Abstract

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$\beta X \setminus X$ is the remainder of the Stone-Čech compactification of a locally compact space X. This paper introduces a lattice which we call $L(X)$ that is constructed using equivalence classes of closed sets of X. We then determine that $St(L(X))$ (the set of ultrafilters on $L(X)$) is homeomorphic to $\beta X \setminus X$. We subsequently give some examples. Most notably, for $X = \mathbf{H}$ this now provides a lattice-theoretic approach for representing $\beta\mathbf{H} \setminus \mathbf{H}$.

In addition, we expand and clarify some aspects of lattice theory related to our constructions. We introduce the term "upwardly nonlinear" as a way to describe lattices with a certain property related to the ultrafilters on it. We also investigate some of the lattice properties of $L(X)$.

THE LATTICE OF EQUIVALENCE CLASSES OF CLOSED SETS AND THE STONE-ČECH COMPACTIFICATION

by

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Biography

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Chapter 1

PRELIMINARIES

1.1 DEFINITIONS – Topological

Knowledge of some of the basic concepts of topology (such as the separation axioms) is assumed. *General Topology* by Willard [14] is a good source for topological definitions not contained herein. Both Willard's book [14] and *Extensions and Absolutes of Hausdorff Spaces* by Porter and Woods [9] are good references for the following definitions.

A subset S of a topological space X is called clopen if S is both open and closed with respect to the topology on X. A space X is called 0-dimensional provided that the set of all clopen sets of X forms a base for the open sets of X.

A collection C of sets of a space X is said to have the finite intersection property if the intersection of any finite subcollection of elements of C is non-empty.

A space X is said to be compact provided that every open cover of X has a finite subcover. Equivalently, X is compact if every collection of closed sets of X with the finite intersection property has non-empty intersection.

A compactification αX of a space X is a compact Hausdorff space that contains X as a dense subset. A space has a compactification if and only if it is Tychonoff. From now on, unless otherwise noted, spaces will be assumed to be Tychonoff. $\alpha X \setminus X$ denotes the remainder (i.e. the compactification αX minus the original space X) of the compactification αX.

A space X is said to be locally compact provided that every element of X has a compact neighborhood. The one-point compactification of X, sometimes denoted ωX, is the compactification of X whose remainder consists of a single point. Locally compact, non-compact spaces are precisely those that have a one-point compactification.

The maximum compactification of a space X is called the Stone-Čech compactification of X, and is denoted βX. The Stone-Čech compactification of X can be characterized as the unique compactification of X such that every bounded continuous function on X can be extended to a continuous function on βX (i.e. X is $C^\star$-embedded in βX). To see varying constructions of βX, we recommend Chandler's *Hausdorff Compactifications* [1], and *Rings of Continuous Functions* by Gillman and Jerison [5].

A space X is called connected if no two disjoint open sets of X cover X. If X is compact and connected, it is called a continuum.

Additionally, a subset S of a space X is called a G_δ if S is the countable intersection of open sets, and S is called an F_σ if S is the countable union of closed sets.

1.2 DEFINITIONS – Lattice-theoretic

Many of the following definitions can be found in Koppelberg's *Handbook of Boolean Algebras* [7].

A partial order $\leq$ is a binary relation that is reflexive, transitive, and antisymmetric. A partially ordered set $(P, \leq)$ is a set P paired with a partial order $\leq$. We will usually just write P instead of $(P, \leq)$.

Given a partially ordered set P and elements $a, b \in P$, the supremum of a and b (denoted $a \vee b$) is the least element that is greater than or equal to both a and b. The infimum of a and b (denoted $a \wedge b$) is the greatest element that is less than or equal to both a and b. If L is a partially ordered set where $a \vee b$ and $a \wedge b$ exist for all $a, b \in L$, then we call L a lattice.

A lattice L is called distributive if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ and $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ for all $a, b, c \in L$.

A bounded lattice is a lattice with both a greatest and a least element (usually denoted 1 and 0, respectively).

A filter p on a lattice L is a subset of L such that

(i). $0 \notin p$

(ii). $a, b \in p$ implies that $a \wedge b \in p$

(iii). $a \in p$ and $b \in L$ with $b \geq a$ implies that $b \in p$.

An ultrafilter is a maximal filter (with respect to set inclusion).

Let $a \in L$. Then $a' \in L$ is said to be the pseudocomplement of a if $a \wedge a' = 0$ and $b \wedge a = 0 \Rightarrow b \leq a'$, $\forall b \in L$. A lattice L is called pseudocomplemented if every $a \in L$ has a pseudocomplement.

Let $a \in L$. Then $a^c \in L$ is said to be the complement of a if $a \wedge a^c = 0$ and $a \vee a^c = 1$. We say "the" complement because it is unique (if it exists). A lattice B where every element of B has a complement is called a Boolean algebra. The *Handbook of Boolean Algebras* [7] is an excellent reference for more background about Boolean algebras.

The power set of ω, denoted $P(\omega)$, is a Boolean algebra. $P(\omega)/fin$ denotes the set of subsets of ω mod the finite sets (also a Boolean algebra).

If B is a Boolean algebra, then the Stone space of B, denoted $St(B)$, is the space formed by the set of all ultrafilters on B. We also use this notation to describe the collection of ultrafilters of a lattice (i.e. if L is a lattice, $St(L)$ is the space of all ultrafilters on L).

We use the notation $CO(St(B))$ to denote the set of clopen subsets of $St(B)$.

1.3 BACKGROUND

M.H. Stone first introduced using Boolean algebras in the study of compact 0-dimensional spaces in the late 1930's. The following theorem is named for him:

Theorem 1.3.1. (Stone's Representation Theorem) Let B be a Boolean Algebra. Then:

1) $St(B)$ is a compact, zero-dimensional space.

2) B is isomorphic to $CO(St(B))$ (the Boolean algebra of the clopen subsets of $St(B)$.

A detailed treatment of Stone's theory can be found in the books of Porter and Woods [9] and of Walker [12]. This theorem led to, in particular, the characterization of $\beta\omega$ as the Stone space of the power set of ω (i.e. $St(P(\omega))$). Additionally, the Stone space of the Boolean algebra $P(\omega)/fin$ is homeomorphic to $\beta\omega \setminus \omega$. In general, Stone-Čech compactifications and their remainders could be studied using Boolean algebras – provided that they were zero-dimensional.

Also in the late 1930's, H. Wallman worked on an alternate approach to studying compact T_1-spaces, but he used lattices in general (without the additional structure of Boolean algebras). In his paper *Lattices and Topological Spaces* [13], Wallman considers the lattice L of closed sets of a T_1-space R. He defines the term "point" (now referred to as "ultrafilter") and shows that the set S of all "points" of a distributive lattice is a bicompact T_1-space. He also shows that R normal if and only if S Hausdorff.

Additionally, he defines in the paper a condition on the lattice L of the closed sets of R that is inspired by the closed set definition of normality. Today, any lattice satisfying this condition is usually referred to as a normal lattice. This version of normality led to the definition of a Wallman base (a discussion about the concept of Wallman bases can be found in Willard's book [14]):

Definition 1.3.2. Let $\mathcal{B}$ be any base for the closed sets of X satisfying the following conditions:

a) for each closed set F and $x \notin F$, there is some $A \in \mathcal{B}$ such that $x \in A$ and $A \cap F = \emptyset$,

b) $\mathcal{B}$ is closed under finite unions and intersections,

c) if $A, B \in \mathcal{B}$ are disjoint, then there exists $C, D \in \mathcal{B}$ such that $A \subset X \setminus C$, $B \subset X \setminus D$, and $(X \setminus C) \cap (X \setminus D) = \emptyset$.

Then $\mathcal{B}$ is called a Wallman base for X.

Given a Wallman base $\mathcal{B}$, the concept of a $\mathcal{B}$-ultrafilter can be defined. Then, the set of all $\mathcal{B}$-ultrafilters on X is called a Wallman compactification of X. It was once believed that every Hausdorff compactification might be a Wallman compactification for some Wallman base.

In the 1960's, O. Frink worked with understanding Wallman bases (which he called normal bases at the time) and the resulting Wallman compactifications. For example, he proved the following about semi-normal spaces (i.e. spaces with at least one Wallman base):

Theorem 1.3.3. A T_1-space is completely regular if and only if it is semi-normal.

The proof of this theorem can be found in his 1964 paper *Compactifications and Semi-Normal Spaces* [4]. Also in this paper, Frink first posed the question of whether every Hausdorff compactification can be obtained by taking the Wallman compactification for some Wallman base.

In 1974, L.B. Šapiro published a paper [10] which considered this question posed by Frink. Šapiro broke this problem down to a set-theoretic one. He did not fully answer the question, but his work paved the way for V.M. Ul'Janov. In Ul'Janov's 1977 paper [11] on the subject, he determined that there are in fact compactifications not of Wallman type.

In the meantime, use for Stone's work had risen, thanks to the importance that Parovičenko found in the space $\beta\omega \setminus \omega$. In his 1963 paper [8] on the subject, Parovičenko proved that every compact set of weight $\leq \omega_1$ is a continuous image of $\beta\omega \setminus \omega$. Assuming the Continuum Hypothesis, he showed that $\beta\omega \setminus \omega$ is a universal compact Hausdorff space of weight ω_1. This led to the definition of a space called a Parovičenko space:

Definition 1.3.4. X is called a Parovičenko space provided that

a) X is a zero-dimensional compact space without isolated points with weight c,

b) every two disjoint open F_σ's in X have disjoint closures,

c) every nonempty G_δ in X has nonempty interior.

Thus, Parovičenko showed that the Continuum Hypothesis implies that every Parovičenko space is homeomorphic to $\beta\omega \setminus \omega$.

E.K. van Douwen and J. van Mill expanded on this work in their 1978 paper [2], where they proved that Parovičenko's characterization of $\beta\omega \setminus \omega$ is equivalent to the Continuum Hypothesis by showing that if every Parovičenko space is homeomorphic to $\beta\omega \setminus \omega$, then the Continuum Hypothesis is true.

In 1991, K.P. Hart and A. Dow published a paper [6] in which they proved a similar result to that of Parovičenko, but this time for the space $\beta\mathbf{H} \setminus \mathbf{H}$. They showed that every continuum of weight ω_1 is a continuous image of $\beta\mathbf{H} \setminus \mathbf{H}$. This characterization provided new motivation for the further study of $\beta\mathbf{H} \setminus \mathbf{H}$.

The characterization of $\beta\omega \setminus \omega$ popularized the use of Boolean algebras in studying compactifications (begun by Stone), as this technique allowed one to approach topological problems from an algebraic angle. On the other hand, the lattice approach introduced by Wallman fell out of favor with the discovery that there are compactifications that are not Wallman type. However, one of the overlooked advantages of Wallman's approach is that it could (at least hypothetically) handle more topological cases, since the theory wasn't restricted to the study of 0-dimensional spaces (as the Boolean algebra approach was).

We use this advantage in our development of the lattice that we call $L(X)$ (see chapter 3), as we are motivated by the work of Hart and Dow to find an algebraic approach with which to study $\beta\mathbf{H} \setminus \mathbf{H}$. Though used differently, the construction of

$L(X)$ was inspired, in part, by a similar lattice construction found in a 1994 paper by Faulkner and Vipera [3].

Chapter 2

GENERAL LATTICES

Suppose that L is a lattice that is bounded (i.e. has a minimum and maximum with respect to the partial order; generally we'll refer to these as 0 and 1) and distributive (we assume all lattices are bounded and distributive unless noted otherwise). An ultrafilter on L is a filter on L that is maximal. A filter on L is said to be prime if $a \vee b \in L$ implies that $a \in L$ or $b \in L$. With the help of the following lemma, we show that every ultrafilter is prime.

Lemma 2.1. Let p be an ultrafilter of a lattice L. Suppose $b \in L$ and $\forall a \in p$, $a \wedge b \neq 0$. Then $b \in p$.

Proof. Let $q =\{c \in L : \exists a \in p, c \geq a \wedge b\}$.(Show that q is a filter.)

(i). $a \wedge b \neq 0$, $\forall a \in p$, so $0 \notin q$.

(ii). Let $c_1, c_2 \in q$. Then $c_1 \geq a_1 \wedge b$ and $c_2 \geq a_2 \wedge b$ for some $a_1, a_2 \in p$. Hence $c_1 \wedge c_2 \geq (a_1 \wedge b) \wedge (a_2 \wedge b) = (a_1 \wedge a_2) \wedge b$. So $c_1 \wedge c_2 \in q$ since $a_1 \wedge a_2 \in p$.

(iii). Let $c_1 \in q$ and $c_2 \geq c_1$. Then $c_1 \geq a \wedge b$ for some $a \in p$. Clearly $c_2 \geq a \wedge b$. Hence $c_2 \in q$.

So, by (i), (ii), and (iii), q is a filter. Let $a \in p$. Since $a \geq a \wedge b$, then $a \in q$. So $p \subseteq q$.

But p is an ultrafilter, so $p = q$. Since for any $a \in p$, $b \geq b \wedge a$, then $b \in q$. But $p = q$ $\Rightarrow b \in p$. □

Proposition 2.2. Suppose that p is a filter of a lattice L. Then p is maximal $\Rightarrow p$ is prime.

Proof. Let p be a maximal filter of L. Let $x, y \in L$ such that $x \vee y \in p$. Then $x \vee y \in p \Rightarrow (x \vee y) \wedge a \neq 0, \forall a \in p \Rightarrow (x \wedge a) \vee (y \wedge a) \neq 0, \forall a \in p$. Suppose that $\exists a, b \in p$ such that $x \wedge a = 0$ and $y \wedge b = 0$. Then $(x \wedge (a \wedge b)) \vee (y \wedge (a \wedge b)) = 0$, which contradicts above since $a \wedge b \in p$. So either $(x \wedge a \neq 0, \forall a \in p)$ or $(y \wedge a \neq 0, \forall a \in p)$. So by lemma 2.1, either $x \in p$ or $y \in p$. □

We denote the set of all ultrafilters on L by $St(L)$ (In the case of a Boolean algebra B, the set of all ultrafilters on B is usually referred to as the Stone space of B – we'll use this notation in tribute to that. We will also occasionally refer to $St(L)$ as the Stone space of L.). Now let's define the topology on $St(L)$ by considering a base for the closed sets.

Notation 2.3. Let $[f] = \{p \in St(L) : f \in p\}$, for each $f \in L$, and let $B(St(L)) = \{[f] : f \in L\}$.

Obviously, $B(St(L))$ is a base for the closed sets of some topology on $St(L)$ (For $f_1, f_2 \in L$, $[f_1] \cup [f_2] = [f_1 \vee f_2] \in B(St(L))$. Also, $\bigcap [f] = \emptyset$ since $[0] = \emptyset$). Let τ_B be the topology that $B(St(L))$ is a base for the closed sets of. It will be understood that $St(L)$ has this topology on it. The following are fundamental properties of these basic closed sets.

Lemma 2.4. $[a] \cup [b] = [a \vee b]$, $\forall a, b \in L$.

Proof. There are two parts:

($\subseteq$) Let $p \in [a] \cup [b]$. Then $a \in p$ or $b \in p$. Since $a \vee b \geq a$ and $a \vee b \geq b$, then $a \vee b \in p$. Hence $p \in [a \vee b]$.

($\supseteq$) Let $p \in [a \vee b]$. Then $a \vee b \in p$. Since p is prime, $a \in p$ or $b \in p$. Hence $p \in [a] \cup [b]$. □

Lemma 2.5. $[a] \cap [b] = [a \wedge b]$, $\forall a, b \in L$.

Proof. There are two parts:

($\subseteq$) Let $p \in [a] \cap [b]$. Then $a \in p$ and $b \in p$. Since p is a filter, $a \wedge b \in p$. Hence $p \in [a \wedge b]$.

($\supseteq$) Let $p \in [a \wedge b]$. Then $a \wedge b \in p$. Since $a \geq a \wedge b$ and $b \geq a \wedge b$, then $a, b \in p$. Hence $p \in [a] \cap [b]$. □

With $St(L)$ defined as a topological space, the obvious next step is to consider it's topological properties. We need a couple of preliminary results first, beginning with a modified version of the Boolean prime ideal theorem, adjusted to suit our needs here. The statement of the original theorem, with proof, can be found in *Handbook of Boolean Algebras* [7]. The proof I provide here, paraphrased directly from the *Handbook*, is included for the sake of completeness.

Theorem 2.6. A subset E of a lattice L has the f.i.p. $\Rightarrow$ E is contained in an ultrafilter of L.

Proof. Assume $E \subseteq L$ has the f.i.p. So, if p_0 is the filter generated by E, then p_0 is proper. Let $P = \{q : q \supseteq p_0, q \; is \; a \; filter \; of \; L\}$. Clearly, P is non-empty and can be partially ordered by set inclusion. For any non-empty chain C in P, $\bigcup C$ is clearly a filter containing each member of C. Finally, apply Zorn's lemma. □

This gives us the following fundamental result about filters:

Corollary 2.7. Any filter p of a lattice L is contained in an ultrafilter.

Proof. Filters have the f.i.p., thus this follows directly from theorem 2.6. $\square$

With this corollary, we prove that $St(L)$ is compact.

Proposition 2.8. $St(L)$ is compact.

Proof. Let C be a collection of basic closed sets of $St(L)$ with the finite intersection property; denote $C = \{[f_\alpha] : \alpha \in \Delta\}$. Let S be the collection of all finite infimums of the f_α's, $\alpha \in \Delta$. (Show that S is a filter base.)

(i). Let $g \in S$. Then $g = f_{\alpha_1} \wedge ... \wedge f_{\alpha_n}$. Since C has the f.i.p., $\exists p \in [f_{\alpha_1}] \cap ... \cap [f_{\alpha_n}] \Rightarrow f_{\alpha_1}, ..., f_{\alpha_n} \in p \Rightarrow f_{\alpha_1} \wedge ... \wedge f_{\alpha_n} \in p \Rightarrow f_{\alpha_1} \wedge ... \wedge f_{\alpha_n} \neq 0$ since $0 \notin p$.

(ii). Let $g_1, g_2 \in S$. Then $g_1 = f_{\alpha_1} \wedge ... \wedge f_{\alpha_n}$ and $g_2 = f_{\beta_1} \wedge ... \wedge f_{\beta_m}$. So $g_1 \wedge g_2 = f_{\alpha_1} \wedge ... \wedge f_{\alpha_n} \wedge f_{\beta_1} \wedge ... \wedge f_{\beta_m} \in S$.

So, by (i) and (ii), S is a base for some filter p, and thus by corollary 2.7, $\exists$ an ultrafilter p' such that $p' \supseteq p$. By construction, $f_\alpha \in p', \forall \alpha \in \Delta$. Hence $p' \in [f_\alpha], \forall \alpha \in \Delta$. So $\bigcap C \neq \emptyset$. Therefore $St(L)$ is compact. $\square$

Next, we'll consider the following lattice property, labeled $(\star)$:

$(\star)$ If $p, q \in L$ such that $p \wedge q = 0$, then $\exists a, b \in L$ such that $p \wedge a = 0$, $q \wedge b = 0$, and $a \vee b = 1$.

This property can be found as a footnote in Wallman's *Lattices and Topological Spaces* ([13], pg.119). It also closely resembles a property used by Wallman in his definition of a Wallman base for a topological space (see Willard, [14], pg. 142). It is used as an analagous condition in the setting of lattices to the condition of normality in the topological setting. Thus, the motivation for our following definition:

Definition 2.9. A lattice L is called *normal* if L has the $(\star)$ property.

The use of the word normal becomes justified when considering the result of two propositions, the first being:

Proposition 2.10. If L is normal, then $St(L)$ is T_2.

Proof. Let $p, q \in St(L)$ such that $p \neq q$. Then $\exists f \in p$ and $g \in q$ such that $f \wedge g = 0$. So by $(\star)$, $\exists a, b \in L$ such that $f \wedge a = 0$, $g \wedge b = 0$, and $a \vee b = 1$. So $p \notin [a]$ and $q \notin [b]$. Also, by lemma 2.4, $[a] \cup [b] = [a \vee b] = [1] = St(L)$, so $St(L)$ is T_2. □

Notice, however, that $St(L)$ being T_2 doesn't guarantee that L is normal, as the following example demonstrates:

Example 2.11. Let L be the lattice with 5 distinct elements defined in the following way:

$L = \{0, a, b, a \vee b, 1\}$ where $a \wedge b = 0$. (See appendix A for proof that L is distributive; see figure B.1 in appendix B for illustration.)

Let $p = \{a,\ a \vee b,\ 1\}$ and $q = \{b,\ a \vee b,\ 1\}$. Because of the simplicity of the lattice, it is easy to check that p and q are each ultrafilters on L, and that there are no other ultrafilters (i.e. $St(L) = \{p,\ q\}$). By construction, p is in the basic closed set $[a]$, and q is in the basic closed set $[b]$. More to the point, $[a] = \{p\}$ and $[b] = \{q\}$. So $[a] \cap [b] = \emptyset$ and $[a] \cup [b] = St(L)$, which implies that $[a]$ and $[b]$ are clopen (so $[a]$ and $[b]$ are disjoint open sets containing p and q, respectively). Hence $St(L)$ is T_2.

L is, however, not normal. To see this, consider $a, b \in L$. By definition, $a \wedge b = 0$. But there doesn't exist two non-one elements in L whose supremum is 1. In other words, if $x \vee y = 1$ then either $x = 1$ or $y = 1$. Since $a \wedge 1 \neq 0$ and $b \wedge 1 \neq 0$, then L clearly can't be normal.

The key distinguishing feature of the above example is the linear nature of the top part of the lattice. If we restrict ourselves to considering lattices that are different from the example in this regard, then we can prove the desired result. We introduce the following term to describe lattices that are different from our example with respect to this feature.

Definition 2.12. A lattice L is called *upwardly nonlinear* provided that 1 is the only element of L that is in every ultrafilter on L.

Although we first considered this property from a purely lattice-theoretic point of view, we later noticed a connection between this property and a topological one. This is explored in chapter 4.

For now, we use the property to prove the second of two propositions that show that our intuitive reasons for using the term normal are well-founded.

Proposition 2.13. Assume that L is upwardly nonlinear. If $St(L)$ is T_2, then L is normal.

Proof. Suppose $a, b \in L$ such that $a \wedge b = 0$. Then $[a] \cap [b] = \emptyset$ (i.e. they have no ultrafilters in common). Since $St(L)$ is T_2 (by assumption) and compact (by proposition 2.8), $St(L)$ is normal.

Note that the base $B = \{[a] : a \in L\}$ for the closed sets of $St(L)$ is closed under finite intersections. This, combined with the fact that $St(L)$ is normal and compact, implies the following:

$\exists [c], [d] \in B$ such that $[a] \cap [c] = \emptyset$, $[b] \cap [d] = \emptyset$, and $[c] \cup [d] = St(L)$.

Suppose $a \wedge c \neq 0$. Then there exists an ultrafilter $p \in St(L)$ such that $a \wedge c \in p$. But then $a, c \in p$ (since $a, c \geq a \wedge c$), which means that $p \in [a] \cap [c]$, a contradiction. Hence $a \wedge c = 0$.

Similarly, $b \wedge d = 0$. Since $St(L) = [c] \cup [d] = [c \vee d]$ (by lemma 2.4), and L is assumed to be upwardly nonlinear, then $c \vee d = 1$. Hence L is normal. □

Thus, since $St(L)$ is always compact, propositions 2.10 and 2.13 can be summed up by the following:

Corollary 2.14. Let L be upwardly nonlinear. L normal iff $St(L)$ normal.

We now pause to make a few observations. Hart and Dow ([6], pg. 3) suggest that Wallman's paper *Lattices and Topological Spaces* [13] proves that if L is a distributive lattice then there is a compact T_1-space X such that X is Hausdorff iff L is normal. However, on reconsidering the Wallman paper, it appears that Wallman did not prove this for all distributive lattices, but only for distributive lattices that are the lattice of closed sets of some T_1-space. This would leave out any distributive lattice that is not equivalent to the lattice of closed sets of a T_1-space (Note that the Birkhoff-Stone Theorem (Porter and Woods, [9], pg.104) does guarantee that every distributive lattice has a set representation, but doesn't specify that the representation is a ring of closed sets.). The disjunction property (as Wallman called it), also mentioned by Hart and Dow [6], does appear to play a role in showing that $St(L)$ is $\mathrm{T}_2 \Rightarrow L$ is normal. In fact, it is noted in the Wallman paper that any lattice of closed sets of a T_1-space has the disjunction property.

We now consider the disjunction property and how it relates to our upwardly nonlinear property. First let's define the disjunction property as it is given by Wallman ([13], pg. 115).

Definition 2.15. A lattice L has the *disjunction property* provided that whenever $a, b \in L$ such that $a \neq b$, there exists an element $c \in L \backslash \{0\}$ such that one of $a \wedge c$ and $b \wedge c$ is 0 and the other is not 0.

Wallman stated that c should be taken out of L, but clearly c would never be 0, thus the reason for the modification. Wallman also stated the following theorem (as lemma 3, [13], pg. 115) concerning the disjunction property.

Theorem 2.16. *L has the disjunction property iff there is a 1-1 correspondence between the elements of L and the elements of $B(St(L)) = \{[f] : f \in L\}$.*

Assume a lattice L has the disjunction property. So by this theorem, if $a, b \in L$ such that $a \neq b$, then $[a] \neq [b]$. In particular, assume $a = 1$. Then $b \in L \backslash \{1\}$ implies that $[b] \neq [1]$. Since $[1] = St(L)$, this is the same as stating that for every $b \in L \backslash \{1\}$, there is an ultrafilter $p \in St(L)$ such that $b \notin p$. In other words, 1 is the only element of L that is in every ultrafilter on L. We have just shown the following:

Proposition 2.17. *L has the disjunction property $\Rightarrow$ L is upwardly nonlinear.*

So if L is a lattice with the disjunction property, then no two distinct elements a and b of L can have their associated closed sets $[a]$ and $[b]$ equal to each other. On the other hand, if L is an upwardly nonlinear lattice, then no two distinct elements a and b of L can have their associated closed sets $[a]$ and $[b]$ both equal to $[1]$ – but that doesn't mean that $[a]$ can't equal $[b]$. Thus, it would intuitively seem that there is an example of an upwardly nonlinear lattice that does not have the disjunction property.

Now, in corollary 2.7, we showed that every filter on a lattice is contained in at least one ultrafilter. The next proposition gives a condition under which a filter will be contained in exactly one ultrafilter.

Proposition 2.18. If p is a prime filter of a normal lattice L , then p is contained in a unique ultrafilter.

Proof. Suppose L is normal. Suppose p is a prime filter of L contained in distinct ultrafilters q_1 and q_2 (i.e. q_1 and q_2 are maximal, $q_1 \neq q_2$, and $p \subseteq q_1 \cap q_2$). Then $\exists a \in q_1 \setminus q_2$ and $\exists b \in q_2 \setminus q_1$ such that $a \wedge b = 0$. Since L is normal, $\exists c,\ d \in L$ such that $a \wedge c = 0$, $b \wedge d = 0$, and $c \vee d = 1$. Since $1 \in p$, then either $c \in p$ or $d \in p$ (by prime). Either is a contradiction (W.L.O.G. assume $c \in p$. Then $p \subseteq q_1 \Rightarrow c \in q_1$, but $c \in q_1$ and $a \wedge c = 0$ contradicts that $a \in q_1$.). □

In the context of Boolean algebras, if there is an embedding g between Boolean algebras B_1 and B_2, then the map $h : St(B_2) \to St(B_1)$ defined by $h(p) = g^{-1}(p)$ is a surjective map. This h is well-defined because $g^{-1}(p)$ is an ultrafilter on B_1. Our first thought was to mimic this construction in the lattice setting to get a similar result. This, however, proved difficult. Whether the inverse of a lattice embedding maps a lattice ultrafilter to a lattice ultrafilter is a question we have not yet been able to answer, at least given the usual assumption that the lattices involved are distributive. The following example arose from an attempt to find a counterexample – it was later noticed that the lattice L' (defined in example 2.19) is not distributive. Subsequent attempts to modify the example to make it distributive have been unsuccessful. It is presented here for the sake of completeness, and that it may eventually inspire a solution to the problem.

Example 2.19. Let L be the lattice with 4 distinct elements defined in the following way:

$L = \{0, x, y, 1\}$ where $x \wedge y = 0$ and $x \vee y = 1$. (See figure B.2 in appendix B for illustration.)

Note that L is trivially normal. Now let L' be the lattice with 5 distinct elements defined as follows:

$L' = \{0', a, b, c, 1'\}$ where $a \wedge b = a \wedge c = b \wedge c = 0'$ and $a \vee b = a \vee c = b \vee c = 1'$.(Note that L' is not distributive.)

Define $f : L \to L'$ by $f(0) = 0'$, $f(1) = 1'$, $f(x) = a$, and $f(y) = b$. Clearly f is well-defined and 1-1. It is a little tedious, but easy, to check that f satisfies the conditions of being a lattice homomorphism (namely that $f(m \vee n) = f(m) \vee f(n)$ and $f(m \wedge n) = f(m) \wedge f(n)$, for all $m, n \in L$). So f is a lattice embedding.

Let $p =\{c, 1'\}$. Much like in example 2.11, the finite setting makes it easy to see that p is an ultrafilter on L'. Notice, though, that $f^{-1}(p) =\{1\}$, which is not an ultrafilter on L since it is properly contained in the ultrafilter $\{x, 1\}$.

Though we are uncertain if the inverse of a lattice embedding would map lattice ultrafilters to lattice ultrafilters (and we suspect that, in general, it wouldn't), we can get the following somewhat weaker result:

Proposition 2.20. Suppose that $f : L \to L'$ is a lattice embedding and that p is a prime filter on L'. Then $f^{-1}(p)$ is a prime filter on L.

Proof. There are two parts:

1. $f^{-1}(p)$ is a filter:

(i). Suppose $0 \in f^{-1}(p)$. Then $f(0) \in p$. But $f(0) = 0$, a contradiction. Hence $0 \notin f^{-1}(p)$.

(ii). Suppose $a, b \in f^{-1}(p)$. Then $f(a), f(b) \in p$. So $f(a) \wedge f(b) = f(a \wedge b) \in p$. Hence $a \wedge b \in f^{-1}(p)$.

(iii). Suppose $b \in L, a \in f^{-1}(p)$, such that $b \geq a$. Since $b \geq a$, $f(b) \geq f(a)$ (lattice homomorphisms preserve order). Since $f(a) \in p$, then $f(b) \in p$. Hence $b \in f^{-1}(p)$.

2. $f^{-1}(p)$ is prime: Let $a \vee b \in f^{-1}(p)$. Then $f(a \vee b) = f(a) \vee f(b) \in p$. Since p is prime, $f(a) \in p$ or $f(b) \in p$. Hence $a \in f^{-1}(p)$ or $b \in f^{-1}(p)$. □

Though weaker, proposition 2.20 is good enough, when used with proposition 2.18, to give us a comparable construction to the Boolean algebra case.

Definition 2.21. Suppose $f : L \to L'$ is a lattice embedding and that L is normal. Define $f^D : St(L') \to St(L)$, the *Stone-dual map of f*, by $f^D(p) = q_p$, where q_p is the unique ultrafilter in L containing $f^{-1}(p)$.

If $p \in St(L')$, then p is also prime (by proposition 2.2). By proposition 2.20, $f^{-1}(p)$ is a prime filter on L. Since L is normal, then proposition 2.18 gives us that $f^{-1}(p)$ is contained in a unique ultrafilter on L. Thus f^D is well defined.

Furthermore,

Proposition 2.22. f^D is onto.

Proof. Let $q \in St(L)$. Since f is 1-1, without loss of generality we can assume that $f = id_L$ (i.e. $L \subseteq L'$). Since $q \in St(L)$, q has the f.i.p and is a subset of L' . By theorem 2.6, q can be extended to an ultrafilter p of L'. Notice that $f^{-1}(p)$ is a prime filter in L and $f^{-1}(p) \supseteq q$. Since $f^{-1}(p)$ is prime, it is contained in a unique ultrafilter q_p (proposition 2.18). So $q_p \supseteq f^{-1}(p) \supseteq q$. But q is maximal, so $q_p = q$ (i.e. $f^D(p) = q$). Hence f^D is onto. □

The following theorem also deals with the effects normality has on the set of ultrafilters of a lattice.

Proposition 2.23. Suppose L is a normal lattice with no complements, L is upwardly nonlinear, and $|L| \leq \omega_1$. Then $St(L)$ is:

(i). compact

(ii). connected

(iii). of weight $\leq \omega_1$

Proof. There are three parts:

(i). $St(L)$ is compact for any L (doesn't need to be normal).

(ii). Suppose there exists F, G that are closed subsets of $St(L)$ such that $F \cap G = \emptyset$ and $F \cup G = St(L)$. Since L is normal, then $St(L)$ is T_2. Since $St(L)$ is also compact, then $St(L)$ is normal. So there exists basic closed sets $[a]$, $[b] \subseteq St(L)$ such that $F \cap [a] = \emptyset$, $G \cap [b] = \emptyset$, and $[a] \cup [b] = St(L)$.

Since $F \cap [a] = \emptyset$ and $F \cup G = St(L)$, then $G \supseteq [a]$. But $[a] \cup [b] = St(L)$ and $G \cap [b] = \emptyset$, so $[a] \supseteq G$. Hence $[a] = G$.

Similarly, $[b] = F$. Therefore $\emptyset = [a] \cap [b] = [a \wedge b]$, which implies that $a \wedge b = 0$.

Since $St(L) = [a] \cup [b] = [a \vee b]$, and since 1 is the only element of L in every ultrafilter on L, then $a \vee b = 1$. This contradicts the assumption that L has no complements, hence L is connected.

(iii). Since $B = \{[a] : a \in L\}$ is a base for the closed sets of $St(L)$ and $|L| \leq \omega_1$, then $|B| \leq \omega_1$. Thus $w(St(L)) \leq \omega_1$. □

Note that example 2.11 is a lattice with no complemented elements, and yet it's Stone space is disconnected (because there are two distinct ultrafilters on the lattice). That is why the other assumptions are present in the statement of proposition 2.23.

Incidentally, it is true that if a lattice L has complemented elements, that is enough to guarantee that $St(L)$ is disconnected (Let $a, b \in L$ such that $a \wedge b = 0$ and $a \vee b = 1$. Thus $[a] \cap [b] = \emptyset$ and $St(L) = [1] = [a \vee b] = [a] \cup [b]$. Therefore, $[a]$ and $[b]$ create a disconnection of $St(L)$).

Chapter 3

$L(X)$

This chapter will begin by defining a lattice based on the closed sets of some topological space X. Ultimately the goal is to use this lattice, which we will call $L(X)$, to characterize $\beta X \setminus X$.

Notation 3.1. Suppose X is a topological space and F is a closed subset of X. Then we define: $[F] = \{G \subseteq X : G \textit{ closed and } Cl_X(F \triangle G) \textit{ is compact}\}$.

Note that the use of brackets to define the above equivalence class is standard, however it is also standard to use brackets to define the collection of ultrafilters containing a given element of a lattice. It should be clear in context which meaning is being used (or when both are being used).

Using the above defined equivalence classes, we can construct a lattice:

Definition 3.2. For a topological space X, let $L(X) = \{[F] : F \textit{ is closed in } X\}$, with the operations $\wedge$ and $\vee$ defined by $[F_1] \wedge [F_2] = [F_1 \cap F_2]$ and $[F_1] \vee [F_2] = [F_1 \cup F_2]$.

It may not be clear that $L(X)$ as defined is a lattice, so a partial proof is presented here:

Proposition 3.3. The operations $\wedge$ and $\vee$ defined above are well defined.

Proof. We'll just show this for $\vee$, as the proof that $\wedge$ is well defined is nearly identical.

Suppose that $[F_1] \vee [F_2] = [F]$ and $[F_1] \vee [F_2] = [G]$ (i.e. $Cl_X(F \triangle (F_1 \cup F_2))$ and $Cl_X(G \triangle (F_1 \cup F_2))$ are compact). We'd like to show that $Cl_X(F \triangle G)$ is compact. We'll start by observing that $F \triangle G = (F \setminus G) \cup (G \setminus F) \subseteq (F \setminus (F_1 \cup F_2)) \cup ((F_1 \cup F_2) \setminus F) \cup (G \setminus (F_1 \cup F_2)) \cup ((F_1 \cup F_2) \setminus G)$.

(To verify this last set containment, suppose W.L.O.G. that $x \in (F \setminus G)$. Then $x \in F$. If $x \notin (F \setminus (F_1 \cup F_2))$ then $x \in (F_1 \cup F_2)$. Since $x \notin G$, then $x \in ((F_1 \cup F_2) \setminus G)$.)

So $Cl_X((F \setminus G) \cup (G \setminus F)) \subseteq Cl_X((F \setminus (F_1 \cup F_2)) \cup ((F_1 \cup F_2) \setminus F) \cup (G \setminus (F_1 \cup F_2)) \cup ((F_1 \cup F_2) \setminus G)) \subseteq Cl_X((F \setminus (F_1 \cup F_2)) \cup ((F_1 \cup F_2) \setminus F)) \cup Cl_X((G \setminus (F_1 \cup F_2)) \cup ((F_1 \cup F_2) \setminus G)) = Cl_X(F \triangle (F_1 \cup F_2)) \cup Cl_X(G \triangle (F_1 \cup F_2))$.

Since the union of two compact sets is compact, then $Cl_X(F \triangle G)$ is a closed subset of a compact set, and therefore compact. So $[F] = [G]$. $\square$

Proposition 3.4. *$L(X)$ with the given operations is a bounded distributive lattice.*

Proof. Since the union or the intersection of two closed sets is closed, $L(X)$ is clearly closed under the operations. It is also clear that the operations are commutative, associative, and idempotent (i.e. $[F] \vee [F] = [F]$ and $[F] \wedge [F] = [F]$). Additionally, $[F] < [G]$ is equivalent to $[F] \vee [G] = [G]$ (or $[F] \wedge [G] = [F]$). The lattice is bounded, as $[X]$ and $[\emptyset]$ are the 1 and 0 elements, respectively. The following equalities prove that $L(X)$ is distributive:

$[F] \wedge ([G] \vee [H]) = [F] \wedge [G \cup H] = [F \cap (G \cup H)] = [(F \cap G) \cup (F \cap H)] = [F \cap G] \vee [F \cap H] = ([F] \wedge [G]) \vee ([F] \wedge [H])$

$[F] \vee ([G] \wedge [H]) = [F] \vee [G \cap H] = [F \cup (G \cap H)] = [(F \cup G) \cap (F \cup H)] = [F \cup G] \wedge [F \cup H] = ([F] \vee [G]) \wedge ([F] \vee [H])$ $\square$

From this point on, unless otherwise stated, we will assume that our topological spaces are locally compact. In what follows, we will show that for any space X, $\beta X \setminus X$ is homeomorphic to $St(L(X))$. We will first introduce some notation and provide proofs of some necessary results related to $\beta X \setminus X$. Then we will define the candidate for our homeomorphism, show that it's well defined, and show that it is, in fact, a homeomorphism.

Notation 3.5. Let A be a subset of X. Then the notation A^* means $(Cl_{\beta X}(A)) \setminus X$.

Lemma 3.6. $(F \cup G)^* = F^* \cup G^*$.

Proof. There are two parts:

($\supseteq$) Let $x \in F^* \cup G^*$. Then $x \in F^*$ or $x \in G^*$. Let U be an open neighborhood (in βX) of x. Then U intersects F or U intersects G. Hence $U \cap (F \cup G) \neq \emptyset$. So $x \in Cl_{\beta X}(F \cup G)$ and $x \notin X$, so $x \in (F \cup G)^*$.

($\subseteq$) Let $x \in \beta X \setminus X$ such that $x \notin (F^* \cup G^*)$ (i.e. $x \notin F^*$ and $x \notin G^*$). So there exists an open (in βX) neighborhood V_1 of x such that $V_1 \cap F = \emptyset$ and there exists an open (in βX) neighborhood V_2 of x such that $V_2 \cap G = \emptyset$. Let $U = V_1 \cap V_2$. Then U is an open neighborhood (in βX) of x such that $U \cap (F \cup G) = \emptyset$. Therefore $x \notin (F \cup G)^*$. □

Lemma 3.7. $(F \cap G)^* = F^* \cap G^*$.

Proof. There are two parts:

($\supseteq$) Let $x \in F^* \cap G^*$. Then $x \in F^*$ and $x \in G^*$. Let U be an open neighborhood (in βX) of x. Then U intersects F and U intersects G. So U intersects $F \cap G$. Hence $x \in Cl_{\beta X}(F \cap G)$ and $x \notin X$, so $x \in (F \cap G)^*$.

($\subseteq$) Let $x \in (F \cap G)^*$. Let U be an open neighborhood (in βX) of x. Then U intersects $F \cap G$, and thus U intersects F and U intersects G. Hence $x \in Cl_{\beta X}(F)$, $x \in Cl_{\beta X}(G)$, and $x \notin X$. Therefore $x \in F^* \cap G^*$. □

Proposition 3.8. $\mathcal{F} = \{F^* : F \textit{ closed in } X\}$ is a base for the closed sets of $\beta X \setminus X$.

Proof. Let C be a closed set in $\beta X \setminus X$ and let $x \in (\beta X \setminus X) \setminus C$. Since X is locally compact, C is closed in βX. Let V and W be open sets in βX such that $x \in V \cap W$, $Cl_{\beta X}(V) \subseteq W$, and $W \cap C = \emptyset$. Note that $V' = V \cap X$ is open in X.

Let $F = X \setminus V'$. Clearly F is closed in X. Since $\beta X \setminus X = X^* = (V' \cup F)^* = (V')^* \cup F^*$ and $(V')^* \cap C = \emptyset$ (because $(V')^* \subseteq (Cl_{\beta X}(V) \setminus X) \subseteq W$), then $C \subseteq F^*$. Additionally, $\beta X \setminus V$ is a closed set in βX containing $F \Rightarrow Cl_{\beta X}(F) \subseteq \beta X \setminus V \Rightarrow F^* \subseteq \beta X \setminus V$. Since $x \in V$, then $x \notin F^*$. Therefore $\mathcal{F}$ is a base for the closed sets of $\beta X \setminus X$. □

Notation 3.9. Let $\mathcal{F}_p = \{F^* : [F] \in p\}$.

Proposition 3.10. For any $p \in St(L(X))$, $|\bigcap \mathcal{F}_p| = 1$.

Proof. We'll present the proof in two parts:

1. For any $p \in St(L(X))$, $\bigcap \mathcal{F}_p \neq \emptyset$:

Let $p \in St(L(X))$. Suppose $F_1^*, F_2^*, ..., F_n^* \in \mathcal{F}_p$. Then $[F_1], [F_2], ..., [F_n] \in p \Rightarrow [F_1] \wedge [F_2] \wedge ... \wedge [F_n] = [F_1 \cap F_2 \cap ... \cap F_n] \in p$. Hence, $F_1^* \cap F_2^* \cap ... \cap F_n^* = (F_1 \cap F_2 \cap ... \cap F_n)^* \in \mathcal{F}_p$. Therefore $\mathcal{F}_p$ is a family of closed sets with the finite intersection property. Since X is locally compact, $\beta X \setminus X$ is closed (Gillman and Jerison, [5], pg. 90) and thus compact, so $\bigcap \mathcal{F}_p \neq \emptyset$.

2. For any $p \in St(L(X))$, $\bigcap \mathcal{F}_p$ contains only one point:

Let $p \in St(L(X))$. Suppose there exists $r, s \in \bigcap \mathcal{F}_p$ with $r \neq s$. Then there exists some basic closed set G^* (with G closed in X) with $r \in G^*$ and $s \notin G^*$. Then $[G] \notin p$ because $s \notin G^*$. Since p is maximal, there exists a $[F] \in p$ such that $[\emptyset] = [F] \wedge [G] = [F \cap G]$. Hence $F \cap G$ is compact. So $\emptyset = (F \cap G)^* = F^* \cap G^* \supseteq \{r\}$, a contradiction. □

We are now ready to define our homeomorphism candidate.

Definition 3.11. Let $h : St(L(X)) \to \beta X \setminus X$ be defined by $h(p) = r_p$, where r_p is the unique point in $\bigcap \mathcal{F}_p$.

By proposition 3.10, it is clear that h is well defined. That it is a homeomorphism comes from the results that follow.

Proposition 3.12. The function $h : St(L(X)) \to \beta X \setminus X$ as defined above is 1-1.

Proof. Suppose that $p \neq q$, and that $h(p) = r_p$, $h(q) = r_q$ (Show that $r_p \neq r_q$). Suppose that $r_q \in F^*$, $\forall [F] \in p$. Let $[G] \in q$. Then $r_q \in F^* \cap G^* = (F \cap G)^*$, $\forall [F] \in p$. So $F \cap G$ is closed in X, but not compact, $\forall [F] \in p$. This implies that $[\emptyset] \neq [F \cap G] = [F] \wedge [G]$, $\forall [F] \in p$. Since p is maximal, $[G] \in p$. Hence $q \subseteq p$. Since q is maximal, $q = p$, a contradiction. Therefore there exists a $[F] \in p$ such that $r_q \notin F^*$. So $r_q \neq h(p) = r_p$. So h is 1-1. □

Proposition 3.13. *h is onto.*

Proof. Let $x \in \beta X \setminus X$. Let $\mathcal{G} = \{G^* : G \textit{ is closed in } X \textit{ and } x \in G^*\}$. Let $p = \{[F] : F^* \in \mathcal{G}\}$.

Show that $p \in St(L(X))$:

1. p is a filter:

(i). Since $x \notin \emptyset = \emptyset^*$, then $\emptyset^* \notin \mathcal{G}$ and $[\emptyset] \notin p$.

(ii). Suppose $[F], [G] \in p$. So $x \in F^* \cap G^* = (F \cap G)^*$. Since $F \cap G$ is closed, $(F \cap G)^* \in \mathcal{G}$. Thus $[F \cap G] \in p$. But $[F \cap G] = [F] \wedge [G]$.

(iii). Suppose $[F] \in p$ and $[G] \in L(X)$ such that $[F] < [G]$. So $[F] = [F] \wedge [G] = [F \cap G]$. So $Cl_X(F \triangle (F \cap G))$ is compact $\Rightarrow Cl_X((F \setminus (F \cap G)) \cup ((F \cap G) \setminus F))$ is compact $\Rightarrow Cl_X((F \setminus G) \cup \emptyset)$ is compact $\Rightarrow Cl_X(F \setminus G)$ is compact $\Rightarrow (F \setminus G)^* = \emptyset$. Thus $F^* = ((F \setminus G) \cup (F \cap G))^* = (F \setminus G)^* \cup (F \cap G)^* = (F \cap G)^* = F^* \cap G^*$. Therefore $F^* \subseteq G^* \Rightarrow x \in G^* \Rightarrow [G] \in p$.

2. p is maximal:

Suppose $[G] \wedge [F] \neq [\emptyset]$, $\forall [F] \in p$ (Show $[G] \in p$). Since $[\emptyset] \neq [G] \wedge [F] = [G \cap F]$, then $\forall [F] \in p$, $G \cap F$ is not compact. Hence $\emptyset \neq (G \cap F)^* = G^* \cap F^*$, $\forall [F] \in p$. Suppose that $x \notin G$. Then there exists a basic closed set of the form J^* (with J closed in X) such that $x \in J^*$ and $J^* \cap G^* = \emptyset$ (because $\beta X \setminus X$ is regular). But $x \in J^* \Rightarrow [J] \in p$, which contradicts that $G^* \cap F^* \neq \emptyset$, $\forall [F] \in p$. Hence $x \in G^*$, i.e. $[G] \in p$. Thus p is maximal.

Therefore by 1. and 2., $p \in St(L(X))$ and $h(p) = x$, so h is onto. □

Notation 3.14. In the following proofs, $[[F]]$ will be used as notation for the set of ultrafilters on $L(X)$ that contain $[F]$ (i.e. $[[F]] = \{p \in St(L(X)) : [F] \in p\}$). Although this notation may look a bit awkward, it is consistent with the notation used in chapter 2 with ultrafilters on a lattice. Additionally, as discussed in chapter 2, the $[[F]]$'s form a base for the closed sets of $St(L(X))$.

Proposition 3.15. *h is continuous.*

Proof. Let F^* be a basic closed set in $\beta X \setminus X$.

The following will show that $h^{-1}(F^*) = [[F]]$:

($\subseteq$) Let $p \in h^{-1}(F^*)$ (so $h(p) \in F^*$) and suppose that $[F] \notin p$. Then there exists a $[G] \in p$ such that $[\emptyset] = [F] \wedge [G] = [F \cap G]$ (since p is maximal). So $Cl_X(F \cap G)$ is compact. Hence, $\emptyset = (F \cap G)^* = F^* \cap G^*$, which contradicts that $h(p) \in F^* \cap G^*$. Hence $[F] \in p$, so $p \in [[F]]$.

($\supseteq$) Let $p \in [[F]]$. Then $[F] \in p \Rightarrow h(p) \in F^*$ (by definition of h) $\Rightarrow p \in h^{-1}(F^*)$.

Therefore, $h^{-1}(F^*) = [[F]]$ and since $[[F]]$ is closed in $St(L(X))$, h is continuous. □

Proposition 3.16. *h is closed.*

Proof. Let $[[F]]$ be a basic closed set in $St(L(X))$.

The following will show that $h([[F]]) = F^*$:

($\subseteq$) Suppose $r \in h([[F]])$. Then $r = h(p)$ for some $p \in [[F]]$. So since $[F] \in p$, then $r \in F^*$.

($\supseteq$) Suppose $r \in F^*$ and $r \notin h([[F]])$. Since h is onto, $\exists q \in St(L(X))$ such that $h(q) = r$. But $r \notin h([[F]]) \Rightarrow q \notin [[F]] \Rightarrow [F] \notin q$.

However, $h(q) = r \Rightarrow r \in G^*$, $\forall [G] \in q$, and so $r \in F^* \Rightarrow \emptyset \neq F^* \cap G^* = (F \cap G)^*$, $\forall [G] \in q$. So $Cl_X(F \cap G) = F \cap G$ (since $F \cap G$ is already closed in X) is not compact, $\forall [G] \in q$. It follows that, for all $[G] \in q$, $[\emptyset] \neq [F \cap G] = [F] \wedge [G]$. Since q is maximal, then $[F] \in q$, a contradiction. Hence $r \in h([[F]])$.

Therefore, $h([[F]]) = F^*$ and since F^* is closed, h is closed. □

So by the previous four propositions, h is a homeomorphism (actually, the proposition that h is closed was unnecessary since any bijective continuous function from a compact space to a Hausdorff space is a homeomorphism). Thus we have the following result:

Big Proposition 3.17. $St(L(X)) \cong \beta X \setminus X$

Chapter 4

PROPERTIES OF $L(X)$

Here we will make various observations concerning our construction $L(X)$. Ideally, we'd like to find properties of the space X that characterize it's associated $L(X)$. For instance, from Faulkner and Vipera [3], we know that if a lattice L is pseudocomplemented, then $St(L)$ is 0-dimensional. Thus it would be useful to know when a space X would have a pseudocomplemented $L(X)$. While not a full characterization, the following will develop a condition on X that will guarantee that $L(X)$ is pseudocomplemented. First, we'll start with some notation.

Notation 4.1. The *frontier* of F will be denoted $Fr(F)$, and is defined by $Fr(F) = Cl_X(F) \cap Cl_X(X \setminus F)$.

The following lemma is used in the proof of proposition 4.3.

Lemma 4.2. Suppose F is a closed subset of X. Then $F \cap (X \setminus F^o) = Cl_X(F) \cap Cl_X(X \setminus F)$ $(= Fr(F))$.

Proof. There are two parts:

($\subseteq$) Let $x \in F \cap (X \setminus F^o)$. Since $x \in F$ and $Cl_X(F) = F$, then $x \in Cl_X(F)$. Suppose that $x \notin Cl_X(X \setminus F)$. So $\exists$ an open neighborhood U of x such that $U \cap (X \setminus F) = \emptyset$ (i.e. $U \subseteq F$). So $x \in F^o$, a contradiction. Hence $x \in Cl_X(X \setminus F)$.

($\supseteq$) Let $x \in Fr(F)$. So $x \in Cl_X(F)$ and $x \in Cl_X(X \setminus F)$. Since F is closed, $Cl_X(F) = F$, so $x \in F$. Also, $X \setminus F \subseteq X \setminus F^o$ implies that $Cl_X(X \setminus F) \subseteq Cl_X(X \setminus F^o) = X \setminus F^o$. So $x \in X \setminus F^o$. Hence $x \in F \cap (X \setminus F^o)$. □

Proposition 4.3. Suppose that $[F] \in L(X)$. $Fr(F)$ is compact $\iff$ $[X \setminus F^o]$ is the pseudocomplement of $[F]$ (i.e. $[F]' = [X \setminus F^o]$).

Proof. There are two parts:

($\Rightarrow$) Suppose that $Fr(F)$ is compact. Then $[F] \wedge [X \setminus F^o] = [F \cap (X \setminus F^o)] = [Fr(F)]$ (by lemma 4.2) $= [\emptyset]$. Let $[V] \in L(X)$ such that $[F] \wedge [V] = [\emptyset]$. Then there exists a compact K such that $(V \setminus K) \cap F = \emptyset$. So $(V \setminus K) \subseteq (X \setminus F) \subseteq (X \setminus F^o)$. So $[V] = [V \setminus K] \leq [X \setminus F^o]$. Hence $[X \setminus F^o]$ is the pseudocomplement of $[F]$.

($\Leftarrow$) Suppose that $[X \setminus F^o]$ is the pseudocomplement of $[F]$. Then $[\emptyset] = [X \setminus F^o] \wedge [F] = [(X \setminus F^o) \cap F]$. So $(X \setminus F^o) \cap F$ (which is $Fr(F)$ by lemma 4.2) is compact. □

The above results obviously lead to the following conclusion:

Corollary 4.4. If X is a space such that $Fr(F)$ is compact for all closed $F \subseteq X$, then $L(X)$ is a pseudocomplemented lattice.

Additionally, the topological condition specified in corollary 4.4 will also lead to $L(X)$ being normal, as we now show:

Proposition 4.5. If X is a space such that $Fr(F)$ is compact for all closed $F \subseteq X$, then $L(X)$ is normal.

Proof. By corollary 4.4, if we assume that $Fr(F)$ is compact for all closed $F \subseteq X$, then $L(X)$ is pseudocomplemented. In particular, for any $[F] \in L(X)$, $[F]' = [X \setminus F^o]$. Suppose that $[F], [G] \in L(X)$ such that $[F] \wedge [G] = [\emptyset]$. Let's look at $[F]'$ and $[G]'$. Suppose that $[X \setminus F^o] \vee [X \setminus G^o] \neq [X]$. Since $[X \setminus F^o] \vee [X \setminus G^o] = [(X \setminus F^o) \cup (X \setminus G^o)] = [X \setminus (F^o \cap G^o)]$, then $Cl_X(F^o \cap G^o)$ is not compact. Also, since $Cl_X(F^o \cap G^o) \subseteq Cl_X(F \cap G)$, then $Cl_X(F \cap G)$ is not compact. But $[\emptyset] = [F] \wedge [G] = [F \cap G] \Rightarrow Cl_X(F \cap G)$ is compact, a contradiction. Hence $[X \setminus F^o] \vee [X \setminus G^o] = [X]$. Additionally, $[F] \wedge [X \setminus F^o] = [\emptyset]$ and $[G] \wedge [X \setminus G^o] = [\emptyset]$ (since they are pseudocomplements), so $L(X)$ is normal. □

So based on Faulkner and Vipera [3], and propositions 2.8 and 2.10, we know that if X has the property that $Fr(F)$ is compact for all closed $F \subseteq X$, then $St(L(X))$ is compact, normal, and 0-dimensional. But chapter 3 culminated with the fact that $St(L(X))$ is really just the remainder of the Stone-Čech compactification of X. So these results combine to give a way to characterize $\beta X \setminus X$ for the aforementioned class of spaces:

Proposition 4.6. If X is a space such that $Fr(F)$ is compact for all closed $F \subseteq X$, then $\beta X \setminus X$ is compact, normal, and 0-dimensional.

At this point, we'd like to insert a remark. Stone-Čech remainders that are 0-dimensional have been studied extensively using Boolean algebras (and to a lesser extent using general lattices). Every compact, Hausdorff space that is 0-dimensional is the Stone space of some Boolean algebra (Koppelberg, [7], pg.100). However, Stone-Čech remainders that are not 0-dimensional cannot be studied in the parallel setting of Boolean algebras, since Stone's representation theorem essentially says that the Stone space of any Boolean algebra is 0-dimensional. That is the motivation for generalizing Boolean algebras to general lattices, and specifically, the advantage of our construction $L(X)$. With $L(X)$, we have a lattice whose Stone space is homeomorphic to $\beta X \setminus X$, regardless of the connectedness of the remainder. How connected, or disconnected, the remainder is seems to have to do with how many of the elements of the lattice have complements or pseudocomplements (Specifically, it seems that the greater the amount of complemented elements in $L(X)$, the more disconnected $\beta X \setminus X$ will be.). If every element of $L(X)$ has a complement (i.e. $L(X)$ is a Boolean algebra), then $\beta X \setminus X$ is 0-dimensional. Also, as we have noted, if every element of $L(X)$ has a pseudocomplement (i.e. $L(X)$ is pseudocomplemented), then again $\beta X \setminus X$ is 0-dimensional. However, from proposition 2.23, if $L(X)$ is upwardly nonlinear, normal, and no element of $L(X)$ has a complement, then $\beta X \setminus X$ is connected. Of course, there are a lot of possibilities in between $L(X)$ having no complements and $L(X)$ being pseudocomplemented. The next two propositions directly lead to corollary 4.9, which gives us a condition on X that will lead to being able to predict the disconnectedness of $\beta X \setminus X$. At the same time, the results will also provide more support to our theory that the more complemented elements in $L(X)$, the more disconnected $\beta X \setminus X$ is.

Proposition 4.7. Let X be a topological space. Assume that $C = \{F_1, F_2, ..., F_n\}$ is a finite collection of unbounded closed subsets of X such that $\bigcup C = X$ and $F_i \cap F_j$ is compact for all $i, j \leq n$, $i \neq j$. Then $L(X)$ has at least n complemented elements.

Proof. By the assumptions and the definition of $L(X)$, it follows that $[F_1] \vee [F_2] \vee ... \vee [F_n] = [X]$ and $[F_1] \wedge [F_2] \wedge ... \wedge [F_n] = [\emptyset]$.

Consider $[F_i]$ for $i \leq n$. Let $[F_i]^c = \bigvee_{j_m \neq i} [F_{j_m}]$ for $j_m \leq n$. Clearly, $[F_i] \vee [F_i]^c = [X]$. It also follows that:

$[F_i] \wedge [F_i]^c = [F_i] \wedge ([F_{j_1}] \vee [F_{j_2}] \vee ... \vee [F_{j_{n-1}}]) = ([F_i] \wedge [F_{j_1}]) \vee ([F_i] \wedge [F_{j_2}]) \vee ... \vee ([F_i] \wedge [F_{j_{n-1}}]) = 0$

Thus $[F_i]^c$ is the complement of $[F_i]$. Hence, $L(X)$ has at least n complemented elements. □

Though we're currently concerned with $L(X)$, the next proposition is true for general lattices:

Proposition 4.8. Suppose that L is a lattice and $S = \{a_1, a_2, ..., a_n\}$ is a finite collection of elements of L such that $a_1 \vee a_2 \vee ... \vee a_n = 1$ and $a_i \wedge a_j = 0$ for all $i, j \leq n$, $i \neq j$. Then $St(L)$ is disconnected with at least n components.

Proof. Since $a_i \wedge a_j = 0$ for all $i, j \leq n$, $i \neq j$, then no ultrafilter on L contains more than one element of S. Note that the element 1 is in every ultrafilter on L and that ultrafilters are prime (proposition 2.2). Now since $a_1 \vee a_2 \vee ... \vee a_n = 1$, then every ultrafilter on L must contain exactly one element of S. Thus $[a_1] \cup [a_2] \cup ... \cup [a_n] = St(L)$ and $[a_i] \cap [a_j] = \emptyset$ for all $i, j \leq n$, $i \neq j$. This demonstrates that $St(L)$ is disconnected with at least n components. □

Propositions 4.7 and 4.8 (and, of course, proposition 3.17) directly lead to the following result concerning $\beta X \setminus X$:

Corollary 4.9. Let X be a topological space. Assume that $C = \{F_1, F_2, ..., F_n\}$ is a finite collection of unbounded closed subsets of X such that $\bigcup C = X$ and $F_i \cap F_j$ is compact for all $i, j \leq n$, $i \neq j$. Then $\beta X \setminus X$ is disconnected with at least n components.

Now let's list a few other conditions that will allow us to determine that $L(X)$ is normal. To do so, we'll first mention a condition on a space X that will guarantee that the associated $L(X)$ is upwardly nonlinear.

Proposition 4.10. Suppose X is a locally compact metric space with the property that for every unbounded $U \subseteq X$, there exists a closed $F \subseteq X$ such that F is unbounded and $F \subseteq U$. Then $L(X)$ is upwardly nonlinear.

Proof. Suppose that $\exists\ [F] \in L(X)$ such that $[F] \neq [X]$ $(= 1)$ and $[F]$ is in every ultrafilter on $L(X)$. Then since $[F] \neq [X]$, $X \setminus F$ is unbounded. So by our assumption, there exists a closed unbounded $G \subseteq X$ such that $G \subseteq X \setminus F$.

Since G is unbounded, $[G] \neq 0$. So there exists an ultrafilter p such that $[G] \in p$. Also, $G \subseteq X \setminus F$ implies that $F \cap G = \emptyset$. So, $0 = [F \cap G] = [F] \wedge [G]$. It follows that $[F] \notin p$, a contradiction. Therefore, $L(X)$ is upwardly nonlinear. □

From this, combined with a fact about general lattices from chapter 2, we have the following result:

Corollary 4.11. Suppose X is a locally compact metric space. Then $St(L(X))$ is $\mathrm{T}_2 \Rightarrow L(X)$ is normal.

Proof. Since X is a locally compact metric space, then for every unbounded $U \subseteq X$, there exists a closed $F \subseteq X$ such that F is unbounded and $F \subseteq U$. By proposition 4.10, $L(X)$ is upwardly nonlinear. Since we also have that $St(L(X))$ is T_2, then by proposition 2.13, $L(X)$ is normal. □

Note that $L(X)$, by construction, is always lower complete, regardless of X. However, $L(X)$ is only upper complete if X is a space such that any arbitrary union of closed sets of X is closed. An example of such a space is **N** with the discrete topology. Many spaces, though, will not have this property. If we do have that $L(X)$

is complete, then with one additional assumption, we can guarantee that $L(X)$ is normal.

Proposition 4.12. Assume that $L(X)$ is complete and that for every element $[F] \in L(X) \setminus \{0, 1\}$, there is some other non-zero element $[G] \in L(X) \setminus \{0\}$ such that $[F] \wedge [G] = 0$. Then $L(X)$ is normal.

Proof. Suppose $[F] \wedge [G] = [\emptyset]$. Then $[[F]] \cap [[G]] = \emptyset$ and $[[F]] \cup [[G]] \subseteq St(L(X))$. For each $p_\alpha \in [[G]]$, let $[F_\alpha] \in p_\alpha$ such that $[F] \wedge [F_\alpha] = [\emptyset]$. Similarly, for each $q_\beta \in [[F]]$, let $[G_\beta] \in q_\beta$ such that $[G] \wedge [G_\beta] = [\emptyset]$.
Let $U = St(L(X)) \setminus ([[F]] \cup [[G]])$.

So for each $p \in U$, neither $[F]$ nor $[G]$ is in p. So for each $p_\gamma \in U$, there exists $[H_\gamma] \in p_\gamma$ such that $[F] \wedge [H_\gamma] = [\emptyset]$ and $[G] \wedge [H_\gamma] = [\emptyset]$ (this can be seen using the following argument: Suppose there is a $p \in U$ such that each $[H] \in p$ has non-zero infimum with either $[F]$ or $[G]$. Then $[H] \wedge ([F] \vee [G]) \neq [\emptyset]$ for all $[H] \in p$. Hence since p is maximal, $[F] \vee [G] \in p$. But ultrafilters are prime, so either $[F] \in p$ or $[G] \in p$, a contradiction.).

So since $L(X)$ is complete, denote $[A] = \bigvee_\alpha [F_\alpha]$ and $[B] = (\bigvee_\beta [G_\beta]) \vee (\bigvee_\gamma [H_\gamma])$. Then,

$[F] \wedge [A] = [F] \wedge (\bigvee_\alpha [F_\alpha]) = \bigvee_\alpha ([F] \wedge [F_\alpha]) = \bigvee_\alpha (0) = 0$

$[G] \wedge [B] = [G] \wedge ((\bigvee_\beta [G_\beta]) \vee (\bigvee_\gamma [H_\gamma])) = ([G] \wedge (\bigvee_\beta [G_\beta])) \vee ([G] \wedge (\bigvee_\gamma [H_\gamma])) = (\bigvee_\beta ([G] \wedge [G_\beta])) \vee (\bigvee_\gamma ([G] \wedge [H_\gamma])) = (\bigvee_\beta (0)) \vee (\bigvee_\gamma (0)) = 0 \vee 0 = 0$

$[A] \vee [B] = (\bigvee_\alpha [F_\alpha]) \vee (\bigvee_\beta [G_\beta]) \vee (\bigvee_\gamma [H_\gamma])$

Since every ultrafilter in $St(L(X))$ is represented by at least one element in the above expression, then $((\bigvee_\alpha [F_\alpha]) \vee (\bigvee_\beta [G_\beta]) \vee (\bigvee_\gamma [H_\gamma]))$ must be in every ultrafilter of $L(X)$. The only way that this expression representing $[A] \vee [B]$ is not equal to $[X]$ is if there exists an element $[D] \in L(X)$ such that $[D] \neq [X]$ and $St(L(X)) = [[D]]$ (i.e. [D] is in every ultrafilter). But if such a $[D]$ exists, then by assumption $\exists [E] \in L(X) \setminus \{0\}$ such that $[D] \wedge [E] = [\emptyset]$. Let p be an ultrafilter containing $[E]$. Then $[D] \notin p$, which contradicts $St(L(X)) = [[D]]$. Thus,

$$[A] \vee [B] = (\textstyle\bigvee_\alpha [F_\alpha]) \vee (\bigvee_\beta [G_\beta]) \vee (\bigvee_\gamma [H_\gamma]) = [X]$$

Hence the defined elements $[A]$ and $[B]$ witness to the $(\star)$ property. Therefore, $L(X)$ is normal. $\square$

Chapter 5

EXAMPLES OF $L(X)$

Example 5.1. $L(\omega_1)$

We use the standard notation of ω_1 to denote the sets of all countable ordinals. In order to study the structure of the lattice $L(\omega_1)$ we first need to have an understanding of the closed sets of ω_1. The following proposition shows how any two closed unbounded subsets of ω_1 must have a substantial intersection, with respect to how the equivalence classes that make up our lattice are defined.

Proposition 5.1.1. Assume $H, K \subseteq \omega_1$ such that H and K are both closed and unbounded. Then $H \cap K$ is not contained in a compact set.

Proof. Suppose H and K are closed and unbounded subsets of ω_1 such that $H \cap K$ is contained in a compact set. This implies that $H \cap K$ is bounded, so there exists some $\sigma \in \omega_1$ such that $H \cap K \subseteq \sigma + 1$.

Look at the tail $\omega_1 - (\sigma + 1) = \{\alpha \in \omega_1 : \alpha \geq \sigma + 1\}$. Let $H_1 = H \cap (\omega_1 - (\sigma + 1))$ and $K_1 = K \cap (\omega_1 - (\sigma + 1))$. By construction, $H_1 \cap K_1 = \emptyset$. Since $\omega_1 - (\sigma + 1)$ is a closed set in ω_1 (Gillman and Jerison, [5], pg. 73), then H_1 and K_1 are closed in ω_1. Since H_1 and K_1 are disjoint closed subsets of ω_1, then one of them must be bounded

(Gillman and Jerison, [5], pg. 74). This is a contradiction since $H_1 \cup (\sigma+1) \supseteq H$, $K_1 \cup (\sigma+1) \supseteq K$, and both H and K were assumed to be unbounded. □

With this result about the closed sets of ω_1, we can conclude the following about the elements of $L(\omega_1)$:

Corollary 5.1.2. For any $[F], [G] \in L(\omega_1) - \{[\emptyset]\}$, $[F] \wedge [G] \neq [\emptyset]$.

Proof. Since $[F]$ and $[G]$ are each nonzero elements of $L(\omega_1)$, then F and G are closed and noncompact. Any closed noncompact subset of ω_1 is unbounded, hence $F \cap G$ is not contained in a compact set (proposition 5.1.1). Thus $[F \cap G] \neq [\emptyset]$. But $[F \cap G] = [F] \wedge [G]$, hence $[F] \wedge [G] \neq [\emptyset]$. □

Note that a consequence of corollary 5.1.2 is that for any $[F] \in L(\omega_1) - \{[\emptyset]\}$, the pseudocomplement of $[F]$ is 0. It turns out that lattices that have this property always have trivial Stone spaces, as the following proposition of general lattices states.

Proposition 5.1.3. If L is a bounded lattice such that every non-zero element of L has 0 as it's pseudocomplement, then $\mid St(L) \mid = 1$.

Proof. Suppose there exists $p, q \in St(L)$ such that $p \neq q$. Then there exists $f \in p$ and $g \in q$ such that $f \wedge g = 0$. But $f' = 0$ by assumption, so since $g \leq f'$, then $g = 0$. This is a contradiction since 0 cannot be in an ultrafilter. Hence $\mid St(L) \mid = 1$. □

On the surface, this says that there is only one ultrafilter on the lattice $L(\omega_1)$. When used in combination with proposition 3.17, it also determines that $|\beta\omega_1 \setminus \omega_1| = 1$. In other words, the Stone-Čech compactification of ω_1 is simply it's one-point compactification. Of course, this is a well-known result, but we now have a lattice-based technique of arriving at it.

Example 5.2. $L(\mathbf{H})$

Let $\mathbf{H}$ denote the halfline $[0,\infty)$ with the usual topology. We now present a couple of facts concerning the associated lattice $L(\mathbf{H})$.

Proposition 5.2.1. $L(\mathbf{H})$ is normal.

Proof. Let $[F],[G] \in L(\mathbf{H}) \setminus \{[\emptyset],[\mathbf{H}]\}$ and $[F] \wedge [G] = [\emptyset]$ (the case where either $[F]$ or $[G]$ is $[\emptyset]$ is trivial). Thus $F \cap G$ is compact, which gives that $F \cap G$ is bounded by some $M \in \mathbf{H}$.

So look at the subset $[M,\infty)$ of $\mathbf{H}$ and denote $F_1 = F \cap [M,\infty)$ and $G_1 = G \cap [M,\infty)$. So $F_1 \cap G_1 \subseteq \{M\}$. Consider F_1^c and G_1^c (the set-theoretic complements of F and G, respectively). Each of these is a countable union of disjoint open intervals in $[M,\infty)$. Since $F_1 \cap G_1 \subseteq \{M\}$, then $F_1^c \cup G_1^c \supseteq (M,\infty)$.

Also note that neither F_1^c nor G_1^c can contain an interval of the form (a,∞) (Without loss of generality, assume $(a,\infty) \subseteq F_1^c$. Then F_1 is bounded by a, which implies that F is bounded by a. Thus since F is closed and bounded in a metric space, F is compact. Therefore $[F] = [\emptyset]$, a contradiction.). It is, however, possible that either or both F_1^c and G_1^c will contain an interval of the form $[M,b)$ (which is open in $[M,\infty)$ with respect to the subspace topology).

So we can write F_1^c and G_1^c in the following way:

$F_1^c = \bigcup_{n\in\omega} U_n$, where each U_n is of the form (a,b) (or possibly $[M,b)$) and $U_i \cap U_j = \emptyset$ for any $i,j \in \omega$.

$G_1^c = \bigcup_{m\in\omega} U_m$, where each U_m is of the form (a,b) (or possibly $[M,b)$) and $U_i \cap U_j = \emptyset$ for any $i,j \in \omega$.

Now look at $F_1^c \cap G_1^c$. It can also be written as a countable union of disjoint open intervals in $[M, \infty)$. Again, these intervals must be of the form (a, b) (or possibly $[M, b)$). When indexing the open intervals that comprise $F_1^c \cap G_1^c$, let's order the intervals in the obvious way – namely, given $m, n \in \omega$ with $V_m = (a_m, b_m)$ and $V_n = (a_n, b_n)$, then $b_m < a_n \Leftrightarrow m < n$.

So let's write $F_1^c \cap G_1^c$ in the following way:

$F_1^c \cap G_1^c = \bigcup_{n\in\omega} V_n$, where each V_n is of the form (a_n, b_n) (or possibly $V_1 = [M, b_1)$), $V_i \cap V_j = \emptyset$ for any $i, j \in \omega$, and the V_n are ordered as described above.

For each $n \in \omega$, let $p_n = (a_n + b_n)/2$. We'll use these p_n's to define the closed intervals $[M, p_1]$, $[p_1, p_2]$, $[p_2, p_3]$, etc. (Note that by the ordering of the V_n's, $\{p_n\}$ is a strictly increasing sequence.). We want to next take these closed intervals and assign each to one of two collections, $\hat{F}$ and $\hat{G}$.

First put $[M, p_1]$ in collection $\hat{F}$. Then, for each $i \geq 2$, put the interval $[p_{i-1}, p_i]$ in $\hat{F}$ provided that $[p_{i-1}, p_i] \cap F_1 = \emptyset$. Put the rest of the intervals $[p_{i-1}, p_i]$, $i \geq 2$, in collection $\hat{G}$. Notice that for each interval $[p_{i-1}, p_i] \in \hat{G}$, $[p_{i-1}, p_i] \cap G_1 = \emptyset$.

In other words, each $[p_{i-1}, p_i]$ for $i \geq 2$ will be disjoint with either F_1 or G_1. The following argument will prove this:

Suppose for some $i \geq 2$, $[p_{i-1}, p_i]$ intersects both F_1 and G_1. Then $\exists p \in F_1 \cap [p_{i-1}, p_i]$ and $\exists q \in G_1 \cap [p_{i-1}, p_i]$. We know that $p \neq q$, because M is the only point that F_1 and G_1 can possibly share and $M \notin [p_{i-1}, p_i]$. So, without loss of generality, assume that $p < q$. Since $p \in F_1$, then $p \notin F_1^c$, and thus $p \notin F_1^c \cap G_1^c$. So since V_{i-1} and V_i are subsets of $F_1^c \cap G_1^c$, then $p \notin V_{i-1}$ and $p \notin V_i$. Similarly, $q \notin F_1^c \cap G_1^c$, so $q \notin V_{i-1}$ and $q \notin V_i$. In particular, using the notation $V_{i-1} = (a_{i-1}, b_{i-1})$ and $V_i = (a_i, b_i)$, we get that p and q are strictly between b_{i-1} and a_i.

Also, $F_1^c \cup G_1^c \supseteq (M, \infty)$ implies that $p \in G_1^c$ and $q \in F_1^c$. So based on the representations of F_1^c and G_1^c as unions of disjoint open intervals, there exists some $n_p, m_q \in \omega$ such that $p \in U_{n_p}$ and $q \in U_{m_q}$. For ease of discussion, let's label the endpoints of these two intervals in the following way: $U_{n_p} = (a_p, b_p)$ and $U_{m_q} = (a_q, b_q)$. Let's analyze the three cases that could occur:

(i). If $a_q < b_p$, then $(a_p, b_p) \cap (a_q, b_q) = (a_q, b_p)$ is a non-empty subset of $F_1^c \cap G_1^c$. As we've already observed, $b_{i-1} < p < q < a_i$ and neither p nor q are in $F_1^c \cap G_1^c$. So $p < a_q < b_p < q$. However, there are no elements of $F_1^c \cap G_1^c$ between b_{i-1} and a_i (since V_{i-1} and V_i are consecutive intervals in the order). So we have a contradiction.

(ii). Suppose that $b_p < a_q$. Since (a_p, b_p) is one of the disjoint open intervals in our union representation of G_1^c, then $b_p \notin G_1^c$ (Suppose $b_p \in G_1^c$. Then b_p would be in one of the U_m from the union $G_1^c = \bigcup_{m \in \omega} U_m$, and that particular U_m would have to intersect (a_p, b_p) since $b_p \in Cl_{[M,\infty)}((a_p, b_p))$. This contradicts that the U_m are disjoint). Hence $b_p \in F_1^c$. So b_p would be in one of the U_n from the union $F_1^c = \bigcup_{n \in \omega} U_n$, and that particular U_n must intersect (a_p, b_p) (again, because $b_p \in Cl_{[M,\infty)}((a_p, b_p))$). So $\emptyset \neq U_n \cap (a_p, b_p) \subseteq F_1^c \cap G_1^c$. $U_n \cap (a_p, b_p)$ is another open interval, so call it (x, y). As in case (i), since neither p nor q are in $F_1^c \cap G_1^c$, then $p < x < y < q$. This implies that there are elements of $F_1^c \cap G_1^c$ between b_{i-1} and a_i, which is a contradiction.

(iii). Suppose that $b_p = a_q$. Since (a_p, b_p) is one of the disjoint open intervals in our union representation of G_1^c, then $b_p \notin G_1^c$ (see case (ii) for proof of this). Similarly, since (a_q, b_q) is one of the disjoint open intervals in our union representation of F_1^c, then $a_q \notin F_1^c$. But $b_p = a_q$, so $b_p \notin F_1^c \cup G_1^c$, which contradicts that $F_1^c \cup G_1^c \supseteq (M, \infty)$.

Since all three possible cases led to contradictions, our assumption that "for some $i \geq 2$, $[p_{i-1}, p_i]$ intersects both F_1 and G_1" must be false. So we have proven that each $[p_{i-1}, p_i]$ for $i \geq 2$ will be disjoint with either F_1 or G_1.

We needed the above fact so that our construction of $\hat{F}$ and $\hat{G}$ will allow us to arrive at candidates in $L(\mathbf{H})$ that will be suitable to demonstrate normality. We can now state the following facts about $\hat{F}$ and $\hat{G}$:

1. $F_1 \cap (\bigcup \hat{F}\ ) \subseteq [M, p_1]$
2. $G_1 \cap (\bigcup \hat{G}\ ) = \emptyset$
3. $(\bigcup \hat{F}\ ) \cup (\bigcup \hat{G}\ ) = [M, \infty)$

To conclude the proof, let

$A = [0, M] \cup (\bigcup \hat{F}\ )$

$B = [0, M] \cup (\bigcup \hat{G}\ )$

Thus $A \cap F \subseteq [0, p_1]$, which is compact in $\mathbf{H}$, so $[A] \wedge [F] = [\emptyset]$. Also, $B \cap G \subseteq [0, M]$, which is also compact, so $[B] \wedge [G] = [\emptyset]$. Additionally, by the construction of $\hat{F}$ and $\hat{G}$, $A \cup B = \mathbf{H}$, hence $[A] \vee [B] = [\mathbf{H}]$. Therefore, by using $[A], [B] \in L(\mathbf{H})$, we have shown that $L(\mathbf{H})$ is normal. □

Proposition 5.2.2. $L(\mathbf{H})$ is not pseudocomplemented.

Proof. Look at the closed set $C \subseteq \mathbf{H}$ given by $C = [1, 2] \cup [3, 4] \cup [5, 6] \cup ... = \bigcup [2i - 1, 2i]\ ,\ \forall i \in \omega$. Clearly C is not compact, hence $[C] \neq [\emptyset]$. Suppose that $[F]$ is the pseudocomplement of $[C]$. Then $C \cap F$ is compact. Since $C \cap F \subseteq \mathbf{H}$, then $C \cap F$ is bounded – call the upper bound $M (\in \mathbf{H})$. Let $G = F \cup [a, a+1] \cup S$, where a is some odd integer greater than M and S is defined in the following way:

For each $[2i - 1, 2i] \subseteq C$ with $i > (a + 1)/2$, we know that $2i \notin F$. So look at the sequence $((2i)_n)$ defined by $(2i)_n = 2i + (1/2n)$. Since $(2i)_n \to 2i$, then there exists an $m_{2i} \in \omega$ such that $(2i)_{m_{2i}} \notin F$. Let $S = \{(2i)_{m_{2i}} : i > (a + 1)/2\}$. Notice that for any $n \in \omega$, $2i < 2i + (1/2n) < 2i + 1$, so $S \cap C = \emptyset$.

Since S is closed, unbounded, and $S \cap F = \emptyset$, then $Cl_{\mathbf{H}}(G \setminus F) = G \setminus F$ is not compact. This implies that $[G] \neq [F]$. Since $G \supseteq F$, then $[G] > [F]$.

But by construction, $G \cap C = (F \cup [a, a+1] \cup S) \cap C = (F \cap C) \cup ([a, a+1] \cap C) \cup (S \cap C) = (F \cap C) \cup [a, a+1] \cup \emptyset = (F \cap C) \cup [a, a+1]$, which is compact. Thus $[G] \wedge [C] = [\emptyset]$, which contradicts that $[F]$ is the pseudocomplement of $[C]$. Therefore C has no pseudocomplement, and thus $L(\mathbf{H})$ is not pseudocomplemented. □

Notice that $L(\mathbf{H})$ gives us an example of a lattice that is normal without being pseudocomplemented. Not only are there elements of $L(\mathbf{H})$ that fail to have pseudocomplements, but we can say something much stronger than that about the elements of $L(\mathbf{H})$. First, we prove the following fact about closed, unbounded subsets of $\mathbf{H}$.

Proposition 5.2.3. There does not exist two closed, unbounded subsets F and G of $\mathbf{H}$ such that $F \cap G$ is compact and $Cl_{\mathbf{H}}(\mathbf{H} \setminus (F \cup G))$ is compact.

Proof. Assume there exists closed, unbounded $F, G \subseteq \mathbf{H}$ such that $F \cap G$ is compact and $Cl_{\mathbf{H}}(\mathbf{H} \setminus (F \cup G))$ is compact. In $\mathbf{H}$, compact sets are bounded. Thus there exists $M, N \in \mathbf{H}$ such that $F \cap G \subseteq [0, M)$ and $Cl_{\mathbf{H}}(\mathbf{H} \setminus (F \cup G)) \subseteq [0, N)$.

Let $x = max\{M, N\}$. Consider $F \cap [x, \infty)$ and $G \cap [x, \infty)$. Clearly $F \cap [x, \infty)$ and $G \cap [x, \infty)$ are closed in the subspace $[x, \infty)$. Since $F \cap G \subseteq [0, x)$, then $F \cap [x, \infty)$ and $G \cap [x, \infty)$ are disjoint. Since $Cl_{\mathbf{H}}(\mathbf{H} \setminus (F \cup G)) \subseteq [0, x)$, then $(F \cap [x, \infty)) \cup (G \cap [x, \infty)) = [x, \infty)$. Therefore $[x, \infty)$ is the disjoint union of closed sets $F \cap [x, \infty)$ and $G \cap [x, \infty)$, which implies that $F \cap [x, \infty)$ and $G \cap [x, \infty)$ are clopen in $[x, \infty)$.

Since $[x, \infty)$ is homeomorphic to $\mathbf{H}$, let $h : [x, \infty) \to \mathbf{H}$ be a homeomorphism. Since h is injective, $h(F \cap [x, \infty)) \cap h(G \cap [x, \infty)) = \emptyset$. Since h is surjective, $h(F \cap [x, \infty)) \cup h(G \cap [x, \infty)) = \mathbf{H}$. Homeomorhisms are closed maps, so $h(F \cap [x, \infty))$ and $h(G \cap [x, \infty))$ are closed in $\mathbf{H}$. So $\mathbf{H}$ is the disjoint union of closed sets $h(F \cap [x, \infty))$

and $h(G \cap [x, \infty))$, thus $h(F \cap [x, \infty))$ and $h(G \cap [x, \infty))$ are clopen in $\mathbf{H}$. This is a contradiction, as $\mathbf{H}$ is connected (Willard, [14], pg. 191). $\square$

Based on our definition of $L(X)$, proposition 5.2.3 gives us that the lattice $L(\mathbf{H})$ has no complemented elements. Now we go about showing that $L(\mathbf{H})$ is upwardly nonlinear.

Proposition 5.2.4. $L(\mathbf{H})$ is upwardly nonlinear.

Proof. First we must note that $\mathbf{H}$ has the property that for every unbounded $U \subseteq \mathbf{H}$, there exists a closed $F \subseteq \mathbf{H}$ such that F is unbounded and $F \subseteq U$ (One way to see this is as follows: Suppose that $U \subseteq \mathbf{H}$ and U is unbounded. Choose any $x_1 \in U$. Now consider $U \setminus [0, x_1 + 1]$, and choose $x_2 \in U \setminus [0, x_1 + 1]$. Then consider $U \setminus [0, x_2 + 1]$, and choose $x_3 \in U \setminus [0, x_2 + 1]$. Continue this process inductively on ω to create the sequence $F = \{x_n : n \in \omega\}$. By construction, $F \subseteq U$ and F is an unbounded sequence of points where successive terms are at least one unit apart. So the complement of F is a union of open intervals, implying that F is closed.). Since $\mathbf{H}$ is a locally compact metric space, we can now apply proposition 4.10 to get that $L(\mathbf{H})$ is upwardly nonlinear. $\square$

This leads to our lattice-based proof that $\beta\mathbf{H} \setminus \mathbf{H}$ is connected.

Proposition 5.2.5. $\beta\mathbf{H} \setminus \mathbf{H}$ is connected.

Proof. By propositions 5.2.1, 5.2.3, and 5.2.4, we have what is required to use proposition 2.23(ii) to conclude that $St(L(\mathbf{H}))$ is connected. With proposition 3.17, we get that $\beta\mathbf{H} \setminus \mathbf{H}$ is connected. $\square$

Example 5.3. $L(\mathbf{R})$

Let $\mathbf{R}$ denote the real line $(-\infty, \infty)$ with the usual topology. The following three results concerning the lattice $L(\mathbf{R})$ are presented without proof; their proofs are nearly identical to the corresponding proofs involving $L(\mathbf{H})$.

Proposition 5.3.1. $L(\mathbf{R})$ is normal.

Proposition 5.3.2. $L(\mathbf{R})$ is not pseudocomplemented.

Proposition 5.3.3. $L(\mathbf{R})$ is upwardly nonlinear.

However, not everything about $L(\mathbf{R})$ and $L(\mathbf{H})$ is the same. For example, whereas $L(\mathbf{H})$ has no complemented elements, $L(\mathbf{R})$ certainly does. To see this, take the subsets $F = (-\infty, 0]$ and $G = [0, \infty)$ of $\mathbf{R}$. These subsets are closed noncompact subsets of $\mathbf{R}$, they are almost disjoint, and their symmetric difference is not compact. That means that in $L(\mathbf{R})$, the corresponding elements $[F]$ and $[G]$ are distinct elements with the properties that $[F] \wedge [G] = [\emptyset]$ and $[F] \vee [G] = [\mathbf{R}]$ (since $F \cup G = \mathbf{R}$). Thus $[F]$ and $[G]$ are complements in $L(\mathbf{R})$. By corollary 4.9, we can thus conclude that $\beta\mathbf{R} \setminus \mathbf{R}$ is disconnected with at least 2 components.

So we have once again used our lattice results to provide an alternate derivation of a fact concerning the Stone-Čech remainder of a space. To see why $\beta\mathbf{R} \setminus \mathbf{R}$ is disconnected with exactly 2 components, consider that $F = (-\infty, 0]$ and $G = [0, \infty)$ are each homeomorphic to $\mathbf{H}$. So $\beta F \setminus F$ and $\beta G \setminus G$ are each connected. Let $S = \beta F \cup \beta G$. S is clearly a compact set containing $\mathbf{R}$. Now let $f \in C^\star(\mathbf{R})$. Thus $f\,|_F \in C^\star(F)$, so it has an extension $(f\,|_F)^\beta \in C^\star(\beta F)$. Similarly, $f\,|_G \in C^\star(G)$, so it has an extension $(f\,|_G)^\beta \in C^\star(\beta G)$. Let $f^S = (f\,|_F)^\beta \cup (f\,|_G)^\beta$ (i.e. $f^S(x) = (f\,|_F)^\beta(x)$ if $x \in \beta F$ and $f^S(x) = (f\,|_G)^\beta(x)$ if $x \in \beta G$). Notice that since $(f\,|_F)^\beta(0) = (f\,|_G)^\beta(0)$, f^S is both well-defined and continuous on S. Thus $f^S \in C^\star(S)$. Since f was chosen arbitrarily, then $\mathbf{R}$ is $C^\star$-embedded in S. Thus, by Porter and Woods ([9], pg. 284), S must be $\beta\mathbf{R}$. So $\beta\mathbf{R} \setminus \mathbf{R} = (\beta F \cup \beta G) \setminus \mathbf{R} = (\beta F \setminus F) \cup (\beta G \setminus G)$, which is the union of two disjoint, connected sets (so there can't be more than 2 components to $\beta\mathbf{R} \setminus \mathbf{R}$). So we can conclude:

Proposition 5.3.4. $\beta\mathbf{R} \setminus \mathbf{R}$ is disconnected with 2 components.

Example 5.4. Non-normal lattices

Back in example 2.11, we defined a special five element lattice $L = \{0, a, b, a \vee b, 1\}$ with $a \wedge b = 0$, and we determined that the lattice L was not normal. This was of interest because $St(L)$ is T_2, which showed the usefulness of the concept of upwardly nonlinear. The example was originally constructed, however, to answer a question concerning the relationship between the concepts of normal and pseudocomplemented. Namely, the question was – does either condition imply the other?

In example 5.2, we showed that the lattice $L(\mathbf{H})$ is normal but not pseudocomplemented. So we know in general that normal lattices aren't necessarily psuedocomplemented. Our five element lattice answers the opposite question. Notice that $1' = 0$, $(a \vee b)' = 0$, $a' = b$, $b' = a$, and $0' = 0$. So L is, in fact, pseudocomplemented. In chapter 2 we showed that L is not normal. Therefore, the conditions of normal and pseudocomplemented are independent in general.

For completeness sake, we present an example of a topological space whose associated $L(X)$ has as a sublattice the five element lattice from example 2.11.

Example 5.4.1. Let ω be the set of all natural numbers with the usual (discrete) topology (Note: $L(\omega) = P(\omega)/fin$). Let A, B, and C be subsets of ω where C is the set of all even natural numbers, A is the set of all evens that are divisible by 4, and B is the set of all evens that are not divisible by 4. Let $S = \{[\emptyset], [A], [B], [C], [\omega]\} \subseteq L(\omega)$.

S is clearly a sublattice of $L(\omega)$, as it is closed under $\vee$ and $\wedge$. It is easy to see that S is really just the five element lattice from example 2.11, where $0 = [\emptyset]$, $a = [A]$, $b = [B]$, $a \vee b = [C]$, and $1 = [\omega]$.

Now note the following:

Proposition 5.4.2. $L(\omega)$ is upwardly nonlinear.

Proof. Suppose that $[Q] \neq [\omega]$ and that $[Q]$ is in every ultrafilter on $L(\omega)$. Let $Q' = \omega \setminus Q$. Since ω has the discrete topology, Q' is closed. Since $[Q] \neq [\omega]$, Q' is infinite (not compact). So $[Q'] \neq [\emptyset]$. Thus there is an ultrafilter $p \in St(L(\omega))$ such that $[Q'] \in p$. Since $[Q'] \wedge [Q] = [Q' \cap Q] = [\emptyset]$, then $[Q] \notin p$, a contradiction. □

So the following conclusion can be drawn:

Corollary 5.4.3. An upwardly nonlinear lattice can have a sublattice that fails to be upwardly nonlinear.

Chapter 6

QUESTIONS

In general, there is much to learn about $L(X)$. Of particular interest is the example $L(\mathbf{H})$, as an understanding of this lattice could lead to a characterization of $\beta\mathbf{H} \setminus \mathbf{H}$ in much the same way that the Boolean algebra $P(\omega)/fin$ has been used to help characterize $\beta\omega \setminus \omega$. As we have mentioned in section 1.3, Hart and Dow [6] showed that $\beta\mathbf{H} \setminus \mathbf{H}$ is a universal continuum of weight ω_1, thus assuring the importance of finding a characterization of $\beta\mathbf{H} \setminus \mathbf{H}$.

Other lattice-oriented questions have arisen in the process of this research. One such question involves the concept of upwardly nonlinear, which we presented as definition 2.12. We have determined that if L is a lattice with the disjunction property then L is upwardly nonlinear, however we don't know whether the converse of this is true or not. We suspect that the converse is not true (i.e. that L upwardly nonlinear does not imply that L has the disjunction property). The intuition for this hypothesis is as follows:

Suppose L is upwardly nonlinear. Since 1 is the only element in every ultrafilter on L, if $a \in L$ such that $a \neq 1$, then $[a] \neq [1]$. However, if we instead suppose that L has the disjunction property, then given distinct $a, b \in L$, $[a] \neq [b]$ (so the upwardly nonlinear case can be thought of as fixing $b = 1$). This would appear to be a stronger assumption.

Another question arose in the process of arriving at definition 2.21; namely, does the inverse of a lattice embedding map lattice ultrafilters to lattice ultrafilters? Answering this question turned out to be unnecessary for what we were trying to do, but the answer may still be of interest.

Also, when considering the disconnectedness of Stone spaces of lattices, we at one point thought to consider whether the Stone space of a complete lattice would be fundamentally different from the Stone space of a complete Boolean algebra. If B is a complete Boolean algebra, then the Stone space of B is extremally disconnected. Porter and Woods [9] present a standard proof of this. The proof is heavily reliant on the presence of complements in the Boolean algebra. In a general lattice, elements would not necessarily have complements, thus it seems reasonable to wonder if the assumption of complements is necessary to arrive at the conclusion that the Stone space is extremally disconnected. If so, perhaps a complete lattice can be constructed whose Stone space is not extremally disconnected (perhaps even connected).

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APPENDICES

Appendix A

Distributivity of Example 2.11

What follows is the proof that the lattice defined in example 2.11 is distributive:

Proposition A.1. Define a lattice L in the following way:

$L = \{0, a, b, a \vee b, 1\}$ where $a \wedge b = 0$.

Then L is distributive.

Proof. For brevity and clarification, let $c = a \vee b$. Then there are two conditions that need to be satisfied for L to be distributive:

1) $z \wedge (x \vee y) = (z \wedge x) \vee (z \wedge y)$, for all $x, y, z \in L$

2) $z \vee (x \wedge y) = (z \vee x) \wedge (z \vee y)$, for all $x, y, z \in L$

We'll begin with the proof of 1):

First note that for any $x, y \in L$, $1 \wedge (x \vee y) = x \vee y = (1 \wedge x) \vee (1 \wedge y)$. Also, $0 \wedge (x \vee y) = 0 = 0 \vee 0 = (0 \wedge x) \vee (0 \wedge y)$ for all $x, y \in L$.

The following list of calculations covers the cases where a, b, and c are equal to "z":

$a \wedge (0 \vee 0) = a \wedge 0 = 0 = 0 \vee 0 = (a \wedge 0) \vee (a \wedge 0)$

$a \wedge (0 \vee a) = a \wedge a = a = 0 \vee a = (a \wedge 0) \vee (a \wedge a)$

$a \wedge (0 \vee b) = a \wedge b = 0 = 0 \vee 0 = (a \wedge 0) \vee (a \wedge b)$

$a \wedge (0 \vee c) = a \wedge c = a = 0 \vee a = (a \wedge 0) \vee (a \wedge c)$

$a \wedge (0 \vee 1) = a \wedge 1 = a = 0 \vee a = (a \wedge 0) \vee (a \wedge 1)$

$a \wedge (a \vee a) = a \wedge a = a = a \vee a = (a \wedge a) \vee (a \wedge a)$

$a \wedge (a \vee b) = a \wedge c = a = a \vee 0 = (a \wedge a) \vee (a \wedge b)$

$a \wedge (a \vee c) = a \wedge c = a = a \vee a = (a \wedge a) \vee (a \wedge c)$

$a \wedge (a \vee 1) = a \wedge 1 = a = a \vee a = (a \wedge a) \vee (a \wedge 1)$

$a \wedge (b \vee b) = a \wedge b = 0 = 0 \vee 0 = (a \wedge b) \vee (a \wedge b)$

$a \wedge (b \vee c) = a \wedge c = a = 0 \vee a = (a \wedge b) \vee (a \wedge c)$

$a \wedge (b \vee 1) = a \wedge 1 = a = 0 \vee a = (a \wedge b) \vee (a \wedge 1)$

$a \wedge (c \vee c) = a \wedge c = a = a \vee a = (a \wedge c) \vee (a \wedge c)$

$a \wedge (c \vee 1) = a \wedge 1 = a = a \vee a = (a \wedge c) \vee (a \wedge 1)$

$a \wedge (1 \vee 1) = a \wedge 1 = a = a \vee a = (a \wedge 1) \vee (a \wedge 1)$

$b \wedge (0 \vee 0) = b \wedge 0 = 0 = 0 \vee 0 = (b \wedge 0) \vee (b \wedge 0)$

$b \wedge (0 \vee a) = b \wedge a = 0 = 0 \vee 0 = (b \wedge 0) \vee (b \wedge a)$

$b \wedge (0 \vee b) = b \wedge b = b = 0 \vee b = (b \wedge 0) \vee (b \wedge b)$

$b \wedge (0 \vee c) = b \wedge c = b = 0 \vee b = (b \wedge 0) \vee (b \wedge c)$

$b \wedge (0 \vee 1) = b \wedge 1 = b = 0 \vee b = (b \wedge 0) \vee (b \wedge 1)$

$b \wedge (a \vee a) = b \wedge a = 0 = 0 \vee 0 = (b \wedge a) \vee (b \wedge a)$

$b \wedge (a \vee b) = b \wedge c = b = 0 \vee b = (b \wedge a) \vee (b \wedge b)$

$b \wedge (a \vee c) = b \wedge c = b = 0 \vee b = (b \wedge a) \vee (b \wedge c)$

$b \wedge (a \vee 1) = b \wedge 1 = b = 0 \vee b = (b \wedge a) \vee (b \wedge 1)$

$b \wedge (b \vee b) = b \wedge b = b = b \vee b = (b \wedge b) \vee (b \wedge b)$

$b \wedge (b \vee c) = b \wedge c = b = b \vee b = (b \wedge b) \vee (b \wedge c)$

$b \wedge (b \vee 1) = b \wedge 1 = b = b \vee b = (b \wedge b) \vee (b \wedge 1)$

$b \wedge (c \vee c) = b \wedge c = b = b \vee b = (b \wedge c) \vee (b \wedge c)$

$$b \wedge (c \vee 1) = b \wedge 1 = b = b \vee b = (b \wedge c) \vee (b \wedge 1)$$
$$b \wedge (1 \vee 1) = b \wedge 1 = b = b \vee b = (b \wedge 1) \vee (b \wedge 1)$$

$$c \wedge (0 \vee 0) = c \wedge 0 = 0 = 0 \vee 0 = (c \wedge 0) \vee (c \wedge 0)$$
$$c \wedge (0 \vee a) = c \wedge a = a = 0 \vee a = (c \wedge 0) \vee (c \wedge a)$$
$$c \wedge (0 \vee b) = c \wedge b = b = 0 \vee b = (c \wedge 0) \vee (c \wedge b)$$
$$c \wedge (0 \vee c) = c \wedge c = c = 0 \vee c = (c \wedge 0) \vee (c \wedge c)$$
$$c \wedge (0 \vee 1) = c \wedge 1 = c = 0 \vee c = (c \wedge 0) \vee (c \wedge 1)$$
$$c \wedge (a \vee a) = c \wedge a = a = a \vee a = (c \wedge a) \vee (c \wedge a)$$
$$c \wedge (a \vee b) = c \wedge c = c = a \vee b = (c \wedge a) \vee (c \wedge b)$$
$$c \wedge (a \vee c) = c \wedge c = c = a \vee c = (c \wedge a) \vee (c \wedge c)$$
$$c \wedge (a \vee 1) = c \wedge 1 = c = a \vee c = (c \wedge a) \vee (c \wedge 1)$$
$$c \wedge (b \vee b) = c \wedge b = b = b \vee b = (c \wedge b) \vee (c \wedge b)$$
$$c \wedge (b \vee c) = c \wedge c = c = b \vee c = (c \wedge b) \vee (c \wedge c)$$
$$c \wedge (b \vee 1) = c \wedge 1 = c = b \vee c = (c \wedge b) \vee (c \wedge 1)$$
$$c \wedge (c \vee c) = c \wedge c = c = c \vee c = (c \wedge c) \vee (c \wedge c)$$
$$c \wedge (c \vee 1) = c \wedge 1 = c = c \vee c = (c \wedge c) \vee (c \wedge 1)$$
$$c \wedge (1 \vee 1) = c \wedge 1 = c = c \vee c = (c \wedge 1) \vee (c \wedge 1)$$

Now we present the proof of 2):

First note that for any $x, y \in L$, $1 \vee (x \wedge y) = 1 = 1 \wedge 1 = (1 \vee x) \wedge (1 \vee y)$. Also, $0 \vee (x \wedge y) = x \wedge y = (0 \vee x) \wedge (0 \vee y)$ for all $x, y \in L$.

The following list of calculations covers the cases where a, b, and c are equal to "z":

$$a \vee (0 \wedge 0) = a \vee 0 = a = a \wedge a = (a \vee 0) \wedge (a \vee 0)$$
$$a \vee (0 \wedge a) = a \vee 0 = a = a \wedge a = (a \vee 0) \wedge (a \vee a)$$
$$a \vee (0 \wedge b) = a \vee 0 = a = a \wedge c = (a \vee 0) \wedge (a \vee b)$$

$$a \vee (0 \wedge c) = a \vee 0 = a = a \wedge c = (a \vee 0) \wedge (a \vee c)$$
$$a \vee (0 \wedge 1) = a \vee 0 = a = a \wedge 1 = (a \vee 0) \wedge (a \vee 1)$$
$$a \vee (a \wedge a) = a \vee a = a = a \wedge a = (a \vee a) \wedge (a \vee a)$$
$$a \vee (a \wedge b) = a \vee 0 = a = a \wedge c = (a \vee a) \wedge (a \vee b)$$
$$a \vee (a \wedge c) = a \vee a = a = a \wedge c = (a \vee a) \wedge (a \vee c)$$
$$a \vee (a \wedge 1) = a \vee a = a = a \wedge 1 = (a \vee a) \wedge (a \vee 1)$$
$$a \vee (b \wedge b) = a \vee b = c = c \wedge c = (a \vee b) \wedge (a \vee b)$$
$$a \vee (b \wedge c) = a \vee b = c = c \wedge c = (a \vee b) \wedge (a \vee c)$$
$$a \vee (b \wedge 1) = a \vee b = c = c \wedge 1 = (a \vee b) \wedge (a \vee 1)$$
$$a \vee (c \wedge c) = a \vee c = c = c \wedge c = (a \vee c) \wedge (a \vee c)$$
$$a \vee (c \wedge 1) = a \vee c = c = c \wedge 1 = (a \vee c) \wedge (a \vee 1)$$
$$a \vee (1 \wedge 1) = a \vee 1 = 1 = 1 \wedge 1 = (a \vee 1) \wedge (a \vee 1)$$

$$b \vee (0 \wedge 0) = b \vee 0 = b = b \wedge b = (b \vee 0) \wedge (b \vee 0)$$
$$b \vee (0 \wedge a) = b \vee 0 = b = b \wedge c = (b \vee 0) \wedge (b \vee a)$$
$$b \vee (0 \wedge b) = b \vee 0 = b = b \wedge b = (b \vee 0) \wedge (b \vee b)$$
$$b \vee (0 \wedge c) = b \vee 0 = b = b \wedge c = (b \vee 0) \wedge (b \vee c)$$
$$b \vee (0 \wedge 1) = b \vee 0 = b = b \wedge 1 = (b \vee 0) \wedge (b \vee 1)$$
$$b \vee (a \wedge a) = b \vee a = c = c \wedge c = (b \vee a) \wedge (b \vee a)$$
$$b \vee (a \wedge b) = b \vee 0 = b = c \wedge b = (b \vee a) \wedge (b \vee b)$$
$$b \vee (a \wedge c) = b \vee a = c = c \wedge c = (b \vee a) \wedge (b \vee c)$$
$$b \vee (a \wedge 1) = b \vee a = c = c \wedge 1 = (b \vee a) \wedge (b \vee 1)$$
$$b \vee (b \wedge b) = b \vee b = b = b \wedge b = (b \vee b) \wedge (b \vee b)$$
$$b \vee (b \wedge c) = b \vee b = b = b \wedge c = (b \vee b) \wedge (b \vee c)$$
$$b \vee (b \wedge 1) = b \vee b = b = b \wedge 1 = (b \vee b) \wedge (b \vee 1)$$
$$b \vee (c \wedge c) = b \vee c = c = c \wedge c = (b \vee c) \wedge (b \vee c)$$
$$b \vee (c \wedge 1) = b \vee c = c = c \wedge 1 = (b \vee c) \wedge (b \vee 1)$$

$b \vee (1 \wedge 1) = b \vee 1 = 1 = 1 \wedge 1 = (b \vee 1) \wedge (b \vee 1)$

$c \vee (0 \wedge 0) = c \vee 0 = c = c \wedge c = (c \vee 0) \wedge (c \vee 0)$

$c \vee (0 \wedge a) = c \vee 0 = c = c \wedge c = (c \vee 0) \wedge (c \vee a)$

$c \vee (0 \wedge b) = c \vee 0 = c = c \wedge c = (c \vee 0) \wedge (c \vee b)$

$c \vee (0 \wedge c) = c \vee 0 = c = c \wedge c = (c \vee 0) \wedge (c \vee c)$

$c \vee (0 \wedge 1) = c \vee 0 = c = c \wedge 1 = (c \vee 0) \wedge (c \vee 1)$

$c \vee (a \wedge a) = c \vee a = c = c \wedge c = (c \vee a) \wedge (c \vee a)$

$c \vee (a \wedge b) = c \vee 0 = c = c \wedge c = (c \vee a) \wedge (c \vee b)$

$c \vee (a \wedge c) = c \vee a = c = c \wedge c = (c \vee a) \wedge (c \vee c)$

$c \vee (a \wedge 1) = c \vee a = c = c \wedge 1 = (c \vee a) \wedge (c \vee 1)$

$c \vee (b \wedge b) = c \vee b = c = c \wedge c = (c \vee b) \wedge (c \vee b)$

$c \vee (b \wedge c) = c \vee b = c = c \wedge c = (c \vee b) \wedge (c \vee c)$

$c \vee (b \wedge 1) = c \vee b = c = c \wedge 1 = (c \vee b) \wedge (c \vee 1)$

$c \vee (c \wedge c) = c \vee c = c = c \wedge c = (c \vee c) \wedge (c \vee c)$

$c \vee (c \wedge 1) = c \vee c = c = c \wedge 1 = (c \vee c) \wedge (c \vee 1)$

$c \vee (1 \wedge 1) = c \vee 1 = 1 = 1 \wedge 1 = (c \vee 1) \wedge (c \vee 1)$

Since conditions 1) and 2) are satisfied, L is distributive. □

Appendix B

Illustrations of Examples

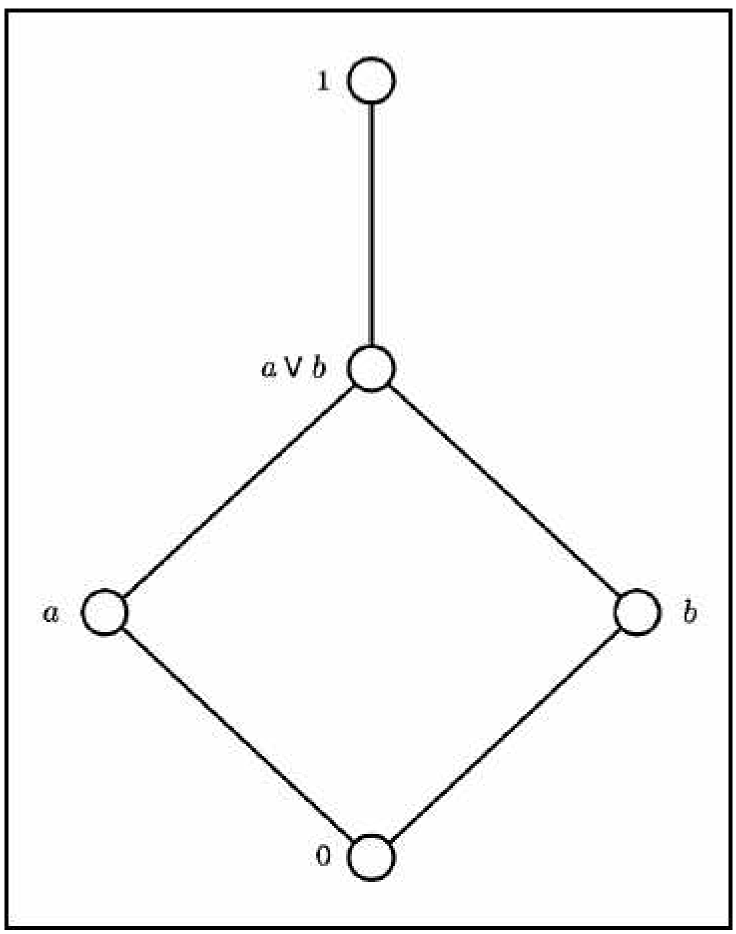

Figure B.1. *Lattice from example 2.11*

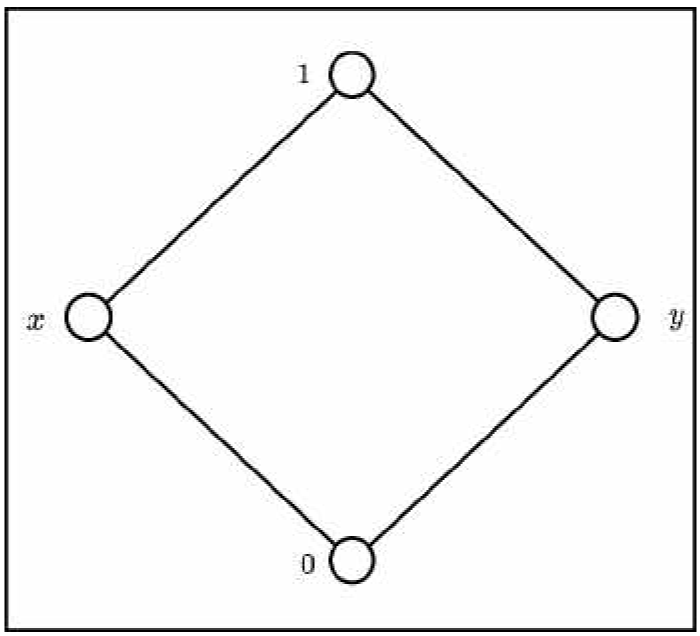

Figure B.2. *Lattice from example 2.19*