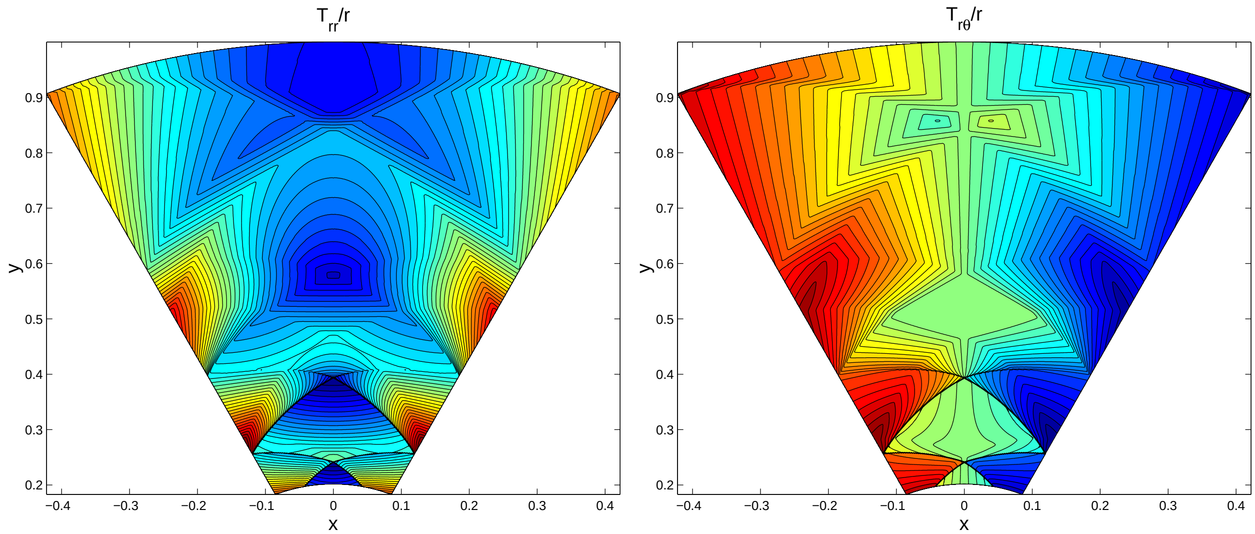

Figure 6.9: Transition from a 30 degree wedge-shaped hopper to a 25 degree wedge-shaped hopper.

that of the conical hopper, but later shows much stronger shocks. It has also been observed that the stress fields in wedge-shaped hoppers are more apt to develop fields that leave the yield surface (as described above for the transition to rougher walls). Consequently, we have made less progress investigating stress fields (and, by extension, velocity fields) in wedge-shaped hoppers.

Finally, we examine the stress field for a wedge shaped hoppers with a transition from a 30° wall angle to a 35° wall angle. See Figure 6.10. Transitions of this type, in wedge-shaped hoppers, develop fields with shocks that, from our observations repeat indefinitely down the hopper. Note however, that in the conical hopper the shocks diminish down the hopper and the field seems to approach a radial field appropriate for the lower hopper.

Unfortunately, it is not clear at this time how to set up appropriate velocity field problems. Initial tests with both the stress and the velocity specified along the top boundary, while initially appealing, failed to satisfy the λ condition that troubled us in the experiments above.

In fact, it is possible to explicitly show that our approach will lead to an unacceptable value of λ for a transition from a shallower to a steeper hopper. In that case the value of v_ϑ in the lower hopper section is the projection of v_r from the upper hopper onto a line of constant ϑ for the lower hopper. This value is, at the left hopper wall, positive, as it will

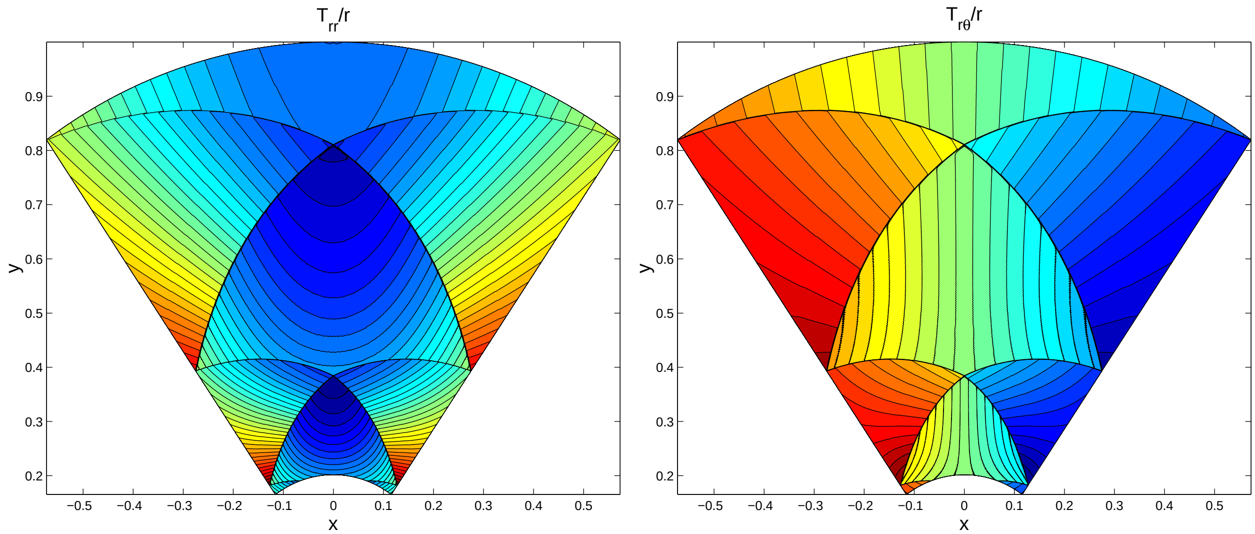

Figure 6.10: Transition from a 30 degree wedge-shaped hopper to a 35 degree wedge-shaped hopper.

point towards the center of the hopper. Yet, the solution will immediately seek $v_\vartheta = 0$ at the wall; thus $\partial_\vartheta v_\vartheta$ will quickly attain a large, positive value under most conditions. Examining the (2,2)-entry of the strain rate V, see (2.10), is enough to show that where $\partial_\vartheta v_\vartheta$ is large and positive, λ will take on a negative value.

Thus, computations of this type must be deferred until we have a greater understanding of how to construct appropriate boundary value problems for Jenike's and Spencer's models.

Chapter 7

Conclusions

In the field of granular materials two models, the Mohr-Coulomb-Jenike and the Mohr-Coulomb-Spencer, are both considered standard approximations of mass flow in a converging hopper. The relative merits of these two models have been debated for years, and there is yet no resolution to the question of which model better captures the qualities of a real granular flow. In this work we have used computational and analytical tools to study these different models of the stress and velocity fields. In particular, we have analyzed the model stress and velocity equations, as well as the naturally arising radial fields and the stability of those radial fields. To our knowledge, we are the first to implement, test, and subsequently apply a high order numerical method to the governing stress and velocity equations in their original conservation form. The results of those numerical experiments have lent insight into the issues surrounding the study of stress and velocity fields arising from discontinuous changes in parameters.

Before this work, research on these problems has focused on the model equations expressed in terms of the Sokolovskii variables. Consequently, only smooth stress and velocity fields could be treated, and that work forms the basis for modern industrial hopper design. Our approach, for the first time, removes the restriction to smooth fields and allows correct calculation of discontinuities in the stress field and subsequent resolution of the related velocity field.

Our study of a variety of boundary value problems has raised some interesting issues that only became apparent once a wider array of fields could be treated. The deceptively simple experiment proposed in Section 6.2, makes the need for meaningful and tractable boundary value problems painfully clear. Further, despite being eliminated from the system, the function λ still bears upon the model and subtly ties the stress field to the velocity field in an essential way. A possible explanation of the appearance of negative values of the function λ could be the formation of rigid zones of material within the hopper. In regions where the material no longer deforms, the model we use breaks down and new equations governing the stress and velocity fields there would have to be found. To our knowledge, it is not known how to handle such zones. Further, if the information propagates in different directions in the hopper for the stress and velocity fields, we will then be required to solve for one field and then the other; this raises the problem of not even knowing that the fields have developed such a region in which λ has assumed negative values until after both computations have been completed. This phenomenon require further research and should lead to a greater understanding of the model itself.

The inclusion of the Mohr-Coulomb-Spencer model in this research serves two purposes. Primarily, we would like to provide some comparative results that may help distinguish one model as capturing more of the features of real granular flows. Our radial field studies have shown that the Spencer model seems, at this stage, to share many of the same stability characteristics as the Jenike model. And even though the Spencer model does not contain this parameter λ that troubles the Jenike model, we may still check the physical feasibility of our results using the sign of work done by friction. Only the first preliminary results of this application are coming to light, but future research should provide further insight. Those preliminary results seem to indicate that Spencer's model can make the sign of work done by friction negative, and thus exhibiting nonphysical features similar to those seen with Jenike's model.

We are actively pursuing several areas of research. Initially, the question of acceptable boundary value problems for these models must be resolved. Consequently, there is a need for acceptable resolution of the stress and velocity fields for both models and a comprehensive study of their features. Ideally this would include some stability studies of the discretized nonlinear stress model to compare with the linear stability studies found in the literature. Ultimately, the work then must be compared with results available from industry and laboratory experiment, for evaluation of the quality of the model and the methods used in numerical simulation. In this last point we hope to involve further our contact with Jenike & Johanson, an engineering firm dedicated to the solution of industrial granular materials handling problems.

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