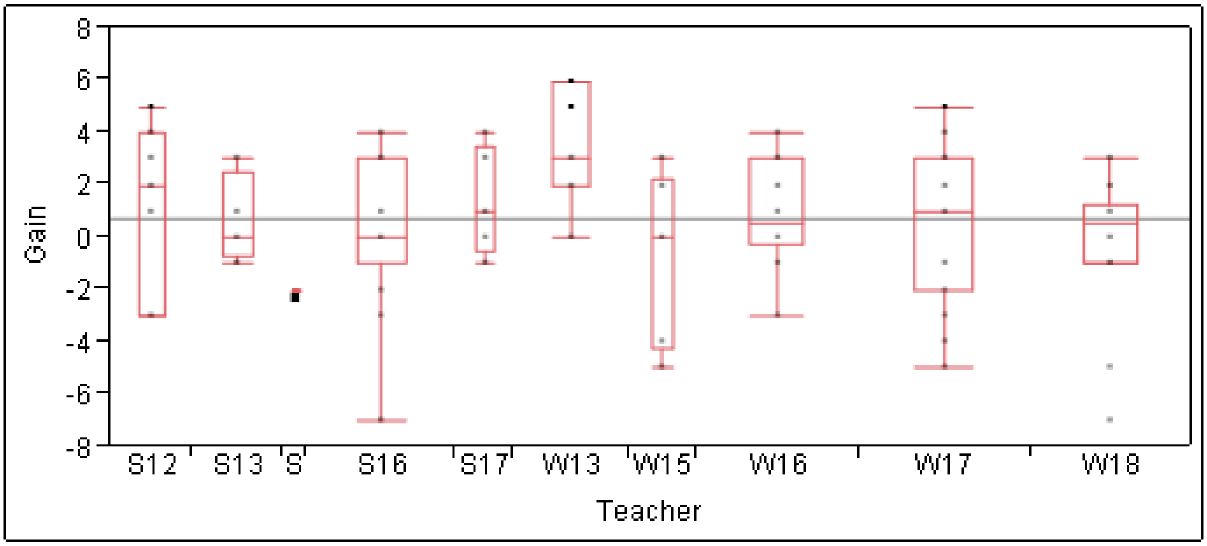


Figure 16. Distribution of student gains by teacher.

The one-way analysis above is useful for determining differences across multiple groups. However, different attributes of the teacher may have affected their instruction, such as prior knowledge or knowledge of the learning trajectory and equipartitioning gained from the ERNR PD. Furthermore, the nested nature of this data may require different statistical techniques to adequately account for its structure. Therefore, before concluding that the student gains were not associated with their teacher, I calculated the intra-class correlation coefficient (ICC) to determine if a hierarchical linear model may help explain some of the variability. Hierarchical linear modeling (HLM) enables researchers to examine the effects of variables at multiple levels by accounting for variation within and among each level (Raudenbush & Bryk, 1992). If appropriate, HLM would be preferable to a one-way analysis for investigating differences in student gain scores among different classes. The ICC value for the data was 0.1081, indicating that only 10.8% of the variability in gain scores can be attributed to class membership. Furthermore, the low number of level 2 classes (10 classrooms) with such a low ICC value indicated that the use of a hierarchical linear model was not warranted. In a final attempt to find a relationship between the students' gains and their teachers, I examined Spearman's rho correlations between student median gain scores with the teachers' posttest scores which are displayed in table 11. These measures indicated that neither the teachers' content nor pedagogical knowledge were significantly associated with students' gains on the items.

Table 11. Spearman's ρ measuring the associations between teachers' content and pedagogical knowledge with their students' median gains.

Variable	by Variable	Spearman ρ	Prob>|ρ|
Post Content Subscore	Median Student Gain	-0.1895	0.6001
Post Pedagogy Subscore	Median Student Gain	-0.2773	0.4380

$n = 10$

From these analyses, I concluded that though statistically significant, the gains in student learning as a result of the teachers' instruction may not have been practically significant in that the median increase across all students was one point. Furthermore, Conjecture 5 of the study suggested that the teachers' knowledge and uses of the learning trajectory would have affected the students' gains. Though technically the professional development goal of advancing student learning of equipartitioning was met, these data do not make a compelling case for the effectiveness of the professional development in terms of student learning. This analysis calls into question whether students' experiences with *Investigations* and the state standards gave them sufficient opportunities to learn equipartitioning.

Sources of Outcome Data

Though statistically significant, the analysis of the teacher and student pretest and posttest did not yield a persuasive argument for the effectiveness of the ERNR PD. However, a closer examination of the remaining outcome data elaborated these findings and detailed other findings not represented in the item analysis above. In addition, this analysis assessed whether the second goal of the ERNR PD concerning equipping teachers with strategies of using students' thinking in instruction was met.

The outcome data were selected from the data corpus to align with the conjectures of the study. Primarily, these include artifacts from the third classroom-based activity and the learning activities from Session V. Other data from Sessions II, III, and IV are used to support these observations, including artifacts from the first and second classroom-based activities and artifacts. The activities from which the data were drawn are summarized in table 12. Included is the number of participating teachers who completed each activity.

Table 12. Learning activities for primary and secondary outcome data.

	Data Source
Conjecture One	Pirate Birthday Interview (*n* = 17) Clinical Interview Activity (*n* = 31) Pirate Treasure Interview (*n* = 29)
Conjecture Two	Task-Ranking Activity (*n* = 31) Task Adaptation (*n* = 24)
Conjecture Three	Student Work Activity (*n* = 31) Student Work Analysis (*n* = 29)
Conjecture Four	Interactions Activity (*n* = 31)

Posts on the website, including written analyses, illustrative video clips and digital photographs, and additional comments and analyses from other teachers served as data for the Pirate Treasure Clinical Interview, Pirate Birthday Clinical Interview, and Task

Adaptation activities. Individual teachers' written notes from the participating teachers attending Session V served as data from the Interview Analysis activity. Audio recordings of grade-level groups, verbatim transcripts of these recordings, and written notes from the participating teachers attending Session V served as data for the Task-Ranking, Student Work, and Interactions activities. Audio recordings and verbatim transcriptions of grade-level group discussions, along with the groups' written notes from the Student Work Analysis activity from participating teachers attending Sessions II, III, and IV served as data.

Analysis

Using the analysis procedure described in Chapter 3, I observed tendencies in the data related to the models that teachers constructed of students' understandings of equipartitioning, the ways in which the models and the learning trajectory influenced their knowledge of equipartitioning and students, and the ways in which the learning trajectory interacted with teachers' knowledge of equipartitioning. The following sections present these observations and interpretations according to the framework of the study. The first three sections organize findings concerning the processes teachers used to construct models of students thinking: Describing, Comparing, and Inferring. The fourth process for building models of students' thinking described in the framework is Restructuring. This section is separated into observations related to teachers' restructuring of two types of knowledge: their knowledge of equipartitioning and students and their knowledge of equipartitioning.

Describing

In the Pirate Treasure Clinical Interview data from early in the ERNR PD before the introduction of the learning trajectory, the teachers' descriptions of students' behaviors and verbalizations were general and imprecise. One teacher wrote in her analysis, "Child demonstrates knowledge of what it means to share. When given the task to share, child explained what it meant to share." Another teacher wrote, "Destiny divided the chips into three groups by essentially passing them out. Then she counted the amount of chips in each group." From these analyses, it is unclear as to how the child demonstrated what it meant to share the treasure fairly or how Destiny divided the chips.

Other teachers' analyses contained information irrelevant to the students' understandings of equipartitioning a collection of discrete objects. For example, one teacher wrote, "He was very much at ease during his entire interview. In fact some of the questions I was to ask him did not get asked because he would answer them as a tag to the previous questions. He did a very good job with the project." Another teacher commented, "She struggled to listen to my directions and didn't answer any of my questions."

After teachers studied the learning trajectory for equipartitioning in the ERNR PD, they engaged in two clinical interview activities that provide a contrast to their initial descriptions. In both the Pirate Birthday clinical interview analysis and in the Interview Analysis activity, many of the teachers' descriptions of students' behaviors and verbalizations were more precise and focused on strategies described in the learning trajectory. For example, in one analysis, a teacher described how her student shared a rectangular birthday cake. She wrote:

> The child tore off the same number of pieces as she had pirates resulting in an extra piece each time. She continued to have one extra piece each time because she was keeping a piece for herself and sharing with the other 2, 4, 3, 6 pirates. At one point, she even tried hiding the extra pieces not knowing how she was to share equally.

The teacher noticed that the student was not exhausting the whole, one of the *three equipartitioning criteria* from the learning trajectory and described the student's actions when she shared. In another interview, a student shared rectangular birthday cakes represented with paper by using scissors to make sequential parallel cuts. Her teacher wrote:

> For the rectangle cake, the student cut down the middle to create 2 equal shares. To create 4 equal parts, the student cut 3 straight lines from left to right. For 3 equal shares, she was able to eye the cake and make 2 cuts to create 3 equal shares. The clip shows her making 6 shares. In the clip, she cuts from left to right. She counted the amount of pieces after each cut until she had 6 strips of cake. The student was able to share equally between 2 and 3 pirates. For 4 or more pirates, she cut the correct amount of pieces, but they were not equal shares.

This teacher precisely described how the student cut from left to right and captures the student's ability to create equal-sized pieces for smaller numbers using the parallel cut strategy.

Data from the Classroom Interactions activity corroborated these findings. In this activity, groups of teachers were asked to describe students' methods for producing four different responses to a task of folding a piece of paper in half three times (see figure 17).

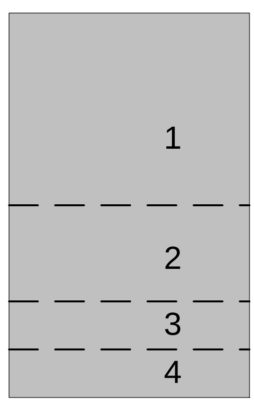


Figure 17. Classroom Interactions item.

Similar to the clinical interview activities, their descriptions were specific and focused on aspects relevant to the students' thinking from the learning trajectory. A group of Kindergarten teachers tried to make sense of one of the hypothetical approaches as follows:

> W1: I thought they folded it in half and then folded the bottom of it, and then again, instead of the whole sheet, just the bottom part maybe.
> S2: They folded and then they unfolded
> W1: And then folded that part up. But it's not equal.
> S2: They said fold it in half. He folded a half, and then he folded the half, and then he folded the half in half. So that is correct.
> S1: Half, half, and half.
> S6: And I think he was thinking, too, about have three folds.
> W1: Fold it in half three times. Yes
> S2: Yes. He was talking about the creases.

One of the second grade groups discussed this same issue:

> W16: Okay. Fold it in half one time. Then it looks like they folded it in halves - that one half - a second half.
> S16: They did this, and then they did it like this…It took me a minute to figure out what they did there.
> W16: So they folded it in half and then from there on out, they just kept it to one side.
> W15: Yeah, they didn't keep the whole in mind.
> S14: They were just focused on those three folds and not on - the quality of parts.
> S16: Not on the whole, you are right…So the question would be are they equal? No. They have an incorrect answer.

Both discussions focused on how the student folded the paper in half three times but managed to create only four regions. They precisely described the actions required to accomplish the outcome, the first group with "unfolding" and the second group with their reference to keeping "the whole in mind."

In these examples and the data, the teachers' descriptions transitioned from being irrelevant & general before experiences with the learning trajectory to descriptions that are specific & focused on behaviors and verbalizations outlined in the learning trajectory. Though descriptions prior to experiences with the trajectory were vague, indistinct, and include superfluous observations, models constructed after experiences with the learning trajectory made clear distinctions among behaviors and verbalizations.

Comparing

As first noted during the ongoing analysis, the teachers' comparisons of students' behaviors and verbalizations before their work with the learning trajectory were based on their experiences with students. In the Pirate Treasure Interview activity, several teachers commented on their student's performance relative to their grade-level expectations of students. For instance, one teacher wrote, "The tasks given to the child were done successfully for a Kindergarten student." Another wrote, "I feel this concept is difficult for a beginning Kindergartener. It is interesting to see how a 5 year old thinks." Other teachers made references to the elementary mathematics curriculum. One teacher wrote, "I was most impressed that a student in this grade level was already using fractional concepts and terms." Another wrote, "He even touched on fractions!" These examples are characteristic of the comparisons teachers made in constructing models of students' thinking prior to the introduction of the learning trajectory.

Yet by the end of the ERNR PD, teachers' comparisons when analyzing clinical interviews included references related to the learning trajectory. These comparisons were implicitly between students' actual behaviors and verbalizations and those described by the trajectory, either with the video exemplars, the progression of task difficulty, or the *three equipartitioning criteria*. For instance, one teacher compared her student's words with those described in the progress variable; "Her terminology is still at the beginning stages because she only knew to refer to each pirate's share as one instead of one-third." In another example, a teacher compared her student's actions based on difficulty of the task in terms of the number of splits required. She wrote. "She gave each pirate a piece of cake, but was unable to do it fairly. This leads me to believe that she understands halves, but is unable to equipartition past halves at this stage developmentally." Other teachers made comparisons based on what they knew about the difficulty of shape of the continuous whole cake. When analyzing her student's work on sharing a circular cake, another teacher wrote, "This was much more difficult than the rectangle cakes. She did try hard to share equally, but was not developmentally ready. Even splitting into halves was difficult for her this time. Rectangles are easier to split than circles."

These observations were supported by data from other learning activities. When making comparisons as they constructed models to accomplish other learning activities, the teachers made references to specific students, to the video exemplars from the learning trajectory, and to generic grade-level students based on their experiences. For instance, in the Student Work Analysis activity, one teacher recalled her own student from the Pirate Birthday clinical interview. In conjecturing what the student may have known about fair sharing based on a work sample, the teacher compared the sample to her student and to the *three equipartitioning criteria*:

W3: Yeah. She understands that each of the four people is supposed to get it.
S5: It seems like she knows a lot about--
S4: She knows about quality, but not quantity. I mean, she knows the number, but not amount.
S5: Yeah.

S4: That's the same thing as our Sam. She can give everybody one but they weren't even.

In another case, a teacher made reference to the video exemplars from the earlier PD sessions for comparison. Because these recordings were shown in the context of relating students' behaviors to their thinking and thus may be regarded more as theoretical examples than the individual observations they had made with students. For example:

S16: I think she understands amounts. She knew there had to be fours. She knew there had to be three.
W16: Numbers. She knows the numbers.
W15: Yeah. She knows that they have to have…
S16: But she didn't really realize what was equal.
S14: She knows that one-to-one correspondence. That there are four people that each have their own piece.
S16: I think what a lot of these kids did in all these things [video exemplars] he showed us, they start before they think it through. And I almost bet that's what she did. And the third one cut and she thought, uh-oh, it's not going to work.

All of the groups used their own experiences with students at different grade levels to make sense of the student's work. One group drew on their knowledge of Kindergarten students in their discussion of what they student may have known about fair sharing:

S14: And especially in Kindergarten when you're talking about concepts of print is a heavy emphasis, they're taught for everything to start on left. And I will bet you she started on left in the circle and drew a cross. And then went like, crap, what do I do now?
W15: Or she might have actually tried to draw it in half and then was like I need one more piece.
W16: That's what I thought of…
S14: The concept of starting in the middle of something, that's not--
W15: You're working your way out and that's hard.
S14: I don't even think that's developmental to a Kindergartner.
S16: I don't think I would do that… I'm not… but I wouldn't.
W15: Yeah. You have to learn, I mean I had to learn those how do you-- because I knew that this wasn't equal, but you had to learn how do you make it equal.

From these examples and others, I concluded that after the ERNR PD, the teachers made empirically-based comparisons to specific students or to generic, composite ones when constructing models of students' thinking, but also to theoretically-based examples from their learning trajectory experiences. Whereas teachers initially made comparisons to students they had encountered by grade-level or comparisons to curriculum based on their professional experiences, later analyses revealed that the teachers included examples from the learning trajectory that were selected and shared based on the *three equipartitioning criteria* and associated behaviors and verbalizations when comparing individual students' work.

Inferring

Clinical interview analyses from early in the ERNR PD contained inferences that were holistic and evaluative. These inferences were broad and nonspecific about what students understood about equipartitioning. For example, in her analysis of the Pirate Treasure interview, one teacher wrote, "The Kindergarten student understands that fair sharing means to separate objects. She doesn't know how to share equally." Another teacher claimed, "I noticed she had no concept of fair share." Also, the analyses judged categorically that students "understood fair sharing" or "did not understand fair sharing." For instance, one teacher wrote, "It appears that this student knows what fair share means and how to apply it." Another stated, "Olivia did not understand what fair share meant." In these cases, the teachers' descriptions of the students' actions were general and did not clearly connect how the actions related to their evaluations of the students' thinking.

After experiences with the learning trajectory, teachers' analyses of clinical interviews were focused on students' coordination of the *three equipartitioning criteria*. These inferences were more discriminating in that they described both what the students did and did not understand. Further, the analyses connected specific evidence from the students' behaviors and verbalizations to students' thinking. Most striking about the analyses was the teachers' use of the *three equipartitioning criteria* as a means of structuring their summaries. Many of their analyses contained statements about which of the three criteria where the foci of the student and which were ignored, such as "She was concerned about the number of cuts she made rather than the size of the pieces," or "In observing Sam dividing the round cake between three pirates, it is obvious that she is only interested in the number of pieces. She was not concerned at all about each pirate having an equal size." When asked to share fairly a circular cake between two people, one student cut two small sectors and left the remaining cake untouched. The student's teacher wrote:

> The student did not understand that the whole cake should be used in the sharing activity. He did not acknowledge that he knew what half meant or could not cut the cake into thirds or fourths. In the video we see that when asked to share the cake fairly between two people he only cut two pieces and deemed that fair. In a way he is right but did not share the whole cake. It might be that the student does not know how to cut things equally in half or he has never had to before. He did not seem frustrated though and I believe he genuinely thought he was doing the correct work

The student may have been satisfied by creating two equal-sized pieces because he did not believe that the whole cake should be shared. In terms of equipartitioning, the teacher inferred that the student had not coordinated the need to exhaust the whole when fair sharing with the correct number of equal-sized pieces. In each of these examples, the teachers use the *three equipartitioning criteria* from the learning trajectory to support their inferences about students' understandings of equipartitioning.

Additionally, many of the teachers made connections between students' behaviors and verbalizations with cognition. In the Pirate Birthday interview, a student cut triangular wedges from a rectangular cake similar to slices of cake or pizza one may cut from circular cakes or pizzas. She made three piles of the triangular slices when sharing among three pirates. Her teacher wrote:

> The student's pieces were cut similar to triangles…The student also cut the rectangle paper cake the same way. She did make three piles, one for each pirate, and she placed a piece on each pile before moving on to the next. After using the entire cake, she recounted the pieces and realized that one pirate had more. The original pieces given were approx. the same size. Yet when she realized that one pirate had seven, she took that piece away and divided just that piece between three pirates. In her eyes, this was equal. She was focused on number of slices
> then.

The teacher used the student's actions of creating piles of the same number and then sharing the left-over piece among the piles as a way determining the focus of the student. In another example from the Interview Analysis activity, a student Emma made two parallel lines on a circular piece of paper when sharing fairly among three pirates. After finishing, she pointed to the center strip of cake and said, "It's bigger than you think." In summarizing what the student knows about fair sharing, one teacher wrote, "She knows there needs to be 3 pieces. She seems to realize the one piece should be bigger since she said 'it's bigger than you think.'" In this example, the teacher is connecting the student's verbalization with the idea that the pieces need to be equal-sized.

As with the clinical interview activities, the teacher's inferences regarding students' thinking in other learning activities were based on the learning trajectory and the *three equipartitioning criteria*. Rather than making a judgment that the student "understands" or "does not understand" fair sharing, the teachers made distinctions in what the student may have known or not known based on the work samples. During the Student Work Analysis activity, one group of teachers discussed:

W3: Yeah. She understands that each of the four people are supposed to get it -
S5: It seems like she knows a lot about-
S4: She knows about quality, but not quantity. I mean, she knows the number, but not amount.
S5: Yeah.
…
S5: So we believe that she knows how to share equally but not fairly?
W3: Yes.
S5: Okay. What do you believe Emma does not know about fair sharing?
W5: That each piece needs to equal in size.
W3: Each piece has to be exactly the same size. Yes. Maybe she should put them on top of each other, you know, like some way of seeing them. Because some people really just don't see it. They're like, oh that's pretty close.
S5: Yeah. Well they would know if this was a real cake and if one person got that middle slice they would understand fair sharing.
W6: That's right.
W3: That's the piece they all grab for.
S5: That's good. Okay. So she does not know about the quantity that each person should get. She does know about, that each person should receive a fair share. What questions do you have about Emma's understanding of fair sharing? The question I have is does she know how to divide evenly among three people or five, or seven people? Any kind of odd number. She probably doesn't.
W3: Yeah, because it said, even with the rectangular cake, she wasn't able to share for three.
S5: Which is, a rectangular cake is a lot easier.
W3: Yeah. It's a lot easier than a circle.

Based on previous samples of the student's work, the teachers inferred that the student knew that when sharing fairly that each person should get the same number of pieces. But based on the sample of sharing for three, the teachers believed that the student did not understand that the pieces had to be of equal size.

Restructuring: Knowledge of Equipartitioning and Students

Taken together, the models of students' thinking that the teachers built and learning trajectory restructured their knowledge of content and students. The learning trajectory allowed teachers to make predictions concerning students' behaviors and thinking about equipartitioning and sensitized them to behaviors indicative of different cognitive orientations. Experiences with the learning trajectory provided them with a language to discuss these behaviors but were largely insufficient to assist them in explaining its interactions with the curriculum. It assisted them in locating students' thinking within the learning trajectory and influenced their next interactions with students. These patterns surfaced during the secondary coding of the data highlighted by the instructional practices portion of the framework.

Predictability

The teachers' models and knowledge of the learning trajectory allowed them to make predictions about the behaviors and thinking of students. For instance, after each segment of the video recording of the Interview Analysis activity, teachers were asked to predict how the student would solve the next task. Many of the teachers suggested strategies consistent with their constructed models and described in the learning trajectory. After viewing a clip of Emma unable to share a rectangular cake fairly among three pirates, one teacher wrote:

> I believe Emma will be able to share among 6 people based on the attempts for 3. She begins with cuts in half which makes her chances better since 6 is an even number

Later, after viewing segments of Emma sharing among three and six pirates using a rectangular cake and for three pirates with a circular cake, another teacher predicted this about Emma's sharing of a circular cake for six:

> She will cut the cake in half diagonally and then again to create 4 parts. She will then draw a line does the middle make size parts. I think she'll do this because she sees things in halves and only makes cuts that result in halving the section.
> Emma will make a diagonal cut through the cake and will probably cut though the mid-line again to make six sections. If she can't make six sections (on a rectangle), she probably won't be able to do three (on a circle)

In both examples, the teachers used the models they were constructing of Emma's understanding of equipartitioning based on her previous sharing strategies with their knowledge of the learning trajectory to make predictions about how she would complete other equipartitioning tasks.

This observation about predictability is supported by data from other learning activities, such as the Student Work Analysis activity from Sessions III. For instance, when analyzing the work sample in figure 6.c, two teachers commented:

> S9: It seems like he wants to throw away that piece. He's ignoring that piece.
> S10: He knows that it doesn't belong there, that that one is too big – so he wants it to be equal.

Based on the work sample, the teachers were predicting that the student was not coordinating the condition of exhausting the whole from the *three equipartitioning criteria*. The teachers' knowledge of the learning trajectory assisted them in anticipating how students might approach new instructional tasks.

Sensitivity

Another way that the learning trajectory restructured the teachers' knowledge of equipartitioning and students was that it sensitized them to students' behaviors and verbalizations that were indicative of particular ways of thinking and were less likely to dismiss actions that they did not understand. For instance, in the Pirate Birthday clinical interview previously mentioned, the student cut triangular wedges as pieces when sharing a rectangular cake. In her analysis, the teacher's partner wrote: "I was surprised to see how she cut the pieces. She did not cut in half or cut into small rectangular pieces. The student's pieces were cut similar to triangles. I realized that she was trying to cut slices like pizza." The other teacher wrote, "When Kimberly was cutting the pirates fair shares of birthday cake, she cut the pieces in triangles as if she was cutting a round birthday cake." In another interview where the student continued to equal-sized parts into more pieces, a pair of teachers wrote in their analysis, "We aren't sure why she kept cutting the pieces into smaller shares unless she wants them to have 'more pieces.'"

Also, the teachers were sensitive to students' verbalizations. In the Interview Analysis activity, many of the teachers commented on Emma's utterance, "it's bigger than you think" believing that this indicated Emma's understanding of the need for equal-sized parts but her inability to create them on a circle. In one of the teachers' interviews, a student shared rectangular cakes among various numbers of pirates but always referred to the pieces as "half." In describing this, the teacher selected a clip showing the student sharing among four. She wrote:

> She also responded by saying that each pirate received half the cake instead of a fourth. Emma knew that each pirate would receive one piece—that's how she knew it was a fair share. Even though Emma doesn't seem to understand

the terminology of fourths etc., she does know how to share fairly. I believe she is just lacking in terminology.

These examples illustrate the teachers' awareness of the subtleties within students' behaviors and verbalizations that may indicate particular understandings and that come to bear on the teachers' models of their thinking.

As before, this observation was supported learning activities other than the clinical interview analyses. In the Student Work Analysis activity for example, some of the teacher groups noticed the students' notation written on the work samples. Rather than dismiss "1 = 4" and "1 = 3" as incorrect, the teachers used this to further refine their model of the students' thinking about equipartitioning. One group stated:

S14: I think she's got some basic fraction skills. I think she's trying to say one of the four instead of - one of three.
S16: Right.
S14: Yeah. That basic kind of ratio. I just think that radial cut is hard -
W15: It is hard.
S16: I don't know that a Kindergartner could do that.
S14: It's hard for me to make!

Another group also noted this, stating:

S17: She wrote on it. She wrote one equals four
W17: That's amazing that she would even know equals.
S17: So she has an idea -
W17: She has a really good sense of it, being a Kindergartener.

Emma may have been expressing a many-to-one relationship (Confrey et al, 2009) between pieces she created and the original whole with her invented notation and discussion. In both dialogues, the teachers noticed this and used that observation to further understand what they student may have known about equipartitioning.

Language

The learning trajectory provided teachers with common language to discuss students' thinking about equipartitioning. Based on their experiences in the ERNR PD, the teachers began to incorporate theoretically-based language addressing students' behaviors, verbalizations, and cognition into their analyses. This allowed them to communicate precisely about behaviors and thinking as they described students' knowledge of equipartitioning. In the Clinical Interview Analysis activity, teachers described different strategies students employed to share cakes fairly, including parallel, diametric, and radial cuts. In anticipating Emma's approach to sharing a circular cake among three people, one teacher wrote, "I think she will attempt a line across (not radial) but I don't think she will accomplish it." In other cases, teachers used language from the *three equipartitioning criteria* to describe their models of students' thinking. Another teacher wrote:

> She understands that the pieces need to equal the number of people sharing, she knows that she must exhaust the whole – she knows that the pieces should be the same size, but worked at getting the correct number. She knows how to halve to get a fair share, but stumbles at sharing 3 and multiples of 3.

Teachers used mathematically precise words, such as "radial," "quantity," and "equipartition", in their discussions in other ERNR PD learning activities. In the Task Ranking activity for instance, one group of teachers even attempted to use the terms "quotitive" and "partitive" in their discussions. They stated:

S16: Is it quotitive… and what was the other?

S14: S16, was it quotitive or quantative?
W16: Quantative… I recognize that word, anyhow.
W15: I mean, I've heard it…
S16: Every time he asked us, I had it wrong. I literally had it wrong.
S14: Was it quotitive or quantitive?

Though they had not fully incorporated the terms into their vocabulary, this is an indication that the teachers were adopting and trying to use words from their work with the learning trajectory to communicate clearly their ideas with one another.

Curriculum

There is evidence in the data that teachers' curricular knowledge interfered with their knowledge of the learning trajectory. Largely in this case, the teachers' experiences with the learning trajectory were not sufficient to restructure their knowledge of the curriculum to incorporate equipartitioning effectively. In the Task Ranking activity, the teachers' curricular knowledge regarding the size of numbers competed with and overwhelmed their knowledge of the learning trajectory. For instance, all of the grade-level groups correctly judged the task of splitting 128 cookies into bags of 12 as difficult, but none of them explicitly noted that this was a quotitive task. Rather, the groups made this judgment based on the size of the number. Grades K-2 focus on developing number sense for numbers up to 30 in Kindergarten, 99 in Gr. 1, and 999 in Gr. 2. The teachers reasoned that 128 was larger than the numbers in the other tasks, therefore the task was more difficult. One group discussed:

S2: Maybe we should find the most difficult one.
S1: A batch of 128 cookies split into 12.
S2: That's just too hard.

Another group used the similar reasoning. They noted that creating groups of 12 was "another whole thing by itself," but it is unclear from her reasoning that this was due to its quotitive nature or the need to count to create a collection of 12.

S5: Well there's that one batch of 128 cookies split into bags of 12. But that's a huge number.
W3: It's such a big number. That's a lot of--
W5: And you know -
W5: And grouping them into groups 12 is another whole thing by itself, I think.
S4: I think so.
W5: Because if they can't count to 12 they are in trouble.
S5: That's right. Alright.

In these examples, the influences of the teachers' curriculum overshadowed what they had learned about dealing as strategy for creating equal-sized groups.

Supporting this observation was data from the Equipartitioning Task Adaptation activity. When adapting tasks from *Investigations*, the teachers used instructional units dealing with modeling problem with arithmetic. Because of the focus on creating tens, counting by grouping, building place value understanding, and ultimately multiplication as repeated addition, approximately 50% of the teachers created problems that were not equipartitioning but quotitive division problems. Furthermore, many of the problems focused on even and odd numbers, related to *Investigations* development of these ideas.

In a few cases, teachers attributed some of their students' behaviors to the influence of the curriculum. In the Pirate Treasure interview, one teacher noted this in her analysis. When the student was presented with a task to share a collection of coins, he promptly counted the collection of 24 coins. After counting 20 coins, he stopped and said, "We can leave these out and make 10 and 10." His teacher wrote in her analysis:

> When given the task to share 24 coins between two pirates, his first step was to find out the total number of coins. When he realized there were 24, he chose to leave 4 coins to the side and just work with 20. I believe he did this because he has a solid understanding of 10's and he knew right away how to evenly divide 20. He quickly divided the 20 into 10 and 10, just in separate piles, not really in any order … In class, I am now curious if he approaches all of his numerical tasks with a "base 10" approach.

Though implicit, the teacher made reference to *Investigations* focus in Gr. 2 on making twos, fives, and tens.

Locating Within the Learning Trajectory

The data suggest that teachers were able to situate a particular student's understanding within a range of understanding described by the learning trajectory. They used their knowledge of the trajectory to describe what ideas a student was likely to already understand and which ideas were likely to develop next. For example, in the Pirate Birthday clinical interview, the student successfully shared a rectangular cake among two and four pirates by making orthogonal cuts. When sharing among six pirates, the student repeated this and cut two of the fourths in half, creating four one-eighth pieces and two one-half pieces. Next, he was unable to create equal-sized parts when sharing among three. His teacher wrote, "The student is not sure how to divide the cake to make equal parts when there are an odd number of pieces. He understands how to divide the cake if it cut into an even number of pieces." This teacher was drawing on her knowledge of the trajectory, recalling that though three is a smaller number, a three-split is difficult. The student's inability to do a three-split prevented him from sharing beyond two and four pirates successfully.

In the Interview Analysis activity, when asked to anticipate how Emma would share a rectangular cake among six pirates, another teacher recalled the increasing difficulty of splits from the trajectory and stated, "She will probably rely on halving and splitting again – winding up with maybe 8 – she does not yet have the skill of equally splitting a continuous whole into thirds." These examples indicate that some of the teachers used the learning trajectory as a means of locating students' understandings within a range of conceptual development.

In the Student Work Analysis activity from Session III, a group of Kindergarten teachers discussed the work sample shown in figure 6.a.

S5: The student was asked to share a key lime pie among three people. What do you think the student understands about the mathematics?
S6: Understand a half - I started with a half. They knew they had to have three pieces….so they halved the other one, so…
S1: The evidence is they had a line down the middle. And they have another line to create three pieces.
S5: Has a diameter cut and a vertical cut.
S2: What do you think the student does not understand about the math?
S5: Equal sharing.
S2: Equal parts.
S5: He doesn't understand how to cut a circle into thirds.

The teachers establish that the student can two-split and that he may understand that there needs to be a one-to-one correspondence between the number of pieces and the number of people sharing the cakes but is still in the process of coordinating the *three equipartitioning criteria* for three-splits.

Movement Through the Learning Trajectory

Finally, teachers were able to make decisions about their next instructional steps based on their constructed models and their knowledge of the learning trajectory. One type of instructional decision involved lateral moves within a level of the progress variable, such as changing the shape or media in order to focus on a particular class of splits or using justification strategies to create dissonance. In the Interview Analysis activity, teachers were asked what they would do next with Emma after she failed to create equal-sized pieces when sharing a circular cake among three. One teacher wrote, "I would work on fair sharing from the stand point of equal. I would show Emma how to divide for a square, rectangle, then circles." Another teacher suggested, "…have her practice sharing with three and six people – she would catch on to this quickly. She needs other ways to share than just halving." Also, teachers suggested vertical moves when planning follow-up activities. Some recommended revisiting previous levels of understanding as a way to overcome current difficulties by relating to more primitive ideas. For Emma, one teacher recommended, "Go back to sharing cookies, gold, etc with 3 people … Maybe with a round cake teach her to fold in half to find the center first before dividing." Similarly, another teacher wrote, "I would give her objects like counters and ask her to fair share among different numbers of people – 2, 3, 4, 5, 6." In both examples, the teachers recommended moving to lower levels of the learning trajectory where Emma may have been successful.

In the Student Work Analysis activity from Session III, teachers were asked what they would do next with the student based on the work sample. Some of the groups of teachers suggested changing from a circle to a rectangle as their next move or changing the number of people sharing from six to four to decrease the difficulty of the task. For example, when suggesting next moves for the student whose work is displayed in figure 6.d, one group discussed:

> S11: Well, you could ask, “do you have four pieces?” He’s going to say, No, you have five, and then…
> S9: And then you need to ask…
> S11: What are you going to do about that last piece? How are you going to make it equal? How will you divide it fairly?
> S9: That kind of answers the next question, too.
> S10: What will you do next with the student? Let them eat cake.
> S9: Ask what are you… right.
> S10: Would you? Give him a new cake? Or slab of play dough or whatever
> S9: And say try again and see if you can divide this, all of the cake, between four people without a left over piece.

The teachers make two suggestions to assist the student. The first is a question that may draw the student’s attention to the piece that they believe he will consider as left-over cake. The second is to have the student try the task again.

Experience with the learning trajectory restructured teachers’ knowledge of equipartitioning and students. It increased their sensitivity to students’ actions and words which strengthened the models of thinking they constructed. These models, taken with knowledge of a learning trajectory, assisted teachers in locating particular students within the range of conceptual development and informed them about what next pedagogical steps might refine the students’ understandings. Their pedagogical moves were accompanied by a sense of predictability based on teachers’ knowledge of a trajectory and their constructed models. Finally, though experiences with the learning trajectory provided teachers with precise language to communicate about students’ behaviors, verbalizations, and cognition, they were largely insufficient for teachers to identify how their curriculum interacted with students' progress along the trajectory.

Restructuring: Teachers' Knowledge of Equipartitioning

In addition to restructuring their knowledge of equipartitioning and students, teachers' models and the learning trajectory influenced the teachers' own mathematical thinking related to equipartitioning. Within the clinical interview activities, there was some evidence of teachers' understandings of mathematical properties described in the learning trajectory. The ways that they described the students' behaviors may indicate their awareness of these properties, though none of them identified the properties by name. For instance, several of the teachers described the essence of composition. One teacher wrote, "she said two, four – so she was taking into account one line could make more than one piece." Another teacher anticipated Emma's strategy for sharing a rectangular cake among six pirates by stating, "I believe that she will divide half and half to get 4 but the other 2 pieces will not be equal or she will come up with 8 equal shares." In these examples, the teachers seem aware of composition of factors and its importance in equipartitioning but are yet to explicitly identify it. They may not have considered how the structure of the two dimensions in the rectangle supports composition or how using composition of factors is linked to arrays and later to multiplication.

Some teachers suggested using a form of equivalence to assist Emma in refining her understandings of equipartitioning. Recall that Emma was unable to create equal-sized pieces when sharing a circular cake among three pirates but was successful at sharing among six. Three teachers suggested covering two pieces of cake in the outcome for six to create a third of a cake. One stated her recommendation as, “I’d work on the round cakes being divided into thirds. Taking the circle she did into 6 equal parts and shading them into 3 sections.” These teachers planned to use the property of equivalence as a means of supporting Emma’s refinement of her own understanding of sharing among six toward an understanding of sharing among three.

Though not emphasized in the version of the learning trajectory shared with the teachers in the ERNR PD, the transitive property has important implications for students’ understandings of the relative nature of rational number. One pair of teachers noticed this in their Pirate Birthday clinical interview. After sharing a rectangular cake between two pirates, the student was asked if there was another way she could have shared. The student successfully shared the cake again and was asked how she would name the result. In her analysis, one of the teachers wrote:

> When asked if she could share the rectangle birthday cake fairly between two pirates, she immediately responded “yes.” She cut the rectangle in half with the long side closest to her. She explained that she cut the rectangle in half and that it was a fair share. She even explained that each pirate would get half of the cake. She said that she could share the cake another way. When W17 gave her another rectangle, she turned the rectangle and began to cut it in half. This time, the short side was closest to her. After W17 asked her, was it still half of the cake even though it looked different from her original rectangle’s half, she replied that it was the same and that it was still half.

This teacher may have noticed an example of transitivity in the student’s work, that is the importance of students’ recognizing the relationship between a piece and the whole in her first sharing as the same as that in her second sharing. These examples indicate that some of the teachers were aware somewhat of the mathematical properties embedded within students’ behaviors and how they could use those in instruction. Given the prominence of mathematical properties in the learning trajectory work from the ERNR PD as well as their importance to a full understanding of equipartitioning and later to rational number, it was disappointing that more teachers did not notice or include these in their work on the analyses activities.

In addition to properties from the learning trajectory, activities from the ERNR PD engaged the teachers' knowledge of equipartitioning as an operation. For instance, at the conclusion of the Task-Ranking activity, teachers were to adapt the following task so that it would have a goal of equipartitioning: 'Kevin wants to put 12 candy canes in the stockings. He puts three in each. How many stockings are there?" In the allotted time, only one group completed the task. Below is their dialogue as the discussed their adaptation:

W16: And adapt this task below. Kevin wants to put 12 candy canes in the stocking. He puts three in each. How many stockings are there? So what? Using pictures?
S14: What?!
W16: It says to adapt it.
S16: To a goal of equal partitions. How many of each? It's already in a goal of equal partitions.
S14: Yes.
S16: No. It's how many stockings?
W16: Yeah. So can't we just show four stockings? You ever seen square stockings?
S14: So we turn it around the other way because it's not… what's that word again?

Initially, S16 believed the task required equipartitioning. However, she realized that it was asking for how many groupings of three were in 12. In this, the teachers' work with the learning trajectory for discrete collections had assisted them in making distinctions between the partitive and quotitive interpretations of division. Through these distinctions are irrelevant when equipartitioning continuous wholes, the teachers work with the trajectory concerning discrete collections supported their recognition of a goal of making equal-sized groups versus measuring the number of groups of a particular size from a collection.

In another instance, some of the teachers' work with the learning trajectory helped them understand a common misconception in early multiplicative reasoning. In the Classroom Interactions activity, the teachers were given hypothetical responses to groups of students solving the task of finding how many parts there would be if they were to fold a rectangular piece of paper in half three times. One of the responses was based on an additive misconception: "We think it's six parts. Three twos make six. We didn't even need to try it out!" Some of the teachers in one group understood that the response indicated an additive approach to the problem. They said:

W16: Number of….What about "C"? "C" understands things being equal but just too lazy to do it?
S16: Well, I know students like "C" where they're just so sure of themselves they -
W16: They don't need to -
S16: Don't realize it's wrong. And I would say prove it. And they would most likely understand it more.
S14: Well because they focus almost on almost that algorithm to two plus two plus--
W15: Yeah – equals six.
S16: That's why if you gave it to them and said to show me, they can do it.
W16: Yeah.
S14: Those are the kids that you have to say prove it to me.
W16: Okay so they understand the equal partitioning. But what?
S16: They used the wrong algorithm.

Whereas S14 believed that the student was not recognizing that each fold in half should double the number of parts, W16 may have believed that since the six was decomposed into a sum of equal numbers, that the student could equipartition. In another group, one teacher, in efforts to understand the response, created six parts with three half-folds by unfolding after each step. They discussed:

> W17: And then half of that. So it can be. Okay. Alright. That's good. So group "B". W18, did you get group "B"?
> W18: I did.
> W17: How did you do that? How did you fold that to get that many?
> W18: I fold it in half and then I fold it in half again and then I folded it a third time.
> W17: Oh. I see what you're saying. Okay. We can get six parts. Three twos make six.
> S17: Because you're using the number two from the half. And three twos would equal six, not three halves would equal eight.

Here, the teachers are using their knowledge of the multiplicative nature of paper folding from the learning trajectory to know that three half-folds would create eight equal-sized parts if the paper is not unfolded.

In addition to restructuring teachers' knowledge of equipartitioning and students, experiences with the learning trajectory engaged their own understandings of equipartitioning and to an extent provided opportunities for them to refine those understandings. Specifically, teachers began to notice the emerging mathematical properties described in the trajectory, identify the distinctions between partitive and quotitive interpretations when sharing discrete collections, and recognize additive misconceptions related to multiplicative reasoning.

This chapter detailed a final analysis of the data from the design study. The teachers' knowledge and uses of the learning trajectory supported the creation of more robust models of students' thinking and restructured their knowledge of equipartitioning and their knowledge of equipartitioning and students within the context of the ERNR PD. Chapter Six will present the findings from the observational study, supporting these findings concerning the construction of models and the restructuring of the teachers' own thinking and elaborating those based on observations from actual classroom practice.

CHAPTER SIX

The preceding chapters described an ongoing and retrospective analysis of the ERNR PD and its influences on evolving conjectures of the study. In this chapter, I report findings from the observational study embedded within the design study. First, I identify and describe the data used for the analysis. I describe patterns in how the teachers used the learning trajectory and their models of students' thinking to restructure their knowledge of equipartitioning and students, outline more evidence concerning the ways that the learning trajectory affected teachers' knowledge of equipartitioning, and describe the ways the learning trajectory assisted teachers to varying degrees in facilitating coherent instruction.

The ten second grade teachers were participants for the observational study. The findings presented in this chapter are to varying degrees. The examples used to illustrate the findings were selected based on clarity rather than representativeness.

Sources of Observational Data

This portion of the study was designed to observe teachers' enacted knowledge of the learning trajectory and to measure gains in their students' knowledge. It was intended to assess the effectiveness of two of the ERNR PD goals related to teachers' uses of the learning trajectory and the advancement of students' learning. Further, it investigates the third research question seeking to understand better the relationships between teachers' knowledge and uses of the learning trajectory for equipartitioning and their students' learning. Second grade teachers were selected for observations because of the timing of the study coincided with units from *Investigations* which could be adapted for equipartitioning. Students' baseline knowledge of equipartitioning was measured using pilot items from the DELTA research group's work to develop a diagnostic assessment system. Ten teachers were observed teaching two lessons on sharing a continuous whole and sharing a discrete collection. After the lessons, students' knowledge of equipartitioning was measured using items parallel to those used for baseline information. In addition, teachers participated in a stimulated recall interview to explain their instructional decisions.

Video recordings and annotated transcripts of one of the two equipartitioning lessons, video recordings and verbatim transcripts of the observation interview, teacher planning documents, and student work samples from the lesson of the $n = 10$ participating second grade teachers served as data for the Classroom Observations and Observation Interviews. The analysis was guided by the study's framework and Conjecture Five concerning the relationship between the teachers' knowledge and uses of the learning trajectory and students' learning.

Restructuring: Knowledge of Equipartitioning and Students

Teachers used the models they constructed and their knowledge of the learning trajectory to restructure their knowledge of equipartitioning and students during instruction. Consistent with the findings from the design study, it assisted the teachers in making predictions as to how students would approach equipartitioning tasks and the likely difficulties students would encounter. Also, the trajectory sensitized the teachers to behaviors and verbalizations discussed in the ERNR PD. Further, data from the observations and interviews indicate that the teachers' knowledge of the learning trajectory competed with their curriculum and its goals in some instances.

Predictability

After experiences with the learning trajectory, teachers were able to make predictions about what students would and would not be able to do when preparing and implementing a lesson on equipartitioning. In the observation interviews, several teachers reported that they had anticipated the difficulties students encountered that were explained by the trajectory, such as circles being more difficult in general than rectangles. They had predicted difficulties related to the mathematics of the lessons, such as coordinating the *three equipartitioning criteria*. While many of the teachers' uses of the learning trajectory were implicit in the observation interviews discussions, one teacher made explicit reference to its use in making predictions about what students might do for the lesson. During her observation interview, W13 discussed that she believed circles were more difficult than rectangles, due in part to the fact that students can use a parallel, sequential cut strategy (W13 refers to these as "straight lines") on a rectangle but not a circle.

> R: What strategies did you think that they would use to share the pies and the cakes?
>
> W13: Well, I knew – I had a feeling that they would use straight lines on the rectangle and I knew that would be the easiest, and it was. Because it's a lot harder to do the circle than it does…With the circles and the rectangles, those kids [from the clinical interview activities and the video exemplars] had a lot of trouble with those circles, circular play-dough things, so I knew that it was going to be hard for the rest of the class because if she had that much trouble

Others had predicted difficulties related to the mathematics of the lesson, such as exhausting the whole and creating equal-sized pieces. During the same interview, W13 stated she had predicted that her students would not consider the whole cake or pie based on her recollection of the video exemplars from the learning trajectory and the clinical interviews. She stated:

> Misconceptions? Well. Yeah, I actually – they seemed to not grasp the idea that all of the whole had to be taken. They kept coming up with these odd little pieces that they wanted to just throw away, get rid of, do something else, so they didn't know they needed to extinguish the whole and that was something that I knew – that I didn't think that they would grasp that from the get-go.

When asked what she anticipated her students would do with the lesson, W12 had predicted that her students would have difficulties creating equal-sized pieces. In the observation interview, she said:

> W18: Well, I know that in first grade they focus on halves. I know that they talked about halves of different shapes and then we kind of go from there with them. And talk about thirds and fourths and – so I knew that they would probably be able to divide something in half, I just didn't know if they would do it equally or not. I didn't know if it would be equal. Maybe halves equal, but I didn't know about thirds and fourths if they would know how to do that.
>
> R: So what did you think your students would do on the cakes and pies lessons?
>
> W18: I thought that they would probably divide it in half and then I was hoping that they would divide it in half again. I was hoping that they would see that if they divide it once in half then they could do it again and make four.
>
> …
>
> W18: I didn't know if they would – I knew they would probably divide it into four pieces, six pieces, whatever. But I didn't think that most of them would – I didn't know if they would do it equally. I figured they would just divide it

While teachers' uses of the learning trajectory are implicit in many of the observation interviews, two of the ten teachers made explicit reference to its use when they predicted how students might approach the tasks in the lesson. S12 said:

> R: So, for this particular lesson with your students, what did you think they would do?
>
> S12: I thought they would be able to share by two, I thought they'd be able to halve and halve again for anything that was divisible by two, but –
>
> R: Why'd you believe that?
>
> S12: Because I just felt that as far as their ability levels, that that's where they were on a learning trajectory, that they had it – they probably had gotten there, but not, not beyond that, not with odd numbers. I thought that they would do the

halves, I thought that they would halve and halve again and I thought that they would try to do that too, for six and then they would realize, "wait," and try to go back. But I don't think that they would– I didn't think that anybody could split into thirds and then halve that.

R: Okay. Why did you think that? Or why do you believe that?

S12: Just based on other mathematical tasks we've done and their performance. I just didn't feel like they had reached that on a learning trajectory.

These examples supported the findings from other data that the learning trajectory assisted teachers in making predictions about students' behaviors and difficulties the may encounter.

Sensitivity

Within the observation data, there was also evidence of teachers' sensitivity to students' behaviors and verbalizations. When students shared their methods in class, some of the teachers noticed behaviors that may have been related to students' attempts to coordinate the *three equipartitioning criteria*. For instance during one of the lessons, Kobe and Mary attempted to modify the sequential, parallel cuts strategy used with rectangles for a circle by creating curved cuts on the pie rather than straight cuts (see figure 18). His teacher, W15, noticed that this may be important and inquired:

W15: Okay. Good … There's four pieces. Show me one person's share. You got one person's share shaded in. So that would be a share, that would be a share, that would be a share, that would be share - Did anyone else think of it this way? How many people thought of this way? Raise your hand.

W15: Why didn't you draw the line straight across, Kobe?

Kobe: Because if I did that then it wouldn't be the same amount for each one of them… and you said that you didn't want leftovers. See. [traces arc]

W15: Did you try to curve a little because it was circle there?

Kobe: [nods yes]

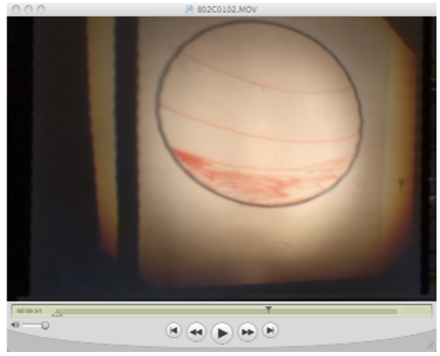


Figure 18. Kobe's pie among four.

I infer that Kobe knew that parallel cuts would yield unequal-size pieces and was attempting to compensate for this by making curved cuts. In the observation interview, I asked W15 about this exchange to see if she was inferring an early form of compensation in the students' work.

R: Why did you ask him that, do you remember?

W15: Because I could see – I was wanting to see where he was going with it. Like, since this was a circular object, if he was trying to get that fair share in there by making the lines curvy or did he know that they would be automatically –

R: How did you know that would be important? If that's an important thing to bring out, how do you know?

W15: Because when you're dealing with circular things, it's different than dealing with straight edges. It makes things a little harder to divide, so I was trying to get to see where they were going with it, to see if that's why there were just drawing it on there just to –

Though it is unclear if W15 realized the potential of this example in developing compensation, her knowledge of the learning trajectory clearly related to her noticing the students' method and her subsequent questions and interactions with Kobe and Mary.

Also during the lessons, teachers were sensitive to verbalizations made by their students. In W18's class, Jason produced three parallel, sequential lines on a rectangle when sharing among four and shared this with the class. He had not judged well and resulting pieces were of unequal sizes. W18 had overheard Anna's discussion about an "invisible line in the middle" while the children were working and called on her to assist Jason.

W18: What if Jason - did anyone else do this on your paper? How did you know what size the pieces needed to be? Anna, how did you know how to divide it? What did you do?
Anna: I made an invisible line and moved it until I could find the right pattern.
W18: So when you say you made an invisible line, in your mind? Did you draw it - where at? Over here? [pointing to transparency] Did you put that invisible line in a certain place?
Anna: Yeah. I tried to make it so - so it would be near the middle.
W18: In the middle? Okay, so she was thinking, so, did that invisible line divide that into two parts then?
Anna: Yes.
W18: [to the class]She did an invisible line in her mind. And so she tried to figure out where the middle would be, and then she wanted that to be her first line.

Because W18 overheard Anna and realized that she was searching for a line of symmetry, she used Anna's idea as a means of supporting Jason's revision of his method for sharing to a more sophisticated strategy that included composition of factors.

However, in other cases some teachers failed to notice behaviors or verbalizations that were important in developing equipartitioning. In W16's class, Jayni introduced the idea of creating multiple pieces for each person sharing. When sharing a pie among four, she created eight equal-sized slices. In the discussion, one child from the class asked the question about how many slices each person would get if there were 20 slices all together. Though the teacher heard this, she ignored the question and moved on to question about the fairness of the pieces.

W16: Alright Jayni, count your pieces that you've got there. Count your pieces out loud for us.
Jayni: One, two, three, four, five, six, seven, eight.

W16: Okay, she has eight pieces, okay. Now can you color in our fair share? We're still serving how many people? [4] Everybody, how many people are we serving?
Class: Four.
Jayni: Each people gets two pieces.
W16: Bingo. Each person is gonna get how many pieces? [2] That's four people, but we've got eight pieces of pie. How many pieces of pie do you get in your fair share?
Class: Two.
W16: Two. Can you do that? [Yes]
Child: How much pieces would you get if you put 20 on there?
W16: Alright now, are those equal? Is everybody gonna get their fair share of that piece of pie?

Following this, the class had small group time to share cakes and pies among six people. Many of the students had adopted Jayni's strategy and were attempting to share by making multiple pieces and were trying to figure out how pieces they would need to cut. When circulating, W16 stated to the class, "Don't make this harder than it is - these are not stained-glass windows!" Though this was an opportunity to explore equivalence with the students, the teacher viewed these solutions as nonessential and tangential, whereas if she had noticed the child's question earlier, she may have made better uses of these ideas from the students. In another case, Laura created six scallop shapes across the top of a rectangle when sharing a cake among six shown in figure 19. My interpretation of Laura's work was that she was cutting six and four "cake squares" because she indicated that she would "save" the rest of the cake, implying that these were actual pieces to allocate among the people having dessert. Another explanation may be that she was focusing on creating six by creating loops and counting intervals. However, in the observation interview, I replayed this clip for W13 and asked her about what she believed that Laura was thinking:

R: Yeah, can you – what do you think she was thinking about? And this is the one where she did the little –
W13: She's an artist. And she likes pretty things. And she thinks little scallops is much prettier than just a straight line going across. If you want to know the honest truth, I think that's the only thing she thought about. I don't think she was thinking about the size of the pieces or anything else. I think she was just looking for something that was pretty.
R: She was decorating the cake?
W14: She was.

W13 believed that Laura was decorating the cake. But the correspondence between the number of "pieces" and the number of people sharing the cake combined with Laura's statement about saving cake leaves this conclusion suspect.

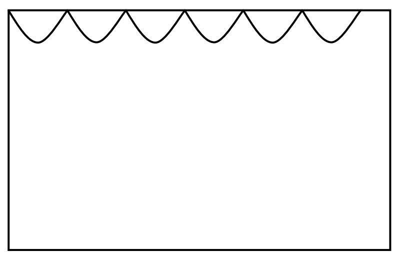


Figure 19. Laura's sharing among six.

In many cases, the teachers' knowledge of the learning trajectory for equipartitioning sensitized them to some of the behaviors and verbalizations outlined in the learning trajectory. Other times, the teachers overlooked some behaviors and verbalizations that could have provided more insight into their students' thinking.

Curriculum

There is evidence in the data that the confluence of teachers' curricular knowledge and their knowledge of the learning trajectory interfered with the goals of the equipartitioning lesson. The Gr. 2 state standards emphasize measurement of length, symmetry and congruence in geometric figures, fraction notation for part-whole relationships, and informal strategies for solving problems through equal-grouping. Also, the unit from Gr. 2 *Investigations* that immediately preceded the equipartitioning lesson emphasizes the geometry of rectangles including symmetries and area measurement. In the teachers' lessons, I observed these curricular goals of measurement, geometry, and numbers competing with the goals of the lesson.

The curricular focus of measurement has potential to assist students in justifying the results of equipartitioning and to make connections to strengthen understanding and coherence. Many of the teachers attempted to use measurement in this way with varying degrees of success. For instance, Andy and Jamie shared a rectangular cake among four by making parallel cuts, one horizontally and the other vertically. They shared the vertical method at the overhead, but the lines were not evenly spaced. W15 used Andy's idea of measuring each strip's length to reinforce the idea that the pieces needed to be equal-sized in order to be fair.

> Andy: Well, I just cut it in strips.
> Jamie: And he did it this way [horizontal] and I changed it around.
> W15: Okay. So--Jenny, look at me, sweetheart. What would you call each person's share right here? Do we have a name for each person's share?
> Andy: A fair share. [unclear] You could just flip and do line, line, line [motioning vertical lines]. Or and you could take a ruler.
> W15: So Andy said if he got a ruler he could even be more precise. Then why would you need a ruler? Why would it be even more important to have a ruler?
> Andy: 'Cause you could like lay it across here and you could and you could lay it across. And do inches. And you could lay it across and see if they got the same amount.
> W15: So it's important for each one to be the same.

Though W15 used the idea of measurement to reinforce the *three equipartitioning criteria*, *Investigations* treatment of measurement requires the import of a standard unit which may or may not be commensurable. However, equipartitioning also creates a unit relative to the whole which can also serve as a measurement.

Three other classrooms explored using area measurement as a means of justifying the results of their equipartitions. In all three examples, the teachers make clear references to activities from the preceding unit in *Investigations* that use one inch square tiles or other objects to cover rectangles to advance an understanding of area measurement. In once class, W15 suggested one-inch square tiles as a way of justifying the fairness of their sharing when Liz and Ally shared a rectangular cake among four by creating two diagonal cuts. Ally mentioned that they could use pieces of candy as well.

> W15: So you guys think this is fair, but just to make sure you would want to measure it with something, kind of like we measured— because we have been doing it with rectangles, and what did we measure the rectangles with to see how much was in each?
> Liz: Colored tiles.
> W15: Those colored tiles. So maybe if we double-check theirs, they might want to take something and measure it, and fill it up to see if each one is the same size. Very good. Okay. So I would be curious to see even if this one looks a little different than this one, doesn't it? [pointing to top share versus side share]
> Ally: They're a little wide than these two.
> W15: And they're a little longer?
> Liz: Yeah.
> W15: So if we measure that, I wonder if they'd be the same size. We'll have to do that sometime. Very good.
> Ally: If we had little tiny pieces of candy, and it had four then -
> W15: The other one had four?
> Liz: Yeah.

In another instance, S14 anticipated that students would make this connection to work from *Investigations* when planning for the lesson and provided one inch tiles for the students to use as tools while equipartitioning. In the observation interview, I asked her if there were tools or resources that she chose to use that may help her understand what her students may or may not know. She responded:

S14: So, I kind of, in my mind, went through what if they do this, what if they do that, what if I give them shapes and they try to build it themselves because they've been building rectangles. So, I thought to maybe build with the shape and find the shape of the pie and find that connecting idea then maybe they can make the leap over to drawing the cuts on the paper.

R: So, when you were thinking about what they might do – what specific strategies did you think they might do?

S14: Well, I thought some of them might – I was surprised a lot of them did not do - I thought they might use the idea where we had built rectangles and looked at the actual area and how much it took up. So, I thought maybe if they could see the idea of the area of a slice of pie then maybe they would be able to connect to that.

In thinking about connections to students' experiences with *Investigations*, S14 did not anticipate that importing a standard unit of area measure, such as the one-inch tiles, may result in incommensurability whereas using the result of equipartitioning would not.

During her lesson after James had presented his method for sharing a rectangular cake among four using diagonal cuts, S14 questioned James and the class to find a justification for the fairness of the non-congruent parts:

S14: Do you think you still have the same amount of cake in this share as you do in this share? Thumbs up if you agree. Does anyone disagree?

James: Yeah, we do.

S14: Why do you disagree? Give me a reason.

James: 'Cause that one – this one looks real long. That one looks so short and it doesn't really look that big.

S14: Can anybody connect this to another activity we've done this year? Can you make a connection here?

Tim: I think we did something about the square thing.

S14: That's right. The rectangle thing. What do you know? Tell me what you know.

James: We know that just because it's longer doesn't always mean it's bigger. In this case, we do think it is.

S14: How would you go about figuring that out then? How would you go about proving it? Think about that activity where we had all the different rectangles and we had to line them up. How did we go about proving that?

James: Uhm, we used the square tiles, and I would use that.

S14: So if that's what we were exploring today, we would be able to prove it by measuring with the square tiles. And that was measuring what? It started with an "A".

Child: Area.

S14: You get the 50 million dollar prize, my friend for figuring out that vocabulary word. Area. Measuring the area. Very good.

S14 used her students' recollection of an activity from *Investigations* to make a connection to area measurement during the lesson.

Another curricular emphasis that affected the lessons was symmetry and attributes of rectangles. One teacher attempted to recall students' experiences with parallel lines in order to confront the misconception that one must make *n* cuts to create *n* pieces. Early in the lesson, the teacher selected a pair of students to show their method for sharing a rectangular cake among four. The students make four parallel, sequential cuts which created five pieces.

S12: Let's look at what they did. Because what they did is a very important thing for us to notice. ... There were four people, so they made how many cuts? [4] Four. And you know what? They made special kinds of cuts because none of these cuts touch each other, do they?

Class: No.

S12: You don't have a cut coming across here that touches any of the other cuts. Those are special cuts. Do you know what we call them? Parallel. Where have you heard that word before?

Child: You.

S12: Yes, but for what? Parallel. Where have you heard that? They don't cross. They're straight. But you know what? We didn't need four, did we? How many cuts do we need? Do you guys see now? … How many cuts did we need to make? Can you figure it out?

Jon: We made four.

S12: You made four cuts. Yes you did. But how many did you need to make? How many pieces did you get when you cut four, when you did four cuts? How many pieces are up here?

Aidan: Five.
S12: Five. So anybody have any idea how many cuts you should make to get four? Tim? Three. So what are we saying? Three cuts gives us how many pieces?
Class: Four

Later, two students showed their method for sharing a pie among four people. S12 carried the discussion of parallel lines over to the new example.

S12: Four pieces. Now is this - are these cuts parallel?
Child: No.
S12: No, they're not. Because what happens?
Child: They touch.
S12: They touch. They cross each other. That's right

In two other examples during the lesson, S12 recalled parallel and intersecting lines in the class discussion. In these, it seems that the influences of other geometric goals became a focus of the lesson rather than serving a supporting role for justifying the results of equipartitioning. Perhaps because of the interactions with the existing curriculum, other concepts tended to either overshadow the goals of the lesson or co-opt the teachers' use of the learning trajectory.

A third curricular influence on the equipartitioning lessons was in terms of number sense. Some of the teachers used the lesson to develop part-whole reasoning and fraction notation. Similar to the previous example related to parallel lines, this became a primary focus for part of the lesson. In her class, W18 used a sequence of students' solutions to sharing a rectangular cake among four people to understand equivalent fractions. Alice and Robert shared the cake by creating eight equal-sized pieces. After naming one piece "one-eighth," the students called a fair share "two-eighths." Later, Denise and Phillip displayed their method of creating four equal-sized pieces and named their solution one-fourth. These methods are shown in figures 20 and 21. W18 uses these two different approaches as a means of addressing equivalent fractions with a part-whole area model and fraction notation written on the board.

W18: So guys, would two of those eight pieces right there be the same thing as Denise's part?
Class: Yes.
W18: Look at that and the look at that. Just ignore Phillip's mark here. Is that the same?
Class: Yes.
W18: Yes. Look how smart you are. You did one-fourth. Alice is also gonna be eating one-fourth of that cake. But they divided it into eight pieces. So she's gonna have two of the eight pieces. Denise is gonna have one of the four pieces. But aren't they the same thing? Now look how smart you are. Look at that. So two of eight pieces, or two-eighths of a piece of cake, or of a whole cake, is the same thing as one-fourth?
Class: Yes.


Figure 20. Alice and Robert's method.


Figure 21. Denise and Phillip's method

Here, other curricular goals such as understanding equivalent fractions and fraction notation became a focus, though the lesson was simply on creating fair shares. One essential idea of the learning trajectory for equipartitioning is for students to make sense of fractions based on experiences creating equal-sized parts. By naming the results of sharing as "one of the four parts" or "two of the eight parts", students could focus on understanding the equivalence of those relationships. In this case, the rush to symbolism supported by the curriculum may not have allowed these foundations to develop fully.

Work on fractions was not the only curricular goal related to number sense that affected the lessons. When sharing a rectangular cake among six people, one teacher selected a student's method to share with the whole class. The student created six by eight grid on the rectangle and had shared eight of the pieces to indicate a share. The teacher used this example to talk about multiplication.

> W17: One of them went like this. 1, 2, 3, 4, 5, 6, 7, 8. 1, 2, 3. This is uneven here. Okay. 1, 2, 3, 4, 5, 6. Okay now, I know… I know and you guys will know later that by looking at this I can count these squares and I can count these squares, and it tells me how many squares are in all. Did you know that?
> Kim: Yeah. Kyle taught me that.
> W17: Okay.
> Kim: Because when we had that math thing a long time ago, I said Kyle, I have to count this, and he said no all you have to do to figure out how many there is to count the top and the side.
> W17: And then if you know your multiplication, if you know times, you can figure that out. So I know that 6 x 8 is 48.

While both part-whole interpretations of fractions and multiplication are curricular goals, they were not the intended goals of the equipartitioning lessons. The learning trajectory for equipartitioning intends for students' understandings of fractions and multiplication to be built from experiences creating fair shares, which was the goal of the lesson. As teachers taught the lesson, influences of *Investigations* related to number sense encouraged teachers to focus on formalism rather than providing local understandings to support deeper understandings of this goal.

Influences from the curriculum such as measurement, geometry, and number ideas could potentially add value to the lessons and support students' development of a more robust understanding of equipartitioning. For many of the teachers however, these other curricular goals preempted the equipartitioning goal to the possible detriment of students.

Restructuring: Teachers' Knowledge of Equipartitioning

The implementation of the lessons further provided a window into the teachers' knowledge of equipartitioning. Three different areas of their knowledge affected the way they used the trajectory in instruction. First, teachers failed to recognize when students were not actually equipartitioning but were assembling equal-sized pieces rather than beginning with a whole. Also, teachers' uncertainty of correct equipartitions that are non-congruent led some to fail to endorse or refute some solutions for the class. Finally, the teachers' uses of the properties described in the learning trajectory varied from incorrect and inappropriate to correct and helpful in instruction.

Assembly and Relating Pieces to the Whole

During the lessons, three of the teachers failed to recognize that students were not equipartitioning but were using other strategies to fairly share the cakes and pies. In one case, Marley made a mistake on the transparency when sharing the rectangular cake among six people and created eight equal-size pieces in a 2 X 4 array. Her teacher S13 suggested that she try again. While Marley was working, S13 attended to classroom management and did not closely observe Marley's work. Using the bottom of the transparency as one side of a new cake, Marley drew a long line parallel to the bottom of the transparency to represent the top of the new cake. Next, she drew a vertical line perpendicular to these to indicate a third side of the cake. Based on her previous attempt, she knew that each vertical cut would create two pieces, so she drew a vertical line and silently mouthed "one, two," drew another line and counted "three, four", and then another line and counted "five, six." She finalized her drawing with a horizontal line connecting the left and right "sides" of her cake. The result is shown in figure 22. When S13 refocused on Marley's cake, she covered up the remainder of the vertical line on the overhead projector:

S13: Okay, Marley, show me the other cake. So we're looking at just this part right here. Let's cover that up. Can everyone see what she did? She redid it 'cause she messed up. So does she have six pieces here? Let's count them. One-- Does she have six pieces? Do you think these pieces are a fair share?

Class: Yes.

S13: Are they equal and the same?

Class: Yes.

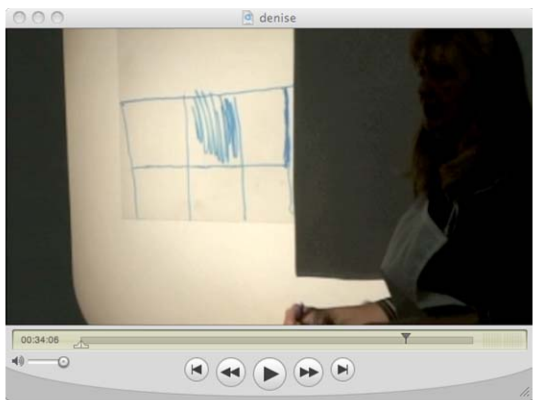


Figure 22. Marley's sharing among six.

In the observation interview, I asked S13 about this episode.

R: So, what do you think she was thinking about?

S13: This was the six. I think she – she knew she needed six pieces and she took these away - Mm-hmm, she understood she needed six pieces.

R: Now, what evidence do you have of that?

S13: Well, you can see the three and three.

R: What does this long line up here indicate to you?

S13: Maybe that she struggles with – I don't exactly know the words to what I'm thinking but – I don't know.

In infer that S13 understood equipartitioning, not as an operation, but as an outcome producing the desired number of pieces. She did not see that in assembling the pieces, she failed to understand the relationship between each of the equal-sized pieces and the original whole.

In another case, the teacher's choice to provide one-inch tiles to her students resulted in a similar misunderstanding. Rather than using the tiles to measure their pieces to ensure equal areas as intended by S14, Deon assembled one inch tiles to a model sharing rectangular cakes. That is, Deon used the tiles to build a cake rather than equipartition an existing rectangle. In this case, S14 became aware that Deon was not equipartitioning. I asked her about Deon's work during the observation interview:

R: So, what are you thinking that he's thinking about?
S14: Well, I was concerned that he would be so focused on - he would forget that those were all the same size and that the whole is what he was supposed to be focused on. Then, I was thinking maybe I shouldn't have put those out, maybe I'll confuse them more.

From these observations, I concluded that the depth of teachers' understandings of equipartitioning was affected by their experiences with the learning trajectory. Some teachers failed to understand that the relationship between the created parts and the original whole is the characteristic of equipartitioning that provides the foundation for rational number reasoning.

Non-Congruent Equal-Sized Parts

As a group, the teachers had some difficulties in judging the correctness of non-congruent equipartitions. In the observations, there were some opportunities to observe teachers' incorrect evaluations of non-congruent equipartitions. The most common example from the lessons was sharing a rectangular cake among four by using two diagonal splits. Though some teachers deemed the example as correct, two of the ten teachers did not make a clear judgment during the class discussion. For example, Kim used the method (see figure 23) and believed that the shares were fair because there was a one-to-one correspondence between the number of people and the number of pieces and that the pieces were of equal-size.

S16: Okay. Let's look at Kim's. Kim, is each person going to get a fair share?
Kim: Yes.
S16: How do you know?
Kim: Because there are four people and four people are coming.
S16: So four people are coming. Are each of the sizes the same?
Kim: Yes.
S16: Does everybody agree?
Class: Yes.

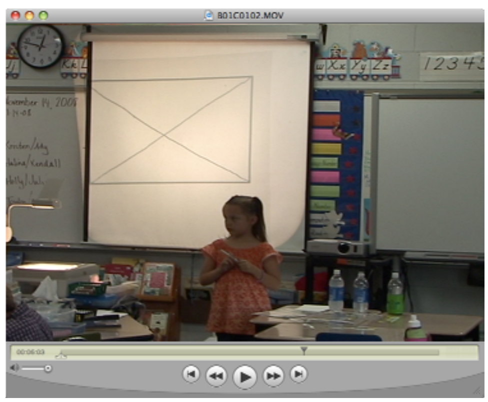


Figure 23. Kim's sharing among four.

From S16's next series of questions, it is unclear if she believed that the pieces were of equal-size or not. She leads the students to using measurement as a justification of equal-sized pieces by cutting the pieces and overlaying them. However, this strategy would likely reinforce the wrong idea as the pieces are non-congruent. This is further exacerbated by her conflation of symmetry, congruence, and equivalence.

S16: So you feel all those pieces are symmetrical? Same size. [no] So if that was your cake, Adam, go point to the piece you would want if that was your cake. That one. Hannah, point to the one you would want. Okay. Alright, Katy, why that one?
Katy: Because…
S16: Someone over here said it, Trey, what did you say? Because it is bigger.
Child: I said it.
S16: Does it look bigger than the others?
Class: No! Yes!
S16: What would be a way we can find out if these were truly congruent, the same size? Hanna, what could we do?
Hanna: We could get a measuring thing and we could measure both of them.
S16: We could. So if we wanted it to be exact, we could get a measuring thing. What do we call that measuring thing?
Class: A ruler.
S16: What's another way if we didn't have a ruler? How could we do it, Mark?
Mark: Measuring tape.
S16: Well, if we didn't have any unit of measurement, how would we do it, Kat?
Kat: You could use a pencil.
S16: You could. But that's kind of using it to measure. Shelly, I'm still looking for something.
Shelly: Use your fingers.
S16: You're still using it to measure. I'm going to give you a hint.
Child: You could put it on top of the other.
S16: You could. What would I have to do first, though? If I cut them out and put them on top of one another, would I know?
Class: Yes.
S16: Do you think they'd all be the same?
Class: No. Yes.
S16: How many say no? How many say yes? You know what? This is a tricky one. The teachers do this, and it fooled us all. I'm not going to tell you. We're going to cut out later and see. Thank you for sharing.

Again, it is unclear whether S16 believes that the method produces fair shares or not. In the observation interview, I asked her about this particular example. She stated, "This one who has done corner-to-corner and corner-to-corner, your mind tells you it's correct, but it really isn't." From this, I conclude that ultimately, S16 does not know whether this method for sharing among four produces equivalent parts or not.

In most cases, teachers correctly evaluated the fairness of the students' solutions. However, in the case of non-congruent equipartitions, some teachers' uncertainty resulted in adequate resolution to the question of the equal-size of the resulting pieces.

Emergent Properties

Another area of teachers' knowledge of equipartitioning that affected instruction was their understanding of the mathematical properties described in the learning trajectory. Some teachers used them in their discussions with students implicitly. Others did not address them at all or missed opportunities to develop these ideas with their students.

Many of the teachers discussed composition tacitly. For instance, it was common during discussions for sharing among four for the teachers to make comments similar to, "it looks like they cut this cake in half and then it looks like they cut the halves in what? In halves again - and they ended up with four different pieces." One teacher drew explicit attention to how composition works on the figure when discussing a student's method for sharing, though she does not use the term composition.

S12: When she cut, how many pieces did she [have an] effect [on]? Did she just cut that piece when she went across here?
Class: Yes.
S12: She just cut this piece?
Child: No.
S12: No. But it also cut part of what? This piece. And what happened when she came down through here? It cut part of this piece and this piece, and this piece. So what happens when you cut all the way across the circle? Are you just cutting one piece at a time?
Child: No.

S12: No. You're cutting more than one, aren't you? You're affecting other shares. Okay.

Another teacher used composition to help students explain their nonstandard solution for sharing a rectangular cake among six (see figure 24). Ally made a three-split vertically and then halved each of the thirds using a diagonal line. S17 covered up two of the three sections and to draw attention to the fact each of the thirds was halved.

S17: Alright, hang on. Ally, can you come up and draw your bottom rectangle. Come draw it. She's got it cool. It's similar to what Abby has got going on. Thank you. Let's see what she's got going on. Keep in mind that Ally's overhead lines won't be great up here. This is six. One, two, three, four, five, six. Are they sort of, kind of equal? Yeah!

Child: No. They're teeth!

S17: Could you also look at this as a rectangle cut in half? [covering two of the thirds and referring to the diagonal two-split]

Child: Yes.

S17: Could you also look at it as another rectangle cut in half? And then a third rectangle cut in half? [ah… I would look at it like teeth] Yeah. Okay, they kind of look like teeth.

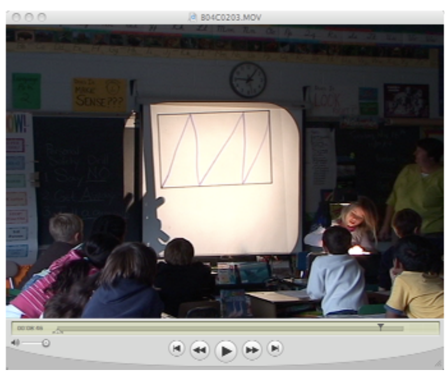


Figure 24. Ally's sharing among six.

The most common property to surface in the lessons was compensation. Many of the teachers observed or questioned students about the inverse relationship between the number of equal-size pieces and the size of those pieces qualitatively. Some of them made reference to this in relation to their work with rectangles from the *Investigations* unit, such as W17:

> Okay, but let's think about when we filled up our rectangles, yesterday. Think about this. This is the exact thing that we did yesterday with our tile squares, filling up our rectangles. Remember how that some rectangles were longer and skinnier. Some were shorter and fatter, but they were the exact same. Remember that, Tyler? Remember that? So even though this is longer and skinner and this is shorter and fatter, if this was measured exactly right and I had a ruler and I measured everything out and did it exactly right, and I took square cubes, square tiles, I am betting that this and this would be the same.

Two other teachers noticed that students tried to use compensation when using a parallel cut strategy on a circle by creating arcs, such as when Kobe's three concentric arcs (figure 18) when sharing a pie among four people as discussed in the previous section. In another case, a teacher made use of her students' work in an interesting way to help students understand the compensation property. When sharing a cake among four, one student composed a horizontal and vertical two-split to create a 2 X 2 array. Another student created a 2 X 3 array when sharing among six. Since both were drawn on transparencies, the teacher overlaid the two and led students in a discussion. Each of the equipartitions and the overlain transparencies are shown in figure 25.

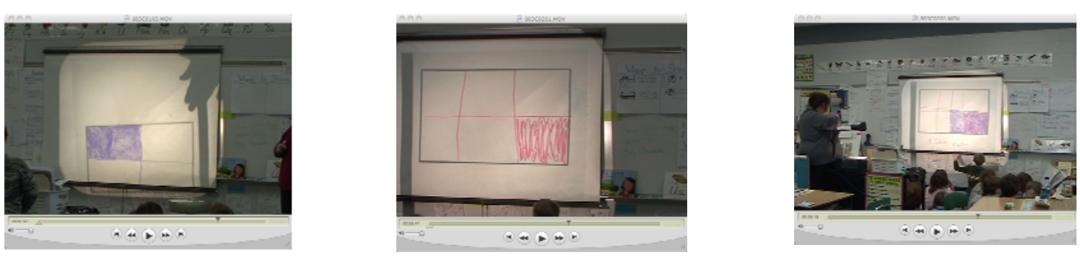


Figure 25. Transparencies of S14's compensation discussion.

> S14: There were lots of interesting ideas, and we were trying to figure out this idea, if there were 6 people now, would they get the same share as four people?

Now, I saw a lot of you chose to cut your cake this way. Would this way have 6 pieces?

Class: Yeah.

S14: Would each person be getting a fair share, assuming the lines are drawn straight?

Class: Yes. No!

S14: Yes. If the lines were drawn nice and straight you would get a fair share. I think this is the drawing that I believe Sam and her partner made. I'm going to lay it on - put it underneath it, and I'm going to see if you notice something. Turn and talk to someone near you about what you notice. [children talking]

S14: Ok. 3, 2, 1 – ready to talk to the group. Okay, here we shared how many pieces? [one] How many pieces were there? [four] There were four people to share and we needed four fair shares. You said this was a fair share. You said, I believe was called it one share or the first piece. Ok, now that story changed a little bit. We had more people come. What did you notice about the share? Marty, what did you notice about the share now? With the six people.

Marty: Well, they are almost the same size except they are -

S14: Do you think they are the same size?

Marty: No. Sort of.

S14: How so?

Marty: Because this one is just a little bit shorter than this one.

S14: It's a little bit shorter – a little bit smaller?

Marty: It's the same except you added two more squares right here.

S14: So this has more squares, but if I'm looking at the fair share, the piece that I am going to hand to someone, how are they different? What would you say about them?

Marty: Uhm, this piece is longer than this piece.

S14: Okay, thumbs up if you agree. Okay, so Marty, what have we learned? If we add more people, what happens to the share? ... If we are adding more people, what happens to the share?

Marty: It gets shorter and shorter so there will be enough for everybody.

S14: So make a prediction. If I decided to make another cake and cut it in 8 pieces, what is going to happen to the share if 8 people come?

Marty: It's going to be about – like that.

S14: Oh, good – so it is going to get even smaller?

Marty: Yeah

S14: Very good.

S14 used this strategy again with the pies shared among four and six people. Other teachers used their knowledge of equipartitioning to draw attention to equivalence, such as W18's sequencing of students' solutions to sharing a rectangular cake among four people to help students understand equivalent fractions described previously.

These examples show some of the ways in which teachers' knowledge of the properties described in the learning trajectory for equipartitioning influenced their instruction. As students are inventive and create new examples of equipartitioning that may not be documented in the learning trajectory, teachers must be able to use the emergent properties flexibly as reasoning tools to support students in meeting their instructional goals. This discussion of properties, along with teachers' knowledge non-congruent equipartitions and their understanding equipartitioning as an operation, illustrates the ways that the teachers' understandings interacted with their knowledge of the learning trajectory to affect instruction.

Restructuring: Teachers' Instructional Practices

Teachers' knowledge of equipartitioning and uses of the learning trajectory affected the coherence of their instruction with students in three ways. It assisted them in the selection of students' ideas to use in whole class discussions around a mathematical goal. Also, it supported the ways they used individual students' ideas in discussions. And to an extent, it helped teachers' organize and relate students' ideas to refine the classes understanding of equipartitioning.

Selection

Some of the teachers used the learning trajectory as a way to select examples to bring to the attention to the class. In the observation interviews, some teachers indicated that when circulating and choosing students to share work, they were looking for both correct and incorrect strategies. For instance, when asked what she was looking for while selecting students to share their work, W13 stated, “Well, the first thing I was looking for, I think, was children that grasped it - that had the strategy down.” Some teachers reported that they were looking for a variety of strategies. W17 said, “I wanted to look for answers in different ways. All these are set up in different ways. Because I wanted them to see a variety of answers.” S16 stated that she was looking for ways that were “most creative.” Still others indicated that they were seeking solutions that would confront misconceptions related to equipartitioning or to coordinate thinking, such as the *n* versus *n – 1* parallel cuts misconception or the *three equipartitioning criteria*. When asked what she was looking for when circulating, S12 noted, “The parallel cuts I definitely wanted to bring up.” W17 was looking for examples to help students coordinate the number of pieces and the need for equal-size pieces. When asked what she was looking for, she stated:

> Just making sure that they understood, number one that they were dividing them into equal parts, and not that they were just – well some of them were dividing them into – say, for instance, the for equal parts. So then some of them were dividing them into four parts, but they weren’t equal parts, so they were understanding the concept of the number of the four, but they weren’t understanding the concept of the equal part.

The teachers’ use of the learning trajectory in selecting students’ work varied. In one sense, the learning trajectory acted as a checklist of strategies to search for, identify, and exhibit to the class for most of the teachers. For a few, it acted as a resource for teachers to examine students’ thinking and to select examples that would assist other students in the class in refining their notions of equipartitioning.

Uses of Individual Students’ Work

Analysis of the observations revealed that the teachers used individual students' work in three different ways during instruction. All of the teachers used selected work samples to address the *three equipartitioning criteria* with the class. Some of them used students' work to confront different misconceptions related to equipartitioning. Many of the teachers explored goals other than equipartitioning with the work samples, including other curricular goals and connections to other curricular ideas.

By far, the most common use of the students' ideas was to address the *three equipartitioning criteria*. W13 asked Laura to present her method for sharing a rectangular cake among six people. She posed questions with Laura through three different attempts to share a rectangular cake among six people. Laura created six scallop-shaped lines along the top edge of the cake shown in figure 26. W13 asked how she had shared among four previously, to which Laura drew similar lines seen in figure X. Based on her response, she then questioned Laura about the rest of the cake, bringing attention to the need to exhaust the whole.

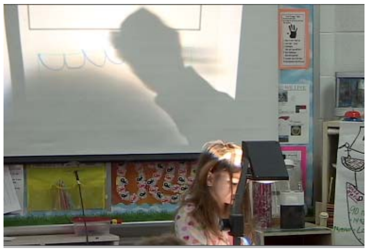
W13: Okay, this is the cake and there are six people there. So let's see, you're gonna give one, two, three, four, five, six pieces. Okay. Well let me ask you a question. Before, when there were four people, how were you gonna divide it? Let's just say there's another rectangle. How were you gonna divide it between four people?

Laura: I was going to one, two, three… [drawing]

W13: You're still gonna do that. Okay. Well let me ask you, what are you gonna do with all that big piece of cake that's left?

Laura: Save it.

W13: Save it? We're not gonna save any cake, guys, okay? We want you to use all of our cake, because we want everybody to have as much as they can, okay. So with that in mind Laura, let's think about this now. We're gonna use the whole cake. How would be a different way you can divide that cake so that every single part of that cake gets a piece? You want to try that again? [child drawing]



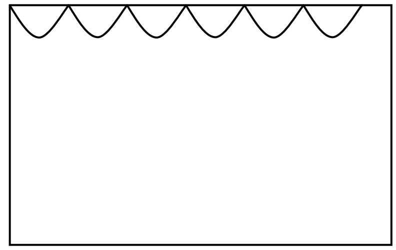


Figure 26. Laura’s initial sharing among six.

Laura completed her second attempt by creating two diagonal lines and then splitting the rectangle vertically into two parts, creating six unequal-sized parts as shown in figure 27. Next, W13 asked Laura which piece she wanted to bring attention to the unequal-sized pieces.

W13: That’s six pieces? Alright. Do you think they are equal pieces? [child nods] Everybody has the same amount? Okay. Which piece would you want? Put an “X” on the piece you want. Why did you pick that piece instead of maybe that piece?
Laura: Because it is bigger.
W13: ‘Cause it’s bigger? Okay. Well what does equal mean? What does equal or fair mean?
Child: Uhm, it has to be the same amount.
W13: Okay. So is this more than that? [child nods yes] So do you think everybody has a fair share? [nods no] Maybe try this here and if you don’t get it right, that’s okay, honey. It doesn’t matter. We’re just learning.

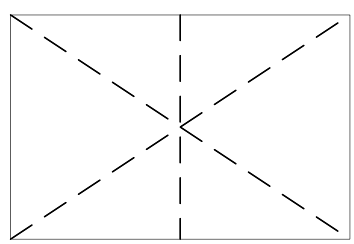


Figure 27. Laura’s second attempt for sharing among six.

Laura's third attempt was to draw a new rectangle at the bottom of the transparency, two-split the rectangle vertically, and then to draw three horizontal splits across creating eight equal-sized pieces. She quickly erased the bottom two pieces creating six equal-sized pieces shown in figure 28. As described above, W13 did not comment that Laura did not understand the relationship between the pieces and the whole because of the change in the unit, and accepted it as correct. Then, she asked questions about both the number of pieces and the size of the pieces.

W13: How else might you do that so that everybody would have the same amount of cake? [child draws three sequential horizontal cuts with a 2-split and counts, erases bottom slice] Okay. Guys, look at that! Give her hand. That was pretty good to me, does it look good to you? How many pieces are there? If this line were a little straighter and we know that it would be more equal. So you've got how many pieces there?
Laura: Six.
W13: Six. Okay. Are they equal? Does everybody get the same? [nodding yes] Does it matter to you if it's this piece or this piece? It doesn't make any difference. If you got this piece, what would you call that piece?
Laura: One-fourth?
W13: One-fourth? How much is in the whole? The whole cake.
Child: 1, 2, 3, 4, 5, 6. One-sixth?
W13: One-sixth. Good job. Wonderful, wonderful work.

Figure 28. Laura's final sharing among six.

In this sequence, W13 used the *three equipartitioning criteria* from the learning trajectory as a means of framing her interactions with Laura to coordinate these three processes for Laura and her class.

Several teachers addressed misconceptions related to equipartitioning with work samples. S12 selected Jon and Aidan to present their method of sharing a rectangular cake among four people. They made four vertical lines on the rectangle creating five approximately equal-sized pieces. S12 used this example as a way to discuss the need to make one cut fewer than the desired number of pieces when using the parallel cut strategy.

S12: Who else had how to share a rectangle? Jon, come and share yours … What did they do? You guys tell us what you did. How did you share this cake? Tell us what you did. Listen.
Jon: When we split it up, there were four people.
S12: Four people, yes. And what did you do? How many times did you cut your cake?
Aidan: Four.
S12: So four people... Alright, he said there are four people, so he cut it four times. How many pieces are there? How many pieces? Five. Okay. How many people are I going to have?
Class: Four.
S12: So does everybody, all four people get a fair share? What do you guys think? Tell me why.
Jenny: Because in order to eat up the whole cake someone is going to get two pieces.
S12: Oh. Okay. Alright, you know what you guys did? Jenny just explained it. She said that four people would have a piece and then there would be an extra piece and somebody would get two. Do you think that's fair?
Class: No.
S12: No. But you know what? Listen. Let's look at what they did. Because what they did is a very important thing for us to notice. … There were four people, so they made how many cuts?
Class: Four.
… [S14 discusses parallel lines as described previously]
S14: How many cuts did we need to make? Can you figure it out?
Jon: We made four.

S12: You made four cuts. Yes, you did. But how many did you need to make? How many pieces did you get when you cut four, when you did four cuts? How many pieces are up here?

Aidan: Five.

S12: Five. So anybody have any idea how many cuts you should make to get four? Tim? Three? So what are we saying? Three cuts give us how many pieces? [Four!] Four. How many pieces would two cuts give us? [Three!] Three. So if you're making these special kinds of cuts and you have four people, how many cuts do you need to make?

Class: Three.

S12: Three. One less - than what?

Class: Four.

S12: The total number of--

Class: Three, four.

S12: Pieces. The total number of pieces.

S12 wanted to confront this particular misconception that some of her students held, so she used Jon and Aidan's work as a means to help the class understand that in order to have fair shares when using a strategy of making parallel cuts, they must create one cut less than the number of desired pieces.

To varying degrees, all of the teachers used the selected work samples to address other goals. Some of these goals were nonmathematical in nature and distracted from the goal of the lesson. For instance, a student shared a rectangular cake among six by composing a horizontal three-split with a vertical two-split creating a 3 X 2 array. W13 asked her students how many lines a student drew and then engaged in a discussion that did not work towards the goal of equipartitioning.

W13: Okay. Well you know guys, that's interesting because how many lines did he draw?

Class: Six.

W13: Did he draw six?

Child: No, three.

W13: It depends on how you add it, doesn't it? You could say well he drew one, two, three, four, five and six. So that would make it seven. Or you could say he drew one, two, three. So maybe… there's a different way that you can divide it. Maybe if you draw the, cut the rectangle like this, straight up and down, then you use how many lines?
Child: Five.

In another case, S17 led students in a discussion as to how a student's solution may be unfair based on the context, even though it met the *three equipartitioning criteria*.

S17: However, being chocolate cake with chocolate frosting, which piece do you think I would want? Do you think I would want one of the middle pieces?
Child: No.
S17: Why not?
Child: Because you don't like chocolate.
S17: Oh yes I do.
Child: You would want one…
S17: Where would most of the frosting on the edges be? On this piece or on that piece, right? 'Cause the frosting is on the outside edges.
Child: Most of it is.
S17: So if you were cutting this cake for your wonderful sister or your wonderful sister, would you let your wonderful sisters [I don't have wonderful sisters]… would you let them have the edges with all the chocolate frosting?
Class: No.

Further, as seen in previous examples, some teachers pursued connections to area measurement to relate to *Investigations*, while others used the examples as a means of addressing fraction notation and equivalent fractions.

In summary, the teachers' uses students' work in instruction fell into three categories. In one way, the teachers used their knowledge of the learning trajectory in concert with students' work to refine notions of equipartitioning. In a related way, they used the learning trajectory as a way to address students' misconceptions related to equipartitioning. However in another sense, the teachers did not use the learning trajectory but pursued other curricular goals through using the students' ideas.

Organizing and Relating Students' Ideas

The teachers used the learning trajectory to relate students' ideas across examples to work towards a goal of equipartitioning to varying degrees. Though some made connections across the examples during instruction, none of the teachers reported any plans of sequencing students' work in the observation interviews. For instance, I asked W17 if she had planned a particular order to the way she had students present, she said, "I don't think I did. I just remembered that – I don't think the order was as important as showing the different ways to me."

However, during instruction, three teachers used the learning trajectory to sequence students' work samples in a way that supported the refinement of equipartitioning. For example, recall that S12 made a point to confront the *n* versus *n* – 1 cut misconception. After the episode described in the previous section, she asked Jon and Aidan, who had used the parallel cut strategy in sharing a pie among six people, to present their work.

> S12: No. I know it's a pie. Alright. I want Aidan to bring his piece up because he did this pie, and also he did it in a different way. Jon, you can come. Put it up there. Alright. They told me that they learned from their last one. On their last one they had four people to share with and they cut it four times, Noah, Jenny, which gave them how many pieces on the last one? Do you remember? They had four people and they cut it four, and they got how many pieces?
> Child: Five.
> S12: Five. That's exactly right. They said we've learned now. How many does that make?
> Aidan: Five.
> Jon: Six!
> S12: How many did you guys make? How many cuts? Five cuts and he gave them how many pieces?

Class: Six.
S12: Is this shared fairly?
Class: Yes, no.
S12: You don't think so? Who agrees that this is a fair share? Which piece would you want?
Child: The biggest.
S12: So there are some that are bigger and smaller?
Class: Yes.
S12: Yes. But they did remember one thing they learned from this is that they made one less cut than the number of people they needed to share with. Okay. Good job, guys. Thank you.

Though the strategy will not create equal-sized parts on a circle, S12 reinforced the students' previous learning with the next example. The next sample selected was of a rectangular cake shared among six people using the parallel cut strategy. Again, the teacher drew the students' attention to the method.

S12: Alright, did anybody share this way? Noah, I want you to come sit right here. Noah. You did that? Anybody else do this? You did that? You did that? Good. A lot of people did. Alright. Tell us how you did this.
Cullen: [inaudible]
S12: Say that louder. Everybody hear what he's saying?
Cullen: We made five slices.
S12: But we needed to share it with six people. Why did you only make five?
Cullen: One, two, three, four, five, six.
S12: Makes how many pieces?
Cullen: Six.
S12: There's that thing again. Did their cuts cross each other?
Class: No.

As previously discussed, S14 used two different students' work on sharing a cake among four and six people to advance a goal of compensation by overlaying transparencies of the cakes to show that as the number of pieces of cake increased, the size of each piece decreased (see figure 25). Also discussed before was W18's use and sequencing of several students' ideas when sharing a rectangular cake among four people towards a goal of understanding equivalent fractions. Two student pairs shared by creating eight equal-sized pieces in two different ways, while another pair created four equal-sized pieces. When asked about her sequencing in the observation interview, W18 stated:

> R: Did you have any order in mind when you did ?
> W18: I kind of – as I was walking around I guess I kind of put those together when I was walking around with them – because I was thinking that would just be a good time to bring up how you and make equal fractions that way-

These examples indicate that a few of the teachers used the learning trajectory as a way of organizing and using students' ideas coherently for whole class instruction.

However, most teachers made sporadic connections across students' presentations if any at all. For instance, eight different methods for sharing cakes and pies among four and six people were shared during S16's lesson. Yet, in each of the eight discussions about the samples, the teacher made no reference to any of the previous examples. Furthermore, four of the eight examples were left without resolution as to whether they were correct or incorrect. In another classroom, fourteen methods for sharing cakes and pies among four and six people were discussed, with three of those methods being from one student who made two attempts before being successful. Of the 11 different discussions about the solutions, the teacher made reference to previously shared solutions only once and this was in terms of fraction notation and not equipartitioning.

For a few teachers, knowledge of the learning trajectory provided a means by which teachers could sequence students' ideas to refine students' understandings of equipartitioning. Largely, however, the results of teachers' selection and lack of sequencing tended to yield a lack of coherence and resolution.

In this chapter, I presented findings for the observational study. The teachers' experiences from the ERNR PD provided an initial foundation in the learning trajectory with which teachers restructured their knowledge of equipartitioning, knowledge of equipartitioning and students, and knowledge of teaching. However, these experiences are not sufficient to assist teachers in observing and interpreting students' novel approaches and in managing their relationships with other curricular goals. Teachers need a means for an ongoing, classroom-based community to continue to observe, interpret, and consider how to use students' inventive approaches to equipartitioning tasks. These results are consistent with many of the findings from the retrospective analysis in Chapter Five and additionally describe the teachers' uses of the learning trajectory in classroom practice.

CHAPTER SEVEN

The previous three chapters reported results outlining the process by which teachers came to know and use a learning trajectory for equipartitioning, the factors which supported its learning and its outcomes, and the results of its use in classroom instruction. In this chapter, I answer the research questions, discuss the ways in which the introduction of a learning trajectory affected teachers' practices, and situate the results within the context of the greater research problem. Next, I present revisions to conjectures of the design study in response to the findings. I use these revisions and the discussion to suggest implications, limitations of the study, and further areas of research related to learning trajectories.

Answers to the Research Questions

The central goal of the study was to understand the effects of the professional development on the learning trajectory for equipartitioning within the context of elementary classrooms. Three research questions guided the study:

1) In what ways and to what extent do teachers use a learning trajectory for equipartitioning to build models of students' thinking?
2) In what ways and to what extent do teachers use a learning trajectory for equipartitioning to inform their adaptations of curricular instructional tasks and evaluation of relative task difficulty, their assessment of students' understandings, and their interactions with students during instruction?
3) What are the relationships between teachers' knowledge of equipartitioning and uses of a learning trajectory for equipartitioning and their students' learning?

The answers to these questions are discussed based on the findings elaborated in Chapters Four, Five, and Six.

Research Question One

Based on the analyses from Chapters Four and Five, I conclude that teachers used the learning trajectory as a tool for coordinating students' behaviors and verbalizations with cognition. The findings suggest that teachers used the trajectory to build more precise and adequate models of students' thinking about equipartitioning. Before the ERNR PD, the teachers constructed models of students' thinking that were vague and yielded little information about students' cognition. After their experiences with the learning trajectory, the teachers' models had greater explanatory and anticipatory power in regards to students' behaviors and verbalizations. These shifts can be characterized through examining the transitions made in the processes teachers used in the construction of their models

Describing. Teachers' descriptions transitioned from being irrelevant & general in before the ERNR PD to descriptions that were specific & focused on behaviors and verbalizations outlined in the learning trajectory. Prior to experiences with the learning trajectory, teachers' descriptions were vague and indistinct, such as noting that a student created two piles when sharing a collection of coins without commenting whether the student had dealt the coins as singletons, dealt with composites, or counted and allocated the coins. Yet after experiences with the learning trajectory, their descriptions made clear distinctions among behaviors and verbalizations, such as a student cutting strips of paper representing a cake from left to right rather than simply stating that the student created a certain number of pieces.

Comparing. The comparisons that teachers made in constructing their models transitioned from referents that were exclusively empirical to comparisons that also included the illustrations of the *three equipartitioning criteria*. Consistent with Black's (1962) notion of theoretical models, teachers used the more familiar domains of their own understanding of equipartitioning and experiences with other students as referents for comparisons initially. These empirical comparisons were based on recollections of specific students, either from their clinical interview activities from the ERNR PD or a student from past experiences with an interesting approach. Later, teachers used their experiences with the learning trajectory as a source of theoretically-based examples to generate general, composite profiles of "students." Along with their empirical-based comparisons, they included these as referents when constructing models. These profiles were based on students from the illustrative video clips included with the learning trajectory for equipartitioning or those examples collected and discussed from clinical interviews that were learning activities from the professional development sessions. Rather than making comparisons strictly based age or grade level, the teachers were more oriented towards progress-based profiles.

Inferring. Teachers' inferences transitioned from holistic judgments of the students' knowledge to more nuanced models that were supported by specific behaviors and verbalizations exhibited by students. Initially, many of the teachers failed to decompose their inferences about students' thinking and simply accepted solutions as correct or dismissed solutions as incorrect. This resulted from a focus on the outcomes of particular tasks rather than the process used by students. Teachers made broad, unqualified statements such as "she understands how to fairly share" or "he does not understand fair sharing" with little or no attention to the method used by the student. After work with the learning trajectory however, the teachers made more specific claims about what was correct or incorrect in the students' thinking due to a shift in focus from outcomes alone to the processes by which the student arrived at the outcome. In particular, teachers used the *three equipartitioning criteria* as a way to organize the inferences they made, often noting when students were unsuccessful at coordinating the *three equipartitioning criteria,* such as stating that a student was focused on creating the correct number of pieces while neglecting the need for equal-sized pieces. Additionally, these claims were supported with specific behaviors and verbalizations described in the learning trajectory, such as using students' folding or descriptions of symmetries as a basis for asserting a focus on the size of pieces or students' allocation of pieces as a focus on the correct number of pieces. Finally, the deficit perspective of many of the teachers from early in the professional development was less prevalent after their work with the learning trajectory.

Restructuring. Teachers used the learning trajectory when they restructured three different domains of their own knowledge. First, their work the learning trajectory in the ERNR PD caused them to restructure modestly their own knowledge of equipartitioning.
Teachers grew in their awareness of mathematical properties emerging from equipartitioning, their understanding of equipartitioning as generalizing to division, and their recognition of the additive misconception. Still, some teachers continued to hold underdeveloped notions of equipartitioning including the uncertainty about the fairness of non-congruent equipartitions, understanding equipartitioning as an operation, and recognizing the essentiality of the relationship of the parts or groups to the original whole. Additionally, the demands of the teaching lessons on equipartitioning suggest that the teachers need to restructure relationships

among mathematical ideas outside of the learning trajectory, such as multiplication, area, and fractions.

Second, the teachers restructured what may relate to what Hill et al. (2008) refer to as of their knowledge of content and students, specifically equipartitioning. It increased teachers' sensitivity to students' actions and words by elucidating particular behaviors or verbalizations documented in the trajectory, like differentiating between counting a collection before allocating as opposed to dealing when sharing fairly a collection or between making cuts from left to right as opposed to splitting splits when sharing a single whole. This sensitivity strengthened the models constructed by teachers and in turn, taken with knowledge of a learning trajectory, assisted teachers in locating particular students within the range of conceptual development, such as suggesting what splits are currently within the students' abilities. The learning trajectory informed teachers' selection of instructional moves to help students refine their understandings and supported their anticipations of how students would approach the task, such as suggesting questions that were targeted based on the *three equipartitioning criteria* and informing predictions about the effects of those questions. To a lesser extent, teachers related these anticipations to the students' curricular experiences or attempted to make those connections for students, like recognizing the effects of the emphasis on place value on a students' dealing or using area measurement for justification as a means to connect to previous or upcoming work with their curriculum.

Third, the learning trajectory restructured portions of teachers' pedagogical knowledge by informing them about what next pedagogical steps might refine students' notions and move them towards a more sophisticated understanding of the particular concept.

Their knowledge of the trajectory affected their selection of students' ideas for class discussion and to a lesser degree, supported the sequencing and relating of those ideas throughout instruction. Additionally, learning trajectories supported teachers' pedagogical work outside of the classroom by providing teachers with precise language to communicate with colleagues about students' behaviors, verbalizations, and cognition.

Thus, teachers used a learning trajectory for equipartitioning to build more precise and adequate models of students' thinking that yielded explanatory and anticipatory power. Further, their uses of the trajectory restructured their own knowledge of the content, knowledge of content and students, and knowledge of teaching. Learning trajectories provide teachers with a means of accessing and understanding what Confrey (1998) calls a student's "voice" while simultaneously altering their own "perspectives" of the content and its teaching.

Research Question Two

The results reported in the previous chapters indicate that teachers used the learning trajectory as a tool for coordinating their instructional practices with their models of students' thinking. Their restructured knowledge manifested in three different instructional practices: their task adaptation and evaluation, their assessment of students' understandings, and their interactions with students.

Task adaptation and evaluation. Teachers used the learning trajectory to coordinate their work with instructional tasks and their students' developing understandings of equipartitioning. For instance, roughly half of the teachers used the trajectory successfully to make adaptations of curricular instructional tasks. Of the teachers who were successful, most used the trajectory either to adapt a task so that its goal was to make equal-sized groups or to select sizes of collections that would permit a composition of factors. One reason for its lack of assistance for the other teachers was their own knowledge of equipartitioning. Most of the teachers who were unsuccessful did not understand fully the differences between equipartitioning and quotitive division. Also contributing to this was the emphasis of the teachers' curriculum on counting by groups and building a notion of multiplication as repeated addition.

However, the results show that teachers used the trajectory to evaluate the relative difficulty of tasks. Teachers used criteria such as the number of splits required, the shapes of continuous wholes, and whether the problem involved collections of discrete wholes, a single continuous whole, or a collection of continuous wholes to order problems from least to most difficult. What they failed to develop adequate sensitivity to was the differences between equipartitioning as a cognitive operation and division as a mathematical operation. Many interpreted equipartitioning as division and adapted tasks that were quotitive division problems when sharing a discrete collection.

Assessment of students' understanding. Additionally, teachers used the learning trajectory to coordinate their analyses of students' work samples with their models of students' thinking about equipartitioning. When analyzing artifacts of students' learning, the teachers used the trajectory to make sense of the students' work. It sensitized them to features indicative of certain approaches and related those approaches to underlying cognition, such as the recognition that students may be focused on creating the correct number of pieces and not equal-sized pieces. With the trajectory, they were able to raise questions for themselves about the students' understanding and provide tasks or pose questions to students that would assist the teacher in answering those questions, like asking which piece a child would want or suggesting that students stack pieces to draw attention to differences in size.

Interactions with students during instruction. Finally, some teachers used the learning trajectory to coordinate their models of different students' understanding of equipartitioning to facilitate instruction. The learning trajectory provided them with different approaches that students take to equipartitioning tasks and related those approaches to what they students may be thinking. This knowledge assisted some teachers in selecting different students' ideas to highlight in instruction, such as selecting an example of a student failing to exhaust the whole to raise the need for that criterion. For all of the teachers, it provided a framework for using individual students' ideas in instruction based on the underlying conceptual coordination of various schema to develop equipartitioning, such as posing questions of allocation or choice of pieces to encourage students to consider multiple parts of the *three equipartitioning criteria*. For some teachers, it acted as a framework for sequencing students' ideas in class discussion to assist all students in refining their understandings and in addressing certain mathematical properties, such as sequencing multiple students' solutions for sharing a continuous whole to support the emergence of equivalence.

Research Question Three

Based on the analysis of the teacher gains provided in Chapter Five, I concluded that the ERNR PD assisted teachers in learning about equipartitioning. The teacher pre/post assessment indicated that the teachers understood the fundamental strategies and emergent properties from the learning trajectory. Though the teachers' performance on items directly related to the additive misconception (Empson & Turner, 2006) at the beginning and end of the PD indicate they differentiated between additive and multiplicative reasoning, there were positive gains on the items in which the misconception was embedded. This suggests that the ERNR PD may have provided opportunities to confront their ideas relating additive and multiplicative reasoning and supported their refinement of these ideas.

Based on the analysis of the student gains provided in Chapter Five, I concluded that though students had very modest statistically significant gains according to the measurement, this learning was not differential among the teachers from whom they received instruction. Further, neither the teachers' content knowledge subscores nor pedagogy subscores were related to the students' gains.

Several factors may explain this. First, the teachers taught a total of two lessons on equipartitioning over four weeks. In the interim, they continued with their usual instruction from *Investigations* and conducted district quarterly assessments for approximately two weeks. The lack of total time and lack of focus devoted to equipartitioning may have contributed to lack of student gains. Second, the influence of *Investigations* was more significant than anticipated at the beginning of the study. This affected the delivery of the equipartitioning lessons by the teachers. Further, it is possible that the students were experiencing the same competing curriculum and equipartitioning goals as their teachers, though the data collected do not address this. Given the difficulties experienced by the teachers, new activities may have been more appropriate than the adaptation of existing curricular activities.

The answers to the research questions above indicate four different areas where the introduction of a learning trajectory affected teachers' practices. First, the use of a trajectory can assist teachers in focusing on students' ideas in their professional practices, providing a tool with which teachers may improve their observations, interpretations, and inferences about cognition. In turn, this focus can provide opportunities for teachers to formulate and revise their knowledge of students and content. Further, it can act as a catalyst for teachers to reexamine and refine their understandings of content. Finally, a trajectory can influence their pedagogical knowledge. Therefore, I conclude that learning trajectories may act as a tool for coordinating: (1) behaviors and verbalizations with cognition within individual students; (2) various understandings of content among multiple students during instruction; and (3) their influences during instruction among their practices, students' ideas, and their own understanding of the content.

Revisions to the Conjectures

According to Cobb et al., design studies require "cycles of invention and revision" (2003, p.10). I began the study with five conjectures concerning the effects of a learning trajectory within the context of elementary classrooms. I designed the professional development and offered the resulting findings outlined above as part of one cycle of inquiry in learning trajectories research. To complete this cycle, I present revisions to the original conjectures based on the results of the study. A side-by-side comparison of the initial and revised conjectures is included in Appendix O.

Revised Conjecture One

A learning trajectory assists teachers in building models of students' thinking that are focused, empirically- and theoretically-based, and nuanced. It sensitizes them to students' behaviors and verbalizations documented in the trajectory, provides support for connecting that evidence to students' cognition, and assists them in locating students' thinking within a range of conceptual development.

Revised Conjecture Two

A learning trajectory supports teachers' judgments of the relative difficulty of instructional tasks. Though a learning trajectory may suggest next pedagogical steps, it interacts significantly with other curricular goals and materials. Teachers need support and guidance in adjusting tasks to align with a learning trajectory.

Revised Conjecture Three

A learning trajectory influences teachers' analysis of students' work on instructional tasks by helping them identify evidence of student thinking within students' work, providing language for teachers to discuss students' thinking, and assisting them in making predications.

Revised Conjecture Four

A learning trajectory affects classroom interactions with students during instruction. It sensitizes teachers to a variety of strategies students may use, provides a framework for using and sequencing students' work, and helps teachers connect and relate mathematical ideas across multiple examples of students' ideas.

Revised Conjecture Five

The relationships between teachers' knowledge and uses of a learning trajectory and their students' learning are complicated by interactions with curriculum. More sensitive measures are needed to understand these relationships.

Implications

Use Model for Learning Trajectories

The study was a part of the larger goals of the DELTA research group to create diagnostic assessments using learning trajectories to provide teachers with instructional guidance. One of the goals of this research was to begin to anticipate how teachers may use trajectories to support their instruction of equipartitioning. The results indicate that trajectories can offer teachers a tool both to understand better students' thinking and the potential to respond appropriately to students' ideas with their practices. This section highlights characteristics of the professional development design that assisted teachers in using the learning trajectory as well as suggests additional needs for the potentials of learning trajectories to be realized.

The findings imply several characteristics of a model for how teachers may use the diagnostic assessment system in practice. First, the teachers benefited from direct connections between students' behaviors and verbalizations described in the progress variable and underlying cognition. During the study, the introduction of a representation linking actions and words with students' thinking was accompanied by increased attention by teachers on cognition rather than holistic evaluations. Whereas the progress variable document tended to act as a list of objectives for teachers, the revised chart acted more as a way of locating students along the trajectory by relating their work with thinking. Also, the video exemplars and student work samples assisted teachers in identifying the behaviors described in the learning trajectory. Some of the teachers made comparisons to these when analyzing students' work, indicating that these artifacts became a way for teachers to enact knowledge of the trajectory within their practice. Thus, the use model should closely align examples of behaviors and verbalizations with cognition and location within the learning trajectory.

The analysis suggests that several additional components of the use model that may have supported teachers as they used the learning trajectory. First, teachers need a diverse portfolio of instructional and assessment tasks to supplement the clinical interview. Though the video exemplars and work samples supported teachers in understanding students' thinking in the context of clinical interviews and analyzing students' work, teachers need additional types of assessments to effectively and efficiently remain informed of students' progress along the trajectory. Recent progress of the DELTA group's assessment work will address partially this need. Additionally, once teachers have an understanding of their students' current conceptions, they need instructional materials that will support students in refining those conceptions, better guidance on how such materials differ from curricular materials, and how to integrate them into their broader instructional program. Finally, the use model should contain more guidance on the selection and sequencing of students' ideas during instruction. The learning trajectory and much of the professional development experience was focused on individual students. In the observational study, transitions from modeling individual students' thinking to modeling collective understanding for a classroom of students proved difficult for teachers and could be better supported in the professional development, perhaps with learning activities simulating these types of classroom situations or through more explicit attention and discussion during the professional development sessions.

Curriculum

Related to the characteristics and needs of the use model, the influence of the students' curricular program warrants more discussion. Though the DELTA learning trajectory definition predicted interactions between learning trajectories and existing curricula, the findings from this study suggest that these interactions were more significant than anticipated. Specifically, students' experiences with *Investigations* did not adequately support students in their work with equipartitioning. Further, the teachers' attempts at adapting curricular materials indicated that this is nontrivial task. Many of the teachers failed to adapt tasks for a goal of equipartitioning; of those who were successful, other curricular goals such as place value competed with the goal. These difficulties suggest a need for new curricular materials rather than working to adapt current ones. Supplemental curricular units with guidance on locating them within teachers' broader instructional program are needed to maximize the benefits of a trajectories approach.

Limitations and Future Research

Limitations

The format and timeframe of the ERNR PD is a limitation of the study. The sessions were conducted after teachers' instructional day and competed with other non-instructional duties. Also, conducting professional development for teachers during the initial part of the instructional year when all other district initiatives were commencing left teachers feeling overwhelmed and somewhat resistant to “one more thing.” The district’s technological infrastructure presented challenges related to the gathering and posting of videos to be shared among the teachers. Finally at the time of the study, the learning trajectory for equipartitioning was still under development. Subsequent versions of the trajectory may have been more effective in meeting the goals of the ERNR PD.

Interactions between curriculum and the learning trajectory were greater than expected. Though the DELTA definition anticipates this, the degree to which alternative goals from *Investigations* supplanted goals for equipartitioning in teachers’ lessons was surprising. This suggests that new curricular materials may be more amenable for professional development related to learning trajectories of constructs not present in existing materials than adapted materials.

Future Research

Given the challenges of integrating new content within an existing curriculum discussed above, one area of need is the research and development of materials for equipartitioning and for learning trajectories in general. The study was concerned with teachers’ uses of a learning trajectory in their practice. However, resources for materials and assessment items provided to the teachers were limited. This research should focus on the ways that teachers incorporate supporting resources, including diagnostic measures and curricular materials, in concert with knowledge of a learning trajectory to affect students’ learning to elaborate the model of use by teachers for learning trajectories.

The ERNR PD produced positive results but was a first attempt at designing professional development for teachers concerning learning trajectories based instruction. A more thorough evaluation of the learning activities, followed by revisions or reconceptualizations based by the results of this study should inform the development of future professional development on the learning trajectory for equipartitioning. Future research should address the ways that teachers develop, use, and share their understandings of learning trajectories.

Finally, this study was concerned with teachers' uses of a single learning trajectory. As previously mentioned, the restructuring of a teacher's understanding of the content associated with a particular learning trajectory requires the restructuring of knowledge of other related mathematical topics. Forthcoming work on a learning trajectories for measurement (Nguyen, in progress) and on multiplication and division (Myers, in progress) that relate to the trajectory for equipartitioning will demand teachers to coordinate multiple trajectories and curriculum. Another area of research should investigate the ways multiple learning trajectories relate with one another and curriculum and the ways that teachers negotiate these interactions.

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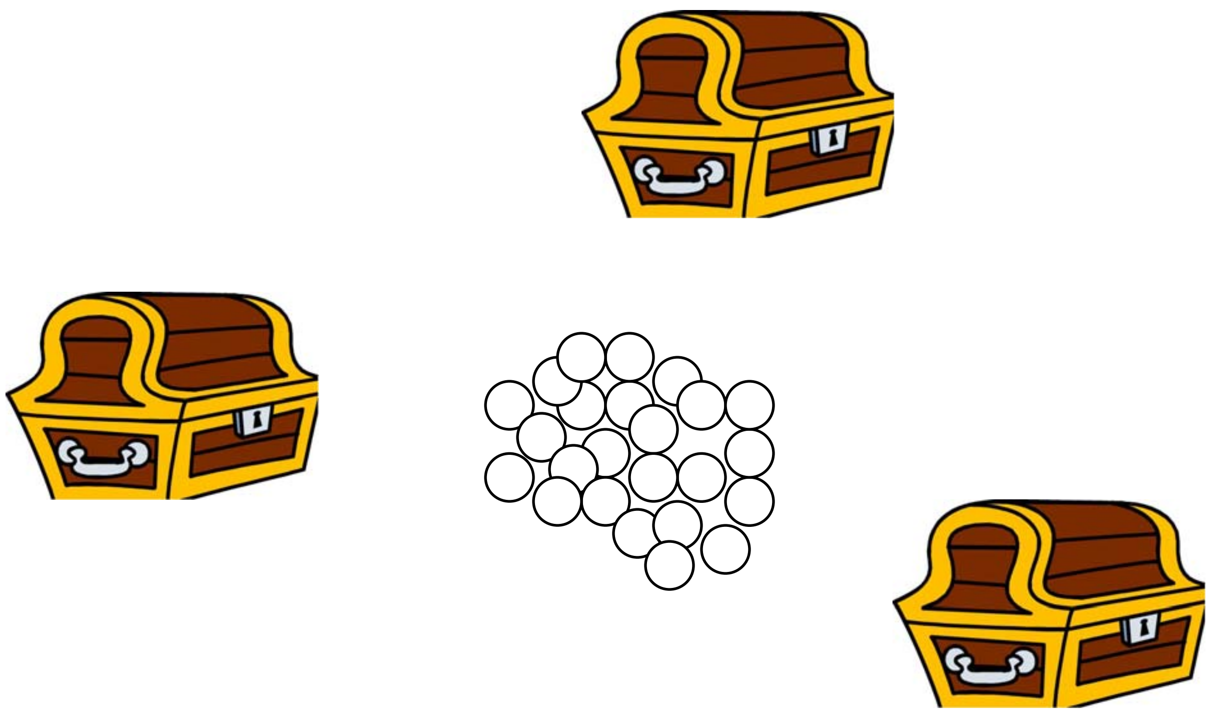
APPENDICES

APPENDIX A

Respond as completely as possible to the questions below. The questions vary from elementary to more difficult. Work as many of the problems as you can. Please show all work and circle your answer.

1. Three pirates found this treasure and want to share it fairly.
    a. Draw a line from *each* coin to the pirates' treasure chests.



    b. What ***mathematical*** name(s) would you give to each pirate's share?

    Answer(s): ____

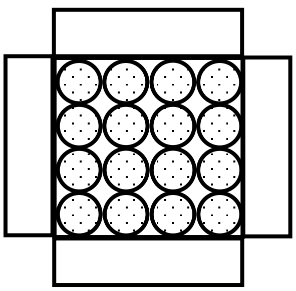
2. In general, if n objects are shared among q people, what is each person's share?

   Answer: ____

3. After fairly sharing an entire deck of Old Maid cards, Erin, Marrielle, and Kenny each have 17 cards. How many cards are in the whole deck of Old Maid cards? Show your work.

   Answer: ____

4. Below is a box of caramels. Four children fairly share 9 boxes of caramels. What is each child's share? Explain your approach. (from Pothier, 1981)



Answer: ____

Explanation:

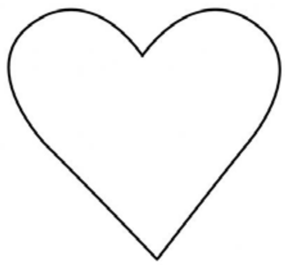
For the comparison below, indicate if the tasks are mathematically equivalent and explain your reasoning. (from Lamon, 1996)

"Three teachers share 4 pepperoni pizzas of the same size. What is each teacher's share?"	- and -	*"Three teachers all like cheese, mushroom, sausage, and pepperoni pizzas equally. They share one cheese, one mushroom, one sausage, and one pepperoni pizza among themselves, where the pizzas are the same size. What is each teacher's share?"*

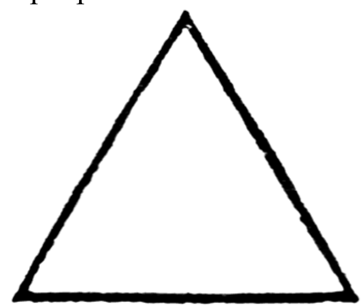
5. Are the two problems mathematically equivalent? ________
   Explanation:

6. Below are different shaped cakes. For each one, a) draw a line where you would cut the cake to share it fairly among the number of people indicated and b) shade one share of the cake.(from Pothier, 1989)

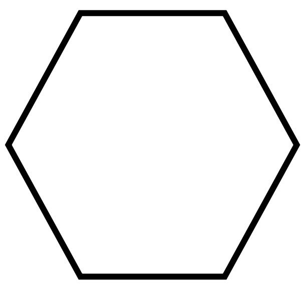
Among 2 people



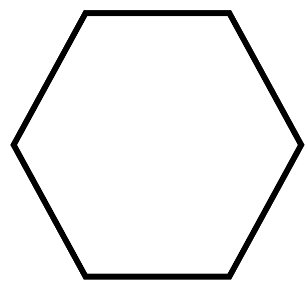
Among 3 people



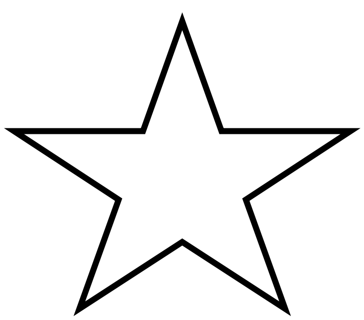
Among 3 people



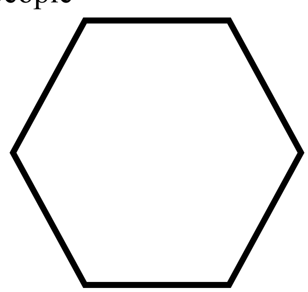
Among 4 people



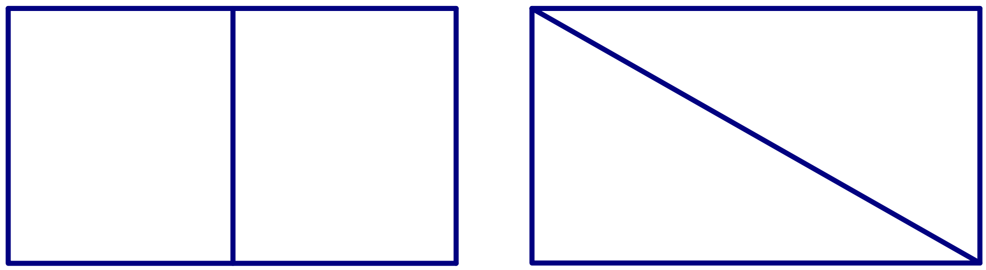
Among 5 people



Among 6 people



7. Yvonne and Pedro each have a rectangular piece of construction paper of the same size. Both children cut their pieces in half. Yvonne's paper is on the left, and Pedro's paper is on the right. Both Yvonne and Pedro keep one piece and trade the other piece. Do Yvonne and Pedro have less than, more than, or the same amount of paper as they did before they traded? Circle your response and explain your reasoning.



Circle: less than the same as more than
cannot tell

Explanation:

8. Mustafa folded a square piece of paper and created 12 equal parts. Describe in steps, in as many was as you can, how he folded the paper. (from Empson & Turner, 2006)

Method One	Method Two	Method Three	Method Four
•	•	•	•
•	•	•	•
•	•	•	•
•	•	•	•

9. Jeni folded a rectangular piece of paper in half four times. How many equal parts did she create? (from Empson & Turner, 2006)

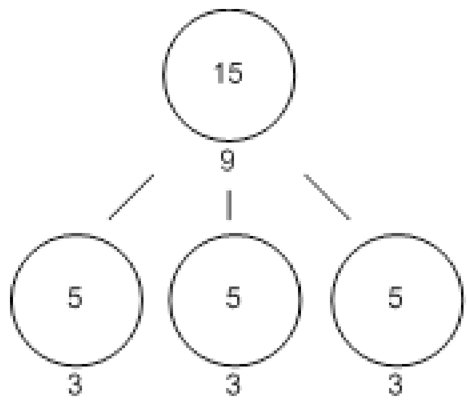
Answer: ________________________

10. Sweet potato pies of the same size are served as dessert for Thanksgiving dinner. There are 6 adults at the adult table and 4 children at the children's table. Assume the pies are cut so that the pieces are of equal size and all of the pies are used. For a - c, indicate whether an adult's piece is larger, a child's piece is larger, or if both an adult and a child get the same size piece of pie. Explain your reasoning.

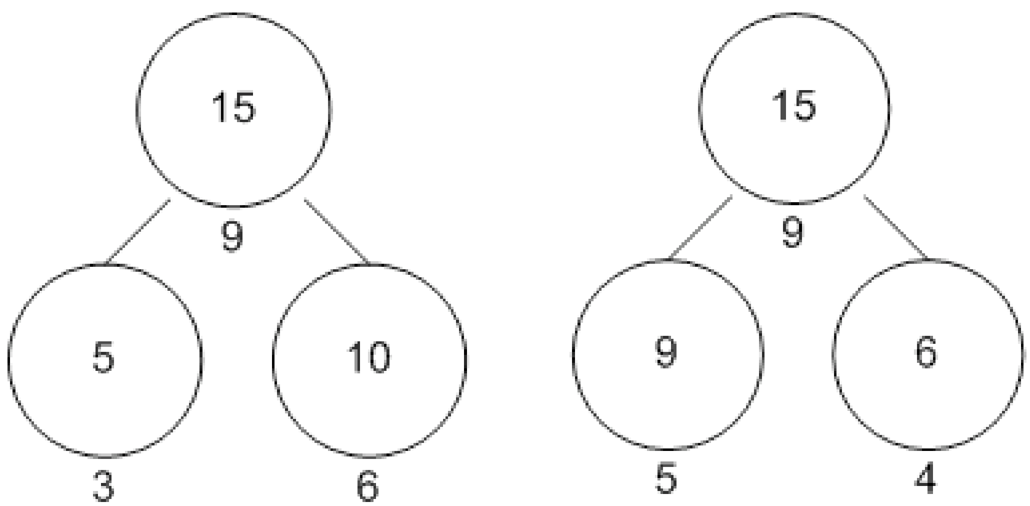
a. Adult table 1 pie for 6 adults Child table 1 pie for 4 children	An adult's piece ________________________ a child's piece. Circle: *is larger than* *is smaller than* *is the same size as* Explanation:
b. Adult table 2 pies for 6 adults Child table 1 pie for 4 children	An adult's piece ________________________ a child's piece. Circle: *is larger than* *is smaller than* *is the same size as* Explanation:
c. Adult table 3 pies for 6 adults Child table 2 pies for 4 children	An adult's piece ________________________ a child's piece. Circle: *is larger than* *is smaller than* *is the same size as* Explanation:

11. The next year, sweet potato pies of the same size are served again for Thanksgiving dinner. This year, there are 7 adults at the adult table and 5 children at the children's table. There are 4 pies at the adult table and 2 pies at the child table. Assume the pies are cut so that the pieces are of equal size and all of the pies are used. Circle whether an adult's piece is larger, a child's piece is larger, or if both an adult and a child get the same size piece of pie. Explain your reasoning.
Circle: "An adult's piece *is larger than* | *is smaller than* | *the same size* as a child's piece." Explanation:

12. Nine people are invited to a birthday party. At the party, 15 pizzas are served. Since there are no tables that sit 9 people, one way for people to share the pizzas equally among themselves is for 5 pizzas to be at served to 3 tables with 3 people at each. This can be represented with the diagram below, where the number of pizzas at a table is inside the circle and the number of people at the table is below the circle:



Below are two other ways to arrange pizzas and people at tables. Do both configurations ensure that all 9 people at the party get a fair share of pizza? Explain your reasoning.



Are they the same?_____________

Explanation:

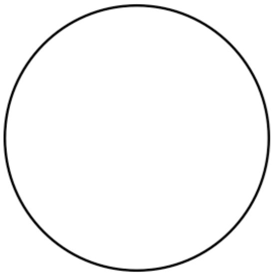
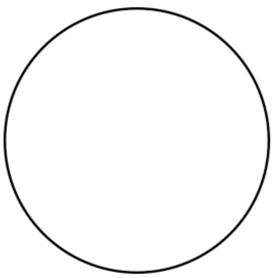
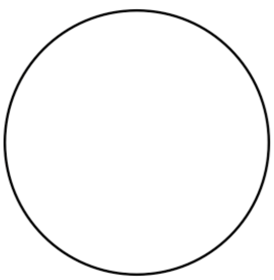
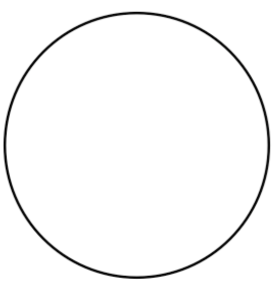
13. Below are pans of brownies to be shared among 4 people.

   a. Draw a variety of strategies *that you anticipate students might use* to solve the task and shade one share.
   b. Describe in words the ways that they may cut to find the solution.
   c. Indicate the level of sophistication of the strategy as I. Unsophisticated, II. Intermediate, III. Sophisticated

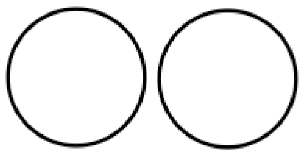
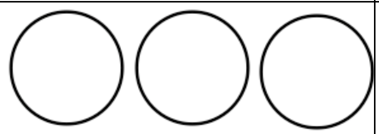
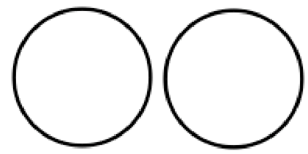
Drawing	Description	Level of Sophistication

14. Below are round birthday cakes to be shared among 6 people.

   a. Draw a variety of strategies *that you anticipate students might use* to solve the task and shade one share.
   b. Describe in words the ways that they may cut to find the solution.
   c. Indicate whether the strategy will yield a correct or incorrect solution.

Drawing	Description	Correct / Incorrect
		
		
		
		

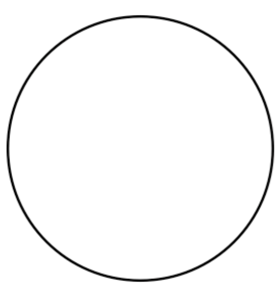
15. Below are three tasks involving numbers of cookies to be shared among various numbers of children. Order the tasks below to indicate the level of difficulty for students, from 1 to 3, where 1 is the *least difficult* and 3 is the *most difficult*. Provide an explanation for your rankings.

Task 1	Task 2	Task 3
 Among 3 children	 Among 5 children	 Among 4 children
Ranking: _______________	Ranking: _______________	Ranking: _____________

Explanation:

16. Draw a picture of how a student might respond to the following task given their understanding indicated. Provide an explanation.

Sharing a round cookie among 3 people if the *focus is on the number of pieces*.



Explanation:

17. For each of the following pairs of tasks, circle the task that you anticipate would to be *more difficult* for K-2 students. Explain your reasoning.

a. *"Sharing 5 cookies between 2 children"* | *"Sharing 2 cookies among 5 people"*

Explanation:

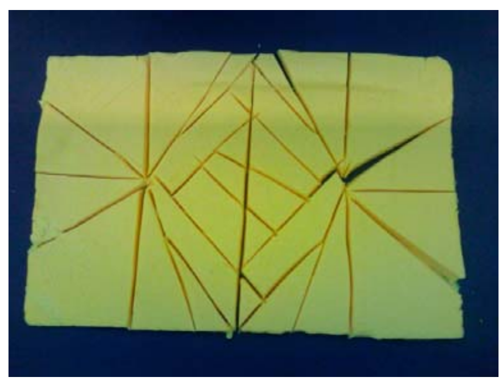
b. *"Sharing a round birthday cake 3 people"* | *"Sharing a round birthday cake among 4 people"*

Explanation:

c. *"Sharing a rectangular birthday cake among 8 people"* | *"Sharing a rectangular birthday cake among 3 people"*

Explanation:

18. Lizzie was asked to share a rectangular birthday cake between two people. Here is a photograph of her work.



a. What might Lizzie understand about fair sharing? Refer to the photograph to justify your claim.

b. What is unclear about Lizzie's understanding about fair sharing? Refer to the photograph to justify your claim.

What question or task might you pose next to Lizzie? Explain your reasoning.

APPENDIX B

1 – Naming of outcome of discrete sharing

Response to 1a is correct if:

- Lines are drawn from each coin or from groups of coins to the chest
  OR
- The number of coins (8) that each pirate gets is explicit.

Correct responses to 1b are categorized as below:

a. Count – 8 coins or numerical expressions (e.g. 24 ÷ 3).
b. Ratio - 8 coins per chest, 16 coins per 2 chests, 24 coins per 3 chests; 8 coins to each pirate.
c. Fraction – 8/24, 1/3, other equivalent fractions, or word equivalents (e.g. "one third").
d. Operator – 8/24 of the coins, 1/3 of the coins.
e. General mathematical name – thirds, fraction, quotient, fair share, equal portions, equipartitions, partitions, portions, equal groups, equal distribution.

Points	Description
3	1a correct AND includes at least three categories from the 1b list
2	1a correct AND includes two categories from the 1b list
1	1a correct AND includes one category from the 1b list
0	1a correct or incorrect AND no attempt or superfluous names OR 1a incorrect OR No response

2 – Quotient construct

Correct Responses:
- "n/q" OR
- "n ÷ q" OR
- "n divided by q" OR
- "1 qth of n" OR
- "n one-qths"

Points	Description
3	Correct response
2	Specific case used as explanation with some mention of generality (e.g., "1/qth" without a reference to the unit or n ÷ q = 1/3) Note: If includes a general statement and then makes a specific case as an example, score as 3.
1	Specific case used as explanation with no mention of generality (e.g. "8" or "1/3")
0	No mention of generality or specific case used OR Specific case used with incorrect generality OR Superfluous answer OR Incorrect Response OR No response

3 – Reversibility of discrete equipartitioning

Correct response: 51 cards or 51

Points	Description
3	Correct response or expression equivalent to the correct response
2	Incorrect response by computational error OR Correct response with incomplete explanation
1	Incorrect response with incomplete explanation
0	Incorrect answer with unreasonable explanation OR No response

4 – Dealing composite units and splitting remainder

Correct responses:
- "36 caramels"
- "2 and ¼ boxes"
- "9/4 boxes"
- "2 boxes and 4 caramels"

Note: does not have to include units for credit

Complete explanations:
- Numerical expressions and computations, simplified or unsimplified, or a written explanations of these computations
- Described dealing and splitting the last box
- Diagrams or models

Incomplete explanations:
- Calculations without a clear solution indicated

Points	Description
3	Correct response AND complete explanation
2	Correct response AND incomplete or no explanation OR Incorrect response but complete explanation (e.g. computational error)
1	Incorrect response with incomplete explanation
0	Incorrect response with unreasonable explanation OR No response

5 – Equivalence of tasks – Case D

Correct response:
Yes – both result in 4/3 or 1 1/3

Complete explanation:

- The first problem can be modeled mathematically as 4 ÷ 3 AND the second problem can be modeled as 1/3 +… +1/3 or 4(1/3) OR
- Uses an area model to illustrate this argument OR
- Both involve three children receiving 1/3 of the total OR
- Explains that pizza type is an extraneous variable

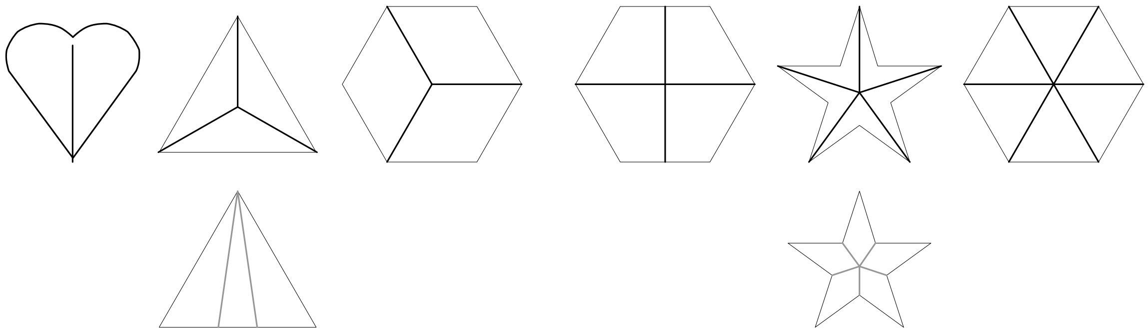
Incomplete explanation:

- Both result in 4/3 or 1 1/3 OR
- Both involve sharing 4 things among 3 people OR
- Both involve the same values/units; may use an area model to illustrate this argument

Points	Description
3	Correct response AND complete explanation
2	Correct response AND incomplete or no explanation OR Correct response and ignores extraneous variable OR Responds “not equivalent” with explanation that the second problem demands 1/3 of each pizza OR Responds “not equivalent” with a complete explanation (e.g. recognized distinction between the two but wrote responded no) Note: If responds “Yes and no” with complete explanations, give 2.
1	Responds “not equivalent” with incomplete explanation OR Responds “equivalent” but to something other than 4/3 or 1 1/3 OR Correct response with unreasonable or no explanation
0	Responds “not equivalent” with unreasonable or no explanation OR No response

6 – Equipartitioning irregular shapes

Correct equipartitioning:



Note: Compositions of these are also correct responses (e.g. equipartitioning the hexagon into 6 equilateral triangles and shading two to split for 3)

Points	Description
3	6 correct AND all shaded
2	5 correct AND all shaded OR 6 correct AND not shaded
1	3 or 4 correct AND all shaded OR 5 correct AND not shaded
0	0, 1 or 2 correct AND all shaded OR 3 or 4 correct AND not shaded OR No response

Note: If > 50% of correct responses are shaded, then count as shaded.
Note: Rotations of the radial segments that preserve angle measure are also correct.

7 – Equivalence of equal-sized pieces

Correct response: “the same as”
Incorrect responses: “less than”, “greater than”, or “cannot tell”

Complete explanation:

- Explains that two halves of the same size unit are equivalent
  OR
- Uses an area argument

Incomplete explanation:

- Does not make reference to the idea that the rectangles are the same size unit

Points	Description
3	Correct response AND complete explanation
2	Correct response AND incomplete or no explanation
1	Incorrect response with complete explanation OR Correct response with unreasonable explanation
0	Incorrect response with unreasonable or no explanation OR No response

8 – Predicting composition of splits

Correct methods:
I: Permutation of half, half, third
II: Permutation of fourth, third
III: Permutation of half, sixth
IV: 11 parallel folds or some permutation involving a 12th

Points	Description
3	Lists method from all 4 categories
2	Lists method from 3 categories
1	Lists method from 2 categories OR Lists 1 method from categories I, II, and III
0	Lists 1 method from categories IV only OR No response

Notes:
- If response lists the results in sequential order but does not describe the folds, then do not award credit (E.g. "fold in thirds, then sixths, then 12ths")
- Subtract one point for the inclusion of an incorrect method.
- Diagrams of I-IV are acceptable

9 – Repeated 2-splits

Correct response: 16 regions

Points	Description
3	Correct response
1	8 regions
0	A number of regions other than 16 or 8 OR No response

10 – Compensation

Correct responses:

a. "is smaller than"
b. "is larger than"
c. "is the same as"

Complete explanations:

a. One-sixth < one-fourth
   OR
   Uses an area model to illustrate this argument with text or symbols
   OR
   Some argument involving compensation
b. Two-sixths > one-fourth or one-third > one-fourth
   OR
   Uses an area model to illustrate this argument with text or symbols
   OR
   Some argument involving compensation
c. Three-sixths = two-fourths or one-half = one-half
   OR
   Uses an area model to illustrate this argument with text or symbols

Incomplete explanations:

- Implied comparisons of fractions that do not use text or symbols (e.g. an area model or fraction notation with out words or "> < =" symbols

Points	Description
3	Correct response and complete explanations for A, B, and C
2	Correct response and complete explanations for A & B, A & C, or B & C OR Correct response and incomplete explanations to A, B, and/or C
1	Correct response and complete explanations for A, B, or C OR Correct response and incomplete explanations for A & B, A & C, or B & C OR Correct response and unreasonable or no explanations for A, B, and C
0	Correct response and incomplete explanations for A, B, or C OR Correct response and unreasonable or no explanations for A & B, A & C, or B & C OR Answers all incorrectly OR No response

11 – Compensation

Correct response: “is larger than”*

Complete explanation:

- Four-sevenths > two-fifths or 20/35 > 14/35
  OR
- Uses an area model to illustrate this argument with text or symbols
  OR
- Some argument involving compensation (E.g. “there are twice as many pies but only 2 more people”)
  OR
- Some argument involving benchmark fractions (e.g. four-sevenths is larger than a half but two-fifths is less than a half)
  OR
- *Note – if respondents create a ratio of people per pie, the relation reverses and yields seven-fourths < five-halves or 1 ¾ < 2 ½

Incomplete explanations:

- Implied comparisons of fractions that do not use text or symbols (e.g. an area model or fraction notation with out words or “> < =” symbols

Points	Description
3	Correct response and complete explanation
2	Correct response with incomplete or no explanation OR No response but complete explanation
1	Responds “is the same as” by using the additive misconception (e.g. the difference between 4 and 2 is 2 which is the same as the distance between 7 and 5) OR Incorrect response with complete explanation (e.g. seven-fourths < five-halves or 1 ¾ < 2 ½) OR
0	Responds “is smaller than” with explanation other than additive misconception or no explanation OR Responds “the same as” with no explanation OR Correct response with unreasonable explanation OR No response

12 – Compensation

Correct response: No

Complete explanations:

- Explains that in the left diagram people at both tables get the same amount of pizza, and the right diagrams shows that people at both tables do not get the same amount of pizza (i.e., people at the table of 5 get more pizza than those sitting at the table of 4 or vice versa)
  OR
- Explains that in the left diagram people at both tables get the same amount of pizza, and the right diagrams shows that people at both tables do not get the same amount of pizza using symbolic representations: $1.67 \neq 1.8 \neq 1.5$, $1\ 2/3 \neq 1\ 4/5 \neq 1\ 1/2$, $36/20 \neq 30/20$, $3/5 \neq 5/9 \neq 4/6$, $.6 \neq .55 \neq .667$, $1.67(5) = 8.33 \neq 9$ and $1.67(4) = 6.67 \neq 6$.

Incomplete Explanation:

- Explains that the shares determined in the right diagram are smaller or are different than the left diagram or vice versa while not explicitly addressing the other.

Points	Description
3	Correct response with complete explanation OR Responds “yes” for the left diagram with complete explanation and “no” for the right diagram with complete explanation
2	Correct response with incomplete explanation OR No response with complete explanation
1	Correct response with no explanation OR Incorrect response with complete explanation for only one diagram OR No response but incomplete explanation
0	Incorrect response with incomplete, unreasonable, or no explanation OR Correct response with unreasonable explanation OR No response

Scoring notes:

- If respondent uses ratios of people per pizza correctly, these count as complete explanations.

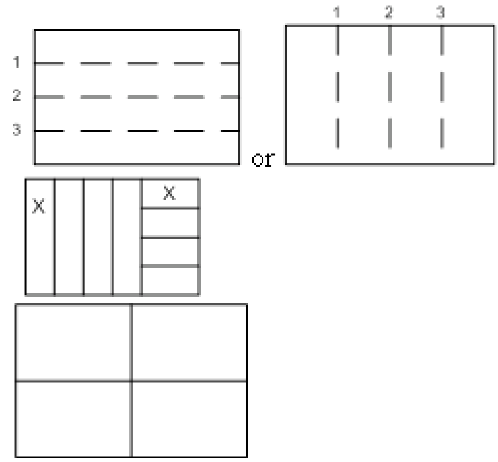
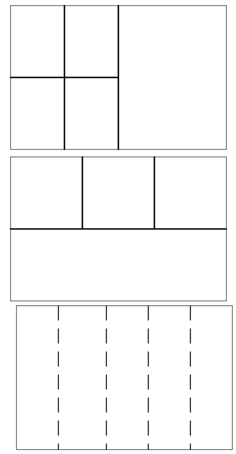
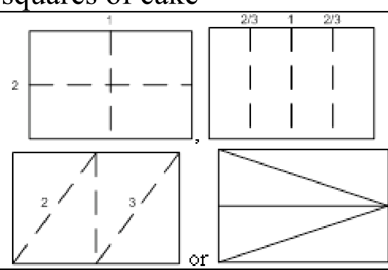
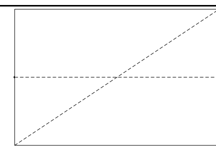
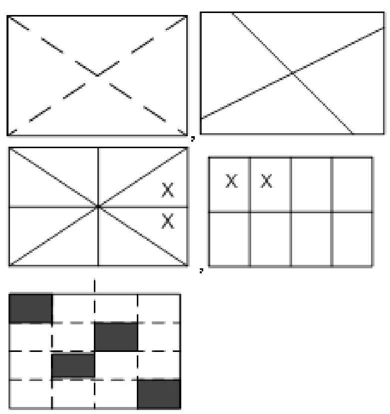
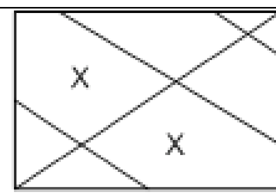
13 – Knowledge of variety and sophistication of strategies

Levels
I: Unsophisticated
- Cannot be done
- Breaking with no attention to creating equal-sized pieces and correct number of equal-sized pieces
- Creating correct number of pieces but of unequal size without composition
- Creating the wrong number of equal-sized pieces
- Failure to exhaust the whole
- n or n – 1 parallel cuts
- Use of landmark fractions and then dividing the remainder because cannot do the split

II: Intermediate
- Must exhaust whole
- A composition of cuts to create all congruent pieces
- Incorrect compositions

III: Sophisticated
- Must exhaust whole
- A composition of cuts to create incongruent pieces
- Correct or incorrect use of equivalence (e.g. creating 8 congruent pieces and giving 2 per person)

Notes:
- Strategies that simply change the orientation of the cuts or similar strategy using equivalence should be counted only once (e.g. three horizontal parallel cuts and three vertical parallel cuts count as only one strategy; a 2 X 8 array with 2 regions shaded and a 2 X 16 array with 4 regions shaded when sharing among four)
- Unless specifically labeled or described as a composition, count a strategy as parallel cuts.
- Repeated examples with the distinction of measuring should only count once (i.e. if three strategies are repeated but say ‘actually measure to find the center’ count all three of these only once).

Level	Correct	Incorrect
I	 Creating 4 "boxes" like drawing 4 squares of cake	"Cannot be done" Breaks 
II		
III	 All uses of equivalence are III	

Locate the number of distinct strategies on the left and the number of levels of sophistication represented across the top. The corresponding number in the table is the raw score for the item. Next, consider the notes below the table to finalize the item's score.

Score	3 Levels	2 Levels	1 Level
4+ strategies	3	2	2
3 strategies	3	2	2
2 strategies	--	2	1
1 strategy	--	--	0

Scoring notes

- Subtract one point if a correct strategy is listed as incorrect
- Subtract one point for incorrect levels of sophistication or no indicated levels
- Subtract one point for ≤ 50% of the strategies have complete descriptions. Descriptions may be written or numbers but specific (e.g. ‘cut into fourths’ is specific, ‘1/4ths’ is not)
- Score as 0 for No Response

14 – Knowledge of variety and correct/incorrect strategies

Levels
I: Unsophisticated –

- Cannot be done
- Breaking with no attention to creating equal-sized pieces and correct number of equal-sized pieces
- Creating correct number of pieces but of unequal size without composition
- Creating the wrong number of equal-sized pieces
- Failure to exhaust the whole
- n or n – 1 parallel cuts
- Sequential radial cuts
- Use of landmark fractions and then dividing the remainder because cannot do the indicated split

II: Intermediate

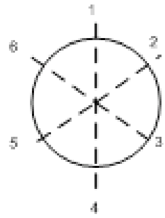
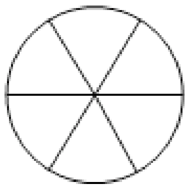
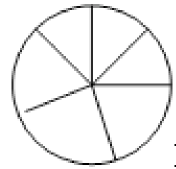
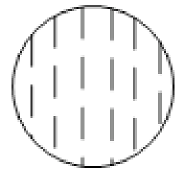
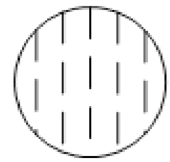
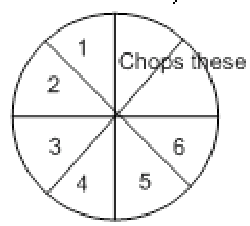
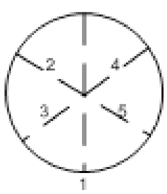
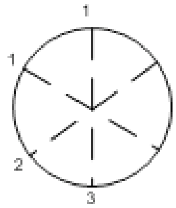
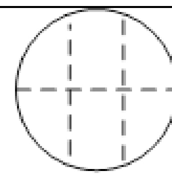
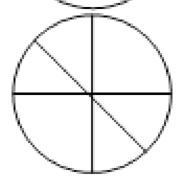
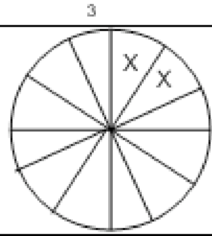
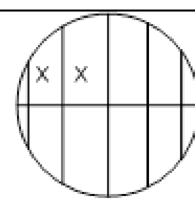
- Must exhaust whole
- A composition of cuts to create all congruent pieces
- Incorrect compositions

III: Sophisticated

- Must exhaust whole
- Correct or incorrect use of equivalence (e.g. creating 12 congruent pieces and giving 2 per person)

Notes:

- Strategies that simply change the orientation of the cuts should be counted only once (e.g. 5 horizontal parallel cuts and 5 vertical parallel cuts count as only one strategy)
- Unless specifically labeled or described as a composition, count a strategy as parallel cuts.
- Repeated examples with the distinction of measuring should only count once (i.e. if three strategies are repeated but say 'actually measure to find the center' count all three of these only once).
- Landmark strategy is different from a composition (e.g. a 3-split on a 2-split) because with a composition, still attending to each piece, whereas with landmarks, the actions focus on remaining piece after distribution.

Level	Correct	Incorrect
I	  Creating 6 sectors like drawing out 6 slices of cake	“Cannot be done”  Breaks  Parallel cuts, either 6 or 5  Landmarks
II	 	 
III		

Points	Description
3	Indicates at least 2 correct strategies and at least 2 incorrect strategies OR Indicates at least 3 correct strategies and 1 incorrect strategy
2	Indicates 2 correct strategies and 1 incorrect strategy OR Indicates 1 correct strategies and 2 incorrect strategy OR Indicates 1 correct strategy and 3 incorrect strategies OR Indicates 3 correct strategies and no incorrect strategies OR Indicates 3 incorrect strategies
1	Indicates 2 correct strategies and no incorrect strategies OR Indicates no correct strategies and 2 incorrect strategies OR Indicates 1 correct strategy and 1 incorrect strategy
0	Indicates 1 correct strategy OR 1 incorrect strategy OR No response

Scoring notes:

- Subtract one point for incorrect or no labeling of correct/incorrect strategies
- Subtract one point for ≤ 50% of the strategies have complete descriptions. Descriptions may be written or numbers but specific (e.g. 'cut into fourths' is specific, '1/4ths' is not)

15 – Knowledge of sophistication

Correct response:
1) Task 3 is the *least difficult*,
2) Task 1 is the *next difficult*, and
3) Task 2 is the *most difficult*

Complete explanation:
- Explains that a split of a 2-split is easiest for students and a 3-split is easier than a 5-split, or vice versa (i.e. must relate all three of the splits, not just the evens versus the odds)

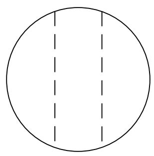
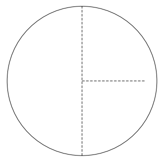
Incomplete explanation:
- Correct area models with no verbal explanations
- Fails to address how pertinent observations relate to one another

Points	Description
3	Correct response with complete response
2	Correct response with incomplete explanation
1	Correct response for exactly one of a, b, or c with complete explanation OR Correct response with unreasonable or no explanation OR Incorrect response with reasonable explanation
0	Incorrect response with unreasonable or no explanation OR Correct response for exactly one of a, b, or c with incomplete, unreasonable, or no explanation OR No response

Notes:
- Explanations about the number of objects being divided do not count.
- Implicit attention to smaller odd numbers counts as incomplete (e.g. "sharing for 3 is less divisions than for 5")

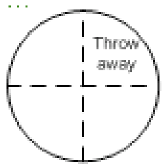
16 – Variety of strategies and knowledge of students' thinking

Correct responses include:
I.

 or  or a configuration where 3 pieces (or a multiple of 3 pieces) are not the same size

OR

II.



OR

III. Chops and deals pieces

Complete explanations:

- Students unable to make radial cuts and difficulty with 3-splits
- Student not using the whole or any other correct composition where student does not use the whole
- Student must fairly deal after creating many pieces or if does not create a multiple of 3 pieces, states that it cannot be done.
- Student finds it easier to half and then halve one-half to get 3 pieces and pieces are not the same size
- Student cuts 3 unequal pieces but ensures that each person gets a piece (these may be radial cuts)

Points	Description
3	Correct response from I – III and complete explanation
2	Correct response with incomplete explanation
1	Correct response with explanation that does not mention number of pieces OR Correct response with no explanation OR Correctly splits circle with complete explanation
0	Incorrect response OR Correctly splits circle with incomplete, unreasonable, or no explanation OR No response

17 – Knowledge of sophistication

Correct response:

a. Circles "2 among 5"
b. Circles "among 3"
c. Circles "among 3"

Complete explanations:

a. States that 2-splits are easier OR
   Dealing is easier OR
   Use symmetry since 2 is an even number
b. States radial cuts are harder OR
   3-splits are harder OR
   Repeated halving is easier OR
   Use symmetry since 4 is an even number
c. States radial cuts are harder OR
   3-splits are harder OR
   Repeated halving is easier OR
   Use symmetry since 8 is an even number

Note:

- Number of cuts is superfluous
- For Gemma's, if there is a distinction of case C being less sophisticated than case D, score as complete explanation.

Points	Description
3	3 correct responses and complete explanation
2	2 correct responses with complete explanations OR 3 correct responses less than 3 complete explanations OR 2 or 3 incorrect responses with 2 or 3 complete explanations (it may be possible that the student circled the easier tasks)
1	1 correct response with a complete explanation OR 2 correct responses less than 2 complete explanations
0	3 incorrect responses with unreasonable or no explanations OR 1 correct response with incomplete, unreasonable, or no explanation OR No response

18 – Knowledge of sophistication

Correct responses with complete explanations:

I. Number of pieces – explains that the child created the same number of pieces on both sides of the 2-split (explicit comparison of the number of pieces on both sides).
II. Size of pieces – explains that the student created two equal parts with a 2-split and notes the symmetric nature of corresponding pieces.
III. Exhaustion of whole – explains that the student used the whole cake.
IV. Allocation – explains that it is unclear how the student will allocate the pieces of cake (i.e. indicating what each person's share is)
V. Related question – includes a question or task that relates to their uncertainties about what they student knows or does not know about fair sharing.

Points	Description
3	Response addresses 4 or 5 points from I, II, III, IV, & V
2	Response addresses points from 3 of I, II, III, IV, & V
1	Response addresses points from 2 of I, II, III, IV, & V
0	Response addresses points from 0 or 1 of I, II, III, IV, & V OR No response

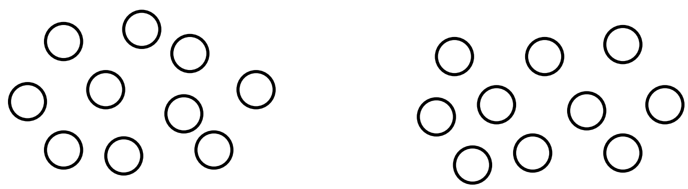
Notes:

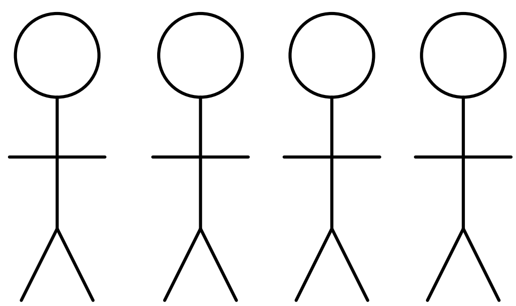
- Demonstrations in response to the next task do not answer the question and should not be counted as addressing IV.

Question 1

Ann and Beth wanted to play marbles. To play, they need to use all the marbles and each to have a fair share of the marbles. So, they made two piles of marbles as shown below. Just before they started to play, two other friends, Carlos and Darian, showed up and wanted to play. Using the circles below to represent the marbles, show Ann, Beth, Carlos, and Darian's share of marbles.

a. How would you help them share the marbles below so that all four children have a fair share of the marbles? Without redrawing any marbles, show each child's share.



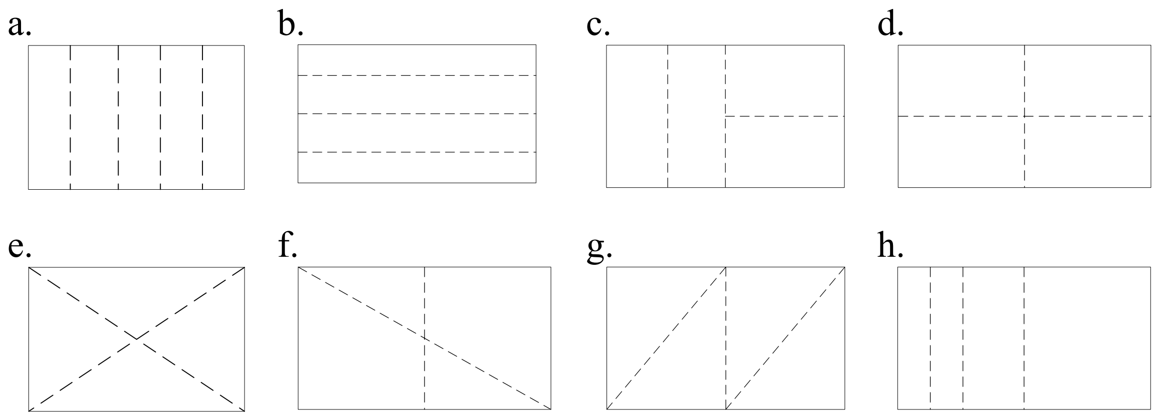


b. Did each child get a fair share of the marbles? Circle one: Yes No
Explain how you know.

c. How much of the marble collection did each child get?

Question 2
A classroom of children was asked to cut a rectangular cake into four fair shares using up all of the cake. Below are some of their pictures.

Circle each picture that is cut into four fair shares and uses up all of the cake.



Easiest   Hardest

Of the correct ones, draw a line from the word “Easiest” to the picture that shows the easiest way to share the cake.

Of the correct ones, draw a line from the word “Hardest” to the picture that shows the hardest way to share the cake.

Question 3
Ann and Beth asked their mother for some jelly beans. Their mother put the jelly beans on the table and began to share them. Watch quietly as I show you how they shared the candy.

a. Did Ann get a fair share of the jelly beans? Circle one: Yes No
Explain how you know.

b. How much of the candy did Ann get?

c. How much of the candy did Beth get?

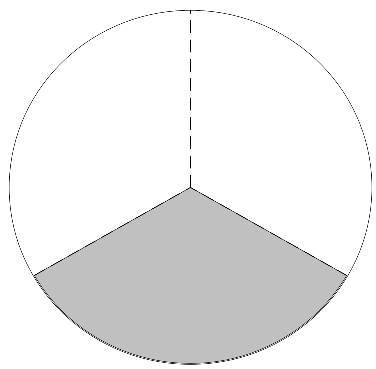
Question 3 (continued)

d. Now, did the children get a fair share? Circle one: Yes No
Explain how you know.

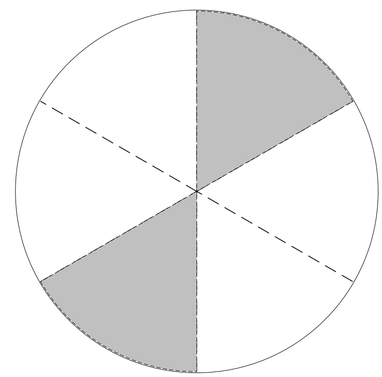
e. Now, how much of the candy did Ann get?

f. Now, how much of the candy did Beth get?

Question 4
Below are two pies that are the same size. The left one is a chocolate pie. The right one is a lemon pie. The pies are cut as shown below. Ann, Beth, and Carlos each want to have a fair share of each flavor of pie they want and to eat all of the pie. The shaded portion below shows Carlos's share of each of the two pies.



Chocolate pie



Lemon pie

a. Does Carlos get the same amount of chocolate pie as lemon pie?

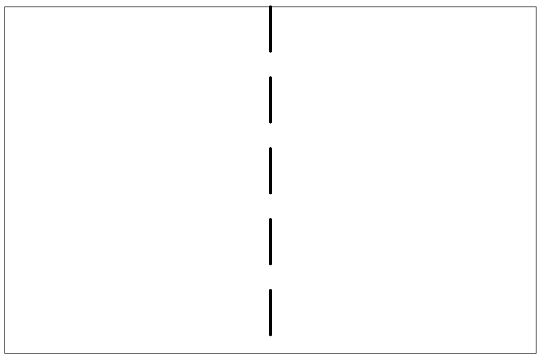
   Circle one: Yes No

   Explain how you know.

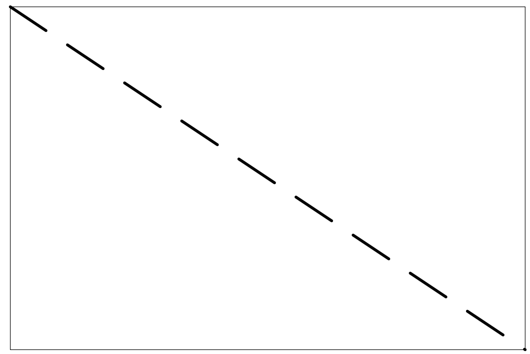
b. How much of the chocolate pie did Carlos get?

c. How much of the lemon pie did Carlos get?

Question 5
Ann and Beth each have brownies that are the same size. Both children cut their brownies into two fair shares. Ann's brownie is on the left, and Beth's brownie is on the right.



Ann's brownie



Beth's brownie

Is a piece of Ann's brownie bigger, smaller, or the same size as a piece of Beth's brownie? Circle one:

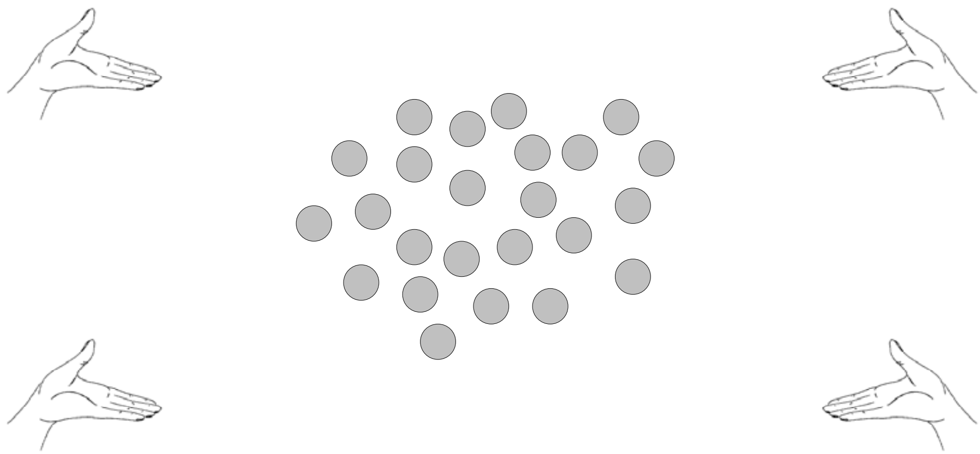
Bigger Smaller The Same Size As

Explain how you know.

Question 6

On Halloween, Ann, Beth, Carlos, and Darian got some M & M's to eat. They want to share them fairly and use them all up.

a. Help these friends share the M & M's fairly by drawing a line from each M & M to the hand of the person who will eat it.



b. Did Ann get a fair share? Circle one. Yes No
Explain how you know.

c. How much of the candy did Ann get?

d. Is there another way to name Ann's share? Circle one. Yes No
If yes, how can it be named?

e. If more children came to share the same amount candy fairly, would Ann's share be more, less, or the same as her share now?

Circle one: More Less The Same Amount
Explain how you know.

Question 7

Ann invited five friends to her birthday party. They decided to skip lunch and eat the whole birthday cake instead, sharing it fairly among the six of them. The birthday cake is the whole rectangle drawn below.

a. Help Ann cut her birthday cake into six fair shares by drawing lines below where you would cut the cake.

b. Shade one person's share.
Explain how you know this is a fair share.

c. How much of the cake does each person get?

APPENDIX D

I. Multiple Methods on Rectangle - A2 B2 C4, D4
Scoring Note: Ignore Easiest/Hardest portion of test. If choices not circled, use discretion to infer how students indicated their choices.

Sharing for **FOUR** on both pretest and posttest
Correct responses b, c, d, e, g Incorrect responses a, f, h

Score	Responses
3	5 correct with 0 incorrect
2	4 correct with 0 incorrect OR 5 correct with 1 incorrect
1	3 correct with 0 incorrect OR 4 correct with 1 incorrect
0	2 or fewer correct with 0 incorrect OR 3 correct and 1 incorrect OR 2 or more incorrect OR No response

Sharing for **SIX** on pretest only
Correct responses a, d, e Incorrect responses b, c, f

Score	Responses
3	3 correct with 0 incorrect
2	2 correct with 0 incorrect OR 3 correct with 1 incorrect
1	1 correct with 0 incorrect OR 2 correct with 1 incorrect
0	0 correct OR 2 or more incorrect OR No response

Sharing for **SIX** on posttest only
Correct responses b, c, d, g Incorrect responses a, e, f, h

Score	Responses
3	4 correct with 0 incorrect
2	3 correct with 0 incorrect OR 4 correct with 1 incorrect
1	2 correct with 0 incorrect OR 3 correct with 1 incorrect
0	1 or fewer correct with 0 incorrect OR 2 correct and 1 incorrect OR 2 or more incorrect OR No response

II. Property and Naming on Circles - A4 B5 C1 D1

a) Property

Score	Response
3	**Yes, developed property reasoning –** Explanation uses compensation or transitivity directly
2	**Yes, developing property reasoning –** Explanation uses informal property reasoning - a reference to "the same"
1	**No, attention to just number or shape –** **OR Yes, with unspecific, unrelated, or no explanation** **-** evidence of counting, such as addition or subtraction facts
0	**No, other, unclear, or no explanations** **OR No response** - changing number of people sharing

b) Naming
Scoring Note – both b and must be answered in some level to receive a nonzero score

Score	Response
3	**Correct Relational Names** Thirds, sixths, fourths, eighths, 1/3 2/6, ¼, 2/8, etc
2	**Unspecific or Incorrect Relational Names** - 1 big piece and 2 little pieces (both responses qualified) -1 correct and 1 incorrect relational name -Any response that equates the total portions from each cookie that is nonspecific
1	**Counts** - 1 piece, 2 pieces, etc - If using a split argument in a, then 2 pieces and 2 pieces - If using an assembly argument in a, then 1 piece and 1 piece
0	**Incorrect Counts or General Names** **OR No Response** - Slices, a share, a strip, etc

III. Properties on Rectangle - A5 B3 C2 D2

Score	Response
3	**The Same As, developed property or reasoning about the whole –** - Splitting and/or assembly in drawing to make congruent portions - Any argument that references the same size whole for each equipartitioning
2	**The Same As, developing property, other, unclear, or unspecific reasoning –**
1	**Bigger or Smaller, any non-holistic or visual argument** - Compares linear or area measurements - Makes reference to where the cuts are - Any argument that rises above a visual/perceptual one
0	**Bigger or Smaller, holistic or visual argument** **OR No Response**

IV. Method, Justification, and Naming for Circle and Rectangle - A7 B4 C6 D6

a) Equipartitioning
Scoring Note – if student solutions are reasonably close, consider as correct (esp. radial cuts)
Scoring Note – if multiple drawings due to mistakes use the most correct for scoring.
Scoring note – read entire explanation to clarify method (e.g. it may be unclear if respondent did not exhaust whole or changed the number of pieces unless one reads entire response)
Scoring note – use justifications & shading from (b) to understand solution in part a.

Score	Response
3	**Correct Equipartitioning** Successfully exhausts whole and creates 6, 12, 18, etc equal sized pieces (i.e. all 3 conditions for EP are met) - if using equivalence, must clearly allocate pieces through sharing / other means
2	**Partially Correct Equipartitioning** Exhausts whole with correct number (6, 12, etc) of unequal size pieces OR Exhausts whole with incorrect number of equal size pieces OR Fails to exhaust whole (i.e. 2 of the 3 conditions are met)
1	**Partially Incorrect Equipartitioning** Exhausts whole with incorrect number of unequal size pieces OR Fails to exhaust whole with incorrect number of equal splits OR Fails to exhaust whole with correct number of unequal splits (i.e. 1 of the 3 conditions are met) - Include parallel cuts using landmark fractions at this level
0	**Incorrect Equipartitioning** OR No discernable answer OR No response

b) Naming

Score	Response
3	**Correct Relational Names** - Thirds, Sixths, Twelfths, 1/6th, 2/12ths
2	**Unspecific or Incorrect Relational Names** - the same amount - names all fractional pieces 'halves'
1	**Counts** - consider strategy
0	**Incorrect Counts or General Names** **OR No Response** - a little of it, a share, a strip, a slice, a fair share - count of an incorrect split

APPENDIX E

Preparation

For each interview, you will need:

- One "treasure chest" (re-sealable plastic bags) containing 24 pieces of "treasure" (coins) - that only 24 coins are used during the activity.
- Instruction sheet -- the instruction sheet is to be reused for all three interviews

The Activity

Preparation:

a. Remove six coins (be sure there are 24 total left in the bag--the "treasure chest"). DO NOT COUNT them in front of the students!
b. Give the student the "treasure chest." (the bag containing the 24 coins) Tell the student that two pirates have found a treasure chest and that the pirates must share the treasure inside.

Ask: "Can you share the treasure fairly between the two pirates?"
Ask: "How did you share the treasure?"
Ask: "Does each pirate have a fair share of the treasure?"

Ask: "How do you know each pirate has a fair share?" (or "How can you be sure each pirate has a fair share?")

Ask: "What would you name each pirate's share?"

Ask: "How much of the total treasure does each pirate get?"

Ask: "How many did each one get?"

Ask: "Is there another way to share the treasure fairly?"

Ask: "If you put all the treasure back together, how would describe the treasure?"

Once the student has completed this task, collect all the coins into a single pile.

Say: "Now, suppose there were <u>four</u> pirates who found the treasure chest."

--Repeat the entire activity again, using <u>four</u> pirates

Say: "Now, suppose there were <u>three</u> pirates who found the treasure chest."

--When the student has completed the activity for four pirates, repeat the activity again for THREE pirates

APPENDIX F

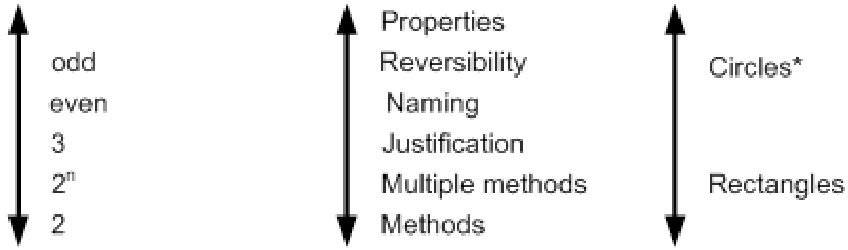
1.1 Partitioning using 2-split (continuous and discrete quantities)			
Performance	**Specific Task Examples**	**Sub-performances / behaviors**	**Quotes and Video Exemplars**
1.1.7 Properties		Equivalence: 2m for 2 people is the same as m for 1 person	
	"How have you done the same thing?" "Can you serve any more people?"	Compensation: Repeated 2-splits	"It's the same way I did it last time." Clip 50 (Daniel, Grade 1) – Repeated halving of cake pieces.
1.1.6 Reversibility: One group or the whole group is the starting point and can be reassembled.	"If you put (all) these back together, what do you have?"	Students are able to state that they will return to the starting place in terms of a whole or a complete collection	["The whole; 1 (whole) cake"] ["32. The original number (of coins)"] Clip 49 (Brian, Grade 3) – Concatenation of equal groups to get the whole.
1.1.5 Naming: Identify one-half as the size of each part resulting from a 2-split	"What would you call the share?" "How would you refer to the share?" "How much of the total would you have?"	Relational number	["Half; one half"] ["Half of the original number of coins"] "one half of a whole" "1 half and a half is 2 equal parts" "because one line in the middle makes half and half"
		Count of a share	"8"
		General names	"a group"
		Cannot name	
1.1.4 Justification: Justify equality of groups	"Did you know they were the same before you counted? How?"	If yes, then same as below…	

	"How do you know that the two shares are the same?" "How do you know the shares are fair?"	Justifies with counting Justifies with dealing Creates and array with or without counting Piles objects in each share, compares the heights, and declares them equal Justifies with folding Justifies by measuring with outside object	Clip 11 (Kate, Grade 1) – Vertical 2-split, stacks to justify. Clip 48 (Annie, Grade 1)- Justification by height and counting. ["Fold it to make sure it's in the right spot"] ["Split it in half and give them an equal amount."] ["Because the piles are equally tall"] "Because they are both equal because I split in the middle" "I folded the corners so they touch… and the sides matched… see how the corners match?" "Make them have the same amount" "I give them one by one" "I'm sorting and I'm making each part have the same amount" "I dealt in groups of 2" "I dealt them out one at a time" I counted them and they were the same"
		Cannot justify	
1.1.3 Multiple Methods Partition a rectangle or circle multiple ways (horizontal, vertical, and diagonal)	"Are there any other ways you can make equal shares for four persons?"	See below for various methods numerous different cuts, all through a common center on the rectangle	"It's a star, afterwards." Clip 47 (Brian, Grade 3) – Fold and rotational symmetry.
		mark or cut the circle or rectangle in half vertically, horizontally, or diagonally, but differently than in 1.1.2	Clip 11 (Kate) - Multiple methods: vertical, horizontal, and diagonal.

1.1.2 Partition a rectangle or circle between 2 people	Share a birthday cake (rectangle, circle) evenly among 2 people. “It’s your birthday, and you and a friend are going to share your birthday cake. How would you share the cake fairly?”	mark, fold, or cut the rectangle in half vertically, horizontally, or diagonally mark, fold, or cut the circle diametrically	Clip 11 (Kate, Grade 1) – Vertical 2-split, stacks to justify. Clip 2 (Alexis) - Vertical 2-split. “Because there are two people and two pieces, so you give one piece to one person and the other piece to the other person so it would be fair.” “Now I have to cut this down the line” and counts 15 pieces of cake on each side of the line.
		Believes it cannot be done.	
1.1.1 Partitioning a set of 2*i* objects between 2 people; *i* up to 16’	2 pirates share 2*i* coins of treasure, *i* up to 16 (dealing): “Two pirates have a treasure of coins. Would you show them how to share their treasure fairly?”	deal two or more objects at a time to each pirate, then one at a time until all are dealt. deal more than one object at a time until all dealt deal one coin to each side simultaneously until all coins are dealt deal one object at a time until all dealt.	Clip 37 (Cameron, Grade 3) - Dealing by 3s. Clip 10 (Kate, Grade 1) - Simultaneous dealing. Clip 46 (Annie, Grade 1) - Creating piles of equal height.
		Breaks whole into 2 [obviously] uneven parts. Separates a set of 2i objects between 2 people unevenly.	
		Believes it cannot be done.	

APPENDIX G

A Learning Trajectory for Equipartitioning: Sharing a Whole among _______

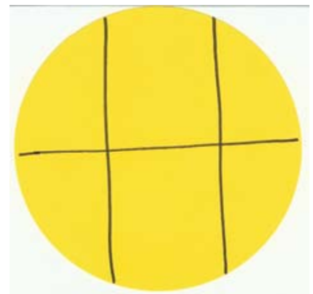


	Actions & Words	**Thinking**
Properties		
Reversibility		
Naming		
Justification		
Methods & Multiple Methods		

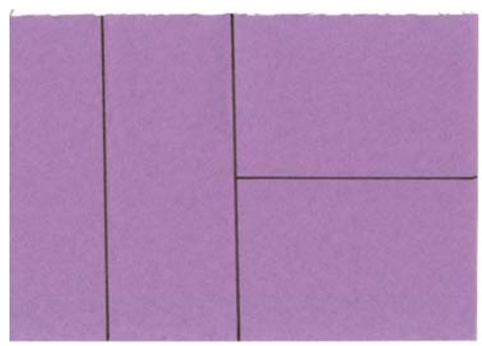
APPENDIX H

Analyzing Students' Work

A student was asked to share the lemon pie fairly among 6 people.



A student was asked to share a pan of brownies fairly among 4 people.



What do you think the student understands about the mathematics of the problem? What evidence supports your claim?

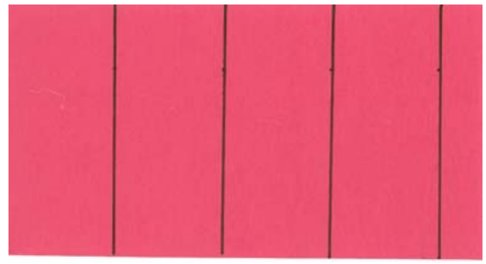
What do you think the student does not understand about the mathematics of the problem? What evidence supports your claim?

What questions about the student's understanding are raised by this work sample?

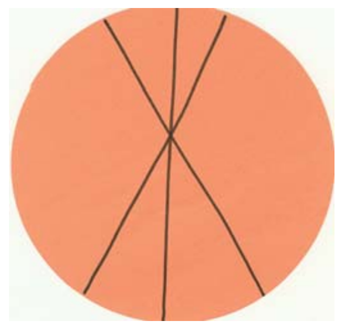
What descriptive feedback would help the student?

What will you do next with the student?

A student was asked to divide the rectangle into 4 equal parts.



A student was asked to share the circular birthday cake fairly among 6 people.



What do you think the student understands about the mathematics of the problem? What evidence supports your claim?

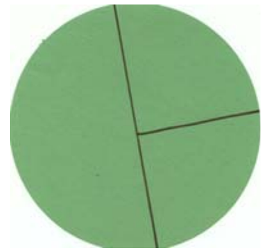
What do you think the student does not understand about the mathematics of the problem? What evidence supports your claim?

What questions about the student's understanding are raised by this work sample?

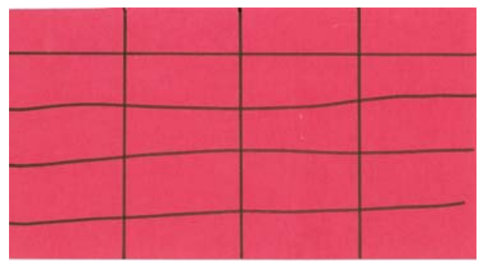
What descriptive feedback would help the student?

What will you do next with the student?

A student was asked to share the key lime pie fairly among 3 people.



A student was asked to share the rectangular birthday cake fairly among 4 people.



What do you think the student understands about the mathematics of the problem? What evidence supports your claim?

What do you think the student does not understand about the mathematics of the problem? What evidence supports your claim?

What questions about the student's understanding are raised by this work sample?

What descriptive feedback would help the student?

What will you do next with the student?

APPENDIX I

Preparation

For each interview, you will need:

- Paper rectangles – up to 36 may be needed (or playdough)
- Paper circles – up to 36 may be needed (or playdough)
- Paper straight-edge (or plastic knife)

The Activity

Tell the student that a pirate is having a birthday party and wants to share his birthday cake with another pirate. The paper rectangle represents the birthday cake. Present the cake to the student so that the longer side is closest to the student. Students may use the straight edge, but do not allow them to use a ruler to measure.

Ask: "Can you share the entire birthday cake fairly between the two pirates? Show me how you would do it." [subsequently, you will repeat the activity with rectangular cakes for 4, 3, and then 6 pirates. Then you will repeat the same set of activities with circular cakes]
Ask: "How did you share the birthday cake?"
Ask: "Does each pirate have a fair share of the cake?"

Ask: "How do you know that each pirate got a fair share?" (or "How can you be sure each pirate has a fair share?")

Ask: "What would you name each pirate's share?"

Ask: "How much of the total cake does each pirate get?"

Ask: "How much did each person get?"

NOTE: If the child responds with a number, for instance, "two," interviewer should ask "Two what?"

If the child then responds "pieces" or "slices", then ask "what is a piece/slice?" and "Are all pieces/slices the same?"

Ask: "Is there another way to share the cake?"

If the student answers "yes", repeat the question series with another rectangle. When student has exhausted all strategies,

Say: "What would they have if they put their pieces back together?"

If student doesn't have an answer for this,

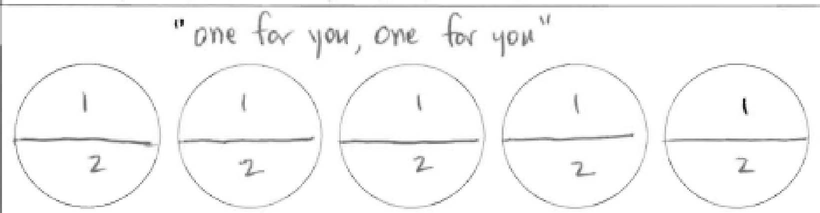
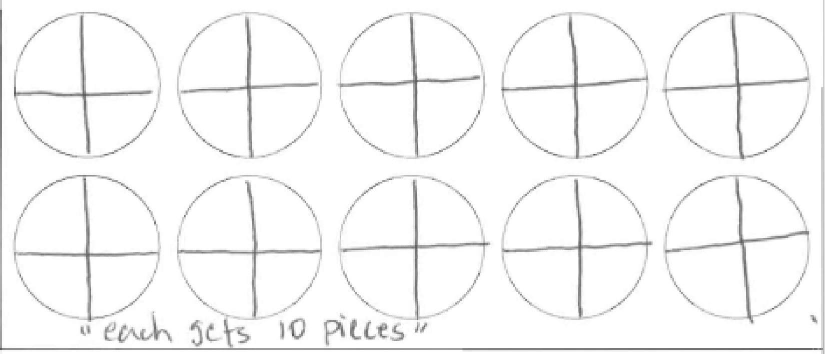
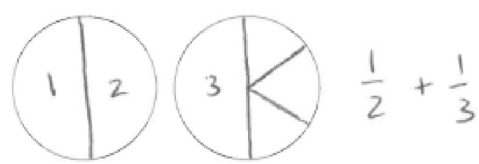
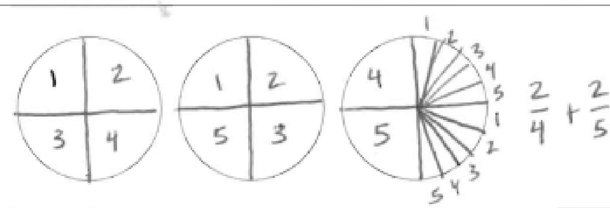
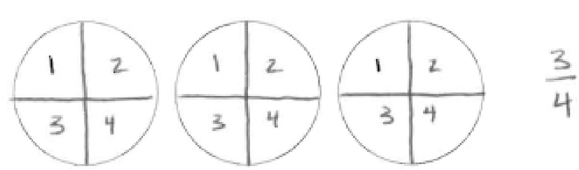
Ask: "If they put all their pieces back together, would they have as much as the whole cake?"

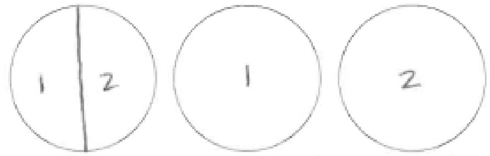
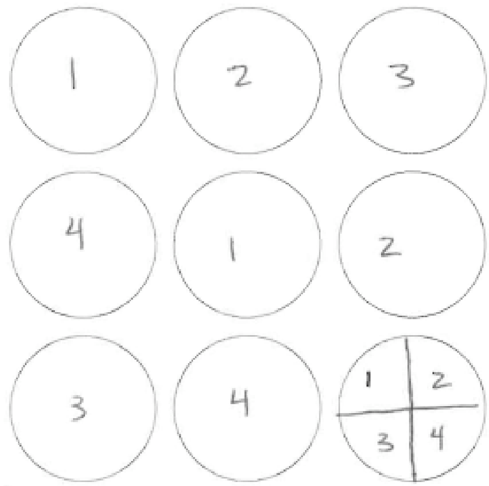
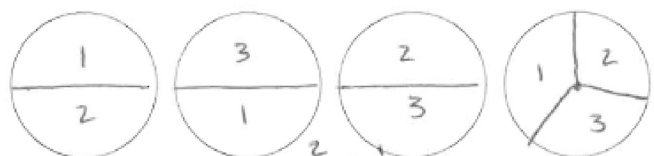
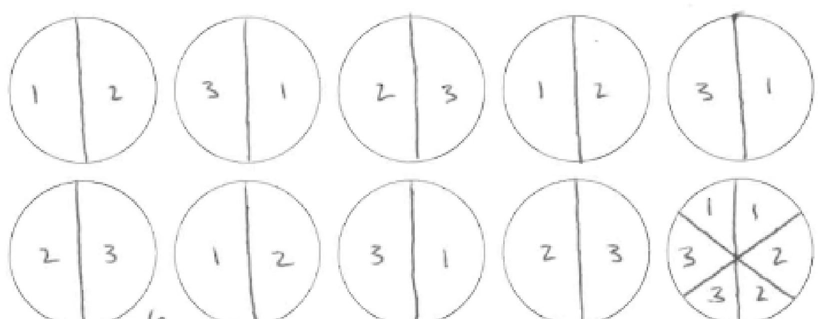
Now, repeat the entire activity using four (then three, then six) pirates using a corresponding record sheet for each iteration.

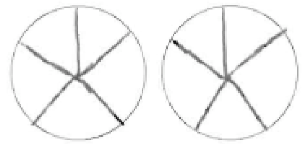
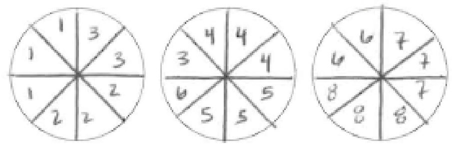
Say: "Now, suppose there were <u>four</u> (then <u>three</u>, then <u>six</u>) pirates at the party."

Say: "Now, suppose that the birthday cakes were *round* rather than rectangular. How could two pirates share the entire cake?"

APPENDIX J

E 5 cookies among 3 people	
B 5 graham crackers among 2 people	
L 10 cookies among 4 people	
K 2 pizzas among 3 people	
F 3 pies among 5 people	
G 3 cakes among 4 people	

A 3 cookies between 2 people	 "we each get one & split the last one in two"
I 9 brownies among 4 people	 "each gets 2 brownies and one fourth"
D 4 graham crackers among 3 people	 $\frac{2}{2} + \frac{1}{3}$
J 10 brownies among 3 people	 $\frac{6}{2} + \frac{2}{6}$

C 2 pizzas among 5 people	 "each gets 2 pieces"
H 3 pies among 8 people	 $\frac{3}{8}$

As a group, sort the tasks with responses into two groups. List the letters of the tasks in the boxes below.

Group 1	Group 2

Explain your criteria for the categories.

As a group, sort the tasks with responses into two groups unlike your previous sort. List the letters of the tasks in the boxes below.

Group 1	Group 2

Explain your criteria for the categories.

As a group, sort the tasks with responses into three groups. List the letters of the tasks in the boxes below.

Group 1	Group 2

Explain your criteria for the categories.

As a group, sort the tasks with responses in any way that makes sense for you. Record the number of groups and the task letters below.

Explain your criteria for the categories.

APPENDIX K

Lesson One: Fair Sharing a Continuous Whole

<u>Math Focus Points</u>

- Creating equal-sized groups with a continuous whole;
- Using geometric properties to justify equivalence of shares;
- Naming the results of an equipartitioning.

<u>Vocabulary</u>

- Shade

<u>SCOS Correlations</u>

- Objectives 1.02, 1.03
- Week by Week Essentials: Week 15 Solve This!
- Indicators: 1.02a – A, B; 1.02b – A, C

<u>*Investigations* Connections</u>

- Unit 2: Shapes, Blocks, & Symmetry
  Investigation Two: What Is a Rectangle?
  Investigation Three: Symmetry

Today's Plan	Materials
Activity 1. Sharing Thanksgiving Desserts Time estimate: 10 minutes Pairs	Handout One (for pairs of students) Several copies of Transparency One Several copies of Transparency Two Overhead markers
Discussion 2. Different Ways of Sharing Time estimate: 10 minutes Whole Class	
Activity 3. More People for Dessert Time estimate: 10 minutes Pairs	Handout One (for pairs of students) Several copies of Transparency One Several copies of Transparency Two Overhead markers
Discussion 4. Different Ways of Sharing Time estimate: 10 minutes Whole Class	
Session Follow-up Many Different People for Dessert Time estimate: 10 minutes	Handout Two (for each student)

1. Activity: Sharing Thanksgiving Desserts
Time estimate: 10 minutes
Pairs

Distribute Handout One to pairs of students.

"Pretend it is Thanksgiving Day. For dessert, there is a cake in the shape of a rectangle and a pecan pie. There are four people at your Thanksgiving dinner and you want everyone to have a fair share of the cake and the pecan pie. With all of the food left over from the meal, there will be no room in the refrigerator for leftover desserts, so you want to use them all up."

"With your partner, draw lines on the cake and pie where you would cut them so that each person at the meal has a fair share of both desserts. Shade one person's share. Show as many ways as you can think of. Think about how you know what you will serve to the guests is a fair share."

Circulate throughout the room and take note of different ways that the pairs of students are partitioning the rectangles and circles. Ask them about how they know that each share they produce is a fair share. Select several pairs of students to present their methods to the class.

For each pair you select, give them a copy of Transparency One [Two] and an overhead pen and ask them to prepare to share the method you selected for the cake [pie].

2. Discussion: Different Ways of Sharing
Time estimate: 10 minutes
Whole Class

Choose several pairs of students to come to the overhead to share their method of cutting the rectangle [circle] with the class on Transparency One [Two].

"Did you all think of the way that [Ann & Beth] shared the cake [pie]?"

"Why or why not?"

"[Ann & Beth], how do you know that each person at dinner gets a fair share of the cake [pie]?"

"What would you call each person's share of the cake [pie]?"

For each presentation, highlight the different points associated with each question above as appropriate:

1. Whether the method produced fair shares or not and why:
    - Fair shares must exhaust the whole
    - Must produce equal size parts
2. The method itself:
    - One less parallel cut always works on rectangle;
    - The use of radial or diametric cuts on a circle;
    - Less cuts if use a composition – emphasize that one cut (action) does the same thing to multiple pieces.
3. The different justifications:
    - Use of symmetry
    - Use of congruency
    - Use of measuring and area
4. The different naming strategies:
    - May name by the number of pieces (1 piece)
    - May name by the number of pieces in relation to the whole (1 of the four pieces)
    - May name as a unit fraction (one-fourth, one-fourth of)

Note: If different pairs of students create different numbers of pieces for their sharing (e.g. 2 of 8 pieces versus 1 of 4 pieces), help students understand that these are equivalent ways of sharing. This can be seen as taking each of the pieces for and splitting in half, for instance.

3. Activity: More People for Dessert
Time estimate: 10 minutes
Pairs

"Just before you cut the cake and pie, two of your neighbors come over and want some dessert, too. Do you think that each person will get more, less, or the same amount of dessert? How do you know?"

Facilitate a discussion about the compensatory principle – as the number of fair shares increases, the size of each share decreases.

Distribute another copy of Handout One to pairs of students.

"With your partner, draw lines on the cake and pie where you would cut them so that each of the six people has a fair share of both desserts. Shade one person's share. Show as many ways as you can think of. Think about how you know what you will serve to the guests is a fair share."

Circulate throughout the room and take note of different ways that the pairs of students are partitioning the rectangles and circles. Ask them about how they know that each share they produce is a fair share. Select several pairs of students to present their methods to the class.

For each pair you select, give them a copy of Transparency One [Two] and an overhead pen and ask them to prepare to share the method you selected for the cake [pie].

4. Discussion: Different Ways of Sharing
Time estimate: 10 minutes
Whole Class

Choose several pairs of students to come to the overhead to share their method of cutting the rectangle [circle] with the students on Transparency One [Two].

"Did you all think of the way that [Ann & Beth] shared the cake [pie]?"
"Why or why not?"
"[Ann & Beth], how do you know that each person at dinner gets a fair share of the cake [pie]?"
"What would you call each person's share of the cake [pie]?"

For each presentation, highlight the different points associated with each question above as appropriate:

5. Whether the method produced fair shares or not and why:
    - Fair shares must exhaust the whole
    - Must produce equal size parts
6. The method itself:
    - One less parallel cut always works on rectangle;
    - The use of radial or diametric cuts on a circle;
    - Less cuts if use a composition – emphasize that one cut (action) does the same thing to multiple pieces.
7. The different justifications:
    - Use of symmetry
    - Use of congruency
    - Use of measuring and area
8. The different naming strategies:
    - May name by the number of pieces (1 piece)
    - May name by the number of pieces in relation to the whole (1 of the four pieces)
    - May name as a unit fraction (one-fourth, one-fourth of)

Note: If different pairs of students create different numbers of pieces for their sharing (e.g. 2 of 8 pieces versus 1 of 4 pieces), help students understand that these are equivalent ways of sharing. This can be seen as taking each of the pieces for and splitting in half, for instance.

5. Session Follow-Up: Many Different People for Dessert

Time estimate: 10 minutes
Individual

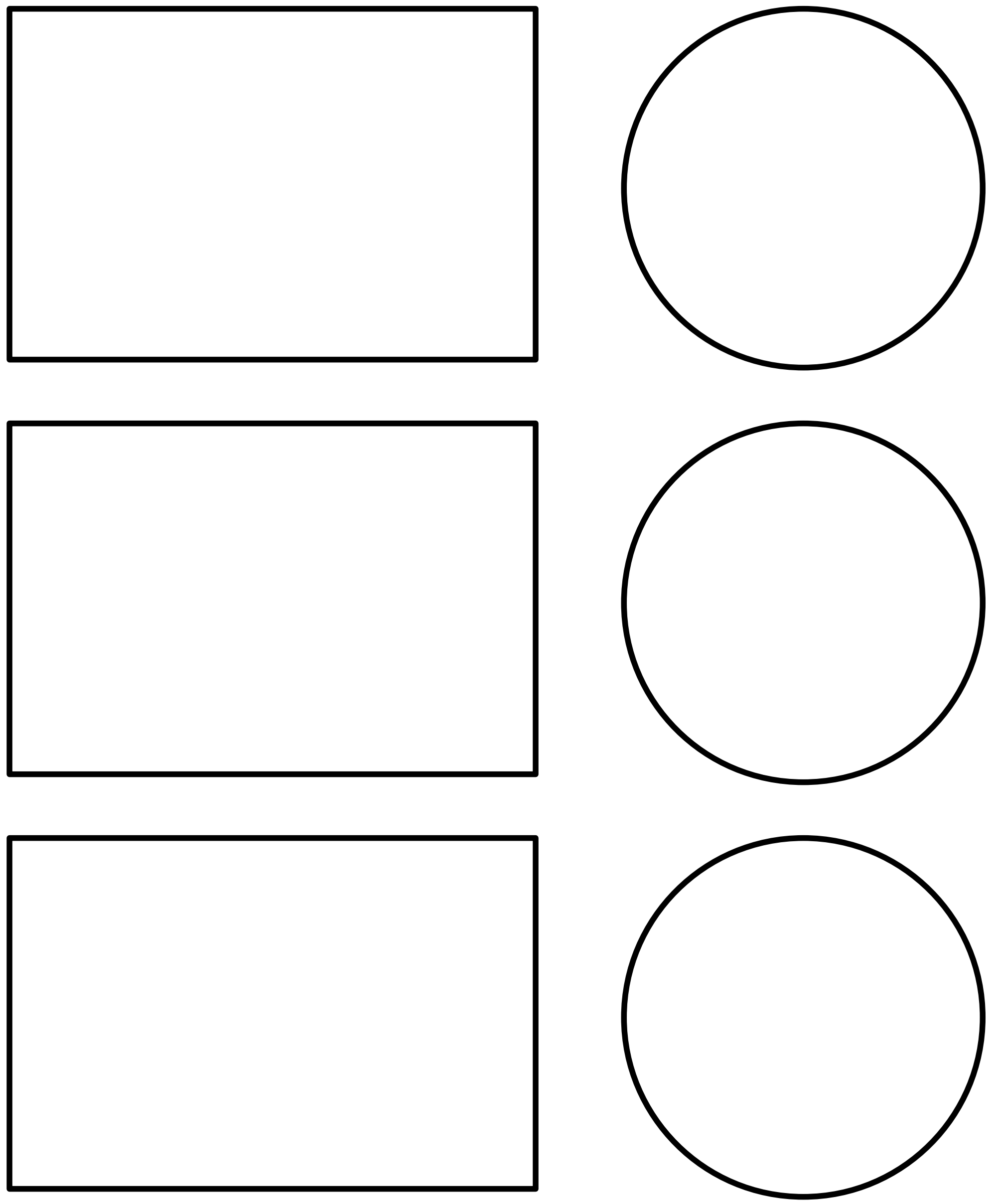
"Now, pretend you are planning dessert for your upcoming Holiday dinner, but you have no idea how many people will be there for dessert! Make a plan for cutting cakes and pies to share fairly among different numbers of guests."

Distribute Handout Two to each student.

Circulate while students are working individually. If students share a pie for three using a radial cut, highlight this for the class.

Collect each student's work at the end of the lesson.

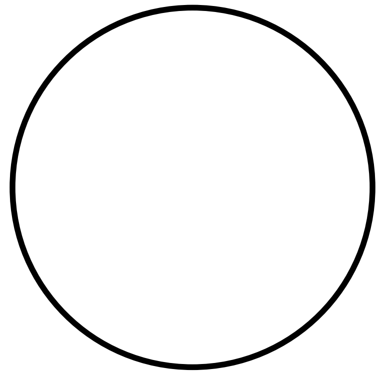
Cakes **Pies**



1.

Share fairly among 8 people

2.

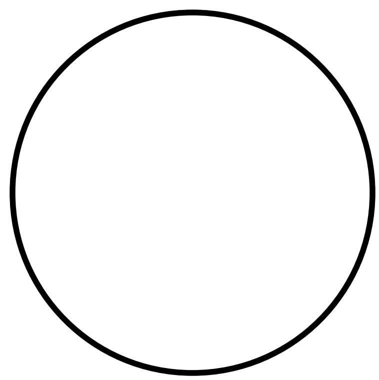


Share fairly among 3 people

3.

Share fairly among 9 people

4.

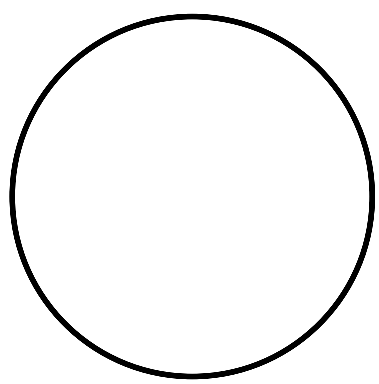


Share fairly among 8 people

5.

Share fairly among 3 people

6.



Share fairly among 5 people

Lesson Two: Fair Sharing a Collection of Objects

Math Focus Points

- Creating equal-sized groups from a collection of objects;
- Using dealing or measuring to justify the equivalence of shares;
- Naming the results of equipartitioning as a unit fraction or a fraction of a group;
- Reversing the process of sharing results in the original collection by either:
    - Counting (perhaps by groups);
    - With “times as many as” (e.g. after sharing a collection among 4, the whole collection would be “4 times as many as one share”);
- Highlight different properties associated with sharing collections, including
    - Compensation – “the more sharers, the smaller the share” or “if twice as many people shared, each share would be half as much”
    - Composition - “if twice as many people shared, each share would be half as much because each person could split their share in half”
    - Equivalence – “sharing 24 for 4 people is the same as sharing 12 for 2 people and 18 for 3 people because each time, a person’s share is 6”

Investigations Connections

- Unit 3: Stickers, Number Strings, and Story Problems
  Investigation Three: Counting by 2s, 5s, and 10s*

** These tasks need adaptation to ensure they are equipartitioning tasks.*

Today’s Plan	Materials
1. Activity: Tell a Story Time estimate: 5 minutes Whole Class	
2. Math Workshop: Sharing Snack Time estimate: 10 minutes Small groups	1 set of counters per group 1 copy of Handout One for each group
3. Discussion: The Results of Sharing Time estimate: 10 minutes Whole Class	
4. Session Follow-up: More Sharing Story Problems Time estimate: 5 minutes Individual	1 copy of Handout Two per person

Based on *Investigations* Lesson 2.6: Story Problems

1. Activity: Tell a Story
Time estimate: 5 minutes
Whole Class

On the board, write the following number sentence and ask the students to generate story problems that go with the following equations:

$$24 = 12 + 12$$

$$24 = 6 + 6 + 6 + 6$$

$$24 = 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3$$

One way to think about splitting numbers into equal-sized groups, like the number sentences above, is the context of fair sharing. Guide students to create fair sharing stories for the above number sentences, such as fairly sharing 24 pennies among 2, 4, or 6 children.

2. Math Workshop: Sharing Snack
Time estimate: 10 minutes
Small Groups

"In your math group, solve these two problems. Make sure you show your work and write an equation like the ones we told stories about at the beginning of class."

Distribute copies of Handout One.

Supply students with counters to model the problem. Circulate and assist students with counters, finding a solution, writing an equation, and explaining how they know that each child gets a fair share of snack.

3. Discussion: The Results of Sharing
Time estimate: 10 minutes
Whole Class

Facilitate a discussion of the groups' work on each problem as they share their solutions to meet the goals of the lesson. This discussion should address the following:

"Explain how you know each child got a fair share for snack."

> Students may respond because each child got the same number of pretzels. Lead the students in recognizing ways of justifying other than counting, such as dealing, by building arrays, or by measuring (lining the up end to end).

"What did you call each child's share?"

> Students may respond with the number of pretzels. Lead the students in recognizing that a share can also be referred to as "one-half (fourth)" or "one-half (fourth) of the pretzels."

"If you were to put all of the pretzels back together, what would you have?"

> Students may respond with "24 pretzels" or with "the bag of pretzels." Lead the students in recognizing that this could be found by counting each group or by saying "it is twice (four times) as many as a share."

Note any references to mathematical properties that may emerge from the students' discussion. If these do not arise, point out the following properties:

<u>Compensation</u> - *"As more children shared the pretzels, what would happen to the size of each child's share?"*

> As a fixed quantity is equipartitioned into more groups, the size of each group smaller.

<u>Compensation</u> - *"After Anna and Kia shared the pretzels, there were twice as many people. How much of the snack did Oscar and Tom get compared to before?"*

> A split of a split results in a predictable change in each group. For instance, doubling the number of groups results in groups of half of their original size. Likewise, halving the number of groups results in groups twice as large.

<u>Composition -</u> *"Rather than re-sharing all 24 pretzels, how else could Oscar and Tom share the pretzels with Anna and Kia?"*

> A split of a split allows one to carry out an action simultaneously for all groups that are equivalent.

<u>Equivalence -</u> *"Sharing 24 pretzels among four children is the same as having 6 pretzels for one child. If a share of pretzels is 6 per child, how many pretzels would there be if there were 3 children?"*

> Fair sharing activities can help students begin to understand ratio equivalence. For instance, 24 among 4 **is the same as** 6 among 1 **is the same as** 12 among 2 **is the same as** 18 among 3…

APPENDIX L

Instructional Planning Questionnaire
Do you plan to supplement the lesson with other tasks? If so, describe them. Why did you select them?

What do your students know about this topic already? How do you know?

What different approaches to the task do you anticipate students will use? Why?

Are there any misconceptions associated with the topic that you will watch for?

What plans for interactions between students have you made for the lesson?

What, if any, questions have you prepared to facilitate a discussion of their ideas?

How will you know if your students have met your instructional goals?

Are there other relevant factors that went into your planning for this lesson? Please describe them.

Interview Protocol
"Thanks for meeting with me today. I enjoyed your lessons and wanted to talk more with you about them. This interview should last approximately 45 minutes. I'll be video recording it and will use an audio recorder as a backup."

LESSON ONE – SHARING CONTINUOUS WHOLES

The first lesson was the Thanksgiving cakes and pies. Take a second to look over the lesson, the instructional planning questionnaire, and some of the students' work samples to refresh your memory of the lesson.

Planning
How did you prepare for the lesson?
- What did you think about?
- What tools/resources did you use?

What prior knowledge did you believe your students would bring to the lesson? How did you know?

What did you believe that your students would do?
- What strategies did you anticipate the students would use?

What misconceptions were you looking for?
How did you know about these?

Teaching
REFRESH LESSON UP TO THE POINT

When you were circulating, what were you looking for?
How did you know what to look for?

How did you select students to share their work?
How did you decide in what order to have your students present?

INDIVIDUAL CLIP

Assessment
What do your students know after this lesson?
How do you know?

SELECTION OF STUDENT'S WORK

What would have been / was your next lesson related to fair sharing?

If not the next observation, describe your work with students between the observations.
What were your goals
How did you select them?

LESSON TWO – SHARING COLLECTIONS

The second lesson was on sharing a collection. Take a second to look over the lesson, the instructional planning questionnaire, and some of the students' work samples to refresh your memory of the lesson.

Planning
How did you prepare for the lesson?
What did you think about?
What tools/resources did you use?

What prior knowledge did you believe your students would bring to the lesson? How did you know?

What did you believe that your students would do?
What strategies did you anticipate the students would use?
What misconceptions were you looking for?
How did you know about these?

Teaching
REFRESH LESSON UP TO THE POINT

When you were circulating, what were you looking for?
    How did you know what to look for?

How did you select students to share their work?
    How did you decide in what order to have your students present?

INDIVIDUAL CLIP

Assessment
What do your students know after this lesson?
    How do you know?

SELECTION OF STUDENT’S WORK

What would have been / was your next lesson related to fair sharing?

APPENDIX M

Analyzing Mathematical Tasks

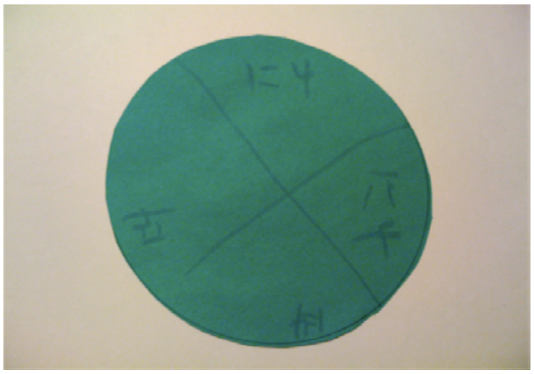
Rank	Problem
	Fairly share a round chocolate cake among 5 people.
	Fairly share 15 gold coins among 3 children
	Make 6 equivalent sectors in a circle.
	Fairly share a pumpkin pie among 8 guests.
	Fairly share 8 holiday cookies among 5 people.
	Tri-fold a New Year's letter to mail to friends and family.
	Fairly share 13 marbles between 2 children.
	Fold a rectangular piece of paper into 12 equal-sized regions
	Fairly share 2 pizzas among 4 siblings.
	Fairly share 5 gingerbread men cookies among 2 children.
	A batch of 128 cookies is split into bags of 12. How many bags of cookies are there?
	Fairly share a sweet potato pie among 2 guests.

Think individually about the questions below concerning the tasks above. After 2 minutes, discuss the questions with your group members. Designate a recorder in case your group is asked to share your results.

1. Order the tasks above according to the level of thinking required to solve correctly. Use "1" for least sophisticated up to "12" for most sophisticated.
2. Provide a rationale for the ordering.
3. Do all of the tasks have a goal of equipartitioning? If so, explain how you know. If not, identify the task(s) that do not and explain how you know.
4. Adapt the task below so that it has a goal of equipartitioning.
   "Kevin wants to put 12 candy canes in the stockings. He puts 3 in each. How many stockings are there?"

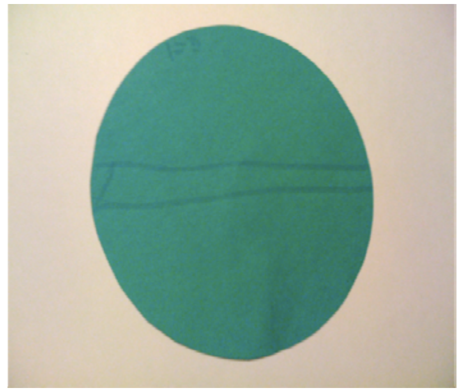
Analyzing Students' Work

Emma sharing a birthday cake for 4 people



The writing in each sector says "1 = 4"

Emma sharing a birthday cake for 3 people



The writing at the top says "1 = 3"

Emma is a Kindergarten student. Prior to these tasks, she had successfully shared rectangular birthday cakes among 2, 4, and 6 people but was unable to share for 3 people. Also, she fairly shared a circular birthday cake between two people.

Think individually about the questions below concerning the tasks above. After 2 minutes, discuss the questions with your group members. Designate a recorder in case your group is asked to share your results.

1. What do you believe Emma knows about fair sharing? Justify with evidence from the work sample.

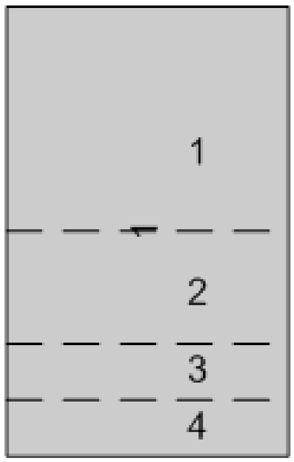
2. What do you believe Emma does no know about fair sharing? Justify with evidence from the work sample.

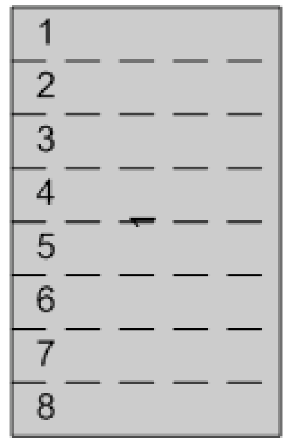
3. What questions do you have about Emma's understanding of fair sharing?

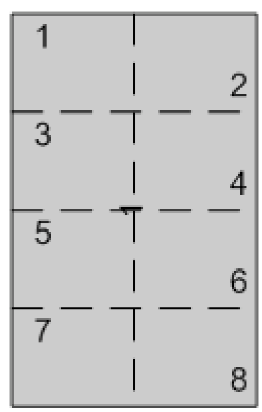
4. What task would you do next with Emma? Explain your reasoning.

Classroom Interactions

The teacher posed the following task to students: *"If a piece of rectangular paper were folded in half 3 times, how many parts would there be?"* While monitoring students' progress, the teacher noticed four different responses in the room shown below:

<table>
<tr>
<td>Group A


"We found four"</td>
<td>Group B


"We think eight parts"</td>
</tr>
<tr>
<td>Group C
"We think it is six parts. Three 2's make 6.
We didn't even need to try it out!"</td>
<td>Group D


"Eight squares is the answer"</td>
</tr>
</table>

After providing time for each group to reach a solution, the teacher plans a whole group discussion.

Think individually about the questions below concerning the tasks above. After 2 minutes, discuss the questions with your group members. Designate a recorder in case your group is asked to share your results.

1. Describe each group's method. Is their solution correct or incorrect?
2. Order the groups' methods from least to most sophisticated and provide a rationale for the ordering.
3. How do you think that the teacher should sequence the presentations from groups A-D? Explain your reasoning.

4. For each group, what questions would you pose the class or the students to facilitate the discussion if you were the teacher?

Group	Questions & Explanation

5. What mathematical goals would you have for the discussion?

6. What task would you post next?

Clinical Interview Analysis Activity

Segment I – Rectangle cake among 3

1. Summarize Emma's sharing of the rectangular cake for 3 people.

2. How do you anticipate Emma will share a rectangle birthday cake among 6 people? Justify your conjecture.

Segment II – Rectangle cake among 6

3. Summarize Emma's sharing the rectangular cake for 6 people.

4. How do you anticipate Emma will share a round birthday cake among 3 people? Justify your conjecture.

Segment III – Round cake among 3

5. Summarize Emma's sharing of the round cake for 3 people.

6. How do you anticipate Emma will share a round birthday cake among 6 people? Justify your conjecture.

Segment IV – Round cake among 6

7. Summarize Emma's sharing of the round cake for 6 people.

8. What does Emma know about equipartitioning? How do you know?

9. If you were Emma's teacher, what ideas related to equipartitioning would you work on next? What learning activities would support that development? Explain why you would choose the ideas and the activities.

APPENDIX N

Coding Glossary

Describing		Data relating to participants' descriptions of students' behaviors and verbalizations
	Superfluous	Descriptions of irrelevant students' behaviors or verbalizations, including other factors (e.g. environmental)
	Affect	Descriptions of students' affective reactions (e.g. frustration)
	Imprecise	General descriptions of students' behaviors or verbalizations
	Focused	Descriptions of students' behaviors or verbalizations that relating to those described in the learning trajectory
	Precise	Articulate descriptions of students' behaviors and verbalizations which may or may not be described by the learning trajectory
Comparing		Data relating to participants' comparisons of students' behaviors and verbalizations to their own work or the work of other students
	Grade level	Comparisons to grade level expectations
	Curriculum	Comparisons to issues addressed explicitly in *Investigations*
	Strategies (LT)	Comparisons to strategies documented in the learning trajectory
	Developmental	Comparisons to what teachers believe to be developmentally appropriate
	Student (specific)	Comparisons to a specific student (e.g. their clinical interview participant)
	Student (composite)	Comparisons to a general student
Inferring		Data relating to participants' inferences of students' cognition based on their behaviors and verbalizations
	Holistic	Inferences about general cognition (e.g. no concept of fair share)
	Evaluative	Inferences about cognition containing judgment
	Three Criteria	Inferences relating to the three equipartitioning criteria
	Discriminating	Inferences that parse students' understandings

Restructuring		Data relating to participants' restructuring of their own knowledge of equipartitioning, equipartitioning and students, and of instruction
	Anticipation	Data relating to participants' predictions of students' behaviors and verbalizations
	Language	Data relating to participants' use of technical language from the learning trajectory
	Sensitivity	Data relating to participants' sensitivity to students' behaviors and verbalizations
	Curriculum	Data relating to the interactions of the learning trajectory with Investigations (e.g. use of tools, mathematical issues other than equipartitioning)
	Location	Data relating to participants' locating of students within a range of conceptual development
	Movement	Data relating to participants' suggestions for assisting students in progressing along the learning trajectory
	EP	Data relating to participants' knowledge of equipartitioning (e.g. mathematical properties, additive misconception)
Learning Trajectory		Data relating to participants' references to or uses of the learning trajectory
	Three Criteria	Data relating to participants' references to the three equipartitioning criteria
	Number Splits	Data relating to participants' references to the number of splits
	Shape	Data relating to participants' references to the shape being equipartitioned
	Progress Variable	Data relating to participants' references to the progress variable
Instructional Tasks		Data relating to participants' adaptations and/or uses of instructional tasks
	EP Knowledge	Data relating to participants' knowledge of equipartitioning, including relating parts to wholes, noncongruent equipartitions, and mathematical properties
	Curriculum	Data relating to the interactions of the learning trajectory and their curriculum, including the use of tools and other mathematical goals
Assessment		Data relating to participants' assessment of students' work

Prediction	Data relating to participants' predictions of students' behaviors and verbalizations
Sensitivity	Data relating to participants' sensitivity to students' behaviors and verbalizations
Movement	Data relating to participants' suggestions for assisting students in progressing along the learning trajectory

Teaching	Data relating to participants' interactions with students during classroom instruction

Selecting	Data relating to participants' selections of students' ideas to share in class discussion
Organizing	Data relating to participants' organization of students' ideas for class discussion
Relating	Data relating to participants' connections among students' ideas to in class discussion
Interacting	Data relating to participants' interactions with students during class discussion (e.g questions)

APPENDIX O

Design Study Conjectures and Revisions

Conjecture One

Initial: The use of a learning trajectory for equipartitioning will assist teachers in building models of students' thinking by highlighting facets of students' behaviors and language and relating those to a particular conception. Whereas initially teachers will not notice distinctions among students' activities and language, the learning trajectory will act as a lens for teachers, accentuating language and actions that are indications of thinking while filtering irrelevant work or words.

Revised: A learning trajectory assists teachers in building models of students' thinking that are focused, empirically- and theoretically-based, and nuanced. It sensitizes them to students' behaviors and verbalizations documented in the trajectory, provides support for connecting that evidence to students' cognition, and assists them in locating students' thinking within a range of conceptual development.

Conjecture Two

Initial: The use of a learning trajectory for equipartitioning will inform the adaptation of instructional tasks by helping them adapt tasks through altering the mathematical goals and assist them in judging the relative difficulty of tasks.

Revised: A learning trajectory supports teachers' judgments of the relative difficulty of instructional tasks. Though a learning trajectory may suggest next pedagogical steps, it interacts significantly with other curricular goals and materials. Teachers need support and guidance in adjusting tasks to align with a learning trajectory.

Conjecture Three

Initial: The use of a learning trajectory for equipartitioning will inform teachers' formative analysis of students' work on instructional tasks by helping them identify evidence of students' thinking, conjecture about what students may or may not know, and raise questions that inform teachers' next instructional steps.

Revised: A learning trajectory influences teachers' analysis of students' work on instructional tasks by helping them identify evidence of student thinking within students' work, providing language for teachers to discuss students' thinking, and assisting them in making predications.

Conjecture Four

Initial: The use of a learning trajectory for equipartitioning will inform interactions with students during instruction by assisting teachers in identifying students' ideas that will support their mathematical goals, by influencing teachers' questions, and in sequencing students' work during whole group discussions.

Revised: A learning trajectory affects classroom interactions with students during instruction. It sensitizes teachers to a variety of strategies students may use, provides a framework for using and sequencing students' work, and helps teachers connect and relate mathematical ideas across multiple examples of students' ideas.

Conjecture Five

Initial: Teachers' knowledge and uses of a learning trajectory for equipartitioning will be positively associated with their students' learning.

Revised: The relationships between teachers' knowledge and uses of a learning trajectory and their students' learning are complicated by interactions with curriculum. More sensitive measures are needed to understand these relationships.