ABSTRACT

BEUN, STACY L. On the Classification of Orbits of Minimal Parabolic k-Subgroups Acting on Symmetric k-Varieties of $\mathrm{SL}(n,k)$. (Under the direction of Dr. Aloysius Helminck).

Symmetric k-varieties are a generalization of symmetric spaces to general fields. They play an important role in many areas, including representation theory, geometry, and singularity theory. Orbits of a minimal parabolic k-subgroup acting on a symmetric k-variety is essential to the study of symmetric k-varieties and their representations. For reductive algebraic groups defined over algebraically closed fields or the real numbers, these orbits have been studied in great detail in the literature. For general fields, Helminck and Wang gave several characterizations of these orbits. However, a classification for specific fields is still needed and is, in fact, quite complicated. These various characterizations of the orbits require one to first classify the orbits of the θ-stable maximal k-split tori under the action of the k-points of the fixed point group, where θ is the defining involution of the symmetric k-variety. In this thesis, we present the classification of these orbits for the group $\mathrm{SL}(2,k)$ and in general for $\mathrm{SL}(n,k)$. We discuss methods and results for a number of base fields k, including finite fields and the $\mathfrak{p}$-adic numbers.

On the Classification of Orbits of Minimal Parabolic k-Subgroups Acting on Symmetric k-Varieties of $\mathrm{SL}(n, k)$

by
Stacy L. Beun

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Approved By:

Dr. Tom Lada	Dr. Amassa Fauntleroy
Dr. Aloysius Helminck Chair of Advisory Committee	Dr. Ernest Stitzinger

BIOGRAPHY

Stacy Leigh Beun was born December 4th, 1979, in Monroe, Wisconsin, to her parents Mike and Sandy Rahberger. She has two sisters and one brother. In 1998 she graduated from New Glarus High School in New Glarus, Wisconsin. She received a Bachelors of Science in Mathematics from the University of Wisconsin at Platteville in May 2002 and married Joshua Beun on June 1, 2002. Together they moved to Raleigh, North Carolina, to begin their graduate studies at North Carolina State University. She received a Masters of Science in Mathematics from NC State in 2004, and she has two young daughters, Emily Elizabeth and April Alyssa.

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Introduction

Symmetric varieties are the homogeneous spaces G/H, where G is a reductive algebraic group and H is the fixed point group of an involution θ of G. When G and θ are defined over a field k which is not algebraically closed, then the homogeneous space $X_k := G_k/H_k$ is called a *symmetric k-variety*. Here G_k and H_k denote the set of k-rational points of G and H respectively. Real symmetric k-varieties are also known as *real reductive symmetric spaces.* The $\mathfrak{p}$-adic symmetric k-varieties are also called reductive $\mathfrak{p}$-adic symmetric spaces or simply $\mathfrak{p}$-adic symmetric spaces.

Example 1.0.1*.* Take $G = \mathrm{GL}_n$ and the involution $\theta(A) = (A^T)^{-1}$, where A^T denotes the transpose of A. This gives the symmetric space $X_k = \mathrm{GL}(n,k)/\,\mathrm{O}(n,k)$, which can be identified with a subset of the symmetric matrices. In fact, if k is algebraically closed, X_k is the entire set of symmetric matrices. Additionally, if $k = \mathbb{R}$, X_k is the set of positive definite symmetric matrices.

Symmetric k-varieties over base fields other than the real numbers occur in many problems in representation theory (see [4] and [37, 38]), geometry (see [9, 10] and [1]), singularity theory (see [24] and [20]), the study of character sheaves (see [13, 25]) and at the study of cohomology of arithmetic subgroups (see [36]). However, it is in the area of representation theory that these symmetric k-varieties are best known.

Over the last few decades many people have studied the representations associated with the real reductive symmetric spaces. Most of the early work was done by Harish-Chandra. Expanding on Harish-Chandra's ideas, the representation theory for the general real reductive symmetric spaces has been carried out by a number of mathematicians including Flensted-Jensen, Ōshima, Sekiguchi, Matsuki, Brylinski, Delorme, Schlichtkrul, and van den Ban (see [3, 8, 11, 12, 14, 27, 28]). Symmetric k-varieties were introduced in the late 1980's as a natural next step to generalize the concept of these real reductive symmetric spaces to similar spaces over the $\mathfrak{p}$-adic numbers and to study the representations associated with these spaces. Some people have already started on this with a number of interesting results (see [15, 22, 29]). Another case of interest is the representations associated with symmetric k-varieties defined over a finite field (see [25] and [13]).

For $k = \mathbb{Q}_p$, the $\mathfrak{p}$-adic numbers, it is natural to study the harmonic analysis of these $\mathfrak{p}$-adic symmetric spaces. The main aim of harmonic analysis is to decompose unitary representations as explicitly as possible into irreducible components, which is called finding the *Plancherel decomposition*. Most of the representations occurring in this decomposition are representations induced from a parabolic k-subgroup. This means that the one essential tool in the study of symmetric k-varieties and their representations is a description of the geometry of the orbits of a minimal parabolic k-subgroup acting on the symmetric k-variety. A description of these orbits naturally leads to a description of most of the fine structure of these symmetric k-varieties, including the restricted root system of the symmetric k-variety.

1.1 Orbits of Parabolic Subgroups Acting on Symmetric Varieties

The orbits of a parabolic k-subgroup P acting on the symmetric k-variety can be characterized in several equivalent ways. First, they can be characterized as the P_k-orbits acting on the symmetric k-variety G_k/H_k by θ-twisted conjugation, meaning that if $x, g \in G_k$ then define $g * x := gx\theta(g)^{-1}$. We can also view them as the number of H_k-orbits acting on the flag variety G_k/P_k by conjugation. Finally, we can view these orbits as the set $P_k \backslash G_k / H_k$ of (P_k, H_k)-double cosets in G_k. This last characterization is the same as the set of $P_k \times H_k$-orbits on G_k.

For k algebraically closed and $P = B$ a Borel subgroup, these orbits were characterized by Springer in [33]. Springer proved several characterizations of the double cosets $B\backslash G/H$, including the orbits of H, B, and $B\times H$. For details, see [33]. For k algebraically closed and P a general parabolic subgroup, these orbits were characterized by Brion and Helminck in [7, 19]. For $k = \mathbb{R}$ and P a minimal parabolic k-subgroup, characterizations were given by Matsuki [26] and Rossmann [30]. For general fields, these orbits were characterized by Helminck and Wang [16].

Let P be a minimal parabolic k-subgroup of G. Helminck and Wang [16] characterized the double coset $B_k\backslash G_k/H_k$ in several different ways, including the orbits of H_k, P_k, and $P_k \times H_k$. First, let's consider the H_k-orbits on G_k/P_k. For a torus T of G_k let $N_{G_k}(T)$ denote the normalizer of T in G_k, $Z_{G_k}(T)$ denote the centralizer of T in G_k, $W_{G_k}(T) = N_{G_k}(T)/Z_{G_k}(T)$ denote the Weyl group of T in G_k, and

$$W_{H_k}(T) = N_{H_k}(T)/Z_{H_k}(T) = \{w \in W_{G_k}(T) \mid w \text{ has a representative in } N_{H_k}(T)\}.$$

All of the above of are θ-stable if T is θ-stable.

Let C_k denote the set of pairs (P', T'), where T' is a θ-stable maximal k-split torus of G_k which is contained in the minimal parabolic k-subgroup P' of G. Let $\mathcal{P}_k$ denote the variety of all minimal parabolic k-subgroups of G. The group H_k acts on C_k and $\mathcal{P}_k$ on the left by conjugation. Let $H_k\backslash C_k$ (respectively $H_k\backslash\mathcal{P}_k$) denote the set of H_k-orbits in C_k (respectively $\mathcal{P}_k$).

The H_k-orbits in C_k can be broken up into two parts. First, we can consider the H_k-conjugacy classes of θ-stable maximal k-split tori in G_k. For each θ-stable maximal k-split torus representing an H_k-conjugacy class, we can then determine the minimal parabolic k-subgroups containing it that are not H_k-conjugate. So, if $\{T_i \mid i \in I\}$ are representatives of the H_k-conjugacy classes of θ-stable maximal k-split tori in G_k, then the H_k-orbits in C_k can be identified with $\cup_{i\in I} W_{G_k}(T_i)/W_{H_k}(T_i)$.

For the P_k-orbits, we identify G_k/H_k with $Q_k = \{g\theta(g)^{-1} \mid g \in G_k\}$ and note that P_k acts on Q_k by the θ-twisted action described above. In this case, one shows that if T is a θ-stable maximal k-split torus and U is the unipotent radical of P, then any θ-twisted U_k-orbit on Q_k meets $N_{G_k}(T)$ (see [16], Proposition 6.6). Denote the set of θ-twisted P_k-orbits on Q_k by $P_k\backslash Q_k$.

For the characterization of the $P_k \times H_k$-orbits in G_k, let T be a θ-stable maximal k-split torus of P_k and let $\mathcal{V}_k = \{x \in G_k \mid \tau(x) \in N_{G_k}(T)\}$. The group $Z_{G_k}(T) \times H_k$ acts on $\mathcal{V}_k$ by $(x,z) \cdot y = xyz^{-1}$, $(x,z) \in Z_{G_k}(T) \times H_k$, $y \in \mathcal{V}_k$. Let V_k be the set of $(Z_{G_k}(T) \times H_k)$-orbits on $\mathcal{V}_k$.

Theorem 1.1.1 ([16]). *Let P be a minimal parabolic subgroup k-subgroup of G and let $\{T_i \mid i \in I\}$ be representatives of the H_k-conjugacy classes of θ-stable maximal k-split tori in G_k. Then,*

$$P_k \backslash G_k / H_k \cong H_k \backslash \mathcal{P}_k \cong \cup_{i \in I} W_{G_k}(T_i)/W_{H_k}(T_i) \cong H_k \backslash C_k \cong P_k \backslash Q_k \cong V_k.$$

Throughout this thesis, we study the orbits of minimal parabolic k-subgroups acting on the symmetric k-variety by using the classification

$$P_k \backslash G_k / H_k \cong \cup_{i \in I} W_{G_k}(T_i)/W_{H_k}(T_i).$$

Essentially, we are studying the H_k-orbits on the flag variety $P_k \backslash G_k$. Thus, to classify the double coset $P_k \backslash G_k / H_k$, we will first need to classify the set of H_k-conjugacy classes of θ-stable maximal k-split tori. Then for each representative θ-stable maximal k-split torus T_i, we need to determine the coset $W_{G_k}(T_i)/W_{H_k}(T_i)$.

1.2 H_k-conjugacy classes of θ-stable maximal k-split tori

Since our chosen characterization centers around tori, we give the following definitions and results. If T is a torus of G defined over k, then there are subtori T^a and T^d of T, where T^a is the largest anisotropic subtorus of T and T^d is the largest k-split subtorus of T defined over k. These tori satisfy: $T = T^a \cdot T^d$ and $T^a \cap T^d$ is finite (see [6, 8.15]).

We have a similar decomposition for θ-stable tori, where θ is an involution of G. Let T be a maximal k-split torus. T is θ-stable if $\theta(T) = T$. Then, $T = T^+T^-$, where $T^+ = \{t \in T \mid \theta(t) = t\}^\circ$, $T^- = \{t \in T \mid \theta(t) = t^{-1}\}^\circ$, and $T^+ \cap T^-$ is finite. The torus T^- is called θ-split. A torus T is called (θ, k)-split if T is k-split and θ-split.

In using this classifcation, we obtain two lattices. The first is the lattice representing the H_k-conjugacy classes of θ-stable maximal k-split tori, which is shown in

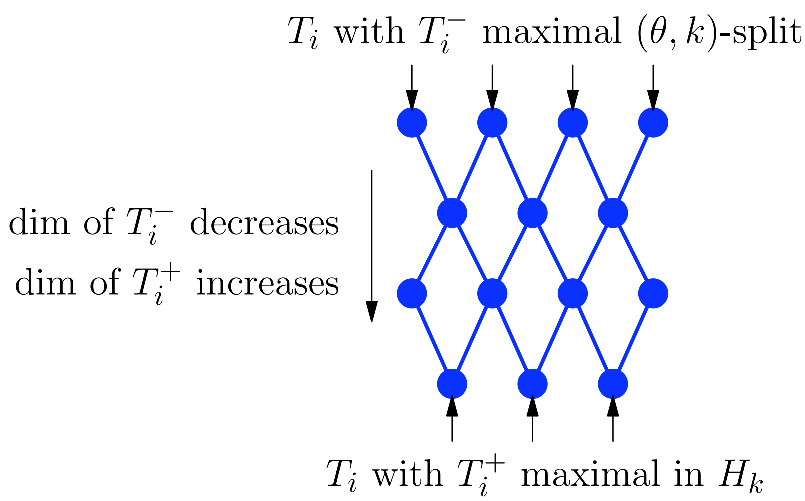

Figure 1.1: The lattice of H_k-conjugacy classes of θ-stable maximal k-split tori

Figure 1.1. The nodes represent the H_k-conjugacy classes of θ-stable maximal k-split tori. As defined above, each of these tori can be written in the form $T_i = T_i^+ T_i^-$. In each level of the lattice of H_k-conjugacy classes of θ-stable maximal k-split tori, the representative tori in that level have the same dimension for their T_i^+ parts and the same dimension for their T_i^- parts. We order the levels in such a way as to have the the tori with T_i^- part of maximal dimension in the top level, and the tori with T_i^+ part of maximal dimension in the bottom level. As you move down the lattice levels, the dimension of the T_i^- part decreases and the dimension of the T_i^+ part increases. Next, at each node in this lattice, we can expand to $W_{G_k}(T_i)/W_{H_k}(T_i)$, giving us our second lattice, which is not pictured. The number of nodes in the second lattice gives $|P_k \backslash G_k / H_k|$. Additionally, the lines connecting the nodes in the second lattice correspond to the Bruhat Ordering.

Example 1.2.1. Let's consider $\mathrm{SL}(3, \mathbb{C})$ with $\theta(A) = (A^T)^{-1}$. Then our first lattice consists of three levels with one H_k conjugacy class of θ-stable k-split tori in each level. In the top and bottom levels, we obtain that $|W_{G_k}(T_i)/W_{H_k}(T_i)| = 1$. In the middle level, we obtain that $|W_{G_k}(T_i)/W_{H_k}(T_i)| = 2$. Then $|P_k \backslash G_k / H_k| = 4$. We can visualize this in Figure 1.2. On the left is the lattice of the H_k-conjugacy classes of θ-stable maximal k-split tori, and on the right is the expansion at each node to

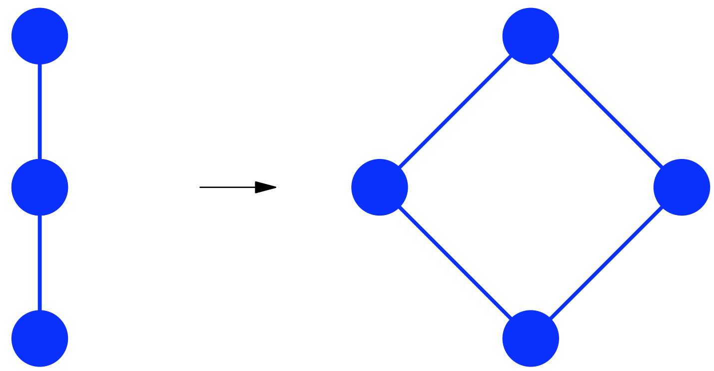

Figure 1.2: Lattices of tori related to $\mathrm{SL}(3, \mathbb{C})$ with $\theta(A) = (A^T)^{-1}$

$W_{G_k}(T_i)/W_{H_k}(T_i)$.

In order to determine the lattice of H_k-conjugacy classes of θ-stable maximal k-split tori and its expansion at each node to $W_{G_k}(T_i)/W_{H_k}(T_i)$, we need to consider each algebraic group G and its associated involutions individually. In this thesis we consider $G_k = \mathrm{SL}(n, k)$. In particular, this thesis involves the following subproblems before a classification of $P_k \backslash G_k / H_k$ can be achieved.

1. Classify the H_k-conjugacy classes of the maximal (θ, k)-split tori.

2. Classify the H_k-conjugacy classes of maximal k-split tori containing a maximal (θ, k)-split torus. These tori are θ-stable, and the number of classes gives us the number of nodes in the top level of the first lattice, whose nodes are the H_k-conjugacy classes of θ-stable maximal k-split tori.

3. Determine the maximal k-split tori in H_k and classify the H_k-conjugacy classes of θ-stable maximal k-split tori which contain a maximal k-split torus in H_k. This gives us the bottom level of the first lattice, whose nodes are the H_k-conjugacy classes of θ-stable maximal k-split tori.

4. Classify the H_k-conjugacy classes of θ-stable maximal k-split tori, meaning we determine all the middle levels of the first lattice, whose nodes are the H_k-conjugacy classes of θ-stable maximal k-split tori.

5. With the classification of the first lattice containing the H_k-conjugacy classes of θ-stable maximal k-split tori now complete, we turn to expanding each node to obtain the second lattice. To do this, first determine $W_{H_k}(T_i)$ for each representative maximal θ-stable k-split torus T_i at each node of the first lattice, meaning, we choose a representative of the H_k-conjugacy class of θ-stable maximal k-split tori represented by each node of the first lattice. Note that $W_{H_k}(T_i)$ means we determine which Weyl elements of $W_{G_k}(T_i)$ are represented in H_k.

6. Finally, expand each node of the lattice of H_k-conjugacy classes of θ-stable maximal k-split tori to $W_{G_k}(T_i)/W_{H_k}(T_i)$ in order to obtain the second lattice, which gives us $|P_k\backslash G_k/H_k|$. Note that each level of this second lattice can have more nodes than the first lattice, and the number of nodes each node of the first lattice expands to is equal to $|W_{G_k}(T_i)/W_{H_k}(T_i)|$.

1.3 Summary of Results for SL(2, k)

A natural first step in my classification is to consider $G_k = \mathrm{SL}(2,k)$. In this case, there are only two levels to the lattice of I. And since tori in $\mathrm{SL}(2,k)$ are one-dimensional, $T_i = T_i^+$ or $T_i = T_i^-$. Therefore, the main problem here is subproblem (1) above. The complete classification for $\mathrm{SL}(2,k)$ has been achieved in the submitted paper [5] and also in Chapters 4 and 5 of this thesis for various fields, including the finite fields and $\mathfrak{p}$-adic numbers. In particular, Chapter 4 focuses solely on the finite fields $\mathbb{F}_p$, and Chapter 5 generalizes that work to all fields k.

Remark 1.3.1. $\mathrm{SL}(2,k)$ is a k-split group, meaning that $\mathrm{SL}(2,k)$ contains a maximal torus that is also maximal k-split, namely $T = \{$ diagonal matrices $\}$. In this case, minimal parabolic k-subgroups are Borel subgroups. Therefore, if we let B be a Borel subgroup containing T, the classification of $B_k\backslash G_k/H_k$ is the exact same as $P_k\backslash G_k/H_k$.

The classification of involutions of $\mathrm{SL}(2,k)$ is given by Helminck and Wu in [18]. They show that all involutions come from bilinear forms. In particular, we have the following classification. Note that $\theta = \mathrm{Inn}_A$ denotes that $\mathrm{Inn}_A(X) = A^{-1}XA$ for any $X \in G$.

Theorem 1.3.2 ([18]). *The number of isomorphy classes of involutions over G equals the order of $k^*/(k^*)^2$. Furthermore, the involutions over G are of the form $\theta = \mathrm{Int}\begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$ for all $m \in k^*/(k^*)^2$. Additionally, if m and q are in the same square class, then $\mathrm{Int}\begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$ is isomorphic to the involution $\mathrm{Int}\begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix}$.*

Since all involutions of $\mathrm{SL}(2,k)$ come from bilinear forms, the orbits of minimal parabolic k-subgroups acting on the symmetric k-variety related to the involution can be classified using quadratic forms.

A summary of the main results for $\mathrm{SL}(2,k)$ is found in the following theorem. Note that we use $\overline{m}$ to denote the entire square class of m, and by abuse of notation, we use $m \in k^*/(k^*)^2$ to denote that m is the representative of the square class $\overline{m}$ of k. Also, T is the set of diagonal matrices in $\mathrm{SL}(2,k)$.

Theorem 1.3.3. *Let $G_k = \mathrm{SL}(2,k)$, and let $\theta = \mathrm{Int}\begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$.*

1. *H_k is k-anisotropic if and only if $m \notin \overline{1}$. If $m \in \overline{1}$, then H_k is a maximal k-split torus.*

2. *Let $U = \{q \in k^*/(k^*)^2 \mid x_1^2 - m^{-1}x_2^2 = q^{-1}$ has a solution in $k\}$. Then the number of H_k-conjugacy classes of (θ,k)-split maximal tori is $|U/\{1,-m\}|$.*

3. *For $y \in U$, let $r,s \in k$ such that $r^2 - m^{-1}s^2 = y^{-1}$ and let $g = \begin{pmatrix} r & sym^{-1} \\ s & ry \end{pmatrix}$. Then $\{T_y = g^{-1}Tg \mid y \in U\}$ is a set of representatives of the H_k-conjugacy classes of maximal (θ,k)-split tori in G_k.*

4. *Let T_i be a (θ,k)-split maximal torus. Then,*

   (a) *$|W_{H_k}(T_i)| = 2$ when $m \in \overline{1}$ and $-1 \in (k^*)^2$.*

   (b) *$|W_{H_k}(T_i)| = 2$ when $m \in \overline{-1}$ and $-1 \notin (k^*)^2$.*

   (c) *$|W_{H_k}(T_i)| = 1$ otherwise.*

Parts (1) and (2) give the means of counting the H_k-conjugacy classes of θ-stable maximal k-split tori. Additionally, part (3) provides the necessary information to find the tori representatives of each H_k-conjugacy class, allowing the calculation of $W_{H_k}(T_i)$ in part (4). In the cases that H_k is a maximal k-split torus, we clearly have $W_{H_k}(H_k) = \{\text{id}\}$, thus $|W_{G_k}(H_k)/W_{H_k}(H_k)| = 2$.

The above results give us a detailed description of the double cosets $B_k \backslash G_k / H_k$ for $G = \text{SL}(2,k)$. Complete results over various fields including the $\mathfrak{p}$-adic numbers and the finite fields can be found in Chapters 4 and 5. An example for the $\mathfrak{p}$-adic numbers is given below.

Example 1.3.4. Let $k = \mathbb{Q}_p$, $p \cong 1 \mod 4$, and for $A \in G$ let $\theta(A) = \text{Inn}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then

$$H_k = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a, b \in k,\ a^2 - b^2 = 1 \right\}.$$

The square class representatives are $\{1, S_p, p, pS_p\}$, where S_p denotes the smallest nonsquare in $\mathbb{F}_p$. Then using Theorem 1.3.3, we have four H_k-conjugacy classes of (θ, k)-split maximal tori, and from Theorem 1.3.3 H_k is also a maximal k-split torus. Hence, our representative maximal (θ, k)-split tori are H, T, and some three other T_1, T_2, and T_3. Here -1 is a square, so following Lemma 1.3.3,

$$|W_{H_k}(T)| = |W_{H_k}(T_1)| = |W_{H_k}(T_2)| = |W_{H_k}(T_3)| = 2.$$

Clearly $|W_{H_k}(H_k)| = 1$. Thus, $|B_k \backslash G_k / H_k| = 6$.

1.4 Summary of Results for SL(n, k).

Next we extend the complete classification of $\text{SL}(2,k)$ to $G_k = \text{SL}(n,k)$. To do this, we use the complete classification of the isomorphism classes of involutions of G presented in [17]. A summary of this classification is given in Chapter 3. The involutions of G are separated into two groups–the inner involutions and the outer involutions. The outer involutions can be written as $\theta(X) = \text{Inn}_M((X^T)^{-1}) = M^{-1}(X^T)^{-1}M$ for all

$X \in G$, where M comes from a bilinear form. There exists a large number of involutions to consider. For example, if $k = \mathbb{Q}_p$ with $p \cong 1 \mod 4$ and n is even, there are a total of $\frac{n}{2} + 9$ isomorphism classes of involutions.

Remark 1.4.1. Like $\mathrm{SL}(2,k)$, $\mathrm{SL}(n,k)$ is a k-split group, meaning that $\mathrm{SL}(n,k)$ contains a maximal torus that is also maximal k-split, namely the set of diagonal matrices T. In this case, minimal parabolic k-subgroups are Borel subgroups. Therefore, if we let B be a Borel subgroup containing T, the classification of $B_k \backslash G_k / H_k$ is the exact same as $P_k \backslash G_k / H_k$.

Currently, we have solved subproblems (1) thru (3) above for some isomorphism classes of involutions. A sampling of our results follows. Note for $X \in G$ that $\theta = \mathrm{Inn}_A$ denotes the inner involution $\mathrm{Inn}_A(X) = A^{-1}XA$. The following results can be found in Chapter 6. Note that the notation $\frac{k^*/(k^*)^2}{\pm 1}$ means we quotient $k^*/(k^*)^2$ by the set containing the square classes of 1 and -1. If -1 is a square, then the square classes of 1 and -1 are the same. Thus, in general, $|\frac{k^*/(k^*)^2}{\pm 1}| = 1/2 * |k^*/(k^*)^2|$ if -1 is not a square and $|\frac{k^*/(k^*)^2}{\pm 1}| = |k^*/(k^*)^2|$ if -1 is a square.

Proposition 1.4.2. *Let Let $\theta = \mathrm{Inn}_{I_{n-i,i}}$ such that $i < n/2$ and*

$$I_{n-i,i} = \begin{pmatrix} I_{n-i \times n-i} & 0 \\ 0 & -I_{i \times i} \end{pmatrix}.$$

Then there exists only one H_k-conjugacy class of maximal (θ, k)-split tori, and there exits only one H_k-conjugacy class of maximal k-split tori cotaining a maximal (θ, k)-split torus. Furthermore, let T_1 be any (θ, k)-split torus. Then $Z_{G_k}(T_1)$ contains only one H_k-conjugacy class of maximal (θ, k)-split tori.

Proposition 1.4.3. *Let $\theta = \mathrm{Inn}_{I_{n/2,n/2}}$ such that*

$$I_{n/2,n/2} = \begin{pmatrix} I_{n/2 \times n/2} & 0 \\ 0 & -I_{n/2 \times n/2} \end{pmatrix}.$$

Note that n must be even. Additionally, let k be $\mathbb{C}$, $\mathbb{R}$, $\mathbb{F}_p$ with $p \neq 2$, or $\mathbb{Q}_p$. Then the number of H_k-conjugacy classes of maximal (θ, k)-split tori equals $|\frac{(k^*/(k^*)^2)}{(\pm 1)}|$.

Proposition 1.4.4. *Let $\theta = \mathrm{Inn}_{I_{n-i,i}}$ with $i \le n/2$, and let H_k be the fixed point group of θ. Then H_k contains a maximal k-split torus that is both maximal k-split and maximal in G_k.*

In Chapter 7, we focus on subproblem (4) and have generalized results for any algebraic group G, not just $\mathrm{SL}(n, k)$. First, assume θ is such that there exists only one H_k-conjugacy class of maximal (θ, k)-split tori, as well as only one H_k-conjugacy class of maximal k-split tori containing a maximal (θ, k)-split torus. (By Proposition 1.4.2, such a θ exists in $\mathrm{SL}(n, k)$.) Let T be the representative torus of this conjugacy class. It can be shown that one can associate each θ-stable maximal k-split torus with an element of $W_{H_k}(T)$. To obtain subproblem (4), one can use the following general result, which translates the problem to $W_{H_k}(T)$.

Theorem 1.4.5. *Suppose G and θ are such that for any (θ, k)-split torus T_1, $Z_{G_k}(T_1)$ contains only one H_k-conjugacy class of maximal (θ, k)-split tori. Let T_2 and T_3 be θ-stable maximal k-split tori with associated Weyl elements $w_2, w_3 \in W_{H_k}(T)$, respectively. Then T_2 and T_3 are H_k-conjugate if and only if w_2 and w_3 are conjugate under $W_{H_k}(T)$.*

Note that Theorem 4 will not apply to $\theta = \mathrm{Inn}_{I_{n/2,n/2}}$ since by Proposition 1.4.3 there exists more than one H_k-conjugacy class of maximal (θ, k)-split tori. Hence, one needs to approach cases like this in another way. In this case, assume that θ is such that there exists only one H_k-conjugacy class of maximal k-split tori containing a torus that is maximal k-split in H_k. (By Proposition 1.4.4, such a θ exists in $\mathrm{SL}(n, k)$.) Let S be the representative torus of this conjugacy class. As before, one can associate each θ-stable maximal k-split torus with an element of $W_{H_k}(S)$, and we can arrive at the following result.

Theorem 1.4.6. *Let Let T_1 and T_2 be θ-stable maximal k-split tori with associated Weyl elements $w_1, w_2 \in W_{H_k}(S)$, respectively. Then T_1 and T_2 are H_k-conjugate if and only if w_1 and w_2 are conjugate under $W_{H_k}(S)$.*

Theorems 4 and 5 significantly simplify subproblem (4) by turning the geometric problem of conjugacy classes of tori into a combinatorial problem within the Weyl group. We are currently working on this Weyl group problem in order to complete our classification, and we give an example of this work in Chapter 7.

Chapter 2

Preliminaries

Our basic references for reductive groups, Borel subgroups, and maximal tori will be the books of Borel [6], Humphreys [21], and Springer [34]. For background on the orbits of a symmetric subgroup acting on a flag variety, our basic reference will be the paper of Helminck-Wang [16]. We will follow their notations and terminology.

Throughout this thesis, k is a field of characteristic not equal to 2, and G is a reductive algebraic group defined over k. Let T be a maximal k-split torus in G, and P be a minimal parabolic subgroup of G. Let θ be an involution of G and $H = G^\theta$ be the fixed point group of θ. Let G_k, H_k and T_k be the k-rational points of G, H and T, respectively. $W_{G_k}(T) = N_{G_k}(T)/Z_{G_k}(T)$ is the Weyl group of T in G_k, where $N_{G_k}(T)$ is the normalizer of T in G_k and $Z_{G_k}(T)$ is the centralizer of T in G_k. For a closed subgroup U of G, we write U° for the connected component containing the identity.

Let $V = k^n$ be a finite-dimensional vector space defined over k, $\overline{V} = \overline{k}^n$, $\mathrm{M}_n(k) = \mathrm{M}(n,k)$ the set of $n \times n$-matrices with entries in k, and $\mathrm{Id} \in \mathrm{M}_n(k)$ the identity automorphism. Let

$$\mathrm{GL}(n,k) = \{A \in \mathrm{M}_n(k) \mid \det(A) \neq 0\},$$

and

$$\mathrm{SL}(n,k) = \{A \in \mathrm{M}_n(k) \mid \det(A) = 1\}.$$

Let k^* denote the product group of all the nonzero elements, and let

$$(k^*)^2 = \{a^2 \mid a \in k^*\}.$$

The following is some notation we will use throughout the rest of this paper.

Notation 2.0.7.

1. We will use $\overline{m}$ to denote the entire square class of m.
2. By abuse of notation, we will use $m \in k^*/(k^*)^2$ to denote that m is the representative of the square class $\overline{m}$ of k.

2.1 Symmetric Spaces

There are several ways of defining symmetric spaces. The definition we choose to use involves an involution of the reductive algebraic group G. Let $\mathrm{Aut}(G)$ denote the set of all automorphisms of G.

Definition 2.1.1. An *involution* of G is an automorphism $\theta \in \mathrm{Aut}(G)$ such that $\theta^2 = \mathrm{Id}$ and $\theta \neq \mathrm{Id}$.

Definition 2.1.2. Given an involution θ of G, the *symmetric space* X is defined as G/H, where $G^\theta = H$ is the fixed point group of the involution θ. One can also characterize this symmetric space as the subvariety $X = \{x\theta(x)^{-1} \mid x \in G\}$ of G. The $X \approx G/H$. For $k = \bar{k}$ algebraically closed, the set X is the connected component containing the identity of $Q = \{x \in G \mid \theta(x) = x^{-1}\}$.

Example 2.1.3. Take $G = \mathrm{GL}_n$ and the involution $\theta(A) = (A^T)^{-1}$, where A^T denotes the transpose of A. This gives the symmetric space $X_k = \mathrm{GL}(n,k)/\mathrm{O}(n,k)$, which can be identified with a subset of the symmetric matrices. In fact, if k is algebraically closed, X_k is the entire set of symmetric matrices.

2.2 Tori

The classification obtained in this thesis involves classifiying conjugacy classes of tori. Here we give the definitions and results for tori needed for our work.

First, if T is a torus of G defined over k, then there are subtori T^a and T^d of T, where T^a is the largest anisotropic subtorus of T and T^d is the largest k-split subtorus of T defined over k. These tori satisfy: $T = T^a \cdot T^d$ and $T^a \cap T^d$ is finite (see [6, 8.15]).

We have a similar decomposition for θ-stable tori, where θ is an involution of G. Let T be a maximal k-split torus.

Definition 2.2.1. T is θ-stable if $\theta(T) = T$. Then, $T = T^+T^-$, where

1. $T^+ = \{t \in T \mid \theta(t) = t\}^\circ$
2. $T^- = \{t \in T \mid \theta(t) = t^{-1}\}^\circ$. The torus T^- is called θ-split.

Moreover, $T^+ \cap T^-$ is finite.

The above definition comes from the fact that the product map

$$\mu : T^+ \times T^- \to T,\ \mu(t_1, t_2) = t_1 t_2$$

is a separable isogeny. In fact, $T^+ \cap T^-$ is an elementary abelian 2-group.

Definition 2.2.2. A torus T is called (θ, k)-split if T is k-split and θ-split. We will let $\mathcal{A}_\theta$ denote the set of maximal (θ, k)-split tori.

All maximal k-split tori of G are G-conjugate. If T is a maximal torus, then $Z_G(T) = T$. By a result of Steinberg [35], any Borel subgroup contains a θ-stable maximal torus.

We will use the following notation to distinguish the tori we will discuss:

- A_i will denote a maximal (θ, k)-split torus
- S_i will denote a θ-stable maximal k-split torus with its S_i^+ part maximal in H_k
- T_i will denote a θ-stable maximal k-split torus

Additionally, $\mathcal{T}$ will denote the set of θ-stable maximal k-split tori, and $\mathcal{A}_\theta$ will denote the set of maximal (θ, k)-split tori.

2.3 Characterization of $P_k \backslash G_k / H_k$

For the rest of this paper, $G = \mathrm{SL}(n, k)$,

$$T = \{\text{diagonal matrices of determinant equal to } 1\},$$

P is a minimal parabolic subgroup containing T, and $H = G^\theta$, where θ is an involution of G. Note that T is both a maximal torus and a maximal k-split torus. Let T_i be any maximal k-split torus.

The following is the general characterization of orbits of a symmetric subgroup acting on a flag variety by Helminck-Wang [16], which was discussed in the introduction.

Theorem 2.3.1 ([16]). *$P_k \backslash G_k / H_k \simeq \bigcup_{T_i \in I} W_{G_k}(T_i)/W_{H_k}(T_i)$, where I is the set of H_k-conjugacy classes of θ-stable maximal k-split tori, and the T_i are representatives of these H_k-conjugacy classes.*

In order to properly use this theorem, we will need to first classify the H_k conjugacy classes of the maximal (θ, k)-split tori, an important classification on its own. The following result, also by Helminck-Wang [16], will aid us in our classification.

Theorem 2.3.2 ([16]). *Let A_1 and A_2 be maximal (θ, k)-split. Let $T_1 \supset A_1$ be maximal k-split. Then there exists $g \in (H \cdot Z_G(T_1))_k$ such that $gA_1g^{-1} = A_2$.*

The above will be the main tool we will use to classify the H_k-conjugacy classes of the maximal (θ, k)-split tori for each involution θ.

2.4 Quadratic Forms

The following comes from Jones [23]. Let $f = \sum_{i,j=1}^{n} a_{ij} x_i x_j$ be a quadratic form with $a_{ij} = a_{ji}$. We define the matrix of the form f to be the n by n matrix A of the coefficients a_{ij}, in other words, $A = (a_{ij})$. The determinant of the form f is the determinant of A, normally denoted d. D_i is the i by i determinant in the upper left hand corner of A. A form (and matrix) is said to be *regular* if no two successive D_i are zero.

Proposition 2.4.1 ([23]). *Every symmetric matrix is congruent to a regular matrix.*

Therefore, we may assumer f is regular. This will be important when we later define and use Hasse's symbol.

Definition 2.4.2. The Legendre symbol $(a|p) = 1$ if a is a square in $\mathbb{F}_p$, and $(a|p) = -1$ if a is a nonsquare.

Definition 2.4.3. For a prime number p, let $\mathbb{Q}_p$ denote the field of $\mathfrak{p}$-adic numbers. We will also write $\mathbb{Q}_\infty$ for $\mathbb{R}$. Recall that in $\mathbb{Q}_p$, a is a unit if $a = a_0 + a_1 * p + a_2 * p^2 + a_3 * p^3 + \ldots$, where $a_0 \neq 0$.

Definition 2.4.4. Let $a \in \mathbb{Q}_p$ be a unit. Then we define $(a|p)$ to be the value of the Legendre symbol $(a_0|p)$ where a_0 is the leading term in the $\mathfrak{p}$-adic expansion of a.

2.4.5 Results needed for Approach 1 in Chapter 5

Definition 2.4.6 (*First definition of the Hilbert Symbol*). For $k = \mathbb{Q}_p$ the *Hilbert symbol* is defined $(a,b)_p = 1$ if the equation $ax_1^2 + bx_2^2 = 1$ has a solution over $\mathbb{Q}_p$, and $(a,b)_p = -1$ if it does not have a solution.

Proposition 2.4.7 ([23]). *Let $a,b,c,d \in \mathbb{Q}_p$ be nonzero.*

1. $(a,b)_\infty = 1$ *unless a and b are both negative.*
2. $(a,b)_p = (b,a)_p$
3. $(ac^2, bd^2)_p = (a,b)_p$
4. $(a,-a)_p = 1$
5. *If $a = p^n a_1, b = p^m b_1$ with a_1 and b_1 units, then*
   (a) *if p is odd,* $(a,b)_p = (-1|p)^{nm}(a_1|p)^m(b_1|p)^n$
   (b) *if $p = 2$,* $(a,b)_p = (2|a_1)^m(2|b_1)^n(-1)^{(a_1-1)(b_1-1)/4}$
6. *If p is prime to $2ab$, $(a,b)_p = 1$, for p finite, a and b in $R(p)$, the ring of $\mathfrak{p}$-adic integers.*

7. $(a,b)_p(a,c)_p = (a,bc)_p$.

8. $(a,a)_p = (a,-1)_p$.

9. $(ac,bc)_p = (a,b)_p(c,-ab)_p$.

Definition 2.4.8. We define *Hasse's Symbol* $c_p(f) = (-1,-D_n)_p \prod_{i=1}^{n-1}(D_i,-D_{i+1})_p$, where the D_i are as defined above. If $D_k = 0$, we define the symbols $(D_k,-D_{k+1})_p$ and $(D_{k-1},-D_k)_p$ as $(\pm 1,-D_{k+1})_p$ and $(D_{k-1},\mp 1)_p$, respectively, with the $\pm$ sign chosen arbitrarily. Since f is taken to be regular, this formula is well-defined.

Proposition 2.4.9 ([23])**.** *If f is a binary form of determinant d and leading coefficient a_{11}, then $f = N$ is solvable for $N \neq 0$ if and only if $c_p(f) = (-d,-N)_p$, that is, $(-a_{11},-d)_p = (-N,-d)_p$.*

Lemma 2.4.10. *Let $m,q \in k^*/(k^*)^2$. For $\mathbb{Q}_p$, $x_1^2 - m^{-1}x_2^2 = q^{-1}$ has a solution if and only if $(m,-qm)_p = 1$.*

Proof. Let $f = x_1^2 - m^{-1}x_2^2$. Then the matrix of f is $\begin{pmatrix} 1 & 0 \\ 0 & -m^{-1} \end{pmatrix}$. By Proposition 2.4.9, $f = q^{-1}$ has a solution $\iff$ $c_p(f) = (-d,-q^{-1})_p$, where d is the determinant of the form f. Now, using Propostion 2.4.7, $(-d,-q^{-1})_p = (m^{-1},-q^{-1})_p = (m,-q)_p$ since $m^{-1} = m{*}m^{-2}$ and $q^{-1} = q{*}q^{-2}$. Also, $c_p(f) = (-1,-D_n)_p \prod_{i=1}^{n-1}(D_i,-D_{i+1})_p = (-1,m^{-1})_p(1,m^{-1})_p = (-1,m^{-1})_p = (-1,m)_p = (m,m)_p$. Thus, $f = q^{-1}$ has a solution if and only if $(m,-q)_p = (m,m)_p \iff (m,-q)_p(m,m)_p = 1 \iff (m,-qm)_p = 1$. □

2.4.11 Results needed for Approach 2 in Chapter 5

The following number theoretic results come from Serre [32]. We assume $k = \mathbb{R}$ or $\mathbb{Q}_p$ for these results.

Definition 2.4.12 (*Alternate Definition of the Hilbert Symbol*)**.** Let $m,q \in k^*$. The *Hilbert symbol* (m,q) is defined $(m,q) = 1$ if the equation $x^2 - my^2 - qz^2 = 0$ has a solution $(x,y,z) \neq (0,0,0) \in k^3$, and $(m,q) = -1$ otherwise.

Proposition 2.4.13. [32] *Let $m, q \in k^*$ and let $l = k[\sqrt{m}]$. Let $N : l \to k$ be the norm map. For $(m, q) = 1$ it is necessary and sufficient that q belongs to $N(l^*)$.*

Theorem 2.4.14. [32] *The Hilbert symbol is a nondegenerate bilinear form on the set $k^*/(k^*)^2$.*

Corollary 2.4.15. [32] *If m is not a square, the group $N(l^*)$ defined in Proposition 2.4.13 is a subgroup of index 2 in k^*.*

Remark 2.4.16. Consider Theorem 5.3.3 and the equation $x^2 - my^2 = qz^2$.

- If m is the square of an element c, then $(c, 1, 0)$ is a solution to $x^2 - my^2 = qz^2$. Hence, $(q, m) = 1$ and $N(l) = k^*$. Then,

$$\mathrm{Im}(N)/(k^*)^2\{1, -m\} = k^*/(k^*)^2\{1, -m\}.$$

- If m is not a square, then by Corollary 2.4.15, $|\mathrm{Im}(N)/(k^*)^2| = \frac{1}{2}|k^*/(k^*)^2|$.

- Let $q = -m$. Then clearly $x^2 - my^2 = -mz^2$ has a solution, and we have $-m \in \mathrm{Im}(N)$. If $m \notin \overline{-1}$, we can let $-m$ be its own coset representative in $\mathrm{Im}(N)/(k^*)^2$.

- If $m \in \overline{-1}$, then $\{1, -m\} = \{1\}$. So $|\mathrm{Im}(N)/(k^*)^2\{1, -1m\}| = \frac{1}{2}|k^*/(k^*)^2|$.

Chapter 3

Involutions of SL(n, k) with $n > 2$

The following is a summary of the classification of the isomorphism classes of involutions of SL(n,k) found in the paper by Dometrius, Helminck, and Wu [17].

Definition 3.0.17 (Isomorphic Involutions)**.** Let $\mathrm{Aut}(G)$ denote the set of all automorphisms of G. For $A \in \mathrm{GL}(n,k)$, let Inn_A denote the inner automorphism defined by $\mathrm{Inn}_A(X) = A^{-1}XA$ for all $X \in \mathrm{GL}(n,k)$. Let $\mathrm{Inn}_k(G) = \{\mathrm{Inn}_A \mid A \in G\}$ denote the set of all *inner* automorphisms of G, and let $\mathrm{Inn}(G)$ denote the set of automorphisms Inn_A of G with $A \in \overline{G}$ such that $\mathrm{Inn}_A(G) = G$.

1. We say that θ and τ in $\mathrm{Aut}(G)$ are $\mathrm{Inn}(G)$*-isomorphic* if there is a ϕ in $\mathrm{Inn}(G)$ such that $\tau = \phi^{-1}\theta\phi$ and write $\theta \approx^{\mathrm{Inn}} \tau$.

2. We say that θ and τ in $\mathrm{Aut}(G)$ are $\mathrm{Aut}(G)$*-isomorphic* if there exists a ϕ in $\mathrm{Aut}(G)$ such that $\tau = \phi^{-1}\theta\phi$ and write $\theta \approx^{\mathrm{Aut}} \tau$.

Remark 3.0.18. Notice that $\theta \approx^{\mathrm{Inn}} \tau \implies \theta \approx^{\mathrm{Aut}} \tau$. The following results will mainly characterize the Inn(G)-isomorphism classes, and we will use the notation $\approx$ instead of $\approx^{\mathrm{Inn}}$.

3.1 Bilinear Forms and Their Induced Involutions

Let $\beta : V \times V \to k$ be a bilinear form on $V = k^n$, $\mathfrak{B} = \{e_i\}$ an ordered basis for V, and $M_{\mathfrak{B}} = (\beta(e_i, e_j))$ the matrix of β with respect to $\mathfrak{B}$. For $x \in V$ let $x_{\mathfrak{B}} = (x_1, \ldots, x_n)$

be the coordinates of x with respect to $\mathfrak{B}$. Then for all $x, y \in V$ we have $\beta(x, y) = x_{\mathfrak{B}}^T M_{\mathfrak{B}} y_{\mathfrak{B}}$. Recall that two matrices M_1 and M_2 are *congruent* over the field k if there exists a matrix $Q \in \mathrm{GL}(n, k)$ such that $M_2 = Q^T M_1 Q$. In this case we will write $M_1 \cong M_2$. The following theorem is a well-known result.

Theorem 3.1.1. [2] *Two matrices M_1 and M_2 represent the same bilinear form with respect to different bases if and only if they are congruent.*

Adjoint of a Matrix

Let M be the matrix of a non-degenerate bilinear form β over $V = k^n$ and let $A \in \mathrm{GL}(n, k)$. The *adjoint of A with respect to β* (denoted A') is defined as the matrix that satisfies the relation $\beta(Ax, y) = \beta(x, A'y)$. It follows from the definition that

$$\beta(Ax, y) = (Ax)^T M y = x^T A^T M y = x^T M A' y = \beta(x, A'y)$$

which implies

$$A' = M^{-1} A^T M \tag{3.1}$$

3.1.2 Using the Adjoint to Construct Involutions

Definition 3.1.3. Given a bilinear form on V with matrix M, we define θ_M by $\theta_M(A) = (A')^{-1}$. We will omit the subscript M when the bilinear form involved is clear. Using Equation (3.1), we see that $\theta(A) = M^{-1}(A^T)^{-1}M$.

Proposition 3.1.4. *If β is either symmetric or skew-symmetric, then θ is an involution of* $\mathrm{GL}(n, k)$*.*

3.2 Outer Involutions of SL(n, k)

Recall that an automorphism θ of G is said to be an *outer automorphism* if $\theta \notin \mathrm{Inn}\, G$. An outer automorphism that is an involution will be called an outer involution.

Lemma 3.2.1. 1. *If k is algebraically closed, then* $|\mathrm{Aut}(G)/\mathrm{Inn}(G)| = 2$.

2. *Any outer automorphism can be written as* $\mathrm{Inn}_M\,\phi$*, where* ϕ *is a fixed outer automorphism.*

Definition 3.2.2. Two bilinear forms on $V = k^n$ with associated matrices M_1 and M_2 are *semi-congruent over* k ($M_1 \cong^s M_2$) if there exists a $Q \in \mathrm{GL}(n,k)$ and an $\alpha \in k$ such that $M_2 = \alpha Q^T M_1 Q$.

Remark 3.2.3. For our purposes we will take for ϕ the involution defined by $\phi(A) = (A^T)^{-1}$. See [6] for details of the above lemma.

Theorem 3.2.4 (Outer Classification Theorem)**.** *If* θ_1 *and* θ_2 *are outer involutions on* $\mathrm{SL}(n,k)$*, then they come from bilinear symmetric or skew-symmetric forms represented by* M_1 *and* M_2 *(respectively), and*

$$\theta_1 \approx \theta_2 \iff \mathrm{Inn}_{M_1}\,\phi \approx \mathrm{Inn}_{M_2}\,\phi \iff M_1 \cong^s M_2.$$

3.2.5 Classification of isomorphism classes of outer involutions

Recall the following results about congruence classes of bilinear forms (see [31]).

Theorem 3.2.6. 1. *Symmetric matrices are congruent to diagonal matrices whose entries are representatives of the square-class group* $k^*/(k^*)^2$.

2. *Skew-Symmetric singular nonsingular matrices are congruent to the* $(2m) \times (2m)$ *matrix* J_{2m}*, where* $n = 2m$ *and*

$$J_{2m} = \begin{pmatrix} 0 & I_{m\times m} \\ -I_{m\times m} & 0 \end{pmatrix}.$$

Remark 3.2.7. Note that a symmetric bilinear form β is non degenerate if and only if M_β is congruent to a diagonal matrix with non-zero diagonal entries.

We will need to introduce the following notation for matrices to complete the classification. Let

$$I_{n-i,i} = \begin{pmatrix} I_{n-i\times n-i} & 0 \\ 0 & -I_{i\times i} \end{pmatrix}$$

Table 3.1: Outer Involutions of G over $k = \bar{k}$, $\mathbb{R}$ and $\mathbb{F}_p$ with $p \neq 2$

Field	Semi-Congruence Class M	Involution Class on G
$k = \bar{k}$		
n odd	$M = I_{n\times n}$	$\theta(A) = (A^T)^{-1}$
n even	$M_1 = I_{n\times n}$ $M_2 = J_{2m}\ \ (n = 2m)$	$\theta_1(A) = (A^T)^{-1}$ $\theta_2(A) = \mathrm{Inn}_{J_{2m}}(A^T)^{-1}$
$k = \mathbb{R}$		
n odd	$M_i = I_{n-i,i}$ $i = 0, 1, 2, \dots, \frac{n-1}{2}$	$\theta_i(A) = \mathrm{Inn}_{M_i}(A^T)^{-1}$
n even	$M_i = I_{n-i,i}$ $i = 0, 1, 2, \dots, \frac{n}{2}$ $M_{\frac{n}{2}+1} = J_{2m}$	$\theta_i(A) = \mathrm{Inn}_{M_i}(A^T)^{-1}$ $\theta_{\frac{n}{2}+1}(A) = \mathrm{Inn}_{J_{2m}}(A^T)^{-1}$
$k = \mathbb{F}_p\ \ p \neq 2$		
n odd	$M_1 = I_{n\times n}$ $M_2 = \begin{pmatrix} I_{(n-1)\times(n-1)} & 0 \\ 0 & S_p \end{pmatrix}$	$\theta_1(A) = (A^T)^{-1}$ $\theta_2(A) = \mathrm{Inn}_{M_2}(A^T)^{-1}$
n even	$M_1 = I_{n\times n}$ $M_2 = \begin{pmatrix} I_{(n-1)\times(n-1)} & 0 \\ 0 & S_p \end{pmatrix}$ $M_3 = J_{2m}$	$\theta_1(A) = (A^T)^{-1}$ $\theta_2(A) = \mathrm{Inn}_{M_2}(A^T)^{-1}$ $\theta_3(A) = \mathrm{Inn}_{M_3}(A^T)^{-1}$

and

$$M_{n,x,y,z} = \begin{pmatrix} I_{n-3\times n-3} & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & z \end{pmatrix}.$$

Table 1 gives the results of the semi-congruence class of matrices M with the corresponding involution class on $\mathrm{SL}(n,k)$ for the fields $k = \bar{k}$, $\mathbb{R}$, and $\mathbb{F}_p$, where $p \neq 2$.

3.2.8 Some results about bilinear forms over $\mathbb{Q}_p$

Definition 3.2.9. Given a matrix M, let D_i be the determinant of the upper left $i \times i$ square submatrix of M. Define the *p-sign* (or Hilbert Symbol) of two non-zero $\mathfrak{p}$-adic numbers to be $(\alpha, \beta)_p = 1$ if $\alpha x^2 + \beta y^2 = 1$ has a solution in $\mathbb{Q}_p$, -1 otherwise. Then the *Hasse Symbol* $c_p(M)$ *of* M is defined by

$$c_p(M) = (-1, -D_n)_p \prod_{i=1}^{n-1} (D_i, -D_{i+1})_p.$$

Note that the Hasse Symbol is limited to 1 or -1.

Lemma 3.2.10. *If* M *is an* $n \times n$ *matrix and* $\alpha \in \mathbb{Q}_p$ *we have the following formula:*

$$c_p(\alpha M) = \begin{cases} (\alpha, -1)_p^{n/2} (\alpha, D_n)_p c_p(M) & \text{if } n \text{ is even} \\ (\alpha, -1)_p^{(n+1)/2} c_p(M) & \text{if } n \text{ is odd} \end{cases}$$

Theorem 3.2.11 ([23])**.** *Two symmetric matrices* M_1 *and* M_2 *with entries in* $\mathbb{Q}_p$ *are congruent if and only if*

$$\det(M_1) = \alpha^2 \det(M_2) \quad and \quad c_p(M_1) = c_p(M_2).$$

To classify the outer involutions over $\mathbb{Q}_p$, we need to consider all combinations $\alpha \in k^*/(k^*)^2$ and the Hasse symbol, which we will call just c for our purposes here. For $\mathbb{Q}_2$, let $k^*/(k^*)^2 = \{\pm 1, \pm 2, \pm 3, \pm 6\}$. For $\mathbb{Q}_p$, $p \neq 2$, let $k^*/(k^*)^2 = \{1, S_p, p, pS_p\}$, where S_p is the smallest nonsquare in $\mathbb{F}_p$. Thus, we have either 16 or 8 possible semi-congruence classes of symmetric matrices over $\mathbb{Q}_p$. We will use the dual value $d_{\alpha,c}$ to denote these classes. Tables 2 and 3 give the resulting isomorphism classes of outer involutions over $\mathbb{Q}_p$.

3.3 Inner Involutions of SL(n, k)

We need to introduce one more matrix notation that will be important. Let

$$L_{n,x} = \begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ x & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & 0 & 1 \\ 0 & 0 & \dots & x & 0 \end{pmatrix}.$$

Notice that $J_{2m} \cong^s L_{n,-1}$. We have the following result.

Table 3.2: Outer Involutions of $\mathrm{SL}(n, \mathbb{Q}_p)$ $(p \neq 2)$

$\dim(V)$	Dual Value $d_{\alpha,c}$ of the Semi-Congruence Class	Matrix M s.t. $\theta(A) = \mathrm{Inn}_M(A^T)^{-1}$
$-1 \notin (\mathbb{Q}_p^*)^2$		
$n = 4k$	$d_{1,1}$	$M_1 = I_{n\times n}$
	$d_{1,-1}$	M_{n,p,S_p,pS_p}
	$d_{p,1}$	$M_{n,1,1,p}$
	$d_{S_p,1}$	$M_{n,1,1,S_p}$
	$d_{pS_p,1}$	$M_{n,1,1,pS_p}$
		$M = J_{2m}$
$n = 4k+1$	$d_{1,1}$	$M = I_{n\times n}$
$n = 4k+2$	$d_{1,1}$	$M_1 = I_{n\times n}$
	$d_{p,1}$	$M_{n,1,1,p}$
	$d_{S_p,1}$	$M_{n,1,p,pS_p}$
	$d_{S_p,-1}$	$M_{n,1,1,S_p}$
	$d_{pS_p,1}$	$M_{n,1,1,pS_p}$
		$M = J_{2m}$
$n = 4k+3$	$d_{1,1}$	$M = I_{n\times n}$
	$d_{1,-1}$	M_{n,p,S_p,pS_p}
$-1 \in (\mathbb{Q}_p^*)^2$		
$n = 4k$	$d_{1,1}$	$M_1 = I_{n\times n}$
	$d_{1,-1}$	$M_{n,1,p,p}$
	$d_{p,1}$	$M_{n,1,1,p}$
	$d_{S_p,1}$	$M_{n,1,1,S_p}$
	$d_{pS_p,1}$	$M_{n,1,1,pS_p}$
		$M = J_{2m}$
$n = 4k+2$	$d_{1,1}$	$M_1 = I_{n\times n}$
	$d_{1,-1}$	M_{n,p,S_p,pS_p}
	$d_{p,1}$	$M_{n,1,1,p}$
	$d_{S_p,1}$	$M_{n,1,1,S_p}$
	$d_{pS_p,1}$	$M_{n,1,1,pS_p}$
		$M = J_{2m}$
n odd	$d_{1,1}$	$M = I_{n\times n}$
	$d_{1,-1}$	M_{n,p,S_p,pS_p}

Table 3.3: Outer Involutions of G over $k = \mathbb{Q}_2$

$\dim(V)$	Dual Value $d_{\alpha,c}$ of the Semi-Congruence Class	Matrix M such that $\theta(A) = \mathrm{Inn}_M(A^T)^{-1}$
$n = 4k$	$d_{1,1}$	$M_1 = I_{n\times n}$
	$d_{1,-1}$	$M_{n,2,3,6}$
	$d_{-1,1}$	$M_{n,1,1,-1}$
	$d_{2,1}$	$M_{n,1,1,2}$
	$d_{-2,1}$	$M_{n,1,1,-2}$
	$d_{3,1}$	$M_{n,1,1,3}$
	$d_{-3,1}$	$M_{n,1,1,-3}$
	$d_{6,1}$	$M_{n,1,1,6}$
	$d_{-6,1}$	$M_{n,1,1,-6}$
		$M = J_{2m}$
$n = 4k + 1$	$d_{1,1}$	$M = I_{n\times n}$
$n = 4k + 2$	$d_{1,1}$	$M_1 = I_{n\times n}$
	$d_{-1,1}$	$M_{n,1,1,-1}$
	$d_{-1,-1}$	$M_{n,2,3,-6}$
	$d_{2,1}$	$M_{n,1,1,2}$
	$d_{-2,1}$	$M_{n,1,1,-2}$
	$d_{3,1}$	$M_{n,1,1,3}$
	$d_{-3,1}$	$M_{n,1,1,-3}$
	$d_{6,1}$	$M_{n,1,1,6}$
	$d_{-6,1}$	$M_{n,1,1,-6}$
		$M = J_{2m}$
$n = 4k + 3$	$d_{1,1}$	$M = I_{n\times n}$
	$d_{1,-1}$	$M_{n,2,3,6}$

Table 3.4: Isomorphism Classes of Inner Involutions of G

Field	Number of Inner Involutions	Representative Matrix Y such that $\theta = \mathrm{Inn}_Y$
n odd, k = any field	$\frac{n-1}{2}$	$Y = I_{n-i,i} \quad i = 1, 2, \ldots, \frac{n-1}{2}$
n even		
$k = \bar{k}$	$\frac{n}{2}$	$Y = I_{n-i,i} \quad i = 1, 2, \ldots, \frac{n}{2}$
$k = \mathbb{R}$	$\frac{n}{2} + 1$	$Y = I_{n-i,i} \quad i = 1, 2, \ldots, \frac{n}{2}$ $Y = L_{n,-1}$
$k = \mathbb{Q}$	∞	$Y = I_{n-i,i} \quad i = 1, 2, \ldots, \frac{n}{2}$ $Y = L_{n,\alpha} \quad \alpha \neq 1 \mod (\mathbb{Q}^*)^2$
$k = \mathbb{F}_p \;\; p \neq 2$	$\frac{n}{2} + 1$	$Y = I_{n-i,i} \quad i = 1, 2, \ldots, \frac{n}{2}$ $Y = L_{n,S_p}$
$k = \mathbb{Q}_2$	$\frac{n}{2} + 7$	$Y = I_{n-i,i} \quad i = 1, 2, \ldots, \frac{n}{2}$ $Y = L_{n,\alpha} \quad \alpha \in \{-1, {}^{\pm}2, {}^{\pm}3, {}^{\pm}6\}$
$k = \mathbb{Q}_p \;\; p \neq 2$	$\frac{n}{2} + 3$	$Y = I_{n-i,i} \quad i = 1, 2, \ldots, \frac{n}{2}$ $Y = L_{n,\alpha} \quad \alpha \in \{p, S_p, pS_p\}$

Theorem 3.3.1. *Suppose the involution $\theta \in \mathrm{Aut}(G)$ is of inner type. Then up to isomorphism θ is one of the following:*

1. $\mathrm{Inn}_Y|_G$, *where* $Y = I_{n-i,i} \in \mathrm{GL}(n,k)$ *where* $i \in \left\{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\right\}$.

2. $\mathrm{Inn}_Y|_G$, *where* $Y = L_{\frac{n}{2},p} \in \mathrm{GL}(n,k)$ *where* $p \in k^*/k^{*2}$, $p \not\equiv 1 \mod k^{*2}$.

Note that (2) *can only occur when n is even.*

Table 4 contains the isomorphism classes of the inner involutions over $k = \bar{k}$, $\mathbb{R}$, $\mathbb{F}_p$ for $p \neq 2$, and $\mathbb{Q}_p$ for all p.

Chapter 4

Classification Results for $SL(2, \mathbb{F}_p)$

4.1 Preliminaries for $SL(2, k)$

For this chapter and the next, $G = SL_2$,

$$T = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \mid x \neq 0 \right\},$$

$$B = \left\{ \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} x \neq 0 \right\},$$

and $H = G^\theta$, where θ is an involution of G. B is a Borel subgroup of G containing T. In $SL(2,k)$, Borel subgroups are minimal parabolic subgroups. Therefore, in Chapters 4 and 5 we use B instead of P. In these chapters, the classification of $B_k \backslash G_k / H_k$ is the same thing as $P_k \backslash G_k / H_k$.

Let T_i be any maximal k-split torus. Since all maximal k-split tori of $SL(2,k)$ are one-dimensional, a maximal (θ, k)-split torus is also a θ-stable maximal k-split torus. Hence, we will only use the T_i notation in the next two chapters instead of the A_i notation for (θ, k)-split tori since there is no distinction between the two types here.

Note that $W_{G_k}(T_i)$ has order two. For $G_k = SL(2,k)$, $W_{G_k}(T) = \{\text{id}, w\}$, where $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ or $w = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ with $i^2 = -1$.

For $A \in G_k$, let $\text{Int}(A)$ denote the inner automorphism defined by $\text{Int}(A)(X) = AXA^{-1}$ for all $X \in G_k$. The paper by Helminck and Wu provides us with the following classification of the isomorphism classes of involutions of $\text{SL}(2,k)$.

Theorem 4.1.1 ([18])**.** *The number of isomorphy classes of involutions over G equals the order of $k^*/(k^*)^2$. Further, the involutions over G are of the form* $\theta = \text{Int}\begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$ *for all $m \in k^*/(k^*)^2$, and if m and q are in the same square class, then* $\text{Int}\begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$ *is isomorphic to the involution* $\text{Int}\begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix}$.

Remark 4.1.2. By Helminck and Wang's result 4.5 [16], there exists a non-trivial (θ, k)-split torus T_1 if $\theta \neq \text{Id}$ and G is not k-anisotropic. Since T_1 is k-conjugate to T, we may assume that T is our (θ, k)-split torus by switching (if necessary) to a $\text{Int}(G_k)$-conjugate of θ.

To classify $B_k \backslash G_k / H_k$, we will use Theorem 2.3.1. Hence, we will need to classify the H_k-conjugacy classes of θ-stable maximal k-split tori. Let $T_i \neq T$ be a θ-stable maximal k-split torus. Since all maximal k-split tori of $\text{SL}(2,k)$ are one-dimensional, $T_i = T^+$ or $T_i = T^-$. If $T_i = T^+$, then $T_i \subset H$, hence we will first check to see under what conditions H contains a θ-stable maximal k-split torus. If H contains a maximal k-split torus, this torus will represent one H_k-conjugacy class of θ-stable maximal k-split tori.

If $T_i = T^-$, then we are looking for maximal (θ, k)-split tori. We will obtain a count for the H_k-conjugacy classes of (θ, k)-split tori in two ways. First, we will make use of Theorem 2.3.2 to obtain these tori and their H_k-conjugacy classes. For our chosen T, T is maximal (θ, k)-split, maximal k-split, and $Z_G(T) = T$, so we have that $T = A_1 = T_1 = Z_G(A)$ in Theorem 2.3.2. Thus, there exists $g \in (H \cdot T)_k$ such that $gTg^{-1} = T_i$. We will use the set $(H \cdot T)_k$ to classify the H_k-conjugacy classes of maximal (θ, k)-split tori. This method gives both a count of the classes and the means of obtaining the representative tori of the H_k-conjugacy classes. The representative torus is necessary for calculating $W_{H_k}(T_i)$. We will use this first method in both

Chapters 4 and 5.

In out second method, which will only be used in Chapter 5, we will use the Lie algebra $\mathfrak{t}$ of T_i to obtain a count via mapping $\mathfrak{t}$ to a quadratic extension and then considering the norm map of this extension. This method allows for easier calculation of the number of H_k conjugacy classes over the $\mathfrak{p}$-adic numbers, but it does not provide a way to obtain the representative tori. Once the H_k-conjugacy classes of θ-stable maximal k-split tori are found, Theorem 2.3.1 we will give us the classification we desire.

In this chapter we will consider the finite fields $\mathbb{F}_p$ with $p \neq 2$ to illustrate the first method. Since $\mathrm{SL}(2, \mathbb{F}_p)$ is finite, I use counting arguments which I later generalize in the next chapter for any field of characteristic not 2.

4.2 Classification of $B_k\backslash G_k/H_k$ for $G = SL_2(\mathbb{F}_p)$, $p \neq 2$

There are two involutions over G: $\phi = \mathrm{Int}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\theta = \mathrm{Int}\begin{pmatrix} 0 & 1 \\ S_p & 0 \end{pmatrix}$, where S_p is the smallest nonsquare in $\mathbb{F}_p$ for $p \cong 1 \bmod 4$ and $S_p = -1$ for $p \cong 3 \bmod 4$. Then

$$H_\phi = G^\phi = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} | a^2 - b^2 = 1, a, b \in \mathbb{F}_p \right\}$$

$$H_\theta = G^\theta = \left\{ \begin{pmatrix} a & b \\ S_p b & a \end{pmatrix} | a^2 - S_p b^2 = 1, a, b \in \mathbb{F}_p \right\}.$$

When no subscript is given for H, the result holds for both H_θ and H_ϕ. We will use $\mathcal{A}_\theta$ to denote the set of maximal (θ, k)-split tori, and we will use $\mathcal{A}_\phi$ to denote the set of maximal (ϕ, k)-split tori.

For $G = SL_2(\mathbb{F}_p), p \neq 2$, a counting argument is used to obtain $B_k\backslash G_k/H_k$. To begin, the following result by Winterhof gives us the necessary means to obtain the size of H_θ and H_ϕ.

Theorem 4.2.1 ([39]). *1. If $p = 4n - 1$, then let $r_1, r_2, \cdots, r_{2n}$ be the squares (including 0) in $\mathbb{F}_p$. For any element $0 \neq c \in \mathbb{F}_p$ the number of squares (including*

0) and the number of nonsquares within $r_1 + c, r_2 + c, \cdots, r_{2n} + c$ is equal.

2. *If $p = 4n + 1$, then let $r_1, r_2, \cdots, r_{2n}$ be the squares (excluding 0) in $\mathbb{F}_p$. For any element $0 \neq c \in \mathbb{F}_p$ the number of squares (including 0) and the number of nonsquares within $r_1 + c, r_2 + c, \cdots, r_{2n} + c$ is equal.*

Lemma 4.2.2. *$|H_\phi| = p - 1 = |T|$ and $|H_\theta| = p + 1$.*

Proof. $|T| = p - 1$ by definition of T. H_ϕ is defined by the solutions to the equation $a^2 - 1 = b^2$ in $\mathbb{F}_p$, and H_θ is defined by the solutions to the equation $a^2 - 1 = S_p b^2$. So we want to know when $a^2 - 1$ is a square, where a^2 is any square, in order to divide the solutions between H_ϕ and H_θ. This is exactly Theorem 4.1 with $c = -1$. Before we continue. let's consider the number of matrices defined by the equations $a^2 - 1 = b^2$ and $a^2 - 1 = S_p b^2$ for each set (a, b). Clearly, each set (a, b) satisfies only one of these equations. If $a^2 \neq 0$, there are two choices for a. The same is true for b. So, if either $a = 0$ or $b = 0$, the the set (a, b) defines two matrices. Otherwise, the set defines four matrices. Another way of saying this is that $a^2 - 1 = 0$ and $a^2 - 1 = -1$ defines two matrices, and anything else defines four matrices. Note that for our purposes here, 0 is considered both a square and nonsquare since if $a^2 = 1$, $b = 0$ satisfies both equations. Theorem 4.1 does not consider 0 to be a nonsquare, so we will be adding it to our result.

Now consider $p = 4n - 1$. Let $r_1, r_2, \cdots, r_{2n}$ be the squares (including 0) in $\mathbb{F}_p$. By part (1) of Theorem 4.1, the number of squares (including 0) and the number of nonsquares within $r_1 - 1, r_2 - 1, \cdots, r_{2n} - 1$ is equal. Since 0 is also a nonsquare here, there are n squares and $n + 1$ nonsquares in all. Notice that -1 is in this set since 0 was originally included, and since $p = 4n - 1$, -1 is in the set of nonsquares. Thus, we have that $|H_\phi| = 4(n-1) + 2 = 4n - 2 = p - 1$, and $|H_\theta| = 4(n-1) + 2(2) = 4n = p + 1$.

Consider $p = 4n + 1$. Let $r_1, r_2, \cdots, r_{2n}$ be the squares (excluding 0) in $\mathbb{F}_p$. By part (2) of Theorem 4.1, the number of squares (including 0) and the number of nonsquares within $r_1 - 1, r_2 - 1, \cdots, r_{2n} - 1$ is equal. Since 0 is also a nonsquare here, there are n squares and $n + 1$ nonsquares in all. In addition, since 0 was initially excluded, -1 is not in this set, yet it does satisfy the equation $a^2 - 1 = b^2$. So, -1

needs to be added to the set of squares. Thus, we have that $|H_\phi| = 4(n-1)+2(2) = 4n = p-1$, and $|H_\theta| = 4(n)+2 = p+1$. □

4.2.3 Case of T^+

In this case, we are looking for θ- (or ϕ-) stable maximal k-split tori in H_θ (or H_ϕ).

Theorem 4.2.4. *H is a maximal k-split torus if and only if $m \in \overline{1}$. In particular,*

1. *If $H = H_\phi$, H is a maximal k-split torus.*

2. *If $H = H_\theta$, H is not a maximal k-split torus when $p > 3$. If $p = 3$, $T \subset H$.*

Proof. To diagonalize H, consider $h = \begin{pmatrix} a & b \\ mb & a \end{pmatrix} \in H$, where $m = 1$ or $m = S_p$, and

$$|h - \lambda \,\mathrm{Id}\,| = \begin{vmatrix} a-\lambda & b \\ mb & a-\lambda \end{vmatrix} = \lambda^2 - 2a\lambda + 1.$$

So, $\lambda = a \pm \sqrt{a^2-1} = a \pm b\sqrt{m}$, since $a^2 - mb^2 = 1$. Therefore, the eigenvalues are in k only when $m \in \overline{1}$. Since m is a square, and all involutions θ with $m \in \overline{1}$ act the same as the involution θ with $m = 1$, we may assume $m = 1$. Thus, the $(a+b)$-eigenspace is $< (1,1) >$, and the $(a-b)$-eigenspace is $< (-1,1) >$. Let $P = \begin{pmatrix} \frac{1}{2} & -1 \\ \frac{1}{2} & 1 \end{pmatrix} \in \mathrm{SL}_2(k)$. Then $P^{-1}HP \subset T$.

Furthermore, if $p = 3$, $T = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right\}$, which is clearly contained in H_θ. □

For later calculations of the Weyl group of tori in H, we will need the following result.

Lemma 4.2.5. *The Weyl element w is*

1. *contained in H_ϕ for $p \cong 1 \bmod 4$*

2. *not contained in H_ϕ for $p \cong 3 \bmod 4$*

3. *not contained in* H_θ *for* $p \cong 1$ *mod* 4

4. *contained in* H_θ *for* $p \cong 3$ *mod* 4*.*

Proof. Recall that $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ or $w = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$. Now, for w to be in H_ϕ, w must be of the form $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$. Hence $w = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix} \in H_\phi \iff \sqrt{-1} \in \mathbb{F}_p$. Thus, $w \in H_\phi$ for $p \cong 1$ mod 4 and $w \notin H_\phi$ for $p \cong 3$ mod 4.

Furthermore, for w to be in H_θ, w must be of the form $\begin{pmatrix} a & b \\ S_p b & a \end{pmatrix}$. Hence, $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in H_\theta \iff b = 1$ and $S_p = -1$. For $p \cong 3$ mod 4, $S_p = -1$, so $w \in H_\theta$. For $p \cong 1$ mod 4, $S_p \neq -1$, so $w \notin H_\theta$. □

4.2.6 Case of T^-

We will refer to θ as any involution in the following discussion. H will be the fixed point group of θ. In the theorems following the discussion, we will go back to the definitions of θ, ϕ, and H for $\mathrm{SL}_2(\mathbb{F}_p)$.

In this case we are looking for maximal (θ, k)-split tori conjugate to T in order to split the set $\mathcal{A}_\theta$ into H-conjugacy classes. For this purpose, we will make use of Theorem 2.8. For our chosen T, T is maximal (θ, k)-split, maximal k-split, and $Z_G(T) = T$, so we have that $T = A_1 = A = Z_G(A)$ in Theorem 2.8. Thus, there exists$g \in (H \cdot T)_k$ such that $gTg^{-1} = T_i$, where T_i is a (θ, k)-split maximal torus.

Consider the following: H acts on $(H \cdot T)_k$ by left multiplication. Each element of $(H \cdot T)_k$ corresponds to a (θ, k)-split maximal torus by conjugating T. So, H acts on the sets $\mathcal{A}_\theta$ by acting on $(H \cdot T)_k$. We are doing this round about way of having $(H \cdot T)_k$ act on $\mathcal{A}_\theta$ since $(H \cdot T)_k$ in not a group in general, and hence we cannot count orbits with $(H \cdot T)_k$. Having H act on $(H \cdot T)_k$ as representatives of $\mathcal{A}_\theta$, we are able to use the group structure of H to count the orbits of $\mathcal{A}_\theta$.

Let $h \in H$, $x \in (H \cdot T)_k$, and $T_i \in \mathcal{A}_\theta$ such that $xTx^{-1} = T_i$. Then $hT_ih^{-1} = hxT(hx)^{-1}$, where clearly $hx \in (H \cdot T)_k$. So hx must give a maximal torus that is H-conjugate to T_i. So h acting on $(H \cdot T)_k$ breaks $(H \cdot T)_k$ into orbits such that the elements of each orbit correspond to maximal tori that are H-conjugate to each other, in other words, corresponding to an H-orbit in $\mathcal{A}_\theta$. By counting the number of orbits of H in $(H \cdot T)_k$ which contain elements corresponding to T, we can count how many elements of $(H \cdot T)_k$ correspond to elements of $\mathcal{A}_\theta$ which are H-conjugate to T, and how many do not. In this way, we will obtain the size of $\mathcal{A}_\theta$, and the H-conjugacy classes we need to classify $B_k \backslash G_k / H_k$.

To begin, we will need to know the size of $|(H \cdot T)_k|$ for each involution.

Lemma 4.2.7. *1. Let $H = H_\phi$, then $|(H \cdot T)_k| = (p-1)^2$.*

2. Let $H = H_\theta$, then $|(H \cdot T)_k| = (p-1)(p+1)$.

Proof. Let $h = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \in H_\phi$ over $\bar{k}$ and $t = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \in T$ over $\bar{k}$. Then, $ht = \begin{pmatrix} ax & bx^{-1} \\ bx & ax^{-1} \end{pmatrix} \in H \cdot T$. So

$$H \cdot T = \left\{ \begin{pmatrix} ax & bx^{-1} \\ bx & ax^{-1} \end{pmatrix} | a^2 - b^2 = 1, a, b, x \in \bar{k} \right\},$$

and

$$(H \cdot T)_k = \left\{ \begin{pmatrix} ax & bx^{-1} \\ bx & ax^{-1} \end{pmatrix} | a^2 - b^2 = 1, a, b, x \in \bar{k}, ax, bx, ax^{-1}, bx^{-1} \in k = \mathbb{F}_p \right\}.$$

To obtain the size of $(H \cdot T)_k$ we will rewrite the set. Let $ax = i \in \mathbb{F}_p$ and $bx = j \in \mathbb{F}_p$. Then $a = i/x$ and $b = j/x$, so $1 = a^2 - b^2 = \frac{i^2 - j^2}{x^2}$. Hence, $x^2 = i^2 - j^2 \Rightarrow x^{-2} = \frac{1}{i^2 - j^2}$. Also, $ax^{-1} = ix^{-2}$ and $bx^{-1} = jx^{-2}$. So,

$$(H \cdot T)_k = \left\{ \begin{pmatrix} i & jy \\ j & iy \end{pmatrix} | i, j \in \mathbb{F}_p, y = \frac{1}{i^2 - j^2} \text{ with } i^2 - j^2 \neq 0 \right\}.$$

Clearly y does not exist when $i = \pm j$. Counting the set shows that there are p choices for i. If $i = 0$, there are $p - 1$ choices for j since both cannot equal 0. For $i \neq 0$, there are $p-2$ choices for j since $j \neq \pm i$. Thus, $|(H \cdot T)_k| = (p-1) + (p-1)(p-2) = (p-1)^2$.

A similar construction for $H = H_\theta$ gives

$$(H \cdot T)_k = \left\{ \begin{pmatrix} i & jyS_p^{-1} \\ j & iy \end{pmatrix} | i, j \in \mathbb{F}_p, y = \frac{1}{i^2 - j^2 S_p^{-1}} \text{ with } i^2 - j^2 S_p^{-1} \neq 0 \right\}.$$

Now $i^2 - j^2 S_p^{-1} = 0 \Rightarrow j = \sqrt{S_p} i$. But $\sqrt{S_p} \notin \mathbb{F}_p$ by definition, so y exists in all cases. Counting on the set gives p choices for i. If $i = 0$, there are $p - 1$ choices for j since both cannot equal 0. For $i \neq 0$, there are p choices for j by previous statement about y. Thus, $|(H \cdot T)_k| = (p - 1) + (p - 1)p = (p - 1)(p + 1)$. □

The next two results use the orbit-stabilizer theorem to determine how many (ϕ, k)- (and (θ, k)) -split maximal tori are H-conjugate to T for both $H = H_\phi$ and $H = H_\theta$.

Lemma 4.2.8. *Let $H = H_\phi$.*

1. *The number of (ϕ, k)-split maximal tori that are H-conjugate to T is equal to $(p - 1)/4$ for $p \cong 1$ mod 4.*

2. *The number of (ϕ, k)-split maximal tori that are H-conjugate to T is equal to $(p - 1)/2$ for $p \cong 3$ mod 4.*

Proof. Recall, $|H| = p - 1$. H acts on the set $\mathcal{A}_\phi$ by conjugation. We know $N_G(T)$ stabilizes T, so $H \cap N_G(T) = \{\pm \text{id}, \pm w\}$, where w is the Weyl element, are the elements of H that stabilize T for $p \cong 1 \mod 4$; and $H \cap N_G(T) = \{\pm \text{id}\}$, are the elements of H that stabilize T for $p \cong 3 \mod 4$. Hence, The number of (ϕ, k)-split maximal tori that are H-conjugate to T is equal to $(p - 1)/4$ for $p \cong 1 \mod 4$ and $(p - 1)/2$ for $p \cong 3 \mod 4$. □

Lemma 4.2.9. *Let $H = H_\theta$.*

1. *The number of (θ, k)-split maximal tori that are H-conjugate to T is equal to $(p + 1)/2$ for $p \cong 1$ mod 4.*

2. *The number of (θ, k)-split maximal tori that are H-conjugate to T is equal to $(p + 1)/4$ for $p \cong 3$ mod 4.*

Proof. Recall, $|H| = p+1$. H acts on the set $\mathcal{A}_\phi$ by conjugation. We know $N_G(T)$ stabilizes T, so $H \cap N_G(T) = \{\pm \mathrm{id}, \pm w\}$, where w is the Weyl element, are the elements of H that stabilize T for $p \cong 3 \mod 4$; and $H \cap N_G(T) = \{\pm \mathrm{id}\}$, are the elements of H that stabilize T for $p \cong 1 \mod 4$. Hence, The number of (ϕ, k)-split maximal tori that are H-conjugate to T is equal to $(p+1)/2$ for $p \cong 1 \mod 4$ and $(p+1)/4$ for $p \cong 3 \mod 4$. □

Next, we need to know how many elements of $(H \cdot T)_k$ correspond to T for both fixed point groups.

Lemma 4.2.10. *The number of elements of $(H \cdot T)_k$ corresponding to T is equal to (i.e., the number of $z \in (H \cdot T)_k$ such that $zTz^{-1} = T$ is equal to) $2(p-1)$ for both $H = H_\phi$ and $H = H_\theta$.*

Proof. From lemma 5.9,

$$(H_\phi \cdot T)_k = \left\{ \begin{pmatrix} i & jy \\ j & iy \end{pmatrix} | i, j \in \mathbb{F}_p, y = \frac{1}{i^2 - j^2} \text{ with } i \neq \pm j \right\}$$

$$\text{and } (H_\theta \cdot T)_k = \left\{ \begin{pmatrix} i & jyS_p^{-1} \\ j & iy \end{pmatrix} | i, j \in \mathbb{F}_p, y = \frac{1}{i^2 - j^2 S_p^{-1}} \right\}.$$

First, let $z = \begin{pmatrix} i & jy \\ j & iy \end{pmatrix} \in (H_\phi \cdot T)k$. Then, $zTz^{-1} = T$
$\iff \begin{pmatrix} i & jy \\ j & iy \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \begin{pmatrix} iy & -jy \\ -j & i \end{pmatrix} = \begin{pmatrix} w & 0 \\ 0 & w^{-1} \end{pmatrix}$ for all $x \in \mathbb{F}_p^*$ and some $w \in \mathbb{F}_p^* \iff \begin{pmatrix} i^2xy - j^2x^{-1}y & -ijxy + ijx^{-1}y \\ ijxy - ijx^{-1}y & -j^2xy + i^2x^{-1}y \end{pmatrix} = \begin{pmatrix} w & 0 \\ 0 & w^{-1} \end{pmatrix}$ for all $x \in \mathbb{F}_p^*$ and some $w \in \mathbb{F}_p^* \iff i^2xy - j^2x^{-1}y = (-j^2xy + i^2x^{-1}y)^{-1}$ and $ijxy = ijx^{-1}y$ for all $x \in \mathbb{F}_p^* \iff ijx = ijx^{-1}$ for all $x \in \mathbb{F}_p^* \iff ij = 0 \iff i = 0$ or $j = 0$. If $i = 0 \Rightarrow -j^2y = 1 \Rightarrow -1 = j^2y = j^2(i^2 - j^2)^{-1} = j^2(j^2)^{-1} = -1$. Thus, all matrices of the form $\begin{pmatrix} 0 & jy \\ j & 0 \end{pmatrix}$ stabilize T. There are $(p-1)$ of them. If $j = 0 \Rightarrow i^2y = 1 \Rightarrow 1 = i^2(i^2 - j^2)^{-1} = i^2(i^2)^{-1} = 1$. Thus, all matrices of the form

$\begin{pmatrix} i & 0 \\ 0 & iy \end{pmatrix}$ stabilize T. There are $(p-1)$ of them. Therefore, $2(p-1)$ matrices of $(H_\phi \cdot T)_k$ stabilize T.

Now, let $z = \begin{pmatrix} i & jyS_p^{-1} \\ j & iy \end{pmatrix} \in (H_\theta \cdot T)_k$. Then a similar calculation to above gives that $zTz^{-1} = T \iff i^2xy - j^2x^{-1}yS_p^{-1} = 1/(-j^2xyS_p^{-1} + i^2x^{-1}y)$ and $ijxy = ijx^{-1}y$ for all $x \in \mathbb{F}_p^* \iff ijx = ijx^{-1}$ for all $x \in \mathbb{F}_p^* \iff ij = 0 \iff i = 0$ or $j = 0$. If $i = 0$, then $-j^2yS_p^{-1} = 1 \Rightarrow -1 = j^2S_p^{-1}(1/(i^2 - j^2S_p^{-1})) = -1$. Thus all matrices of the form $\begin{pmatrix} 0 & jyS_p^{-1} \\ j & 0 \end{pmatrix}$ stabilize T. There are $(p-1)$ of them. If $j = 0 \Rightarrow i^2y = 1 \Rightarrow 1 = i^2(1/(i^2 - j^2S_p^{-1})) = 1$. Thus, all matrices of the form $\begin{pmatrix} i & 0 \\ 0 & iy \end{pmatrix}$ stabilize T. There are $(p-1)$ of them. Therefore, $2(p-1)$ matrices of $(H_\theta \cdot T)_k$ that stabilize T. □

Finally, the next two results give us the number of H-conjugacy classes of (ϕ, k)-split maximal tori and (θ, k)-split maximal tori.

Lemma 4.2.11. *Let $H = H_\phi$. The number of H-conjugacy classes of (ϕ, k)-split maximal tori is equal to*

1. 2 *for* $p \cong 1 \mod 4$.
2. 1 *for* $p \cong 3 \mod 4$.

Proof. Recall from Lemma 5.9, $|(H \cdot T)_k| = (p-1)^2$, and by Lemma 5.12, $2(p-1)$ elements of $(H \cdot T)_k$ correspond to T.

(1) By Lemma 5.10, four elements of H stabilize T. Thus, there are $2(p-1)/4 = (p-1)/2$ H-conjugacy classes of T in $(H \cdot T)_k$. Hence, $(p-1)/2 * |H| = (p-1)^2/2$ elements of $(H \cdot T)_k$ (half of the elements) correspond to maximal (ϕ, k)-split tori that are H-conjugate to T.

Let $T_1 \in \mathcal{A}_\phi$ be not H-conjugate to T, and let $z = \begin{pmatrix} k & ly \\ l & ky \end{pmatrix} \in (H \cdot T)_k$ such that $zTz^{-1} = T_1$, where $y = (k^2 - l^2)^{-1}$. Note that $k \neq 0$ and $l \neq 0$ by lemma 5.12.

Let $M = \{y_1, \ldots, y_{2(p-1)}\}$ be the $2(p-1)$ distinct elements of $(H \cdot T)_k$ corresponding to T, ie, $y_i T y_i^{-1} = T$ for all $y_i \in M$. Recall from lemma 5.12 that for $y_i \in M$, $y_i = \begin{pmatrix} i & 0 \\ 0 & i^{-1} \end{pmatrix}$ or $y_i = \begin{pmatrix} 0 & -j^{-1} \\ j & 0 \end{pmatrix}$. Now, $zy_i = \begin{pmatrix} ki & lyi^{-1} \\ li & kyi^{-1} \end{pmatrix} = \begin{pmatrix} a & bw \\ b & aw \end{pmatrix} \in (H \cdot T)_k$, where $a = ki$, $b = li$, and $w = (a^2 - b^2)^{-1}$; or $zy_i = \begin{pmatrix} lyj & -kj^{-1} \\ kyj & -lj^{-1} \end{pmatrix} = \begin{pmatrix} a & bw \\ b & aw \end{pmatrix} \in (H \cdot T)_k$, where $a = lyj$, $b = kyj$, and $w = (a^2 - b^2)^{-1}$. Hence, the set $zM = \{zy_1, \ldots, zy_{2(p-1)}\} \subset (H \cdot T)_k$, and clearly corresponds to T_1. Thus, $2(p-1)$ elements of $(H \cdot T)_k$ correspond to T_1. Since T_1 was arbitrary, $2(p-1)$ elements of $(H \cdot T)_k$ correspond to each maximal (ϕ, k)-split torus not H-conjugate to T. Hence, there are $[(p-1)^2/2] * [1/2(p-1)] = (p-1)/4$ maximal (ϕ, k)-split tori not H-conjugate to T.

Since $|H| = p-1$, at most four elements of H stabilize T_1. Clearly, id, $(p-1) * \text{id} \in H$ stabilize T_1. Now, by calculating,

$$T_1 = zTz^{-1} = \left\{ \begin{pmatrix} k^2xy - l^2x^{-1}y & -klxy + klx^{-1}y \\ klxy - klx^{-1}y & -l^2xy + k^2x{-}1y \end{pmatrix} | k, l, x \in \mathbb{F}_p^* \right\}.$$

Two simple calculations show that $\pm w = \pm \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix} \in H$ stabilize T_1 as well. Hence, all maximal (ϕ, k)-split tori not H-conjugate to T are H-conjugate to each other. Therefore, there are two H-conjugacy classes of maximal (ϕ, k)-split tori.

(2) By Lemma 5.10, two elements of H stabilize T. Thus, there are $2(p-1)/2 = (p-1)$ H-conjugacy classes of T in $(H \cdot T)_k$. Hence, all of the $(p-1) * |H| = (p-1)^2$ elements of $(H \cdot T)_k$ correspond to maximal (ϕ, k)-split tori that are H-conjugate to T. Therefore, there is just one H-conjugacy class of maximal (ϕ, k)-split tori. □

Lemma 4.2.12. *Let $H = H_\theta$. The number of H-conjugacy classes of (θ, k)-split maximal tori is equal to*

1. 1 *for* $p \cong 1 \mod 4$

2. 2 *for* $p \cong 3 \mod 4$

Proof. Recall from Lemma 5.9, $|(H\cdot T)_k| = (p-1)(p+1)$, and by Lemma 5.12, $2(p-1)$ elements of $(H\cdot T)_k$ correspond to T.

(1) By Lemma 5.11, for $p \cong 1 \mod 4$ two elements of H stabilize T. Thus, there are $2(p-1)/2 = (p-1)$ H-conjugacy classes of T in $(H\cdot T)_k$. Hence, all of the $(p-1) * |H| = (p-1)(p+1)$ elements of $(H\cdot T)_k$ correspond to maximal (ϕ, k)-split tori that are H-conjugate to T. Therefore, there is just one H-conjugacy class of maximal (ϕ, k)-split tori.

(2) In the case $p \cong 3 \mod 4$ $S_p = -1$. By Lemma 5.10, four elements of H stabilize T. Thus, there are $2(p-1)/4 = (p-1)/2$ H-conjugacy classes of T in $(H\cdot T)_k$. Hence, $(p-1)/2 * |H| = (p-1)(p+1)/2$ elements of $(H\cdot T)_k$ (half the elements) correspond to maximal (ϕ, k)-split tori that are H-conjugate to T.

Let $T_1 \in \mathcal{A}_\phi$ be not H-conjugate to T, and let $z = \begin{pmatrix} k & -ly \\ l & ky \end{pmatrix} \in (H\cdot T)_k$ such that $zTz^{-1} = T_1$, where $y = (k^2+l^2)^{-1}$. Note that $k \neq 0$ and $l \neq 0$ by lemma 5.12. Let $M = \{y_1, \dots, y_{2(p-1)}\}$ be the $2(p-1)$ distinct elements of $(H\cdot T)_k$ corresponding to T, ie, $y_i T y_i^{-1} = T$ for all $y_i \in M$. Recall from lemma 5.12 that for $y_i \in M$, $y_i = \begin{pmatrix} i & 0 \\ 0 & i^{-1} \end{pmatrix}$ or $y_i = \begin{pmatrix} 0 & -j^{-1} \\ j & 0 \end{pmatrix}$. Now, $zy_i = \begin{pmatrix} ki & -lyi^{-1} \\ li & kyi^{-1} \end{pmatrix} = \begin{pmatrix} a & -bw \\ b & aw \end{pmatrix} \in (H\cdot T)_k$, where $a = ki$, $b = li$, and $w = (a^2-b^2)^{-1}$; or $zy_i = \begin{pmatrix} -lyj & -kj^{-1} \\ kyj & -lj^{-1} \end{pmatrix} = \begin{pmatrix} a & -bw \\ b & aw \end{pmatrix} \in (H\cdot T)_k$, where $a = -lyj$, $b = kyj$, and $w = (a^2+b^2)^{-1}$. Hence, the set $zM = \{zy_1, \dots, zy_{2(p-1)}\} \subset (H\cdot T)_k$, and clearly corresponds to T_1. Thus, $2(p-1)$ elements of $(H\cdot T)_k$ correspond to T_1. Since T_1 was arbitrary, $2(p-1)$ elements of $(H\cdot T)_k$ correspond to each maximal (ϕ, k)-split torus not H-conjugate to T. Hence, there are $[(p-1)(p+1)/2] * [1/2(p-1)] = (p+1)/4$ maximal (ϕ, k)-split tori not H-conjugate to T.

Since $|H| = p+1$, at most four elements of H stabilize T_1. Clearly, id, $(p-1) * \text{id} \in H$ stabilize T_1. Now, by calculating,

$$T_1 = zTz^{-1} = \left\{ \begin{pmatrix} k^2xy + l^2x^{-1}y & klxy - klx^{-1}y \\ klxy - klx^{-1}y & l^2xy + k^2x{-}1y \end{pmatrix} | k, l, x \in \mathbb{F}_p^* \right\}.$$

Two simple calculations show that $\pm w = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in H$ stabilize T_1 as well. Hence, all maximal (ϕ, k)-split tori not H-conjugate to T are H-conjugate to each other. Therefore, there are two H-conjugacy classes of maximal (ϕ, k)-split tori. □

Corollary 4.2.13. *Let T_1 be the representative of the H-conjugacy class of ϕ-stable (or θ-stable) maximal k-split tori when such a class exists. Then, $W_G(T_1) = W_G(T)$ when T_1 exists.*

Proof. Result follows from the proofs of Lemmas 4.2.11 and 4.2.12, where the Weyl element w stabilizes T_1. □

We now have all the information we need to calculate $B_k \backslash G_k / H_k$ in the next two theorems.

Theorem 4.2.14. *For $H = H_\phi$, $|B_k \backslash G_k / H_k| = 4$.*

Proof. (1) For $p \cong 3 \mod 4$, the H-conjugacy class representative of ϕ-stable maximal k-split tori are T and H by Lemmas 4.2.4 and 4.2.11. By Lemma 4.2.5, the Weyl element $w \notin H$. Hence, $W_H(T) = \{\text{id}\}$. Thus, $|B_k \backslash G_k / H_k| = 4$.

(2) For $p \cong 1 \mod 4$, the H-conjugacy class representatives of ϕ-stable maximal k-split tori are H, T, and T_1 by Lemmas 4.2.4 and 4.2.11. By Lemma 4.2.5, the Weyl element $w \in H$. Hence, $W_H(T) = W_G(T)$ and $W_H(T_1) = W_G(T_1)$. Also, $W_H(H) = \{\text{id}\}$. Thus, $|B_k \backslash G_k / H_k| = 4$.

□

Theorem 4.2.15. *(B) For $H = H_\theta$, $|B_k \backslash G_k / H_k| = 2$ for all $p \neq 2$.*

Proof. For $p \cong 1 \mod 4$, the H-conjugacy class representative of ϕ-stable maximal k-split tori is just T by Lemmas 4.2.4 and 4.2.12. By Lemma 4.2.5, the Weyl element $w \notin H$. Hence, $W_H(T) = \{\text{id}\}$. Thus, $|B_k \backslash G_k / H_k| = 2$.

For $p \cong 3 \mod 4$, the H-conjugacy class representatives of ϕ-stable maximal k-split tori are T and T_1 by Lemmas 4.2.4 and 4.2.12. By Lemma 4.2.5, the Weyl element $w \in H$. Hence, $W_H(T) = W_G(T)$ and $W_H(T_1) = W_G(T_1)$. Thus, $|B_k \backslash G_k / H_k| = 2$. □

Chapter 5

Classification Results for SL(2, k)

In this chapter let $m \in k^*/(k^*)^2$ and $\theta = \text{Int}\begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$. Let H be the fixed point group of θ. Then,

$$H = \left\{ \begin{pmatrix} a & b \\ mb & a \end{pmatrix} | a^2 - mb^2 = 1,\ a, b \in k \right\}.$$

Let T and B be as previously defined, and let $\mathcal{A}_\theta$ be the set of maximal (θ, k)-split tori. These definitions will remain fixed throughout the rest of this Chapter. Denote the subspace of k^n spanned by $v_1, \ldots, v_r$ by $< v_1, \ldots, v_r >$. For $A \in \text{GL}_2(k)$ we will also write $|A|$ for $\det(A)$.

Recall from the previous chapter that in SL$(2, k)$ all θ-stable maximal k-split tori T_i are one-dimensional. Hence, $T_i = T_i^+$ or $T_i = T_i^-$.

5.1 Maximal k-split Tori contained in H_k

First, we will determine when H_k is a maximal k-split torus. When H_k is maximal k-split, it will be the representative of one of the H_k-conjugacy classes of θ-stable maximal k-split tori.

Theorem 5.1.1. *H_k is k-anisotropic if and only if $m \notin \overline{1}$. If $m \in \overline{1}$, then H_k is a maximal k-split torus.*

Proof. To diagonalize H_k, consider $h = \begin{pmatrix} a & b \\ mb & a \end{pmatrix} \in H_k$, and

$$|h - \lambda \,\mathrm{Id}\,| = \begin{vmatrix} a-\lambda & b \\ mb & a-\lambda \end{vmatrix} = \lambda^2 - 2a\lambda + 1.$$

So, $\lambda = a \pm \sqrt{a^2-1} = a \pm b\sqrt{m}$, since $a^2 - mb^2 = 1$. Therefore, the eigenvalues are in k only when $m \in \overline{1}$. Since m is a square, and all involutions θ with $m \in \overline{1}$ act the same as the involution θ with $m = 1$, we may assume $m = 1$. Thus, the $(a+b)$-eigenspace is $< (1,1) >$, and the $(a-b)$-eigenspace is $< (-1,1) >$. Let $P = \begin{pmatrix} \frac{1}{2} & -1 \\ \frac{1}{2} & 1 \end{pmatrix} \in \mathrm{SL}_2(k)$. Then $P^{-1}H_kP \subset T$. □

5.2 H_k-conjugacy classes of (θ, k)-split maximal tori-Approach 1

Next, we are looking for the H_k-conjugacy classes of (θ,k)-split maximal tori in the set $\mathcal{A}_\theta$. Recall from previous discussion of Theorem 2.3.2, there exists $g \in (H \cdot T)_k$ such that $gTg^{-1} = T_i$, where T_i is a (θ,k)-split maximal torus.

Consider the following: H_k acts on $(H \cdot T)_k$ by left multiplication. Each element of $(H \cdot T)_k$ corresponds to a (θ,k)-split maximal torus by conjugating T. So, H_k acts on the sets $\mathcal{A}_\theta$ by acting on $(H \cdot T)_k$. By grouping $(H \cdot T)_k$ into left H_k-cosets, we are grouping $\mathcal{A}_\theta$ into H_k-conjugacy classes at the same time.

Lemma 5.2.1.

$$(H \cdot T)_k = \left\{ \begin{pmatrix} i & jym^{-1} \\ j & iy \end{pmatrix} \mid i,j \in k, \textit{ both not equal to 0, } y = \frac{1}{i^2 - j^2m^{-1}} \right\}.$$

If $m \in \overline{1}$, *then* $i \neq \pm j\sqrt{m^{-1}}$.

Proof. Let $h = \begin{pmatrix} a & b \\ mb & a \end{pmatrix} \in H$ and $t = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \in T$. So

$$(H \cdot T)_k = \left\{ \begin{pmatrix} ax & bx^{-1} \\ mbx & ax^{-1} \end{pmatrix} \mid a^2 - mb^2 = 1, a,b,x \in \overline{k}, ax, bx, ax^{-1}, bx^{-1} \in k \right\}.$$

We can rewrite the set as follows: Let $ax = i \in k$ and $mbx = j \in k$. Then $a = ix^{-1}$ and $b = jx^{-1}m^{-1}$, so $1 = a^2 - mb^2 = \frac{i^2 - j^2m^{-1}}{x^2}$. Hence, $x^2 = i^2 - j^2m^{-1} \Rightarrow x^{-2} = \frac{1}{i^2 - j^2m^{-1}}$. Also, $ax^{-1} = ix^{-2}$ and $bx^{-1} = jx^{-2}m^{-1}$. So,

$$(H \cdot T)_k = \left\{ \begin{pmatrix} i & jym^{-1} \\ j & iy \end{pmatrix} \mid i, j \in k,\ y = \frac{1}{i^2 - j^2m^{-1}} \text{ with } i^2 - j^2m^{-1} \neq 0 \right\},$$

where clearly both i and j are not equal to zero by definition of H and T. Notice that $i^2 - j^2m^{-1} = 0 \to i^2 = j^2m^{-1}$, which is only possible if $m \in \overline{1}$. Thus if $m \in \overline{1}$, we need the condition that $i \neq \pm j\sqrt{m^{-1}}$. □

Remark 5.2.2. In the set

$$(H \cdot T)_k = \left\{ \begin{pmatrix} i & jym^{-1} \\ j & iy \end{pmatrix} \mid i, j \in k,\ y = \frac{1}{i^2 - j^2m^{-1}} \text{ with } i^2 - j^2m^{-1} \neq 0 \right\},$$

the y term is the key element that will be used to break $(H \cdot T)_k$ into left H_k-cosets, which in turn, breaks $\mathcal{A}_\theta$ into H_k-conjugacy classes.

Lemma 5.2.3.

$$(H \cdot T)_k \cap N_G(T) = N_{G_k}(T) = \left\{ \begin{pmatrix} 0 & -j^{-1} \\ j & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & i^{-1} \end{pmatrix} \mid i, j \in k^* \right\}.$$

Proof. First, let $z \in (H \cdot T)_k$. Then, $zTz^{-1} = T \iff$

$$z \in N_{G_k}(T) = \left\{ \begin{pmatrix} 0 & -j^{-1} \\ j & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & i^{-1} \end{pmatrix} \mid i, j \in k^* \right\}.$$

By definition of $(H \cdot T)_k$, $N_{G_k}(T) \subset (H \cdot T)_k$ by setting either i or j equal to 0 in the previous lemma. □

Lemma 5.2.4. *Let* $z = \begin{pmatrix} k & lym^{-1} \\ l & ky \end{pmatrix} \in (H \cdot T)_k$ *and let* $y_i \in N_{G_k}(T)$. *Then* $zy_i \in (H \cdot T)_k$.

Proof. By calculating, we get

$$zy_i = \begin{pmatrix} ki & lym^{-1}i^{-1} \\ li & kyi^{-1} \end{pmatrix} = \begin{pmatrix} a & bwm^{-1} \\ b & aw \end{pmatrix} \in (H \cdot T)_k,$$

where $a = ki$, $b = li \in k$, and $w = (a^2 - b^2m^{-1})^{-1}$; or

$$zy_i = \begin{pmatrix} lyim^{-1} & -ki^{-1} \\ kyi & -li^{-1} \end{pmatrix} = \begin{pmatrix} a & bwm^{-1} \\ b & aw \end{pmatrix} \in (H \cdot T)_k,$$

where $a = lyim^{-1}$, $b = kyi \in k$, and $w = (a^2 - b^2m^{-1})^{-1}$. □

Lemma 5.2.5. *1. Let $T_1 \in \mathcal{A}_\theta$ and $z_1 = \begin{pmatrix} k & lym^{-1} \\ l & ky \end{pmatrix} \in (H \cdot T)_k$ such that $z_1Tz_1^{-1} = T_1$ and $y = (k^2 - l^2m^{-1})^{-1}$. Let $z_s = \begin{pmatrix} i & jwm^{-1} \\ j & iw \end{pmatrix} \in (H \cdot T)_k$ such that $z_sTz_s^{-1} = T_1$ and $w = (i^2 - j^2m^{-1})^{-1}$. Then $w \in \overline{y}$ or $w \in \overline{-my}$.*

2. Moreover, for every $s \in k^*$ there exists a $z_s = \begin{pmatrix} i & jwm^{-1} \\ j & iw \end{pmatrix} \in (H \cdot T)_k$ such that $z_sTz_s^{-1} = T_1$ and $w = s^2y$, and if $w \in \overline{-my}$, then there exists another $z_s \in (H \cdot T)_k$ such that $z_sTz_s^{-1} = T_1$ and $w = -mys^2$.

Proof. Since $z_1Tz_1^{-1} = T_1$ and $z_sTz_s^{-1} = T_1$, we have that $z_1Tz_1^{-1} = z_sTz_s^{-1}$, and so $T = z_1^{-1}z_sTz_s^{-1}z_1$. Hence, $z_1^{-1}z_s \in N_{G_k}(T) \Rightarrow z_1^{-1}z_s = y_s$ where $y_s \in N_{G_k}(T)$. Therefore, $z_s = z_1y_s$.

Let $y_s = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}$. Then by Lemma 5.2.3, $y_sTy_s^{-1} = T \Rightarrow z_1y_sTy_s^{-1}z_1^{-1} = T_1$. By Lemma 5.2.4, $z_s = z_1y_s \in (H \cdot T)_k$ and $z_s = z_1y_s = \begin{pmatrix} ks & lswm^{-1} \\ ls & ksw \end{pmatrix}$ with characteristic $w = ys^{-2} \in \overline{y}$, giving us (1). Since this is true for all $s \in k^*$, we obtain (2).

Now, let $y_s = \begin{pmatrix} 0 & -s^{-1} \\ s & 0 \end{pmatrix}$. Then by Lemma 5.2.3, $y_sTy_s^{-1} = T \Rightarrow z_1y_sTy_s^{-1}z_1^{-1} = T_1$. By Lemma 5.2.4, $z_s = z_1y_s \in (H \cdot T)_k$ and $z_s = z_1y_s = \begin{pmatrix} lysm^{-1} & kyswm^{-1} \\ kys & lysm^{-1}w \end{pmatrix}$ with characteristic $w = -y^{-1}s^{-2}m \in \overline{-my}$, giving us (1). Since this is true for all $s \in k^*$, we obtain (2). □

Remark 5.2.6. This lemma states for every $s \in k^*$ there exists a

$$z_s = \begin{pmatrix} i & jwm^{-1} \\ j & iw \end{pmatrix} \in (H \cdot T)_k$$

such that $z_s T z_s^{-1} = T_1$ and $w = s^2 y$, and another z_s (same conditions) with $w = -mys^2$. Note that $w = s^2 y = (i^2 - j^2 m^{-1})^{-1}$ can have more than one solution. Same is true for $w = -mys^2 = (i^2 - j^2 m^{-1})^{-1}$. Not all matrices of the form $z_s = \begin{pmatrix} i & jwm^{-1} \\ j & iw \end{pmatrix} \in (H \cdot T)_k$ with $w = s^2 y$ or $w = -mys^2$ correspond to T_1. Rather, we have the following result.

Theorem 5.2.7. *Let $q \in k^*/(k^*)^2$, $T_1 \in \mathcal{A}_\theta$, and $z_1 = \begin{pmatrix} k & lym^{-1} \\ l & ky \end{pmatrix} \in (H \cdot T)_k$ such that $z_1 T z_1^{-1} = T_1$ and $y \in \overline{q}$. Then the set of matrices in $(H \cdot T)_k$ that correspond to (θ, k)-split maximal tori that are H_k-conjugate to T_1 is precisely*

$$\left\{ \begin{pmatrix} i & jwm^{-1} \\ j & iw \end{pmatrix} \mid w \in \overline{q} \text{ or } w \in \overline{-mq} \text{ with } i, j \in k \text{ and } w = (i^2 - j^2 m^{-1})^{-1} \neq 0 \right\}.$$

Proof. Let $z_2 = \begin{pmatrix} i & jwm^{-1} \\ j & iw \end{pmatrix} \in (H \cdot T)_k$ such that $z_2 T z_2^{-1} = T_1$. Then $w \in \overline{q}$ or $w \in \overline{-mq}$ by Lemma 5.2.5. Let $h = \begin{pmatrix} a & b \\ mb & a \end{pmatrix} \in H_k$. Then $hz_2 T z_2^{-1} h^{-1} = hT_1 h^{-1}$, so hz_2 corresponds to a (θ, k)-split maximal torus that is H_k-conjugate to T_1. Now,

$$hz_2 = \begin{pmatrix} a & b \\ mb & a \end{pmatrix} \begin{pmatrix} i & jwm^{-1} \\ j & iw \end{pmatrix} = \begin{pmatrix} ai + bj & (aj + bim)wm^{-1} \\ aj + bim & (ai + bj)w \end{pmatrix} \in (H \cdot T)_k$$

since $[(ai + bj)^2 - (aj + mbi)^2 m^{-1}]^{-1} = w$. Hence, hz_2 has characteristic $w \in \overline{q}$ or $w \in \overline{-mq}$, and therefore, any matrix that corresponds to a (θ, k)-split maximal torus that is H_k-conjugate to T_1 has characteristic $w \in \overline{q}$ or $w \in \overline{-mq}$.

Now, let $z_3 = \begin{pmatrix} c & dvm^{-1} \\ d & cv \end{pmatrix} \in (H \cdot T)_k$ with $v = (c^2 - d^2 m^{-1})^{-1} \in \overline{q}$ or $\in \overline{-mq}$. Let z_2 be as above, and suppose the characteristic w of z_2 equals the characteristic v of z_3. Such a z_2 exists by Lemma 5.2.5. The terms i, j, m, c, d, v, and w are fixed. In order to show that z_3 corresponds to a (θ, k)-split maximal torus that is H_k-conjugate to T_1, we have that $z_3 = hz_2$ for some $h \in H_k \iff \begin{pmatrix} c & dvm^{-1} \\ d & cv \end{pmatrix} = \begin{pmatrix} ai+bj & (aj+bim)wm^{-1} \\ aj+bim & (ai+bj)w \end{pmatrix} \iff c = ai+bj,\ d = aj+mbi, \iff c/(ai+bj) = d/(aj+mbi) \iff a = b(dj-cmi)/(cj-di)$. We know $a^2 - mb^2 = 1$, so by substituting, we have $1 = b^2(dj-cmi)^2/(cj-di)^2 - b^2m \Rightarrow b = \pm\sqrt{\frac{1}{(dj-cmi)^2/(cj-di)^2-m}}$. Such a b exists since $(dj-cmi)^2/(cj-di)^2 - m = m^2w^2/(cj-di)^2$ is a square in k, and if $(dj-cmi)^2/(cj-di)^2 = m \Rightarrow (dj-cmi)/(cj-di) = \pm\sqrt{m}$, which could only be possible for $m \in \overline{1}$ by definition of m. So if $m \in \overline{1}$, then we have $dj - ci = \pm\sqrt{m}(cj-di) \Rightarrow c = \pm d\sqrt{m^{-1}}$, a contradiction. Thus, all $\begin{pmatrix} c & dvm^{-1} \\ d & cv \end{pmatrix} \in (H \cdot T)_k$ with $v = (c^2 - d^2m^{-1})^{-1} \in \overline{q}$ or $\in \overline{-mq}$ correspond to a (θ, k)-split maximal torus that is H_k-conjugate to T_1. □

Theorem 5.2.8. *Let $G_k = \mathrm{SL}(2,k)$, and let $\theta = \mathrm{Int}\begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$.*

1. *Let $U = \{q \in k^*/(k^*)^2 \mid x_1^2 - m^{-1}x_2^2 = q^{-1}$ has a solution in $k\}$. Then the number of H_k-conjugacy classes of (θ, k)-split maximal tori is $|U/\{1, -m\}|$.*
2. *For $y \in U$, let $r, s \in k$ such that $r^2 - m^{-1}s^2 = y^{-1}$ and let $g = \begin{pmatrix} r & sym^{-1} \\ s & ry \end{pmatrix}$. Then $\{T_y = g^{-1}Tg \mid y \in U\}$ is a set of representatives of the H_k-conjugacy classes of maximal (θ, k)-split tori in G_k.*

Proof. (1) By previous theorem, we can split the H_k-conjugacy classes of $\mathcal{A}_\theta$ according to the square classes, ie, $\overline{q}$ and $\overline{-mq}$ correspond to the same H_k-conjugacy class. Thus, all we need to do is figure out what kind of y we can obtain, where $\begin{pmatrix} k & lym^{-1} \\ l & ky \end{pmatrix} \in (H \cdot T)_k$ with $y = (k^2 - l^2m^{-1})^{-1}$. This is the same as asking if $x_1^2 - m^{-1}x_2^2 = q^{-1}$ has a solution for each $q \in k^*/(k^*)^2$, by letting $y = q$, $k = x_1$, and $l = x_2$. Thus, we get the set U defined above. Since $\overline{q}$ and $\overline{-mq}$ correspond to the same H_k-conjugacy class, we need to look at the cosets $U/\{1, -m\}$ to obtain an accurate count of the H_k-conjugacy classes. (2) follows immediately from (1). □

5.3 H_k-conjugacy classes of (θ, k)-split maximal tori-Approach 2

Since k is a field of characteristic $\neq 2$, a maximal torus T is the centralizer of its Lie algebra $\mathfrak{t}$. T is θ-split if and only if θ acts on $\mathfrak{t}$ as -1. Then,

$$\begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & m^{-1} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} \iff d = -a,\ c = -mb.$$

So, if T is defined over k, $\mathfrak{t}(k)$ is spanned by a semisimple element of the form

$$X = \begin{pmatrix} a & b \\ -mb & -a \end{pmatrix}$$

with $-(a^2 - mb^2) \neq 0$. Moreover, T is k-split if and only if $-(a^2 - mb^2) \in (k^*)^2$. In fact, since $-(a^2 - mb^2) \in (k^*)^2$, we can say $\mathfrak{t}(k)$ is spanned by a semisimple element of the form

$$X = \begin{pmatrix} x & y \\ -my & -x \end{pmatrix} \tag{5.1}$$

with $-(x^2 - my^2) = 1$.

Now, let $l = k[\sqrt{m}]$, a quadratic extension over k. Let $\sigma : l \to l$ be the automorphism $\sigma(a + b\sqrt{m}) = a - b\sqrt{m}$. Let $N : l \to k$ be the norm map $N(a + b\sqrt{m}) = (a + b\sqrt{m})\sigma(a + b\sqrt{m}) = (a + b\sqrt{m})(a - b\sqrt{m}) = a^2 - mb^2$. Let l_1 be the kernel of N. Then $l_1 = \{a + b\sqrt{m} \mid a, b \in k,\ a^2 - mb^2 = 1\}$. Let X be as in 5.1, and let $h = \begin{pmatrix} a & b \\ mb & a \end{pmatrix} \in H$. Define $\lambda(X) = x + y\sqrt{m}$ and $\lambda(h) = a + b\sqrt{m}$. This map leads to the following results.

Lemma 5.3.1. *1. $H \simeq l_1$*

2. $\mathcal{A}_\theta \simeq l_1/(\pm 1)$

Proof. (1) $\lambda : H \to l_1$ is an isomorphism by definitions of H and l_1. (2) Let $T \in \mathcal{A}_\theta$, where the Lie algebra $\mathfrak{t}(k)$ is spanned by X as in 5.1. Then, $x^2 - my^2 = -1 \Rightarrow \lambda : X \to z * \ker(N) = z * l_1$, where z is such that $N(z) = -1$. Notice that $-X \in \mathfrak{t}(k)$.

Let $\Sigma = \{X \mid \text{for all } T \in \mathcal{A}_\theta \text{ the Lie algebra } \mathfrak{t}(k) = \text{span}(X)\}$. This gives that $\lambda : \Sigma \to z * l_1/(\pm 1)$ is an isomorphism. Thus, $z * l_1/(\pm 1) \simeq \mathcal{A}_\theta$, and since $z * l_1 \simeq l_1$, we obtain the above result. □

Lemma 5.3.2. *1. $\lambda(hXh^{-1}) = \sigma(\lambda(h)^2)\lambda(X)$.*

2. There is a bijection from $\mathcal{A}_\theta/H$ onto $l_1/(\pm(l_1)^2)$.

Proof. (1) Straightforward calculation. (2) From Lemma 5.3.1 and the equality in (1), $H\mathcal{A}_\theta H^{-1} \simeq (l_1)^2(l_1/(\pm 1))$. The isomorphism of cosets follows immediately. □

Theorem 5.3.3. *The number of H_k-conjugacy classes of (θ, k)-split maximal tori is equal to*

$$|Im(N)/(k^*)^2\{1, -m\}|.$$

Proof. We have an exact sequence

$$1 \to k^* \to l^* \to l_1 \to 1,$$

where $\alpha : l^* \to l_1$ is defined by $\alpha(x) = \sigma(x)x^{-1}$. Then, $\alpha(l_1) = l_1^2$ since for all $x \in l_1$, $x^{-1} = \sigma(x)$. Also, $\alpha(\sqrt{m}) = -1$. It follows that $l_1/(\pm l_1^2) \simeq l^*/k^*l_1\{1, \sqrt{m}\}$. By applying N, we obtain $l_1/(\pm l_1^2) \simeq \text{Im}(N)/(k^*)^2\{1, -m\}$. Combining with Lemma 5.3.2 we obtain the desired result. □

Remark 5.3.4. Theorem 5.3.3 is an equivalent way of stating Theorem 5.2.8. Below, we will use both descriptions to give the count of H_k-conjugacy classes of (θ, k)-split tori over the $\mathfrak{p}$-adic numbers. Theorem 5.3.3 makes the counting the number of conjugacy classes easier; however, the description given in Theorem 5.2.8 is needed to find the Weyl elements of the representative (θ, k)-split maximal tori with representatives in G and H, which is necessary to obtain $B_k\backslash G_k/H_k$. Using Theorem 5.2.8 is necessary since the Weyl groups of the Lie group and its Lie algebra are not always the same.

5.4 Weyl element representatives in H

The following results will be needed to complete the classification of $B_k \backslash G_k / H_k$ and the classification of the H_k-conjugacy classes of θ-stable maximal k-split tori in the case that k is algebraically closed, the real numbers, the $\mathfrak{p}$-adic numbers and finite fields. First we will calculate $W_{H_k}(T_i)$, where T_i is a (θ, k)-split maximal torus.

Lemma 5.4.1. *Let T_i be a (θ, k)-split maximal torus. Then,*

1. $|W_{H_k}(T_i)| = 2$ *when* $m \in \overline{1}$ *and* $-1 \in (k^*)^2$.
2. $|W_{H_k}(T_i)| = 2$ *when* $m \in \overline{-1}$ *and* $-1 \notin (k^*)^2$.
3. $|W_{H_k}(T_i)| = 1$ *otherwise.*

Proof. By definition of (θ, k)-split, $\begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix} \in N_{\overline{H_k}}(T_i)$, and thus we can place it in $N_{H_k}(T_i)$ if and only if $\sqrt{-m} \in k$, corresponding to the results above.

Now, let $z = \begin{pmatrix} k & lym^{-1} \\ l & ky \end{pmatrix} \in (H \cdot T)_k$ such that $zTz^{-1} = T_i$. Clearly, id $\in W_{H_k}(T_i)$. Let $w \in W_{G_k}(T), w \neq \text{id}$. Then zwz^{-1} is the nonidentity element of $W_{G_k}(T_i)$ since $(zwz^{-1})T_i(zwz^{-1})^{-1} = zwz^{-1}(zTz^{-1})zw^{-1}z^{-1} = zwTw^{-1}z^{-1} = zTz^{-1} = T_i$. Recall, $W_{G_k}(T) = \{\text{id}, wT\}$, where for w we can choose either $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ or $w = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ with $i^2 = -1$.

Consider,

$$zwz^{-1} = \begin{pmatrix} k & lym^{-1} \\ l & ky \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} ky & -lym^{-1} \\ -l & k \end{pmatrix}$$
$$= \begin{pmatrix} -(kly^2m^{-1} + kl) & l^2y^2m^{-2} + k^2 \\ -(k^2y^2 + l^2) & kly^2m^{-1} + kl \end{pmatrix}.$$

Then $zwz^{-1} \in H \iff zwz^{-1} = \begin{pmatrix} 0 & b \\ mb & 0 \end{pmatrix}$ for some $b \in k$ and $\sqrt{-mb^2} \in k$. Thus, we get the same result. □

Remark 5.4.2. Clearly, $W_{H_k}(H) = \{\text{id}\}$, thus $|W_{G_k}(H)/W_{H_k}(H)| = 2$.

5.5 *k* algebraically closed

Let $G_k = \mathrm{SL}(2,k)$ with k algebraically closed. There is only one involution over $G = G_k$, namely $\theta(A) = \mathrm{Int}\begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$ for all $A \in G$, with $m = 1$. Then by Theorem 5.1.1, $H = G^\theta$ is a maximal k-split torus. Clearly, the quadratic form $x_1^2 - x_2^2 = 1$ has a solution over k. Thus, by Theorem 5.2.8, there is one H_k-conjugacy class of (θ, k)-split maximal tori. So our representative θ-stable maximal k-split tori of the H_k-conjugacy classes are H and T. By Lemma 5.4.1, $|W_{H_k}(T)| = 2$. Thus, by Theorem 2.3.1, $|B_k\backslash G_k/H_k| = |\bigcup_{T_i \in I} W_{G_k}(T_i)/W_{H_k}(T_i)| = |W_{G_k}(H)/W_{H_k}(H) \bigcup W_{G_k}(T)/W_{H_k}(T)| = 3$.

5.6 Real Numbers, $\mathbb{R}$

Let $G_k = \mathrm{SL}(2,\mathbb{R})$. $\mathbb{R}^*/(\mathbb{R}^*)^2 = \{\pm 1\}$, so there are two involutions over G, namely, $\theta(A) = \mathrm{Int}\begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$ for all $A \in G$, with $m = \pm 1$.

First, let $m = 1$. Then $H = G^\theta$ is a maximal k-split torus by Theorem 5.1.1. For the H_k-conjugacy classes of (θ, k)-split maximal tori we will use Theorem 5.3.3. By Remark 2.4.16, $\mathrm{Im}(N)/(k^*)^2\{1, -m\} = \{\pm 1\}/\{\pm 1\} = \{1\}$. So our representative θ-stable maximal k-split tori of the H_k-conjugacy classes are H and T. By Lemma 5.4.1, $|W_{H_k}(T)| = 1$. Thus, by Theorem 2.3.1, $|B_k\backslash G_k/H_k| = |\bigcup_{T_i \in I} W_{G_k}(T_i)/W_{H_k}(T_i)| = |W_{G_k}(H)/W_{H_k}(H) \bigcup W_{G_k}(T)/W_{H_k}(T)| = 4$.

Now, let $m = -1$. Then $H = G^\theta$ does not contain a maximal k-split torus by Theorem 5.1.1. By Remark 2.4.16, $\mathrm{Im}(N)/(k^*)^2\{1, -m\} = \{1\}$, so there is only one H_k-conjugacy class of (θ, k)-split maximal tori with representative T. By Lemma 5.4.1, $|W_{H_k}(T)| = 2$. Thus, by Theorem 2.3.1, $|B_k\backslash G_k/H_k| = |\bigcup_{T_i \in I} W_{G_k}(T_i)/W_{H_k}(T_i)| = |W_{G_k}(T)/W_{H_k}(T)| = 1$.

5.7 Rational Numbers, $\mathbb{Q}$

Let $G_k = \mathrm{SL}(2, \mathbb{Q})$. Then $|\mathbb{Q}^*/(\mathbb{Q}^*)^2| = \infty$. So there are an infinite number of involutions over G.

In general, $|B_k \backslash G_k / H_k| = \infty$. For example, let $m = 1$ and $q \in \mathbb{Q}^*/(\mathbb{Q}^*)^2$. Then by Hasse's principle, the quadratic form $x_1^2 - x_2^2 m^{-1} = q^{-1}$ has a solution over $\mathbb{Q}$ if and only if $x_1^2 - x_2^2 m^{-1} = q^{-1}$ has a solution over every $\mathfrak{p}$-adic field $\mathbb{Q}_p$, including $p = 2$, and $\mathbb{R}$. By Remark 2.4.16, this equation does have a solution for all $\mathbb{Q}_p$ and $\mathbb{R}$, so $U = k^*/(k^*)^2$, where U is as defined in Theorem 5.2.8. Hence, there is an infinite number of H_k-conjugacy classes of maximal (θ, k)-split tori in G under this involution. Hence, $|B_k \backslash G_k / H_k| = \infty$.

5.8 Finite Fields, $\mathbb{F}_p$

Let $G_k = \mathrm{SL}(2, \mathbb{F}_p)$. $\mathbb{F}_p^*/(\mathbb{F}_p^*)^2 = \{1, S_p\}$ where S_p is the smallest nonsquare in $\mathbb{F}_p$. So there are two involutions over G, namely, $\theta(A) = \mathrm{Int}\begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$ for all $A \in G$, with $m = 1$ or $m = S_p$. Let $H = G^\theta$. Then by Theorem 5.1.1, H is a maximal k-split torus in G if $m = 1$.

Remark 5.8.1. Consider the equation $z^2 - ax^2 - b = 0$ over over $\mathbb{F}_p$, $p \neq 0$. Theorem 4.2.1 tells us that this always has a solution for $a, b \in \{1, S_p\}$, where S_p is the smallest nonsquare in $\mathbb{F}_p$. Indeed, $z^2 - ax^2 - b = 0 \Rightarrow ax^2 = z^2 - b$, so by Theorem 4.2.1, ax^2 can be either a square or a nonsquare, giving us a solution in either case.

For the H_k-conjugacy classes of (θ, k)-split maximal tori, we have the following:

Proposition 5.8.2. *Let $k = \mathbb{F}_p$, where $p > 2$ is an odd prime. Let $m \in k^*/(k^*)^2 = \{1, S_p\}$ such that $\theta = \mathrm{Int}\begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$ and S_p is the smallest nonsquare. The number of H_k-conjugacy classes of (θ, k)-split maximal tori is equal to*

1. 2 *for* $p \cong 1 \mod 4$ *and* $m = 1$.

2. 1 *for* $p \cong 3 \mod 4$ *and* $m = 1$.

3. 1 *for* $p \cong 1 \mod 4$ *and* $m = S_p$*.*

4. 2 *for* $p \cong 3 \mod 4$ *and* $m = S_p$*.*

Proof. We will use Theorem 5.2.8. Let $m, q \in k^*/(k^*)^2$ and let

$$U = \{q \in k^*/(k^*)^2 \mid x_1^2 - m^{-1}x_2^2 = q^{-1} \text{ has a solution in } k\}.$$

Now, $x_1^2 - m^{-1}x_2^2 = q^{-1} \Rightarrow mx_1^2 = mq^{-1} + x_2^2$. By Remark 5.8.1, this quadratic form has a solution for all $m, q \in k^*/(k^*)^2$. Thus, $U = \{1, S_p\}$. If $p \cong 1 \mod 4$ and $m = 1$, or if $p \cong 3 \mod 4$ and $m = S_p = -1$, then the number of H_k-conjugacy classes of (θ, k)-split maximal tori is equal to $|U| = 2$. Otherwise, the number of H_k-conjugacy classes of (θ, k)-split maximal tori is equal to $|U|/2 = 1$. □

Theorem 5.8.3. *Let* $k = \mathbb{F}_p$*, where* $p > 2$ *is an odd prime and let* $H = G^\theta$ *where* $\theta(A) = \text{Int}\begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$ *for all* $A \in G$*, with* $m = 1$ *or* $m = S_p$*. Then*

1. $|B_k \backslash G_k / H_k| = 2$ *for* $m = S_p$

2. $|B_k \backslash G_k / H_k| = 4$ *for* $m = 1$*.*

Proof. We will use Proposition 5.8.2, Lemma 5.4.1, Theorem 2.3.1, and previous remarks about maximal tori in H. Recall, $|B_k \backslash G_k / H_k| = |\bigcup_{T_i \in I} W_{G_k}(T_i)/W_{H_k}(T_i)|$, where I is the set of H_k-conjugacy classes of θ-stable maximal k-split tori and the T_i are the representatives of these classes.

1. Let $m = 1$. Then if T_i is a maximal (θ, k)-split torus, we know by Lemma 5.4.1 that $|W_{H_k}(T_i)| = 2 \iff -1 \in (k^*)^2$.

   (a) Let $p \cong 1 \mod 4$. Then our representative tori making up the set I are H, T, and some other T_1. Here -1 is a square, so $|W_{H_k}(T)| = |W_{H_k}(T_1)| = 2$. Thus, $|B_k \backslash G_k / H_k| = 4$.

   (b) Let $p \cong 3 \mod 4$. Then our representative tori making up the set I is H and T. Here -1 is not a square, so $|W_{H_k}(T)| = 1$. Thus, $|B_k \backslash G_k / H_k| = 4$.

2. Let $m = S_p$. Then if T_i is a maximal (θ, k)-split torus, we know by Lemma 5.4.1 that $|W_{H_k}(T_i)| = 2 \iff -1 \notin (k^*)^2$.

   (a) Let $p \cong 1 \mod 4$. Then our representative torus making up the set I is only T. Here -1 is a square, so $|W_{H_k}(T)| = 1$. Thus, $|B_k \backslash G_k / H_k| = 2$.

   (b) Let $p \cong 3 \mod 4$. Then our representative tori making up the set I are T and some other T_1. Here -1 is not a square, so $|W_{H_k}(T)| = |W_{H_k}(T_1)| = 2$. Thus, $|B_k \backslash G_k / H_k| = 2$.

□

5.9 $\mathfrak{p}$-adic numbers, $\mathbb{Q}_p$ - Approach 1

In this Section we will use Theorem 5.2.8 to obtain the number of H_k-conjugacy classes of maximal (θ, k)-split tori, referred previously to as Approach 1.

5.9.1 $\mathfrak{p}$-adic numbers, $\mathbb{Q}_p$, with $p \neq 2$

Let $G_k = \mathrm{SL}(2, \mathbb{Q}_p)$. $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2 = \{1, S_p, p, pS_p\}$ where S_p is the smallest non-square in $\mathbb{F}_p$. So there are four involutions over G, namely, $\theta(A) = \mathrm{Int}\begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$ for all $A \in G$, with $m \in \{1, S_p, p, pS_p\}$. Let $H = G^\theta$. Then by Theorem 5.1.1, H is a maximal k-split torus in G if $m = 1$.

For the H_k-conjugacy classes of (θ, k)-split maximal tori, we have the following:

Lemma 5.9.2. *Let $k = \mathbb{Q}_p$, $p \neq 2$. Let $m \in k^*/(k^*)^2 = \{1, S_p, p, pS_p\}$ such that $\theta = \mathrm{Int}\begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$ and S_p is the smallest nonsquare in $\mathbb{F}_p$. The number of H_k-conjugacy classes of (θ, k)-split maximal tori is equal to*

1. *1 for $m = p, pS_p$.*

2. *1 for $p \cong 1 \mod 4$ and $m = S_p$.*

3. 2 *for* $p \cong 3 \mod 4$ *and* $m = S_p$.

4. 2 *for* $p \cong 3 \mod 4$ *and* $m = 1$.

5. 4 *for* $p \cong 1 \mod 4$ *and* $m = 1$.

Proof. We will use Lemma 2.4.10, Proposition 2.4.7, and Corollary 5.2.8. Let $m, q \in k^*/(k^*)^2$ and let $U = \{q \in k^*/(k^*)^2 | x_1^2 - m^{-1}x_2^2 = q^{-1}$ has a solution in $k\}$. We will determine if $x_1^2 - m^{-1}x_2^2 = q^{-1}$ has a solution by seeing if $(m, -qm)_p = 1$.

1. Let $m = 1$.

   (a) If $q = 1$, then $(m, -qm)_p = (1, -1)_p = 1$.

   (b) If $q = S_p$, then $(m, -qm)_p = (1, -S_p)_p = (1p^0, -S_p p^0)_p$
   $= (-1|p)^0(1|p)^0(-S_p|p)^0 = 1$.

   (c) If $q = p$, then $(m, -qm)_p = (1, -p)_p = (1p^0, -1p^1)_p$
   $= (-1|p)^0(1|p)^1(-1|p)^0 = 1$.

   (d) If $q = pS_p$, then $(m, -qm)_p = (1, -pS_p)_p = (1p^0, -S_p p^1)_p$
   $= (-1|p)^0(1|p)^1(-S_p|p)^0 = 1$.

   Then $U = k^*/(k^*)^2$. If $p \cong 1 \mod 4$, then we have that the number of H_k-conjugacy classes of (θ, k)-split maximal tori is equal to $|U| = 4$. If $p \cong 3 \mod 4$, then we have that the number of H_k-conjugacy classes of (θ, k)-split maximal tori is equal to $|U/(-1)| = |\{1, p)\}| = 2$ since $-1 \in \overline{S_p}$.

2. Let $m = S_p$.

   (a) If $q = 1$, then $(m, -qm)_p = (S_p, -S_p)_p = 1$.

   (b) If $q = S_p$, then $(m, -qm)_p = (S_p, -S_p^2)_p = (S_p, -1)_p = (S_p p^0, -1p^0)_p$
   $= (-1|p)^0(S_p|p)^0(-1|p)^0 = 1$.

   (c) If $q = p$, then $(m, -qm)_p = (S_p, -pS_p)_p = (S_p p^0, -S_p p^1)_p$
   $= (-1|p)^0(S_p|p)^1(-S_p|p)^0 = -1$.

   (d) If $q = pS_p$, then $(m, -qm)_p = (S_p, -pS_p^2)_p = (S_p, -p)_p = (S_p p^0, -1p^1)_p$
   $= (-1|p)^0(S_p|p)^1(-1|p)^0 = -1$.

Then $U = \{1, S_p\}$. If $p \cong 1 \mod 4$, then we have that the number of H_k-conjugacy classes of (θ, k)-split maximal tori is equal to $|U/(S_p)| = 1$. If $p \cong 3 \mod 4$, then we have that the number of H_k-conjugacy classes of (θ, k)-split maximal tori is equal to $|U| = 2$.

3. Let $m = p$.

   (a) If $q = 1$, then $(m, -qm)_p = (p, -p)_p = 1$.

   (b) If $q = S_p$, then $(m, -qm)_p = (p, -pS_p)_p = (1p^1, -S_p p^1)_p$
   $= (-1|p)^1(1|p)^1(-S_p|p)^1 = -1$.

   (c) If $q = p$, then $(m, -qm)_p = (p, -p^2)_p = (p, -1)_p = (1p^1, -1p^0)_p$
   $= (-1|p)^0(1|p)^0(-1|p)^1 = -1$ if $p \cong 3 \mod 4$, and $(m, -qm)_p = 1$ if $p \cong 1 \mod 4$.

   (d) If $q = pS_p$, then $(m, -qm)_p = (p, -p^2S_p)_p = (p, -S_p)_p = (1p^1, -S_p p^0)_p$
   $= (-1|p)^0(1|p)^0(-S_p|p)^1 = -1$ if $p \cong 1 \mod 4$, and $(m, -qm)_p = 1$ if $p \cong 3 \mod 4$..

   Then $U_1 = \{1, p\}$ for $p \cong 1 \mod 4$, and $U_2 = \{1, pS_p\}$ for $p \cong 3 \mod 4$. If $p \cong 1 \mod 4$, then we have that the number of H_k-conjugacy classes of (θ, k)-split maximal tori is equal to $|U_1/(p)| = |\{1\}| = 1$. If $p \cong 3 \mod 4$, then we have that the number of H_k-conjugacy classes of (θ, k)-split maximal tori is equal to $|U_2/(-p)| = |\{1\}| = 1$ since $-p \in \overline{pS_p}$.

4. Let $m = pS_p$.

   (a) If $q = 1$, then $(m, -qm)_p = (pS_p, -pS_p)_p = 1$.

   (b) If $q = S_p$, then $(m, -qm)_p = (pS_p, -pS_p^2)_p = (pS_p, -p)_p = (S_p p^1, -1p^1)_p$
   $= (-1|p)^1(S_p|p)^1(-1|p)^1 = -1$.

   (c) If $q = p$, then $(m, -qm)_p = (pS_p, -p^2S_p)_p = (pS_p, -S_p)_p$
   $= (S_p p^1, -S_p p^0)_p = (-1|p)^0(S_p|p)^0(-S_p|p)^1 = 1$ if $p \cong 3 \mod 4$, and $(m, -qm)_p = -1$ if $p \cong 1 \mod 4$.

(d) If $q = pS_p$, then $(m, -qm)_p = (pS_p, -p^2S_p^2)_p = (pS_p, -1)_p$
$= (S_p p^1, -1p^0)_p = (-1|p)^0(S_p|p)^0(-1|p)^1 = 1$ if $p \cong 1 \mod 4$, and $(m, -qm)_p = -1$ if $p \cong 3 \mod 4$..

Then $U_1 = \{1, pS_p\}$ for $p \cong 1 \mod 4$, and $U_2 = \{1, p\}$ for $p \cong 3 \mod 4$. If $p \cong 1 \mod 4$, then we have that the number of H_k-conjugacy classes of (θ, k)-split maximal tori is equal to $|U_1/(pS_p)| = |\{1\}| = 1$. If $p \cong 3 \mod 4$, then we have that the number of H_k-conjugacy classes of (θ, k)-split maximal tori is equal to $|U_2/(-pS_p)| = |\{1\}| = 1$ since $-pS_p \in \overline{p}$.

□

Theorem 5.9.3. *Let $H = G^\theta$ where $\theta(A) = \text{Int}\begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$ for all $A \in G$, with $m \in \{1, S_p, p, pS_p\}$.*

1. *$|B_k \backslash G_k / H_k| = 2$ for $m = S_p$, $m = p$, and $m = pS_p$.*
2. *$|B_k \backslash G_k / H_k| = 6$ for $m = 1$.*

Proof. We will use Lemma 5.9.2, Lemma 5.4.1, Theorem 2.3.1, and previous remarks about maximal tori in H. Recall, $|B_k \backslash G_k / H_k| = |\bigcup_{T_i \in I} W_{G_k}(T_i)/W_{H_k}(T_i)|$, where I is the set of H_k-conjugacy classes of θ-stable maximal k-split tori.

1. Let $m = 1$. Then if T_i is a maximal (θ, k)-split torus, we know by Lemma 5.4.1 that $|W_{H_k}(T_i)| = 2 \iff -1$ is a square and $x_1^2 - x_2^2 = 1$ has a solution in $\mathbb{Q}_p$, which it does for all p by letting $x_1 = 1$ and $x_2 = 0$.

   (a) Let $p \cong 1 \mod 4$. Then our representative tori making up the set I are H, T, and some three other T_1, T_2, and T_3. Here -1 is a square, so $|W_{H_k}(T)| = |W_{H_k}(T_1)| = |W_{H_k}(T_2)| = |W_{H_k}(T_3)| = 2$. Thus, $|B_k \backslash G_k / H_k| = 6$.

   (b) Let $p \cong 3 \mod 4$. Then our representative tori making up the set I are H, T, and some other T_1. Here -1 is not a square, so $|W_{H_k}(T)| = |W_{H_k}(T_1)| = 1$. Thus, $|B_k \backslash G_k / H_k| = 6$.

2. Let $m = S_p$. Then if T_i is a maximal (θ, k)-split torus, we know by Lemma 5.4.1 that $|W_{H_k}(T_i)| = 2 \iff -1$ is not a square and $x_1^2 - x_2^2 S_p = \sqrt{-S_p}$ has a solution in $\mathbb{Q}_p$.

   (a) Let $p \cong 1 \mod 4$. Then our representative torus making up the set I is only T. Here -1 is a square, so $|W_{H_k}(T)| = 1$. Thus, $|B_k \backslash G_k / H_k| = 2$.

   (b) Let $p \cong 3 \mod 4$. Then our representative tori making up the set I are T and some other T_1. Here -1 is not a square, so $\sqrt{-S_p} \in \mathbb{Q}_p$. Hence, by Lemma 5.8.1, $x_1^2 - x_2^2 S_p = \sqrt{-S_p}$ has a solution in $\mathbb{F}_p \subset \mathbb{Q}_p$, and therefore, $|W_{H_k}(T)| = |W_{H_k}(T_1)| = 2$. Thus, $|B_k \backslash G_k / H_k| = 2$.

3. Let $m = p$. Then our representative torus making up the set I is only T for all p. Note that $|W_{H_k}(T)| = 1$. Thus, $|B_k \backslash G_k / H_k| = 2$.

4. Let $m = pS_p$. Then our representative torus making up the set I is only T for all p. Note that $|W_{H_k}(T)| = 1$. Thus, $|B_k \backslash G_k / H_k| = 2$.

□

5.9.4 2-adic numbers $\mathbb{Q}_2$

Let $G_k = \mathrm{SL}(2, \mathbb{Q}_2)$.

Then $\mathbb{Q}_2^* / (\mathbb{Q}_2^*)^2 = \{\pm 1, \pm 2, \pm 3, \pm 6\}$. So there are eight involutions over G, namely, $\theta(A) = \mathrm{Int}\begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$ for all $A \in G$, with $m \in \{\pm 1, \pm 2, \pm 3, \pm 6\}$. Let $H = G^\theta$. Then by Theorem 5.1.1, H is a maximal k-split torus in G for $m = 1$.

For the H_k-conjugacy classes of (θ, k)-split maximal tori, we have the following:

Lemma 5.9.5. *Let $k = \mathbb{Q}_2$. Let $m \in k^*/(k^*)^2 = \{\pm 1, \pm 2, \pm 3, \pm 6\}$ such that $\theta = \mathrm{Int}\begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$. The number of H_k-conjugacy classes of (θ, k)-split maximal tori is equal to*

1. *2 for $m = \pm 2, \pm 3, \pm 6$.*

2. 4 *for* $m = \pm 1$.

Proof. We will use Lemma 2.4.10, Proposition 2.4.7, and Corollary 5.2.8. Let $m, q \in k^*/(k^*)^2$ and let $U = \{q \in k^*/(k^*)^2 | x_1^2 - m^{-1}x_2^2 = q^{-1}$ has a solution in $k\}$. We will determine if $x_1^2 - m^{-1}x_2^2 = q^{-1}$ has a solution by seeing if $(m, -qm)_2 = 1$. Note that the Legendre symbol is done modulo 8, meaning, $(2|-1) = (2|7) = 1$ and $(2|-3) = (2|5) = -1$.

1. Let $m = 1$.

   (a) Let $q = 1$. Then, $(m, -qm)_2 = (1, -1)_2 = 1$.

   (b) Let $q = -1$. Then $(m, -qm)_2 = (1, 1)_2 = (1, -1)_2 = 1$.

   (c) Let $q = 2$. Then $(m, -qm)_2 = (1, -2)_2 = (1 * 2^0, -1 * 2)_2$
   $= (2|1)^1(2|-1)^0(-1)^{(1-1)(-1-1)/4} = 1$.

   (d) Let $q = -2$. Then $(m, -qm)_2 = (1, 2)_2 = (1 * 2^0, 1 * 2)_2$
   $= (2|1)^1(2|1)^0(-1)^{(1-1)(1-1)/4} = 1$.

   (e) Let $q = 3$. Then $(m, -qm)_2 = (1, -3)_2 = (1 * 2^0, -3 * 2^0)_2$
   $= (2|1)^0(2|-3)^0(-1)^{(1-1)(-3-1)/4} = 1$.

   (f) Let $q = -3$. Then $(m, -qm)_2 = (1, 3)_2 = (1 * 2^0, 3 * 2^0)_2$
   $= (2|1)^0(2|3)^0(-1)^{(1-1)(3-1)/4} = 1$.

   (g) Let $q = 6$. Then $(m, -qm)_2 = (1, -6)_2 = (1 * 2^0, -3 * 2)_2$
   $= (2|1)^1(2|-3)^0(-1)^{(1-1)(-3-1)/4} = 1$.

   (h) Let $q = -6$. Then $(m, -qm)_2 = (1, 6)_2 = (1 * 2^0, 3 * 2)_2$
   $= (2|1)^1(2|3)^0(-1)^{(1-1)(3-1)/4} = 1$.

   Then $U = k^*/(k^*)^2$, and we have that the number of H_k-conjugacy classes of (θ, k)-split maximal tori is equal to $|U/(-1)| = |\{1, 2, 3, 6\}| = 4$.

2. Let $m = -1$.

   (a) Let $q = 1$. Then, $(m, -qm)_2 = (-1, 1)_2 = 1$.

(b) Let $q = -1$. Then $(m, -qm)_2 = (-1, -1)_2 = (-1 * 2^0, -1 * 2^0)_2$
$= (2|-1)^0 (2|-1)^0 (-1)^{(-1-1)(-1-1)/4} = (-1)^1 = -1.$

(c) Let $q = 2$. Then $(m, -qm)_2 = (-1, 2)_2 = (-1 * 2^0, 1 * 2)_2$
$= (2|-1)^1 (2|1)^0 (-1)^{(-1-1)(1-1)/4} = 1.$

(d) Let $q = -2$. Then $(m, -qm)_2 = (-1, -2)_2 = (-1 * 2^0, -1 * 2)_2$
$= (2|-1)^1 (2|-1)^0 (-1)^{(-1-1)(-1-1)/4} = (-1)^1 = -1.$

(e) Let $q = 3$. Then $(m, -qm)_2 = (-1, 3)_2 = (-1 * 2^0, 3 * 2^0)_2$
$= (2|-1)^0 (2|3)^0 (-1)^{(-1-1)(3-1)/4} = (-1)^{-1} = -1.$

(f) Let $q = -3$. Then $(m, -qm)_2 = (-1, -3)_2 = (-1 * 2^0, -3 * 2^0)_2$
$= (2|-1)^0 (2|-3)^0 (-1)^{(-1-1)(-3-1)/4} = (-1)^2 = 1.$

(g) Let $q = 6$. Then $(m, -qm)_2 = (-1, 6)_2 = (-1 * 2^0, 3 * 2)_2$
$= (2|-1)^1 (2|3)^0 (-1)^{(-1-1)(3-1)/4} = (-1)^{-1} = -1.$

(h) Let $q = -6$. Then $(m, -qm)_2 = (-1, -6)_2 = (-1 * 2^0, -3 * 2)_2$
$= (2|-1)^1 (2|-3)^0 (-1)^{(-1-1)(-3-1)/4} = (-1)^2 = 1.$

Then $U = \{1, 2, -3, -6\}$, and we have that the number of H_k-conjugacy classes of (θ, k)-split maximal tori is equal to $|U| = 4$.

3. Let $m = 2$.

(a) Let $q = 1$. Then, $(m, -qm)_2 = (2, -2)_2 = 1$.

(b) Let $q = -1$. Then $(m, -qm)_2 = (2, 2)_2 = (2, -1)_2 = (-1, 2)_2 = 1$.

(c) Let $q = 2$. Then $(m, -qm)_2 = (2, -4)_2 = (2, -1)_2 = 1$.

(d) Let $q = -2$. Then $(m, -qm)_2 = (2, 4)_2 = (2, 1)_2 = (1, 2)_2 = 1$.

(e) Let $q = 3$. Then $(m, -qm)_2 = (2, -6)_2 = (1 * 2, -3 * 2)_2$
$= (2|1)^1 (2|-3)^1 (-1)^{(1-1)(-3-1)/4} = -1.$

(f) Let $q = -3$. Then $(m, -qm)_2 = (2, 6)_2 = (1 * 2, 3 * 2)_2$
$= (2|1)^1 (2|3)^1 (-1)^{(1-1)(3-1)/4} = -1.$

(g) Let $q = 6$. Then $(m, -qm)_2 = (2, -12)_2 = (2, -3)_2 = (1 * 2, -3 * 2^0)_2$
$= (2|1)^0 (2|-3)^1 (-1)^{(1-1)(-3-1)/4} = -1.$

(h) Let $q = -6$. Then $(m, -qm)_2 = (2, 12)_2 = (2, 3)_2 = (1 * 2, 3 * 2^0)_2$
$= (2|1)^0(2|3)^1(-1)^{(1-1)(3-1)/4} = -1$.

Then $U = \{\pm 1, \pm 2\}$, and we have that the number of H_k-conjugacy classes of (θ, k)-split maximal tori is equal to $|U/(-2)| = |\{\pm 1\}| = 2$.

4. Let $m = -2$.

   (a) Let $q = 1$. Then, $(m, -qm)_2 = (-2, 2)_2 = 1$.

   (b) Let $q = -1$. Then $(m, -qm)_2 = (-2, -2)_2 = (-2, -1)_2 = (-1, -2)_2 = -1$.

   (c) Let $q = 2$. Then $(m, -qm)_2 = (-2, 4)_2 = (-2, 1)_2 = (1, -2)_2 = 1$.

   (d) Let $q = -2$. Then $(m, -qm)_2 = (-2, -4)_2 = (-2, -1)_2 = -1$.

   (e) Let $q = 3$. Then $(m, -qm)_2 = (-2, 6)_2 = (-1 * 2, 3 * 2)_2$
   $= (2|-1)^1(2|3)^1(-1)^{(-1-1)(3-1)/4} = -1 * (-1)^{-1} = 1$.

   (f) Let $q = -3$. Then $(m, -qm)_2 = (-2, -6)_2 = (-1 * 2, -3 * 2)_2$
   $= (2|-1)^1(2|-3)^1(-1)^{(-1-1)(-3-1)/4} = -1 * (-1)^2 = -1$.

   (g) Let $q = 6$. Then $(m, -qm)_2 = (-2, 12)_2 = (-2, 3)_2 = (-1 * 2, 3 * 2^0)_2$
   $= (2|-1)^0(2|3)^1(-1)^{(-1-1)(3-1)/4} = -1 * (-1)^{-1} = 1$.

   (h) Let $q = -6$. Then $(m, -qm)_2 = (-2, -12)_2 = (-2, -3)_2$
   $= (-1*2, -3*2^0)_2 = (2|-1)^0(2|-3)^1(-1)^{(-1-1)(-3-1)/4} = -1*(-1)^2 = -1$.

   Then $U = \{1, 2, 3, 6\}$, and we have that the number of H_k-conjugacy classes of (θ, k)-split maximal tori is equal to $|U/(2)| = |\{1, 3\}| = 2$.

5. Let $m = 3$.

   (a) Let $q = 1$. Then, $(m, -qm)_2 = (3, -3)_2 = 1$.

   (b) Let $q = -1$. Then $(m, -qm)_2 = (3, 3)_2 = (3, -1)_2 = (-1, 3)_2 = -1$.

   (c) Let $q = 2$. Then $(m, -qm)_2 = (3, -6)_2 = (3 * 2^0, -3 * 2)_2$
   $= (2|3)^1(2|-3)^0(-1)^{(3-1)(-3-1)/4} = -1 * (-1)^{-2} = -1$.

   (d) Let $q = -2$. Then $(m, -qm)_2 = (3, 6)_2 = (3 * 2^0, 3 * 2)_2$
   $= (2|3)^1(2|3)^0(-1)^{(3-1)(3-1)/4} = -1 * (-1)^1 = 1$.

(e) Let $q = 3$. Then $(m, -qm)_2 = (3, -3^2)_2 = (3, -1)_2 = -1$.

(f) Let $q = -3$. Then $(m, -qm)_2 = (3, 3^2)_2 = (3, 1)_2 = (1, 3)_2 = 1$.

(g) Let $q = 6$. Then $(m, -qm)_2 = (3, -18)_2 = (3, -2)_2 = (-2, 3)_2 = 1$.

(h) Let $q = -6$. Then $(m, -qm)_2 = (3, 18)_2 = (3, 2)_2 = (2, 3)_2 = -1$.

Then $U = \{1, -2, -3, 6\}$, and we have that the number of H_k-conjugacy classes of (θ, k)-split maximal tori is equal to $|U/(-3)| = |\{1, -2\}| = 2$.

6. Let $m = -3$.

(a) Let $q = 1$. Then, $(m, -qm)_2 = (-3, 3)_2 = 1$.

(b) Let $q = -1$. Then $(m, -qm)_2 = (-3, -3)_2 = (-3, -1)_2 = (-1, -3)_2 = 1$.

(c) Let $q = 2$. Then $(m, -qm)_2 = (-3, 6)_2 = (-3 * 2^0, 3 * 2)_2$
$= (2|-3)^1 (2|3)^0 (-1)^{(-3-1)(3-1)/4} = -1 * (-1)^{-2} = -1$.

(d) Let $q = -2$. Then $(m, -qm)_2 = (-3, -6)_2 = (-3 * 2^0, -3 * 2)_2$
$= (2|-3)^1 (2|-3)^0 (-1)^{(-3-1)(-3-1)/4} = -1 * (-1)^4 = -1$.

(e) Let $q = 3$. Then $(m, -qm)_2 = (-3, 3^2)_2 = (-3, 1)_2 = (1, -3)_2 = 1$.

(f) Let $q = -3$. Then $(m, -qm)_2 = (-3, -3^2)_2 = (-3, -1)_2 = 1$.

(g) Let $q = 6$. Then $(m, -qm)_2 = (-3, 18)_2 = (-3, 2)_2 = (2, -3)_2 = -1$.

(h) Let $q = -6$. Then $(m, -qm)_2 = (-3, -18)_2 = (-3, -2)_2 = (-2, -3)_2 = -1$.

Then $U = \{\pm 1, \pm 3\}$, and we have that the number of H_k-conjugacy classes of (θ, k)-split maximal tori is equal to $|U/(3)| = |\{\pm 1\}| = 2$.

7. Let $m = 6$.

(a) Let $q = 1$. Then, $(m, -qm)_2 = (6, -6)_2 = 1$.

(b) Let $q = -1$. Then $(m, -qm)_2 = (6, 6)_2 = (6, -1)_2 = (-1, 6)_2 = -1$.

(c) Let $q = 2$. Then $(m, -qm)_2 = (6, -12)_2 = (6, -3)_2 = (-3, 6)_2 = -1$.

(d) Let $q = -2$. Then $(m, -qm)_2 = (6, 12)_2 = (6, 3)_2 = (3, 6)_2 = 1$.

(e) Let $q = 3$. Then $(m, -qm)_2 = (6, -18)_2 = (6, -2)_2 = (-2, 6)_2 = 1$.

(f) Let $q = -3$. Then $(m, -qm)_2 = (6, 18)_2 = (6, 2)_2 = (2, 6)_2 = -1$.

(g) Let $q = 6$. Then $(m, -qm)_2 = (6, -6^2)_2 = (6, -1)_2 = -1$.

(h) Let $q = -6$. Then $(m, -qm)_2 = (6, 6^2)_2 = (6, 1)_2 = (1, 6)_2 = 1$.

Then $U = \{1, -2, 3, -6\}$, and we have that the number of H_k-conjugacy classes of (θ, k)-split maximal tori is equal to $|U/(-6)| = |\{1, 3\}| = 2$.

8. Let $m = -6$.

(a) Let $q = 1$. Then, $(m, -qm)_2 = (-6, 6)_2 = 1$.

(b) Let $q = -1$. Then $(m, -qm)_2 = (-6, -6)_2 = (-6, -1)_2 = (-1, -6)_2 = 1$.

(c) Let $q = 2$. Then $(m, -qm)_2 = (-6, 12)_2 = (-6, 3)_2 = (3, -6)_2 = -1$.

(d) Let $q = -2$. Then $(m, -qm)_2 = (-6, -12)_2 = (-6, -3)_2 = (-3, -6)_2 = -1$.

(e) Let $q = 3$. Then $(m, -qm)_2 = (-6, 18)_2 = (-6, 2)_2 = (2, -6)_2 = -1$.

(f) Let $q = -3$. Then $(m, -qm)_2 = (-6, -18)_2 = (-6, -2)_2 = (-2, -6)_2 = -1$.

(g) Let $q = 6$. Then $(m, -qm)_2 = (-6, 6^2)_2 = (-6, 1)_2 = (1, -6)_2 = 1$.

(h) Let $q = -6$. Then $(m, -qm)_2 = (-6, -6^2)_2 = (-6, -1)_2 = 1$.

Then $U = \{\pm 1, \pm 6\}$, and we have that the number of H_k-conjugacy classes of (θ, k)-split maximal tori is equal to $|U/(6)| = |\{1, 6\}| = 2$.

□

Theorem 5.9.6. *Let $H = G^\theta$ where $\theta(A) = \mathrm{Int}\begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$ for all $A \in G$, with $m \in \{\pm 1, \pm 2, \pm 3, \pm 6\}$.*

1. $|B_k \backslash G_k / H_k| = 10$ for $m = 1$.

2. $|B_k \backslash G_k / H_k| = 4$ otherwise.

Proof. We will use Lemma 5.9.5, Lemma 5.4.1, Theorem 2.3.1, and previous remarks about maximal tori in H. Recall, $|B_k \backslash G_k / H_k| = |\bigcup_{T_i \in I} W_{G_k}(T_i)/W_{H_k}(T_i)|$, where I is the set of H_k-conjugacy classes of θ-stable maximal k-split tori. Note that $-1 \notin (\mathbb{Q}_2^*)^2$, so if T_i is a maximal (θ, k)-split torus, $|W_{H_k}(T_i)| = 2$ only for $m = -1$ since $x_1^2 + x_2^2 = 1$ has a solution by letting $x_1 = 1$ and $x_2 = 0$.

1. Let $m = 1$. Then our representative tori making up the set I are H, T and some three other T_1, T_2, and T_3. Thus, $|B_k \backslash G_k / H_k| = 10$.

2. Let $m = -1$. Then our representative tori making up the set I are T and some three other T_1, T_2, and T_3. Thus, $|B_k \backslash G_k / H_k| = 4$, since in this case, $|W_{H_k}(T)| = |W_{H_k}(T_1)| = |W_{H_k}(T_2)| = |W_{H_k}(T_3)| = 2$.

3. Let $m \in \{\pm 2, \pm 3, \pm 6\}$. Then our representative tori making up the set I are T and some other T_1 for each m. Thus, $|B_k \backslash G_k / H_k| = 4$.

$\square$

5.10 p-adic numbers, $\mathbb{Q}_p$ - Approach 2

In this Section we will use Theorem 5.3.3 to obtain the number of H_k-conjugacy classes of maximal (θ, k)-split tori, referred previously to as Approach 2.

5.10.1 p-adic numbers, $\mathbb{Q}_p$, with $p \neq 2$

Let $G_k = \mathrm{SL}(2, \mathbb{Q}_p)$. $\mathbb{Q}_p^* / (\mathbb{Q}_p^*)^2 = \{1, S_p, p, pS_p\}$ where S_p is the smallest non-square in $\mathbb{F}_p$. So there are four involutions over G, namely, $\theta(A) = \mathrm{Int}\begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$ for all $A \in G$, with $m \in \{1, S_p, p, pS_p\}$. Let $H = G^\theta$. Then by Theorem 5.1.1, H is a maximal k-split torus in G if $m = 1$.

For the H_k-conjugacy classes of (θ, k)-split maximal tori, we have the following:

Proposition 5.10.2. *Let $k = \mathbb{Q}_p$, $p \neq 2$. Let $m \in k^*/(k^*)^2 = \{1, S_p, p, pS_p\}$ such that $\theta = \mathrm{Int}\begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$ and S_p is the smallest nonsquare in $\mathbb{F}_p$. The number of H_k-conjugacy classes of (θ, k)-split maximal tori is equal to*

1. *1 for $m = p, pS_p$.*

2. *1 for $p \cong 1 \mod 4$ and $m = S_p$.*

3. *2 for $p \cong 3 \mod 4$ and $m = S_p$.*

4. *2 for $p \cong 3 \mod 4$ and $m = 1$.*

5. *4 for $p \cong 1 \mod 4$ and $m = 1$.*

Proof. We will use Theorem 5.3.3.

1. Let $m = 1$. By Remark 2.4.16, $\mathrm{Im}(N)/(k^*)^2\{1, -m\} = \{1, S_p, p, pS_p\}$ if $p \cong 1 \mod 4$ and $m = 1$ since in this case $-m \in \overline{1}$. Hence, there are four H_k-conjugacy classes of (θ, k)-split maximal tori. Otherwise $\mathrm{Im}(N)/(k^*)^2\{1, -m\} = \{1, S_p, p, pS_p\}/\{1, S_p\} = \{1, p\}$, giving two H_k-conjugacy classes of (θ, k)-split maximal tori.

2. Let $m = S_p$. By Remark 2.4.16, $|\mathrm{Im}(N)/(k^*)^2| = 2$. If $p \cong 3 \mod 4$, $-m \in \overline{1}$, hence there are two H_k-conjugacy classes of (θ, k)-split maximal tori. If $p \cong 1 \mod 4$, then by Remark 2.4.16 we can let $\mathrm{Im}(N)/(k^*)^2 = \{1, -m\}$, so there is just one H_k-conjugacy class of (θ, k)-split maximal tori.

3. Let $m = p$ or $m = pS_p$. By Remark 2.4.16, $|\mathrm{Im}(N)/(k^*)^2| = 2$ and we can let $\mathrm{Im}(N)/(k^*)^2 = \{1, -m\}$. Thus, there is just one H_k-conjugacy class of (θ, k)-split maximal tori.

□

Theorem 5.10.3. *Let $k = \mathbb{Q}_p$, $p \neq 2$ and let $H = G^\theta$ where $\theta(A) = \mathrm{Int}\begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$ for all $A \in G$, with $m \in \{1, S_p, p, pS_p\}$.*

1. *$|B_k \backslash G_k / H_k| = 2$ for $m = S_p$, $m = p$, and $m = pS_p$.*

2. *$|B_k \backslash G_k / H_k| = 6$ for $m = 1$.*

Proof. We will use Proposition 5.10.2, Lemma 5.4.1, Theorem 2.3.1, and previous remarks about maximal tori in H. Recall, $|B_k \backslash G_k / H_k| = |\bigcup_{T_i \in I} W_{G_k}(T_i)/W_{H_k}(T_i)|$, where I is the set of H_k-conjugacy classes of θ-stable maximal k-split tori.

1. Let $m = 1$. Then if T_i is a maximal (θ, k)-split torus, we know by Lemma 5.4.1 that $|W_{H_k}(T_i)| = 2 \iff -1$ is a square.

   (a) Let $p \cong 1 \mod 4$. Then our representative tori making up the set I are H, T, and some three other T_1, T_2, and T_3. Here -1 is a square, so $|W_{H_k}(T)| = |W_{H_k}(T_1)| = |W_{H_k}(T_2)| = |W_{H_k}(T_3)| = 2$. Thus, $|B_k \backslash G_k / H_k| = 6$.

   (b) Let $p \cong 3 \mod 4$. Then our representative tori making up the set I are H, T, and some other T_1. Here -1 is not a square, so $|W_{H_k}(T)| = |W_{H_k}(T_1)| = 1$. Thus, $|B_k \backslash G_k / H_k| = 6$.

2. Let $m = S_p$. Then if T_i is a maximal (θ, k)-split torus, we know by Lemma 5.4.1 that $|W_{H_k}(T_i)| = 2 \iff -1$ is not a square.

   (a) Let $p \cong 1 \mod 4$. Then our representative torus making up the set I is only T. Here -1 is a square, so $|W_{H_k}(T)| = 1$. Thus, $|B_k \backslash G_k / H_k| = 2$.

   (b) Let $p \cong 3 \mod 4$. Then our representative tori making up the set I are T and some other T_1. Here -1 is not a square, so $|W_{H_k}(T)| = |W_{H_k}(T_1)| = 2$. Thus, $|B_k \backslash G_k / H_k| = 2$.

3. Let $m = p$. Then our representative torus making up the set I is only T for all p. Note that $|W_{H_k}(T)| = 1$. Thus, $|B_k \backslash G_k / H_k| = 2$.

4. Let $m = pS_p$. Then our representative torus making up the set I is only T for all p. Note that $|W_{H_k}(T)| = 1$. Thus, $|B_k \backslash G_k / H_k| = 2$.

□

5.10.4 2-adic numbers $\mathbb{Q}_2$

Let $G_k = \mathrm{SL}(2, \mathbb{Q}_2)$.

Then $\mathbb{Q}_2^*/(\mathbb{Q}_2^*)^2 = \{\pm 1, \pm 2, \pm 3, \pm 6\}$. So there are eight involutions over G, namely, $\theta(A) = \mathrm{Int}\begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$ for all $A \in G$, with $m \in \{\pm 1, \pm 2, \pm 3, \pm 6\}$. Let $H = G^\theta$. Then by Theorem 5.1.1, H is a maximal k-split torus in G for $m = 1$.

For the H_k-conjugacy classes of (θ, k)-split maximal tori, we have the following:

Proposition 5.10.5. *Let $k = \mathbb{Q}_2$. Let $m \in k^*/(k^*)^2 = \{\pm1, \pm2, \pm3, \pm6\}$ such that $\theta = \text{Int}\begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$. The number of H_k-conjugacy classes of (θ, k)-split maximal tori is equal to*

1. *2 for $m = \pm2, \pm3, \pm6$.*

2. *4 for $m = \pm1$.*

Proof. We will use Theorem 5.3.3.

1. Let $m = 1$. By Remark 2.4.16,

$$|\text{Im}(N)/(k^*)^2\{1, -m\}| = |\{\pm1, \pm2, \pm3, \pm6\}/\{\pm1\}| = 4,$$

   so there are 4 H_k-conjugacy classes of (θ, k)-split maximal tori.

2. Let $m = -1$. Then $-m$ is a square. By Remark 2.4.16,

$$|\text{Im}(N)/(k^*)^2\{1, -m\}| = |\frac{1}{2}|k^*/(k^*)^2| = 4,$$

   so there are 4 H_k-conjugacy classes of (θ, k)-split maximal tori.

3. Let $m = \pm2, \pm3$ or ±6. Then by Remark 2.4.16,

$$|\text{Im}(N)/(k^*)^2| = |\frac{1}{2}|k^*/(k^*)^2| = 4.$$

   Hence $|\text{Im}(N)/(k^*)^2\{1, -m\}| = 2$, and there are 2 H_k-conjugacy classes of (θ, k)-split maximal tori in these cases.

□

Theorem 5.10.6. *Let $k = \mathbb{Q}_2$ and let $H = G^\theta$ where $\theta(A) = \text{Int}\begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$ for all $A \in G$, with $m \in \{\pm1, \pm2, \pm3, \pm6\}$.*

1. *$|B_k \backslash G_k / H_k| = 10$ for $m = 1$.*

2. *$|B_k \backslash G_k / H_k| = 4$ otherwise.*

Proof. We will use Proposition 5.10.5, Lemma 5.4.1, Theorem 2.3.1, and previous remarks about maximal tori in H. Recall, $|B_k\backslash G_k/H_k| = |\bigcup_{T_i\in I} W_{G_k}(T_i)/W_{H_k}(T_i)|$, where I is the set of H_k-conjugacy classes of θ-stable maximal k-split tori. Note that $-1 \notin (\mathbb{Q}_2^*)^2$, so if T_i is a maximal (θ,k)-split torus, $|W_{H_k}(T_i)| = 2$ only if $m = -1$.

1. Let $m = 1$. Then our representative tori making up the set I are H, T and some three other T_1, T_2, and T_3. Thus, $|B_k\backslash G_k/H_k| = 10$.

2. Let $m = -1$. Then our representative tori making up the set I are T and some three other T_1, T_2, and T_3. Thus, $|B_k\backslash G_k/H_k| = 4$, since in this case, $|W_{H_k}(T)| = |W_{H_k}(T_1)| = |W_{H_k}(T_2)| = |W_{H_k}(T_3)| = 2$.

3. Let $m \in \{\pm 2, \pm 3, \pm 6\}$. Then our representative tori making up the set I are T and some other T_1 for each m. Thus, $|B_k\backslash G_k/H_k| = 4$.

□

5.11 Example of a maximal (θ, k)-split torus not H_k-conjugate to T

Using Theorem 5.2.8, one can obtain representatives of the different H_k-conjugacy classes of (θ,k)-split tori. As an example, we give the results for $k = \mathbb{Q}_5$ and $\theta = \mathrm{Int}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ below. We also depict the lattice of I and its expansionto the cosets $W_{G_k}(T_i)/W_{H_k}(T_i)$.

Example 5.11.1. Let $G = \mathrm{SL}(2,\mathbb{Q}_5)$ and $\theta = \mathrm{Int}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. By Proposition 5.10.2, there are four H_k-conjugacy classes of (θ,k)-split tori. Here $m = 1$, and we will use $S_p = 3$. Let $U = \{1, 1/3, 1/5, 1/15\}$, where U is as described in Theorem 5.2.8. Note that the term $1/a$ is the $\mathfrak{p}$-adic number such that $a * (1/a) = 1$ in $\mathbb{Q}_5$. Let $y \in U$. For each y, we need to find solution (k,l) such that $y^{-1} = k^2 - l^2$. Then we conjugate T by

Table 5.1: Results for $\mathrm{SL}(2,k)$

Field	Involution Class	Number of H_k-conjugacy classes of max (θ,k)-split tori	$\lvert B_k\backslash G_k/H_k\rvert$
$k=\bar{k}$	$m=1$	1	3
$k=\mathbb{R}$	$m=1$	1	4
	$m=-1$	1	1
$k=\mathbb{F}_p$, $p\cong 1 \mod 4$	$m=1$	2	4
	$m=S_p$	1	2
$k=\mathbb{F}_p$, $p\cong 3 \mod 4$	$m=1$	1	4
	$m=S_p$	2	2
$k=\mathbb{Q}_p$, $p\cong 1 \mod 4$	$m=1$	4	6
	$m=S_p,p,pS_p$	1	2
$k=\mathbb{Q}_p$, $p\cong 3 \mod 4$	$m=1$	2	6
	$m=S_p$	2	2
	$m=p,pS_p$	1	2
$k=\mathbb{Q}_2$	$m=1$	4	10
	$m=-1$	4	4
	$m=\pm 2,\pm 3,\pm 6$	2	4

the matrix $\begin{pmatrix} k & ly \\ l & ky \end{pmatrix}$ to get our representative tori. Solving $y^{-1} = k^2 - l^2$ for each $y \in U$, gives the four matrices,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1/3 \\ 1 & 2/3 \end{pmatrix}, \begin{pmatrix} 3 & 2/5 \\ 2 & 3/5 \end{pmatrix}, \begin{pmatrix} 4 & 1/15 \\ 1 & 4/15 \end{pmatrix}.$$

Conjugating T by these matrices gives the representative tori

$$T = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \mid x \in k^* \right\},$$

$$T_1 = \left\{ \begin{pmatrix} 1/3(4x - x^{-1}) & 2/3(x^{-1} - x) \\ -2/3(x^{-1} - x) & 1/3(4x^{-1} - x) \end{pmatrix} \mid x \in k^* \right\},$$

$$T_2 = \left\{ \begin{pmatrix} 1/5(9x - 4x^{-1}) & 6/5(x^{-1} - x) \\ -6/5(x^{-1} - x) & 1/5(9x^{-1} - 4x) \end{pmatrix} \mid x \in k^* \right\},$$

and

$$T_3 = \left\{ \begin{pmatrix} 1/15(16x - x^{-1}) & 4/15(x^{-1} - x) \\ -4/15(x^{-1} - x) & 1/15(16x^{-1} - x) \end{pmatrix} \mid x \in k^* \right\}.$$

Now, $|W_{G_k}(T_i)/W_{H_k}(T_i)| = 1$ for all maximal (θ, k)-split tori T_i, and $|W_{G_k}(H_k)/W_{H_k}(H_k)| = 2$. Thus, we obtain in Figure 5.1 lattice of I of the H_k-conjugacy classes of θ-stable maximal k-split tori and the expansion to the cosets $W_{G_k}(T_i)/W_{H_k}(T_i)$. Hence, $|P_k \backslash G_k / H_k|$=6.

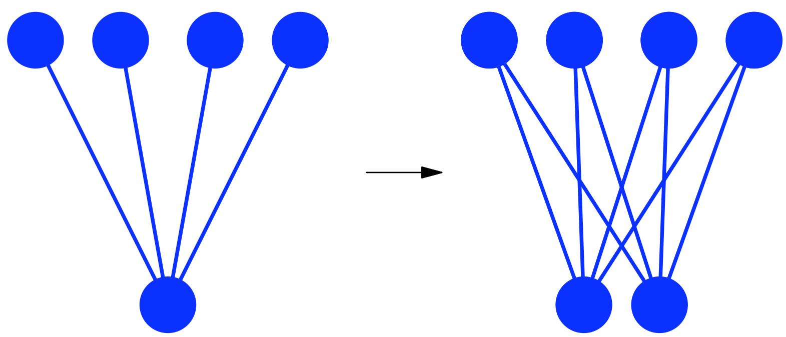

Figure 5.1: Lattice of I for $\mathrm{SL}(2, \mathbb{Q}_5)$ and its expansion

Chapter 6

Some Classification Results of SL(n, k)

In this Chapter, we begin the process of extending our classification of SL$(2, k)$ to SL(n, k). In particular, we focus on classifying the H_k-conjugacy classes of maximal (θ, k)-split tori for some isomorphism classes of involutions of SL(n, k) as well as the H_k-conjugacy classes of θ-stable maximal k-split tori containing a maximal (θ, k)-split torus. This is the top level of our tori lattice I. We will also determine the maximal k-split tori contained in H_k and classify the H_k-conjugacy classes of θ-stable maximal k-split tori containing a maximal k-split torus of H_k. This is the bottom level of our tori lattice I.

In the following, we will use the notation A_i to denote a maximal (θ, k)-split torus and the notation T_i to denote a θ-stable maximal k-split torus containing a maximal (θ, k)-split torus. Note that the notation $\frac{k^*/(k^*)^2}{\pm 1}$ means we quotient $k^*/(k^*)^2$ by the set containing the square classes of 1 and -1. If -1 is a square, then the square classes of 1 and -1 are the same. Thus, in general, $|\frac{k^*/(k^*)^2}{\pm 1}| = 1/2 * |k^*/(k^*)^2|$ if -1 is not a square and $|\frac{k^*/(k^*)^2}{\pm 1}| = |k^*/(k^*)^2|$ if -1 is a square.

Like SL$(2, k)$, SL(n, k) is a k-split group, meaning that SL(n, k) contains a maximal torus that is also maximal k-split, namely the set of diagonal matrices T. In this case, minimal parabolic k-subgroups are Borel subgroups. Therefore, if we let B be a Borel subgroup containing T, the classification of $B_k\backslash G_k/H_k$ is the exact same as $P_k\backslash G_k/H_k$.

6.1 Computing maximal (θ, k)-split tori

In order to determine the H_k-conjugacy classes of the maximal (θ, k)-split tori, we first need to find a maximal (θ, k)-split torus. Recall that T is the set of diagonal matrices of determinant equal to 1. Since T is a nice torus to work with, we would like our representative maximal (θ, k)-split torus to be contained in T for each involution θ. In particular, for each isomorphism class of involutions classified in the previous section, we will choose to use the θ in each class such that there exists a maximal (θ, k)-split torus A_1 contained in T.

Proposition 6.1.1. *Let* $I^*_{n-i,i} = \begin{pmatrix} A & 0 \\ 0 & I_{(n-2i)x(n-2i)} \end{pmatrix}$, *where*

$$A = \begin{pmatrix} 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \text{ a } 2i \text{ by } 2i \text{ matrix.}$$

Let $\theta = \mathrm{Inn}_{I_{n-i,i}}$ *and* $\tilde{\theta} = \mathrm{Inn}_{I^*_{n-i,i}}$. *Then* $\theta \approx \tilde{\theta}$. *Moreover,*

$$A_1 = \{\mathrm{diag}(a_1, \ldots, a_i, a_i^{-1}, \ldots, a_1^{-1}, 1, \ldots, 1) \mid a_i \in k^*\}$$

is a maximal $(\tilde{\theta}, k)$*-split torus.*

Proof. Let A_2 be a maximal $(\mathrm{Inn}_{I_{n-i,i}}, k)$-split torus. We know there exists $x \in G$ such that $x^{-1}A_1x = A_2$, where $A_1 \subset T$. Let $t_1 \in A_1$ and $t_2 \in A_2$ such that $x^{-1}t_1x = t_2$. Then

$$\begin{aligned} \mathrm{Inn}_{I_{n-i,i}}(t_2) &= t_2^{-1} \\ \iff \mathrm{Inn}_{I_{n-i,i}}(x^{-1}t_1x) &= (x^{-1}t_1x)^{-1} \\ \iff I_{n-i,i}^{-1}x^{-1}t_1xI_{n-i,i} &= x^{-1}t_1^{-1}x \\ \iff t_1(xI_{n-i,i}x^{-1}) &= (xI_{n-i,i}x^{-1})t_1^{-1} \end{aligned}$$

Since t_1 is conjugate to t_1^{-1}, t_1 has dimension $\leq \frac{n}{2}$. Let

$$t_1 = \operatorname{diag}(a_1, \ldots, a_i, a_i^{-1}, \ldots, a_1^{-1}, 1, \ldots, 1)$$

and let $I^*_{n-i,i}$ be as above. Then it can be easily calculated that $t_1 I^*_{n-i,i} = I^*_{n-i,i} t_1^{-1}$. Therefore, if the (θ, k)-split torus has dimension of i, the corresponding $I_{n-j,j}$ has to be conjugate to $I^*_{n-i,i}$. Hence, a (θ, k)-split torus has dimension i if and only if the corresponding $I_{n-j,j}$ is such that $j \geq i$, meaning that the maximal (θ, k)-split torus for $I_{n-i,i}$ has dimension i. Thus, there exists $x \in G$ such that $xI_{n-i,i}x^{-1} = I^*_{n-i,i}$. Then, $\tilde{\theta} = \operatorname{Inn}_{I^*_{n-i,i}} = \operatorname{Inn}_{xI_{n-i,i}x^{-1}} = \operatorname{Inn}_x \operatorname{Inn}_{I_{n-i,i}} \operatorname{Inn}_{x^{-1}}$. Therefore, $\theta \approx \tilde{\theta}$, and clearly $A_1 = \{\operatorname{diag}(a_1, \ldots, a_i, a_i^{-1}, \ldots, a_1^{-1}, 1, \ldots, 1) \mid a_i \in k^*\}$ is a maximal $(\tilde{\theta}, k)$-split torus. □

Remark 6.1.2. We will use $\theta = \operatorname{Inn}_{I^*_{n-i,i}}$ in place of $\theta = \operatorname{Inn}_{I_{n-i,i}}$ for the rest of this thesis since it gives a representative maximal $(\tilde{\theta}, k)$-split torus contained in T.

Proposition 6.1.3. *Let $\theta = \operatorname{Inn}_{L_{n,x}}$. Then*

$$A_1 = \{\operatorname{diag}(a_1, a_1^{-1}, \ldots, a_{n/2}, a_{n/2}^{-1}) \mid a_i \in k^*\}$$

is a maximal (θ, k)-split torus.

Proof. Let A_2 be a maximal (θ, k)-split torus. We know there exists $x \in G$ such that $x^{-1}A_1x = A_2$, where $A_1 \subset T$. Let $t_1 \in A_1$ and $t_2 \in A_2$ such that $x^{-1}t_1x = t_2$. Then

$$\begin{aligned} \theta(t_2) &= t_2^{-1} \\ \iff \theta(x^{-1}t_1x) &= (x^{-1}t_1x)^{-1} \\ \iff L_{n,x}^{-1}x^{-1}t_1xL_{n,x} &= x^{-1}t_1^{-1}x \\ \iff t_1(xL_{n,x}x^{-1}) &= (xL_{n,x}x^{-1})t_1^{-1} \end{aligned}$$

Since t_1 is conjugate to t_1^{-1}, t_1 has dimension $\leq \frac{n}{2}$. In fact, for $x = I$, the dimension of A_1 is maximal and equal to $n/2$. In particular, A_1 is the above maximal (θ, k)-split torus. □

Proposition 6.1.4. *Let θ be one of the following:*

1. $\theta(A) = (A^T)^{-1}$ *for all* $A \in G$

2. $\theta(A) = \mathrm{Inn}_{I_{n-i,i}}(A^T)^{-1}$ *for all* $A \in G$

3. $\theta(A) = \mathrm{Inn}_{M_{n,x,y,z}}(A^T)^{-1}$ *for all* $A \in G$.

Then T *is a maximal* (θ, k)*-split torus.*

Proof. For (1), clearly $\theta(t) = t^{-1}$ for all $t \in T$, and we know T is of maximal dimension. For (2) and (3), we observe that $I_{n-i,i}$ and $M_{n,x,y,z}$ are in T, hence, $\mathrm{Inn}_{I_{n-i,i}}$ and $\mathrm{Inn}_{M_{n,x,y,z}}$ fix T. □

Proposition 6.1.5. *Let* $\theta(A) = \mathrm{Inn}_{J_{2m}}(A^T)^{-1}$ *for all* $A \in G$*. Then a maximal* (θ, k)*-split torus is*

$$A_2 = \{\mathrm{diag}(a_1, a_2, \ldots, a_n) \mid a_i \in k \text{ and } a_1 = a_{n/2+1}, a_2 = a_{n/2+2}, \ldots, a_{n/2} = a_n\}.$$

Proof. Let A_2 be a maximal (θ, k)-split torus. We know there exists $x \in G$ such that $x^{-1}A_1x = A_2$, where $A_1 \subset T$. Let $t_1 \in A_1$ and $t_2 \in A_2$ such that $x^{-1}t_1x = t_2$. Then

$$\begin{aligned} \theta(t_2) &= t_2^{-1} \\ \Longleftrightarrow \theta(x^{-1}t_1x) &= (x^{-1}t_1x)^{-1} \\ \Longleftrightarrow J_{2m}^{-1}x^Tt_1^{-1}(x^T)^{-1}J_{2m} &= x^{-1}t_1^{-1}x \\ \Longleftrightarrow t_1(xJ_{2m}^{-1}x^T) &= (xJ_{2m}^{-1}x^T)t_1 \end{aligned}$$

For $x = I$, the dimension of A_2 is maximal and equal to $n/2$. In particular, A_2 is the above maximal (θ, k)-split torus. □

6.2 Computing Fixed Point Groups

Next, we need to know what the fixed point group H_k is for every above involution.

Lemma 6.2.1. *Let $\theta = \mathrm{Inn}_{I^*_{n-i,i}}$. Then the fixed point group H_k consists of the matrices* $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SL}(n,k)$ *such that* $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,2i} \\ \vdots & & & \vdots \\ a_{i1} & \cdots & \cdots & a_{i,2i} \\ a_{i,2i} & \cdots & \cdots & a_{i1} \\ \vdots & & & \vdots \\ a_{1,2i} & \cdots & a_{12} & a_{11} \end{pmatrix}$ *is $2i$ by $2i$,* $B = \begin{pmatrix} B_1 & \cdots & B_i & B_i & \cdots & B_1 \end{pmatrix}^T$ *is $2i$ by $n-2i$ with B_j denoting the rows of B,* $C = \begin{pmatrix} C_1 & \cdots & C_i & C_i & \cdots & C_1 \end{pmatrix}$ *is $n-2i$ by $2i$ with C_j denoting the columns of C, and $D \in \mathrm{GL}(n-2i,k)$.*

Proof. Let $h = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SL}(n,k)$ where A is $2i$ by $2i$, B is $2i$ by $n-2i$, C is $n-2i$ by $2i$, and D is $n-2i$ by $n-2i$. Consider conjugating h by $I^*_{n-i,i}$. This action reverses A's rows and columns, reverses B's rows, reverses C's columns, and leaves D fixed. Thus, $I^*_{n-i,i}h(I^*_{n-i,i})^{-1} = h$ if and only if $h \in H_k$ described above. □

Lemma 6.2.2. *Let $\theta = \mathrm{Inn}_{L_{n,x}}$. Then H_k consists of the elements*

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{pmatrix},$$

where $m = n/2$ and $A_{ij} = \begin{pmatrix} a_{ij} & b_{ij} \\ xb_{ij} & a_{ij} \end{pmatrix}$.

Proof. Let $A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{pmatrix}$ and assume $\mathrm{Inn}_{L_{n,x}}(A) = A$. Let $L_2 =$

$\begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix}$. Then,

$$A = \begin{pmatrix} L_2^{-1} & 0 & \cdots & 0 \\ 0 & L_2^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & L_2^{-1} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{pmatrix} \begin{pmatrix} L_2 & 0 & \cdots & 0 \\ 0 & L_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & L_2 \end{pmatrix}.$$

That is $L_2^{-1}A_{ij}L_2 = A_{ij}$ for all $i, j = 1, \ldots, n$. Let $A_{ij} = \begin{pmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{pmatrix}$. Then $L_2^{-1}A_{ij}L_2 = A_{ij}$ implies that $a_{ij} = d_{ij}$ and $c_{ij} = xb_{ij}$. □

Lemma 6.2.3. *Let $\theta(A) = (A^T)^{-1}$ for all $A \in G$. Then the fixed point group of θ is the group* $\mathrm{SO}(n, k) = \{A \in G \mid A^T A = \mathrm{Id}\}$.

Proof. The result follows immediately from the definition of θ. □

Lemma 6.2.4. *Let $\theta(A) = \mathrm{Inn}_{I_{n-i,i}}(A^T)^{-1}$ for all $A \in G$. Then the fixed point group of θ is the group* $\mathrm{SO}(p, q, k) = \{A \in G \mid A^T I_{n-i,i} A = I_{n-i,i}\}$, *where $p = n - i$ and $q = i$.*

Proof. The result follows immediately from the definitions of θ and the $\mathrm{SO}(p, q, k)$ groups. □

Lemma 6.2.5. *Let $\theta(A) = \mathrm{Inn}_{J_{2m}}(A^T)^{-1}$ for all $A \in G$. Then the fixed point group of θ is the group* $\mathrm{Sp}(2m, k) = \{A \in G \mid A^T J_{2m} A = J_{2m}\}$. *In particular, if $A = (a_{ij})$, then the entries of A must satisfy the equations:*

$$a_{1i}a_{\frac{n}{2}+1,j} + a_{2i}a_{\frac{n}{2}+2,j} + \cdots + a_{\frac{n}{2},i}a_{nj} - a_{\frac{n}{2}+1,i}a_{1j} - a_{\frac{n}{2}+2,i}a_{2j} - \cdots$$
$$\cdots - a_{ni}a_{\frac{n}{2},j} = \begin{cases} 1 & i = s \text{ and } j = \frac{n}{2} + s, \text{ where } 1 \leq s \leq \frac{n}{2} \\ 0 & \text{otherwise} \end{cases}$$

Proof. The result follows immediately from the definitions of θ and the $\mathrm{Sp}(2m, k)$ group. □

6.3 H_k-conjugacy classes of (θ, k)-split maximal tori

We are looking for the H_k-conjugacy classes of (θ, k)-split maximal tori in the set $\mathcal{A}_\theta$. Recall that we can always choose $A_1 \subset T$, where A_1 is a (θ, k)-split maximal torus. Recall that T is a maximal k-split torus, a maximal torus, and $Z_G(T) = T$. Thus by Theorem 2.3.2, for any other maximal (θ, k)-split torus A_2, there exists $g \in (H \cdot T)_k$ such that $gA_1g^{-1} = A_2$.

Consider the following: H_k acts on $(H \cdot T)_k$ by left multiplication. Each element of $(H \cdot T)_k$ corresponds to a (θ, k)-split maximal torus by conjugating T_1. So, H_k acts on the sets $\mathcal{A}_\theta$ by acting on $(H \cdot T)_k$. By grouping $(H \cdot T)_k$ into H_k-left multiplication classes, we are grouping $\mathcal{A}_\theta$ into H_k-conjugacy classes at the same time.

Lemma 6.3.1. *The set of elements of* $(H \cdot T)_k$ *corresponding to* A_1 *is*

$$(H \cdot T)_k \cap N_{G_k}(A_1).$$

Proof. Let $z \in (H \cdot T)_k$. Then $zA_1z^{-1} = A_1 \iff z \in N_{G_k}(A_1)$. □

Lemma 6.3.2. *Let* $z \in (H \cdot T)_k$ *and let* $n \in (H \cdot T)_k \cap N_{G_k}(A_1)$. *Then* $zn \in (H \cdot T)_k$.

Proof. By definition, let $z = h_1t_1$ and $n = h_2t_2$, with $h_1, h_2 \in H$ and $t_1, t_2 \in A_1$. Since $n \in N_{G_k}(A_1)$, we can assume h_2 is a permutation matrix (with an appropriate negative sign or $\sqrt{-1}$ entries to make the determinant 1). Thus, $h_2^{-1}t_1h_2 = t_3$, where t_3 is the permutation of t_1 under h_2. Hence,

$$zn = h_1t_1h_2t_2 = h_1h_2t_3t_1 = h_3t_4 \in (H \cdot T)_k.$$

□

Lemma 6.3.3. *Let* $A_2 \in \mathcal{A}_\theta$ *and* $z_1, z_2 \in (H \cdot T)_k$ *such that* $z_1A_1z_1^{-1} = A_2$ *and* $z_2A_1z_2^{-1} = A_2$. *Then* $z_2 = z_1n$ *where* $n \in (H \cdot T)_k \cap N_{G_k}(A_1)$.

Proof. Since $z_1A_1z_1^{-1} = A_2$ and $z_2A_1z_2^{-1} = A_2$, then $z_1A_1z_1^{-1} = z_2A_1z_2^{-1} \Rightarrow A_1 = z_1^{-1}z_2A_1z_2^{-1}z_1$. Hence, $z_1^{-1}z_2 \in N_{G_k}(A_1) \Rightarrow z_1^{-1}z_2 = n$ for some $n \in N_{G_k}(A_1)$. Therefore, $z_2 = z_1n$. □

Remark 6.3.4. Let $A_2 \in \mathcal{A}_\theta$ and $z_1 \in (H \cdot T)_k$ such that $z_1 A_1 z_1^{-1} = A_2$. If we can find $n \in N_{G_k}(A_1)$ such that $z_1 n = h$ for some $h \in H_k$, then A_2 is H_k-conjugate to A_1. This is the idea used in Theorems 6.3.6 and 6.3.10 to prove that there exists only one H_k-conjugacy class of maximal (θ, k)-split tori.

Lemma 6.3.5. *Let $\hat{\theta}$ be the extension of θ to GL_n, $\hat{H}$ be the fixed point group of $\hat{\theta}$ in GL_n, and $\hat{T}$ be the set of diagonal matrices in GL_n. Suppose every $z \in (H \cdot T)_k$ can be written as $z = ht$, where $h \in \hat{H}_k$ and $t \in \hat{T}_k$. Then $(H \cdot T)_k = \hat{H}_k \cdot \hat{T}_k \cap \mathrm{SL}(n, k)$.*

Proof. Clearly by assumption, $(H \cdot T)_k \subset \hat{H}_k \cdot \hat{T}_k \cap \mathrm{SL}(n, k)$. Let $ht \in \hat{H}_k \cdot \hat{T}_k \cap \mathrm{SL}(n, k)$. Since $\det(z) = 1$, $\det(h) = \det(t)^{-1}$. Let $d = \det h$. Then, $h_1 = \frac{1}{\sqrt[n]{d}} h \in H$ and $t_1 = \sqrt[n]{d}\, t \in T$. Hence, $ht = h_1 t_1 \in (H \cdot T)_k$. Thus, we have equality. □

Theorem 6.3.6. *Let $\theta = \mathrm{Inn}_{I^*_{n-i,i}}$ such that $i < n/2$. Then there exists only one H_k-conjugacy class of maximal (θ, k)-split tori with representative*

$$A_1 = \{\mathrm{diag}(a_1, \dots, a_i, a_i^{-1}, \dots, a_1^{-1}, 1, \dots, 1) \mid a_i \in k^*\}.$$

Proof. We know A_1 is a maximal (θ, k)-split torus. Note that $T \subset (H \cdot T)_k \cap N_{G_k}(A_1)$. Let $A_2 \neq A_1$ be a maximal (θ, k)-split torus, and let $z_1 \in (H \cdot T)_k$ such that $z_1 A_1 z_1^{-1} = A_2$. By Lemma 6.3.3, we know $z_1 n$, where $n \in (H \cdot T)_k \cap N_{G_k}(A_1)$, also corresponds to A_2 via conjugating A_1.

Let $h = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in H$ be as in Lemma 6.2.1 with a_{ij} representing the coefficients of A and let $t = \mathrm{diag}(x_1, \dots, x_n) \in T$ such that $z_1 = ht$. Then we know $a_{11}x_1, \dots, a_{1i}x_i \in k$ and $a_{11}x_{2i}, \dots, a_{1i}x_{i+1} \in k$ where the a_{ij} are the coefficients of the matrix A. Let's construct $y = \mathrm{diag}(y_1, \dots, y_n) \in T$ as $y_1 = a_{11}x_{2i}, \dots, y_i = a_{1i}x_{i+1}, y_{i+1} = a_{1i}x_i, \dots, y_{2i} = a_{11}x_1$ and $y_{2i+1}, \dots, y_n$ can be anything to make $\det(y) = 1$. Then for ty we have that $x_1 y_1 = x_{2i} y_{2i}$, $x_2 y_2 = x_{2i-1} y_{2i-1}$, $\cdots$, $x_j y_j = x_{2i-j+1} y_{2i-j+1}$, $\cdots$, $x_i y_i = x_{i+1} y_{i+1} \Rightarrow ty \in H \Rightarrow z_1 y \in H_k \Rightarrow T_2$ is H_k conjugate to A_1. Since A_2 was arbitrary, we get the above result.

□

Theorem 6.3.7. *Let $\theta = \mathrm{Inn}_{I^*_{n-i,i}}$ such that $i = n/2$. Note that n must be even. Then the number of H_k-conjugacy classes of maximal (θ, k)-split tori is at most $|\frac{(k^*/(k^*)^2)}{(\pm 1)}|$.*

Proof. In this case,

$$H_k = \left\{ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,n} \\ \vdots & & & \vdots \\ a_{i1} & \cdots & \cdots & a_{i,n} \\ a_{i,n} & \cdots & \cdots & a_{i1} \\ \vdots & & & \vdots \\ a_{1,n} & \cdots & a_{12} & a_{11} \end{pmatrix} \mid a_{ij} \in k \text{ and } \det = 1 \right\}.$$

Thus,

$$(H \cdot T)_k = \left\{ \begin{pmatrix} a_{11}x_1 & a_{12}x_2 & \cdots & a_{1,n}x_n \\ \vdots & & & \vdots \\ a_{i1}x_1 & \cdots & \cdots & a_{i,n}x_n \\ a_{i,n}x_1 & \cdots & \cdots & a_{i1}x_n \\ \vdots & & & \vdots \\ a_{1,n}x_1 & \cdots & a_{12}x_{n-1} & a_{11}x_n \end{pmatrix} \mid a_{ij} \in \bar{k},\ x_j \in \bar{k}^*, \text{ and } a_{ij}x_l \in k \right\}.$$

Let $z_{11} = a_{11}x_1 \Rightarrow a_{11} = z_{11}/x_1 \Rightarrow a_{11}x_n = z_{11}x_n/x_1 = z_{11}k_1$, where $k_1 = x_n/x_1 \in k$. Let $z_{21} = a_{21}x_1 \Rightarrow a_{21} = z_{21}/x_1 \Rightarrow a_{21}x_n = z_{21}x_n/x_1 = z_{21}k_1$, where $k_1 = x_n/x_1 \in k$. Continue in this fashion for $j \le n/2$, obtaining that if $a_{j1}x_1 = z_{j1} \Rightarrow a_{ji}x_n = z_{j1}k_1$ and if $a_{j,n}x_1 = z_{n-j+1,1} \Rightarrow a_{j,n}x_n = z_{n-j+1,1}k_1$. Repeat this process for pair of columns containing x_2 and x_{n-1}, where $k_2 = x_{n-1}/x_2$, and continue repeating through pair of columns containing $x_{n/2}$ and $x_{n/2+1}$, where $k_{\frac{n}{2}} = x_{\frac{n}{2}+1}/x_{\frac{n}{2}}$. This rewriting of the elements of $(H \cdot T)_k$ means that we are able to rewrite $(H \cdot T)_k$ as two k-point subsets in $\mathrm{GL}(n,k)$. Let $\hat{H} = \{A \in \mathrm{GL}_n \mid \theta(A) = A\}$ and $\hat{T} = \{\mathrm{diag}(t_1, \ldots, t_n) \mid \det \ne 0\} \subset \mathrm{GL}_n$. Then, $(H \cdot T)_k$ is contained in the set

$$\left\{ \hat{h}\hat{t} \mid \hat{h} = \begin{pmatrix} z_{11} & z_{12} & \cdots & z_{1,n} \\ \vdots & & & \vdots \\ z_{i1} & \cdots & \cdots & z_{i,n} \\ z_{i,n} & \cdots & \cdots & z_{i1} \\ \vdots & & & \vdots \\ z_{1,n} & \cdots & z_{12} & z_{11} \end{pmatrix} \in \hat{H},\ \hat{t} = \mathrm{diag}(1, \ldots, 1, k_{n/2}, \ldots, k_1) \in \hat{T} \right\}. \quad (6.1)$$

Equality follows from Lemma 6.3.5.

Let $A_1 = \{\mathrm{diag}(a_1, \ldots, a_{n/2}, a_{n/2}^{-1}, \ldots, a_1^{-1}) \mid a_i \in k^*\}$. We know A_1 is a maximal (θ, k)-split torus. Let $A_2 \neq A_1$ be a maximal (θ, k)-split torus, and let $z_1 \in (H \cdot T)_k$ such that $z_1 A_1 z_1^{-1} = A_2$. By Lemma 6.3.3, we know that if $z_2 \in (H \cdot T)_k$ such that $z_2 A_1 z_2^{-1} = A_2$, then $z_2 = z_1 n$, where $n \in (H \cdot T)_k \cap N_{G_k}(A_1)$. Now we can write each $n \in (H \cdot T)_k \cap N_{G_k}(A_1)$ in the form $n = Pt$, where P is a permutation matrix such that $P \in \hat{H}$ and $t = \mathrm{diag}(y_1, \ldots, y_n) \in \hat{T}$ with $\det(P) = \pm 1$ and $\det(t) = \det(P)^{-1} = \pm 1$. Note that $y_n = \pm \frac{1}{y_1 \cdots y_{n-1}}$. Also, observe that P acts on A_1 by either interchanging a_j and a_j^{-1}, interchanging the pairs (a_j, a_j^{-1}) and (a_i, a_i^{-1}), or any combination of these.

Now let $z_1 = \hat{h}_1 \hat{t}_1$ as in (6.1), $d = \det(\hat{t}_1) = k_1 \cdots k_{n/2}$, and

$$n = Pt \in (H \cdot T)_k \cap N_{G_k}(A_1).$$

Consider $z_1 n = \hat{h}_1 \hat{t}_1 P t$. We know P permutes $\hat{t}_1$ via conjugation, so let $\hat{t}_2 = P^{-1} \hat{t}_1 P$. Note that because of how P acts on T_1, we have that

$$\hat{t}_2 = \mathrm{diag}(w_1, \ldots, w_{n/2}, w_{n/2}^*, \ldots, w_1^*),$$

where the pair (w_i, w_i^*) is some pair $(1, k_j)$ in any order. Also, note that $\det(\hat{t}_2) = \det(\hat{t}_1)$. So, we have that $z_1 n = \hat{h}_1 \hat{t}_1 P t = \hat{h}_1 P \hat{t}_2 t = \hat{h}_2 \hat{t}_2 t$, where $\hat{h}_2 = \hat{h}_1 P \in \hat{H}$. Hence, in order to get $z_1 n$ in the form of (6.1), we need to just rewrite $\hat{t}_2 t$ in the form of (6.1).

Now,

$$\hat{t}_2 t = \mathrm{diag}(w_1 y_1, \ldots, w_{n/2} y_{n/2}, w_{n/2}^* y_{n/2+1}, \ldots, w_1^* y_n),$$

where we recall that the pair (w_i, w_i^*) is some pair $(1, k_j)$ from $\hat{t}_1$ and $y_n = \pm (y_1 \cdots y_{n-1})^{-1}$ by previous definitions. We need to write $\hat{t}_2 t$ as

$$\mathrm{diag}(b_1, \ldots, b_{n/2}, b_{n/2}, \ldots, b_1)\, \mathrm{diag}(1, \ldots, 1, c_{n/2}, \ldots, c_1) \in (H \cdot T)_k$$

for some $b_i, c_j \in k$. We can do this by the following method. Let $w_i y_i = b_i \Rightarrow w_i = b_i / y_i$. We know $w_i = 1$ or $w_i = k_j$ for some j determined by P. If $w_i = 1$, then $1 = b_i / y_i$ and $w_i^* = k_j$. Hence, $w_i^* y_{n-i+1} = (b_i k_j y_{n-i+1}) / y_i = b_i c_i$ where $c_i = k_j (y_{n-i+1} / y_i)$. Also, if $w_i = k_j$, then $w_i^* = 1$ and $k_j = b_i / y_i \Rightarrow 1 = b_i / (k_j y_i)$.

Hence, $w_i^* y_{n-i+1} = (b_i y_{n-i+1})/(k_j y_i) = b_i c_i$ where $c_i = y_{n-i+1}/(k_j y_i)$. Therefore, we obtain that

$$\hat{t}_2 t = \mathrm{diag}(w_1 y_1, \ldots, w_{n/2} y_{n/2}, w_{n/2} y_{n/2}, \ldots, w_1, y_1)\,\mathrm{diag}(1, \ldots, 1, c_{n/2}, \ldots, c_1)$$

where $c_i = k_j(y_{n-i+1}/y_i)$ or $c_i = y_{n-i+1}/(k_j y_i)$ as determined by the permutation P.

Thus, we have that $z_1 n = \hat{h}_3 \hat{t}_3$, where

$$\hat{h}_3 = \hat{h}_2 \cdot \mathrm{diag}(w_1 y_1, \ldots, w_{n/2} y_{n/2}, w_{n/2} y_{n/2}, \ldots, w_1, y_1)$$

and

$$\hat{t}_3 = \mathrm{diag}(1, \ldots, 1, c_{n/2}, \ldots, c_1) \in (H \cdot T)_k,$$

where $c_i = k_j^*(y_{n-i+1}/y_i)$ where $k_j^* = k_j$ or k_j^{-1} as determined by the permutation P. Let $\prod k_i$ represent the product of k_j^*'s such that $k_j^* = k_j$, and let $\prod k_j$ represent the product of k_j^*'s such that $k_j^* = k_j^{-1}$. Now,

$$\begin{aligned}
\det(\hat{t}_3) &= c_1 \cdots c_{n/2} \\
&= (k_1^* \cdots k_i^*) \frac{y_n \cdots y_{n/2+1}}{y_1 \cdots y_{n/2}} \\
&= \pm \frac{k_1^* \cdots k_{\frac{n}{2}}^*}{(y_1 \cdots y_{n/2})^2} \\
&= \frac{\prod k_i}{\prod k_j} \frac{1}{(y_1 \cdots y_{n/2})^2} \in \pm\overline{\det(\hat{t}_1)}
\end{aligned}$$

since $y_n = \pm \frac{1}{y_1 \cdots y_{n-1}}$ and $\det(\hat{t}_1) = \prod k_i \prod k_j \Rightarrow \frac{\prod k_i}{\prod k_j} = \frac{\det(\hat{t}_1)}{(\prod k_j)^2}$.

Since the y_j's can be anything, there exists an $n \in (H \cdot T)_k \cap N_{G_k}(A_1)$ such that $z_1 n = \hat{h}_3 \hat{t}_3 \in (H \cdot T)_k$ with $\det(\hat{t}_3) = s$ for each $s \in \pm\overline{\det(\hat{t}_1)}$. Therefore, all elements of $(H \cdot T)_k$ that correspond to T_2 are of the form $\hat{h}_3 \hat{t}_3$ with

$$\hat{t}_3 = \mathrm{diag}(1, \ldots, 1, c_{n/2}, \ldots, c_1) \in (H \cdot T)_k$$

and $\det(\hat{t}_3) = s \in \pm\overline{\det(\hat{t}_1)}$ and there exists such a matrix for each $s \in \pm\overline{\det(\hat{t}_1)}$.

Now, let $h \in H$ and let $g = \hat{h}\hat{t} \in (H \cdot T)_k$ with $\hat{t} = \mathrm{diag}(1, \ldots, 1, k_{n/2}, \ldots, k_1)$ and $d = \det(\hat{t}_1) = k_1 \cdots k_{n/2}$ be an element corresponding to T_2. Then $hg = h\hat{h}\hat{t} = \hat{h}_1 \hat{t}$

corresponds to a maximal (θ, k)-split torus H-conjugate to T_2. Note that $\hat{t}$ remains unchanged by the left multiplication. Hence, all elements of $(H \cdot T)_k$ that correspond to maximal (θ, k)-split tori H-conjugate to T_2 are of the form $\hat{h}_3\hat{t}_3$ with $\hat{t}_3 = \operatorname{diag}(1, \ldots, 1, c_{n/2}, \ldots, c_1) \in (H \cdot T)_k$ and $\det(\hat{t}_3) = s \in \pm\overline{\det(\hat{t}_1)}$ and there exists such a matrix for each $s \in \pm\overline{\det(\hat{t}_1)}$.

Let $s \in \pm\overline{\det(\hat{t})}$. Suppose there exists a $g_1 = \hat{h}_4\hat{t}_4 \in (H \cdot T)_k$ such that $\hat{t}_4 = \operatorname{diag}(1, \ldots, 1, d_{n/2}, \ldots, d_1)$ and $\det(\hat{t}_4) = s$. We know there exists $g_2 = \hat{h}_3\hat{t}_3 \in (H \cdot T)_k$ corresponding to T_2 with $\hat{t}_3 = \operatorname{diag}(1, \ldots, 1, c_{n/2}, \ldots, c_1) \in (H \cdot T)_k$ and $\det(\hat{t}_3) = s$. Let $n_1 = \operatorname{diag}(1, \ldots, 1, c_{n/2}/d_{n/2}, \ldots, c_1/d_1) \in (H \cdot T)_k \cap N_{G_k}(T_1)$. Then let $g_3 = g_1 n = \hat{h}_4\hat{t}_4 n_1 = \hat{h}_4\hat{t}_3$, which corresponds to the same maximal (θ, k)-split torus as g_1. Now, $g_3 g_2^{-1} = \hat{h}_4\hat{t}_3\hat{t}_3^{-1}\hat{h}_3^{-1} = \hat{h}_4\hat{h}_3^{-1} = h_1 \in H$. Thus, $g_3 = h_1 g_2$, and so g_3 corresponds to a maximal (θ, k)-split torus H-conjugate to T_2. Hence, g_1 corresponds to a maximal (θ, k)-split torus H-conjugate to T_2. Therefore, the set of all elements of $(H \cdot T)_k$ corresponding to a maximal (θ, k)-split torus H-conjugate to T_2 is exactly

$$\left\{\hat{h}_1\hat{t}_1 \mid \hat{h}_1 \in \hat{H},\ \hat{t}_1 = \operatorname{diag}(1, \ldots, 1, c_{n/2}, \ldots, c_1) \in \hat{T} \text{ and } \det(\hat{t}_1) \in \pm\overline{\det(\hat{t})}\right\}.$$

Hence, $|\mathcal{A}_\theta|$ is at most $|\frac{(k^*/(k^*)^2)}{(\pm 1)}|$. □

Corollary 6.3.8. *Let $\theta = \operatorname{Inn}_{I^*_{n-i,i}}$ such that $i = n/2$. Note that n must be even. Additionally, let k be $\mathbb{C}$, $\mathbb{R}$, $\mathbb{F}_p$ with $p \neq 2$, or $\mathbb{Q}_p$. Then the number of H_k-conjugacy classes of maximal (θ, k)-split tori equals $|\frac{(k^*/(k^*)^2)}{(\pm 1)}|$.*

Proof. We need to now determine what $\det(\hat{t}_1)$ in (6.1) can equal. In particular, we need to figure out the square classes $m \in k^*/(k^*)^2$ such that $\det(\hat{t}_1) \in \overline{m}$. Observe that we can embed $\begin{pmatrix} a & by \\ b & ay \end{pmatrix}$ with $y = \frac{1}{a^2 - b^2}$ from Lemma 5.2.1 into $(H \cdot T)_k$. Indeed,

$$\begin{pmatrix} \mathrm{Id}_{(n-2)/2 \times (n-2)/2} & 0 & 0 & 0 \\ 0 & a & by & 0 \\ 0 & b & ay & 0 \\ 0 & 0 & 0 & \mathrm{Id}_{(n-2)/2 \times (n-2)/2} \end{pmatrix} \in (H \cdot T)_k.$$

We can then separate into $h_1 \in \hat{H}$ and $t_1 \in \hat{T}$ such that $t_1 = \operatorname{diag}(1, \ldots, 1, y, 1, \ldots, 1)$, where y is in the $n/2 + 1$ place. By Propositions 5.8.2, 5.9.2 and 5.9.5, we can obtain a y for any square class $m \in k^*/(k^*)^2$ for both $k = \mathbb{F}_p$ with $p \neq 2$ and $k = \mathbb{Q}_p$ (any p). Clearly, we can do the same for $\mathbb{C}$ and $\mathbb{R}$ as discussed in Sections 5.5 and 5.6. □

Theorem 6.3.9. *Let $\theta = \mathrm{Inn}_{L_{n,x}}$. Note that n must be even and $x \in k^*/(k^*)^2$, $x \neq 1$ mod $(k^*)^2$. Then the number of H_k-conjugacy classes of maximal (θ, k)-split tori is at most $|\frac{(k^*/(k^*)^2)}{(\pm 1)}|$.*

Proof. Let H_k be as in 6.2.2. Then

$$(H \cdot T)_k = \left\{ \begin{pmatrix} a_{11}x_1 & a_{12}x_2 & \cdots & a_{1,n-1}x_{n-1} & a_{1,n}x_n \\ xa_{12}x_1 & a_{11}x_2 & \cdots & xa_{1,n}x_{n-1} & a_{1,n-1}x_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1}x_1 & a_{n-1,2}x_2 & \cdots & a_{n-1,n-1}x_{n-1} & a_{n-1,n}x_n \\ xa_{n-1,2}x_1 & a_{n-1,1}x_2 & \cdots & xa_{n-1,n}x_{n-1} & a_{n-1,n-1}x_n \end{pmatrix} \mid a_{ij} \in \bar{k},\ x_j \in \bar{k}^*, \text{ and } a_{ij}x_l \in k \right\}.$$

Let $z_{11} = a_{11}x_1 \Rightarrow a_{11} = z_{11}/x_1 \Rightarrow a_{11}x_2 = z_{11}x_2/x_1 = z_{11}k_1$, where $k_1 = x_2/x_1 \in k$. Let $z_{12} = a_{12}x_1 \Rightarrow a_{12} = z_{12}/x_1 \Rightarrow a_{12}x_2 = z_{12}x_2/x_1 = z_{12}k_1$, where $k_1 = x_2/x_1 \in k$. Continue in this fashion for $j \leq n$, obtaining that if $a_{j1}x_1 = z_{j1} \Rightarrow a_{j1}x_2 = z_{j1}k_1$ and if $z_{j2} = a_{j2}x_1 \Rightarrow a_{j2}x_2 = z_{j2}k_1$. Repeat this process for pair of columns containing x_3 and x_4 with $k_2 = x_4/x_3$ and continue repeating through pair of columns containing x_{n-1} and x_n with $k_{n/2} = x_n/x_{n-1}$. This rewriting of the elements of $(H \cdot T)_k$ gives the following, where $\hat{H} = \{A \in \mathrm{GL}(n,k) \mid \theta(A) = A\}$ and $\hat{T} = \{\mathrm{diag}(t_1, \ldots, t_n) \mid t_i \in k, \det \neq 0\}$.

$$(H \cdot T)_k \subset \left\{ \hat{h}\hat{t} \mid \hat{h} = \begin{pmatrix} z_{11} & z_{12} & z_{13} & z_{14} & \cdots & z_{1,n-1} & z_{1,n} \\ xz_{12} & z_{11} & xz_{14} & z_{13} & \cdots & xz_{1,n} & z_{1,n-1} \\ z_{31} & z_{32} & \ddots & & & \vdots & \vdots \\ xz_{32} & z_{31} & & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & & \ddots & \vdots & \vdots \\ z_{n-1,1} & z_{n-1,2} & \cdots & \cdots & \cdots & z_{n-1,n-1} & z_{n-1,n} \\ xz_{n-1,2} & z_{n-1,1} & \cdots & \cdots & \cdots & xz_{n-1,n} & z_{n-1,n-1} \end{pmatrix} \in \hat{H}, \hat{t} = \mathrm{diag}(1, k_1, 1, k_2, \ldots, 1, k_{n/2}) \in \hat{T} \right\}.$$

Equality follows from Lemma 6.3.5.

Let $A_1 = \{\text{diag}(a_1, a_1^{-1}, \ldots, a_{n/2}, a_{n/2}^{-1}) \mid a_i \in k^*\}$. We know A_1 is a maximal (θ, k)-split torus. Let $A_2 \neq A_1$ be a maximal (θ, k)-split torus, and let $z_1 \in (H \cdot T)_k$ such that $z_1 A_1 z_1^{-1} = A_2$. By Lemma 6.3.3, we know that if $z_2 \in (H \cdot T)_k$ such that $z_2 A_1 z_2^{-1} = A_2$, then $z_2 = z_1 n$, where $n \in (H \cdot T)_k \cap N_{G_k}(A_1)$. Now we can write each $n \in (H \cdot T)_k \cap N_{G_k}(A_1)$ in the form $n = Pt$, where P is a permutation matrix such that $P \in \hat{H}$ and $t = \text{diag}(y_1, \ldots, y_n) \in \hat{T}$ with $\det(P) = \pm 1$ and $\det(t) = \det(P)^{-1} = \pm 1$. Note that $y_n = \pm \frac{1}{y_1 \cdots y_{n-1}}$. Also, observe that P acts on A_1 by either interchanging a_j and a_j^{-1}, interchanging the pairs (a_j, a_j^{-1}) and (a_i, a_i^{-1}), or any combination of these.

Now let $z_1 = \hat{h}_1 \hat{t}_1$ as in (6.2), $d = \det(\hat{t}_1) = k_1 \cdots k_{n/2}$, and

$$n = Pt \in (H \cdot T)_k \cap N_{G_k}(A_1).$$

Consider $z_1 n = \hat{h}_1 \hat{t}_1 P t$. We know P permutes $\hat{t}_1$ via conjugation, so let $\hat{t}_2 = P^{-1} \hat{t}_1 P$. Note that because of how P acts on T_1, we have that

$$\hat{t}_2 = \text{diag}(w_1, w_1^*, \ldots, w_{n/2}, w_{n/2}^*),$$

where the pair (w_i, w_i^*) is some pair $(1, k_j)$ in any order. Also, note that $\det(\hat{t}_2) = \det(\hat{t}_1)$. So, we have that $z_1 n = \hat{h}_1 \hat{t}_1 P t = \hat{h}_1 P \hat{t}_2 t = \hat{h}_2 \hat{t}_2 t$, where $\hat{h}_2 = \hat{h}_1 P \in \hat{H}$. Hence, in order to get $z_1 n$ in the form of (6.2), we need to just rewrite $\hat{t}_2 t$ in the form of (6.2).

Now,

$$\hat{t}_2 t = \text{diag}(w_1 y_1, w_1^* y_2, \ldots, w_{n/2} y_{n-1}, w_{n/2}^* y_n),$$

where we recall that the pair (w_i, w_i^*) is some pair $(1, k_j)$ from $\hat{t}_1$ and $y_n = \pm(y_1 \cdots y_{n-1})^{-1}$ by previous definitions. We need to write $\hat{t}_2 t$ as

$$\text{diag}(b_1, b_1, \ldots, b_{n/2}, b_{n/2}) \,\text{diag}(1, c_1, \ldots, 1, c_{n/2}) \in (H \cdot T)_k$$

for some $b_i, c_j \in k$. We can do this by the following method. Let $w_i y_i = b_i \Rightarrow w_i = b_i / y_i$. We know $w_i = 1$ or $w_i = k_j$ for some j determined by P. If $w_i = 1$, then $1 = b_i / y_i$ and $w_i^* = k_j$. Hence, $w_i^* y_{i+1} = (b_i k_j y_{i+1}) / y_i = b_i c_i$ where $c_i =$

$k_j(y_{i+1}/y_i)$. Also, if $w_i = k_j$, then $w_i^* = 1$ and $k_j = b_i/y_i \Rightarrow 1 = b_i/(k_j y_i)$. Hence, $w_i^* y_{i+1} = (b_i y_{i+1})/(k_j y_i) = b_i c_i$ where $c_i = y_{i+1}/(k_j y_i)$. Therefore, we obtain that

$$\hat{t}_2 t = \operatorname{diag}(w_1 y_1, w_1, y_1 \ldots, w_{n/2} y_{n/2}, w_{n/2} y_{n/2}) \operatorname{diag}(1, c_1, \ldots, 1, c_{n/2})$$

where $c_i = k_j(y_{i+1}/y_i)$ or $c_i = y_{i+1}/(k_j y_i)$ as determined by the permutation P.

Thus, we have that $z_1 n = \hat{h}_3 \hat{t}_3$, where

$$\hat{h}_3 = \hat{h}_2 \cdot \operatorname{diag}(w_1 y_1, \ldots, w_{n/2} y_{n/2}, w_{n/2} y_{n/2}, \ldots, w_1, y_1)$$

and $\hat{t}_3 = \operatorname{diag}(1, c_1, \ldots, 1, c_{n/2}) \in (H \cdot T)_k$ where $c_i = k_j^*(y_{i+1}/y_i)$ where $k_j^* = k_j$ or k_j^{-1} as determined by the permutation P. Let $\prod k_i$ represent the product of k_j^*'s such that $k_j^* = k_j$, and let $\prod k_j$ represent the product of k_j^*'s such that $k_j^* = k_j^{-1}$. Now,

$$\begin{aligned}
\det(\hat{t}_3) &= c_1 \cdots c_{n/2} \\
&= (k_1^* \cdots k_i^*) \frac{y_n \cdots y_{n/2+1}}{y_1 \cdots y_{n/2}} \\
&= \pm \frac{k_1^* \cdots k_{\frac{n}{2}}^*}{(y_1 \cdots y_{n/2})^2} \\
&= \frac{\prod k_i}{\prod k_j} \frac{1}{(y_1 \cdots y_{n/2})^2} \in \pm \overline{\det(\hat{t}_1)}
\end{aligned}$$

since $y_n = \pm \frac{1}{y_1 \cdots y_{n-1}}$ and $\det(\hat{t}_1) = \prod k_i \prod k_j \Rightarrow \frac{\prod k_i}{\prod k_j} = \frac{\det(\hat{t}_1)}{(\prod k_j)^2}$.

Since the y_j's can be anything, there exists an $n \in (H \cdot T)_k \cap N_{G_k}(A_1)$ such that $z_1 n = \hat{h}_3 \hat{t}_3 \in (H \cdot T)_k$ with $\det(\hat{t}_3) = s$ for each $s \in \pm \overline{\det(\hat{t}_1)}$. Therefore, all elements of $(H \cdot T)_k$ that correspond to T_2 are of the form $\hat{h}_3 \hat{t}_3$ with $\hat{t}_3 = \operatorname{diag}(1, c_1, \ldots, 1, c_{n/2}) \in (H \cdot T)_k$ and $\det(\hat{t}_3) = s \in \pm \overline{\det(\hat{t}_1)}$ and there exists such a matrix for each $s \in \pm \overline{\det(\hat{t}_1)}$.

Now, let $h \in H$ and let $g = \hat{h}\hat{t} \in (H \cdot T)_k$ with $\hat{t} = \operatorname{diag}(1, k_1, \ldots, 1, k_{n/2})$ and $d = \det(\hat{t}_1) = k_1 \cdots k_{n/2}$ be an element corresponding to T_2. Then $hg = h\hat{h}\hat{t} = \hat{h}_1 \hat{t}$ corresponds to a maximal (θ, k)-split torus H-conjugate to T_2. Note that $\hat{t}$ remains unchanged by the left multiplication. Hence, all elements of $(H \cdot T)_k$ that correspond to maximal (θ, k)-split tori H-conjugate to T_2 are of the form $\hat{h}_3 \hat{t}_3$ with $\hat{t}_3 = \operatorname{diag}(1, c_1 \ldots, 1, c_{n/2}) \in (H \cdot T)_k$ and $\det(\hat{t}_3) = s \in \pm \overline{\det(\hat{t}_1)}$ and there exists such a matrix for each $s \in \pm \overline{\det(\hat{t}_1)}$.

Let $s \in \pm\overline{\det(\hat{t})}$. Suppose there exists a $g_1 = \hat{h}_4\hat{t}_4 \in (H \cdot T)_k$ such that $\hat{t}_4 = \operatorname{diag}(1, \dots, 1, d_{n/2}, \dots, d_1)$ and $\det(\hat{t}_4) = s$. We know there exists some $g_2 = \hat{h}_3\hat{t}_3 \in (H \cdot T)_k$ corresponding to T_2 with $\hat{t}_3 = \operatorname{diag}(1, c_1 \dots, 1, c_{n/2}) \in (H \cdot T)_k$ and $\det(\hat{t}_3) = s$. Let $n_1 = \operatorname{diag}(1, c_1/d_1 \dots, 1, c_{n/2}/d_{n/2}) \in (H \cdot T)_k \cap N_{G_k}(T_1)$. Then let $g_3 = g_1 n = \hat{h}_4\hat{t}_4 n_1 = \hat{h}_4\hat{t}_3$, which corresponds to the same maximal (θ, k)-split torus as g_1. Now, $g_3 g_2^{-1} = \hat{h}_4\hat{t}_3\hat{t}_3^{-1}\hat{h}_3^{-1} = \hat{h}_4\hat{h}_3^{-1} = h_1 \in H$. Thus, $g_3 = h_1 g_2$, and so g_3 corresponds to a maximal (θ, k)-split torus H-conjugate to T_2. Hence, g_1 corresponds to a maximal (θ, k)-split torus H-conjugate to T_2. Therefore, the set of all elements of $(H \cdot T)_k$ corresponding to a maximal (θ, k)-split torus H-conjugate to T_2 is exactly

$$\left\{\hat{h}_1\hat{t}_1 \mid \hat{h}_1 \in \hat{H},\ \hat{t}_1 = \operatorname{diag}(1, \dots, 1, c_{n/2}, \dots, c_1) \in \hat{T} \text{ and } \det(\hat{t}_1) \in \pm\overline{\det(\hat{t})}\right\}.$$

Hence, $|\mathcal{A}_\theta|$ is at most $|\frac{(k^*/(k^*)^2)}{(\pm 1)}|$.

□

Theorem 6.3.10. *Let $\theta(A) = \operatorname{Inn}_{J_{2m}}(A^T)^{-1}$ for all $A \in G$. Then there exists one H_k-conjugacy class of maximal (θ, k)-split tori with representative*

$$A_2 = \{\operatorname{diag}(a_1, a_2, \dots, a_n) \mid a_i \in k \text{ and } a_1 = a_{n/2+1}, a_2 = a_{n/2+2}, \dots, a_{n/2} = a_n\}.$$

Proof. Note that n is even. We know A_2 is a maximal (θ, k)-split torus. Note that $T \subset (H \cdot T)_k \cap N_{G_k}(A_2)$. Let $A_3 \neq A_2$ be a maximal (θ, k)-split torus, and let $z_1 \in (H \cdot T)_k$ such that $z_1 A_2 z_1^{-1} = A_3$. By Lemma 6.3.3, we know $z_1 n$, where $n \in (H \cdot T)_k \cap N_{G_k}(A_2)$, also corresponds to A_3 via conjugating A_2.

Let $z_1 = ht$, where $h = (a_{ij}) \in H$ and $t = \operatorname{diag}(x_1, x_2, \cdots, x_n) \in T$, with $x_n = (x_1 \cdots x_{n-1})^{-1}$. By Lemma 6.2.5, the entries of h satisfy the equations

$$\begin{aligned} &a_{1i}a_{\frac{n}{2}+1,j} + a_{2i}a_{\frac{n}{2}+2,j} + \cdots + a_{\frac{n}{2},i}a_{nj} - a_{\frac{n}{2}+1,i}a_{1j} - a_{\frac{n}{2}+2,i}a_{2j} - \cdots \\ &\cdots - a_{ni}a_{\frac{n}{2},j} = \begin{cases} 1 & i = s \text{ and } j = \frac{n}{2} + s, \text{ where } 1 \leq s \leq \frac{n}{2} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Now the entries of z_1 are $a_{ij}x_j$. Using the above equations, we obtain that $x_s x_{\frac{n}{2}+s} \in k$ for all $1 \leq s \leq \frac{n}{2}$. Construct

$$n = \operatorname{diag}((x_1 x_{\frac{n}{2}+1})^{-1}, (x_2 x_{\frac{n}{2}+2})^{-1}, \dots, (x_{\frac{n}{2}} x_n)^{-1}, 1, \dots, 1).$$

Then the entries of $z_1 n$ are $a_{ij}x^{-1}_{j+\frac{n}{2}}$ for $j \leq \frac{n}{2}$ and $a_{ij}x_j$ for $j > \frac{n}{2}$. It can be easily shown that the entries of $z_1 n$ satisfy the equations of H_k, hence A_3 is H_k-conjugate to A_2. Since A_3 was arbitrary, the result follows. □

The classification results for the rest of the outer involutions still needs to be completed and is not given here.

Example 6.3.11. Let $\theta = \mathrm{Inn}_{I^*_{n-i,i}}$ such that $i = n/2$. Note that n must be even. Recall that the number of H_k-conjugacy classes of maximal (θ, k)-split tori equals $|\frac{(k^*/(k^*)^2)}{(\pm 1)}|$. This means that if -1 is a square, the number of H_k-conjugacy classes equals $|k^*/(k^*)^2|$. If -1 is not a square, then the number of H_k-conjugacy classes equals $|k^*/(k^*)^2|/2$. So for the various fields k, we have that the number of H_k-conjugacy classes of maximal (θ, k)-split tori equals

- 1 for $k = \mathbb{C}$, $\mathbb{R}$, or $\mathbb{F}_p$ with $p \cong 3 \mod 4$.
- 2 for $k = \mathbb{F}_p$ with $p \cong 1 \mod 4$ or $k = \mathbb{Q}_p$ with $p \cong 3 \mod 4$.
- 4 for $k = \mathbb{Q}_p$ with $p \cong 1 \mod 4$ or $k = \mathbb{Q}_2$.

6.4 H_k-conjugacy classes of maximal θ-stable k-split tori containing a maximal (θ, k)-split torus

Let A_1 be one the representative maximal (θ, k)-split torus from Section 6.2 for each involution. In order to continue our classification, we need to determine the H_k-conjugacy classes of the maximal k-split tori containing A_1 for each involution from the previous section. To do this, we will make use of the $Z_{G_k}(A_1)$.

Lemma 6.4.1. *1. Let $\theta = \mathrm{Inn}_{I^*_{n-i,i}}$ with $i \neq n/2$ or $(n+1)/2$. Then the $Z_{G_k}(A_1)$ consists of the matrices*

$$\begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & \vdots \\ \vdots & 0 & x_{2i} & 0 \\ 0 & \cdots & 0 & \mathrm{GL}(n-2i,k) \end{pmatrix}.$$

2. Let $\theta = \mathrm{Inn}_{L_{n,x}}$ or $\theta = \mathrm{Inn}_{I^*_{n-i,i}}$ with $i = (n+1)/2$ or $i = n/2$. Then $Z_{G_k}(A_1) = T$.

Proof. Let $A = (a_{ij}) \in Z_{G_k}(A_1)$. We know $At = tAt$ for all $\in A_1$. The entries on the diagonal of A_1 greatly restrict the possibilities for A, creating zeros in the off-diagonal entries of A. In fact, the only way to obtain more than T in the $Z_{G_k}(A_1)$ is when there are multiple ones on the diagonal. This only happens for $\theta = \mathrm{Inn}_{I^*_{n-i,i}}$ with $i \neq n/2$ or $(n+1)/2$. In this case, we obtain $\mathrm{GL}(n-2i, k)$ in the lower right-hand corner. □

Proposition 6.4.2. *Let $\theta = \mathrm{Inn}_{I^*_{n-i,i}}$ with $i < n/2$, and let*

$$A_1 = \{\mathrm{diag}(a_1, \ldots, a_i, a_i^{-1}, \ldots, a_1^{-1}, 1, \ldots, 1) \mid a_i \in k^*\}$$

be the representative maximal (θ, k)-split torus as described in Section 5. Then there exists only one H_k-conjugacy class of maximal k-split tori containing A_1.

Proof. For $\theta = \mathrm{Inn}_{I^*_{n-i,i}}$ with $i \neq n/2$ or $(n+1)/2$, we know from Lemma 6.4.1 that $Z_{G_k}(A_1)$ consists of the matrices

$$\begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & \vdots \\ \vdots & 0 & x_{2i} & 0 \\ 0 & \cdots & 0 & \mathrm{GL}(n-2i, k) \end{pmatrix}. \tag{6.2}$$

Clearly, $A_1 \subset T \subset Z_{G_k}(A_1)$. Now, if $T_1 \neq T$ is a maximal k-split torus containing A_1, then $T_1 \subset Z_{G_k}(A_1)$. Thus, there exists $g \in Z_{G_k}(A_1)$ such that $gTg^{-1} = T_1$. Let $z \in T$. Then clearly $gzT(gz)^{-1} = T_1$. Assume g is as in (6.2) and construct

$$z = \mathrm{diag}(x_1^{-1}, \ldots, x_{2i}^{-1}, (x_1 \cdots x_{2i}), 1, \ldots, 1) \in T.$$

Then $gz \in H_k$ where H_k is as defined in Lemma 6.2.1, and hence, T_1 is H_k-conjugate to T. Since T_1 was arbitrary, we obtain only one H_k-conjugacy class.

□

Theorem 6.4.3. *Let $\theta(A) = \mathrm{Inn}_{J_{2m}}(A^T)^{-1}$ for all $A \in G$, and let*

$$A_2 = \{\mathrm{diag}(a_1, a_2, \ldots, a_n) \mid a_i \in k \text{ and } a_1 = a_{n/2+1}, a_2 = a_{n/2+2}, \ldots, a_{n/2} = a_n\}$$

be the representative maximal (θ, k)-split torus as described in Section 5. Then there exists one H_k-conjugacy class of maximal θ-stable k-split tori containing a maximal (θ, k)-split torus with representative T.

Proof. Let

$$A_2 = \{\mathrm{diag}(a_1, a_2, \dots, a_n) \mid a_i \in k \text{ and } a_1 = a_{n/2+1}, a_2 = a_{n/2+2}, \dots, a_{n/2} = a_n\}$$

be the representative maximal (θ, k)-split torus. It can be easily computed that $Z_{G_k}(A_2)$ consists of the matrices

$$\begin{pmatrix} a_{11} & 0 & \cdots & 0 & a_{1,\frac{n}{2}+1} & 0 & \cdots & 0 \\ 0 & a_{22} & & & 0 & a_{2,\frac{n}{2}+2} & & \vdots \\ \vdots & & \ddots & & & \ddots & \ddots & 0 \\ 0 & & & \ddots & & & 0 & a_{\frac{n}{2},n} \\ a_{\frac{n}{2}+1,1} & 0 & & & \ddots & & & 0 \\ 0 & a_{\frac{n}{2}+2,2} & \ddots & & & \ddots & & \vdots \\ \vdots & \ddots & \ddots & 0 & & & \ddots & 0 \\ 0 & \cdots & 0 & a_{n,\frac{n}{2}} & 0 & \cdots & 0 & a_{nn} \end{pmatrix}. \qquad (6.3)$$

Clearly, $A_1 \subset T \subset Z_{G_k}(A_1)$. Now, if $T_1 \neq T$ is a maximal k-split torus containing A_1, then $T_1 \subset Z_{G_k}(A_1)$. Thus, there exists $g \in Z_{G_k}(A_1)$ such that $gTg^{-1} = T_1$. Let $z \in T$. Then clearly $gzT(gz)^{-1} = T_1$. Assume g is as in (6.3) and consider g separated into $\frac{n}{2}$ by $\frac{n}{2}$ blocks A, B, C, and D, written $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Then

$$1 = \det(g) = \det(AD - BC) = \prod_{i=1}^{\frac{n}{2}} (a_{ii}a_{\frac{n}{2}+i,\frac{n}{2}+i} - a_{\frac{n}{2}+i,i}a_{i,\frac{n}{2}+i}).$$

Let $b_i = a_{ii}a_{\frac{n}{2}+i,\frac{n}{2}+i} - a_{\frac{n}{2}+i,i}a_{i,\frac{n}{2}+i}$. Then $1 = \prod_{i=1}^{\frac{n}{2}} b_i \Rightarrow b_{\frac{n}{2}} = (b_1 \cdots b_{\frac{n}{2}-1})^{-1}$.

Construct

$$z = \mathrm{diag}(b_1^{-1}, \dots, b_{2i}^{-1}, 1, \dots, 1) \in T.$$

Then the entries of gz satisfy the equations $1 = \frac{a_{ii}a_{\frac{n}{2}+i,\frac{n}{2}+i} - a_{\frac{n}{2}+i,i}a_{i,\frac{n}{2}+i}}{b_i}$ for all $1 \leq i \leq \frac{n}{2}$. Thus, $gz \in H_k$ where H_k is as defined in Lemma 6.2.5, and hence, T_1 is H_k-conjugate to T. Thus, there is only one H_k-conjugacy class of maximal k-split tori containing a maximal (θ, k)-split torus. □

6.5 H_k-conjugacy classes of θ-stable maximal k-split tori containing a maximal k-split torus in H_k

Proposition 6.5.1. *Let $\theta = \mathrm{Inn}_{I^*_{n-i,i}}$, and let H_k be the fixed point group of θ. Then H_k contains a maximal k-split torus that is both maximal k-split and maximal in G_k. Therefore, there exists only one H_k-conjugacy class.*

Proof. Recall the fixed point group of θ as defined in Lemma 6.2.1. It is clear here that

$$T^+ = \{\mathrm{diag}(a_1, \ldots, a_i, a_i, \ldots, a_1, b_1, \ldots, b_{n-2i}) \mid a_1^2 \cdots a_i^2 b_1 \cdots b_{n-2i} = 1\}.$$

Now, $Z_{H_k}(T^+)$ consists of matrices of the form

$$\begin{pmatrix} B & 0 & \cdots & 0 \\ 0 & x_1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & x_{n-2i} \end{pmatrix}$$

where B is the $2i$ by $2i$ matrix

$$B = \begin{pmatrix} a_1 & 0 & \cdots & \cdots & 0 & b_1 \\ 0 & \ddots & & & \iddots & 0 \\ \vdots & & a_i & b_i & & \vdots \\ \vdots & & b_i & a_i & & \vdots \\ 0 & \iddots & & & \ddots & 0 \\ b_1 & 0 & \cdots & \cdots & 0 & a_1 \end{pmatrix}.$$

Clearly $\dim(Z_{H_k}(T^+) = n - 1$. Further, the characteristic polynomial is

$$c(\lambda) = (\lambda - (a_1 + b_1))(\lambda - (a_1 - b_1) \cdots (\lambda - (a_i + b_i))(\lambda - (a_i - b_i)(\lambda - x_1) \cdots (\lambda - x_{2n-i}).$$

Thus, $Z_{H_k}(T^+)$ is conjugate T, and we have the above result. □

Theorem 6.5.2. *Let $\theta = \mathrm{Inn}_{L_{n,x}}$, and let H_k be the fixed point group of θ. Then H_k contains a maximal k-split torus, namely*

$$S = \{\mathrm{diag}(a_1, a_1, \ldots, a_{\frac{n}{2}}, a_{\frac{n}{2}}) \mid a_1^2 \cdots a_{\frac{n}{2}}^2 = 1\}.$$

Proof. Recall the fixed point group of θ as defined in Lemma 6.2.2. It is clear here that

$$T^+ = \{\operatorname{diag}(a_1, a_1, \ldots, a_{\frac{n}{2}}, a_{\frac{n}{2}}) \mid a_1^2 \cdots a_{\frac{n}{2}}^2 = 1\}.$$

Now, $Z_{H_k}(T^+)$ consists of matrices of the form

$$\begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & A_{\frac{n}{2}} \end{pmatrix}, \tag{6.4}$$

where $A_i = \begin{pmatrix} a_i & b_i \\ xb_i & a_i \end{pmatrix}$. The eigenvalues of (6.4) are $a_i \pm b_i\sqrt{x}$ for all $1 \le i \le \frac{n}{2}$. Since $x \neq 1 \mod (k^*)^2$, the only part of $Z_{H_k}(T^+)$ that is k-split is T^+. Thus, T^+ is not contained in any bigger k-split torus in H_k. Hence, we cannot increase the size of T^+, so we obtain the above result. □

6.6 Classification of $P_k \backslash G_k / H_k$ for $\theta = \operatorname{Inn}_{J_{2m}}$

Theorem 6.6.1. *Let $\theta(A) = \operatorname{Inn}_{J_{2m}}(A^T)^{-1}$ for all $A \in G$, and let*

$$A_2 = \{\operatorname{diag}(a_1, a_2, \ldots, a_n) \mid a_i \in k \textit{ and } a_1 = a_{n/2+1}, a_2 = a_{n/2+2}, \ldots, a_{n/2} = a_n\}$$

be the representative maximal (θ, k)-split torus as described in Section 5. Then there exists one H_k-conjugacy class of maximal θ-stable k-split tori with representative T.

Proof. Let

$$A_2 = \{\operatorname{diag}(a_1, a_2, \ldots, a_n) \mid a_i \in k \text{ and } a_1 = a_{n/2+1}, a_2 = a_{n/2+2}, \ldots, a_{n/2} = a_n\}$$

be the representative maximal (θ, k)-split torus.

Now, we know $T = T^+T^-$, where clearly $T^- = A_2$ here. By Theorem 6.4.3, there is only one H_k-conjugacy class of maximal θ-stable k-split tori containing a maximal (θ, k)-split torus with representative T. Since $T^+ \in H_k$, we easily compute that

$$T^+ = \{\operatorname{diag}(x_1, x_2, \ldots, x_n) \mid x_{\frac{n}{2}+s} = x_s^{-1} \text{ for all } 1 \le s \le \frac{n}{2}\}.$$

It can also be easily computed that $Z_{G_k}(T^+) = T$. Thus, T^+ is maximal k-split in H_k. Thus, we cannot increase the size of T^+, and hence we obtain only one H_k-conjugacy class of θ-stable maximal k-split tori with representative T. □

Chapter 7

Standard Tori Results

The results from Chapter 6 were specifically for classifying the H_k-conjugacy classes of either the maximal $(\theta.k)$-split tori (top level of tori lattice) or the θ-stable maximal k-split tori with maximal T^+ in H_k (bottom level of tori lattice). Next we need to determine the H_k-conjugacy of θ-stable maximal k-split tori in all the levels between. In fact, it would be beneficial to be able to determine the middle levels just by knowing the top level or bottom level classification. To do this, we will introduce the concept of *standard tori*, and prove many general results which will aid in our classification.

Remark 7.0.2. In all of the following, G is any reductive connected linear algebraic group. These results do apply to $\mathrm{SL}(n,k)$, but they are not specific to $\mathrm{SL}(n,k)$.

Let $\mathcal{T}$ denote the set of all θ-stable maximal k-split tori of G. Let $T \in \mathcal{T}$. Recall that $T = T^+T^-$, where $T^+ = T^+_\theta$ and $T^- = T^-_\theta$. For any other involutions of T we shall keep the subscript to avoid confusion. We will continue to use the notation that A_i is maximal (θ, k)-split torus, T_i is a θ-stable maximal k-split torus, and S_i is a θ-stable maximal k-split torus containing a maximal k-split torus in H_k.

Definition 7.0.3. Let $T_1, T_2 \in \mathcal{T}$ be such that $T_1^- \subset T_2^-$ and $T_2^+ \subset T_1^+$. Then the pair (T_1, T_2) is called *standard*, and we say T_1 is standard with respect to T_2.

7.1 One H_k-conjugacy class of θ-stable maximal k-split tori in the top level

Let's assume there exists only one H_k-conjugacy class of θ-stable maximal k-split tori containing a maximal (θ, k)-split torus. Note that we do have cases for which this happens, see Theorems 6.4.2 and 6.6.1. Let T be the representative torus of this H_k-conjugacy class. Let $A \subset T$ be a maximal (θ, k)-split torus. Note $A = T^-$.

Lemma 7.1.1. *Let $T_1 \in \mathcal{T}$. Then there exists $h_1, h_2 \in H_k$ such that $h_1 T_1^- h_1^{-1} \subset A$ and $h_2 T^+ h_2^{-1} \subset T_1^+$, ie, we can make T_1 standard with respect to T.*

Proof. We know $T_1 = T_1^+ T_1^-$. We also know that $T_1^- \subset A_1$, where A_1 is some maximal (θ, k)-split torus. Since there is only one H_k-conjugacy class of maximal (θ, k)-split tori, there exists an $h_1 \in H_k$ such that $h_1 A_1 h_1^{-1} = A$. Thus, $h_1 T_1^- h_1^{-1} \subset h_1 A_1 h_1^{-1} = A$, so we may assume $T_1 \subset A$.

Now, let $G_1 = [Z_{G_k}(h_1 T_1^- h_1^{-1}), Z_{G_k}(h_1 T_1^- h_1^{-1})]$. Then, $h_1 T_1^+ h_1^{-1} \subset G_1$ is maximal k-split in $H_k \cap G_1$. Also, $T^+ \subset G_1$ since $h_1 T_1^- h_1^{-1} \subset A$, and $T^+ \subset S_1$, where S_1 is some maximal k-split torus in $H_k \cap G_1$. Since maximal k-split tori are conjugate, there exists $h_2 \in H_k \cap G_1$ such that $h_2 S_1 h_2^{-1} = h_1 T_1^+ h_1^{-1} \Rightarrow h_1^{-1} h_2 S_1 h_2^{-1} h_1 = T_1^+ \Rightarrow (h_1^{-1} h_2) T^+ (h_1^{-1} h_2)^{-1} \subset T_1^+$, so we may assume $T^+ \subset T_1^+$. Therefore, we can assume every $T_1 \in \mathcal{T}$ is standard with respect to T. □

So for every $T_1 \in \mathcal{T}$, (T_1, T) is a standard pair. To visualize this pair, see Figures 7.1 and 7.2. Each of these pairs gives rise to an involution in $W_{G_k}(T)$ as follows.

Lemma 7.1.2. *Let $T_1 \in \mathcal{T}$. Then,*

1. *There exists $g_1 \in Z_{G_k}(T_1^- T^+)$ such that $g_1 T_1 g_1^{-1} = T$.*
2. *If $n_1 = \theta(g_1) g_1^{-1}$, then $n_1 \in N_{G_k}(T)$.*
3. *If w_1 is the image of n_1 in $W_{G_k}(T)$, then $w_1^2 = \mathrm{Id}$ and $T_{w_1}^+ = (T_1^- T^+)$.*

Proof. 1. T_1 and T are maximal k-split tori of $Z_{G_k}(T_1^- T^+)$, hence this statement is clear since maximal k-split tori are conjugate.

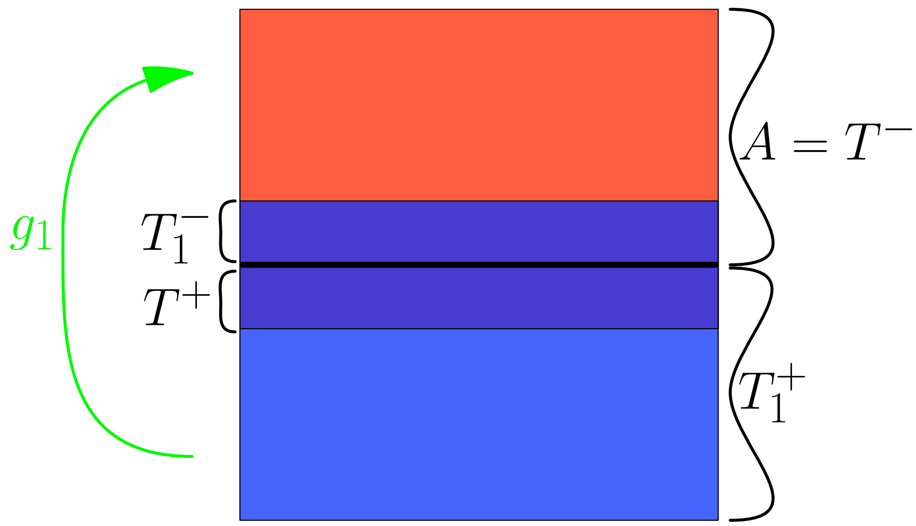

Figure 7.1: Illustration of Lemma 7.1.2 and the map g_1

2. We know $\theta(T_1) = T_1 = g_1^{-1}Tg_1$ and $\theta(T_1) = \theta(g_1^{-1}Tg_1) = g_1^{-1}Tg_1$. Also,

$$\begin{aligned} \theta(g_1)^{-1}\theta(T)\theta(g_1) &= g_1^{-1}Tg_1 \\ \Rightarrow g_1\theta(g_1)^{-1}T\theta(g_1)g_1^{-1} &= T \\ \Rightarrow \theta(g_1)g_1^{-1} &\in N_{G_k}(T). \end{aligned}$$

3. Note that since T is θ-stable, $Z_{G_k}(T_1^-T^+)$ is θ-stable, hence $\theta(g_1) \in Z_{G_k}(T_1^-T^+)$. Now $T = (T_1^-T^+)(g_1T_1^+g_1^{-1} \cap T^-)$. Let $x \in (T_1^-T^+)$. We have $n_1xn_1^{-1} = \theta(g_1)g_1^{-1}xg_1\theta(g_1)^{-1} = \theta(g_1)x\theta(g_1)^{-1} = x$.

   Now, let $x \in (g_1T_1^+g_1^{-1} \cap T^-)$ and write $x = g_1tg_1^{-1}$ for some $t \in T_1^+$. Then,

$$\begin{aligned} n_1xn_1^{-1} &= \theta(g_1)g_1^{-1}xg_1\theta(g_1)^{-1} \\ &= \theta(g_1)g_1^{-1}g_1tg_1^{-1}g_1\theta(g_1)^{-1} \\ &= \theta(g_1)t\theta(g_1)^{-1} \\ &= \theta(g_1)\theta(t)\theta(g_1)^{-1} \\ &= \theta(g_1tg_1^{-1}) = \theta(x) = x^{-1}. \end{aligned}$$

   It follows that

$$\mathrm{Inn}(n_1)|_{T_1^-T^+} = 1 \text{ and } \mathrm{Inn}(n_1)|_{g_1T_1^+g_1^{-1}\cap T^-} = -1.$$

Hence, $w_1^2 = \text{Id}$ and $T^+_{w_1} = T_1^- T^+$.

□

Figures 7.1 and 7.2 illustrate how the standard pair (T_1, T) are contained in each other. T consists of the red and purple parts, and T_1 consists of the blue and purple parts. The purple part is their intersection. We can see in Figure 7.1 that g_1 maps $T_1^+ - T^+$ to $A - T_1^-$ and fixes T_1^- and T^+. And in Figure 7.2, we can see that $\theta(g_1)^{-1}$ maps $A - T_1^-$ to $T_1^+ - T^+$ and fixes T_1^- and T^+. Therefore, we can see that $n_1 = \theta(g_1)g_1^{-1} \in N_{G_k}(T)$, and if w_1 is the image of n_1 in $W_{G_k}(T)$, then $w_1^2 = \text{Id}$ and $T^+_{w_1} = (T_1^- T^+)$.

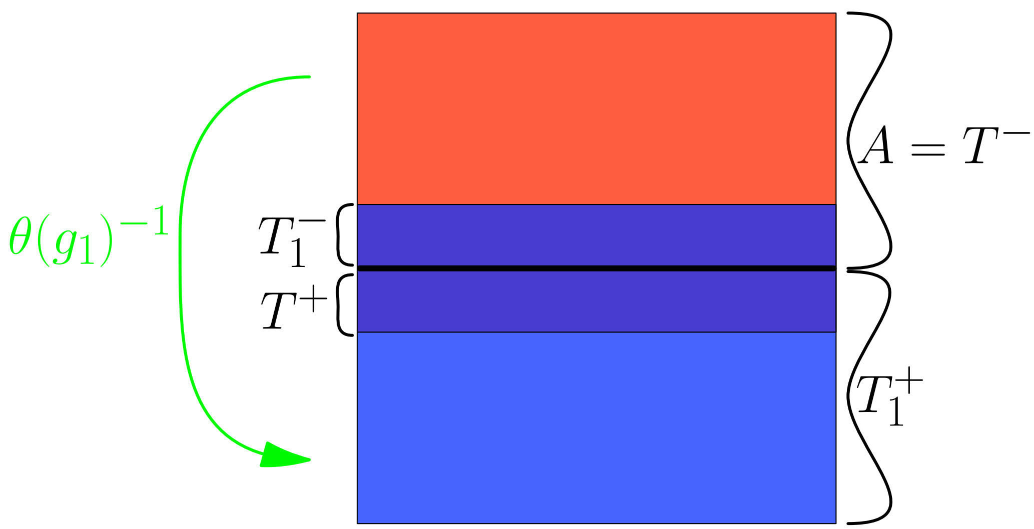

Figure 7.2: Illustration of Lemma 7.1.2 and the map $\theta(g_1)^{-1}$

Definition 7.1.3. Let T_1 and $w_1 \in W_{G_k}(T)$ be as in Lemma 7.1.2. We call w_1 the T_1-standard involution of $W_{G_k}(T)$.

Definition 7.1.4. We say G_k has the (θ, k)*-split reduction property* if there exists only one H_k-conjugacy class of maximal (θ, k)-split tori in $Z_{G_k}(A_1)$ for every (θ, k)-split torus A_1 in G_k.

Theorem 7.1.5. *Suppose G_k has the (θ, k)*-split reduction property. *Let $T_1, T_2 \in \mathcal{T}$ such that they are standard with respect to T. Let w_1 and w_2 be the T_1-standard and*

T_2-standard involutions in $W_{G_k}(T)$, respectively. Then T_1 and T_2 are H_k-conjugate if and only if w_1 and w_2 are conjugate under $W_{H_k}(T)$.

Proof. ($\Rightarrow$) Let $h \in H_k$ such that $hT_1h^{-1} = T_2$. Then $hT_1^-h^{-1} = T_2^-$ and $hT_1^+h^{-1} = T_2^+$.

Let $M = Z_{G_k}(T_2^-)$. Then T^- and hT^-h^{-1} are maximal (θ, k)-split tori of M, so by our assumption, there exists $h_1 \in M \cap H_k$ such that $h_1hT^-h^{-1}h_1^{-1} = T^-$. Since $h_1hT^+h^{-1}h_1^{-1}$ and T^+ are conjugate under $Z_{G_k}(T^-) \cap H_k$, there exists $h_2 \in H_k$ such that $h_2h_1hT^+h^{-1}h_1^{-1}h_2^{-1} = T^+$. Thus, $h_3 = h_2h_1h \in N_{H_k}(T)$. Then,

$$\begin{aligned} h_3T_1^-h_3^{-1} &= h_2h_1hT_1^-h^{-1}h_1^{-1}h_2^{-1} \\ &= h_2h_1T_2^-h_1^{-1}h_2^{-1} \\ &= h_2T_2^-h_2^{-1} \text{ since } h_1 \in Z_{G_k}(T_2^-) \\ &= T_2^- \text{ since } h_2 \in Z_{G_k}(T^-) \text{ and } T_2^- \subset T^- \end{aligned}$$

Thus, $h_3T_{w_1}^+h_3^{-1} = h_3(T^+T_1^-)h_3^{-1} = T^+T_2^- = T_{w_2}^+$. Hence, w_1 and w_2 are conjugate under $W_{H_k}(T)$.

($\Leftarrow$) Let $w \in W_{H_k}(T)$ such that $ww_1w^{-1} = w_2$, and let $h \in N_{H_k}(T^-)$ be a representative of w. Then $hT^+h^{-1} = T^+$ and $hT_{w_1}^+h^{-1} = T_{w_2}^+$. Since $T_{w_i}^+ = T^+T_i^-$ for $i = 1$ and 2, we have that $hT_1^-h^{-1} = T_2^-$.

Now recall that $T^+ \subset T_1^+$, so $hT^+h^{-1} \subset T_1^+$, and also $T^+ \subset T_2^+$, so $hT^+h^{-1} \subset T_2^+$. Let $M = Z_{G_k}(T_2^-) \cap H_k$. Then $hT_1^+h^{-1}$ and T_2^+ are maximal k-split in M, so there exists $h_1 \in M$ such that $h_1hT_1^+h^{-1}h_1^{-1} = T_2^+$. Thus, $h_1hT_1h^{-1}h_1^{-1} = T_2$. □

Remark 7.1.6*.* Theorem 7.1.5 translates the difficult problem of classifying the H_k-conjugacy classes of θ-stable maximal k-split tori to the combinatorial problem of classifying the conjugacy classes within $W_{H_k}(T)$. To proceed with this new problem one must:

1. First determine which involutions in $W_{G_k}(T)$ are T_i-standard for some T_i θ-stable maximal k-split torus.

2. Then restrict the involutions to $W_{H_k}(T)$ and classify the conjugacy classes.

Example 7.1.7. Let $\theta(X) = AXA^{-1}$ where

$$A \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then there exists one H_k-conjugacy class of θ-stable maximal k-split tori containing a maximal (θ, k)-split torus with representative

$$T = \{\text{diagonal matrices of } \det = 1\},$$

which contains the maximal (θ, k)-split torus

$$A = \{\text{diag}(a_1, a_2, a_2^{-1}, a_1^{-1}, 1) \mid a_1, a_2 \in k^*\}.$$

Also, H_k contains a maximal k-split torus, and $\mathrm{SL}(5, k)$ has the (θ, k)-*split reduction property* in this case. Thus, we can translate the problem to the Weyl group.

We find $\mathrm{Id}, w_1, w_2, w_3 \in W_{G_k}(T)$ are standard involutions, where

$$w_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \ w_2 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \text{ and } w_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

and w_1 and w_2 are $W_{H_k}(T)$-conjugate.

Thus, there are three H_k-conjugacy classes of θ-stable maximal k-split tori. And because of dimensions, there is one H_k-conjugacy class on each level of the lattice. This is depicted in Figure 7.3. The expansion to the second lattice is a separate problem that is still being worked on.

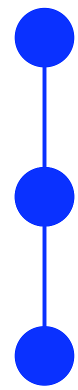

Figure 7.3: The lattice of I for $\mathrm{SL}(5,k)$ example

7.2 One H_k-conjugacy class of θ-stable maximal k-split tori in the bottom level

The previous translation only works if there is one H_k-conjugacy class on top. But what if there exists more than one H_k-conjugacy class of θ-stable maximal k-split tori on the top level? Recall that this is the case in $\mathrm{SL}(n,k)$ if $\theta = \mathrm{Inn}_{I^*_{n-i,i}}$ such that $i = n/2$. In a case like this, we can consider the bottom row instead.

Now suppose there exists only one H_k-conjugacy class of θ-stable maximal k-split tori with S_i^+ part maximal in H_k. Note that we do have a case in which this happens, see Proposition 6.5.1. Let S be the representative of this class.

Definition 7.2.1. We say G_k has the (θ,k)*-split centralizer property* if there exists only one H_k-conjugacy class of maximal (θ,k)-split tori in $Z_{G_k}(S_1)$ for every k-split torus S_1 in H_k.

Lemma 7.2.2. *Let $T_1 \in \mathcal{T}$. Assume G_k has the (θ,k)-split centralizer property. Then there exists $h_1, h_2 \in H_k$ such that $h_1 T_1^- h_1^{-1} \subset S^+$ and $h_2 S^- h_2^{-1} \subset T_1^-$, ie, we can make (S,T_1) a standard pair.*

Proof. We can write $T_1 = T_1^+ T_1^-$ and $S = S^+ S^-$. We know $T_1^+ \subset S_1$ for some maximal k-split torus S_1 in H_k. Since all maximal k-split tori in H_k are conjugate, there exists an $h_1 \in H_k$ such that $h_1 S_1 h_1^{-1} = S$. Thus, $h_1 T_1^+ h_1^{-1} \subset S$.

Consider $M = Z_{G_k}(h_1 T_1^+ h_1^{-1})$. Clearly, $h_1 T_1^+ h_1^{-1}$ is maximal (θ, k)-split in M and S^- is (θ, k)-split in M. Let A be maximal (θ, k)-split such that $S^- \subset A$. Then there exists an $h_2 \in M \cap H_k$ such that $h_2 A h_2^{-1} = h_1 T_1^+ h_1^{-1}$. Hence, $h_1^{-1} h_2 A h_2^{-1} h_1 = T_1^+$, and we have that $h_1^{-1} h_2 S^- h_2^{-1} h_1 \subset T_1^+$. □

So for every $T_1 \in \mathcal{T}$, (S, T_1) is a standard pair. Each of these pairs gives rise to an involution in $W_{G_k}(S)$ as follows.

Lemma 7.2.3. *Let $T_1 \in \mathcal{T}$. Then,*

1. *There exists $g_1 \in Z_{G_k}(S^- T_1^+)$ such that $g_1 S g_1^{-1} = T_1$.*
2. *If $n_1 = \theta(g_1)^{-1} g_1$, then $n_1 \in N_{G_k}(S)$.*
3. *If w_1 is the image of n_1 in $W_{G_k}(S)$, then $w_1^2 = \mathrm{Id}$ and $S_{w_1}^+ = (S^- T_1^+)$.*

Proof.

1. T_1 and S are maximal k-split tori of $Z_{G_k}(T_1^- T^+)$, hence this statement is clear since maximal k-split tori are conjugate.

2. We know $\theta(T_1) = T_1 = g_1 S g_1^{-1}$ and $\theta(T_1) = \theta(g_1 S g_1^{-1})$. So,

$$\begin{aligned} \theta(g_1)\theta(S)\theta(g_1)^{-1} &= g_1 S g_1^{-1} \\ \Rightarrow S &= \theta(g_1)^{-1} g_1 S g_1^{-1} \theta(g_1) \\ \Rightarrow \theta(g_1)^{-1} g_1 &\in N_{G_k}(S). \end{aligned}$$

3. Note that since S and T_1 are θ-stable, $Z_{G_k}(S^- T_1^+)$ is θ-stable, hence $\theta(g_1) \in Z_{G_k}(S^- T_1^+)$. Now $S = (S^- T_1^+)(S^+ \cap g_1^{-1} T_1^- g_1)$. Let $x \in (S^- T_1^+)$. We have $n_1 x n_1^{-1} = \theta(g_1)^{-1} g_1 x g_1^{-1} \theta(g_1) = \theta(g_1) x \theta(g_1)^{-1} = x$.

   Now, let $x \in (S^+ \cap g_1^{-1} T_1^- g_1)$ and write $x = g_1^{-1} t g_1$ for some $t \in T_1^-$. Then,

$$\begin{aligned} n_1 x n_1^{-1} &= \theta(g_1)^{-1} g_1 x g_1^{-1} \theta(g_1) \\ &= \theta(g_1)^{-1} g_1 g_1^{-1} t g_1 g_1^{-1} \theta(g_1) \\ &= \theta(g_1^{-1}) t \theta(g_1) \\ &= \theta(g_1)^{-1} \theta(t^{-1}) \theta(g_1) \\ &= \theta(g_1^{-1} t^{-1} g_1) = \theta(x)^{-1} = x^{-1}. \end{aligned}$$

It follows that

$$\mathrm{Inn}(n_1)|_{S^-T_1^+} = 1 \text{ and } \mathrm{Inn}(n_1)|_{S^+\cap g_1^{-1}T_1^-g_1} = -1.$$

Hence, $w_1^2 = \mathrm{Id}$ and $S_{w_1}^+ = S^-T_1^+$.

□

Definition 7.2.4. Let T_1 and $w_1 \in W_{G_k}(S)$ be as in Lemma 7.2.3. We call w_1 the T_1-standard involution of $W_{G_k}(S)$.

Theorem 7.2.5. *Suppose G has the (θ,k)-split centralizer property. Let $T_1, T_2 \in \mathcal{T}$ such that they are standard with respect to S. Let w_1 and w_2 be the T_1-standard and T_2-standard involutions in $W_{G_k}(S)$, respectively. Then T_1 and T_2 are H_k-conjugate if and only if w_1 and w_2 are conjugate under $W_{H_k}(S)$.*

Proof. ($\Rightarrow$) Let $h \in H_k$ such that $hT_1h^{-1} = T_2$. Then, $hT_1^-h^{-1} = T_2^-$ and $hT_1^+h^{-1} = T_2^+$.

Let $M = Z_{G_k}(T_2^+)$. Now, S^+ and hS^+h^{-1} are maximal k-split tori of $M \cap H_k$. Thus, there exists an $h_1 \in M \cap H_k$ such that $h_1hS^+h^{-1}h_1^{-1} = S^+$. By assumption, $h_1hS^-h^{-1}h_1^{-1}$ and S^- are conjugate under $Z_{G_k}(S^+) \cap H_k$, thus there exists $h_2 \in H_k$ such that $h_2h_1hS^-h^{-1}h_1^{-1}h_2^{-1} = S^-$. Thus, $h_3 = h_2h_1h \in N_{H_k}(S)$. Then,

$$\begin{aligned} h_3T_1^+h_3^{-1} &= h_2h_1hT_1^+h^{-1}h_1^{-1}h_2^{-1} \\ &= h_2h_1T_2^+h_1^{-1}h_2^{-1} \\ &= h_2T_2^+h_2^{-1} \text{ since } h_1 \in Z_{G_k}(T_2^+) \\ &= T_2^+ \text{ since } h_2 \in Z_{G_k}(S^+) \text{ and } T_2^+ \subset S^+ \end{aligned}$$

Thus, $h_3S_{w_1}^+h_3^{-1} = h_3(S^-T_1^+)h_3^{-1} = S^-T_2^+ = T_{w_2}^+$. Hence, w_1 and w_2 are conjugate under $W_{H_k}(S)$.

($\Leftarrow$) Let $w \in W_{H_k}(S)$ such that $ww_1w^{-1} = w_2$, and let $h \in N_{H_k}(S)$ be a representative of w. Then $hS^-h^{-1} = S^-$ and $hS_{w_1}^+h^{-1} = S_{w_2}^+$. Since $S_{w_i}^+ = S^-T_i^+$ for $i = 1$ and 2, we have that $hT_1^+h^{-1} = T_2^+$.

Now recall that $S^- \subset T_1^-$, so $hS^-h^{-1} \subset T_1^-$, and also $S^- \subset T_2^-$, so $hS^-h^{-1} \subset T_2^-$. Let $M = Z_{G_k}(T_2^+) \cap H_k$. Then $hT_1^-h^{-1}$ and T_2^- are maximal (θ,k)-split in M, so by assumption, there exists $h_1 \in M$ such that $h_1hT_1^-h^{-1}h_1^{-1} = T_2^-$. Thus, $h_1hT_1h^{-1}h_1^{-1} = T_2$. □

Remark 7.2.6*.* Theorem 7.2.5 translates the difficult problem of classifying the H_k-conjugacy classes of θ-stable maximal k-split tori to the combinatorial problem of classifying the conjugacy classes within $W_{H_k}(S)$. To proceed with this new problem one must:

1. First determine which involutions in $W_{G_k}(S)$ are T_i-standard for some T_i θ-stable maximal k-split torus.

2. Then restrict the involutions to $W_{H_k}(S)$ and classify the conjugacy classes.

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