ABSTRACT

TAYLOR, MONIQUE R. Dafermos Regularization of a Modified KdV–Burgers Equation. (Under the direction of Dr. Stephen Schecter).

This project involves Dafermos regularization of a partial differential equation of order higher than 2. The modified Korteweg de Vries–Burgers equation is

$$u_T + f(u)_X = \alpha u_{XX} + \beta u_{XXX},$$

where the flux is $f(u) = u^3$, $\alpha > 0$, and $\beta \neq 0$. We show the existence of Riemann–Dafermos solutions near a given Riemann solution composed of shock waves using geometric singular perturbation theory. When $\beta > 0$, there is a possibility that the Riemann solution is composed of two shock waves as opposed to one. In addition, we use linearization to study the stability of the Riemann–Dafermos solutions.

Dafermos Regularization of a Modified KdV–Burgers Equation

by
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DEDICATION

I dedicate this dissertation work to my mom. I hope that I made you proud and that you are smiling from heaven. I would like to also dedicate my work to my daughter Robin Pooh. You have such a sweet spirit. You are everything that I need you to be. God bless you little one!

BIOGRAPHY

Monique Taylor was born in Henderson, NC. That is where she attended Pinkston Street Elementary, Henderson Middle School, and Southern Vance High School. She was the first of her extended family to attend a 4-year college. She attended Fayetteville State University and obtained a Bachelor of Science in Mathematics with a minor in Computer Science. She graduated magna cum laude!

At the conclusion of her undergraduate experience she was an SI (supplemental instruction) Leader. Though tutoring and participating in this program, she realized that she found joy in helping others understand mathematics. Being encouraged by the mathematics faculty at Fayetteville State University, she headed off to graduate school to earn her PhD in Mathematics with a secret desire to be called Dr. Momo, (a childhood nickname). Immediately upon finishing her undergraduate studies, she made her way to North Carolina State University.

Currently, she is married to Robert Taylor Jr. and they have a beautiful, smart little girl, Robin Chelsea Taylor. She is currently an Assistant Professor at Ferrum College in Virginia.

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Chapter 1

Introduction

The modified KdV–Burgers equation that we study is

$$u_T + f(u)_X = \alpha u_{XX} + \beta u_{XXX}, \tag{1.1}$$

where $f(u) = u^3$, $\alpha > 0$, and $\beta \neq 0$. We are interested in solutions that satisfy a given initial condition $u(0, X) = u_0(X)$ and boundary conditions

$$u(T, -\infty) = u^\ell \text{ and } u(T, \infty) = u^r, \tag{1.2}$$

for $-\infty < X < \infty$, $0 \leq T < \infty$, and $u^\ell \neq u^r$. Please note that (1.1) is invariant under the transformation of replacing u by $-u$. Thus, we will consider $u^\ell > 0$. We know from Shearer et al. [6] that when $\beta > 0$, this equation admits travelling waves that are both compressive and undercompressive. These individual waves were shown to be stable by Dodd in [5]. If we consider our equation with constant boundary conditions, only certain values of u^ℓ and u^r generate travelling wave solutions. Our concern is to determine the behavior of the solutions for other values of u^ℓ and u^r.

Numerics suggest that solutions of (1.1)–(1.2) approach a Riemann solution of

$$u_T + \left(u^3\right)_X = 0, \tag{1.3}$$

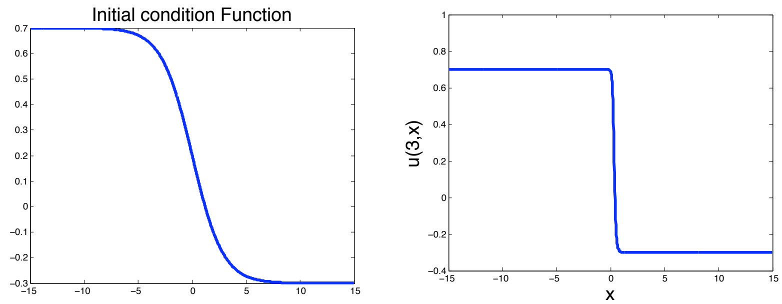
with shock waves that satisfy the viscous profile criterion for the regularization (1.1). We observe that convergence is evident in the compressed variable $x = \frac{X}{T}$. See Figures 1.1 and 1.2. We introduce the variable $t = \ln T$ for convenience. With these coordinates, (1.1) becomes

$$u_t + (3u^2 - x)u_x = \alpha e^{-t} u_{xx} + \beta e^{-2t} u_{xxx}. \tag{1.4}$$

As $t \to \infty$, (1.4) approaches

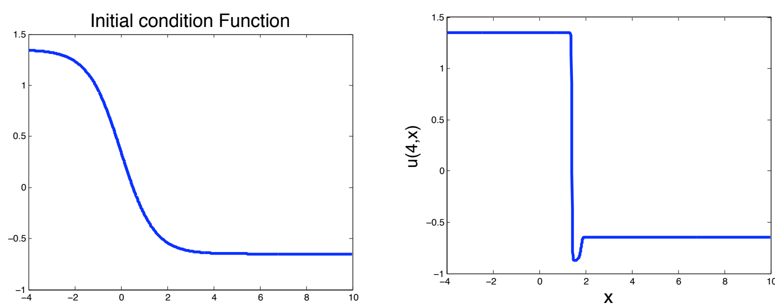
$$u_t + (3u^2 - x)u_x = 0, \tag{1.5}$$

which is simply the hyperbolic equation (1.3) in these new coordinates. The steady state solutions of (1.5) are solutions of (1.3) that only depend on $\frac{X}{T}$. These steady state solutions approach constants as $x = \frac{X}{T} \to \pm\infty$ and correspond to Riemann solutions of (1.3).



(a) Initial Condition (b) Solution at t=3

Figure 1.1: Above is a numerical solution of (1.1)–(1.2) subject to $\beta = \alpha = 1, t = 3$, and the indicated initial condition. This solution was obtained by the method of lines and it converges to a Riemann solution with left and right states given by 0.7 and -0.3 respectively. The speed of this Riemann solution derived from the Rankine–Hugoniot condition is 0.37.



(a) Initial Condition (b) Solution at t=4

Figure 1.2: Here is another numerical solution of (1.1)–(1.2) subject to $\beta = \alpha = 1, t = 4$, and indicated initial condition. This solution was obtained by the method of lines and it converges to a Riemann solution consisting of two shock waves with left, middle, and right states given by 1.35, -0.8726, and -0.65 respectively. The Rankine–Hugoniot condition gives the speed of the first shock wave as 1.40592 and the speed of the second wave as 1.75112. From the formulas derived in [6], the middle state and speed of the first shock wave is calculated to respectively be -0.878595 and 1.40833. So the numerical results are accurate up two decimal places.

Notice that for large t, e^{-t} is small. We shall attempt to gain insight into the long-time behavior of the solutions of (1.4) by replacing the exponential term, e^{-t}, with a small positive constant, ϵ. This substitution transforms (1.4) into

$$u_t + (3u^2 - x)u_x = \alpha\epsilon u_{xx} + \beta\epsilon^2 u_{xxx}. \tag{1.6}$$

This freezing of time is a way to begin to study a time-dependent system. Equation (1.6) can be regarded as a Dafermos regularization of (1.3). Dafermos regularization refers to a technique introduced in 1973 by Constantine Dafermos. This process of regularization gives rise to stationary solutions, called Riemann–Dafermos solutions, that converge almost everywhere as $\epsilon \to 0$ to a weak solution of a corresponding Riemann problem. Lin and Schecter in [7] have used this type of regularization to study conservation laws with a second-order viscous term. However, no previous work has been done with higher order equations. Furthermore, the stability of Riemann–Dafermos solutions with undercompressive waves has not been looked at even for second order equations. Our plan is to study the stability of some of the steady state solutions of (1.6).

Chapter 2

Background

Recall the conservation law $u_T + \left(u^3\right)_X = 0$, equation (1.3). Special solutions of interest of the conservation law are of the form $u(x)$, where $x = \frac{X}{T}$. Substituting $u(x)$ into (1.3) yields $(3u^2 - x)u_x = 0$. This implies that $u(x)$ is constant or $\pm\sqrt{\frac{x}{3}}$; the latter are rarefaction waves. In (X,T) coordinates, rarefaction waves fan out, or expand with time.

Shock waves are weak solutions of (1.3) of the form

$$u(X,T) = \begin{cases} u^- & \frac{X}{T} < s \\ u^+ & \frac{X}{T} > s \end{cases} .$$

The numbers u^-, u^+, and s must satisfy the Rankine-Hugoniot condition,

$$s = \frac{f\left(u^+\right) - f\left(u^-\right)}{u^+ - u^-} = \frac{\left(u^+\right)^3 - \left(u^-\right)^3}{u^+ - u^-} = \left(u^+\right)^2 + u^- u^+ + \left(u^-\right)^2.$$

It turns out that the speed, given by s, will always be positive for all of our considered shock waves. In this thesis, admissible shock waves will be those that satisfy the viscous profile criterion for (1.1). That is, we consider shock waves that correspond to travelling wave solutions of (1.1) with the same speed satisfying the boundary conditions $u(-\infty) = u^-$ and $u(\infty) = u^+$. Travelling waves with speed s are solutions of (1.1) of the form $u(\eta)$, where $\eta = X - sT$. Substituting $u(\eta)$ in (1.1) produces

$$(3u^2 - s)u_\eta = \alpha u_{\eta\eta} + \beta u_{\eta\eta\eta}. \tag{2.1}$$

The equation (2.1) is referred to as the travelling wave ordinary differential equation. We are interested in solutions of this ordinary differential equation that satisfy the boundary conditions

$$u(-\infty) = u^-, u(\infty) = u^+, u_\eta(\pm\infty) = 0, \text{ and } u_{\eta\eta}(\pm\infty) = 0. \tag{2.2}$$

A Riemann problem is a Cauchy problem for (1.3) with the piecewise constant initial condition

$$u(X,0) = \begin{cases} u^\ell, & X < 0 \\ u^r, & X > 0 \end{cases} . \tag{2.3}$$

We look for Riemann solutions of the form $u(x)$, $x = \frac{X}{T}$. This implies that $u(x)$ must satisfy

$$u(x) \to \begin{cases} u^\ell \text{ as } x \to -\infty \\ u^r \text{ as } x \to \infty \end{cases} . \tag{2.4}$$

Riemann solutions are composed of constant parts, rarefaction waves, and/or shock waves. We will only consider Riemann solutions that are composed of constant states and shock waves.

In chapter 3, we use [6] to discuss the travelling waves of (1.1),(2.2). We then introduce the system equations for the Riemann-Dafermos solutions, and show there exists a Riemann–Dafermos solution near the considered Riemann solution. For $\beta > 0$, we were able to find Riemann–Dafermos solutions that converge to a Riemann solution consisting of two shock waves, one undercompressive and one compressive. In the succeeding chapters, we investigate the stability of that Riemann–Dafermos solution by linearization. We were able to discover properties that could potentially lead to stability results.

Chapter 3

The Riemann–Dafermos Solution

3.1 Travelling Wave

The travelling wave ordinary differential equation is $(3u^2 - s)u_\eta = \alpha u_{\eta\eta} + \beta u_{\eta\eta\eta}$. Recall that admissible travelling waves should satisfy the four boundary conditions (2.2). Considering the boundary conditions and integrating from $-\infty$ to η yields

$$u^3 - su - (u^-)^3 + su^- = \alpha u_\eta + \beta u_{\eta\eta}. \tag{3.1}$$

Written as a system, (3.1) becomes

$$u_\eta = v, \tag{3.2}$$

$$v_\eta = \frac{u^3 - su - z}{\beta} - \frac{\alpha}{\beta}v, \tag{3.3}$$

for $z = (u^-)^3 - su^-$. This is a 2-dimensional system parameterized by z and s for each fixed $\alpha > 0$ and $\beta \neq 0$. There are at most 3 equilibria for this system. We consider pairs (s, z) with $s > 0$ so that 3 distinct equilibria are produced for this system. Since $z = su^- - (u^-)^3$, one equilibrium is $(u^-, 0)$. Given the s and u^- one can determine the other equilibria by solving $u_\eta = v_\eta = 0$.

Linearizing the system (3.2)-(3.3) about an equilibrium $(u, 0)$ will give rise to two eigenvalues

$$r_{1,2}(u) = \frac{-\alpha}{2\beta} \pm \sqrt{\left(\frac{\alpha}{2\beta}\right)^2 + \frac{3u^2 - s}{\beta}}. \tag{3.4}$$

If the eigenvalues (3.4) have opposite signs, the corresponding equilibrium is called a saddle. When the eigenvalues have negative real part, the equilibrium is referred to as an attractor. When the eigenvalues have positive real part, the equilibrium is said to be a repeller.

For $\beta < 0$, if $s < 3u^2$, the eigenvalues (3.4) have positive real part. However, if $3u^2 < s$, the eigenvalues have opposite signs. For $\beta > 0$, if $s < 3u^2$, the eigenvalues (3.4) have opposite signs. If $3u^2 < s$, eigenvalues with negative real part are produced. Please note that $s = 3u^2$ is the transitional case. As s increase through $3u^2$, a positive eigenvalue becomes 0, then negative.

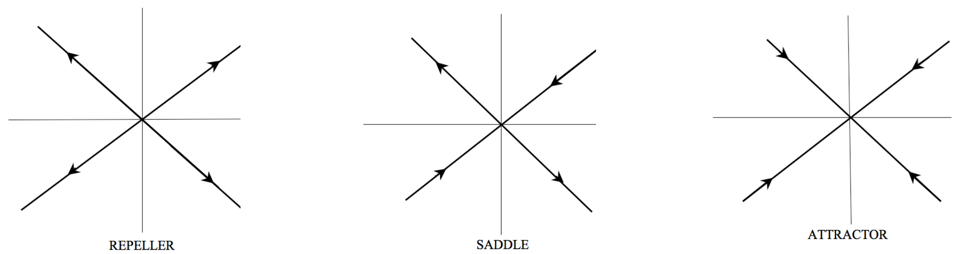


Figure 3.1: Graphical representation of the types of equilibria.

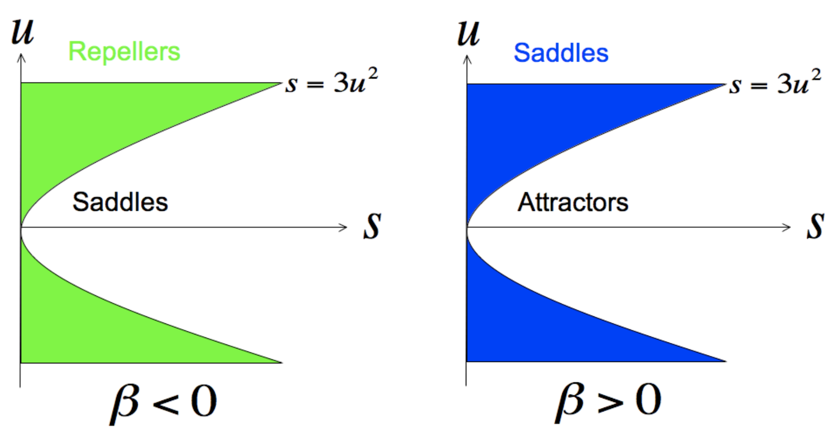


Figure 3.2: Equilibria types in the xu-plane.

3.1.1 Connecting orbits that start from $(u^-, 0)$ and their corresponding Riemann Solutions

The information revealed in this section comes from Jacobs, McKinney, and Shearer in [6]. In [6], the equilibria of the system (3.2)-(3.3) and connecting orbits are described. The corresponding shock waves are used to construct Riemann solutions of (1.3), (2.3). We begin by discussing orbits that connect the equilibrium $(u^-, 0)$ to another equilibrium $(u^+, 0)$.

When $\beta < 0$, the equilibria are given by two repellers and one saddle for $s > 0$. The only way to have a connection is to have the equilibrium $(u^-, 0)$ as one of the repellers and the equilibrium $(u^+, 0)$ as the saddle. See Figure 3.3. In [6], shock waves that correspond to the different types of travelling wave orbits from $(u^-, 0)$ to $(u^+, 0)$ are described. In this case, the travelling wave orbit corresponds to a compressive shock wave with left and right states given by u^- and u^+ respectively. A shock wave is referred to as compressive if it satisfies the Lax entropy condition. The Lax entropy condition states that the left and right states of a shock wave must satisfy $f'(u^-) > speed > f'(u^+)$. A compressive shock is one in which characteristics must enter the shock curve on both sides but not emanate from it. These types of shock waves are often referred to as classical shocks. An undercompressive shock is one in which characteristics enter the shock curve and pass through it. These shocks satisfy one of the following conditions,

1. $f'(u^-) > speed$ and $f'(u^+) > speed$, or
2. $f'(u^-) < speed$ and $f'(u^+) < speed$.

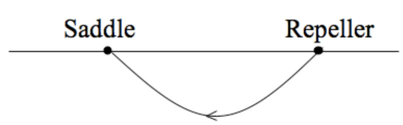


Figure 3.3: Solution that connects the repelling equilibrium to the saddle equilibrium.

When $\beta > 0$, the equilibria are given by two saddles and one attractor for $s > 0$. There are two ways in which one can find a solution that starts at $(u^-, 0)$ and ends at $(u^+, 0)$. In either case, $(u^-, 0)$ must be a saddle. See Figure 3.4(a). According to [6], if $(u^+, 0)$ is a saddle point, then the travelling wave orbit corresponds to an undercompressive shock wave. However, if $(u^+, 0)$ is an attractor, then the corresponding shock wave is compressive. In [6], it is explicitly determined that a saddle-to-saddle connection occurs when $|u^-| > \frac{2\alpha\sqrt{2}}{3\sqrt{\beta}}$ and with a speed of $s^* = (u^-)^2 - \frac{\alpha\sqrt{2}}{3\sqrt{\beta}}|u^-| + \frac{2\alpha^2}{9\beta}$.

Now consider a Riemann problem (1.3), (2.3) for $\beta > 0$. The Riemann solution is a classical solution for some pairs (u^ℓ, u^r). Those pairs results in a saddle-to-attractor connection from $(u^\ell, 0)$ to $(u^r, 0)$ for a certain s value. There is then a Riemann solution of

(1.3), (2.3) that is a classical shock wave and takes the form

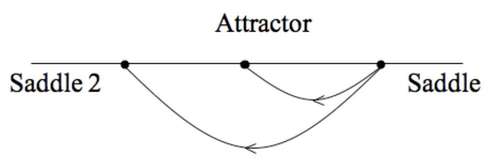
$$u(x) = \begin{cases} u^\ell \text{ for } x < s \\ u^r \text{ for } x > s \end{cases},$$

for $x = \frac{X}{T}$. According to [6], this occurs for $0 < u^\ell < \frac{2\alpha\sqrt{2}}{3\sqrt{\beta}}$ and $-\frac{u^\ell}{2} < u^r < u^\ell$.

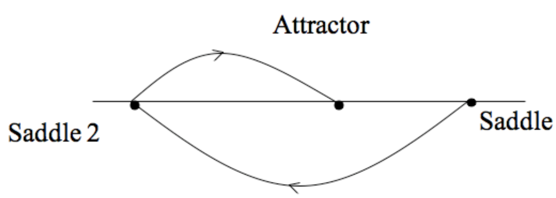
Also, there are some pairs (u^ℓ, u^r) that result in a Riemann solution consisting of two shock waves. There is a possible saddle-to-saddle connection from $(u^\ell, 0)$ to the third equilibrium $(u^m, 0)$ for $s = s_2$, and a saddle-to-attractor connection from $(u^m, 0)$ to $(u^r, 0)$ with a speed $s = s_3 > s_2$. See Figure 3.4(b). According to [6], this occurs when $u^\ell > \frac{2\alpha\sqrt{2}}{3\sqrt{\beta}}$ and $-u^\ell + \frac{\alpha\sqrt{2}}{3\sqrt{\beta}} < u^r < -\frac{u^\ell}{2}$. Furthermore, the corresponding Riemann solution consists of an undercompressive shock followed by a faster classical or compressive shock. This Riemann solution is given as

$$u(x) = \begin{cases} u^\ell \text{ for } x < s_2 \\ u^m \text{ for } s_2 < x < s_3 \\ u^r \text{ for } x > s_3 \end{cases}.$$

We will focus on $\beta > 0$ and the equilibrium $(u^r, 0)$ as an attractor.



(a) Direct Connection



(b) Indirect Connection

Figure 3.4: Connections from $(u^-, 0)$ to $(u^+, 0)$ when $\beta > 0$.

3.2 Riemann–Dafermos Equations

Riemann–Dafermos solutions are stationary solutions of (1.6) that satisfy the boundary conditions,

$$u(-\infty) = u^\ell \text{ and } u(\infty) = u^r. \tag{3.5}$$

Being stationary solutions of (1.6) imply that they satisfy

$$(3u^2 - x)u_x = \alpha\epsilon u_{xx} + \beta\epsilon^2 u_{xxx}. \tag{3.6}$$

Written as a system, (3.6) becomes

$$\epsilon u_x = v, \tag{3.7}$$
$$\epsilon v_x = w, \tag{3.8}$$
$$\epsilon w_x = \frac{1}{\beta}(3u^2 - x)v - \frac{\alpha}{\beta}w, \tag{3.9}$$
$$x_x = 1. \tag{3.10}$$

In singular perturbation theory, this system is referred to as the slow form of a slow-fast system. The change of variable $x = \epsilon\xi$ converts (3.7)–(3.10) into

$$u_\xi = v, \tag{3.11}$$
$$v_\xi = w, \tag{3.12}$$
$$w_\xi = \frac{1}{\beta}(3u^2 - x)v - \frac{\alpha}{\beta}w, \tag{3.13}$$
$$x_\xi = \epsilon. \tag{3.14}$$

This is referred to as the fast form. Letting $\epsilon \to 0$, the fast form becomes

$$u_\xi = v, \tag{3.15}$$
$$v_\xi = w, \tag{3.16}$$
$$w_\xi = \frac{1}{\beta}(3u^2 - x)v - \frac{\alpha}{\beta}w, \tag{3.17}$$
$$x_\xi = 0. \tag{3.18}$$

This is commonly referred to as the fast limit system. We begin by studying the fast limit system.

3.3 The Fast Limit System

The set of equilibria of (3.15)–(3.18) is the ux-plane. The linearization of the fast limit system at one of these equilibria has the matrix

$$J(u, 0, 0, x) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{3u^2 - x}{\beta} & -\frac{\alpha}{\beta} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The corresponding characteristic polynomial is $C(r) = r^2\left(r^2 + \frac{\alpha}{\beta}r - \frac{1}{\beta}(3u^2 - x)\right)$. The eigenvalues are 0, 0, and $\dfrac{-\alpha}{2\beta} \pm \sqrt{\left(\dfrac{\alpha}{2\beta}\right)^2 + \dfrac{3u^2 - x}{\beta}}$.

Notice that as a system, the travelling wave ordinary differential equation (2.1) is written as

$$\begin{aligned} u_\eta &= v, &(3.19)\\ v_\eta &= w, &(3.20)\\ w_\eta &= \frac{1}{\beta}(3u^2 - s)v - \frac{\alpha}{\beta}w. &(3.21)\end{aligned}$$

This system has the same form as (3.15)–(3.18). Thus, the dynamics of the solutions of (3.15)–(3.18) will be exactly the dynamics of the solutions of (3.19)–(3.21) when $x = s$.

We will now take what was observed in the 2-dimensional setting when $\beta > 0$ and relate it to the dynamics of (3.15)–(3.18) in $uvwx$-space. A connecting orbit of (3.2)–(3.3) from $(u^-, 0)$ to $(u^+, 0)$ corresponds to a connecting orbit of (3.15)–(3.18) from $(u^-, 0, 0, x)$ and $(u^+, 0, 0, x)$ with $x = s$.

Let $z = \beta w + xu - u^3 + \alpha v$. Differentiating with respect to ξ results in

$$\begin{aligned} z_\xi &= \frac{d}{d\xi}\left(\beta w + xu - u^3 + \alpha v\right)\\ &= \beta\dot{w} + xv - 3u^2 v + \alpha w\\ &= 0.\end{aligned}$$

We are now able to write the fast limit system as

$$\begin{aligned} u_\xi &= v, &(3.22)\\ v_\xi &= \frac{1}{\beta}(u^3 - xu + z) - \frac{\alpha}{\beta}v, &(3.23)\\ z_\xi &= 0, &(3.24)\\ x_\xi &= 0. &(3.25)\end{aligned}$$

Since $z_\xi = 0$, z is constant in the system (3.22)-(3.25). Since the equilibria for the fast limit system is the ux-plane, we may define $z = xu^* - (u^*)^3$ for each given u^* and x. Please observe that when $x = s$, the system (3.2)–(3.3) is the same as the system (3.22)–(3.23). We will now show that the connections of interest of (3.15)–(3.18) can be viewed as transversal intersections of invariant manifolds that persist for small $\epsilon > 0$.

3.3.1 Manifolds

Recall that the system (3.15)–(3.18) has a 2-dimensional plane of equilibria, the ux-space. For $x < 3u^2$ the linearization of (3.15)–(3.18) at one of these equilibria has one negative eigenvalue, one positive eigenvalue, and two 0 eigenvalues. For $x > 3u^2$ the linearization of (3.15)-(3.18) at one of these equilibria has two eigenvalues with real part negative and two 0 eigenvalues. We will describe the stable and unstable manifolds of certain 1- and 2- dimensional subsets of the ux-space.

$\mathcal{W}_0^s(P)$ - Stable manifold of P ={all saddle equilibria in the ux-plane such that $x < 3u^2 - \frac{\alpha^2}{3\beta}$}

On the parabola, $x = 3u^2$, another 0 eigenvalue emerges. By considering $x < u^2 - \frac{\alpha^2}{3\beta}$, we insure that x stays away from $3u^2$. Each point in P has a 1-dimensional stable eigenspace. Each of these stable spaces has an invariant curve tangent to it, which is referred to as the stable fiber for its corresponding point. We will denote $\mathcal{W}_0^s(P)$ as the manifold constructed by the union of all of the 1-dimensional stable fibers of each saddle point in the ux-plane. Locally this manifold is given by:

$$uvwx\text{-space}\begin{cases} u & \in & \text{an open set,} \\ v & = & G_1(u,w,x), \\ w & \in & \text{an open set,} \\ x & \in & \text{an open set.} \end{cases} \quad uvzx\text{-space}\begin{cases} u & \in & \text{an open set,} \\ v & \in & G(u,z,x), \\ z & \in & \text{an open set,} \\ x & \in & \text{an open set.} \end{cases}$$

$\mathcal{W}_0^u(P)$ - Unstable manifold of P

Consider $\mathcal{W}_0^u(P)$ to be the manifold constructed by the union of the 1-dimensional unstable fibers of each saddle point in P. This manifold is locally given by:

$$uvwx\text{-space}\begin{cases} u & \in & \text{an open set,} \\ v & = & F_1(u,w,x), \\ w & \in & \text{an open set,} \\ x & \in & \text{an open set.} \end{cases} \quad uvzx\text{-space}\begin{cases} u & \in & \text{an open set,} \\ v & \in & F(u,z,x), \\ z & \in & \text{an open set,} \\ x & \in & \text{an open set.} \end{cases}$$

$\mathcal{W}_0^u(\mathcal{L}_{u^\ell})$ - Unstable manifold of the line $\mathcal{L}_{u^\ell} = \left\{(u^\ell, 0, 0, x) | x < 3(u^\ell)^2 - \frac{\alpha^2}{3\beta}\right\}$

The collection of the 1-dimensional unstable fibers of each point on $\mathcal{L}_{u^\ell}$ will foliate $\mathcal{W}_0^u(\mathcal{L}_{u^\ell})$. Locally, this manifold is given by:

$$uvwx\text{-space}\begin{cases} u & \in & \text{a neighborhood of } u^\ell, \\ v & = & F_1(u, f_1(u,x), x), \\ w & = & f_1(u,x), \\ x & \in & \text{an open set.} \end{cases} \quad uvzx\text{-space}\begin{cases} u & \in & \text{a neighborhood of } u^\ell, \\ v & = & F(u, z(x), x), \\ z & = & xu^\ell - (u^\ell)^3, \\ x & \in & \text{an open set.} \end{cases}$$

$\mathcal{W}_0^u(\mathcal{L}_{u^m})$ - Unstable manifold of the line $\mathcal{L}_{u^m} = \left\{(u^m, 0, 0, x) | x < 3(u^m)^2 - \frac{\alpha^2}{3\beta}\right\}$

The u^ℓ and u^r are chosen so that the Riemann solution consists of two shock waves. This determines the u^m value. After u^m is determined, we are able to consider the unstable fibers of each point located on the line $\mathcal{L}_{u^m}$. The collection of these fibers will be used to foliate the manifold, $\mathcal{W}_0^u(\mathcal{L}_{u^m})$. Locally this manifold is given by:

$$uvwx\text{-space}\begin{cases} u & \in & \text{a neighborhood of } u^m, \\ v & = & G_1(u, g_1(u,x), x), \\ w & = & g_1(u,x), \\ x & \in & \text{an open set.} \end{cases} \qquad uvzx\text{-space}\begin{cases} u & \in & \text{a neighborhood of } u^m, \\ v & = & G(u, z(x), x), \\ z & = & xu^m - (u^m)^3, \\ x & \in & \text{an open set.} \end{cases}$$

$\mathcal{W}_0^s(\mathcal{L}_{u^r})$ **- Stable manifold of the line** $\mathcal{L}_{u^r} = \left\{ (u^r, 0, 0, x) | x > 3(u^r)^2 + \frac{\alpha^2}{3\beta} \right\}$

Each point on the line $\mathcal{L}_{u^r}$ is associated with a 2-dimensional stable eigenspace. The 2-dimensional stable space at each point has a two-dimensional invariant manifold that is tangent to it. Each of these 2-dimensional manifolds are considered to be a stable fiber of its corresponding point. The collection of these fibers will foliate $\mathcal{W}_0^s(\mathcal{L}_{u^r})$. The point $(u^r, 0, 0, x^*)$ in $uvxx$-space corresponds to the point $(u^r, 0, u^r x^* - (u^r)^3, x^*)$ in $uvzx$-space, and the stable fiber of the latter is clearly the space (u, v, z^*, x^*) where $z^* = u^r x^* - (u^r)^3$. Therefore, the manifold is locally given by:

$$uvwx\text{-space}\begin{cases} u & \in & \text{a neighborhood of } u^r, \\ v & \in & \text{an open set,} \\ w & = & \frac{1}{\beta}\left(x(u^r - u) + u^3 - (u^r)^3 - \alpha v\right), \\ x & \in & \text{an open set.} \end{cases}$$

$$uvzx\text{-space}\begin{cases} u & \in & \text{a neighborhood of } u^r, \\ v & \in & \text{an open set,} \\ z & = & xu^r - (u^r)^3, \\ x & \in & \text{an open set.} \end{cases}$$

Theorem 1 *For some fixed* $\hat{s} > 0$, *let* $L_1 = \{x : |x - s_1| > \hat{s}\}$, *and let* $L_2 = \{x : |x - s_2| > \hat{s}$ *and* $|x - s_3| > \hat{s}\}$. *For* $\beta > 0$, *consider* $\Gamma_1 = \left\{ (u^\ell, u^r) \middle| 0 < u^\ell < \frac{2\alpha\sqrt{2}}{3\sqrt{\beta}} \text{ and } -\frac{u^\ell}{2} < u^r < u^\ell \right\}$ *and* $\Gamma_2 = \left\{ (u^\ell, u^r) \middle| u^\ell > \frac{2\alpha\sqrt{2}}{3\sqrt{\beta}} \text{ and } u^r > -u^\ell + \frac{\alpha\sqrt{2}}{3\sqrt{\beta}} \right\}$. *Let* $(u^\ell, u^r) \in \Gamma_i$. *Then for small* $\epsilon > 0$ *there is a Riemann–Dafermos solution* $\hat{u}_\epsilon(x)$ *of* (1.6)*; i.e., a solution of* (3.6), (3.5). *As* $\epsilon \to 0$, $\hat{u}_\epsilon(x)$ *approaches the Riemann solution of* (1.3), (2.3) *in the* C^1 *sense on* L_i.

This theorem relies on information about the transversality of the manifolds, Fenichel's Theorems, [14], and the Exchange Lemma. For convenience we prove the supporting lemmas of this theorem in the $uvzx$-coordinates. After the supporting lemmas are verified, we then prove Theorem 1, in the $uvwx$-coordinates.

3.3.2 Saddle-to-Attractor Connections

Observe that the equilibrium (u, v, w, x)=$(u^*, 0, 0, x)$ corresponds to (u, v, z, x)= $(u^*, 0, xu^* - (u^*)^3, x)$. For $\epsilon = 0$, given $(u^\ell, u^r) \in \Gamma_1$, there may exist a x-value, s_1, such that there is a solution to (3.22)-(3.25) that serves as a direct connection from $(u^\ell, 0, x_1 u^\ell - (u^\ell)^3, s_1)$ to $(u^r, 0, s_1 u^r - (u^r)^3, s_1)$. Let's denote this solution by the curve $(\bar{u}^1(\xi), \bar{v}^1(\xi), z, s_1)$. Please note that $z = s_1 u^\ell - (u^\ell)^3$ takes the same value as $z = s_1 u^r - (u^r)^3$; i.e. we have $s_1 u^\ell - (u^\ell)^3 = s_1 u^r - (u^r)^3$. This follows from the fact that z is constant on solutions of (3.22)-(3.25).

Similarly, after discovering the value for u^m for the given $(u^\ell, u^r) \in \Gamma_2$, there may exist a x-value, s_3, such that there is a solution to (3.22)-(3.25) that serves as a direct connection from $(u^m, 0, s_3 u^m - (u^m)^3, s_3)$ to $(u^r, 0, s_3 u^r - (u^r)^3, s_3)$ for $\epsilon = 0$. Denote this curve by $(\bar{u}^3(\xi), \bar{v}^3(\xi), z, s_3)$, where $z = s_3 u^m - (u^m)^3 = s_3 u^r - (u^r)^3$.

Lemma 1 *$\mathcal{W}_0^u(\mathcal{L}_{u^\ell})$ intersects $\mathcal{W}_0^s(\mathcal{L}_{u^r})$ transversally along the curve $(\bar{u}^1(\xi), \bar{v}^1(\xi), z, s_1)$ when the underlying Riemann solution consist of one shock wave.*

Lemma 2 *$\mathcal{W}_0^u(\mathcal{L}_{u^m})$ intersects $\mathcal{W}_0^s(\mathcal{L}_{u^r})$ transversally along the curve $(\bar{u}^3(\xi), \bar{v}^3(\xi), z, s_3)$ when the underlying Riemann solution consist of two shock waves.*

Proof (Lemma 1). Since the proofs of both lemmas are very similar, we will just prove one. The manifolds $\mathcal{W}_0^u(\mathcal{L}_{u^\ell})$ and $\mathcal{W}_0^s(\mathcal{L}_{u^r})$ are 2 and 3 dimensional respectively. In a 4 dimensional space, if the two intersect transversally, then they will intersect in a 1-dimensional manifold. The two manifolds intersect along the curve $(\bar{u}^1(\xi), \bar{v}^1(\xi), z, s_1)$. We will show that for any point on this curve, the tangent vectors of $\mathcal{W}_0^u(\mathcal{L}_{u^\ell})$ and $\mathcal{W}_0^s(\mathcal{L}_{u^r})$ together will span $\mathbb{R}^4$. We will show this result in the $uvzx$-space.

The tangent space to $\mathcal{W}_0^u(\mathcal{L}_{u^\ell})$ at a point is the span of

$$\left\{ \begin{pmatrix} 1 \\ F_u \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ F_x + u^\ell F_z \\ u^\ell \\ 1 \end{pmatrix} \right\} = \{q_1, q_2\}.$$

The tangent space to $\mathcal{W}_0^s(\mathcal{L}_{u^r})$ at a point is the span of

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u^r \\ 1 \end{pmatrix} \right\} = \{q_3, q_4, q_5\}.$$

Since the vector $(1, F_u, 0, 0)^T$ is dependent on the other vectors, we only need to verify that the other four vectors are linearly independent. Assume that $\sum_{i=1}^4 c_i q_{i+1} = 0$.

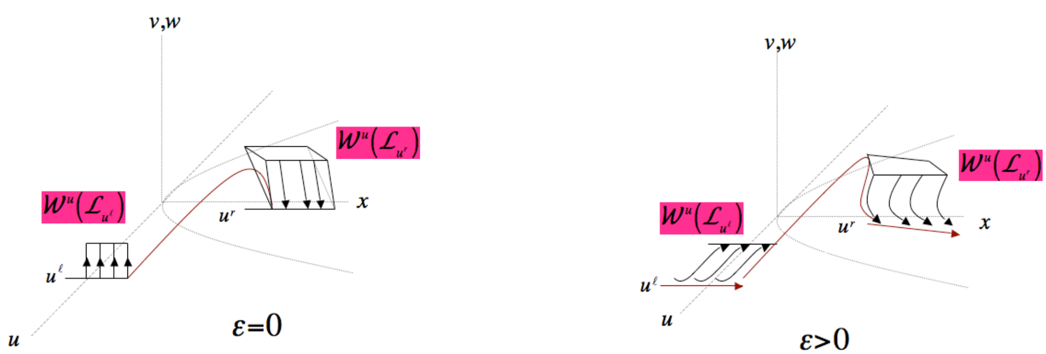
$$\Rightarrow \left\{ \begin{array}{ccc} c_2 & = & 0, \\ c_1(F_x + u^\ell F_z) + c_3 & = & 0, \\ c_1 u^\ell + c_4 u^r & = & 0, \\ c_1 + c_4 & = & 0. \end{array} \right. \qquad \Rightarrow \left\{ \begin{array}{ccc} c_2 & = & 0, \\ c_3 & = & -c_1(F_x + u^\ell F_z), \\ c_4 & = & -c_1, \\ c_1(u^\ell - u^r) & = & 0. \end{array} \right.$$

Since $u^\ell \neq u^r$, $c_1 = 0$. Hence, all $c_i = 0$ for $i = 1, 2, 3, 4$. Therefore, $\{q_i\}_{i=2}^5$ will span $\mathbb{R}^4$. □

Theorem 2 (Fenichel's First and Second Theorem) *Let $\Pi \in \mathbb{R}^n$ and $\epsilon \in \mathbb{R}$. Let M_0 be an m-dimensional normally hyperbolic locally invariant manifold with compact closure for the differential equation $\Pi' = f(\Pi, \epsilon)$ when $\epsilon = 0$. Let $\mathcal{W}^u(M_0)$ and $\mathcal{W}^s(M_0)$ be the unstable and stable manifolds of M_0 respectively, with dimensions $m+k$ and $m+\ell$, where $m+k+\ell = n$. Then for small $\epsilon > 0$ there exist locally invariant manifolds M_ϵ, $\mathcal{W}^u(M_\epsilon)$, and $\mathcal{W}^s(M_\epsilon)$ of dimensions m, $m+k$, and $m+\ell$ respectively, that are respectively C^1 near $M_0, \mathcal{W}^u(M_0)$ and $\mathcal{W}^s(M_0)$.*

Theorem 3 (Fenichel's Third Theorem) *For $\epsilon = 0$ and $\Pi \in \mathbb{R}^n$, let M_0 be a normally hyperbolic locally invariant manifold for $\Pi' = f(\Pi, \epsilon)$. When $\epsilon \neq 0$, the nearby manifold given by Theorem 2 is denoted M_ϵ. Consider the yab-space where the y-, a-, and b- spaces are m-, k-, and ℓ- dimensional respectively with $m+k+\ell = n$. In appropriate yab-coordinates, M_ϵ is given by $a = b = 0$, the unstable manifold, $\mathcal{W}^u(M_\epsilon)$, is given by $b = 0$, and the stable manifold, $\mathcal{W}^s(M_\epsilon)$, is given by $a = 0$. Also in these coordinates, y' depends only on y and ϵ when either a or b equals 0. If M_0 consists of only equilibria, then the differential equation on M_ϵ can be written as*

$$y' = \epsilon f(y, \epsilon) + g(y, a, b, \epsilon)ab, \tag{3.26}$$
$$a' = A(y, a, b, \epsilon)a, \tag{3.27}$$
$$b' = B(y, a, b, \epsilon)b. \tag{3.28}$$



(a) $\epsilon = 0$ (b) $\epsilon > 0$

Figure 3.5: Three dimensional representation of the direct connection from $\mathcal{L}_{u^\ell}$ to $\mathcal{L}_{u^r}$. When $\epsilon > 0$, the solution asymptotically converges back to $\mathcal{L}_{u^\ell}$ and forward to $\mathcal{L}_{u^r}$ for large $|x|$.

For the system (3.26)-(3.28), for any small ϵ, the point $(y^0, 0, 0) \in M_\epsilon$ has for its unstable fiber the set $\{(y, a, b) | y = y^0, b = 0\}$, and for its stable fiber the set $\{(y, a, b) | y = y^0, a = 0\}$. These sets are independent of ϵ. Hence, for $\Pi' = f(\Pi, \epsilon)$, the unstable and stable fibers depend in a C^1 manner on the base point and ϵ.

Proof of Theorem 1 (One Wave Riemann Solution). For $\epsilon = 0$, the set P defined earlier is a normally hyperbolic manifold of equilibria for (3.11)-(3.14). By [14], for small $\epsilon > 0$ there is a nearby normally hyperbolic locally invariant manifold. This manifold is again P with its points not given as equilibria. The line $\mathcal{L}_{u^\ell}$ is contained in P. It is invariant for each ϵ, and its unstable manifold $\mathcal{W}^u_\epsilon(\mathcal{L}_{u^\ell})$ is the union of the unstable fibers of its points. Therefore $\mathcal{W}^u_\epsilon(\mathcal{L}_{u^\ell})$ is C^1 close to $\mathcal{W}^u_0(\mathcal{L}_{u^\ell})$. Similarly, for small ϵ, we conclude that $\mathcal{W}^s_\epsilon(\mathcal{L}_{u^r})$ is C^1 close to $\mathcal{W}^s_0(\mathcal{L}_{u^r})$.

Since $\mathcal{W}^u_0(\mathcal{L}_{u^\ell})$ and $\mathcal{W}^s_0(\mathcal{L}_{u^r})$ meet transversally along the solution $(\bar{u}^1(\xi), \bar{v}^1(\xi), \bar{w}^1(\xi), s_1)$, for small $\epsilon > 0$, $\mathcal{W}^u_\epsilon(\mathcal{L}_{u^\ell})$ and $\mathcal{W}^s_\epsilon(\mathcal{L}_{u^r})$ meet transversally as well. The intersection is a solution $(\bar{u}^1_\epsilon(\xi), \bar{v}^1_\epsilon(\xi), \bar{w}^1_\epsilon(\xi), \epsilon\xi)$ that asymptotically approaches $\mathcal{L}_{u^\ell}$ as $\xi \to -\infty$ and asymptotically approaches $\mathcal{L}_{u^r}$ as $\xi \to \infty$. The Riemann–Dafermos solution is $\hat{u}_\epsilon(x) = \bar{u}^1_\epsilon(\frac{x}{\epsilon})$. $\square$

3.3.3 Saddle-to-Saddle Connection

There exists a unique $x = s_2$ that provides a connection of (3.22)-(3.25) from $\mathcal{L}_{u^\ell}$ to P for $\epsilon = 0$. This connection goes from $(u^\ell, 0, s_2u^\ell - (u^\ell)^3, s_2)$ to $(u^m, 0, s_2u^m - (u^m)^3, s_2)$. Denote this curve as $(\bar{u}^2(\xi), \bar{v}^2(\xi), z, s_2)$, where $z = s_2u^\ell - (u^\ell)^3 = s_2u^m - (u^m)^3$.

Lemma 3 *$W^u_0(\mathcal{L}_{u^\ell})$ intersects $\mathcal{W}^s_0(P)$ tranversally along the curve $(\bar{u}^2(\xi), \bar{v}^2(\xi), z, s_2)$ when the underlying Riemann solution consists of two shock waves.*

Proof. Based on the dimensions of the two manifolds, we expect an intersection of dimension 1. The intersection is given by the curve $(\bar{u}^2(\xi), \bar{v}^2(\xi), z, s_2)$. The tangent space to $W^u_0(\mathcal{L}_{u^\ell})$ at a point is given by the span of

$$\left\{ \begin{pmatrix} 1 \\ F_u \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ F_x + u^\ell F_z \\ u^\ell \\ 1 \end{pmatrix} \right\} = \{p_1, p_2\}.$$

Please note that for a given x value, $z = xu^\ell - (u^\ell)^3$. The tangent space to $W^s_0(P)$ at a point is given by the span of

$$\left\{ \begin{pmatrix} 1 \\ G_u \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ G_z \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ G_x \\ 0 \\ 1 \end{pmatrix}, \right\} = \{q_1, p_3, p_4\}.$$

Remark 1 *On the curve in the intersection of the two manifolds of concern, $v = F(u, z, x) = F(u, s_2u^\ell - (u^\ell)^3, s_2)$ must be the same as $v = G(u, z, x) = G(u, s_2u^\ell - (u^\ell)^3, s_2)$. Thus, on this common curve, $F_u = G_u$.*

Suppose $\sum_{i=1}^4 c_ip_i = 0$.

$$\Rightarrow \left\{ \begin{array}{rcl} c_1 & = & 0, \\ c_2(F_x + u^\ell F_z) + c_3G_z + c_4G_x & = & 0, \\ c_3 & = & -c_2u^\ell, \\ c_4 & = & -c_2. \end{array} \right. \Rightarrow \ c_2(F_x + u^\ell F_z - G_x - G_zu^\ell) \ = \ 0.$$

Remark 2 *To analyze the equation $c_2(F_x + u^\ell F_z - G_x - G_z u^\ell) = 0$, we need to consider a separation function denoted by $S(x)$. In the uvzx-space, our system of equations are defined on the uv-space since both x and z are constant in the system. In the uv-space, we can consider the line $\mathcal{L}_{u^0} = \{(u^0, v)\}$ for some fixed u^0. The separation function $S(x)$ is designed to measure the distance between $\mathcal{W}_0^u(\mathcal{L}_{u_\ell})$ and $\mathcal{W}_0^s(P)$. That is, it considers the v coordinate values of each manifold intersected with the line $\mathcal{L}_{u^0}$, then measures the difference of the two v-values. So $S(x) = F(u^0, xu^\ell - (u^\ell)^3, x) - G(u^0, xu^\ell - (u^\ell)^3, x)$. The derivative is given by $S'(x) = (F_x + u^\ell F_z) - (G_x + u^\ell G_z)$. Since there is only one x-value, s_2, that gives a connection between the two manifolds, we know that $S(s_2) = 0$. Thus, we are interested in $S'(s_2)$. $S'(s_2)$ can be computed explicitly as a positive multiple of the Melnikov integral. Thus, if the Melinkov integral is nonzero, $S'(s_2)$ is nonzero as well.*

Recall the system (3.22)-(3.23)

$$\begin{aligned} u_\xi &= v, \\ v_\xi &= \frac{1}{\beta}(u^3 - xu + z) - \frac{\alpha}{\beta}v, \end{aligned}$$

where $z = xu^\ell - (u^\ell)^3$. This system may be viewed as

$$\begin{aligned} u_\xi &= f(u, v, x), \\ v_\xi &= g(u, v, x). \end{aligned}$$

The matrix of the linerization at the arbitrary point (u, v) is

$$\hat{J} = \begin{bmatrix} 0 & 1 \\ \frac{1}{\beta}(3u^2 - x) & -\frac{\alpha}{\beta} \end{bmatrix}.$$

So the trace$(\hat{J}) = -\frac{\alpha}{\beta}$, $f_x = 0$, and $g_x = \frac{1}{\beta}(u^\ell - u)$. Let's compute $S'(s_2)$ by a positive multiple of the Melnikov integral.

$$\begin{aligned} S'(s_2) &= E \int_{-\infty}^{\infty} e^{-\int_0^\xi trace\hat{J}(\tau)d\tau} \begin{vmatrix} u_\xi & f_x(u, v, s_2) \\ v_\xi & g_x(u, v, s_2) \end{vmatrix} d\xi \\ &= E \int_{-\infty}^{\infty} e^{-\int_0^\xi trace\hat{J}(\tau)d\tau} (u_\xi g_x(u, v, s_2) - v_\xi f_x(u, v, s_2)) d\xi \\ &= E \int_{-\infty}^{\infty} e^{\int_0^\xi \frac{\alpha}{\beta}d\tau} \left(v \frac{1}{\beta}(u^\ell - u) \right) d\xi \\ &= \frac{E}{\beta} \int_{-\infty}^{\infty} e^{\frac{\alpha}{\beta}\xi} \left(v \left(u^\ell - u \right) \right) d\xi. \end{aligned}$$

We know from [6], that the connecting orbit from $(u^\ell, 0)$ is a parabola below the u-axis. This implies that $u_\xi = v < 0$, so we are moving along the parabola from right to left. Therefore, $(u^\ell, 0)$ is at the right of any point on the part of the parabola that we are interested in. Hence, $u^\ell - u > 0$. This gives us a negative integrand, which produces a negative integral, since the integrand is an increasing function. Therefore, $S'(s_2) \neq 0$.

Thus, $c_i = 0$ for $i = 1, 2, 3, 4$, and $\{p_i\}_1^4$ is a linearly independent set. Therefore, span $\{p_i\}_1^4 = \mathbb{R}^4$. □

Lemma 4 (Exchange Lemma) *Consider the system* $(3.26) - (3.28)$ *on yab-space where* $y, a,$ *and* b *are* $m-, k-,$ *and* $\ell-$ *dimensional respectively with* $m+k+\ell = n$*. In* $\mathbb{R}^n$*, for* $\epsilon = 0$ *let* Σ_0 *be an invariant manifold of dimension* $k+1$ *that transversally intersects the yb-plane at* $(y_0, 0, b_0)$*. Let* Σ_ϵ *be an invariant manifold of dimension* $k+1$ *as well for* $\epsilon > 0$*. Assume that as* $\epsilon \to 0, \Sigma_\epsilon \to \Sigma_0$*, in the* C^1 *sense. Then* Σ_ϵ *meets the yb-space at the point,* $(y_\epsilon, 0, b_\epsilon)$*, near* $(y_0, 0, b_0)$*, such that as* $\epsilon \to 0$*,* $(y_\epsilon, 0, b_\epsilon) \to (y_0, 0, b_0)$*.*

For $\epsilon = 0$*, consider the linear differential equation* $y' = f(y, 0)$ *on the y-space, and assume that* $f(y_0, 0) \neq 0$*. Let* $\phi(t)$ *be the solution of* $y' = f(y, 0)$ *with* $\phi(0) = y_0$*. For some* $t_0 > 0$ *and small* $\upsilon > 0$*, define* $I = [t_0 - \upsilon, t_0 + \upsilon]$ *and* $U_0 = \{(y, a, 0) \,|y = \phi(t), t \in I, \|a\| \leq \upsilon\}$*. Then for small* $\epsilon > 0$*,part of* Σ_ϵ *is* C^1 *close to* U_0*.*

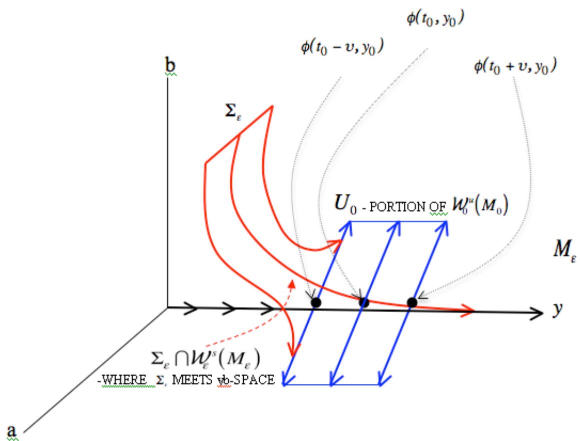


Figure 3.6: Graphical representation of the Exchange Lemma.

When the given Riemann solution consists of 2 shock waves for $\epsilon = 0$, there is no connection from $\mathcal{L}_{u^\ell}$ to $\mathcal{L}_{u^r}$. See Figure 3.7(a). When $\epsilon > 0$, we rely on the Exchange Lemma to track solutions on $\mathcal{W}_\epsilon^u(\mathcal{L}_{u^\ell})$ as ξ increase and on $\mathcal{W}_\epsilon^u(\mathcal{L}_{u^r})$ as ξ decrease. See Figure 3.3.3. This is done to find the one solution that is on both manifolds. This particular solution is the Riemann–Dafermos solution for $\xi = \frac{x}{\epsilon}$.

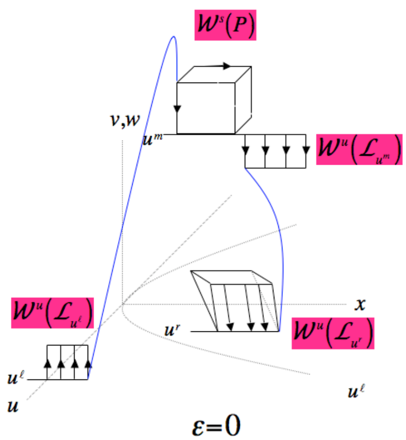
Proof of Theorem 1 (Two Waves Riemann Solution). We shall work in $uvwx$-space. Please note that $\mathcal{L}_{u^\ell}, P, \mathcal{L}_{u^m}$, and $\mathcal{L}_{u^r}$ are all contained in the ux-space which consists of only equilibria. From [14] and Fenichel's Theorems, we can conclude that for $\epsilon > 0$ sufficiently small, $\mathcal{W}_\epsilon^u(\mathcal{L}_{u^\ell}), \mathcal{W}_\epsilon^s(P)$, and $\mathcal{W}_\epsilon^s(\mathcal{L}_{u^r})$ are respectively C^1 close to $\mathcal{W}_0^u(\mathcal{L}_{u^\ell}), \mathcal{W}_0^s(P)$, and $\mathcal{W}_0^s(\mathcal{L}_{u^r})$.

In Fenichel's Third Theorem, let M_ϵ be P. Then in the Exchange Lemma, $P, \mathcal{W}_\epsilon^u(P)$, and $\mathcal{W}_\epsilon^s(P)$ correspond respectively to y-space, ya-space, and yb-space. In addition, let $\mathcal{W}_\epsilon^u(\mathcal{L}_{u^\ell})$ correspond to Σ_ϵ. Recall that the dimensions of $\mathcal{W}_0^u(\mathcal{L}_{u^\ell})$ and $\mathcal{W}_0^s(P)$ are 2 and 3 respectively. So the two spaces intersect in a 1-dimentional space. By Lemma 3, $\mathcal{W}_0^u(\mathcal{L}_{u^\ell})$ intersects $\mathcal{W}_0^s(P)$ transversally along the curve $(\bar{u}^2(\xi), \bar{v}^2(\xi), \bar{w}^2(\xi), x_2)$, which is the stable fiber of $(u^m, 0, 0, s_2)$. By Fenichel's Theorems, $\mathcal{W}^u_\epsilon(\mathcal{L}_{u^\ell})$ intersect $\mathcal{W}^s_\epsilon(P)$ transversally along the curve $(\bar{u}^2_\epsilon(\xi), \bar{v}^2_\epsilon(\xi), \bar{w}^2_\epsilon(\xi), \epsilon\xi)$.

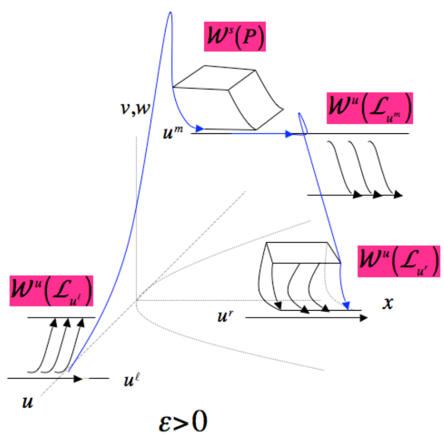
The system (3.11)–(3.14) on P, which is a portion of ux-space, is just $\dot{u} = 0$, $\dot{x} = \epsilon$. Now the solution of $\dot{u} = 0, \dot{x} = 1$, with $(u, x)(0) = (u^m, 0)$, is $(u, x)(\tau) = (u^m, \tau)$. Thus the solution runs along the line $\mathcal{L}_{u^m}$ and the x-coordinate increases as τ increases.

For a small $c^* > 0$, let I be the portion of $\mathcal{L}_{u^m}$ with $s_2 - c^* < x < s_3 + c^*$. Since $s_3 > s_2$, the Exchange Lemma implies that for small $\epsilon > 0$, part of $\mathcal{W}^u_\epsilon(\mathcal{L}_{u^\ell})$ is C^1 close to $\mathcal{W}^u_0(I)$. Now $\mathcal{W}^u_0(\mathcal{L}_{u^m})$, and hence $\mathcal{W}^u_0(I)$, meets $\mathcal{W}^s_0(\mathcal{L}_{u^r})$ transversally in a curve located in the unstable fiber of $(u_m, 0, 0, s_3)$.

Observe that the Exchange Lemma gives the result that for small $\epsilon > 0$, part of $\mathcal{W}^u_\epsilon(\mathcal{L}_{u^\ell})$ is C^1 close to $\mathcal{W}^u_0(I)$. Recall that for small $\epsilon > 0, \mathcal{W}^s_\epsilon(\mathcal{L}_{u^r})$ is C^1-close to $\mathcal{W}^s_0(\mathcal{L}_{u^r})$. This implies that $\mathcal{W}^u_\epsilon(\mathcal{L}_{u^\ell})$ meets $\mathcal{W}^s_\epsilon(\mathcal{L}_{u^r})$ transversally. The u-component of the curve of intersection, after the substitution $\xi = \frac{x}{\epsilon}$, is the Riemann–Dafermos solution. □



(a) $\epsilon = 0$



(b) $\epsilon > 0$

Figure 3.7: When $\epsilon = 0$, there is no connection from $\mathcal{L}_{u^\ell}$ to $\mathcal{L}_{u^r}$. For $\epsilon > 0$, shown in blue are the corresponding curves in the two pairs of intersecting manifolds.

Chapter 4

Linearization and Large Eigenvalues

When studying the stability of dynamical systems, it is common to use the linearization method. We begin by linearizing (3.6) about the Riemann–Dafermos solution. We will then study the large eigenvalues associated with this linearization.

Let $h(\epsilon, x, t) = \hat{u}_\epsilon(x) + \hat{U}(x, t)$, where $\hat{u}_\epsilon(x)$ is the Riemann–Dafermos solution. Plugging $h(\epsilon, x, t)$ into the reguarization (1.6) yields

$$\hat{U}_t + (6\hat{u}_\epsilon\hat{U} + 3\hat{U}^2)\hat{u}'_\epsilon + (3\hat{u}_\epsilon^2 - x)\hat{U}_x + (6\hat{u}_\epsilon\hat{U} + 3\hat{U}^2)\hat{U}_x \quad = \quad \alpha\epsilon\hat{U}_{xx} + \beta\epsilon^2\hat{U}_{xxx}.$$

Disregarding the nonlinear terms with respect to $\hat{U}$ produces

$$\hat{U}_t + 6\hat{u}_\epsilon\hat{u}'_\epsilon\hat{U} + (3\hat{u}_\epsilon^2 - x)\hat{U}_x = \alpha\epsilon\hat{U}_{xx} + \beta\epsilon^2\hat{U}_{xxx}. \tag{4.1}$$

Equation (4.1) is the linearization of the partial differential equation (3.6) at our Riemann–Dafermos solution $\hat{u}_\epsilon(x)$. We look for solutions of the form $\hat{U}(x, t) = e^{\lambda t}U(x)$.

Substituting in the linear partial differential equation gives rise to

$$\lambda U + (3\hat{u}_\epsilon^2 - x)U_x + 6\hat{u}_\epsilon\hat{u}'_\epsilon U \quad = \quad \alpha\epsilon U_{xx} + \beta\epsilon^2 U_{xxx}. \tag{4.2}$$

The corresponding system is

$$\begin{aligned}
\epsilon U_x &= V, \\
\epsilon V_x &= W, \\
\epsilon W_x &= \frac{1}{\beta}[\epsilon\lambda U + (3\hat{u}_\epsilon^2 - x)V + 6\epsilon\hat{u}_\epsilon\hat{u}'_\epsilon U - \alpha W].
\end{aligned}$$

Substituting $x = \epsilon\xi$ into the system, produces

$$\begin{aligned}
U_\xi &= V, \\
V_\xi &= W, \\
W_\xi &= \frac{1}{\beta}(\epsilon\lambda U + (3u^2 - x)V + 6uu_\xi U - \alpha W).
\end{aligned}$$

Introducing $\rho = \epsilon\lambda$, we can merge together the linear system with the fast system, (3.11)–(3.14) and obtain the following 8-dimensional system

$$\begin{aligned}
u_\xi &= v, && (4.3)\\
v_\xi &= w, && (4.4)\\
w_\xi &= \frac{1}{\beta}(3u^2 - x)v - \frac{\alpha}{\beta}w, && (4.5)\\
x_\xi &= \epsilon, && (4.6)\\
U_\xi &= V, && (4.7)\\
V_\xi &= W, && (4.8)\\
W_\xi &= \frac{1}{\beta}[\rho U + (3u^2 - x)V + 6uvU - \alpha W], && (4.9)\\
\rho_\xi &= 0 && (4.10)
\end{aligned}$$

For $\epsilon > 0$, let $x = \epsilon\xi$, and define $u_\epsilon(\xi) = \hat{u}_\epsilon(\epsilon\xi)$ where $\hat{u}_\epsilon(x)$ is a solution of (3.6),(3.5), a Riemann–Dafermos solution of (1.6). Then for each $\rho \in \mathbb{C}$, the system (4.3)–(4.10) has the solution $(u_\epsilon(\xi), v_\epsilon(\xi), w_\epsilon(\xi), \epsilon\xi, 0, 0, 0, \rho)$. If for some ρ the system (4.3)–(4.10) has another solution $(u_\epsilon(\xi), v_\epsilon(\xi), w_\epsilon(\xi), \epsilon\xi, U(\xi), V(\xi), W(\xi), \rho)$ such that $(U, V, W) \to (0, 0, 0)$ as $\xi \to \pm\infty$ with $U(\xi) \not\equiv 0$, then for $\lambda = \frac{\rho}{\epsilon}$, $\Big(\lambda, U(\epsilon\xi)\Big)$ is an eigenpair for the linear equation (4.2).

For $\epsilon = 0$, the equilibria of this system are dependent on the value of ρ. For $\rho \neq 0$, the equilibria are the points in $ux\rho$-space. For $\rho = 0$, the equilibria are the points in uxU-space. In addition when $\epsilon = 0$, $u(\xi)$ is a travelling wave solution for (1.1) with the speed x. Dodd shows in [5] that the only eigenvalue with real part nonnegative of the linearization of (1.1) at a traveling wave is the simple eigenvalue 0. Thus, for $\epsilon = 0$ and each ρ with real part nonpositive, the system (4.3)–(4.10) has the solution $(u(\xi), v(\xi), w(\xi), x, U(\xi), V(\xi), W(\xi), \rho)$ such that $(U, V, W) \to (0, 0, 0)$ as $\xi \to \pm\infty$ with $U(\xi) \not\equiv 0$. For ρ with positive real part, the solution of (4.3)–(4.10) is $(u(\xi), v(\xi), w(\xi), x, 0, 0, 0, \rho)$.

4.0.4 Jacobian

The Jacobian evaluated at one of the equilibria for $\rho \neq 0$ is

$$J(u, 0, 0, x, 0, 0, 0, \rho) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{3u^2 - x}{\beta} & -\frac{\alpha}{\beta} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & \frac{\rho}{\beta} & \frac{3u^2 - x}{\beta} & -\frac{\alpha}{\beta} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

Notice that this matrix is in the form of

$$\begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix},$$

where $A_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{\beta}(3u^2 - x) & -\frac{\alpha}{\beta} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, and $A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\rho}{\beta} & \frac{3u^2-x}{\beta} & -\frac{\alpha}{\beta} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. We will denote the matrix of the principal minor $[A_1]_{4,4}$ as $A = A(\rho, u, x) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{\rho}{\beta} & \frac{3u^2-x}{\beta} & -\frac{\alpha}{\beta} \end{pmatrix}$.
The spectrum of $J(u, 0, 0, x, 0, 0, 0, \rho)$ is simply the union of the spectra of A_0 and of A_1. Since the spectrum of A_0 is already computed, we only need to determine the spectrum of A_1. The characteristic polynomial of A_1 is given by $\mathcal{C}_{A_1}(\mu) = \mu\left(\mu^3 + \frac{\alpha}{\beta}\mu^2 - \frac{3u^2-x}{\beta}\mu - \frac{\rho}{\beta}\right)$. The nonzero eigenvalues of A_1 and the eigenvalues of A are given as the roots of the cubic $g(\mu) = \mu^3 + \frac{\alpha}{\beta}\mu^2 - \frac{3u^2-x}{\beta}\mu - \frac{\rho}{\beta}$.

4.1 Eigenvalues of $A(\rho, u, x)$

We first find conditions on ρ for which $A(\rho, u, x)$ has an eigenvalue with 0 real part. To do this we let $\mu = bi$ with b real, and solve the equation $g(bi) = 0$ for ρ. We obtain $\rho = -\alpha b^2 - ib(\beta b^2 + 3u^2 - x)$. Therefore $Re(\rho) < 0$ unless $b = 0$, in which case $\rho = 0$.

This tells us that when $\mathrm{Re}(\rho) \geq 0$ and $\rho \neq 0$, $A(\rho, u, x)$ will have no pure imaginary eigenvalues. Therefore for $Re(\rho) \geq 0$ and $\rho \neq 0$, the number of eigenvalues of $A(\rho, u, x)$ with negative (respectively positive) real part never changes. We shall determine these numbers by considering $\rho \in \mathbb{R}^+$.

If the three roots of $g(\mu)$ are given by μ_1, μ_2, and μ_3, we can compare the product of the factors to the expression for $g(\mu)$ and obtain the following relationships

$$\begin{aligned} -\frac{\alpha}{\beta} &= \mu_1 + \mu_2 + \mu_3, \\ \frac{x - 3u^2}{\beta} &= \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3, \\ \frac{\rho}{\beta} &= \mu_1\mu_2\mu_3. \end{aligned} \tag{4.11}$$

Since both the leading coefficient of $g(\mu)$ and ρ are positive, then the vertical intercept is negative and one of the roots must be positive. If we assume that $\mu_1 > 0$, then $-\frac{\alpha}{\beta} - \mu_1 = \mu_2 + \mu_3$ and $\frac{\rho}{\mu_1\beta} = \mu_2\mu_3$. Since the sum of the remaining roots is negative and their product is positive, those two roots must have negative real part.

So for all $Re(\rho) > 0$, two eigenvalues of $A(\rho, u, x)$ will have real part negative and one will have real part positive. Dodd tells us in [5], we know that $\frac{\rho}{\epsilon}$ such that $Re(\rho) > 0$ is not an eigenvalue of the linearization of (1.1) at a travelling wave. We use this information to show that the $\frac{\rho}{\epsilon}$ values with $Re(\rho) > 0$ are not eigenvalues of the linear equation (4.2). For $\epsilon = 0$, we will investigate the behavior of the solutions of (4.3)–(4.10) when $\rho > 0$. We begin by discussing appropriate manifolds in $uvwxUVW\rho$-space, and their transversal intersections for $\rho > 0$.

4.2 Transversal Intersections for $\rho > 0$

For $\epsilon = 0$ and fixed $\rho = \rho^0 > 0$, we define the following manifolds of equilibria:

$$\mathcal{K} = \left\{ (u, 0, 0, x, 0, 0, 0, \rho^0) \Big| x < 3u^2 - \frac{\alpha^2}{3\beta} \right\},$$

$$\mathcal{K}_{u^{\ell/m}} = \left\{ (u^{\ell/m}, 0, 0, x, 0, 0, 0, \rho^0) \Big| x < 3(u^{\ell/m})^2 - \frac{\alpha^2}{3\beta} \right\},$$

$$\mathcal{K}_{u^r} = \left\{ (u^r, 0, 0, x, 0, 0, 0, \rho^0) \Big| x > 3(u^r)^2 + \frac{\alpha^2}{3\beta} \right\}.$$

Observe that if $\epsilon = 0$ and the $u(\xi)$ component of a solution of (4.3)–(4.6) approaches constants on the boundary, then $u(\xi)$ is a travelling wave solution of (1.1). Consider the point $(u^0, 0, 0, x^0, 0, 0, 0, \rho^0)$ where x^0 is contained either in $\{x|x < 3u^2 - \frac{\alpha^2}{3\beta}\}$ or $\{x|x > 3u^2 + \frac{\alpha^2}{3\beta}\}$. We will first describe the stable and unstable fibers of this point. We then will describe the relevant stable and unstable manifolds of those manifolds of equilibria mentioned above.

For $(u^1, v^1, w^1, x^0) \in \mathcal{W}_0^s(u^0, 0, 0, x^0)$, let $(u(\xi), v(\xi), w(\xi))$ be the solution of (4.3)–(4.5), with $(u(0), v(0), w(0)) = (u^1, v^1, w^1)$ and $x \equiv x^0$. Define $\mathcal{S}_0^s(u^1, v^1, w^1, x^0, \rho^0)$ as the set of all (U^1, V^1, W^1) such that if $(U(\xi), V(\xi), W(\xi))$ is the solution of (4.7)–(4.9) with $(U(0), V(0), W(0)) = (U^1, V^1, W^1)$ and $(u, v, w, x, \rho) = (u(\xi), v(\xi), w(\xi), x^0, \rho^0)$, then $(U(\xi), V(\xi), W(\xi)) \to (0, 0, 0)$ as $\xi \to \infty$. Each $\mathcal{S}_0^s(u^1, v^1, w^1, x^0, \rho^0)$ is a vector space of dimension 2, and $\mathcal{W}_0^s(u^0, 0, 0, x^0, 0, 0, 0, \rho^0)$ is the set of all points

$$(u^1, v^1, w^1, x^0, U^1, V^1, W^1, \rho^0) \text{ such that } (u^1, v^1, w^1, x^0) \in \mathcal{W}_0^s(u^0, 0, 0, x^0)$$

$$\text{and } (U^1, V^1, W^1) \in \mathcal{S}_0^s(u^1, v^1, w^1, x^0, \rho^0).$$

For $(u^1, v^1, w^1, x^0) \in \mathcal{W}_0^u(u^0, 0, 0, x^0)$, let $(u(\xi), v(\xi), w(\xi))$ be the solution of (4.3)–(4.5), with $(u(0), v(0), w(0)) = (u^1, v^1, w^1)$ and $x \equiv x^0$. Let $\mathcal{S}_0^u(u^1, v^1, w^1, x^0, \rho^0)$ be the set of all (U^1, V^1, W^1) such that if $(U(\xi), V(\xi), W(\xi))$ is the solution of (4.7)–(4.9) with $(U(0), V(0), W(0)) = (U^1, V^1, W^1)$ and $(u, v, w, x, \rho) = (u(\xi), v(\xi), w(\xi), x^0, \rho^0)$, then $(U(\xi), V(\xi), W(\xi)) \to (0, 0, 0)$ as $\xi \to -\infty$. Each $\mathcal{S}_0^u(u^1, v^1, w^1, x^0, \rho^0)$ is a vector space of dimension 1, and $\mathcal{W}_0^u(u^0, 0, 0, x^0, 0, 0, 0, \rho^0)$ is the set of all points

$$(u^1, v^1, w^1, x^0, U^1, V^1, W^1, \rho^0) \text{ such that } (u^1, v^1, w^1, x^0) \in \mathcal{W}_0^u(u^0, 0, 0, x^0)$$

$$\text{and } (U^1, V^1, W^1) \in \mathcal{S}_0^u(u^1, v^1, w^1, x^0, \rho^0).$$

The stable fiber of a point $(u,0,0,x,0,0,0,\rho^0)\in\mathcal{K}$ is given by $\mathcal{W}_0^s(u,0,0,x,0,0,0,\rho^0)$. The collection of all of the stable fibers of each point in $\mathcal{K}$ will foliate the stable manifold of $\mathcal{K}, \mathcal{W}_0^s(\mathcal{K})$. For a fixed ρ value, the dimension of $\mathcal{W}_0^s(\mathcal{K})$ is 5.

The unstable fiber of each point $(u^{\ell/m},0,0,x,0,0,0,\rho^0)\in\mathcal{K}_{u^{\ell/m}}$ is given by $\mathcal{W}_0^u(u^{\ell/m},0,0,x,0,0,0,\rho^0)$. The collection of all of the unstable fibers of each point in $\mathcal{K}_{u^{\ell/m}}$ will foliate the unstable manifold of $\mathcal{K}_{u^{\ell/m}}, \mathcal{W}_0^u(\mathcal{K}_{u^{\ell/m}})$. For a fixed ρ value, the dimension of $\mathcal{W}_0^u(\mathcal{K}_{u^{\ell/m}})$ is 3.

The stable fiber of a point $(u^r,0,0,x,0,0,0,\rho^0)\in\mathcal{K}_{u^r}$ is given by $\mathcal{W}_0^s(u^r,0,0,x,0,0,0,\rho^0)$. The collection of all of the stable fibers of each point in $\mathcal{K}_{u^r}$ will foliate the stable manifold of $\mathcal{K}_{u^r}, \mathcal{W}_0^s(\mathcal{K}_{u^r})$. For a fixed ρ value, the dimension of $\mathcal{W}_0^s(\mathcal{K}_{u^r})$is 5.

Theorem 4 *If ρ has positive real part, then for small $\epsilon>0$, the linear equation* (4.1) *does not have $\frac{\rho}{\epsilon}$ as an eigenvalue with an eigenfunction that approaches 0 as $x\to\pm\infty$.*

Proof. Recall when $\beta>0$ that the system (3.15)–(3.18) has 1 or 2 travelling wave solution(s). There is a possible travelling wave solution that starts out $\mathcal{W}_0^u(\mathcal{L}_{u^\ell})$ and ends in $\mathcal{W}_0^s(\mathcal{L}_{u^r})$. Alternatively, there are possible travelling wave solutions in which one starts in $\mathcal{W}_0^u(\mathcal{L}_{u^\ell})$ and ends in $\mathcal{W}_0^s(P)$, and the other starts in $\mathcal{W}_0^u(\mathcal{L}_{u^m})$ and ends in $\mathcal{W}_0^s(\mathcal{L}_{u^r})$.

When $\rho>0$, ρ is not an eigenvalue of the linearization of (1.1) at a travelling wave, (Dodd, [5]). Thus, $\mathcal{S}_0^u(u(\xi),v(\xi),w(\xi),s_*,\rho^0)$ meets $\mathcal{S}_0^s(u(\xi),v(\xi),w(\xi),s_*,\rho^0)$ transversally at the origin. When there is only one shock wave for the related Riemann solution, $\mathcal{W}_0^u(\mathcal{K}_{u^\ell})$ will meet $\mathcal{W}_0^s(\mathcal{K}_{u^r})$ transversally at the curve $(\bar{u}^1(\xi),\bar{v}^1(\xi),\bar{w}^1(\xi),s_1,0,0,0,\rho^0)$. When there are two shock waves for the corresponding Riemann solution, there will be two pairs of transverse intersections. $\mathcal{W}_0^u(\mathcal{K}_{u^\ell})$ will transversally intersect $\mathcal{W}_0^s(\mathcal{K})$ along the curve $(\bar{u}^2(\xi),\bar{v}^2(\xi),\bar{w}^2(\xi),s_2,0,0,0,\rho)$ and $\mathcal{W}_0^u(\mathcal{K}_{u^m})$ will transversally intersect $\mathcal{W}_0^s(\mathcal{K}_{u^r})$ along the curve $(\bar{u}^3(\xi),\bar{v}^3(\xi),\bar{w}^3(\xi),s_3,0,0,0,\rho)$.

It was shown with Theorem 1 that when $\epsilon>0$ is sufficiently small, there exists a curve, $(u_\epsilon(\xi),v_\epsilon(\xi),w_\epsilon(\xi),\epsilon\xi)$, which lies in the intersection of $\mathcal{W}_\epsilon^u(\mathcal{L}_{u^\ell})$ and $\mathcal{W}_\epsilon^s(\mathcal{L}_{u^r})$. The u component for $x=\epsilon\xi, \hat{u}_\epsilon(x)=u_\epsilon\left(\frac{x}{\epsilon}\right)$, is the Riemann–Dafermos solution. Using the Exchange Lemma, we can track solutions on a portion of $\mathcal{W}_\epsilon^u(\mathcal{K}_{u^\ell})$ to a portion of $\mathcal{W}_0^u(\mathcal{K}_{u^m})$. Recall that $\mathcal{W}_0^u(\mathcal{K}_{u^m})$ meets $\mathcal{W}_0^s(\mathcal{K}_{u^r})$ transversally. Also, we know by Fenichel's Theorems that $\mathcal{W}_\epsilon^s(\mathcal{K}_{u^r})$ is C^1 close to $\mathcal{W}_0^s(\mathcal{K}_{u^r})$. So $\mathcal{W}_\epsilon^s(\mathcal{K}_{u^r})$ can be tracked backwards to $\mathcal{W}_0^u(\mathcal{K}_{u^m})$. This leads us to the conclusion that $\mathcal{W}_\epsilon^u(\mathcal{K}_{u^\ell})$ meets $\mathcal{W}_\epsilon^s(\mathcal{K}_{u^r})$ transversally. Based on the dimensions and descriptions of these two manifolds in $ux\rho$-space, the intersection of $\mathcal{W}_\epsilon^u(\mathcal{K}_{u^\ell})$ with $\mathcal{W}_\epsilon^s(\mathcal{K}_{u^r})$ contains only the curve $(u_\epsilon(\xi),v_\epsilon(\xi),w_\epsilon(\xi),\epsilon\xi,0,0,0,\rho^0)$. Since $\lambda=\frac{\rho^0}{\epsilon}>0$ is associated with $U(\xi)\equiv 0$, λ is not an eigenvalue for the linear equation (4.2). $\square$

Chapter 5

Small Eigenvalues

Substituting $\rho = \epsilon\lambda$ back into the system (4.3)–(4.10) gives rise to the following system

$$\begin{aligned}
u_\xi &= v, && (5.1)\\
v_\xi &= w, && (5.2)\\
w_\xi &= \frac{1}{\beta}(3u^2 - x)v - \frac{\alpha}{\beta}w, && (5.3)\\
x_\xi &= \epsilon, && (5.4)\\
U_\xi &= V, && (5.5)\\
V_\xi &= W, && (5.6)\\
W_\xi &= \frac{1}{\beta}[\epsilon\lambda U + (3u^2 - x)V + 6uvU - \alpha W], && (5.7)\\
\lambda_\xi &= 0. && (5.8)
\end{aligned}$$

Studying (5.1)–(5.8) is equivalent to studying (4.3)–(4.10) for $\rho = \epsilon\lambda$, which is near 0 for small enough ϵ. Suppose the system (5.1)–(5.8) has a family of solutions

$$(u_\epsilon(\xi), v_\epsilon(\xi), w_\epsilon(\xi), \epsilon\xi, U(\epsilon,\xi), V(\epsilon,\xi), W(\epsilon,\xi), \lambda(\epsilon)), \qquad (5.9)$$

where we recall that $\hat{u}_\epsilon(x) = u_\epsilon(\frac{x}{\epsilon})$ is the Riemann–Dafermos solution. Then $\Big(\lambda(\epsilon), U\Big(\epsilon, \frac{x}{\epsilon}\Big)\Big)$ is an eigenpair for the linear equation (4.1) on any space to which $U\Big(\epsilon, \frac{x}{\epsilon}\Big)$ belongs.

Without loss of generality, we assume that the small eigenvalues take the form $\lambda_0 + \epsilon\lambda_1 + \mathcal{O}(\epsilon^2)$ for small $\epsilon > 0$. So for sufficently small $\epsilon < 1$, the eigenvalues are contained in a neighborhood about λ_0. We were able to calculate the only possible value of λ_0. If the Riemann solution consists of one shock wave and $\lambda \neq -1 + \epsilon\lambda_1 + \mathcal{O}(\epsilon^2)$, then the linear equation (4.1) does not have λ as an eigenvalue with an eigenfunction that approaches 0 rapidly as $x \to \pm\infty$. If the Riemann solution consists of two shocks and $\lambda \neq \lambda_0 + \epsilon\lambda_1 + \mathcal{O}(\epsilon^2)$ where λ_0 is given by the expression (5.66), the linear equation (4.1) does not have λ as an eigenvalue with an eigenfunction that approaches 0 rapidly as $x \to \pm\infty$. We will clarify what it means to "approach 0 rapidly" in subsection 5.1.2. By looking for solutions of (5.1)-(5.8) of the form (5.9), with the property that $(U, V, W) \to (0, 0, 0)$ rapidly as $\xi \to \pm\infty$, we were able to find more conditions on the eigenfunctions of (4.1). This is revealed at the end of the chapter.

5.1 Manifolds of equilibria

When $\epsilon = 0$, the space of equilibria is given by $uxU\lambda$-space. The Jacobian evaluated at one of those equilibria is given as the block $\text{diag}(A_0, A_0)$. The nonzero eigenvalues are the exact same ones as from (3.4), each having algebraic multiplicity of 2. So for $x < 3u^2 - \frac{\alpha^2}{3\beta}$, we have 4 zero eigenvalues, 2 negative eigenvalues, and 2 positive eigenvalues. When $x > 3u^2 + \frac{\alpha^2}{3\beta}$, we have 4 zero eigenvalues and 4 eigenvalues with negative real part.

Therefore, for $\epsilon = 0$ the manifolds of equilibria

$$\begin{aligned} \mathcal{M}_0^- &= \left\{(u, 0, 0, x, U, 0, 0, \lambda)\middle| x < 3u^2 - \frac{\alpha^2}{3\beta}\right\}, \\ \mathcal{M}_0^+ &= \left\{(u, 0, 0, x, U, 0, 0, \lambda)\middle| x > 3u^2 + \frac{\alpha^2}{3\beta}\right\}, \end{aligned}$$

are normally hyperbolic. By Fenichel's theorem these manifolds perturb to normally hyperbolic invariant manifolds $\mathcal{M}_\epsilon^\pm$ for small $\epsilon > 0$. The equations of these manifolds must take the form

$$\begin{aligned} v &= v(u, x, U, \lambda, \epsilon), \\ w &= w(u, x, U, \lambda, \epsilon), \\ V &= V(u, x, U, \lambda, \epsilon), \\ W &= W(u, x, U, \lambda, \epsilon). \end{aligned}$$

Of course we have

$$v(u, x, U, \lambda, 0) = w(u, x, U, \lambda, 0) = V(u, x, U, \lambda, 0) = W(u, x, U, \lambda, 0) = 0.$$

In fact we know $v(u, x, U, \lambda, \epsilon) = w(u, x, U, \lambda, \epsilon) = 0$ for any ϵ. We also know that $V(u, x, U, \lambda, \epsilon)$ and $W(u, x, U, \lambda, \epsilon)$ must be linear in U for fixed $(u, x, \lambda, \epsilon)$. Therefore, on $\mathcal{M}_\epsilon^\pm$, $V = A(u, x, \lambda, \epsilon)U$ and $W = B(u, x, \lambda, \epsilon)U$, where

$$\begin{aligned} A(u, x, \lambda, \epsilon) &= \Big(A^0(u, x, \lambda) + \epsilon A^1(u, x, \lambda) + \mathcal{O}(\epsilon^2)\Big), \\ B(u, x, \lambda, \epsilon) &= \Big(B^0(u, x, \lambda) + \epsilon B^1(u, x, \lambda) + \mathcal{O}(\epsilon^2)\Big). \end{aligned}$$

When $\epsilon = 0, A(u, x, \lambda, 0) = \mathcal{A}^0(u, x, \lambda)$ and $B(u, x, \lambda, 0) = \mathcal{B}^0(u, x, \lambda)$. Since on $\mathcal{M}_0^\pm$, $V(u, x, \lambda, 0)$=$W(u, x, \lambda, 0) \equiv 0$, both $A^0(u, x, \lambda)$ and $B^0(u, x, \lambda)$ are 0. So actually the equations of $\mathcal{M}_\epsilon^\pm$ are

$$\begin{aligned} V &= \Big(\epsilon A^1(u, x, \lambda) + \mathcal{O}(\epsilon^2)\Big)U, \\ W &= \Big(\epsilon B^1(u, x, \lambda) + \mathcal{O}(\epsilon^2)\Big)U. \end{aligned}$$

5.1.1 Differential Equation on $\mathcal{M}_\epsilon^\pm$

Let $Q^\pm(\xi,\epsilon)$ be respective solutions on $\mathcal{M}_\epsilon^\pm$. Assume that $Q^\pm(\xi,\epsilon)$ take the form

$$Q^\pm(\xi,\epsilon) = (u(\xi,\epsilon),(\xi,\epsilon),(\xi,\epsilon),x(\xi,\epsilon),U(\xi,\epsilon),V(\xi,\epsilon),W(\xi,\epsilon),\lambda(\epsilon)).$$

We can expand the U component of $Q^\pm(\xi,\epsilon)$ in powers of ϵ by $U = U_0(\xi)+\epsilon U_1(\xi)+\mathcal{O}(\epsilon^2)$. After differentiating, we obtain $\dot{U} = \dot{U}_0(\xi)+\epsilon\dot{U}_1(\xi)+\mathcal{O}(\epsilon^2)$. From (5.1)–(5.8), we know that $\dot{U} = V$. Thus,

$$\begin{aligned}
\dot{U}_0+\epsilon\dot{U}_1(\xi)+\mathcal{O}(\epsilon^2) &= \big(\epsilon A^1(u,x,\lambda)+\mathcal{O}(\epsilon^2)\big)U \\
&= \big(\epsilon A^1(u,x,\lambda)+\mathcal{O}(\epsilon^2)\big)\big(U_0(\xi)+\epsilon U_1(\xi)+\mathcal{O}(\epsilon^2)\big) \\
&= \epsilon A^1(u,x,\lambda)U_0(\xi)+\mathcal{O}(\epsilon^2) \qquad .
\end{aligned}$$

From this, we have

$$\begin{aligned}
\dot{U}_0 &= 0, \\
\dot{U}_1(\xi) &= A^1(u,x,\lambda)U_0(\xi).
\end{aligned}$$

Therefore, $U_0(\xi) = U_0$ is constant in ξ. In addition,

$$\begin{aligned}
\dot{V} &= \epsilon\big(D_uA^1u_\xi + D_xA^1x_\xi\big)U + \epsilon A^1(u,x,\lambda)\big(\dot{U}_0+\epsilon\dot{U}_1\big)+\mathcal{O}(\epsilon^2) \\
&= \epsilon\big(D_uA^1v + \epsilon D_xA^1\big)U + \epsilon A^1(u,x,\lambda)\dot{U}_0+\mathcal{O}(\epsilon^2) \\
&= \epsilon^2 D_xA^1U + \epsilon A^1(u,x,\lambda)\dot{U}_0+\mathcal{O}(\epsilon^2) \\
&= \epsilon A^1(u,x,\lambda)\dot{U}_0+\mathcal{O}(\epsilon^2) \\
&= 0+\mathcal{O}(\epsilon^2).
\end{aligned}$$

Similarly, $\dot{W} = 0+\mathcal{O}(\epsilon^2)$. From (5.1)–(5.8), we are able equate the two equations for V_ξ and obtain,

$$0+\mathcal{O}(\epsilon^2) = W = \epsilon B^1(u,x,\lambda)U+\mathcal{O}(\epsilon^2).$$

This reveals to us that $B^1(u,x,\lambda) = 0$. Therefore, the W components of $Q^\pm(\xi,\epsilon)$ are 0 up to the order of ϵ. Equating the two equations for $\dot{W}$, we obtain

$$\begin{aligned}
0+\mathcal{O}(\epsilon^2) &= \frac{1}{\beta}\Big(\epsilon\lambda U + (3u^2-x)V + 6uvU - \alpha W\Big) \\
&= \frac{1}{\beta}\Big(\epsilon\lambda U + (3u^2-x)\epsilon A^1(u,x,\lambda)U + 6uvU - \alpha\epsilon B^1(u,x)U + \mathcal{O}(\epsilon^2)\Big) \\
&= \frac{1}{\beta}\Big(\big(\lambda + (3u^2-x)A^1(u,x,\lambda)\big)\epsilon U + \mathcal{O}(\epsilon^2)\Big).
\end{aligned}$$

This informs us that $A^1(u,x,\lambda) = \frac{\lambda}{x-3u^2}$. Thus, the equations of $\mathcal{M}_\epsilon^\pm$ are

$$\begin{aligned}
v &= 0, \\
w &= 0, \\
V &= \frac{\epsilon\lambda}{x-3u^2}U+\mathcal{O}(\epsilon^2), \\
W &= \mathcal{O}(\epsilon^2).
\end{aligned}$$

The differential equations on $\mathcal{M}_\epsilon^{\pm}$ are

$$\begin{aligned} u_\xi &= 0, &(5.10)\\ x_\xi &= \epsilon, &(5.11)\\ U_\xi &= \frac{\epsilon\lambda}{x-3u^2}U + \mathcal{O}(\epsilon^2), &(5.12)\\ \lambda_\xi &= 0. &(5.13)\end{aligned}$$

5.1.2 The connection with solutions of (5.1)–(5.8).

Let

$$\mathcal{M}^{u^\ell} = \left\{ (u^\ell, 0, 0, x, 0, 0, 0, \lambda) \middle| x < 3(u^\ell)^2 - \frac{\alpha^2}{3\beta} \right\}.$$

Notice that $\mathcal{M}^{u^\ell}$ has dimension 2 and is an invariant subset of $\mathcal{M}_\epsilon^-$ for every ϵ. (In making dimension counts, we shall ignore the fact that λ is complex and not real and that $(U, V, W) \in \mathbb{C}^3$ and not $\mathbb{R}^3$. Considering this information simply doubles the dimension counts for λ and (U, V, W).) Let

$$\mathcal{M}^{u^r} = \left\{ (u^r, 0, 0, x, 0, 0, 0, \lambda) \middle| x > 3(u^r)^2 + \frac{\alpha^2}{3\beta} \right\}.$$

Similarly, $\mathcal{M}^{u^r}$ has dimension 2 and is an invariant subset of $\mathcal{M}_\epsilon^+$ for every ϵ. We are interested in solutions (5.9) of (5.1)–(5.8) that approach $\mathcal{M}^{u^\ell}$ as $\xi \to -\infty$ and $\mathcal{M}^{u^r}$ as $\xi \to \infty$. If we can find one with $U(\epsilon, \xi) \not\equiv 0$, then $U(\epsilon, \xi)$ is an eigenfunction.

For $(u^2, v^2, w^2, x^0) \in \mathcal{W}_0^s(u^r, 0, 0, x^0)$, let $(u(\xi), v(\xi), w(\xi))$ be the solution of (5.1)–(5.3), with $(u(0), v(0), w(0)) = (u^2, v^2, w^2)$ and $x \equiv x^0$. Let $\mathcal{S}_0^s(u^2, v^2, w^2, x^0)$ be the set of all (U^2, V^2, W^2) such that if $(U(\xi), V(\xi), W(\xi))$ is the solution of (5.5)–(5.7) when $\epsilon = 0$, with $(U(0), V(0), W(0)) = (U^2, V^2, W^2)$ and $(u, v, w, x) = (u(\xi), v(\xi), w(\xi), x^0)$, then $(U(\xi), V(\xi), W(\xi)) \to (0, 0, 0)$ as $\xi \to \infty$. Each $\mathcal{S}_0^s(u^2, v^2, w^2, x^0)$ is a vector space of dimension 2, and $\mathcal{W}_0^s(u^r, 0, 0, x^0, 0, 0, 0, \lambda_0)$ is the set of all points

$$(u^2, v^2, w^2, x^0, U_2, V_2, W_2, \lambda_0) \text{ such that } (u^2, v^2, w^2, x^0) \in \mathcal{W}_0^s(u^r, 0, 0, x^0)$$

$$\text{and } (U^2, V^2, W^2) \in \mathcal{S}_0^s(u^2, v^2, w^2, x^0).$$

The stable fiber of a point $(u^r, 0, 0, x, 0, 0, 0, \lambda) \in \mathcal{M}^{u^r}$ is a 4-dimensional manifold and is given by $\mathcal{W}_0^s(u^r, 0, 0, x, 0, 0, 0, \lambda)$. The collection of these fibers foliates the 6-dimensional manifold $\mathcal{W}_0^s(\mathcal{M}^{u^r})$.

For $(u^2, v^2, w^2, x^0) \in \mathcal{W}_0^u(u_2, 0, 0, x^0)$, let $(u(\xi), v(\xi), w(\xi))$ be the solution of (5.1)–(5.3), with $(u(0), v(0), w(0)) = (u^2, v^2, w^2)$ and $x \equiv x^0$. Let $\mathcal{S}_0^u(u^2, v^2, w^2, x^0)$ be the set of all (U^2, V^2, W^2) such that if $(U(\xi), V(\xi), W(\xi))$ is the solution of (5.5)–(5.7) with $(U(0), V(0), W(0)) = (U^2, V^2, W^2)$ and $(u, v, w, x) = (u(\xi), v(\xi), w(\xi), x^0)$, then $(U(\xi), V(\xi), W(\xi)) \to (0, 0, 0)$ as $\xi \to -\infty$. Each $\mathcal{S}_0^u(u^2, v^2, w^2, x^0)$ is a vector space of dimension 1, and $\mathcal{W}_0^u(u^0, 0, 0, x^0, 0, 0, 0, \lambda^0)$ is the set of all points

$$(u^2, v^2, w^2, x^0, U^2, V^2, W^2) \text{ such that } (u^2, v^2, w^2, x^0) \in \mathcal{W}_0^u(u^0, 0, 0, x^0)$$

$$\text{and } (U^2, V^2, W^2) \in \mathcal{S}_0^u(u^2, v^2, w^2, x^0).$$

The unstable fiber of a point $(u^\ell, 0, 0, x, 0, 0, 0, \lambda) \in \mathcal{M}^{u^\ell}$ is a 2-dimensional manifold given by $\mathcal{W}_0^u(u^\ell, 0, 0, x, 0, 0, 0, \lambda)$. The collection of these fibers foliates the 4-dimensional manifold $\mathcal{W}_0^u(\mathcal{M}^{u^\ell})$.

$\mathcal{M}_\epsilon^-$ is a normally hyperbolic invariant manifold, so each of its points has an unstable fiber. $\mathcal{W}_\epsilon^u(\mathcal{M}^{u^\ell})$ is the union of the unstable fibers of the points of $\mathcal{M}^{u^\ell}$, which is a subset of $\mathcal{M}_\epsilon^-$. So $\mathcal{W}_\epsilon^u(\mathcal{M}^{u^\ell})$ is the set of points whose backwards orbit approaches $\mathcal{M}^{u^\ell}$ rapidly as $\xi \to -\infty$. There may be orbits in $\mathcal{M}_\epsilon^-$ for small $\epsilon > 0$ that approach $\mathcal{M}^{u^\ell}$ slowly as $\xi \to -\infty$. Then points in their unstable fibers would also approach $\mathcal{M}^{u^\ell}$ slowly as $\xi \to -\infty$. We will not consider such solutions in this chapter. In the next chapter we consider solutions that go to zero at at least a certain rate. By considering solutions that go to zero at at least a certain rate, we again omit any solutions that very slowly approach $\mathcal{M}^{u^\ell}$.

In $uvwxUVW\lambda$-space, we expect that for small $\epsilon > 0$, the intersection of $\mathcal{W}_\epsilon^u(\mathcal{M}^{u^\ell})$ with $\mathcal{W}_\epsilon^s(\mathcal{M}^{u^r})$ to be 2 dimensional. We will show that the set

$$\left\{(u_\epsilon(\xi), \dot{u}_\epsilon(\xi), \ddot{u}_\epsilon(\xi), \epsilon\xi, 0, 0, 0, \lambda) \middle| \lambda = \lambda_0 + \epsilon\lambda_1 + \mathcal{O}(\epsilon^2)\right\}, \tag{5.14}$$

is contained in the intersection. If $\mathcal{W}_\epsilon^u(\mathcal{M}^{u^\ell})$ and $\mathcal{W}_\epsilon^s(\mathcal{M}^{u^r})$ meet transversally at a point on this 2-dimensional manifold with $\lambda = \lambda_0 + \epsilon\lambda_1 + \mathcal{O}(\epsilon^2)$, there are no other solutions nearby. We now consider the possibility of having additional solutions.

The solutions of interest are those that rapidly approach a solution in $\mathcal{M}^{u^\ell}$ as $\xi \to -\infty$ and a solution in $\mathcal{M}^{u^r}$ as $\xi \to \infty$. On $\mathcal{M}^{u^\ell}$, a solution takes the form

$$Q_{u^\ell}^-(\xi, \epsilon) = (u^\ell, 0, 0, x(\xi, \epsilon), 0, 0, 0, \lambda),$$

and a solution on $\mathcal{M}^{u^r}$ takes the form

$$Q_{u^r}^+(\xi, \epsilon) = (u^r, 0, 0, x(\xi, \epsilon), 0, 0, 0, \lambda).$$

In the two waves case, we must consider an additional condition for the solutions of interest. We expect that when $\frac{s_2}{\epsilon} \leq \xi \leq \frac{s_3}{\epsilon}$, the solutions are near

$$Q_{u^m}^*(\xi, \epsilon) = (u(\epsilon), v(\xi, \epsilon), w(\xi, \epsilon), x(\xi, \epsilon), U(\xi, \epsilon), V(\xi, \epsilon), W(\xi, \epsilon), \lambda(\epsilon)),$$

a solution of $\mathcal{M}_\epsilon^{u^m}$ where

$$\mathcal{M}_\epsilon^{u^m} = \left\{(u(\epsilon), v(\xi, \epsilon), w(\xi, \epsilon), x(\xi, \epsilon), U(\xi, \epsilon), V(\xi, \epsilon), W(\xi, \epsilon), \lambda(\epsilon)) \middle| x(\xi, \epsilon) < 3u^2(\epsilon) - \frac{\alpha^2}{3\beta}\right\}.$$

Please note that $\mathcal{M}_\epsilon^{u^m}$ is a 2-dimensional invariant subset of $\mathcal{M}_\epsilon^-$ for every ϵ. Please note that when $\epsilon = 0$,

$$\mathcal{M}_0^{u^m} = \left\{(u^m, 0, 0, x, 0, 0, 0, \lambda) \middle| x < 3(u^m)^2 - \frac{\alpha^2}{3\beta}\right\}.$$

5.2 Solution of Interest of System (5.1)-(5.8)

Consider the differential equation

$$\dot{X} = F(X). \tag{5.15}$$

Suppose that $X(\xi)$ is a solution to (5.15). The corresponding linearized differential equation at $X(\xi)$ is

$$\dot{Y} = DF(X(\xi))Y. \tag{5.16}$$

Differentiating both sides of (5.15) gives rise to $\ddot{X} = DF(X)\dot{X}$. This implies that $\dot{X}$ solves (5.16), the linearized differential equation. So if we let

$$(u, v, w) = (u^*(\xi), v^*(\xi), w^*(\xi), x^*)$$

be a solution of (5.1)–(5.3) for $\epsilon = 0$, then one solution of (5.5)–(5.7) for $\epsilon = 0$ is

$$(U, V, W) = (\dot{u}^*(\xi), \dot{v}^*(\xi), \dot{w}^*(\xi)).$$

Since a multiple of a solution is again a solution, for the constant k, we have

$$(U, V, W) = (k\dot{u}^*(\xi), k\dot{v}^*(\xi), k\dot{w}^*(\xi))$$

as a solution.

Recall from Chapter 3 that when there is only one travelling wave present, it is given by $(\bar{u}^1(\xi), \bar{v}^1(\xi), \bar{w}^1(\xi))$ with speed s_1. When there are two travelling waves present, they are respectively given by $(\bar{u}^2(\xi), \bar{v}^2(\xi), \bar{w}^2(\xi))$ and $(\bar{u}^3(\xi), \bar{v}^3(\xi), \bar{w}^3(\xi))$ with respective speeds s_2 and s_3. In $(u, v, w, x, U, V, W, \lambda)$-space with $\lambda = \lambda_0$ fixed and with k_j given as constants such that $k_1 \neq 0$ and k_2 and k_3 are not both 0, we define the following 8 curves:

$$\begin{aligned}
C_1 &= \left\{(u^\ell, 0, 0, x, 0, 0, 0, \lambda_0)\middle| x \le s_1\right\}, \\
C_2 &= \left\{(\bar{u}^1(\xi), \bar{v}^1(\xi), \bar{w}^1(\xi), s_1, k_1\bar{u}^1_\xi, k_1\bar{v}^1_\xi, k_1\bar{w}^1_\xi, \lambda_0)\middle| -\infty < \xi < \infty\right\}, \\
C_3 &= \left\{(u^r, 0, 0, x, 0, 0, 0, \lambda_0)\middle| x \ge s_1\right\}, \\
C_4 &= \left\{(u^\ell, 0, 0, x, 0, 0, 0, \lambda_0)\middle| x \le s_2\right\}, \\
C_5 &= \left\{(\bar{u}^2(\xi), \bar{v}^2(\xi), \bar{w}^2(\xi), s_2, k_2\bar{u}^2_\xi, k_2\bar{v}^2_\xi, k_2\bar{w}^2_\xi, \lambda_0)\middle| -\infty < \xi < \infty\right\}, \\
C_6 &= \left\{(u^m, 0, 0, x, 0, 0, 0, \lambda_0)\middle| s_2 \le x \le s_3\right\}, \\
C_7 &= \left\{(\bar{u}^3(\xi), \bar{v}^3(\xi), \bar{w}^3(\xi), s_3, k_3\bar{u}^3_\xi, k_3\bar{v}^3_\xi, k_3\bar{w}^3_\xi, \lambda_0)\middle| -\infty < \xi < \infty\right\}, \\
C_8 &= \left\{(u^r, 0, 0, x, 0, 0, 0, \lambda_0)\middle| x \ge s_3\right\}.
\end{aligned}$$

In the case of a one-wave Riemann solution, we will look for values of λ_0 for which (5.1)–(5.8) has, for small $\epsilon > 0$, a solution near $\{C_1 \cup C_2 \cup C_3\}$. In the case of a two-wave Riemann solution, we will look for values of λ_0 for which (5.1)–(5.8) has, for small $\epsilon > 0$, a solution near $\{C_4 \cup C_5 \cup C_6 \cup C_7 \cup C_8\}$.

Let $(u^*(\xi), v^*(\xi), w^*(\xi), x^*)$ denote either

$$(\bar{u}^1(\xi), \bar{v}^1(\xi), \bar{w}^1(\xi), s_1), (\bar{u}^2(\xi), \bar{v}^2(\xi), \bar{w}^2(\xi), s_2), \text{ or } (\bar{u}^3(\xi), \bar{v}^3(\xi), \bar{w}^3(\xi), s_3).$$

Let $R(\xi, 0) = (u^*(\xi), v^*(\xi), w^*(\xi), x^*, k_j\dot{u}^*(\xi), k_j\dot{v}^*(\xi), k_j\dot{w}^*(\xi), \lambda_0)$ be a solution of (5.1)-(5.8) for $\epsilon = 0$ that parameterizes the curve C_2, C_5 or C_7. Let

$$R(\xi, \epsilon) = (u(\xi, \epsilon), v(\xi, \epsilon), w(\xi, \epsilon), x(\xi, \epsilon), U(\xi, \epsilon), V(\xi, \epsilon), W(\xi, \epsilon), \lambda(\epsilon))$$

be a perturbation of this solution. Then

$$\begin{aligned}
u(\xi, \epsilon) &= u^*(\xi) + \epsilon u_1(\xi) + \mathcal{O}(\epsilon^2), && (5.17)\\
v(\xi, \epsilon) &= v^*(\xi) + \epsilon v_1(\xi) + \mathcal{O}(\epsilon^2), && (5.18)\\
w(\xi, \epsilon) &= w^*(\xi) + \epsilon w_1(\xi) + \mathcal{O}(\epsilon^2), && (5.19)\\
x(\xi, \epsilon) &= x^* + \epsilon x_1(\xi), && (5.20)\\
U(\xi, \epsilon) &= k_j\dot{u}^*(\xi) + \epsilon U_1(\xi) + \mathcal{O}(\epsilon^2), && (5.21)\\
V(\xi, \epsilon) &= k_j\dot{v}^*(\xi) + \epsilon V_1(\xi) + \mathcal{O}(\epsilon^2), && (5.22)\\
W(\xi, \epsilon) &= k_j\dot{w}^*(\xi) + \epsilon W_1(\xi) + \mathcal{O}(\epsilon^2), && (5.23)\\
\lambda &= \lambda_0 + \epsilon\lambda_1 + \mathcal{O}(\epsilon^2). && (5.24)
\end{aligned}$$

We investigate the solution of (5.1)–(5.8) when $\epsilon = 0$. Differentiating (5.17)–(5.23) produces

$$\begin{aligned}
\dot{u}(\xi, \epsilon) &= \dot{u}^*(\xi) + \epsilon\dot{u}_1(\xi) + \mathcal{O}(\epsilon^2), && (5.25)\\
\dot{v}(\xi, \epsilon) &= \dot{v}^*(\xi) + \epsilon\dot{v}_1(\xi) + \mathcal{O}(\epsilon^2), && (5.26)\\
\dot{w}(\xi, \epsilon) &= \dot{w}^*(\xi) + \epsilon\dot{w}_1(\xi) + \mathcal{O}(\epsilon^2), && (5.27)\\
\dot{x}(\xi, \epsilon) &= \epsilon\dot{x}_1(\xi), && (5.28)\\
\dot{U}(\xi, \epsilon) &= k_j\ddot{u}^*(\xi) + \epsilon\dot{U}_1(\xi) + \mathcal{O}(\epsilon^2), && (5.29)\\
\dot{V}(\xi, \epsilon) &= k_j\ddot{v}^*(\xi) + \epsilon\dot{V}_1(\xi) + \mathcal{O}(\epsilon^2), && (5.30)\\
\dot{W}(\xi, \epsilon) &= k_j\ddot{w}^*(\xi) + \epsilon\dot{W}_1(\xi) + \mathcal{O}(\epsilon^2). && (5.31)
\end{aligned}$$

From (5.1)–(5.8) with substitutions from (5.17)–(5.23), we can conclude that

$$
\begin{aligned}
\dot{u}(\xi,\epsilon) &= v^*(\xi) + \epsilon v_1(\xi) + \mathcal{O}(\epsilon^2), &(5.32)\\
\dot{v}(\xi,\epsilon) &= w^*(\xi) + \epsilon w_1(\xi) + \mathcal{O}(\epsilon^2), &(5.33)\\
\dot{w}(\xi,\epsilon) &= \frac{1}{\beta}(3(u^*(\xi))^2 v^*(\xi) + 3\epsilon(u^*(\xi))^2 v_1(\xi) + 6\epsilon u^*(\xi)u_1(\xi)v^*(\xi) &(5.34)\\
&\quad -x^*v^*(\xi) - \epsilon x^* v_1(\xi) - \epsilon x_1(\xi)v^*(\xi) - \alpha w^*(\xi) - \epsilon\alpha w_1(\xi) + \mathcal{O}(\epsilon^2)),\\
\dot{x}(\xi,\epsilon) &= \epsilon, &(5.35)\\
\dot{U}(\xi,\epsilon) &= k_j\dot{v}^*(\xi) + \epsilon V_1(\xi) + \mathcal{O}(\epsilon^2), &(5.36)\\
\dot{V}(\xi,\epsilon) &= k_j\dot{w}^*(\xi) + \epsilon W_1(\xi) + \mathcal{O}(\epsilon^2), &(5.37)\\
\dot{W}(\xi,\epsilon) &= \frac{1}{\beta}\Bigg(\epsilon k_j\lambda_0\dot{u}^*(\xi) + 3k_j(u^*(\xi))^2\dot{v}^*(\xi) + 3\epsilon(u^*(\xi))^2 V_1(\xi) &(5.38)\\
&\quad +6k_j\epsilon u^*(\xi)u_1(\xi)\dot{v}^*(\xi) - k_j x^*\dot{v}^*(\xi) - \epsilon x^* V_1(\xi) - \epsilon k_j x_1(\xi)\dot{v}^*(\xi)\\
&\quad +6k_j u^*(\xi)v^*(\xi)\dot{u}^*(\xi) + 6\epsilon u^*(\xi)v^*(\xi)U_1(\xi) + 6\epsilon k_j u^*(\xi)v_1(\xi)\dot{u}^*(\xi)\\
&\quad +6\epsilon k_j u_1(\xi)v^*(\xi)\dot{u}^*(\xi) - \alpha k_j\dot{w}^*(\xi) - \alpha\epsilon W_1(\xi) + \mathcal{O}(\epsilon^2)\Bigg).
\end{aligned}
$$

Equating the common expressions from (5.25)–(5.31) and (5.32)–(5.38) yields

$$
\begin{aligned}
\dot{u}_1(\xi) &= v_1(\xi), &(5.39)\\
\dot{v}_1(\xi) &= w_1(\xi), &(5.40)\\
\dot{w}^*(\xi) + \epsilon\dot{w}_1(\xi) &= \frac{1}{\beta}\Bigg(3(u^*(\xi))^2 v^*(\xi) - x^*v^*(\xi) - \alpha w^*(\xi) &(5.41)\\
&\quad +\frac{\partial}{\partial\xi}\Big[3\epsilon(u^*(\xi))^2 u_1(\xi)\Big] - \epsilon\Big(x^*v_1(\xi) + x_1(\xi)v^*(\xi) + \alpha w_1(\xi)\Big)\Bigg),\\
\dot{x}_1(\xi) &= 1, &(5.42)\\
\dot{U}_1(\xi) &= V_1(\xi), &(5.43)\\
\dot{V}_1(\xi) &= W_1(\xi), &(5.44)\\
k_j\ddot{w}^*(\xi) + \epsilon\dot{W}_1(\xi) &= \frac{k_j}{\beta}\left(\frac{\partial}{\partial\xi}\Big[3(u^*(\xi))^2 v^*(\xi)\Big] - x^*\dot{v}^*(\xi) - \alpha\dot{w}^*(\xi)\right) &(5.45)\\
&\quad +\frac{\epsilon}{\beta}\Bigg(\lambda_0 k_j\dot{u}^*(\xi) - x^*V_1(\xi) - k_j x_1(\xi)\dot{v}^*(\xi) - \alpha W_1(\xi))\\
&\quad +\frac{\partial}{\partial\xi}\Big[3(u^*(\xi))^2 U_1(\xi) + 6k_j u^*(\xi)v^*(\xi)u_1\Big]\Bigg).
\end{aligned}
$$

We are able to derive the following two equations from (5.39)–(5.45).

$$\beta \dot{w}_1(\xi) = \frac{\partial}{\partial \xi}\Big[3(u^*(\xi))^2 u_1(\xi)\Big] - x^* v_1(\xi) - x_1(\xi) v^*(\xi) - \alpha w_1(\xi) \tag{5.46}$$

$$\begin{aligned} \beta \dot{W}_1(\xi) &= \lambda_0 k_j \dot{u}^*(\xi) - x^* V_1(\xi) - k_j x_1(\xi) \dot{v}^*(\xi) - \alpha W_1(\xi)) \\ &\quad + \frac{\partial}{\partial \xi}\Big[3(u^*(\xi))^2 U_1(\xi) + 6k_j u^*(\xi) v^*(\xi) u_1)\Big]. \end{aligned} \tag{5.47}$$

Before we can investigate the equations (5.46)–(5.47), we need to determine the behavior of V_1 and W_1 on the solution of interest. To do this, we must track this solution of interest and determine the behavior of its limiting solutions. Since it is approaching these limiting solutions, its long time behavior will be close to that of the limiting solutions.

5.3 Behavior of $R(\xi, \epsilon)$ near $\mathcal{M}_\epsilon^\pm$

For small $\epsilon > 0$, let $Q^-_{u^\ell}(\xi, \epsilon)$ and $Q^+_{u^r}(\xi, \epsilon)$ be respectively solutions on $\mathcal{M}^{u^\ell}$ and $\mathcal{M}^{u^r}$. So the nonzero coordinates of $Q^-_{u^\ell}(\xi, \epsilon)$ take the form

$$\begin{aligned} u(\xi, \epsilon) &= u^\ell, \\ x(\xi, \epsilon) &= \big(s_* + \epsilon x_1^* + \mathcal{O}(\epsilon^2)\big) + \epsilon \xi, \\ \lambda(\epsilon) &= \lambda_0 + \epsilon \lambda_1 + \mathcal{O}(\epsilon^2), \end{aligned}$$

where s_* is s_1 or s_2. The nonzero coordinates of $Q^+_{u^r}(\xi, \epsilon)$ take the form

$$\begin{aligned} u(\xi, \epsilon) &= u^r, \\ x(\xi, \epsilon) &= \big(s_j + \epsilon x_1^* + \mathcal{O}(\epsilon^2)\big) + \epsilon \xi, \\ \lambda(\epsilon) &= \lambda_0 + \epsilon \lambda_1 + \mathcal{O}(\epsilon^2), \end{aligned}$$

where s_j is s_1 or s_3. In addition, for small $\epsilon > 0$ on $\mathcal{M}_\epsilon^-$, we consider the solution $Q^*_{u^m}(\xi, \epsilon)$ with its nonzero coordinates given by

$$\begin{aligned} u(\xi, \epsilon) &= u^m + \epsilon u_1^m + \mathcal{O}(\epsilon^2), \\ x(\xi, \epsilon) &= \big(s + \epsilon x_1^* + \mathcal{O}(\epsilon^2)\big) + \epsilon \xi, \\ U(\xi, \epsilon) &= \epsilon U_1 + \mathcal{O}(\epsilon^2), \\ V(\xi, \epsilon) &= \mathcal{O}(\epsilon^2), \\ W(\xi, \epsilon) &= \mathcal{O}(\epsilon^2), \\ \lambda(\epsilon) &= \lambda_0 + \epsilon \lambda_1 + \mathcal{O}(\epsilon^2), \end{aligned}$$

where $s_2 \le s \le s_3$.

We want to track $\mathcal{W}^u_\epsilon(\mathcal{M}^{u^\ell})$ as it passes near C_5 and then arrives near the middle of C_6. Similarly we want to track $\mathcal{W}^s_\epsilon(\mathcal{M}^{u^r})$ backwards as it passes near C_7 and then arrives near the middle of C_6. Define the portion of a solution of (5.1)–(5.7) that is located in $\mathcal{W}^u_\epsilon(\mathcal{M}^{u^\ell})$ as

$$R_-(\xi,\epsilon) = \Big(u(\xi,\epsilon), v(\xi,\epsilon), w(\xi,\epsilon), x^* + \epsilon\xi, U(\xi,\epsilon), V(\xi,\epsilon), W(\xi,\epsilon), \lambda(\epsilon)\Big),$$

such that $\big(U(\xi), V(\xi), W(\xi)\big) \to (0,0,0)$ as $\xi \to -\infty$. So we have $R_-(\xi,\epsilon) \to Q^-_{u^\ell}(\xi,\epsilon)$ as $\xi \to -\infty$ and $R_-(\xi,\epsilon) \to Q^*_{u^m}(\xi,\epsilon)$ as ξ increases. Define the portion of a solution of (5.1)–(5.7) that is located in $\mathcal{W}^s_\epsilon(\mathcal{M}^{u^r})$ as

$$R_+(\xi,\epsilon) = \Big(u(\xi,\epsilon), v(\xi,\epsilon), w(\xi,\epsilon), x^* + \epsilon\xi, U(\xi,\epsilon), V(\xi,\epsilon), W(\xi,\epsilon), \lambda(\epsilon)\Big),$$

such that $\big(U(\xi), V(\xi), W(\xi)\big) \to (0,0,0)$ as $\xi \to \infty$. Thus, we have $R_+(\xi,\epsilon) \to Q^+_{u^r}(\xi,\epsilon)$ as $\xi \to \infty$ and $R_+(\xi,\epsilon) \to Q^*_{u^m}(\xi,\epsilon)$ as ξ decreases.

As $\xi \to -\infty$, $R_-(\xi,\epsilon)$ will converge to $Q^-_{u^\ell}(\xi,\epsilon)$ component-wise.

$$\begin{aligned}
u(\xi,\epsilon) &\to u^\ell,\\
v(\xi,\epsilon) &\to 0,\\
w(\xi,\epsilon) &\to 0,\\
x(\xi,\epsilon) &\to s_* + \epsilon x_1^* + \epsilon\xi + \mathcal{O}(\epsilon^2),\\
U(\xi,\epsilon) &\to 0,\\
V(\xi,\epsilon) &\to 0,\\
W(\xi,\epsilon) &\to 0.
\end{aligned}$$

As $\xi \to \infty$, $R_+(\xi,\epsilon)$ will converge to $Q^+_{u^r}(\xi,\epsilon)$ component-wise.

$$\begin{aligned}
u(\xi,\epsilon) &\to u^r,\\
v(\xi,\epsilon) &\to 0,\\
w(\xi,\epsilon) &\to 0,\\
x(\xi,\epsilon) &\to s_j + \epsilon x_1^* + \epsilon\xi + \mathcal{O}(\epsilon^2),\\
U(\xi,\epsilon) &\to 0,\\
V(\xi,\epsilon) &\to 0,\\
W(\xi,\epsilon) &\to 0.
\end{aligned}$$

In the two waves case, we must consider how close $\mathcal{W}^u_\epsilon(\mathcal{M}^{u^\ell})$ and $\mathcal{W}^s_\epsilon(\mathcal{M}^{u^r})$ are to $\mathcal{M}^-_\epsilon$. We will show in section 5.6.2 that as ξ increases, $R_-(\xi,\epsilon)$ gets close to a solution on $\mathcal{M}^-_\epsilon$

for a span of time, component-wise in the following manner

$$
\begin{aligned}
u(\xi,\epsilon) &\rightarrow u^m + \epsilon u_1^m + \mathcal{O}(\epsilon^2),\\
v(\xi,\epsilon) &\rightarrow 0,\\
w(\xi,\epsilon) &\rightarrow 0,\\
x(\xi,\epsilon) &\rightarrow s_2 + \epsilon x_1^* + \epsilon\xi + \mathcal{O}(\epsilon^2),\\
U(\xi,\epsilon) &\rightarrow \epsilon U_1 + \mathcal{O}(\epsilon^2),\\
V(\xi,\epsilon) &\rightarrow \mathcal{O}(\epsilon^2),\\
W(\xi,\epsilon) &\rightarrow \mathcal{O}(\epsilon^2).
\end{aligned}
$$

Recall in subsection 5.1.1 we expanded U as $U_0 + \epsilon U_1 + \mathcal{O}(\epsilon^2)$. Also recall that we have shown in section 5.2 that for small $\epsilon > 0, U = \dot{u} = v$. This implies that $U_0 = v$. On $\mathcal{M}_\epsilon^\pm, v = 0$ for all ϵ. Thus, $U_0 \equiv 0$. Since $\dot{U}_1 = A^1(u,x,\lambda)U_0, U_1$ must be a constant. So we have both u_1^m and U_1 as constants. We will show in section 5.6.2 that as ξ decreases, $R_+(\xi,\epsilon)$ gets close to a solution on $\mathcal{M}_\epsilon^-$ for a span of time, component-wise in the following manner

$$
\begin{aligned}
u(\xi,\epsilon) &\rightarrow u^m + \epsilon u_1^m + \mathcal{O}(\epsilon^2),\\
v(\xi,\epsilon) &\rightarrow 0,\\
w(\xi,\epsilon) &\rightarrow 0,\\
x(\xi,\epsilon) &\rightarrow s_3 + \epsilon x_1^* + \epsilon\xi + \mathcal{O}(\epsilon^2),\\
U(\xi,\epsilon) &\rightarrow \epsilon U_1 + \mathcal{O}(\epsilon^2),\\
V(\xi,\epsilon) &\rightarrow \mathcal{O}(\epsilon^2),\\
W(\xi,\epsilon) &\rightarrow \mathcal{O}(\epsilon^2).
\end{aligned}
$$

5.4 Riemann Solution: One Wave

Please observe that when the Riemann solution consists of one wave, we have $v(\pm\infty,\epsilon) = w(\pm\infty,\epsilon) = U(\pm\infty,\epsilon) = V(\pm\infty,\epsilon) = W(\pm\infty,\epsilon) = 0$. Integrating equations (5.46) and (5.47) over $\mathbb{R}$ will produce the following results:

From Equation (5.46) we have,

$$\begin{aligned}
\beta\int_{-\infty}^{\infty}\dot{w}_1(\xi)d\xi &= \int_{-\infty}^{\infty}\Bigg(\frac{\partial}{\partial\xi}\Big[3(u^*(\xi))^2u_1(\xi)\Big] \\
&\quad -x^*v_1(\xi)-x_1(\xi)v^*(\xi)-\alpha w_1(\xi)\Bigg)d\xi \\
0 &= -\int_{-\infty}^{\infty}x_1(\xi)v^*(\xi)d\xi \\
0 &= -\int_{-\infty}^{\infty}(x_1^*+\xi)v^*(\xi)d\xi \\
0 &= -x_1^*(u^r-u^\ell)-\int_{-\infty}^{\infty}\xi v^*(\xi)d\xi \\
\Rightarrow x_1^* &= \frac{\int_{-\infty}^{\infty}\xi v^*(\xi)d\xi}{u^\ell-u^r}.
\end{aligned}$$

From Equation (5.47), we have

$$\begin{aligned}
\beta\int_{-\infty}^{\infty}\dot{W}_1(\xi)d\xi &= \int_{-\infty}^{\infty}\Bigg(\lambda_0\dot{u}^*(\xi)-x^*V_1(\xi)-x_1(\xi)\dot{v}^*(\xi) \\
&\quad -\alpha W_1(\xi))+\frac{\partial}{\partial\xi}\Big[3(u^*(\xi))^2U_1(\xi)+6u^*(\xi)v^*(\xi)u_1(\xi))\Big]\Bigg)d\xi \\
0 &= \lambda_0\int_{-\infty}^{\infty}\dot{u}^*d\xi-\int_{-\infty}^{\infty}x_1(\xi)\dot{v}^*(\xi)d\xi \\
0 &= \lambda_0(u^r-u^\ell)-x_1(\xi)v^*(\xi)\Big|_{-\infty}^{\infty}+\int_{-\infty}^{\infty}v^*(\xi)\dot{x}_1d\xi \\
0 &= \lambda_0(u^r-u^\ell)+\int_{-\infty}^{\infty}v^*(\xi)d\xi \\
0 &= \lambda_0(u^r-u^\ell)+(u^r-u^\ell) \\
\Rightarrow \lambda_0 &= -1.
\end{aligned}$$

5.4.1 Discussion

Recall the descriptions of $\mathcal{W}_0^u(\mathcal{M}^{u^\ell})$ and $\mathcal{W}_0^s(\mathcal{M}^{u^r})$ in subsection 5.1.2. For a fixed λ_f, $\mathcal{W}_0^u(\mathcal{M}^{u^\ell})$ and $\mathcal{W}_0^s(\mathcal{M}^{u^r})$ are 3- and 5-dimensional respectively. To be transverse in $\mathbb{R}^7$, the intersection must have dimension 1. Their intersection consists of the family of curves C_2, defined in section 5.2. The union of these curves is a manifold parameterized by $\xi, k_1 \in \mathbb{R}$, so it is 2-dimensional. Having a 2-dimensional intersection implies that the tangent spaces of $\mathcal{W}_0^u(\mathcal{M}^{u^\ell})$ and $\mathcal{W}_0^s(\mathcal{M}^{u^r})$ will not span $\mathbb{R}^7$. Thus, the two spaces will not be transverse.

For small $\epsilon > 0$ and $\lambda_f \neq -1+\epsilon\lambda_1+\mathcal{O}(\epsilon^2)$, the intersection of $\mathcal{W}_\epsilon^u(\mathcal{M}^{u^\ell})$ and $\mathcal{W}_\epsilon^s(\mathcal{M}^{u^r})$ is given by (5.9). Having a 1-dimensional intersection implies that the tangent spaces of $\mathcal{W}^u_\epsilon(\mathcal{M}^{u^\ell})$ and $\mathcal{W}^s_\epsilon(\mathcal{M}^{u^r})$ will span $\mathbb{R}^7$. So the interpretation of the calculation in this section is that for small $\epsilon > 0$ and fixed $\lambda_f \neq -1 + \epsilon\lambda_1 + \mathcal{O}(\epsilon^2)$, the intersection of $\mathcal{W}^u_\epsilon(\mathcal{M}^{u^\ell})$ and $\mathcal{W}^s_\epsilon(\mathcal{M}^{u^r})$ becomes transverse.

5.5 The Adjoint Solution

Consider the system (3.22)–(3.23) with $x = x^*$. We assume that for $u = u^-$ and $z^* = xu^- - (u^-)^3$, there is a solution $(u^*(\xi), v^*(\xi))$ of (3.22)–(3.23) that goes from the equilibrium $(u^-, 0)$ to a second equilibrium $(u^+, 0)$ with $u^+ \neq u^-$. Then $(u^*(\xi), v^*(\xi), z^*)$ is a solution of (3.22)–(3.23) that goes from the equilibrium $(u^-, 0, z^*)$ to the equilibrium $(u^+, 0, z^*)$. The linearization of (3.22)–(3.23) along this solution is

$$\dot{X} = \hat{A}(\xi)X \tag{5.48}$$

where $X = \begin{pmatrix} U_1 \\ V_1 \\ Z_1 \end{pmatrix}$ and $\hat{A} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{3(u^*(\xi))^2 - x^*}{\beta} & -\frac{\alpha}{\beta} & \frac{1}{\beta} \\ 0 & 0 & 0 \end{pmatrix}$.

In this section we consider adjoint solutions of (5.48). This information is vital to understanding the solution of (5.1)–(5.8) when the Riemann solution consists of two waves.

The adjoint system corresponding to the given homogeneous system is $\Psi' = -\Psi\hat{A}(\xi)$, where Ψ is in the form of a row vector. An equivalent system for Ψ given as a column vector is $\Psi' = -\hat{A}(\xi)^{\mathsf{T}}\Psi$. Either system can be written as

$$\begin{aligned} \dot{\Psi}_1 &= \frac{x^* - 3u^{*2}}{\beta}\Psi_2, \\ \dot{\Psi}_2 &= -\Psi_1 + \frac{\alpha}{\beta}\Psi_2, \\ \dot{\Psi}_3 &= -\frac{1}{\beta}\Psi_2. \end{aligned}$$

Since $\dot{\Psi}_1$ and $\dot{\Psi}_2$ are decoupled from Ψ_3, one can try to use $\dot{\Psi}_1$ and $\dot{\Psi}_2$ to solve for Ψ_1 and Ψ_2. Afterwards, one can use Ψ_2 to find Ψ_3. Furthermore, $\Psi_3 = -\dfrac{1}{\beta}\displaystyle\int \Psi_2(\tau)d\tau$. Another solution is given by the constant vector $E_3 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$. For $i = 1, 2$, the adjoint solutions are in the form of $c_0E_3 + \sum_{i=1}^2 c_i \begin{pmatrix} \Psi_1^i & \Psi_2^i & -\frac{1}{\beta}\int \Psi_2^i(\tau)d\tau \end{pmatrix}$. We now consider the 2-dimensional system

$$(\dot{\Psi}_1, \dot{\Psi}_2) = (\Psi_1, \Psi_2)D(\xi), \tag{5.49}$$

where

$$D(\xi) = \begin{pmatrix} 0 & -1 \\ \frac{x - 3u^2}{\beta} & \frac{\alpha}{\beta} \end{pmatrix}.$$

5.5.1 $\Psi_1(\xi)$ and $\Psi_2(\xi)$

Consider the differential equation $\dot{X} = C(\xi)X$, where $X \in \mathbb{R}^2$ and $C(\xi) = \begin{pmatrix} a(\xi) & b(\xi) \\ c(\xi) & d(\xi) \end{pmatrix}$. Suppose a solution to the differential equation is given by $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$. The corresponding adjoint system is given by $\dot{\Psi} = -\Psi C(\xi)$.

Proposition 1 *A solution to the adjoint system is*

$$\Psi(\xi) = \begin{pmatrix} \Psi_1(\xi) & \Psi_2(\xi) \end{pmatrix} = \begin{pmatrix} -e^{-\int_0^\xi trace\{A(s)\}ds} X_2 & e^{-\int_0^\xi trace\{A(s)\}ds} X_1 \end{pmatrix}.$$

Proof. Differentiating Ψ produces the following equations:

$$\begin{aligned} \frac{d\Psi_1}{d\xi} &= (a(\xi)+d(\xi))e^{-\int_0^\xi (a(s)+d(s))ds} X_2 - e^{-\int_0^\xi (a(s)+d(s))ds} \dot{X}_2 \\ &= (a(\xi)+d(\xi))e^{-\int_0^\xi (a(s)+d(s))ds} X_2 - e^{-\int_0^\xi (a(s)+d(s))ds} (c(\xi)X_1 + d(\xi)X_2) \\ &= a(\xi)e^{-\int_0^\xi (a(s)+d(s))ds} X_2 - e^{-\int_0^\xi (a(s)+d(s))ds} c(\xi)X_1, \end{aligned}$$

$$\begin{aligned} \frac{d\Psi_2}{d\xi} &= -(a(\xi)+d(\xi))e^{-\int_0^\xi (a(s)+d(s))ds} X_1 + e^{-\int_0^\xi (a(s)+d(s))ds} \dot{X}_1 \\ &= -(a(\xi)+d(\xi))e^{-\int_0^\xi (a(s)+d(s))ds} X_1 + e^{-\int_0^\xi (a(s)+d(s))ds} (a(\xi)X_1 + b(\xi)X_2) \\ &= -d(\xi)e^{-\int_0^\xi (a(s)+d(s))ds} X_1 + e^{-\int_0^\xi (a(s)+d(s))ds} b(\xi)X_2. \end{aligned}$$

The differential equation for the adjoint system is given by

$$\begin{aligned} \dot{\Psi} &= -\Psi A(\xi) \\ &= \begin{pmatrix} e^{-\int_0^\xi (a(s)+d(s))ds} X_2 & -e^{-\int_0^\xi (a(s)+d(s))ds} X_1 \end{pmatrix} \begin{pmatrix} a\xi & b(\xi) \\ c(\xi) & d(\xi) \end{pmatrix} \\ &= \begin{pmatrix} e^{-\int_0^\xi (a(s)+d(s))ds} X_2 a(\xi) - e^{-\int_0^\xi (a(s)+d(s))ds} X_1 c(\xi) \\ e^{-\int_0^\xi (a(s)+d(s))ds} X_2 b(\xi) - e^{-\int_0^\xi (a(s)+d(s))ds} X_1 d(\xi) \end{pmatrix}^{\mathsf{T}}. \end{aligned}$$

This is the same result as the one found by calculating the derivatives directly. □

Let

$$U^* = \dot{u}^* = v^* \text{ and } V^* = \dot{v}^* = \frac{1}{\beta}\Big(z - x^*u^*(\xi) + (u^*)^3(\xi) - \alpha v^*(\xi)\Big).$$

Then $(U^*, V^*, 0)$ is a solution of (5.48). Hence by Proposition 1,

$$(\Psi_1, \Psi_2) = e^{-\int_0^\xi traceD(\tau)d\tau}(-\dot{v}^*, v^*) \tag{5.50}$$

is a solution of the adjoint equation. Please note,

$$e^{-\int_0^\xi traceD(\tau)d\tau}(-\dot{v}^*, v^*) = e^{-\int_0^\xi (\lambda_1(\tau)+\lambda_2(\tau))d\tau}(-\dot{v}^*, v^*),$$

where $\lambda_1(\tau), \lambda_2(\tau)$ are the eigenvalues of $D(\tau)$. As $\tau \to \pm\infty$, $\lambda_i(\tau) \to \lambda_i^{\pm}$, for some nonzero $\lambda_i^{\pm}$'s. Furthermore, the λ_i^- values will have opposing signs and λ_i^+ values will be negative.

When $\tau \to -\infty$, the equilibria are saddles, which indicates $\lambda_1^- < 0 < \lambda_2^-$. If for nonzero constant d_1 $U^*(\xi) \approx d_1 e^{\lambda_2^- \xi}$, then there exists a nonzero constant d_2 so that $V^*(\xi) \approx d_2 e^{\lambda_2^- \xi}$ when ξ near $-\infty$. This implies that for ξ near $-\infty$,

$$\begin{aligned} e^{-\int_0^\xi traceD(\tau)d\tau}(-\dot{v}^*, v^*) &\approx e^{-\int_0^\xi(\lambda_1^- + \lambda_2^-)d\tau}(-\dot{v}^*, v^*) \\ &= e^{(-\lambda_1^- - \lambda_2^-)\xi}(-\dot{v}^*, v^*) \\ &= e^{(-\lambda_1^- - \lambda_2^-)\xi}(-d_2 e^{\lambda_2^- \xi}, d_1 e^{\lambda_2^- \xi}) \\ &= (-d_2 e^{-\lambda_1^- \xi}, d_1 e^{-\lambda_1^- \xi}). \end{aligned}$$

Notice that as $\xi \to -\infty$, $(-d_2 e^{-\lambda_1^- \xi}, d_1 e^{-\lambda_1^- \xi}) \to (0,0)$. However, if we consider $U^*(\xi) \approx d_1 e^{-\lambda_2^- \xi}$ when ξ near ∞, then as $\xi \to \infty$ both $U^*(\xi)$ and $V^*(\xi)$ will converge to 0. Therefore, if we consider a solution that decays to 0 when $\xi \to \pm\infty$, then its adjoint solution will also decay to 0 when $\xi \to \pm\infty$.

We approximate the adjoint solutions to be

$$\Psi = (\Psi_1, \Psi_2, \Psi_3) = \left(-e^{-\frac{\alpha}{\beta}\xi}\dot{v}^*, e^{-\frac{\alpha}{\beta}\xi}v^*, \frac{1}{\beta}\int_\xi^\infty e^{-\frac{\alpha}{\beta}\xi}v^* d\xi\right). \tag{5.51}$$

5.5.2 Inhomogeneous Equation

Consider the inhomogeneous equation

$$\dot{\Psi} = -\Psi\hat{A}(\xi) + H, \tag{5.52}$$

where $H = \begin{pmatrix} 0 \\ 0 \\ \frac{d}{d\xi}\left(6u^*(\xi)u_1(\xi)v(\xi) + \lambda_0 u(\xi) - x_1^* v^*(\xi)\right) - \xi w^*(\xi) \end{pmatrix}$.

Let X be a solution of (5.52), and let Ψ be a solution of $\dot{\Psi} = -\Psi\hat{A}(\xi)$. Then

$$\begin{aligned} \int_{-\infty}^{\infty} \Psi H d\tau &= \int_{-\infty}^{\infty} \Psi(\dot{X} - \hat{A}(\xi)X)d\xi \\ &= \int_{-\infty}^{\infty} (\Psi\dot{X} - \Psi\hat{A}(\xi)X)d\xi \\ &= \int_{-\infty}^{\infty} (\Psi\dot{X} + \dot{\Psi}X)d\xi \\ &= \int_{-\infty}^{\infty} (\Psi X)' d\xi \\ &= (\Psi X)(\infty) - (\Psi X)(-\infty) \end{aligned}$$

Remark 3 *This integral exists if one of the following occurs:*

1. *$\Psi(\xi)$ must be bounded when $X(\infty) \neq 0$ or $X(-\infty) \neq 0$,*

2. *$\Psi(\xi)$ must not grow faster than $X(\xi)$ decays to 0 at $\pm\infty$.*

Observe that the first two components of H are 0 in (5.52). This implies $\int_{-\infty}^{\infty} \Psi H d\xi = \int_{-\infty}^{\infty} \Psi_3 H_3 d\xi$. Alternatively, for $i = 1, 2, 3$,

$$\int_{-\infty}^{\infty} \Psi H d\xi = (\Psi X)(\infty) - (\Psi X)(-\infty) = \sum_{i=1}^{3} (\Psi_i X_i)(\infty) - (\Psi_i X_i)(-\infty).$$

The previous subsection gives us a good candidate for the first 2 components of our adjoint system. If we choose our solution $U^*(\xi)$ that converges to 0 on the boundary, $V^*(\xi)$ will also go to 0 on the boundary due to the nature of our problem. Moreover, the corresponding first two components of our chosen adjoint solution will converge to 0 on the boundary as well. Thus, $(\Psi_i X_i)(\infty) = (\Psi_i X_i)(-\infty) = 0$, for $i = 1, 2$. Therefore,

$$\int_{-\infty}^{\infty} \Psi H d\xi = (\Psi_3 X_3)(\infty) - (\Psi_3 X_3)(-\infty). \tag{5.53}$$

5.6 Riemann Solution: Two Waves

We denote RD1 as the portion of the solution of (5.1)-(5.3) that asymptotically approaches $\mathcal{M}_0^{u^\ell}$ as $\xi \to -\infty$ and stays near $\mathcal{M}_0^{u^m}$ as ξ increases from $-\infty$. Let RD2 be the portion of the solution that asymptotically approaches $\mathcal{M}_0^{u^r}$ as $\xi \to \infty$ and stays near $\mathcal{M}_0^{u^m}$ as ξ decreases from ∞. Similarly, we denote by LRD1, the portion of the solution of (5.5)-(5.8) that is associated with RD1 and approaches zero as $\xi \to -\infty$, and LRD2 as the portion of the solution that is associated with RD2 and approaches zero as $\xi \to \infty$.

5.6.1 Riemann–Dafermos solution portion

Under the change of coordinates

$$u_1 = u_1, v_1 = v_1, \text{ and } z_1(\xi) = w_1 - \alpha v_1 - (3(u^*)^2 - x^*)u_1, \tag{5.54}$$

(5.39)–(5.41) becomes

$$\dot{u}_1(\xi) = v_1(\xi), \tag{5.55}$$

$$\dot{v}_1(\xi) = \frac{1}{\beta}\left(z_1(\xi) + (3(u^*(\xi))^2 - x^*)u_1(\xi) - \alpha v_1(\xi)\right), \tag{5.56}$$

$$\dot{z}_1(\xi) = -x_1(\xi)v^*(\xi). \tag{5.57}$$

This representation can be written in the matrix form,

$$\Theta_\xi = \hat{A}(\xi)\Theta + H(\xi),$$

where $\Theta = \begin{pmatrix} u_1 \\ v_1 \\ z_1 \end{pmatrix}$ and $H = \begin{pmatrix} 0 \\ 0 \\ -x_1(\xi)v^*(\xi) \end{pmatrix}$.
Let Ψ be a solution of the adjoint equation $\dot{\Psi} = -\Psi\hat{A}(\xi)$.

$$\int_{-\infty}^{\infty} \Psi(\xi)H(\xi)d\xi = \int_{-\infty}^{\infty} \Psi_3(\xi)H_3(\xi)d\xi = \int_{-\infty}^{\infty} -\Psi_3(\xi)(x_1^* + \xi)v^*(\xi)d\xi.$$

RD1

Let us first consider RD1. Take $\Psi = E_3$.

$$-\int_{-\infty}^{\infty} \Psi_3(\xi)x_1(\xi)v^*(\xi)d\xi = -\int_{-\infty}^{\infty} (x_1^* + \xi)v^*(\xi)d\xi = -x_1^*(u^m - u^\ell) - \int_{-\infty}^{\infty} \xi v^*(\xi)d\xi.$$

From equation (5.53) we have

$$-\int_{-\infty}^{\infty} \Psi_3(\xi)x_1(\xi)v^*(\xi)d\xi = \Theta_3(\infty) - \Theta_3(-\infty) = z_1(\infty) - z_1(-\infty).$$

Based on (5.54),

$$z_1(\infty) - z_1(-\infty) = -(3(u^m)^2 - x^*)u_1(\infty).$$

Therefore,

$$-\int_{-\infty}^{\infty} \Psi_3(\xi)x_1(\xi)v^*(\xi)d\xi = -(3(u^m)^2 - x^*)u_1(\infty) \tag{5.58}$$

$$-x_1^*(u^m - u^\ell) - \int_{-\infty}^{\infty} \xi v^*(\xi)d\xi = -(3(u^m)^2 - x^*)u_1(\infty) \tag{5.59}$$

Recall that $\Psi = (\Psi_1, \Psi_2, \Psi_3)$, given by equation (5.51). Then,

$$-\int_{-\infty}^{\infty} \Psi_3(\xi)x_1(\xi)v^*(\xi)d\xi = -x_1^*\int_{-\infty}^{\infty} \Psi_3(\xi)v^*(\xi)d\xi - \int_{-\infty}^{\infty} \Psi_3(\xi)\xi v^*(\xi)d\xi.$$

From equation (5.53),

$$-\int_{-\infty}^{\infty} \Psi_3(\xi)x_1(\xi)v^*(\xi)d\xi = -\Psi_3(\infty)(3(u^m)^2 - x^*)u_1(\infty).$$

Hence,

$$-x_1^*\int_{-\infty}^{\infty} \Psi_3(\xi)v^*(\xi)d\xi - \int_{-\infty}^{\infty} \Psi_3(\xi)\xi v^*(\xi)d\xi = -\Psi_3(\infty)(3(u^m)^2 - x^*)u_1(\infty).$$

The obtained information gives rise to the following equations

$$-(3(u^m)^2 - x^*)u_1(\infty) = -x_1^*(u^m - u^\ell) - \int_{-\infty}^{\infty} \xi v^*(\xi)d\xi, \tag{5.60}$$

$$-\Psi_3(\infty)(3(u^m)^2 - x^*)u_1(\infty) = -x_1^* \int_{-\infty}^{\infty} \Psi_3(\xi)v^*(\xi)d\xi - \int_{-\infty}^{\infty} \Psi_3(\xi)\xi v^*(\xi)d\xi. \tag{5.61}$$

Substituting (5.60) into (5.61) gives

$$x_1^* = \frac{\int_{-\infty}^{\infty} \Big(\Psi_3(\xi)\xi v^*(\xi) - \Psi_3(\infty)\xi v^*(\xi)\Big)d\xi}{\Psi_3(\infty)(u^m - u^\ell) - \int_{-\infty}^{\infty} \Psi_3(\xi)v^*(\xi)d\xi} = -\frac{\int_{-\infty}^{\infty} (\Psi_3(\xi) - \Psi_3(\infty))\,\xi v^*(\xi)d\xi}{\int_{-\infty}^{\infty} (\Psi_3(\xi) - \Psi_3(\infty))\,v^*(\xi)d\xi}.$$

Since $\Psi_3(\xi)$ is any antiderivative for $-\frac{1}{\beta}\Psi_2(\xi)$, we may choose the antiderivative so that $\Psi_3(\infty) = 0$. With this choice of Ψ_3,

$$x_1^* = -\frac{\int_{-\infty}^{\infty} \Psi_3(\xi)\xi v^*(\xi)d\xi}{\int_{-\infty}^{\infty} \Psi_3(\xi)v^*(\xi)d\xi}.$$

Recall from (5.50) that a good choice for the second component of our adjoint system is $e^{-\frac{\alpha}{\beta}\xi}v^*(\xi)$. Also, observe that $v^*(\xi)$ is the derivative of a travelling wave solution of (1.1). On any trajectory that connects a point on the space of saddle equilibria to another saddle point on that space, Jacob, McKinney, Shearer [6] points out that $v^*(\xi) < 0$. (Refer back to subsection 3.3.3.) Thus,

$$\Psi_3(\xi) = -\frac{1}{\beta}\int_{\infty}^{\xi} \Psi_2(\tau)d\tau = \frac{1}{\beta}\int_{\xi}^{\infty} e^{-\frac{\alpha}{\beta}\tau}v^*(\tau)d\tau.$$

Since $\Psi_3(\xi) \not\equiv 0$, x_1^* is well defined. Knowing the value of x_1^* will allow us to find the value of $u_1(\infty)$ from (5.60),

$$u_1(\infty) = \frac{x_1^*(u^m - u^\ell) + \int_{-\infty}^{\infty} \xi v^*(\xi)d\xi}{3(u^m)^2 - x^*}.$$

RD2

Now we will consider RD2. Take $\Psi = E_3$.

$$\begin{aligned} -\int_{-\infty}^{\infty} \Psi_3(\xi)x_1(\xi)v^*(\xi)d\xi &= -\int_{-\infty}^{\infty} x_1(\xi)v^*(\xi)d\xi \\ &= -\int_{-\infty}^{\infty} x_1^* v^*(\xi)d\xi - \int_{-\infty}^{\infty} \xi v^*(\xi)d\xi \\ &= x_1^*(u^m - u^r) - \int_{-\infty}^{\infty} \xi v^*(\xi)d\xi. \end{aligned}$$

Equation (5.53) implies that,

$$\begin{aligned} x_1^*(u^m - u^r) - \int_{-\infty}^{\infty} \xi v^*(\xi)d\xi &= \Theta_3(\infty) - \Theta_3(-\infty) \\ &= z_1(\infty) - z_1(-\infty) \\ &= (3(u^m)^2 - x^*)u_1(-\infty). \end{aligned}$$

Remark 4 *Although* $u_1(\xi)$ *is defined to be two different functions for* $RD1$ *and* $RD2$, $u_1(\infty)$ *defined for RD1 will be the same value as* $u_1(-\infty)$ *for RD2. So from the above calculation*

$$x_1^* = \frac{(3(u^m)^2 - x^*)u_1(-\infty) + \int_{-\infty}^{\infty} \xi v^*(\xi)d\xi}{u^m - u^r}.$$

5.6.2 Linearized portion

Defining $Z_1(\xi) = -(3(u^*(\xi))^2 - x^*)U_1(\xi) + \alpha V_1(\xi) + \beta W_1(\xi)$ produces a new representation of (5.43)–(5.45),

$$\begin{aligned}
\dot{U}_1(\xi) &= V_1(\xi), \\
\dot{V}_1(\xi) &= \frac{1}{\beta}\left(Z_1(\xi) + (3(u^*(\xi))^2 - x^*)U_1(\xi) - \alpha V_1(\xi)\right), \\
\dot{Z}_1(\xi) &= k_j\left(\lambda_0 v^*(\xi) - x_1(\xi)w^*(\xi) + \frac{d}{d\xi}(6u^*(\xi)u_1(\xi)v^*(\xi))\right).
\end{aligned}$$

This representation can be written in the following matrix form $\Omega' = \hat{A}(\xi)\Omega + H(\xi)$, where $\Omega = \begin{pmatrix} U_1 \\ V_1 \\ Z_1 \end{pmatrix}$, $\hat{A}(\xi) = \begin{pmatrix} 0 & 1 & 0 \\ \frac{3(u^*(\xi))^2 - x^*}{\beta} & -\frac{\alpha}{\beta} & \frac{1}{\beta} \\ 0 & 0 & 0 \end{pmatrix}$, and

$$H(\xi) = k_j\begin{pmatrix} 0 \\ 0 \\ \lambda_0 v^*(\xi) - x_1(\xi)w^*(\xi) + \frac{d}{d\xi}(6u^*(\xi)u_1(\xi)v^*(\xi)) \end{pmatrix}.$$

The corresponding adjoint system for the homogeneous system is given by $\Psi' = -\Psi A(\xi)$. Let $\Psi(\xi)$ be a solution to the adjoint system. Then
$\int_{-\infty}^{\infty} \Psi(\xi)H(\xi)d\xi = \int_{-\infty}^{\infty} \Psi_3(\xi)H_3(\xi)d\xi$.

$$\begin{aligned}
\int_{-\infty}^{\infty} \Psi_3(\xi)H_3(\xi)d\xi &= k_j\int_{-\infty}^{\infty} \Psi_3(\xi)\Bigg(-\xi w^*(\xi) + \frac{d}{d\xi}\Big(\lambda_0 u^*(\xi) + 6u^*(\xi)u_1(\xi)v^*(\xi) \\
&\quad -x_1^* v^*(\xi)\Big)\Bigg)d\xi \\
&= k_j\Psi_3(\xi)\Big(\lambda_0 u^*(\xi) + 6u^*(\xi)u_1(\xi)v^*(\xi) - x_1^* v^*(\xi)\Big)\Big|_{-\infty}^{\infty} - \\
&\quad k_j\int_{-\infty}^{\infty} \dot{\Psi}_3(\xi)\Big(\lambda_0 u^*(\xi) + 6u^*(\xi)u_1(\xi)v^*(\xi) - x_1^* v^*(\xi)\Big) \\
&\quad +\Psi_3(\xi)\xi w^*(\xi)d\xi \\
&= k_j\lambda_0\Big(\Psi_3(\infty)u^*(\infty) - \Psi_3(-\infty)u^*(-\infty)\Big) \\
&\quad -k_j\int_{-\infty}^{\infty} \dot{\Psi}_3(\xi)\Big(\lambda_0 u^*(\xi) + 6u^*(\xi)u_1(\xi)v^*(\xi) - x_1^* v^*(\xi)\Big) \\
&\quad +\Psi_3(\xi)\xi w^*(\xi)d\xi
\end{aligned}$$

LRD1

Let us first consider the portion of the linearized solution that corresponds to RD1, LRD1. Taking $\Psi(\xi) = E_3$ implies

$$\begin{aligned}\int_{-\infty}^{\infty} \Psi_3(\xi) H_3(\xi) d\xi &= \Omega_3(\infty) - \Omega_3(-\infty) \\ &= Z_1(\infty) - Z_1(-\infty) \\ &= -(3(u^m)^2 - x^*) U_1(\infty).\end{aligned} \tag{5.62}$$

In addition,

$$\begin{aligned}\int_{-\infty}^{\infty} \Psi_3(\xi) H_3(\xi) d\xi &= k_j \lambda_0 \left(u^m - u^\ell\right) - k_j \int_{-\infty}^{\infty} \xi w^*(\xi) d\xi \\ &= k_j \lambda_0 \left(u^m - u^\ell\right) - k_j \left(u^\ell - u^m\right) \\ &= k_j \left(\lambda_0 + 1\right) \left(u^m - u^\ell\right).\end{aligned}$$

Equating this to the equation (5.62), we are able to conclude that $U_1(\infty)$ has a finite value and $k_j \left(\lambda_0 + 1\right) \left(u^m - u^\ell\right) = -(3(u^m)^2 - x^*) U_1(\infty)$.

Now, take $\Psi(\xi)$ to be given by (5.51).

$$\begin{aligned}\int_{-\infty}^{\infty} \Psi_3(\xi) H_3(\xi) d\xi = {} & k_j \lambda_0 \left(\Psi_3(\infty) u^m - \Psi_3(-\infty) u^\ell + \frac{1}{\beta} \int_{-\infty}^{\infty} \Psi_2(\xi) u^*(\xi)\right) \\ & + k_j \int_{-\infty}^{\infty} \Big(\frac{1}{\beta} \Psi_2(\xi) \left(6u^*(\xi) u_1(\xi) v^*(\xi) - x_1^* v^*(\xi)\right) \\ & + \xi \Psi_3(\xi) w^*(\xi)\Big) d\xi\end{aligned}$$

Please note

$$\begin{aligned}\int_{-\infty}^{\infty} \Psi_3(\xi) H_3(\xi) d\xi &= \Psi_3(\infty) \Omega_3(\infty) - \Psi_3(-\infty) \Omega_3(-\infty) \\ &= \Psi_3(\infty) Z_1(\infty) - \Psi_3(\infty) Z_1(-\infty) \\ &= \Psi_3(\infty) Z_1(\infty) \\ &= -\Psi_3(\infty)(3(u^m)^2 - x^*) U_1(\infty).\end{aligned} \tag{5.63}$$

From equation (5.63) we obtain

$$\begin{aligned}-\Psi_3(\infty)(3(u^m)^2 - x^*) U_1(\infty) = {} & k_j \lambda_0 \left(\Psi_3(\infty) u^m - \Psi_3(-\infty) u^\ell + \frac{1}{\beta} \int_{-\infty}^{\infty} \Psi_2(\xi) u^*(\xi)\right) \\ & + k_j \int_{\infty}^{\infty} \Big(\frac{1}{\beta} \Psi_2(\xi) \left(6u^*(\xi) u_1(\xi) v^*(\xi) - x_1^* v^*(\xi)\right) \\ & + \xi \Psi_3(\xi) w^*(\xi)\Big) d\xi.\end{aligned}$$

Thus, the following equations are generated,

$$\begin{aligned} k_j\left(\lambda_0+1\right)\left(u^m-u^\ell\right) &= -(3(u^m)^2-x^*)U_1(\infty), && (5.64)\\ -\Psi_3(\infty)(3(u^m)^2-x^*)U_1(\infty) &= k_j\lambda_0\Big(\Psi_3(\infty)u^m-\Psi_3(-\infty)u^\ell \\ &\quad +\frac{1}{\beta}\int_{-\infty}^{\infty}\Psi_2(\xi)u^*(\xi)d\xi\Big) \\ &\quad +k_j\int_{-\infty}^{\infty}\Big(\frac{1}{\beta}\Psi_2(\xi)\big(6u^*(\xi)u_1(\xi)v^*(\xi) \\ &\quad -x_1^*v^*(\xi)\big)+\xi\Psi_3(\xi)w^*(\xi)\Big)d\xi && (5.65)\\ &= k_j\lambda_0\int_{-\infty}^{\infty}\Psi_3(\xi)v^*(\xi)d\xi-k_j\int_{-\infty}^{\infty}\Psi_3(\xi)\xi w^*(\xi)d\xi \\ &\quad +k_j\int_{-\infty}^{\infty}\frac{1}{\beta}\Psi_2(\xi)\left(6u^*(\xi)u_1(\xi)v^*(\xi)-x_1^*v^*(\xi)\right)d\xi. \end{aligned}$$

Substituting (5.64) into (5.65) produces,

$$\begin{aligned} \Psi_3(\infty)(\lambda_0+1)\left(u^m-u^\ell\right) &= \lambda_0\int_{-\infty}^{\infty}\Psi_3(\xi)v^*(\xi)d\xi-\int_{-\infty}^{\infty}\Psi_3(\xi)\xi w^*(\xi)d\xi \\ &\quad +\frac{1}{\beta}\int_{-\infty}^{\infty}\Psi_2(\xi)(6u^*(\xi)u_1(\xi)v^*(\xi)-x_1^*v^*(\xi))d\xi. \end{aligned}$$

This implies that

$$\lambda_0=\frac{\Psi_3(\infty)(u^\ell-u^m)+\int_{-\infty}^{\infty}\frac{1}{\beta}\Psi_2(\xi)\left(6u^*(\xi)u_1(\xi)v^*(\xi)-x_1^*v^*(\xi)\right)-\Psi_3(\xi)\xi w^*(\xi)d\xi}{\Psi_3(\infty)(u^m-u^\ell)-\int_{-\infty}^{\infty}\Psi_3(\xi)v^*(\xi)d\xi}.$$

We may choose $\Psi(\xi)$ so that $\Psi_3(\infty)=0$. This produces,

$$\lambda_0 = \frac{\int_{-\infty}^{\infty}-\frac{1}{\beta}\Psi_2(\xi)\left(6u^*(\xi)u_1(\xi)v^*(\xi)-x_1^*v^*(\xi)\right)+\Psi_3(\xi)\xi w^*(\xi)d\xi}{\int_{-\infty}^{\infty}\Psi_3(\xi)v^*(\xi)d\xi}.$$

Please observe that

$$\begin{aligned} \int_{-\infty}^{\infty}\Psi_3(\xi)\xi w^*(\xi)d\xi &= \Psi_3(\xi)(\xi v^*(\xi)-u^*(\xi))\Big|_{-\infty}^{\infty}+\frac{1}{\beta}\int_{-\infty}^{\infty}\Psi_2(\xi)(\xi v^*(\xi)-u^*(\xi)d\xi \\ &= \Psi_3(-\infty)u^\ell-\Psi_3(\infty)u^m+\frac{1}{\beta}\int_{-\infty}^{\infty}\Psi_2(\xi)(\xi v^*(\xi)-u^*(\xi)d\xi \\ &= \Psi_3(-\infty)u^\ell-\Psi_3(\infty)u^m+\Psi_3(\xi)u^*(\xi)\Big|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}\Psi_3(\xi)v^*(\xi)d\xi \\ &\quad +\frac{1}{\beta}\int_{-\infty}^{\infty}\Psi_2(\xi)\xi v^*(\xi)d\xi \\ &= -\int_{-\infty}^{\infty}\Psi_3(\xi)v^*(\xi)d\xi+\frac{1}{\beta}\int_{-\infty}^{\infty}\Psi_2(\xi)\xi v^*(\xi)d\xi. \end{aligned}$$

So we have,

$$\lambda_0 = \frac{\int_{-\infty}^{\infty} -\frac{1}{\beta}\Psi_2(\xi)\left(6u^*(\xi)u_1(\xi)v^*(\xi) - x_1^*v^*(\xi)\right) + \Psi_3(\xi)\xi w^*(\xi)d\xi}{\int_{-\infty}^{\infty} \Psi_3(\xi)v^*(\xi)d\xi} \tag{5.66}$$

$$= \frac{\int_{-\infty}^{\infty} -\frac{1}{\beta}\Psi_2(\xi)\left(6u^*(\xi)u_1(\xi)v^*(\xi) - x_1(\xi)v^*(\xi)\right)d\xi}{\int_{-\infty}^{\infty} \Psi_3(\xi)v^*(\xi)d\xi} - 1.$$

Observe that $v^*(\xi)$ is the derivative of a travelling wave solution of (1.1). It was noted in subsection 3.3.3 that on any trajectory that connects a point on the space of saddle equilibria to another saddle point on that space, we have $v^*(\xi) < 0$. Thus, using (5.51),

$$\Psi_3(\xi) = -\frac{1}{\beta}\int_{\infty}^{\xi} \Psi_2(\tau)d\tau = \frac{1}{\beta}\int_{\xi}^{\infty} e^{-\frac{\alpha}{\beta}\tau} v^*(\tau)d\tau \not\equiv 0.$$

Since $\Psi_3(\xi) \not\equiv 0$, λ_0 is well defined.

$$U_1(\infty) = \frac{k_j(\lambda_0+1)(u^\ell - u^m)}{3(u^m)^2 - x^*}.$$

LRD2

Now we will consider the portion of the linearized solution that corresponds to RD2, LRD2. Please notice that,

$$\begin{aligned}\int_{-\infty}^{\infty} \Psi_3(\xi)H_3(\xi)d\xi &= \Psi_3(\infty)\Omega_3(\infty) - \Psi_3(-\infty)\Omega_3(-\infty) \\ &= \Psi_3(\infty)Z_1(\infty) - \Psi_3(\infty)Z_1(-\infty) \\ &= -\Psi_3(-\infty)Z_1(-\infty) \\ &= \Psi_3(-\infty)(3(u^m)^2 - x^*)U_1(-\infty).\end{aligned} \tag{5.67}$$

When $\Psi_3(\xi) = E_3$, we have

$$\begin{aligned}\int_{-\infty}^{\infty} \Psi_3(\xi)H_3(\xi)d\xi &= k_j\lambda_0\left(u^r - u^m\right) - k_j\int_{-\infty}^{\infty} \xi w^*(\xi)d\xi \\ &= k_j(\lambda_0+1)\left(u^r - u^m\right).\end{aligned}$$

Equating this equation with equation (5.67), we have the results that $U(-\infty)$ has a finite value and that $k_j(\lambda_0+1)\left(u^r - u^m\right) = \Psi_3(-\infty)(3(u^m)^2 - x^*)U_1(-\infty)$. Since λ_0 was determined when viewing LRD1, $U_1(-\infty)$ is determined as $\frac{k_j(\lambda_0+1)(u^r-u^m)}{3(u^m)^2-x^*}$, a finite value.

5.6.3 Eigenfuncton for λ

Recall that on $\mathcal{M}_\epsilon^\pm$ we have the system (5.10)–(5.13), or equivalently

$$u_x = 0, \tag{5.68}$$

$$U_x = \frac{\lambda}{x - 3u^2}U + \mathcal{O}(\epsilon), \tag{5.69}$$

$$\lambda_x = 0. \tag{5.70}$$

The solutions of (5.68)–(5.70) to lowest order are

$$u = u_0, \tag{5.71}$$
$$U = \tilde{C}(x - 3u_0^2)^{\lambda_0}, \tag{5.72}$$
$$\lambda = \lambda_0. \tag{5.73}$$

Recall that $x = s_2$ and $x = s_3$ are the speeds of the two shock waves in the Riemann solution, and recall that for the real number λ_0 given by (5.66), we have found $U_1(\infty)$ for LRD1 and $U_1(-\infty)$ for LRD2. Recall the curves from section 5.2, $\{C_1, C_2, ..., C_8\}$. Notice that the trajectory of reference for LRD1 when $\epsilon > 0$ is simply a solution near $\{C_4, C_5, C_6\}$. The trajectory of reference for LRD2 when $\epsilon > 0$ is near $\{C_6, C_7, C_8\}$. Equating (5.72) with the $U(\pm\infty)$ values derived earlier enables us to obtain the two constants. Let $\tilde{C}_j$ denote the constant $\tilde{C}$ in (5.72) for the two solutions of concern. Then,

$$\tilde{C}_1 = \frac{k_2(\lambda_0+1)(u^m - u^\ell)}{(s_2 - 3(u^m)^2)^{\lambda_0+1}} \qquad \text{and} \qquad \tilde{C}_2 = \frac{k_3(\lambda_0+1)(u^m - u^r)}{(s_3 - 3(u^m)^2)^{\lambda_0+1}}$$

Since we are interested in the continuous solution, we look to see when the two solutions lie on the same trajectory (5.72). That is, we find conditions in which $\tilde{C}_1 = \tilde{C}_2$.

$$\begin{aligned} \frac{k_2(\lambda_0+1)(u^m - u^\ell)}{(s_2 - 3(u^m)^2)^{\lambda_0+1}} &= \frac{k_3(\lambda_0+1)(u^m - u^r)}{(s_3 - 3(u^m)^2)^{\lambda_0+1}} \\ \Rightarrow \frac{k_2\left(u^m - u^\ell\right)}{k_3(s_2 - 3(u^m)^2)^{\lambda_0+1}} &= \frac{(u^m - u^r)}{(s_3 - 3(u^m)^2)^{\lambda_0+1}}, \text{ for } k_3 \neq 0 \\ \Rightarrow \frac{k_2\left(u^m - u^\ell\right)}{k_3\left(u^m - u^r\right)} &= \left(\frac{s_2 - 3(u^m)^2}{s_3 - 3(u^m)^2}\right)^{\lambda_0+1} \quad \text{for } k_3 \neq 0. \end{aligned}$$

Please notice that since $0 < s_2 < s_3 < 3(u^m)^2$, $\frac{s_2 - 3(u^m)^2}{s_3 - 3(u^m)^2} > 1$. Then

$$\left(\frac{s_2 - 3(u^m)^2}{s_3 - 3(u^m)^2}\right)^{\lambda_0+1} > 0,$$

for all $\lambda_0 \in \mathbb{R}$. This implies that $\frac{k_2\left(u^m - u^\ell\right)}{k_3(u^m - u^r)} > 0$.

In [6], it was proved that when $u^\ell > \frac{2\alpha\sqrt{2}}{3\sqrt{\beta}}$, we have the ordering $u^m < u^r < u^\ell$. This gives the result that both $u^m - u^\ell$ and $u^m - u^r$ are negative, so the quotient is positive. Since $\frac{k_2\left(u^m - u^\ell\right)}{k_3(u^m - u^r)} > 0$, we have $\frac{k_2}{k_3} > 0$ and $\ln\left(\frac{k_2\left(u^m - u^\ell\right)}{k_3(u^m - u^r)}\right) \in \mathbb{R}$. Thus,

$$\lambda_0 = \frac{\ln\left(\frac{k_2\left(u^m - u^\ell\right)}{k_3(u^m - u^r)}\right)}{\ln\left(\frac{s_2 - 3(u^m)^2}{s_3 - 3(u^m)^2}\right)} - 1. \tag{5.74}$$

We can observe the following from (5.74):

1. If $\lambda_0 \le -1$, then $0 < \frac{k_2}{k_3} \le \frac{u^m - u^r}{u^m - u^\ell}$.
2. If $\lambda_0 > -1$, then $\frac{k_2}{k_3} > \frac{u^m - u^r}{u^m - u^\ell} > 0$.

We are able to define the range of values for k_2 and k_3 in terms of the other, for the given λ_0 value (5.66). Since the signs of k_2 and k_3 are the same, the related eigenfunction will have a consistent sign as x varies.

5.6.4 Discussion

Recall the description of $\mathcal{W}_0^u(\mathcal{M}^{u^\ell})$ and $\mathcal{W}_0^s(\mathcal{M}_0^-)$ in section 5.1. For a fixed λ_f, $\mathcal{W}_0^u(\mathcal{M}^{u^\ell})$ and $\mathcal{W}_0^s(\mathcal{M}_0^-)$ are 3 and 5 dimensional respectively. Their intersection consists of the curve C_5, defined in section 5.2. The curve C_5 is defined for all $\xi, k_2 \in \mathbb{R}$, making it 2-dimensional. Having a 2-dimensional intersection implies that the two spaces are not transverse. The calculation of λ_0 given by (5.66), leads us to conclude that for small $\epsilon > 0$ and fixed $\lambda_f \neq \lambda_0 + \epsilon\lambda_1 + \mathcal{O}(\epsilon^2)$, the intersection of $\mathcal{W}_\epsilon^u(\mathcal{M}^{u^\ell})$ and $\mathcal{W}_\epsilon^s(\mathcal{M}_0^-)$ becomes transverse. This intersection is given by the curve (5.9).

Recall that for a fixed $\lambda_f, \mathcal{W}_0^s(\mathcal{M}^{u^r})$ and $\mathcal{W}_0^u(\mathcal{M}_0^-)$ are both 5-dimensional. So $\mathcal{W}_0^s(\mathcal{M}^{u^r})$ and $\mathcal{W}_0^u(\mathcal{M}_0^-)$ are transverse if their intersection is 3-dimensional. Observe that for each x, there is saddle-to-attractor connection denoted by $u(\xi)$. The intersection contains the family of curves C_7. Since there are multiples of $\dot{u}, \dot{v}$, and $\dot{w}$ in the linear portion, another dimension is added to the intersection. So the intersection is parameterized by the triple (x, ξ, k_3). Thus, the two manifolds are transverse for fixed λ_f and small $\epsilon > 0$.

Notice that the dimensions of both $\mathcal{W}_\epsilon^u(\mathcal{M}^{u^\ell})$ and $\mathcal{W}_0^u(\mathcal{M}^{u^m})$ are 3 for a fixed λ_f. To verify that a portion of $\mathcal{W}_\epsilon^u(\mathcal{M}^{u^\ell})$ is C^1 close to a portion of $\mathcal{W}_0^u(\mathcal{M}^{u^m})$ we need an extended version of the Exchange Lemma. Schecter creates in [12] an extended version of the Exchange Lemma. We should be able to use this extended Exchange Lemma to track $\mathcal{W}_\epsilon^u(\mathcal{M}^{u^\ell})$ as ξ increases and track $\mathcal{W}_\epsilon^s(\mathcal{M}^{u^r})$ as ξ decreases. As we track $\mathcal{W}_\epsilon^s(\mathcal{M}^{u^r})$ as ξ decreases, we will see that it will be close to $\mathcal{W}_0^s(\mathcal{M}_0^-)$. As we follow $\mathcal{W}_\epsilon^u(\mathcal{M}^{u^\ell})$ as ξ increases, we will find that it becomes close to $\mathcal{W}_0^u(\mathcal{M}^{u^m}) \subset \mathcal{W}_0^u(\mathcal{M}_0^-)$. Since $\mathcal{W}_0^u(\mathcal{M}^{u^m})$ and $\mathcal{W}_0^s(\mathcal{M}_0^-)$ are transverse, C^1 spaces near by the two manifolds will be transverse as well. So for small $\epsilon > 0$ and fixed λ_f, $\mathcal{W}_\epsilon^u(\mathcal{M}^{u^\ell})$ and $\mathcal{W}_\epsilon^s(\mathcal{M}^{u^r})$ are transverse. The intersection of $\mathcal{W}_\epsilon^u(\mathcal{M}^{u^\ell})$ and $\mathcal{W}_\epsilon^s(\mathcal{M}^{u^r})$ is a curve that starts out near C_4 and ends near C_8. It is given by (5.14), which has a 0 U-component. Thus, λ_f is not an eigenvalue of (4.1). This means that when $\lambda_f \neq \lambda_0 + \epsilon\lambda_1 + \mathcal{O}(\epsilon^2)$, then $\mathcal{W}_\epsilon^u(\mathcal{M}^{u^\ell})$ and $\mathcal{W}_\epsilon^s(\mathcal{M}^{u^r})$ becomes transverse. The details of this have not been worked out.

We will now examine the linear system (4.7)–(4.10) further. Analyzing this system will assist us in finding properties possessed by the eigenpairs of (4.1).

Chapter 6

Resolvent Set

Recall the linear equation (4.2),

$$\lambda U + (3\hat{u}_\epsilon^2 - x)U_x + 6\hat{u}_\epsilon \hat{u}_\epsilon' U = \alpha\epsilon U_{xx} + \beta\epsilon^2 U_{xxx}.$$

We can write this as the system $Y_\xi = BY$, where $Y = (U, V, W)^T$ and

$$B(\rho, u, v, x) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{\rho+6uv}{\beta} & \frac{3u^2-x}{\beta} & -\frac{\alpha}{\beta} \end{pmatrix}.$$

In addition, recall from Chapter 4 that

$$A(\rho, u, x) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{\rho}{\beta} & \frac{3u^2-x}{\beta} & -\frac{\alpha}{\beta} \end{pmatrix}.$$

From Chapter 5, we know that $u_\epsilon(\xi), \xi = \frac{x}{\epsilon}$, is the Riemann–Dafermos solution, and $v_\epsilon(\xi) = \dot{u}_\epsilon(\xi)$. So we have

$$\hat{B}(\rho, \epsilon, \xi) = B(\rho, u_\epsilon, v_\epsilon, \epsilon\xi) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{\rho+6u_\epsilon v_\epsilon}{\beta} & \frac{3u_\epsilon^2-\epsilon\xi}{\beta} & -\frac{\alpha}{\beta} \end{pmatrix}.$$

Let u^* be either u^ℓ or u^r. We define

$$\hat{A}(\rho, \epsilon, \xi) = A(\rho, u^*, \epsilon\xi) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{\rho}{\beta} & \frac{3(u^*)^2-\epsilon\xi}{\beta} & -\frac{\alpha}{\beta} \end{pmatrix}.$$

For $x = \epsilon\xi$, the linear equation (4.2), can also be written as

$$Y_\xi = \hat{B}(\rho, \epsilon, \xi)Y. \tag{6.1}$$

For the sparse matrix

$$\tilde{A}(\epsilon,\xi) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{6u_\epsilon v_\epsilon}{\beta} & \frac{3\left(u_\epsilon^2-(u^*)^2\right)}{\beta} & 0 \end{pmatrix},$$

$\hat{B}(\rho,\epsilon,\xi)$ is a sum of $\hat{A}(\rho,\epsilon,\xi)$ and $\tilde{A}(\epsilon,\xi)$.

We begin by studying the matrix $A(\rho,u^*,x)$. Information discovered about the system,

$$Y_\xi = A(\rho,u^*,x)Y, \tag{6.2}$$

when $x=\epsilon\xi$ will lead to discovering vital properties of (6.1). We show that there exists a spectral gap between the eigenvalues of $A(\rho,u^*,x)$ for $|x|$ large. Consider H such that its columns constitutes the eigenspace of $A(\rho,u^*,x)$. This spectral gap property leads to the system $Y_\xi = H^{-1}\hat{A}(\rho,\epsilon,\xi)HY$ having a pseudoexponential dichotomy. We then are able to show that the system $Y_\xi = H^{-1}\hat{B}(\rho,\epsilon,\xi)HY$ has a pseudoexponential dichotomy as well with a nearby projection matrix. Consequently, we are able to locate the eigenvalues with geometric multiplicity one and resolvent values of the operator,

$$\mathcal{T}^\xi[U] = \beta U_{\xi\xi\xi} + \alpha U_{\xi\xi} - (3u_\epsilon^2 - \epsilon\xi)U_\xi - 6u_\epsilon\dot{u}_\epsilon U.$$

This allows us to determine the location the corresponding eigenvalues and resolvent values of the operator form of (4.1),

$$\mathcal{E}[U] = \beta\epsilon^2 U_{xxx} + \alpha\epsilon U_{xx} - (3\hat{u}^2 - x)U_x - 6\hat{u}\hat{u}_x U,$$

which is our interest.

6.1 Spectral Gap

For $\theta,\omega,a,b\in\mathbb{R}$, let $\rho=\theta+i\omega$ and let $\mu=a+ib$ be an eigenvalue of $A(\rho,u^*,x)$. Recall that the characteristic equation of $A(\rho,u^*,x)$ is

$$0 = \mu^3 + \frac{\alpha}{\beta}\mu^2 + \frac{x-3u^{*2}}{\beta}\mu - \frac{\rho}{\beta}.$$

Substituting the expressions for μ and ρ into this equation results in

$$\begin{aligned} 0 &= (a+bi)^3 + \frac{\alpha}{\beta}(a+bi)^2 + \frac{x-3u^{*2}}{\beta}(a+bi) - \frac{\theta+i\omega}{\beta}, \\ 0 &= \left[a^3 - 3ab^2 + \frac{\alpha}{\beta}(a^2-b^2) + \frac{a(x-3u^{*2})}{\beta} - \frac{\theta}{\beta}\right] \\ &\quad +i\left[3a^2b - b^3 + \frac{2ab\alpha}{\beta} + \frac{b(x-3u^{*2})}{\beta} - \frac{\omega}{\beta}\right]. \end{aligned}$$

This gives rise to the two equations

$$0 = a^3 - 3ab^2 + \frac{\alpha}{\beta}(a^2 - b^2) + \frac{a(x - 3u^{*2})}{\beta} - \frac{\theta}{\beta}, \tag{6.3}$$

$$0 = 3a^2b - b^3 + \frac{2ab\alpha}{\beta} + \frac{b(x - 3u^{*2})}{\beta} - \frac{\omega}{\beta}. \tag{6.4}$$

From (6.3) we have, $b^2 = \frac{\beta a^3 + \alpha a^2 + a(x - 3u^{*2}) - \theta}{3a\beta + \alpha}$. If $a > -\frac{\alpha}{3\beta}$, then $\beta a^3 + \alpha a^2 + a(x - 3u^{*2}) - \theta > 0$. This implies that $\theta < -a(-\beta a^2 - \alpha a + 3u^{*2} - x)$. Consider $m > 0$ such that $m >> \frac{\alpha^2}{3\beta}$. We consider m sufficiently large so that

1. if $x \leq 3u^{*2} - m$, then x is close to $-\infty$, or
2. if $x \geq 3u^{*2} + m$, then x is close to ∞.

Please notice that if $x \leq 3u^{*2} - m$, then $x < 3u^{*2} - \frac{\alpha^2}{3\beta}$ and if $x \geq 3u^{*2} + m$, then $x > 3u^{*2} + \frac{\alpha^2}{3\beta}$.

Lemma 5 *For* $0 < \delta < \frac{\alpha^3}{192\beta^2}$*, if* $a \in I_1 = [a_1, b_1] = \left[\frac{48\beta\delta}{\alpha^2}, \frac{\alpha}{4\beta}\right]$ *and* $x \in J_1 = \{x | x \leq 3u^{*2} - m\}$*, then* $\theta < -\delta$*.*

Proof. For each $\delta > 0$, the assumptions $a \in I_1$ and $x \in J_1$ gives us the following conclusion:

$$\delta = \frac{48\beta\delta}{\alpha^2}\left(\frac{\alpha^2}{48\beta}\right) \leq a\left(\frac{\alpha^2}{48\beta}\right) = a\left(-\frac{\alpha^2}{16\beta} - \frac{\alpha^2}{4\beta} + \frac{\alpha^2}{3\beta}\right) < a(-\beta a^2 - \alpha a + 3u^{*2} - x) < -\theta.$$

□

Lemma 6 *For* $0 < \delta < \frac{\alpha^3}{192\beta^2}$*, if* $a \in I_2 = [a_2, b_2] = \left[-\frac{\alpha}{4\beta}, -\frac{48\beta\delta}{\alpha^2}\right]$ *and* $x \in J_2 = \{x | x \geq 3u^{*2} + m\}$*, then* $\theta < -\delta$*.*

Proof. For each $\delta > 0$, the assumptions $a \in I_2$ and $x \in J_2$ gives us the following conclusion:

$$\begin{aligned} -\delta &\geq a\left(\frac{\alpha^2}{48\beta}\right) \\ &= -a\left(-\frac{\alpha^2}{48\beta}\right) \\ &= -a\left(\frac{\alpha^2}{16\beta} + \frac{\alpha^2}{4\beta} - \frac{\alpha^2}{3\beta}\right) \\ &> -a\left(-\beta a^2 - \alpha a + 3u^{*2} - x\right) \\ &> \theta. \end{aligned}$$

□

Theorem 5 *For $\delta > 0$ define $K_\delta = \{\rho | Re(\rho) \geq -\delta\}$. Then for $0 < \delta < \frac{\alpha^3}{192\beta^2}$ and $x \in J_j$, two eigenvalues of $A(\rho, u^*, x)$ have real part less than a_j, and one eigenvalue of $A(\rho, u^*, x)$ has real part greater than b_j when $\rho \in K_\delta$.*

Proof. Note that $0 \in K_\delta$. The eigenvalues of $A(0, u^*, x)$ satisfy the following for $u = u^*$:

$$0 = \mu^3 + \frac{\alpha}{\beta}\mu^2 + \frac{x - 3u^{*2}}{\beta}\mu = \mu\left(\mu^2 + \frac{\alpha}{\beta}\mu + \frac{x - 3u^{*2}}{\beta}\right) = \mu(\mu - \mu_-)(\mu - \mu_+).$$

The nonzero roots are given by

$$\mu_\pm = -\frac{\alpha}{2\beta} \pm \sqrt{\left(\frac{\alpha}{2\beta}\right)^2 + \frac{3u^{*2} - x}{\beta}}.$$

First we consider $x \in J_1$. This will result in μ_- and μ_+ having opposing signs. In addition, we have the following result:

$$\begin{aligned} 3u^{*2} - x &> \frac{\alpha^2}{3\beta}, \\ \Rightarrow \frac{\alpha^2}{4\beta^2} + \frac{3u^{*2} - x}{\beta} &> \frac{7\alpha^2}{12\beta^2} > \frac{9\alpha^2}{16\beta^2}, \\ \Rightarrow \sqrt{\frac{\alpha^2}{4\beta^2} + \frac{3u^{*2} - x}{\beta}} &> \frac{3\alpha}{4\beta}, \\ \Rightarrow -\frac{\alpha}{2\beta} + \sqrt{\frac{\alpha^2}{4\beta^2} + \frac{3u^{*2} - x}{\beta}} &> \frac{\alpha}{4\beta}, \\ \Rightarrow \mu_+ &> \frac{\alpha}{4\beta}. \end{aligned}$$

Hence for $x \in J_1$, the eigenvalues of $A(0, u^*, x)$ divide into two sets:

Set 1: $\{0, \mu_-\}$ – The real parts are less than $\frac{48\beta\delta}{\alpha^2} = a_1$.
Set 2: $\{\mu_+\}$ – The real part is greater than $\frac{\alpha}{4\beta} = b_1$.

Now consider $x \in J_2$. The assumption $x > 3u^{*2} + \dfrac{\alpha^2}{3\beta}$ implies that $Re(\mu_\pm) = -\frac{\alpha}{2\beta} < -\frac{\alpha}{4\beta}$. When $x \in J_2$, the eigenvalues of $A(0, u^*, x)$ split into the following sets:

Set 1: $\{\mu_-, \mu_+\}$ – The real parts are less than $-\frac{\alpha}{4\beta} = a_2$.
Set 2: $\{0\}$ – The real part is greater than $-\frac{48\beta\delta}{\alpha^2} = b_2$.

We have shown that for $x \in J_j$, two eigenvalues of $A(0, u^*, x)$ have real part less than a_j and one has real part greater than b_j. Moreover, from Lemmas 5 and 6, we know that when $x \in J_j$ and $\rho \in K_\delta$, no eigenvalue has real part in I_j. Since the eigenvalues depend continuously on (ρ, u^*, x), two eigenvalues of $A(\rho, u^*, x)$ have real part less than a_j and one has real part greater than b_j, for all $x \in J_j$ and $\rho \in K_\delta$. □

6.2 Pseudoexponential Dichotomy

Let the eigenvalues of $A(\rho, u^*, x)$ be denoted as $\left\{\mu_j = \mu_j(\rho, u^*, x)\right\}_{j=1}^3$. We will first look at the behavior of these eigenvalues for $|x|$ large. This is done to determine the appropriate scale of the eigenvectors so that they remain bounded as $|x| \to \infty$. This in turn will lead to (6.1) having a pseudoexponential dichotomy on J_j.

6.2.1 Eigenvalues of $A(\rho, u^*, x)$ for $|x|$ Large

Let $q = \frac{1}{x}$. As $x \to \pm\infty, q \to 0^{\pm}$. Multipying (6.3) and (6.4) by q generates

$$\begin{aligned} 0 &= -qa^3 + 3aqb^2 - \frac{q\alpha}{\beta}(a^2 - b^2) + \frac{3qau^{*2}}{\beta} - \frac{a}{\beta} + \frac{q\theta}{\beta} \\ &= f_1(a, b, q), \\ 0 &= -3qa^2b + qb^3 - \frac{2qab\alpha}{\beta} + \frac{bq3u^{*2}}{\beta} - \frac{b}{\beta} + \frac{q\omega}{\beta} \\ &= f_2(a, b, q). \end{aligned} \tag{6.5}$$

Denote $f(a, b, q) = \big(f_1(a, b, q), f_2(a, b, q)\big)$. One solution to $f(a, b, q) = (0, 0)$ is given by $(a, b, q) = (0, 0, 0)$. The Jacobian evaluated at this solution is

$$Df(0, 0, 0) = \begin{pmatrix} -\frac{1}{\beta} & 0 & \frac{\theta}{\beta} \\ 0 & -\frac{1}{\beta} & \frac{\omega}{\beta} \end{pmatrix}.$$

Notice that $D_{(a,b)}f(0, 0, 0) = \begin{pmatrix} -\frac{1}{\beta} & 0 \\ 0 & -\frac{1}{\beta} \end{pmatrix}$ is nonsingular. The Implicit Function Theorem tells us that near $(0, 0, 0)$, the solutions of the equation $f(a, b, q) = 0$ are given by $(a, b) = (\pi_1(q), \pi_2(q)) = \pi(q)$, with $\pi(0) = (0, 0)$. So we have

$$0 = f_1(a, b, q) \approx -\frac{a}{\beta} + \frac{q\theta}{\beta},$$

$$0 = f_2(a, b, q) \approx -\frac{b}{\beta} + \frac{q\omega}{\beta}.$$

This implies that $a \approx q\theta$ and $b \approx q\omega$. This tells us that one of the eigenvalue is given by $\mu = a + bi = (\theta + i\omega)q + \mathcal{O}(q^2) = \rho q + \mathcal{O}(q^2)$. Without loss of generality, we will let $\mu_3 = \rho q + \mathcal{O}(q^2)$. This eigenvalue is close to zero as $x \to \pm\infty$ for any given ρ.

From the relationships (4.11) we have

$$\begin{aligned} \mu_1 + \mu_2 &= -\frac{\alpha}{\beta} - \mu_3 = -\frac{\alpha}{\beta} - \beta q + \mathcal{O}(q^2), \\ \mu_1\mu_2 &= \frac{\rho}{\beta\mu_3} = \frac{\rho}{\beta q + \mathcal{O}(q^2)}. \end{aligned}$$

This tells us that as $|x| \to \infty$, $\mu_1 + \mu_2$ and $|\mu_1\mu_2|$ approaches $-\frac{\alpha}{\beta}$ and ∞ respectively. This reveals that $\frac{1}{|\mu_1\mu_2|} \to 0$ as $x \to \pm\infty$. Also from the equations (4.11), we can conclude that $\frac{\rho}{\beta\mu_1\mu_3} = -\frac{\alpha}{\beta} - \mu_1 - \mu_3$. This implies that $0 = \beta\mu_3\mu_1^2 + (\alpha + \beta\mu_3)\mu_3\mu_1 + \rho$. This forces the values of μ_1 and μ_2 to be

$$\begin{aligned}
\mu_{1,2} &= -\frac{\alpha + \beta q\rho}{2\beta} + \mathcal{O}(q^2) \pm \sqrt{\left(\frac{\alpha + \beta q\rho}{2\beta}\right)^2 + \mathcal{O}(q^2) - \frac{1}{\beta q + \mathcal{O}(q^2)}} \\
&= \mathcal{O}(q^0) + \mathcal{O}(q^1) + \mathcal{O}(q^2) \pm \sqrt{\mathcal{O}(q^0) + \mathcal{O}(q^1) + \mathcal{O}(q^2) + \mathcal{O}(q^{-1})} \\
&= \mathcal{O}(q^0) \pm \sqrt{\mathcal{O}(q^{-1})} \\
&= \mathcal{O}(q^{-\frac{1}{2}}).
\end{aligned}$$

Please note that as $|x| \to \infty$, $|\mu_1|, |\mu_2| \to \infty$ for any given ρ.

6.2.2 Pseudostable Projector

Lemma 7 *Let* $0 < \delta < \frac{\alpha^3}{192\beta^2}$. *If* $x \in J_j$ *and* $\rho \in K_\delta$, *the eigenvalues of* $A(\rho, u^*, x)$ *are distinct.*

Proof. We know from Theorem 5 that if $\rho \in K_\delta = \{\rho | Re(\rho) \geq -\delta, \delta > 0\}$ and $x \in J_j$, the eigenvalues will split into two disjoint sets. Thus, all 3 eigenvalues will not be the same. Assume that μ_1 has algebraic multiplicity 2.

Observe from 4.11 that,

$$\begin{aligned}
\frac{\rho}{\beta} &= \mu_1^2\mu_3 \\
\frac{\alpha}{\beta} = -2\mu_1 - \mu_3 \;\Rightarrow\; & \mu_3 = -\frac{\alpha}{\beta} - 2\mu_1, \\
\Rightarrow \frac{x - 3u^{*2}}{\beta} &= -3\mu_1^2 - \frac{2\alpha}{\beta}\mu_1, \\
\Rightarrow 0 &= \mu_1^2 + \frac{2\alpha}{3\beta}\mu_1 + \frac{x - 3u^{*2}}{3\beta}.
\end{aligned}$$

Hence, the two possibilities for μ_1 are $-\frac{\alpha}{3\beta} \pm \sqrt{\left(\frac{\alpha}{3\beta}\right)^2 + \frac{3u^{*2} - x}{3\beta}}$. This gives rise to two corresponding possibilities for μ_3

$$-\frac{\alpha}{\beta} - 2\mu_1 = -\frac{\alpha}{3\beta} \mp 2\sqrt{\left(\frac{\alpha}{3\beta}\right)^2 + \frac{3u^{*2} - x}{3\beta}}.$$

For $x \in J_2, x \geq 3u^{*2} + m > 3u^{*2} + \frac{\alpha^2}{3\beta}$, we have

$$\begin{aligned} x - 3u^{*2} &> \frac{\alpha^2}{3\beta}, \\ \Rightarrow \frac{x - 3u^{*2}}{3\beta} &\geq \left(\frac{\alpha}{3\beta}\right)^2, \\ \Rightarrow 0 &\geq \left(\frac{\alpha}{3\beta}\right)^2 + \frac{3u^{*2} - x}{3\beta}, \\ \Rightarrow Re(\mu_k) &= -\frac{\alpha}{3\beta} \text{ for } k = 1, 2, 3 \text{ with } \mu_1 = \mu_2. \end{aligned}$$

This contradicts the result of Theorem 5. One of the eigenvalues must have real part greater than b_2.

For $x \in J_1$, x is close to $-\infty$. we know that when x is close to $-\infty$, there is one eigenvalue of A that is close to 0. However, as x gets close to $-\infty$, μ_1 gets close to $\pm\infty$ and the related μ_3 gets close to $\mp\infty$. This contradicts the fact that one eigenvalue is close to 0 as x gets close to $-\infty$.

Since the assumption that an eigenvalue of $A(\rho, u^*, x)$ has multiplicity greater than one leads to a contradiction, all eigenvalues of $A(\rho, u^*, x)$ are distinct on J_j when $\rho \in K_\delta$. $\square$

The eigenvectors of $A(\rho, u^*, x)$ are

$$\left\{ \begin{pmatrix} 1 \\ \mu_1 \\ \mu_1^2 \end{pmatrix}, \begin{pmatrix} 1 \\ \mu_2 \\ \mu_2^2 \end{pmatrix}, \begin{pmatrix} 1 \\ \mu_3 \\ \mu_3^2 \end{pmatrix} \right\}.$$

Please note that when $x \in J_1$, we can order the eigenvalue by $Re(\mu_1) < Re(\mu_3) < a_1 < b_1 < Re(\mu_2)$. However, for $x \in J_2$, we can order the eigenvalue by $Re(\mu_1) < Re(\mu_2) < a_2 < b_2 < Re(\mu_3)$. The pseudostable space is the space spanned by the eigenvectors associated with the eigenvalues of $A(\rho, u^*, x)$ with real part less than a_j. The pseudounstable space is spanned by the eigenvector associated with the eigenvalue with real part greater than b_j.

For a fixed $\rho \in K_\delta$ and $x \in J_1$, we define

$$H_1(\rho, u^*, x) = \begin{pmatrix} \frac{1}{\mu_1^2} & 1 & \frac{1}{\mu_2^2} \\ \frac{1}{\mu_1} & \mu_3 & \frac{1}{\mu_2} \\ 1 & \mu_3^2 & 1 \end{pmatrix}.$$

Its inverse is given by

$$H_1^{-1}(\rho, u^*, x) = \begin{pmatrix} \frac{\mu_1^2\mu_2\mu_3}{(\mu_3-\mu_1)(\mu_2-\mu_1)} & -\frac{\mu_1^2(\mu_2+\mu_3)}{(\mu_3-\mu_1)(\mu_2-\mu_1)} & \frac{\mu_1^2}{(\mu_3-\mu_1)(\mu_2-\mu_1)} \\ \frac{\mu_1\mu_2}{(\mu_3-\mu_2)(\mu_3-\mu_1)} & -\frac{\mu_1+\mu_2}{(\mu_3-\mu_2)(\mu_3-\mu_1)} & \frac{1}{(\mu_3-\mu_2)(\mu_3-\mu_1)} \\ -\frac{\mu_1\mu_2^2\mu_3}{(\mu_3-\mu_2)(\mu_2-\mu_1)} & \frac{\mu_2^2(\mu_1+\mu_3)}{(\mu_3-\mu_2)(\mu_2-\mu_1)} & -\frac{\mu_2^2}{(\mu_3-\mu_2)(\mu_2-\mu_1)} \end{pmatrix}.$$

For a fixed $\rho \in K_\delta$ and $x \in J_2$, define

$$H_2(\rho, u^*, x) = \begin{pmatrix} \frac{1}{\mu_1^2} & \frac{1}{\mu_2^2} & 1 \\ \frac{1}{\mu_1} & \frac{1}{\mu_2^2} & \mu_3 \\ 1 & 1 & \mu_3^2 \end{pmatrix},$$

and its inverse as

$$H_2^{-1}(\rho, u^*, x) = \begin{pmatrix} \frac{\mu_1^2\mu_2\mu_3}{(\mu_3-\mu_1)(\mu_2-\mu_1)} & -\frac{\mu_1^2(\mu_2+\mu_3)}{(\mu_3-\mu_1)(\mu_2-\mu_1)} & \frac{\mu_1^2}{(\mu_3-\mu_1)(\mu_2-\mu_1)} \\ -\frac{\mu_1\mu_2^2\mu_3}{(\mu_3-\mu_2)(\mu_2-\mu_1)} & \frac{\mu_2^2(\mu_1+\mu_3)}{(\mu_3-\mu_2)(\mu_2-\mu_1)} & -\frac{\mu_2^2}{(\mu_3-\mu_2)(\mu_2-\mu_1)} \\ \frac{\mu_1\mu_2}{(\mu_3-\mu_2)(\mu_3-\mu_1)} & -\frac{\mu_1+\mu_2}{(\mu_3-\mu_2)(\mu_3-\mu_1)} & \frac{1}{(\mu_3-\mu_2)(\mu_3-\mu_1)} \end{pmatrix}.$$

The columns of H_j are the eigenvectors of A on J_j. The projection for the pseudostable space is given as $\mathcal{P}(\rho, u^*, x) = H_j P_2 H_j^{-1}$, where $P_2 = diag(1, 1, 0)$. For fixed $\rho \in K_\delta$, each entry of $\mathcal{P}(\rho, u^*, x)$ is well defined and is a continuous function of x. Therefore, $\mathcal{P}(\rho, u^*, x)$ is continuous with respect to x.

Proposition 2 *Let Ω be a compact set of $\mathbb{C}$. Then there exists a constant $k > 0$ such that for $0 < \delta < \frac{\alpha^3}{192\beta^2}, \rho \in \left(K_\delta \cap \Omega\right)$, and $x \in J_j$, $\|\mathcal{P}(\rho, u^*, x)(I - P_2)\| \leq k$.*

Proof. Observe that as $\xi \to -\infty$,

$$H_1 \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \text{ and } H_1^{-1}(I - P_2) = \begin{pmatrix} 0 & 0 & \frac{\mu_1^2}{(\mu_3-\mu_1)(\mu_2-\mu_1)} \\ 0 & 0 & \frac{1}{(\mu_3-\mu_2)(\mu_3-\mu_1)} \\ 0 & 0 & -\frac{\mu_2^2}{(\mu_3-\mu_2)(\mu_2-\mu_1)} \end{pmatrix} \to \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

As $x \to \infty$, we have

$$H_2 \to \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \text{ and } H_2^{-1}(I - P_2) = \begin{pmatrix} 0 & 0 & \frac{\mu_1^2}{(\mu_3-\mu_1)(\mu_2-\mu_1)} \\ 0 & 0 & -\frac{\mu_2^2}{(\mu_3-\mu_2)(\mu_2-\mu_1)} \\ 0 & 0 & \frac{1}{(\mu_3-\mu_2)(\mu_3-\mu_1)} \end{pmatrix} \to \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}.$$

Since both H_j and $H_j^{-1}(I - P_2)$ approach constant matrices when $|x| \to \infty$, there exists nonzero constants $k_{1j}, k_{2j} > 0$ such that $\|H_j\| \leq k_{1j}$ and $\|H_j^{-1}(I - P_2)\| \leq k_{2j}$. Let $k = \sqrt{2} \max\{k_{1j}^2, k_{2j}^2, k_{1j}k_{2j}\}$. Since $\mathcal{P}(\rho, u^*, x) = H_j P_2 H_j^{-1}$, then

$$\|\mathcal{P}(\rho, u^*, x)(I - P_2)\| = \|H_j P_2 H_j^{-1}(I - P_2)\| \leq \|H_j\|\|P_2\|\|H_j^{-1}(I - P_2)\| \leq k.$$

Therefore, for all $x \in J_j$ and for fixed $\rho \in K_\delta \cap \Omega$, we have $\|\mathcal{P}(\rho, u^*, x)(I - P_2)\|$ bounded by k. □

6.2.3 Pseudoexponential Dichotomy Property

Definition 1 *Suppose* $T(\xi,\zeta)$ *be a family of state transition matrices for the given system* $Y_\xi = D(\xi)Y$*. That system is said to have* **pseudoexponential dichotomy** *on* J *with spectral gap* $(\hat{a},\hat{b})$ *if there exist* $C > 0$ *and projections* $\mathcal{P}(\xi)$*, for* $\xi \in J$*, such that the following holds:*

(i) $\mathcal{P}(\xi)$ *is continuous on* J *and uniformly bounded on* J.

(ii) $T(\xi,\zeta)\mathcal{P}(\zeta) = \mathcal{P}(\xi)T(\xi,\zeta)$.

(iii) If $y_s \in \mathcal{R}(\mathcal{P}(\zeta))$*, and* $\xi > \zeta$*, then* $\|T(\xi,\zeta)y_s\| \le Ce^{\hat{a}(\xi-\zeta)}\|y_s\|$.

(iv) If $y_u \in \mathcal{R}(I - \mathcal{P}(\zeta))$*, and* $\xi < \zeta$*, then* $\|T(\xi,\zeta)y_u\| \le Ce^{\hat{b}(\xi-\zeta)}\|y_u\|$.

Theorem 6 (Coppel's Roughness Theorem) *Suppose* $Y_\xi = D(\xi)Y$ *has a pseudoexponential dichotomy on* J *with given projections* $P(\xi)$*. Let* C_0 *and* γ *be positive numbers. Then there exist positive numbers* ϵ_0 *and* L *such that the following is true. Suppose the pseudoexponential dichotomy has constants* $C > 0$ *and* $\hat{a} < \hat{b}$ *so that* $C < C_0$ *and* $\hat{b} - \hat{a} > \gamma$*. Let* $0 < \epsilon < \epsilon_0$*. If* $\|E(\xi)\| < \epsilon$ *for all* $\xi \in J$*, then the linear differential equation* $Y_\xi = (D(\xi) + E(\xi))Y$ *has a pseudoexponential dichotomy on* J *with projections* $\tilde{P}(\xi)$ *close to* $P(\xi)$*, constant* $\tilde{C}$ *close to* C*, and exponents* $\tilde{a}$ *and* $\tilde{b}$ *with* $\tilde{a} < \tilde{b}$ *and*

$$\begin{aligned}
&\|\tilde{P}(\xi) - P(\xi)\| < \epsilon L \text{ for all } \xi \in J,\\
&|\tilde{C} - C| < \epsilon L,\\
&|\tilde{a} - \hat{a}| < \epsilon L,\\
&|\tilde{b} - \hat{b}| < \epsilon L.
\end{aligned}$$

Theorem 7 *For each compact subset* Ω *of* $\mathbb{C}$*, there exist positive constants* ϵ_0 *and* L *such that the following is true. If* $0 < \delta < \frac{\alpha^3}{192\beta^2}, \rho \in \left(K_\delta \cap \Omega\right)$*, and* $0 < \epsilon < \epsilon_0$ *then*

$$Z_\xi = H_j^{-1}(\rho, u^*, \epsilon\xi)\hat{B}(\rho, \epsilon, \xi)H_j(\rho, u^*, \epsilon\xi)Z$$

has a pseudoexponential dichotomy on both $\bar{J}_j = \{\xi | \epsilon\xi \in J_j\}$ *with projections near* P_2*, constant* C *near 1, and exponents* $\tilde{a}_j < \tilde{b}_j$ *with* $|\tilde{a}_j - a_j| < \epsilon L$ *and* $|\tilde{b}_j - b_j| < \epsilon L$.

Lemma 8 *For* $0 < \delta < \frac{\alpha^3}{192\beta^2}, \rho \in K_\delta$*, and any* $\epsilon > 0$*, the system*

$$Z_\xi = H_j^{-1}(\rho, u^*, \epsilon\xi)\hat{A}(\rho, \epsilon, \epsilon\xi)H_j(\rho, u^*, \epsilon\xi)Z$$

has an pseudoexponential dichotomy on $\bar{J}_j$*. The projection in the dichotomy is* P_2*. The constant* C *is 1. The constants* $\hat{a}$ *and* $\hat{b}$ *for the intervals* J_j *are respectively* a_j *and* b_j*.*

Proof (Lemma). Choose $0 < \delta < \frac{\alpha^3}{192\beta^2}, \rho \in K_\delta$ and $\epsilon > 0$. Consider $x \in J_j$. Let $\mu_k(\rho, u^*, x)$, $k = 1, 2, 3$, be the eigenvalues of $A(\rho, u^*, x)$. Denote $\mu_k(x) = \mu_k(\rho, u^*, x)$. For $x = \epsilon\xi$, the eigenvalues take the form $\mu_k(\epsilon\xi) = \mu_k(\rho, u^*, \epsilon\xi)$. Let $N = H_j^{-1} A H_j$. We have

$$N|_{\bar{J}_1} = \begin{pmatrix} \mu_1(\epsilon\xi) & 0 & 0 \\ 0 & \mu_3(\epsilon\xi) & 0 \\ 0 & 0 & \mu_2(\epsilon\xi) \end{pmatrix},$$

$$N|_{\bar{J}_2} = \begin{pmatrix} \mu_1(\epsilon\xi) & 0 & 0 \\ 0 & \mu_2(\epsilon\xi) & 0 \\ 0 & 0 & \mu_3(\epsilon\xi) \end{pmatrix}.$$

Notice that the eigenvalues of N are the same as those of $\hat{A}(\rho, \epsilon, \xi)$. Consider the system

$$Z_\xi = N(\rho, \epsilon, \xi)Z. \tag{6.6}$$

Consider the system (6.6) on $\bar{J}_1$. The solution of (6.6) with constant initial condition $Z(\xi_0) = (Z_1(\xi_0), Z_2(\xi_0), Z_3(\xi_0))$ is

$$\begin{aligned} Z_1(\xi, \epsilon) &= Z_1(\xi_0, \epsilon) e^{\int_{\xi_0}^{\xi} \mu_1(\epsilon\tau) d\tau}, \\ Z_2(\xi, \epsilon) &= Z_2(\xi_0, \epsilon) e^{\int_{\xi_0}^{\xi} \mu_3(\epsilon\tau) d\tau}, \\ Z_3(\xi, \epsilon) &= Z_3(\xi_0, \epsilon) e^{\int_{\xi_0}^{\xi} \mu_2(\epsilon\tau) d\tau}. \end{aligned}$$

The state transition matrix for (6.6) is given by

$$R(\xi, \xi_0, \epsilon) = diag\left(e^{\int_{\xi_0}^{\xi} \mu_1(\epsilon\tau) d\tau}, e^{\int_{\xi_0}^{\xi} \mu_3(\epsilon\tau) d\tau}, e^{\int_{\xi_0}^{\xi} \mu_2(\epsilon\tau) d\tau} \right).$$

Denote $\vec{0}$ as a column vector of zeros and $R_{i,*}$ and $R_{*,j}$ as the i^{th} row and j^{th} column of $R(\xi, \xi_0, \epsilon)$ respectively. Observe that the two matrices $\begin{pmatrix} R_{1,*} \\ R_{2,*} \\ \vec{0}^\top \end{pmatrix}$ and $\begin{pmatrix} R_{*,1} & R_{*,2} & \vec{0} \end{pmatrix}$ are equal. Thus, the equation $R(\xi, \xi_0, \epsilon)P_2 = P_2 R(\xi, \xi_0, \epsilon)$ is satisfied.

For $\xi \in \bar{J}_1$ and $\xi > \xi_0$, we observe the following:

$$\begin{aligned} |Z_1(\xi, \epsilon)| &= \left| Z_1(\xi_0, \epsilon) e^{\int_{\xi_0}^{\xi} \mu_1(\epsilon\tau) d\tau} \right| \\ &= |Z_1(\xi_0, \epsilon)| e^{\int_{\xi_0}^{\xi} Re\left(\mu_1(\epsilon\tau)\right) d\tau} \\ &\leq |Z_1(\xi_0, \epsilon)| e^{\int_{\xi_0}^{\xi} \frac{48\beta\delta}{\alpha^2} d\tau} \\ &= |Z_1(\xi_0, \epsilon)| e^{\frac{48\beta\delta}{\alpha^2}(\xi - \xi_0)} \end{aligned}$$

$$\begin{aligned} |Z_2(\xi,\epsilon)| &= \left| Z_2(\xi_0,\epsilon) e^{\int_{\xi_0}^{\xi} \mu_3(\epsilon\tau)d\tau} \right| \\ &= |Z_2(\xi_0,\epsilon)| e^{\int_{\xi_0}^{\xi} Re\left(\mu_3(\epsilon\tau)\right)d\tau} \\ &\leq |Z_2(\xi_0,\epsilon)| e^{\int_{\xi_0}^{\xi} \frac{48\beta\delta}{\alpha^2} d\tau} \\ &= |Z_2(\xi_0,\epsilon)| e^{\frac{48\beta\delta}{\alpha^2}(\xi-\xi_0)} \end{aligned}$$

For $\xi < \xi_0$,

$$\begin{aligned} |Z_3(\xi,\epsilon)| &= |Z_3(\xi_0,\epsilon)| e^{\int_{\xi_0}^{\xi} Re\left(\mu_2(\epsilon\tau)\right)d\tau} \\ &= |Z_3(\xi_0,\epsilon)| e^{\int_{\xi}^{\xi_0} -Re\left(\mu_2(\epsilon\tau)\right)d\tau} \\ &\leq |Z_3(\xi_0,\epsilon)| e^{\int_{\xi}^{\xi_0} -\frac{\alpha}{4\beta} d\tau} \\ &= |Z_3(\xi_0,\epsilon)| e^{\frac{\alpha}{4\beta}(\xi-\xi_0)} \end{aligned}$$

For $j = 1, 2$ and $\xi > \xi_0$,

$$|Z_j(\xi,\epsilon)| \leq |Z_j(\xi_0,\epsilon)| e^{\frac{48\beta\delta}{\alpha^2}(\xi-\xi_0)}.$$

We have for $\xi < \xi_0$,

$$|Z_3(\xi,\epsilon)| \leq |Z_3(\xi_0,\epsilon)| e^{\frac{\alpha}{4\beta}(\xi-\xi_0)}.$$

Now consider $\xi \in \bar{J}_2$. The solution of (6.6) with constant initial condition $Z(\xi_0,\epsilon) = \Big(Z_1(\xi_0,\epsilon), Z_2(\xi_0,\epsilon), Z_3(\xi_0,\epsilon)\Big)$ is

$$\begin{aligned} Z_1(\xi,\epsilon) &= Z_1(\xi_0,\epsilon) e^{\int_{\xi_0}^{\xi} \mu_1(\epsilon\tau)d\tau}, \\ Z_2(\xi,\epsilon) &= Z_2(\xi_0,\epsilon) e^{\int_{\xi_0}^{\xi} \mu_2(\epsilon\tau)d\tau}, \\ Z_3(\xi,\epsilon) &= Z_3(\xi_0,\epsilon) e^{\int_{\xi_0}^{\xi} \mu_3(\epsilon\tau)d\tau}. \end{aligned}$$

The state transition matrix for (6.6) is given by

$$R(\xi,\xi_0,\epsilon) = diag\left(e^{\int_{\xi_0}^{\xi} \mu_1(\epsilon\tau)d\tau}, e^{\int_{\xi_0}^{\xi} \mu_2(\epsilon\tau)d\tau}, e^{\int_{\xi_0}^{\xi} \mu_3(\epsilon\tau)d\tau} \right).$$

Clearly, $\begin{pmatrix} R_{1,*} \\ R_{2,*} \\ \vec{0}^\top \end{pmatrix}$ and $\begin{pmatrix} R_{*,1} & R_{*,2} & \vec{0} \end{pmatrix}$ are equal. This implies that the equation $R(\xi,\xi_0,\epsilon)P_2 = P_2 R(\xi,\xi_0,\epsilon)$ is satisfied.

For $j = 1, 2$ and $\xi > \xi_0$

$$|Z_j(\xi,\epsilon)| \leq |Z_j(\xi_0,\epsilon)| e^{-\frac{\alpha}{4\beta}(\xi-\xi_0)}.$$

When $\xi < \xi_0$,

$$|Z_3(\xi,\epsilon)| \leq |Z_3(\xi_0,\epsilon)| e^{-\frac{48\beta\delta}{\alpha^2}(\xi-\xi_0)}.$$

We conclude that the system (6.6) has a pseudoexponential dichotomy on each $\bar{J}_j$. The projection is P_2. The constants are given in the statement of the lemma. □

Proof (Theorem). Recall that $\hat{B}(\rho,\epsilon,\xi) = \hat{A}(\rho,\epsilon,\xi) + \tilde{A}(\epsilon,\xi)$. Under the change of variable $Y = H_j Z$, the system (6.1) becomes

$$Z_\xi = (H_j^{-1}Y)_\xi = (H_j^{-1}\hat{A}(\rho,\epsilon,\xi)H_j + H_j^{-1}\tilde{A}(\epsilon,\xi)H_j - H_j^{-1}H_\xi)Z. \qquad (6.7)$$

The derivative of H_j is given by

$$H_\xi|_{\bar{J}_1} = \begin{pmatrix} -\frac{2\dot{\mu}_1}{\mu_1^3} & 0 & -\frac{2\dot{\mu}_2}{\mu_2^3} \\ -\frac{\dot{\mu}_1}{\mu_1^2} & \dot{\mu}_3 & -\frac{\dot{\mu}_2}{\mu_2^2} \\ 0 & 2\mu_3\dot{\mu}_3 & 0 \end{pmatrix} \text{ and } H_\xi|_{\bar{J}_2} = \begin{pmatrix} -\frac{2\dot{\mu}_1}{\mu_1^3} & -\frac{2\dot{\mu}_2}{\mu_2^3} & 0 \\ -\frac{\dot{\mu}_1}{\mu_1^2} & -\frac{\dot{\mu}_2}{\mu_2^2} & \dot{\mu}_3 \\ 0 & 0 & 2\mu_3\dot{\mu}_3 \end{pmatrix}.$$

Please note that $\dot{\mu}_j = \frac{d\mu_j}{d\xi} = \frac{d\mu_j}{dq}\frac{dq}{dx}\frac{dx}{d\xi} = -\epsilon q^2 \frac{d\mu_j}{dq}$. This give rise to the following derivatives:

$$\begin{aligned}
\frac{d\mu_{1,2}}{d\xi} &= -\epsilon q^2\Bigg(-\frac{\rho}{2} + \mathcal{O}(q) \pm \\
&\quad \left(\frac{\rho(\alpha+\beta q\rho)}{4\beta} + \mathcal{O}(q) - \frac{1}{2q^2(\beta+\mathcal{O}(q))}\right)\Bigg(\left(\frac{\alpha+\beta q\rho}{2\beta}\right)^2 + \mathcal{O}(q^2) \\
&\quad -\frac{1}{\beta q + \mathcal{O}(q^2)}\Bigg)^{-\frac{1}{2}}\Bigg) \\
&= -\epsilon q^2\left(\mathcal{O}(q^0) \pm \mathcal{O}(q^{-2})(\mathcal{O}(q^{-1}))^{-\frac{1}{2}}\right) \\
&= -\epsilon q^2 \mathcal{O}(q^{-\frac{3}{2}}) \\
&= \epsilon\mathcal{O}(q^{\frac{1}{2}}), \\
\frac{d\mu_3}{d\xi} &= -\epsilon q^2(\rho + \mathcal{O}(q)) \\
&= \epsilon\mathcal{O}(q^2).
\end{aligned}$$

So as $|x| \to \infty$, $H_\xi|_{\bar{J}_j}$ approaches the zero matrix. This implies that there exists k_3, such that $\left\|H_\xi|_{\bar{J}_j}\right\| < k_3$. Please observe that although H_j^{-1} are not bounded on $\bar{J}_j$, both

$$H_1^{-1}H_\xi|_{\bar{J}_1} = \begin{pmatrix} \frac{\dot{\mu}_1(-2\mu_3\mu_2+\mu_1(\mu_2+\mu_3))}{\mu_1(\mu_2-\mu_1)(\mu_3-\mu_1)} & \frac{\dot{\mu}_3\mu_1^2(\mu_3-\mu_2)}{(\mu_3-\mu_1)(\mu_2-\mu_1)} & \frac{\dot{\mu}_2\mu_1^2(\mu_2-\mu_3)}{\mu_2^2(\mu_3-\mu_1)(\mu_2-\mu_1)} \\ \frac{\dot{\mu}_1(\mu_1-\mu_2)}{(\mu_3-\mu_1)(\mu_3-\mu_2)} & \frac{-\dot{\mu}_3(\mu_1+\mu_2-2\mu_3)}{(\mu_3-\mu_1)(\mu_3-\mu_2)} & \frac{\dot{\mu}_2(\mu_2-\mu_1)}{\mu_2^2(\mu_3-\mu_1)(\mu_3-\mu_2)} \\ \frac{\dot{\mu}_1\mu_2^2(\mu_3-\mu_1)}{\mu_1^2(\mu_3-\mu_2)(\mu_2-\mu_1)} & \frac{\dot{\mu}_3\mu_2^2(\mu_1-\mu_3)}{(\mu_3-\mu_2)(\mu_2-\mu_1)} & \frac{-\dot{\mu}_2(-2\mu_1\mu_3+\mu_2(\mu_3+\mu_1))}{\mu_2(\mu_3-\mu_2)(\mu_2-\mu_1)} \end{pmatrix},$$

$$H_2^{-1}H_\xi|_{\bar{J}_2} = \begin{pmatrix} \frac{\dot{\mu}_1(-2\mu_3\mu_2+\mu_1\mu_2+\mu_1\mu_3)}{\mu_1(\mu_3-\mu_1)(\mu_2-\mu_1)} & -\frac{\dot{\mu}_2\mu_1^2(\mu_3-\mu_2)}{\mu_2^2(\mu_3-\mu_1)(\mu_2-\mu_1)} & \frac{\dot{\mu}_3\mu_1^2(\mu_3-\mu_2)}{\mu_2^2(\mu_3-\mu_1)(\mu_2-\mu_1)} \\ \frac{\dot{\mu}_1\mu_2^2(\mu_3-\mu_1)}{\mu_1^2(\mu_3-\mu_1)(\mu_3-\mu_2)} & \frac{\dot{\mu}_2(2\mu_1\mu_3-\mu_1\mu_2-\mu_2\mu_3)}{\mu_2(\mu_3-\mu_2)(\mu_2-\mu_1)} & \frac{\dot{\mu}_3\mu_2^2(\mu_1-\mu_3)}{(\mu_3-\mu_2)(\mu_2-\mu_1)} \\ \frac{\dot{\mu}_1(\mu_1-\mu_2)}{\mu_1^2(\mu_3-\mu_2)(\mu_3-\mu_1)} & \frac{\dot{\mu}_2(\mu_2-\mu_1)}{\mu_2^2(\mu_3-\mu_2)(\mu_3-\mu_1)} & -\frac{\dot{\mu}_3(\mu_1+\mu_2-2\mu_3))}{(\mu_3-\mu_2)(\mu_3-\mu_1)} \end{pmatrix}$$

approach the zero matrix as $|\xi| \to \infty$. Moreover, the product $H_j^{-1}H_\xi = \mathcal{O}(\epsilon)$ on $\bar{J}_j$.

The matrix product $H_1^{-1}\tilde{A}(\epsilon,\xi)H_1$ on $\bar{J}_1$ is

$$H_1^{-1}\tilde{A}(\epsilon,\xi)H_1 = \begin{pmatrix} \frac{6uv+3\mu_1^2(u^2-(u^*)^2)}{\beta(\mu_3-\mu_1)(\mu_2-\mu_1)} & \frac{6uv\mu_1^2+3\mu_1^2\mu_3(u^2-(u^*)^2)}{\beta(\mu_3-\mu_1)(\mu_2-\mu_1)} & \frac{6uv\mu_1^2+3\mu_2(u^2-(u^*)^2)}{\beta\mu_2^2(\mu_3-\mu_1)(\mu_2-\mu_1)} \\ \frac{6uv+3\mu_1(u^2-(u^*)^2)}{\beta\mu_1^2(\mu_3-\mu_2)(\mu_3-\mu_1)} & \frac{6uv+3\mu_3(u^2-(u^*)^2)}{\beta(\mu_3-\mu_2)(\mu_3-\mu_1)} & \frac{6uv+3\mu_2(u^2-(u^*)^2)}{\beta\mu_2^2(\mu_3-\mu_2)(\mu_3-\mu_1)} \\ \frac{-6\mu_2^2uv-3\mu_1(u^2-(u^*)^2)}{\beta\mu_1^2(\mu_3-\mu_2)(\mu_2-\mu_1)} & \frac{-6\mu_2^2uv-3\mu_3(u^2-(u^*)^2)}{\beta(\mu_3-\mu_2)(\mu_2-\mu_1)} & \frac{-6uv-3\mu_2(u^2-(u^*)^2)}{\beta(\mu_3-\mu_2)(\mu_2-\mu_1)} \end{pmatrix}.$$

On $\bar{J}_2$,

$$H_2^{-1}\tilde{A}(\epsilon,\xi)H_2 = \begin{pmatrix} \frac{6uv+3\mu_1^2(u^2-(u^*)^2)}{\beta(\mu_3-\mu_1)(\mu_2-\mu_1)} & \frac{6uv\mu_1^2+3\mu_1^2\mu_2(u^2-(u^*)^2)}{\beta\mu_2^2(\mu_3-\mu_1)(\mu_2-\mu_1)} & \frac{6uv\mu_1^2+3\mu_1^2\mu_3(u^2-(u^*)^2)}{\beta(\mu_3-\mu_1)(\mu_2-\mu_1)} \\ -\frac{6uv\mu_2^2+3\mu_1\mu_2^2(u^2-(u^*)^2)}{\beta\mu_1^2(\mu_3-\mu_2)(\mu_2-\mu_1)} & -\frac{6uv+3\mu_2(u^2-(u^*)^2)}{\beta(\mu_3-\mu_2)(\mu_2-\mu_1)} & -\frac{6uv\mu_2^2+3\mu_2^2\mu_3(u^2-(u^*)^2)}{\beta(\mu_3-\mu_2)(\mu_2-\mu_1)} \\ \frac{6uv+3\mu_1(u^2-(u^*)^2)}{\beta\mu_1^2(\mu_3-\mu_2)(\mu_3-\mu_1)} & \frac{6uv+3\mu_2(u^2-(u^*)^2)}{\beta\mu_2^2(\mu_3-\mu_2)(\mu_3-\mu_1)} & \frac{6uv+3\mu_3(u^2-(u^*)^2)}{\beta(\mu_3-\mu_2)(\mu_3-\mu_1)} \end{pmatrix}.$$

As $|\xi| \to \infty$, $H_j^{-1}\tilde{A}(\epsilon,\xi)H_j$ approach the zero matrix. This implies that the product $H_j^{-1}\tilde{A}(\epsilon,\xi)H_j$ is uniformly bounded on $\bar{J}_j$; i.e. there exists $l > 0$ so that $\|H_j^{-1}\tilde{A}(\epsilon,\xi)H_j\| \leq l$. Observe that $\tilde{A}(\epsilon,\xi)$ decrease exponentially on $\bar{J}_j$. Although H_j^{-1} increases algebraically on $\bar{J}_1$, the product $H_j^{-1}\tilde{A}(\epsilon,\xi)$ will still decrease exponentially on $\bar{J}_j$. Thus, for a sufficiently small $\epsilon > 0$, $H_j^{-1}\tilde{A}(\epsilon,\xi)$ is small in norm. Moreover, for a sufficiently small $\epsilon > 0$, $\|H_j^{-1}\tilde{A}(\epsilon,\xi)H_j\| = \mathcal{O}(\epsilon)$. Thus, there exists a constant $M > 0$ so that for all $\xi \in \bar{J}_j$, we have

$$\|H_j^{-1}\tilde{A}(\epsilon,\xi)H_j - H_j^{-1}H_\xi\| \leq \epsilon M.$$

The result follows from Coppel's Roughness Theorem and Lemma 8. The system (6.7) has a pseudoexponential dichotomy on $\bar{J}_j$ with projections $\tilde{\mathcal{Q}}_j(\rho,\epsilon,\xi)$ that are near P_2. Since $\tilde{\mathcal{Q}}_j(\rho,\epsilon,\xi)$ are near P_2, $\tilde{\mathcal{Q}}(\rho,\epsilon,\xi)$ will be uniformly bounded on $\bar{J}_j\}$. □

We denote the projection for the pseudoexponential dichotomy of (6.7) on $\bar{J}_j$ by $Q_j(\rho,\epsilon,\xi)$. To simplify the notation, we suppress the ϵ and ρ. Since (6.7) has a pseudoexponential dichotomy, for small $\delta > 0$, there exists constants $C > 0$ and $a_j(\delta) < b_j(\delta)$ so that for $\xi > \zeta$

$$\|\Phi(\xi,\zeta)\tilde{\mathcal{Q}}_j(\zeta)\| \leq Ce^{a_j(\delta)(\xi-\zeta)}, \tag{6.8}$$

and for $\xi < \zeta$

$$\|\Phi(\xi,\zeta)(I - \tilde{\mathcal{Q}}_j(\zeta))\| \leq Ce^{b_j(\delta)(\xi-\zeta)}. \tag{6.9}$$

Observe that a solution of (6.7) is represented by $Z(\xi) = \Phi(\xi,\zeta)Z(\zeta)$. Since $Y(\zeta) = H_j(\zeta)Z(\zeta)$, the solution representation is actually

$$H_j^{-1}(\xi)Y(\xi) = \Phi(\xi,\zeta)H_j^{-1}(\zeta)Y(\zeta).$$

This implies that

$$Y(\xi) = H_j(\xi)\Phi(\xi,\zeta)H_j^{-1}(\zeta)Y(\zeta).$$

So the state transition matrix for (6.7) in Y-space is

$$T(\xi,\zeta) = H_j(\xi)\Phi(\xi,\zeta)H_j^{-1}(\zeta). \tag{6.10}$$

Since (6.7) has a pseudoexponential dichotomy, $\Phi(\xi,\zeta)\tilde{\mathcal{Q}}_j(\zeta) = \tilde{\mathcal{Q}}_j(\xi)\Phi(\xi,\zeta)$. This implies that

$$H_j(\xi)\Phi(\xi,\zeta)H_j^{-1}(\zeta)H_j(\zeta)\tilde{\mathcal{Q}}_j(\zeta)H_j^{-1}(\zeta) = H_j(\xi)\tilde{\mathcal{Q}}_j(\xi)H_j^{-1}(\xi)H_j(\xi)\Phi(\xi,\zeta)H_j^{-1}(\zeta).$$

Therefore, for

$$\mathcal{Q}_j(\rho,\epsilon,\xi) = \mathcal{Q}_j(\xi) = H_j(\xi)\tilde{\mathcal{Q}}_j(\xi)H_j^{-1}(\xi), \tag{6.11}$$

$T(\xi,\zeta)\mathcal{Q}_j(\zeta) = \mathcal{Q}_j(\xi)T(\xi,\zeta)$.

This information will help us to explore the solutions of both the homogeneous and nonhomogeneous versions of (6.1). Knowing the behavior of these solutions combined with using the information revealed about λ for $\rho = \epsilon\lambda$ will help us to locate the eigenvalues of the operator form of (4.1).

6.3 Resolvent Set

For u given as the Riemann–Dafermos solution $\hat{u}_\epsilon(x)$, we consider the following linear operator

$$\mathcal{T}^x[U] = \beta\epsilon^3 U_{xxx} + \alpha\epsilon^2 U_{xx} - \epsilon(3\hat{u}^2 - x)U_x - 6\epsilon\hat{u}\hat{u}_x U.$$

For $x = \epsilon\xi$, this operator is

$$\mathcal{T}^\xi[U] = \beta U_{\xi\xi\xi} + \alpha U_{\xi\xi} - (3u^2 - \epsilon\xi)U_\xi - 6uu_\xi U.$$

We consider both the eigenvalue problem

$$\big(\mathcal{T}^\xi - \rho I\big)U = 0, \tag{6.12}$$

and the nonhomogeneous problem

$$\big(\mathcal{T}^\xi - \rho I\big)U = f. \tag{6.13}$$

We will denote $\mathcal{T}^\xi$ and $\mathcal{T}^\xi - \rho I$ as $\mathcal{T}$ and $\mathcal{T}_\rho$ respectively. The weighted space, $C(\epsilon\gamma, \mathbb{R}_\xi)$, is the space of continuous functions on $\mathbb{R}$ such that the weighted norm is given by $||U||_{\epsilon\gamma} = \sup_\xi |U(\xi)|e^{\epsilon\gamma|\xi|} < \infty$. The resolvent set of $\mathcal{T}$ consists all the ρ's for which the following are satisfied:

1. $\mathcal{T}_\rho^{-1}$ exists.
2. $\mathcal{T}_\rho^{-1}$ is bounded.
3. $C(\epsilon\gamma, \mathbb{R}_\xi)$ is the domain of $\mathcal{T}_\rho^{-1}$.

Notice that the system

$$Y_\xi = B(\rho,\epsilon,\xi)Y + F \tag{6.14}$$

is equivalent to (6.13) with $F = \begin{pmatrix} 0 & 0 & f \end{pmatrix}^T$. We will use properties of the system (6.1) to locate the resolvent set of $\mathcal{T}$.

Theorem 8 *Let Ω be a compact subset of $\mathbb{C}$, ϵ_0 and L be the positive constants given by Theorem 7, and $\delta_0 = \min\left(\frac{\epsilon_0}{L}, \frac{\alpha^3}{192\beta^2}\right)$. Let $\gamma > L + \frac{48\beta}{L\alpha^2}$. Let $0 < \epsilon < L\delta_0$ and $\delta = \frac{\epsilon}{L}$. Then if $\rho \in \left(K_\delta \cap \Omega\right)$, ρ is either a eigenvalue of $\mathcal{T}^\xi$ with geometric multiplicity 1 or in the resolvent set of $\mathcal{T}^\xi$, where $\mathcal{T}^\xi$ operates on $C(\epsilon\gamma, \mathbb{R}_\xi)$.*

Proof. Fix $\rho \in K_\delta \cap \Omega$. Notice that $\delta = \frac{\epsilon}{L} < \delta_0 \leq \frac{\alpha^3}{192\beta^2}$ and $\epsilon < L\delta_0 \leq \epsilon_0$. Hence by Theorem 7, (6.7) has a pseudoexponential dichotomy on $\bar{J}_j$, with projections $\mathcal{Q}_j(\xi) = \mathcal{Q}_j(\rho, \epsilon, \xi)$. According to [7], this result can be extended back from $\pm\infty$ to 0.

Let $\Lambda = R(\mathcal{Q}_2(0)) \cap R(I - \mathcal{Q}_1(0))$. Since $R(\mathcal{Q}_2(0))$ has dim 2 and $R(I - \mathcal{Q}_1(0))$ has dim 1, Λ has dim 0 or 1. First suppose Λ has dim 1. If $Y_0 \in \Lambda$, let $Y(\xi) = (U(\xi), V(\xi), W(\xi))$ be the solution of $Y_\xi = B(\rho, \epsilon, \xi)Y$ with $Y(0) = Y_0$. Then $U(\xi)$ is an eigenfunction of $\mathcal{T}^\xi$ for the eigenvalue ρ, and all eigenfunctions arise in this way. Since Λ has dim 1, ρ is an eigenvalue of $\mathcal{T}^\xi$ of geometric multiplicity one.

Now we will assume that Λ only contains the zero vector. Since $R(\mathcal{Q}_2(0))$ has dimension 2 and $R(I - \mathcal{Q}_1(0))$ has dimension 1, we have $R(\mathcal{Q}_2(0)) \oplus R(I - \mathcal{Q}_1(0)) = \mathbb{R}^3$. Let $f \in C(\epsilon\gamma, \mathbb{R})$. Assume that $Y(\xi)$ is a solution to (6.14) in $C(\epsilon\gamma, \mathbb{R})$. Let $Y_s = \mathcal{Q}_2(0)Y(0)$, and let $Y_u = (I - \mathcal{Q}_1(0))Y(0)$. Let $T(\xi, \zeta)$ denote the family of state transition matrices for the homogeneous system (6.1). We will show that when Λ has dimension 0 and $f \in C(\epsilon\gamma, \mathbb{R}_\xi)$, $\rho \in K_\delta \cap \Omega$ is a resolvent value for $\mathcal{T}^\xi$ if and only if $Y \in C(\epsilon\gamma, \mathbb{R}_\xi)$.

Let $0 \leq \xi < \infty$ and $\tau > \xi$. Using the variation of parameters formula and then allowing τ to approach infinity, we can write

$$\begin{aligned}
Y(\xi) &= \mathcal{Q}_2(\xi)Y(\xi) + (I - \mathcal{Q}_2(\xi))Y(\xi) \\
&= \mathcal{Q}_2(\xi)T(\xi,0)Y(0) + \int_0^\xi \mathcal{Q}_2(\xi)T(\xi,\zeta)F(\zeta)d\zeta + (I - \mathcal{Q}_2(\xi))T(\xi,\tau)Y(\tau) \\
&\quad + \int_\tau^\xi (I - \mathcal{Q}_2(\xi))T(\xi,\zeta)F(\zeta)d\zeta \\
&= T(\xi,0)\mathcal{Q}_2(0)Y(0) + \int_0^\xi T(\xi,\zeta)\mathcal{Q}_2(\zeta)F(\zeta)d\zeta + \int_\infty^\xi T(\xi,\zeta)(I - \mathcal{Q}_2(\zeta))F(\zeta)d\zeta \\
&= T(\xi,0)Y_s + \int_0^\xi T(\xi,\zeta)\mathcal{Q}_2(\zeta)F(\zeta)d\zeta + \int_\infty^\xi T(\xi,\zeta)(I - \mathcal{Q}_2(\zeta))F(\zeta)d\zeta.
\end{aligned}$$

Please notice that as $\tau \to \infty$, the term $(I - \mathcal{Q}_2(\xi))T(\xi,\tau)Y(\tau) \to 0$. This can be seen by looking at its norm. Based on the fact that (6.7) has a pseudoexponential dichotomy on $\bar{J}_2$, and on the equations (6.10), (6.11), we have

$$\begin{aligned}
\|(I - \mathcal{Q}_2(\xi))T(\xi,\tau)Y(\tau)\| &= \|T(\xi,\tau)(I - \mathcal{Q}_2(\xi))Y(\tau)\| \\
&= \|H_2(\xi)\Phi(\xi,\tau)H_2^{-1}(\tau)(I - H_2(\tau)\tilde{\mathcal{Q}}_2(\tau)H_2^{-1}(\tau))Y(\tau)\| \\
&= \|H_2(\xi)\Phi(\xi,\tau)H_2^{-1}(\tau)\Big(H_2(\tau)(I - \tilde{\mathcal{Q}}_2(\tau))H_2^{-1}(\tau)\Big)Y(\tau)\| \\
&= \|H_2(\xi)\Phi(\xi,\tau)(I - \tilde{\mathcal{Q}}_2(\tau))H_2^{-1}(\tau)Y(\tau)\| \\
&\leq \|H_2(\xi)\|\|\Phi(\xi,\tau)(I - \tilde{\mathcal{Q}}_2(\tau))\|\|H_2^{-1}(\tau)\|\|Y(\tau)\|
\end{aligned}$$

Recall that for $\xi \leq \tau, \|\Phi(\xi,\tau)(I - \tilde{\mathcal{Q}}_2(\tau)\| \leq Ce^{\tilde{b}_2(\xi-\tau)}$ with $\tilde{b}_2$ given in the statement of Theorem 7. Then

$$\begin{aligned}
\epsilon\gamma + \tilde{b}_2 > \epsilon\gamma + b_2 + L\epsilon &= \epsilon\gamma - \frac{48\beta\delta}{\alpha^2} + L\epsilon \\
&= \epsilon\gamma - \frac{48\beta\epsilon}{L\alpha^2} + L\epsilon \\
&= \epsilon\left(\gamma - \left(\frac{48\beta}{L\alpha^2} - L\right)\right) > 0.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|(I - \mathcal{Q}_2(\xi))T(\xi,\tau)Y(\tau)\| &\leq \|H_2(\xi)\|\|\Phi(\xi,\tau)(I - \tilde{\mathcal{Q}}_2(\tau))\|\|H_2^{-1}(\tau)\|\|Y(\tau)\| \\
&\leq \tilde{D}e^{\tilde{b}_2(\xi-\tau)-\epsilon\gamma\tau}\|H_2^{-1}(\tau)\|\|Y(\tau)\| \\
&\leq \tilde{D}e^{-(\epsilon\gamma+\tilde{b}_2)\tau}\|H_2^{-1}(\tau)\|\|Y(\tau)\|
\end{aligned}$$

Since $\|H_2^{-1}(\tau)\|$ grows algebraically as $\tau \to \infty$, the decaying exponential term will dominate the behavior of the product as $\tau \to \infty$. Thus as $\tau \to \infty$, $\|(I - \mathcal{Q}_2(\xi))T(\xi,\tau)Y(\tau)\| \to 0$.

Similarly, for $\tau < \xi$ and allowing $\tau \to -\infty$, we can write $Y(\xi)$ on $-\infty < \xi \leq 0$ as

$$\begin{aligned}
Y(\xi) &= \mathcal{Q}_1(\xi)Y(\xi) + (I - \mathcal{Q}_1(\xi))Y(\xi) \\
&= \mathcal{Q}_1(\xi)T(\xi,\tau)Y(\tau) + \int_\tau^\xi \mathcal{Q}_1(\xi)T(\xi,\zeta)F(\zeta)d\zeta + (I - \mathcal{Q}_1(\xi))T(\xi,0)Y(0) \\
&\quad + \int_0^\xi (I - \mathcal{Q}_1(\xi))T(\xi,\zeta)F(\zeta)d\zeta \\
&= \int_{-\infty}^\xi T(\xi,\zeta)\mathcal{Q}_1(\zeta)F(\zeta)d\zeta + T(\xi,0)(I - \mathcal{Q}_1(0))Y(0) \\
&\quad + \int_0^\xi T(\xi,\zeta)(I - \mathcal{Q}_1(\zeta))F(\zeta)d\zeta \\
&= \int_{-\infty}^\xi T(\xi,\zeta)\mathcal{Q}_1(\zeta)F(\zeta)d\zeta + T(\xi,0)Y_u + \int_0^\xi T(\xi,\zeta)(I - \mathcal{Q}_1(\zeta))F(\zeta)d\zeta.
\end{aligned}$$

As $\tau \to -\infty, \|\mathcal{Q}_1(\xi)T(\xi,\tau)Y(\tau)\| \to 0$. This can be seen by observing that

$$\begin{aligned}
\|\mathcal{Q}_1(\xi)T(\xi,\tau)Y(\tau)\| &= \|T(\xi,\tau)\mathcal{Q}_1(\tau)Y(\tau)\| \\
&= \|H_1(\xi)\Phi(\xi,\tau)H_1^{-1}(\tau)H_1(\tau)\tilde{\mathcal{Q}}_1(\tau)H_1^{-1}(\tau)Y(\tau)\| \\
&\leq \|H_1(\xi)\|\|\Phi(\xi,\tau)\tilde{\mathcal{Q}}_1(\tau)\|\|H_1^{-1}(\tau)\|\|Y(\tau)\|.
\end{aligned}$$

Recall that for $\xi \geq \tau, \|\Phi(\xi,\tau)\tilde{\mathcal{Q}}_1(\tau)\| \leq Ce^{\tilde{a}_1(\xi-\tau)}$ with $\tilde{a}_1$ given in the statement of Theorem 7. Then

$$\begin{aligned}
\epsilon\gamma - \tilde{a}_1 > \epsilon\gamma - (a_1 + L\epsilon) &= \epsilon\gamma - \frac{48\beta\delta}{\alpha^2} - L\epsilon \\
&= \epsilon\gamma - \frac{48\beta\epsilon}{L\alpha^2} - L\epsilon \\
&= \epsilon\left(\gamma - \left(\frac{48\beta}{L\alpha^2} + L\right)\right) > 0.
\end{aligned}$$

So we have,

$$\begin{aligned}\|\mathcal{Q}_1(\xi)T(\xi,\tau)Y(\tau)\| &\leq \|H_1(\xi)\|\|\Phi(\xi,\tau)\tilde{\mathcal{Q}}_1(\tau)\|\|H_1^{-1}(\tau)\|\|Y(\tau)\| \\ &\leq \tilde{C}e^{(\tilde{a}_1(\xi-\tau)-\epsilon\gamma|\tau|)}\|H_1^{-1}(\tau)\|\|Y(\tau)\|_{\epsilon\gamma} \\ &\leq \tilde{C}e^{(\epsilon\gamma-\tilde{a}_1)\tau}\|H_1^{-1}(\tau)\|\|Y(\tau)\|_{\epsilon\gamma}.\end{aligned}$$

Since $\|H_1^{-1}(\tau)\|$ grows algebraically as $\tau \to -\infty$, the decaying exponential term will dominate the behavior of the product as $\tau \to -\infty$. Therefore $\|\mathcal{Q}_1(\xi)T(\xi,\tau)Y(\tau)\| \to 0$ as $\tau \to -\infty$.

We conclude that if equation (6.14), with $f \in C(\epsilon\gamma, \mathbb{R}_\xi)$, has a solution $Y \in C(\epsilon\gamma, \mathbb{R}_\xi)$, then Y must be given by expression

$$\int_{-\infty}^{\xi} T(\xi,\zeta)\mathcal{Q}_1(\zeta)F(\zeta)d\zeta + T(\xi,0)Y_u + \int_0^{\xi} T(\xi,\zeta)(I-\mathcal{Q}_1(\zeta))F(\zeta)d\zeta$$

for $\xi \leq 0$ and the expression

$$T(\xi,0)Y_s + \int_0^{\xi} T(\xi,\zeta)\mathcal{Q}_2(\zeta)F(\zeta)d\zeta + \int_{\infty}^{\xi} T(\xi,\zeta)(I-\mathcal{Q}_2(\zeta))F(\zeta)d\zeta$$

for $\xi \geq 0$. It is easy to check that these formulas define solutions of (6.14) on $\xi \leq 0$ and $\xi \geq 0$ respectively. If $Y \in C(\epsilon\gamma, \mathbb{R}_\xi)$, then it must be continuous at 0. We will find a condition on Y_u and Y_s that guarantees that Y is continuous at 0. In addition, we will show that the $\|Y(\xi)\|_{\epsilon\gamma} \leq K\|f\|_{\epsilon\gamma}$, for some constant K. This will aid in verifying that the ρ values in $K_\delta \cap \Omega$ for the 0-dimensional Λ are the resolvent values of $\mathcal{T}^\xi$.

Using the fact that $(I-P_2)F = F$, we observe when $\xi \geq 0$, the norm of $Y(\xi)$ is bounded in the following way,

$$\begin{aligned}
\|Y(\xi)\| &\le \|T(\xi,0)Y_s\| + \int_0^{\xi} \|T(\xi,\zeta)\mathcal{Q}_2(\zeta)(I-P_2)F(\zeta)\|d\zeta \\
&\quad + \int_{\xi}^{\infty} \|T(\xi,\zeta)(I-\mathcal{Q}_2(\zeta))(I-P_2)F(\zeta)d\zeta\| \\
&= \|H_2(\xi)\Phi(\xi,0)H_2^{-1}(0)Y_s\| \\
&\quad + \int_0^{\xi} \|H_2(\xi)\Phi(\xi,\zeta)H_2^{-1}(\zeta)H_2(\zeta)\tilde{\mathcal{Q}}_2(\zeta)H_2^{-1}(\zeta)(I-P_2)F(\zeta)\|d\zeta \\
&\quad + \int_{\xi}^{\infty} \|H_2(\xi)\Phi(\xi,\zeta)H_2^{-1}(\zeta)H_2(\zeta)(I-\tilde{\mathcal{Q}}_2(\zeta))H_2^{-1}(\zeta)(I-P_2)F(\zeta)\|d\zeta \\
&= \|H_2(\xi)\Phi(\xi,0)H_2^{-1}(0)Y_s\| + \int_0^{\xi} \|H_2(\xi)\Phi(\xi,\zeta)\tilde{\mathcal{Q}}_2(\zeta)H_2^{-1}(\zeta)(I-P_2)F(\zeta)\|d\zeta \\
&\quad + \int_{\xi}^{\infty} \|H_2(\xi)\Phi(\xi,\zeta)(I-\tilde{\mathcal{Q}}_2(\zeta))H_2^{-1}(\zeta)(I-P_2)F(\zeta)d\zeta\| \\
&\le \|H_2(\xi)\|\|\Phi(\xi,0)\|\|H_2^{-1}(0)\|\|Y_s\| \\
&\quad + \int_0^{\xi} \|H_2(\xi)\|\Phi(\xi,\zeta)\tilde{\mathcal{Q}}_2(\zeta)\|\|H_2^{-1}(\zeta)(I-P_2)\|\|F(\zeta)\|d\zeta \\
&\quad + \int_{\xi}^{\infty} \|H_2(\xi)\|\|\Phi(\xi,\zeta)(I-\tilde{\mathcal{Q}}_2(\zeta))\|\|H_2^{-1}(\zeta)(I-P_2)\|\|F(\zeta)d\zeta\| \\
&\le \hat{D}_1 e^{\tilde{a}_2\xi}\|Y_s\| + \hat{D}_2\|F\|_{\epsilon\gamma}\int_0^{\xi} e^{\tilde{a}_2(\xi-\zeta)-\epsilon\gamma\zeta}d\zeta + \hat{D}_3\|F\|_{\epsilon\gamma}\int_{\xi}^{\infty} e^{\tilde{b}_2(\xi-\zeta)-\epsilon\gamma\zeta}d\zeta \\
&\le \hat{D}_1 e^{\tilde{a}_2\xi}\|Y_s\| + \hat{D}_4\|F\|_{\epsilon\gamma}(e^{-\epsilon\gamma\xi}-e^{\tilde{a}_2\xi}) + \hat{D}_5\|F\|_{\epsilon\gamma}e^{-\epsilon\gamma\xi}.
\end{aligned}$$

Multiplying $\|Y(\xi)\|$ by $e^{\epsilon\gamma\xi}$, gives $e^{\epsilon\gamma\xi}\|Y(\xi)\| \le C_2\|Y_s\| + c\|F\|_{\epsilon\gamma}$.

Again using the fact that $(I-P_2)F = F$, we observe when $\xi \le 0$, the norm of $Y(\xi)$ is bounded in the following manner,

$$
\begin{aligned}
\|Y(\xi)\| = \quad \leq \quad & \int_{-\infty}^{\xi} \|T(\xi,\zeta)\mathcal{Q}_1(\zeta)F(\zeta)\|d\zeta + \|T(\xi,0)Y_u\| + \int_{\xi}^{0} \|T(\xi,\zeta)(I-\mathcal{Q}_1(\zeta))F(\zeta)\|d\zeta \\
= \quad & \int_{-\infty}^{\xi} \|H_1(\xi)\Phi(\xi,\zeta)H_1^{-1}(\zeta)H_1(\zeta)\tilde{\mathcal{Q}}_1(\zeta)H^{-1}(\zeta)(I-P_2)F(\zeta)\|d\zeta \\
& +\|H_1(\xi)\Phi(\xi,0)H_1^{-1}(0)Y_u\| \\
& + \int_{\xi}^{0} \|H_1(\xi)\Phi(\xi,\zeta)H_1^{-1}(\zeta)H_1(\zeta)(I-\tilde{\mathcal{Q}}_1(\zeta))H^{-1}(\zeta)(I-P_2)F(\zeta)\|d\zeta \\
= \quad & \int_{-\infty}^{\xi} \|H_1(\xi)\Phi(\xi,\zeta)\tilde{\mathcal{Q}}_1(\zeta)H^{-1}(\zeta)(I-P_2)F(\zeta)\|d\zeta + \|H_1(\xi)\Phi(\xi,0)H_1^{-1}(0)Y_u\| \\
& + \int_{\xi}^{0} \|H_1(\xi)\Phi(\xi,\zeta)(I-\tilde{\mathcal{Q}}_1(\zeta))H^{-1}(\zeta)(I-P_2)F(\zeta)\|d\zeta \\
\leq \quad & \int_{-\infty}^{\xi} \|H_1(\xi)\|\|\Phi(\xi,\zeta)\tilde{\mathcal{Q}}_1(\zeta)\|\|H^{-1}(\zeta)(I-P_2)\|\|F(\zeta)\|d\zeta \\
& +\|H_1(\xi)\|\|\Phi(\xi,0)\|\|H_1^{-1}(0)\|\|Y_u\| \\
& + \int_{\xi}^{0} \|H_1(\xi)\|\|\Phi(\xi,\zeta)(I-\tilde{\mathcal{Q}}_1(\zeta))\|\|H^{-1}(\zeta)(I-P_2)\|\|F(\zeta)\|d\zeta \\
= \quad & \hat{C}_1\|F\|_{\epsilon\gamma} \int_{-\infty}^{\xi} e^{\tilde{a}_1(\xi-\zeta)-\epsilon\gamma|\zeta|}d\zeta + \hat{C}_2 e^{\tilde{b}_1\xi}\|Y_u\| + \hat{C}_3\|F\|_{\epsilon\gamma} \int_{\xi}^{0} e^{\tilde{b}_1(\xi-\zeta)-\epsilon\gamma|\zeta|}d\zeta \\
= \quad & \hat{C}_4\|F\|_{\epsilon\gamma}e^{\epsilon\gamma\xi} + \hat{C}_2 e^{\tilde{b}_1\xi}\|Y_u\| + \hat{C}_5\|F\|_{\epsilon\gamma}(e^{\epsilon\gamma\xi} - e^{\tilde{b}_1\xi}).
\end{aligned}
$$

Multiplying by $e^{\epsilon\gamma|\xi|}$ gives $e^{\epsilon\gamma|\xi|}\|Y(\xi)\| \leq C_1\|Y_s\| + k\|F\|_{\epsilon\gamma}$.

The values Y_s and Y_u are chosen so that $Y(\xi)$ is continuous at 0. So for $\xi = 0$, we have

$$
Y_s - Y_u = \int_{-\infty}^{0} T(0,\zeta)\mathcal{Q}_1(\zeta)F(\zeta)d\zeta + \int_{0}^{\infty} T(0,\zeta)(I-\mathcal{Q}_2(\zeta))F(\zeta)d\zeta.
$$

Since $R(\mathcal{Q}_2(0))$ and $R(I-\mathcal{Q}_1(0))$ are complementary, we can define a projection $\hat{Q}$ on $\mathbb{R}^3$ with $R(\hat{Q}) = R(\mathcal{Q}_2(0))$ and $R(I-\hat{Q}) = R(I-\mathcal{Q}_1(0))$. Let $\hat{Y}$ denote the right hand side of the previous equation. Then $\hat{Q}\hat{Y} = Y_s$ and $(I - \hat{Q})\hat{Y} = -Y_u$. Therefore:

$$\begin{aligned}
\|\hat{Q}\hat{Y}\| &\le \int_{-\infty}^{0} \|\hat{Q}\mathcal{Q}_1(0)T(0,\zeta)F(\zeta)\|d\zeta + \int_{0}^{\infty} \|\hat{Q}(I-\mathcal{Q}_2(0))T(0,\zeta)F(\zeta)\|d\zeta \\
&= \int_{-\infty}^{0} \|\hat{Q}H_1(0)\tilde{\mathcal{Q}}_1(0)H_1^{-1}(0)H_1(0)\Phi(0,\zeta)H_1^{-1}(\zeta)(I-P_2)F(\zeta)\|d\zeta \\
&\quad + \int_{0}^{\infty} \|\hat{Q}H_2(0)(I-\tilde{\mathcal{Q}}_2(0))H_2^{-1}(0)H_2(0)\Phi(0,\zeta)H_2^{-1}(\zeta)(I-P_2)F(\zeta)\|d\zeta \\
&= \int_{-\infty}^{0} \|\hat{Q}H_1(0)\tilde{\mathcal{Q}}_1(0)\Phi(0,\zeta)H_1^{-1}(\zeta)(I-P_2)F(\zeta)\|d\zeta \\
&\quad + \int_{0}^{\infty} \|\hat{Q}H_2(0)(I-\tilde{\mathcal{Q}}_2(0))\Phi(0,\zeta)H_2^{-1}(\zeta)(I-P_2)F(\zeta)\|d\zeta \\
&\le \int_{-\infty}^{0} |\|\hat{Q}\|\|H_1(0)\|\|\tilde{\mathcal{Q}}_1(0)\Phi(0,\zeta)\|\|H_1^{-1}(\zeta)(I-P_2)\|\|F(\zeta)|d\zeta \\
&\quad + \int_{0}^{\infty} \|\hat{Q}\|\|H_2(0)\|\|(I-\tilde{\mathcal{Q}}_2(0))\Phi(0,\zeta)\|\|H_2^{-1}(\zeta)(I-P_2)\|\|F(\zeta)\|d\zeta \\
&\le \hat{E}_1\|F\|_{\epsilon\gamma}\int_{-\infty}^{0} e^{-\tilde{a}_1\zeta-\epsilon\gamma|\zeta|}d\zeta + \hat{E}_2\|F\|_{\epsilon\gamma}\int_{0}^{\infty} e^{-\tilde{b}_2\zeta-\epsilon\gamma\zeta}d\zeta \\
&= E_1\|F\|_{\epsilon\gamma}
\end{aligned}$$

$$\begin{aligned}
\|(I-\hat{Q})\hat{Y}\| &\le \int_{-\infty}^{0} \|(I-\hat{Q})\mathcal{Q}_1(0)T(0,\zeta)F(\zeta)\|d\zeta + \int_{0}^{\infty} \|(I-\hat{Q})(I-\mathcal{Q}_2(0))T(0,\zeta)F(\zeta)\|d\zeta \\
&= \int_{-\infty}^{0} \|(I-\hat{Q})H_1(0)\mathcal{Q}_1(0)H_1^{-1}(0)H_1(0)\Phi(0,\zeta)H_1^{-1}(\zeta)(I-P_2)F(\zeta)\|d\zeta \\
&\quad + \int_{0}^{\infty} \|(I-\hat{Q})H_2(0)(I-\mathcal{Q}_2(0))H_2^{-1}(0)H_2(0)\Phi(0,\zeta)H_2^{-1}(\zeta)(I-P_2)F(\zeta)\|d\zeta \\
&= \int_{-\infty}^{0} \|(I-\hat{Q})H_1(0)\mathcal{Q}_1(0)\Phi(0,\zeta)H_1^{-1}(\zeta)(I-P_2)F(\zeta)\|d\zeta \\
&\quad + \int_{0}^{\infty} \|(I-\hat{Q})H_2(0)(I-\mathcal{Q}_2(0))\Phi(0,\zeta)H_2^{-1}(\zeta)(I-P_2)F(\zeta)\|d\zeta \\
&\le \int_{-\infty}^{0} \|I-\hat{Q}\|\|H_1(0)\|\|\mathcal{Q}_1(0)\Phi(0,\zeta)\|\|H_1^{-1}(\zeta)(I-P_2)\|\|F(\zeta)\|d\zeta \\
&\quad + \int_{0}^{\infty} \|I-\hat{Q}\|\|H_2(0)\|\|(I-\mathcal{Q}_2(0))\Phi(0,\zeta)\|\|H_2^{-1}(\zeta)(I-P_2)\|\|F(\zeta)\|d\zeta \\
&\le \hat{E}_3\|F\|_{\epsilon\gamma}\int_{-\infty}^{0} e^{-\tilde{a}_1\zeta-\epsilon\gamma|\zeta|}d\zeta + \hat{E}_4\|F\|_{\epsilon\gamma}\int_{0}^{\infty} e^{-\tilde{b}_2\zeta-\epsilon\gamma\zeta}d\zeta \\
&= E_2\|F\|_{\epsilon\gamma}
\end{aligned}$$

So we have, $\|Y_s\| \le E_1\|F\|_{\epsilon\gamma}$ for $\xi \ge 0$ and $\|Y_u\| = \|-Y_u\| \le E_2\|F\|_{\epsilon\gamma}$ for $\xi < 0$. This implies that, $\|Y(\xi)\|_{\epsilon\gamma} = e^{\epsilon\gamma\xi}|Y(\xi)|$ is bounded by a positive multiple of $\|F\|_{\epsilon\gamma}$.

Therefore, $Y(\xi) \in C(\epsilon\gamma, \mathbb{R}_\xi)$ for all $\xi \in \mathbb{R}$. Since solving $\mathcal{T}_\rho U = f$ is equivalent to solving (6.14), U is the first component of the solution of (6.14), which is $Y_1(\xi)$. Thus, there exists a constant $K > 0$ so that $\|U\|_{\epsilon\gamma} \leq K\|F\|_{\epsilon\gamma} = K\|f\|_{\epsilon\gamma}$. Therefore, $\mathcal{T}_\rho^{-1}$ exists and is bounded on $C(\epsilon\gamma, \mathbb{R}_\xi)$. □

6.3.1 Discussion

The purpose of the section is to analyze the eigenpairs of the linear operator given by the equation (4.1) denoted by

$$\mathcal{E}_\epsilon(U) = U_t = \frac{1}{\epsilon}\mathcal{T}^x.$$

The eigenpairs of $\mathcal{E}_\epsilon$ satisfy (4.2). Please observe that $U(\xi) \in C(\epsilon\gamma, \mathbb{R}_\xi)$ implies that the function $\tilde{U}(x) = U\left(\frac{x}{\epsilon}\right)$ is in $C(\gamma, \mathbb{R}_x)$. $\tilde{U}(x) \in C(\gamma, \mathbb{R}_x)$ implies that $\sup_x |\tilde{U}(x)|e^{\gamma|x|} < \infty$. As a result, ρ is an eigenvalue or resolvent value of $\mathcal{T}^\xi$ on $C(\epsilon\gamma, \mathbb{R}_\xi)$ is equivalent to ρ being an eigenvalue or resolvent value of $\mathcal{T}^x$ on $C(\gamma, \mathbb{R}_x)$. Then the corresponding $\lambda(\epsilon) = \frac{\rho}{\epsilon}$ is an eigenvalue or resolvent value of $\mathcal{E}_\epsilon$ on $C(\gamma, \mathbb{R}_x)$.

Please note that since δ is defined to be $\frac{\epsilon}{L}$ in the Theorem 8, $\frac{1}{\epsilon}K_\delta$ is simply the set of complex numbers with real part at least $-\frac{1}{L}$. Theorem 8 tells us that if $\rho = \epsilon\lambda \in K_\delta \cap \Omega$, then ρ is an eigenvalue of $\mathcal{T}^x$ with geometric multiplicity 1 and eigenfunction located in $C(\gamma, \mathbb{R}_x)$ or a resolvent value of $\mathcal{T}^x$. Since $\rho = \epsilon\lambda$, we have

$$\lambda \in \frac{1}{\epsilon}\left(K_\delta \cap \Omega\right) = \left(K_{\frac{1}{L}} \cap \frac{1}{\epsilon}\Omega\right).$$

Since Ω is given as any arbitrary compact set in $\mathbb{C}$, the space $\left(K_{\frac{1}{L}} \cap \frac{1}{\epsilon}\Omega\right)$ is relatively large. The result from Theorem 8 simply implies that the above λ values are either eigenvalues of $\mathcal{E}_\epsilon$ with geometric multiplicity 1 and eigenfunctions in $C(\gamma, \mathcal{R}_x)$ or resolvent values of $\mathcal{E}_\epsilon$.

In Chapter 5, we mentioned for sufficiently small $\epsilon > 0$ that the small eigenvalues of the operator $\mathcal{E}_\epsilon$ whose eigenfunctions go to 0 rapidly as $x \to \pm\infty$, were contained in a small neighborhood about the calculated λ_0. The corresponding eigenfunctions that go to 0 rapidly as $x \to \pm\infty$ are those whose values at 0 are in the space Λ. Thus the eigenpairs discussed in Chapter 5 are precisely those in which the related eigenfunctions are located in $C(\gamma, \mathbb{R}_x)$. So these eigenpairs are the very ones mentioned in Theorem 8.

We pointed out in Chapter 5 that $\mathcal{W}_\epsilon^u(\mathcal{M}^{u^\ell})$ and $\mathcal{W}_\epsilon^s(\mathcal{M}^{u^r})$ are transverse for fixed $\lambda \neq \lambda_0 + \epsilon\lambda_1 + \mathcal{O}(\epsilon^2)$. We are able to make an inference that if λ is a resolvent value of $\mathcal{E}_\epsilon$, then $\mathcal{W}_\epsilon^u(\mathcal{M}^{u^\ell})$ and $\mathcal{W}_\epsilon^s(\mathcal{M}^{u^r})$ are transverse for each λ. So the resolvent values in $\left(K_{\frac{1}{L}} \cap \frac{1}{\epsilon}\Omega\right)$, will produce a solution that starts in $\mathcal{W}_\epsilon^u(\mathcal{M}^{u^\ell})$ and ends in $\mathcal{W}_\epsilon^s(\mathcal{M}^{u^r})$.

Chapter 7

Concluding Thoughts and Future Endeavors

7.1 Conclusions

7.1.1 Existence of a Riemann–Dafermos Solution

We began by considering a modified version of the KdV-Burgers equation (1.1),

$$u_T + 3u^2 u_X = \alpha u_{XX} + \beta u_{XXX}.$$

Motivated by the suggestion of numerical approximations, we applied the change of variables

$$x = \frac{X}{T} \qquad t = \ln T, \tag{7.1}$$

to (1.1) to produce equation (1.4),

$$u_t + (3u^2 - x)u_x = \alpha e^{-t} u_{xx} + \beta e^{-2t} u_{xxx}.$$

Instead of focusing on (1.4) directly, we replaced the time dependent coefficients in (1.4) with small positive constants and obtained (1.6),

$$u_t + (3u^2 - x)u_x = \alpha\epsilon u_{xx} + \beta\epsilon^2 u_{xxx}.$$

One can view those substituted constants as the exponentials evaluated at some large time t_0. So when time is near t_0, the solutions (1.6) are near those of (1.4) for $\epsilon = e^{-t_0}$.

Motivated by (1.1) subject to

$$u(T, -\infty) = u^\ell \text{ and } u(T, \infty) = u^r,$$

we turned our interests to the steady state solution of (1.6), subject to the conditions

$$u(-\infty) = u^\ell, \text{ and } u(\infty) = u^r.$$

We refer to this solution as the Riemann–Dafermos solution.

We learned from [6] that select pairs of boundary conditions give rise to shock waves,

$$u(x) = \begin{cases} u^\ell & \text{for } x < s \\ u^r & \text{for } x > s \end{cases},$$

that are either compressive or undercompressive. These shock waves solutions are weak solutions of (1.3), the underlying conservation law of (1.1) in (x,t)-coordinates

$$u_t + (3u^2 - x)u_x = 0.$$

For $\beta > 0$, it was shown by Jacobs, McKinney, and Shearer in [6] that the shock wave was compressive when

$$u^\ell \in \left(0, \frac{2\alpha\sqrt{2}}{3\sqrt{\beta}}\right) \text{ and } u^r \in \left(-\frac{u^\ell}{2}, u^\ell\right).$$

It was also shown in [6] that a dispersion phenomenon occurred when $\beta > 0$,

$$u^\ell > \frac{2\sqrt{2}\alpha}{3\sqrt{\beta}}, \text{ and } u^r > -u^\ell + \frac{\alpha\sqrt{2}}{3\sqrt{\beta}}.$$

This phenomenon is one in which a compressive shock splits into an undercompressive shock followed by a faster compressive shock. For $\beta > 0$ and these pairs of boundary conditions, we were able to show, for a Riemann solution consisting of either 1 or 2 shock wave(s) and for small $\epsilon > 0$, the existence of a Riemann–Dafermos solution that is C^1 close to the corresponding Riemann solution away from its discontinuities.

7.1.2 Stability Properties

We studied the stability of the Riemann–Dafermos solution by means of linearization. After finding the corresponding linear partial differential equation (4.1),

$$U_t + 6\hat{u}_\epsilon \hat{u}'_\epsilon U + (3\hat{u}_\epsilon^2 - x)U_x = \alpha\epsilon U_{xx} + \beta\epsilon^2 U_{xxx},$$

we used Dodd results in [5] to be able to conclude that for $\epsilon > 0$ and $Re(\lambda) > 0, \lambda$ will not be an eigenvalue of (4.1). The operator form of (4.1) is given by $U_t = \mathcal{E}_\epsilon U$, where

$$\mathcal{E}_\epsilon U = \beta\epsilon^2 U_{xxx} + \alpha\epsilon U_{xx} - 6\hat{u}_\epsilon \hat{u}'_\epsilon U - (3\hat{u}_\epsilon^2 - x)U_x.$$

We consider Ω as any given compact subset of $\mathbb{C}$ and $K_{\frac{1}{L}} = \left\{\lambda \middle| Re\,(\lambda) \geq -\frac{1}{L}, L > 0\right\}$. For sufficiently small $\epsilon > 0$, we were able to show that the set $\left(K_{\frac{1}{L}} \cap \frac{1}{\epsilon}\Omega\right)$ contains only resolvent values of $\mathcal{E}_\epsilon$ and eigenvalues of $\mathcal{E}_\epsilon$ with geometric multiplicity of one.

We assume that the eigenvalues take the form $\lambda_0 + \epsilon\lambda_1 + \mathcal{O}(\epsilon^2)$. Thus for sufficiently small $\epsilon > 0$, the eigenvalues are contained in a neighborhood about λ_0. Let N denote the portion of the neighborhood about λ_0 that contains all small eigenvalues of $\mathcal{E}_\epsilon$ with geometric multiplicity 1. Then $\left(K_{\frac{1}{L}} \cap \frac{1}{\epsilon}\Omega\right) \setminus N$ would contain the resolvent set of $\mathcal{E}_\epsilon$.

As $\epsilon > 0$ gets smaller, N gets smaller and $\left(K_{\frac{1}{L}} \cap \frac{1}{\epsilon}\Omega\right)$ gets larger. Thus, as $\epsilon > 0$ gets smaller, $\left(K_{\frac{1}{L}} \cap \frac{1}{\epsilon}\Omega\right) \setminus N$ gets larger. The intersection $\left(K_{\frac{1}{L}} \cap \frac{1}{\epsilon}\Omega\right)$ is simply a subset that lies completely in $K_{\frac{1}{L}}$. As $\epsilon \to 0$, the intersection $\left(K_{\frac{1}{L}} \cap \frac{1}{\epsilon}\Omega\right) \to K_{\frac{1}{L}}$. This implies that as $\epsilon \to 0$, $K_{\frac{1}{L}}$ will only contain resolvent values. That is, as $\epsilon \to 0$, the number of eigenvalues with geometric multiplicity 1 goes to 0. There is no way to guarantee the existence of a sufficiently small ϵ^* value so that for every fixed ϵ smaller than ϵ^*, the intersection $\left(K_{\frac{1}{L}} \cap \frac{1}{\epsilon}\Omega\right)$ contains only resolvent values. If we could provide this, then any of those sufficiently small ϵ values less than ϵ^* would create that neighborhood about λ_0 so that if λ is in it then $Re(\lambda) < -\frac{1}{L} < 0$.

7.2 Next

The natural next step is to investigate the spectrum more to get those stability results and determine how to relate our results about solutions of (1.6) to those of (1.4), solutions of (1.1) under the change of variables (7.1). Another equation of interest is a thin film model studied by Bertozzi, Münch, Shearer, (1999). This model is given by

$$h_T(h^2 - h^3)_X = -\epsilon^3(h^3 h_{XXX})_X. \tag{7.2}$$

This equation models the movement of a thin liquid film of thickness h up an inclined plane due to a temperature gradient on the plane. This equation has been studied subject to the conditions:

$$h(-\infty, T) = h_\infty \qquad h(\infty, T) = b \qquad 0 < b < h_\infty \qquad \epsilon > 0 \text{ (small) }. \tag{7.3}$$

The constant h_∞ represents the depth of the film at the bottom of the plane. The constant b is the depth of a thin precursor layer above the climbing liquid. Numerical simulations of (7.2) – (7.3), as well as experiments, show that after a large time the solution is close to a Riemann solution of the conservation law of concern. A future project is to use Dafermos regularization to investigate this model further.

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