Abstract

MATTHEWS, JOHN VIVIAN. An Analytical and Numerical Study of Granular Flows in Hoppers. (Under the direction of P. A. Gremaud.)

This work investigates the characteristics of a steady state flow of granular material, under the influence of gravity, in two and three dimensional hoppers of simple geometry. Simulations of such flows are of particular interest to various industries, such as the food and mining industries, where the handling of large quantities of granular materials in hoppers and silos is routine. While understanding and simulation of time-dependent phenomena are the ultimate goals in this field, those phenomena are still poorly understood and thus their study is beyond the scope of this research. It has been observed that steady flows can provide reasonable approximations, and the corresponding steady state model has consequently been the focus of a great deal of research. Historically, these steady state models have been approached using only smooth radial fields, and even today most practical hopper design uses these fields as their basis. Our work represents the first time that quality numerical methods have been brought to bear on the model equations in their original form, without assuming smoothness of the resulting fields. Two different, yet related, models for stress/velocity consisting of systems of hyperbolic conservation laws and algebraic relations are considered and discussed. The radial stress and velocity fields, and the stability of those fields, are studied briefly with both analytical and numerical results presented. More importantly, a Runge-Kutta Discontinuous Galerkin method is implemented and applied to various boundary value problems involving perturbed stress and velocity fields arising from discontinuous changes in parameters such as hopper wall angle or hopper wall friction.
Biography

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Chapter 1

Introduction

In the course of normal business, many industries regularly struggle to find effective methods of storage and handling of granular materials. Common examples of difficulties involve regions of stagnant material adhering to a hopper wall, the formation of poorly flowing rat holes, and severe vibrations that can damage or even collapse a containing structure. Despite the importance of remediating these costly problems, the phenomena behind them are poorly understood.

In Industry, many types of granular materials are handled, ranging from the extremely fine powders as one might find in the Pharmaceutical Industry to the large grains of raw ore in the Mining Industry. Besides grain size, these materials may differ in other respects. For example, some fine powders like cement exhibit some degree of cohesion in which the material can support some measure of tensile stress. Other materials, such as seeds or beads do not show this quality. Yet other materials can show variations in bulk density, that is the density per unit of volume in a container, and others can demonstrate compressibility.

Not only can the materials themselves differ in their physical properties, but the properties of the flow of the materials can vary as well. Restricting our attention to hoppers with inclined walls, there are, speaking generally, two types of flow: mass flow and funnel flow. The former describes a material moving such that the grains initially higher up in the hopper exit only after the grains below them do. With funnel flow, material at the wall may move more slowly or not at all while the material in a channel at the center of the hopper exits more quickly, permitting grains initially higher in the hopper to exit ahead of grains that were closer to the exit. These flows, especially funnel flow, can exhibit time dependent
phenomena, requiring more complicated models to describe.

And as the types of material vary, so do the containers that Industry uses to hold them. Size is an issue. Industry uses silos, bunkers, and hoppers, ranging from a handheld device for handling powders to giant bunkers on the scale of several meters for holding raw ore. And the shapes of these can be complicated, from simple box-shaped bunkers to complex silos made of a section with vertical walls and a converging conical section with an insert to guide or regulate material flow [16].

Of primary importance is the understanding of the time dependent problems; however the current general understanding (or lack of understanding) of those models makes a sound numerical approach difficult at best. It has been shown that the time dependent equations for incompressible, purely plastic granular flows are linearly ill posed, and show similarities to the mixed forward/backward heat equation [35]. The question of adding elastic effects has been studied and those effects were not found to regularize sufficiently the equations to prevent blow up similar to the incompressible, purely elastic case [28]. Addition of compressibility does regularize the equation enough to make the time dependent system linearly well posed [28].

As a first step much work has been done on the steady state stress equations. As the model equations involve two partial differential equations (actually a pair of hyperbolic conservation laws) and an algebraic constraint, the steady state equations are commonly expressed and then analyzed using the well-known Sokolovskii variables While this substitution has the fortunate consequence of “solving away” the algebraic constraint, leaving only a pair of partial differential equations, it also destroys the conservation form of the system.

Our focus here is on the determination of the properties of steady state flows of granular materials in converging hoppers. These flows have been the focus of intense study since Jenike’s original work [22] in the 1960s, and to a large extent modern design and modeling of converging hoppers is based on the results of that original work. Our goal is to bring a new approach to the problem of stress and velocity determination, and in so doing further the understanding of how these flows behave in the presence of a discontinuous change in the physical parameters.

In real hopper configurations, one commonly finds a transition from a hopper of one
wall angle to a hopper of a different wall angle. It has been hypothesized that in these situations, nonlinear phenomena such as discontinuities and/or rarefactions may occur in the resulting fields. Dealing with discontinuities is impossible if the model equations are not in conservative form, as the notion of weak solution is no longer uniquely defined.

While much effort has been expended upon the understanding of these models with smooth data \cite{15}, \cite{22}, \cite{26}, \cite{27}, \cite{29}, \cite{38}, \cite{39} it is not clear how far those results can be extended for handling an abrupt change in the angle of inclination of the hopper wall or a discontinuous change in the coefficient of friction between the hopper wall and the granular material. Consequently, a main contribution of this work is to approach the model stress equations in their original form, as a pair of hyperbolic conservation laws coupled with an algebraic constraint. Further, we consider model equations for the velocity field within this flow; those equations are also pairs of conservation laws. By doing so, stress and velocity may be conserved in the presence of discontinuous fields. Specifically, the stress equations are kept in their conservation form; the equations for velocity, while not in conservation form, are linear and thus can still be handled correctly numerically.

This research also provides an opportunity to compare in a meaningful way two competing models for the velocity field for these types of granular flows. The relative merits of these models, the Jenike and Spencer models, have been widely debated \cite{19}, \cite{38} with neither having a clear claim as the best choice. In the course of our work we have had the opportunity to incorporate with our stress fields each velocity model separately and compare the resulting fields. We have also compared the smooth radial velocity fields with experimental work by Nedderman \cite{5}. It is our hope that these comparisons may increase the understanding of this problem and, in conjunction with laboratory experiments, perhaps even resolve whether one, both, or perhaps even neither model is an accurate representation of real velocity fields.

Our tool for these computations is a high order numerical solver based on the Runge-Kutta Discontinuous Galerkin method \cite{7}, \cite{9} that has been developed, tested, and applied to the above problems. This code has first been tested to confirm the preservation of the radial fields in the discretized versions of the model equations and is then used for numerical experiments for which a discontinuous change in parameters is presumed. Both changes in hopper wall angle and changes in hopper wall material (leading to a change in the coefficient
of sliding friction between the wall and the granular material) are explored.

Several unforeseen difficulties have arisen in the course of this work. For the computation of the stress field, we assume that all of the stress components must lie on a manifold known as the yield surface. However, it has been found that certain choices of the parameters lead to stress components that leave the yield surface. Once that occurs, computations cannot continue with the current approach. Further, while the direction of propagation of stress and velocity information in a granular material has been debated for some time, this work encounters that difficulty in a real and troubling manner. For the first time full, perturbed stress and velocity fields can be numerically computed starting from an initial surface and following the field either up or down the hopper. Those computations lead to strikingly different results and it is not clear, without further research, which is the more physically meaningful, if indeed one has more meaning than the other. Additionally, calibration of this work with real parameters from problems of interest to engineers has proved difficult. In addition to a dearth in the literature of experimental results that apply to hoppers of the type we examine, the model itself becomes more difficult to handle for realistic values of the parameters. In particular, it is easy to find parameters for which there is no corresponding radial field that can be treated by our approach. Finally, it has been found that certain computations of the stress and related velocity fields can produce results that aren’t physically feasible. In particular, the function \( \lambda \), a function which links the stress tensor with the strain rate tensor and whose values are equivalent to the work done by friction, is found to assume negative values in some problems.

Let us provide a brief outline of this work. In Chapter 2 the two basic models to be studied are described, along with initial/boundary conditions, alternative formulations, and alternative models. Then in Chapter 3, the radial stress and velocity fields are examined analytically and numerically for both models. A brief, but interesting, comparison of the radial velocity fields to some experimental results from the literature is given. Chapter 4 gives a detailed analysis of the model equations, covering such aspects as eigenvalues, model limitations, and a discussion of boundary conditions. The Runge-Kutta Discontinuous Galerkin method is described in Chapter 5, with implementation details, as well as how it is applied to the two models for both stress and velocity. Finally the results of the application
of the RKDG method are reported in Chapter 6 and conclusions are summarized in Chapter 7.
Chapter 2

Models

2.1 Opening Remarks

It is not yet known which model best describes the motion of a granular material through a hopper, under the action of gravity. Nor is it known whether it is even reasonable to expect, as is for instance the case in Fluid Dynamics, that one kind of model can cover a wide range of problems and applications. So while there is yet some doubt on the proper assumptions one should choose as the basis of a model, we opt to forego that debate here and instead examine in detail two basic, well-known models. Both presume a cohesionless granular material and then use a balance of forces, the Haar-Von Karman assumption, and the Mohr-Coulomb Yield Condition to arrive at a common model for the stress field. However, while both models use incompressibility as essential to the velocity field and thus trivially conserve mass, they use different flow rules to arrive at different overall models. The first model presented here, which we refer to by the name Mohr-Coulomb-Jenike, arises from a flow rule known as Jenike’s Principle of Coaxiality; the second, called the Mohr-Coulomb-Spencer or Spencer’s Double-Shearing model, uses a flow rule that relates strain rate tensor to both the stress and the stress rates. Note that while we restrict out attention to the two models above, we make reference to and comparison with a model using the von Mises Yield Condition. The system for stress arising from this alternative yield condition has been previously studied in comparison to the Mohr-Coulomb yield condition [15].
2.2 The Mohr-Coulomb-Jenike Model

We consider a steady flow of an incompressible granular material through a conical or wedge-shaped hopper. See Figure (2.1). For the case of the conical hopper, we assume that the flowing particles undergo no motion in the axial (or $\varphi$) direction; the analogous assumption in the wedge-shaped hopper is that the particles undergo no motion perpendicular to the intersecting plane.

These assumptions, while perhaps contrary to the intuition of a reader familiar with fluid flows, is borne out by observations and experiments [33]. They also reduce these three dimensional problems to essentially two dimensional problems, and both may be described using the polar coordinates $(r, \vartheta)$. First we will present only the equations for the case of a granular material in a conical hopper; the model for a wedge-shaped hopper is summarized afterwards.

Under these assumptions, the dependent variables reduce to four components of the stress tensor and two components of velocity:

\[
T = \begin{bmatrix}
T_{rr} & T_{r\vartheta} & 0 \\
T_{r\vartheta} & T_{\vartheta\vartheta} & 0 \\
0 & 0 & T_{\varphi\varphi}
\end{bmatrix}
\]  

(2.1)

\[
v = \begin{bmatrix}
v_r \\
v_\vartheta
\end{bmatrix}.
\]  

(2.2)

The corresponding equations of motion are

\[
\nabla \cdot T = \rho g,
\]  

(2.3)

and

\[
\nabla \cdot v = 0.
\]  

(2.4)
Chapter 2. Models

Here \( \rho \) is the bulk density of the granular material (assumed constant) and the vector \( g \) is the acceleration due to gravity. Note that equation (2.4) is a statement about the incompressibility of the material while equation (2.3) expresses a balance of forces; the nonlinear inertial terms can be neglected at these low speeds [35]. As \( g \) has no component in the axial (\( \varphi \)) direction, we see that \( T \) is independent of \( \varphi \). Thus, by the form of \( T \) given in (2.1), (2.3) reduces to only two meaningful equations, the third being the trivial \( 0 = 0 \). Consequently, the system (2.3), (2.4) constitutes only three equations in six unknowns. A variety of constitutive laws will provide the necessary extra relations needed to close this system.

We first assume that the granular material experiences plastic deformation everywhere. Such constitutive models based on plasticity are easily expressed in terms of the eigenvalues (also known as the principal stresses) \( \sigma_i, i = 1, 2, 3 \) of the stress tensor \( T \). Here we will always assume that the principal stresses are ordered \( \sigma_1 \geq \sigma_2 \geq \sigma_3 \geq 0 \). Note that compressive stresses are taken as positive. One such model, known as the Mohr-Coulomb yield condition [26], can then be expressed as

\[
\frac{\sigma_1}{\sigma_3} = \frac{1 + \sin \delta}{1 - \sin \delta}, \tag{2.5}
\]

where \( \delta \) is the angle of internal friction of the granular material. Often known as the angle of repose for noncohesive materials, the angle of internal friction can be thought of as the greatest angle that a pile of granular material naturally makes with a horizontal surface [26]. Typically, \( \delta \) can range from 30° to 45° [27]. For notational convenience, we hereafter use \( s \)
Figure 2.2: The Mohr-Coulomb yield surface.

to represent sin δ and similarly c for cos δ, as this is the mode by which δ most often presents itself. The yield condition may be derived as a consequence of the law of sliding friction as outlined in [26], Chapter 3. Instead of the above form, it is more natural in the context of this work to express (2.5) in terms of the components of the stress tensor (2.1)

\[(T_{rr} - T_{\theta\theta})^2 + 4T_{r\theta}^2 = s^2 (T_{rr} + T_{\theta\theta})\].

(2.6)

Thus, the Mohr-Coulomb Yield Condition gives us our first constitutive relation.

A second constitutive relation is found in the Haar-Von Karman assumption. It expresses the assumption that the axial stress $T_{\varphi\varphi}$ for a material in a conical hopper may be written in terms of the other stress components. While the Mohr-Coulomb analysis states only that $\sigma_1 \geq T_{\varphi\varphi} \geq \sigma_3$, the Haar-Von Karman assumption sharpens this statement to say that $T_{\varphi\varphi}$ is in fact the major principal stress [26], [29]

\[T_{\varphi\varphi} = \sigma_1 = \frac{1}{2} (T_{rr} + T_{\theta\theta}) (1 + s)\].

(2.7)

A final constitutive relation, one that closes the system and express a relationship between the stresses and velocities, is the flow rule. For present, we consider only Jenike’s Principle of Coaxiality [26], p. 254, which states that the principal axes of stress and strain rate must
be coincident. In other words, the eigenvectors of the strain rate tensor

\[ V = -\frac{1}{2} \left( \nabla v + (\nabla v)^T \right) \] (2.8)

and the stress tensor (2.1) are parallel. Algebraically, we express this relationship as the existence of a function \( \lambda \) such that

\[ V = \lambda (T - pI), \] (2.9)

where \( p = \frac{1}{3} (\sigma_1 + \sigma_2 + \sigma_3) \) is the average of the principal stresses. In the polar coordinates we use for these problems, \( V \) has the explicit form

\[ V = \begin{bmatrix}
-\partial_r v_r & -\frac{1}{2} \left( \frac{1}{r} \partial_\theta v_r + \partial_r v_\theta - \frac{1}{r} v_\theta \right) & 0 \\
-\frac{1}{2} \left( \frac{1}{r} \partial_\theta + \partial_r v_\theta - \frac{1}{r} v_\theta \right) & -\frac{1}{r} (\partial_\theta v_\theta + v_r) & 0 \\
0 & 0 & -\frac{\rho g \sin \theta + v_\theta \cos \theta}{r \sin \theta}
\end{bmatrix} \] (2.10)

It can be shown that the sign of the function \( \lambda \) is the same as the sign of work done by friction, and thus \( \lambda \) must be positive. Specifically, Physics tells us that the work done by friction is \( T \cdot V \geq 0 \) [18], p.35, and given the forms of \( T, V \) above, this implies that \( \lambda \) is positive. The sign of \( \lambda \) has greater importance in Chapter 6 where we present our numerical simulations.

The combined relations (2.3), (2.4), (2.6), (2.7), and (2.9) form a closed system for the stresses and velocities, which we summarize here:

\[ \partial_r T_{rr} + \frac{1}{r} \partial_\theta T_{r\theta} + \frac{2}{r} T_{rr} + \frac{1}{r} \cot \theta T_{r\theta} - \frac{1}{r} (T_{\theta\theta} - T_{\phi\phi}) = -\rho g \cos \theta \] (2.11)

\[ \partial_r T_{r\theta} + \frac{1}{r} \partial_\theta T_{r\theta} + \frac{3}{r} T_{r\theta} + \frac{1}{r} \cot \theta (T_{\theta\theta} - T_{\phi\phi}) = \rho g \sin \theta \] (2.12)

\[ (T_{rr} - T_{\theta\theta})^2 + 4T_{r\theta}^2 = s^2 (T_{rr} + T_{\theta\theta}). \] (2.13)
\[ T_{rr} = \frac{1}{2} (T_{rr} + T_{\theta\theta}) (1 + s). \] (2.14)

\[ \partial_r v_r + \frac{1}{r} \partial_\theta v_\theta + \frac{2}{r} v_r + \frac{1}{r} \cot \theta v_\theta = 0 \] (2.15)

\[
\left( \partial_r v_r - \frac{1}{r} \partial_\theta v_\theta - \frac{1}{r} v_r \right) \frac{2 T_{r\theta}}{T_{rr} - T_{\theta\theta}} - \left( \partial_r v_\theta + \frac{1}{r} \partial_\theta v_r - \frac{1}{r} v_\theta \right) = 0
\] (2.16)

Finally, for a wedge-shaped hopper, the equations simplify to

\[ \partial_r T_{rr} + \frac{1}{r} \partial_\theta T_{r\theta} + \frac{1}{r} (T_{rr} - T_{\theta\theta}) = -\rho g \cos \theta \] (2.17)

\[ \partial_r T_{r\theta} + \frac{1}{r} \partial_\theta T_{r\theta} + \frac{2}{r} T_{r\theta} = \rho g \sin \theta \] (2.18)

\[ (T_{rr} - T_{\theta\theta})^2 + 4 T_{r\theta}^2 = s^2 (T_{rr} + T_{\theta\theta}) \] (2.19)

\[ \partial_r v_r + \frac{1}{r} \partial_\theta v_\theta + \frac{1}{r} v_r = 0 \] (2.20)

\[
\left( \partial_r v_r - \frac{1}{r} \partial_\theta v_\theta - \frac{1}{r} v_r \right) \frac{2 T_{r\theta}}{T_{rr} - T_{\theta\theta}} - \left( \partial_r v_\theta + \frac{1}{r} \partial_\theta v_r - \frac{1}{r} v_\theta \right) = 0
\] (2.21)

It should be pointed out that the above collection of equations shows that the function \( \lambda \) has been eliminated from the system. This raises an interesting question, especially in the context of the results shown in Chapter 6. Can the system know that it is retaining positive values of \( \lambda \) throughout the entire domain? The values of \( \lambda \) can be recovered after stress and velocity solutions have been obtained, but it is not clear that the equations that produced those solutions have built in any facility to ensure that \( \lambda \) is everywhere positive. And, as is shown in Chapter 6, our numerical solutions to these systems exhibit cases for which \( \lambda \) does
indeed locally become negative.

2.3 The Mohr-Coulomb-Spencer Model

In addition to the above widely used model, we also consider Spencer’s Double-Shearing model [38], [39], which arises from a different choice of flow rule. Objecting to differences between experimentally observed phenomena and those predicted by various other models, Spencer created a model to express the observed quality that a granular material deforms in plane strain by shear along stress characteristics. This key difference between Spencer’s model and that obtained through Jenike’s Principle of Coaxiality means that the strain rate depends not only on the stress itself but also on the stress rate. Through an involved derivation, Spencer arrives at

\[-\left( \partial_r v_r \frac{1}{r} \partial_{\theta} v_{\theta} - \frac{1}{r} v_r \right) \sin 2\psi + \left( \partial_r v_r \frac{1}{r} \partial_{\theta} v_{\theta} - \frac{1}{r} v_r \right) \cos 2\psi + s \left( \frac{1}{r} \partial_{\theta} v_r - \partial_r v_{\theta} + \frac{1}{r} v_{\theta} + 2\Omega \right) = 0,\]

(2.22)

The $\psi$ above is one of the Sokolovskii variables $(\sigma, \psi)$. The Sokolovskii variables provide a parameterization of the Mohr-Coulomb yield surface and thus a nonlinear change of variables from the components of the stress tensor (2.1). Their importance and meaning is outlined below. Also, $\Omega$ in (2.22) is defined to be

$$\Omega = v_r \partial_r \psi + \frac{1}{r} v_{\theta} \partial_{\theta} \psi.$$ 

The function $\Omega$ is a measure of the misalignment of the stress tensor with the strain rate tensor. One can think of it as a sort of angular velocity for the stress characteristics relative to the fixed coordinate system in which the model is set.

We note here that (2.22) bears great similarity to (2.16). Together, (2.11) - (2.15) and (2.22) form an alternative, closed system for the unknowns in a conical hopper. For a wedge-shaped hopper, one would use (2.17) - (2.20) and (2.22).
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2.4 Alternative Models

While the above two models are of primary interest here and provide an ideal starting point for our investigations, many other theories have been developed to model the flow of granular materials. In particular, the von Mises model bears a brief mention within the context of this work.

In the von Mises model, one replaces the Mohr-Coulomb Yield Condition with the von Mises Yield Condition:

\[(T_{rr} - p)^2 + (T_{\theta\theta} - p)^2 + (T_{\phi\phi} - p)^2 + 2T_{r\phi}^2 = 2s^2p^2.\] (2.23)

Here \(p\) is the average of the principal stresses \(p = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)\). Note the distinct similarity between (2.23) and (2.6). The unfortunate consequence of using the von Mises Yield Condition is that, for a conical hopper, the stresses and velocities are coupled in an essential way, making far more difficult any analytical or computational approach [15].

2.5 Boundary Conditions

The law of sliding friction is used to develop appropriate boundary conditions for the stresses at the hopper walls. At a given point on the wall, the tangential stress \(|T_T|\) is proportional to the magnitude of the normal stress \(|T_N|\), that is

\[|T_T| = \mu|T_N|,\]

where \(\mu = \tan \phi > 0\) is the coefficient of sliding friction between the granular material and the wall material; \(\phi\) is known as the angle of wall friction. Typical values of \(\mu\) are 10° to 25° [27]. In our chosen radial geometry, this condition simplifies to

\[T_{r\theta} = \pm \mu T_{\theta\theta} \quad \text{at} \quad \theta = \pm \theta_w\] (2.24)
and may be simplified even more, as shown below in (4.8). Further, it should be mentioned that the relative sizes of $\mu$ and $\delta$ are also important and that we discuss this later in our analysis of the model; see specifically (4.10) and (4.11) below.

It is also natural to take advantage of the symmetry of the problem and consider solving on just a half-domain, either $[-\vartheta_w, 0]$ or $[0, \vartheta_w]$. Subsequently, a boundary condition at $\vartheta = 0$ must be formulated; here we propose two slightly different choices. By the symmetry of the radial problem, it is relatively easy to see that $T_{r\vartheta}$ has odd parity and, if $T_{r\vartheta}$ is a continuous function, then naturally $T_{r\vartheta}$ is zero at $\vartheta = 0$. Thus, a first choice for a boundary condition might be

$$T_{r\vartheta} = 0 \quad \text{at} \quad \vartheta = 0. \quad (2.25)$$

This has exactly the form of the relation for $T_{r\vartheta}$ in (2.24) with $\mu = 0$, i.e. a frictionless wall along the center of the hopper. However, the types of problems we wish to consider are such that discontinuities may arise in a stress field, and the assumption of continuity of $T_{r\vartheta}$ at $\vartheta = 0$ may not be appropriate. Consequently, we observe that the symmetry of the problem dictates only a parity condition on $T_{rr}$ and $T_{r\vartheta}$ such that

$$T_{rr}(0^-) = T_{rr}(0^+),$$
$$T_{r\vartheta}(0^-) = -T_{r\vartheta}(0^+). \quad (2.26)$$

A boundary condition for the system of velocities can be found by observing that there can be no lateral ($\vartheta$-direction) movement at the hopper walls, i.e.

$$v_\vartheta = 0 \quad \text{at} \quad \vartheta = \pm \vartheta_w. \quad (2.27)$$

As with the stresses, we may formulate the problem on the half-domain by noting that, for continuous velocities, symmetry dictates

$$v_\vartheta = 0 \quad \text{at} \quad \vartheta = 0, \quad (2.28)$$

since $v_\vartheta$ is odd. If, however, we wish to allow for discontinuous velocities, it is enough to
specify the parity of $v_\theta$ with

$$v_\theta(0^-) = -v_\theta(0^+) \quad (2.29)$$

## 2.6 Sokolovskii Variables

The relationship between $T_{\theta\theta}$ and the other stress variables, given in (2.13), is not a proper functional relation but rather constrains the variables $(T_{rr}, T_{r\theta}, T_{\theta\theta})$ to lie on a cone in stress space. See Figure 2.2. This cone, however, admits a rather simple parameterization in terms of the two variables $(\sigma, \psi)$, more commonly known as the Sokolovskii variables [26]. Using this parameterization, the relationship between the components of the stress tensor and the Sokolovskii variables is given by

$$
T_{rr} = \sigma(1 - s \cos 2\psi) \\
T_{r\theta} = -\sigma s \sin 2\psi \\
T_{\theta\theta} = \sigma(1 + s \cos 2\psi) \\
T_{\varphi\varphi} = \sigma(1 + s) \quad (\text{conical hopper only}) \quad (2.30)
$$

While this nonlinear change of variables conveniently disposes of the algebraic constraint (2.13), it also destroys the conservation form of the partial differential equations (2.11),(2.12) and, consequently, the ability to compute shocks in a reliable way. Without conservation form, the speed of a shock is no longer uniquely determined. (That the partial differential equations here form a system of hyperbolic conservation laws is a fact that we shall establish in Chapter 4.) As our interest here lies with problems for which a discontinuous change in parameters occurs, we expect shocks and so must consider the model in its original conservative form. Nevertheless, this form of the stress problem, using the Sokolovskii variables, has been extensively studied [26],[27], is useful for smooth field analyses, and can offer intuitive insight.

Physically, the angle $\psi$ represents the angle between the position vector of a point in the hopper and the direction of the minor principal stress, $\sigma_3$. By extension, this is also the angle between a line of constant $r$ and the direction of the major principal stress, $\sigma_1$ as seen in Figure 2.3. (Please note that some writers [38] and engineers [33] choose as $\psi$ in their work the complement of our $\psi$.) This physical interpretation of $\psi$ implies that $|\psi| < \pi/2$. The
quantity $\sigma$ also admits a physical interpretation: it is the average stress of the material.

The values $|\psi| < \pi/4$ correspond to the so-called passive state of stress [26], [29] while the values $\pi/4 < |\psi|$ are the active state. For the converging hoppers we remark that as a result of our chosen constitutive relations the granular material is always to be in the passive state; this is in accordance with experimental evidence [29], [33]. The passive state can be thought of as a state in which, generally speaking, the principal stress points more in a horizontal ($\theta$) direction. For hoppers with vertical walls, it has been observed that the active state is realized, i.e. one in which the principal stress has a more vertical direction. This presents an interesting and puzzling open question. In practical applications, one often finds a vertical bunker or silo atop a hopper with inclined walls. Given that the stress realizes the active state in the section with vertical walls and the passive state in the section with inclined walls, how does the material transition from one to the other?

In addition to the above information that the Sokolovskii variables can provide, the characteristics of the systems for both the stress and velocities are easily described in terms of $\sigma$ and $\psi$. So while computations with only the Sokolovskii variables must be restricted to regimes in which the stress fields are smooth, the physical insight they provide for these problems is of great value.
2.7 Reformulation

As the model equations primarily involve pairs of hyperbolic conservation laws (see Chapter 4), it is natural to consider one variable as a time-like variable and the other as spatial. Bear in mind, however, that the flows we compute are steady state flows and so no concept of real time is involved. To this end, we consider the change of variable

\[
\tau = -\ln r
\]

(2.31)

which allows one to think of traveling down a hopper towards its apex as going “forward” in time in a natural way. As most computations of the stress and velocity fields will in this work start at \( r = 1 \), and proceed towards \( r = \varepsilon \ll 1 \), this causes \( \tau \) to start at 0 and proceed to a large positive value \(-\ln \varepsilon\), just as if \( \tau \) were time. If, as will be desired for certain computations, we wish to start at some point \( r = \varepsilon \) near the apex of the hopper and solve upwards towards \( r = 1 \), the change of variable \( \tau = -\ln \varepsilon + \ln r \) is used which starts at \( \tau = 0 \) and ends at some large positive value of \( \tau \), again in a very natural time-like manner.

Considering the case where \( \tau = -\ln r \), the equations (2.11) and (2.12) corresponding to the stress field in a conical hopper become

\[
\partial_\tau T_{rr} - \partial_\vartheta T_{r\vartheta} = f(\tau, \vartheta, T_{rr}, T_{r\vartheta}) \\
\partial_\tau T_{r\vartheta} - \partial_\vartheta T_{\vartheta \vartheta} = g(\tau, \vartheta, T_{rr}, T_{r\vartheta})
\]

(2.32)

where

\[
f(\tau, \vartheta, T_{rr}, T_{r\vartheta}) = \frac{3 - s}{2} T_{rr} + \cot \vartheta T_{r\vartheta} - \frac{3 + s}{2} T_{\vartheta \vartheta} + \rho g e^{-\tau} \cos \vartheta
\]

\[
g(\tau, \vartheta, T_{rr}, T_{r\vartheta}) = -\frac{1 + s}{2} \cot \vartheta T_{rr} + 3 T_{r\vartheta} + \frac{1 - s}{2} \cot \vartheta T_{\vartheta \vartheta} - \rho g e^{-\tau} \sin \vartheta
\]

(2.33)

In a like manner, one may rewrite the equations for the stress field in a wedge-shaped hopper.
In that case, the source terms \( f, g \) become

\[

dfrac{\partial}{\partial \tau} v_r - \dfrac{\partial}{\partial \varphi} v_\varphi = p(\vartheta, v_r, v_\varphi)
\]

\[
\dfrac{\partial}{\partial \tau} v_\varphi - \dfrac{\partial}{\partial \varphi} v_r = q(\vartheta, \psi, v_r, v_\varphi)
\]

The equations for the velocities may be similarly rewritten. First, the equations arising from Jenike’s Principle of Coaxiality from (2.15), (2.16) can be rewritten as

\[
\tan 2\psi = \dfrac{2T_{r\varphi}}{T_{rr} - T_{r\varphi}}.
\]

The functions \( p, q \) are given by:

\[
p(\vartheta, v_r, v_\varphi) = (d - 1)v_r + (d - 2)\cot \vartheta v_\varphi
\]

\[
q(\vartheta, \psi, v_r, v_\varphi) = d\tan 2\psi v_r + ((d - 2)\cot \vartheta \tan 2\psi - 1)v_\varphi.
\]

Here \( d = 2 \) corresponds to the wedge-shaped hopper (referred to sometimes as the “two dimensional case”) and \( d = 3 \) to the conical hopper (the “three dimensional case”). The use of \( d \) will arise again in our work on stability below.

When considering Spencer’s Double-Shearing Model (replacing (2.16) with (2.22)) the system changes significantly:

\[
\dfrac{\partial}{\partial \tau} v_r - \dfrac{\partial}{\partial \varphi} v_\varphi = \hat{p}(\vartheta, v_r, v_\varphi)
\]

\[
\dfrac{\partial}{\partial \tau} v_\varphi + \dfrac{s + \cos 2\psi}{s - \cos 2\psi} \dfrac{\partial}{\partial \varphi} v_r + \dfrac{2\sin 2\psi}{s - \cos 2\psi} \dfrac{\partial}{\partial \varphi} v_\varphi = \hat{q}(\vartheta, \psi, v_r, v_\varphi)
\]

for which we define

\[
\hat{p}(\vartheta, v_r, v_\varphi) = (d - 1)v_r + (d - 2)\cot \vartheta v_\varphi
\]

\[
\hat{q}(\vartheta, \psi, v_r, v_\varphi) = \dfrac{-d\sin 2\psi}{s - \cos 2\psi} v_r - \left( 1 + \dfrac{(d - 2)\cot \vartheta \sin 2\psi}{s - \cos 2\psi} \right) v_\varphi + \dfrac{2s}{s - \cos 2\psi} \hat{\Omega}.
\]
where

\[ \hat{\Omega} = v_r \partial_\tau \psi - v_\vartheta \partial_\vartheta \psi. \]

The same convention on \( d \) as above still holds. Here \( \hat{\Omega} \) is treated as a source term in spite of having derivatives in both \( \tau \) and \( \vartheta \). Further it is written separately from the other source terms, despite containing the quantities \( v_r, v_\vartheta \), to emphasize that this term should be handled differently numerically when solving this system.
Chapter 3

Radial Solutions

One feature common to all the model equations presented, both for stresses and velocities, is the existence of similarity solutions. For these solutions, discovered in the early 1960s by Jenike [22], the paths of individual grains are radial lines converging at the vertex of the hopper; for this reason, such solutions are commonly called radial fields. All of the computational testing and experimentation in our work depends on obtaining radial fields and then perturbing them in some way to approximate a particular real problem.

We consider primarily radial stress fields in a conical hopper, as the analogous results for the wedge-shaped hopper are quite similar. Where appropriate, the differences will be noted. For the velocity field, both types of hoppers and both models are discussed. In [27], the radial stress and velocity fields for the Mohr-Coulomb-Jenike model were discussed in great detail, and our present work complements and extends those results.

3.1 Radial Stress Fields

We begin by considering a solution for the stress equations of the form proposed by Jenike

\[ T_{ij}(r, \theta) = rT_{ij}(\theta) \]  

(3.1)
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Instead of working directly with the components of the stress tensor $T$, we choose to rewrite (2.32) using the Sokolovskii variables $\sigma, \psi$ since the radial field is described by smooth functions and the form is slightly more convenient. As a result, we arrive at the system

$$
\mathcal{A} \begin{pmatrix} \sigma \\ \psi \end{pmatrix}' = \mathcal{F}(\vartheta, \sigma, \psi) \quad \vartheta \in (-\vartheta_w, \vartheta_w),
$$

(3.2)

where $\mathcal{A}$ is given by

$$
\mathcal{A} = \begin{pmatrix} -s \sin 2\psi & -2s\sigma \cos 2\psi \\ 1 + s \cos 2\psi & -2s\sigma \sin 2\psi \end{pmatrix}
$$

and $\psi_w = \frac{1}{2} \left( \phi + \arcsin \left( \frac{\sin \phi}{s} \right) \right)$.

Note that $\mathcal{F}$ for the conical hopper differs from that used for a wedge-shaped hopper. The function $\mathcal{F}$ for the conical hopper is

$$
\mathcal{F}(\vartheta, \sigma, \psi) = \begin{pmatrix} -\sigma + \sigma s (1 + 4 \cos 2\psi + \cot \vartheta \sin 2\psi) - \rho g \cos \vartheta \\ \sigma s (1 + 4 \sin 2\psi - \cot \vartheta \cos 2\psi) + \rho g \sin \vartheta \end{pmatrix}
$$

and for a wedge-shaped hopper it is

$$
\mathcal{F}(\vartheta, \sigma, \psi) = \begin{pmatrix} -\sigma (1 - 3s \cos 2\psi) - \rho g \cos \vartheta \\ -3\sigma s \sin 2\psi + \rho g \sin \vartheta \end{pmatrix}.
$$

Here, as before, the angle $\phi$ is the angle of wall friction, $\mu = \tan \phi$. The above system is nonsingular provided that

$$
|\psi| < \frac{\pi}{4} + \frac{\delta}{2}.
$$

Except when simplifying assumptions are made [27],[15] this system cannot be solved exactly and must therefore be solved numerically for $\sigma, \psi$. In the context of our work, those values are then converted into appropriate values of $T_{rr}, T_{r\vartheta}$.

Roughly speaking, we can consider a granular material as having properties of both solids
and fluids. This intuition can, however, lead one to make false assumptions about how the material behaves. In particular, if one associates fluid pressure that a fluid in a conical container experiences with the stress that a granular material would experience in that same container, the two behave in exactly the opposite manner. As one measures fluid pressure at various points from the top of the fluid down to the bottom of the conical container (i.e. decreasing $r$), the pressure increases. However, a granular material in this situation will actually show a drop in the measured average stress; that is, the stress felt by grains at the top of the hopper is greater than those grains at the bottom. The typical explanation is that granular materials in such a container support themselves in arches of grains. As the arches at the bottom of the hopper are smaller, they induce less stress. The arches near the top of the granular material in a conical hopper are larger and induce a greater stress.

A typical stress field, visualized in terms of the Sokolovskii variables $\sigma$ and $\psi$ is given in Figure 3.1.
3.2 Radial Velocity Fields

A radial velocity field can be found in much the same manner as the stress field was above. We start by considering a solution for the coaxiality model (2.35). We choose a solution of the form

\[ v(r, \vartheta) = -e^{(d-1)\tau} u(\vartheta) e_r, \]

(3.4)

where \( e_r \) is the unit vector in the \( \tau \) direction and \( u \) is a scalar function. Note that \( d \) as above refers to the spatial dimension. Then the velocity equations (2.35) collapse to

\[ u'(\vartheta) = -d u(\vartheta) \tan 2\psi(\vartheta) \]

(3.5)

Naturally, \( \psi \) is obtained from the corresponding radial field for the stresses.

For Spencer’s double-shearing model we instead arrive at the differential equation

\[ u'(\vartheta) = -d u(\vartheta) \frac{\sin 2\psi(\vartheta)}{s + \cos 2\psi(\vartheta)} \]

(3.6)

Note that in the derivation of the above, we have used the fact that the stress field is radial and therefore \( \partial_r \psi = 0 \).

Figures 3.7 and 3.8 give examples of radial velocity fields, both from Jenike’s model and from Spencer’s model, for specific parameters along with experimental results. These figures will be discussed in Section 3.6.

3.3 Stability of the Radial Field: Stress

In the case of a Mohr-Coulomb material both in two- and three-dimensional configurations, the stress equations decouple from the velocity equations, making possible a stability study of each system separately. This differs, for example, from the situation with a von Mises material, see Section 2.4, where the stress and velocity equations for a three-dimensional model must be treated as a whole, resulting in a much more difficult problem [15]. For
two-dimensional configurations, the von Mises and Mohr-Coulomb models coincide.

Stability here refers to the long time behavior of a perturbation of the radial solution. In particular, we have occasion to refer to a radial field being “stable going down the hopper.” By this, we mean that a perturbation of the radial field at the top of the hopper is damped out further down the hopper, in the direction of decreasing \( r \). Similarly, “unstable going up the hopper” means that a perturbation in the radial field far down in the hopper grows without bound as one follows the field in the direction of increasing \( r \). This differs from the concept of numerical stability for the computational method in Chapter 5.

We begin by linearizing the equations (2.32) about the radial stress field. Following the example above, instead of the usual stress components \( T_{r\tau}, T_{r\psi} \), we express the stress field and the governing equations in terms of the Sokolovskii variables. That is, we substitute into (2.11), (2.12) the expressions

\[
\sigma(\tau, \vartheta) = e^{-\tau} \left( \sigma^0(\vartheta) + \varepsilon \sigma^1(\tau, \vartheta) \right)
\]

\[
\psi(\tau, \vartheta) = \psi^0(\vartheta) + \varepsilon \psi^1(\tau, \vartheta)
\]

where \( \sigma^0, \psi^0 \) are the radial field as derived above and \( \sigma^1, \psi^1 \) are perturbations. As usual, we assume that \( \varepsilon > 0 \) is small. Consequently, after the substitution we obtain

\[
\partial_\tau \begin{pmatrix} \sigma^1 \\ \psi^1 \end{pmatrix} - \mathcal{A} \partial_\vartheta \begin{pmatrix} \sigma^1 \\ \psi^1 \end{pmatrix} - \mathcal{B} \begin{pmatrix} \sigma^1 \\ \psi^1 \end{pmatrix} - (d - 2) \mathcal{C} \begin{pmatrix} \sigma^1 \\ \psi^1 \end{pmatrix} = 0. \tag{3.7}
\]

The matrices \( \mathcal{A}, \mathcal{B}, \mathcal{C} \) are given by

\[
\mathcal{A} = \frac{1}{2\sigma^0 s(s - \cos 2\psi^0)} \begin{pmatrix} -2s\sigma^0 \sin 2\psi^0 & 8s^2(\sigma^0)^2 \\ (1 - s^2)/2 & -2s\sigma^0 \sin 2\psi^0 \end{pmatrix},
\]

\[
\mathcal{B} = \frac{1}{2\sigma^0 s(s - \cos 2\psi^0)} \begin{pmatrix} 2\sigma^0 s(s(3 + (\psi^0)' - \cos 2\psi^0)) - s(1 + (\psi^0)' \sin 2\psi^0) & 8s^2\sigma^0(\sigma^0)' \\ -s(1 + (\psi^0)' \sin 2\psi^0) & 2s\sigma^0(s - \cos 2\psi^0)(3 + (\psi^0)' - 2s(\sigma^0)' \sin 2\psi^0) \end{pmatrix},
\]

\[
\mathcal{C} = \frac{s}{2\sigma^0 s(s - \cos 2\psi^0)} \begin{pmatrix} 2\sigma^0 s(1 + \cos 2\psi^0 + \cot \vartheta \sin 2\psi^0) & 8(\sigma^0)^2 s \cos \vartheta \\ -(1 + s)(\sin 2\psi^0 + (1 - \cos 2\psi^0) \cot \vartheta)/2 & 2\sigma^0(s - \cos 2\psi^0 - \cot \vartheta \sin 2\psi^0) \end{pmatrix}.
\]
We separate variables

\[ \sigma^1(\theta, \tau) = e^{\lambda \tau} \Sigma(\theta) \quad \psi^1(\theta, \tau) = e^{\lambda \tau} \Psi(\theta), \]

and note that stability will correspond to eigenvalues \( \lambda \) with negative real parts. We now discretize in \( \theta \)-space the unknowns,

\[ \Sigma_j = \Sigma(\theta_j), \Psi_j = \Psi(\theta_j). \]

Upon substitution into the above equations, and with appropriate discretization of the derivative, we arrive at a discretized eigenvalue problem

\[ (\mathcal{A} \mathcal{D} + \mathcal{B} + (d - 2) \mathcal{C}) \begin{pmatrix} \Sigma \\ \Psi \end{pmatrix} = \lambda \begin{pmatrix} \Sigma \\ \Psi \end{pmatrix}, \quad (3.8) \]

where

\[ \mathcal{D} = \begin{pmatrix} \mathcal{D}_\Sigma & 0 \\ 0 & \mathcal{D}_\Psi \end{pmatrix}. \]

Here, \( \mathcal{D}_\Sigma \) is a discretization the operator \( \frac{d}{d\theta} \) without boundary conditions and \( \mathcal{D}_\Psi \) is a discretization of \( \frac{d}{d\theta} \) with homogeneous Dirichlet boundary conditions. For our particular implementation, a pseudospectral Chebyshev collocation method was used to construct the matrices \( \mathcal{D}_\Sigma, \mathcal{D}_\Psi \) [30].

Under the certain conditions, an exact spectrum can be found [15], [27]. As an example, we consider a “smooth wall” case with the condition that \( \psi \) is small Then we may make the approximations \( \sin 2\psi \approx 2\psi \) and \( \cos 2\psi \approx 1 \). With the substitution \( \chi = \sigma \psi \) in (3.7), we arrive at the system

\[ \partial_\tau \begin{pmatrix} \sigma^1 \\ \chi^1 \end{pmatrix} - \hat{\mathcal{A}} \partial_\theta \begin{pmatrix} \sigma^1 \\ \chi^1 \end{pmatrix} - \hat{\mathcal{B}} \begin{pmatrix} \sigma^1 \\ \chi^1 \end{pmatrix} - (d - 2) \hat{\mathcal{C}} \begin{pmatrix} \sigma^1 \\ \chi^1 \end{pmatrix} = 0. \quad (3.9) \]
where the matrices $\hat{A}, \hat{B}, \hat{C}$ are given by

\[
\hat{A} = \begin{pmatrix} 0 & -\frac{2s}{1-s} \\ -\frac{1+s}{2s} & 0 \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} \frac{1-3s}{1-s} & 0 \\ 0 & 3 \end{pmatrix}, \quad \hat{C} = \begin{pmatrix} -\frac{2s}{1-s} & -\frac{2s\cot\vartheta}{1-s} \\ 0 & 1 \end{pmatrix}.
\]

We separate variables again

\[
\sigma^1(\vartheta, \tau) = e^{\lambda \tau \hat{\Sigma}(\vartheta)}, \quad \chi^1(\vartheta, \tau) = e^{\lambda \tau \hat{X}(\vartheta)},
\]

and exploiting the structure of the system we derive an equation in a single variable. For two dimensions,

\[
(1 + s)\hat{X}'' - \left((1 - s)\hat{\lambda}^2 + (-4 + 6s)\hat{\lambda} + 3(1 - 3s)\right)\hat{X} = 0; \tag{3.10}
\]

and for three dimensions

\[
(1 + s)\hat{X}'' + (1 + s)(\hat{X} \cot \vartheta)' - \left((1 - s)\hat{\lambda}^2 + (-5 + 9s)\hat{\lambda} + 4 - 20s\right)\hat{X} = 0. \tag{3.11}
\]

A smooth wall is represented in the problem through the boundary condition

\[
\hat{X}(\pm \vartheta_w) = 0. \tag{3.12}
\]

For (3.10),(3.12) one easily obtains the eigenfunctions

\[
\left\{ \sin \left( n\pi \frac{\vartheta}{\vartheta_w} \right) \right\}_{n=1,2,...} \quad \left\{ \cos \left( n + \frac{1}{2} \right) \frac{\pi}{2} \frac{\vartheta}{\vartheta_w} \right\}_{n=0,1,2,...}
\]

provided that

\[
\hat{\lambda} = \frac{2 - 3s}{1-s} \pm \frac{1}{1-s} \sqrt{1 - (1 - s^2) \left( \frac{n\pi}{\vartheta_w} \right)^2}, \quad n = 1, 2, \ldots \tag{3.13}
\]

or

\[
\hat{\lambda} = \frac{2 - 3s}{1-s} \pm \frac{1}{1-s} \sqrt{1 - (1 - s^2) \left( \frac{(2n + 1)\pi}{2\vartheta_w} \right)^2}, \quad n = 0, 1, 2, \ldots \tag{3.14}
\]

These are pairs of points along a vertical line in the complex plane passing through the point
The value $\frac{3-2s}{1-s} = 0$, namely $\delta \approx 41.81^\circ$ corresponds to neutral stability. Below that value of $\delta$ the two dimensional smooth wall problem is predicted to be unstable. Note that, by parity, only the sine eigenfunctions are relevant.

A similar analysis may be used for the three dimensional problem, albeit with the small angle approximation $\cot \vartheta \approx \frac{1}{\vartheta}$. Then (3.11) becomes

$$\vartheta^2 \ddot{X}'' + \vartheta \dot{X}' + (\alpha \vartheta^2 - 1) \dot{X} = 0 \quad (3.15)$$

where

$$\alpha = - \left( \frac{1 - s}{1 + s} \lambda^2 + \frac{-5 + 9s}{1 + s} \hat{\lambda} + \frac{4 - 20s}{1 + s} \right).$$

Solutions here can be shown to be

$$\{J_1(\sqrt{\alpha_n \vartheta})\}_{n=1,2,...}, \quad \{Y_1(\sqrt{\alpha_n \vartheta})\}_{n=1,2,...},$$

Bessel’s functions of the first and second, respectively. That the latter are singular at $\vartheta = 0$ implies the $J_1$ are the only relevant solutions. Again using a smooth wall analysis, we apply (3.12) and find that $J_1(\sqrt{\alpha_n \vartheta}) = 0$. Consequently, there is a sequence of admissible $\alpha_n$ given by

$$\alpha_n = \left( \frac{x_n}{\vartheta_w} \right)^2, \quad n = 1, 2, \ldots$$

where $x_n$ is the $n^{th}$ positive root of $J_1$. Thus, the equation for the eigenvalues $\hat{\lambda}_n$ is

$$\frac{1 - s}{1 + s} \hat{\lambda}_n^2 + \frac{-5 + 9s}{1 + s} \hat{\lambda}_n + \frac{4 - 20s}{1 + s} + \alpha_n = 0. \quad (3.16)$$

A careful analysis [15] shows that the eigenvalues are complex for all realistic values of $\delta$ and, similar to the two dimensional case, all lie along a vertical line in the complex plane.
In fact
\[ \Re(\lambda_n) = \frac{1}{2} \frac{-9s^2 - 4s + 5}{1 - s^2} \]
and from this, we deduce that neutral stability corresponds to \( \delta \approx 33.75^\circ \). Consequently, for values of \( \delta \) greater than \( 33.75^\circ \) the solution of the three-dimensional smooth wall problem is shown to be stable.

This analysis has been shown to agree quite well with numerical solutions to (3.8) [15]. Using these results, low values of \( \delta \) lead to instability in both problems with the three dimensional problem being the more unstable. Similarly, for higher values of \( \delta \) both problems become stable, with the three dimensional problem stable for lower values of \( \delta \). Through numerical studies of (3.8), it has also been shown that increasing the wall friction can add stability (i.e. move the vertical line of eigenvalues so that the real part is negative), but for large values of \( \mu \) this effect is reversed, making the solution unstable. The hopper wall angle also affects the stability, with steep walls having more stable radial solutions [15].

### 3.4 Stability of the Radial Field: Velocity

As we have presented two different systems of equations for velocity, we study the stability of each separately. Others have studied the stability problem for Jenike’s model [27] and found the radial field to be stable (when solving down the hopper) using analytic methods on simplified problems and using numerical methods on the full problem. For our work, we developed our numerical approach to solve the problem for Jenike’s model and, once confirmed against the work of others, extended that approach to Spencer’s model.

#### 3.4.1 Models

Starting with the velocity equations (2.35), we first note that the equations are linear in the velocities. Assuming that the stress terms are from a radial stress field, we may linearize the velocities about the appropriate radial velocity field (which, in this case, depends on the
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\[ v_r(\tau, \vartheta) = e^{(d-1)\tau} \left( v_0(\vartheta) + \varepsilon v_1^r(\vartheta, \tau) \right), \]
\[ v_\vartheta(\tau, \vartheta) = e^{(d-1)\tau} \left( 0 + \varepsilon v_1^\vartheta(\vartheta, \tau) \right), \]

where \( d \) and \( \varepsilon \) are as above. Substituting into (2.35) we arrive at the following equations for the perturbations \( v_1^r, v_1^\vartheta \):

\[ \frac{\partial}{\partial \tau} \begin{pmatrix} v_1^r \\ v_1^\vartheta \end{pmatrix} - \mathcal{A} \frac{\partial}{\partial \vartheta} \begin{pmatrix} v_1^r \\ v_1^\vartheta \end{pmatrix} - \mathcal{B} \begin{pmatrix} v_1^r \\ v_1^\vartheta \end{pmatrix} = 0 \]

(3.17)

where

\[ \mathcal{A} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \tan 2\psi \end{pmatrix}, \]
\[ \mathcal{B} = \begin{pmatrix} 0 & (d - 2) \cot \vartheta \\ d \tan 2\psi & -(d - (d - 2) \cot \vartheta \tan 2\psi) \end{pmatrix}. \]

We separate variables

\[ v_1^r(\vartheta, \tau) = e^{\lambda \tau} U(\vartheta), \quad v_1^\vartheta(\vartheta, \tau) = e^{\lambda \tau} V(\vartheta), \]

Substitution of these into (3.17) leads us to the following discretized stability problem

\[ (\mathcal{A} \mathcal{D} + \mathcal{B}) \begin{pmatrix} U \\ V \end{pmatrix} = \lambda \begin{pmatrix} U \\ V \end{pmatrix}, \]

(3.18)

where \( \mathcal{D} \) is as above since the boundary conditions on \( v_1^\vartheta \) are the same as those on \( T_1^{r\vartheta} \). Numerical results of this approach are presented below in Section 3.4.3.

For Spencer’s model (2.37), we follow the same approach and arrive at the following
equation for the perturbations \( v_r^1, v_\theta^1 \)

\[
\partial_\tau \begin{pmatrix} v_r^1 \\ v_\theta^1 \end{pmatrix} - \hat{A} \partial_\theta \begin{pmatrix} v_r^1 \\ v_\theta^1 \end{pmatrix} - \hat{B} \begin{pmatrix} v_r^1 \\ v_\theta^1 \end{pmatrix} = 0
\]  

(3.19)

where

\[
\hat{A} = \begin{pmatrix} 0 & 1 \\ -\frac{s + \cos 2\psi}{s - \cos 2\psi} & -\frac{2 \sin 2\psi}{s - \cos 2\psi} \end{pmatrix},
\]

\[
\hat{B} = \begin{pmatrix} 0 & (d - 2) \cot \vartheta \\ -d \sin 2\psi & (d - (d - 2) \cot \vartheta \sin 2\psi) - \frac{2s \partial_\vartheta \psi}{s - \cos 2\psi} \end{pmatrix}.
\]

Then, separating as before, we arrive at the discretized stability problem

\[
(\hat{A}D + \hat{B}) \begin{pmatrix} U \\ V \end{pmatrix} = \lambda \begin{pmatrix} U \\ V \end{pmatrix},
\]  

(3.20)

Numerical results are presented in Section 3.4.3.

### 3.4.2 Analysis

Following [15],[27] we may find exact stability results under simplifying assumptions. For example, consider the case of radial gravity in a wedge-shaped hopper where the radial stress field is \( \psi = 0 \) and \( \sigma \) is constant. The assumption that gravity acts along radial lines has been used before [34] and provides a reasonable approximation for steep hoppers. Then the system (3.19) becomes

\[
\partial_\tau \begin{pmatrix} v_r^1 \\ v_\theta^1 \end{pmatrix} + \partial_\theta \begin{pmatrix} -v_r^1 \\ \frac{s+1}{s-1} v_\theta^1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2v_\theta^1 \end{pmatrix}
\]
which after separation of variables, as above, can be simplified to a second order equation for $V$

$$V'' = \left(\frac{1 + s}{1 - s}\right) \lambda (\lambda + 2)V.$$  

Coupled with the boundary conditions $V(\pm \vartheta_w) = 0$, we arrive at the conclusion that the eigenvalues are

$$\lambda = -1 \pm \sqrt{1 - \left(\frac{1 + s}{1 - s}\right) \left(\frac{k\pi}{\vartheta_w}\right)^2}, \quad k = 1, 2, \ldots$$

As the quantity under the radical is always negative, the eigenvalues all lie on a vertical line in the complex plane with negative real parts; consequently the modes are all stable.

For radial gravity in a conical hopper, the problem is a bit more delicate. The system (3.19) becomes

$$\partial_\tau \left( \begin{array}{c} v_1^r \\ v_1^\vartheta \end{array} \right) + \partial_\vartheta \left( \begin{array}{c} -v_1^r \\ \frac{s+1}{s-1} v_1^\vartheta \end{array} \right) = \left( \begin{array}{c} \cot \vartheta v_\vartheta \\ -3v_\vartheta \end{array} \right)$$

If we assume a relatively steep hopper and make the approximation $\cot \vartheta \approx \frac{1}{\vartheta}$, then we arrive at the second order equation

$$\vartheta^2 V'' + \vartheta V' + \left(\alpha \vartheta^2 - 1\right)V = 0$$

where $\alpha = \lambda (\lambda + 3) \left(\frac{s-1}{s+1}\right)$. We recognize this as exactly the form of the equation from (3.15) for which we found [15], [17] the eigenfunctions to be Bessel’s functions of the first kind

$$V_k = J_1(\sqrt{\alpha_k} \vartheta) \quad k = 1, 2, \ldots$$

As the boundary conditions $V(\pm \vartheta_w) = 0$ must be satisfied, we conclude that $\alpha_k = \left(\frac{\vartheta_k}{\vartheta_w}\right)^2$
Figure 3.2: Numerically computed eigenvalues for a 10 degree hopper, with delta = 32.1 degrees and mu = tan 11.7 degrees.

where \( x_k \) is the \( k \)th positive root of \( J_1 \). Therefore,

\[
\lambda_k(\lambda_k + 3) - \alpha_k \left( \frac{s + 1}{s - 1} \right) = 0
\]

and solving this for \( \lambda_k \) we find a discriminant

\[
\Delta_k = 9 + 4\alpha_k \left( \frac{s + 1}{s - 1} \right).
\]

Setting \( \Delta_k < 0 \) leads to the inequality

\[
\vartheta_w < \frac{2}{3} x_k \left( \frac{1 + s}{1 - s} \right).
\]

Observe that the smallest positive root of \( J_1 \) is \( x_1 \approx 2.4 \) and \( \frac{1 + s}{1 - s} \) is always greater than 1. Then the right hand side of (3.21) is greater than 1.6, thus greater than \( \frac{\pi}{2} \). Therefore, since \( \vartheta_w < \frac{\pi}{2} \) for all \( k \), the eigenvalues \( \lambda_k \) are always complex and line along a vertical line, \( \Re(\lambda_k) = -\frac{3}{2} \), in the complex plane. Consequently, the modes are all stable.
3.4.3 Numerics

Our numerical approach was tested first on problems involving only radial gravity under the Mohr-Coulomb-Jenike assumptions, so that the numerical results could be compared with the analytical results above. In particular, we take a conical hopper of half opening angle $10^\circ$ filled with a granular material having parameters $\delta = 35^\circ$ and smooth walls ($\mu = 0$). The analysis of the radial gravity problem above and that found in [27] show that the eigenvalues for this problem satisfy

$$\Re(\lambda_n) = -\frac{3}{2}.$$ 

Indeed, Figure 3.2 indicates that our code, like the numerical code whose results are shown in [27], Table 4.1, finds those eigenvalues. This confirmation, not only with the analysis but also with numerical results gathered by our predecessors indicates some degree of correctness in our code.

The code includes the ability to compute eigenvalues for the problems corresponding to Spencer’s models. To test this portion of the code, we also compared with the analysis we did for granular materials with the Mohr-Coulomb-Spencer assumptions under the influence of radial gravity. As with the Mohr-Coulomb-Jenike model, the numerical results agreed with the analytical results. For comparison, consider the same material parameters as above: a material with an internal angle of friction of $\delta = 35^\circ$ in a smooth wall ($\mu = 0$) hopper of opening angle $10^\circ$. The theory above predicts that for a wedge-shaped hopper the eigenvalues satisfy

$$\Re(\lambda_n) = -1$$

while for a conical hopper, we should see

$$\Re(\lambda_n) = -\frac{3}{2}.$$ 

Figure 3.3 shows the results of our numerical computation and indeed the eigenvalues lie along the appropriate vertical lines.
Figure 3.3: Numerical computation of eigenvalues corresponding to the radial gravity velocity problem; wedge-shaped hopper, left; conical hopper, right.

Now, we consider the material parameters of a standard problem that we have used repeatedly in our later work with normal gravity. Specifically, we focus on a material with angle of internal friction $\delta = 32.1^\circ$ in a conical hopper with wall friction given by $\mu = \tan 11.7^\circ$. Results for hoppers with wall inclinations $\vartheta_w$ of $20^\circ$ and $30^\circ$ are shown in Figure 3.4. These graphs indicate that, like the radial gravity problems tested previously, these radial velocity fields are stable when considered as problems of propagation down the hopper.

3.5 Using the Radial Solution in Experiments

For a hopper and granular material with fixed parameters, the radial fields for both the stress and velocities are relatively easy to obtain numerically [15]. As our interest lies with problems that perturb the radial fields, those numerical radial fields are then valuable in several specific ways.

First, any code designed to solve these systems, like (2.32) or (2.35), may be tested with the radial solution. In particular, it should be shown that in problems with “full radial symmetry” for all data, the radial structure of the solution is preserved as the code solves up or down the hopper. Further, perturbing a radial solution and then using it as a boundary condition could potentially be used to observe the stability properties of the discretized model. One might hope to observe the predictions about stability made in [15],[27], and
Chapter 3. Radial Solutions

Figure 3.4: Numerical computation of eigenvalues corresponding to the full conical hopper velocity problem with standard gravity; 20 degree hopper, left; 30 degree hopper, right.

Figure 3.5: Transition in hopper wall roughness.
above with a numerical code and a perturbed radial field.

Additionally, we use the radial fields to construct initial conditions for supposed “real” experiments. As an example, consider a hopper of constant wall angle but with a discontinuity in wall material. One might presume that above the transition in hopper wall material that radial stress and velocity fields have been achieved and that, as we consider these essentially problems of propagation, the abrupt change in parameters induces a perturbation which propagates from this transition. Consequently, some of our experiments start from a radial field and solve down past a transition. See Figure 3.5. One might well question the validity of this situation, as the appropriate direction for solving either stresses or velocities is far from clear; future research will investigate the development of more meaningful boundary value problems.

In a second type of experiment, we attempt to calculate the stress and velocity fields after a transition in hopper wall angle, all other physical parameters being held constant. See Figure 3.6 for the types of transitions that we wish to simulate. These experiments, too, rely upon radial fields, since we will typically assume that the radial stress field, will have been achieved in the upper section of the hopper (above the wall angle transition) and then use that fact to calculate an appropriate initial condition for the lower section of the hopper (below the wall angle transition). Our computational method for doing so is outlined in Chapter 5.
3.6 Comparison with Experiment

In [5] are found the results of experiments designed to measure the radial velocity field for kale seeds in several different hoppers. We note that the radial velocity fields obtained below through calculations using the Mohr-Coulomb-Jenike and Mohr-Coulomb-Spencer models differ significantly. In fact, comparison with the fields obtained experimentally in [5] suggest that velocity field of the Mohr-Coulomb-Spencer model more accurately approximates real-life fields.

In these experiments, a hopper was constructed with a set of rails along the top. Along these rails, a thin tube could be aligned to deposit a pellet to a desired point in the hopper. Thus, after filling the hopper with a granular material and allowing it to settle, a marker pellet could be deposited into the material at a desired point in the filled hopper. Subsequently, allowing the material to flow out of the hopper, the velocity field within the hopper could be determined by measuring when the marker pellet exited the hopper.

Naturally several questions arise from these experiments and the reliability of the results. To increase the strength of these results, various tests were carried out to establish that not only were the tests reproducible but they represented the actual velocity field for a grain in a material experiencing a sustained radial flow. Two types of marker pellets (steel and ceramic) were tested to show that the marker pellet moved at the same rate, independent of the material making up the pellet itself. Further, the marker was inserted into a hopper of material that had been flowing and had been stopped. Then, after removing the tube, the material flow was restarted. The validity of this start/stop/start method was investigated and found to produce physically acceptable results. Among the other parameters investigated were the initial height of the material in the hopper, the time allowed for a flow to develop before inserting a tracer, and variations in the discharge rate during the tests.

As an example, we turn to [5], Figure 8 (represented by the square data points, marked Experimental, in Figures 3.7 and 3.8 of this document), for results involving kale seeds in a hopper with $\theta_w = 30^\circ$. Three hopper wall materials are tested (corresponding to three different angles of wall friction) and for two of those wall materials, aluminum and woodchip paper, we present results here. Figures 3.7 and 3.8 reveal that the velocities near the wall
Figure 3.7: Comparison of the radial velocity fields of Mohr-Coulomb-Jenike, Mohr-Coulomb-Spencer, and von Mises with experimental results; kale seeds in an aluminum hopper; angles are in radians; square data points are taken from [5], Figure 8.

predicted by the Jenike and von Mises models are far lower than indicated by the experiments in [5] while Spencer’s model seems more appropriate.

No explicit numerical data was given in [5], so the values in 3.7 and 3.8 represent an estimation taken from [5], Figure 8. Further, no error bars were given to indicate an estimated error from the laboratory trials.

The results for the von Mises case are used here courtesy of Pierre Gremaud. They were obtained using a Chebyshev collocation method [30].
Figure 3.8: Comparison of the radial velocity fields of Mohr-Coulomb-Jenike, Mohr-Coulomb-Spencer, and von Mises with experimental results; kale seeds in a woodchip paper hopper; angles are in radians; square data points are taken from [5], Figure 8.
Chapter 4

Analysis

4.1 General Remarks

In addition to the radial stress and velocity field analyses, much can be learned from a straight analysis of the systems (2.11)–(2.16) and (2.11)–(2.15),(2.22). Here we will observe that the systems of interest are hyperbolic conservation laws and subsequently choose our numerical approach based on that observation. However, it should be noted that the type of problems does not necessarily correspond to hyperbolic equations. In particular, the steady state equations for a von Mises material have been shown to be elliptic under some circumstances [35].

4.2 Observations about the Stresses

As the stresses decouple from the velocities, we may study the stress problem independently of that for the velocities. Further, despite appearances, (2.32) can be considered as a system in only two variables if one “solves” the yield condition (2.13) for $T_{\vartheta\vartheta}$ as a function of $T_{rr}, T_{r\vartheta}$.

In stress component space, $(T_{rr}, T_{r\vartheta}, T_{\vartheta\vartheta})$, the yield condition forms a cone with its axis lying in the $(T_{rr}, T_{r\vartheta})$-plane. See again Figure 2.2. Solving for $T_{\vartheta\vartheta}$ we obtain

$$T_{\vartheta\vartheta} = \frac{1 + s^2}{1 - s^2} T_{rr} \pm \frac{2}{1 - s^2} \sqrt{s^2 T_{rr}^2 - (1 - s^2) T_{r\vartheta}^2} \quad (4.1)$$

Note that there are two parts to this solution, one corresponding in some sense to the “top”
of the cone and the other to the “bottom” of the cone.

As explained below, many, but by no means all, problems can be treated using just the part of the yield surface given by

\[ T_{\vartheta\vartheta} = \mathcal{Y}(T_{rr}, T_{r\vartheta}) \equiv \frac{1 + s^2}{1 - s^2} T_{rr} + \frac{2}{1 - s^2} \sqrt{s^2 T_{rr}^2 - (1 - s^2) T_{r\vartheta}^2}. \]  \hspace{1cm} (4.2)

We find the limitation of this approach by setting the third component of the gradient of (2.13) equal to zero. Starting with

\[-2(T_{rr} - T_{\vartheta\vartheta}) - 2s^2(T_{rr} + T_{\vartheta\vartheta}) = 0,\]

and using the relation (4.2) we arrive at the statement

\[ |T_{r\vartheta}| = \tan \delta T_{rr}. \]

Clearly, this implies that the range of \( T_{rr}, T_{r\vartheta} \) values for which our function \( \mathcal{Y} \) is valid is given by

\[ |T_{r\vartheta}| < \tan \delta T_{rr}. \]  \hspace{1cm} (4.3)

Note that this range also coincides with the range of \( T_{rr}, T_{r\vartheta} \) values for which the quantity under the radical in (4.2) is positive and for which \( |\psi| < \frac{\pi}{4} - \frac{\delta}{2} \). Computationally, we observe systems for which the stresses do approach the limits of (or even attempt to violate) this condition, see Chapter 6.

Recall that, as parametrized by the Sokolovskii variables, the yield surface is divided into a section for which the stresses are called *passive*, \( |\psi| < \pi/4 \), and a section for which the stresses are called *active*, \( |\psi| > \pi/4 \). While it would be useful if the choice for \( T_{\vartheta\vartheta} \) above corresponded to the active and passive states of stress, we are not so fortunate. Rather, as seen in Figure 4.1, the “top” (taking the + in (4.1)) represents a portion, but not all, of the passive state of stress.
Turning our attention to the pair of equations (2.32), we rewrite them (again) in the form

$$\partial_t U + \partial_\theta F(U) = G,$$  \hspace{1cm} (4.4)

with the obvious notation. The eigenvalues of the Jacobian $F'$ are then found to be

$$\lambda_{1,2} = \pm \tan \delta \pm \frac{1}{c} \sqrt{\frac{sT_{rr} \pm cT_{r\theta}}{sT_{rr} \pm cT_{r\theta}}},$$  \hspace{1cm} (4.5)

and the corresponding right and left eigenvectors (both of which will be utilized later in the numerical approach) are

$$r_{1,2} = \begin{pmatrix} 1 \\ \lambda_{1,2} \end{pmatrix}, \quad \ell_{1,2} = \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} \pm \lambda_{2,1} \\ \mp 1 \end{pmatrix}.$$  \hspace{1cm} (4.6)

Provided that the stresses $T_{rr}, T_{r\theta}$ remain within the limit (4.3) described above, i.e. $|T_{r\theta}| < \tan \delta T_{rr}$, the eigenvalues are indeed real and distinct with $\lambda_1 < 0 < \lambda_2$. Thus, the equations (2.32), and the definition (4.2) form a strictly hyperbolic system of nonlinear conservation laws with source terms. Indeed, this form underscores the approach that we have taken to the stress and velocity problems here: that these are problems of propagation.

Investigating further, we observe that the eigenvalues may become unbounded, even for
bounded values of the stress. Holding $T_{rr}$ fixed, for example, we note that

$$
\begin{align*}
\lambda_1 &\rightarrow -\tan \delta \quad \text{when } T_{r\theta} \uparrow \tan \delta T_{rr}, \\
\lambda_1 &\rightarrow -\infty \quad \text{when } T_{r\theta} \downarrow -\tan \delta T_{rr}, \\
\lambda_2 &\rightarrow \infty \quad \text{when } T_{r\theta} \uparrow \tan \delta T_{rr}, \\
\lambda_2 &\rightarrow \tan \delta \quad \text{when } T_{r\theta} \downarrow -\tan \delta T_{rr}.
\end{align*}
$$

With some manipulations the above eigenvalues (4.5) may be rewritten in terms of the Sokolovskii variables $(\sigma, \psi)$. One finds

$$
\lambda_{1,2} = \cot \left( \psi \mp \left( \frac{\pi}{4} - \frac{\delta}{2} \right) \right).
$$

(4.7)

Physically this means that the stress characteristics are inclined at angles $\pm \left( \frac{\pi}{4} - \frac{\delta}{2} \right)$ to the direction of major principal stress. See Figure 4.2. As noted at the beginning of this section, when $|\psi| \geq \frac{\pi}{4} - \frac{\delta}{2}$ our definition of $T_{\vartheta \vartheta}$ breaks down. Here the same condition would lead to further problems since, in our chosen coordinate system, the range of influence would be pointing partially into “the past”, as determined by the flow of our time-like variable $\tau$.

We have discussed two obstacles: one related to stresses reaching the edge of the yield
surface and one related to unbounded eigenvalues. These are indeed distinct problems that happen to coincide in this coordinate system. Keeping the unknowns \( T_{rr}, T_{r\theta} \) and using a different coordinate system for which the characteristics do not point back in time might alleviate the problem of unbounded speeds, but would not make any more of the passive portion of the yield surface available. Consequently, such an approach would not prevent the stresses from reaching the edge of the yield surface, resulting in a breakdown of computations.

Turning to the boundary condition as expressed in (2.24), at first glance it appears to be nonlinear if we assume \( T_{\phi\phi} \) as defined above by (4.2). However, basic algebra allows one to rewrite (2.24) as

\[
T_{r\phi} = \pm \eta T_{rr}.
\]

(4.8)

where the constant \( \eta \) is given by

\[
\eta = \frac{(1 + s^2 + 2\sqrt{s^2 - \mu^2(1 - s^2)})}{1 - s^2 + 4\mu^2}.
\]

(4.9)

About this relation, we make two observations. First, the boundary condition is actually linear in the unknowns, rather than nonlinear as it first appeared. Second, to ensure that the quantity under the radical in (4.9) is non-negative, it must be that

\[
\delta \geq \phi.
\]

(4.10)

That is, the angle of wall friction can not be more than the angle of internal friction. Equality corresponds to perfectly rough walls, i.e. the granular material is only sliding upon itself instead of upon the wall. This is the meaning of perfectly rough used when comparing the radial solutions with those in [5].

However, a more stringent condition may be derived. At the wall, we know that

\[
\tan \phi = \frac{T_{r\phi}(\vartheta_w)}{T_{\phi\phi}(\vartheta_w)} = \frac{\sin \delta \sin 2\psi}{1 - \sin \delta \cos 2\psi}.
\]

Since we know \( 0 \leq |\psi| \leq \pi/4 \) it is easy so see that at \( \psi = \pi/4 \) this expression achieves its
maximum value, and we may conclude that

\[ \sin \delta \geq \tan \phi = \mu. \]  

(4.11)

This statement may be interpreted to mean that the maximum shear stress that a granular material can sustain while in motion is somewhat less than what it can sustain at rest [26].

4.3 Observations about the Velocities

4.3.1 The Mohr-Coulomb-Jenike Model

Following our work with the stresses above, we strive to analyze the equations describing the velocities in the same manner. We rewrite the equations from Jenike’s Coaxiality Model (2.35) as

\[ \partial_t V + \partial_\text{\textphi} X(V) = Y, \]  

(4.12)

again with the natural notation and consequently find the eigenvalues of \(X'\) to be

\[ \lambda_{1,2} = \pm \frac{1 \pm \sin 2\psi}{\cos 2\psi}. \]  

(4.13)

The corresponding eigenvectors are

\[ r_{1,2} = \begin{pmatrix} \lambda_{1,2} \\ 1 \end{pmatrix}, \quad \ell_{1,2} = \frac{\cos 2\psi}{2} \begin{pmatrix} \pm 1 \\ \mp \lambda_{2,1} \end{pmatrix}. \]  

(4.14)

Here, in the velocity problem, we again find a range of admissible stresses, as the eigenvalues are real and distinct with \( \lambda_1 < 0 < \lambda_2 \) only for \( |\psi| < \frac{\pi}{4} \). This corresponds to the range of stresses we call passive. Practically this limit will never be reached provided that we take the approach outlined above for the stresses, since there we are constrained by the more stringent \( |\psi| < \frac{\pi}{4} - \frac{\delta}{2} \). We also remark that, under these constraints, the equations form a system of strictly hyperbolic linear conservation laws with source terms and that they are
Figure 4.3: Visualizing the velocity characteristics for Jenike’s Coaxiality Model.

coupled to the stress equations through the appearance of terms involving $\psi$.

It is instructive to rewrite (4.13) in the equivalent form

$$
\lambda_{1,2} = \cot \left( \psi \pm \frac{\pi}{4} \right). \quad (4.15)
$$

In this form, one easily sees that the velocity characteristics are inclined at an angle of $\pi/4$ from the direction of major principal stress. See Figure 4.3.

4.3.2 The Mohr-Coulomb-Spencer Model

Here it is interesting to compare the results arising from Jenike’s model with those from Spencer’s Double-Shearing Model. Rewriting in the obvious way

$$
\partial_r V + \partial_\theta \hat{X}(V) = \hat{Y}, \quad (4.16)
$$

one finds the eigenvalues of the Jacobian $\hat{X}'$ to be

$$
\lambda_{1,2} = \pm \frac{c \pm \sin 2\psi}{s - \cos 2\psi}. \quad (4.17)
$$
and the respective eigenvectors as
\[
\begin{align*}
\mathbf{r}_{1,2} &= \begin{pmatrix} \pm \frac{c \mp \sin 2\psi}{s + \cos 2\psi} \\ 1 \end{pmatrix}, \\
\ell_{1,2} &= \frac{1}{2c} \begin{pmatrix} s + \cos 2\psi \\ c \pm \sin 2\psi \end{pmatrix}.
\end{align*}
\]
(4.18)

As with the system of stresses, the eigenvalues are real and distinct provided that (4.3) is satisfied, and in that case we have \(\lambda_1 < 0 < \lambda_2\) so that Spencer’s model leads to a system of strictly hyperbolic linear conservation laws with source terms. Note that the limitation on \(\psi\) here coincides exactly with that of the stresses, whereas in the Mohr-Coulomb-Jenike model the limitations placed on \(\psi\) were different between the velocity and stress equations.

For reference, we remark here upon the following four inequalities that can be useful when determining the sign of a quantity (as in the eigenvalues):
\[
\begin{align*}
c - \sin 2\psi &> 0, \\
s - \cos 2\psi &< 0.
\end{align*}
\]

We tacitly assume here that \(|\psi| < \frac{\pi}{4} - \frac{\delta}{2}\) and \(0 < \delta < \frac{\pi}{2}\).

Recall that Spencer’s model purports to align the characteristic directions of the velocities and the stresses, and indeed it is easily seen that the eigenvalues (4.5) are exactly the same as (4.17) with the help of a few trigonometric identities. Consequently, the figure (4.2) actually represents the characteristic directions and domain of influence for both stresses and velocities when using Spencer’s model.

4.4 Observations about the Boundary Conditions

We now turn our attention to the boundary conditions for the stress and velocity fields. As has been pointed out above, the law of sliding friction imposes conditions on the stresses on the hopper walls, seen in (2.24) and (4.8). Similarly, the boundary condition at the wall for the velocity field is rooted in the physical concept that grains cannot pass through the solid hopper walls. Since our models consist of systems of hyperbolic conservation laws, we need only a boundary condition for each component of the stresses (and respectively velocities) along a noncharacteristic surface, in addition to the boundary conditions at the walls, to
form a valid mathematical system. Here, this extra boundary condition can be thought of as an “initial condition” since we consider $\tau$ as if it were timelike and imagine the solution propagating forward in time. If we consider a line of constant $\tau$ (or $r$) as a boundary along which to specify a boundary condition, then we are faced with a crucial question: in which direction shall we solve? That is, should we solve towards or away from the vertex of the hopper?

It is generally assumed among those that study granular flows that the direction of propagation for stress information is in the “downward” direction, i.e. towards the vertex. Intuitively, one thinks that when more stress is added to the system from the top, say, through the addition of more granular material, the grains near the bottom will feel this change. Additionally, if one considers adjusting the outlet velocity for a particular hopper (say through the use of a piston below the exit) then changes in the velocity at the exit of the hopper will no doubt affect the velocity of the grains further up in the hopper. This, in turn, suggests that information for the velocities travels “upwards”, away from the vertex. As there is no purely mathematical reason to choose one direction over the other, we must rely on such thought experiments or on experimental evidence to guide us.

So, choosing to solve the stress field downwards and subsequently the velocity field upwards, it would seem as though the problem had been adequately stated and that solutions could be easily and confidently generated. Unfortunately, there are a few problems with this arrangement. Most importantly, let’s consider the function $\lambda$ from the flow rule for Jenike’s model, found in (2.9), whose sign coincides with the work done by friction and thus must remain positive, see Section 2.2. In the derivation of (2.16), this function $\lambda$ is eliminated from the model equations. As the system can no longer “see” $\lambda$, it cannot know whether the positivity of $\lambda$ is preserved or not. Yet, the values of $\lambda$ can be determined a posteriori from the calculated stress and velocity fields, and in some experiments (given in Chapter 6) these values of $\lambda$ do indeed locally change sign from positive to negative. This suggests that the stress and velocity equations may be linked in some more subtle way than is currently understood; see the conclusions in Chapter 7.

Even though the Mohr-Coulomb-Spencer model arises from a different flow rule, one for which there is no parameter $\lambda$, the sign of work done by friction can be computed for both
models. Consequently, solutions to the Mohr-Coulomb-Spencer model should be tested to ensure that this sign stays positive. The correct computation of the work done by friction for Spencer’s model presents a challenge in that it requires the derivative with respect to $\tau$ of the function $\psi$, a function that is piecewise polynomial in $\theta$ but is known only at distinct points in the $\tau$ direction. This aspect is currently under investigation and could show a distinct and measurable difference between the behavior of these two models of velocity fields.

The stability results derived in Chapter 3 point to other potential problems with solving the system in these given directions. For certain choices of the material parameters, the stress model has been shown to be linearly unstable. Further, it is clear that the velocity model seems to be stable always when solving downwards, but this in turn means that solving up will be unstable. Consequently, as has been observed experimentally (again, see Chapter 6) a perturbation in an initial condition for the velocities near the bottom of the hopper can lead to disastrous results. For example, when solved upwards through a known stress field, the velocity field can show instabilities that seem to be one of the driving factors in pushing $\lambda$ negative.
Chapter 5

Solver

As noted above, the problems we wish to solve are essentially systems of hyperbolic conservation laws. As the system for the stresses is non-linear, it would not be unexpected to see the evolution of discontinuities while solving a perturbed, non-radial system. Further, it is our goal to solve problems that involve discontinuous changes in parameters, such as the angle of wall friction or the angle of inclination of the hopper walls. For these reasons, the original system of stresses will be kept in conservative form.

We have chosen to use the Runge-Kutta Discontinuous Galerkin (RKDG) method as developed in [9],[10],[12]. This scheme lends itself easily to the present work, in several ways. Primarily, it allows us to treat our problem in the original conservation form, thus allowing for accurate handling of discontinuities. Further, the scheme is explicit and consequently relatively easy to implement. Implementation of the boundary conditions is also straightforward, even in the case of a system. And, if higher accuracy is required, a method of arbitrarily high (formal) order can be constructed in a natural way by adding more moments within each cell. This higher order, however, comes at the price of a more stringent stability condition requiring smaller timesteps to maintain that accuracy.

While both velocity systems are linear in the unknowns, source terms are present which may lead to the formation of arbitrarily steep slopes (but not shocks) in the evolution of a solution. Naturally, the shock-capturing ability of the RKDG method will just as easily resolve these features, should they arise.
5.1 Overview of Discontinuous Galerkin Methods

In 1973, the discontinuous Galerkin method was introduced by Reed and Hill as a method for solving the neutron transport equation [31], a linear hyperbolic problem. The method’s importance was recognized and subsequently analyzed by LeSaint and Raviart in 1974 [24]. Shortly thereafter, applications to ordinary differential equations [24], wave propagation in elastic media [42], and optimal control [13] were studied. More recently, authors have explored applications of the discontinuous Galerkin method in the areas of parabolic equations discretized in time [21], viscoelastic flows [14], and Maxwell’s equations [41].

Having initially been successful with linear hyperbolic problems, authors naturally sought an extension to the more difficult nonlinear hyperbolic problems. However, a nonlinear flux makes it impossible to solve for the characteristics explicitly; thus at each timestep a new system of nonlinear equations must be solved [2]. Some of this difficulty was mitigated with some local time discretization methods that permitted local element-by-element computations [32].

With the development of the Runge-Kutta Discontinuous Galerkin methods, however, efficient application to hyperbolic problems became a reality. The first step in this direction solved a single scalar conservation law with a piecewise linear discretization in space and a forward Euler integration in time. Unfortunately, without restricting the ratio $\frac{\Delta t}{\Delta x}$ to the order of $\sqrt{\Delta x}$ the method was unconditionally unstable [4]. This scheme was modified to incorporate a specially defined slope limiter, and the resulting method was proven to be total variation bounded in the means provided that the CFL number $f'\frac{\Delta t}{\Delta x}$ was less than or equal to 1/2. Despite convergence of numerical solutions to the correct entropy solution, the order of the approximation was only first order accurate in time. Further, the heavy-handed slope limiter affected the quality of the solution in smooth regions [3].

Finally, combining the piecewise linear DG method for the spatial discretization, a special explicit total variation diminishing (TVD) second order Runge-Kutta discretization in time, and a modified slope limiter man of these difficulties were overcome [8]. The result was an explicit scheme that was formally uniformly second order accurate in space and time, total variation bounded in the means, and linearly stable for CFL numbers less than 1/3.
Numerically, this method proved to be quite desirable, with second order convergence in smooth regions, sharp resolution of discontinuities without oscillations, and convergence to the proper entropy solution, even for non-convex fluxes. This was the first Runge-Kutta Discontinuous Galerkin (RKDG) method.

After this breakthrough, the method was quickly generalized to incorporate formally high-order accurate RKDG methods for scalar hyperbolic conservation laws [9]. The achieve a method of formal order $k + 1$, the scheme used a discontinuous Galerkin spatial discretization with polynomials of degree $k$, a TVD Runge-Kutta $(k + 1)$-th order accurate explicit time discretization, and a generalized slope limiter (the natural extension of the slope limiter from the original RKDG method). Then the schemes were extended to one-dimensional systems [10] and then to multidimensional scalar equations [11], culminating with an extension to multidimensional systems [12].

To date, the analysis of the discontinuous Galerkin method for nonlinear problems has been difficult, and there are only three primary results. First, it has been shown [23] that for one-dimensional nonlinear conservation laws with strictly convex (or concave) nonlinearities, if the approximating DG solution of arbitrary polynomial degree converges then it converges to the entropy solution. The other results apply only to a version of the space-time DG method for scalar nonlinear conservation laws with an additional term called a shock-capturing term. In this case it has been shown that if the approximating solution converges then it converges to the entropy solution [20]. Further, a posteriori error estimates have been obtained. That work proved not only convergence, but also an error on the order of convergence for possibly discontinuous solutions [6]. Unfortunately, these results have little bearing upon the use of the RKDG method in this work.

The method used in this work is based on the approach in [9] and [10]. We use piecewise linear and piecewise parabolic elements with a three stage Runge-Kutta method. Consequently, our approach is formally second order accurate (for linear elements) or third order accurate (for parabolic elements) in space and always third order accurate in time.
5.2 A Scalar Equation and Finite Elements

For simplicity and clarity, we explain the algorithm as for a single scalar equation. Extension to a system is relatively simple and will be explained subsequently. More detailed work can be found in [9],[10],[11],[12]

Consider the domain $I = (0, \vartheta_w)$ and let $I_j = (\vartheta_{j-1/2}, \vartheta_{j+1/2})$, $j = 0, \ldots, N - 1$ be a partition of $I$ with $\Delta \vartheta = \vartheta_{j+1/2} - \vartheta_{j-1/2} = \vartheta_w/N$ and $\vartheta_{j-1/2} = j \Delta \vartheta$. We define the space of functions

$$U_h = V_h^k = \{v \in L^\infty(I) : v|_{I_j} \in \mathcal{P}^k(I_j), j = 0, \ldots, N - 1\}$$

(5.1)

where $\mathcal{P}^k(K)$ is the space of polynomials of degree at most $k$ over the interval $K$. Note that elements of $U_h$ are allowed to be multivalued at cell boundaries. This feature, besides giving the Runge-Kutta Discontinuous Galerkin method its name, distinguishes this method from other finite element methods and makes this method explicit in semidiscrete ODE form.

To work with $U_h$ we provide a basis

$$\varphi_j^0(\vartheta) = 1, \quad \varphi_j^1(\vartheta) = \vartheta - \vartheta_j, \quad \varphi_j^2(\vartheta) = (\vartheta - \vartheta_j)^2 - \frac{\Delta \vartheta^2}{12}, \ldots$$

(5.2)

which one readily recognizes as the Legendre polynomials over the interval $I_j$. Now we may write a discretized solution as

$$u^h(\vartheta, \tau) = \sum_{\ell=0}^k a_\ell u_j^\ell(\tau) \varphi_j^\ell(\vartheta) \quad \text{for } \vartheta \in I_j$$

(5.3)

where the components $u_j^\ell$ and coefficients $a_\ell$ are given by

$$u_j^\ell = u_j^\ell(\tau) = \frac{1}{\Delta \vartheta^{\ell+1}} \int_{I_j} u(\vartheta, \tau) \varphi_j^\ell(\vartheta) d\vartheta$$

(5.4)

and

$$a_\ell = \frac{\Delta \vartheta^{\ell+1}}{\int_{I_0} \varphi_j^\ell(\vartheta)^2 d\vartheta}. \quad (5.5)$$
Note that $u^0_j$ is the mean of $u$ in cell $j$.

Now, consider the scalar equation

$$
\partial_\tau u + \partial_\vartheta f(u) = g(\vartheta, \tau, u) \tag{5.6}
$$

We substitute $u^h$ in for $u$, multiply by $\varphi^\ell_j$ and formally integrate (by parts) over the cell $I_j$ to obtain an ODE

$$
\frac{d}{d\tau} u^\ell_j = -\frac{1}{\Delta+} \left( \Delta+ \left( \varphi^\ell_j \left( \vartheta_{j-1/2} \right) h_{j-1/2} \right) \right) - \int_{I_j} f \left( u^h(\vartheta, \tau) \right) \frac{d}{d\vartheta} \varphi^\ell_j(\vartheta) d\vartheta
$$

where $\Delta+ u_j = u_{j+1} - u_j$ and we have replaced the ambiguous $f \left( u_{j-1/2} \right)$ with a flux $h_{j-1/2} = h(u^-_{j-1/2}, u^+_{j-1/2})$.

In our work, we have used a local Lax-Friedrichs flux, which for a scalar equation is

$$
h_{j-1/2} = \frac{1}{2} \left( f^+_{j-1/2} + f^-_{j-1/2} - \alpha_{j-1/2} \left( u^+_{j-1/2} - u^-_{j-1/2} \right) \right)
$$

Here,

$$
\alpha_{j-1/2} = \max_{\min(u^+_{j-1/2}, u^-_{j-1/2}) \leq u \leq \max(u^+_{j-1/2}, u^-_{j-1/2})} |f'(u)|.
$$

In general, the only requirements on a flux for accuracy are consistency $h(u, u) = f(u)$ and Lipschitz continuity of $h(\cdot, \cdot)$. Other choices for the flux might be slightly less diffuse, but for ease of implementation and reasonable accuracy, local Lax-Friedrichs is sufficient. In our testing, changing the flux to something less diffusive like the Godonov flux did not have a discernible impact upon the quality of the solutions obtained and was substantially more work to implement and maintain.

The change from a scalar equation to a system is primarily one of notation. We opt here to let capital letters denote vector quantities and matrices. Further, extension to a system involves moving some of the calculations over into characteristic space. For example, for
calculating the flux for a system, one would use (instead of the above)

\[ \tilde{H}_{j-1/2} = \frac{1}{2} \left( \tilde{F}^{+}_{j-1/2} + \tilde{F}^{-}_{j-1/2} - \alpha_{j-1/2} \left( \tilde{U}^{+}_{j-1/2} - \tilde{U}^{-}_{j-1/2} \right) \right) \]

where \( U \) is the vector analog of \( u \) and \( \tilde{U} \) is \( U \) in characteristic space. Here we use the matrices

\[
R = \begin{pmatrix} r_1 & r_2 \end{pmatrix} \\
L = \begin{pmatrix} \ell^T_1 \\ \ell^T_2 \end{pmatrix} = R^{-1}
\]

with (respectively) right and left eigenvectors \( r_j, \ell_j \) of the Jacobian \( \mathcal{F}'(U) \) to convert quantities to and from characteristic space:

\[
\tilde{U} = LU \\
U = R\tilde{U}
\]

### 5.3 Solving the ODE

We collect all the unknowns \( U^j \) into a column and rewrite the equation (5.7) as

\[
\frac{d}{d\tau} U = \mathcal{F}(U) + \mathcal{G}(\tau, U) \tag{5.8}
\]

where \( \mathcal{F}, \mathcal{G} \) come respectively from the discretizations of \( f, g \). One might well suggest applying a split approach, such as those given in [25]. However, we note that the boundary conditions interact with \( \mathcal{G} \) in an essential manner, making a split method more difficult to implement. Further, elementary calculations show that the term \( \mathcal{G} \) is not stiff, and so we are free to use a non-split approach.

Our time discretization combines a third order TVD Runge-Kutta method [9] with a local slope limiting process, \( \Pi\Lambda \). Slope limiting is a “trick”, common in numerical methods for hyperbolic conservation laws, that attempts to correct or control spurious oscillations introduced by a finite element scheme by potentially altering that the slope in a cell relative to the means in neighboring cells [12].
The initial condition $U_h(\tau_0, \cdot)$ is supplied at $\tau_0 = 0$ and, given $\Delta \tau > 0$, integration proceeds through $N_\tau$ timesteps $\tau^n = \tau_0 + n\Delta \tau$. Finally, we may give an outline of the process:

- Initialize $U^0_h = \Pi \Lambda(U_h(\tau_0, \cdot))$
- For $n = 0, \ldots, N_\tau - 1$, compute $U^{n+1}_h$:
  1. Set $U^0 = U^n_h$
  2. For $i = 1, \ldots, 3$ compute intermediate stages
     $$U_i = \Pi \Lambda \left( \sum_{k=0}^{i-1} \alpha_{ik} U_k + \Delta \tau \beta_{ik} \left( \mathcal{F}(U_k) + \mathcal{G}(\tau^n + d_k \Delta \tau, U_k) \right) \right);$$
  3. Set $U^{n+1}_h = U^3$.

The numerical parameters, $\{\alpha_{ik}\}, \{\beta_{ik}\}, \{d_k\}$ are specially chosen to give the integration its TVD quality [9], [36], [37]:

\[
\begin{align*}
\alpha_{10} &= 1 & \beta_{10} &= 1 & d_0 &= 0 \\
\alpha_{20} &= \frac{3}{4} & \alpha_{21} &= \frac{1}{4} & \beta_{20} &= 0 & \beta_{21} &= \frac{1}{4} & d_1 &= 1 \\
\alpha_{30} &= \frac{1}{3} & \alpha_{31} &= 0 & \alpha_{32} &= \frac{2}{3} & \beta_{30} &= 0 & \beta_{31} &= 0 & \beta_{32} &= \frac{2}{3} & d_2 &= \frac{1}{2}
\end{align*}
\]

The slope limiter, $\Pi \Lambda$, is based on the modified minmod function and is required for the stability of this method. Also, note that the minmod function also requires transformation into characteristic coordinates. The normal minmod function alters the slope in a cell relative to the means of the cells around it. However, this condition may be too strong and tends to flatten out numerical solutions around local extrema. Consequently, the modified minmod presumes that the slopes are small and approximately correct near a local extremum where near is a parameter that can be tweaked to fit the situation. Here we chose to go with the standard values provided in the literature [9],[12].

While there is only one convergence result for these types of methods [6], these methods are generally accepted as reliable and accurate, provided that one obeys a few rules. Among these are the use of the slope limiter (a crucial part of our implementation) and the CFL
condition:

\[ \lambda_{\text{max}} \frac{\Delta \tau}{\Delta \vartheta} < \frac{1}{2k + 1} \]

where \( k \) is the degree of polynomial used in the finite elements [9].

### 5.4 Numerical Flux and Boundary Conditions

The construction of the numerical flux \( H \) at the domain boundaries requires a bit of explanation. As an example, recall from (4.8) that the boundary condition for the stresses may be expressed as

\[ T_{r\vartheta} = \pm \eta T_{rr} \quad \text{at} \quad \vartheta = \pm \vartheta_w \]

Let \( R, L \) be as above and transform \( U = \begin{pmatrix} T_{rr,N-1/2}^- & T_{r\theta,N-1/2}^- \end{pmatrix} \) to \( \tilde{U} = \begin{pmatrix} u_{1,N-1/2}^- \\ u_{2,N-1/2}^- \end{pmatrix} \) by left multiplication by \( L \). We set \( u_{2,N-1/2}^+ = u_{2,N-1/2}^- \), since \( \lambda_2 \) indicates that the direction of propagation of \( u_2 \) is towards the cell boundary and it is assumed that this will provide an appropriate approximation of \( u_{2,N-1/2}^+ \). Now, the boundary condition (4.8) written in characteristic coordinates allows us to find the corresponding value of \( u_{1,N-1/2}^+ \):

\[ u_{1,N-1/2}^+ = \frac{r_{22} - \eta r_{12}}{\eta_{11} - r_{21}} u_{2,N-1/2}^+ = \frac{\lambda_2 - \eta}{\eta - \lambda_1} u_{2,N-1/2}^+ \]

Then the values actually used in the computation of the numerical flux at \( \vartheta = \vartheta_w \) are

\[ \begin{pmatrix} T_{rr,N-1/2}^+ \\ T_{r\theta,N-1/2}^+ \end{pmatrix} = R \begin{pmatrix} u_{1,N-1/2}^+ \\ u_{2,N-1/2}^+ \end{pmatrix} \]

Though the above is specific to the boundary treatment for the stresses, the two velocity models are handled in a very similar way, essentially taking the corresponding constant \( \eta \) to be zero, i.e. \( v_{\vartheta} = 0 \).

At the center of the hopper, the boundary condition may be treated as above with
\( T_{r\theta} = 0 \). Alternatively, as suggested earlier, the symmetry of the model may be used, thus avoiding the change to characteristic variables altogether. As an example, we consider the stresses at the centerline of the hopper where it is assumed that \( U = \begin{pmatrix} T_{rr,-1/2}^+ \\ T_{r\theta,-1/2}^+ \\ -T_{rr,-1/2}^- \end{pmatrix} \) is known. Then the symmetry of the stresses indicates taking

\[
\begin{pmatrix} T_{rr,-1/2}^- \\ T_{r\theta,-1/2}^- \end{pmatrix} = \begin{pmatrix} T_{rr,-1/2}^+ \\ -T_{r\theta,-1/2}^+ \end{pmatrix}
\]

for the values used to compute the flux at the center. As the symmetries of the velocities \( v_r, v_\theta \) in both velocity models mirror those of \( T_{rr}, T_{r\theta} \) this also extends in a very natural way to the boundary condition for the velocities at \( \psi = 0 \).

### 5.5 Implementation Details

The RKDG method was implemented from scratch entirely in C and utilized the Meschach linear algebra library \[40\] for its data structures. The current form consists of some 1700 lines of code, neglecting the Meschach library code. Initially on a Sun Solaris machine, the code was eventually migrated to an Intel-based machine running Linux. Initial conditions were created with a FORTRAN routine based on COLSYS [1] and then converted to polynomial form in Matlab.

The C code outputs files in a form native to Matlab and all visualization was performed there. Helper routines, written in Matlab’s scripting language, assisted with the post-processing of the data.
Chapter 6

Results

Here we introduce and attempt to solve several initial boundary value problems for (2.32), (2.35), (2.37) with the RKDG method outlined in Chapter 5. Most experiments are performed with $100 \mathcal{P}^1$ elements across the domain $(0, \vartheta_{w})$ as numerical studies we’ve conducted don’t show enough of an increase in detail to justify the extra work that comes with more cells or more moments per cell.

6.1 Radial Fields Forever

Although our ultimate goal is to compute non-radial fields, we may first test our numerical approach by confirming that it preserves the radial fields with the proper scaling.

For this purpose, a FORTRAN code `hopper.f` utilizing COLSYS [1], based on collocation at Gaussian points, was developed. It solves the systems (3.2) for the quantities $\sigma, \psi$. Approximating polynomials of $\sigma, \psi$ were then placed into our RKDG code as initial conditions. As $\psi$ and $\sigma$ are smooth and simple for a radial field, approximation by polynomial is adequate.

For the radial solution to be preserved, the interplay between the boundary conditions and the source terms must be captured by the scheme. Further, we remark that the radial solution itself may be unstable [15]. Nevertheless, our RKDG solver does preserve the radial solution to a high degree of accuracy as can be seen in Figure 6.1. For these numerical experiments, we used $\delta = 32.1^\circ$ and $\mu = \tan 11.7^\circ$ with $\vartheta_{w} = 30^\circ$. Starting at $r = 0$, the integration was carried out to roughly $r = 0.05$ or 95% of the way down the cone. One
observes that, although the stability properties and magnitudes of the errors in the solutions are appreciably different, both are acceptably accurate even after a very long integration. The large periodic perturbations visible in the conical hopper case arises from an inaccuracy in our polynomial approximation of the initial condition at the boundary. These travel from the boundary to the center of the hopper and back and are amplified when they reach the center and interact with waves coming in from all other directions.

For the velocities in both the Jenike and Spencer models, we see similar preservation of the radial solution. The situation with Spencer’s model is a bit more sensitive, mainly due to the terms involving the derivative of $\psi$, but with care the radial solution can be preserved nearly as well as with the Jenike model.

### 6.2 Wall Roughness Transitions

Here we consider experiments in a hopper with constant wall angle but with a sharp transition in wall material, resulting in a discontinuous change in the angle of wall friction. Again, see Figure 3.5. While this situation arises infrequently in real applications, we find it a simple yet puzzling experiment from which we have arrived at some surprising results.

Before turning to the numerical results, we recall the discussion in Section 4.4 about the

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Figure 6.1: Preservation of the radial field by the PDE solver. Left: wedge-shaped hopper, right: conical hopper. For each value of tau the error is represented by the maximum percentage of difference between the radial field and the numerical PDE solution.
Figure 6.2: Stress field with a transition from a rough hopper wall, \( \mu = \tan(11.7) \), to a smooth hopper wall, \( \mu = \tan(6.7) \); solution obtained by solving down the hopper.

function \( \lambda \) and that it should remain positive for the model to reflect a physically meaningful stress and velocity field.

In our first numerical example, we consider a granular material having an angle of internal friction \( \delta = 32.1^\circ \) passing a transition from a rough wall with \( \mu = \tan 11.7^\circ \) to a smoother wall \( \mu = \tan 6.7^\circ \) in a conical hopper. See Figure 6.2 for the stress field and Figure 6.3 for the velocity field using the Jenike model. Note that, in this particular instance the velocity calculations were in fact initialized along the same surface as the stress calculations, and both solved down through the hopper towards the vertex. As explained previously, a radial stress and velocity field was presumed to exist above this transition. Also, at each step down the hopper, the solution has been divided by the radius \( r \), thus scaling it as if it were a radial solution.

There are a few features here that deserve mention. First, note the rarefaction originating from the point of transition in hopper wall material; this rarefaction interacts at the center of the hopper with other waves from the wall and in turn induces a sharpening that becomes a shock as one moves down the hopper. Also note that in the velocity component \( v_\theta \) the solution maintains zero lateral velocity until it “feels” the change in stresses along a stress characteristic.

The most important feature in this experiment, however can’t be visualized with just
Figure 6.3: Velocity field with a transition from a rough hopper wall, \( \mu = \tan(11.7) \), to a smooth hopper wall, \( \mu = \tan(6.7) \); solution obtained by solving down the hopper.

Figure 6.4: Values of lambda in the stress-velocity field with a transition from a rough hopper wall, \( \mu = \tan(11.7) \), to a smooth hopper wall, \( \mu = \tan(6.7) \); on the right is highlighted the region in which lambda becomes negative; both stress and velocity were solved down the hopper.
Chapter 6. Results

Figure 6.5: Velocity field with a transition from a rough hopper wall, $\mu = \tan(11.7)$, to a smooth hopper wall, $\mu = \tan(6.7)$; solution obtained by solving up the hopper.

these graphs. Instead we consider Figure 6.4 where we see the values of $\lambda$ throughout this experiment and, in the right graph, we highlight the region in which $\lambda$ becomes negative. So, despite first appearances, this solution actually exhibits nonphysical properties. Note that, as the velocities and stresses are scaled appropriately, we know that the values of $\lambda$ should be proportional to $r^4$.

Naturally, then, one questions whether starting at the top for both the stresses and velocities is the appropriate approach for this problem. Consequently, we solved the same problem by specifying a radial velocity field at the bottom of the hopper and then solving up through the computed stress field. The results here were no more encouraging, as shown in Figures 6.5 and 6.6. Here the stress field acts to perturb the velocity field and, as the velocity field is unstable going up the hopper, the perturbations only get larger.

In other experiments we have solved the stress field far enough down the hopper that it becomes essentially radial, a property that nearly all stress computations in conical hoppers demonstrated. As the stress field was nearly radial, we then specified a radial velocity field at this lower boundary and solved up through the stress field. In all cases of significant change in wall material parameters, the velocities eventually developed perturbations that drove the values of $\lambda$ negative.

Experiments involving transitions to rougher walls are equally challenging, in fact pushing
the limits of the numerical approach we have chosen. Here, we have found that a transition to too rough of a wall can lead the stresses beyond the range of the yield surface for which our approach is suited; in effect the stresses want to “jump off” of the yield surface to values of $T_{rr}, T_{r\theta}$ for which there is no corresponding value of $T_{\theta\theta}$. Even for those transitions for which the stresses remained well-behaved, the same “negative $\lambda$” difficulties that plagued the experiments outlined above returned, manifesting themselves in practically the same manner.

As an example of the transition from smooth to rough, Figure 6.7 shows half of the characteristic field in a hopper after a transition from a wall where $\mu = \tan(11.7^\circ)$ to a slightly rough wall where $\mu = \tan(13.0^\circ)$. This figure clearly shows a shock with characteristics entering it. Despite the relatively small transition in friction, the shock induced is strong enough to push the ensuing field off the yield surface.

Finally, our work with Spencer’s model is still in progress. The inclusion of terms involving the derivative of $\psi$ has proved problematic computationally, so while computation of the radial velocity field is currently feasible, the determination of more complicated velocity fields is still being investigated.
These experiments, and others that we’ve performed, indicate that developing a consistent and useful boundary value problem, even for a physical arrangement as simple as presented here, is actually far from trivial. We have not found a single experiment that gave satisfying, physically meaningful results. In Chapter 7, the ramifications of the negative values of $\lambda$ are discussed.

### 6.3 Wall Angle Transitions

In real life, hoppers sometimes are made in two sections with a transition in wall angle. Many such transitions are from a silo (hopper with vertical walls) to a hopper with inclined walls. However, it is generally accepted that stresses in a silo are in the active state while those in a hopper are in the passive state, and the appropriate method for switching between the two states is, to our knowledge, an open problem. Thus, we choose here to avoid that difficulty and consider only structures made from two converging hoppers. See Figure 3.6 for the geometrical situation. Here we try to determine the stress and velocity fields after a transition in hopper angle, given some assumptions about how those fields stand high up in
the hopper or far down in the hopper.

From Figure 3.6 note that the point \( P \) admits two representations: \( P = (R, \Theta) \) in the upper hopper and \( P = (r, \vartheta) \) in the lower hopper. Given that the material parameters \( \delta \) and \( \mu \) remain fixed, our numerical approach for solving for the stress field consists of

- generating the radial stress field \( T \) in the upper hopper \([15],[26]\)

\[
\{(R, \Theta) : R > 0, |\Theta| \leq \Theta_w\};
\]

by its radial nature, the radial stress field at any point in the upper hopper is \( RT(\Theta)\);

- interpolating the radial stress field along the curve

\[
\Gamma_0 = \{(r, \vartheta) : r = r_0, |\vartheta| \leq \vartheta_w\},
\]

leading to a stress tensor \( S_{\Theta_w} \);

- changing to the coordinate system for the lower hopper via

\[
S_{\vartheta_w} = R^T(\Theta - \vartheta)S_{\Theta_w}R(\Theta - \vartheta),
\]

where \( R(\alpha) \) is the rotation matrix of angle \( \alpha \);

- solving in the lower hopper \( \{(r, \vartheta) : 0 < r < r_0, 0 < \vartheta < \vartheta_w\} \) using the RKDG method outlined in Chapter 5 using \( S_{\vartheta_w} \) as the initial condition.

Naturally the boundary conditions will be treated as indicated in the previous chapter. A similar approach can be applied to the case of a velocity field. To that end, we note that instead of rotation, we use a projection of \( v_R \) for the upper hopper into the two components \( v_r, v_\vartheta \) for the lower hopper.

Tacit in the above approach is the assumption that, above the transition, the radial stress (or, respectively, radial velocity) field is sought and attained by the granular material. Further, we are assuming that this radial field reaches the curve \( \Gamma_0 \) unperturbed by the change in wall angle. Both of these rely on the downward propagation of information in the field, be it velocity or stress. Assuming downward propagation, the second point is clearly
satisfied for a transition to a flatter hopper, i.e. $\Theta_w < \vartheta_w$, see Figure 3.6, left. However, for transitions of the type $\vartheta_w < \Theta_w$, see Figure 3.6, right, an analysis based on the characteristic curves shows that the difference in angles should not be too large. Specifically, we can say

$$\Theta_w - \vartheta_w < \arctan \frac{1}{|\lambda_{\text{max}}|}, \quad \vartheta_w, \Theta_w > 0,$$

where $\lambda_{\text{max}}$ is the largest eigenvalue in modulus of $F'$, from (4.4), evaluated along the curve $\Gamma_0$. An illustration of this result is given in [17].

We now apply our RKDG solver to these types of problems. First we examine a configuration consisting of a conical $30^\circ$ hopper above a conical $25^\circ$ hopper. As usual, $\delta = 32.1^\circ$ and $\mu = \tan 11.7^\circ$. The results are represented in Figure 6.8. The same sort of rarefaction and subsequent sharpening into a shock as seen above for the wall material transition is seen again here. Also, the stress field for $T_{r\vartheta}$ illustrates that, for longer integrations, the stress field seems to lose its perturbations and settle down to the radial field appropriate for the lower hopper. This phenomenon is not well understood at this time, since for these material parameters, the linear stability analysis in described in Chapter 3 indicates that this field should be unstable.

Next, we present the computed stress field for the same configuration, but for wedge-shaped hoppers. See Figure 6.9. Here, the initial development of the rarefaction mirrors