Abstract

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The purpose of the research has been to develop a decomposition for the $\mathcal{J}$-classes of a reductive monoid. The reductive monoid $M_n(K)$ is considered first. A $\mathcal{J}$-class in $M_n(K)$ consists of elements of the same rank. Lower and upper staircase matrices are defined and used to decompose a matrix x of rank r into the product of a lower staircase matrix, a matrix with a rank r permutation matrix in the upper left hand corner, and an upper staircase matrix, each of which is of rank r. The choice of permutation matrix is shown to be unique. The primary submatrix of a matrix is defined. The unique permutation matrix from the decomposition above is seen to be the unique permutation matrix from Bruhat's decomposition for the primary submatrix. All idempotent elements and regular $\mathcal{J}$-classes of the lower and upper staircase matrices are determined. A decomposition for the upper and lower staircase matrices is given as well.

The above results are then generalized to an arbitrary reductive monoid by first determining the analogue of the components for the decomposition above. Then the decomposition above is shown to be valid for each $\mathcal{J}$-class of a reductive monoid. The analogues of the upper and lower staircase matrices are shown to be semigroups and all idempotent elements and regular $\mathcal{J}$-classes are determined. A decomposition for each of them is discussed.

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THE STAIRCASE DECOMPOSITION FOR REDUCTIVE MONOIDS

BY

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Dedication

To my family

Biography

The author was born in Siler City, North Carolina in 1964. She received a B.S. in Mathematics and a B.S. in Mathematics Education from North Carolina State University in 1988, after which she taught high school for three years. She earned an M.Ed. in Mathematics Education in 1994. She currently resides in Sanford, North Carolina with her husband and three children.

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Chapter 0

Introduction

In Chapter 1, the author begins with a review of general semigroup theory. Introductory definitions and theorems are presented. These introductory concepts are followed by a discussion of Green's relations, regular semigroups, 0-simple semigroups, Rees' Theorem and inverse semigroups.

In Chapter 2, the author gives an extension of Renner's decomposition as applied to $M_n(K)$. The definitions of a lower and upper staircase matrix are given then used to show that each matrix x in $M_n(K)$ can be written as the product of a lower staircase matrix, a permutation matrix, and an upper staircase matrix, each of which has the same rank as x. The choice of permutation matrix is shown to be unique. The primary submatrix of x is defined. It is shown that the permutation matrix needed for the above decomposition of x is the same as the permutation matrix needed for Bruhat's decomposition of the primary submatrix. All idempotent elements of the upper and lower staircase matrices are determined. All $\mathcal{J}$-classes are found as well.

In Chapter 3, a brief introduction to algebraic geometry is given, sufficient to introduce the concept of an algebraic variety.

In Chapter 4, reductive groups are defined. Definitions and examples of Borel subgroups, unipotent subgroups and the Weyl group are given. Bruhat's decomposition for all reductive groups is stated. Parabolic and Levi subgroups are defined and examples of each are given.

In Chapter 5, the ideas of Chapter 4 are extended to Reductive monoids by way of a cross-section lattice, Λ. The concepts of Chapter 4 are redefined in terms of elements from Λ. A proof of Renner's decomposition for reductive monoids which utilizes the notation introduced in this chapter is given.

In Chapter 6, the results of Chapter 2 are generalized to arbitrary reductive monoids. In particular, the decomposition of a matrix into a product of matrices of the same rank is shown to generalize to a decomposition for an element of a $\mathcal{J}$-class of a reductive monoid into the product of elements from that $\mathcal{J}$-class. The analogue of the upper and lower staircase matrices is given and each is shown to be a semigroup. All idempotent elements and regular $\mathcal{J}$-classes of these semigroups are determined. A decomposition for each is discussed as well.

Chapter 1

General Semigroup Theory

In this chapter a brief introduction to general semigroup theory is presented. The elementary concepts needed for work presented later in this paper are defined here. For a deeper understanding of general semigroup theory the reader may refer to books by Clifford [3] and Howie [4]. The development of much of the following material closely follows that of Howie [4].

In this chapter and in the remainder of this paper a large number of examples will need to be generated. At this point, let us recall some familiar sets from which we will draw many examples.

If $X = \{1, 2, \ldots, n\}$, then $\mid X \mid = n$. Denote by S_X or S_n the symmetric group; i.e., the set of bijective transformations on n objects. Recall that $\mid S_n \mid = n!$.

Example 1.1. If $X = \{1, 2\}$, then using cyclic notation

$$S_2 = \{id, (12)\}$$

and $\mid S_2 \mid = 2! = 2$. S_2 is isomorphic to a subgroup of $GL_2(K)$, so

$$S_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

as well. When S_n is represented in this manner, the elements are called permutation matrices. We will switch freely between the two different notations.

Denote by $\mathcal{T}_X$ or $\mathcal{T}_n$ the full transformation semigroup; that is, the set of all transformations from X into X. Recall that $\mid \mathcal{T}_n \mid = n^n$.

EXAMPLE 1.2. Let $X = \{1, 2\}$. Then

$$\mathcal{T}_2 = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \right\}$$

where $\begin{pmatrix} 1 & 2 \\ i & j \end{pmatrix}$ stands for $1 \to i$, $2 \to j$. $\mathcal{T}_2$ may be thought of as matrices as well, so

$$\mathcal{T}_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

and

$$\mid \mathcal{T}_2 \mid = 2^2 = 4.$$

Denote by $\mathcal{P}_X$ or $\mathcal{P}_n$ the partial maps from X into X; that is, the maps from a subset of X to a subset of X. Recall that $\mid \mathcal{P}_n \mid = (n+1)^n$.

EXAMPLE 1.3. Let $X = \{1, 2\}$. Then

$$\mathcal{P}_2 = \left\{ 0, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \right\}$$

or

$$\begin{aligned} \mathcal{P}_2 \quad = \quad & \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \right. \\ & \left. \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \end{aligned}$$

and $\mid \mathcal{P}_2 \mid = (2+1)^2 = 9$.

Denote by $\mathcal{I}_X$ or $\mathcal{I}_n$ the symmetric inverse semigroup; i.e., all $1-1$ maps from a subset of X to a subset of X. Recall that

$$\mid \mathcal{I}_n \mid = \sum_{r=0}^{n} \binom{n}{r}^2 r!$$

EXAMPLE 1.4. Let $X = \{1, 2\}$. Then

$$\mathcal{I}_2 = \left\{0, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\right\}$$

or

$$\mathcal{I}_2 = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\},$$

and $\mid \mathcal{I}_2 \mid = \binom{2}{0}^2 0! + \binom{2}{1}^2 1! + \binom{2}{2}^2 2! = 7$. When $\mathcal{I}_n$ is thought of as a set of matrices the elements of $\mathcal{I}_n$ are called **partial permutation matrices** since they may be obtained from the elements of S_n by exchanging some or all of the 1's for 0's.

1.1 Introductory Concepts

A **semigroup** is a pair $(S, \cdot)$ consisting of a set of objects S and an associative operation, $\cdot$.

EXAMPLE 1.5. Let $S = \{2, 4, 6, \ldots\} = \mathbb{E}$. Then $(S, +)$ is a semigroup.

If a semigroup S has an identity element, denoted 1_S, then it is unique. In order to see this, suppose that 1_S and $1_{S'}$ are both identity elements for a semigroup S. Then $1_S = 1_S 1_{S'} = 1_{S'}$. A semigroup that has an identity element is called a **monoid**. If a semigroup S does not have an identity element, it is an easy matter to adjoin one to S. Let

$$S^1 = \begin{cases} S & \text{if } S \text{ has an identity} \\ S \cup \{1\} & \text{otherwise} \end{cases}$$

and define multiplication in the following way:

$$1 \cdot a = a \cdot 1 = a \qquad \forall a \in S.$$

It is left to the reader to verify associativity.

EXAMPLE 1.6. $M_3(K) = \{x \mid x \text{ is a } 3 \times 3 \text{ matrix with entries from a field } K\}$ together with the usual matrix multiplication is a monoid.

A semigroup S is said to contain a 0 if $\mid S \mid > 1$ and $0 \cdot a = a \cdot 0 = 0$, $\forall a \in S$. $\mid S \mid$ is required to be greater than 1 to eliminate the trivial case when $S = \{1\}$ and 1 is it's own zero!

EXAMPLE 1.7. Let $S = \{0, a, b\}$ have the following multiplication table.

	0	a	b
0	0	0	0
a	0	0	0
b	0	0	0

Then S is a semigroup containing a zero. In fact, since $x \cdot y = 0 \ \forall x, y \in S$ S is called a **null** or **zero semigroup**.

EXAMPLE 1.8. Let $S = M_2(K)$ with the usual matrix multiplication. Then S is a semigroup with $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

If S does not contain a zero, it is an easy matter to adjoin a zero to S. Let

$$S^0 = \begin{cases} S & \text{if } S \text{ has a zero} \\ S \cup \{0\} & \text{otherwise} \end{cases}$$

with multiplication defined as

$$0 \cdot a = a \cdot 0 = 0 \qquad \forall a \in S.$$

It is wise at this point to caution the reader. Although it is easy to adjoin a zero or an identity element to a semigroup S, some fundamental property of S may be lost in doing so. For example, if S is a semigroup that is a group, then S^0 is a semigroup that is not a group.

EXAMPLE 1.9. Let $S = \left\{ x = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in K \right\}$ with ordinary matrix multiplication. S is a group, but S^0 is not since $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ does not have an inverse.

The following example shows that semigroups without 2-sided identity or zero elements may indeed have 1-sided identity as well as 1-sided zero elements.

EXAMPLE 1.10. Let $S = \{a, b, c\}$ have the following multiplication table

	a	b	c
a	a	a	a
b	b	b	b
c	c	c	c

Then S is a semigroup with a zero or an identity element. Define a **left zero** to be an element $x \in S$ so that $xy = x\ \forall y \in S$. Clearly, every element of S is a left-zero. Define a **right identity** to be an element $x \in S$ so that $yx = y\ \forall y \in S$. So each element of S is a right identity. Such a semigroup is called a **left-zero semigroup**. A **right zero**, **left identity**, and **right zero semigroup** are defined in an analogous manner.

A non-empty subset T of a semigroup S is a **subsemigroup** if $T^2 \subset T$; i.e., if it is closed under multiplication.

EXAMPLE 1.11. Let $S = M_2(K)$ with the usual matrix multiplication and

$$T = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

$T^2 \subset T$, so T is a subsemigroup of S.

EXAMPLE 1.12. Let $e \in S$ so that $e^2 = e$. Then e is said to be an **idempotent**, and $\{e\}$ is a subsemigroup of S. Denote by $E(S)$ the subsemigroup of S consisting of all the idempotents in S.

A subsemigroup that is a group is called a **subgroup**.

EXAMPLE 1.13. Let $S = M_2(K)$ and

$$T = \left\{ x \in M_2(K) \mid x = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, a \in K \right\}.$$

Then $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in T$ is the identity. Since

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

each element has an inverse; therefore, T is a subsemigroup of S that is a group. Hence, T is a subgroup.

A semigroup need not contain a subgroup; for example, $(\mathbb{E}, +)$ is a semigroup that has no subgroup. In fact, a semigroup will contain a subgroup if and only if it contains an idempotent.

The identity element, e_T, of a subgroup T is an idempotent since $e_T \cdot e_T = e_T$. It remains to show that if a semigroup contains an idempotent, then the idempotent is the identity of some subgroup. If e is an idempotent of S, then $eS = \{a \in S \mid ea = a\}$ since $ex = a$ implies $ea = e^2x = ex = a$. Similarly, $Se = \{a \in S \mid ae = a\}$. In other words, eSe is the subsemigroup of S where e is a two-sided identity. Thus, it makes sense to talk about the group of units of eSe. Denote this group of units by H_e. H_e is the desired subgroup of S containing e. It is interesting to note that if $f \neq e$ are both idempotents, then H_e and H_f are disjoint. Otherwise, if $H_e \cap H_f \neq \emptyset$, then there exists an $a \in H_e \cap H_f$. Therefore, there exists an $x \in H_e$ so that $xa = ax = e$ and there exists $y \in H_f$ so that $ya = ay = f$. Thus,

$$e = xa = xaf = ef = eay = ay = f$$

which contradicts our assumption that $f \neq e$.

In group theory we often study subgroups generated by subsets of the group. When studying semigroups, it is often instructive to look at subsemigroups generated by subsets of a semigroup. The development of the following material closely follows that of Howie.[4]

If $\Phi \neq X \subseteq S$, for some semigroup S, then $< X >$, the subsemigroup of S generated by X, is the smallest subsemigroup containing X. That is to say, $< X >$

is the intersection of all subsemigroups that contain X. When $X = \{a\}$, $< X >=< a >= \{a, a^2, \ldots\}$ is known as a **monogenic** or **cyclic subgroup**. If $a, a^2, a^3, \ldots$ are distinct, i.e., if

$$a^m = a^n \Rightarrow m = n,$$

then $(< a >, \cdot) \cong (\mathbb{N}, +)$. In this case a is said to have infinite order in S. If $\mid a \mid < \infty$, then $\{a, a^2, \ldots\}$ will have repetitions; therefore

$$\{x \in \mathbb{N} : (\exists y \in \mathbb{N}) a^x = a^y, x \neq y\}$$

is a non-empty set, so it has a least element. Define that least element to be the **index** of the element a. Hence, the set

$$\{x \in \mathbb{N} : a^{m+x} = a^m\}$$

is also non-empty, and so it also has a least element r. Define r to be the period of a. Thus, in general,

$$a^m = a^{m+qr}, \forall q \in \mathbb{N}$$

and $\mid a \mid = m+r-1 =$ (index of a)+(period of a)-1. The set $K_a = \{a^r, a^{r+1}, \ldots, a^{m+r-1}\}$ is a cyclic subgroup of $< a >$ called the kernel of $< a >$. A semigroup is said to be **periodic** if all its elements are of finite order. A proof of the following may be found in [4].

THEOREM 1.1. *In a periodic semigroup every element has a power which is idempotent. Hence, in every periodic semigroup–in particular, in every finite semigroup–there is at least one idempotent.*

EXAMPLE 1.14. Let $X = \{1, 2, \ldots, 7\}$ and consider the element

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 & 5 \end{pmatrix}$$

of $\mathcal{T}_X$. The notation here means that $\alpha(1) = 2, \alpha(2) = 3, \ldots, \alpha(7) = 5$. By making a few calculations we see that

$$\alpha^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 & 7 & 5 & 6 \end{pmatrix} \quad , \quad \alpha^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 5 & 6 & 7 \end{pmatrix}$$
$$\alpha^4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 5 & 6 & 7 & 5 \end{pmatrix} \quad , \quad \alpha^5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 7 & 5 & 6 & 7 & 5 & 6 \end{pmatrix}$$
$$\alpha^6 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 5 & 6 & 7 & 5 & 6 & 7 \end{pmatrix} \quad , \quad \alpha^7 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 5 & 6 & 7 & 5 \end{pmatrix}.$$

Thus, α has index 4 and period 3. The kernel, K_α, is equal to $\{\alpha^4, \alpha^5, \alpha^6\}$, and has Cayley table

	α^4	α^5	α^6
α^4	α^5	α^6	α^4
α^5	α^6	α^4	α^5
α^6	α^4	α^5	α^6

and so α^6 is the identity of K_α. Notice that α^4 is a generator of K_α since $(\alpha^4)^2 = (\alpha^5)$ and $(\alpha^4)^3 = \alpha^6$. Since $\alpha^6 \cdot \alpha^6 = \alpha^6$, α^6 is an idempotent of $< \alpha >$, and since $(\alpha^2)^3 = \alpha^6$, α^2 has a power that is an idempotent. It is left to the reader to verify that the remaining elements of $< a >$ all have a power which is an idempotent.

A non-empty subset I of a semigroup S is called a **left ideal** if $SI \subseteq I$, a **right ideal** if $IS \subseteq I$, and an **ideal** if $SIS \subseteq I$. Every ideal is a subsemigroup, but each subsemigroup need not be an ideal.

EXAMPLE 1.15. If

$$S = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\},$$

then

$$I_1 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

is a right ideal,

$$I_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

is a left ideal,

$$I_3 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

is an ideal, and

$$T = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

is a subsemigroup that is not an ideal.

Let $K(S)$, known as the kernel of S, be the intersection of all ideals of S. Then $K(S)$ is itself an ideal. Notice that this definition of the kernel of a semigroup does not contradict the definition of the kernel of a periodic semigroup. They are the same when S is a periodic semigroup. If $0 \in S$, then $0 = K(S)$. A semigroup is said to be **simple** if it contains no proper ideals.

EXAMPLE 1.16. $(\mathbb{E}, +)$ is a simple semigroup.

Similarly, a semigroup is said to be left (right) simple if it contains no proper left (right) ideals.

The principal right (left) ideal generated by an element a in a semigroup is aS (Sa). The principal ideal generated by a is SaS. Note: If $1 \notin S$, then a may not be an element of the ideal generated by it.

The **direct product** of two semigroups, S and T, denoted $S \times T$, is a semigroup if we define multiplication as follows

$$(s, t)(s', t') = (ss', tt').$$

Associativity is easily seen.

A map $\varphi : S \to T$, where S and T are semigroups, is called a **morphism** if

$$\varphi(xy) = \varphi(x)\varphi(y) \qquad \forall x, y \in S.$$

If S and T are monoids with identity elements 1_S and 1_T respectively, then $\varphi : S \to T$ is a morphism if and only if $\varphi(1_S) = 1_T$ as well.

EXAMPLE 1.17. Let $S = M_2(K)$ and $T = GL_2(K) \cup \{0\}$. Define $\varphi : S \to T$ by $\varphi(A) = \det A \cdot A$. Then φ is a morphism.

A monomorphism is an injective morphism. An epimorphism is a surjective morphism. An isomorphism is a bijective morphism. If there exists an isomorphism $\varphi : S \to T$, then S and T are said to be **isomorphic**, written $S \cong T$. If a semigroup S is, for some $X = \{1, 2, \ldots, n\}$, a subsemigroup of $\mathcal{T}_X$, we say that S is a **transformation semigroup**.

THEOREM 1.2. *Every semigroup is isomorphic to a transformation semigroup.*

For a proof of the above theorem see [5].

EXAMPLE 1.18. Let $a \in S$ be an element of a semigroup S. Let $\rho_s : S \to S$ be given by $x \to xs\ \forall s \in S$. Then ρ_s is a homomorphism known as a right translation. Define $\lambda_s : S \to S$ by $x \to sx$. λ_s is also a homomorphism. It is called a left translation.

A semigroup S is called a **right group** if it is right simple; i.e., if it contains no proper right ideals and it is left cancelative. In other words, for every pair of elements a and b of S, there is a unique x so that $ax = b$.

A semigroup S is called a **left group** if it is left simple; i.e., if it contains no proper left ideals and it is right cancelative. In other words, for every pair of elements a and b of S, there is a unique y so that $ya = b$.

The following theorem determines the structure of all left groups. An analogous theorem exists for determining the structure of all right groups. For a proof, see [3].

THEOREM 1.3. *The following assertions concerning a semigroup S are equivalent.*

(i). S is a left group.

(ii). S is left simple and contains an idempotent.

(iii). S is the direct product $G \times E$, of a group G and a left-zero semigroup.

1.2 Ordered Sets and Lattices

Recall that a relation, $\leq$, on a set X is called a partial order when for every $a \in X$

(i). $a \leq a$ (Reflexive Property),

(ii). $a \leq b$ and $b \leq a$ implies $a = b$ (Anti-Symmetric Property),

(iii). $a \leq b$ and $b \leq c$ implies $a \leq c$ (Transitive Property).

Example 1.19. In a semigroup S let $E = \{e \in S \mid e^2 = e\}$. Define $\leq$ on E by $e \leq f$ when $ef = fe = e$. Then $\leq$ is a partial order on E.

An element b of a partially ordered set X is called an upper bound of a subset Y of X if $y \leq b\ \forall y \in Y$. The element b is said to be the least upper bound or **join** if $b \leq z$ for every upper bound z. If Y has a join, it is unique. An element b of a partially ordered set X is called a lower bound of a subset Y of X if $b \leq y$, $\forall y \in Y$. The element b is said to be the greatest lower bound or **meet** if $z \leq b$ for every lower bound z. If Y has a meet, it is unique. An **upper** [**lower**] **semilattice** is a partially ordered set X in which each two-element subset has a join [meet]. We will denote the join [meet] of $\{x, y\}$ as $x \vee y$ [$x \wedge y$].

Example 1.20. Let

$$X = \left\{ e_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Define a partial order on X by $e_i \leq e_j$ if $e_i e_j = e_j e_i = e_i$. For $Y = \{e_1, e_2\}$, e_0 and e_1 are lower bounds; therefore, $e_1 \wedge e_2 = e_1$. Similarly, e_2 and e_3 are upper bounds of Y; therefore, $e_1 \vee e_2 = e_2$. X is both an upper and lower semilattice.

The following proposition gives the relationship between a lower semilattice and a commutative semigroup made up entirely of idempotents also known as a commutative **band**. A proof may be found in [4].

THEOREM 1.4. *Let $(E, \leq)$ be a lower semilattice. Then $(E, \wedge)$ is a commutative semigroup consisting entirely of idempotents, and*

$$(\forall a, b \in E)\ a \leq b \iff a \wedge b = a.$$

Conversely, suppose that $(E, \cdot)$ is a commutative semigroup of idempotents. Then the relation, $\leq$, on E defined by

$$a \leq b \iff ab = a$$

is a partial order on E, with respect to which $(E, \leq)$ is a lower semilattice. In $(E, \leq)$, the meet of a and b is their product ab.

Let S be a semigroup and Ω a semilattice. If S_α, $\alpha \in \Omega$ is a subsemigroup and $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ for all $\alpha, \beta \in \Omega$, then

$$S = \bigsqcup_{\alpha \in \Omega} S_\alpha.$$

S is said to be a **semilattice union** of S_α, $\alpha \in \Omega$ [12].

Much of our later work will involve equivalence classes. It is at this point that we will introduce some basic definitions and results. Recall that a binary relation on a set X is a subset of $X \times X$. For a given relation, ρ, we often write $a\ \rho\ b$ instead of $(a, b) \in \rho \subseteq X \times X$. Denote by B_X the set of all binary relations on X.

If we define $\circ$ on B_X by the rule that for $\rho, \sigma \in B_X$

$$x(\rho \circ \sigma)y \iff \exists z \in X \text{ such that } x\rho z \text{ and } z\sigma y,$$

then it is easy to see that $\circ$ is a well-defined associative operation, thus $(B_X, \circ)$ is a semigroup. Notice that an equivalence relation is an idempotent element of B_X.

If ρ is any relation on a set X, then the family of equivalence relations containing ρ is non-empty since $X \times X$ itself contains ρ. Thus it makes sense to talk about the intersection of the collection of equivalence relations containing ρ. It is precisely this intersection that is said to be the equivalence relation generated by ρ.

By the join, $\rho \vee \sigma$, of two equivalence relations ρ and σ we mean the equivalence relation generated by $\rho \cup \sigma$. In general, determining $\rho \vee \sigma$ is very difficult, but when

ρ and σ commute our job becomes much more manageable. A proof of the following lemma may be found in [3].

LEMMA 1.1. *If ρ and σ are equivalence relations on a set X, and if $\rho \circ \sigma = \sigma \circ \rho$, then $\rho \circ \sigma$ is an equivalence relation on X, and is the join, $\rho \vee \sigma$, of ρ and σ.*

1.3 Congruences

A relation ρ on a semigroup S is said to be **right compatible** if $a\ \rho\ b$ implies $ac\ \rho\ bc$ for every $c \in S$. A relation ρ on a semigroup S is said to be **left compatible** if $a\ \rho\ b$ implies $ca\ \rho\ cb$ for all $c \in S$. An equivalence relation on S that is right [left] compatible will be called a **right [left] congruence**. An equivalence relation on S that is both right and left compatible will be called a **congruence**.

If ρ is a congruence on a semigroup S, then a binary operation on S/ρ may be defined in the following natural way,

$$(a\rho)(b\rho) = (ab)\rho.$$

Thus, S/ρ becomes a semigroup. Define $\rho^\natural : S \to S/\rho$ by $\rho^\natural(x) = x\rho$. We see that $\rho^\natural$ is a homomorphism since $x, y \in S$ implies $\rho^\natural(xy) = xy\rho = x\rho y\rho = \rho^\natural(x)\rho^\natural(y)$. We call it the **natural** or **canonical homomorphism**.

EXAMPLE 1.21. Let S be a semigroup and I an ideal of S. If we define $\equiv$ as

$$a \equiv b \qquad \text{if } a = b \text{ or } a, b \in I,$$

then $\equiv$ is a congruence. We see that

$$S/\equiv\ = S/I.$$

The quotient $S/\equiv$ is known as a **quotient semigroup** or a **Rees factor semigroup**. The relationship between a congruence and a homomorphism is seen in the following theorem. A proof may be found in [4].

THEOREM 1.5. *Let S be a semigroup, and let ρ be a congruence on S. Then S/ρ is a semigroup if multiplication is defined as*

$$(x\rho)(y\rho) = (xy)\rho, \qquad \forall x, y \in S.$$

The map $\rho^\natural : S \to S/\rho$ given by

$$\rho^\natural(x) = x\rho$$

is a homomorphism. Now let T be a semigroup and let $\Phi : S \to T$ be a morphism. Then the relation

$$\ker\Phi = \{(a,b) \in S \times S \mid \Phi(a) = \Phi(b)\}$$

is a congruence on S, and there is a monomorphism $\alpha : S/\ker\Phi \to T$ such that im $\alpha =$ im Φ and the diagram

$$\begin{tikzcd} S \arrow[r, "\Phi"] \arrow[d, "(\ker\Phi)^\natural"'] & T \\ S/\ker\Phi \arrow[ur, "\alpha"'] & \end{tikzcd}$$

commutes.

1.4 Green's Relations

The following equivalence relations, first introduced and studied by J.A. Green, have proven very useful in the study of the structure of semigroups. Two elements, a and b, of a semigroup S are said to be $\mathcal{R}$-related if they generate the same principal right ideal; that is,

$$a \,\mathcal{R}\, b \qquad \text{if and only if } aS^1 = bS^1.$$

The element a is said to be $\mathcal{L}$-related to the element b if they generate the same principal left ideal; that is,

$$a \,\mathcal{L}\, b \qquad \text{if and only if } S^1a = S^1b.$$

The element a is said to be $\mathcal{J}$-related to the element b if they generate the same principal ideal; that is,

$$a \mathcal{J} b \qquad \text{if and only if } S^1aS^1 = S^1bS^1.$$

EXAMPLE 1.22. In the semigroup $M_3(K)$, two elements, x and y, are $\mathcal{J}$-related when the rank of x equals the rank of y. The elements x and y are $\mathcal{R}$-related when x and y are column equivalent. The elements x and y are $\mathcal{L}$-related when x and y are row equivalent.

Denote by J_a, R_a, and L_a the $\mathcal{J}$-class, $\mathcal{R}$-class, and $\mathcal{L}$-class respectively of $a \in S$. Since $\mathcal{L}, \mathcal{R}$, and $\mathcal{J}$ are defined in terms of ideals, set inclusion defines a particular order on the equivalence classes.

$$\begin{aligned} L_a \leq L_b &\iff S^1a \subseteq S^1b \\ R_a \leq R_b &\iff aS^1 \subseteq bS^1 \\ J_a \leq J_b &\iff S^1aS^1 \subseteq S^1bS^1 \end{aligned}$$

Notice also that for all $a \in S$ and $x, y \in S^1$

$$L_{ax} \leq L_a, \quad R_{ax} \leq R_a, \quad J_{xay} \leq J_a$$

and

$$L_a \leq L_b \text{ implies } J_a \leq J_b$$

and

$$R_a \leq R_b \text{ implies } J_a \leq J_b$$

An element a is said to **divide** b, denoted $\mathbf{a} \mid \mathbf{b}$, if $xay = b$ for some $x, y \in S^1$. If $a \mid b$, then $J_a \geq J_b$. If $a \mid b \mid a$, then $a\mathcal{J}b$.

Previously we saw that the intersection and the join of two equivalence relations is again an equivalence relation. Let $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$ and $\mathcal{D} = \mathcal{R} \vee \mathcal{L}$. As we noted earlier, determining the join of two equivalence classes is a bit tricky, but we are saved by the following theorem [4].

THEOREM 1.6. *The relations $\mathcal{L}$ and $\mathcal{R}$ commute.*

Thus, $\mathcal{D} = \mathcal{R} \vee \mathcal{L} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$. Notice that $\mathcal{R} \subseteq \mathcal{J}$ and $\mathcal{L} \subseteq \mathcal{J}$. Therefore, $\mathcal{J}$ is an equivalence relation that contains both $\mathcal{R}$ and $\mathcal{L}$. $\mathcal{D}$ is the smallest equivalence relation containing $\mathcal{R}$ and $\mathcal{L}$, so, in general, $\mathcal{D} \subseteq \mathcal{J}$.

A useful technique when studying a $\mathcal{D}$-class is to imagine the elements of a $\mathcal{D}$-class arranged in a rectangular pattern with the rows corresponding to $\mathcal{R}$-classes and the columns corresponding to $\mathcal{L}$-classes. The individual cells of this "egg-box" correspond to the $\mathcal{H}$-classes of $\mathcal{D}$. Notice that no cell is empty since, by definition of $\mathcal{D}$,

$$a \,\mathcal{D}\, b \iff R_a \cap L_b \neq \emptyset \iff R_b \cap L_a \neq \emptyset.$$

Also notice that the egg-box may be square or rectangular. In fact, it may contain only one column or row of cells. It is also possible for the egg-box to have an infinite number of rows or columns.

An interesting fact about these egg-boxes is that all $\mathcal{R}$-classes have the same-number of elements. Similarly, all $\mathcal{L}$-classes have the same number of elements. These statements are a direct result of Green's Lemma. A proof of the following lemma may be found in [4].

Lemma 1.2 (Green's Lemma). *Let a, b be $\mathcal{R}$-equivalent elements in a semigroup S, and let s, s' in S be such that*

$$as = b, bs' = a.$$

Then the right translations ${}^{\rho_s}|_{L_a}$ and ${}^{\rho_s}|_{L_b}$ are mutually inverse $\mathcal{R}$-class preserving bijections from L_a onto L_b and L_b onto L_a, respectively.

If the right translations ρ_s and $\rho_{s'}$ are replaced by left translations λ_s and $\lambda_{s'}$, an analogous result is obtained for R_a and R_b. By applying both of these results one can show that $\mid H_a \mid = \mid H_b \mid$ whenever $a \,\mathcal{D}\, b$. Thus, the cells in the egg-box have the same number of elements.

Example 1.23. Let $S = \mathcal{I}_3$. Let $D_2 = \{x \in S \mid \text{rank of } x = 2\}$. Then D_2 is a $\mathcal{D}$-class of S. D_2 has 3 $\mathcal{R}$-classes and 3 $\mathcal{L}$-classes. The egg-box for D_2 is as follows:

	L_1	L_2	L_3
R_1	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
R_2	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
R_3	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

1.5 Regular Semigroups

An element a of a semigroup S is called **regular** if $a = axa$ for some x in S. A semigroup S is called **regular** if every element of S is regular. In the previous section we saw that semigroups are often studied by looking at their $\mathcal{D}$-classes.

THEOREM 1.7. *If a is a regular element of a semigroup S, then every element of D_a is regular.*

Proof. If a is a regular element of a semigroup S, then $axa = a$ for some $x \in S$. If $b \in L_a$, then there exists $u, v \in S$ such that $ua = b$ and $vb = a$, so

$$b = ua = uaxa = bxa = b(xv)b$$

implies b is regular. A similar argument shows the result for any element in R_a. ■

Since idempotents are regular, it follows that every $\mathcal{D}$-class which contains an idempotent is regular. In fact, we can show the following.

Theorem 1.8. *In a regular $\mathcal{D}$-class, each $\mathcal{L}$-class and each $\mathcal{R}$-class contain an idempotent.*

Proof. If $a = axa$ is a regular element of D_a, then xa is an idempotent since $xaxa = xa$. Thus, $xa \mathcal{L} a$. Similarly, ax is an idempotent and $a \mathcal{R} ax$. ■

Example 1.24. Let $S = \mathcal{I}_3$ and $D_2 = \{x \in \mathcal{I}_3 \mid \text{ rank of } x = 2\}$. D_2 contains the following idempotents:

$$E(D_2) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

Each $\mathcal{R}$-class and $\mathcal{L}$-class of $\mathcal{I}_3$ contains one of the above.

If S is a regular semigroup and ρ is a congruence on S, then S/ρ is regular. This is a direct result of the following lemma. A proof may be found in [4].

Lemma 1.3. *Let $\Phi : S \longrightarrow T$ be a morphism from a regular semigroup S into a semigroup T. Then im Φ is regular. If f is an idempotent in im Φ, then there exists an idempotent e in S such that $\Phi(e) = f$.*

Example 1.25. Define $\Phi : S_3 \longrightarrow M_3(K)$ by $\Phi(A) = \det(A) \cdot A$. Then

$$\Phi \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in E(M_3(K)).$$

An element a in a semigroup S is said to be **completely regular** if $a^2x = a$ and $ya^2 = a$ for some x, y in S. In other words, $a\mathcal{H}a^2$. If an element a in S is completely regular, then it is regular.

EXAMPLE 1.26. Let $S = M_n(K)$. $A \in M_n(K)$ is completely regular if and only if the rank of $A =$ rank of A^2. Recall, $A\mathcal{H}A^2$ implies that A and A^2 are in the same $\mathcal{D}$-class. Therefore, they have the same rank. Thus,

$$A \sim \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$$

for some invertible matrix B. Hence, A is regular since

$$A = Q \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} R$$

for some invertible matrices Q and R. $M_n(K)$ is not completely regular, since each element of $M_n(K)$ is not completely regular. When $n = 3$, the element

$$x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

is not completely regular since

$$x^2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and the rank of $x = 2 \neq 1 =$ rank of x^2. Each element of $M_n(K)$ is regular since each $\mathcal{D}$-class of $M_n(K)$ contains an idempotent.

In the above example we saw that $M_n(K)$ is not completely regular. However, a power of each element is completely regular. In order to see this, recall that each matrix $A \in M_n(K)$ may be reduced to a matrix of the form

$$\begin{pmatrix} B & 0 \\ 0 & N \end{pmatrix},$$

where B is invertible and N is nilpotent; i.e., $N^t = 0$ for some $t \in \mathbb{Z}^+$. Thus,

$$\begin{pmatrix} B & 0 \\ 0 & N \end{pmatrix}^t = \begin{pmatrix} B^t & 0 \\ 0 & N^t \end{pmatrix} = \begin{pmatrix} B^t & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence,

$$A^t \sim \begin{pmatrix} B^t & 0 \\ 0 & 0 \end{pmatrix},$$

which implies that A^t is completely regular. The property that each element a in a semigroup S has a power that is completely regular is called **strongly π-regular**, and is denoted sπr.

THEOREM 1.9. *Let S is a $s\pi r$-semigroup and $a, b, c \in S$.*

(i). $a \mathcal{J} ab$ implies $a \mathcal{R} ab$; $a \mathcal{J} ba$ implies $a \mathcal{L} ba$.

(ii). $ab \mathcal{J} b \mathcal{J} bc$ implies $b \mathcal{J} abc$.

(iii). If $e \in E(S)$, then $J_e \cap eSe = H_e$.

(iv). $\mathcal{J} = \mathcal{D}$ on S.

(v). $a \mathcal{J} a^2$ implies that the $\mathcal{H}$-class of a is a group.

For a proof see [12].

LEMMA 1.4. *Let S be a semigroup and $e \in S$ be an idempotent with $(J_e)^2 \subseteq J_e$. Let $a, b \in S$. If $a \mid e$ and $b \mid e$, then $ab \mid e$.*

Proof. Let a, b, e be as above. Since $a \mid e$, there exist elements x and y in S so that $e = xay$. Similarly, since $b \mid e$, there exist elements s and t in S so that $e = sbt$.

$$e \mid ayexa \mid xayexay = e$$

implies

$$ayexa \in J_e$$

Similarly,

$$e \mid btesb \mid sbtesbt = e$$

implies

$$btesb \in J_e$$

Thus, $(ayexa)(btesb) \in J_e$ since $(J_e)^2 \subseteq J_e$. Therefore there exist elements z and w in S so that

$$e = z(ayexa)(btesb)w.$$

Hence,

$$ab \mid ayexabtesb \mid e$$

■

THEOREM 1.10. *Let S be a sπr semigroup such that $J^2 \subseteq J$ for all regular $\mathcal{J}$-classes, J. Let $\Omega = \{J \subseteq S \mid J$ is a regular $\mathcal{J}$-class$\}$ and $S_J = \{a \in S \mid a^k \in J$ for some $K \in \mathbb{Z}^+\}$. Then*

(i). *Ω is a semilattice.*

(ii). $S = \bigsqcup_{J \in \Omega} S_J$.

(iii). $S_{J_1}, S_{J_2} \subseteq S_{J_1 \wedge J_2}$.

(iv). *S_J is a subsemigroup.*

(v). *J is an ideal of S_J.*

(vi). *S_J/J is a nil semigroup.*

Proof.

(i). Let $J_1, J_2 \in \Omega$ be arbitrary and let $e_i \in E(J_i), \forall i$. Since S is sπr, $(e_1e_2)^k \mathcal{H}_e$ for some $k \in \mathbb{Z}$, $e \in E(S)$. Therefore, $e_1 \mid e$ and $e_2 \mid e$. If we let $J = J_e$, $J_1 \geq J$ and $J_2 \geq J$ then J is a lower bound for $\{J_1, J_2\}$. Is it the greatest lower bound? Suppose $J_0 \in \Omega$ is such that $J_1 \geq J_0$ and $J_2 \geq J_0$. Let $e_0 \in E(J_0)$. Then $e_1 \mid e_0$ and $e_2 \mid e_0$, so $(e_1e_2)^k \mid e_0$ for all $k \in \mathbb{Z}^+$. But $e \mid (e_1e_2)^k \mid e_0$ implies $e \mid e_0$, so $J \geq J_0$. Hence, $J = J_1 \wedge J_2$.

(ii). By definition of S_J, $\bigcup_{J \in \Omega} S_J \subseteq S$. Let $a \in S$, then as S is sπr, $a^k \mathcal{H} e$ for some $e \in E(S)$. Thus, $a^k \in J_e \in \Omega$; i.e., $a \in S_{J_e}$. Therefore, $S \subseteq \bigcup_{J \in \Omega} S_J$. To show disjointness, let $a \in S_{J_1} \cap S_{J_2}$ then for some $m, n \in \mathbb{Z}^+, a^m \in J_1$ and $a^n \in J_2$ which implies $a^{m+n} \in J_1 \cap J_2$. A contradiction, since $J_1 \cap J_2 = \emptyset$.

(iii). Let $a_1 \in S_{J_1}, a_2 \in S_{J_2}, e_1 \in E(J_1)$, and $e_2 \in E(J_2)$. Suppose $J = J_1 \wedge J_2$. Let $e \in E(J)$. Since $a_1 \in S_{J_1}$, there exists an $n \in \mathbb{Z}$ so that $a_1^n \in J_1$. Similarly, there exists an $m \in \mathbb{Z}$ so that $a_2^m \in J_2$. So $a_1 \mid e_1 \mid e$ and $a_2 \mid e_2 \mid e$. J is sπr. So, there exists $k \geq \max\{m, n\}$ so that $(a_1a_2)^k \mathcal{H} f$ for some $f \in E(S)$. Since $a_1 \mid e$ and $a_2 \mid e$, $(a_1a_2)^k \mid e$. Therefore, $f \mid e$, which implies $J_f \geq J$. Since $a_1 \mid f$, $(a_1)^n \mid f$. Therefore, $e_1 \mid f$. Similarly $a_2 \mid f$ implies $(a_2)^m \mid f$, which implies $e_2 \mid f$. Therefore, $J_1 \geq J_f$ and $J_2 \geq J_f$, but $J = J_1 \wedge J_2$, so $J \geq J_f$. Therefore, $J = J_f$ and $(a_1a_2)^k \in J$. Hence, $a_1a_2 \in S_J$. In particular,

$$S_{J_1}S_{J_2} \subseteq S_{J_1 \wedge J_2}.$$

(iv). From (iii) $S_JS_J \subseteq S_{J\wedge J}$ but $J \wedge J = J$ so $S_JS_J \subseteq S_J$.

(v). Let $a \in S_J, b \in J$. Then $(ab)^n \in J$ for some n. Thus, $b \mid ab \mid (ab)^n \mid b$. Therefore, $ab \, J \, b$. So $ab \in J$. Similarly $ba \in J$. Hence, J is an ideal of S_J.

(vi). Let $a \in S_J$. Then $a^n \in J$ for some $n \in \mathbb{Z}$ so $a^n = 0 \in S_J/J$; i.e. S_J/J is nil.

1.6 0-Simple Semigroups and Rees' Theorem

Recall that a simple semigroup is one which contains no proper two-sided ideals. Thus, any semigroup that has a zero cannot be simple since $K(S) = \{0\}$ is a proper ideal of S. Define a **0-simple semigroup** to be a semigroup S with zero element, 0, where 0 and S are the only ideals and $S^2 \neq 0$.

Example 1.27. Let $J_2 = \{x \in M_3(K) \mid \text{rank of } x = 2\}$. J_2^0 is a 0-simple semigroup.

A semigroup that is sπr and 0-simple is called **completely 0-simple**.

Example 1.28. Let $I_2 = \{x \in \mathcal{I}_3 \mid \text{rank of } x = 2\}$, then I_2^0 is a completely 0-simple semigroup.

An important factorization of general semigroups given by Rees, decomposes a semigroup into 0-simple factors. A principal ideal series of a semigroup S is a chain of ideals

$$S = I_n \supset I_{n-1} \supset \ldots \supset I_0 = K(S)$$

such that, for $j = 1, 2, \ldots, n$ each I_j is maximal in I_{j-1}. Then we can see that S is the disjoint union of $(I_n \setminus I_{n-1}), (I_{n-1} \setminus I_{n-2}), \ldots, (I_1 \setminus I_0), (I_0 \setminus \emptyset)$ and

$$I_{j-1}/I_j = \{I_j \setminus I_{j-1}\} \cup \{0\}$$

is known as a **Rees factor semigroup**. Each Rees factor semigroup is 0-simple.

EXAMPLE 1.29. If $S = M_3(K)$, then

$$M_3(K) = I_3 \supset I_2 \supset I_1 \supset I_0 = 0$$

where $I_0 = \{0\}, I_1 = \{x \in S \mid \text{rank of } x \leq 1\}, I_2 = \{x \in S \mid \text{rank of } x \leq 2\}$ is a principal ideal series with Rees factors,

$$\begin{aligned} J_3^0 &= I_3/I_2 = \{x \in S \mid \text{rank of } x = 3\} \cup \{0\}, \\ J_2^0 &= I_2/I_1 = \{x \in S \mid \text{rank of } x = 2\} \cup \{0\}, \\ J_1^0 &= I_1/I_0 = \{x \in S \mid \text{rank of } x = 1\} \cup \{0\}, \\ J_0^0 &= I_0/\emptyset = \{x \in S \mid \text{rank of } x = 0\}. \end{aligned}$$

Rees gave the following method for constructing a completely 0-simple semigroup. Let G be a group with identity element e and let I and Λ be non-empty sets. Let $P = (p_{\lambda j})$ be a $\Lambda \times I$ matrix with entries in G^0 and suppose that P is regular; i.e., no row or column of P consists entirely of zeroes. Then $S = (I \times G \times \Lambda)$ with multiplication defined as

$$(i, a, \lambda)(j, b, \mu) = \begin{cases} (i, a p_{\lambda j} b, \mu) & , \text{if } p_{\lambda j} \neq 0 \\ 0 & , \text{if } p_{\lambda j} = 0 \end{cases}$$

and

$$(i, a, \lambda)0 = 0(i, a, \lambda) = 0 \cdot 0 = 0$$

is a completely 0-simple semigroup. Notice that $S \setminus \{0\}$ is in $1-1$ correspondence with the set of $I \times \Lambda$ matrices $(a)_{i\lambda}$ where $a \in G$ and $(a)_{i\lambda}$ denotes the matrix with entry a in the (i, λ) position and zeros elsewhere.

$T = \{(a)_{i\lambda} \mid a \in G^0, i \in I, \lambda \in \Lambda\}$ is denoted by $M^0[G; I, \Lambda, P]$ and is called the $I \times \Lambda$ Rees matrix semigroup over the 0-group G^0 with regular sandwich matrix P. A proof of the following may be found in [4].

Theorem 1.11 (Rees Theorem). *All completely 0-simple semigroups are isomorphic to some Rees matrix semigroup, $M^0(G, I, \Lambda, P)$ with regular matrix P.*

Example 1.30. Let $S = \mathcal{I}_3$. The Rees factors are $J_3^0 = \{x \in S \mid \text{rank of } x = 3\} \cup \{0\}$, $J_2^0 = \{x \in S \mid \text{rank of } x = 2\} \cup \{0\}$, $J_1^0 = \{x \in S \mid \text{rank of } x = 1\} \cup \{0\}$and $J_0^0 = \{0\}$. So each of these Rees factors is completely 0-simple. Hence, for J_2^0 we can find a Rees Matrix Semigroup that is isomorphic to it. Let $G = H_{11}$ from Example 1.23. Then

$$H_{11} = \left\{ e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\},$$

$I = \{1, 2, 3\}, \Lambda = \{1, 2, 3\}$, and

$$P = \begin{pmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e \end{pmatrix} \cong \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

so $M^0(H_{11}, I, \Lambda, P) \cong J_2^0$. For instruction on computing P, see [4].

1.7 Inverse Semigroups

Two elements a and b of a semigroup S are said to be inverses of each other if

$$aba = a \qquad \text{and} \qquad bab = b$$

Clearly, a and b are both regular elements of S. Hence, in a regular semigroup each element has at least one inverse. Inverses are not necessarily unique. A semigroup where each element has a unique inverse is known as an **inverse semigroup**.

EXAMPLE 1.31. $\mathcal{I}_3$ is an inverse semigroup.

For a proof of the following, see [3].

THEOREM 1.12. *The following three conditions on a semigroup S are equivalent:*

(i). S is regular, and any two idempotent elements of S commute with each other.

(ii). every principal right ideal and every principal left ideal has a unique idempotent generator.

(iii). S is an inverse semigroup.

Recall that if a has an inverse, then a is regular, which implies D_a will be a regular $\mathcal{D}$-class. We can now use the egg-box picture of D_a to find all the inverses of an element a. A proof of the following may be found in [3].

THEOREM 1.13. *Let a be a regular element of a semigroup S. Then*

(i). Every inverse of a lies in D_a.

(ii). An $\mathcal{H}$-class H_b contains an inverse of a if an only if both of the $\mathcal{H}$-classes $R_a \cap L_b$ and $R_b \cap L_a$ contain idempotents.

(iii). No $\mathcal{H}$-class contains more than one inverse of a.

EXAMPLE 1.32. Recall the egg-box for $J_2 \in \mathcal{I}_3$ from Example 1.23.

$$e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in R_a \cap L_b \qquad a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in R_a \cap L_a$$

$$b = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in R_b \cap L_b \qquad f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in R_b \cap L_a$$

and a simple calculation shows that $aba = a$ and $bab = b$, thus a and b are inverses of each other. For proofs of the following two theorems, see [4].

THEOREM 1.14. *Let S be an inverse semigroup. Let T be a semigroup. Let Φ be a semigroup morphism from S onto T. Then T is an inverse semigroup.*

THEOREM 1.15. *Every inverse semigroup can be embedded in a symmetric inverse semigroup.*

Chapter 2

Staircase Decomposition

The purpose of this chapter is to introduce a decomposition for the J-classes of $M_n(K)$. Recall that the J-classes of $M_n(K)$ are of the form

$$J_r = \{x \in M_n(K) \mid \text{ rank of } x = r\}$$

and may be considered semigroups by adjoining zero; $J_r^0 = J_r \cup \{0\}$, and defining multiplication as follows:

$$x \cdot y = \begin{cases} xy & \text{if} \quad xy \in J_r \\ 0 & \text{otherwise} \end{cases} .$$

The decomposition discussed in this chapter will utilize only elements from each J-class.

Recall that invertible matrices whose leading principal minors are non-zero can be decomposed into the product of a lower triangular and an upper triangular matrix. Bruhat's decomposition [6] for $GL_n(K)$ extends this idea in the following way. Let B denote the subgroup of all upper triangular matrices, B^- denote the subgroup of all lower triangular matrices and P denote the subgroup of permutation matrices. Bruhat's decomposition tells us that

$$GL_n(K) = \bigsqcup_{\sigma \in P} B\sigma B \ .$$

By letting $w_0 = \begin{pmatrix} 0 & \cdots & 1 \\ 1 & \ddots & 0 \end{pmatrix}$, we may rewrite the decomposition to achieve an alternate form:

$$\begin{aligned} GL_n(K) &= w_0 GL_n(K) \\ &= w_0 \bigsqcup_{\sigma \in P} B\sigma B \\ &= \bigsqcup_{\sigma \in P} w_0 B\sigma B \\ &= \bigsqcup_{\sigma \in P} w_0 B w_0 w_0 \sigma B \\ &= \bigsqcup_{\sigma \in P} B^- \sigma B. \end{aligned}$$

Thus, the invertible matrices with LU decompositions are precisely those matrices with Bruhat decompositions where σ = identity.

Renner [13] extended Bruhat's work to give a decomposition for all of $M_n(K)$. If we let B and B^- be as above and let R denote the subsemigroup of all partial permutation matrices, then

$$M_n(K) = \bigsqcup_{\rho \in R} B\rho B \ .$$

Using an argument similar to the one above, it is once again possible to replace one or both B's with B^- to achieve the alternate form:

$$M_n(K) = \bigsqcup_{\rho \in R} B^- \rho B.$$

If we let $\hat{R}$ denote all the $n \times n$ partial permutation matrices of rank r then

$$J_r = \bigsqcup_{\rho \in \hat{R}} B^- \rho B$$

We now have a decomposition for each $\mathcal{J}$ -class of $M_n(K)$, but we have gone outside of J_r to obtain it. We would like to have a decomposition that lies entirely in J_r. In order to achieve this goal, we will need to replace B and B^- with subsemigroups in J_r.

Define a **lower staircase matrix**, denoted $\mathcal{A}_r^-$, to be an $n \times r$ matrix such that $r \le n$ and if $a_{s_1 1}, a_{s_2 2}, \cdots a_{s_r r}$ are the first non-zero entries in each column, then $s_1 < s_2 < \cdots < s_m$.

EXAMPLE 2.1. The following matrices are lower staircase:

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 4 & 0 \\ 1 & 0 & 8 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 4 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}, \quad \begin{pmatrix} 3 & 0 & 0 \\ 2 & 4 & 0 \\ 1 & 9 & 8 \\ 2 & 0 & 6 \end{pmatrix}.$$

Counterexample: The following matrices are not lower staircase:

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \\ 6 & 9 & 8 \end{pmatrix}, \quad \begin{pmatrix} 0 & 3 & 0 \\ 4 & 0 & 0 \\ 2 & 3 & 4 \end{pmatrix}.$$

Notice that lower staircase matrices are almost in column-echelon form. And lower staircase matrices of the same rank are row equivalent.

Define an **upper staircase matrix**, denoted $\mathcal{A}_r$, to be an $r \times n$ matrix such that $r \le n$ and if $a_{1s_1}, a_{2s_2}, \cdots a_{rs_r}$ are the first non-zero entries in each row, then $s_1 < s_2 < \cdots s_r$.

EXAMPLE 2.2. The following are upper staircase matrices:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 5 \end{pmatrix}, \quad \begin{pmatrix} 2 & 3 & 4 & 5 \\ 0 & 6 & 0 & 2 \\ 0 & 0 & 0 & 8 \end{pmatrix}.$$

Counterexample: The following are not upper staircase matrices:

$$\begin{pmatrix} 0 & 0 & 3 & 4 \\ 0 & 5 & 6 & 8 \\ 0 & 0 & 2 & 9 \end{pmatrix}, \quad \begin{pmatrix} 3 & 4 & 0 & 8 \\ 0 & 5 & 6 & 9 \\ 0 & 4 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 0 & 2 & 8 \\ 0 & 2 & 0 & 0 \end{pmatrix}.$$

Notice that upper staircase matrices are almost in row-echelon form. Upper staircase matrices of the same rank are column-equivalent.

At this point it is necessary to point out that these lower and upper staircase matrices are not necessarily square. Each of the lower staircase matrices may be made square by adding additional zero columns to the right of the matrix. Similarly, each upper staircase matrix may be made square by adding additional zero rows to the bottom of the matrix.

Example 2.3.

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 4 \end{pmatrix} \quad \text{becomes} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 4 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 & 4 \\ 0 & 0 & 8 \end{pmatrix} \quad \text{becomes} \quad \begin{pmatrix} 2 & 3 & 4 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{pmatrix}$$

Denote by $\boldsymbol{\mathcal{A}}$ the set of $n \times n$ augmented upper staircase matrices and by $\boldsymbol{\mathcal{A}}^-$ the set of $n \times n$ augmented lower staircase matrices.

Theorem 2.1. *$\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{A}}^-$ are subsemigroups of $M_n(K)$.*

Proof. We will prove this in the general case in Chapter 6. ■

In order to simplify calculations, we will use the notation of the lower and upper staircase matrices instead of their augmented counterparts for the remainder of this chapter, keeping in mind that we can always recover a square matrix from a staircase matrix by adding zero rows or columns.

Lemma 2.1. *Let B and B^- denote the upper and lower triangular matrices respectively. Let $\mathcal{A}_r^-$ and $\mathcal{A}_r$ be as defined above. Then*

(i) $\mathcal{A}_r^- B^- \subseteq \mathcal{A}_r^-$.

(ii) $B^- \mathcal{A}_r^- \subseteq \mathcal{A}_r^-$.

(iii) $\mathcal{A}_r B \subseteq \mathcal{A}_r$.

(iv) $B \mathcal{A}_r \subseteq \mathcal{A}_r$.

Proof. (i) Let $x \in \mathcal{A}_r^-$ and $y \in B^-$. Let $x_{i_1 1}, \ldots x_{i_r r}$ denote the first non-zero entries in each column of x. Then by the definition of $\mathcal{A}_r^-$, $i_1 < i_2, \ldots < i_r$. Let $z = xy$. Then for $k = 1, 2, \ldots, r$,

$$z_{i_k k} = \sum_{j=1}^{n} z_{i_k j} y_{jk} \neq 0$$

since $z_{i_k k} \neq 0$ and $y_{kk} \neq 0$. If $m \leq i_k$ for some $k = 1, 2, \ldots, r$, then

$$z_{mk} = \sum_{j=1}^{n} x_{mj} y_{jk} = 0$$

since $x_{mj} = 0$ for $j = 1, 2, \ldots, n$. Therefore, $z_{i_1 1}, \ldots, z_{i_r r}$ are the first non-zero entries in each column of z. Hence, $z \in \mathcal{A}_r^-$. ∎

THEOREM 2.2. *Let $\mathcal{A}_r^-$ and $\mathcal{A}_r$ denote the lower and upper staircase matrices of rank r, $I_r = \{x \in M_n(K) \mid$ rank of $x \leq r\}$ and R denote the $r \times r$ partial permutation matrices. Then*

$$I_r = \bigcup_{\rho \in R} \mathcal{A}_r^- \rho \mathcal{A}_r \ .$$

Proof. If $x \in \bigcup_{\rho \in R} \mathcal{A}_r^- \rho \mathcal{A}_r$, then $x = w\rho z$ for some $w \in \mathcal{A}_r^-, \rho \in R$, and $z \in \mathcal{A}_r$.

The rank of w = rank of $z = r$ and the rank of $\rho = t < r$. Hence, the rank of $x = \min\{r, t\} \leq r$. Therefore, $x \in I_r$.

To show inclusion in the opposite direction, assume $x \in I_r$. Then either the rank of $x = r$ or the rank of $x = t < r$.

Case 1: Suppose the rank of $x = r$. Then there exists invertible matrices Q_1 and

Q_2 so that

$$\begin{aligned} x &= Q_1 \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} Q_2 \\ &= Q_1 \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} Q_2 \\ &= \begin{pmatrix} C & 0 \end{pmatrix} \begin{pmatrix} D \\ 0 \end{pmatrix} \\ &= CD \end{aligned}$$

where C is $n \times r$ of rank r and D is $r \times n$ of rank r. C may be reduced to lower staircase form so, there exists $T_1 \in GL_r(K)$ and $Y \in \mathcal{A}_r^-$ such that $C = YT_1$. Similarly, D may be reduced to upper staircase form so, there exists $T_2 \in GL_r(K)$ and $Z \in \mathcal{A}_r$ such that $D = T_2 Z$. Hence,

$$\begin{aligned} x &= YT_1T_2Z \\ &= YT_3Z \end{aligned}$$

where $T_3 \in GL_r(K)$. Therefore, from Bruhat's decomposition we know that $T_3 = L\sigma U$ for some $L \in B^-, \sigma \in P = \{x \in M_r(K) \mid x \text{ is a permutation matrix}\}$ and $u \in B$. So

$$x = YL\sigma UZ,$$

and by Lemma 2.1

$$x = Y^{'}\sigma Z^{'}$$

for some $Y^{'} \in \mathcal{A}_r^-, \sigma \in P \subset R, Z^{'} \in \mathcal{A}_r$. Hence,

$$x \in \bigcup_{\rho \in R} \mathcal{A}_r^- \rho \mathcal{A}_r$$

Case 2: If the rank of $x = t < r$, then there exists invertible matrices Q_1 and Q_2

such that

$$\begin{aligned} x &= Q_1 \begin{pmatrix} I_t & 0 \\ 0 & 0 \end{pmatrix} Q_2 \\ &= Q_1 \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I_t & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} Q_2 \\ &= \begin{pmatrix} C & 0 \end{pmatrix} \begin{pmatrix} I_t & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} D & 0 \end{pmatrix} \\ &= C \begin{pmatrix} I_t & 0 \\ 0 & 0 \end{pmatrix} D \end{aligned}$$

where C is $n \times r$ of rank r, D is $r \times n$ of rank r and $\begin{pmatrix} I_t & 0 \\ 0 & 0 \end{pmatrix}$ is $r \times r$ of rank t. The matrix C may be reduced to lower staircase form, so there exists $T_1 \in GL_r(K)$ and $Y \in \mathcal{A}_r^-$ such that $C = YT_1$. Similarly, there exists $T_2 \in GL_r(K)$ and $Z \in \mathcal{A}_r$ such that $D = T_2 Z$. Thus,

$$\begin{aligned} x &= YT_1 \begin{pmatrix} I_t & 0 \\ 0 & 0 \end{pmatrix} T_2 Z \\ &= YT^{'} Z \end{aligned}$$

where $T^{'} \in M_r(K)$. From Renner's decomposition, we know that $T^{'} = L\rho U$ for some $L \in B^-, \rho \in R, U \in B$. Hence,

$$x = YL\rho UZ.$$

From Lemma 2.1,

$$x = Y^{'} \rho Z^{'}$$

where $Y^{'} \in \mathcal{A}_r^-, \rho \in R$, and $Z^{'} \in \mathcal{A}_r$.

Therefore,

$$x \in \bigcup_{\rho \in R} \mathcal{A}_r^- \rho \mathcal{A}_r$$

■

Uniqueness fails in general, but if we replace R with its unit group P, the permutation matrices, we see that

$$J_r = \bigcup_{\sigma \in P} \mathcal{A}_r^- \sigma \mathcal{A}_r \tag{2.1}$$

Is the choice of σ unique? As it turns out, the answer is yes. Before we are able to prove uniqueness, we will need the following lemmas.

Lemma 2.2. *Let $X, Y \in \mathcal{A}_r^-$, $W, Z \in \mathcal{A}_r$, and $\sigma, \gamma \in P$. If $X\sigma W = Y\gamma Z$, then X is column equivalent to Y and W is row equivalent to Z.*

Proof. Since the rank of X = rank of $X\sigma W$, $(X \mid 0)$ and $X\sigma W$ are column equivalent. Therefore, there exists $Q_1 \in GL_n(K)$ so that $(X \mid 0)Q_1 = X\sigma W$. Similarly, since the rank of Y = the rank of $Y\gamma Z$, there exists $Q_2 \in GL_n(K)$ so that $(Y \mid 0)Q_2 = Y\gamma Z$. Hence, $(X \mid 0)Q_1 = (Y \mid 0)Q_2$ implies $(X \mid 0) = (Y \mid 0)Q_2Q_1^{-1}$ implies $(X \mid 0)$ is column equivalent to $(Y \mid 0)$. Hence, X is column equivalent to Y.

The rank of W = rank of $X\sigma W$ implies that $\left(\frac{W}{0}\right)$ and $X\sigma W$ are row equivalent; therefore, there exists $Q_1 \in GL_n(K)$ so that $Q_1\left(\frac{W}{0}\right) = X\sigma W$. The rank of Z equals rank of $Y\gamma Z$ implies that $\left(\frac{Z}{0}\right)$ and $Y\gamma Z$ are row equivalent; therefore, there exists $Q_2 \in GL_n(K) sothat$ $Q_2\left(\frac{Z}{0}\right) = Y\gamma Z$. Hence, $Q_1\left(\frac{W}{0}\right) = Q_2\left(\frac{Z}{0}\right)$ implies $\left(\frac{W}{0}\right) = Q_1^{-1}Q_2\left(\frac{Z}{0}\right)$, which implies $\left(\frac{W}{0}\right)$ is row equivalent to $\left(\frac{Z}{0}\right)$. Hence, W is row equivalent to Z. ∎

Lemma 2.3. *Let $X, Y \in \mathcal{A}_r^-$, $W, Z \in \mathcal{A}_r$ and σ, $\gamma \in \mathcal{R}$. If $X\sigma W = Y\gamma Z$ with X and Y column equivalent and W and Z row equivalent, then $\sigma = \gamma$.*

Proof. Since X and Y are column equivalent, there exists a lower triangular matrix $Q_1 \in GL_r(K)$ so that $Y = XQ_1$. Since W and Z are row equivalent there exists an upper triangular matrix $Q_2 \in GL_r(K)$ so that $Z = Q_2W$. Hence,

$$\begin{aligned} X\sigma W &= Y\gamma Z \\ &= XQ_1\gamma Q_2 W. \end{aligned}$$

X has full column rank implies that X has a left inverse, denoted X^l. W has full row rank implies that W has a right inverse, denoted W^r. Multiplying each side of

$$X\sigma W = XQ_1\gamma Q_2 W$$

on the left by X^l and on the right by W^r yields

$$\sigma = Q_1\gamma Q_2$$

and the uniqueness of Bruhat's decomposition implies

$$\sigma = \gamma.$$

■

By applying the two previous lemmas, we have shown uniqueness of choice of $\sigma \in P$ in 2.1. Hence, we have proved the following theorem.

THEOREM 2.3 (The Staircase Decomposition). *If $\mathcal{A}_r^-$, $\mathcal{A}_r$, and P denote the rank r lower staircase, rank r upper staircase, and $r \times r$ permutation matrices respectively, then*

$$J_r = \bigsqcup_{\sigma \in P} \mathcal{A}_r^- \sigma \mathcal{A}_r \ .$$

We now have a decomposition for each $\mathcal{J}$ -class of $M_n(K)$ that is unique. At this point let us take note of the fact that we have lost the ability to interchange $\mathcal{A}_r^-$ and $\mathcal{A}_r$ as in Bruhat's and Renner's decompositions. A brief calculation when $r = 1$ will show that $\mathcal{A}_r \sigma \mathcal{A}_r$ does not even exist.

Let us now consider which matrices have a decomposition where ρ=identity. In other words, which matrices will have an $\mathcal{A}^-\mathcal{A}$ decomposition?

We will need to define the $r \times r$ **primary submatrix** of a matrix X, denoted $X_{[r]}$. The primary submatrix of $X \in M_n(K)$ of rank r is obtained by choosing the first r linearly independent columns and rows and deleting the remaining $n - r$ rows and columns, hence $X_{[r]} \in GL_r(K)$.

EXAMPLE 2.4.

$$X = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 4 & 7 \\ 3 & 6 & 13 \end{pmatrix} \quad X_{[2]} = \begin{pmatrix} 1 & 5 \\ 2 & 7 \end{pmatrix}$$

THEOREM 2.4. *Let $D \in M_n(K)$. If $D = Y\sigma_1 Z$ for some $Y \in \mathcal{A}_r^-, Z \in \mathcal{A}_r, \sigma_1 \in P$ and $D_{[r]} = L\sigma_2 U$ for some $L \in B^-, U \in B$, and $\sigma_2 \in P$, then $\sigma_1 = \sigma_2$.*

Proof. Let $D \in M_n(K)$. Then $D = X\sigma_1 Y$ for some $X \in \mathcal{A}_r^-, Y \in \mathcal{A}_r$ and $\sigma_1 \in P$ by Theorem 2.3. $D_{[r]} \in GL_r(K)$ so, by Bruhat's Decomposition $D_{[r]} = L\sigma_2 U$ for some $L \in B^-, U \in B$, and $\sigma_2 \in P$. If the rank of $D = n$, then $X\sigma_1 Y = D = D_{[r]} = L\sigma_2 U$, therefore $\sigma_1 = \sigma_2$ by the uniqueness of Bruhat's decomposition. If the rank of $D = r < n$, then D has $n - r$ linearly dependent columns and rows. Multiplying D on the left by matrices of the form $X_{ij}(c) = I + ce_{ij}$ where $i > j$ will add rows from top to bottom. Thus, we can zero out all linearly dependent rows of D in the following manner. Keep row 1. If row 2 is linearly independent from 1 proceed to row 3. If row 2 is a multiple of row 1, then multiply D on the left by $X_{21}(c) = I + ce_{21}$ which adds c times row 1 to row 2. Continue in this manner until we arrive at a matrix $D' = L_1 D$ where L_1 is lower triangular and in D' the linearly dependent rows of D are now zero rows. Multiplying D on the right by matrices of the form $X_{ij}(c) = I + ce_{ij},\ i < j$ will add columns from left to right. Thus, there exists an upper triangular matrix U_1 so that $D'' = L_1 D U_1$; i.e., $D = L_1^{-1} D'' U_1^{-1}$.

By Theorem 2.3, $D'' = Z\gamma W$, which implies $X\sigma_1 Y = L^{-1} Z\gamma W U^{-1}$. Thus, $\sigma_1 = \gamma$ by uniqueness. Therefore, $D'' = Z\sigma_1 W$ $D_{[r]}$ may be obtained from D'' by deleting the zero rows and columns of D''. Therefore, $D'_{[r]} = Z'\sigma_1 W'$ where Z' is obtained from Z by deleting the rows in Z corresponding to the zero rows in D''. Similarly, W' is obtained from W by deleting the columns of W which correspond to the zero columns in D''. Thus, we have $Z\sigma_1 W' = L\sigma_2 U$. From the uniqueness of Bruhat's decomposition, $\sigma_1 = \sigma_2$. ∎

Let us now take a closer look at $\boldsymbol{\mathcal{A}}^-$ and $\boldsymbol{\mathcal{A}}$. Recall that an element $e \in M$, a monoid, is an idempotent if $e^2 = e$. What are the idempotents of $\boldsymbol{\mathcal{A}}^-$? Of $\boldsymbol{\mathcal{A}}$? Let

$$\Lambda = \{e_i\}_{i=1}^n$$

where e_r is the diagonal matrix with $e_{ii} = 1$ for $i = 1, 2, \cdots, r$ and $e_{ii} = 0$ for $i > r$. Since $e_r^2 = e_r\ \forall\ r = 1, 2, \cdots, n,\ e_r$ is an idempotent. Do $\boldsymbol{\mathcal{A}}^-$ and $\boldsymbol{\mathcal{A}}$ have other idempotents? Let us consider the case where $n = 3$. If $x \in \boldsymbol{\mathcal{A}}^{-}$, then

$$x = \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix}$$

for some $a, b, c, d, e, f \in K$. If x is an idempotent, then

$$\begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix} = \begin{pmatrix} a^2 & 0 & 0 \\ ab + cb & c^2 & 0 \\ ab + eb + df & ec + fe & f^2 \end{pmatrix}.$$

Case 1: Suppose the rank of $x = 1$. Then $a^2 = a$ implies $a = 0$ or $a = 1$. If $a = 0$, then $c = f = 0$ since $x \in \mathcal{A}^{-}$. Therefore,

$$x = \begin{pmatrix} 0 & 0 & 0 \\ b & 0 & 0 \\ d & e & 0 \end{pmatrix}$$

is nilpotent, contradicting the fact that x is idempotent. Hence, $a = 1$. Since $a = 1$, $c = f = 0$; otherwise, the rank of $x > 1$. Therefore,

$$x = \begin{pmatrix} 1 & 0 & 0 \\ b & 0 & 0 \\ d & e & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 & 0 \\ b & 0 & 0 \\ d & e & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ b & 0 & 0 \\ d + eb & 0 & 0 \end{pmatrix}$$

implies $e = 0$. Hence,

$$x = \begin{pmatrix} 1 & 0 & 0 \\ b & 0 & 0 \\ d & 0 & 0 \end{pmatrix}.$$

Case 2: Suppoes the rank of $x = 2$. Then $a^2 = a$ implies $a = 0$ or $a = 1$. But $a \neq 0$ since x would then be nilpotent as in Case 1. Therefore $a = 1$, which implies

$$x = \begin{pmatrix} 1 & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ b+bc & c^2 & 0 \\ d+eb+df & ec+fe & f^2 \end{pmatrix}.$$

Let $c^2 = c$. Then $c = 0$ or $c = 1$. If $c = 0$, then $f = 0$. Therefore,

$$x = \begin{pmatrix} 1 & 0 & 0 \\ b & 0 & 0 \\ d & e & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 & 0 \\ b & 0 & 0 \\ d & e & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ b & 0 & 0 \\ d+be & 0 & 0 \end{pmatrix}.$$

implies $e = 0$, a contradiction of the assumption that rank of x is 2. Hence, $c = 1$. Thus,$a = 1$ and $c = 1$, which implies $f = 0$; otherwise, the rank of $x > 2$. Therefore,

$$x = \begin{pmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ d & e & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ d & e & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2b & 1 & 0 \\ d+eb & e & 0 \end{pmatrix}.$$

Note, $b = 2b$ implies $b = 0$. Hence,

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ d & e & 0 \end{pmatrix}.$$

Case 3. Suppose the rank of $x = 3$. Then $a = b = c = 1$. Therefore,

$$x = \begin{pmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ d & e & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ d & e & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2b & 1 & 0 \\ 2d + eb & 2e & 0 \end{pmatrix}.$$

Note, $2b = b$ implies $b = 0$, and $2e = e$ implies $e = 0$. Thus, $2d + eb = 2d = d$, which implies $d = 0$. Hence,

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We have shown that when $n = 3$

$$E(\boldsymbol{\mathcal{A}}^-) = \left\{ (0), \begin{pmatrix} 1 & 0 & 0 \\ b & 0 & 0 \\ d & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ d & e & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

If we let $U_{e_r}^- = \left\{ x \mid x = \left(\begin{array}{c|c} I_r & 0 \\ \hline Y & I_{n-r} \end{array} \right) \right\}$, then

$$E(\boldsymbol{\mathcal{A}}^-) = \bigcup_{r=1}^{3} U_{e_r}^- e_r.$$

Similarly,

$$E(\boldsymbol{\mathcal{A}}) = \bigcup_{r=1}^{3} U_{e_r} e_r$$

when

$$U_{e_r} = \left\{ x \mid x = \left(\begin{array}{c|c} I_r & Y \\ \hline 0 & I_{n-r} \end{array} \right) \right\} .$$

The proof for any n is done in the general case later in the paper.

To find the regular $\mathcal{J}$ -classes of $\boldsymbol{\mathcal{A}}^-$ and $\boldsymbol{\mathcal{A}}$, we will need to introduce a new set. Let $B^-_{e_r} = \{x \in \boldsymbol{\mathcal{A}}^- \mid x = be_r,\ b \in B^-\}$.

THEOREM 2.5. *$B^-_{e_r}$ is a regular subsemigroup of $\boldsymbol{\mathcal{A}}^-$.*

Proof. If x and $y \in B^-_{e_r}$, then $x = b_1e_r$ and $y = b_2e_r$ for some $b_1, b_2 \in B^-$. $xy = b_1e_rb_2e_r = b_1b_2e_re_r = b_3e_r \in B^-e_rxb_1^{-1}x = (b_1e_r)b_1^{-1}(b_1e_r) = b_1e_re_r = b_1e_r = x$.

■

All regular $\mathcal{J}$ -classes of $\boldsymbol{\mathcal{A}}^-$ are of the form $B^-_{e_r}$. Similarly, all regular $\mathcal{J}$ -classes of $\boldsymbol{\mathcal{A}}$ are of the form B_{e_r}. The proof is given in Chapter 6.

Let $\boldsymbol{\mathcal{A}}^-_{(r)} = \{a \in \boldsymbol{\mathcal{A}}^- \mid a^t \in B^-_{e_r} \text{ for some } t \in \mathbb{Z}^+\}$

THEOREM 2.6. $\boldsymbol{\mathcal{A}}^- = \bigsqcup_{0 \le r \le n} \boldsymbol{\mathcal{A}}^-_{(r)}$

Proof. By definition

$$\bigsqcup_{0 \le r \le n} \boldsymbol{\mathcal{A}}^-_{(r)} \subseteq \boldsymbol{\mathcal{A}}^-.$$

To show the reverse inclusion, let $a \in \boldsymbol{\mathcal{A}}^-$ be an arbitrary element. If $a = 0$, then $a \in \boldsymbol{\mathcal{A}}^-_{(0)}$. If $a \neq 0$, then $a_{ii} \neq 0$ for $i = 1, 2, \cdots, k < n$. If $a_{ii} \neq 0\ \forall\ i$, then $a \in \boldsymbol{\mathcal{A}}^-_{(n)}$. If $a_{ii} = 0$ for some k, $1 \le k < n$, then $a_{ii} = 0\ \forall\ i > k$. Let $a_{k+1\ k+1}$ denote the first diagonal entry which is equal to zero. Our matrix may be partitioned as follows:

$$a = \left(\begin{array}{c|c} B & 0 \\ \hline X & N \end{array} \right),$$

where B is a $k \times k$ lower triangular matrix, N is an $(n-k) \times (n-k)$ nilpotent matrix and X is an $(n-k) \times k$ matrix. Since N is nilpotent, there exists $t \in \mathbb{Z}^+$ such that

$N^t = 0$. Therefore,

$$\begin{aligned} a^t &= \left(\begin{array}{c|c} B & 0 \\ \hline X & N \end{array}\right)^t \\ &= \left(\begin{array}{c|c} B^t & 0 \\ \hline * & N^t \end{array}\right) \\ &= \left(\begin{array}{c|c} B^t & 0 \\ \hline * & 0 \end{array}\right) \end{aligned}$$

implies $a^t \in \boldsymbol{\mathcal{A}}^-_{(r)}$.

The union is disjoint since if we suppose $\boldsymbol{\mathcal{A}}^-_{(r)} \cap \boldsymbol{\mathcal{A}}^-_{(s)} \neq 0$, then $\exists\ x \in \boldsymbol{\mathcal{A}}^-_{(r)} \cap \boldsymbol{\mathcal{A}}^-_{(s)}$. Thus, $x = \left(\begin{array}{c|c} B_1 & 0 \\ \hline X_1 & N_1 \end{array}\right)$ where B_1 is an $r \times r$ lower triangular matrix, N_1 is $(n-r) \times (n-r)$ with zeros on the diagonal and $x = \left(\begin{array}{c|c} B_2 & 0 \\ \hline X_2 & N_2 \end{array}\right)$ where B_2 is an $s \times s$ lower triangular matrix, and N_2 is an $(n-s) \times (n-s)$ nilpotent matrix with zeros on the diagonal. If $r < s$, then $B_2 = \left(\begin{array}{c|c} B_1 & 0 \\ \hline X_3 & N_3 \end{array}\right)$, a contradiction. Similarly, if $s < r$, $B_1 = \left(\begin{array}{c|c} B_2 & 0 \\ \hline X_4 & N_4 \end{array}\right)$, a contradiction as well. Hence, $\boldsymbol{\mathcal{A}}^-_{(r)} \cap \boldsymbol{\mathcal{A}}^-_{(s)} = \emptyset$. ■

THEOREM 2.7. $\boldsymbol{\mathcal{A}}^-_{(r)}\boldsymbol{\mathcal{A}}^-_{(s)} \subseteq \boldsymbol{\mathcal{A}}^-_{(\min\{r,s\})}$

Proof. Let $x \in \boldsymbol{\mathcal{A}}^-_{(r)}$ and $y \in \boldsymbol{\mathcal{A}}^-_{(s)}$. Then

$$x = \left(\begin{array}{c|c} B_1 & 0 \\ \hline X_1 & N_1 \end{array}\right) \quad \text{and} \quad y = \left(\begin{array}{c|c} B_2 & 0 \\ \hline X_2 & N_2 \end{array}\right)$$

where B_1 is $r \times r$, B_2 is $s \times s, N_1$ is $(n-r) \times (n-r)$ and N_2 is $(n-s) \times (n-s)$. Without loss of generality, assume $r < s$. Then y can be repartitioned so that

$$y = \left(\begin{array}{c|c} B_3 & 0 \\ \hline X_2 & \begin{array}{c|c} B_4 & 0 \\ \hline X_3 & N_2 \end{array} \end{array}\right),$$

where B_3 is $r \times r$ and B_4 is $(s-r) \times (s-r)$. Therefore,

$$xy = \left(\begin{array}{c|c} B_1 & 0 \\ \hline X_1 & \begin{array}{c|c} N_3 & 0 \\ \hline X_5 & N_4 \end{array} \end{array}\right) \left(\begin{array}{c|c} B_3 & 0 \\ \hline X_2 & \begin{array}{c|c} B_4 & 0 \\ \hline X_3 & N_2 \end{array} \end{array}\right) = \left(\begin{array}{c|c} B_1 B_3 & 0 \\ \hline Y & N_5 \end{array}\right) \in \boldsymbol{\mathcal{A}}^-_{(r)}$$

■

Chapter 3

Algebraic Geometry

The subject of Algebraic Geometry is vast. We shall give only a brief overview sufficient for later applications. The reader who wishes to gain a deeper knowledge of the subject may refer to books by Zariski [15], Jacobson [9], or Humphreys [6].

3.1 Affine Varieties

If K is a field, then $K^n = K^*$ x K^* x $\cdots$ x K^* is known as **affine n-space** and is denoted $\mathbb{A}^n$. An **affine variety** is the set of common zeros in $\mathbb{A}^n$ of a finite collection of polynomials in $K[T] = K[T_1, T_2, \cdots, T_n]$.

EXAMPLE 3.1. For $K = \mathbb{C}$, $\mathbb{A}^2 = \mathbb{C} \times \mathbb{C}$ is affine 2-space. Let $f(x, y) \in K[x, y]$ be given by $f(x, y) = xy - 1$. Then

$$\begin{aligned} X &= \{(x, y) \in \mathbb{C} \times \mathbb{C} \mid f(x, y) = 0\} \\ &= \{(x, y) \in \mathbb{C} \times \mathbb{C} \mid xy = 1\} \end{aligned}$$

is an affine variety.

Hilbert's Basis Theorem states that since K is a field, $K[T]$ is Noetherian and is finitely generated. Therefore, each ideal $I \subseteq K[T]$ is generated by a collection of polynomials, $\{f_\alpha(T)\}_{\alpha=1}^m$. Notice that $I = (\{f_\alpha(T)\}_{\alpha=1}^m)$ has the same zeros as $\{f_\alpha(T)\}_{\alpha=1}^m$. Hence, each ideal corresponds to an affine variety.

EXAMPLE 3.2. Define $f(x,y) \in \mathbb{C}[x,y]$ as $f(x,y) = xy - 1$. Then $X = \{(x,y) \mid xy = 1\}$ is an affine variety. Let $I = (f(x,y))$ be the ideal generated by $f \in \mathbb{C}[x,y]$. If $g(x,y) \in I$, then $g(x,y) = \alpha f^n(x,y)$ for some $\alpha \in \mathbb{C}, n \in \mathbb{Z}^+$. Let $X' = \{(x,y) \mid g(x,y) = 0,\ \forall g \in I\}$. If $(x',y') \in X'$, then

$$\begin{aligned} 0 &= g(x',y') \\ &= \alpha f^n(x',y') \\ &= \alpha(x'y'-1)^n \end{aligned}$$

implies $f(x',y') = 0$. Hence, $(x',y') \in X$. Clearly, $X \subseteq X'$, thus $\{f\}$ and (f) correspond to the same affine variety in $\mathbb{C} \times \mathbb{C}$.

Notice that the correspondence between ideals in $K[T]$ and the affine varieties is not 1-1. For example, the ideals generated by T and T^2 are distinct in $K[T]$, but they have the same set of zeros, $\{0\}$, in $\mathbb{A}^1$. To achieve a 1-1 correspondence, we will need to refine our choice of ideals to eliminate this repetition in the correspondence.

Define the **radical** of an ideal I, denoted $\sqrt{I}$, as follows:

$$\sqrt{I} = \{f(T) \in K[T] \mid f(T)^r \in I \text{ for some } r \geq 0\}.$$

EXAMPLE 3.3. Let $I = T^2$. Then $\sqrt{I} = (T) \neq I$.

We will define a **radical ideal** to be an ideal which is equal to its radical. Hilbert's Nullstellensatz [2] implies that there is a 1-1 correspondence between the set of all radical ideals in $K[T]$ and the set of all affine varieties in $\mathbb{A}^n$.

What topology should be used? If we let K be the field of complex numbers, the usual topology of complex n-space could be imposed on $\mathbb{A}^n$. The zeros of polynomials would be closed in this topology since they are the inverse image of a closed set, $\{0\}$, in $\mathbb{C}$ under the continuous map $x \longmapsto f(x)$. Thus, the set of zeros of a finite collection of polynomials would also be closed. Unfortunately, not all of the closed sets in the usual topology would be captured in this manner. We must find a new topology for $\mathbb{A}^n$. Let the closed sets be the affine varieties. Then it is easy to verify that we do indeed have a topology on $\mathbb{A}^n$. It is known as the Zariski Topology. What type of topology is the Zariski Topology? Points are closed. $\mathbb{A}^n$ is not Hausdorff since proper closed sets are finite. Thus, no two open sets can be disjoint. $\mathbb{A}^n$ is compact; but since it is compact not Hausdorff, we will say $\mathbb{A}^n$ is quasi-compact, reserving the term compact for compact Hausdorff spaces.

A topological space is said to be **irreducible** if it cannot be written as the union of two proper, non-empty, closed subsets. Equivalently, any non-empty open subset, $\mathcal{O}$, is dense; i.e., $\overline{\mathcal{O}} = X$. A subspace Y is irreducible if and only if $\overline{Y}$ is irreducible. The continuous image of an irreducible set is irreducible.

Example 3.4. $\mathbb{A}^1$ is irreducible.

Example 3.5. $X = \{(x, y) \mid xy = 0\} \subseteq \mathbb{A}^2$ is reducible since $X = \{(x, y) \mid x = 0\} \cup \{(x, y) \mid y = 0\}$

A topological space X is said to be **disconnected** if and only if there are two sets, H and K, with $H \cap K = \emptyset$, $H \neq \emptyset$, and $K \neq \emptyset$ so that $X = H \cup K$. When no such disconnection exists, the space X is said to be **connected**. If X is irreducible then X is connected.

Example 3.6. $X = \{(x, y) \mid xy = 0\} \subseteq \mathbb{A}^2$ is connected since X is not the disjoint union of two closed sets.

When X is a Noetherian topological space, i.e., each non-empty collection of closed sets has a minimal element, the space has a finite number of maximal irreducible subspaces. Hence,

$$X = X_1 \cup X_2 \cup \cdots \cup X_n,$$

where each X_i is closed. Each $X_i,\ i = 1, \cdots, n$ is called an **irreducible component** of X. In $\mathbb{A}^n$, a set X is irreducible if and only if the ideal consisting of the collection of all polynomials vanishing on X, denoted $\mathcal{I}(X)$, is prime. Hence, $\mathbb{A}^n$ is irreducible [6].

Example 3.7. Let

$$X = \{(x, y) \mid x^2 - x + xy + y = 0, xy - y^2 = 0\}.$$

Then $X = X_1 \cup X_2$, where $X_1 = \{(x, y) \mid x = y\}$ and $X_2 = \{(1, 0)\}$ are closed and irreducible. $\mathcal{I}(X)$ is not prime since $x^2 - x + xy + y = (x - 1)(x - y)$.

3.2 Dimension of an Affine Variety

If $X \subseteq \mathbb{A}^n$, then $K[X] = K[T_1, \cdots, T_n]/\mathcal{I}(X)$ is called the **coordinate ring** of X.

EXAMPLE 3.8. Let $X = \{(x, y) \ : \ xy = 1\}$. Then

$$\begin{aligned} K[X] &= K[x,y] \Big/ (xy-1) \\ &\cong K[x, x^{-1}] \\ &\cong \left\{ \frac{f(x)}{x^m} \middle| f(x) \in K[x] \right\}. \end{aligned}$$

If X is an irreducible affine variety, then its coordinate ring $K[X]$ is an integral domain and can be embedded in its quotient field $K(X)$. $K(X)$ is a finitely generated field extention of K, thus it has a finite transcendence degree over K, denoted

$$\mathrm{tr.deg}_K\, K(X) \ .$$

Define the **dimension of** X, denoted $\dim X$, as follows:

$$\dim X = \mathrm{tr.deg}_K\, K(X) \ .$$

If $X \subseteq \mathbb{A}^n$ is not irreducible, then

$$X = X_1 \cup X_2 \cup \cdots \cup X_t,$$

where X_i is closed for each $i = 1, 2, \cdots, t$. We then define

$$\dim X = \max\{\dim X_1, \dim X_2, \cdots, \dim X\}.$$

EXAMPLE 3.9. Let X be as in Example 3.7. Then

$$\begin{aligned} \dim X &= \max\{\dim X_1, \dim X_2\} \\ &= \max\{1, 0\} \\ &= 1. \end{aligned}$$

EXAMPLE 3.10. If X, Y are irreducible varieties of dimension m and n respectively, then

$$\dim X \times Y = m + n.$$

A polynomial map $f : X \to Y$ is called a **morphism**. When X is irreducible and $\overline{f(X)} = Y$, f is said to be **dominant**.

EXAMPLE 3.11. Let $X = Y = M_2(K)$. Then $f : X \to Y$, given by $f(A) = \det A \cdot A$, is a dominant morphism. Note that f is clearly a polynomial map and

$$f(M_2(K)) = GL_2(K) \cup \{0\},$$

so $\overline{f(M_2(K))} = M_2(K)$

THEOREM 3.1 (Dimension Theorem). *If $\varphi : X \to Y$ is dominant and X and Y are irreducible with dimensions n and m respectively, then*

(1) $n \geq m$.

(2) there exists a non-empty open subset U of Y contained in $\varphi(X)$ such that for any closed irreducible subset W and Y with $W \cap U \neq \phi$ and any component V of $\varphi^{-1}(W)$ with $\varphi(V) = W$ we have $\dim V = \dim W + n - m$.

(3) For any closed irreducible subset W of Y and any component V of $\varphi^{-1}(W)$ with $\varphi(V) = W$, we have $\dim V \geq \dim W + n - m$.

EXAMPLE 3.12. Let f be as in the previous example. Then $f^{-1}(0) = \{x \in M_2(K) \mid x \text{ is singular } \}$ implies $\dim\{f^{-1}(0)\} = 3$ If $A \in GL_2(K)$, then $f^{-1}(A) = A_0 \cdot f^{-1}(I_n)$ is finite, which implies $\dim\{f^{-1}(A)\} = 0$.

3.3 A sheaf of functions

Let X be a topological space and suppose that for each open subset U of X we have a K-algebra $\mathcal{O}_U$ of functions from U to K. The set $\{\mathcal{O}_U\}$ is called a sheaf of K-valued functions if the following conditions are satisfied:

(i) If $U_1 \subseteq U_2$ are open subsets of X and $f \in \mathcal{O}_{U_2}$, then $f|_{U_1} \in \mathcal{O}_{U_1}$,

(ii) If $U = \underset{i}{\cup} U_i$ where U, U_i are open subsets of X, and $f : U \to K$ satisfies $f|_{U_1} \in \mathcal{O}_{U_i}$, then $f \in \mathcal{O}_U$.

Let S be an affine variety. For each open set U of X define $\mathcal{O}_U$ by

$$\begin{aligned}\mathcal{O}_U \;=\; & \{f \mid U \to K;\ \text{for each } x \in X \text{ there exists an open subset} \\ & U(x) \text{ of } U \text{ containing } x \text{ and functions } g, h \in K[X] \text{ such that} \\ & h \text{ does not vanish at any point of } U(x) \text{ and } f = g/h \text{ on } U(x)\ \}\ [2].\end{aligned}$$

$\{\mathcal{O}_U\}$ will be a sheaf of K-valued functions for X.

3.4 Algebraic Varieties

The concept of an algebraic variety is broader than that of an affine variety. To be able to explain it we must introduce an intermediate concept, that of a pre-variety. A **pre-variety** over K is a Noetherian topological space with a sheaf of K-valued functions. As we saw above, every affine variety has a sheaf of K-valued functions hence it is a pre-variety. Is it true that every pre-variety is affine? A counterexample is $\mathbb{P}^n$, the projective variety. $\mathbb{P}^n$ is the set of one dimensional subspaces of $\mathbb{A}^{n+1}$. It is often useful to think of $\mathbb{P}^n$ as the set of lines in $\mathbb{A}^{n+1}$ through the origin. Closed sets in $\mathbb{P}^n$ will be defined by a finite collection of homogeneous polynomials. $\mathbb{P}^n$ is a pre-variety since if we define

$$\begin{aligned}\mathcal{O}_U \;=\; & \{f : U \to K \mid \ \text{for each } v \in U \text{ there exists an open} \\ & \text{subset } U(v) \text{ of } U \text{ containing } v \text{ and functions } g, h \text{ in the} \\ & \text{homogeneous coordinate ring of } \mathbb{P}^n \text{ of degree } i \text{ for} \\ & \text{some } i \text{ such that } h \text{ does not vanish at any point of} \\ & U(v) \text{ and } f = g/h \text{ on } U(v)\}\end{aligned}$$

then $\{\mathcal{O}_U\}$ is a sheaf of K-valued function.

If X and X' are pre-varieties then their product $X \times X'$ is also a pre-variety in the following way:

For isomorphisms $\phi : V \to U$ and $\phi' : V' \to U'$ where V and V' are affine varieties and U and U' are open subsets of $X,\ X'$ respectively, the map

$$\phi \times \phi' : V \times V' \to U \times U'$$

is an isomorphism between the affine variety $V \times V'$ and the open subset $U \times U'$ of $X \times X'$. (The isomorphisms here are the usual isomorphisms of topological spaces with a sheaf of K-valued functions.) [2] We have defined the product of two pre-varieties and are now able to define an algebraic variety. An **algebraic variety** is a pre-variety X such that the set $\Delta(X) = \{(x, x) : x \in X\}$ is closed in $X \times X$. Hence, both projective and affine varieties are algebraic varieties.

Example 3.13. It is here that we can see a connection between algebraic varieties and semigroup theory. Let

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(K).$$

Then

$$R_e = \{x \in M_2(K) \mid x = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}; a \neq 0 \text{ and } b \neq 0\},$$

$$L_e = \{x \in M_2(k) \mid x = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}; a \neq 0 \text{ and } c \neq 0\},$$

$$J_e = \{x \in M_2(k) \mid \text{ the rank of } x = 1\}$$

and $\mathcal{R}/\mathcal{L} = \mathcal{J}/\mathcal{L} \cong \mathbb{P}^1$.

A **morphism** of algebraic varieties is a map $\phi : X \to X'$ such that ϕ is continuous and for each open subset U' of X' and each $f \in \mathcal{O}_{U'}$ we have $f \circ \phi \in \mathcal{O}_{\phi^{-1}(U')}$. Notice that this definition coincides with the previous definition of a morphism given for affine varieties. An **isomorphism** of algebraic varieties is a bijective map ϕ such that ϕ and ϕ^{-1} are both morphisms.

Chapter 4

Reductive Groups

As we saw in the last chapter, an algebraic variety is a prevariety X such that the set $\Delta(X) = \{(x,x) : x \in X\}$ is closed in $X \times X$. Thus, all affine varieties are algebraic varieties. An **algebraic group** over K is a set which is an algebraic variety and also a group such that the maps $\varphi : G \times G \to G$ and $\psi : G \times G \to G$ given by $\varphi(x,y) = xy$ and $\psi(x) = x^{-1}$ respectively are morphisms of varieties. An algebraic group, G, is called an **affine algebraic group** if the variety of G is affine.

EXAMPLE 4.1. Let us consider $G = GL_n(K)$. G may be regarded as a subset of K^{n^2} in the following way:

$$GL_n(K) = \{(a_{11}, a_{12}, \ldots, a_{1n}) \mid a_{ij} \in K\}.$$

As a subset of K^{n^2}, $GL_n(K)$ is not closed. However, if we consider $GL_n(K)$ as a subset of K^{n^2+1} in the following way:

$$GL_n(K) = \left\{(a_{11}, \ldots, a_{nn}, b) \mid b = \frac{1}{\det a_{ij}}\right\},$$

then $GL_n(K)$ is a closed subset of K^{n^2+1}. Hence, $GL_n(K)$ is an affine algebraic group.

For a proof of the following theorem, see [2].

THEOREM 4.1.

(i). If G is an algebraic group and H is a closed subgroup of G, then H is an algebraic group.

(ii). If G_1 and G_2 are algebraic groups then their direct product, $G_1 \times G_2$, is also an algebraic group.

So every closed subgroup of $GL_n(K)$ is also an affine algebraic group. These closed subgroups of $GL_n(K)$ for various n are called **linear algebraic groups**. All linear algebraic groups are affine algebraic groups. The converse is also true.

THEOREM 4.2. *Every affine algebraic group is isomorphic to a closed subgroup of $GL_n(K)$.*

For a proof of the above theorem, see [2].

If we think of a linear algebraic group, G, as a topological space, then

$$G = G_0 \cup G_1 \cup \ldots \cup G_k, \tag{4.1}$$

where each G_i is connected. But if we consider G as an affine variety, then

$$G = G_0' \cup \ldots \cup G_m', \tag{4.2}$$

where each G_i is irreducible. When G is a group, G is connected if and only if G is irreducible [2]. So the decompositions in (4.1) and (4.2) coincide; i.e., $k = m$ and $G_i = G_i'$.

Let G^c denote the component of G that contains 1. G^c is a closed normal subgroup of G with finite index [2]. Thus,

$$G = G^c x_1 \cup G^c x_2 \cup \ldots \cup G^c x_k.$$

The cosets are the irreducible components from (4.2). G^c is itself a connected linear algebraic group. It is called the connected component of G.

EXAMPLE 4.2. There are two linear algebraic groups of dimension 1. Let

$$G_a = \left\{ x \in GL_2(K) \mid x = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, a \in K \right\}.$$

Then G_a is a closed subgroup of $GL_2(K)$; hence, it is an algebraic group. It is isomorphic to the additive group $(K, +)$ since

$$\begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a_1 + a_2 \\ 0 & 1 \end{pmatrix}.$$

Any algebraic group isomorphic to this one is called the **additive group**.

Let $G_m = GL_1(K)$. G_m contains all matrices of the form (λ) where $\lambda \in K^*$. G_m is isomorphic to the multiplicative group K^*. Any algebraic group isomorphic to this one is called the **multiplicative group**.

Recall the following definitions:

(1) the **normalizer** of a subgroup H in a group G is defined as

$$N_G(H) = \left\{g \in G \mid ghg^{-1} \in H, \forall h \in H\right\} \text{ and}$$

(2) the **centralizer** of a subgroup H in a group G is defined as

$$C_G(H) = \{g \in G \mid gh = hg, \forall h \in H\}.$$

For the proofs of the following theorems, see [6].

THEOREM 4.3. *G is an algebraic group.*

(i). If $H \subseteq G$ is a subgroup, then $\overline{H}$ is a subgroup.

(ii). If H_1 and H_2 are closed subgroups of G with $H_2 \subseteq N_G(H_1)$, then H_1H_2 is a closed subgroup of G.

(iii). If $G = < \bigcup_{\lambda\in\Lambda} H_\lambda >$, for H_λ closed and connected, then G is connected.

THEOREM 4.4. *If $\Phi : G \to G'$, where G and G' are algebraic groups, is a homomorphism, then*

(i). $Ker\Phi$ is a closed normal subgroup of G.

(ii). $Im\Phi$ is a closed subgroup of G'.

(iii). $\dim(Ker\Phi) + \dim(Im\Phi) = \dim(G)$.

4.1 Nilpotent and Solvable Groups

When x and y are elements of a group, denote by (x, y) the commutator $xyx^{-1}y^{-1}$. Denote by (X, Y) the subgroup generated by all the $(x, y), x \in X$ and $y \in Y$. The **derived series** of a group G is defined as a chain of subgroups, G_i, determined in the following way. Let $G_0 = G, G_1 = (G, G), G_2 = (G_1, G_1), \ldots, G_{i+1} = (G_i, G_i)$. If $G_k = \{1_G\}$ for some k, then G is said to be **solvable**. Notice that if G is a solvable group, then $(G_i, G_i) \subset G_{i+1}$ and G_i/G_{i-1} is abelian.

The **descending central series** of a group G is defined as a chain of subgroups, G^i, determined in the following way. Let $G^0 = G, G^1 = (G, G), G^2 = (G, G^1), \ldots, G^i = (G, G^{i-1})$. If $G^k = \{1_G\}$ for some k, then G is said to be **nilpotent**. Notice that $G^{i+1} \lhd G^i, \forall i$. Also, since $G^i \subseteq G_i, \forall i$, every nilpotent group is solvable. However, there are solvable groups which are not nilpotent.

Example 4.3. $T_n(K) = \{x \in GL_n(K) \mid x \text{ is upper triangular}\}$ is solvable but not nilpotent.

Example 4.4. $U_n(K) = \{x \in GL_n(K) \mid x$ is upper triangular with 1's on the diagonal$\}$ is both solvable and nilpotent.

If G is a closed subgroup of $GL_n(K)$, i.e., G is an affine algebraic group, such that G is connected and solvable, then the Lie-Kolchin Theorem tells us that G is conjugate in $GL_n(K)$ to a subgroup of $T_n(K)$ [2]. $T_n(K)$ is a maximal connected solvable subgroup of $GL_n(K)$ [2].

Let $x \in G$, an algebraic group. If the only eigenvectors of x are equal to 1, then x is said to be **unipotent**. A subgroup of G is said to be **unipotent** if all of its elements are unipotent.

Example 4.5. $U_n(K)$ is a unipotent subgroup of $GL_n(K)$.

A subgroup $T = K^* \times K^* \times \ldots \times K^*$ of an algebraic group G is called a **torus**. If T is a torus then it is isomorphic to $D_n(K) = \{x \in GL_n(K) \mid x \text{ is a diagonal matrix}\}$.

4.2 Semisimple and Reductive Groups

An arbitrary algebraic group G possesses a unique largest normal solvable subgroup which is automatically closed [2]. It is called the **radical** of G and is denoted $R(G)$. If $R(G) = 1$ and $G \neq \{1\}$, then G is said to be **semisimple**. If G is semisimple, then

$$G = G_0 \cup \ldots \cup G_k,$$

where G_i is simple for each $i = 1, 2, \ldots, k$ [6].

EXAMPLE 4.6. $SL_n(K) = \{x \in GL_n(K) \mid \det x = 1\}$ is semisimple

The subgroup of $R(G)$ consisting of all its unipotent elements is called the **unipotent radical** of G and is denoted $R_U(G)$[6]. $R_U(G)$ is normal in G. If $G \neq \{1\}$ is a connected algebraic group and $R_U(G) = \{1\}$, then G is said to be **reductive**. For the remainder of the paper, we will be concerned primarily with reductive groups.

EXAMPLE 4.7. $GL_n(K)$ is reductive since $R(GL_n(K)) = \{\alpha I \mid \alpha \in K^*\}$ and $R_U(GL_n(K)) = \{I\}$. Any torus is reductive. Any semisimple group is reductive.

4.3 Borel Subgroups and Unipotent Subgroups

A maximal closed connected solvable subgroup of a reductive group G is called a **Borel subgroup**. It will be denoted as B.

EXAMPLE 4.8. Let $G = GL_n(K)$. Then $B = T_n(K)$. A proof of the following is found in [6].

THEOREM 4.5. *All Borel subgroups are conjugate.*

All maximal tori, T, are subgroups of some Borel group. Hence, all tori are conjugate. Fix T, a maximal torus in G. Then $T \subseteq B$ for some Borel subgroup. There exists a unique Borel subgroup, denoted B^-, so that

$$B \cap B^- = T.$$

B and B^- are called **opposite Borel subgroups**.

EXAMPLE 4.9. Let $G = GL_3(K)$, $T = D_3(K)$. Then $B = T_3(K)$ and $B^- = \{x \in GL_n(K) \mid x$ is a lower triangular matrix$\}$. So, $B \cap B^- = T$.

A proof of the following theorem is found in [6].

THEOREM 4.6. *Let G be a reductive group and $B \subseteq G$, a Borel subgroup. Then*

$$G = \bigcup_{x \in G} x^{-1}Bx.$$

Let T be a maximal torus and B be a Borel subgroup containing T. Let $U = R_U(B)$ and $U^- = R_U(B^-)$. Then B is a semidirect product of U and T; i.e., $B = UT$, $U \lhd B$ and $U \cap T = 1$. Similarly, $B^- = U^-T$, $U^- \lhd B^-$ and $U^- \cap T = 1$. U and U^- are connected groups, normalized by T, and $U \cap U^- = I$. They are maximal unipotent subgroups of G [6].

4.4 Weyl Group

Let G be a reductive group and $T \subseteq G$ a maximal torus. Then $C_G(T) = T$ and

$$W = N_G(T)/T$$

is a finite group called the **Weyl group** of G. Every Weyl group is isomorphic to an abstract group known as a Coxeter group. The development of the material that follows parallels that of [7].

A group W is called a **Coxeter Group** if W is generated by a set of elements S subject to the following relations:

$$(ss^{'})^{m(s,s^{'})} = 1 \qquad \text{where } m(s,s^{'}) \in \mathbb{Z}\ \forall\ s, s^{'} \in S,$$
$$m(s,s^{'}) = m(s^{'},s) \geq 2 \qquad \text{whenever } s \neq s^{'}.$$

$\mid S \mid$ is called the **rank** of the Coxeter group. Much of the following theory is true when $\mid S \mid = \infty$, but we will only consider the case when $\mid S \mid < \infty$.

EXAMPLE 4.10. S_4 is a familiar example of a Coxeter group. In fact, every symmetric group is a Coxeter group, but we will use S_4 to illustrate the concepts discussed below. To see that S_4 meets the criteria for being a Coxeter group, recall that S_4 is generated by the transpositions $(12), (23),$ and (34). If we let $S = \{(12), (23)$ and $(34)\}$ and define $m : S \times S \to \mathbb{Z}$ to be the order of the product of each pair of elements in S; that is, $m[(12), (23)] = m[(23), (34)] = 3$, $m[(12), (34)] = 2$ and $m[(12), (12)] = m[(23), (23)] = m[(34), (34)] = 1$, we can see that all requirements have been met in order for S_4 to be a Coxeter group.

4.4.1 Length Function

Each $w \in$ W can be written in the form $w = s_1 s_2 \ldots s_r$ for some $s_i \in S$. The expression of $w \in W$ as a product of elements in S is not unique. If r is as small as possible, we call it the **length** of w written as $l(w)$. By convention, $l(1) = 0$. The element of longest length is denoted by w_0. The length function has the following elementary properties:

[L1] $l(w) = l(w^{-1})$.

[L2] $l(w) = 1$ if and only if $w \in S$.

[L3] $l(ww') \leq l(w) + l(w')$.

[L4] $l(ww') \geq l(w) - l(w')$.

[L5] $l(w) - 1 \leq l(ws) + 1$ for $s \in S$ and $w \in W$.

The properties above may be illustrated by using the information from the table in example 4.11.

THEOREM 4.7. *Let $w \in W$ and $s \in S$. Then $l(ws) = l(w) \pm 1$. Similarly, $l(sw) = l(w) \pm 1$.*

In order to show the above proposition, use the homomorphism $\epsilon : W \to \{1, -1\}$ defined by $\epsilon(w) = (-1)^{l(m)}$ together with L4 and L5.

EXAMPLE 4.11. The following table shows the length of each element of S_4.

Element	Decomposition	Length
1		0
(12)		1
(23)		1
(34)		1
(12)(34)		2
(123)	(12)(23)	2
(132)	(23)(12)	2
(234)	(23)(34)	2
(243)	(34)(23)	2
(13)	(12)(23)(12)	3
(24)	(23)(34)(23)	3
(1234)	(12)(23)(34)	3
(1243)	(12)(34)(23)	3
(1342)	(23)(34)(12)	3
(1432)	(34)(23)(12)	3
(13)(24)	(23)(12)(34)(23)	4
(124)	(12)(34)(23)(34)	4
(142)	(34)(23)(12)(34)	4
(134)	(12)(23)(12)(34)	4
(143)	(34)(23)(12)(23)	4
(1423)	(23)(34)(12)(23)(12)	5
(1324)	(12)(23)(12)(34)(23)	5
(14)	(12)(23)(34)(23)(12)	5
(14)(23)	(12)(23)(34)(23)(12)(23)	6

Since $l[(14)(23)] = 6$, it is the element of longest length; that is, $\mathrm{w}_0 = (14)(23)$. Note, while the length of each element is unique, the decomposition is not.

$$\begin{aligned}(14) &= (34)(12)(23)(34)(12)\\ &= (12)(23)(34)(23)(34).\end{aligned}$$

4.4.2 Roots, Reflections, and A Geometric Representation

To determine precisely when $l(ws)$ is smaller or larger than $l(w)$, we need to introduce the concept of a root system Φ of W. Begin with a vector space over $\mathbb{R}$ with a basis $\{\alpha_s \mid s \in S\}$ in 1-1 correspondence to S. Define a bilinear form B on V by

$$B(\alpha_s, \alpha_{s'}) = -cos\frac{\pi}{m(s, s')}.$$

$B(\alpha_s, \alpha_{s'}) = 1$ when $s = s'$, and $B(\alpha_s, \alpha_{s'}) \leq 0$ when $s \neq s'$. If we define a **reflection** to be a linear transformation that fixes a hyperplane pointwise and sends some non-zero vector to its negative, the transformation $\sigma_s : V \to V$, given by

$$\sigma_s\lambda = \lambda - 2B(\alpha_s, \lambda)\alpha_s,$$

is a reflection. For a proof of the following, see Humphreys[7].

Theorem 4.8. *There exists a unique homomorphism $\sigma : W \to GL(V)$ sending s to σ_s, and the group $\sigma(W)$ preserves the form B on V. Moreover, for each pair, s and s', the order of ss' is $m(s, s')$.*

We shall refer to σ as the geometric representation of W. We will define a **root system Φ** to be the set of unit vectors that are permuted by W. Φ is the set of all vectors $\sigma(w)(\alpha_s)$ where $w \in W$ and $s \in S$. (To simplify notation we will write $w(\alpha_s)$ for $\sigma(w)(\alpha_s)$.) So

$$\Phi = \{w(\alpha_s) \mid w \in W, s \in S\}.$$

Let $\alpha \in \Phi$. Then α may be written as

$$\alpha = \sum_{s \in S} c_s\alpha_s (c_s \in \mathbb{R}).$$

The reflection α is said to be positive (negative) and written $\alpha > 0$ ($\alpha < 0$) if all $c_s > 0$ ($c_s < 0$). Let Φ^+ denote all positive roots, and Φ^- denote all negative roots.

Theorem 4.9. *Let $w \in W$ and $s \in S$. If $l(ws) > l(w)$, then $w(\alpha_s) > 0$. If $l(ws) < l(w)$, then $w(\alpha_s) < 0$.*

By applying this theorem, we can show that every $\alpha \in \Phi$ is either positive or negative. Thus, $\Phi^+ \cup \Phi^- = \Phi$.

By the way we defined σ, each $s \in S$ is a reflection. We call them **simple reflections** in W. The simple reflections are not the only reflections in W. Conjugation of a simple reflection by an element in W gives another reflection. Let $T \subset W$ denote the reflections. Then

$$T = \bigcup_{w \in W} wSw^{-1}$$

is the set of all reflections in W. Therefore, if $t_\alpha \in T$, then $t_\alpha(\alpha) = -\alpha$. We began with the task of finding a way to determine when $l(ws)$ is larger or smaller than $l(w)$. We can now complete that task. For $t_\alpha \in T$, $l(wt_\alpha) > l(w)$ if and only if $w(\alpha) > 0$. Hence, $l(sw) = l(w) + 1$ when $w(\alpha_s) > 0$, and $l(sw) = l(w) - 1$ when $w(\alpha_s) < 0$.

Example 4.12. If we let V be the three dimensional subspace of $\mathbb{R}^4$ spanned by $\{(1000), (0100), (0010)\}$, then by performing the required computations we discover that $T = \{(12), (23), (34), (12)(34)\}$.

4.4.3 Parabolic Subgroups

Each subset of S generates a subgroup of W. Subgroups generated in this way together with their conjugates are called **parabolic subgroups**.

Example 4.13. Let $S = \{(12), (23), (34)\}$. We have already seen that S generates S_4 . S has six nonempty proper subsets. These subsets and the subgroups of S_4 they generate are listed below.

$I_1 = \{(12)\}$	$W_{I_1} = \{ 1, (12) \}$
$I_2 = \{(23)\}$	$W_{I_2} = \{ 1, (23) \}$
$I_3 = \{(34)\}$	$W_{I_3} = \{ 1, (34) \}$
$I_4 = \{(12),(23)\}$	$W_{I_4} = \{ 1, (12), (23), (13), (123), (132) \}$
$I_5 = \{(12),(34)\}$	$W_{I_5} = \{ 1, (12), (34), (12)(34)\}$
$I_6 = \{(23),(34)\}$	$W_{I_6} = \{ 1, (23), (34), (234), (24), (243) \}$

4.4.4 Strong Exchange Condition

An important fact concerning the nature of reduced expressions is that if $w = s_1 s_2 \dots s_r$ is a reduced expression and $t \in T$ is a reflection with $l(tw) < l(w)$ then $tw = s_1 \dots \hat{s_j} \dots s_r$ where $\hat{s_j}$ indicates that the factor s_j has been deleted from the product. This idea is known as the **strong exchange condition**. If we know that the expression is reduced then j is unique.

As a result of the strong exchange condition, if $w = s_1 s_2 \dots s_r$ with $l(w) < r$, then for some $i < j$, $w = s_1 \dots \hat{s_i} \dots \hat{s_j} \dots s_r$. This idea is called the **deletion condition**.

EXAMPLE 4.14. The element $(13)(24) = (\hat{12})(23)(12)(\hat{23})(34)(23)$ can be used to illustrate the deletion condition. If we multiply $(12)(23)(12)(34)(23)$ by (23) on the left we get $(23)(12)(23)(12)(34)(23) = (12)(23)(34)(23)$ which is an example of the strong exchange condition.

4.4.5 Partial Ordering

The most useful of the many ways to partially order the group W that is consistent with the length function is the Bruhat ordering, defined as follows. Write $w' \to w$ if there is a $t \in T$, the set of reflections, such that $w = w't$, with $l(w') < l(w)$. Then define $w' \leq w$ if there is a sequence $w' = w_0 \to w_1 \to \dots \to w_n = w$. The resulting relation is a partial ordering of W. (It is easy to verify that the relation is reflexive, antisymmetric and transitive.) 1 is the unique element of minimal length.

A proof of the following may be found in Humphreys [7].

THEOREM 4.10. *Let $w' \leq w$ and $s \in S$. Then either $w's \leq w$ or $w's \leq ws$ (or both).*

4.4.6 Cosets and D_I

Define $D_I = \{w \in W \mid l(ws) > l(w) \text{ for all } s \in I\}$. For each $w \in W$, there is a unique $u \in D_I$ and a unique $v \in W_I$ such that $w = uv$. Their lengths satisfy $l(w) = l(u) + l(v)$. Moreover, u is the unique element of smallest length in the coset wW_I.

EXAMPLE 4.15. Let's consider the cosets of the parabolic subgroup W_{I_5}. They are as follows:

1W$_{I_5}$ = {1, (12), (34), (12)(34)},
(23)W$_{I_5}$ = {(23), (132), (234), (1342)},
(13)W$_{I_5}$ = {(13),(123),(134),(1234)},
(14)W$_{I_5}$ = {(14), (124), (143), (1243)},
(24)W$_{I_5}$ = {(24), (142), (243), (1432)},
(13)(24)W$_{I_5}$ = {(13)(24), (1423), (1324), (14)(23)}.
Note: The coset representative has been chosen to be the element of smallest length; i.e., u in the previous theorem.

4.5 Bruhat's Decomposition

In Chapter 2 we saw an application of Bruhat's decomposition for the reductive group $GL_n(K)$. We give here the decomposition for any reductive group. A proof may be found in [6].

THEOREM 4.11. *Let G be a reductive group with Weyl group W and Borel subgroup B. Then*

$$G = \bigsqcup_{\sigma \in W} B\sigma B.$$

As we saw in Chapter 2, there are alternate forms for this decomposition. To show this, we will need the following Lemma, the proof of which is to be found in [6].

LEMMA 4.1. *Let G be a reductive group with Weyl group W. Let T be a maximal torus of G. If $T \subseteq B_1$ and $T \subseteq B_2$, then $x^{-1}B_1x = B_2$ for some $x \in W$.*

COROLLARY 4.1. *Let G be a reductive group with Weyl group W. Let $T \subseteq G$ be a maximal torus. If B_1 and B_2 are Borel subgroups containing T, then*

$$G = \bigsqcup_{w \in W} B_1 w B_2.$$

In particular, if $B_1 = B$ and $B_2 = B^-$, then

$$G = \bigsqcup_{w \in W} B^- w B.$$

Proof. Let B, B_1 and B_2 be Borel subgroups of G. From Lemma 4.1, $B_1 = \alpha B \alpha^{-1}$ and $B_2 = \beta B \beta^{-1}$ for some $\alpha, \beta \in W$. Thus,

$$\begin{aligned} G &= \alpha G \beta^{-1} \\ &= \bigsqcup_{x \in W} \alpha B x B \beta^{-1} \qquad \text{(Theorem 4.11)} \\ &= \bigsqcup_{x \in W} \alpha B \alpha^{-1} \alpha x \beta^{-1} \beta B \beta^{-1} \\ &= \bigsqcup_{x \in W} B_1 \alpha x \beta^{-1} B_2 \\ &= \bigsqcup_{\sigma \in W} B_1 \sigma B_2 \end{aligned}$$

■

4.6 Parabolic and Levi Subgroups

Recall that a Weyl group W is generated by a set S of reflections, and for each $I \subseteq S$, $W_I =< I >$ is a subgroup of W. Fix a maximal torus T in a reductive group G.

Let B and B^- be opposite Borel subgroups such that $B \cap B^- = T$. For each $I \subseteq S$, define P_I and P_I^- in the following way:

$$P_I = BW_IB,$$

and

$$P_I^- = B^-W_IB^-.$$

P_I and P_I^- are both subgroups of G containing B. Such subgroups are called **opposite parabolic subgroups**. Define L_I as follows:

$$L_I = P_I \cap P_I^-.$$

L_I is also a subgroup of G. It is called a **Levi subgroup** of G. Its Weyl group is W_I [2]. Opposite Borel subgroup of L_I, denoted B_L and B_L^-, are found in the following way:

$$B_L = L_I \cap B,$$

and

$$B_L^- = L_I \cap B^-.$$

EXAMPLE 4.16. Let $G = GL_n(K)$. Then $W = \{x \in GL_n(K) \mid x \text{ is a permutation matrix}\}$. Let

$$I = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right\}.$$

Then

$$W_I = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right\},$$

$$\begin{aligned}
P_I^- =& B^- W_i B^- \\
=& \left\{ x \in GL_4(K) \mid x = \left(\begin{array}{cc|cc} a & b & 0 & 0 \\ c & d & 0 & 0 \\ \hline x & y & e & f \\ z & w & g & h \end{array}\right), a,b,c,d,e,f,g,h,x,y,z,w \in K \right\},
\end{aligned}$$

$$\begin{aligned}
P_I =& B W_I B \\
=& \left\{ x \in GL_4(K) \mid x = \left(\begin{array}{cc|cc} a & b & x & y \\ c & d & z & w \\ \hline 0 & 0 & e & f \\ 0 & 0 & g & h \end{array}\right), a,b,c,d,e,f,g,h,x,y,z,w \in K \right\},
\end{aligned}$$

$$\begin{aligned}
L_I =& P_I^- \cap P_I \\
=& \left\{ x \in GL_4(K) \mid x = \left(\begin{array}{cc|cc} a & b & 0 & 0 \\ c & d & 0 & 0 \\ \hline 0 & 0 & e & f \\ 0 & 0 & g & h \end{array}\right), a,b,c,d,e,f,g,h \in K \right\},
\end{aligned}$$

$$\begin{aligned}
B_L =& P_I \cap B \\
=& \left\{ x \in GL_4(K) \mid x = \left(\begin{array}{cc|cc} a & b & 0 & 0 \\ 0 & d & 0 & 0 \\ \hline 0 & 0 & e & f \\ 0 & 0 & 0 & h \end{array}\right), a,b,d,e,f,h \in K \right\},
\end{aligned}$$

$$\begin{aligned}
B_L^- =& P_I \cap B^- \\
=& \left\{ x \in GL_4(K) \mid x = \left(\begin{array}{cc|cc} a & 0 & 0 & 0 \\ c & d & 0 & 0 \\ \hline 0 & 0 & e & 0 \\ 0 & 0 & g & h \end{array}\right), a,c,d,e,g,h \in K \right\}.
\end{aligned}$$

The roots of W, denoted Φ, are in $1-1$ correspondence with $C = \{(i,j) \mid i \neq j \ ; 1 \leq i, j \leq n\}$. In particular, there is a $1-1$ correspondence between Φ^+, the set of all positive roots, and $C^+ = \{(i,j) \mid i < j\}$. Similarly, there is a $1-1$ correspondence between Φ^-, the set of all negative roots, and $C^- = \{(i,j) \mid i > j\}$. Thus for each $\alpha \in \Phi^+$, let $X_\alpha = I + \beta e_{ij}$ for some $\beta \in K^*$ and $i < j$. For each $\alpha \in \Phi^-$, let $X_\alpha = I + \beta e_{ij}$ for some $\beta \in K^*$ and $i > j$.

Example 4.17. Let $G = GL_4(K)$. Then $\mid \Phi^+ \mid = 6 = \mid \Phi^- \mid$. Let $\Phi^+ = \{\alpha_1, \alpha_2, \ldots, \alpha_6\}$ and $\Phi^- = \{-\alpha_1, -\alpha_2, \ldots, -\alpha_6\}$. Then we can make the following assignments.

$$X_{\alpha_1} = \begin{pmatrix} 1 & * & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad X_{\alpha_2} = \begin{pmatrix} 1 & 0 & * & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$X_{\alpha_3} = \begin{pmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad X_{\alpha_4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$X_{\alpha_5} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad X_{\alpha_6} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$X_{-\alpha_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad X_{-\alpha_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ * & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$X_{-\alpha_3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & * & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad X_{-\alpha_4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ * & 0 & 0 & 1 \end{pmatrix}$$

$$X_{-\alpha_5} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & * & 0 & 1 \end{pmatrix} \quad X_{-\alpha_6} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & * & 1 \end{pmatrix}$$

Each X_α above is a subgroup of dimension 1, called a **root subgroup**. Recall that maximal unit subgroups of G, U and U^-, exist with $U \subseteq B$ and $U^- \subseteq B^-$ for some opposite Borel subgroups B and B^-. $U \cong U_n(K)$ [2]. Thus, from example 4.17, we can see that

$$U = \prod_{\alpha \in \Phi^+} X_\alpha, \qquad \text{in any order}$$

and

$$U^- = \prod_{\alpha \in \Phi^-} X_\alpha, \qquad \text{in any order.}$$

We are now in a position to find the unipotent subgroups of P_I and P_I^-. Let

$$U_I = \prod_{\alpha \in \Phi^+ \setminus \Phi_I} X_\alpha$$

and

$$U_I^- = \prod_{\alpha \in \Phi^- \setminus \Phi_I} X_\alpha,$$

where $\Phi_I = \{\alpha \in \Phi \mid X_\alpha \subseteq L_I\}$.

Example 4.18. Let $G = GL_n(K)$ and I be as in example 4.16. Then

$$U_I = \left\{ x \in GL_4(K) \mid x = \left(\begin{array}{cc|cc} 1 & 0 & a & b \\ 0 & 1 & c & d \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) ; a, b, c, d \in K \right\}$$

and

$$U_I^- = \left\{ x \in GL_4(K) \mid x = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline a & b & 1 & 0 \\ c & d & 0 & 1 \end{array} \right) ; a,b,c,d \in K^* \right\}.$$

The following theorems are needed for later work. Proofs may be found in [6] and [2].

THEOREM 4.12. *Let $x \in W$. If $x^{-1}Bx = B$, then $x = 1$.*

THEOREM 4.13. *$P_I = L_I U_I$ and $P_I^- = L_I U_I^-$.*

THEOREM 4.14. *$B = B_L U_I$ and $B^- = U_I^- B_L$.*

THEOREM 4.15. *The following are equivalent:*

(i). $x \in D_I$.

(ii). $x B_L x^{-1} \subseteq B$.

(iii). $x B_L^- x^{-1} \subseteq B^-$.

(iv). $x^{-1} B x \subseteq B U_I^-$.

(v). $x^{-1} B^- x \subseteq B^- U_I$.

Chapter 5

Reductive Monoids

In this chapter we will extend the ideas of the previous chapter to study a special class of monoids called reductive monoids. The development of the following material follows that of Putcha [12].

An **algebraic semigroup** S is an affine variety S together with a map $\theta : S \times S \to S$ that is a morphism of varieties. An **algebraic monoid** is an algebraic semigroup that has an identity element. A **homomorphism of algebraic semigroups** is a semigroup homomorphism which is also a morphism of varieties.

Example 5.1. Let $S = M_n(K)$. Let $X \subseteq S$ be a subsemigroup. Then X is an algebraic semigroup. Let $T = D_n(K)$. Then $\overline{T}$ is an algebraic monoid. Let $\varphi : S \to S$ be defined by $\varphi(A) = \det A \cdot A$. Then φ is a semigroup homomorphism.

In the previous chapter we saw that an algebraic semigroup is isomorphic to a subgroup of $GL_n(K)$. The following result is well-known [12].

Theorem 5.1. *Let M be an algebraic monoid. Then M is isomorphic to a closed submonoid of some $M_n(K)$.*

Theorem 5.2. *Let S be a closed subsemigroup of $M_n(K)$. Then for all $a \in S$, a^n lies in a subgroup of S.*

Proof. Let $a \in S$, $b = a^n$. By the Fitting decomposition, $b\mathcal{H}e$ for some $e = e^2 \in M_n(K)$. Let

$$\begin{aligned} S_1 &= \{x \in S | xe = ex = x\} \\ &= eM_n(K)e \cap S \end{aligned}$$

Then there exists $c \in M_n(K)$ such that $bc = cb = e$ and $ec = ce = c$. For $i \geq 1$,

$$b^i S_1 = \{x \in S_1 | c^i x \in S_1\}.$$

To see this, we will show set inclusion in either direction. Let $x \in b^i S_1$. Then $x = b^i y$ for some $y \in S$. Therefore, $c^i x = c^i b^i y = ey = y \in S_1$. Let $x \in S_1$ with $c^i x \in S_1$. Then $x = ex = b^i c^i x \in b^i S_1$. For every i, $b^i S_1$ is a closed set. Thus, since $b \in S_1$,

$$S_1 \supseteq bS_1 \supseteq b^2 S_1 \supseteq \cdots$$

is a descending chain of closed sets. By Hilbert's Basis Theorem, $b^i S_1 = b^{i+1} S_1$ for some i. Therefore, $S_1 = eS_1 = c^i b^i S_1 = c^i b^{i+1} S_1 = bS_1$. Similarly, $S_1 = S_1 b$. So there exists an $x \in S_1$ such that $b = bx$. Therefore, $x = ex = cbx = cb = e$. Thus, $e \in S_1$. Then there exists $y, z \in S_1$ such that $by = e = zb$; i.e., b has both a left and right inverse. Therefore, $b\mathcal{H}e$ in $S_1 \subseteq S$. ■

From the above theorem, we see that algebraic monoids are strongly π-regular. The following theorem gives us another useful piece of information about how to determine when elements of an irreducible algebraic monoid are $\mathcal{J}$-related, $\mathcal{R}$-related, or $\mathcal{L}$-related. The proof is due to Putcha[12].

THEOREM 5.3. *Let M be an irreducible monoid with group of units G, and let a, $b \in M$. Then*

(i). $a\mathcal{J}b$ *iff* $\overline{MaM} = \overline{MbM}$ *iff* $b \in GaG$.

(ii). $a\mathcal{R}b$ *iff* $\overline{aM} = \overline{bM}$ *iff* $b \in aG$.

(iii). $a\mathcal{L}b$ *iff* $\overline{Ma} = \overline{Mb}$ *iff* $b \in Ga$.

Proof. We will give the proof of (i). The proof of (ii) and (iii) will follow by similar arguments. $a\mathcal{J}b$ implies $\overline{MaM} = \overline{MbM}$ implies $\overline{MaM} = \overline{MbM}$. Suppose $\overline{MaM} = \overline{MbM}$. Define $\phi : G \times G \to MaM$ as $\phi(g_1, g_2) = g_1ag_2$. Then ϕ is a dominant map. So $\overline{GaG} = \overline{MaM}$. Therefore, there exists a non-empty open set $U \subseteq \overline{MaM}$ such that $U \subseteq GaG$. Similarly, there exists a non-empty open subset $V \subseteq \overline{MbM}$ such that $V \subseteq GbG$. $\overline{MaM} = \overline{MbM}$ is irreducible, so $U \cap V \neq \emptyset$. Therefore, $GaG \cap GbG \neq \emptyset$. So $g_1ag_2 = g_3bg_4$ for some g_1, g_2, g_3, $g_4 \in G$ and $g_3^{-1}g_1ag_2g_4^{-1} = b$. Hence, $b \in GaG$. If $b \in GaG$, then $b = g_1ag_2$ for some g_1, $g_2 \in G$. So $MbM = Mg_1ag_2M = MaM$. Hence, $a\mathcal{J}b$. ∎

Recall that semigroups are often studied by using their idempotents. The following two results due to Putcha [12] give useful information about the idempotents of an irreducible monoid.

THEOREM 5.4. *Let M be an irreducible monoid with group of units G. Let e, $f \in E(M)$. Then*

(i). $e\mathcal{J}f$ iff $x^{-1}ex = f$ for some $x \in G$.

(ii). $e\mathcal{R}f$ iff there exists $x \in G$ such that $ex = x^{-1}ex = f$.

(iii). $e\mathcal{L}f$ iff there exists $x \in G$ such that $xe = xex^{-1} = f$.

THEOREM 5.5. *Let M be an irreducible monoid with group of units G. Then for any maximal torus T of G,*

$$E(M) = \bigcup_{x \in G} x^{-1}E(\overline{T})x.$$

An irreducible monoid M with unit group G is said to be **reductive** if G is reductive.

The following theorem relates reductive monoids and regular monoids. A proof may be found in [12].

THEOREM 5.6. *Let M be an irreducible monoid with* 0*. M is reductive if and only if M is regular.*

COROLLARY 5.1. *Let M be a reductive monoid with* 0*. Let $e \in E(M)$. Then H_e is reductive.*

Proof. M is reductive implies M is regular by Theorem 5.6. Let $a \in eMe$. Then $a = axa$ for some $x \in M$. Since $ea = ae = a$, $a = aexea$. So, a is regular in eMe. Thus, eMe is regular. Therefore, H_e is reductive by Theorem 5.6. ■

For the remainder of this chapter we will let M be a reductive monoid , G be its unit group, and W be the Weyl group of G. Fix a maximal torus $T \subseteq G$. Let B and B^- be the opposite Borel subgroups such that $B \cap B^- = T$. The idempotents, $E(\overline{T})$, are called **diagonal idempotents**.

A **cross-section lattice**, Λ, is a subset of $E(M)$ such that

(i). $|\Lambda \cap J| = 1$ for all $\mathcal{J}$-classes.

(ii). If $e_1 \in \Lambda \cap J_1$, $e_2 \in \Lambda \cap J_2$, then $J_1 \leq J_2 \Leftrightarrow e_1 \leq e_2$

THEOREM 5.7.

(i). Cross-section lattices exist.

(ii). Any two are conjugate.

(iii). There is a 1-1 correspondence between the set of all cross-section lattices and with pairs of opposite Borel subgroups.

(iv). Any two cross-section lattices contained in $E(\overline{T})$ are conjugate by an element of W.

For a proof of the above see [12]. Cross-section lattices may be found in the following manner:

$$\begin{aligned} \Lambda &= \{e \in E(T) | Be = eBe\} \\ &= \{e \in E(T) | eB^- = eB^-e\}. \end{aligned}$$

Example 5.2. Let $M = M_4(K)$. Then

$$\Lambda = \left\{ \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix} \right\}$$

is a cross-section lattice. From Theorem 5.7 part iv we see that we can find another cross-section lattice by conjugating Λ with $w_0 \in W$. Recall that

$$w_0 = \begin{pmatrix} 0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0 \end{pmatrix}.$$

Then

$$\begin{aligned} \Lambda' &= w_0 \Lambda w_0 \\ &= \left\{ \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix} \right\} \end{aligned}$$

is a cross-section lattice as well.

EXAMPLE 5.3. Let

$$M = \{A \otimes B | A, B \in M_3(K), A^t B = B A^t = \alpha I\}.$$

Then

$$\Lambda = \left\{ I, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

is a cross-section lattice.

5.1 Notation

In the previous chapter, the Weyl group of G was studied in terms of its parabolic subgroups W_I, where $I \subseteq S$, the set of simple reflections of W. We would like to relate what we did in the previous chapter to our cross-section lattice, Λ. Let $e \in \Lambda$. We will say that $I \subseteq S$ is of type e, denoted $I(e)$, if $xe = ex$, for all $x \in I$. Denote by $W(e)$ the subgroup of W generated by $I(e)$; i.e., $W(e) = W_{I(e)}$.

EXAMPLE 5.4. Let $M = M_4(K)$. Then $G = GL_4(K)$. Let Λ be as in Example 5.2 and

$$e = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$I(e) = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \right\}$$

$$\text{and } W(e) = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$$

Notice that in $M = M_n(K)$ we do not recover all of the parabolic subgroups of W in this manner. However, there exist reductive monoids where all possible subgroups of W are recovered in this manner. An example follows.

Example 5.5. Let M and Λ be as in Example 5.3. Let $G_0 = \{A \otimes (A^{-1})^t | A \in SL_3(k)\}$. Then the unit group of M is $G = kG_0$ with Weyl group $W = \{P \otimes P | P \text{ is a permutation matrix }\}$. Therefore,

$$S = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\}.$$

S has the following four subgroups, each of which corresponds to an idempotent.

$$\begin{aligned} I(0) &= \{S\} \\ I(e) &= \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \\ I(f) &= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\} \\ I(h) &= \emptyset \end{aligned}$$

Let $D(e) = \{w \in W | w \text{ is of shortest length in } wW(e)\}$ and $D(e)^{-1} = \{w \in W | w$ is of shortest length in $W(e)w\}$.

EXAMPLE 5.6. Let M, G, W, Λ, and e be as in Example 5.4. Then

$$\begin{aligned} D(e) \;=\; & \left\{ \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&1&0\\1&0&0&0\\0&1&0&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&1&0\\1&0&0&0\\0&0&0&1\\0&1&0&0 \end{pmatrix}, \right.\\ & \left. \begin{pmatrix} 1&0&0&0\\0&0&1&0\\0&0&0&1\\0&1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&1\\1&0&0&0\\0&0&1&0\\0&1&0&0 \end{pmatrix} \right\} \end{aligned}$$

and

$$\begin{aligned} D(e)^{-1} \;=\; & \left\{ \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&0&0&1\\0&1&0&0\\0&0&1&0 \end{pmatrix}, \right.\\ & \left. \begin{pmatrix} 0&1&0&0\\0&0&0&1\\1&0&0&0\\0&0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&1\\1&0&0&0\\0&0&1&0\\0&1&0&0 \end{pmatrix} \right\}. \end{aligned}$$

Let us take a closer look at $W(e)$. Let $W_e = \{x \in W(e) | xe = e\}$. $W_e = W_K$ for some $K \subseteq S$. In particular, since $W_e \subseteq W(e)$, $K \subseteq I(e)$. Hence, $W(e) = W_K \times W_{I(e)\setminus K}$. Let $W^e = W_{I(e)\setminus K}$

Denote by P_e and P_e^- the parabolic subgroups $P_{I(e)}$ and $P_{I(e)}^-$ respectively. Then

$$P_e \;=\; \{x \in G | xe = exe, \quad e \in \Lambda\}$$

and

$$P_e^- \quad = \quad \{x \in G | ex = exe, \quad e \in \Lambda\}.$$

Let $L_e = P_e \bigcap P_e^-$. Then

$$L_e \quad = \quad \{x \in G | xe = ex\}.$$

Let B_e and B_e^- denote opposite Borel subgroups of L_e. They are found as follows:

$$B_e \quad = \quad B \bigcap L_e$$

and

$$B_e^- \quad = \quad B^- \bigcap L_e.$$

Let $U_e = U_{I(e)}$ be as in the previous chapter. Notice that $eU_e^- = U_e e = e$.

EXAMPLE 5.7. See Example 4.16. $I(e) = I$ in that example, so $L_e = L_I$, etc.

THEOREM 5.8. *Let $e \in \Lambda$.*

(i) $H_e = L_e e = eL_e$.

(ii) If B_1 is any Borel subgroup of G_1 containing T, then eB_1e is a Borel subgroup of H_e.

(iii) H_e has Weyl group $W^e e = eW^e$.

For a proof, see [12].

5.2 Renner's Decomposition

Let $\mathcal{R} = \overline{N_G(T)}/T$. $\mathcal{R}$ is an inverse monoid with idempotents $E(\overline{T})$. It is known as Renner's Monoid.

EXAMPLE 5.8. Let $M = M_2(K)$. Then

$$\mathcal{R} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

THEOREM 5.9.

$$\mathcal{R} = \bigsqcup_{e \in \Lambda} WeW$$

For a proof of the above theorem, see[12] or [13].

From the above theorem, when $\sigma \in \mathcal{R}$, $\sigma = zev$ for some z, $v \in W$ and $e \in \Lambda$. As $z \in W$, $z = xw_1$ for some $x \in D(e)$, $w_1 \in W(e)$. Similarly, $v = w_2y$ for some $y \in D(e)^{-1}$, $w_2 \in W(e)$. Thus, $\sigma = xw_1ew_2y$. Now, $w_1 = a_1b_1$ for some $a_1 \in W_e$ and $b_1 \in W^e$. Similarly, $w_2 = a_2b_2$ for some $a_2 \in W_e$ and $b_2 \in W^e$. Hence,

$$\begin{aligned} \sigma &= xa_1b_1ea_2b_2y \\ &= xa_1b_1eb_2y \\ &= xa_1eb_1b_2y. \end{aligned}$$

Let $b_1b_2 = w \in W^e$. Then

$$\begin{aligned} \sigma &= xewy \\ &= xwey. \end{aligned}$$

The choice of x, w, e, and y is unique [11],[12]. We will call this the **standard form** for $\sigma \in R$.

THEOREM 5.10 (Renner's Decomposition).

$$M = \bigsqcup_{r \in R} BrB$$

Proof. From Corollary 5.1, H_e is reductive. Bruhat's decomposition applied to H_e yields

$$H_e = \bigcup_{w \in W^e} B_e^- w e B_e.$$

From Corollary 4.1,

$$G = \bigcup_{x \in W} B^- x B,$$

and since

$$W = \bigcup_{x \in D(e)} xW(e)$$

and

$$W = \bigcup_{y \in D(e)^{-1}} W(e)y,$$

we see that

$$G = \bigcup_{x \in D(e)} B^- xW(e)B$$

or

$$G = \bigcup_{y \in D(e)^{-1}} B^- W(e)yB.$$

Let $e \in \Lambda$. Then

$$\begin{aligned} J_e &= GeG \\ &= \bigcup_{\substack{x \in D(e) \\ y \in D(e)^{-1}}} B^- x W(e) B e B^- W(e) y B \\ &= \bigcup_{\substack{x \in D(e) \\ y \in D(e)^{-1}}} B^- x W(e) e B e e B^- e W(e) y B. \end{aligned}$$

Notice that e, $W(e)e$, Be, eB^-, and $eW(e)$ are all contained in H_e. Therefore,

$$\begin{aligned}
J_e &\subseteq \bigcup_{\substack{x \in D(e) \\ y \in D(e)^{-1}}} B^- x H_e y B^- \\
&= \bigcup_{\substack{x \in D(e) \\ y \in D(e)^{-1} \\ w \in W^e}} B^- x B_e^- we B_e y B \\
&= \bigcup_{\substack{x \in D(e) \\ y \in D(e)^{-1} \\ w \in W^e}} B^- x B_e^- x^{-1} weyy^{-1} B_e y B \\
&\subseteq \bigcup_{\substack{x \in D(e) \\ y \in D(e)^{-1} \\ w \in W^e}} B^- xweyB.
\end{aligned}$$

But $B^-xw \subseteq G$ and $yB \subseteq G$. Therefore,

$$GeG = \bigcup_{\substack{x \in D(e) \\ y \in D(e)^{-1} \\ w \in W^e}} B^- xweyB.$$

Hence,

$$\begin{aligned}
M &= \bigcup_{e \in \Lambda} J_e \\
&= \bigcup_{e \in \Lambda} GeG \\
&= \bigcup_{\substack{e \in \Lambda \\ x \in D(e) \\ y \in D(e)^{-1} \\ w \in W^e}} B^- xweyB \\
&= \bigcup_{r \in \mathcal{R}} B^- rB.
\end{aligned}$$

To show uniqueness, suppose

$$B^- xweyB \cap x_1 w_1 e y_1 B \neq \emptyset.$$

Then

$$bxweyb_0 = x_1 w_1 e y_1 b_1$$

for some $b \in B^-$, b_0 and $b_1 \in B$. Therefore,

$$xex^{-1}bxewyb_G = xex^{-1}x_1 e w_1 y_1 b_y.$$

Since $x \in D(e)$ and $b \in B^-$,

$$xex^{-1}bxex^{-1}\mathcal{J}e$$

implies

$$ex^{-1}x_1e\mathcal{J}e.$$

Therefore, $x^{-1}x_1 \in W(e)$ since $\mathcal{R}$ is an inverse semigroup. Hence, $x = x_1$ since x, $x_1 \in D(e)$. Similarly, $y = y_1$.
Therefore,

$$B^-xwey \cap xw_1eyB \neq \emptyset$$

implies

$$x^{-1}B^-xwe \cap w_1eyBy^{-1} \neq \emptyset$$

implies

$$ex^{-1}B^-xwe \cap w_1eyBy^{-1}e \neq \emptyset$$

implies

$$(ex^{-1}B^-)we \cap w_1e(eyBy^{-1}e) \neq \emptyset$$

implies $we = w_1w$. Hence, $w = w_1$. ■

Chapter 6

General Case

In this chapter we will generalize the results of Chapter 2 to an arbitrary reductive monoid. Recall that two important subsemigroups of the reductive monoid $M = M_n(K)$ were the the following:

$$\mathcal{A}^- \quad = \quad \{x \in M_n(k) | x \text{ is an augmented lower staircase matrix}\}$$

and

$$\mathcal{A} \quad = \quad \{x \in M_n(k) | x \text{ is an augmented upper staircase matrix.}\}$$

To generalize them, we will need to consider them in a new way. Opposite Borel subgroups of $GL_n(k)$, B and B^- were seen to be the upper and lower triangular matrices respectively. If we define two new sets, A and A^- as

$$A^- \quad = \quad \{a \in M | aB^- \subseteq B^- a\}$$

and

$$A \quad = \quad \{a \in M | Ba \subseteq aB\},$$

then we will have a useful generalization of $\boldsymbol{\mathcal{A}}^-$ and $\boldsymbol{\mathcal{A}}$. The above idea is an extension of Renner's generalization of row reduction[13].

THEOREM 6.1. *If $M = M_n(K)$, then $\boldsymbol{\mathcal{A}}^- = A^-$ and $\boldsymbol{\mathcal{A}} = A$.*

Proof. We will show that $\boldsymbol{\mathcal{A}}^- = A^-$ by showing set inclusion in both directions. Let $a \in \boldsymbol{\mathcal{A}}^-$ and $x \in B^-$. Then $ax \in \boldsymbol{\mathcal{A}}^-$ by Lemma 2.1. Notice that ax is column equivalent to a since $a = (ax)x^{-1}$. Therefore, a may be transformed into ax by row operations from the top down without interchanging rows. So there exists $z \in B^-$ such that $za = ax$. Therefore, $ax \in B^-a$. Hence, $aB^- \subseteq B^-a$.

If $a \in A^-$, then $aB^- \subseteq B^-a$. Therefore, for every $x \in B^-$, there exists $y \in B^-$ such that $ax = ya$. Choose $x \in B^-$ such that $x_{ij} \neq 0$ for every $i \geq j$ and such that each x_{ij} is distinct for $i \geq j$. That is to say, no two entries in the lower triangular part of x are equal. Let $z = y^{-1}$. Then $a = y^{-1}ax = zax$, where z, $x \in B^-$. If $a_{11} = 0$, then $a_{1j} = 0$, $\forall j \geq 2$. Otherwise, if $a_{1j} = 0$ for some $j \geq 2$, say $a_{1m} = 0$, then $0 = a_{11} = z_{11}a_{1m}x_{m1}$ implies that $z_{11} = 0$ or $x_{m1} = 0$, but $z_{11} \neq 0$ since $z \in B^-$, and $x_{m1} \neq 0$ since $m \geq 2$. Therefore, $a_{1m} = 0$. If $a_{11} \neq 0$, then $a_{1j} = 0$, $\forall j \geq 2$. Otherwise, if $a_{1m} \neq 0$ for some $m \geq 2$, then $a_{12} = z_{11}a_{1m}x_{m2}$ and $a_{1m} = z_{11}a_{1m}x_{mm}$ imply that $x_{m2} = x_{mm}$, which is a contradiction.

If the first row of a is a zero row proceed to row 2. An argument similar to the one above will show that $a_{21} = 0$ implies $a_{2j} = 0$ for $j = 2, \cdots, n$, and $a_{21} \neq 0$ implies $a_{2j} = 0$ for $j = 2, \cdots, n$. Therefore, let $a_{k1} \neq 0$ for some k be the first non-zero entry in the first column. Then $a_{ij} = 0\ \forall i < k$ and $j \geq 1$, and $a_{kj} = 0\ \forall j \geq 2$. Without loss of generality, let's assume that $k = 1$. Therefore, $a_{11} \neq 0$ is the first non-zero entry in column one. We will now find the first non-zero entry in column two. If $a_{22} = 0$, then $a_{2j} = 0\ \forall j \geq 3$. Otherwise, if $a_{2m} = 0$ for some $m \geq 3$, then $0 = a_{22} = z_{22}a_{2m}x_{m2}$ implies that either $z_{22} = 0$ or $x_{m2} = 0$, a contradiction since $z_{22} \neq 0$ because $z \in B^-$ and $x_{m2} \neq 0$ by choice of x. Therefore, $a_{2m} = 0$. If $a_{22} \neq 0$, then $a_{2j} = 0$, $\forall j \geq 3$. Otherwise, if $a_{2m} \neq 0$ for some $m \geq 3$, then $a_{2m} = z_{22}a_{2m}x_{mm}$ and $a_{22} = z_{22}a_{2m}x_{m2}$ imply that $x_{mm} = x_{m2}$ contradicting the choice of x. Proceeding in this manner we see that a is indeed lower staircase. Hence, $A^- = \boldsymbol{\mathcal{A}}^-$. $A = \boldsymbol{\mathcal{A}}$ is shown by using a similar argument. ■

We will now revisit the results of Chapter 2. Let B, B^-, W, U_e, U_e^-, B_e, B_e^-,

$D(e)$, $D(e)^{-1}$, $W(e)$, W_e, W^e, $\mathcal{R}$, and $\overline{T}$ be as defined in the previous chapter.

THEOREM 6.2. *A^- and A are subsemigroups of M.*

Proof. $A^- \subseteq M$ by definition. For any x and y in A^-,

$$xyB^- \subseteq xB^-y \subseteq B^-xy.$$

Therefore, $xy \in A^-$. A similar argument shows the result for A. ■

THEOREM 6.3. *B and B^- are the unit groups of A and A^- respectively.*

Proof. If $x \in B^-$, then $xB^- = B^- = B^-x$. Thus, $B^- \subseteq A^-$. Let $x \in A^-$ be a unit. Then

$$B^- = xx^{-1}B^- \subseteq xB^-x^{-1}$$

and

$$xB^-x^{-1} \subseteq B^-xx^{-1} = B^-.$$

Hence,

$$B^- = xB^-x^{-1}.$$

B^- is its own normalizer, therefore $x \in B^-$. The remaining result is shown similarly.
■

The following two results were not shown directly in Chapter 2.

THEOREM 6.4.

$$A^- = \bigcup_{\substack{x \in D(e) \\ e \in \Lambda}} B^-xe$$

Proof. Suppose $z = bxe$ for some $b \in B^-$, $x \in D(e)$, $e \in \Lambda$. Then

$$\begin{aligned} bxeB^- &= bxe(U_e^- B_e^-) && \text{(Theorem 4.14)} \\ &= bxB_e^- e \\ &= bxBx^{-1}xe \\ &\subseteq bB^-xe && \text{(Theorem 4.15)} \\ &= B^-(bxe). \end{aligned}$$

Thus, $z \in A^-$. To show the reverse inclusion, let $a \in A^-$. By Renner's decomposition, Theorem 5.10,

$$a = b_1 xweyb_2$$

for some b_1, $b_2 \in B^-$, $x \in D(e)$, $w \in W(e)$, $e \in \Lambda$ and $y \in D(e)^{-1}$. A^- is closed under multiplication so $xwey \in A^-$. Therefore,

$$xweyB^- \subseteq B^- xwey.$$

Then there exist b_1, $b_2 \in B^-$ such that

$$xweyb_1 = b_2 xwey, \tag{6.1}$$

which implies

$$\begin{aligned} xweyb_1(y^{-1}ey) &= b_2 xwey(y^{-1}ey) \\ &= b_2 xwey \\ &= xweyb_1 \qquad\qquad \text{(by 6.1).} \end{aligned}$$

Thus,

$$xweyB^-(y^{-1}ey) \subseteq xweyB^-.$$

Multiplying on the left by $w^{-1}x^{-1}$, we see that

$$eyB^- y^{-1}ey \subseteq eyB^-$$

which implies,

$$(y^{-1}ey)B^-(y^{-1}ey) \subseteq (y^{-1}ey)B^-.$$

Since $e \in \Lambda$ and $y^{-1}ey = e$, $y^{-1}ey \in \Lambda$. But, $y \in W(e)$, so $y = 1$ and $xywe = xwe$.

Therefore,

$$xweB^- \subseteq B^-xwe,$$

which implies

$$xewB_e^- \subseteq B^-xew \quad \text{(Theorem 4.14)}.$$

Multiply on the left by x^{-1} and on the right by w^{-1}. Thus,

$$ewB_e^-w^{-1} \subseteq x^{-1}B^-xe,$$

which implies

$$\begin{aligned} eewB_{L(e)}^-w^{-1} &\subseteq ex^{-1}B^-xe \\ &\subseteq eB^-U_ee && \text{(Theorem 4.15)} \\ &= eU_e^-B_e^-U_ee && \text{(Theorem 4.14)} \\ &= eB_e^-e \\ &= eB_e^- && \text{(since } e \in \Lambda). \end{aligned}$$

Thus,

$$ewB_e^-ew^{-1} \subseteq eB_e^-,$$

which implies $ew = e$. Hence,

$$\begin{aligned} a \in B^-xeB^- &= B^-xeU_e^-B_e^- && \text{(Theorem 4.14)} \\ &= B^-xeB_e^- \\ &= B^-xB_e^-eee \\ &= B^-xB_e^-x^{-1}xe \\ &\subseteq B^-(B^-xe) \\ &= B^-xe. && \blacksquare \end{aligned}$$

An analogous argument will prove the following theorem.

THEOREM 6.5.

$$A = \bigcup_{\substack{y \in D(e)^{-1} \\ e \in \Lambda}} eyB$$

We can now generalize the staircase decomposition for any $\mathcal{J}$-class in a reductive monoid.

THEOREM 6.6. *Let $e \in \Lambda$. Let $\Gamma = \{\gamma | \gamma = \tilde{w}e, \tilde{w} \in W^e\}$. Then*

$$J_e = \bigsqcup_{\gamma \in \Gamma} A^- \gamma A.$$

Proof. From Theorem 4.11, we see that

$$\begin{aligned} J_e &= \bigcup_{\sigma \in WeW} B^- \sigma B \\ &= \bigcup_{g,h \in W} B^- g e h B \\ &= \bigcup_{\substack{x \in D(e) \\ y \in D(e)^{-1} \\ w_1, w_2 \in W(e)}} B^- x w_1 e w_2 y B \end{aligned}$$

$$= \bigcup_{\substack{x \in D(e) \\ y \in D(e)^{-1} \\ w \in W(e)}} B^- xweyB$$

$$= \bigcup_{\substack{x \in D(e) \\ y \in D(e)^{-1} \\ \hat{w} \in W_e \\ \tilde{w} \in W^e}} B^- x\hat{w}\tilde{w}eyB$$

$$= \bigcup_{\substack{y \in D(e)^{-1} \\ \hat{w} \in W_e \\ \tilde{w} \in W^e}} B^- x\hat{w}e\tilde{w}eyB$$

$$= \bigcup_{\substack{y \in D(e)^{-1} \\ \tilde{w} \in W^e}} (B^- xe)\tilde{w}(eyB)$$

$$= \bigcup_{\substack{x \in D(e) \\ y \in D(e)^{-1}}} (B^- xe)\tilde{w}e(eyB)$$

$$= \bigcup_{\tilde{w} \in W^e} A^- \tilde{w} e A$$

$$= \bigcup_{\gamma \in \Gamma} A^- \gamma A$$

Let $y \in A^-$. Suppose that

$$y = b_1 x_1 e \gamma_1 e y_1 \hat{b}_1$$

where $\gamma_1 = \tilde{w}_1 e$, and

$$y = b_2 x_2 e \gamma_2 e y_2 \hat{b}_2$$

where $\gamma_2 = \tilde{w}_2 e$. Then

$$b_1 x_1 e \tilde{w}_1 e e y_1 \hat{b}_1 = b_2 x_2 e \tilde{w}_2 e e y_2 \hat{b}_2$$

implies

$$b_1 x_1 \tilde{w}_1 e y_1 \hat{b}_1 = b_2 x_2 \tilde{w}_2 e y_2 \hat{b}_2.$$

From the uniqueness of Renner's decomposition,

$$x \tilde{w}_1 e y_1 = x_2 \tilde{w}_2 e y_2.$$

From the uniqueness of the standard form for an element in $\mathcal{R}$,

$$\tilde{w}_1 e = \tilde{w}_2 e.$$

Therefore,

$$\gamma_1 = \gamma_2.$$

■

Let us now continue our investigation of A^- and A. Just as in the matrix example from Chapter 2, we are interested in finding the idempotents and regular elements of A^- and A.

THEOREM 6.7.

(i). $E(A^-) = \bigcup_{e\in\Lambda} U_e^- e$

(ii). $E(A) = \bigcup_{e\in\Lambda} eU_e$

Proof. We will show (i). Part (ii) will follow by a similar argument. Let $ue \in U_e^- e$ for some $e \in \Lambda$. Then $ueue = ue \cdot e = ue$. To show the reverse inclusion, let $a \in E(A^-)$. Since $a \in A^-$, $a = bxe$ for some $x \in D(e)$, $e \in \Lambda$, and $b \in B^-$. As $a \in A^-$,

$$bxebxe = bxe.$$

Therefore,

$$ebxe = e,$$

which implies $exe\mathcal{J}e$. Hence, $exex^{-1}\mathcal{J}e$, which implies $e = xex^{-1}$. Since $x \in D(e)$, $x = 1$. Since $x = 1$, $a \in B^-e = U_e^- B_e^- e = U_e^- e B_e^-$. Therefore, $a = ueb$ for some $b \in B_e^-$, $u \in U_e^-$. Thus,

$$\begin{aligned} ueb &= uebueb \\ &= ebeueb \\ &= ubeeb \\ &= ubeb \\ &= uebb. \end{aligned}$$

Multiplying on the right by b^{-1} yields

$$ue = ueb.$$

Hence, $a \in U_e^- e$. ■

Before we can determine the regular $\mathcal{J}$-classes of A^- and A, we need the following Lemma.

LEMMA 6.1. *$\phi : A^- \to \mathcal{R}$ given by $\phi(bxe) = xe$ is a semigroup homomorphism.*

Proof. For $a_1 = b_1xe$ and $a_2 = b_2yf$ in A^-,

$$\begin{aligned}\phi(a_1a_2) &= \phi(b_1xeb_2xyf)\\ &= \phi(b_1xeub_3yf),\end{aligned}$$

where $b_2 \in B^- = U_e^- B_e^-$ may be written as $b_2 = ub_3$ for some $u \in U$ and $b_3 \in B_e^-$. Therefore,

$$\begin{aligned}\phi(a_1a_2) &= \phi(b_1xb_3eyf)\\ &= \phi(b_1xb_3x^{-1}xeyf).\end{aligned}$$

By Theorem 4.15, there exists $b_4 \in B^-$ so that $b_4 = xb_3x^{-1}$. Therefore,

$$\begin{aligned}\phi(a_1a_2) &= \phi(b_1b_4xeyf)\\ &= \phi(bxeyf)\\ &= xeyf\\ &= \phi(b_1xe)\phi(b_2ye).\end{aligned}$$

∎

THEOREM 6.8.

(i). The regular $\mathcal{J}$-classes of A^- are of the form B^-e, $e \in \Lambda$.

(ii). Let $e \in \Lambda$. Then B^-e is a subsemigroup.

(iii). $B^-e \cong eB^- \times U_e^-e$

Proof.

(i) $y \in B^-e$ is regular since $y = be = be(b^{-1})be = yb^{-1}y$.
Let $z \in A^-$. Then, by Theorem 6.4, $z = bxe$ for some $b \in B^-$, $x \in D(e)$ and $e \in \Lambda$. Let ϕ be as in Lemma 6.1. Then $\phi(z) = \phi(bxe) = xe$ implies that xe is regular in $\mathcal{R}$. Thus, $xe = f$ for some $f \in \Lambda$. Hence, $z = bxe = bf$ for some $f \in \Lambda$. If $Z = \{a \in A^- | a \text{ is regular}\}$, then

$$Z = \bigcup_{e \in \Lambda} B^- e.$$

Let $e \in \Lambda$. Then $B^-e \subseteq J$ for some regular $\mathcal{J}$-class J. In particular, $B^-e \subseteq J_e$ since $b^-(be)e = e$ implies $be\mathcal{J}e$. Let $x \in J_e \setminus B^-e$. Then x is regular, so $x \in B^-f$ for some $f \in \Lambda$. Therefore, $x = bf$, which means that $bf\mathcal{J}e$. Therefore, $f\mathcal{J}e$, a contradiction since $e \neq f \in \Lambda$. Therefore, $B^-e = J_e$.

(ii) Let $x, y \in B^-e$. Then $x = b_1e$ and $y = b_2e$ for some b_1, $b_2 \in B^-$, and

$$\begin{aligned} xy &= b_1eb_2e \\ &= b_1eub_3e \qquad \text{for some } b_3 \in B_e^- \text{ and } u \in U_e^- \\ &= b_1eb_3e \\ &= b_1b_3e \\ &= be. \end{aligned}$$

Hence, $(B^-e)^2 \subseteq B^-e$.

(iii) $e \in B^-e$, so B^-e contains an idempotent. Let $be \in B^-e$. Since $eB^-e = eB^-$,

$$\begin{aligned} B^-ebe &= B^-eb_0 \qquad \text{for some } \ b_0 \in B^-, \\ &= B^-eub_1 \qquad \text{for some } u \in U_e^- and b_1 \in B_e^-, \\ &= B^-eb_1 \\ &= B^-b_1e \\ &= B^-e. \end{aligned}$$

Thus, B^-e is left-simple and by Theorem 1.3 the assertion is true. ■

THEOREM 6.9. *For $e \in \Lambda$, let*

$$A_e^- = \{a \in A^- | a^k \in B^- e, \text{ for some } k \in \mathbb{Z}^+\}.$$

Then

(i) $A^- = \bigsqcup_{e \in \Lambda} A_e^-$.

(ii) $A_e^- A_f^- \subseteq A_{ef}^-$.

(iii) A_e *is a subsemigroup.*

(iv) $B^- e$ *is an ideal of* A_e^-.

(v) $A_e^- / B^- e$ *is a nil semigroup.*

Proof. Follows from (Theorem 1.10)

In conclusion, the analogues of the lower and upper staircase matrices, A^- and A respectively, from Chapter 2 were determined. They were used to generalize the staircase decomposition to any $\mathcal{J}$-class of a reductive monoid. A^- and A were shown to be a semigroups and a decomposition for each of them was given. All idempotent elements and regular $\mathcal{J}$-classes of A^- and A were determined.

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