This study investigates the implications of firm’s demand for real balances from the production and monetary economics perspective. Money is a component of production as a factor increasing efficiency in exchange in input markets. Firms may improve efficiency in production by holding real money to decrease their transactions cost in input market. Feenstra (1986) equivalence states that transactions cost as a constraint in a firm’s budget is functionally equivalent to money in the production function. Monetary expansion may change transactions cost, hence can affect the efficiency in both goods and input markets. If a monetary model is set up in accordance with Feenstra equivalence it can be used to examine the impact of the change in money supply over net consumption and net input. Monetary expansion is no longer super-neutral over net consumption and net input because it has the ability to alter efficiency in exchange. The study concludes with an empirical test of the theory by applying the stochastic production frontier approach over 12 European Union countries. The test verifies the significance of real money in determining the technical inefficiency in production.
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Introduction

It has been thirty years since Fisher (1974) recognized real balances as a valid factor of production. Real balances is recognized as a variable in the production function due to the fact that it economizes on the use of other factors of production rather than being their substitute. In other words, money enters the production function as a factor increasing efficiency in production. It is a tool to allocate resources in production and exchange.

Beside the theoretical framework established by Fisher (1974), there is important evidence justifying the growing importance of real balances in real economic activities. First, in 12 European Union (EU) countries\(^1\) the rate of increase in demand deposits was 18.3% higher than the rate of increase in the real GDP, on the average, between 1995-2002. Second, in these countries for the same period, the proportion of money (coins + demand deposit) held by firms to total demand deposits increased from

---

\(^1\)Austria, Belgium, Denmark, Finland, France, Germany, Italy, Netherland, Portugal, Spain, Sweden, UK
13.8% to 16.2%, on the average.

The link between efficiency in production and holding real balances can be explained in at least two ways. First, as it is pointed out by Fisher (1974), money is used for exchange in factor markets and higher holding of real balances reduces intermediation cost which enables firms to transfer the factor hired to perform transactions activities into real production. In other words, firms can increase productive factors by reallocating resources from transactions services into production.

Secondly, high liquidity could enable firms to have better and longer term relationship with their customers. Liquid firms can pay their debt on time and can produce and deliver their output within the just-in-time principle. All these can cause firms to work with a better pool of customers. Thus, firms with a better pool of customers can operate smoothly with a higher rate of profitability which in turn could lead firms to exploit their factors of production better.

According to classical monetary theory, money is super-neutral over real macroeconomic variables. However, recognition of the significance of real balances in determining efficiency in production implies that there is room for monetary authorities to increase physical output. It is vital to note that the success of monetary policies and efficiency in use of real balances depends on firms’ real balances/physical input ratio.
In other words, firms may get benefits from expansionary monetary policy as long as the rate of return to money is higher than the rate of return to physical inputs. Expansionary monetary policy reduces the rate of return on money therefore decreases the opportunity / transactions costs for a firm performing exchange and production. Since there is no substitution between real balances and other physical factors of productions, returns would be equalized when firms hold the optimum quantity of money with their physical factors of production. Monetary authorities should predict this ratio and design their policies in a way to reach the optimum.

This dissertation first sets up a theoretical model to include Fisher’s recognition of real balances as a factor improving the efficiency in production in a monetary model. Proper set up of the model requires a production function that includes money, but at the same time functionally equivalent to the classical Cobb-Douglas production function. Once the equivalence of a Cobb-Douglas production function is set up then it is straightforward to show the impact of monetary actions over production.

In the first chapter, a gross production function is developed by using the Feenstra (1986) approach. Feenstra shows the functional equivalence between money in the budget constraint and the utility of money. The chapter applies the same approach and shows that for a profit maximizing firm, demand for real balances is functionally
equivalent to money in the production function. The production function with money is called the gross production function. It shows the gross production, which includes the physical factors used in production and the money held to perform exchange in input markets.

The second chapter explains the way that monetary authorities have power to affect net consumption and allocation of resources between transactions cost and production. It criticizes Sidrauski (1967) and offers a detailed discussion on the monetary super-neutrality over the macroeconomic variables. In this simple version of the Baumol-Tobin model, if transaction cost is high, firms have to use some of their inputs in managing its transactions cost. Firms hold real balances to decrease their transactions cost, hence could transfer their resources from transactions sector to increase production. Therefore, by changing the quantity of real balances in the economy, monetary authorities may decrease transactions cost and may help to shift resources from transactions to production.

The third chapter presents an empirical examination of the significance of real balances on the level of (in)efficiency in production for the case of 12 European Union countries for the period 1995-2002. The classical approach in the empirical literature on this topic is the direct test on the significance of the coefficient of real balances in
a Cobb-Douglas form production function. In this chapter, this approach is criticized as the method treating money as an ordinary physical factor of production. Rather it applies the stochastic production frontier approach. This approach aims to estimate the level of inefficiency in production, jointly with the production frontier. It allows one to test the significance of possible factors, including money, affecting inefficiency. The classical approach takes money as a substitute for other factors of production. The theory explained above clearly states that money is an element in production affecting efficiency in use of other factors of production. The classical approach fails to recognize this property. From the empirical point of view, regarding money as the same as other physical factors of production causes the well known statistical problem of multi-collinearity. The stochastic production frontier approach could be applied to tackle this problem.

Another contribution of the chapter to the existing empirical literature is that it considers money held by firms and money held by households separately as a factor affecting efficiency in production. The current literature uses the total stock of money (M1, M2, Divisia index) in their analysis. Results of our analysis confirm the theoretical expectation that money holdings of firms contribute positively and significantly to efficiency in production. Money holdings of households affect efficiency
positively and significantly as well. However, the impact of firm actions is stronger than that of households.
Chapter 1

Transactions Demand for Money by Firms is Functionally Equivalent to Money in the Production Function

1.1 Introduction

Fisher (1974) argues that a firm holds real cash balances at the cost of forgone interest, therefore real cash balances have to be considered as a factor of production. Increases in real cash balances save up resources in transaction services and directly increase resources for physical production.

The purpose of this chapter is to show that the transactions demand for money by firms is functionally equivalent to money in the production function. The functional equivalence between liquidity costs in the budget constraint and money in the utility function has been shown in Feenstra (1986). Transactions cost in production is a constraint on a firm’s budget. It implies that the firm produces less than an optimal
level of output. Inefficiency arises if the firm is unable to employ all of its resources in production. Firms hold real balances to decrease their transactions cost allowing them to reallocate resources from transactions services to production. The relationship between transactions cost and the quantity of factors employed in production implies that money has to be considered as a factor in production function as an element increasing efficiency.

The organization of the chapter is as follows. The second section describes the model and sets up the firm problem. It explains the similarity between the households’ problem in Feenstra (1986) and the firms’ problem. The third section, after examining the duality in expressing inefficiency in production as a loss in output and the loss in quantity of capital in production, shows that a duality between ‘uses’ minimization problem and the ‘sources’ maximization problem for a profit maximizing firm arises due to the earlier duality. The second duality is analogous to the duality of cost minimization and output maximization in microeconomics analysis where, in a simple economic world a firm’s income consists of sales revenue and the cost is the expenditure on factors of production. In our analysis the firm engages in both production and cash management. Therefore its sources consist of sale receipts, borrowing and cash holding, while its uses are expenditure on factors of production.
and borrowing. The fourth section demonstrates that due to these duality properties, transactions demand for money in the budget constraint of the firm can be interpreted equivalently as money in the production function. Section five presents concluding remarks and the proof of the proposition 1 can be found in the appendix.

1.2 Model

There are two agents: a representative household and a representative firm. Money is accepted as a medium of exchange both by the household and the firm. The household maximizes utility from the consumption of goods, subject to the budget constraint. The utility function, $U(c_t)$, where $c_t$ is consumption, satisfies $U_c > 0$ and $U_{cc} \leq 0$. The household earns income and pays for consumption. The income of the household consists of income from renting physical capital $k_t$ to the firm and interest income from holding bonds. Therefore the household’s utility maximization problem can be summed up as

$$\max U(c_t)$$

subject to

$$c_t + b_t + m_t = r k_t + [(1 + i_{t-1})b_{t-1} + m_{t-1}] \frac{p_{t-1}}{p_t}$$
Note that household non-financial income consists of receipts from renting capital only. In this simple model we assume that labor is not used in production and therefore it is equal to zero.

### 1.2.1 Transactions Cost

Consider a profit maximizing firm engaging with production and cash management decisions. Assume period \( t \) production is \( y(k_t) \), where \( k_t \) denotes employment of capital, and \( y_k > 0 \) and \( y_{kk} \leq 0 \). The firm issues bonds to borrow from households to finance its production and holds money to repay its debt. If \( m_t \) and \( b_t \) represents real money and bond holding in period \( t \), then the firm’s problem is to choose \( k_t, m_t \) and \( b_t \) to maximize its profit:

\[
\max \Pi(r_t, R_t, \kappa) = y_t - \Theta(y_t, m_t) + b_t + m_t - r_t k_t - [(1 + R_{t-1})b_{t-1} + m_{t-1}] \frac{p_{t-1}}{p_t} \tag{1.1}
\]

where \( r_t \) and \( R_t \) are rates of return to capital and nominal interest rate on borrowing respectively. \( \Theta(y_t, m_t) \) is real transactions cost and represents the loss in output due to some friction in the production. According to Fisher (1974), the firm cannot produce at the maximum level of output with the given stock of resources and technology because it allocates some of those resources in the transactions to perform...
cash management. The equation (1.1) represents the profit function for a firm engaging in production and exchange, and competition among firms leads the economic profit to be equal to zero. The income of the firm consists of receipts from sales, loans from households and stock of money holding. The firm spends its income on the employment of capital and repayment of old loans.

Feenstra (1986) investigates the meaning of the liquidity cost in household budget constraint and shows that there is a functional equivalence between liquidity cost in the budget constraint and the utility of money. The main objective of this chapter is to analyze the role of transactions cost in production and examine the similar relationship between transactions cost and money in the production function. In order to make the analysis simple it is assumed that transactions cost exists in production only, therefore only firms hold cash for transactions purposes.

1.2.2 Transactions Model for a Firm
Similarity of Households’ and Firms’ Cash Management Decisions

Feenstra (1986) provides three models of the demand for money; transactions model, generalized transactions model, and precautionary model, to show how the liquidity cost arises for a household engaging in consumption and cash management decisions. Our analysis will discuss the Baumol-Tobin type transactions demand for money by
firms briefly. The objective is to illustrate the similarity between the model for a household in Feenstra (1986) and the transactions model for a firm in our analysis. The figure 1.1 shows the household cash management for the Baumol-Tobin type model. In this model the household has regular visits to banks to withdraw money to finance its consumption. In the Feenstra (1986) version of the Baumol-Tobin model, the household holds both money and bonds and has to pay a liquidity cost in terms of forgone interest and the waste of time at the bank in order to convert some of its assets into money.

Figure 1.1: Household Cash Management in Feenstra for the Baumol-Tobin Type Model

In the Feenstra (1986) model, there is no firm. Here we introduce a representative
firm to the model. To simplify the analysis, in our model we assume that there are frictions in the form of transactions cost only in the production sector, and the firm bears the burden and wastes some resources in transactions that otherwise would be used up in production, unavoidably.

In our two agent model, the household lends to the firm, such that the household’s assets are equal to the firm’s debt. At the same time, the household’s spending on consumption is equal to the firm’s sales receipts. The firm’s debt is decreased each time the household converts some of its assets into money. The firm accumulates income from sales and at the end of the period pays back some of its debt to the household at a transactions cost of $\kappa$. Figure (1.2) shows the firm cash management over the course of a period. Note the similarity between household and firms cash management decision. It is clear from figures (1.2) and (1.1) that the firm’s problem is just a mirror of the household’s problem. In the following section, we will apply the same approach as in Feenstra to the firm and show that there is a functional equivalence between the transactions demand for money and money in the production function.
Baumol-Tobin Type Transactions Demand Model for a Firm

The firm optimizes production and the required cash management by hiring capital from the household. If the employment within each period is $k_t$, total production would be $y(k_t)$. Production function is assumed to satisfy $y(0) = 0$, $y_k > 0$ and $y_{kk} < 0$. Capital is paid at the time it is used.

Assume that the firm borrows $b_t$ from the household to start up production. Within the discrete time period $t$, as the firm realizes sales, it receives money from the household in the amount of $p_t y_h$. Here $p_t$ represents exogenous price and it is assumed to be constant within a period but vary across periods. During a period,
the firm accumulates cash and at the end of each period pays back a portion of its
debt. The firm holds money at the transactions cost of $\kappa$.

Similar to the household’s regular visits to banks in the Feenstra analysis, assume
that the household has regular visits to the firm and the firm makes $N_t$ of these
transactions within the period $t$, and further assume that transactions are evenly
spaced to minimize transactions cost as in Baumol (1952) inventory model.

Therefore, analogous to the Baumol-Tobin analysis, the firm receives $\mu_t = \frac{\bar{y}}{N_t}$
in each period to make its factor payments and loan repayments. Then, the average
money holdings over a period would be $\bar{m}_t = \frac{\mu_t}{2} = \frac{\bar{y}}{2N_t} \ t$. The total cost of transaction
would be $\kappa N_t = \kappa \frac{\bar{y}}{2N_t}$.

In order to keep the analysis as simple as possible, as in Feenstra (1986), assume
that the firm issues a constant amount of debt $\bar{b}_t$ at the beginning of the production
and over a period pays $\frac{b}{N}$ back to the household. The firm can not borrow more than
its income, $\bar{y}_t$. It holds $\mu_t$ money and wastes $\kappa$ amount on each transaction. Then,
in each period $(\mu_t - \kappa)$ will be laid out on the average to the repayment of debt to
the household. Thus $(\mu_t - \kappa) = \frac{b}{N}(1 + R_t)$, where $R_t$ is the rate of nominal interest
on borrowing, is given to the firm. Hence the firm’s debt holdings are

\[ \bar{b}_t = \bar{y}_t - \frac{1}{N_t} \left[ \left( \mu_t - \kappa \right) + 2\left( \mu_t - \kappa \right) + 3\left( \mu_t - \kappa \right) \right] \]

Equation (1.2) corresponds to equation (3) in Feenstra (1986). Here \( \bar{b}_t \) represents the remaining debt holdings of a firm, which corresponds to the remaining asset holdings of the household in Feenstra (1986). Terms in parentheses show the decrease in the amount of debt in the first, second ...Nth periods.

Note that \( \frac{\mu_t N_t}{2} = \frac{\bar{y}}{2} \) and \( \bar{m}_t = \frac{\mu}{2} \). By using these relationships, equation (1.2) can be expressed as

\[ \bar{y}_t = \bar{b}_t + \left[ \frac{\bar{y}}{2} - \frac{\kappa N_t}{2} \right] + \left( \bar{m}_t - \frac{\kappa}{2} \right) \] (1.3)

The firm uses its income to pay its factors, previous periods’ interest, principal payments, and transfer dividends to the household. Assume zero interest on money holdings and no depreciation on capital. Hence, the firm’s total expenditure may be summed up as

\[ \bar{y}_t = r \bar{k}_t + \left[ (1 + i_{t-1})\bar{b}_{t-1} + \bar{m}_{t-1} \right] \frac{p_{t-1}}{p_t} \] (1.4)

Note that changes in prices are reflected in the value of the firm’s debt. Inflation decreases the value of debt in favor of the firm. The firm’s income will be equal to
its expenditures. Then we can equate (1.3) to (1.4) and rearrange to get

\[
\bar{y}_t + \bar{b}_t + \bar{m}_t = r\bar{k}_t + [(1 + i_{t-1})\bar{b}_{t-1} + \bar{m}_{t-1}]\frac{p_{t-1}}{p_t} + \frac{\Theta(\bar{y}_t, \bar{m}_t)}{2} + \frac{\kappa}{2}
\] (1.5)

where \( \Theta(\bar{y}, \bar{m}) = \frac{\kappa N_t}{2} = \kappa \frac{\bar{y}}{2\bar{m}} \), the transactions cost for the firm.

Thus, the firm’s economic profit will be the difference between the firm’s income and expenses. Income consists of revenue from sales, borrowing from households and the part of the income that is held in the form of cash to perform transactions. The firm spends its income on hiring factors of production and the cost of borrowing. The cost of borrowing includes the total cost of cash management, which includes old principal plus interest payments and transactions costs. Therefore, the economic profit of the firm is

\[
\Pi(r_t, i_t, \kappa) = \frac{\bar{y}_t}{2} - \frac{\Theta(\bar{y}_t, \bar{m}_t)}{2} + \bar{b}_t + \bar{m}_t - \frac{\kappa}{2} - r_t\bar{k}_t - [(1 + i_{t-1})\bar{b}_{t-1} + \bar{m}_{t-1}]\frac{p_{t-1}}{p_t} = 0
\] (1.6)

The profit function above is written for a Baumol-Tobin type model and it is very similar in form to the general profit function in the equation (1.1).

### 1.3 Duality in Production

The previous section sets up the objective of a firm engaging in production and exchange as maximizing economic profit. There is a duality between uses minimization
and sources maximization for a profit maximizing firm. In this section we use the properties of duality to show that the transactions demand for money is indistinguishable from money in the production function.

Up to now we have shown that transactions cost in production causes the firm to produce less than the optimal level of output: the level of output expected with the given technology, quality and quantity of factors. Inefficiency in production arises due to the fact that production involves frictions and the firm has to waste some resources in transactions. The existence of transactions costs results in unavoidable output loss in production. We measure the productive inefficiency in terms of output loss. We have not shown the reason why transactions demand for money is functionally equivalent to money in the production function. In order to demonstrate the equivalence first we need to express productive inefficiency in terms of input loss. This rearrangement does not change the content of the problem, but helps to show that duality between cost minimization and output maximization arises due to the duality in explaining the inefficiency in production from output and input perspectives.

Note that we measure the inefficiency when technology, quality and quantity of inputs are given. Inefficiency arises because firms can not utilize all of theirs available resources in production. Some of the resources are used in the transactions sector.
Therefore, an increase in efficiency implies an increase in output as a result of a shift of resources from transactions to production.

1.3.1 Transactions Cost from the Output and Input Perspective

Transactions cost can be expressed either as a reduction in the level of output or as a loss of resources in production. In order to understand the equivalence of the definition, let’s use a simple production function as in Figure (1.3). Figure (1.3) shows the path of production with and without transactions cost. Points at the production function show all optimal level of output can be produced given factors of production.

For a moment, assume there is no transactions cost for a firm in an economy. On the figure, if $z_t$ is the total stock of capital given quality, quantity and the level of technology, the firm that employs $z_t$ can produce at a point on the production function, $y(z_t)$. $y(z_t)$ is the maximum level of output that can be produced with $z_t$. Let’s call it the “ideal output”.

A firm incurring transactions cost produces less than the ideal level of output. The distance between the ideal level of output and the actual level of output is interpreted as an inefficiency in production. If we know the ideal level of output, the actual level of output can be computed as the difference between the ideal level of
output and the value of inefficiency. Since in this simple model the only disturbance in production is the transactions cost, we have to subtract the value of transactions cost in terms of output from the ideal level of output to obtain the actual level of output: \( y(z_t) - \Theta(y(z_t), m_t) \). This point is shown on the figure (1.3). The problem here is ideal output \( y(z_t) \) can not be observed in the real world. The firm can not produce at this point because \( \Theta(y(z_t), m_t) \) is the unavoidable loss to frictions, i.e the firm spends time getting information from the market.

Note that an efficient firm can produce the same non-optimal level of output, \( y(z_t) - \Theta(y(z_t), m_t) \) with less capital. Let \( k_t < z_t \) be the quantity of capital an
efficient firm can use to produce $y(k_t)$, such that

$$y(k_t) = y(z_t) - \Theta(y(z_t), m_t) \quad (1.7)$$

This means that an inefficient firm is unable to employ $(z_t - k_t)$ in the production due to the transactions cost. Transactions cost can be observed as a restriction in the employment of the capital in production. Let $\chi(k_t, m_t)$ be the transactions cost that limits a firm to employ $z_t$. Then

$$z_t = k_t + \chi(k_t, m_t) \quad (1.8)$$

It is clear from the figure (1.3) and equation (1.7) that

$$\chi(k_t, m_t) = y^{-1}[y(k_t) - \Theta(y(z_t), m_t)] - k_t \quad (1.9)$$

Equation (1.7) states an inefficiency in production due to transactions cost in terms of loss in output. On the other hand, equation (1.9) indicates the inefficiency in terms of loss in units of capital used in production. Both equations point out that transactions cost in production results in waste of resources in production and causes inefficiency.

Consequently, the profit function can be expressed in two ways: If we use equation (1.7),

$$\Pi_1(r_t, i_t, \kappa) = y(z) - \Theta(y(z_t), m_t) + A_t - rz_t - L_t \quad (1.10)$$
or according to the equation (1.8) as

\[
\Pi_2(r_t, i_t, \kappa) = y(k_t) + A_t - r[k_t + \chi(k_t, m_t)] - L_t
\]  

(1.11)

where \( A_t = b_t + m_t \) and \( L_t = [(1 + \iota_{t-1})b_{t-1} + m_{t-1}]\frac{p_{t-1}}{p_t} \).

### 1.3.2 General Properties of the Transactions Cost Function: \( \chi(k_t, m_t) \)

We have expressed the efficiency loss in production in terms of units of output, \( \Theta(y(z_t), m_t) \) and alternatively in terms of units of capital, \( \chi(k_t, m_t) \). Both transactions cost functions have to have certain properties in order to use each of them interchangeably. Our objective in this study is to show that there is a functional equivalence between the transactions demand for money and money in the production function. As we pointed out above, transactions cost in factors market results in inefficiency. Firms hold money to decrease their transactions cost. The role money plays in the production process is to improve efficiency by increasing the quantity of factors used in production. Therefore, from now on we will focus on the properties of the transactions cost in the factors market, namely \( \chi(k_t, m_t) \). We need to impose the following set of assumptions on the transactions cost function, knowing that these assumptions have to be imposed on \( \Theta(\cdot) \) as well, due to the equation (1.9):
**Assumption 1:** For all $k_t \geq 0$ and $m_t > 0$, $\chi(k_t, m_t)$ is twice continuously differentiable and satisfies:

1. $\chi(k_t, m_t) \geq 0$, $\chi(0, m_t) = 0$;

2. Homogenous of degree 0;

3. $\chi_k \geq 0$, $\chi_m \leq 0$;

4. $\chi_{kk} \geq 0$, $\chi_{mm} \geq 0$, $\chi_{km} \leq 0$;

5. $\chi(k_t, m_t)$ is quasi-convex with expansion path of non-negative slope.

The first property assures that transactions cost occurs as long as firm employs capital. If the firm does not employ capital there will be no production. Therefore the firm does not need to hold money to minimize its transactions cost. If the firm can not borrow, there will be no production and no transactions cost. The third and fourth properties together state that cost of cash management is positively related to employment of capital and negatively related to cash holding. As the employment increases the marginal cost of cash management stays the same or increases. On the other hand, higher cash balances either keep the marginal cost of cash management constant or decrease it. The cross derivative of the transaction cost is negative because
if the firm holds money for transaction purposes, some of the resources assigned to perform transactions can be shifted into production, thus causing a decrease in the resources used in transaction. Assumption (1.5) states that iso-curves have rising slope and the expansion paths are lines through the origin. A constant or rising slope of the iso-curves implies a quasi-convex $\chi$ function; thus $k + \chi$ is also quasi-convex, and has an upward sloping expansion paths.

Now we are ready to start the discussion of the duality between sources maximization and uses minimization. Splitting the profit function into income and cost components allows us investigate the firm’s budget and expenditure functions separately. The duality is relevant to our analysis, due to the fact that it helps to demonstrate our main objective of the equivalence between transactions demand for money and money in the production function.

### 1.3.3 Uses Minimization

In most microeconomic analysis it is more convenient to break up the profit maximization problem into two more general statements such as uses minimization and sources maximization. Cost of production must include all implicit and explicit costs that a firm has to pay to be in business. The firm minimizes the following uses
function given its sources:

$$\min_{y,m,b} TC(y_t, m_t, b_t) = r[k_t + \chi(k_t, m_t)] + L_t$$ (1.12)

subject to the given

$$y(k_t) + A_t$$

Equation (1.12) states that a firm should minimize its total uses that arise from the transactions cost and the cost of capital in production at each given level of income. The solution to this problem will be some choice of $y^*, m^*$ and $b^*$ that minimizes the costs of the firm’s actions. At this optimum level of output, the firm hires an optimum number of factors.

### 1.3.4 Duality of Uses Minimization: Sources Maximization

The duality of uses minimization is maximization of sources for a profit maximizing firm. By using this property, we want to show that money in the production function is equivalent to the firm’s transactions demand for real money. The firm’s sources maximization problem is:

$$\max_{k,m,b} y(k_t) + A_t$$ (1.13)

subject to the given

$$r[k_t + \chi(k_t, m_t)] + L_t$$ (1.14)
The firm choose $k^*$, $m^*$ and $b^*$ to maximize its sources. The duality holds if the choice of $k^*$, $m^*$ and $b^*$ at the same time minimizes the uses at $y^*$, $m^*$ and $b^*$.

1.4 Money in the Production Function

We have demonstrated the duality between expressing the transactions cost for a firm in terms of output and loss in utilization of factors in production. Then we have examined the duality between uses minimization and sources maximization for a profit maximizing firm by using the earlier duality between expressing the transactions cost as an input loss and output loss. Now we can show the equivalence between transactions cost in the budget constraint and entering money in the production function. We can use equation (1.13) and (1.14) to examine the equivalence. In order to do that, first we need to restate the firms sources maximization problem. Let choices of the firm be $z_t$, $m_t \geq 0$ and $b_t$ to

$$\max f(z_t, m_t) + A_t,$$  

(1.15)

subject to

$$rz_t + L_t$$  

(1.16)

where $f(z_t, m_t)$ is the real output, depending on capital $z_t$ and real money balances $m_t$. 

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Equation (1.13) is equivalent to (1.15) if for all \( k_t, m_t \) and \( z_t \),

\[
y(k_t) \equiv f(z_t, m_t)
\] (1.17)

The equation (1.17) together with (1.7) also states that:

\[
y(z_t) - \Theta(y(z_t), m_t) \equiv f(z_t, m_t)
\] (1.18)

At the same time equation (1.14) is equivalent to (1.16) if

\[
k_t + \chi(k_t, m_t) \equiv z_t
\] (1.19)

Analogous to Feenstra (1986), \( z_t \) can be interpreted as ‘gross capital’, including both ‘net capital’ \( k_t \) and transactions cost.

Equation (1.17) together with (1.19) implies that

\[
y(k_t) \equiv f(k_t + \chi(k_t, m_t), m_t)
\] (1.20)

Therefore, as in Feenstra (1986) we may say that \((y, \chi)\) are ‘equivalent’ to \( f \), if (1.20) holds for all \( k_t \) and \( m_t \). This implies that if a firm demands money for transactions purposes, the problem can be explained either as a cost in the budget constraint or money in the production function. These expressions differ only in their functional notations. \( f(\cdot) \) is the gross output that maximizes profit. It has been shown that
when a firm demands money for transactions purposes, gross output can be expressed as a function of gross factor of production and real balances. Using equation (1.17) and (1.19), profit function (1.11) can be simplified as

$$\max \prod_3 (r, R, \kappa) = f(z_t, m_t) + A_t - rz_t - L_t$$  \hspace{1cm} (1.21)

Following the steps in Feenstra (1986), let’s define another function, $W(z_t, m_t)$ such that

$$W(z_t, m_t) = y^{-1}[f(x_t, m_t)]$$

Then, we can express net capital $k_t$ as

$$k_t \equiv W(k_t + \chi(k_t, m_t), m_t)$$  \hspace{1cm} (1.22)

Once again similar to Feenstra (1986), we may say that $\chi$ is ‘equivalent’ to $W$, if (1.22) holds for all $k_t$ and $m_t$. Equation (1.22) states that given the production function, $W_t$, $f_t$ is simply a concave transformation of $y_t$: $f(z_t, m_t) \equiv y(W(z_t, m_t))$.

Following Feenstra we need to impose the following conditions on $W$:

**Assumption 2:** For all $z_t \geq 0$ and $m > 0$, $W(.)$ is twice continuously differentiable and satisfies:

1. $W(z_t, m_t) \geq 0, W(0, m_t) = 0;$
2. \( W(.) \) is homogenous of degree 1 in \( z_t \) and \( m_t \);

3. \( 0 < W_z \leq 1, W_m \geq 0 \);

4. \( W_{zz} \leq 0, W_{zm} \geq 0 \) and \( W_{mm} \leq 0 \);

5. \( W(z_t, m_t) \) is quasi-concave in \( z_t \) given \( m_t \) and quasi-concave in \( m_t \) given \( z_t \) with an expansion path of non-negative slop.

The first assumption states that production occurs as long as physical factors are hired. Money does not have value by itself as a productive resource. On the other hand if the firm can not borrow money from households, it can not hire factors for production, therefore there will be no transactions cost. The second assumption together with the third point out a constant return to scale technology where marginal product of input is bounded from above. The fourth assumption is consistent with Fisher(1974), and states that increases in real cash balances increase the use of some resources that would otherwise be tied up in transaction. There is a positive relationship between holding money and the use of resources in real production. Assumption (2.5) is the outcome of increasing and concave transformation of \( W \) into \( f \). It states that iso-curves are line through the origin and with the upward sloping expansion paths.
Now, we can state Feenstra (1986)'s “Proposition 1” as it applies in our analysis.

**Proposition:** If there is a transactions cost $\chi(k_t, m_t)$ satisfying Assumption 1, then there exists an equivalent production function $W(z_t, m_t)$ satisfying Assumptions 2, and conversely.

A formal proof of this proposition is given in the appendix. Intuitively this proposition states that $f$ is obtained as a concave transformation of $W$. According to the Assumption 2, $W$ is quasi-concave; it does not need to be concave. If the net production function is sufficiently concave, the non-concavity of $W$ may be overcome after the transformation $y$ is applied. The crucial point here is that after the transformation, the gross production function $f$ is strictly concave: $f_z > 0$, $f_m > 0$, $f_{zz} < 0$, $f_{mm} < 0$ and $f_{zz}f_{mm} - f_{zm}^2 \geq 0$. Feenstra (1986) imposes the concavity of the transformed utility function as an additional assumption. Note that we need to impose this additional assumption since concave transformation can result in $f_{zm} < 0$. Following Feenstra, in order to determine the sign of $f_{zm}$, we need to find its components as follows:

First, recall that $f = y(W) \equiv y(k)$ therefore $f_z = y_k W_z$. Taking the second
derivative with respect to $m_t$ we get

$$f_{zm} = y_{kk} W_m W_z + y_k W_{zm}$$

Second, we need to find $W_m$, $W_z$ and $W_{zm}$. These derivatives can be found by using $k_t = W(k_t + \chi(k_t, m_t), m_t)$. Then, the first order derivative of $W$ with respect to $z_t$ is

$$1 = W_z (1 + \chi_k) \Rightarrow W_z = \frac{1}{(1 + \chi_k)} \geq 0 \quad (1.23)$$

where $\chi_k \geq 0$. The second derivative with respect to $m_t$ is

$$0 = W_{zm} (1 + \chi_k) + W_z \frac{d}{dm} \chi_k |_{k+\chi} \quad (1.24)$$

Feenstra (1986) shows that

$$\frac{d}{dm} \chi_k |_{k+\chi} = \chi_{km} - \frac{\chi_{kk} \chi_m}{1 + \chi_k} \leq 0 \quad (1.25)$$

Thus

$$W_{zm} = -W_z \chi_{km} + \frac{\chi_{kk} \chi_m}{1 + \chi_k} \geq 0 \quad (1.26)$$

On the other hand, the derivative of $W$ with respect to $m$ is

$$W_m = -\frac{\chi_m}{(1 + \chi_k)} \quad (1.27)$$

Using equations (1.23), (1.26) and (1.27) we have

$$f_{zm} = y_k \frac{\chi_m}{k(1 + \chi_m)^2} \left[ -k y_{kk} \frac{y_k}{y_k} - k \chi_{mk} \frac{\chi_m}{\chi_m} + k \chi_{kk} \frac{1}{1 + \chi_k} \right] \quad (1.28)$$
where $\chi_m \leq 0$, $\chi_k \geq 0$ and $\chi_{km} \leq 0$. $-\frac{k\chi_{kk}}{yk}$ measures the concavity of the net production function. The difference between the last two terms in the square bracket is non-positive. The cross derivative of the production function, $f_{zm}$ is non-negative if the difference exceeds or at least is equal to $-\frac{k\chi_{kk}}{yk}$. On the other hand, if

$$\frac{-k\chi_{kk}}{yk} \geq \frac{-k\chi_{mk}}{\chi_m} + \frac{k\chi_{kk}}{(1 + \chi_k)} \quad (1.29)$$

then $f_{zm}$ tends to be non-positive. The condition above implies that over some region $f_{zm}$ can be negative. Equation (1.29) states that if we have a sufficiently concave production function $y$, non-concavity of $W$ may be overcome after the transformation $y$ is applied, such that $f$ is a concave function of $z_t$ and $m_t$.

Consider a Cobb-Douglas production function, $y(k) = k^\alpha$ and a Baumol- Tobin type transactions demand for money, $\chi(k_t, m_t) = \kappa \frac{k_t}{2m_t}$. Then

$$z_t = k_t + \kappa \frac{k_t}{2m_t}$$

so

$$k_t = \frac{2z_t m_t}{2m_t + \kappa}$$

The gross production function is then

$$f(z_t, m_t) = y(\frac{2z_t m_t}{2m_t + \kappa}) = [\frac{2z_t m_t}{2m_t + \kappa}]^\alpha$$
The cross derivative of \( f \) is 

\[
f_{zm} = \alpha^2 z^{\alpha-1} \frac{2\kappa}{(2m + \kappa)^2} \left[ \frac{kykk}{yk} - 1 \right]^{\alpha-1} > 0
\]

In general, the cross derivative of the production function is \( f_{zm} > (\prec)0 \) as \( \frac{kykk}{yk} < (\succ)1 \).

1.4.1 Duality in Utility

Before we conclude this chapter it would be interesting to note that the duality discussed in this analysis applies to duality in consumption in Feenstra (1986) as well. The figure (1.4) shows the relationship between utility and the level of consumption. It assumes that the good is normal, therefore as consumption increases utility increases at a decreasing rate. Assume that without any liquidity cost a household can consume \( x_t \) and the maximum utility it gets is \( U(x_t) \). Analogous to the production analysis, let’s call \( U(x_t) \) the “ideal utility”. Liquidity cost in consumption implies that household could not get the maximum utility it is supposed to get. The difference between the actual level of utility and the ideal utility can be described as an unavoidable wasted utility, \( \beta(U(x_t), m_t) \).

A household that exist in an economy with no liquidity cost can get the same level of utility by consuming less than \( x_t \), such as at \( c_t \). According to Feenstra (1986)
$x_t - c_t = \phi(c_t, m_t)$. $\beta(\cdot)$ expresses the inefficiency in consumption in terms of loss in utility while $\phi(\cdot)$ uses units of consumption.

The duality between expressing inefficiency in consumption in different units allows us to define another relationship as

$$U(c_t) = U(x_t) - \beta(U(x_t), m_t) \equiv V(x_t, m_t) \quad (1.30)$$

Equation (1.30) argues that net utility equals gross utility minus unavoidable wasted utility. This relationship is analogous to the equation (1.7). The equation (1.30) also states that gross utility can be expressed equivalently as a function of gross consumption, $x_t$ and real money, $m_t$. 

Figure 1.4: Utility Function
1.5 Concluding Remarks

In this study, we have shown that the transaction demand for money by firms is functionally equivalent to money in the production function. At the beginning of the production process the firm borrows from the household. Household total interest bearing assets are equal to firm total debt. The firm receives money from the household continuously, such that the firm’s total receipts are equal to the household’s expenditures on consumption. The firm accumulates cash over a particular period and then at the end of each period pays its factors of production, incurring a transactions cost. The purpose of the firm is to maximize its profit given income and cost of cash management and factor employment. This scenario is analogous to Feenstra (1986)’s household problem. There is a functional equivalence between the transactions demand for money and money in the production function for the same reason that there is a functional equivalence between the transactions demand for money and utility of money.

Money in the production function implies efficient use of other resources, rather than considering money as a physical factor of production used in the production process directly. For this reason, the role of money has to be in the total factor
productivity component of the production function. It is important to identify the
differences between technical change related to the quality of factors and technical
change related to the efficient use of physical factors. This study emphasizes the
second property. Reinterpretation of the production function with money allows one
to include this missing variable in monetary models.

Once money enters the production function as a factor that increases the efficiency
of capital in factor markets and efficiency in consumption in goods markets, the
monetary authorities would have room to improve production. The study can be
extended in at least two different ways. First, it is possible to show that monetary
expansion would not be super-neutral over the growth rate of consumption and capital
in the long run. Second, an empirical test is required to support implications of this
study. The second chapter examines the first question while the third chapter applies
a stochastic production frontier approach to examine the significance of real balances
held by firms and households separately, in determining the technical inefficiency in
production for 12 European Union countries.
Appendix

Proof of the Proposition

In this section it is shown that all proofs in Feenstra (1986) can be used to prove proposition 1 in our analysis.

First we start with the first derivatives of \( W_t \) and \( \chi_t \) and show the relationship at the first order between these functions:

Beginning with \( \chi \) satisfying assumption (1.1) and \( z_t = k_t + \chi(k_t, m_t) \) where both \( x_t \in [0, \infty) \) for \( k_t \in [0, \infty) \) given \( m_t > 0 \) we can invert equation 1.19 to obtain (1.22) as \( k_t = W(z_t, m_t) \) as long as \( (1 + \chi_c) > 0 \). If \( W(k_t + \chi(k_t, m_t), m_t) = k_t \) then

\[
W(0 + \chi(0, m_t), m_t) = W(0, m_t) = 0 \quad (1.31)
\]

\[
W_z(1 + \chi_k) = 1 \Rightarrow W_z = \frac{1}{(1 + \chi_k)} \quad (1.32)
\]

\[
W_z\chi_m + W_m = 0 \Rightarrow W_m = -W_z\chi_m = \frac{\chi_m}{(1 + \chi_k)} \quad (1.33)
\]

therefore \( 0 < W_z < 1 \) and \( W_m > 0 \) since \( \chi_k > 0 \) and \( \chi_m < 0 \).
To prove that the proposition 1 is true we need to show that everything above holds conversely as well. Then, start with $k_t = W(z_t, m_t)$ satisfying Assumptions (4.1) where $k_t \in [0, \infty]$ with $z_t \in [0, \infty)$ and fixed $m_t > 0$. Since $W_x > 0$ we can invert $k_t$ to have $z_t = H(k_t, m_t)$ so that $\chi(k_t, m_t) = z_t - k_t = H(k_t, m_t) - k_t$ Then

$$\chi(0, m_t) = H(0, m_t) - 0 = 0 \quad (1.34)$$

$$\chi_k = H_k - 1$$

If we take the derivative of $z_t = H(k_t, m_t)$ with respect to $z_t$ we have $1 = H_k k_W W_z$ that gives $H_k = \frac{1}{W_z}$ therefore

$$\chi_k = \frac{1}{W_z} - 1 \geq 0 \quad (1.35)$$

$$\chi_m = H_m = \frac{W_m}{W_z} \leq 0 \quad (1.36)$$

Second, the relationship between second derivatives of $W$ and $\chi_t$ are as follow:

First we are going to differentiate the equation (1.32) with respect to $k_t$. Before that let express the equation as $1 = W_z(1 + \chi_k)$. Then

$$0 = W_{zz}(1 + \chi_k)^2 + W_z \chi_{kk}$$

therefore

$$W_{zz} = -\frac{W_z \chi_{kk}}{(1 + \chi_k)^2}$$
Second differentiate (1.32) with respect to \( m_t \), taking \( z_t = k_t + \chi \) fixed,

\[
0 = W_{zm}(1 + \chi_k) + W_z \frac{d\chi_{km}}{dm}|_{k+\chi}
\]

where

\[
\frac{d\chi_{km}}{dm}|_{k+\chi} = \chi_{km} - \frac{\chi_{kk}\chi_{km}}{(1 + \chi_k)} = (1 + \chi_k)\left[\frac{\chi_{km}}{1 + \chi_k} - \frac{\chi_{kk}\chi_{km}}{(1 + \chi_k)^2}\right]
\]

In order to understand the equality above first we need to express \( \chi_k(k, m) = \chi_k(W(z, m), m) \). Then \( \chi_{kk}W_m + \chi_{km} \). Substituting \( W_m \) we will get the equation above which can be further simplified as

\[
\frac{d\chi_{km}}{dm}|_{k+\chi} = (1 + \chi_k)\frac{\partial(\frac{\chi_{km}}{1 + \chi_k})}{\partial k}
\]

Therefore

\[
W_{zm} = -W_z \frac{\partial(\frac{\chi_{km}}{1 + \chi_k})}{\partial k}
\]

Third differentiate (1.36) with respect to \( z_t \),

\[
\frac{\partial(-\frac{W_{zm}}{W_z})}{\partial z} = \chi_{mk} \frac{dk}{dz}|_m = \chi_{mk}W_z = \frac{\chi_{mk}}{1 + \chi_k}
\]

since \( k_t = W(z_t, m_t) \).

Fourth Differentiate (1.36) with respect to \( m_t \), holding \( z_t = k_t + \chi \) fixed,

\[
\frac{\partial(-\frac{W_{zm}}{W_z})}{\partial m} = \frac{d\chi_m}{dm}|_{k+\chi} = -\chi_{mk}W_m + \chi_{mm} = \frac{\chi_{mk}}{1 + \chi_k} + \chi_{mm}
\]
Similar to Feenstra (1986) the equation above further simplified as

\[
\frac{\partial (-\frac{W_m W_z}{W_z})}{\partial m} = (1 + \chi_m) \frac{\partial (-\frac{W_m}{W_z})}{\partial m}
\]

Fifth, following Feenstra steps differentiate (1.36) with respect to \( m_t \), holding \( k_t \) fixed,

\[
\frac{\partial \chi_m}{\partial m} = \chi_{mm} = \frac{d(-\frac{W_m}{W_z})}{dm}|_c = \frac{\partial (-\frac{W_m}{W_z})}{\partial m} + \chi_m \frac{\partial (-\frac{W_m}{W_z})}{\partial z}
\]

Using second and the fourth property we have

\[
\frac{\partial \left(\frac{W_m}{1+\chi_k}\right)}{\partial k} - \frac{(1 + \chi_k) \frac{\partial \left(\frac{W_m}{1+\chi_k}\right)}{\partial m}}{\chi_m} = -\frac{W_z}{W_z} + \frac{W_z}{W_m} \frac{\partial (-\frac{W_m}{W_z})}{\partial m} = -\frac{W_m W_z}{W_z}
\]

As in Feenstra (1986) first to fifth properties shows that conditions in Assumptions 1 implies conditions in Assumptions 2, and conversely. Q.E.D.
Chapter 2

Sidrauski with Feenstra Equivalence and Money in the Production Function

2.1 Introduction

Sidrauski (1967) uses money as an alternative government non-interest bearing asset to real capital in order to explore certain aspects of the interaction of these factors in a model of economic growth. The study supports super-neutrality of monetary expansion in the long run.

This study shows that if Sidrauski’s analysis is set up in accordance with Feenstra (1986) its main conclusions will be changed. Feenstra explains that when money is in the utility function together with consumption, consumption must be in gross terms, which includes both net consumption and the liquidity cost. Gross consumption is a composite aggregate and shows the combination of net consumption and liquidity cost. The first section of this chapter discusses problems with the Sidrauski model.
in the context of Feenstra equivalence and concludes that change in the growth rate of money supply is super-neutral over the gross consumption in a model of money in the utility function where both money and gross consumption is used, but not super-neutral over net consumption.

After showing the properties of the money-in-the-utility function models, the second section performs the same exercise for a model where money appears as a factor of production. The first chapter shows the functional equivalence between the transactions demand for money by firms and money in the production function. It is noted that if one wants to use money in the production function, the physical factor has to be in gross terms, which shows the relationship between the net physical factor and transactions cost. The model supports non super-neutrality of monetary expansion on net consumption.

These results deliver two important messages. First, for both money in the utility functions and money in the production function models, money is not super-neutral over net consumption. The impact of the growth rate of money on real aggregates depends on the relationship between the consumption/money ratio in the liquidity costs function in the goods market and the capital/money ratio in the transactions costs function in the input market.
Second, when a monetary model is set up in accordance with Feenstra equivalence, it is possible to show the impact of transactions costs on the determination of inflation. For a simple Baumol-Tobin type model in which unit transactions cost is assumed to be constant there is no optimal quantity of real money. The same reason explains the non-optimality of inflation as well. In both cases we show that Sidrauski’s constant money supply rule is invalid. Sidrauski ignores the fact that the main service of money in an economy is to improve efficiency in exchange in both goods and factor markets.

The organization of the study is as follows. The second section summarizes the Sidrauski (1967) model and points out its important conclusions. Then it criticizes Sidrauski by using the Feenstra analysis and explains the misinterpretation in Sidrauski (1967). The third section presents a monetary model with money in the production function. The reason why money is justified as a productive factor is explained in the first chapter. This section analyzes the implications of such a production function from the monetary policy perspective. Implications of a model where money is used in both utility and production functions are examined at the end of the third section as well. The final section concludes.
2.2 Sidrauski within the Feenstra Framework

The purpose of this section is to summarize the Sidrauski analysis to explain money in the utility models and then criticize Sidrauski (1967) by using Feenstra (1986) equivalence.

2.2.1 Money in the Utility Function

Sidrauski Model

A representative household gets utility from consumption as well as holding real cash balances for consumption purposes. Thus

\[
\max_{c,g} \sum_{t=0}^{\infty} U(c_t, m_t)(1 + \rho)^{-(t+1)}, 0 < \rho < 1
\]  

(2.1)

where \(c_t, m_t\) and \(\rho\) represents consumption, real balances and a constant rate of time preference respectively. It is assumed that the utility function is strictly concave with continuous first and the second derivatives; \(U_c > 0, U_{cc} < 0, U_m > 0\) and \(U_{mm} < 0\).

The Household’s real non-human wealth, \(a_t\), is allocated among capital, \(k_t\), and real cash balances, \(m_t\) as

\[a_t = m_t + k_t\]  

(2.2)

The household spends its disposable income, \(y(k)\), on real consumption and gross...
real savings, \( s_t \):

\[
y(k) + \tau = c_t + s_t
\]  \hspace{1cm} (2.3)

where \( \tau \) is the net transfer from the government, \( y(k) \) is production and is a function of the capital/labor ratio \( k_t \), where labor is normalized to unity. Gross real savings consists of gross capital accumulation, \( i_t \), plus the gross addition to holdings of real cash balances for consumption, \( g_t \), such that

\[
s_t = i_t + g_t
\]  \hspace{1cm} (2.4)

where

\[
g_t = \frac{M_{t+1} - M_t}{p_t}
\]  \hspace{1cm} (2.5)

and \( M \) stands for stock of nominal money supply.

Gross capital accumulation is equal to the net addition to the capital stock and the replacement of the depreciated capital, \( \delta k_t \). Therefore

\[
k_t - k_{t-1} = \Delta k = i_t - \delta k_t
\]  \hspace{1cm} (2.6)

Gross accumulation of real cash balances for consumption is equal to net addition to cash holding for consumption plus deterioration in the value of cash.

\[
m_t - m_{t-1} = \Delta m = (1 + \pi_t)^{-1}(g_t - \pi_t m_t)
\]  \hspace{1cm} (2.7)
where $\pi_t = \frac{P_t}{P_{t-1}}$ and $g_t$ is the gross accumulation of real balances.

Thus, the household flow budget constraint is:

$$y(k_t) + \tau - i_t - g_t - c_t = 0 \quad (2.8)$$

which is equivalent to

$$a_t - a_{t-1} = \Delta a = y(k) + \tau - g_t - i_t - c_t \quad (2.9)$$

where

$$\Delta a = \Delta k + \Delta m \quad (2.10)$$

There are three choice variables; $c_t$, $i_t$ and $g_t$. Two of these variables can be determined independently. Let the independent choice variables be $c_t$, and $g_t$. Then,

$$i_t = y(k) + \tau - c_t - g_t \quad (2.11)$$

therefore, substituting (2.11) into (2.6), we obtain an equation for gross capital accumulation as

$$\Delta k = y(k) + \tau - c_t - g_t - \delta k_t \quad (2.12)$$

Equations (2.7) and (2.12) are stock and flow budget constraints for the household to maximize utility.

**Central Planner**
The central planner chooses sequences for \( c_t, g_t \) so as to maximize household utility, equation (2.1), subject to its stock and flow budget constraints, equations (2.7) and (2.12), given initial conditions \( a(0), \tau, \delta, \) and \( \pi_t \).

The Hamiltonian for this problem is:

\[
H = U(c_t, m_t)(1+\rho)^{-(t+1)} + \mu_{t+1}[y(k) + \tau - c_t - g_t - \delta k_t] + \omega_{t+1}(1+\pi_t)^{-1}[g_t - \pi m_t] \tag{2.13}
\]

Defining \( \lambda_{t+1} = \mu_{t+1}(1+\rho)^{t+1} \) and \( \psi_{t+1} = \omega_{t+1}(1+\rho)^{t+1} \), the transformed Hamiltonian would be

\[
\aleph = U(c_t, m_t) + \lambda_{t+1}[y(k) + \tau - c_t - g_t - \delta k_t] + \psi_{t+1}(1 + \pi_t)^{-1}[g_t - \pi m_t] \tag{2.14}
\]

where \( \lambda \) is the Lagrange multiplier attached to the flow constraint (2.12) and \( \psi \) is the Lagrange multiplier attached to the stock constraint (2.7).

Conditions for a maximum are given by the Euler equations for \( c_t \) and \( g_t \) respectively, dynamic equations of \( \Delta m, \Delta k, \Delta \lambda, \Delta \psi \) and end point conditions, given \( a(0), \delta, \tau, \) and \( \pi_t \). Therefore

\[
\frac{\partial H}{\partial c} = U_c(c_t, m_t) - \lambda_{t+1} = 0 \Rightarrow U_c = \lambda_{t+1} \tag{2.15}
\]

\[
\frac{\partial H}{\partial g} = \psi_{t+1}(1 + \pi_t)^{-1} - \lambda_{t+1} = 0 \Rightarrow \psi_{t+1} = (1 + \pi_t)\lambda_{t+1}, \tag{2.16}
\]

\[
\Delta \lambda = -\lambda_{t+1}(1+\rho)^{-1}[y_k - \delta - \rho], \tag{2.17}
\]

41
\[ \Delta \psi = -U_m(1 + \rho)^{-1} + \psi_{t+1}(\rho(1 + \pi_t) + \pi_t)(1 + \rho)^{-1}(1 + \pi_t)^{-1} \]  
\hspace{1cm} (2.18) 

and

\[ \lim_{t \to \infty} \psi_t m_t(1 + \rho)^{-t} = 0 \]  
\hspace{1cm} (2.19)

\[ \lim_{t \to \infty} \lambda_t k_t(1 + \rho)^{-t} = 0 \]  
\hspace{1cm} (2.20)

**General Equilibrium and the Steady State**

Steady state requires:

1. \( \Delta \lambda = 0 \), then, the marginal product of capital equals its user cost:

\[ y_k = \delta + \rho \]  
\hspace{1cm} (2.21)

2. \( \Delta \psi = 0 \), then, \( U_m = \psi_{t+1}(1 + \pi_t)^{-1}(\rho(1 + \pi_t) + \pi_t) = \lambda_{t+1}(\rho(1 + \pi_t) + \pi_t) = U_c(\rho(1 + \pi_t) + \pi_t) = U_c[y_k - \delta + \pi_t(1 + \rho)] \)

Therefore, marginal benefit of holding money is equal to its marginal cost, nominal interest rate:

\[ \frac{U_m}{U_c} = y_k - \delta + \pi_t(1 + \rho) = (1 + \rho)(1 + \pi_t) - 1 = i_t \]
\hspace{1cm} (2.22)

3. In order to find the steady state growth rate of the nominal money supply, \( \theta^* \) assume that the monetary authority applies an exogenous nominal money
supply rule and increases the nominal money supply, $M_t$, by $\theta_t$.

\[
\frac{M_{t+1} - M_t}{M_t} M_t = \theta_t M_t = \mu
\]  

(2.23)

where $\theta_t = \frac{M_{t+1} - M_t}{M_t}$. We assume that government maintains a constant rate of monetary expansion thus $\theta_t = \theta_0$. Further assume that the money supply increases to finance transfers from government, $\tau_t$, such that

\[
\tau_t = \theta_0 m_t = g_t
\]  

(2.24)

where $m_t = \frac{M_t}{P_t}$. Monetary policy implies that the goods market equilibrium is

\[
\Delta k = y(k_t) - c_t - \delta k_t
\]  

(2.25)

Then, the steady state growth rate of nominal money is

\[
\frac{m_{t+1} - m_t}{m_t} = \left[ \frac{g_t}{m_t} - \pi_t \right] (1 + \pi_t)^{-1} = \left[ \theta_0 - \pi_t \right] (1 + \pi_t)^{-1} = 0
\]  

(2.26)

At the steady state, the inflation rate is equal to the growth rate of nominal money supply:

\[
\pi^* = \theta_0
\]  

(2.27)

4. $\Delta k = 0$ then, the steady state consumption is equal to the steady state production after the expenses on replacement of depreciated capital.

\[
c^* = y(k^*) - \delta k^*
\]  

(2.28)
The steady state consumption depends on steady state capital stock only. Monetary aggregates do not explicitly show up in the equation for the steady state consumption.

5. Steady state capital stock, \( k^* \) from equation (2.21) is

\[ k^* = y_k^{-1}(\delta + \rho) \]  

(2.29)

The equation (2.29) tells that \( k^* \) is independent of all monetary variables. That implies \( c^* \) is also independent of all monetary variables. Considering equation (2.28) and (2.29), Sidrauski (1967) concludes that money is neutral and superneutral on both capital and consumption.

6. Steady state \( m^* \) can be obtained from the equation (2.22) is

\[ U_m(c^*, m^*) = U_c(c^*, m^*)[y_k(k^*) - \delta + \theta_0(1 + \rho)] \]  

(2.30)

**Steady State Effects of Change in the Rate of Nominal Money Growth**

The next question is how demand for real money, consumption and capital are effected from the change in monetary expansion. The time derivative of (2.30) can be used to analyze the relationship between demand for real balances \( m \) and growth
rate of money $\theta_0$. Then,

$$U_{mm}dm^* + U_{mc}dc^* = D_\theta d\theta_0 U_c + D(\theta_0)[U_{cc}dc^* + U_{cm}dm^*] \quad (2.31)$$

where $D(\theta_0) = [y_k(k^*) - \delta + \theta_0(1 + \rho)]$. At the steady state $c$ is fixed at $c^*$ then $dc^* = 0$, therefore

$$U_{mm}dm^* = D_\theta d\theta_0 U_c + D(\theta_0)U_{cm}dm^*$$

Rearranging the equation above we have

$$\frac{dm^*}{d\theta_0} = \frac{D_\theta U_c}{U_{mm} - D(\theta_0)U_{cm}} \quad (2.32)$$

According to Sidrauski, $\frac{dm}{d\theta_0} = 0$ at the steady state. However, note that $D_\theta = 1 + \rho > 0$, $U_c > 0$, $U_{mm} < 0$, $D(\theta) > 0$ but $U_{cm} \leq 0$, and the sign of $\frac{dm}{d\theta_0}$ depends on the sign and the magnitude of $U_{cm}$. The impact of the growth rate of nominal money on the demand for money depends on the relationship between money and consumption, i.e if money is a substitute for or a complement to consumption.

If $U_{cm} \geq 0$, money and consumption are complements, then $\frac{dm}{d\theta_0} \leq 0$. An increase in the growth rate of nominal money decreases the demand for real money. When the household gets money from the government it wants to spend on consumption as soon as possible. The household does not want to hold money. Also, if we have $U_{cm} \leq 0$ but $U_{mm} > D(\theta)U_{cm}$, the negative relationship between the growth rate of
money and the demand for real money still holds. On the other hand, if $U_{cm} \leq 0$ and $U_{mm} < D(\theta)U_{cm}$ then the negative relationship will be broken and we will have a positive relationship implying that growth rate of money leads household to hold more money. If consumption and holding real balances are substitutes, when, the money supply increases the household does not want to spend extra income on consumption. Instead, it wants to hold money for transactions purposes. This brief analysis states that we can not clarify the relationship between the growth rate of money and the demand for money by the concavity of the utility function.

Also, from the steady state condition (4), the steady state impact of growth rate of nominal money on consumption is

$$dc^* = y_k dk^* - \delta$$

Dividing both sides by $d\theta$ and imposing steady state condition $dc = 0$ we get

$$\frac{dc^*}{d\theta_0} = 0 \quad (2.33)$$

and from equation (2.29)

$$\frac{dk^*}{d\theta_0} = 0 \quad (2.34)$$

Sidrauski (1967) uses equation (2.33) and (2.34) and concludes that monetary expansion is super-neutral over consumption. Real balances do not have impact on
consumption because growth rate of consumption is independent of monetary variables. Growth rate of capital depends on rate of time preference and the rate of its depreciation, and is therefore independent of monetary expansion.

**Optimal Quantity of Money**

According to Sidrauski the optimal quantity of money can be obtained by setting $U_m = 0$ therefore the nominal interest rate is zero. According to equation (2.22), $U_m = 0$ if and only if

$$y_k - \delta + \pi_t (1 + \rho) = 0$$

(2.35)

hence

$$\pi^*_S = -[y_k - \delta](1 + \rho)^{-1} = -\rho(1 + \rho)^{-1} = \theta^* < 0$$

(2.36)

The model produces deflation at a constant rate as an optimal monetary policy. In other words, optimal monetary policy should decrease money supply at a constant rate. This outcome is related to the assumption of $\frac{dc^*}{d\theta} = 0$ and $\frac{dk^*}{d\theta} = 0$ discussed above. If $\frac{dc^*}{d\theta} = 0$ households prefer real goods to money if money does not provide any services required for the household to increase consumption.

Also note that according to the model

$$\frac{dm^*}{d\pi^*_S} = 0$$

(2.37)
verifies Sidrauski (1967). In the long run, change in inflation does not change the demand for real money, therefore all real variables are independent of change in inflation.

The following section shows that Sidrauski (1967) misinterprets money in the utility functions. In Sidrauski’s analysis money does not possess any value to affect household consumption decisions. In Feenstra (1986), exchange in the goods market involves liquidity cost, and the household holds money to minimize liquidity cost therefore increase consumption. Costly exchange in the goods market implies that there is a difference between actual consumption and gross consumption. The cost can be shown either in the budget constraint or in the utility function.

### 2.2.2 Sidrauski with Feenstra Critique

Feenstra (1986) shows that money in the utility function is functionally equivalent to a liquidity cost in the budget constraint. This section aims to clarify a common misinterpretation of money in the utility function models and the impact of changes in the growth of money supply on the growth rate of consumption. Feenstra states that the utility function $V(q_t, m_t)$ is equivalent to $U(c_t)$, if $q_t$ is the gross consumption that includes both cost of transactions and net consumption, $c_t$. Households hold money
to decrease liquidity cost in the goods market.

Feenstra (1986) defines gross consumption as

\[ q_t = c_t + \phi(c_t, m_t) \]

That provides equation for the net consumption:

\[ c_t = W(q_t, m_t) \]

Then the equivalence states that

\[ U(c_t) \equiv V(q_t, m_t) \]

As an example Feenstra (1986) calculates a liquidity cost function for a Baumol-Tobin transactions model. It is shown that if the liquidity cost \( \phi(c_t, m_t) \) is

\[ \phi(c_t, m_t) \equiv \frac{\kappa c_t}{2m_t} \]  \hspace{1cm} (2.38)

where \( \kappa \) is the constant cost of transaction, then equivalence of \( V(.) \) and \( U(.) \) requires

\[ c_t \equiv \frac{2m_t q_t}{2m_t + \kappa} \equiv W(q_t, m_t) \]  \hspace{1cm} (2.39)

Hence,

\[ V(q_t, m_t) \equiv U(\frac{2m_t q_t}{2m_t + \kappa}) \]  \hspace{1cm} (2.40)
According to equation (2.40), a study has to use either $V(q_t, m_t)$ or equivalently $U(\frac{2mq_t}{2m_t+\kappa})$ but not $U(c_t, m_t)$. Sidrauski’s household problem, then, must be revised considering the equivalence and the model set up as a maximization of $V(q_t, m_t)$ subject to the budget constraint, therefore:

$$\max_{q, g} \sum_{t=0}^{\infty} V(q_t, m_t)(1 + \rho)^{-t}(1 + \rho), 0 < \rho < 1$$

subject to

$$\Delta m = (1 + \pi_t)^{-1}(g_t - \pi_t m_t)$$

and

$$\Delta k = y(k_t) + \tau - q_t - \delta k_t$$

It is clear from the setup of the problem that solutions for $q_t$ and $k_t$ are the same as solutions for $c^*$ and $k^*$ in the Sidrauski (1967) analysis. As long as $k_t$ is independent of $m_t$, $q^*$ will be independent of all monetary variables.

Sidrauski (1967) treats $U(c_t, m_t)$ as $V(q_t, m_t)$ and concludes that money is super-neutral on consumption. However, as it will be shown below, this is a misinterpretation of functional forms of utility functions and, when this is corrected, money will be no longer super-neutral on net consumption.

**Central Planner**

Central planner chooses sequences for $q_t$, $g_t$ so as to maximize the household’s
utility, equation (2.41), subject to its flow and stock budget constraints.

The conditions for a maximum are given by the Euler equations for \( q_t \), and \( g_t \) respectively, dynamic equations of \( \Delta m, \Delta k, \Delta \lambda, \Delta \psi \) and end point conditions, given \( a(0), \tau_t, \lambda_t, \) and \( \pi_t \). Then the first order conditions (2.15),(2.16) and (2.17) must be corrected as

\[
\frac{\partial H}{\partial q} = V_q(q_t, m_t) - \lambda_{t+1} = 0 \Rightarrow U_q = \lambda_{t+1}
\]

\[
\frac{\partial H}{\partial g} = \psi_{t+1}(1 + \pi_t)^{-1} - \lambda_{t+1} = 0 \Rightarrow \psi_{t+1} = (1 + \pi_t)\lambda_{t+1},
\]

\[
\Delta \lambda = -\lambda_{t+1}(1 + \rho)^{-1}[y_k - \delta - \rho],
\]

\[
\Delta \psi = -V_m(1 + \rho)^{-1} + \psi_{t+1}(\rho(1 + \pi_t) + \pi_t)(1 + \rho)^{-1}(1 + \pi_t)^{-1}
\]

These conditions are similar to the conditions in Sidrauski (1967) above, except the utility function we have here is \( V(q_t, m_t) \) instead of \( U(c_t, m_t) \).

**General Equilibrium and the Steady State**

Steady state conditions for this problem are

1. \( \Delta \lambda = 0 \) then marginal product of capital equals to the user cost of capital:

\[
y_k = \delta + \rho
\]

(2.43)
2. $\Delta \psi = 0$ then,

$$V_m = \psi_{t+1} \frac{(1 + \rho)(1 + \pi_t) - 1}{(1 + \pi_t)}$$

Since $\psi_{t+1} = (1 + \pi_t)\lambda_{t+1}$

$$V_m = \lambda_{t+1}[(1 + \rho)(1 + \pi_t) - 1]$$

Next, substituting $\lambda_{t+1}$ and $\rho = y_k - \delta$ the marginal utility of money equals to the cost of holding money:

$$\frac{V_m}{V_q} = y_k + \pi_t(1 + \rho) - \delta$$  (2.44)

3. Steady state growth rate of nominal money supply is $\pi^* = \theta_0$.

4. $\Delta k = 0$ then goods market equilibrium is

$$q^* = y(k^*) - \delta k^*$$  (2.45)

5. Steady state $k^*$ by using equation (2.43) is

$$k^* = y_k^{-1}(\delta + \rho)$$  (2.46)

$k^*$ is independent of all monetary variables. This implies, $q^*$ is independent of all monetary variables as well.
6. Steady state $m^*$ can be obtained implicitly by using the following equation:

$$V_m(q^*, m^*) = V_q(q^*, m^*)[y_k(k^*) - \delta + \theta_0(1 + \rho)]$$  \hspace{1cm} (2.47)

**Steady State Effects of Change in the Rate of Nominal Money Growth**

The steady state relationship between demand for real balances $m$ and the growth rate of monetary expansion $\theta_0$ is obtained by taking the derivative of (2.44) with respect to $\theta_0$:

$$V_{mm}dm^* + V_{mq}dq^* = D(\theta_0)[V_{qq}dq^* + V_{qm}dm^*] + V_q[D \theta d\theta_0]$$  \hspace{1cm} (2.48)

where $D(\theta_0) = [y_k(k^*) - \delta + \theta_0(1 + \rho)]$, since, at the steady state, $q$ is equal to $q^*$, then $dq^* = 0$. Thus we have an equality already familiar

$$\frac{dm^*}{d\theta_0} = \frac{D_\theta V_q}{V_{mm} - D(\theta_0)V_{qm}}$$ \hspace{1cm} (2.49)

The sign of $\frac{dm^*}{d\theta_0}$ depends on the sign and the magnitude of the summation in the denominator, particularly on $V_{qm}$. Feenstra (1986) shows that $V_{qm} \gtrless 0$ and $V_{mm} \leq 0$, then $\frac{dm}{d\theta_0} \gtrless 0$. If money and gross consumption are complements then $V_{qm} \geq 0$ and $\frac{dm}{d\theta_0} \leq 0$. There will be a negative relationship between growth rate of money and the demand for real balances if $V_{qm} \leq 0$ but $V_{mm} \geq D(\theta)V_{qm}$. On the other hand if $V_{qm} \leq 0$ and $V_{mm} \leq D(\theta)V_{qm}$, then the growth rate of the money supply will
increase the demand for money. The result is inconclusive. The concavity of the utility function with respect to $q_t$ is not the main determinant of the relationship between the demand for money and the growth rate of the money supply.

On the other hand, the relationship between $q$ and $\theta_0$ from the steady state condition 3 is

$$dq^* = (y_k + \delta)dk^*$$

(2.50)

There is no relationship between $k^*$ and $\theta_0$ because production is a function of capital only. Since $k^*$ is independent of $\theta_0$, $q^*$ is also independent of the growth rate of money. This is the main explanation for the inconclusive result above. However, if $q^*$ is independent of monetary variables, $c^*$ cannot be. $q^*$ shows the gross consumption which includes the cost of net consumption and loss in the value of consumption due to the frictions in the exchange in the goods market. The household actual consumption is $c$. In order to analyze the impact of the liquidity cost on household wellbeing we need to consider $c^*$ not $q^*$. If we have a liquidity cost in an economy a household cannot get the ideal utility due to the unavoidable waste in consumption. Household actual consumption will be less than the ideal consumption. Therefore
using the equation for the relationship between gross and the net consumption,

\[ dq^* = dc^* + \phi_c dc^* + \phi_m dm^* \] (2.51)

In the steady state, equation (2.51) is equal to zero, therefore

\[ \frac{dc^*}{dm^*} = -\frac{\phi_m}{1 + \phi_c} \geq 0 \] (2.52)

since Feenstra (1986) shows that \( \phi_m \leq 0 \) and \( \phi_c \geq 0 \). A household holds cash to minimize its liquidity cost. An increase in cash holding decreases liquidity cost, and allows the household to consume more goods and services. Dividing both sides by \( d\theta_0 \) we get

\[ \frac{dc^*}{d\theta_0} = -\frac{\phi_m}{1 + \phi_c} \frac{dm^*}{d\theta_0} \geq 0 \] (2.53)

If equation (2.52) is true, then \( \frac{dm^*}{d\theta_0} \) has the same sign as \( \frac{dc^*}{d\theta_0} \geq 0 \). The explanation is similar to the explanation in Sidrauski (1967): government creates money to finance government net transfers to the household. An increase in the rate of money supply increases the government net transfers to the household, and therefore immediately raises household disposable income. The increase in real income raises the demand for consumption and real cash balances. Increase in the growth rate of real money implies that a household can hold more real balances to minimize its liquidity cost. As the growth rate of real money supply increases, demand for money for liquidity
services increases. If we have a Baumol-Tobin type transactions demand for money model, the cost of holding money is constant. Liquidity cost gets lower and lower the more the household holds money. There is no limit on household cash holding. The household’s liquidity cost approaches zero if the cash holding goes to infinity.

More importantly, equation (2.52) together with (2.53) state that the relationship between the growth rate of money and the demand for real money is determined by the liquidity cost, in other words by the relationship between consumption and real money. Concavity of $V$ with respect to $z_t$ is not sufficient to figure out the sign of $\frac{dn^*}{dn_0}$ at the steady state.

### 2.2.3 Optimal Quantity of Money

The quantity of money is optimal when the first order condition (2.44) is equal to zero, $V_m = 0$. $V_m = 0$ if and only if

$$y_k + \pi_t (1 + \rho) - \delta = 0$$

hence

$$\pi_S^* = -[y_k - \delta](1 + \rho)^{-1} = -\rho(1 + \rho)^{-1} = \theta^*$$

If we take $q$ as a the household decision variable, the model produces the same optimal quantity of money as in Sidrauski. However this policy does not reflect the impact
of liquidity cost over consumption. Indeed, it produces an impractical policy for the Baumol-Tobin type models. In the real world, a household wants to maximize net consumption, not gross consumption. An example can be used to explain the impracticality.

Let \( U(c) = \frac{c^{1-\alpha} - 1}{1-\alpha} \) and \( \phi(c_t, m_t) = \frac{k_c}{2m_t} \) such that \( c_t = \frac{2mq_t}{k + 2m} = W(q_t, m_t) \). Then the utility function

\[
V(q_t, m_t) = U(W(q_t, m_t)) = U\left(\frac{2mq_t}{2m + k}\right) = \frac{\left[\frac{2mq_t}{2m + k}\right]^{1-\alpha} - 1}{1 - \alpha}
\]

(2.56)

\[
V_m = (1 - \alpha)^2 \kappa q^{1-\alpha} \frac{(2m + k)^{\alpha - 2}}{m^{\alpha}}
\]

(2.57)

\( m_t \) shows up in both the numerator and the denominator. In order to have \( V_m \to 0 \) we must have \( m \to \infty \). Such a goal is impractical. There can not be such an optimal quantity of money. There is no limit on holding real money for the Baumol-Tobin type model of demand for money.

The problem in Sidrauski’s analysis is it assumes household actual consumption is \( q \). However \( q \) shows the gross expenditure of household on actual consumption \( c \). The central planner’s objective function has to be set up in a way to maximize household actual consumption. Therefore, in general, optimal monetary policy will be affected by transactions cost and should maximize net consumption instead of
gross consumption. The correct setup of the policy should be as follows:

\[ Vq\phi_m + V_m = 0 \Rightarrow \frac{V_m}{V_q} = -\phi_m \]  

(2.58)

Then, from steady state condition 2 we know that

\[ \frac{V_m}{V_q} = y_k + \pi_t(1 + \rho) - \delta = -\phi_m \]

That implies

\[ \pi_v^* = \left[ -(y_k - \delta) - \phi_m \right](1 + \rho)^{-1} = (-\rho - \phi_m)(1 + \rho)^{-1} \]  

(2.59)

According to the equation (2.59), inflation depends on the marginal impact of money on the liquidity cost and the rate of time preference. The rate of time preference is assumed to be constant in general. The message of equation (2.59) is that the optimal inflation varies with respect to the response of the liquidity cost function to the change in the money supply. Policy makers have to consider transactions cost in an economy to determine the optimal quantity of money, because transaction cost has an impact on the household net consumption.

Also, recall that \( \phi_m \leq 0 \). This implies that the optimal inflation in a model that considers liquidity cost will be higher than the optimal inflation noted in Sidrauski (1967). We show that Sidrauski’s constant money supply rule is invalid. Sidrauski assumes that \( \phi_m = 0 \).
The relationship between rate of inflation and the demand for real balances can be analyzed by taking the total derivative of (2.59) and holding the quantity of $k$ constant as

$$\frac{dm^*}{d\pi^*} = -\frac{(1 + \rho)}{\phi_{mm}} \leq 0$$  \hspace{1cm} (2.60)

The demand for real balances decreases if there is an increase in the rate of inflation at the steady state. Higher expected inflation increases the opportunity cost of holding money, thereby decreases the demand for real balances.

The impact of the change in the rate of inflation on the net consumption is clear from equation (2.52). Equation (2.60) and (2.52) state that $\frac{dc^*}{d\pi^*}$ has the same sign as $\frac{dm^*}{d\pi^*}$. 

$$\frac{dc^*}{d\pi^*} = \frac{\phi_m}{1 - \phi_c} \left(1 + \rho \right) \frac{1}{\phi_{mm}} \leq 0$$  \hspace{1cm} (2.61)

Increase in the rate of inflation decreases consumption due to the fact that inflation depreciates the value of real income, even though equation (2.46) states that change in inflation has no impact on the steady state capital.
Summary

In this section Sidrauski (1967) is summarized and the main conclusions of the study are examined. Feenstra justifies the use of money in the utility function as a factor increasing efficiency in exchange in the goods market. This section shows that Sidrauski (1967) ignores this particular function of money and concludes that money is super-neutral over real variables. If money is important as a factor increasing efficiency in exchange of goods and services then it is hard to believe it is super-neutral.

The second part of the section recognizes the importance of money in exchange and shows that money is not super-neutral over net consumption even if, at the steady state, it is super-neutral over gross consumption.

2.3 Money in the Production Function

There are two agents: a firm and a household. Both are representative. The household uses money to buy goods from the firm. The firm buys capital services with money from the household. Money is accepted as a medium of exchange by both parties. In order to start up production the firm has to borrow from the household and hire its capital service. The household gives the money back to the firm in return for the output. As the firm realizes sales, it accumulates money and holds money for
transactions purposes. The earlier section assumes that only the household holds money at the end of the period. To make the analysis simple, in this section, first assume that only the firm holds money. The more general case, where both households and firms hold money, will be analyzed at the end of the chapter.

The first chapter demonstrates the functional equivalence between the demand for money by a firm and money in the production function. The firm holds money to economize on the use of physical factors of production. Money is used as a factor improving efficiency in production and representing the specialization in exchange in the factor market. A production function can be represented either as a function of gross physical capital and real balances or as a function of net capital alone. Gross capital is a composite factor containing a combination of net capital and real balances in production. The definition of capital is identical to the relationship between gross and net consumption in Feenstra (1986)’s analysis. By using these properties, this section applies Sidrauski (1967)’s analysis to examine the monetary implications of such a model over the net capital and consequently over consumption.
2.3.1 The Model

A representative household maximizes utility with respect to consumption subject to
its budget constraint. Thus the problem can be set up as

$$\max U(c_t)$$

subject to

$$c_t + i_t = r_t k_t - g_t$$

(2.62)

where $i_t$ is the gross investment, $g_t$ is the gross addition to the holding of the gross real
money for consumption, $c_t$ is the consumption and $k_t$ is the net capital/labor ratio.

Also, since we have a representative household model, labor supply is assumed to be
one and constant. Details of the household problem are given in the first chapter.

Here only the intuition for the budget constraint (2.62) is presented. The firm hires
capital from the household. However because of the transactions cost some of the
resources are either wasted or used in the transactions sector as in the Fisher (1974)
analysis. Therefore the household could not consume the amount it is supposed
to consume because of frictions in production. The impact of the efficiency loss in
production can be shown as a decrease in consumption for the household.

According to the first chapter, the household budget constraint can be simplified
by substituting \( z_t = k_t + \chi(k_t, m_t) \) into the budget constraint such that

\[
c_t + i_t = r_t z_t - r_t \chi(k_t, m_t)
\]  

(2.63)

Here \( z_t \) is the gross capital and \( \chi(k_t, m_t) \) is the transactions cost. A representative firm uses the physical factor together with real balances in production to maximize its profit\(^1\),

\[
\max \Pi(r) = y(k_t) - r(k_t + \chi(k_t, m_t) - \delta k_t + (\tau - g)
\]  

(2.64)

where \( g \) and \( \tau \) are gross accumulation of real money and lump sum net transfer from the government. The profit function equivalently can be simplified as

\[
\max \Pi(r) = f(z_t, m_t) - r(z_t + \delta) + (\tau - g)
\]  

(2.65)

where \( f(z_t, m_t) \equiv y(k_t) \) and \( z = k_t + \chi(k_t, m_t) \).

### 2.3.2 Central Planners Problem

A central planner chooses sequences for \( c_t \) and \( g_t \) so as to maximize household utility,

\[
\max_{c,g} \sum_{t=0}^{\infty} U(c_t)(1 + \rho)^{-(t+1)}, 0 < \rho < 1
\]  

(2.66)

subject to

\[
\Delta m = (1 + \pi_t)^{-1}(g_t - \pi_t m_t)
\]

\(^1\)Here to simplify the analysis we assume that firm income consists of sale receipts and money holding only; \( b_t = 0 \).
and

$$\Delta k = i_t - \delta k_t$$ (2.67)

Then substitute $i_t$ from equation (2.62) into (2.67) to get

$$\Delta k = r k_t + r \chi(k_t, m_t) - c_t - \delta k_t$$ (2.68)

We want to simplify the central planner’s problem in a way that we did for the household. First recall that $k_t = z_t - \chi(k_t, m_t)$, thus

$$\Delta z - \Delta \chi = r(z_t - \chi(k_t, m_t)) - c_t - \delta(z_t - \chi(k_t, m_t)).$$

Rearranging provides

$$\Delta z = rz_t - \delta z_t - c_t - (r - \delta)\chi(k_t, m_t) + \Delta \chi(k_t, m_t).$$ (2.69)

Equation (2.69) argues that it is impossible to eliminate both $k_t$ and $\chi(k_t, m_t)$ from the central planner’s problem. Whenever the central planner changes $k_t$, $\chi(k_t, m_t)$ will change accordingly. It is impossible to separate these changes. This may be interpreted as an externality of transactions cost over the net capital.

### 2.3.3 Externality and the Transactions Cost

Externality of the transactions cost over private capital accumulation is another important topic. Here we summarize the argument of externality to show that the
central planner’s problem in these models is not the same as the central planner’s problem in a classical model.

Barro and Salai-Martin (1999) explain that in an economy where there is an externality, the average product of capital will exceed its marginal product. Recall from chapter one that

\[ f(z, m) \equiv y(k) = y(W(z, m)) \]

where \( W(z, m) = k_t \), \( W_z > 0 \) and \( W_m < 0 \). \( f(z, m) \) is called gross production because it is expressed as a function of gross capital \( z_t \) and real money \( m_t \). \( y(k) \) or equivalently \( y(W(z, m)) \) is called net output or actual output that the household can consume and actual output must be equal to gross output. Then, the average product of gross capital is

\[ \frac{f(z, m)}{z} \equiv \frac{y(W(z, m))}{z} = y(W(1, \frac{m}{z})) \quad (2.70) \]

and the marginal product of gross capital is

\[ \frac{\partial f(z, m)}{\partial z} \equiv \frac{y(W(1, \frac{m}{z}))}{z} + y_W W_m \frac{m}{z} \quad (2.71) \]

Note that existence of \( W_m < 0 \), on the right hand side makes the marginal product of capital less than the average product of capital. If there is an externality in an economy, setting up the central planner problem in accordance with the marginal
product will produce Pareto non-optimal outcomes (Barro and Salai-Martin 1999). We have a competitive market, therefore any new information about cash management immediately spreads over the economy. An externality arises due to the fact that improvement in the transactions cost for each firm increases its output and adds to the total capital stock. Therefore externality improves the productivity of all firms in the economy. The central planner has to recognize the externality.

2.3.4 Externality and the Central Planners Problem

The existence of an externality makes the decentralized solution Pareto non-optimal. The economy’s total stock of capital is \( z_t \) therefore the central planner has to set up the problem considering the total stock of the economy. Hence, the constraint (2.68) has to be set up as

\[
\Delta z = f(z_t, m_t) + \tau - c_t - g_t - \delta z \tag{2.72}
\]

given \( a(0), \tau, \delta, \) and \( \pi \).

The transformed Hamiltonian is

\[
\mathcal{H} = U(c_t) + \lambda_{t+1}[f(z_t, m_t) + \tau - c_t - g_t - \delta z_t] + \psi_{t+1}(1 + \pi_t)^{-1}[g_t - \pi_t m_t] \tag{2.73}
\]

where \( \lambda \) and \( \phi \) are the Lagrange multipliers attached to flow and stock constraints respectively.
Conditions for a maximum are given by the Euler equations for \( c_t \), and \( g_t \) respectively, dynamic equations of \( \Delta m \), \( \Delta z \) \( \Delta \lambda \), \( \Delta \psi \) and end point conditions, given \( k(0), \tau, \delta \), and \( \pi \). Therefore

\[
\frac{\partial H}{\partial c} = 0 = U_c - \lambda_{t+1} \Rightarrow U_c = \lambda_{t+1} \tag{2.74}
\]

\[
\frac{\partial H}{\partial g} = 0 = \psi_{t+1}(1 + \pi_t)^{-1} - \lambda_{t+1} \Rightarrow \psi_{t+1} = (1 + \pi_t)\lambda_{t+1}, \tag{2.75}
\]

\[
\Delta \lambda = -\lambda_{t+1}(1 + \rho)^{-1}[f_z - \delta - \rho], \tag{2.76}
\]

\[
\Delta \psi = \psi_{t+1}\left[\frac{(1 + \rho)(1 + \pi_t) - 1}{(1 + \rho)(1 + \pi_t)}\right] - \lambda_{t+1} \frac{f_m}{(1 + \rho)}
\]

\[
= \lambda_{t+1}\left[\frac{(1 + \rho)(1 + \pi_t) - 1 - f_m}{(1 + \rho)}\right] \tag{2.77}
\]

and

\[
\lim_{t \to \infty} \psi_t m_t (1 + \rho)^{-t} = 0 \tag{2.78}
\]

\[
\lim_{t \to \infty} \lambda_t z_t (1 + \rho)^{-t} = 0 \tag{2.79}
\]

**General Equilibrium and the Steady State**

Steady state requires:

1. \( \Delta \lambda = 0 \) then, the marginal product of gross capital is equal to it user cost,

\[
f_z = \delta + \rho \tag{2.80}
\]
2. $\Delta \psi = 0$ then

$$f_m = (1 + \rho)(1 + \pi_t) - 1 = (1 + f_z - \delta)(1 + \pi_t) - 1 = i_t \quad (2.81)$$

where $(1+\rho)$ is the gross real interest from the equation (2.80) and $(1+\rho)(1+\pi_t)$ is the gross nominal rate. The equation (2.81) states marginal product of money must be equal to the marginal cost, net nominal interest rate.

3. Assuming the monetary authority applies an exogenous nominal money supply policy as before, then steady state growth rate of the nominal money supply is

$$\pi^* = \theta_0$$

4. $\Delta z = 0$ then

$$c^* = f(z^*, m^*) - \delta z^* \quad (2.82)$$

Steady state consumption is gross output minus the gross steady state replacement investment, $\delta z^*$

5. Steady state gross capital, $z^*$, from equation (2.80) is

$$z^* = f_z^{-1}[\delta + \rho] \quad (2.83)$$
6. Equation (2.81) provides the implicit solution for steady state $m^*$ as a function of net nominal rate:

$$f_m(z^*, m^*) = (1 - \delta + f_z(z^*, m^*))(1 + \pi_t) - 1 = i_t \quad (2.84)$$

**Steady State Effects of Change in the Rate of Money Growth**

Steady state relationship between $m$ and $\pi_t = \theta_0$ can be analyzed by taking the time derivative of equation (2.84):

$$f_mz^* dz^* + f_mm^* dm^* = (1 - \delta) d\theta_0 + (f_{zz}dz^* + f_{zm}dm^*)(1 + \theta_0) \quad (2.85)$$

At the steady state, $z$ is equal to $z^*$ then $dz = 0$. Then,

$$\frac{dm^*}{d\theta_0} = \frac{1 - \delta + f_z}{f_{mm} - f_{zm}(1 + \theta_0)} = \frac{1 + \rho}{f_{mm} - f_{zm}(1 + \theta_0)} \quad (2.86)$$

We know that $f_{mm} < 0$ then the sign of $\frac{dm^*}{d\theta_0}$ depends on the sign and the magnitude of $f_{zm}$. From the first chapter we know that $f_{zm} \geq 0$. This implies that the sign of $\frac{dm^*}{d\theta_0}$ depends on whether money is a complement to gross capital or a substitute. If $f_{zm} \geq 0$ then money and gross capital are complements and $\frac{dm^*}{d\theta_0} \leq 0$. In this case, an increase in the growth rate of money decreases the demand for holding money. On the other hand, if money and gross capital are substitutes, $f_{zm} \leq 0$, but $f_{mm} \geq f_{zm}(1 + \theta_0)$ we still have $\frac{dm^*}{d\theta_0} \leq 0$. The relationship is broken up when
\( f_{zm} \leq 0 \) but \( f_{mm} \leq f_{zm}(1+\theta_0) \) and we have \( \frac{dm^*}{d\theta_0} \geq 0 \). This analysis is not sufficient to determine the sign of the relationship between the growth rate of nominal money and the demand for real money. The result is inconclusive and indicates that concavity of the production function with respect to \( z_t \) may not be the main determinant of the relationship. Because at the steady state, \( z^* \) is constant therefore independent of any variable.

The inconclusive result above is related to the steady state relationship between \( z_t \) and \( m_t \). Time derivative of equation (2.80) with respect to \( m_t \) is

\[
dz^* = -\frac{f_{zm}}{f_{mm}} dm^* 
\]  

(2.87)

At the steady state \( dz^* = 0 \), hence \( z^* \) is independent of the money supply. It indicates that \( m_t \) and \( z_t \) are unrelated at the steady state. However this is a misleading indicator. What increases output in this model is the increase in the net capital \( k_t \). We need to analyze the relationship between \( m_t \) and \( k_t \).

The relationship between \( k_t \) and \( m^* \) can be found by using the relationship between \( z_t \) and \( k_t \). The first chapter shows that, in an economy with transactions costs in the input market, the firm can not employ all of its resources in real production. There will be a difference between the stock of capital and the quantity of capital actually
utilized in the production. How close the firm is to its ideal level of production depends on the transactions cost. Therefore the main decision variable of the central planner is the net, not the gross, capital. The first chapter defines gross capital as

\[ z_t = k_t + \chi (k_t, m_t) \]

Taking the time derivative of \( z_t \) we have

\[ dz^* = dk^* + \chi_k dk^* + \chi_m dm^* \]  \hspace{1cm} (2.88)

In the steady state, equation (2.88) equals zero, therefore

\[ \frac{dk^*}{dm^*} = -\frac{\chi_m}{1 + \chi_k} \geq 0 \]  \hspace{1cm} (2.89)

because, from the first chapter \( \chi_m < 0 \) and \( \chi_k > 0 \). Equation (2.89) argues that holding money will decrease transactions costs and allows the firm to hire more capital in production. Dividing both sides by \( d\theta \) we have

\[ \frac{dk^*}{d\theta_0} = -\frac{\chi_m}{1 + \chi_k} \frac{dm^*}{d\theta_0} \geq 0 \]  \hspace{1cm} (2.90)

To be consistent with the sign of equation (2.89), it is the case that \( \frac{dm^*}{d\theta_0} \) has the same sign as \( \frac{dk^*}{d\theta_0} \). There is a positive relationship between the growth rate of nominal money supply and the demand for real money holding. The intuition behind the positive
relationship between the growth rate of money and the demand for the real money
is analogous to the previous explanation. Government creates money to finance its
net transfer to the firm. Therefore increase in the monetary expansion implies that
the firm’s real income is increasing. The more money held by a firm, the lower the
transactions cost. This implies that there may be no optimal quantity of money.

The relationship between \( c^* \) and \( \theta_0 \) from the steady state condition (4) is

\[
dc^* = f_z d\theta_0 + f_m dm^* - \delta dz^* 
\]

(2.91)

Divide both sides by \( d\theta_0 \)

\[
\frac{dc^*}{d\theta_0} = (f_z - \delta) \frac{dz^*}{d\theta_0} + f_m \frac{dm^*}{d\theta_0} 
\]

(2.92)

and we have \( \frac{dz^*}{d\theta_0} = 0 \) and \( \frac{dm^*}{d\theta_0} \geq 0 \), at the steady state then

\[
\frac{dc^*}{d\theta_0} = f_m \frac{dm^*}{d\theta_0} \geq 0
\]

(2.93)

Thus, money in the production function produces the same result as in section
(2.2). An increase in the growth rate of nominal money increases the demand for real
balances to minimize transactions cost, and therefore increases net capital in produc-
tion. Physical capital used in transactions services now can be utilized in production
causing output to increase. An increase in output in turn increases consumption. If
equation (2.90) is substituted in equation (2.93) we have

\[ \frac{dc^*}{d\theta_0} = -f_m \left( \frac{\chi_m}{1 + \chi_k} \right) - \frac{1}{1} \frac{dk^*}{d\theta_0} \geq 0 \]  \hspace{1cm} (2.94)

Equation (2.94) states that the transactions costs have an indirect impact on consumption via their direct impact on the net capital.

### 2.3.5 Non-Optimal Quantity of Money

Central planner maximizes households utility with respect to the net consumption. Using first order conditions (2.74), (2.77) we have

\[ \Delta \psi = U(c)\left( \frac{(1 + \rho)(1 + \pi_t) - 1 - f_m)}{1 - \rho} \right) \]  \hspace{1cm} (2.95)

At the steady state \( \Delta \psi = 0 \). \( \Delta \psi = 0 \) only if

\[ (1 + \rho)(1 + \pi_t) - 1 - f_m = 0 \Rightarrow f_m = i \]  \hspace{1cm} (2.96)

Therefore if we ignore the impact of transactions cost on capital and solve the problem in a regular way, the optimum inflation would be

\[ \pi^* = \frac{-\rho + f_m}{1 + \rho} \]  \hspace{1cm} (2.97)

If there is no cost to creating money, \( f_m = 0 \) then we will have standard result, \( \pi^* = \frac{-\rho}{1 + \rho} \). And the impact of the change in inflation over the demand for real money holding is
\begin{equation}
\frac{dm^*}{d\pi^*} = \frac{1 + \rho}{f_{mm}} \leq 0
\end{equation}

Higher inflation increases the opportunity of holding money thereby decreasing demand for real money.

However the household can consume more if the firm can increase its output. In this model, the only way to produce more goods and services is to employ more capital in production\(^2\). Therefore the first order conditions have to be taken with respect to the net capital, instead of the gross. Then

\begin{equation}
\Delta \psi = U_c \left[ \frac{(1 + \rho)(1 + \pi_t) - 1 - f_z z_{\chi} \chi_m - f_m + \delta \chi_m}{1 + \rho} \right] = 0
\end{equation}

where \(z_{\chi} = 1\) and inflation at the steady state is

\begin{equation}
\pi_f^* = \frac{-\rho - \delta \chi_m + f_m + f_z \chi_m}{(1 + \rho)} = \frac{(f_m - \rho) + (f_z - \delta) \chi_m}{(1 + \rho)}
\end{equation}

Equation (2.100) points out that there is an additional factor which is ignored in equation (2.98) in the determination of inflation. According to the equation (2.100) optimal policy has to consider transactions cost. When the money supply changes, the transactions cost for firms will be changed as well.

\(^2\)We assume there is no technological change, no change in quality and quantity of gross capital.
Also equation (2.100) argues that Sidrauski (1967)'s constant money supply rule requires $\chi_m = 0$ and $f_m = 0$. For the Baumol- Tobin type transactions cost, $\chi_m = 0$ when $m_t$ goes to infinity. This is not a practical policy to apply.

The steady state effect of the change in inflation on the demand for real balances can be analyzed by taking the total derivative of equation (2.100) as follows.

$$d\pi_f(1 + \rho) = -\delta[\chi_mdk + \chi_{mm}dm^*] + f_mdz^* + f_{mm}dm^* + [f_{zz}dz^* + f_{zm}dm^*] \chi_m$$

$$+ f_z[\chi_mdk + \chi_{mm}dm^*]$$

At the steady state, $dz^* = 0$ and holding $dk^*$ constant we have

$$\frac{dm^*}{d\pi_f} = \frac{1 + \rho}{-\delta \chi_{mm} + f_{mm} + f_{zm}\chi_m + f_z\chi_{mm}} \leq 0$$  \hspace{1cm} (2.101)$$

We have $\chi_m \leq 0$, $\chi_{mm} \geq 0$, $f_z \geq 0$, $f_{zm} \leq 0$ and $f_{mm} \leq 0$. Therefore the sign of $\frac{dm^*}{d\pi_f}$ is ambiguous. There are two opposing effects on the demand for money due to the change in inflation. An increase in inflation implies that value of real money decreases. Therefore the firm tends to hold less money. Less money means less production for the given level of net capital. However, less money implies that there is more money to spend on net capital. Therefore firms substitute capital for real money. Firm tends to convert real money into capital as soon as possible. The increase in net capital means increase in net output. Due to this income effect there
will be an increase in demand for money. Net outcome of the change in inflation depends on the relative magnitude of slope of the convexity of $\chi$ and concavity of $f$ functions\(^{3}\). In other words, if substitution effect is greater than the income effect, increase in inflation will decrease the demand for money. On the other, hand if income effect is greater, the relationship will be positive.

Equations (2.89) and (2.101) provide an equation for the relationship between change in inflation and capital as

$$\frac{dk^*}{d\pi_f^*} = \frac{\chi_m}{1 + \chi_k} \frac{dm^*}{d\pi_f^*}$$  \hspace{1cm} (2.102)

and state that the sign of $\frac{dk^*}{d\pi_f^*}$ depends on the sign of $\frac{dm^*}{d\pi_f^*}$. Since the sign of $\frac{dm^*}{d\pi_f^*}$ is ambiguous the impact of the inflation on the net capital is inconclusive. Similarly, equations (2.93), and (2.101) indicate an inconclusive relationship between consumption and inflation.

**2.3.6 Summary**

Use of money in the production function can be justified as a factor increasing efficiency of exchange of the physical factor in the input market. We have shown that putting money in the production function does not simplify the central planner’s

\(^{3}\)We know from the first chapter that after the transformation $y$ is applied, convexity of $W$ function is overcome to make $f$ concave. However, the absolute value of the slope $f$ may or may not be greater than the absolute value of the slope of $W$. 

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problem. The externality of the transactions cost over the net capital is explained as a source of the difficulty in setting up the problem for the central planner. Then, we suggest that the central planner has to recognize the externality and set up the problem in accordance with the gross capital. By solving the problem in this way we have shown that there is a way to show the impact of a transactions cost over the determination of inflation. The central planner takes $z_t$ as a policy variable because $z_t$ is the total stock of capital that is available but implicitly aims to accumulate net capital $k_t$ that is the actual quantity of capital used in production. The central planner can increase output by increasing net capital by decreasing transactions costs. Transactions costs are an important factor in production and have a direct effect on the determination of the inflation rate.

Change in both the growth rate of money and change in inflation have impact over consumption and net capital. The growth rate of money tends to increase accumulation of capital and consumption by decreasing cost of transactions, however the impact of the growth rate of inflation over the real net variables is inconclusive.
2.3.7 Money in both Utility and Production Function

Section two examines the properties of money in the utility functions while section three does the same exercise for models where money is used as a factor of production. The money in the utility model assumes that only the household holds money at the end of the period, while money in the production function model assumes only the firm holds money at the end of the period. These cases are special cases, but they are simple to analyze. The next question, after analyzing the impact of monetary expansion on money only in the utility function and only in the production function models, is how these properties may change or are similar to a model where money is used in both utility and production functions. Therefore we assume both the household and the firm hold money for transactions purposes. Money improves exchange in both goods and factor markets. Thus money in the utility function is interpreted as a specialization of exchange in the goods market, while money in the production function can be interpreted as a specialization in exchange in the factor market.

In this case the central planner maximizes utility of the form

\[
\max_{q, \xi} \sum_{t=0}^{\infty} V(q_t, m_t)(1 + \rho)^{-(t+1)}, 0 < \rho < 1
\]  

\hspace{1cm} (2.103)
subject to

\[ \Delta m = (1 + \pi_t)^{-1}(g_t - \pi_t m_t) \]

and

\[ \Delta k = y(k) + \tau - q_t - g_t - \delta k_t - \delta \chi(k_t, m_t) \]

Here we have the same problem. The central planner cannot eliminate both \( k_t \) and \( \chi(k_t, m_t) \) at the same time to simplify the problem. If we follow the same steps as above and assume that implicitly the central planner aims to accumulate net capital the problem may be overcome. Therefore, if

\[ \Delta z = f(z_t, m_t) + \tau - q_t - g_t - \delta z_t \]

given \( a(0), \tau, \delta \) and \( \pi_t \), the steady state relationship between \( m \) and \( \theta_0 \) would be

\[
\frac{dm^*}{d\theta_0} = \frac{V_q(1 - \delta + f_z)}{V_{mm} - V_{qm} B + V_q[f_{mm} - f_{zm}(1 + \theta_0)]} \quad (2.104)
\]

According to equation (2.104) the relationship between \( m \) and \( \theta_0 \) depends not only on the sign and the magnitude of \( V_{qm} \gtrless 0 \), but also the sign and the magnitude of \( f_{zm} \gtrless 0 \) as well. It is difficult to come up with a specific conclusion for the sign of \( \frac{dm^*}{d\theta_0} \). If we use the intuition of the earlier sections we may conclude that neither the concavity of utility function with respect to \( q_t \) nor the concavity of the production

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function with respect to $z_t$ is sufficient to determine the relationship between the demand for money and the growth rate of nominal money supply.

The relationship between $z^*$ and $\theta_0$ is the same as in section (3) and equation (2.90) shows the impact of money on net capital. Even if there is no relationship between $z$ and $m$, the growth rate of money will change the demand for net capital.

The relationship between $q^*$ and $\theta_0$ is the same as in section (2) and equation (2.53) shows the impact of money growth on net consumption. If equation (2.90) is substituted for $\frac{dm^*}{d\theta_0}$ in equation (2.53), we have

$$\frac{dc^*}{d\theta_0} = \frac{\phi_m}{\chi_m} \left( \frac{1 + \chi_k}{1 + \phi_c} \right) f_m \frac{dk^*}{d\theta_0} \geq 0 \quad (2.105)$$

A change in the growth rate of money changes net consumption via the changes in the liquidity cost as well as changes in transactions cost and productivity of capital.

### 2.3.8 Non-Optimal Quantity of Money

Express the utility function as $V(c_t + \phi(c_t, m_t), m_t)$, and the production function as $f(k_t + \chi(k_t, m_t), m_t)$. Then, take the first derivative with respect to $m$ to get

$$V_q \phi_m + V_m = 0 \Rightarrow \frac{V_m}{V_q} = -\phi_m$$

Optimal policy maximizes net utility. Also, we know that the central planner has to build up net capital to increase the income of the household. The steady state
condition 2 can be rearranged for this problem as

\[(1 + \rho)(1 + \pi^*_v) - 1 - f_m - f_z \chi_m + \delta \chi_m = -\phi_m \quad (2.106)\]

If we rearrange the equation above for \(\pi_v\) we have

\[\pi^* = \frac{-\rho - \phi_m - \delta \chi_m + f_m + f_z \chi_m}{(1 + \rho)} = \frac{-\phi_m + (f_z - \delta) \chi_m + (f_m - \rho)}{(1 + \rho)} \quad (2.107)\]

Equation (2.107) argues that both liquidity cost in the goods market and transactions cost in the factor market together with the marginal product of money in the factor market are important in determination of optimal inflation. Sidrauski (1967)’s optimal monetary rule requires both \(\phi_m = 0\), \(\chi_m = 0\) and \(f_m = 0\). For a Baumol-Tobin type transactions cost these conditions imply that \(m_t\) goes to infinity. This is not a practical policy to apply. A monetary policy has to consider the impact of the liquidity and the transactions cost over inflation. Both transactions cost in the factor market and liquidity cost in the goods market are important and they tend to make inflation higher than the standard result.

The impact of the change in the rate of inflation over the demand for real balances at the steady state is

\[\frac{dn^*}{d\pi^*_v} = \frac{1 + \rho}{-\phi_{mm} - \delta \chi_{mm} + f_{mm} + f_{zmm} \chi_m + f_{zmm} \chi_m \chi_m} \Bigg|_{\Delta \pi^*_v = 0} \quad (2.108)\]
We have the same conclusion as in section (2.3.5). The relationship is ambiguous even though the liquidity cost in the goods market tends to make the relationship negative. The net impact of the change in inflation on the demand for money depends on the magnitude of the convexity of both transactions cost in input the market and liquidity cost in goods the market and the concavity of gross production function.

The impact of change in inflation on net consumption and net capital depends on the relationship between demand for money and the change in inflation. Since the impact of inflation on demand for money is ambiguous, the effect of the change in inflation on the real net variables is also inconclusive.

2.4 Concluding Remarks

This study contributes to the literature in two ways: first, it examines Sidrauski (1967) within the Feenstra (1986) critique and shows that Sidrauski (1967) fails to recognize the functional equivalence between liquidity cost and money in the utility function. Money is included in the utility function as a factor affecting efficiency in exchange in the goods market. The Feenstra analysis states that Sidrauski ignores the function of money in exchange in goods markets. We reexamine Sidrauski by using a model where households maximize utility with respect to money and gross
consumption. We show that even if gross consumption is assumed to be independent of money, net consumption cannot be. Therefore, money is not super-neutral over net consumption.

Second, we extend the discussion, first to analyze the properties of a model in which money is used in the production function and then to a model where money is used in both production and utility functions. For both models, we have shown that it is impossible to eliminate both $k_t$ and $\chi_t$ in the central planner’s problem when money is used as a factor increasing efficiency in exchange in the input market. It is difficult to isolate the impact of transactions cost on net capital if the central planner wants to set up the problem according to net capital. The difficulty arises due to the externality of the transactions cost over the net capital.

We suggest that the central planner setup the problem in accordance with gross capital, because gross capital is the total stock of capital that is available to firms. If the central planner sets up the problem in accordance with gross capital and implicitly considers the build up of net capital in forming monetary policy, it is possible to show the impact of the change in transactions cost on the determination of the optimal quantity of nominal money supply. Net capital measures the amount of capital actually used in production. The central planner can increase net capital by changing
the transactions cost in the factor market. Total output can increase without any change either in technology, quality or quantity of inputs.

Monetary policy has to be formed in a way to increase the net capital stock. When the money supply increases, the firm can increase its money holding to decrease transactions cost. A decrease in transactions cost allows the firm to employ more net capital in production. Non super-neutrality of money is verified in all models. The impact of the change in the growth rate of money depends on the marginal impact of money on the transactions cost in the factor market and the liquidity cost in the goods market.

The other significant finding of the analysis is that the impact of the rate of change in expected inflation on the demand for real money is ambiguous in the model where there is a transactions cost in the factor market. Increase in inflation implies that opportunity cost of holding money for firm is increasing. Therefore the firm tends to hold less money. However, less money means more money spend on net capital. Intuitively, higher inflation tends to decrease output because it increases the opportunity cost of holding money therefore increasing the transactions cost in production. Firms want to convert their money into net capital as soon as possible, causing output to increase and inflation to decrease. The final impact depends on
the relative concavity of gross production function and convexity of transactions cost function in both goods and input markets.
Chapter 3

Transactions Cost as a Source of Technical (In)efficiency

3.1 Introduction

It is important to distinguish two different factors causing an increase in productivity: technological change and technical efficiency. Technological change describes a shift in the production frontier. An increase in the quality of factors (embodied technical change) or developments in the technological knowledge (disembodied) make a shift in the production frontier. On the other hand, efficiency is obtained when an economy produces the most, given economic resources. Coelli, Rao, and Battese (1999) states that efficiency rises if an economy experiences either technical efficiency or/and allocative efficiency. The former reflects the ability of a firm to obtain maximum output given economic resources, while the latter points out the use of factors in optimal proportions given their respective prices. The first approach measures the inefficiency
from the output perspective while the second describes the inefficiency from the cost perspective.

Technical inefficiency may arise in production due to many different factors including work effort, insufficient information, misallocation of resources, weak financial system etc. The model developed in the first chapter assumes for the sake of simplicity that the transactions costs are the unique disturbance in production, and that firms hold real balances to tackle their transactions cost. It builds a theoretical foundation for inclusion of real balances in the production function as a factor improving efficient use of other physical factors. Inclusion of real balances in production implies that firms hold real balances to save on time and resources that otherwise would be allocated to transactions services. Therefore, holding of real balances is justified as a factor changing efficient use of the physical factors of production. This study aims to support this theory empirically.

This is not the first attempt to test the significance of real balances in the production function. The empirical literature follows two different approaches. The first approach plugs real balances directly into a Cobb-Douglas production function and then tests the significance of the coefficient on real balances\(^1\). The second approach

\(^1\)Nguyen (1986), Short (1979), and Sinai and Stokes (1972) are a few studies that use money in the production function directly and test the significance of real balances in production function.
uses a stochastic production frontier and aims to test the importance of real balances in explaining economic inefficiency\(^2\). The main difference between these approaches is that the latter treats real balances as a factor improving efficiency in production. Neither of these studies invalidates the significance of real balances in production.

This study follows the second approach and examines the role of real balances in allocating resources efficiently in production. To my knowledge there are two empirical studies using this approach. Delorme, Thompson, and Warren (1995) uses a two-stage approach, contrary to Nourzad (2002), where parameters of the production function and the effect of real balances on inefficiency are jointly estimated. The first study aims to explain inefficiency for the US economy, while the other uses a panel approach over 10 developed and 10 developing countries. Both studies use either M1, M2 or Divisia aggregates as an explanatory variable.

The two-stage approach specifies and estimates a stochastic production frontier in the first stage and predicts technical inefficiency under the assumption that these inefficiency effects are identically distributed. Then, in the second stage, predicted technical inefficiencies are explained by some other parameters. The second stage

contradicts the assumptions of the first stage. If technical inefficiencies are identi-
cally distributed as assumed in the first stage, they can not be explained by other
factors. Battese and Coelli (1995) suggests a one-stage approach in which parameters
of the stochastic production frontier and the technical inefficiency model are esti-
mated simultaneously, given appropriate distributional assumptions associated with
panel data.

The aim of this study is to test the impact of real balances on inefficiency among
12 European Union countries. It considers the Battese and Coelli (1995) critique,
and similar to Nourzad (2002) it uses a one-stage estimation where the production
frontier, country, and time varying inefficiency in production are estimated jointly.
Instead of using M1, M2 or the Divisia index as a monetary aggregate, the study uses
deposits held by firms as an explanatory variable in analyzing determinants of country
and time varying technical inefficiency. Also, it uses real balances held by households
to investigate the relative importance of real balances held by households and real
balances held by firms in determining the productive inefficiency. The structure of
the transactions cost depends on some other financial variables that may vary across
countries. Therefore, contrary to both Delorme, Thompson, and Warren (1995) and
Nourzad (2002), the technical inefficiency model includes short-term interest rates to
proxy the differences in the financial markets structure among countries in explaining inefficiency. The choice of short-term rates as an explanatory variable in the technical inefficiency model is based on the objective of the European Central Banks (ECB) monetary policy. The monetary objective of the ECB is to maintain medium term price stability by steering short term interest rates. The policy defines price stability as a year-to-year increase in the Harmonized Index of Consumer Prices below 2%.

The organization of the chapter is as follows. Section two explains how to distinguish efficiency in a stochastic frontier approach, and then expands the discussion with the money in the production function proposed theoretically in the first chapter. Section three discusses the data and variables. Section four presents the empirical results. Section five includes concluding remarks.

3.2 Theoretical Framework

3.2.1 Stochastic Production Frontier Approach

In the first chapter we have claimed that firms hold real balances to decrease their transactions cost in the input market. Existence of transactions cost in production implies that firms are wasting some of their resources that otherwise would be used in production, thus causing them to operate at a level of output less than the ideal
level. Firms hold money to manage their transactions cost to increase their resources used in production. Therefore, real balances are a factor in determining efficiency.

This study adopts a stochastic frontier approach as an econometric method. This method first predicts inefficiency in production and then allows one to run a simultaneous regression to investigate the possible factors causing inefficiency. Indeed, this method allows one to distinguish productive factors and factors improving use of these factors. Real balances are not a factor used up in production directly such as capital or labor. The stochastic production frontier method, then, is the method that has to be used to test the significance of the real balances in determining technical inefficiency.

Firm level efficiency measurement using stochastic frontiers can be found in Battese and Coelli (1988). Application of the model to panel data is explained in Battese and Coelli (1995), Coelli (1996) and Coelli, Rao, and Battese (1999) as follows. The Cobb-Douglas production function can be specified formally as

\[
\ln(y_{it}) = x_{it}\beta + t\varphi + v_{it} - u_{it}, \quad i = 1, 2, ..., N; \quad t = 1, 2, ..., T
\]  

(3.1)

where \(i(= 1..N)\) denotes countries and \(t(= 1..T)\) denotes time. \(\ln(y_{it})\) is the natural logarithm of output for the \(i\)-th country at the \(t\)-th period; \(x_{it}\) is a \(1 \times (s+1)\) row vector
whose first element is 1 and whose remaining elements are the natural logarithms of
input quantities; \( \beta = (\beta_0, \beta_1, ..., \beta_n)' \) is an \((s+1)\) column vector of unknown parameters
to be estimated; \( t \) is the time trend to capture a shift in the production frontier;
\( v_{it} \) is random error, and captures measurement error or variations in level of output
associated with anything other than technical inefficiency, such as recession in the rest
of the world and assumed to be \( iid., \mathcal{N}(0, \sigma^2_v) \). On the other hand, the random error,
\( u_{it} \), is a non-negative random variable and assumed to capture technical inefficiency in
production, which is independently distributed and assumed to be truncated normal
such that \( u_{it} \sim \mathcal{N}(\mu_{it}, \sigma^2_u) \).

Battese and Coelli (1995) states that the mean of technical inefficiency can be
specified as,

\[
\mu_{it} = w_{it}\delta
\]  

(3.2)

where \( w_{it} \) is a \((1 \times p)\) vector of country specific variables that and expected to deter-
mine the technical efficiency and \( \delta \) is an \((p \times 1)\) vector of parameters to be estimated.

In this study, firm demand for money, household demand for money and interest rates
are used as country specific explanatory variables in explaining technical inefficiency.

According to Battese and Coelli (1995), if inefficiencies are stochastic, explanatory
variables in the inefficiency model may include some input variables in the stochastic
frontier. Therefore, the technical inefficiency effect, $u_{it}$, could be specified as,

\[ u_{it} = w_{it}\delta + \epsilon_{it} \quad (3.3) \]

where the random variable $\epsilon_{it}$ is defined by the truncated normal distribution with zero mean and variance, $\sigma^2$, such that the point of truncation is $-w_{it}\delta$, i.e., $\epsilon_{it} \geq -w_{it}\delta$. These assumptions state that $u_{it}$ is truncated normal and distributed as $u_{it} \sim N(w_{it}\delta, \sigma^2_u)$. Specification of $u_{it} \geq 0$ in equation (3.1) guarantees that all observations lie on or under the stochastic production frontier.

Maximum likelihood is proposed for the simultaneous estimation of the parameters of both the stochastic production frontier and the technical inefficiency model. Variance terms are parameterized according to Battese and Coelli (1995) by replacing $\sigma^2_v$ and $\sigma^2_u$ with $\sigma^2 = \sigma^2_v + \sigma^2_u$ and $\gamma = \frac{\sigma^2_u}{\sigma^2_v+\sigma^2_u}$.

Technical efficiency (TE) relative to the production frontier (3.1) for the $i-th$ observation is

\[ L(\theta; y) = -\frac{1}{2}\sum_{i=1}^{N} T_i \{\ln2\pi + \ln\sigma^2\} - \frac{1}{2} \sum_{i=1}^{N} T_i \sum_{i=1}^{T_i} \left( \frac{(y_{it} - x_{it}\beta + w_{it}\delta)^2}{\sigma^2} \right) - \sum_{i=1}^{N} T_i \left\{ \ln\Phi(d_{it}) - \ln\Phi(d_{it}^*) \right\} \]

where $d_{it} = \frac{w_{it}\delta}{(\gamma\sigma^2)^{\frac{1}{2}}}$, $d_{it}^* = \frac{\mu_{it}^*}{\gamma(1-\gamma)^{\frac{1}{2}}}$, $\mu_{it}^* = (1-\gamma)w_{it}\delta - \gamma(y_{it} - x_{it}\beta)$, $\sigma_u^* = \frac{\gamma(1-\gamma)}{\gamma(1-\gamma)^{\frac{1}{2}}}$, and $\theta = (\beta', \delta', \sigma^2, \gamma)$.
country and the $t-th$ period can be defined as:

$$TE = \frac{E(ln(y_{it})|u_{it}, x_{it})}{E(ln(y_{it})|u_{it} = 0, x_{it})} = E(-u_{it}) = E(-w_{it}\delta - \epsilon_{it})$$  \hspace{1cm} (3.4)$$

where $E$ is the usual expectations operator. Equation (3.4) is bounded between zero and one and states that the loss in production is partly explained by technical inefficiency. $u_{it}$ is not observable, thus $TE$ relies on the conditional expectations of $u_{it}$, conditional upon the observed value of $v_{it} - u_{it}$ and the maximum value of $y_{it}$ is conditional upon $u_{it} = 0$.

The parameter $\gamma = \frac{\sigma^2_u}{\sigma^2}$ reports the rate of variation in random errors that can be explained by technical inefficiency. If $\gamma = \frac{\sigma^2_u}{\sigma^2} = 0$, the expected value of $TE$ is one and there is no deviation due to technical inefficiency, $\sigma^0_u = 0$. If $\gamma = 1$ then deviations in output are due entirely to technical inefficiency. On the other hand, if $1 > \gamma > 0$ technical inefficiencies are partially responsible for the variation in the random error. In other words, output deviations are due to both technical inefficiency and random error.
### 3.2.2 Transactions Cost and Technical Inefficiency

**Production Frontier with Money in the Production Function**

The first chapter shows that the transactions cost in the budget constraint for a firm maximizing output is functionally equivalent to putting money in the production function. The production function with money represents a production function for a firm incurring transactions cost. Recall the production function, Figure (3.1), from the first chapter. The relationship between technical inefficiency and the transactions cost can be observed clearly in the figure.

![Production Function II](image)

**Figure 3.1: Production Function II**

Equivalence implies that

$$y(z_{it}) - \Theta(y(z_{it}), m_{it}) \equiv f(z_{it}, m_{it})$$  \hspace{1cm} (3.5)
where \( z_t \) is the stock of capital, and \( m_t \) is real money balances. Therefore the inefficiency in the production is represented as a loss in the quantity of output. On the other hand, the technical efficiency (\( TE \)) is measured as a ratio of the observed level of output to the maximum level of output:

\[
TE = \frac{f(z_{it}, m_{it})}{y(z_{it})}
\]

After taking the natural logarithm we may rearrange the equation above as

\[
\ln f(z_{it}, m_{it}) = \ln y(z_{it}) + \ln TE
\]

(3.6)

**Stochastic Production Frontier Application**

Application of the stochastic production frontier approach over the problem above can be explained as follow: The classical Cobb-Douglas production function can be estimated by using equation (3.1) as

\[
\ln y(z_{it}) \equiv \beta \ln z_{it} + \varphi t + v_{it} - u_{it}
\]

(3.7)

For estimation purposes we want to explain inefficiency as a function of transactions cost. Then, the technical inefficiency distribution parameter can be arranged as

\[
\mu_{it} = \phi \ln \Theta(y_{it}, m_{it})
\]

(3.8)
Inefficiency is assumed to be a function of $\Theta(y_{it}, m_{it})$. The transactions cost $\Theta(y(z_{it}), m_{it})$ can be specified as

$$\Theta(y(z_{it}), m_{it}) = \kappa \left[ \frac{y(z_{it})}{m_{it}} \right]^\alpha$$

(3.9)

where $\kappa$ is the unit transactions cost. Note that equation (3.9) is consistent with Feenstra (1986).

In Feenstra, $\kappa$ is assumed to be constant. It includes both implicit and explicit costs that firms have to pay for each transaction. Constant unit cost implies that firms in different countries have the same transactions cost. This is an unrealistic assumption. In general, we may relax the assumption of fixed transactions cost and add some other variables into our regressions to account for differences in transactions costs for different countries. Transactions cost depends on expected inflation. Expected inflation is not an observable variable. The best proxy may be to use nominal short term interest rates to make transactions costs differ for firms and households in different countries. Thus, instead of assuming transactions costs to be the same across countries, we may use

$$\kappa(R_{it}) = (1 + R_{it})^\rho$$

(3.10)

where $R_{it}$ is the nominal interest rate. Interest rates are not the unique opportunity
cost that a firm and a household have to pay. Tax rate differences, different monetary policies, etc. can be counted to define heterogeneity among countries. The impact of these differences or other forms of opportunity costs will show up in random variable $\epsilon_{it}$ or $v_{it}$.

According to the equation (3.8), variable $y_{it}$ explains both production frontier and inefficiency, therefore the technical inefficiency effect can be modelled as

$$u_{it} = \phi \ln \Theta(y_{it}, m_{it}) + \epsilon_{it}$$

Considering the specific functional form of the transactions cost;

$$u_{it} = \delta_0 + \delta_1 \ln(1 + R_{it}) + \delta_2 \ln \left( \frac{y_{it}}{m_{it}} \right) + \epsilon_{it}$$

where $\delta_1 = \rho \phi$ and $\delta_2 = \alpha \phi$. The model includes a constant term due to the note in Battese and Coelli (1995), stating that missing $\delta_0$ causes biased estimates of other $\delta_i$ and unnecessarily restricts the distribution of technical inefficiency effect.

### 3.3 Data

Yearly volume indexes for the values of output, labor and capital services for 12 EU countries\(^4\) are taken from OECD productivity releases for 8 years, 1995-2002. The

\(^4\)Austria, Belgium, Denmark, France, Finland, Germany, Ireland, Italy, Portugal, Spain, Sweden and UK.
major advantage in using this data set is that measure of inputs are corrected for quality changes and rates of utilization. Also, in order to use a standard measures among countries, data are deflated by harmonized price indexes. Output index and capital service index are divided by the labor service, before entering them into stochastic production frontier estimations.

Currency and deposits held by non-financial corporations are used as a proxy for the transactions demand for money by firms. In addition, in order to investigate the relative importance of transactions cost on households’ and firms’ decisions in determining the inefficiency in production, money held by household and money held by firms enter into the estimations separately as explanatory variables. These data are available for 1995-2002 in “Eurostat”. Before entering these monetary aggregates into the regressions, both demand by firms and households are indexed by taking 1995 as the base year and then dividing by the output index.

Nominal interest rates are 3-month money market rates, where available, or rates on similar financial instruments taken from OECD financial statistics. Interest rates are also converted to an index series before putting them into estimations.
3.4 Empirical Results

3.4.1 Empirical Specification

The stochastic production frontier to be estimated is

$$lny_{it} = \beta_0 + \beta_1 ln(k_{it}) + \beta_2 t + v_{it} - u_{it}$$  \hspace{1cm} (3.11)

where $y_{it}$ is the value of output per labor, $k_{it}$ is the capital input per labor, and $t$ is the time trend. Time trend is added to the regression to capture the shift in the production function. It counts for the Hicksian neutral technological change. Also, $ln$ denotes natural logarithm.

Technical inefficiency effects are assumed to be defined by

$$u_{it} = \delta_0 + \delta_1 ln(nf_{it}) + \delta_2 ln(hh_{it}) + \delta_3 ln(1 + R_{it}) + \epsilon_{it}$$ \hspace{1cm} (3.12)

where $nf$ is the non-financial corporations’ real demand deposits and coins as a percentage of output, $hh$ is the households’ real demand deposits and coins as a percentage of output, and $R_{it}$ is the 3-month nominal money market rates. If we want to compare specifications in this section to section (2.1), $w_{it}$ is a vector consisting of $(ln(nf_{it}), ln(hh_{it}), ln(1 + R_{it}))'$.  

Equation (3.12) investigates the impact of real balances held by firms and real balances held by households separately. Signs of the coefficients of money holding
are expected to be negative, and significant to be consistent with the theory. If coefficients are negative then we may say that money holdings by households and firms contribute to the inefficiency negatively. Therefore money holding improves efficiency. The sign of the nominal interest rates is expected to be positive due to the fact that nominal interest rates should increase transactions cost for firms, thus may increase inefficiency.

A series of tests can be conducted to test the specification of the technical inefficiency model. Likelihood ratio tests are used to determine the significance of the technical inefficiency model restrictions. The generalized likelihood ratio statistic is given by

\[ LR = -2\{lnL(H_0) - lnL(H_1)\} \]

where \( L(H_0) \) and \( L(H_1) \) are the values of the likelihood functions under the null and the alternative hypothesis respectively. The test statistics \( LR \) has a \( \chi^2 \) distribution with the degree of freedom equal to the number of parameters assumed to be zero in the null hypothesis, \( H_0 \). Table(3.2) summarizes the tests of the hypothesis for parameter restrictions in the technical inefficiency models. The null hypothesis that technical inefficiency is absent can be specified as \( \gamma = \delta_0 = \delta_1 = \delta_2 = \delta_3 = 0 \). The
null hypothesis of no stochastic technical inefficiency can be specified as $\gamma = \delta_0 = 0$. If the parameter $\gamma$ is zero, then $\sigma_u^2$ is zero, and the model implies that all variables in the technical inefficiency model have to be included in the production function. These restrictions are the one-sided generalized likelihood ratio test of Battese and Coelli (1995). The alternative hypothesis is $0 < \gamma < 1$, the test has an asymptotic mixture of $\chi^2$ distribution with the critical values given by Kodde and Palm (1986). If the hypothesis is not rejected, there is no evidence of underutilization of factors that causes countries to produce below the production frontier and the production frontier is identical to a standard production function.

The null hypothesis that inefficiency is not a linear function of money and interest rates can be constructed as $\delta_1 = \delta_2 = \delta_3 = 0$. The significance of this hypothesis can be tested using standard $\chi^2$ tables.

### 3.4.2 Results

Single stage maximum likelihood estimates of the model are obtained using FRONTIER 4.1 (Coelli 1996). The program follows three-step estimation methods in estimating maximum likelihood estimates of the parameters. In the first step, OLS

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5Battese and Coelli (1993) states that when $\gamma$ is zero then $\delta_0$ must be zero. If the model does not have random technical inefficiency $\delta_0$ would not be identified.
estimates of the functions are obtained. At this step all OLS estimators, except for \( \beta_0 \) are unbiased. Therefore, the second step does the two face grid search of \( \gamma \), with adjustments to OLS estimates of \( \beta_0 \) and \( \sigma^2 \) parameters while other \( \beta \) parameters set to the OLS values. During this grid search, parameters \((\mu, \delta)\) are set to zero as well. In the third stage, the values selected in the grid search are used as starting values in a quasi-Newton iterative procedure to obtain the final maximum likelihood estimates (Coelli 1996).

Results of the models (3.11) and (3.12) are reported in Table (3.1). Table (3.1) presents both OLS and MLE estimates of the production function. As expected, MLE results vary from OLS results for the input variable. The sign of the coefficient on \( k_t \) (= capital/labor) is as expected and highly significant. The coefficient of the time trend indicates that the value of the output has tended to decrease by a small but significant rate over the eight-year period.

For the purpose of our study the estimated coefficients of the inefficiency model are of particular interest. Results indicate that both the demand for money by firms and the demand for money by households contribute negatively to the inefficiency. The impact of the firm demand for money is negative and highly significant (significant at the 1% significance level). Household demand for money is also negative but
Table 3.1: Estimates of SPF and Technical Inefficiency Models

<table>
<thead>
<tr>
<th></th>
<th>Coeff.</th>
<th>Asym. t</th>
<th>Coeff.</th>
<th>Asym. t</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>MLE</td>
<td>OLS</td>
<td>MLE</td>
<td>OLS</td>
</tr>
<tr>
<td>constant</td>
<td>2.56***</td>
<td>9.92</td>
<td>3.06***</td>
<td>10.45</td>
</tr>
<tr>
<td></td>
<td>(0.25)</td>
<td></td>
<td>(0.29)</td>
<td></td>
</tr>
<tr>
<td>lnk</td>
<td>0.5***</td>
<td>15.4</td>
<td>0.33***</td>
<td>5.23</td>
</tr>
<tr>
<td></td>
<td>(0.032)</td>
<td></td>
<td>(0.064)</td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>-0.009**</td>
<td>-2.96</td>
<td>0.01***</td>
<td>4.86</td>
</tr>
<tr>
<td></td>
<td>(0.0031)</td>
<td></td>
<td>(0.0020)</td>
<td></td>
</tr>
<tr>
<td>Tech Ineff</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>constant</td>
<td>6.24***</td>
<td>7.47</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.83)</td>
<td></td>
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<td></td>
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<tr>
<td>lnnf</td>
<td>-0.68***</td>
<td>-8.07</td>
<td></td>
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</tr>
<tr>
<td></td>
<td>(0.085)</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>lnhh</td>
<td>-0.04*</td>
<td>1.31</td>
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<tr>
<td></td>
<td>(0.033)</td>
<td></td>
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<tr>
<td>ln R</td>
<td>-0.57***</td>
<td>-6.92</td>
<td></td>
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</tr>
<tr>
<td></td>
<td>(0.083)</td>
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</tr>
<tr>
<td>$\sigma^2$</td>
<td>0.0086***</td>
<td>7.01</td>
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<tr>
<td></td>
<td>(0.001)</td>
<td></td>
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<tr>
<td>$\gamma$</td>
<td>0.868***</td>
<td>11.02</td>
<td></td>
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</tr>
<tr>
<td></td>
<td>(0.078)</td>
<td></td>
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</tr>
<tr>
<td>Log LH</td>
<td>216.67</td>
<td>189.5</td>
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<tr>
<td>LR</td>
<td>54.36</td>
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</tr>
</tbody>
</table>

Note: *, ** and *** show statistical significance at the 0.10, 0.05 and 0.01 level respectively. Numbers in parenthesis are asymptotic standard errors.

significant at the 10% significance level. Results suggests that firm demand for money has a stronger impact over the determination of the technical inefficiency compared to the impact of household demand for money. Those countries whose firms hold more money were technically more efficient during the period 1995-2002.

There is a significant and negative relationship between inefficiency and nominal
short term interest rates. This implies that higher interest rates lowers the inefficiency. At first, we might think that the sign of the interest rate coefficient contradicts expectations. The unexpected relationship may be explained with the monetary policy applied during this period. As noted in the introduction, the objective of the monetary policy is to maintain medium-term price stability by targeting short-term interest rates. If we examine the path of the nominal rates, we observe that rates do not vary much after 1999 across countries, consistent with this monetary policy. This is a result of implementation of common monetary policy across countries in the sample. Countries decreased their nominal rates consistently until 1999 when all rates were almost equalized. Therefore, if we take the nominal rates as a proxy for monetary policy, indeed the negative and significant coefficient of interest rates states that monetary policy tended to decrease inefficiency across 12 EU countries over the eight-year period.

Next, first a constant and then an interaction dummy for interest rates for each country is added to the estimations to analyze whether the sign of interest rates vary across countries. However, addition of dummy variables did not improve the result. First, not only the coefficients dummy variables but also none of the explanatory variables in the technical inefficiency model were significant. Second, the modification
did not correct the sign of the parameter for interest rates.

The estimate for the variance parameter is 0.868 and is highly significant. It indicates that part of the variation in the total residual is due to the technical inefficiency effect. The impact of the variance of $v_t$ still matters. Table (3.2) presents tests of hypotheses for the inefficiency model.

<table>
<thead>
<tr>
<th>Null Hypothesis</th>
<th>$\chi^2$ - statistics</th>
<th>$\chi^2_{0.95}$ - value</th>
<th>Decision</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_0: \gamma = \delta_0 = \delta_1 = \delta_2 = \delta_3 = 0$</td>
<td>54.36</td>
<td>10.37</td>
<td>Reject $H_0$</td>
</tr>
<tr>
<td>$H_1: 0 &lt; \gamma &lt; 1, \delta_0, \delta_1, \delta_2, \delta_3 \neq 0$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$H_0: \gamma = \delta_0 = 0$</td>
<td>41.78</td>
<td>5.13</td>
<td>Reject $H_0$</td>
</tr>
<tr>
<td>$H_1: 0 &lt; \gamma &lt; 1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$H_0: \delta_1 = \delta_2 = \delta_3 = 0$</td>
<td>54.34</td>
<td>7.81</td>
<td>Reject $H_0$</td>
</tr>
<tr>
<td>$H_1: \delta_1, \delta_2, \delta_3 \neq 0$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$H_0: \delta_1 = \delta_2 = 0$</td>
<td>34.96</td>
<td>5.13</td>
<td>Reject $H_0$</td>
</tr>
<tr>
<td>$H_1: \delta_1, \delta_2 \neq 0$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$H_0: \delta_3 = 0$</td>
<td>45.36</td>
<td>3.84</td>
<td>Reject $H_0$</td>
</tr>
<tr>
<td>$H_1: \delta_3 \neq 0$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: These hypotheses jointly test equality and inequality constraints. According to Kodde and Palm (1986), Battese and Coelli (1993) and Coelli (1996) the large sample statistics under the null hypothesis is a mixture of $\chi^2$ distribution. $\chi^2$ statistics are LR test of one-sided errors. Therefore the critical values for the first two hypotheses are obtained from Table 1 of Kodde and Palm (1986).

Generalized likelihood ratio tests of the null hypothesis that the inefficiency effect is absent from the model reject the null hypothesis. Table (3.2) also rejects the second null hypothesis of no stochastic inefficiency effect in the model. Therefore inefficiency is stochastic. The third null hypothesis specifies that the inefficiency effects are not
a linear function of firm and household demand for money and interest rates. This null hypothesis is also strongly rejected at the 5% significance level, indicating that the joint effects of these three explanatory variables on the technical inefficiency is significant. The fourth null hypothesis tests the significant of joint impact of firm and household demand for money on technical inefficiency. The test finds that these two variables are significant joint impact on determination of technical inefficiency. The fifth null hypothesis of no interest rate effect on inefficiency in Table (3.2), is also strongly rejected.

To summarize, results point out that the joint effects of these three explanatory variables on the inefficiency of production is significant. Production significantly suffers from technical inefficiency and inefficiency is not independent of the money holdings of firms and households and nominal short term interest rates.

### 3.5 Concluding Remarks

This chapter examines the impact of transactions demand for money on productive efficiency with an annual panel of 12 EU countries over the period of 1995-2002. It applies a stochastic production frontier approach to determine the significance of alternative monetary aggregates on country-level technical inefficiency.
Results point out that the joint effects of these three explanatory variables on the inefficiency of production is significant. Stochastic production frontiers significantly suffer from technical inefficiency, and inefficiency is not independent of money holding by firms and households and nominal short term interest rates.

The study can be improved in several ways. The first and the most important improvement would be to apply the test on the firm level. This is a micro level analysis, however it is also important to understand the impact of financial variables over firm level inefficiencies. The main problem in such a study would be finding data to perform such an examination.

Secondly, more explanatory monetary variables can be added to the technical inefficiency model to identify country-specific variables that may have a significant impact on inefficiency. Among these variables there might be some other variables that can be used to define the structure of financial sectors for different countries.

The third extension may apply the same test on developing countries, compare results, and investigate if the results are sensitive to the economic development level. Here again the main problem would be finding particularly firm-level data for developing countries.
List of References


