ABSTRACT

Zhang, Ying. Testing for Unit Roots in Seasonal Time Series With Long Period. (Under the direction of Professor David Dickey).

Testing for seasonal unit roots has been discussed extensively in the literature. However, the test will be difficult if the time series has a long period, where the critical values for the test statistics are not available. We modify the seasonal unit roots test of Dickey, Hasza, and Fuller (1984) to investigate results for less typical, long period cases, and present some asymptotic normality properties. We also suggest an empirical adjustment to improve the normal approximation when the seasonal period is not sufficiently long.

The basic idea is to use a double-index form \( \{ Y_{ij}: 1 \leq i \leq m, 1 \leq j \leq d, md = n \} \) for the seasonal time series with a long period, where \( d \) denotes the large lag number, so that the \( d \) “channels” will be independent for each \( i \). By applying the Classical Central Limit Theorem for iid random variables, we can obtain the asymptotic result. The convergence is proved to be order independent with respect to \( m \) and \( d \).

An advantage of this technique is that one can make the adjustment and use a standard normal as a reference distribution instead of looking into the seasonal percentile tables when doing the seasonal unit roots test, no matter what kind of deterministic terms are included in the model as long as the number of the regressors is fixed. We also show that for an AR(\( p \)) model we still obtain the asymptotic normality of the unit root statistics.
Testing For Unit Roots In Seasonal Time Series With Long Period

by

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DEDICATION

To my parents

To my husband Shuo
and our daughter Grace
Ying Zhang was born on January 24th, 1979 to Daxin Zhang and Chuchu Qiu in Hefei, Anhui Province, P. R. China. She received her Bachelor of Science degree in Management Information System from University of Science and Technology of China in Hefei, China in 2001. From August 2001 to July 2003, she studied in the Department of Mathematics and Statistics at Oakland University for her Master of Science degree in Applied Statistics. She then joined the graduate program in the Department of Statistics at North Carolina State University in August 2003. She is currently pursuing her Ph.D. degree in Statistics under the guidance of Professor David Dickey.
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Chapter 1

Introduction and Overview

1.1 Motivation

Time series quite often show patterns that repeat periodically, for example monthly retail sales data. If seasonality is very regular, seasonal dummy variables can be used to fit the model and explain the monthly effects. An alternate model that is useful when the seasonality changes over the years is the seasonal unit root model. Seasonal ARMA error terms can be added to make some local modifications. Motivated by Box and Jenkins’s approach to model international airline ticket sales, quite a few studies have discussed seasonal unit roots test approaches, and the tables of critical values for testing seasonal unit roots are available (Dickey, Hasza, Fuller, 1984, henceforth “DHF”) for seasonal periods \( d = 2, 4, \) and 12. It is well known that the limiting distributions of the seasonal unit root tests are non-normal, as in ordinary \((d=1)\) unit root tests, and the nonstandard distributions shift when typical deterministic inputs are included in the model. However, it is very possible that a user may want to test for unit roots at a longer seasonal period which is not included in the DHF tables. For example one might suspect periodicity 24 in hourly data or 365 in daily data and hence might ask if unit roots at those lags give an appropriate model. We work on the large \(d\) case and obtain the asymptotic normality for univariate unit root tests in this dissertation.
This chapter is organized as follows. In section 2 we briefly review the existing approaches to test unit roots in seasonal time series and discuss their advantages and disadvantages. In section 3 we give an outline of the dissertation.

1.2 Literature Review

Since Dickey and Fuller’s (1979) seminal work, unit roots test has been more and more discussed. Testing for unit roots in seasonal time series has also seen a revived interest in recent years.

Dickey, Hasza, and Fuller (1984) have investigated unit roots test in seasonal time series. In their paper, the regression estimators of coefficients in seasonal autoregressive models were studied. The limiting distributions for the estimators of coefficients were proposed. The percentiles of the distributions for the time series that have unit roots at the seasonal lag were obtained by Monte Carlo simulation. The test was also proved to be appropriate for higher-order models. However, a major drawback is that it does not allow for unit roots at all seasonal frequencies.

Hylleberg, Engle, Granger, and Yoo (1990, henceforth “HEGY”) extended the unit roots test to the seasonal frequencies. They studied quarterly data and developed a general test which takes care of all the seasonal frequencies as well as the zero frequency. The test was applied to the UK consumption and income data to examine cointegration at different frequencies. Beaulieu and Miron (1993) derived the mechanics and asymptotic distributions of the HEGY procedure for monthly data and applied the quarterly and monthly procedure to aggregate US data. However, Taylor (1997) showed that the HEGY procedure is highly sensitive to the form of deterministic components and the lag structure as well.

Ghysels, Lee, and Noh (1994, henceforth “GLN”) conducted a Monte Carlo study to investigate the properties of the DHF and HEGY tests. They concluded that the HEGY procedure is more useful in the sense that it could test separate unit roots at each frequency without assuming other unit roots are present, while DHF test does not. Taylor (2003) confirmed the validity of this conjecture. GLN’s study
also showed that the two tests both suffer from size distortion, or low power against plausible alternatives when the size is adequate.

Burridge and Taylor (2004) proposed bootstrap version of the HEGY tests, which were shown to have comparable power to the exact HEGY tests, and therefore recommended.

The tests discussed above were all regression-based tests. Based on Hansen’s (1992) general theory on parameter stability testing, Canova and Hansen (1995, henceforth “CH”) derived score-based Lagrange multiplier tests which test unit roots at some or all of the seasonal frequencies, except for zero frequency. This test could be considered as a complement to the existing DHF and HEGY etc, since CH take the null hypothesis to be stationary seasonality.

Hylleberg (1995) conducted Monte Carlo experiments and compared the HEGY test and the CH test. He noted that the size and power properties of the CH test could be adversely affected when there exists a unit root at other frequencies. Taylor (2003) also addressed this problem in his paper and proposed a practical solution: to prefilter out unit roots at all the frequencies not directly under test before carrying out the test.

By extending HEGY’s procedures and percentiles for quarterly data, Smith and Taylor (1998) obtained new tests and additional asymptotic results and critical values, which replaced the single time trend variable in the HEGY procedure by trend variables that can differ across the seasons.

Burridge and Taylor (2001) analyzed the regression-based tests for seasonal unit roots with higher-order serial correlation, and showed that for the quarterly case, the common assumption that the serial correlation could be accommodated by including appropriate lagged seasonal differences in the regression is only partially correct for the $t$ tests. Thus they recommended to use the pivotal $F$ statistic. Rodrigues and Taylor (2004) also generalized the existing regression-based seasonal unit root tests, and derived the asymptotic distributions for them. Their results from the simulation lend further weight to the advice that the $F$-statistic is preferable for inference, because of the significant dependence on serial correlation nuisance parameters for the
Lee and Dickey (2004) proposed seasonal unit root test approaches based on unconditional maximum likelihood. They studied the limiting distributions of the unconditional maximum likelihood statistics and produced percentiles in five different models. Their test was shown to have substantial power advantages comparing with ordinary least square (henceforth “OLS”) statistics.

Breitung and Franses (1998) and Rodrigues (2002) extended Schmidt and Phillips’s (1992) first-difference (FD)-based de-trending method to the HEGY regression and found power gains over the existing OLS-de-trending tests.

While existing methods de-mean and de-trend the series over the sample data, Taylor (2002) proposed a similar test with respect to initial value and the drift parameters, but using recursive seasonal de-meaning and de-trending technique. This test was proved to have the best size and power properties compared with all the other regression-based statistics computed by OLS, simple symmetric least square (henceforth “SSLS”), and weighted symmetric least square (henceforth “WSLS”).

Motivated by Leybourne’s (1995) idea of using forward and reverse regression in regular unit roots test, Leybourne and Taylor (2003) suggested a set of regression-based statistics which combine conventional HEGY-type statistics computed from forward and reverse estimation of the auxiliary regression equation. This set of statistics were also showed to be better than other regression-based statistics in terms of size and power properties.

Taylor (2003) introduced a class of locally optimal invariant tests. He adopted King and Wu’s (1997) framework to develop a locally mean most powerful invariant (LMMPI) test, which allowed for deterministic linear seasonal trend functions. Non-parametric modifications of the test statistics whose limiting distributions were free of nuisance parameters were also proposed.

Motivated by the SSLS and WSLS estimators in the non-seasonal unit roots test, Rodrigues and Taylor (2004) derived new tests by applying the SSLS and WSLS estimators to the HEGY test regression. The new tests were proved to dominate those tests based on the HEGY statistics in terms of asymptotic local power.
One drawback of the unit roots test is its low power. Although some approaches such as the weighted symmetric estimator (Pantula, Gonzalez-Farias and Fuller, 1994 and Park and Fuller, 1995) have been developed to improve this problem, there exists no uniformly most powerful test. Elliott, Rothenberg, and Stock (1996, henceforth “ERS”) derived an asymptotic power envelope for point-optimal tests. They claimed that the power of the tests family whose asymptotic power functions are tangent to the power envelope at one point are never far below the envelope. They also proposed a modified version of the Dickey-Fuller $t$ test—DF-GLS (Generalized Least Square) test, and got great power improvement when there is an unknown mean or a linear trend in the model. The DF-GLS test is shown to be approximately uniformly most powerful invariant. Rodrigues and Taylor (2007) applied ERS’s local GLS-de-trending method for non-seasonal unit root tests. They derived asymptotic local power bounds for seasonal unit root tests and proposed near-efficient regression-based (HEGY) seasonal unit roots tests. The tests were shown to have significant power gains over existing OLS-de-trending tests.

1.3 Outline

The remainder of this dissertation is organized as follows.

In Chapter 2, we introduce the study for the unit roots test in seasonal time series with a long period. We consider three commonly used models and derive the asymptotic results for the testing statistics.

In Chapter 3, we investigate the key properties and generalize our findings. We also prove that the theorem works for the higher order model.

In Chapter 4, we propose some suggested adjustment to improve our normal approximation when the series period is not sufficiently long. Suggestions are proposed for some commonly used models.

In chapter 5, we present our Monte Carlo study to show the finite sample behavior for our theorem. We show the normal probability plots and percentiles for the test statistics. Size and power are also obtained from the simulation.
Chapter 6 includes an application of proposed asymptotic normality result and suggested adjustment to Natural Gas Storage data reported on the Department of Energy’s Energy Information Agency web page.

The last chapter contains conclusions of our research study and some perspectives on future works.
Chapter 2

Asymptotic Results: Some Commonly Used Models

2.1 Model I: No Mean Model

Testing for unit roots in seasonal time series \( Y_t = \rho Y_{t-d} + e_t \), when \( d = 2, 4 \) and 12 has been discussed in the literature. We investigate the situation when the time series has a long period, say \( d = 365 \). We know that the periodicity for the seasonality does not change (for example, we cannot change the number of days in a year), but we would still like to study the asymptotic properties, so that we can use them to approximate the finite sample behavior when the period is reasonably long, i.e., \( d \) is reasonably large.

The OLS estimator for \( \rho \) is

\[
\hat{\rho} = \frac{\sum_{t=1}^{n} Y_{t-d}Y_t}{\sum_{t=1}^{n} Y_{t-d}^2}.
\]

(2.1)

The model can also be written in double index form as

\[
Y_{i,j} = \rho Y_{i-1,j} + e_{i,j},
\]

(2.2)

where the \( e_{i,j} \) are iid \((0, \sigma^2)\), \( i = 1, \ldots, m \) (ith year), \( j = 1, \ldots, d \) (jth day), \( n = md \).
We assume \( Y_{0,j} \) to be 0, for \( j = 1, \ldots d \).

The OLS estimator for \( \rho \) is

\[
\hat{\rho} = \frac{\sum_{i=1}^{m} \sum_{j=1}^{d} Y_{i-1,j}Y_{i,j}}{\sum_{i=1}^{m} \sum_{j=1}^{d} Y_{i-1,j}^2}. \tag{2.3}
\]

A properly normalized test statistic under the null hypothesis that \( \rho = 1 \) is:

\[
m\sqrt{d}(\hat{\rho} - 1) = \frac{1}{m\sqrt{d}} \sum_{i=1}^{m} \sum_{j=1}^{d} Y_{i-1,j}e_{i,j}. \tag{2.4}
\]

**Theorem 2.1** Under model (2.2), we have \( m\sqrt{d}(\hat{\rho} - 1) \overset{d}{\longrightarrow} N(0, 2) \) as \( m \to \infty \) and \( d \to \infty \).

**Proof:**

Recall that the estimator is a ratio of two sums. The numerator is the sum of \( d \) independent and identically distributed pieces, \( N_j = \sum_{i=1}^{m} Y_{i-1,j}e_{i,j} \) (in this simple model, days are independent of each other in the same year), divided by \( \sqrt{d} \), and by the classical central limit theorem (for iid random variables with zero mean and finite variance), this normalized sum converges to a normal variate with mean zero and some variance. The denominator is the sum of \( d \) independent and identically distributed pieces, \( D_j = \sum_{i=1}^{m} Y_{i-1,j}^2 \), divided by \( d \), and converges in probability to the expectation of the \( D_j \)'s. The following is a detailed proof:

A normalized \( \rho \) statistic is

\[
m\sqrt{d}(\hat{\rho} - 1) = \frac{1}{m\sqrt{d}} \sum_{i=1}^{m} \sum_{j=1}^{d} Y_{i-1,j}e_{i,j} = \frac{1}{m\sqrt{d}} \sum_{j=1}^{d} \sum_{i=1}^{m} Y_{i-1,j}e_{i,j}.
\]

Define

\[
N_j = \sum_{i=1}^{m} Y_{i-1,j}e_{i,j},
\]
\[ D_j = \sum_{i=1}^{m} Y_{i-1,j}^2, \]

\[ j = 1, \ldots, d, \text{ } N_j \text{'s are independent and identically distributed, and } D_j \text{'s are independent and identically distributed, with second moments (Fuller, 1996, p. 547):} \]

\[
\begin{align*}
E(N_j) &= 0, \\
\text{Var}(N_j) &= \frac{1}{2} m(m - 1)\sigma^4,
\end{align*}
\]

\[
\begin{align*}
E(D_j) &= \frac{1}{2} m(m - 1)\sigma^2, \\
\text{Var}(D_j) &= \frac{1}{3} m(m - 1)(m^2 - m + 1)\sigma^4.
\end{align*}
\]

When \( m \) and \( d \) increase, we have:

\[
\frac{1}{d} \sum_{j=1}^{d} \frac{1}{m} \frac{1}{\sigma^2} N_j \overset{d}{\rightarrow} N(0, \frac{1}{2}),
\]

and

\[
\frac{1}{d} \sum_{j=1}^{d} \frac{1}{m^2} \frac{1}{\sigma^2} D_j \overset{p}{\rightarrow} \frac{1}{2}.
\]

The appropriately normalized estimator is

\[ m\sqrt{d}(\hat{\rho} - 1) = \frac{1}{m\sqrt{d}} \sum_{j=1}^{d} N_j \frac{1}{m\sqrt{d}} \sum_{j=1}^{d} D_j. \]

Therefore, by Slutsky’s theorem,

\[ m\sqrt{d}(\hat{\rho} - 1) \overset{d}{\rightarrow} N(0, 2). \]
Alternatively,

\[
m\sqrt{d}(\hat{\rho} - 1) = \frac{1}{m\sqrt{d}} \sum_{i=1}^{m} \sum_{j=1}^{d} Y_{i-1,j} c_{i,j} = \frac{1}{d\sqrt{d}} \sum_{j=1}^{d} \left( \frac{1}{m} \sum_{i=1}^{m} Y_{i-1,j} c_{i,j} \right),
\]

and we consider sequential limits. When \( m \to \infty \),

\[
m\sqrt{d}(\hat{\rho} - 1) \xrightarrow{d} \frac{1}{\sqrt{d}} \frac{1}{d} \sum_{j=1}^{d} (W_j^2(1) - 1) \int_{0}^{1} W_j^2(r) dr,
\]

(2.10)

where \( \int_{0}^{1} W_j^2(r) dr \) are well defined random variables by Fuller (1996) and \( W_j \)'s are independent of each other. Thus the sum in the numerator is distributed as \( (\chi_d^2 - d)\sigma^2 \). As \( d \to \infty \),

\[
\frac{1}{d} \sum_{j=1}^{d} \int_{0}^{1} W_j^2(r) dr \xrightarrow{p} E\left( \int_{0}^{1} W_j^2(r) dr \right) = \frac{1}{2},
\]

(2.11)

and, since \( (\chi_d^2 - d)/\sqrt{d} \xrightarrow{d} N(0, 2) \) as \( d \to \infty \), we have

\[
\frac{1}{2\sqrt{d}} \sum_{j=1}^{d} (W_j^2(1) - 1) \xrightarrow{d} N(0, \frac{1}{2}).
\]

(2.12)

Again, by Slutsky’s theorem and the central limit theorem for iid random variables, \( m\sqrt{d}(\hat{\rho} - 1) \xrightarrow{d} N(0, 2) \).

The Studentized regression statistic for testing the hypothesis \( H_0: \rho = 1 \) is

\[
\tau = \frac{\hat{\rho} - 1}{\sqrt{\frac{1}{\sigma^2} \sum_{i=1}^{m} \sum_{j=1}^{d} Y_{i-1,j}^2}},
\]

(2.13)

where

\[
\sigma^2 = \frac{1}{n} \sum_{t=1}^{n} (Y_t - \hat{\rho}Y_{t-d})^2.
\]

(2.14)

It is automatically computed by most statistical software packages, thus we would also like to study its asymptotic properties.

**Corollary 2.1** Under model (2.2), the Studentized regression statistic \( \tau \xrightarrow{d} N(0, 2) \).
\( N(0,1) \) as \( m \to \infty \) and \( d \to \infty \).

**Proof:**

The \( \tau \)-statistic is:

\[
\tau = \frac{\hat{\rho} - 1}{\sqrt{\hat{\sigma}^2} \sum_{i=1}^{m} \sum_{j=1}^{d} Y_{i-1,j}^2} = \frac{\sum_{i=1}^{m} \sum_{j=1}^{d} Y_{i-1,j} e_{i,j}}{\hat{\sigma} \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{d} Y_{i-1,j}^2}} = \frac{\sum_{j=1}^{d} N_j}{\hat{\sigma} \sqrt{\sum_{j=1}^{d} D_j}}. \tag{2.15}
\]

Also, under the null hypothesis that \( \rho = 1 \), we have

\[
\hat{\sigma}^2 = \frac{1}{n} \sum_{t=1}^{n} (Y_t - \hat{\rho} Y_{t-d})^2
\]
\[
= \frac{1}{md} \sum_{i=1}^{m} \sum_{j=1}^{d} (Y_{i,j} - \hat{\rho} Y_{i-1,j})^2
\]
\[
= \frac{1}{md} \sum_{i=1}^{m} \sum_{j=1}^{d} (e_{i,j} - (\hat{\rho} - \rho) Y_{i-1,j})^2
\]
\[
= \frac{1}{md} \sum_{i=1}^{m} \sum_{j=1}^{d} e_{i,j}^2 - (\hat{\rho} - \rho) \frac{1}{md} \sum_{i=1}^{m} \sum_{j=1}^{d} Y_{i-1,j} e_{i,j}
\]
\[
= \frac{1}{md} \sum_{i=1}^{m} \sum_{j=1}^{d} e_{i,j}^2 - (\hat{\rho} - \rho) \frac{1}{md^2} \sum_{i=1}^{m} \sum_{j=1}^{d} Y_{i-1,j} e_{i,j}
\]
\[
= \frac{1}{md} \sum_{i=1}^{m} \sum_{j=1}^{d} e_{i,j}^2 - O_p(\frac{1}{md}) \frac{1}{md} O_p(1)
\]
\[
= \frac{1}{md} \sum_{i=1}^{m} \sum_{j=1}^{d} e_{i,j}^2 + O_p(\frac{1}{md}) \xrightarrow{p} \sigma^2.
\]

Therefore, by Slutsky’s theorem and the central limit theorem for iid random variables,

\[
\tau = \frac{\sum_{j=1}^{d} N_j}{\hat{\sigma} \sqrt{\sum_{j=1}^{d} D_j}} = \frac{\frac{1}{d} \sum_{j=1}^{d} \frac{1}{m} \frac{1}{\sigma^2} N_j}{\sqrt{\frac{1}{d} \sum_{j=1}^{d} \frac{1}{m^2} \frac{1}{\sigma^2} \sum_{j=1}^{d} D_j}} \xrightarrow{d} N(0,1).
\]
2.2 Model II: Seasonal Mean Model

Consider the model with seasonal mean, $Y_{i,j} = \mu_j + \rho Y_{i-1,j} + e_{i,j}$, where the $e_{i,j}$ are iid $(0, \sigma^2)$, $i = 1, \ldots, m$, $j = 1, \ldots, d$, $n = md$. We assume $Y_{0,j}$ to be 0 for $j = 1, \ldots, d$. The OLS estimator for $\rho$ is:

$$\hat{\rho}_\mu = \frac{\sum_{i=1}^{m} \sum_{j=1}^{d} (Y_{i,j} - \bar{Y}_{0,j})(Y_{i-1,j} - \bar{Y}_{(-1),j})}{\sum_{i=1}^{m} \sum_{j=1}^{d} (Y_{i-1,j} - \bar{Y}_{(-1),j})^2},$$

(2.16)

where $\bar{Y}_{r,j} = \frac{1}{m} \sum_{i=1}^{m} Y_{i+r,j}$.

Thus we have:

$$m \sqrt{d} (\hat{\rho}_\mu - 1) = \frac{1}{m \sqrt{d}} \sum_{i=1}^{m} \sum_{j=1}^{d} (Y_{i-1,j} - \bar{Y}_{(-1),j})e_{i,j} = \frac{1}{m \sqrt{d}} \sum_{j=1}^{d} N_j,$$

(2.17)

Throughout we will use generic symbols $N$ and $D$ for numerator and denominator terms, redefining them for each model considered. At this point we redefine the numerator, $N_j$, and denominator, $D_j$, terms for the seasonal means model as $N_j = \sum_{i=1}^{m} (Y_{i-1,j} - \bar{Y}_{(-1),j})e_{i,j}$ and $D_j = \sum_{i=1}^{m} (Y_{i-1,j} - \bar{Y}_{(-1),j})^2$ with first and second moments:

$$E(N_j) = -\frac{m-1}{2} \sigma^2,$$

(2.18)

$$\text{Var}(N_j) = \frac{(m-1)(m+7)}{12} \sigma^4,$$

(2.19)

$$E(D_j) = \frac{m^2 - 1}{6} \sigma^2,$$

(2.20)

and

$$\text{Var}(D_j) = \frac{(m-1)(m+1)(2m^2 + 7)}{90} \sigma^4.$$
For a fixed $j$, since (Fuller, 1996, p. 239)

\[
\frac{1}{m} \cdot \frac{1}{\sigma^2} N_j = \frac{1}{m} \cdot \frac{1}{\sigma^2} \sum_{i=1}^{m} Y_{i-1,j} e_{i,j} - \frac{1}{m^{3/2}} \cdot \frac{1}{\sigma} \sum_{i=1}^{m} Y_{i-1,j} - \frac{1}{\sigma} \sum_{i=1}^{m} e_{i,j}
\]

\[
\xrightarrow{d} \int_0^1 W_j(r) dW_j(r) - \int_0^1 W_j(r) dr \int_0^1 dW_j(r)
\]

\[
= \frac{1}{2} (W_j^2(1) - 1) - W_j(1) \int_0^1 W_j(r) dr
\]

and

\[
\frac{1}{m^2} \cdot \frac{1}{\sigma^2} D_j = \frac{1}{m^2} \cdot \frac{1}{\sigma^2} \left( \sum_{i=1}^{m} Y_{i-1,j} - m \bar{Y}_{(-1),j}^2 \right)
\]

\[
= \frac{1}{m^2} \cdot \frac{1}{\sigma^2} \sum_{i=1}^{m} Y_{i-1,j} - \left( \frac{1}{m^{3/2}} \cdot \frac{1}{\sigma} \sum_{i=1}^{m} Y_{i-1,j} \right)^2
\]

\[
\xrightarrow{d} \int_0^1 W_j^2(r) dr - \left( \int_0^1 W_j(r) dr \right)^2,
\]

we have, for a fixed $d$,

\[
m \sqrt{d} (\hat{\rho} - 1) = \frac{1}{m^d} \sum_{j=1}^{d} \frac{1}{\sigma^2} N_j
\]

\[
= \frac{1}{m^d} \sum_{j=1}^{d} \frac{1}{m} \cdot \frac{1}{\sigma^2} N_j
\]

\[
= \frac{1}{d} \sum_{j=1}^{d} \frac{1}{m} \cdot \frac{1}{\sigma^2} D_j
\]

\[
\xrightarrow{d} \frac{1}{d} \sum_{j=1}^{d} \left[ \frac{1}{2} (W_j^2(1) - 1) - W_j(1) \int_0^1 W_j(r) dr \right] - \left( \int_0^1 W_j(r) dr \right)^2,
\]

as $m \to \infty$.

**BIAS CORRECTION:**

The numerator in (2.17) has expected value $-(m-2)\sqrt{d} \sigma^2$. The denominator has expected value $m^{-2} \sigma^2$. Thus a bias correction would be needed to do large $d$ asymptotics. We do not pursue this here as it seems unreasonable to fit seasonal means
when \( d \) is large, as would happen with daily data. Instead we work with smooth functions of time.

### 2.3 Model III: Slowly Evolving Mean Model

Now we look at models with a slowly evolving mean function. We claim that under such a condition, the limit distribution of the statistic is exactly the same as that in Model I. We start with three cases of specific interest and then generalize.

Throughout we use some standard results on projections. Let \( Y = X\beta + Z\gamma + e \), where \( Z \) is a single column and \( e \) is a vector of iid variates with mean 0 and finite variance. The regression of \( Y \) on \((X : Z)\) can be done in three steps if \( \gamma \) is the focus of our study.

1. Regress \( Y \) on \( X \) getting \( \hat{\beta} \) and residuals \( R_y \)
2. Regress \( Z \) on \( X \) getting an estimate and residuals \( R_z \)
3. Regress \( R_y \) on \( R_z \) getting \( \hat{\gamma} \)

#### 2.3.1 Single Mean Model

To illustrate, we now look at a simple model with a single mean:

\[
Y_t = \mu + W_t, \tag{2.22}
\]

where \( W_t = \rho_d W_{t-d} + e_t \), and the \( e_t \) are iid \((0, \sigma^2)\), \( t = 1, \ldots, n \). Using the double index notation we write

\[
Y_{i,j} = \mu + W_{i,j}, \tag{2.23}
\]

where \( W_{i,j} = \rho_d W_{i-1,j} + e_{i,j} \), and the \( e_{i,j} \) are iid \((0, \sigma^2)\), \( i = 1, \ldots, m \), \( j = 1, \ldots, d \), \( n = md \). We assume \( W_{0,j} \) to be 0 for \( j = 1, \ldots, d \). The model can also be written as:

\[
Y = X\beta + W, \tag{2.24}
\]
with \( X = 1 \), a column of 1’s. Take span \( d \) differences and regress \( e \) on 1, \( W_{(-1)} \), where \( e \) is the vector of \( e_t \)’s \((W_t - W_{t-d})\) which are iid with mean 0 and finite variance, and \( W_{(-1)} \) is the vector of \( W_{t-d} \) (lagged \( W_t \)). The \( \rho \) statistic is:

\[
\hat{\rho}_d - 1 = \frac{e'(I - 1(1')^{-1}1')W_{(-1)}}{W_{(-1)}'(I - 1(1')^{-1}1')W_{(-1)}}. \tag{2.25}
\]

A properly normalized statistic is:

\[
m\sqrt{d}(\hat{\rho}_d - 1) = \frac{-\frac{1}{m\sqrt{d}}e'(I - 1(1')^{-1}1')W_{(-1)}}{m\sqrt{d}W_{(-1)}'(I - 1(1')^{-1}1')W_{(-1)}} \frac{-\frac{1}{m\sqrt{d}}W_{(-1)'}}{m\sqrt{d}W_{(-1)}'(I - \frac{1}{n}1'1')W_{(-1)}} \frac{-\frac{1}{m\sqrt{d}}e'W_{(-1)} - \frac{1}{n}(e'1)(1'W_{(-1)})}{m\sqrt{d}[W_{(-1)}'W_{(-1)} - \frac{1}{n}(W_{(-1)'1}(1'W_{(-1)}))].}
\]

**Lemma 2.1** Using the definitions above, \( 1'W_{(-1)} = O_p(\sqrt{m^3d}) \).

**Proof:**

We know that \( W_{i,j} = \sum_{t=1}^{i} e_{t,j} \), thus, \( 1'W_{(-1)} = \sum_{i=1}^{m} \sum_{j=1}^{d} W_{i-1,j} \) with mean 0 and variance

\[
\sum_{j=1}^{d} \text{Var} \left( \sum_{i=1}^{m} W_{i-1,j} \right) = d \cdot E \left( \sum_{i=1}^{m} W_{i-1,1} \right)^2
\]

\[
= d \cdot E( (m-1)e_{11} + (m-2)e_{21} + \cdots + e_{m-1,1})^2
\]

\[
= d\sigma^2 \sum_{t=1}^{m-1} t^2
\]

\[
= d \frac{(m-1)m(2m-1)}{6} \sigma^2
\]

\[
= O(dm^3).
\]

This completes the proof.

**Theorem 2.2** Under model (2.23), we have \( m\sqrt{d}(\hat{\rho}_d - 1) \overset{d}{\longrightarrow} N(0,2) \) as \( m \to \infty \) and \( d \to \infty \).
Proof:

By Lemma 2.1,

\[
\frac{1}{m^2d}[\mathbf{W}'(-1)\mathbf{W}(-1) - \frac{1}{md}(\mathbf{W}'(-1)\mathbf{1})(\mathbf{I}'\mathbf{W}(-1))] = \frac{1}{m^2d}\mathbf{W}'(-1)\mathbf{W}(-1) - \frac{1}{m^3d^2}O_p(m^3d)
\]

\[
= \frac{1}{m^2d}\mathbf{W}'(-1)\mathbf{W}(-1) + O_p\left(\frac{1}{d}\right)
\]

and \(\mathbf{e}'\mathbf{1} = O_p(\sqrt{md})\). Thus, the numerator

\[
\frac{1}{m\sqrt{d}}[\mathbf{e}'\mathbf{W}(-1) - \frac{1}{md}(\mathbf{e}'\mathbf{1})(\mathbf{I}'\mathbf{W}(-1))] = \frac{1}{m\sqrt{d}}\mathbf{e}'\mathbf{W}(-1) - \frac{1}{m^2\sqrt{d^3}}O_p(m\sqrt{d} \cdot \sqrt{m^3d})
\]

\[
= \frac{1}{m\sqrt{d}}\mathbf{e}'\mathbf{W}(-1) + O_p\left(\frac{1}{\sqrt{d}}\right)
\]

Therefore, by Slustky’s theorem, the difference between \(m\sqrt{d}(\hat{\rho}_d - 1)\) and \(m\sqrt{d}(\hat{\rho} - 1)\) goes to 0, where \(m\sqrt{d}(\hat{\rho} - 1)\) is the normalized \(\rho\) statistic that we have studied in model I (the no mean model).

The Studentized regression statistic for testing the hypothesis \(H_0: \rho = 1\) is

\[
\tau = \frac{\hat{\rho} - 1}{\sqrt{\hat{\sigma}^2 W'_{(-1)}(\mathbf{I} - \mathbf{1}'(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1})^{-1}W_{(-1)}}},
\]

where

\[
\hat{\sigma}^2 = \frac{1}{n} \left\{ \mathbf{e}'\mathbf{e} - \frac{[\mathbf{e}'(\mathbf{I} - \mathbf{1}'(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1})\mathbf{W}(-1)]^2}{\mathbf{W}'_{(-1)}(\mathbf{I} - \mathbf{1}'(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1})^{-1}\mathbf{W}(-1)} \right\}.
\]

Corollary 2.2 Under model (2.23), We have \(\tau \overset{d}{\rightarrow} N(0, 1)\) as \(m \rightarrow \infty\) and \(d \rightarrow \infty\).

Proof:
We have

\[
\hat{\sigma}^2 = \frac{1}{n} \left\{ \mathbf{e}' \mathbf{e} - \frac{[\mathbf{e}'(\mathbf{I} - \mathbf{1}'\mathbf{1}^{-1}\mathbf{I}^{-1})\mathbf{W}_{(-1)}]^2}{\mathbf{W}'_{(-1)}(\mathbf{I} - \mathbf{1}'\mathbf{1}^{-1}\mathbf{I}^{-1})\mathbf{W}_{(-1)}} \right\}
\]

\[
= \frac{1}{n} \left\{ \mathbf{e}' \mathbf{e} - \frac{[\mathbf{e}'(\mathbf{I} - \frac{1}{n}\mathbf{1}1')\mathbf{W}_{(-1)}]^2}{\mathbf{W}'_{(-1)}(\mathbf{I} - \frac{1}{n}\mathbf{1}1')\mathbf{W}_{(-1)}} \right\}
\]

\[
= \frac{1}{n} \left\{ \mathbf{e}' \mathbf{e} - [\mathbf{e}'(\mathbf{I} - \frac{1}{n}\mathbf{1}1')\mathbf{W}_{(-1)}]^2 \cdot [\mathbf{W}'_{(-1)}(\mathbf{I} - \frac{1}{n}\mathbf{1}1')\mathbf{W}_{(-1)}]^{-1} \right\}
\]

\[
= \frac{1}{n} (\mathbf{e}' \mathbf{e}) - \frac{1}{md} \cdot O_p(m^2d^2) \cdot O_p\left(\frac{1}{m^2d}\right)
\to \frac{\sigma^2}{m_d},
\]

so the \( \tau \)-statistic is also unaffected. This completes the proof.

### 2.3.2 Sinusoidal Mean Model

Now we adopt a model with a sinusoidal function of period \( d \), a model useful in seasonal time series:

\[
Y_t = \mu + A_1 \sin\left(\frac{2\pi}{d} t\right) + A_2 \cos\left(\frac{2\pi}{d} t\right) + W_t,
\]

(2.28)

\( W_t = W_{t-d} + e_t \), where the \( e_t \) are iid \((0, \sigma^2)\), \( t = 1, \ldots, n \). We assume \( W_t \) to be 0 for \( t < 1 \). The model can also be written as:

\[
\mathbf{Y} = \mathbf{X}\beta + \mathbf{W},
\]

(2.29)

where \( \mathbf{X} = \{1, \sin\left(\frac{2\pi}{d} t\right), \cos\left(\frac{2\pi}{d} t\right)\} \), \( \mathbf{W} = \mathbf{W}_{(-1)} + \mathbf{e} \), \( \mathbf{e} \) is the vector of \( e_t \)'s \((W_t - W_{t-d})\) which are iid with mean 0 and finite variance, and \( \mathbf{W}_{(-1)} \) is the vector of \( W_{t-d} \) (lagged \( W_t \)).

The \( \rho \) statistic is

\[
\hat{\rho}_d - 1 = \frac{\mathbf{e}'(\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')\mathbf{W}_{(-1)}}{\mathbf{W}'_{(-1)}(\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')\mathbf{W}_{(-1)}},
\]
and we have

\[
m\sqrt{d}(\hat{\rho}_d - 1) = \frac{1}{m^\top d}e'(I - X(X'X)^{-1}X')W_{(-1)} - \frac{1}{m^\top d}[e'W_{(-1)} - (e'X)(X'X)^{-1}(X'W_{(-1)})]
\]

**Theorem 2.3** Under model (2.29), we have \(m\sqrt{d}(\hat{\rho}_d - 1) \overset{d}{\rightarrow} N(0, 2)\) as \(m \to \infty\) and \(d \to \infty\).

**Proof:**

By the orthogonality of the Fourier columns, we have:

\[
(X'X)^{-1} = \text{diag}\{\frac{1}{n}, \frac{2}{n}, \frac{2}{n}\} = O\left(\frac{1}{n}\right).
\]

Also,

\[
X'e = \begin{pmatrix}
\sum e_t \\
\sum \sin\left(\frac{2\pi}{d}t\right)e_t \\
\sum \cos\left(\frac{2\pi}{d}t\right)e_t
\end{pmatrix}
\]

with

\[
E[X'e] = 0
\]

\[
\text{Var}[X'e] = \sigma^2 \begin{pmatrix}
\sum_{t=1}^{n} \sin\left(\frac{2\pi}{d}t\right) & \sum_{t=1}^{n} \sin^2\left(\frac{2\pi}{d}t\right) & \sum_{t=1}^{n} \cos\left(\frac{2\pi}{d}t\right) \\
\sum_{t=1}^{n} \sin\left(\frac{2\pi}{d}t\right) & \sum_{t=1}^{n} \sin^2\left(\frac{2\pi}{d}t\right) & \sum_{t=1}^{n} \cos\left(\frac{2\pi}{d}t\right) \\
\sum_{t=1}^{n} \cos\left(\frac{2\pi}{d}t\right) & \sum_{t=1}^{n} \sin\left(\frac{2\pi}{d}t\right) & \sum_{t=1}^{n} \cos^2\left(\frac{2\pi}{d}t\right)
\end{pmatrix}
\]

\[
= O(n) = O(dm).
\]
Thus,

\[ X'e = O_p(\sqrt{dm}). \] (2.32)

\[
X'W_{(-1)} = \begin{pmatrix}
\sum W_{t-d} \\
\sum \sin\left(\frac{2\pi}{d}t\right)W_{t-d} \\
\sum \cos\left(\frac{2\pi}{d}t\right)W_{t-d}
\end{pmatrix} = \begin{pmatrix}
\sum_{j=1}^{d} \sum_{i=1}^{m} W_{i-1,j} \\
\sum_{i=1}^{d} \sin\left(\frac{2\pi}{d}t\right) \sum_{i=1}^{m-1} W_{il} \\
\sum_{i=1}^{d} \cos\left(\frac{2\pi}{d}t\right) \sum_{i=1}^{m-1} W_{il}
\end{pmatrix}
\]

with

\[ E[X'W_{(-1)}] = 0 \]

\[
\text{Var}[X'W_{(-1)}] = \begin{pmatrix}
V[\sum_{j=1}^{d} \sum_{i=1}^{m} W_{i-1,j}] & \text{Cov} & \text{Cov} \\
\text{Cov} & V[\sum_{i=1}^{d} \sin\left(\frac{2\pi}{d}t\right) \sum_{i=1}^{m-1} W_{il}] & \text{Cov} \\
\text{Cov} & \text{Cov} & V[\sum_{i=1}^{d} \cos\left(\frac{2\pi}{d}t\right) \sum_{i=1}^{m-1} W_{il}]
\end{pmatrix},
\]

where Cov’s denote the covariance terms. Since the order of the covariance is always less than or equal to the geometric mean of the two corresponding variances, we only need to consider the order of the variances, and we don’t write the detail for the covariance terms here.

For the first element, by Lemma 2.1, we have:

\[
\text{Var}\left[\sum_{j=1}^{d} \left(\sum_{i=1}^{m} W_{i-1,j}\right)\right] = O(dm^3)
\] (2.33)

For the other elements: \( \sum_{i=1}^{d} \sin\left(\frac{2\pi}{d}i\right) \sum_{i=1}^{m-1} W_{il} \), the variances are:
For any given $l$, we have:

$$\sin^2 \left( \frac{2\pi}{d} l \right) E \left[ \sum_{i=1}^{m-1} W_{il} \right]^2 = \sin^2 \left( \frac{2\pi}{d} l \right) \sigma^2 \frac{(m-1)m(2m-1)}{6} < \sigma^2 \frac{(m-1)m(2m-1)}{6} = O(m^3)$$

For the cosine term, we have the same result. Thus, the variance is $O(dm^3)$, and

$$X'W_{(-1)} = O_p(\sqrt{dm^3}). \quad (2.34)$$

Therefore, the numerator of $m\sqrt{d}(\hat{\rho}_d - 1)$ is:

$$\frac{1}{m\sqrt{d}} [e'W_{(-1)} - (e'X)(X'X)^{-1}(X'W_{(-1)})] = \frac{1}{m\sqrt{d}} e'W - \frac{1}{m\sqrt{d}} \cdot O_p(\sqrt{md}) \cdot O(\frac{1}{md}) \cdot O_p(\sqrt{dm^3}) = \frac{1}{m\sqrt{d}} e'W_{(-1)} - O_p(\frac{1}{\sqrt{d}}).$$

The denominator of $m\sqrt{d}(\hat{\rho}_d - 1)$ is:
\[
\frac{1}{m^2 d} [W'_(-1) W(-1) - (W'(-1) X)(X'X)^{-1}(X'W(-1))] \\
= \frac{1}{m^2 d} W(-1) W(-1) - \frac{1}{m^2 d} \cdot O_p(\sqrt{md^3}) \cdot O_p(\frac{1}{md}) \cdot O_p(\sqrt{md^3}) \\
= \frac{1}{m^2 d} W(-1) W(-1) - O_p\left(\frac{1}{d}\right).
\]

By Slustky’s theorem, when \(d\) gets large, the difference between \(m^2 d(\hat{\rho}_d - 1)\) and \(m\sqrt{d}(\hat{\rho} - 1)\) goes to 0.

The Studentized regression statistic for testing the hypothesis \(H_0: \rho = 1\) is

\[
\tau = \frac{\hat{\rho} - 1}{\sqrt{\hat{\sigma}^2} \cdot \frac{1}{W(-1)'(I - X(X'X)^{-1}X')W(-1)}}
\]

where

\[
\hat{\sigma}^2 = \frac{1}{n} \left\{ e'e - \frac{[e'(I - X(X'X)^{-1}X')W(-1)]^2}{W(-1)'(I - X(X'X)^{-1}X')W(-1)} \right\}.
\]

(2.35)

(2.36)

**Corollary 2.3** Under model (2.29), We have \(\tau \xrightarrow{d} N(0, 1)\) as \(m \to \infty\) and \(d \to \infty\).

**Proof:**

\[
\hat{\sigma}^2 = \frac{1}{n} \left\{ e'e - \frac{[e'(I - X(X'X)^{-1}X')W(-1)]^2}{W(-1)'(I - X(X'X)^{-1}X')W(-1)} \right\} \\
= \frac{1}{n} \left\{ e'e - [e'(I - X(X'X)^{-1}X')W(-1)]^2 \cdot [W(-1)'(I - X(X'X)^{-1}X')W(-1)]^{-1} \right\} \\
= \frac{1}{n} (e'e) - \frac{1}{md} \cdot O_p(m^2 d^2) \cdot O_p\left(\frac{1}{m^2 d}\right) \\
\xrightarrow{p} \sigma^2,
\]

so the \(\tau\)-statistic is also unaffected by the regression correction. This completes the proof.
2.3.3 Linear Trend Model

Another interesting case is the model with linear trend. Consider model

\[ Y_t = \alpha + \beta t + W_t, \]  

\[ (2.37) \]

\( W_t = \rho_d W_{t-d} + e_t, \) where the \( e_t \) are iid \((0, \sigma^2)\), \( t = 1, \ldots, n \). Since \( t = d(i - 1) + j \), written in double index, the model becomes:

\[ Y_{i,j} = \alpha + \beta [d(i - 1) + j] + W_{i,j}, \]  

\[ (2.38) \]

\( W_{i,j} = \rho_d W_{i-1,j} + e_{i,j}, \) where the \( e_{i,j} \) are iid \((0, \sigma^2)\), \( i = 1, \ldots, m, j = 1, \ldots, d, n = md \).

We assume \( W_{0,j} \) to be 0 for \( j = 1, \ldots, d \). Take span \( d \) differences and regress \( e \) on 1, \( t \), and \( W(-1) \), where \( e \) is the vector of \( e_t \)’s \((W_t - W_{t-d})\), \( W(-1) \) is the vector of \( W_{t-d} \) (lagged \( W_t \)). The \( \rho \) statistic is:

\[ \hat{\rho}_d - 1 = \frac{\mathbf{e}'(\mathbf{I} - \mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{W}(-1)}{\mathbf{W}'(-1)(\mathbf{I} - \mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{W}(-1)}, \]  

\[ (2.39) \]

where \( \mathbf{X} = \left( \begin{array}{cc} 1 & t - \bar{t} \end{array} \right)_{(n \times 2)} \).

A properly normalized statistic is:

\[ m\sqrt{d}(\hat{\rho}_d - 1) = \frac{\frac{1}{m \sqrt{d}} \mathbf{e}'(\mathbf{I} - \mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{W}(-1)}{\frac{1}{m \sqrt{d}} \mathbf{W}'(-1)(\mathbf{I} - \mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{W}(-1) - \frac{1}{m \sqrt{d}} (\mathbf{W}'(-1)\mathbf{X}'\mathbf{X}^{-1}(\mathbf{X}'\mathbf{W}(-1))). \]

**Theorem 2.4** Under model (2.38), we have \( m\sqrt{d}(\hat{\rho}_d - 1) \xrightarrow{d} N(0, 2) \) as \( m \to \infty \) and \( d \to \infty \).

**Proof:**
\[(X'X)^{-1} = \begin{pmatrix} n & 0 \\ 0 & \frac{(n-1)n(n+1)}{12} \end{pmatrix}^{-1} \quad (2.40)\]

Also,
\[e'X = \left( \sum_{i=1}^m \sum_{j=1}^d e_{i,j} \quad \sum_{i=1}^m \sum_{j=1}^d (t - \bar{t})e_{i,j} \right) \quad (2.41)\]

where \(t = d(i - 1) + j\).

For the first element,
\[E\left( \sum_{i=1}^m \sum_{j=1}^d e_{i,j} \right) = 0, \]
and
\[\text{Var}\left( \sum_{i=1}^m \sum_{j=1}^d e_{i,j} \right) = md, \]
thus,
\[\sum_{i=1}^m \sum_{j=1}^d e_{i,j} = O_p(\sqrt{md}). \quad (2.42)\]

For the second element, since \(t = d(i - 1) + j\) and \(\bar{t} = d(\bar{i} - 1) + \bar{j}\), we have
\[\sum_{i=1}^m \sum_{j=1}^d (t - \bar{t})e_{i,j} = \sum_{i=1}^m \sum_{j=1}^d (d(i - \bar{i}) + (j - \bar{j}))e_{i,j}, \]
then
\[E\left[ \sum_{i=1}^m \sum_{j=1}^d (t - \bar{t})e_{i,j} \right] = 0, \]
and since
\[2d \sum_{i=1}^m \sum_{j=1}^d (i - \bar{i})(j - \bar{j})\sigma^2 = 0, \]
\[
\text{Var} \left[ \sum_{i=1}^{m} \sum_{j=1}^{d} (t - \bar{t})e_{i,j} \right] = \text{Var} \left[ \sum_{i=1}^{m} \sum_{j=1}^{d} (d(i - \bar{i}) + (j - \bar{j}))e_{i,j} \right] \\
= d^2 \sum_{i=1}^{m} \sum_{j=1}^{d} (i - \bar{i})^2 \text{Var}(e_{i,j}) + \sum_{i=1}^{m} \sum_{j=1}^{d} (j - \bar{j})^2 \text{Var}(e_{i,j}) \\
= \sigma^2 \left[ d^2 \sum_{i=1}^{m} (i - \bar{i})^2 + m \sum_{j=1}^{d} (j - \bar{j})^2 \right] \\
= O(d^3 m^3) + O(d^3 m) \\
= O(d^3 m^3).
\]

Thus,
\[
\sum_{i=1}^{m} \sum_{j=1}^{d} (t - \bar{t})e_{i,j} = O_p(\sqrt{d^3 m^3}) \quad (2.43)
\]

Next consider
\[
X'W_{(-1)} = \left( \begin{array}{c} \sum_{i=1}^{m} \sum_{j=1}^{d} W_{i-1,j} \\ \sum_{i=1}^{m} \sum_{j=1}^{d} (t - \bar{t})W_{i-1,j} \end{array} \right) \quad (2.44)
\]

For the first element, by Lemma 2.1, we have
\[
\sum_{i=1}^{m} \sum_{j=1}^{d} W_{i-1,j} = O_p(\sqrt{dm^3}). \quad (2.45)
\]

For the second element,
\[
\sum_{i=1}^{m} \sum_{j=1}^{d} (t - \bar{t})W_{i-1,j} = \sum_{i=1}^{m} \sum_{j=1}^{d} (d(i - \bar{i}) + (j - \bar{j}))W_{i-1,j}, \quad (2.46)
\]

\[
E \left[ \sum_{i=1}^{m} \sum_{j=1}^{d} (t - \bar{t})W_{i-1,j} \right] = 0
\]
\[
\begin{align*}
\text{Var} \left[ \sum_{i=1}^{m} \sum_{j=1}^{d} (t - \bar{t}) W_{i-1,j} \right] &= \text{Var} \left[ \sum_{i=1}^{m} \sum_{j=1}^{d} (d(i - \bar{i}) + (j - \bar{j})) W_{i-1,j} \right] \\
&= d^2 \sum_{j=1}^{d} \text{Var} \left[ \sum_{i=1}^{m} (i - \bar{i}) W_{i-1,j} \right] + \sum_{j=1}^{d} (j - \bar{j})^2 \text{Var} \left[ \sum_{i=1}^{m} W_{i-1,j} \right] \\
&= d^3 \text{Var} \left[ \sum_{i=1}^{m} (i - \frac{m+1}{2}) W_{i-1,j} \right] + \sum_{j=1}^{d} (j - \bar{j})^2 \text{Var} \left[ \sum_{i=1}^{m} W_{i-1,j} \right] .
\end{align*}
\]

For the first part, \(\text{Var}[\sum_{i=1}^{m} (i - \frac{m+1}{2}) W_{i-1,j}]\) can be written as a Riemann sum:

\[
\begin{align*}
\text{Var} \left[ \sum_{i=1}^{m} \left( i - \frac{m+1}{2} \right) W_{i-1,j} \right] &= \sigma^2 m^5 \sum_{i} \sum_{i'} \left( \frac{i - \frac{1}{2}}{m} - \frac{1}{2} \right) \left( \frac{i' - \frac{1}{2}}{m} - \frac{1}{2} \right) \min \left( \frac{i - \frac{1}{2}}{m}, \frac{i' - \frac{1}{2}}{m} \right) \frac{1}{m} \frac{1}{m} \\
&\sim \sigma^2 m^5 \int_{0}^{1} \int_{0}^{1} (x - \frac{1}{2}) \left( y - \frac{1}{2} \right) \min(x, y) dxdy \\
\end{align*}
\]

So the first part is \(O(d^3 m^5)\).

By Lemma 2.1, the second part is:

\[
\sum_{j=1}^{d} (j - \bar{j})^2 \text{Var} \left[ \sum_{i=1}^{m} W_{i-1,j} \right] = O(d^3)O(m^3) = O(d^3 m^3)
\]

Therefore, the second element is \(O_p(\sqrt{d^3 m^3})\).

Now, for the numerator

\[
\frac{1}{m\sqrt{d}} \mathbf{e}' \mathbf{W}_{(-1)} - \frac{1}{m\sqrt{d}} (\mathbf{e}' \mathbf{X})(\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{W}_{(-1)}),
\]

because \((\mathbf{X}'\mathbf{X})^{-1}\) is diagonal, the second part can be written as \((a) + (b)\) corresponding
to the two diagonal elements.

The order of \((a)\) is:

\[
\frac{1}{m\sqrt{d}} O_p(\sqrt{dm})O_p\left(\frac{1}{dm}\right)O_p(\sqrt{dm^3}) = O_p\left(\frac{1}{\sqrt{d}}\right),
\]

and the order of \((b)\) is:

\[
\frac{1}{m\sqrt{d}} O_p(\sqrt{d^3m^5})O_p\left(\frac{1}{d^3m^3}\right)O_p(\sqrt{d^3m^5}) = O_p\left(\frac{1}{\sqrt{d}}\right),
\]

so the second term in the numerator is \(O_p\left(\frac{1}{\sqrt{d}}\right)\) while the first is \(O_p(1)\).

For the denominator

\[
\frac{1}{m^2d} W'_{(-1)} W_{(-1)} - \frac{1}{m^2d} (W'_{(-1)} X)(X'X)^{-1}(X'W_{(-1)}),
\]

the second part can be written as \((c) + (d)\) in the same fashion as the numerator.

The order of \((c)\) is:

\[
\frac{1}{m^2d} O_p(\sqrt{d^3m^3})O_p\left(\frac{1}{dm}\right)O_p(\sqrt{dm^3}) = O_p\left(\frac{1}{d}\right),
\]

and the order of \((d)\) is:

\[
\frac{1}{m^2d} O_p(\sqrt{d^3m^5})O_p\left(\frac{1}{d^3m^3}\right)O_p(\sqrt{d^3m^5}) = O_p\left(\frac{1}{d}\right),
\]

so the second term in the denominator is \(O_p\left(\frac{1}{d}\right)\) while the first is \(O_p(1)\).

As \(d\) increases, by Slustky’s theorem, the difference between

\[
m\sqrt{d}(\hat{\rho}_d - 1) = \frac{\frac{1}{m\sqrt{d}} e'(I - X(X'X)^{-1}X')W_{(-1)}}{\frac{1}{m^2d} W'_{(-1)} (I - X(X'X)^{-1}X')W_{(-1)}}
\]

\[
= \frac{\frac{1}{m\sqrt{d}} e'W_{(-1)} - \frac{1}{m\sqrt{d}} (e'X)(X'X)^{-1}(X'W_{(-1)})}{\frac{1}{m^2d} W'_{(-1)} W_{(-1)} - \frac{1}{m^2d} (W'_{(-1)} X)(X'X)^{-1}(X'W_{(-1)})}
\]
and \( m\sqrt{d}(\hat{\rho} - 1) \) goes to 0.

The Studentized regression statistic for testing the hypothesis \( H_0: \rho = 1 \) is

\[
\tau = \frac{\hat{\rho} - 1}{\sqrt{\hat{\sigma}^2 \frac{1}{W_{(-1)}(I-X'X)^{-1}X'} W_{(-1)}}},
\]

where

\[
\hat{\sigma}^2 = \frac{1}{n} \left\{ e' e - \frac{[e'(I - X'X)^{-1}X'] W_{(-1)}]^2}{W_{(-1)}(I - X'X)^{-1}X' W_{(-1)}} \right\}.
\]

**Corollary 2.4** Under model (2.38), we have \( \tau \overset{d}{\to} N(0, 1) \) as \( m \to \infty \) and \( d \to \infty \).

**Proof:**

\[
\hat{\sigma}^2 = \frac{1}{n} \left\{ e' e - \frac{[e'(I - X'X)^{-1}X'] W_{(-1)}]^2}{W_{(-1)}(I - X'X)^{-1}X' W_{(-1)}} \right\}
\]

\[= \frac{1}{n} \{ e' e - [e'(I - X'X)^{-1}X'] W_{(-1)}^2 \cdot [W_{(-1)}(I - X'X)^{-1}X'] W_{(-1)}^{-1} \}
\]

\[= \frac{1}{n} \{ e' e - \frac{1}{md} \cdot O_p(m^2 d^2) O_p \left( \frac{1}{m^2 d} \right) \}
\]

\[\overset{p}{\to} \sigma^2
\]

Therefore, the \( \tau \) statistic is not affected by the regression correction. This completes the proof.
Chapter 3

Asymptotic Results:
Generalizations

3.1 A Generalization

Now we know that the large $d$ asymptotic normal approximation works in the sense that the $\tau$ statistics converge to the standard normal distribution for some cases. For example, asymptotic normality is obtained for the single mean case, the sine wave case, and the linear trend case. Clearly, a model with sine wave and linear trend combined works too. Unless we adjust for bias, however, we do not obtain a limiting normal distribution with mean 0 in some other cases, like the seasonal mean case. What are the key properties of the deterministic terms that make the $\tau$ statistics converge to standard normals?

We claim that the large $d$ asymptotic normal approximation works as long as the number of regressors in our model is fixed.

Theorem 3.1 Let $Y = X\beta + W$, $W = W_{(-1)} + e$. Here $W$ is a random walk, expressed as a vector, with elements $W_t$ satisfying $W_t = W_{t-d} + e_t$, $e$ is the vector of an iid sequence of $e_t$’s with mean 0 and finite variance, and $W_{(-1)}$ the vector of lagged values. It follows that $m\sqrt{d}(\hat{\rho} - 1) \xrightarrow{d} N(0, 2)$ as long as $X$ contains a fixed number of regressors.
Proof:

For clarity, assume we have daily data, and we use “years” and “days” instead of the terminology for $m$ and $d$ throughout the proof, but the result holds for any $m$ and $d$. The model is $Y = X\beta + W$, $W = W_{(-1)} + e$. Here $W$ is a random walk, expressed as a vector, $W_{(-1)}$ the vector of lagged values, and $e$ the vector of iid $e_i$'s with mean 0 and finite variance.

1. Let $V$ be the variance-covariance matrix of the $m$ Jan $1st$ values. The largest eigenvalue of $V$ is $\lambda_{max}$. We have (Fuller, 1996, p. 548):

$$
\lambda_{max} = \frac{1}{4} \sec^2 \left( \frac{(m - 1)\pi}{2m - 1} \right)
= \frac{1}{4} \cos^2 \left( \frac{(m - 1)\pi}{2m - 1} \right)
= \frac{1}{4} \cos^2 \left( \frac{\pi}{2} - \frac{\pi}{2(2m - 1)} \right)
= \frac{1}{4} \sin^2 \left( \frac{\pi}{2(2m - 1)} \right)
$$

As $m \to \infty$

$$
\frac{\pi}{2(2m - 1)} \to 0
\Rightarrow \frac{\sin \left( \frac{\pi}{2(2m - 1)} \right)}{\frac{\pi}{2(2m - 1)}} \to 1
$$
Thus

\[
\lambda_{max} = \frac{1}{4} \frac{1}{\sin^2 \left( \frac{\pi}{2(2m-1)} \right)} \left( \sin \left( \frac{\pi}{2(2m-1)} \right) \right)^2 \cdot \frac{1}{4} \cdot \frac{1}{\left( \frac{\pi}{2(2m-1)} \right)^2} = O(m^2)
\]

2. Rearrange the \( md \) observations so that all January 1\text{st} values appear first, followed by all January 2\text{nd} values, etc., and set \( Y \) as \( [Y_{11}, Y_{21}, ..., Y_{m1}, Y_{12}, Y_{22}, ..., Y_{m2}, ..., Y_{1d}, Y_{2d}, ..., Y_{md}] \). We then have a \( md \) by \( md \) block diagonal variance-covariance matrix \( A \) with \( d \) identical \( m \) by \( m \) blocks \( V \) as follows:

\[
A = \begin{pmatrix}
V & 0 & \cdots & 0 \\
0 & V & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & V
\end{pmatrix}
\]
The eigenvalues of matrix $A$ are solutions for:

$$\det(A - \lambda I_{md\times md}) = 0$$

$$\Rightarrow \det \begin{pmatrix}
V & 0 & \cdots & 0 \\
0 & V & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & V
\end{pmatrix} - \lambda I_{md\times md} = 0$$

$$\Rightarrow \det \begin{pmatrix}
V - \lambda I_{m\times m} & 0 & \cdots & 0 \\
0 & V - \lambda I_{m\times m} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & V - \lambda I_{m\times m}
\end{pmatrix} = 0$$

$$\Rightarrow \det(V - \lambda I_{m\times m}) \det(V - \lambda I_{m\times m}) \cdots \det(V - \lambda I_{m\times m}) = 0$$

$$\Rightarrow \det(V - \lambda I_{m\times m}) = 0$$

Thus, the eigenvalues of $A$ are the $m$ eigenvalues of $V$, each repeated $d$ times.

3. Let $X$ be just one column, i.e.

$$X = [X_1, X_2, \ldots, X_n]'.$$

After normalization, we have

$$x = \frac{X}{\sqrt{X'X}} = [x_1, x_2, \ldots, x_n]$$

which is a vector of length 1.

Consider the regression adjusted test statistic using just this column for the adjustment. For the denominator $\frac{1}{m^2d}\{W'W - W'xx'W\}$ of the adjusted test statistic, we know that $\frac{1}{m^2d}W'W = O_p(1)$ and we want to show that the scalar random variable $\frac{1}{m^2d}W'xx'W \xrightarrow{p} 0$ as $d$ increases.

Since $\frac{1}{m^2d}W'xx'W \geq 0$, if $E\{\frac{1}{m^2d}W'xx'W\} \to 0$, then $\frac{1}{m^2d}W'xx'W \xrightarrow{p} 0$. 
$$E \left\{ \frac{1}{m^2d} W'xx'W \right\} = \frac{1}{m^2d} x'E\{WW'\}x$$
$$= \frac{1}{m^2d} x'Ax$$
$$= \frac{1}{m^2d} x'\Gamma \Lambda \Gamma'x, \quad (3.1)$$

where $\Gamma$ contains the eigenvectors and $\Lambda$ (diagonal) the eigenvalues of $A$. Now $x'\Gamma \Gamma'x = x'x = 1$, so the maximum possible value of (3.1) is the maximum eigenvalue of $A$, $\lambda_{\text{max}}$, which is $O(m^2)$ and thus (3.1) is $O\left(\frac{1}{d}\right)$.

The limit behavior of the denominator is thus the same as that of $\frac{1}{m^2d}W'W$, which we have already studied.

For the numerator, $\frac{1}{m\sqrt{d}}\{e'W - e'xx'W\}$, we want to show that $\frac{1}{m\sqrt{d}}e'xx'W \xrightarrow{p} 0$ as $d$ increases. Note that

$$E\{e'x\} = 0$$

and

$$\text{Var}\{e'x\} = E\{e'xx'e\} = \text{tr}\{E\{e'xx'e\}\} = \text{tr}\{E\{x'e'e'x\}\} = x'\{I\sigma^2\}x = x'\sigma^2x = \sigma^2.$$ 

We have $e'x = O_p(1)$, and we know $x'W = O_p(m)$ from the above.

Therefore,

$$\frac{1}{m\sqrt{d}}e'xx'W = \frac{1}{m\sqrt{d}}O_p(1)O_p(m) = O_p\left(\frac{1}{\sqrt{d}}\right),$$

which goes to 0 as $d$ increases.

Later on we will look at this $\frac{1}{\sqrt{d}}$ order and get a finite sample expression for our
approximation to the limit distribution.

4. Now let us take a look at the more general case: the model with a fixed number of regressors. Suppose there are \( k \) independent variables in the model, \( k \) is a fixed number as \( d \to \infty \). Let \( P_X \) be the projection matrix for \( X \). Then it has eigenvalues 1 (repeated \( k \) times) and 0 (repeated \( md - k \) times). Corresponding to the unit eigenvalues, it has eigenvectors \( z_1, z_2, \ldots, z_k \), which span the same space as \( X_1, X_2, \ldots, X_k \). We know that regressing \( Y \) on \( X \) gives the same residuals as regressing \( Y \) on \( z_1, z_2, \ldots, z_k \). Let \( Z = [z_1: z_2: \ldots: z_k] \), thus for the denominator, we have

\[
\frac{1}{m^2d} [W'W - W'P_XW] \equiv \frac{1}{m^2d} [W'W - W'P_ZW],
\]

where \( P_Z = Z(Z'Z)^{-1}Z' = ZZ' \), since \( Z'Z = I \).

Because the columns of \( Z \) are mutually orthogonal, we can apply our single column results to each column of \( Z \) separately.

For each column \( z \) of \( Z \) we have shown (with \( z \) replacing \( x \)) that

\[
E \left\{ \frac{1}{m^2d} W'zz'W \right\} = O_p \left( \frac{1}{d} \right).
\]

Because we have a fixed number of columns \( k \), the difference between the denominator and \( \frac{1}{m^2d} W'W \) goes to 0 as \( d \to \infty \).

For the numerator, similarly we have

\[
\frac{1}{m^2d} [e'W - e'P_XW] \equiv \frac{1}{m^2d} [e'W - e'P_ZW],
\]

the difference between the numerator and \( \frac{1}{m^2d} e'W \) goes to 0 as \( d \to \infty \).

Note that the number of regressors could also increase as long as \( \frac{k}{\sqrt{d}} \to 0 \).

**Corollary 3.1** Let \( Y = X\beta + W, W = W_{(-1)} + e \). Here \( W \) is a random walk, expressed as a vector, \( W_{(-1)} \) the vector of lagged values, and \( e \) is a vector of iid variates with mean 0 and finite variance. Let \( R \) be the column of the least square residuals \( r_t \) from the regression of \( Y \) on \( X \). The studentized statistic \( \tau \) for \( r_{t-1} \) in the regression of \( r_t - r_{t-1} \) on \( r_{t-1} \) satisfies \( \tau \to \mathcal{N}(0, 1) \) when \( m \to \infty \) and \( d \to \infty \) as
long as $X$ contains a fixed number of regressors.

**Proof:**

The $\tau$ statistic is:

$$\tau = \frac{1}{n} \left\{ e' e - \frac{[e'(I - X'X)^{-1}X']W_{(-1)}]^2}{W'_{(-1)}(I - X'X)^{-1}X'W_{(-1)}} \right\}, \quad (3.2)$$

where

$$\sigma^2 = \frac{1}{n} \left\{ e' e - \frac{[e'(I - X'X)^{-1}X']W_{(-1)}]^2}{W'_{(-1)}(I - X'X)^{-1}X'W_{(-1)}} \right\}. \quad (3.3)$$

We have

$$\begin{align*}
\sigma^2 & = \frac{1}{n} \left\{ e' e - \frac{[e'(I - X'X)^{-1}X']W_{(-1)}]^2}{W'_{(-1)}(I - X'X)^{-1}X'W_{(-1)}} \right\} \\
& = \frac{1}{n} \{e' e - [e'(I - X'X)^{-1}X']W_{(-1)}]^2 \cdot [W'_{(-1)}(I - X'X)^{-1}X'W_{(-1)}]^{-1}\} \\
& = \frac{1}{n} (e' e) - \frac{1}{md} \cdot O_p(m^2d^2)O_p(\frac{1}{m^2d}) \\
& \xrightarrow{p} \sigma^2,
\end{align*}$$

so the $\tau$-statistic is also unaffected by such regression correction. This completes the proof.

### 3.2 Order Independent Convergence

In this section, we want to show that no matter in what order $m$ and $d$ get large, we have the same asymptotic result.

**Theorem 3.2** Let $Y = X\beta + W$, $W = W_{(-1)} + e$. Here $W$ is a random walk, expressed as a vector, $W_{(-1)}$ the vector of lagged values, and $e$ is a vector of iid variates with mean 0 and finite variance. We have $m\sqrt{d}(\hat\beta - 1) \xrightarrow{d} N(0, 2)$ as $m \to \infty$ and $d \to \infty$, regardless in which pattern $m$ and $d$ increase.

**Proof:**

For the numerator:
\( \frac{1}{m} \sum_{i=1}^{m} Y_{i-1,j} e_{i,j} \) has mean 0. For simplicity, we assume \( e_{i,j}'s \) are normal. The variance of \( \frac{1}{m} \sum_{i=1}^{m} Y_{i-1,j} e_{i,j} \) is

\[
\frac{1}{m^2} \sum_{t=1}^{m} \sum_{s=1}^{m} E[Y_{t-1} Y_{s-1} e_t e_s]
\]

\[
= \frac{1}{m^2} \sum_{t=1}^{m} \sum_{s=1}^{m} [E(Y_{t-1} Y_{s-1}) E(e_t e_s) + E(Y_{t-1} e_s) E(Y_{s-1} e_t) + E(Y_{t-1} e_t) E(Y_{s-1} e_s)]
\]

\[
= \frac{1}{m^2} \sum_{t=1}^{m} E(Y_{t-1}^2) E(e_t^2)
\]

Expression (3.4)

The simplification above occurs because
(a) if \( t > s \) then \( E(Y_{s-1} e_t) = 0 \); if \( t < s \) then \( E(Y_{t-1} e_s) = 0 \);
(b) \( E(e_t e_s) = 0 \) if \( t \neq s \), so \( t = s \) is the only nonzero contribution;
(c) \( E(Y_{t-1} e_t) = 0 \) always.

Expression (3.2) is

\[
\frac{1}{m^2} \sum_{t=1}^{m} (t-1) \sigma^4 = \frac{m(m-1)}{2m^2} \sigma^4 = \frac{m-1}{2m} \sigma^4.
\]

Let \( X_{dm} = \frac{1}{\sqrt{d}} \sum_{i=1}^{m} \sum_{j=1}^{d} Y_{i-1,j} e_{i,j} \). Thus \( X_{dm} \) is the sum of \( d \) iid variables with variance \( \frac{\sigma^4}{2} \), divided by \( \sqrt{d} \). Note that the variance of \( X_{dm} \) does not involve \( m \).

It follows that \( X_{dm} \to N(0, \frac{\sigma^4}{2}) \) as \( d \) gets large. By Slutsky’s theorem, when \( m \) increases \( \frac{1}{\sqrt{dm}} \sum_{i=1}^{m} \sum_{j=1}^{d} Y_{i-1,j} e_{i,j} \) converges to the same limit as \( X_{dm} \) because \( \frac{m}{m-1} \to 1 \).

For the Denominator:
Let

\[ Y_{md} = \frac{1}{m^2d} \sum_{i=1}^{m} \sum_{j=1}^{d} W_{i-1,j}^2 \]

\[ = \frac{1}{d} \sum_{j=1}^{d} \left\{ \frac{\sum_{i=1}^{m} W_{i-1,j}^2}{m^2} - E \left( \frac{\sum_{i=1}^{m} W_{i-1,j}^2}{m^2} \right) \right\} \]

\[ + \frac{1}{d} \sum_{j=1}^{d} \left\{ E \left( \frac{\sum_{i=1}^{m} W_{i-1,j}^2}{m^2} \right) - \frac{1}{2} \right\} + \frac{1}{d} \sum_{j=1}^{d} \frac{1}{2} \]

\[ = \frac{1}{d} \sum_{j=1}^{d} \left\{ \frac{\sum_{i=1}^{m} W_{i-1,j}^2}{m^2} - \frac{m(m-1)}{2m^2} \right\} + \frac{1}{d} \sum_{j=1}^{d} \left\{ \frac{m(m-1)}{2m^2} - \frac{1}{2} \right\} + \frac{1}{2} \]

\[ \xrightarrow{p} \frac{1}{2}. \]

Since \( \frac{\sum_{i=1}^{m} W_{i-1,j}^2}{m^2} - \frac{m(m-1)}{2m^2} \) is a random variable with mean 0 and finite variance, and independent over \( j \) as \( d \) increases,

\[ \frac{1}{d} \sum_{j=1}^{d} \left\{ \frac{\sum_{i=1}^{m} W_{i-1,j}^2}{m^2} - \frac{m(m-1)}{2m^2} \right\} \xrightarrow{p} 0. \]

As \( m \) increases,

\[ \frac{m(m-1)}{2m^2} \xrightarrow{} \frac{1}{2}. \]

Therefore, our normalized \( \rho \) statistic converges to \( N(0, 2) \) regardless in which pattern \( m \) and \( d \) grow.

### 3.3 Higher Order Model

A multiplicative model

\[ (1 - \rho_d B^d)(1 - \theta_1 B - \ldots - \theta_p B^p) Y_t = e_t, \quad (3.5) \]
where $B$ is the backshift operator defined by $B(Y_t) = Y_{t-1}$ and $e_t$’s are iid $(0, \sigma^2)$, has seen interest in the seasonal time series since it nicely isolates the seasonal factor in a way that allows testing for seasonal unit roots. We assume that all roots of

$$m^p - \theta_1m^{p-1} - \ldots - \theta_p = 0 \quad (3.6)$$

are less than one in absolute value. This model can always be written as

$$(1 - \beta_1B - \beta_2B^2 - \ldots - \beta_{p+d}B^{p+d})Y_t = e_t, \quad (3.7)$$

for the right $\beta$’s.

Dickey, Hasza, and Fuller (1984) discussed the multiplicative model (3.5) in regular seasonal time series. Following their argument, we can show that the asymptotic normality also holds for higher order models.

By Taylor expansion of (3.5), we get

$$e_t(\hat{\rho}_d, \hat{\theta}) = e_t(\rho_d, \theta) - (1-\theta_1B - \ldots - \theta_pB^p)Y_t(\hat{\rho}_d - \rho_d) - \sum_{i=1}^{p}(Y_{t-i} - \hat{\rho}_dY_{t-d-i})(\hat{\theta}_i - \theta_i) + r_t, \quad (3.8)$$

where $r_t$ is the Taylor series remainder.

This suggests the following estimation procedure:

1. Regress $\nabla Y_t$ on $\nabla Y_{t-1}, \nabla Y_{t-2}, \ldots, \nabla Y_{t-p}$ to get an initial estimator of $\theta$, which is consistent for $\theta$ when $\rho_d = 1$.

2. Compute $e_t(1, \hat{\theta})$ and regress $e_t(1, \hat{\theta})$ on $[(1 - \hat{\theta}_1B - \hat{\theta}_2B^2 - \ldots - \hat{\theta}_pB^p)Y_{t-d}, \nabla Y_{t-1}, \nabla Y_{t-2}, \ldots, \nabla Y_{t-p}]$ to get estimates of $(\rho_d - 1, \theta - \hat{\theta})$.

**Theorem 3.3** Let $d$ denote the length of a period, $m$ denote the number of seasons and $n = md$ be the number of available data points. If $\rho_d = 1$ in model (3.5), the regression procedure suggested earlier results in an estimator $\hat{\rho}_d$ and a corresponding statistic with the same limit distribution as in the previous study.

**Proof:**

Define $\hat{e}_t = e_t(1, \hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_p) = (1 - \rho_dB^d)(1 - \hat{\theta}_1B - \ldots - \hat{\theta}_pB^p)Y_t$. The series
\( \hat{e}_t \) regressed on \((1 - \hat{\theta}_1 B - \ldots - \hat{\theta}_p B^p)Y_{t-d} \) yields an estimator of \( \lambda = \rho_d - 1 \), denoted by \( \hat{\lambda} \), and \( e_t \) on \((1 - \theta_1 B - \ldots - \theta_p B^p)Y_{t-d} \) yields the estimator \( \bar{\lambda} \), where the limit distribution of \( m\sqrt{d\lambda} \) is known.

Now we want to show that \( m\sqrt{d(\hat{\lambda} - \bar{\lambda})} \to 0 \). We will use identity

\[
\hat{\theta}_i \hat{\theta}_j = \theta_i \theta_j + (\hat{\theta}_i - \theta_i)\theta_j + (\hat{\theta}_j - \theta_j)\theta_i + (\hat{\theta}_i - \theta_i)(\hat{\theta}_j - \theta_j).
\]

Recall that \( n = md \) and note that

\[
\sqrt{n}(\hat{\theta}_i - \theta_i) = O_p(1).
\]

Therefore,

\[
(\hat{\theta}_i \hat{\theta}_j - \theta_i \theta_j) = O_p\left(\frac{1}{\sqrt{n}}\right) + O_p\left(\frac{1}{\sqrt{n}}\right) + O_p\left(\frac{1}{n}\right) = O_p\left(\frac{1}{\sqrt{n}}\right).
\]

For the lag 1 model, we have shown that

\[
\sum_{i=1}^{m} \sum_{j=1}^{d} Y_{i-1-k,j}^2 = O_p(m^2d),
\]

and

\[
\sum_{i=1}^{m} \sum_{j=1}^{d} Y_{i-1-k,j} e_{i,j} = O_p(m\sqrt{d}),
\]

\( k = 1, 2, \ldots, p \).

Following Fuller’s (Fuller, 1996, p. 555-556) argument, we know the two order results also hold for the higher order model.
Thus, for the denominator, we have

\[
\frac{1}{m^2 d} \sum_{i=1}^{m} \sum_{j=1}^{d} [(1 - \hat{\theta}_1 B - \ldots - \hat{\theta}_p B^p) Y_{i-1,j}]^2
\]

\[
- \frac{1}{m^2 d} \sum_{i=1}^{m} \sum_{j=1}^{d} [(1 - \theta_1 B - \ldots - \theta_p B^p) Y_{i-1,j}]^2
\]

\[
= \frac{1}{m^2 d} O_p(m^{\frac{3}{2}} d^{\frac{1}{2}})
\]

\[
= O_p(m^{-\frac{1}{2}} d^{-\frac{1}{2}}).
\]

For the numerator, we have

\[
\frac{1}{m \sqrt{d}} \sum_{i=1}^{m} \sum_{j=1}^{d} [(1 - \hat{\theta}_1 B - \ldots - \hat{\theta}_p B^p) Y_{i-1,j} e_{i,j}]
\]

\[
- \frac{1}{m \sqrt{d}} \sum_{i=1}^{m} \sum_{j=1}^{d} [(1 - \theta_1 B - \ldots - \theta_p B^p) Y_{i-1,j} e_{i,j}]
\]

\[
= \frac{1}{m \sqrt{d}} O_p(m^{\frac{1}{2}})
\]

\[
= O_p(m^{-\frac{3}{2}} d^{-\frac{1}{2}}).
\]

Let \( \hat{\theta} \) and \( \hat{\theta} \) be the numerator and denominator of \( \hat{\lambda} \), and \( \hat{\theta} \) and \( \hat{\theta} \) be the numerator and denominator of \( \tilde{\lambda} \), we have

\[
m \sqrt{d} (\hat{\lambda} - \tilde{\lambda}) = \frac{\hat{N}}{\hat{D}} - \frac{N}{D}
\]

\[
= \frac{N}{D^2} (\hat{D} - D) + \frac{1}{D} (\hat{N} - N) + r
\]

\[
= O_p \left( \frac{1}{\sqrt{m \hat{d}}} \right) + O_p \left( \frac{1}{\sqrt{m \hat{d}}} \right) + r
\]

\[
= O_p \left( \frac{1}{\sqrt{m \hat{d}}} \right).
\]

So by Slutsky’s theorem, the limit distribution for \( m \sqrt{d} \hat{\lambda} \) is the same as that of \( m \sqrt{d} \tilde{\lambda} \).
Chapter 4

Improving the Normal Approximation

4.1 Mappings to the Normal

While the normal limit is a very nice result, it is clear from the tables of Dickey, Hasza, and Fuller that the \( d \) values for quarterly or monthly data are not sufficiently large for these limit results to be used; that is, those tables are far from \( N(0, 1) \) even for large \( m \). To study the finite sample behavior when \( d \) is not sufficiently large, we construct an approximate mapping between the three models (the no mean model, the single mean model, and the seasonal mean model) and the normal distribution. We find the mean \( \mu \) and the standard deviation \( \sigma \) so that \( \frac{\tau - \mu}{\sigma} \) has the same 5th and 95th percentiles as the standard normal, i.e:

\[
\tau(0.95) = \mu + \sigma z(0.95), \quad (4.1)
\]
\[
\tau(0.05) = \mu + \sigma z(0.05). \quad (4.2)
\]

Thus, \( \mu \) and \( \sigma \) are computed by:

\[
\mu = \frac{1}{2}(\tau(0.95) + \tau(0.05)), \quad (4.3)
\]
Table 4.1: Mapping Table for No Mean Model

<table>
<thead>
<tr>
<th>d=2</th>
<th>n= 20</th>
<th>$\mu = -0.23$</th>
<th>$\sigma = 1.040$</th>
<th>0.01</th>
<th>0.05</th>
<th>0.50</th>
<th>0.95</th>
<th>0.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>d=2</td>
<td>n= 30</td>
<td>$\mu = -0.25$</td>
<td>$\sigma = 1.021$</td>
<td>0.010</td>
<td>0.05</td>
<td>0.477</td>
<td>0.95</td>
<td>0.992</td>
</tr>
<tr>
<td>d=2</td>
<td>n= 40</td>
<td>$\mu = -0.255$</td>
<td>$\sigma = 1.018$</td>
<td>0.011</td>
<td>0.05</td>
<td>0.475</td>
<td>0.95</td>
<td>0.992</td>
</tr>
<tr>
<td>d=2</td>
<td>n= 100</td>
<td>$\mu = -0.265$</td>
<td>$\sigma = 1.006$</td>
<td>0.011</td>
<td>0.05</td>
<td>0.47</td>
<td>0.95</td>
<td>0.991</td>
</tr>
<tr>
<td>d=2</td>
<td>n= 200</td>
<td>$\mu = -0.27$</td>
<td>$\sigma = 1.003$</td>
<td>0.011</td>
<td>0.05</td>
<td>0.468</td>
<td>0.95</td>
<td>0.991</td>
</tr>
<tr>
<td>d=2</td>
<td>n= 400</td>
<td>$\mu = -0.27$</td>
<td>$\sigma = 1.003$</td>
<td>0.012</td>
<td>0.05</td>
<td>0.468</td>
<td>0.95</td>
<td>0.991</td>
</tr>
<tr>
<td>d=2 n= $\infty$</td>
<td>$\mu = -0.27$</td>
<td>$\sigma = 1.003$</td>
<td>0.012</td>
<td>0.05</td>
<td>0.468</td>
<td>0.95</td>
<td>0.991</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>d=4</th>
<th>n= 40</th>
<th>$\mu = -0.17$</th>
<th>$\sigma = 1.033$</th>
<th>0.01</th>
<th>0.05</th>
<th>0.488</th>
<th>0.95</th>
<th>0.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>d=4</td>
<td>n= 60</td>
<td>$\mu = -0.20$</td>
<td>$\sigma = 1.027$</td>
<td>0.011</td>
<td>0.05</td>
<td>0.496</td>
<td>0.95</td>
<td>0.99</td>
</tr>
<tr>
<td>d=4</td>
<td>n= 80</td>
<td>$\mu = -0.21$</td>
<td>$\sigma = 1.021$</td>
<td>0.011</td>
<td>0.05</td>
<td>0.496</td>
<td>0.95</td>
<td>0.99</td>
</tr>
<tr>
<td>d=4</td>
<td>n= 200</td>
<td>$\mu = -0.225$</td>
<td>$\sigma = 1.018$</td>
<td>0.011</td>
<td>0.05</td>
<td>0.498</td>
<td>0.95</td>
<td>0.991</td>
</tr>
<tr>
<td>d=4</td>
<td>n= 400</td>
<td>$\mu = -0.23$</td>
<td>$\sigma = 1.015$</td>
<td>0.011</td>
<td>0.05</td>
<td>0.496</td>
<td>0.95</td>
<td>0.991</td>
</tr>
<tr>
<td>d=4</td>
<td>n= 800</td>
<td>$\mu = -0.23$</td>
<td>$\sigma = 1.015$</td>
<td>0.011</td>
<td>0.05</td>
<td>0.496</td>
<td>0.95</td>
<td>0.991</td>
</tr>
<tr>
<td>d=4 n= $\infty$</td>
<td>$\mu = -0.23$</td>
<td>$\sigma = 1.015$</td>
<td>0.011</td>
<td>0.05</td>
<td>0.496</td>
<td>0.95</td>
<td>0.991</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>d=12</th>
<th>n= 120</th>
<th>$\mu = -0.11$</th>
<th>$\sigma = 1.009$</th>
<th>0.009</th>
<th>0.05</th>
<th>0.504</th>
<th>0.95</th>
<th>0.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>d=12</td>
<td>n= 180</td>
<td>$\mu = -0.12$</td>
<td>$\sigma = 1.003$</td>
<td>0.009</td>
<td>0.05</td>
<td>0.50</td>
<td>0.95</td>
<td>0.99</td>
</tr>
<tr>
<td>d=12</td>
<td>n= 240</td>
<td>$\mu = -0.12$</td>
<td>$\sigma = 1.003$</td>
<td>0.009</td>
<td>0.05</td>
<td>0.496</td>
<td>0.95</td>
<td>0.99</td>
</tr>
<tr>
<td>d=12</td>
<td>n= 600</td>
<td>$\mu = -0.135$</td>
<td>$\sigma = 1.006$</td>
<td>0.01</td>
<td>0.05</td>
<td>0.498</td>
<td>0.95</td>
<td>0.99</td>
</tr>
<tr>
<td>d=12</td>
<td>n= 1200</td>
<td>$\mu = -0.135$</td>
<td>$\sigma = 1.006$</td>
<td>0.01</td>
<td>0.05</td>
<td>0.498</td>
<td>0.95</td>
<td>0.99</td>
</tr>
<tr>
<td>d=12</td>
<td>n= 2400</td>
<td>$\mu = -0.14$</td>
<td>$\sigma = 1.009$</td>
<td>0.01</td>
<td>0.05</td>
<td>0.50</td>
<td>0.95</td>
<td>0.99</td>
</tr>
<tr>
<td>d=12 n= $\infty$</td>
<td>$\mu = -0.14$</td>
<td>$\sigma = 1.009$</td>
<td>0.01</td>
<td>0.05</td>
<td>0.50</td>
<td>0.95</td>
<td>0.99</td>
<td></td>
</tr>
</tbody>
</table>

\[
\sigma = \frac{1}{2 \times 1.645} (\tau(0.95) - \tau(0.05)). \tag{4.4}
\]

Results are shown in Tables 4.1-4.3. Due to the limited space, we present only the important part of each table. The complete tables are available in the Appendices (Tables 8.1-8.3).

The first line shows the normal percentiles, while the rest are the empirical $\tau$ probabilities, with $m$, $d$, $\mu$ and $\sigma$ accordingly. From Table 4.1 and Table 4.2, we can see that in the no mean model and the single mean model, the $\tau$ probabilities shown get quite close to those of the normal scores. Also, the mean goes to 0 in the limit as we expect. The variance is around 1. In the seasonal mean model (Table 4.3), the mean does not appear to converge to 0, which also coincides with our conclusion.
Table 4.2: Mapping Table for Single Mean Model

| d=2  | n=20  | \mu = -0.9 | \sigma = 1.1064 | 0.009 | 0.05 | 0.475 | 0.95 | 0.992 |
| d=2  | n=30  | \mu = -0.905 | \sigma = 1.0851 | 0.01  | 0.05 | 0.461 | 0.95 | 0.991 |
| d=2  | n=40  | \mu = -0.91  | \sigma = 1.0760 | 0.01  | 0.05 | 0.459 | 0.95 | 0.991 |
| d=2  | n=100 | \mu = -0.91  | \sigma = 1.0578 | 0.012 | 0.05 | 0.447 | 0.95 | 0.991 |
| d=2  | n=200 | \mu = -0.905 | \sigma = 1.0547 | 0.012 | 0.05 | 0.445 | 0.95 | 0.991 |
| d=2  | n=400 | \mu = -0.91  | \sigma = 1.0517 | 0.012 | 0.05 | 0.443 | 0.95 | 0.991 |
| d=2  | n=\infty | \mu = -0.905 | \sigma = 1.0486 | 0.013 | 0.05 | 0.441 | 0.95 | 0.991 |
| d=4  | n=40  | \mu = -0.595 | \sigma = 1.0851 | 0.01  | 0.05 | 0.498 | 0.95 | 0.991 |
| d=4  | n=60  | \mu = -0.615 | \sigma = 1.0729 | 0.01  | 0.05 | 0.494 | 0.95 | 0.991 |
| d=4  | n=80  | \mu = -0.625 | \sigma = 1.0669 | 0.01  | 0.05 | 0.494 | 0.95 | 0.991 |
| d=4  | n=200 | \mu = -0.635 | \sigma = 1.0608 | 0.011 | 0.05 | 0.491 | 0.95 | 0.991 |
| d=4  | n=400 | \mu = -0.635 | \sigma = 1.0608 | 0.011 | 0.05 | 0.487 | 0.95 | 0.991 |
| d=4  | n=800 | \mu = -0.64  | \sigma = 1.0578 | 0.011 | 0.05 | 0.489 | 0.95 | 0.991 |
| d=4  | n=\infty | \mu = -0.64 | \sigma = 1.0578 | 0.011 | 0.05 | 0.489 | 0.95 | 0.991 |
| d=12 | n=120 | \mu = -0.315 | \sigma = 1.0304 | 0.01  | 0.05 | 0.502 | 0.95 | 0.991 |
| d=12 | n=180 | \mu = -0.335 | \sigma = 1.0243 | 0.01  | 0.05 | 0.502 | 0.95 | 0.991 |
| d=12 | n=240 | \mu = -0.34  | \sigma = 1.0213 | 0.01  | 0.05 | 0.5   | 0.95 | 0.991 |
| d=12 | n=600 | \mu = -0.35  | \sigma = 1.0274 | 0.01  | 0.05 | 0.5   | 0.95 | 0.991 |
| d=12 | n=1200| \mu = -0.36  | \sigma = 1.0274 | 0.011 | 0.05 | 0.504 | 0.95 | 0.991 |
| d=12 | n=2400| \mu = -0.365 | \sigma = 1.0304 | 0.011 | 0.05 | 0.502 | 0.95 | 0.991 |
| d=12 | n=\infty | \mu = -0.365 | \sigma = 1.0304 | 0.011 | 0.05 | 0.502 | 0.95 | 0.989 |


<table>
<thead>
<tr>
<th>d</th>
<th>n</th>
<th>$\mu$</th>
<th>$\sigma$</th>
<th>prob($\tau$)</th>
<th>0.01</th>
<th>0.05</th>
<th>0.50</th>
<th>0.95</th>
<th>0.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>20</td>
<td>-1.945</td>
<td>1.0061</td>
<td>0.006</td>
<td>0.05</td>
<td>0.51</td>
<td>0.95</td>
<td>0.991</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>30</td>
<td>-1.93</td>
<td>0.9544</td>
<td>0.008</td>
<td>0.05</td>
<td>0.492</td>
<td>0.95</td>
<td>0.991</td>
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<td>2</td>
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<td>0.9271</td>
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<td>0.05</td>
<td>0.485</td>
<td>0.95</td>
<td>0.992</td>
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</tr>
<tr>
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<td>100</td>
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<td>0.8875</td>
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<td>0.05</td>
<td>0.469</td>
<td>0.95</td>
<td>0.992</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>200</td>
<td>-1.92</td>
<td>0.8754</td>
<td>0.01</td>
<td>0.05</td>
<td>0.468</td>
<td>0.95</td>
<td>0.992</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>400</td>
<td>-1.92</td>
<td>0.8693</td>
<td>0.01</td>
<td>0.05</td>
<td>0.463</td>
<td>0.95</td>
<td>0.992</td>
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<td>d=2 n=∞</td>
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<td>0.8632</td>
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<td>0.463</td>
<td>0.95</td>
<td>0.992</td>
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<td>-2.575</td>
<td>0.9939</td>
<td>0.007</td>
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<td>400</td>
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<td>0.01</td>
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<td>0.95</td>
<td>0.992</td>
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<td>0.992</td>
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<td>12</td>
<td>600</td>
<td>-4.325</td>
<td>0.9088</td>
<td>0.012</td>
<td>0.05</td>
<td>0.493</td>
<td>0.95</td>
<td>0.991</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>1200</td>
<td>-4.34</td>
<td>0.8997</td>
<td>0.012</td>
<td>0.05</td>
<td>0.496</td>
<td>0.95</td>
<td>0.99</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>2400</td>
<td>-4.345</td>
<td>0.8967</td>
<td>0.012</td>
<td>0.05</td>
<td>0.493</td>
<td>0.95</td>
<td>0.99</td>
<td></td>
</tr>
<tr>
<td>d=12 n=∞</td>
<td></td>
<td>-4.35</td>
<td>0.8936</td>
<td>0.013</td>
<td>0.05</td>
<td>0.496</td>
<td>0.95</td>
<td>0.989</td>
<td></td>
</tr>
</tbody>
</table>
4.2 Suggested Empirical Adjustment

We now try to find some relationship between $\mu$, $\sigma$ and $m$, $d$. If $\mu$ and $\sigma$ have nice functional forms, we can suggest a simple way to compute p-values for $\tau$ from those of the normal distribution instead of looking into the seasonal percentile tables.

4.2.1 No Mean Model

Note that in Table 4.1 and Table 4.2, the variance is around 1 throughout. By exploring the table of Dickey, Hasza and Fuller (hereafter “DHF”) for the no mean model, we find that the difference between the 5th and the 95th percentiles is very close to that in the normal table. For example, for large samples in monthly ($d=12$) data, the 5th and 95th percentiles are -1.80 and 1.52, which differ by 3.32. For the standard normal distribution, the difference is $2 \times 1.645 = 3.290$. We claim that the spreads of the two distributions are about the same, and only $\mu$ is related to $m$ and $d$. The DHF tables are far from $N(0,1)$ even for large $m$. We therefore conclude that the normality is coming mostly from increasing $d$, not $m$, i.e., a simple location adjustment related with $d$ might be appropriate and give a distribution with 5th and 95th percentiles very close to those of a normal.

In a normal distribution, the median is half way between the 5th and 95th percentiles. The average of the 5th and 95th percentiles for $\tau$ is $(-1.80+1.52)/2 = -0.14$ which is the median from the DHF $\tau$ tables. This provides further motivation for investigating a normal approximation for $\tau$.

The medians of the $\tau$ statistic’s distribution for $d=2, 4, 12$ are -0.35, -0.24, and -0.14 for the limit cases, according to DHF, table 3. Corresponding values of $\frac{1}{\sqrt{d}}$ are -0.7072, -0.5000 and -0.28868. We plot DHF medians versus $\frac{1}{\sqrt{d}}$. The figure shows a remarkably linear relationship.

A regression of the medians on $\frac{1}{\sqrt{d}}$ indicates an intercept near 0 and slope near -0.5 (-0.49311 from the SAS output), suggesting $\frac{1}{2\sqrt{d}}$ as a simple median bias correction.

We can also motivate a median adjustment by looking at Taylor’s series expansion of the test statistic $\tau$. Let
Figure 4.1: DHF Median vs \( \frac{1}{\sqrt{d}} \) (for No Mean Model)

\[
\bar{N} = \frac{1}{d} \sum_{j=1}^{d} N_{j0} = O_p\left(\frac{1}{\sqrt{d}}\right), \tag{4.5}
\]

\[
\bar{D} = \frac{1}{d} \sum_{j=1}^{d} D_{j0} = E(D_1) + O_p\left(\frac{1}{\sqrt{d}}\right), \tag{4.6}
\]

where

\[
N_{j0} = \frac{N_j}{\sigma^2 m} = \sum_{i=1}^{m} \frac{Y_{i-1,j} e_{i,j}}{\sigma^2 m} = O_p(1), \tag{4.7}
\]

\[
D_{j0} = \frac{D_j}{\sigma^2 m^2} = \sum_{i=1}^{m} \frac{Y_{i-1,j}^2}{\sigma^2 m^2} = O_p(1). \tag{4.8}
\]
We have

\[
\tau = \sqrt{\bar{d}} \cdot \frac{\bar{N}}{\sqrt{\bar{D}}}
\]

\[
= \sqrt{\bar{d}} \cdot \left[ \frac{N_0}{\sqrt{D_0}} + \frac{1}{\sqrt{D_0}}(\bar{N} - N_0) - \frac{N_0}{2\sqrt{D_0}}(\bar{D} - D_0) + \frac{0}{2}(\bar{N} - N_0)^2 \right.
\]

\[
+ \frac{3N_0}{8\sqrt{D_0}}(\bar{D} - D_0)^2 - \frac{1}{2\sqrt{D_0^3}}(\bar{D} - D_0)(\bar{N} - N_0) + r \]

where \( r \) is a Taylor series remainder. For the no mean model we have

\[
E(\bar{N}) = N_0 = 0, \quad (4.9)
\]

\[
E(\bar{D}) = D_0 = \frac{m - 1}{2m}, \quad (4.10)
\]

and note that

\[
E(\bar{D} - D_0) = 0, \quad (4.11)
\]

then we have

\[
E(\tau) = E\left(\sqrt{\bar{d}} \cdot \frac{\bar{N}}{\sqrt{\bar{D}}}\right) \approx -\sqrt{\bar{d}} \cdot \frac{1}{2\sqrt{D_0^3}} E(\bar{D} - D_0)(\bar{N} - N_0). \quad (4.12)
\]

Recalling the moment result (Dickey, 1976) \( E\{(\bar{N} - N_0)(\bar{D} - D_0)\} = \frac{1}{3}(m - 1)(m - 2)\frac{1}{m^2\bar{d}} \), we can easily get the approximation:

\[
E(\tau) \approx -\sqrt{\bar{d}} \cdot \frac{1}{2\sqrt{D_0^3}} E\{(\bar{D} - D_0)(\bar{N} - N_0)\} \quad (4.13)
\]

\[
= -\sqrt{\bar{d}} \cdot \frac{1}{2 \cdot \frac{1}{2m}(m - 1) \sqrt{\frac{1}{2m}(m - 1)}} \cdot \frac{1}{3}(m - 1)(m - 2)\frac{1}{m^2\bar{d}} \quad (4.14)
\]

\[
= -\frac{\sqrt{2}(m - 2)}{3\sqrt{m(m - 1)\bar{d}}} \quad (4.15)
\]

\[
\approx -\frac{1}{2.17\sqrt{\bar{d}}}. \quad (4.16)
\]
This approximation is close to our suggested \( \frac{1}{2\sqrt{d}} \) median adjustment.

When \( d=4 \), we have \( \mu=-0.23 \) from Table 4.1. Substituting 4 for \( d \) in equation (4.16), we also get -0.23. When \( d=12 \), we have \( \mu=-0.14 \) from Table 4.1. Substituting 12 for \( d \) in equation (4.16), we get -0.13, which is very close to -0.14.

Under the seasonal mean model, we redefine the numerator, \( N_j \), and denominator, \( D_j \), as:

\[
N_j = \sum_{i=1}^{m} (Y_{i-1,j} - \bar{Y}_{(-1),j}) e_{i,j}
\]

and

\[
D_j = \sum_{i=1}^{m} (Y_{i-1,j} - \bar{Y}_{(-1),j})^2
\]

where \( \bar{Y}_{r,j} = \frac{1}{m} \sum_{i=1}^{m} Y_{i+r,j} \). Substituting the redefined \( N_j \) and \( D_j \) in (4.5) - (4.8), we get the new \( \bar{N} \) and the new \( \bar{D} \) for the seasonal mean model, and also their moments:

\[
E(\bar{N}) = N_0 = -\frac{m-2}{2m}, \quad (4.17)
\]

\[
E(\bar{D}) = D_0 = \frac{m-2}{6m}, \quad (4.18)
\]

\[
E(\bar{D} - D_0)^2 = \text{Var}(\bar{D}) = \frac{1}{90m^3} (m-2)(2m^2-4m+9), \quad (4.19)
\]

and

\[
E\{(\bar{N} - N_0)(\bar{D} - D_0)\} = \text{Cov}(\bar{N}, \bar{D}) = \frac{m-2}{6m^2d}. \quad (4.20)
\]

Therefore, for reasonable large \( m \), the Taylor Expansion approximation becomes:

\[
E(\tau) \approx \sqrt{d} \left[ \frac{N_0}{\sqrt{D_0}} + \frac{3N_0}{8\sqrt{D_0}} \text{Var}(\bar{D}) - \frac{1}{2\sqrt{D_0}} E\{(\bar{D} - D_0)(\bar{N} - N_0)\} \right] \quad (4.21)
\]

\[
\approx -\frac{\sqrt{6d}}{2} - \frac{3\sqrt{6}}{20\sqrt{d}} + 0 \quad (4.22)
\]

\[
= -\frac{\sqrt{6}(10d+3)}{20\sqrt{d}}. \quad (4.23)
\]

From this expression, we can see that \( E(\tau) \) does not converge to 0 as \( d \) gets large, which coincides with our conclusion and mapping table. Substituting \( d \) by 4 and 12 in equation (4.23), we get -2.6 and -4.35, which approximate the respective \( \mu \) values.
in Table 4.3.

For the single mean model and the sinusoidal mean model, the Taylor expansion changes as the regressors are added.

### 4.2.2 Single mean model

Let $\tilde{N}$ denote the numerator with the regression adjustment (hereafter “correction term”) applied, and $\tilde{D}$ denote the denominator with the “correction term”. The Taylor series for $\tau$ is:

$$\frac{\tau}{\sqrt{d}} = \frac{\tilde{N}}{\sqrt{\tilde{D}}}
= \frac{N_0}{\sqrt{D_0}} + \frac{1}{\sqrt{D_0}}(\tilde{N} - N_0) - \frac{N_0}{2\sqrt{D_0}}(\tilde{D} - D_0) - \frac{1}{2\sqrt{D_0}}(\tilde{D} - D_0)(\tilde{N} - N_0) + r,$$

where $N_0$ is the expectation of $\tilde{N}$ and $D_0$ is the expectation of $\tilde{D}$.

For the numerator, we have:

$$\tilde{N} = \frac{1}{\sigma^2_d} e^e W_{(-1)} - \frac{1}{\sigma^2_d} e^e 1' (1')^{-1} 1' W_{(-1)} = \tilde{N} - \nabla N. \quad (4.24)$$
The expectation of the correction term is:

\[
E(\nabla N) = E\left[ \frac{1}{\sigma^2 m d} \mathbf{e}' \mathbf{1} (\mathbf{1}')^{-1} \mathbf{1}' \mathbf{W}_{(-1)} \right]
\]

\[
= \frac{1}{\sigma^2 m d} \frac{1}{md} E \left[ (\mathbf{e}' \mathbf{1}) (\mathbf{1}' \mathbf{W}_{(-1)}) \right]
\]

\[
= \frac{1}{\sigma^2 m d} \frac{1}{md} \int E \left[ \sum_{i=1}^{m} e_{i,j} (\sum_{j=1}^{m} W_{i-1,j}) \right] dE
\]

\[
= \frac{1}{\sigma^2 m d} \frac{1}{md} \int [(e_{i,j} + e_{2j} + \ldots + e_{m,j})((m-1)e_{1j} + (m-2)e_{2j} + \ldots + e_{m-1,j})] dE
\]

\[
= \frac{1}{\sigma^2 m d} \frac{1}{md} \int \frac{(m-1)m}{2} \sigma^2 dE
\]

\[
= \frac{m-1}{2md}.
\]

The expectation of the numerator is:

\[
E(\tilde{N}) = E(N) - E(\nabla N) = -\frac{m-1}{2md}.
\]  

For the denominator, we have:

\[
\tilde{D} = \frac{1}{\sigma^2 d} \mathbf{W}'_{(-1)} \mathbf{W}_{(-1)} - \frac{1}{\sigma^2 d} \mathbf{W}'_{(-1)} \mathbf{1}(\mathbf{1}')^{-1} \mathbf{1}' \mathbf{W}_{(-1)} = \tilde{D} - \nabla D.
\]  

The expectation of the correction term is:

\[
E(\nabla D) = E\left[ \frac{1}{\sigma^2 m^2 d} \mathbf{W}'_{(-1)} \mathbf{1}(\mathbf{1}')^{-1} \mathbf{1}' \mathbf{W}_{(-1)} \right]
\]

\[
= \frac{1}{\sigma^2 m^2 d md} E[(\mathbf{W}'_{(-1)} \mathbf{1})(\mathbf{1}' \mathbf{W}_{(-1)})]
\]

\[
= \frac{1}{\sigma^2 m^2 d md} E[(\sum_{i=1}^{m} \sum_{j=1}^{d} W_{i-1,j})^2]
\]

\[
= \frac{1}{\sigma^2 m^2 d md} \frac{1}{md} (m-1)m(2m-1) \frac{\sigma^2}{6}
\]

\[
= \frac{(m-1)(2m-1)}{6m^2 d}.
\]
The expectation of the denominator is:

\[
E(\tilde{D}) = E(\tilde{D}) - E(\nabla D) = \frac{m - 1}{2m} - \frac{(m - 1)(2m - 1)}{6m^2d}
\]  

(4.27)

Therefore the leading term of the \( \tau \) statistic is:

\[
\sqrt{d} \frac{E(\tilde{N})}{\sqrt{E(\tilde{D})}} = \frac{N_0}{\sqrt{D_0}} = \frac{-\sqrt{d} \cdot \frac{m - 1}{2md}}{\sqrt{\frac{m - 1}{2m} - \frac{(m - 1)(2m - 1)}{6m^2d}}},
\]  

(4.28)

which suggests \(-\frac{\sqrt{d}}{2\sqrt{d}}\) for reasonably large \( m \).

Now the orders of the two correction terms on the numerator and denominator are:

\[
\nabla N = \tilde{N} - \bar{N} = -\frac{1}{\sigma^2md} e'1(1'1)^{-1}1'W(-1) = O_p\left(\frac{1}{d}\right),
\]  

(4.29)

and

\[
\nabla D = \tilde{D} - \bar{D} = -\frac{1}{\sigma^2m^2d} e'1(1'1)^{-1}1'W(-1) = O_p\left(\frac{1}{d}\right).
\]  

(4.30)

Because \( \tilde{N} \) and \( \tilde{D} \) are \( O_p\left(\frac{1}{\sqrt{d}}\right) \), they are the dominant terms in \( \tilde{N} = \bar{N} + \nabla N \) and in \( \tilde{D} = \bar{D} + \nabla D \). The crossproduct term is:

\[
(\tilde{D} - D_0)(\tilde{N} - N_0) = [\tilde{D} - E(\tilde{D})][\tilde{N} - E(\tilde{N})]
\]

\[
= [\tilde{D} - E(\bar{D}) + \nabla D - E(\nabla D)][\tilde{N} - E(\bar{N}) + \nabla N - E(\nabla N)]
\]

\[
= [\tilde{D} - E(\bar{D})][\tilde{N} - E(\bar{N})] + O_p\left(\frac{1}{d\sqrt{d}}\right).
\]

There is no further contribution to the \( O(\frac{1}{\sqrt{d}}) \) adjustment term beyond that of \([\tilde{D} - E(\bar{D})][\tilde{N} - E(\bar{N})]\). Therefore, for the single mean model, the suggested adjustment is \( \frac{\sqrt{d}}{2\sqrt{d}} + \frac{\sqrt{d}}{3\sqrt{d}} \).
4.2.3 Sinusoidal and Other Periodic Functions

If we fit a Fourier sine or cosine function of frequency $\frac{2\pi t}{d}$, the corresponding numerator and denominator with correction term become:

$$\tilde{N} = \frac{1}{\sigma^2 d} e' W_{(-1)} - \frac{1}{\sigma^2 d} e' X' X^{-1} X' W_{(-1)} = \tilde{N} - \nabla N,$$  \hspace{1cm} (4.31)

and

$$\tilde{D} = \frac{1}{\sigma^2 d} W'_{(-1)} W_{(-1)} - \frac{1}{\sigma^2 d} W'_{(-1)} X' X^{-1} X' W_{(-1)} = \tilde{D} - \nabla D.$$  \hspace{1cm} (4.32)

Let $X_j = c_j 1$ be the regression variable for season $j$ where $c_j$ is 1 for an overall mean, or $\sin\left(\frac{2\pi t}{d} j\right)$ or $\cos\left(\frac{2\pi t}{d} j\right)$ for the trigonometric functions. Then

$$X' X = m \sum_{j=1}^{d} c_j^2$$  \hspace{1cm} (4.33)

is $md$ for an overall intercept column or Fourier cosine at the Nyquist frequency, and $\frac{md}{2}$ for any other Fourier sine or cosine column.

For the numerator, the expectation of the correction term is:

$$E(\nabla N) = E\left[\frac{1}{\sigma^2 md} e' X' X^{-1} X' W_{(-1)}\right]$$

$$= \frac{1}{\sigma^2 md} \cdot \frac{1}{m} \sum_{j=1}^{d} c_j^2 E\left[\sum_{j=1}^{d} c_j^2 (e' 1)(1' W_{(-1)})\right]$$

$$= \frac{1}{\sigma^2 md} \cdot \frac{1}{m} \sum_{j=1}^{d} c_j^2 \sum_{j=1}^{d} c_j^2 E[(e' 1)(1' W_{(-1)})]$$

$$= \frac{1}{\sigma^2 md} \cdot \frac{1}{m} \sum_{j=1}^{d} c_j^2 \sum_{j=1}^{d} c_j^2 (m - 1)m \frac{\sigma^2}{2}$$

$$= \frac{m - 1}{2md}.$$
The expectation of the numerator is:

\[ E(\tilde{N}) = E(\tilde{N}) - E(\nabla N) = - \frac{m - 1}{2md}. \]  

(4.34)

Similar arguments apply to the denominator. Therefore, if we fit a mean function and several Fourier sines and cosines, the orthogonality of the resulting design matrix ensures that the numerator adjustment will be the adjustment for the single mean multiplied by the number of regressors. Since \( \sum_{j=1}^{d} c_j^2 \) cancels out in the computation, the result holds for any \( c_j \)'s (bounded and not all equal to 0). Therefore, for any periodic function mean, the adjustment is \( \frac{\sqrt{2k}}{2\sqrt{d}} + \frac{\sqrt{3}}{3\sqrt{d}} \), with \( k \) being the number of periodic regressors.
Chapter 5

Simulation Study

The goal of the simulation study is to investigate the finite sample behavior of the \( \rho \) and the \( \tau \) statistics in seasonal time series with a long period, and to have a rough idea how large the lag number should be to use the normal distribution as the reference distribution.

5.1 Normal Probability Plots

Using the Monte Carlo methods, we generate Normal Probability Plots for both \( \rho \) and \( \tau \) statistics discussed in this dissertation.

Monte Carlo datasets (pseudonormal variates \( e_t', e_t \sim N(0, 1) \) for \( t = 1, \ldots, md \)) of replication 10000 are randomly generated for all combinations of \( m=20 \) and \( d=12, 24, 52, 365 \) under the four different models: no mean model, single mean model, sinusoidal model and linear trend model.

There are four sets of plots (Figure 5.1-Figure 5.4), for four different models. Each set contains two columns with four plots in each column. The plots in the left column are for \( \tau \) statistics and the ones in the right column are for \( \rho \). So we can easily compare the performance of \( \tau \) and \( \rho \) by comparing the right column and left column in each set. In each column, the four plots are for different \( d \) values: 12, 24, 52, 365. So the performance change of the test statistics can be easily seen as \( d \) gets large.
From the plots, we can see that overall the \( \tau \) statistics behave better than the \( \rho \) statistics.

Under the no mean model, the Normal Probability plot for the \( \tau \) statistics approximates a straight line, even near the end, for \( d = 12 \). The one for the \( \rho \) statistics approximates a straight line near the center. For \( d = 52 \), the Normal Probability plot for the \( \rho \) statistics performs reasonably well, and for \( d = 365 \), it is almost a straight line. Under the single mean model, also, the \( \tau \) statistics perform nicely for even \( d = 12 \). The Normal Probability plot for the \( \rho \) statistics approximates a straight line near the center. For \( d = 52 \), the Normal Probability plot for the \( \rho \) performs reasonably well, and for \( d = 365 \), it is almost a straight line. For the Sinusoidal model and the linear trend model, we need \( d = 52 \) or larger to get a nice approximation. Thus we conclude when \( d = 52 \) or larger, the normal approximation holds for all the models we discussed in this paper.

### 5.2 Percentiles

The same Monte Carlo datasets are used to construct the percentile tables for the four different models: no mean model, single mean model, sinusoidal model and linear trend model.

Standard errors for the estimates of the percentiles are obtained as following:

1. Get the estimate of the percentile;
2. Compute \( \sqrt{np(1-p)} \);
3. From the estimate (1) in the ordered list of simulated statistics, count up \( 2\sqrt{np(1-p)} \) and get the number, which we call \( b \), and count down \( 2\sqrt{np(1-p)} \) and get the number, which we call \( a \);
4. The approximate standard error is \( \frac{b-a}{4} \).

Due to the limited space, we only present the important part of each table. The complete tables are available in the Appendices (Tables 8.4-8.11).

From the percentile tables, we can see that for all four models, the estimated percentiles for both \( \rho \) statistic and \( \tau \) statistic fall to the left of the corresponding
Table 5.1: \( \rho \) Percentiles for the Zero Mean Model (\( m = 20 \))

<table>
<thead>
<tr>
<th>Percentiles</th>
<th>0.05</th>
<th>0.1</th>
<th>0.5</th>
<th>0.9</th>
<th>0.95</th>
</tr>
</thead>
<tbody>
<tr>
<td>d=12</td>
<td>-3.45(0.05)</td>
<td>-2.58(0.04)</td>
<td>-0.17(0.02)</td>
<td>1.51(0.02)</td>
<td>1.93(0.02)</td>
</tr>
<tr>
<td>d=24</td>
<td>-3.01(0.05)</td>
<td>-2.28(0.04)</td>
<td>-0.16(0.02)</td>
<td>1.57(0.02)</td>
<td>1.99(0.02)</td>
</tr>
<tr>
<td>d=52</td>
<td>-2.81(0.04)</td>
<td>-2.13(0.03)</td>
<td>-0.09(0.02)</td>
<td>1.68(0.02)</td>
<td>2.11(0.03)</td>
</tr>
<tr>
<td>d=60</td>
<td>-2.77(0.04)</td>
<td>-2.14(0.03)</td>
<td>-0.06(0.02)</td>
<td>1.70(0.02)</td>
<td>2.10(0.02)</td>
</tr>
<tr>
<td>d=365</td>
<td>-2.50(0.03)</td>
<td>-1.94(0.03)</td>
<td>-0.03(0.02)</td>
<td>1.78(0.03)</td>
<td>2.29(0.03)</td>
</tr>
<tr>
<td>normal(0,2)</td>
<td>-2.33</td>
<td>-1.81</td>
<td>0</td>
<td>1.81</td>
<td>2.33</td>
</tr>
</tbody>
</table>

Table 5.2: \( \tau \) Percentiles for the Zero Mean Model (\( m = 20 \))

<table>
<thead>
<tr>
<th>Percentiles</th>
<th>0.05</th>
<th>0.1</th>
<th>0.5</th>
<th>0.9</th>
<th>0.95</th>
</tr>
</thead>
<tbody>
<tr>
<td>d=12</td>
<td>-1.82(0.02)</td>
<td>-1.46(0.02)</td>
<td>-0.12(0.01)</td>
<td>1.19(0.02)</td>
<td>1.55(0.02)</td>
</tr>
<tr>
<td>d=24</td>
<td>-1.74(0.02)</td>
<td>-1.37(0.02)</td>
<td>-0.11(0.01)</td>
<td>1.19(0.02)</td>
<td>1.56(0.02)</td>
</tr>
<tr>
<td>d=52</td>
<td>-1.72(0.02)</td>
<td>-1.36(0.02)</td>
<td>-0.06(0.01)</td>
<td>1.24(0.02)</td>
<td>1.58(0.02)</td>
</tr>
<tr>
<td>d=60</td>
<td>-1.71(0.02)</td>
<td>-1.36(0.02)</td>
<td>-0.04(0.01)</td>
<td>1.24(0.02)</td>
<td>1.57(0.02)</td>
</tr>
<tr>
<td>d=365</td>
<td>-1.66(0.02)</td>
<td>-1.30(0.02)</td>
<td>-0.02(0.01)</td>
<td>1.26(0.02)</td>
<td>1.62(0.02)</td>
</tr>
<tr>
<td>normal(0,1)</td>
<td>-1.64</td>
<td>-1.28</td>
<td>0</td>
<td>1.28</td>
<td>1.64</td>
</tr>
</tbody>
</table>

Table 5.3: \( \rho \) Percentiles for the Single Mean Model (\( m = 20 \))

<table>
<thead>
<tr>
<th>Percentiles</th>
<th>0.05</th>
<th>0.1</th>
<th>0.5</th>
<th>0.9</th>
<th>0.95</th>
</tr>
</thead>
<tbody>
<tr>
<td>d=12</td>
<td>-3.92(0.05)</td>
<td>-3.02(0.04)</td>
<td>-0.49(0.02)</td>
<td>1.30 (0.02)</td>
<td>1.73(0.03)</td>
</tr>
<tr>
<td>d=24</td>
<td>-3.33(0.04)</td>
<td>-2.58(0.03)</td>
<td>-0.32(0.02)</td>
<td>1.44 (0.02)</td>
<td>1.88(0.02)</td>
</tr>
<tr>
<td>d=52</td>
<td>-2.98(0.04)</td>
<td>-2.29(0.03)</td>
<td>-0.23(0.02)</td>
<td>1.56 (0.02)</td>
<td>1.99(0.03)</td>
</tr>
<tr>
<td>d=60</td>
<td>-2.91(0.03)</td>
<td>-2.24(0.03)</td>
<td>-0.18(0.02)</td>
<td>1.58 (0.02)</td>
<td>2.01(0.03)</td>
</tr>
<tr>
<td>d=365</td>
<td>-2.59(0.03)</td>
<td>-1.99(0.03)</td>
<td>-0.08(0.02)</td>
<td>1.76 (0.03)</td>
<td>2.28(0.03)</td>
</tr>
<tr>
<td>normal(0,2)</td>
<td>-2.33</td>
<td>-1.81</td>
<td>0</td>
<td>1.81</td>
<td>2.33</td>
</tr>
</tbody>
</table>

Table 5.4: \( \tau \) Percentiles for the Single Mean Model (\( m = 20 \))

<table>
<thead>
<tr>
<th>Percentiles</th>
<th>0.05</th>
<th>0.1</th>
<th>0.5</th>
<th>0.9</th>
<th>0.95</th>
</tr>
</thead>
<tbody>
<tr>
<td>d=12</td>
<td>-2.03(0.02)</td>
<td>-1.65(0.02)</td>
<td>-0.33(0.01)</td>
<td>0.99 (0.02)</td>
<td>1.38(0.02)</td>
</tr>
<tr>
<td>d=24</td>
<td>-1.91(0.02)</td>
<td>-1.54(0.02)</td>
<td>-0.22(0.01)</td>
<td>1.05 (0.02)</td>
<td>1.45(0.02)</td>
</tr>
<tr>
<td>d=52</td>
<td>-1.82(0.02)</td>
<td>-1.44(0.02)</td>
<td>-0.16(0.01)</td>
<td>1.16 (0.02)</td>
<td>1.51(0.02)</td>
</tr>
<tr>
<td>d=60</td>
<td>-1.81(0.02)</td>
<td>-1.45(0.02)</td>
<td>-0.15(0.01)</td>
<td>1.13 (0.01)</td>
<td>1.51(0.02)</td>
</tr>
<tr>
<td>d=365</td>
<td>-1.72(0.02)</td>
<td>-1.32(0.02)</td>
<td>-0.06(0.01)</td>
<td>1.22 (0.02)</td>
<td>1.63(0.02)</td>
</tr>
<tr>
<td>normal(0,1)</td>
<td>-1.64</td>
<td>-1.28</td>
<td>0</td>
<td>1.28</td>
<td>1.64</td>
</tr>
</tbody>
</table>
Table 5.5: $\rho$ Percentiles for the Linear Trend Model ($m = 20$)

<table>
<thead>
<tr>
<th>Percentiles</th>
<th>0.05</th>
<th>0.1</th>
<th>0.5</th>
<th>0.9</th>
<th>0.95</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d=12$</td>
<td>-4.05(0.06)</td>
<td>-3.06(0.05)</td>
<td>-0.49(0.02)</td>
<td>1.32(0.02)</td>
<td>1.77(0.02)</td>
</tr>
<tr>
<td>$d=24$</td>
<td>-3.42(0.05)</td>
<td>-2.65(0.03)</td>
<td>-0.38(0.02)</td>
<td>1.45(0.02)</td>
<td>1.88(0.02)</td>
</tr>
<tr>
<td>$d=52$</td>
<td>-2.97(0.04)</td>
<td>-2.28(0.03)</td>
<td>-0.22(0.02)</td>
<td>1.54(0.02)</td>
<td>2.01(0.03)</td>
</tr>
<tr>
<td>$d=60$</td>
<td>-2.93(0.04)</td>
<td>-2.26(0.03)</td>
<td>-0.17(0.02)</td>
<td>1.61(0.02)</td>
<td>2.03(0.03)</td>
</tr>
<tr>
<td>$d=365$</td>
<td>-2.55(0.03)</td>
<td>-1.97(0.03)</td>
<td>-0.06(0.02)</td>
<td>1.78(0.02)</td>
<td>2.26(0.02)</td>
</tr>
<tr>
<td>normal(0,2)</td>
<td>-2.33</td>
<td>-1.81</td>
<td>0</td>
<td>1.81</td>
<td>2.33</td>
</tr>
</tbody>
</table>

Table 5.6: $\tau$ Percentiles for the Linear Trend Model ($m = 20$)

<table>
<thead>
<tr>
<th>Percentiles</th>
<th>0.05</th>
<th>0.1</th>
<th>0.5</th>
<th>0.9</th>
<th>0.95</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d=12$</td>
<td>-2.18(0.02)</td>
<td>-1.74(0.02)</td>
<td>-0.33(0.01)</td>
<td>1.03(0.02)</td>
<td>1.44(0.02)</td>
</tr>
<tr>
<td>$d=24$</td>
<td>-1.98(0.02)</td>
<td>-1.60(0.02)</td>
<td>-0.26(0.01)</td>
<td>1.10(0.02)</td>
<td>1.48(0.02)</td>
</tr>
<tr>
<td>$d=52$</td>
<td>-1.84(0.03)</td>
<td>-1.45(0.02)</td>
<td>-0.15(0.01)</td>
<td>1.14(0.02)</td>
<td>1.51(0.02)</td>
</tr>
<tr>
<td>$d=60$</td>
<td>-1.82(0.02)</td>
<td>-1.45(0.02)</td>
<td>-0.12(0.01)</td>
<td>1.19(0.02)</td>
<td>1.53(0.02)</td>
</tr>
<tr>
<td>$d=365$</td>
<td>-1.69(0.02)</td>
<td>-1.32(0.02)</td>
<td>-0.04(0.01)</td>
<td>1.27(0.02)</td>
<td>1.61(0.02)</td>
</tr>
<tr>
<td>normal(0,1)</td>
<td>-1.64</td>
<td>-1.28</td>
<td>0</td>
<td>1.28</td>
<td>1.64</td>
</tr>
</tbody>
</table>

Table 5.7: $\rho$ Percentiles for the Sinusoidal Model ($m = 20$)

<table>
<thead>
<tr>
<th>Percentiles</th>
<th>0.05</th>
<th>0.1</th>
<th>0.5</th>
<th>0.9</th>
<th>0.95</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d=12$</td>
<td>-5.23(0.06)</td>
<td>-4.19(0.05)</td>
<td>-1.27(0.03)</td>
<td>0.79(0.03)</td>
<td>1.25(0.03)</td>
</tr>
<tr>
<td>$d=24$</td>
<td>-4.02(0.06)</td>
<td>-3.24(0.04)</td>
<td>-0.83(0.02)</td>
<td>1.11(0.02)</td>
<td>1.60(0.02)</td>
</tr>
<tr>
<td>$d=52$</td>
<td>-3.38(0.04)</td>
<td>-2.67(0.03)</td>
<td>-0.55(0.02)</td>
<td>1.30(0.02)</td>
<td>1.73(0.02)</td>
</tr>
<tr>
<td>$d=60$</td>
<td>-3.32(0.04)</td>
<td>-2.61(0.03)</td>
<td>-0.50(0.02)</td>
<td>1.32(0.02)</td>
<td>1.82(0.03)</td>
</tr>
<tr>
<td>$d=365$</td>
<td>-2.70(0.03)</td>
<td>-2.16(0.03)</td>
<td>-0.19(0.02)</td>
<td>1.61(0.03)</td>
<td>2.11(0.03)</td>
</tr>
<tr>
<td>normal(0,2)</td>
<td>-2.33</td>
<td>-1.81</td>
<td>0</td>
<td>1.81</td>
<td>2.33</td>
</tr>
</tbody>
</table>

Table 5.8: $\tau$ Percentiles for the Sinusoidal Model ($m = 20$)

<table>
<thead>
<tr>
<th>Percentiles</th>
<th>0.05</th>
<th>0.1</th>
<th>0.5</th>
<th>0.9</th>
<th>0.95</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d=12$</td>
<td>-3.01(0.04)</td>
<td>-2.49(0.02)</td>
<td>-0.86(0.02)</td>
<td>0.61(0.02)</td>
<td>1.02(0.02)</td>
</tr>
<tr>
<td>$d=24$</td>
<td>-2.40(0.02)</td>
<td>-1.97(0.02)</td>
<td>-0.57(0.01)</td>
<td>0.84(0.02)</td>
<td>1.22(0.02)</td>
</tr>
<tr>
<td>$d=52$</td>
<td>-2.09(0.03)</td>
<td>-1.70(0.02)</td>
<td>-0.38(0.01)</td>
<td>0.95(0.02)</td>
<td>1.29(0.02)</td>
</tr>
<tr>
<td>$d=60$</td>
<td>-2.07(0.02)</td>
<td>-1.67(0.02)</td>
<td>-0.34(0.01)</td>
<td>0.97(0.02)</td>
<td>1.35(0.02)</td>
</tr>
<tr>
<td>$d=365$</td>
<td>-1.79(0.02)</td>
<td>-1.45(0.02)</td>
<td>-0.13(0.01)</td>
<td>1.14(0.02)</td>
<td>1.50(0.02)</td>
</tr>
<tr>
<td>normal(0,1)</td>
<td>-1.64</td>
<td>-1.28</td>
<td>0</td>
<td>1.28</td>
<td>1.64</td>
</tr>
</tbody>
</table>
Table 5.9: Proportion of Rejection for $\rho$-statistic in No Mean Model at a Nominal Significance Level of 5% (Using N(0,2) as Reference Distribution, $m = 20$)

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\rho = 1$</th>
<th>$\rho = 0.99$</th>
<th>$\rho = 0.95$</th>
<th>$\rho = 0.7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>0.12 ( 0.0032 )</td>
<td>0.21 ( 0.0041 )</td>
<td>0.78 ( 0.0041 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
<tr>
<td>24</td>
<td>0.099 ( 0.0030 )</td>
<td>0.24 ( 0.0042 )</td>
<td>0.95 ( 0.0021 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
<tr>
<td>52</td>
<td>0.088 ( 0.0028 )</td>
<td>0.31 ( 0.0046 )</td>
<td>1.00 ( 0.0024 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
<tr>
<td>365</td>
<td>0.065 ( 0.0024 )</td>
<td>0.84 ( 0.0036 )</td>
<td>1.00 ( 0.00 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
</tbody>
</table>

standard normal percentiles. They get closer to the theoretical limits monotonically as $d$ gets large. The upper tail is much closer to the limit than the lower tail. The $\tau$ percentiles are closer to the limit than those of the $\rho$.

Most standard errors for the $\rho$ percentiles are less than 0.05, with just a few around 0.1. Almost all the standard errors for the $\tau$ percentiles are less than 0.05.

5.3 Size and Power for the OLS Estimators

Power estimates were also obtained from the same Monte Carlo datasets, for the four different models: no mean model, single mean model, sinusoidal model and linear trend model. In all cases, the hypothesis that $\rho_d = 1$ was tested at the 0.05 significance level against the alternative that $\rho_d < 1$. The normal distribution is used as the reference distribution. The power estimate is the fraction of the samples in which the statistic was in the rejection region. The number of replications was still 10000, and $m = 20$.

Under the null hypothesis that $\rho_d = 1$, the empirical size for all statistics gets closer to the significance level (0.05) as $d$ gets large. Under the alternative, the power increases with $d$. We see the power gets larger when the true value for $\rho$ goes further away from the null: When $\rho = 0.95$, all statistics have power exceeding 0.9 for $d \geq 24$. When $\rho = 0.7$, all statistics have power 1.

The $\tau$ statistic is better than the $\rho$ in terms of level.
Table 5.10: Proportion of Rejection for $\tau$-statistic in No Mean Model at a Nominal Significance Level of 5% (Using N(0,1) as Reference Distribution, $m = 20$)

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\rho = 1$</th>
<th>$\rho = 0.99$</th>
<th>$\rho = 0.95$</th>
<th>$\rho = 0.7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>0.067 ( 0.0025 )</td>
<td>0.13 ( 0.0034 )</td>
<td>0.64 ( 0.0048 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
<tr>
<td>24</td>
<td>0.058 ( 0.0024 )</td>
<td>0.16 ( 0.0037 )</td>
<td>0.90 ( 0.0030 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
<tr>
<td>52</td>
<td>0.056 ( 0.0024 )</td>
<td>0.24 ( 0.0043 )</td>
<td>1.00 ( 0.00044 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
<tr>
<td>365</td>
<td>0.051 ( 0.0022 )</td>
<td>0.81 ( 0.0039 )</td>
<td>1.00 ( 0.00 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
</tbody>
</table>

Table 5.11: Proportion of Rejection for $\rho$-statistic in Single Mean Model at a Nominal Significance Level of 5% (Using N(0,2) as Reference Distribution, $m = 20$)

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\rho = 1$</th>
<th>$\rho = 0.99$</th>
<th>$\rho = 0.95$</th>
<th>$\rho = 0.7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>0.16 ( 0.0037 )</td>
<td>0.28 ( 0.0040 )</td>
<td>0.83 ( 0.0039 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
<tr>
<td>24</td>
<td>0.12 ( 0.0033 )</td>
<td>0.29 ( 0.0041 )</td>
<td>0.96 ( 0.0019 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
<tr>
<td>52</td>
<td>0.10 ( 0.0029 )</td>
<td>0.35 ( 0.0044 )</td>
<td>1.00 ( 0.00020 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
<tr>
<td>365</td>
<td>0.067 ( 0.0025 )</td>
<td>0.85 ( 0.0034 )</td>
<td>1.00 ( 0.00 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
</tbody>
</table>

Table 5.12: Proportion of Rejection for $\tau$-statistic in Single Mean Model at a Nominal Significance Level of 5% (Using N(0,1) as Reference Distribution, $m = 20$)

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\rho = 1$</th>
<th>$\rho = 0.99$</th>
<th>$\rho = 0.95$</th>
<th>$\rho = 0.7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>0.10 ( 0.0030 )</td>
<td>0.18 ( 0.0038 )</td>
<td>0.71 ( 0.0042 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
<tr>
<td>24</td>
<td>0.078 ( 0.0027 )</td>
<td>0.21 ( 0.0040 )</td>
<td>0.92 ( 0.0019 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
<tr>
<td>52</td>
<td>0.068 ( 0.0025 )</td>
<td>0.28 ( 0.0043 )</td>
<td>1.00 ( 0.00021 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
<tr>
<td>365</td>
<td>0.055 ( 0.0023 )</td>
<td>0.83 ( 0.0037 )</td>
<td>1.00 ( 0.00 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
</tbody>
</table>

Table 5.13: Proportion of Rejection for $\rho$-statistic in Linear Trend Model at a Nominal Significance Level of 5% (Using N(0,2) as Reference Distribution, $m = 20$)

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\rho = 1$</th>
<th>$\rho = 0.99$</th>
<th>$\rho = 0.95$</th>
<th>$\rho = 0.7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>0.16 ( 0.0037 )</td>
<td>0.29 ( 0.0045 )</td>
<td>0.84 ( 0.0037 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
<tr>
<td>24</td>
<td>0.12 ( 0.0033 )</td>
<td>0.28 ( 0.0045 )</td>
<td>0.96 ( 0.0020 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
<tr>
<td>52</td>
<td>0.093 ( 0.0029 )</td>
<td>0.35 ( 0.0048 )</td>
<td>1.00 ( 0.00017 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
<tr>
<td>365</td>
<td>0.066 ( 0.0025 )</td>
<td>0.86 ( 0.0035 )</td>
<td>1.00 ( 0.00 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
</tbody>
</table>
Table 5.14: Proportion of Rejection for $\tau$-statistic in Linear Trend Model at a Nominal Significance Level of 5% (Using N(0,1) as Reference Distribution, $m = 20$)

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\rho = 1$</th>
<th>$\rho = 0.99$</th>
<th>$\rho = 0.95$</th>
<th>$\rho = 0.7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>0.11 ( 0.0032 )</td>
<td>0.21 ( 0.0041 )</td>
<td>0.74 ( 0.0044 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
<tr>
<td>24</td>
<td>0.084 ( 0.0028 )</td>
<td>0.22 ( 0.0041 )</td>
<td>0.92 ( 0.0026 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
<tr>
<td>52</td>
<td>0.069 ( 0.0025 )</td>
<td>0.28 ( 0.0045 )</td>
<td>1.00 ( 0.0030 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
<tr>
<td>365</td>
<td>0.055 ( 0.0023 )</td>
<td>0.83 ( 0.0037 )</td>
<td>1.00 ( 0.00 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
</tbody>
</table>

Table 5.15: Proportion of Rejection for $\rho$-statistic in Sinusoidal Model at a Nominal Significance Level of 5% (Using N(0,2) as Reference Distribution, $m = 20$)

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\rho = 1$</th>
<th>$\rho = 0.99$</th>
<th>$\rho = 0.95$</th>
<th>$\rho = 0.7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>0.17 ( 0.0037 )</td>
<td>0.28 ( 0.0045 )</td>
<td>0.83 ( 0.0037 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
<tr>
<td>24</td>
<td>0.12 ( 0.0033 )</td>
<td>0.28 ( 0.0045 )</td>
<td>0.96 ( 0.0018 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
<tr>
<td>52</td>
<td>0.099 ( 0.0030 )</td>
<td>0.36 ( 0.0048 )</td>
<td>1.00 ( 0.00024 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
<tr>
<td>365</td>
<td>0.072 ( 0.0026 )</td>
<td>0.85 ( 0.0035 )</td>
<td>1.00 ( 0.00 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
</tbody>
</table>

Table 5.16: Proportion of Rejection for $\tau$-statistic in Sinusoidal Model at a Nominal Significance Level of 5% (Using N(0,1) as Reference Distribution, $m = 20$)

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\rho = 1$</th>
<th>$\rho = 0.99$</th>
<th>$\rho = 0.95$</th>
<th>$\rho = 0.7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>0.10 ( 0.0030 )</td>
<td>0.19 ( 0.0039 )</td>
<td>0.71 ( 0.0045 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
<tr>
<td>24</td>
<td>0.075 ( 0.0026 )</td>
<td>0.20 ( 0.0040 )</td>
<td>0.92 ( 0.0026 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
<tr>
<td>52</td>
<td>0.070 ( 0.0025 )</td>
<td>0.29 ( 0.0045 )</td>
<td>1.00 ( 0.00037 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
<tr>
<td>365</td>
<td>0.060 ( 0.0023 )</td>
<td>0.83 ( 0.0037 )</td>
<td>1.00 ( 0.00 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
</tbody>
</table>
Table 5.17: Proportion of Rejection for Adjusted $\tau$-statistic in No Mean Model at a Nominal Significance Level of 5% (Using N(0,1) as Reference Distribution, $m = 20$)

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\rho = 1$</th>
<th>$\rho = 0.99$</th>
<th>$\rho = 0.95$</th>
<th>$\rho = 0.7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>0.05 ( 0.0022 )</td>
<td>0.10 ( 0.0030 )</td>
<td>0.58 ( 0.0049 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
<tr>
<td>24</td>
<td>0.05 ( 0.0022 )</td>
<td>0.14 ( 0.0034 )</td>
<td>0.88 ( 0.0032 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
<tr>
<td>52</td>
<td>0.052 ( 0.0022 )</td>
<td>0.23 ( 0.0042 )</td>
<td>1.00 ( 0.00054 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
<tr>
<td>365</td>
<td>0.051 ( 0.0022 )</td>
<td>0.81 ( 0.0039 )</td>
<td>1.00 ( 0.00 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
</tbody>
</table>

Table 5.18: Proportion of Rejection for Adjusted $\tau$-statistic in Single Mean Model at a Nominal Significance Level of 5% (Using N(0,1) as Reference Distribution, $m = 20$)

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\rho = 1$</th>
<th>$\rho = 0.99$</th>
<th>$\rho = 0.95$</th>
<th>$\rho = 0.7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>0.053 ( 0.0022 )</td>
<td>0.11 ( 0.0031 )</td>
<td>0.55 ( 0.00497 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
<tr>
<td>24</td>
<td>0.046 ( 0.0021 )</td>
<td>0.14 ( 0.0035 )</td>
<td>0.87 ( 0.00335 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
<tr>
<td>52</td>
<td>0.047 ( 0.0021 )</td>
<td>0.22 ( 0.0041 )</td>
<td>1.00 ( 0.00067 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
<tr>
<td>365</td>
<td>0.048 ( 0.0021 )</td>
<td>0.81 ( 0.0039 )</td>
<td>1.00 ( 0.00000 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
</tbody>
</table>

5.4 Size and Power after the Adjustment

We run the simulations again to get the percentage of rejection for the adjusted test statistics under a nominal significance level of 5%. The adjustment is the one developed in section 4.2. From Table 5.17-5.20, we can see that the adjustment works pretty well. Under the null hypothesis we get closer to the nominal significance level than before (compared with Table 5.10, 5.12, 5.14 and 5.16, especially when lag number $d$ is small). Also we reject less often under the alternative. For $\rho \geq 0.95$, we get pretty nice power.

Table 5.19: Proportion of Rejection for Adjusted $\tau$-statistic in Linear Trend Model at a Nominal Significance Level of 5% (Using N(0,1) as Reference Distribution, $m = 20$)

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\rho = 1$</th>
<th>$\rho = 0.99$</th>
<th>$\rho = 0.95$</th>
<th>$\rho = 0.7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>0.092 ( 0.0029 )</td>
<td>0.17 ( 0.0038 )</td>
<td>0.68 ( 0.0047 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
<tr>
<td>24</td>
<td>0.069 ( 0.0025 )</td>
<td>0.19 ( 0.0039 )</td>
<td>0.91 ( 0.0029 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
<tr>
<td>52</td>
<td>0.060 ( 0.0023 )</td>
<td>0.26 ( 0.0043 )</td>
<td>1.00 ( 0.00040 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
<tr>
<td>365</td>
<td>0.052 ( 0.0022 )</td>
<td>0.83 ( 0.0038 )</td>
<td>1.00 ( 0.00 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
</tbody>
</table>
Table 5.20: Proportion of Rejection for Adjusted \( \tau \)-statistic in Sinusoidal Model at a Nominal Significance Level of 5\% (Using N(0,1) as Reference Distribution, \( m = 20 \))

<table>
<thead>
<tr>
<th>( d )</th>
<th>( \rho = 1 )</th>
<th>( \rho = 0.99 )</th>
<th>( \rho = 0.95 )</th>
<th>( \rho = 0.7 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>0.076 ( 0.0027 )</td>
<td>0.15 ( 0.0036 )</td>
<td>0.65 ( 0.0048 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
<tr>
<td>24</td>
<td>0.061 ( 0.0024 )</td>
<td>0.18 ( 0.0038 )</td>
<td>0.91 ( 0.0029 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
<tr>
<td>52</td>
<td>0.057 ( 0.0023 )</td>
<td>0.25 ( 0.0043 )</td>
<td>1.00 ( 0.00050 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
<tr>
<td>365</td>
<td>0.052 ( 0.0022 )</td>
<td>0.82 ( 0.0038 )</td>
<td>1.00 ( 0.00 )</td>
<td>1.00 ( 0.00 )</td>
</tr>
</tbody>
</table>
Figure 5.1: Normal Probability Plots for $\rho$ and $\tau$: Zero Mean Model
Figure 5.2: Normal Probability Plots for $\rho$ and $\tau$: Single Mean Model
Figure 5.3: Normal Probability Plots for $\rho$ and $\tau$: Linear Trend Model
Figure 5.4: Normal Probability Plots for $\rho$ and $\tau$: Sinusoidal Mean Model
Chapter 6

Application: Natural Gas Storage

Data Analysis

In this application, our goal is to apply our theorem to analyze some weekly Natural Gas Storage data.

6.1 Data

The data that we use to illustrate our asymptotic result is reported on the Department of Energy’s Energy Information Agency web page. It is weekly data on working natural gas in underground storage in billions of cubic feet. The data is from 31st December, 1993 to 22nd August, 2008, 765 observations in total. Figure 6.1 is a plot of the gas storage data by date.

6.2 Analysis

In order to investigate the seasonality of the data, we perform spectral analysis. There are two big spikes in the smoothed periodogram in Figure 6.2, suggesting a fundamental sinusoid of period 52 and one harmonic. We thus work with an overall mean, a linear trend and the four sinusoidal regressors, and then analyze the residuals.
Model fitting was done in SAS. From the SAS output (not shown), an AR(2) model seems to fit the span 52 differences of the residuals: \( Y_t = r_t - 1.38r_{t-1} + 0.39r_{t-2} \). Following DHF’s suggestion, we want to test whether the filtered series \( Y_t \) is a seasonal random walk or not. We run a regression of the span 52 difference \( (Y_t - Y_{t-52}) \) on the lagged series \( (Y_{t-52}) \) and lagged differences of residuals \( (r_{t-1} - r_{t-53}) \) and \( r_{t-2} - r_{t-54} \). The coefficient estimate for the lagged series \( (Y_{t-52}) \) gives a test of the seasonal unit root \( (H_0: \rho=1) \).

The \( \tau \) statistic for the lagged series \( (Y_{t-52}) \) is -26.51. Following our discussion for adjustment in Subsection 4.2.3, the adjusted \( \tau \) statistic for 5 periodic regressors (1 overall mean and 4 sinusoidal regressors) is:
\[ \tau_{adj} = -26.51 + \frac{\sqrt{2}}{3\sqrt{d}} + \frac{5\sqrt{2}}{2\sqrt{d}} \]
\[ = -26.51 + \frac{\sqrt{2}}{3\sqrt{52}} + \frac{5\sqrt{2}}{2\sqrt{52}} \]
\[ \approx -25.95 \]

Compared to \(N(0,1)\), it is highly significant. We reject the null hypothesis that there is a seasonal unit root. The sinusoid and trend may have accounted for all seasonality.

The SAS analysis shows that the coefficient on the lagged series \(Y_{t-52}\), which estimates the \(\rho - 1\), is -1.00953. The actual seasonal AR coefficient may be near 0. The updates for the \(AR(2)\) parameters are -0.01273 and 0.01609. All of the Box-Ljung p-values are greater than 0.05, so this is a good fit. We can use this model to
make predictions.

Since the ACF of the residuals dies off very slowly, we also consider the popular “airline model” of Box and Jenkins, which takes into account the first order difference. In this case the first few of the Box-Ljung statistics are highly significant, therefore we decide not to choose this model.

To check our fit, we refit the model. Data starting from January 1st, 2007 were withheld to do the post-sample validation. A forecast is made based on the fitted model. Figure 6.3 shows a plot of the data forecast and forecast error bands. From the plot we can see that the forecast is nice, with tight forecast error bands.

Figure 6.3: Post-sample Validation
Chapter 7

Conclusion

7.1 Summary

In this dissertation, we modified the seasonal unit roots test of Dickey, Hasza, and Fuller, and proved the asymptotic normality result which can be used for testing unit roots in seasonal time series with a long period.

We studied several commonly used models in seasonal time series, and derived the limiting distribution for the test statistics as the seasonal period increases. We proved that the asymptotic normality can be generalized to regression adjusted models as long as the number of regressors in the deterministic terms is fixed. The convergence was proved to be order independent, and holds in the higher order model as well. We also proposed an empirical adjustment to improve the normal approximation.

We conducted a simulation study to generate the percentiles for the test statistics in seasonal time series. The distribution of the percentiles gets closer to normal as the seasonal period increases. We also ran a simulation to investigate the performance of the asymptotic normality and the suggested adjustment. The results show that our approximation is appealing with the right size and nice power. We used the suggested adjustment and normal reference distribution in an application to Natural Gas Storage data analysis. This application showed that the proposed method leads to an appropriate model fitting and promising prediction. In practice, the proposed
approach is very attractive because of the simplicity.

7.2 Future Research

In this dissertation, we only presented empirical adjustment suggestions for no mean model, single mean model, and sinusoidal and periodic function mean model. We will try to deduce the Taylor expansion expression for the linear trend model and thus obtain the suggested adjustment in our future study.

When applying our asymptotic normality theorem in practice, we might confront with some real data that does not have constant lag numbers. For example, for daily data, the lag number is 365. However there is a leap year every four years, where you cannot get the right data point if you still lag the data by 365. How to deal with this kind of real problem is worth investigation.

We would also like to apply the Elliott, Rothenberg, and Stock’s DF-GLS detrending approach to our long period seasonal time series model to see whether a power improvement can be obtained.
Bibliography


Time Series with a Unit Root. Journal of the American Statistical Association, 


[12] Dickey, D. A., Further Results on Seasonal Time Series. JSM Presentation, 


[14] Elliott, G., Rothenberg, T.J., and Stock, J. H., Efficient tests for an autoregres-


York, 2nd ed, 1996.

Seasonal Unit Roots. Journal of American Statistical Association, 91:1551-1559, 
1996.


Appendices
APPENDIX A

Table A.1: Mapping Table for No Mean Model

<table>
<thead>
<tr>
<th>prob(τ)</th>
<th>0.01</th>
<th>0.025</th>
<th>0.05</th>
<th>0.10</th>
<th>0.50</th>
</tr>
</thead>
<tbody>
<tr>
<td>d=2 n=20</td>
<td>μ=-.23</td>
<td>σ=1.040</td>
<td>0.894</td>
<td>0.95</td>
<td>0.977</td>
</tr>
<tr>
<td>d=2 n=30</td>
<td>μ=-.25</td>
<td>σ=1.021</td>
<td>0.897</td>
<td>0.95</td>
<td>0.977</td>
</tr>
<tr>
<td>d=2 n=40</td>
<td>μ=-.255</td>
<td>σ=1.018</td>
<td>0.897</td>
<td>0.95</td>
<td>0.976</td>
</tr>
<tr>
<td>d=2 n=100</td>
<td>μ=-.265</td>
<td>σ=1.006</td>
<td>0.897</td>
<td>0.95</td>
<td>0.976</td>
</tr>
<tr>
<td>d=2 n=200</td>
<td>μ=-.27</td>
<td>σ=1.003</td>
<td>0.897</td>
<td>0.95</td>
<td>0.976</td>
</tr>
<tr>
<td>d=2 n=600</td>
<td>μ=-.27</td>
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<td>0.897</td>
<td>0.95</td>
<td>0.975</td>
</tr>
<tr>
<td>d=2 n=1200</td>
<td>μ=-.27</td>
<td>σ=1.003</td>
<td>0.898</td>
<td>0.95</td>
<td>0.975</td>
</tr>
<tr>
<td>d=4 n=60</td>
<td>μ=-.20</td>
<td>σ=1.027</td>
<td>0.897</td>
<td>0.95</td>
<td>0.975</td>
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<td>0.975</td>
</tr>
<tr>
<td>d=4 n=200</td>
<td>μ=-.225</td>
<td>σ=1.018</td>
<td>0.898</td>
<td>0.95</td>
<td>0.975</td>
</tr>
<tr>
<td>d=4 n=400</td>
<td>μ=-.23</td>
<td>σ=1.015</td>
<td>0.90</td>
<td>0.95</td>
<td>0.976</td>
</tr>
<tr>
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<td>0.976</td>
</tr>
<tr>
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<td>μ=-.23</td>
<td>σ=1.015</td>
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<tr>
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<td>μ=-.11</td>
<td>σ=1.003</td>
<td>0.901</td>
<td>0.95</td>
<td>0.975</td>
</tr>
<tr>
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<td>0.975</td>
</tr>
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<td>d=12 n=600</td>
<td>μ=-.135</td>
<td>σ=1.006</td>
<td>0.899</td>
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<td>0.975</td>
</tr>
<tr>
<td>d=12 n=1200</td>
<td>μ=-.135</td>
<td>σ=1.006</td>
<td>0.899</td>
<td>0.95</td>
<td>0.975</td>
</tr>
<tr>
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<td>0.95</td>
<td>0.975</td>
</tr>
<tr>
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<td>μ=-.14</td>
<td>σ=1.009</td>
<td>0.899</td>
<td>0.95</td>
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</tr>
</tbody>
</table>

(a) Part 1 of the mapping table

<table>
<thead>
<tr>
<th>prob(τ)</th>
<th>0.90</th>
<th>0.95</th>
<th>0.975</th>
<th>0.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>d=2 n=20</td>
<td>μ=-.23</td>
<td>σ=1.040</td>
<td>0.894</td>
<td>0.95</td>
</tr>
<tr>
<td>d=2 n=30</td>
<td>μ=-.25</td>
<td>σ=1.021</td>
<td>0.897</td>
<td>0.95</td>
</tr>
<tr>
<td>d=2 n=40</td>
<td>μ=-.255</td>
<td>σ=1.018</td>
<td>0.897</td>
<td>0.95</td>
</tr>
<tr>
<td>d=2 n=100</td>
<td>μ=-.265</td>
<td>σ=1.006</td>
<td>0.897</td>
<td>0.95</td>
</tr>
<tr>
<td>d=2 n=200</td>
<td>μ=-.27</td>
<td>σ=1.003</td>
<td>0.897</td>
<td>0.95</td>
</tr>
<tr>
<td>d=2 n=600</td>
<td>μ=-.27</td>
<td>σ=1.003</td>
<td>0.897</td>
<td>0.95</td>
</tr>
<tr>
<td>d=2 n=1200</td>
<td>μ=-.27</td>
<td>σ=1.003</td>
<td>0.898</td>
<td>0.95</td>
</tr>
<tr>
<td>d=4 n=60</td>
<td>μ=-.20</td>
<td>σ=1.027</td>
<td>0.897</td>
<td>0.95</td>
</tr>
<tr>
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<td>0.898</td>
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<td>0.898</td>
<td>0.95</td>
</tr>
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<td>d=4 n=400</td>
<td>μ=-.23</td>
<td>σ=1.015</td>
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<td>0.95</td>
</tr>
<tr>
<td>d=4 n=800</td>
<td>μ=-.23</td>
<td>σ=1.015</td>
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<td>0.95</td>
</tr>
<tr>
<td>d=4 n=∞</td>
<td>μ=-.23</td>
<td>σ=1.015</td>
<td>0.90</td>
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</tr>
<tr>
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<td>μ=-.11</td>
<td>σ=1.003</td>
<td>0.901</td>
<td>0.95</td>
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<td>d=12 n=240</td>
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<td>σ=1.003</td>
<td>0.899</td>
<td>0.95</td>
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<td>σ=1.006</td>
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<td>0.95</td>
</tr>
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</table>

(b) Part 2 of the mapping table
Table A.2: Mapping Table for Single Mean Model

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<tr>
<th>d=n</th>
<th>n</th>
<th>μ</th>
<th>σ</th>
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<th>0.025</th>
<th>0.05</th>
<th>0.10</th>
<th>0.50</th>
</tr>
</thead>
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<td>-0.9</td>
<td>1.064</td>
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<td>0.024</td>
<td>0.05</td>
<td>0.1</td>
<td>0.475</td>
<td></td>
</tr>
<tr>
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<td>1.0851</td>
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<td>0.026</td>
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<td>0.098</td>
<td>0.461</td>
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<td>0.026</td>
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<td>0.097</td>
<td>0.459</td>
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<tr>
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</tr>
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</tr>
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(a) Part 1 of the mapping table

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<td>1.0274</td>
<td>0.901</td>
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<td>0.975</td>
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</table>

(b) Part 2 of the mapping table
Table A.3: Mapping Table for Seasonal Mean Model

<table>
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<tr>
<th>d=2</th>
<th>n=20</th>
<th>μ = -1.945</th>
<th>σ = 1.0061</th>
<th>0.01</th>
<th>0.025</th>
<th>0.05</th>
<th>0.10</th>
<th>0.50</th>
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</thead>
<tbody>
<tr>
<td>d=2</td>
<td>n=30</td>
<td>μ = -1.93</td>
<td>σ = 0.9544</td>
<td>0.008</td>
<td>0.023</td>
<td>0.05</td>
<td>0.10</td>
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<tr>
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<td>σ = 0.9271</td>
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<td>0.023</td>
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<td>σ = 0.8693</td>
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<td>0.026</td>
<td>0.05</td>
<td>0.097</td>
<td>0.463</td>
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</tbody>
</table>

(a) Part 1 of the mapping table

<table>
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<th>d=2</th>
<th>n=20</th>
<th>μ = -1.945</th>
<th>σ = 1.0061</th>
<th>0.90</th>
<th>0.95</th>
<th>0.975</th>
<th>0.99</th>
</tr>
</thead>
<tbody>
<tr>
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<td>μ = -1.93</td>
<td>σ = 0.9544</td>
<td>0.899</td>
<td>0.95</td>
<td>0.976</td>
<td>0.991</td>
</tr>
<tr>
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<td>n=40</td>
<td>μ = -1.925</td>
<td>σ = 0.9271</td>
<td>0.896</td>
<td>0.95</td>
<td>0.976</td>
<td>0.991</td>
</tr>
<tr>
<td>d=2</td>
<td>n=100</td>
<td>μ = -1.92</td>
<td>σ = 0.8875</td>
<td>0.894</td>
<td>0.95</td>
<td>0.977</td>
<td>0.992</td>
</tr>
<tr>
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<td>n=200</td>
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<td>σ = 0.8754</td>
<td>0.893</td>
<td>0.95</td>
<td>0.977</td>
<td>0.992</td>
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<td>σ = 0.8693</td>
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<td>0.95</td>
<td>0.977</td>
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<tr>
<td>d=2</td>
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<td>σ = 0.8632</td>
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<td>0.95</td>
<td>0.977</td>
<td>0.992</td>
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(b) Part 2 of the mapping table
Table A.4: ρ Percentiles for the Zero Mean Model ($m = 20$)

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<tr>
<th>Percentiles</th>
<th>0.01</th>
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<th>0.1</th>
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</thead>
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<tr>
<td>d=12</td>
<td>-5.27 ( 0.09 )</td>
<td>-4.33 ( 0.08 )</td>
<td>-3.45 ( 0.05 )</td>
<td>-2.58 ( 0.04 )</td>
</tr>
<tr>
<td>d=24</td>
<td>-4.59 ( 0.07 )</td>
<td>-3.69 ( 0.06 )</td>
<td>-3.01 ( 0.05 )</td>
<td>-2.28 ( 0.04 )</td>
</tr>
<tr>
<td>d=52</td>
<td>-4.15 ( 0.09 )</td>
<td>-3.36 ( 0.06 )</td>
<td>-2.81 ( 0.04 )</td>
<td>-2.13 ( 0.03 )</td>
</tr>
<tr>
<td>d=60</td>
<td>-4.03 ( 0.05 )</td>
<td>-3.39 ( 0.05 )</td>
<td>-2.77 ( 0.04 )</td>
<td>-2.14 ( 0.03 )</td>
</tr>
<tr>
<td>d=365</td>
<td>-3.61 ( 0.05 )</td>
<td>-2.95 ( 0.04 )</td>
<td>-2.50 ( 0.03 )</td>
<td>-1.94 ( 0.03 )</td>
</tr>
<tr>
<td>normal(0,2)</td>
<td>-3.29</td>
<td>-2.77</td>
<td>-2.33</td>
<td>-1.81</td>
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(a) Part 1 of the Percentiles table

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<th>0.25</th>
<th>0.5</th>
<th>0.75</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>d=12</td>
<td>-1.30 ( 0.03 )</td>
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</tr>
<tr>
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<td>-0.16 ( 0.02 )</td>
<td>0.81 ( 0.02 )</td>
<td>1.57 ( 0.02 )</td>
</tr>
<tr>
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<td>-0.09 ( 0.02 )</td>
<td>0.88 ( 0.02 )</td>
<td>1.68 ( 0.02 )</td>
</tr>
<tr>
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<td>-1.10 ( 0.02 )</td>
<td>-0.06 ( 0.02 )</td>
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<td>1.70 ( 0.02 )</td>
</tr>
<tr>
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<td>-0.03 ( 0.02 )</td>
<td>0.92 ( 0.02 )</td>
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<tr>
<td>normal(0,2)</td>
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<td>0</td>
<td>0.95</td>
<td>1.81</td>
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(b) Part 2 of the Percentiles table

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<th>0.99</th>
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</thead>
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<tr>
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<td>1.93 ( 0.02 )</td>
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<td>2.69 ( 0.03 )</td>
</tr>
<tr>
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<td>2.35 ( 0.03 )</td>
<td>2.76 ( 0.05 )</td>
</tr>
<tr>
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<td>2.11 ( 0.03 )</td>
<td>2.49 ( 0.03 )</td>
<td>2.91 ( 0.05 )</td>
</tr>
<tr>
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</tr>
<tr>
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<td>3.29</td>
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(c) Part 3 of the Percentiles table
Table A.5: \( \tau \) Percentiles for the Zero Mean Model \((m = 20)\)

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<th>0.025</th>
<th>0.05</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>d=12</td>
<td>-2.50 (0.04)</td>
<td>-2.14 (0.03)</td>
<td>-1.82 (0.02)</td>
<td>-1.46 (0.02)</td>
</tr>
<tr>
<td>d=24</td>
<td>-2.44 (0.03)</td>
<td>-2.05 (0.02)</td>
<td>-1.74 (0.02)</td>
<td>-1.37 (0.02)</td>
</tr>
<tr>
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<td>-2.42 (0.05)</td>
<td>-2.01 (0.03)</td>
<td>-1.72 (0.02)</td>
<td>-1.36 (0.02)</td>
</tr>
<tr>
<td>d=60</td>
<td>-2.40 (0.03)</td>
<td>-2.06 (0.03)</td>
<td>-1.71 (0.02)</td>
<td>-1.36 (0.02)</td>
</tr>
<tr>
<td>d=365</td>
<td>-2.35 (0.04)</td>
<td>-1.94 (0.03)</td>
<td>-1.66 (0.02)</td>
<td>-1.30 (0.02)</td>
</tr>
<tr>
<td>normal(0,1)</td>
<td>-2.33</td>
<td>-1.96</td>
<td>-1.64</td>
<td>-1.28</td>
</tr>
</tbody>
</table>

(a) Part 1 of the Percentiles table

<table>
<thead>
<tr>
<th>Percentiles</th>
<th>0.25</th>
<th>0.5</th>
<th>0.75</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>d=12</td>
<td>-0.81 (0.01)</td>
<td>-0.12 (0.01)</td>
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<td>1.19 (0.02)</td>
</tr>
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<td>-0.11 (0.01)</td>
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<td>1.19 (0.02)</td>
</tr>
<tr>
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<td>-0.75 (0.01)</td>
<td>-0.06 (0.01)</td>
<td>0.63 (0.01)</td>
<td>1.24 (0.02)</td>
</tr>
<tr>
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<td>1.24 (0.02)</td>
</tr>
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<td>-0.02 (0.01)</td>
<td>0.64 (0.01)</td>
<td>1.26 (0.02)</td>
</tr>
<tr>
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<td>-0.67</td>
<td>0</td>
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<td>1.28</td>
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</table>

(b) Part 2 of the Percentiles table

<table>
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<th>0.975</th>
<th>0.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>d=12</td>
<td>1.55 (0.02)</td>
<td>1.89 (0.02)</td>
<td>2.24 (0.04)</td>
</tr>
<tr>
<td>d=24</td>
<td>1.56 (0.02)</td>
<td>1.85 (0.02)</td>
<td>2.21 (0.04)</td>
</tr>
<tr>
<td>d=52</td>
<td>1.58 (0.02)</td>
<td>1.92 (0.03)</td>
<td>2.28 (0.04)</td>
</tr>
<tr>
<td>d=60</td>
<td>1.57 (0.02)</td>
<td>1.91 (0.03)</td>
<td>2.26 (0.04)</td>
</tr>
<tr>
<td>d=365</td>
<td>1.62 (0.02)</td>
<td>1.93 (0.02)</td>
<td>2.37 (0.05)</td>
</tr>
<tr>
<td>normal(0,1)</td>
<td>1.64</td>
<td>1.96</td>
<td>2.32</td>
</tr>
</tbody>
</table>

(c) Part 3 of the Percentiles table
Table A.6: \( \rho \) Percentiles for the Single Mean Model \((m = 20)\)

<table>
<thead>
<tr>
<th>Percentiles</th>
<th>0.01</th>
<th>0.025</th>
<th>0.05</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>d=12</td>
<td>-5.92 (0.13)</td>
<td>-4.77 (0.08)</td>
<td>-3.92 (0.05)</td>
<td>-3.02 (0.04)</td>
</tr>
<tr>
<td>d=24</td>
<td>-4.97 (0.08)</td>
<td>-4.04 (0.06)</td>
<td>-3.33 (0.04)</td>
<td>-2.58 (0.03)</td>
</tr>
<tr>
<td>d=52</td>
<td>-4.29 (0.07)</td>
<td>-3.59 (0.05)</td>
<td>-2.98 (0.04)</td>
<td>-2.29 (0.03)</td>
</tr>
<tr>
<td>d=60</td>
<td>-4.27 (0.07)</td>
<td>-3.50 (0.05)</td>
<td>-2.91 (0.03)</td>
<td>-2.24 (0.03)</td>
</tr>
<tr>
<td>d=365</td>
<td>-3.66 (0.06)</td>
<td>-3.09 (0.04)</td>
<td>-2.59 (0.03)</td>
<td>-1.99 (0.03)</td>
</tr>
<tr>
<td>normal(0,2)</td>
<td>-3.29</td>
<td>-2.77</td>
<td>-2.33</td>
<td>-1.81</td>
</tr>
</tbody>
</table>

(a) Part 1 of the Percentiles table

<table>
<thead>
<tr>
<th>Percentiles</th>
<th>0.25</th>
<th>0.50</th>
<th>0.75</th>
<th>0.90</th>
</tr>
</thead>
<tbody>
<tr>
<td>d=12</td>
<td>-1.67 (0.03)</td>
<td>-0.49 (0.02)</td>
<td>0.53 (0.019)</td>
<td>1.30 (0.02)</td>
</tr>
<tr>
<td>d=24</td>
<td>-1.44 (0.02)</td>
<td>-0.32 (0.02)</td>
<td>0.67 (0.019)</td>
<td>1.44 (0.02)</td>
</tr>
<tr>
<td>d=52</td>
<td>-1.28 (0.02)</td>
<td>-0.23 (0.02)</td>
<td>0.77 (0.017)</td>
<td>1.56 (0.02)</td>
</tr>
<tr>
<td>d=60</td>
<td>-1.24 (0.02)</td>
<td>-0.18 (0.02)</td>
<td>0.78 (0.020)</td>
<td>1.58 (0.02)</td>
</tr>
<tr>
<td>d=365</td>
<td>-1.07 (0.02)</td>
<td>-0.08 (0.02)</td>
<td>0.91 (0.018)</td>
<td>1.76 (0.03)</td>
</tr>
<tr>
<td>normal(0,2)</td>
<td>-0.95</td>
<td>0</td>
<td>0.95</td>
<td>1.81</td>
</tr>
</tbody>
</table>

(b) Part 2 of the Percentiles table

<table>
<thead>
<tr>
<th>Percentiles</th>
<th>0.95</th>
<th>0.975</th>
<th>0.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>d=12</td>
<td>1.73 (0.03)</td>
<td>2.08 (0.03)</td>
<td>2.48 (0.04)</td>
</tr>
<tr>
<td>d=24</td>
<td>1.88 (0.02)</td>
<td>2.21 (0.03)</td>
<td>2.59 (0.04)</td>
</tr>
<tr>
<td>d=52</td>
<td>1.99 (0.03)</td>
<td>2.36 (0.03)</td>
<td>2.80 (0.04)</td>
</tr>
<tr>
<td>d=60</td>
<td>2.01 (0.03)</td>
<td>2.41 (0.03)</td>
<td>2.77 (0.03)</td>
</tr>
<tr>
<td>d=365</td>
<td>2.28 (0.03)</td>
<td>2.71 (0.04)</td>
<td>3.13 (0.04)</td>
</tr>
<tr>
<td>normal(0,2)</td>
<td>2.33</td>
<td>2.77</td>
<td>3.29</td>
</tr>
</tbody>
</table>

(c) Part 3 of the Percentiles table
Table A.7: \( \tau \) Percentiles for the Single Mean Model \((m = 20)\)

<table>
<thead>
<tr>
<th>Percentiles</th>
<th>(0.01)</th>
<th>(0.025)</th>
<th>(0.05)</th>
<th>(0.1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(d=12)</td>
<td>-2.80 ( 0.05 )</td>
<td>-2.38 ( 0.03 )</td>
<td>-2.03 ( 0.02 )</td>
<td>-1.65 ( 0.02 )</td>
</tr>
<tr>
<td>(d=24)</td>
<td>-2.67 ( 0.04 )</td>
<td>-2.23 ( 0.03 )</td>
<td>-1.91 ( 0.02 )</td>
<td>-1.54 ( 0.02 )</td>
</tr>
<tr>
<td>(d=52)</td>
<td>-2.50 ( 0.04 )</td>
<td>-2.16 ( 0.03 )</td>
<td>-1.82 ( 0.02 )</td>
<td>-1.44 ( 0.02 )</td>
</tr>
<tr>
<td>(d=60)</td>
<td>-2.50 ( 0.04 )</td>
<td>-2.12 ( 0.03 )</td>
<td>-1.81 ( 0.02 )</td>
<td>-1.45 ( 0.02 )</td>
</tr>
<tr>
<td>(d=365)</td>
<td>-2.37 ( 0.04 )</td>
<td>-2.03 ( 0.03 )</td>
<td>-1.72 ( 0.02 )</td>
<td>-1.32 ( 0.02 )</td>
</tr>
<tr>
<td>normal(0,1)</td>
<td>-2.33</td>
<td>-1.96</td>
<td>-1.64</td>
<td>-1.28</td>
</tr>
</tbody>
</table>

(a) Part 1 of the Percentiles table

<table>
<thead>
<tr>
<th>Percentiles</th>
<th>(0.25)</th>
<th>(0.5)</th>
<th>(0.75)</th>
<th>(0.9)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(d=12)</td>
<td>-1.01 ( 0.01 )</td>
<td>-0.33 ( 0.01 )</td>
<td>0.38 ( 0.01 )</td>
<td>0.99 ( 0.02 )</td>
</tr>
<tr>
<td>(d=24)</td>
<td>-0.92 ( 0.01 )</td>
<td>-0.22 ( 0.01 )</td>
<td>0.48 ( 0.01 )</td>
<td>1.05 ( 0.02 )</td>
</tr>
<tr>
<td>(d=52)</td>
<td>-0.84 ( 0.01 )</td>
<td>-0.16 ( 0.01 )</td>
<td>0.54 ( 0.01 )</td>
<td>1.16 ( 0.02 )</td>
</tr>
<tr>
<td>(d=60)</td>
<td>-0.83 ( 0.01 )</td>
<td>-0.15 ( 0.01 )</td>
<td>0.52 ( 0.01 )</td>
<td>1.13 ( 0.01 )</td>
</tr>
<tr>
<td>(d=365)</td>
<td>-0.72 ( 0.01 )</td>
<td>-0.06 ( 0.01 )</td>
<td>0.64 ( 0.01 )</td>
<td>1.22 ( 0.02 )</td>
</tr>
<tr>
<td>normal(0,1)</td>
<td>-0.67</td>
<td>0</td>
<td>0.67</td>
<td>1.28</td>
</tr>
</tbody>
</table>

(b) Part 2 of the Percentiles table

<table>
<thead>
<tr>
<th>Percentiles</th>
<th>(0.95)</th>
<th>(0.975)</th>
<th>(0.99)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(d=12)</td>
<td>1.38 ( 0.02 )</td>
<td>1.70 ( 0.02 )</td>
<td>2.12 ( 0.05 )</td>
</tr>
<tr>
<td>(d=24)</td>
<td>1.45 ( 0.02 )</td>
<td>1.71 ( 0.03 )</td>
<td>2.07 ( 0.03 )</td>
</tr>
<tr>
<td>(d=52)</td>
<td>1.51 ( 0.02 )</td>
<td>1.80 ( 0.03 )</td>
<td>2.18 ( 0.03 )</td>
</tr>
<tr>
<td>(d=60)</td>
<td>1.51 ( 0.02 )</td>
<td>1.86 ( 0.03 )</td>
<td>2.23 ( 0.04 )</td>
</tr>
<tr>
<td>(d=365)</td>
<td>1.63 ( 0.02 )</td>
<td>1.91 ( 0.03 )</td>
<td>2.27 ( 0.03 )</td>
</tr>
<tr>
<td>normal(0,1)</td>
<td>1.64</td>
<td>1.96</td>
<td>2.32</td>
</tr>
</tbody>
</table>

(c) Part 3 of the Percentiles table
Table A.8: $\rho$ Percentiles for the Linear Trend Model ($m = 20$)

<table>
<thead>
<tr>
<th>Percentiles</th>
<th>0.01</th>
<th>0.025</th>
<th>0.05</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d=12$</td>
<td>-6.16 (0.13)</td>
<td>-4.94 (0.07)</td>
<td>-4.05 (0.06)</td>
<td>-3.06 (0.05)</td>
</tr>
<tr>
<td>$d=24$</td>
<td>-5.02 (0.08)</td>
<td>-4.19 (0.06)</td>
<td>-3.42 (0.05)</td>
<td>-2.65 (0.03)</td>
</tr>
<tr>
<td>$d=52$</td>
<td>-4.37 (0.07)</td>
<td>-3.56 (0.07)</td>
<td>-2.97 (0.04)</td>
<td>-2.28 (0.03)</td>
</tr>
<tr>
<td>$d=60$</td>
<td>-4.12 (0.07)</td>
<td>-3.49 (0.05)</td>
<td>-2.93 (0.04)</td>
<td>-2.26 (0.03)</td>
</tr>
<tr>
<td>$d=365$</td>
<td>-3.62 (0.07)</td>
<td>-3.04 (0.04)</td>
<td>-2.55 (0.03)</td>
<td>-1.97 (0.03)</td>
</tr>
<tr>
<td>normal(0.2)</td>
<td>-3.29</td>
<td>-2.77</td>
<td>-2.33</td>
<td>-1.81</td>
</tr>
</tbody>
</table>

(a) Part 1 of the Percentiles table

<table>
<thead>
<tr>
<th>Percentiles</th>
<th>0.25</th>
<th>0.5</th>
<th>0.75</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d=12$</td>
<td>-1.73 (0.03)</td>
<td>-0.49 (0.02)</td>
<td>0.54 (0.02)</td>
<td>1.32 (0.02)</td>
</tr>
<tr>
<td>$d=24$</td>
<td>-1.47 (0.02)</td>
<td>-0.38 (0.02)</td>
<td>0.64 (0.02)</td>
<td>1.45 (0.02)</td>
</tr>
<tr>
<td>$d=52$</td>
<td>-1.22 (0.02)</td>
<td>-0.22 (0.02)</td>
<td>0.73 (0.02)</td>
<td>1.54 (0.02)</td>
</tr>
<tr>
<td>$d=60$</td>
<td>-1.25 (0.02)</td>
<td>-0.17 (0.02)</td>
<td>0.77 (0.02)</td>
<td>1.61 (0.02)</td>
</tr>
<tr>
<td>$d=365$</td>
<td>-1.03 (0.02)</td>
<td>-0.06 (0.02)</td>
<td>0.93 (0.02)</td>
<td>1.78 (0.02)</td>
</tr>
<tr>
<td>normal(0.2)</td>
<td>-0.95</td>
<td>0</td>
<td>0.95</td>
<td>1.81</td>
</tr>
</tbody>
</table>

(b) Part 2 of the Percentiles table

<table>
<thead>
<tr>
<th>Percentiles</th>
<th>0.95</th>
<th>0.975</th>
<th>0.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d=12$</td>
<td>1.77 (0.02)</td>
<td>2.13 (0.03)</td>
<td>2.48 (0.03)</td>
</tr>
<tr>
<td>$d=24$</td>
<td>1.88 (0.02)</td>
<td>2.21 (0.03)</td>
<td>2.64 (0.04)</td>
</tr>
<tr>
<td>$d=52$</td>
<td>2.01 (0.03)</td>
<td>2.37 (0.04)</td>
<td>2.84 (0.05)</td>
</tr>
<tr>
<td>$d=60$</td>
<td>2.03 (0.03)</td>
<td>2.44 (0.04)</td>
<td>2.87 (0.05)</td>
</tr>
<tr>
<td>$d=365$</td>
<td>2.26 (0.02)</td>
<td>2.66 (0.03)</td>
<td>3.13 (0.05)</td>
</tr>
<tr>
<td>normal(0.2)</td>
<td>2.33</td>
<td>2.77</td>
<td>3.29</td>
</tr>
</tbody>
</table>

(c) Part 3 of the Percentiles table
Table A.9: $\tau$ Percentiles for the Linear Trend Model ($m = 20$)

<table>
<thead>
<tr>
<th>Percentiles</th>
<th>0.01</th>
<th>0.025</th>
<th>0.05</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d=12$</td>
<td>-3.04 (0.04)</td>
<td>-2.54 (0.04)</td>
<td>-2.18 (0.02)</td>
<td>-1.74 (0.02)</td>
</tr>
<tr>
<td>$d=24$</td>
<td>-2.71 (0.03)</td>
<td>-2.33 (0.03)</td>
<td>-1.98 (0.02)</td>
<td>-1.60 (0.02)</td>
</tr>
<tr>
<td>$d=52$</td>
<td>-2.56 (0.04)</td>
<td>-2.15 (0.03)</td>
<td>-1.84 (0.03)</td>
<td>-1.45 (0.02)</td>
</tr>
<tr>
<td>$d=60$</td>
<td>-2.48 (0.03)</td>
<td>-2.12 (0.03)</td>
<td>-1.82 (0.02)</td>
<td>-1.45 (0.02)</td>
</tr>
<tr>
<td>$d=365$</td>
<td>-2.35 (0.05)</td>
<td>-2.00 (0.02)</td>
<td>-1.69 (0.02)</td>
<td>-1.32 (0.02)</td>
</tr>
<tr>
<td>normal(0,1)</td>
<td>-2.33</td>
<td>-1.96</td>
<td>-1.64</td>
<td>-1.28</td>
</tr>
</tbody>
</table>

(a) Part 1 of the Percentiles table

<table>
<thead>
<tr>
<th>Percentiles</th>
<th>0.25</th>
<th>0.5</th>
<th>0.75</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d=12$</td>
<td>-1.07 (0.01)</td>
<td>-0.33 (0.01)</td>
<td>0.40 (0.01)</td>
<td>1.03 (0.02)</td>
</tr>
<tr>
<td>$d=24$</td>
<td>-0.95 (0.01)</td>
<td>-0.26 (0.01)</td>
<td>0.47 (0.01)</td>
<td>1.10 (0.02)</td>
</tr>
<tr>
<td>$d=52$</td>
<td>-0.81 (0.01)</td>
<td>-0.15 (0.01)</td>
<td>0.53 (0.01)</td>
<td>1.14 (0.02)</td>
</tr>
<tr>
<td>$d=60$</td>
<td>-0.83 (0.01)</td>
<td>-0.12 (0.01)</td>
<td>0.55 (0.02)</td>
<td>1.19 (0.02)</td>
</tr>
<tr>
<td>$d=365$</td>
<td>-0.70 (0.01)</td>
<td>-0.04 (0.01)</td>
<td>0.65 (0.01)</td>
<td>1.27 (0.02)</td>
</tr>
<tr>
<td>normal(0,1)</td>
<td>-0.67</td>
<td>0</td>
<td>0.67</td>
<td>1.28</td>
</tr>
</tbody>
</table>

(b) Part 2 of the Percentiles table

<table>
<thead>
<tr>
<th>Percentiles</th>
<th>0.95</th>
<th>0.975</th>
<th>0.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d=12$</td>
<td>1.44 (0.02)</td>
<td>1.77 (0.03)</td>
<td>2.12 (0.04)</td>
</tr>
<tr>
<td>$d=24$</td>
<td>1.48 (0.02)</td>
<td>1.76 (0.03)</td>
<td>2.14 (0.04)</td>
</tr>
<tr>
<td>$d=52$</td>
<td>1.51 (0.02)</td>
<td>1.82 (0.03)</td>
<td>2.21 (0.04)</td>
</tr>
<tr>
<td>$d=60$</td>
<td>1.53 (0.02)</td>
<td>1.85 (0.02)</td>
<td>2.24 (0.04)</td>
</tr>
<tr>
<td>$d=365$</td>
<td>1.61 (0.02)</td>
<td>1.91 (0.03)</td>
<td>2.28 (0.04)</td>
</tr>
<tr>
<td>normal(0,1)</td>
<td>1.64</td>
<td>1.96</td>
<td>2.32</td>
</tr>
</tbody>
</table>

(c) Part 3 of the Percentiles table
Table A.10: $\rho$ Percentiles for the Sinusoidal Model ($m = 20$)

<table>
<thead>
<tr>
<th>Percentiles</th>
<th>0.01</th>
<th>0.025</th>
<th>0.05</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>d=12</td>
<td>-7.59 (0.13)</td>
<td>-6.37 (0.07)</td>
<td>-5.23 (0.06)</td>
<td>-4.19 (0.05)</td>
</tr>
<tr>
<td>d=24</td>
<td>-5.92 (0.13)</td>
<td>-4.88 (0.08)</td>
<td>-4.02 (0.06)</td>
<td>-3.24 (0.04)</td>
</tr>
<tr>
<td>d=52</td>
<td>-4.79 (0.08)</td>
<td>-3.98 (0.05)</td>
<td>-3.38 (0.04)</td>
<td>-2.67 (0.03)</td>
</tr>
<tr>
<td>d=60</td>
<td>-4.86 (0.11)</td>
<td>-3.95 (0.05)</td>
<td>-3.32 (0.04)</td>
<td>-2.61 (0.03)</td>
</tr>
<tr>
<td>d=365</td>
<td>-3.79 (0.05)</td>
<td>-3.18 (0.04)</td>
<td>-2.70 (0.03)</td>
<td>-2.16 (0.03)</td>
</tr>
<tr>
<td>normal(0,2)</td>
<td>-3.29</td>
<td>-2.77</td>
<td>-2.33</td>
<td>-1.81</td>
</tr>
</tbody>
</table>

(a) Part 1 of the Percentiles table

<table>
<thead>
<tr>
<th>Percentiles</th>
<th>0.25</th>
<th>0.5</th>
<th>0.75</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>d=12</td>
<td>-2.69 (0.03)</td>
<td>-1.27 (0.03)</td>
<td>-0.12 (0.02)</td>
<td>0.79 (0.03)</td>
</tr>
<tr>
<td>d=24</td>
<td>-1.99 (0.03)</td>
<td>-0.83 (0.02)</td>
<td>0.22 (0.02)</td>
<td>1.11 (0.02)</td>
</tr>
<tr>
<td>d=52</td>
<td>-1.63 (0.02)</td>
<td>-0.55 (0.02)</td>
<td>0.45 (0.02)</td>
<td>1.30 (0.02)</td>
</tr>
<tr>
<td>d=60</td>
<td>-1.59 (0.02)</td>
<td>-0.50 (0.02)</td>
<td>0.48 (0.02)</td>
<td>1.32 (0.02)</td>
</tr>
<tr>
<td>d=365</td>
<td>-1.20 (0.02)</td>
<td>-0.19 (0.02)</td>
<td>0.77 (0.02)</td>
<td>1.61 (0.03)</td>
</tr>
<tr>
<td>normal(0,2)</td>
<td>-0.95</td>
<td>0.00</td>
<td>0.95</td>
<td>1.81</td>
</tr>
</tbody>
</table>

(b) Part 2 of the Percentiles table

<table>
<thead>
<tr>
<th>Percentiles</th>
<th>0.95</th>
<th>0.975</th>
<th>0.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>d=12</td>
<td>1.25 (0.03)</td>
<td>1.64 (0.03)</td>
<td>2.07 (0.04)</td>
</tr>
<tr>
<td>d=24</td>
<td>1.60 (0.02)</td>
<td>1.91 (0.03)</td>
<td>2.34 (0.05)</td>
</tr>
<tr>
<td>d=52</td>
<td>1.73 (0.02)</td>
<td>2.13 (0.03)</td>
<td>2.59 (0.05)</td>
</tr>
<tr>
<td>d=60</td>
<td>1.82 (0.03)</td>
<td>2.21 (0.03)</td>
<td>2.60 (0.05)</td>
</tr>
<tr>
<td>d=365</td>
<td>2.11 (0.03)</td>
<td>2.50 (0.03)</td>
<td>2.97 (0.04)</td>
</tr>
<tr>
<td>normal(0,2)</td>
<td>2.33</td>
<td>2.77</td>
<td>3.29</td>
</tr>
</tbody>
</table>

(c) Part 3 of the Percentiles table
Table A.11: \( \tau \) Percentiles for the Sinusoidal Model \((m = 20)\)

<table>
<thead>
<tr>
<th>Percentiles</th>
<th>0.01</th>
<th>0.025</th>
<th>0.05</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>d=12</td>
<td>-4.13 ( 0.07 )</td>
<td>-3.51 ( 0.04 )</td>
<td>-3.01 ( 0.04 )</td>
<td>-2.49 ( 0.02 )</td>
</tr>
<tr>
<td>d=24</td>
<td>-3.25 ( 0.05 )</td>
<td>-2.77 ( 0.03 )</td>
<td>-2.40 ( 0.02 )</td>
<td>-1.97 ( 0.02 )</td>
</tr>
<tr>
<td>d=52</td>
<td>-2.81 ( 0.03 )</td>
<td>-2.43 ( 0.03 )</td>
<td>-2.09 ( 0.03 )</td>
<td>-1.70 ( 0.02 )</td>
</tr>
<tr>
<td>d=60</td>
<td>-2.87 ( 0.06 )</td>
<td>-2.42 ( 0.03 )</td>
<td>-2.07 ( 0.02 )</td>
<td>-1.67 ( 0.02 )</td>
</tr>
<tr>
<td>d=365</td>
<td>-2.45 ( 0.04 )</td>
<td>-2.10 ( 0.03 )</td>
<td>-1.79 ( 0.02 )</td>
<td>-1.45 ( 0.02 )</td>
</tr>
<tr>
<td>normal(0,1)</td>
<td>-2.33</td>
<td>-1.96</td>
<td>-1.64</td>
<td>-1.28</td>
</tr>
</tbody>
</table>

(a) Part 1 of the Percentiles table

<table>
<thead>
<tr>
<th>Percentiles</th>
<th>0.25</th>
<th>0.5</th>
<th>0.75</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>d=12</td>
<td>-1.69 ( 0.02 )</td>
<td>-0.86 ( 0.02 )</td>
<td>-0.08 ( 0.02 )</td>
<td>0.61 ( 0.02 )</td>
</tr>
<tr>
<td>d=24</td>
<td>-1.29 ( 0.01 )</td>
<td>-0.57 ( 0.01 )</td>
<td>0.16 ( 0.02 )</td>
<td>0.84 ( 0.02 )</td>
</tr>
<tr>
<td>d=52</td>
<td>-1.08 ( 0.01 )</td>
<td>-0.38 ( 0.01 )</td>
<td>0.32 ( 0.01 )</td>
<td>0.95 ( 0.02 )</td>
</tr>
<tr>
<td>d=60</td>
<td>-1.05 ( 0.01 )</td>
<td>-0.34 ( 0.01 )</td>
<td>0.34 ( 0.01 )</td>
<td>0.97 ( 0.02 )</td>
</tr>
<tr>
<td>d=365</td>
<td>-0.82 ( 0.01 )</td>
<td>-0.13 ( 0.01 )</td>
<td>0.54 ( 0.02 )</td>
<td>1.14 ( 0.02 )</td>
</tr>
<tr>
<td>normal(0,1)</td>
<td>-0.67</td>
<td>0</td>
<td>0.67</td>
<td>1.28</td>
</tr>
</tbody>
</table>

(b) Part 2 of the Percentiles table

<table>
<thead>
<tr>
<th>Percentiles</th>
<th>0.95</th>
<th>0.975</th>
<th>0.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>d=12</td>
<td>1.02 ( 0.02 )</td>
<td>1.35 ( 0.03 )</td>
<td>1.72 ( 0.05 )</td>
</tr>
<tr>
<td>d=24</td>
<td>1.22 ( 0.02 )</td>
<td>1.53 ( 0.03 )</td>
<td>1.89 ( 0.04 )</td>
</tr>
<tr>
<td>d=52</td>
<td>1.29 ( 0.02 )</td>
<td>1.63 ( 0.03 )</td>
<td>2.06 ( 0.04 )</td>
</tr>
<tr>
<td>d=60</td>
<td>1.35 ( 0.02 )</td>
<td>1.68 ( 0.02 )</td>
<td>2.01 ( 0.04 )</td>
</tr>
<tr>
<td>d=365</td>
<td>1.50 ( 0.02 )</td>
<td>1.81 ( 0.03 )</td>
<td>2.15 ( 0.03 )</td>
</tr>
<tr>
<td>normal(0,1)</td>
<td>1.64</td>
<td>1.96</td>
<td>2.32</td>
</tr>
</tbody>
</table>

(c) Part 3 of the Percentiles table