

Abstract

Mawhinney, Katherine Joyner. Characteristics of Complexity within the Lattice of Compactifications (Under the direction of Dr. Gary D. Faulkner)

The purpose of this research is to determine some topological characteristics that may be used to classify a Hausdorff compactification of a topological space as a complex compactification, within the lattice of compactifications. The Stone-Čech compactification is the supremum of the lattice, the Alexandroff one-point compactification the infimum. We look to characteristics that the Stone-Čech compactification holds and whether or not those properties are found in compactifications “close” to it. The idea of a complex compactification has not been strictly defined and there are numerous properties that could be used in a definition.

Beginning with mappings with a finite number of nontrivial fibers, we find that F-space is invariant. F-space can not be guaranteed for all finite-to-one mappings. The characteristic we call G-int is invariant under any finite-to-one, irreducible mapping and the continuous image of a nowhere F space is nowhere F, a characteristic of compactifications that are simple. We also consider the mappings on $\beta\mathbb{N}$ that are simple mappings, proving that if a simple mapping is finite-to-one, then so is its generator and vice versa.

CHARACTERISTICS OF COMPLEXITY WITHIN THE
LATTICE OF COMPACTIFICATIONS

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Biography

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Chapter 1

Introduction

1.1 The Stone-Čech Compactification

A Hausdorff compactification of a topological space X is defined to be the pairing (Y, e) of a compact, Hausdorff space Y with a mapping e , that embeds X into Y , so that the image of X in Y is dense. A topological space will have a compactification if and only if the space is Tychonoff. Thus, all spaces considered in this paper are Tychonoff as well as Hausdorff. Usually there is more than one type of compactification of a noncompact topological space. The simplest compactification of a space X is the Alexandroff one-point compactification, which we will denote ΩX . Since we are interested in those compactifications that are Hausdorff, in order to guarantee ΩX Hausdorff, X must be locally compact. Assuming our spaces to be locally compact would also provide Tychonoff.

A more complex and interesting compactification is the Stone-Čech compactification of a topological space X , denoted βX . βX is a well-known and widely studied compactification. Many characteristics of βX have been investigated, some more interesting than others, and some of which are determined by the structure of X . However, there are characteristics of the Stone-Čech compactification that hold true in every space X . For example, if a space X is the continuous, pre-image of any compact space K , there exists a continuous extension of that function from βX to

K , equal to the original function when restricted to X . In fact, any compactification of X is a continuous image of βX under a mapping that leaves points of X fixed. The Stone-Čech compactification of a space is also the compactification to which all bounded, continuous, real-valued functions on the space extend. We label this characteristic of βX using the ring of bounded, real-valued continuous functions on X , saying that the space X is C^* -embedded in βX . It is known that this property is unique to βX . Thus, when X is C^* -embedded in a compactification K , K is (homeomorphic to) the Stone-Čech compactification of X .

There is more than one construction of βX . We begin with the classic construction involving the embedding of a topological space into a product of closed intervals. If we let I_f be the range of every bounded, real-valued, continuous function on X , we can embed X into the product of the intervals I_f . βX is the closure of the copy of X embedded in the product. It is also known that βX is homeomorphic to the set of all maximal ideals of the ring of bounded, continuous, real-valued functions on X , $C^*(X)$, when the set of ideals is given the appropriate topology which is determined by the ring structure of $C^*(X)$. Moreover, the set of all maximal ideals of $C^*(X)$ with this topology is homeomorphic to the set of all z -ultrafilters of X , again with a natural topology. So, βX can be realized from the set of all z -ultrafilters on X . The way that we will relate to βX for this work is the well known set-theoretic construction of βX as the set of all z -ultrafilters on X .

1.2 The Lattice of Hausdorff Compactifications

The collection $K(X)$ of all Hausdorff compactifications of a locally compact space X , forms a complete lattice. We here forth use the notation $K(X)$ to represent not just the collection of compactifications, but the lattice itself with the partial ordering \leq on the lattice described as follows. For any two compactifications αX and γX , $\alpha X \leq \gamma X$ if and only if there exists a continuous mapping from γX onto αX that leaves points of X fixed. Any two compactifications are considered equivalent if and

only if there exists a homeomorphism from one onto the other that leaves points of X fixed. For this work we will not distinguish between equivalent compactifications.

βX is the supremum of the lattice and ΩX is the infimum. Though these extremes of the lattice are known, there is still a lot that is not known about the entire structure. A lattice of compactifications can be quite large. It is known that the lattice is of cardinality at most 2^{2^κ} , where κ denotes the cardinality of X . Further, if X is an infinite, discrete space the cardinality of $K(X)$ is exactly 2^{2^κ} .

We want to be able to look at a compactification of a space X and classify that compactification as either simple or complex, with the hope that our labeling reflects the position of the compactification within the lattice. A complex compactification we would expect to find “closer” in the lattice to the Stone-Čech compactification than the one-point compactification, simple compactifications the opposite. Being complex or “close” to βX also means sharing some of its characteristics. So we look for properties that are found in the most complex compactification βX for indications of complexity. Since βX is the largest and most complex compactification within the lattice, the topological characteristics held by βX are the best candidates for traits of a complex compactification. We know that there is a collection of compactifications called the *dual points* of a lattice, that are lesser only to βX under the partial ordering. Beginning in βX and following a characteristic by mapping into a dual point is a preliminary test for a characteristic of complexity.

We begin by looking at mappings from one topological space onto another, that may cause little change in the structure of the space. Focusing on one group of characteristics, we determine which of them are invariant under our mappings. As previously stated, the mappings used within the lattice of compactifications of a space X keep points of X fixed. Thus, when we apply these mappings to a compactification we can focus on changes made to the remainder of a compactification γX , denoted $\gamma X - X$. The remainder of the Stone-Čech compactification of a space X , which is $\beta X - X$, we denote with the usual symbol X^* . This remainder is our focus for finding characteristics of complexity.

Wanting to start with the "simplest" mapping, involves some choosing. We cannot use perfect mappings, the obvious generalization of homeomorphism, since all of the mappings have compact domain, all are perfect. We choose to begin questioning the invariance of certain characteristics under finite-to-one mappings. Even this involves us in consistency issues. Though we may not explicitly note it, some of our results are only meaningful under \neg CH. Even the nature of the finite-to-one mapping in this case, depends on set theory.

1.3 $\beta\mathbb{N}$: The Motivation

One of the most famous Stone-Čech compactifications is $\beta\mathbb{N}$, the Stone-Čech compactification of the set of natural numbers. It intrigues many mathematicians, because the compactification creates a space that is extremely complex, from a space that is very simple. We know that $\beta\mathbb{N}$ has cardinality $2^{\mathfrak{c}}$, as well as a clopen base of size \mathfrak{c} , much larger than the size and weight of \mathbb{N} . As mentioned before, the construction of $\beta\mathbb{N}$ we will employ is that of z -ultrafilters on \mathbb{N} . Since \mathbb{N} is discrete, the z -ultrafilters are just the ultrafilters on \mathbb{N} , so that points of $\beta\mathbb{N}$ are considered ultrafilters of subsets of \mathbb{N} . The points of \mathbb{N} within the compactification correspond to the fixed or principle ultrafilters on \mathbb{N} . Points in the remainder \mathbb{N}^* correspond to free ultrafilters on \mathbb{N} .

An intriguing fact about $\beta\mathbb{N}$ is that there exist numerous copies of $\beta\mathbb{N}$ within itself. They are found within the most interesting piece of $\beta\mathbb{N}$, its remainder. It is the remainder that provides $\beta\mathbb{N}$ its enormous size. Since \mathbb{N} is countable, the cardinality of \mathbb{N}^* is $2^{\mathfrak{c}}$. Plus, it has a clopen base of size \mathfrak{c} , which is in fact the weight of \mathbb{N}^* . It is within each of those basic clopen sets of \mathbb{N}^* that a subset homeomorphic to $\beta\mathbb{N}$ is found.

Since this work will be applied to $\beta\mathbb{N}$, and more specifically to $K(\mathbb{N})$, we begin with a small set of topological characteristics of $\beta\mathbb{N}$. \mathbb{N}^* is one example of a Parovičenko space, though Parovičenko's theorem states that under the Continuum Hypothesis (CH) it is the only Parovičenko space up to homeomorphism. We start by focusing

on the characteristics of a Parovičenko space, testing their invariance under various mappings, and determining whether or not they may be used in a description of a compactification that is complex.

Chapter 2

Background

2.1 Notation and Cardinal Functions

We begin by defining notation that will be seen throughout this paper. As usual \mathbb{N} denotes the set of natural numbers, \mathbb{Q} the set of rational numbers, and \mathbb{R} the set of all real numbers. \mathbb{I} will denote the closed unit interval. Upper case letters such as X or Y will denote sets. Lower case letters x or y will denote elements of sets. A set X is *transitive* if every element of X is also a subset of X . A set X is also said to be *well-ordered* if there is a total ordering on the set in which every nonempty subset of X has a smallest element.

Definition 2.1.1. *A set X is an ordinal if and only if X is transitive and well-ordered by \in .*

Two sets are said to be equipotent if there exists a bijection from one onto the other. The ordinals are the order types of the well ordered sets, so every well ordered set is equipotent to some ordinal. We use the previous definition to define a cardinal, denoting cardinals with lower case Greek letters such as α or δ .

Definition 2.1.2. *An ordinal α is said to be a cardinal if and only if it is not equipotent to any smaller ordinal.*

For a set Y the cardinality of Y , written $|Y|$, is the smallest cardinal δ with which there exists a surjection from δ onto Y . By the well ordering theorem (axiom of choice) such a cardinal exists. The cardinality of \mathbb{N} is denoted by the cardinal ω . So we write $|\mathbb{N}| = \omega$. A set of cardinality ω is said to be *countable*. ω_1 will denote the first uncountable cardinal. It is customary for \mathfrak{c} or 2^ω to denote the cardinality of the \mathbb{R} . A mapping f from a set S to a set P maps S *cofinally* into P if and only if $\{f(x) : x \in S\}$ is unbounded in P .

Definition 2.1.3. *If κ is a cardinal, then the cofinality of κ , denoted $cf(\kappa)$, is the smallest cardinal which will map cofinally into κ .*

Definition 2.1.4. *For a set X , $[X]^\omega$ denotes the set of countable subsets of X . Similarly $[X]^{<\omega}$ denotes the set of finite subsets of X .*

Definition 2.1.5. *The **Continuum Hypothesis** (CH) is the statement $\omega_1 = \mathfrak{c}$.*

It is known that CH is independent of the Zermelo-Frankel set theory with the Axiom of Choice (ZFC). That is, both CH and \neg CH are consistent with ZFC. \mathbb{N}^* , the topological space in which we are most interested, is very different under the assumption of CH than it is under \neg CH [vM84].

Any collection of non-empty, disjoint open subsets of a space X is said to be a *cellular family* of X . A *cardinal function* is loosely a mapping h from the class of all topological spaces to the class of cardinals that satisfies, if X and Y are homeomorphic topological spaces, then $h(X) = h(Y)$. The following are the cardinal functions necessary for this work.

2.1.1 Cardinal Functions

For a given topological space X

1. The weight of X , $w(X)$ is given by:

$$w(X) = \min\{\alpha : |B| = \alpha \text{ and } B \text{ is a base for } X\} + \omega$$

2. The density of X , $d(X)$ is given by:

$$d(X) = \min\{\gamma : |D| = \gamma \text{ and } D \text{ is dense in } X\} + \omega$$

3. The cellularity of X , $c(X)$ is given by:

$$c(X) = \sup\{\delta : |F| = \delta \text{ and } F \text{ is a cellular family of } X\} + \omega$$

4. The character of p in X , $\chi(p, X)$ is given by:

$$\chi(p, X) = \min\{\eta : |B| = \eta \text{ and } B \text{ is a neighborhood base of } p\} + \omega$$

5. The character of X , $\chi(X)$ is given by:

$$\chi(X) = \sup\{\chi(p, X) : p \in X\} + \omega$$

Since we will focus on \mathbb{N} in future examples, it is important that we also mention several cardinal numbers associated with it. To do this we must define a quasi-ordering \subset^* on the power set of \mathbb{N} , $P(\mathbb{N})$, by $A \subset^* B$ if and only if $|A - B| < \omega$. For a family of sets $\mathcal{A} \subseteq P(\mathbb{N})$ we say that B is a *pseudointersection* of \mathcal{A} if and only if $B \subset^* A$, for every $A \in \mathcal{A}$. A *tower*, T is a subset of $[\mathbb{N}]^\omega$ that is well-ordered by \subset^* and has no infinite pseudointersection. $\mathcal{S} \subseteq P(\mathbb{N})$ is called a *splitting family* if for any $A \in [\mathbb{N}]^\omega$ there is an $S \in \mathcal{S}$ such that $|A \cap S| = |A - S| = \omega$.

2.1.2 Cardinal Numbers

1. $\mathfrak{s} = \min\{|S| : S \text{ is a splitting family in } [\mathbb{N}]^\omega\}$
2. $\mathfrak{t} = \min\{|T| : T \text{ is a tower}\}$

Readers interested in set theory are referred to [K80] and for more background on cardinal functions see [H84] and [J84].

2.2 General Topology

We will assume the reader has a basic knowledge of topology, but will begin with a few preliminaries. A topological space X is said to be *locally compact* if for every $x \in X$ there exists a compact subset of X that contains a neighborhood of x . The space X is called *Tychonoff* or *completely regular* if it is T_1 and if for any $x \in X$ and any closed set $U \subset X$ that misses x , there exists a continuous function $f : X \rightarrow I$ such that $f(x) = 0$ and $f(y) = 1$ for every $y \in U$. All spaces considered in this paper are Hausdorff and locally compact. This implies our spaces are Tychonoff.

Proposition 2.2.1. *Locally compact spaces are Tychonoff.*[E89]

Proof. Let X be a locally compact space. Let $x \in X$ and U be a closed subset of X that misses x . Since X is locally compact we have a neighborhood V of x such that \bar{V} compact and hence normal. Then the set $S = (\bar{V} - V) \cup (\bar{V} \cap U)$ is a closed subset of \bar{V} that does not contain x . So by Urysohn's lemma there exists a function $f : \bar{V} \rightarrow \mathbb{I}$ such that $f(x) = 0$ and $f(S) \subset \{1\}$. Extend f to $(X - V)$ defining $f(y) = 1$, for all $y \in (X - V)$. It is easy to see that f is continuous and thus $f : X \rightarrow \mathbb{I}$ is the desired function. \square

A subset Z of X is a *zero-set* of X if there is a continuous, real-valued function f of X with $Z = \{x \in X : f(x) = 0\}$. We write the ring of all real-valued, continuous functions on a topological space X as standard, $C(X)$. The subset consisting of those functions that are bounded is denoted $C^*(X)$.

Proposition 2.2.2. *A space X is Tychonoff if and only if the collection of all the zero-sets of the space, denoted $Z(X)$, is a base for the closed sets.* [GJ61]

Proof. First assume X is Tychonoff. So for any closed set $C \subset X$ and point $x \notin C$, there exists an $f \in C(X)$ such that $f(x) = 1$ and $f(C) = \{0\}$. Then C is contained in the zero set of f and x is not. Thus, the sets of $Z(X)$ form a base for the closed sets of X .

Now suppose $Z(X)$ forms a base. So let C be a closed set in X and $x \notin C$. Then there is a zero-set from $Z(X)$, call it $Z(f)$, such that $C \subset Z(f)$ and $x \notin Z(f)$. So, $f(x) = r$, $r \neq 0$. Let \mathbf{r} denote the constant function of value r . Then $h = f\mathbf{r}^{-1}$ (where $\mathbf{r}^{-1} = \frac{1}{r}$) is a function from $C(X)$ with $h(x) = 1$ and $h(C) = \{0\}$. \square

Tychonoff spaces are of great importance, because they are precisely the subspaces of compact spaces. Hence they are the spaces which have compactifications. In particular, they have a Stone-Čech compactification. We will address this compactification further in coming sections. We continue now with some necessary definitions.

If, in X , a singleton $\{x\}$ is an open set, then x is called an *isolated point* of X . A topological space X is called *totally disconnected* if and only if the only nonempty, connected subsets of the space are singletons. Since our spaces are Hausdorff and locally compact, if X is totally disconnected, then X is also *zero-dimensional*. This is equivalent to saying that X has a base of open-closed, or clopen sets. A space X is *Lindelöf* if every open cover of the space has a countable subcover. So certainly compact spaces are Lindelöf. A Lindelöf and zero-dimensional space is *strongly zero-dimensional*. Tychonoff X is *strongly zero-dimensional* if and only if for every pair of completely separated sets A, B there exists a clopen U such that $A \subset U \subset (X - B)$. In other words, completely separated sets can be separated by disjoint clopen sets.

Within a topological space, a subset of the space is called an F_σ if it is a countable union of closed sets. A G_δ set of a space is a subset that is a countable intersection of open sets. Zero-sets of a space are examples of G_δ sets. Certainly, the complement of an F_σ is a G_δ and vice versa. Further, the inverse image of an F_σ and a G_δ under a continuous mapping is again an F_σ and a G_δ , respectively.

Definition 2.2.3. *A space X is an F-space if and only if for any pair of disjoint open F_σ sets F_1, F_2 in X , $\overline{F_1} \cap \overline{F_2} = \emptyset$.*

The extreme opposite of being an F-space is to be *nowhere F*. A space is nowhere F if and only if it does not contain any subspaces that are F-spaces.

Next we define the pair (Y, e) to be a Hausdorff *compactification* of the space X if and only if Y is a compact, Hausdorff space and e is a mapping which embeds X into Y , so that the image of X , $e(X)$ is dense in Y . The simplest compactification is known as the Alexandroff or one-point compactification. For a space X we will denote this compactification with ΩX .

Definition 2.2.4. *Let X be a locally compact, noncompact Hausdorff space. Define the space $\Omega X = X \cup \{y\}$, where y is not a point of X . Define the open sets of ΩX to be the collection of open sets of X and the sets $\{p\} \cup (X - L)$, with L a compact set in X . ΩX is the Alexandroff compactification of X .*

Two references for more information about these and other concepts in general topology are [E89] and [Wi70].

2.3 The Stone -Čech Compactification

The very well known Hausdorff compactification of a topological space X is the Stone-Čech compactification, written βX . Mathematicians Stone and Čech independently proved the existence of this compactification. Stone employed Boolean algebra techniques to construct the compactification, while Čech's characterization involved the embedding of X into a compact space. Another construction of βX using zero-sets, can be found in [GJ61]. Though Čech's work is an extension of compactification work completed by A.N. Tychonoff, it is Čech's name that is associated with βX . In βX , X is embedded so that every bounded, real-valued continuous function on X will continuously extend to βX . For any $f \in C^*(X)$ we denote the extension of f to βX with βf . We say that X is C^* -embedded in βX . βX is unique in this, so that any other compactification in which X is C^* -embedded, is equivalent to βX .

Theorem 2.3.1. *(Stone-Čech) Every completely regular space X has a Hausdorff compactification βX in which it is C^* -embedded.*

Proof of this fact can be found in [C76] and [Wa74]. The key is that Tychonoff spaces can be embedded in a product of compact intervals. For a space X , let I_f denote the range of f , for every function $f \in C^*(X)$. Then X can be embedded in the product of the intervals I_f . βX is the closure of the embedded X within the product.

There are other constructions of βX besides embedding X into a product of closed intervals. A *z-ultrafilter* on a topological space X is a maximal filter on X whose elements are zero-sets of X , i.e. from $Z(X)$. We say that a *z-ultrafilter* is *fixed* if it has a cluster point. In a compact space all *z-ultrafilters* are fixed. In a noncompact space there are *z-ultrafilters* without cluster points, that are called *free*. One can think of the process of compactification in the following way: If X is a noncompact space we could adjoin to X points to act as a limit points for each of its closed, noncompact subsets, in order to create a compactification of X . One might more easily do this considering a base for the closed sets. Since $Z(X)$ forms a base for the closed sets in Tychonoff spaces, we focus this action to the zero-sets of the space. βX can be constructed by adjoining to X one new point for each of its free *z-ultrafilters*. With the appropriate topology, there exists a one-to-one correspondence between βX and the set of all *z-ultrafilters* on X . More information about this construction can be found in [GJ61].

Since $C(X)$ is a *ring*, we can describe ideals and maximal ideals in $C(X)$. From above, we know that points of βX are *z-ultrafilters* on X . It is also known that the set of all *z-ultrafilters* on a set X is in one-to-one correspondence with the set of all maximal ideals in $C(X)$. Thus, another construction of βX is that for every point of βX there is a corresponding maximal ideal in $C(X)$. The interested reader is referred to [GJ61].

We also note a fourth way of relating to βX , important to our motivating example $\beta\mathbb{N}$. When a space X is discrete, the set of all *z-ultrafilters* on X is homeomorphic to the set of all *ultrafilters* on X . In this case, one may think of points of βX as *ultrafilters* on X . Fixed or principal *ultrafilters* correspond to points of X in βX ,

the free ultrafilters to the points in the remainder X^* . Now consider the following characteristics of βX .

Theorem 2.3.2. *(Stone) Every completely regular space X has a compactification βX , to which any mapping of X to a compact space K will extend uniquely. [Wa74]*

Corollary 2.3.3. *Any compactification of X is a continuous image of βX under a mapping that leaves points of X fixed. [Wa74]*

From the corollary we begin to see βX as maximum in the collection of all compactifications of X . We will define a partial ordering of this collection in the next section. βX is the compactification in which completely separated sets in X are completely separated in βX , and in which disjoint zero-sets of X have disjoint closures. Thus, for normal spaces, βX is the unique compactification in which disjoint closed sets in X have disjoint closures in βX .

Theorem 2.3.4. *For any space X , the remainder $\beta X - X = X^*$, is compact if and only if X is locally compact. [Wa74]*

Proof. Suppose X^* is a compact subspace of βX . Thus, X^* is closed and X is open. Open subspaces of compact spaces are locally compact.

Now assume X is locally compact. We can create the Alexandroff compactification ΩX , which is a continuous image of βX . Then X^* is the inverse image of the added point of ΩX , and hence a closed subspace of βX . So, X^* is compact. \square

2.4 The Lattice of Hausdorff Compactifications

The collection of all Hausdorff compactifications of a topological space X forms a complete upper semilattice with βX the supremum. Since our spaces are locally compact, we are guaranteed that ΩX is also Hausdorff. In these cases, the lattice of Hausdorff compactifications of X , $K(X)$, is a complete lattice with ΩX the infimum. The partial order on elements of the lattice is as follows.

Definition 2.4.1. For compactifications αX and γX , $\alpha X \leq \gamma X$ if and only if αX is the continuous image of γX under a mapping that leaves points of X fixed. Compactifications are equivalent if there exists a homeomorphism between them that leaves points of X fixed.

Within $K(X)$ we do not distinguish between equivalent compactifications. $K(X)$ may be a very large structure, of size at most 2^{2^α} , where $|X| = \alpha$. Its cardinality is exactly 2^{2^α} when X is an infinite, discrete space. X^* plays a large role in determining the structure of $K(X)$. Magill has proven that for two locally compact, non-compact topological spaces X and Y , the lattices $K(X)$ and $K(Y)$ are isomorphic if and only if X^* and Y^* are homeomorphic. The interested reader is referred to [M68] for proof of this fact. Magill also defines *dual points* of the lattice as follows.

Definition 2.4.2. The compactification αX of X is a dual point of the lattice $K(X)$, if $\alpha X \neq \beta X$ and there exists no compactification γX different from both αX and βX such that $\alpha X \leq \gamma X \leq \beta X$.

He goes on to show that the dual points of a lattice are precisely those compactifications that are mapped onto from βX , by a mapping that has only one nontrivial fiber of size two. This mapping is collapsing two points from the remainder X^* , leaving points of X fixed. When looking for characteristics of a complex compactification it is reasonable to begin by verifying that the characteristics remain in the dual points.

2.5 The Stone-Čech Compactification of the Natural Numbers

$\beta\mathbb{N}$ is a very complex space. It is an interesting example to study for set theorists, Boolean algebraists, combinatorists, and analysts, as well as for point-set topologists. For the topologists, it is a space that contains many interesting subspaces and often appears within the Stone-Čech compactifications of other spaces. Many things are now known about $\beta\mathbb{N}$.

Proposition 2.5.1. *Facts about $\beta\mathbb{N}$*

1. *The cardinality of $\beta\mathbb{N}$ is $2^{\mathfrak{c}}$.*
2. *The weight of $\beta\mathbb{N}$ is \mathfrak{c} .*
3. *Any infinite, closed subset and any non-empty open subset of \mathbb{N}^* contains a copy of $\beta\mathbb{N}$.*
4. *$\beta\mathbb{N}$ is extremely disconnected and \mathbb{N}^* is totally disconnected.*

Following Rudin's example we consider the structure of $\beta\mathbb{N}$ as the collection of all ultrafilters on \mathbb{N} . This is possible since \mathbb{N} is discrete so that every set is a zero-set. Thus the z-ultrafilters are the ultrafilters on \mathbb{N} . Hence every point of $\beta\mathbb{N}$ corresponds to an ultrafilter of \mathbb{N} . We now define a base for $\beta\mathbb{N}$. Let U be a subset of \mathbb{N} and $[U] = \{\rho \in \beta\mathbb{N} : U \in \rho\}$. $\{[U] : U \subset \mathbb{N}\}$ is in fact, a clopen base for $\beta\mathbb{N}$, since $\beta\mathbb{N} - [U] = [\mathbb{N} - U]$. So $\beta\mathbb{N}$ is zero-dimensional.

As mentioned previously, topological space X is called an *F-space* if for any pair of disjoint open F_σ sets, their closures remain disjoint. There are many other equivalences to this definition. X is an F-space if finitely generated ideals in $C(X)$ are principal, and if every cozero set in X is C^* -embedded. Plus, if X is an F-space, then βX is too. Hence, $\beta\mathbb{N}$ is an F-space. Since \mathbb{N}^* is a closed, normal subspace of $\beta\mathbb{N}$, it is also an F-space [GJ61]. We are only interested in F-spaces that are infinite and compact. So for this work the term *F-space* will imply infinite, compact F-space.

Theorem 2.5.2. *Every countable set in an F-space is C^* -embedded. [Wa74]*

Proof. Let X be an F-space and D a countable subspace of X . Let $\{x_n\}$ and $\{y_n\}$ be completely separated subsets of D . Then each subset can be contained in a cozero-set of X , U and V respectively, with U and V disjoint. Since U and V are completely separated in X , $\{x_n\}$ and $\{y_n\}$ are completely separated in X . So, by Urysohn's Extension Theorem, D is C^* -embedded in X . \square

As mentioned earlier, copies of $\beta\mathbb{N}$ and its remainder have been found in many other compactifications as well as within $\beta\mathbb{N}$ itself. We say that a space X *contains a copy of $\beta\mathbb{N}$* if it contains a subspace that is homeomorphic to $\beta\mathbb{N}$.

Lemma 2.5.3. *Every infinite, compact F-space contains a copy of $\beta\mathbb{N}$.*

Proof. This lemma is an exercise in [GJ61]. Suppose X is a compact (infinite) F-space. Let E be an infinite, closed subspace of X , thus E is also compact and an F-space. Since every infinite Hausdorff space contains a copy of \mathbb{N} , there exists a countably infinite, discrete set D in E . From the previous theorem 2.5.2 we know that every countable set in an F-space is C^* -embedded. So, E is a compact space to which every function in $C^*(D)$ extends. Thus, E is homeomorphic to βD , or $E \cong \beta D$, which is homeomorphic to $\beta\mathbb{N}$. \square

Another fact about $\beta\mathbb{N}$ that is of interest in this work is that $\beta\mathbb{N}$ is *contrasequential*. A topological space is *contrasequential* if and only if it contains no nontrivial convergent subsequences [FV94]. In other words, no point of the space is the limit of a sequence of distinct points. This is also a characteristic of F-spaces in general. When the space X is contrasequential we will write $\omega + 1 \not\rightarrow X$, and $\omega + 1 \hookrightarrow X$ when X has a nontrivial convergent sequence.

Lemma 2.5.4. *F-spaces are contrasequential.*

Proof. Again, this proof is an answer to an exercise found in [GJ61]. Another definition of F-space given in [GJ61] is that if X is an F-space then for any $f \in C(X)$, $posf$ and $negf$ are completely separated ($posf$ denotes the subset of X on which $f > 0$, etc.). So assume X is an F-space and that $\{x_n\}_{n \in \mathbb{N}}$ is a sequence of distinct points of X with the limit $x \in X$. Define the continuous function

$$f(x_n) = \frac{(-1)^n}{n}.$$

Then $posf$ and $negf$ are disjoint, open cozero sets of X , thus they are F_σ sets of X . But $\{x_{2j}\}_{j \in \mathbb{N}}$ and $\{x_{2k-1}\}_{k \in \mathbb{N}}$ are subsequences of $posf$ and $negf$ respectively, that have the same limit x . Thus, \overline{posf} and \overline{negf} intersect (a contradiction). \square

2.6 The Remainder, \mathbb{N}^*

It is the remainder of the Stone-Čech compactification of the natural numbers, \mathbb{N}^* , that is the most interesting piece of $\beta\mathbb{N}$. Since \mathbb{N} is countable and $|\beta\mathbb{N}| = 2^{\mathfrak{c}}$, $|\mathbb{N}^*| = 2^{\mathfrak{c}}$. Also, $w(\mathbb{N}^*) = \mathfrak{c}$. The clopen base defined for \mathbb{N}^* is $\{U^* = [U] - \mathbb{N} : U \subset \mathbb{N}\}$.

We know that every countable subspace of $\beta\mathbb{N}$ is C^* -embedded in $\beta\mathbb{N}$ and that \mathbb{N}^* has cellularity \mathfrak{c} . So that a countable set of distinct points can be chosen from countably many disjoint sets. Thus, this is a copy of \mathbb{N} , C^* -embedded in \mathbb{N}^* , whose closure is homeomorphic with $\beta\mathbb{N}$. Further, one of the most intriguing facts about \mathbb{N}^* is the third item in the previous list, 2.5.1, that every infinite closed subset of \mathbb{N}^* contains a copy of $\beta\mathbb{N}$. This comes from the previous lemma, 2.5.3. Also, following from the cellularity of \mathbb{N}^* , is the fact that $d(\mathbb{N}^*) = \mathfrak{c}$, though $\beta\mathbb{N}$ is separable.

A topological space is called *homogeneous* if and only if for any pair of points in the space there exists a homeomorphism of the space, that maps one of the pair of points to the other. A point p of a space X is called a *p-point* if every countable intersection of neighborhoods of p contains a neighborhood of p . Rudin found that it is not the case that if X is homogeneous, then X^* is also homogeneous, with the example $\beta\mathbb{N}$ [R56]. He did this by showing that under CH, \mathbb{N}^* has $2^{\mathfrak{c}}$ p-points, the set of which is dense in \mathbb{N}^* , and that if every point of a compact Hausdorff space is a p-point, then the space is finite. Since \mathbb{N}^* is not finite it must also contain points that are not p-points. A homeomorphism will map p-points only to other p-points.

A point p of a topological space X is called a *weak p-point* if and only if p is not contained in the closure of any countable subset of X . Under CH every compact F-space has a weak p-point [vM84].

As mentioned earlier \mathbb{N}^* is one example of an F-space. A topological space X is said to be *G-int* if and only if every non-empty G_δ in X has non-empty interior [GJ61]. Any sequence of clopen sets from \mathbb{N}^* having the finite intersection property has non-empty interior [R56]. So certainly, \mathbb{N}^* is both an F-space and a G-int space. In fact, these properties almost characterize \mathbb{N}^* .

Definition 2.6.1. A Parovičenko Space is defined by the following properties:

- compact, Hausdorff space
- with no isolated points and weight $\leq \omega_1$
- an F-space
- and G-int.

Parovičenko proved that under CH all such spaces are homeomorphic [P63].

Theorem 2.6.2. CH is equivalent to the statement that \mathbb{N}^* is the only Parovičenko space up to homeomorphism.

Proof. We now outline how sufficiency is demonstrated in [vM84]. We begin by looking at two examples of Parovičenko spaces that would not be homeomorphic under \neg CH. First, take a decreasing ω_1 sequence of clopen subsets in \mathbb{N}^* , $\{U_\alpha : \alpha < \omega_1\}$, and let $A = \bigcap_{\alpha < \omega_1} U_\alpha$. Then define the quotient map $f : \mathbb{N}^* \rightarrow \mathbb{N}^*/A$ which collapses A to one point. Take a clopen base C of \mathbb{N}^* and define the collection $B = \{S \in C : A \subset S \text{ or } A \cap S = \emptyset\}$. Then one can show that \mathbb{N}^*/A is a Parovičenko space with $f(B) = \{f(K) : K \in B\}$ a base for the space. Note the character, $\chi(\mathbb{N}^*/A) = \omega_1$.

Secondly, consider the space $(\omega \times 2^\mathfrak{c})^*$. Since $(\omega \times 2^\mathfrak{c})$ is zero-dimensional, $\beta(\omega \times 2^\mathfrak{c})$ is as well. $(\omega \times 2^\mathfrak{c})$ is σ -compact and locally compact which implies that $(\omega \times 2^\mathfrak{c})^*$ is an F-space [GJ61]. Then in $(\omega \times 2^\mathfrak{c})$ the combination of real compact and locally compact provide that $(\omega \times 2^\mathfrak{c})^*$ is G-int [FG60]. van Douwen and van Mill define π_α to be the α^{th} projection from $2^\mathfrak{c}$ onto 2. Then let

$$K(\alpha, i) = (\omega \times 2^\mathfrak{c})^* \bigcap \overline{(\omega \times \pi_\alpha\{i\})}$$

where $i = 0$ or 1 . Each $K(\alpha, i)$ is a clopen subset of $(\omega \times 2^\mathfrak{c})^*$. Next define

$$\mathbf{K} = \{K(\alpha, i) : \alpha < \mathfrak{c}, i = 0 \text{ or } 1\}.$$

They show that any intersection of ω_1 distinct members of \mathbf{K} has empty interior. It follows that for any $x \in (\omega \times 2^\mathfrak{c})^*$ and any neighborhood base U of x , there is some $K \in \mathbf{K}$ containing x within each $V \in U$. Forcing $\chi(x, (\omega \times 2^\mathfrak{c})^*) = \mathfrak{c}$.

Therefore, under $\neg\text{CH}$ these two Parovičenko spaces are not homeomorphic. \square

2.7 The Lattice $K(\mathbb{N})$

We assume that compactifications “closer” in the lattice $K(\mathbb{N})$ to $\beta\mathbb{N}$ would be like $\beta\mathbb{N}$ in some of their characteristics. These characteristics could be used to determine the complexity of a compactification of \mathbb{N} since $\beta\mathbb{N}$ is the most complex. Faulkner and Vopera have begun by examining the existence of compactifications that contain copies of $\beta\mathbb{N}$ and of compactifications that are contrasequential. Such compactifications would be considered complex compactifications. We continue a similar discussion using other topological characteristics in the next chapter.

As mentioned earlier for any two compactifications $\alpha\mathbb{N}$ and $\gamma\mathbb{N}$, $\alpha\mathbb{N} \leq \gamma\mathbb{N}$ if and only if there exists a mapping $\pi_{\gamma\alpha}$ from $\gamma\mathbb{N}$ onto $\alpha\mathbb{N}$ that leaves points of \mathbb{N} fixed. We denote $\alpha\mathbb{N} \vee \gamma\mathbb{N}$ as the supremum of the two compactifications.

Lemma 2.7.1. *For any compactification $\alpha\mathbb{N}$ of \mathbb{N} , if for some $p \in (\alpha\mathbb{N} - \mathbb{N})$, $\pi_{\beta\alpha}^{-1}(p)$ has nonempty interior in \mathbb{N}^* , then $\alpha\mathbb{N}$ is not contrasequential. (i.e. or $\omega + 1 \hookrightarrow \alpha\mathbb{N}$) [FV94]*

Proof. Suppose $\pi_{\beta\alpha}^{-1}(p)$ has nonempty interior in \mathbb{N}^* . So there exists a clopen $U^* \subset \pi_{\beta\alpha}^{-1}(p)$, with $U \subset \mathbb{N}$. Then when that clopen set gets mapped to the point p in the remainder of $\alpha\mathbb{N}$, p will be the limit of the distinct sequence of numbers found in U . \square

Proposition 2.7.2. *If $\omega + 1 \hookrightarrow \alpha\mathbb{N}$, then there exists a compactification $\gamma\mathbb{N}$, strictly larger than $\alpha\mathbb{N}$, for which $\omega + 1 \hookrightarrow \gamma\mathbb{N}$. [FV94]*

Proof. Let $\{p_n\}_{n \in \mathbb{N}}$ be a nontrivial convergent sequence with limit p in $\alpha\mathbb{N}$. Then let $D = \{q_n : q_n \in \pi_{\beta\alpha}^{-1}(p_n)\}$, a countable, discrete subset of $\beta\mathbb{N}$. Thus, D is C^* -embedded in $\beta\mathbb{N}$, and $Cl_{\beta\mathbb{N}}(D) \cong \beta\mathbb{N}$. Now there is a copy F , of \mathbb{N}^* in $\beta\mathbb{N}$ such that $\pi_{\beta\alpha}(F) = p$. Take a copy of $\beta\mathbb{N}$ in F and collapse the remainder of that copy

to a single point. This compactification of \mathbb{N} will not be contrasequential and will be strictly larger than $\alpha\mathbb{N}$. \square

When comparing two compactifications, Faulkner and Vopera show that if the lesser compactification is not contrasequential then the fibers of the mapping from one onto the other may help determine if the larger compactification also fails to be contrasequential. Recall the cardinal numbers \mathfrak{t} and \mathfrak{s} previously defined in section 2.1.2.

Theorem 2.7.3. *Assume $\alpha\mathbb{N} \leq \gamma\mathbb{N}$ and suppose for each $p \in \alpha\mathbb{N}$, one of the following is true:*

1. $\pi_{\gamma\alpha}^{-1}(p)$ is first-countable;
2. $|\pi_{\gamma\alpha}^{-1}(p)| < 2^{\mathfrak{t}}$;
3. $w(\pi_{\gamma\alpha}^{-1}(p)) < \mathfrak{s}$.

Then $\omega + 1 \hookrightarrow \alpha\mathbb{N} \implies \omega + 1 \hookrightarrow \gamma\mathbb{N}$. [FV94]

Proof. Begin by assuming that each fiber of $\pi_{\gamma\alpha}$ is finite and let $\{p_n\}_{n \in \mathbb{N}}$ be a non-trivial convergent sequence in $\alpha\mathbb{N}$, that converges to p in $\alpha\mathbb{N}$. For each p_n choose $q_n \in \pi_{\gamma\alpha}^{-1}(p_n)$ and let Q be the set of all limit points of $\{q_n\}$. Since $Q \subset \pi_{\gamma\alpha}^{-1}(p)$, Q is finite. Thus, some subsequence of $\{q_n\}$ converges. Now assume $A = \pi_{\gamma\alpha}^{-1}(p)$ is an infinite fiber. Then by 1. A contains a nontrivial convergent sequence since the fiber is non-empty and closed along with first-countable. A would contain a non-trivial convergent subsequence by 2. and 3., because the fiber would be sequentially compact. \square

We now consider the characteristic of containing a copy of $\beta\mathbb{N}$. Even stronger, is the characteristic that every infinite closed subset of a space contains a copy of $\beta\mathbb{N}$. We begin with a proposition similar to the previous proposition concerning contrasequential, but for compactifications of any space X , not just for \mathbb{N} .

Proposition 2.7.4. *If αX contains a copy of $\beta\mathbb{N}$ and $\alpha X \leq \gamma X$, then γX also contains a copy of $\beta\mathbb{N}$. [FV94]*

It is well known that if any continuous image of a space contains a copy of $\beta\mathbb{N}$, then the space must as well. Faulkner and Vopera prove that for a closed set C of \mathbb{N}^* , the compactification resulting from identifying C to one point is contrasequential. They also show the following.

Proposition 2.7.5. *Let C be a closed subset of \mathbb{N}^* , such that $\omega + 1 \not\leq \alpha\mathbb{N}$, where $\alpha\mathbb{N}$ is the compactification resulting from collapsing C to one point. Then every infinite closed subset of $\alpha\mathbb{N}$ contains a copy of $\beta\mathbb{N}$. [FV94]*

Proof. Let $\pi_{\beta\alpha}(C) = \{p\}$ and let F be an infinite closed subset in $\alpha\mathbb{N}$. If $p \notin F$, then $\pi_{\beta\alpha}^{-1}$ homeomorphic on F with $\pi_{\beta\alpha}^{-1}(F)$ infinite and closed in \mathbb{N}^* so that it contains a copy of $\beta\mathbb{N}$. Then F contains a copy of $\beta\mathbb{N}$. Now suppose that $p \in F$ and let $\mathbf{F} = \pi_{\beta\alpha}^{-1}(F)$. If there exists an open set U in $\beta\mathbb{N}$ with $C \subset U$ and $\mathbf{F} - U$ infinite, then $\mathbf{F} - U$ must contain a copy B of $\beta\mathbb{N}$ and $\pi_{\beta\alpha}(B)$ would be the copy of $\beta\mathbb{N}$ in F . Now assume that $\mathbf{F} - U$ is finite for every open U containing C . Since $\mathbf{F} - C$ is infinite it contains a countable, discrete subset $\{p_n\}_{n \in \mathbb{N}}$. Let $q_n = \pi_{\beta\alpha}(p_n)$ for every n . By our assumption of U all but finitely many p_n are in U . Thus, $\{q_n\}$ converges to p in $\alpha\mathbb{N}$, a contradiction. \square

The following facts are needed for the next theorem.

2.7.1 Facts

- If $\prod_{\alpha < \delta} X_\alpha$ contains a copy of $\beta\mathbb{N}$ and $\delta < cf(\mathfrak{c})$ then there is a $\lambda < \delta$ such that X_λ contains a copy of $\beta\mathbb{N}$. [S80]
- Let X be Tychonoff and $\alpha X = \sup\{\alpha_\lambda \mathbf{X} : \lambda < \delta\}$ with $\delta < cf(\mathfrak{c})$. If $\alpha_\lambda \mathbf{X}$ does not contain a copy of $\beta\mathbb{N}$ for all λ , then neither does αX . [FV94]

Containing a copy of $\beta\mathbb{N}$ within every infinite closed subset is a quality of a compactification that is complex. If the supremum of two compactifications within $K(\mathbb{N})$ is $\beta\mathbb{N}$, at least one of those compactifications must be complex.

Theorem 2.7.6. *If $\beta\mathbb{N} = \alpha\mathbb{N} \vee \gamma\mathbb{N}$ and $\gamma\mathbb{N}$ does not contain any copies of $\beta\mathbb{N}$, then every infinite closed subset of $\alpha\mathbb{N}$ contains a copy of $\beta\mathbb{N}$. [FV94]*

Proof. From the second of the previous facts we have that $\alpha\mathbb{N} \times \gamma\mathbb{N}$ contains a copy of $\beta\mathbb{N}$. Note that $\pi_{\beta\alpha}$ is the restriction of the first projection. Let F be an infinite closed subset of $\alpha\mathbb{N}$ and $G = \pi_{\beta\alpha}^{-1}(F)$. Then $G \subset F \times \gamma\mathbb{N}$. Then since G contains a copy of $\beta\mathbb{N}$ the first fact previously listed demands that F contain a copy of $\beta\mathbb{N}$. \square

The complexity of at least one of a pair of compactifications, whose supremum is $\beta\mathbb{N}$, can be seen in another situation. Consider the following.

Theorem 2.7.7. *Suppose $\beta\mathbb{N} = \alpha\mathbb{N} \vee \gamma\mathbb{N}$ and that $\omega + 1 \hookrightarrow \alpha\mathbb{N}$. Then $\gamma\mathbb{N}$ must contain a copy of $\beta\mathbb{N}$. [FV94]*

Proof. If $\omega + 1 \hookrightarrow \alpha\mathbb{N}$ then some fiber $A = \pi_{\beta\alpha}^{-1}(p)$, $p \in \alpha\mathbb{N}$, is infinite. Then A is both closed and infinite in $\beta\mathbb{N}$ so it must contain a copy of $\beta\mathbb{N}$. Faulkner and Vipera show that $\pi_{\beta\gamma}$ is injective on the fibers of $\pi_{\beta\alpha}$. So the restriction of $\pi_{\beta\gamma}$ to A is an embedding. Thus, $\gamma\mathbb{N}$ contains a copy of $\beta\mathbb{N}$. \square

We direct the interested reader to [FV94] for further reading on these characteristics within $K(\mathbb{N})$. Our goal in the next chapter is to consider a new list of characteristics of $\beta\mathbb{N}$ and to determine if they could also be used as characteristics of complexity.

Chapter 3

Results and Discussion

3.1 Zero-dimensional

Beginning with the characteristics found in Parovicenko spaces, we first test the invariance of those characteristics under finite-to-one mappings. We know that it is not always the case that the image of a compact zero-dimensional space under a finite-to-one mapping is zero-dimensional. We give the following example.

Example 3.1.1. *Consider the closed unit interval \mathbb{I} , and divide it into two closed subintervals $I_0 = [0, \frac{1}{2}]$ and $I_1 = [\frac{1}{2}, 1]$. Then in a similar manner create four closed subintervals $I_{00} = [0, \frac{1}{4}]$, $I_{01} = [\frac{1}{4}, \frac{1}{2}]$, $I_{10} = [\frac{1}{2}, \frac{3}{4}]$, and $I_{11} = [\frac{3}{4}, 1]$. Continue in this manner halving each newly created subinterval for the next step. Now define the map $f : 2^\omega \rightarrow \mathbb{I}$, so that for any $x \in 2^\omega$, where $x = a_1 a_2 a_3 a_4 \dots$*

$$f(x) = I_{a_1} \cap I_{a_1 a_2} \cap I_{a_1 a_2 a_3} \cap I_{a_1 a_2 a_3 a_4} \cap \dots,$$

where for all n , a_n is either 1 or 0. The mapping is certainly finite-to-one, in fact it is ≤ 2 -to-one and also surjective. To see that it is continuous, let y be an element in \mathbb{I} and (a, b) an open interval in \mathbb{I} that contains y . Within (a, b) there exists a closed subinterval as created above that also contains y . Call this subinterval J and note that $J = I_{a_1 a_2 \dots a_n}$ for some $n \in \omega$. Then

$$f^{-1}(J) = a_1 \times a_2 \times \dots \times a_n \times 2 \times 2 \times \dots$$

This inverse image is a basic open set in the product topology on 2^ω .

However, we can restrict our mapping further and zero-dimensionality is maintained.

Lemma 3.1.2. *The image of a compact zero-dimensional space under a mapping with a finite number of non-trivial fibers is zero-dimensional.*

Proof. Let X be a compact, zero-dimensional space and f a continuous mapping from X onto Y , with which the set of points $A = \{y_1, y_2, \dots, y_k\}$ in Y are the only points with nontrivial fibers. Thus, for each element $y_i \in A$, $|f^{-1}(y_i)| > 1$. Let \mathbf{C} be a clopen base of X and define

$$\mathbf{B} = \{C \in \mathbf{C} : C \cap f^{-1}(y_i) = \emptyset \text{ or } f^{-1}(y_i) \subset C, \text{ for } i = 1, \dots, k\}.$$

Note that since \mathbf{C} is a clopen base, then there is a basic clopen set that contains each of the fibers $f^{-1}(y_i)$, for every y_i , since each fiber is closed in a compact space. Thus, \mathbf{B} is a collection of clopen sets of X .

The collection $W = \{f(B) : B \in \mathbf{B}\}$ is a clopen base for Y . For the sets $C \in \mathbf{C}$ that miss the nontrivial fibers, f is homeomorphic. Thus for such a set C , $f(C)$ is closed since f is perfect and $f(X - C) = Y - f(C)$ is also closed. So that $f(C)$ is clopen. Now let D be one of the other members of \mathbf{C} with $f^{-1}(y_i) \subset D$ for some $i \in \{1, 2, \dots, k\}$. Since D is clopen, $f(D)$ is closed in Y and since D can be chosen so that it is saturated, the quotient map takes open, saturated D to open $f(D)$. To see that W is a base for Y , take any point p in Y and let U be an open set about p . Regardless of whether or not $f^{-1}(p)$ is nontrivial, there will be a $C \in \mathbf{B}$ containing $f^{-1}(p)$ inside $f^{-1}(U)$. The image of which contains p , is clopen, and is contained in U . □

3.2 G-int and F space

The statement that the characteristics of G-int and F-space can be maintained under any finite-to-one mapping is false. In [vD93] we see that there exists an image of \mathbb{N}^*

under a ≤ 2 -to-one mapping, that is separable. This image is not G-int.

Example 3.2.1. *Suppose S is the separable image of \mathbb{N}^* under a ≤ 2 -to-one mapping. Let D be a countable dense set in S . Then for each $d \in D$, $U_d = S - \{d\}$ is a dense, open set of S . By the Baire Category Theorem $\bigcap_{d \in D} U_d$ is not empty. Thus, it is a nonempty G_δ set with empty interior. The interior is empty since if it were not, there would be a basic open set in the intersection that would contain an element of D .*

However, if the finite-to-one mapping is irreducible we prove that G-int remains. For compact, Hausdorff spaces X and Y a surjection f is *irreducible* if there is no proper, compact subset $S \subset X$, such that $f(S) = Y$. Otherwise f is *reducible*.

Lemma 3.2.2. *The finite-to-one, irreducible image of a compact, zero-dimensional, G-int space is G-int.*

Proof. Let X be a compact, zero-dimensional, G-int space and f a finite-to-one, irreducible mapping from X onto Y . Let \mathbf{C} be a clopen base for X . We begin by showing that the collection

$$W = \{Y - f(X - C) : C \in \mathbf{C}\}$$

is a base for Y . So let U be an open set in Y and y a point in U . Then $f^{-1}(y) \subset f^{-1}(U)$ and $f^{-1}(U)$ is an open set in X . Since f is finite-to-one, $|f^{-1}(y)| < \omega$. Thus for each $x \in f^{-1}(y)$ there is a $C_x \in \mathbf{C}$ such that $x \in C_x \subset f^{-1}(U)$. Now let $B_y = \bigcup_{x \in f^{-1}(y)} C_x$, a finite union of clopen basis elements, so note that it is an element of our base \mathbf{C} along with $X - B_y$.

Now we have that $y \in Y - f(X - B_y) \subset U$. We know the subset relationship holds since

$$X - f^{-1}(U) \subset X - B_y$$

which implies that

$$Y - U \subset f(X - f^{-1}(U)) \subset f(X - B_y)$$

and thus

$$Y - f(X - B_y) \subset U.$$

Now that we have established a base for Y let G be a nonempty G_δ set in Y . Then $f^{-1}(G)$ is a nonempty G_δ set in X . Since X is G-int, there exists a basic clopen set C from the clopen base \mathbf{C} , such that $C \subset f^{-1}(G)$. Then

$$X - f^{-1}(G) \subset X - C$$

which, as with our previous arguments, gives that

$$Y - f(X - C) \subset G.$$

Since f is irreducible and $X - C$ compact, $f(X - C) \neq Y$. Thus, G contains a nonempty, basic open set. \square

As mentioned earlier, a space X is referred to as an F-space if it is a compact, infinite F-space, so that we may avoid the trivial discrete spaces. To guarantee the image of an F-space is still an F-space we must use an almost one-to-one mapping, since F-space is not always maintained by finite-to-one mappings.

Example 3.2.3. *Back to van Douwen's example, let K be the separable image of \mathbb{N}^* under f , a ≤ 2 -to-one mapping. Assume $D \subset K$ is a countable, dense subset of K . Suppose K is an F-space. van Mill proved that under CH every compact F-space contains a weak p -point [vM79]. So let $p \in K$ be a weak p -point of K . But then $p \in \overline{D} = K$. Thus, no point of K can be a weak p -point, and so K can not be an F-space.*

Lemma 3.2.4. *The image of a compact, zero-dimensional F-space under a mapping with a single, non-trivial fiber is an F-space.*

Proof. Let X be a compact F-space and f a continuous mapping from X onto Y , with $y \in Y$ the only point of Y with a non-trivial fiber. Let

$$F = \bigcup_{n \in \mathbb{N}} F_n, G = \bigcup_{n \in \mathbb{N}} G_n, \text{ with } F_n \text{ and } G_n \text{ closed for all } n \in \mathbb{N},$$

be two open, disjoint F_σ sets in Y . If, without loss of generality, $y \in F$ then $f^{-}(F)$ and $f^{-}(G)$ are disjoint open F_σ sets in X , thus their closures are disjoint. So, there are two disjoint clopen sets of X that contain $f^{-}(F)$ and $f^{-}(G)$ respectively, whose images would contain F and G while still being disjoint and clopen, since f is a perfect mapping and the sets are saturated.

So if y is not an element of either F_σ set, we need only to make sure that the closures of those sets do not intersect at y . For each $n \in \mathbb{N}$, let

$$K_n = f^{-}(F_n) \cup f^{-}(G_n),$$

a closed set and let L_n be a clopen set of X containing $f^{-}(y)$ that does not intersect K_n . Such a set exists since X is normal and both K_n and $f^{-}(y)$ are closed. Now let

$$P_1 = L_1, P_2 = L_1 \cap L_2, \dots, P_n = L_1 \cap \dots \cap L_n, \dots$$

Then define the sets

$$K = \bigcup_{n \in \mathbb{N}} K_n \text{ and } L = \bigcup_{n \in \mathbb{N}} P_n.$$

Both sets are open F_σ sets in X and thus, have disjoint closures. It is the closure of L , \bar{L} , that contains $f^{-}(y)$, separating it from $f^{-}(F)$ and $f^{-}(G)$. Then within X there are disjoint clopen sets that contain K and L respectively, whose images will also contain $F \cup G$ and y respectively. Thus, the union of the F_σ sets is completely separated from a set that contains y . So that

$$\overline{(F \cup G)} \cap f(\bar{L}) = \emptyset,$$

implying that

$$(\bar{F} \cup \bar{G}) \cap f(\bar{L}) = \emptyset.$$

Therefore, neither \bar{F} nor \bar{G} contain y , and they do not intersect. \square

The next theorem follows inductively from the previous lemma, however we include the alternative proof. We refer to the lemma 3.2.4 in the proof of the claim.

Proposition 3.2.5. *The image of a compact, zero-dimensional F-space under a mapping with a finite number of, non-trivial fibers is an F-space.*

Proof. Suppose that X is a compact, zero-dimensional F-space and Y is the image of X under f , a mapping with a finite number of non-trivial fibers. Let $A = \{y_1, \dots, y_m\}$ be the set of points of Y with non-trivial fibers, for some $m \in \mathbb{N}$. Now let

$$F = \bigcup_{n \in \mathbb{N}} F_n, G = \bigcup_{n \in \mathbb{N}} G_n, \text{ with } F_n \text{ and } G_n \text{ closed for all } n \in \mathbb{N},$$

be two open, disjoint F_σ sets in Y .

There are three cases to consider, the last two with similar consequences. If the points of A are contained in F , G , or divided between the two, then $f^{-}(F)$ and $f^{-}(G)$ are disjoint open F_σ sets in X , thus their closures are disjoint. So, there are two disjoint clopen sets of X that contain $f^{-}(F)$ and $f^{-}(G)$ respectively, whose images would contain F and G while still being disjoint and clopen, since f is a perfect mapping and the sets are saturated.

The case where some points of A are neither in F nor G has the same result if $A \cap F \cap G = \emptyset$. So without loss of generality, assume $A \cap F \cap G = \emptyset$. We need only to make sure that the closures of F and G do not intersect at any points of A . As in the previous proof of lemma 3.2.4, let

$$K_n = f^{-}(F_n) \bigcup f^{-}(G_n),$$

a closed set and let Y_{in} be a clopen set of X containing $f^{-}(y_i)$ that does not intersect K_n , for each $i = 1, 2, \dots, m$. Then again, we can create sets, Y_1, Y_2, \dots, Y_m , where $Y_i = \bigcup_{n \in \mathbb{N}} Y_{in}$ for every $i = 1, \dots, m$, that are open F_σ sets in X and thus, have disjoint closures. Then the closures are completely separated in X , separating each of the nontrivial fibers from $f^{-}(F)$ and $f^{-}(G)$, and the images of the disjoint, clopen sets containing each will be disjoint and clopen in Y since f is perfect and the sets are saturated. \square

Theorem 3.2.6. *Suppose $f : X \rightarrow Y$ is onto and finite-to-one, and that X is a compact F-space. Then Y is contrasequential.*

Proof. Suppose $\{y_n\}_{n \in \mathbb{N}}$ is a nontrivial sequence of points that converges to y in Y . Then $Z = f^{-1}(\{y_n : n \in \mathbb{N}\})$ is countable in X , and hence C^* -embedded in X . Now $\bar{Z} \cong \beta\mathbb{N}$ so that $(\bar{Z}) - Z \cong \mathbb{N}^*$. Moreover $f((\bar{Z}) - Z) = y$ which contradicts finite-to-one. Thus, Y must be contrasequential. \square

We also include another result about F-spaces, not related to finite-to-one mappings. Recall the following.

- If $\alpha < cf(\mathfrak{c})$ and $\prod_{\lambda < \alpha} X_\lambda$ contains a copy of $\beta\mathbb{N}$, then X_γ contains a copy of $\beta\mathbb{N}$, for some $\gamma < \alpha$. [S80]
- Every infinite compact F-space contains a copy of $\beta\mathbb{N}$. [GJ61]

From these facts we have the following result.

Theorem 3.2.7. *If $\alpha < cf(\mathfrak{c})$ and $\prod_{\lambda < \alpha} X_\lambda$ contains a compact F-space, then X_γ contains a compact F-space, for some $\gamma < \alpha$.*

Proof. Suppose that $\alpha < cf(\mathfrak{c})$ and $\prod_{\lambda < \alpha} X_\lambda$ contains a compact F-space. Then from the previous facts we know that $\prod_{\lambda < \alpha} X_\lambda$ contains a copy of $\beta\mathbb{N}$. Thus, there exists some $\gamma < \alpha$ such that X_γ contains a copy of $\beta\mathbb{N}$. Then the copy of \mathbb{N}^* within X_γ is a compact F-space in X_γ . \square

3.3 P-points

We know that under CH, \mathbb{N}^* is not homogeneous [R56] because it has points that are p-points and points that are not. We prove the following for spaces that are images of finite-to-one mappings.

Proposition 3.3.1. *Let X be a zero-dimensional G -int space and f a finite-to-one mapping from X onto Y . If p is a p-point of X , then $f(p)$ is a p-point of Y .*

Proof. As seen in previous results if we let \mathbf{C} be a clopen base of X we can create a set

$$W = \{Y - f(X - C) \mid C \in \mathbf{C}\},$$

that is a base for Y . Now let $p \in X$ be a p -point. Assume G is a G_δ in Y that contains $f(p)$. So then $f^{-}(f(p)) \in f^{-}(G)$ a G_δ in X . Hence with $|f^{-}(f(p))| < \omega$ there exists a basic clopen set $C \in \mathcal{C}$ such that $f^{-}(f(p)) \in C \subset f^{-}(G)$. Then as before, $f(p) \in Y - f(X - C) \subset G$ since

$$X - f^{-}(G) \subset X - C$$

which implies that

$$Y - G \subset f(X - f^{-}(G)) \subset f(X - C)$$

and so

$$Y - f(X - C) \subset G.$$

□

3.4 Nowhere F

We defined a nowhere F space to be a space that contains no infinite, compact subspaces that are F-spaces. There are compactifications within the lattice of compactifications of \mathbb{N} with remainders that are nowhere F. Most simply, any compactification with a finite remainder is nowhere F. We prove the following.

Lemma 3.4.1. *The continuous image of a nowhere F space is nowhere F.*

Proof. Let X be a nowhere F space and f a continuous function from X onto Y . Suppose Y is not nowhere F. So, there exists some $K \subset Y$ that is an infinite compact F-space. Thus, K contains a copy of $\beta\mathbb{N}$ embedded into it. Since the continuous image of X contains a copy of $\beta\mathbb{N}$, so must X . Thus, X contains a subspace that is an F-space, a contradiction. □

To be a compactification of \mathbb{N} within the lattice $K(\mathbb{N})$ that is nowhere F, is to be a simple compactification that is not “close” to $\beta\mathbb{N}$. Thus, if the supremum of two compactifications is $\beta\mathbb{N}$ and one of those compactifications is nowhere F, it would be reasonable to expect the other compactification to be complex.

Lemma 3.4.2. *For compactifications $\alpha\mathbb{N}$ and $\gamma\mathbb{N}$, if $\alpha\mathbb{N} \vee \gamma\mathbb{N} = \beta\mathbb{N}$ and $\alpha\mathbb{N}$ is nowhere F , then $\gamma\mathbb{N}$ is contrasequential.*

Proof. Suppose $\alpha\mathbb{N} \vee \gamma\mathbb{N} = \beta\mathbb{N}$ and that $\alpha\mathbb{N}$ is nowhere F . Also assume that there exists a sequence $\{a_n\}_{n \in \mathbb{N}}$ of distinct points that converge to a point p in $\gamma\mathbb{N}$. Then $\pi_{\beta\gamma}^{-1}(p)$, the pre-image of p under this map must be infinite and is closed, thus contains a basic clopen set of $\beta\mathbb{N}$. Then $\pi_{\beta\alpha}$ is one-to-one on the fibers of $\pi_{\beta\gamma}$, so that a copy of a basic clopen set of $\beta\mathbb{N}$ is contained in $\alpha\mathbb{N}$. Thus, the copy of \mathbb{N}^* within that set is a compact F -space inside of $\alpha\mathbb{N}$, a contradiction. \square

Lemma 3.4.3. *For compactifications $\alpha\mathbb{N}$ and $\gamma\mathbb{N}$, if $\alpha\mathbb{N} \vee \gamma\mathbb{N} = \beta\mathbb{N}$, then either $\alpha\mathbb{N}$ or $\gamma\mathbb{N}$ will not be nowhere F . In other words, one must contain an F -space.*

Proof. Suppose for compactifications $\alpha\mathbb{N}$ and $\gamma\mathbb{N}$ that $\alpha\mathbb{N} \vee \gamma\mathbb{N} = \beta\mathbb{N}$. The product of the compactifications $\alpha\mathbb{N}$ and $\gamma\mathbb{N}$ contains their supremum. Thus, from Shapirovski [S80] one of the compactifications will contain $\beta\mathbb{N}$. \square

3.5 Simple Mappings

Automorphisms (homeomorphisms) of \mathbb{N}^* are special cases of finite-to-one mappings. The structure of the group $Aut(\mathbb{N}^*)$ of automorphisms of \mathbb{N}^* depends on set theoretic assumptions beyond ZFC. Walter Rudin [R56] proved that under the assumption of CH, $|Aut(\mathbb{N}^*)| = 2^{\mathfrak{c}}$. An element $f \in Aut(\mathbb{N}^*)$ is said to be *simple* if there exists a bijection $g : \mathbb{N} \rightarrow \mathbb{N}$ so that $\beta g = f$. There can only be \mathfrak{c} such automorphisms. Consequently under CH most of the elements of $Aut(\mathbb{N}^*)$ are not simple. However there are models of set theory in which every element of $Aut(\mathbb{N}^*)$ is simple [Ve86].

This divergence of structure appears as well in the finite-to-one maps defined on \mathbb{N} . If f is a mapping, $f : \mathbb{N}^* \rightarrow K$, where K is compact, then K is the remainder in many compactifications of \mathbb{N} . So for a mapping f on $\beta\mathbb{N}$ with $f|_{\mathbb{N}^*} = K$, f is *simple* if there is a compactification $\alpha\mathbb{N}$ with remainder K and a finite-to-one mapping $g : \mathbb{N} \rightarrow \mathbb{N}$ so that $\beta g = f$.

We will now take a more detailed look at the structure of general mappings f from \mathbb{N}^* to compact K . At times it will not be necessary to have all of the properties of $\beta\mathbb{N}$ to obtain the results. We will always assume that K is a compact Hausdorff space and that $\alpha\mathbb{N}$ is a compactification of \mathbb{N} with $K = \alpha\mathbb{N} - \mathbb{N}$. We will use the notation $f : \mathbb{N} \rightarrow \mathbb{N} \hookrightarrow \alpha\mathbb{N}$ to denote a mapping f on \mathbb{N} and the compactification $\alpha\mathbb{N}$ that has as its remainder the range of βf .

Remark 3.5.1. *Suppose $f : \mathbb{N} \rightarrow \mathbb{N}$ is surjective. We could say that f is $\leq^* n$ -to-1, if it is $\leq n$ -to-1 except on a finite set, in order to eliminate a finite, and hence invisible, violation of just being $\leq n$ -to-1. This has no impact on our results, so we will assume that any such finite set has already been removed.*

Theorem 3.5.2. *Suppose $f : \mathbb{N} \rightarrow \mathbb{N} \hookrightarrow \alpha\mathbb{N}$ is surjective onto \mathbb{N} and $\leq n$ -to-1. Then $\beta f : \mathbb{N}^* \rightarrow K$ is $\leq n$ -to-1, where $K = \alpha\mathbb{N} - \mathbb{N}$.*

Proof. For any $n \in \mathbb{N}$, let $E_k = f^{-1}(k)$ for each $k \in \mathbb{N}$, so that $|E_k| \leq n$. Define the sets $A_1, A_2 \dots A_n$ as follows. A_1 consists of exactly one point chosen from each E_k . Since f is surjective, $|A_1| = \omega$. For A_2 repeat this process by choosing exactly one point from $E_k - A_1$, when it is not empty. If $\bigcup_{k \in \mathbb{N}} E_k - (A_1 \cup A_2)$ is infinite, then repeat with A_3 . If at some stage $\bigcup_{k \in \mathbb{N}} E_k - (A_1 \cup \dots \cup A_m)$ is finite, terminate the process. Note that the family $\{A_j : j < m\}$ is a family of disjoint sets and that $\mathbb{N} - \bigcup_{j < m} A_j$ is finite.

Since f restricted to each A_j is a bijection, βf is an embedding from A_j^* onto a subset of K . Since βf restricted to A_1^* is surjective, K must be homeomorphic to \mathbb{N}^* . Moreover, $\mathbb{N}^* = \bigcup_{j < m} A_j^*$. So, $|(\beta f)^{-1}(x)| \leq m \leq n$ for each $x \in K$. \square

The next theorem that we include is an addendum to the previous theorem.

Theorem 3.5.3. *Suppose that $f : \mathbb{N} \rightarrow \mathbb{N} \hookrightarrow \alpha\mathbb{N}$ is finite-to-one, but not $\leq n$ -to-1 for any $n \in \mathbb{N}$. Then βf has at least one infinite fiber.*

Proof. Note in the above proof 3.5.2, that the sets A_k have are chosen so that $A_k^* \cap A_j^* = \emptyset$ whenever $j < k$, but $f(A_k) \subset f(A_j)$, so that $\beta f(A_k^*) \subset \beta f(A_j^*)$. If the size

of the fibers of f is unbounded then the process of constructing the sets A_k never terminates. Since these closed compact sets $\beta f(A_k^*)$ are nested and no $\beta f(A_k^*)$ is empty for any $k \in \mathbb{N}$, $\bigcap_{k \in \mathbb{N}} \beta f(A_k^*) \neq \emptyset$, by the finite intersection property. So there exists an $x \in \bigcap_{k \in \mathbb{N}} \beta f(A_k^*)$, whose fiber $\beta f^{-1}(x)$ would be infinite. \square

Theorem 3.5.4. *The two preceding theorems prove that if $f : \mathbb{N} \rightarrow \mathbb{N} \hookrightarrow \alpha\mathbb{N}$ is surjective, then f is $\leq n$ -to-1 if and only if βf is $\leq n$ -to-1.*

The proofs of 3.5.2 and 3.5.3 completely capture the structure of a simple finite-to-one mapping of \mathbb{N}^* , that they are extensions of mappings on \mathbb{N} , that their ranges are \mathbb{N}^* , and that they are reducible. Note that simple mappings on \mathbb{N}^* are reducible, but that mappings that are not simple do not have to be irreducible.

We can construct examples of mappings on \mathbb{N}^* which are not simple. Consider the following.

Example 3.5.5. *Assume CH. Let $h : \mathbb{N}^* \rightarrow \mathbb{N}^*$ and $g : \mathbb{N}^* \rightarrow \mathbb{N}^*$ be two nonsimple automorphisms of \mathbb{N}^* . Decompose \mathbb{N}^* into two disjoint copies of itself A^* and B^* . Define $f^* : \mathbb{N}^* \rightarrow \mathbb{N}^*$ by*

$$f^*(x) = \begin{cases} h(x) & \text{if } x \in A^*, \\ g(x) & \text{if } x \in B^*. \end{cases}$$

Then f^ is not simple, for otherwise h would be an extension of a bijection from $A \cong \mathbb{N}$ to \mathbb{N} and g would be an extension of a bijection from $B \cong \mathbb{N}$ to \mathbb{N} . By assumption, that is not possible. Note that in this example, every fiber is nontrivial.*

Example 3.5.6. *If trivial fibers of a mapping are present, then any map having finitely many nontrivial fibers cannot be simple. Note this requires no set theoretic assumptions. We can make this more complicated using a lemma in [FV93], [Lemma 2.13]. Let A and B be two disjoint copies of $\beta\mathbb{N}$ in \mathbb{N}^* , both of which are nowhere dense. Let $h : A \rightarrow B$ be a homeomorphism. Make a quotient space K with map f , by identifying the point $x \in A$ with its image $h(x) \in B$. K is a compact Hausdorff space and the quotient space is 2-to-1 on $2^{\mathfrak{c}}$ points.*

In the previous example, since A and B do not have interior, f is irreducible. Thus, f can not be simple. Finally, we have the following result by similar techniques seen in the proof of Theorem 3.5.3.

Theorem 3.5.7. *Suppose $f : X \rightarrow Y$ is onto and finite-to-one, and X is a compact F -space. Then there is some $n \in \mathbb{N}$ so that f is $\leq n$ -to-one.*

Proof. Mimicking 3.5.3, suppose that the size of the fibers of f is unbounded. Letting $E_y = f^{-1}(y)$ for every $y \in Y$, we have that $|E_y|$ is finite. We could again create a collection of disjoint sets A_1, A_2, A_3, \dots , that would not terminate, so that $f(A_k) \subset f(A_j)$ whenever $j < k$. Then there would be some $x \in \bigcap_{n \in \mathbb{N}} f(A_n)$ with an infinite fiber, a contradiction. \square

Chapter 4

Conclusions and Remaining Questions

4.1 Conclusions

When beginning the search for other characteristics of complexity we wanted to know if having a remainder that is a Parovičenko space would be a characteristic of complexity, particularly within $K(\mathbb{N})$. Under CH it certainly would be, since then that remainder would be homeomorphic to \mathbb{N}^* . Yet, we've discovered that a finite-to-one mapping on \mathbb{N}^* can cause great changes to the space and that in general, the finite-to-one image of a Parovičenko space is not necessarily a Parovičenko space. As seen in the previous chapter, zero-dimensional, F-space, and G-int are not invariant under all finite-to-one mappings.

We found that it was not sufficient to discuss finite-to-one mappings on \mathbb{N}^* without considering the simple mappings. There are finite-to-one mappings on \mathbb{N}^* that are reducible and others that are irreducible. We know that a mapping f^β on $\beta\mathbb{N}$ that has a generating, finite-to-one surjection f on \mathbb{N} (i.e. simple), will be reducible and finite-to-one. But, not all mappings on $\beta\mathbb{N}$ are related to mappings on \mathbb{N} . If there are trivial fibers of a mapping on $\beta\mathbb{N}$ and finitely many nontrivial fibers, then the mapping can not be simple.

We have shown that dual points in $K(\mathbb{N})$ created from an irreducible mapping that only collapses two points from within \mathbb{N}^* , are Parovičenko spaces. Thus, under CH

these remainders are homeomorphic to \mathbb{N}^* . Unfortunately, this does not characterize all dual points of $\beta\mathbb{N}$.

Finally, we know that for a compactification to contain a copy of $\beta\mathbb{N}$, that the compactification must have some level of complexity. Having a copy of $\beta\mathbb{N}$ in every closed, infinite subset would admit an even more complex space. Oppositely, if a compactification is nowhere F , it may not contain any copies of $\beta\mathbb{N}$ and so would be simple. Nowhere F is invariant under any continuous mapping.

4.2 Questions

There are many questions that can be asked about $K(\mathbb{N})$ and its elements. We only mention questions that could relate to this work. Two of the most interesting questions that remain unanswered are as follows.

Question 4.2.1. *With only ZFC (without CH), are the remainders of the dual points of $\beta\mathbb{N}$ homeomorphic to \mathbb{N}^* ?*

Question 4.2.2. *With only ZFC, how could one characterize all the compactifications of \mathbb{N} whose remainders are homeomorphic to \mathbb{N}^* ?*

Eric van Douwen's work on ≤ 2 -to-one images of \mathbb{N}^* has pointed out other characteristics of compactifications within the lattice $K(\mathbb{N})$ [vD93].

Question 4.2.3. *Is irresolvability a characteristic of simple compactifications?*

Question 4.2.4. *Is perfectly disconnected a characteristic of simple compactifications?*

Question 4.2.5. *What characteristics would the image of \mathbb{N}^* under an n -to-one mapping hold?*

4.3 Correction

During this work we discovered an article published in the American Mathematical Society journal *Proceedings*, entitled *Function Algebras and the Lattice of Compactifications* by Franklin Mendivil [Me99]. In this article Mendivil discussed conditions for which $K(X) \cong K(Y)$, for two spaces X and Y . He also proves that $K(X) \cong K(Y)$ if and only if $C^*(X)/C_0(X) \cong C^*(Y)/C_0(Y)$, where $C_0(X) = \{f \in C^*(X) : f^\beta|_{(\beta X - X)} = 0\}$. In the introductory section he makes the following observation.

Remark 4.3.1. *Define $W(\alpha) = \{\delta : \delta < \alpha\}$ for any ordinal α , and place on it the order topology. Then $K(W(\omega_\alpha)) \cong K(W(\omega_\delta))$ if $\alpha, \delta \geq 1$ since $K(W(\omega_\alpha))$ is trivial for $\alpha \geq 1$.*

We note that this observation will not hold for $W(\omega_\omega)$. For ω_α where $cf(\omega_\alpha) > \omega$, $K(W(\omega_\alpha)) = \{\beta(W(\omega_\alpha))\} = \{\Omega(W(\omega_\alpha))\}$, thus the lattices are trivial. We know that the cofinality of ω_ω is ω . It is also known that if there exists a perfect map from X to Y then for $R(X) = \{\text{remainders of } X\}$, $R(Y) \subset R(X)$.

Example 4.3.2. *Let $U_1 = [0, \dots, \omega + 1]$, $U_2 = [\omega + 2, \dots, \omega_1 + 1]$, and continuing in this manner,*

$$U_n = [\omega_{n-2} + 2, \dots, \omega_{n-1} + 1].$$

Note that each U_n is clopen and compact, and that

$$\bigcup_{n \in \mathbb{N}} U_n = \omega_\omega.$$

Now define a map h from ω_ω to ω , so that $h(U_n) = n$. This mapping is continuous, since points map back to clopen sets, and it is also perfect. Thus, $R(\omega) \subset R(\omega_\omega)$ and so $K(W(\omega_\alpha)) \not\cong K(W(\omega_\omega))$, when $\alpha \geq 1$ and $\alpha \neq \omega$.

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