ABSTRACT

ZHANG, MIN. Joint Distributions of Time to Default with Application to the Pricing of Credit Derivatives. (Under the direction of Dr. Peter Bloomfield.)

Modeling portfolio credit risk involves the default dependencies between the individual securities in a portfolio. The copula is a common approach to construct it. It parameterizes the joint distribution of individual defaults independently of their marginal distributions. The current market standard model is the Gaussian copula. It assumes the event of default depends on both the common systematic risk factor and the individually idiosyncratic risk factor. The dependence of the defaults is induced by modeling a vector of latent variables that have a multivariate normal distribution. First we study the loss distribution of the large portfolio by using the Gaussian copula. We derive several asymptotic approximations to the loss distribution of the Gaussian copula. Every approximation is compared with the exact loss distribution by using the Hellinger distance between their probability density functions.

However, the tail dependence of the extreme events such as credit defaults can not be captured by the Gaussian copula. We develop the Student's t copula process to define an appropriate dependence structure of the defaults. The dependence of the default events is induced by modeling a vector of latent variables that have a multivariate Student's t distribution. We study the loss distribution of the large portfolio by using the Student's t copula. We also derive several asymptotic approximations to the loss distribution of the Student's t copula. We compare every approximation to the exact loss distribution by using the Hellinger distance between their probability density functions.

The market data of iTraxx Europe Series 4 (5-year) is investigated by using both the Gaussian copula and the Student's t copula. We say the Student's t copula works better than the Gaussian copula to describe the dependence of the extreme events with

an extra parameter, the degrees of freedom of the Student's t copula. The parameters are estimated by using the weighted least squares of the mark to market. The base correlation curve implied from the Gaussian copula is skewed. The base correlation curve implied from the Student's t copula is also skewed unless we allow varying degrees of freedom. Since the implied base correlations of the individual tranches are not consistent, it is impossible to interpolate the base correlation for a non-standard tranche. A less skewed implied base correlation curve will be one of our interests in the future study.

Joint Distributions of Time to Default with Application to the Pricing of Credit Derivatives

by
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A dissertation submitted to the Graduate Faculty of
North Carolina State University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Statistics

Raleigh, North Carolina
2008

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DEDICATION

To my husband and my family in China: my grandparents, Zonghan and Ruilian; my parents, Xuanbo and Lihong; and my sister, Wei

BIOGRAPHY

Min Zhang was born in Lanzhou, China on December 12th, 1975. She obtained her B.S. degree in Industrial Management (minor in Accounting) from Shanghai University of Finance and Economics, Shanghai, China in June 1998. She worked for HARTING Limited in Zhuhai, China, and then for HSBC (the Hongkong and Shanghai Banking Corporation Limited) in Shanghai, China. In August 2003, she enrolled the master program of the Statistics Department at North Carolina State University in Raleigh, NC. From May 2005 to December 2005, she worked as an intern at the Exposure Modeling Research Branch of U.S. Environmental Protection Agency (EPA) in Research Triangle Park, NC. She started her Ph.D. study since January 2006. Her research is under the direction of Dr. Peter Bloomfield. Her research interest is multivariate statistical analysis, credit risk and credit derivative pricing. From May 2006 to May 2008, she worked as an research assistant on the project from U.S. Department of Agriculture at North Carolina State University in Raleigh, NC.

ACKNOWLEDGMENTS

First I would like to thank Dr. Peter Bloomfield for guiding my research. I started doing research under the direction of Dr. Bloomfield when I was still in the master program. He always welcomes my questions and helps me out. I admire his expertise not only in Statistics but also in other areas such as Mathematics and Finance. He always challenges me with interesting problems. I have learned to be patient, gentle, kind and optimistic from him. I am very lucky to have Dr. Bloomfield as my advisor.

I am grateful to Dr. David A. Dickey, Dr. Jason Osborne and Dr. Tao Pang for serving on my committee. All of their help make me to work on my thesis smoothly. I also learned a lot from their lectures. Especially Dr. Dickey gave me lots of help on statistical analysis and SAS programming. I would like to thank Dr. Cavell Brownie and Dr. George Kennedy for giving me the chance to work on the research project from U.S. Department of Agriculture, and also for their financial support. Dr. Brownie taught me a lot in consulting. From her, I know how to communicate with my clients and answer the questions in the way they are comfortable. I appreciate Dr. Bibhuti Bhattacharyya's help to build up my statistical knowledge. His love in teaching impressed me a lot. I also want to thank Dr. Anastasios Tsiatis for his wonderful lectures. I would like to thank Mr. Adrian Blue for his hard work for our department and Mr. Terry Byron to help us with all the messy problems of software and hardware.

I would thank my parents whose optimism and confidence encourage me not to be afraid of hard time. Without their support and encouragement, I could not make any achievement in my study. My husband gives me a lot of encouragement, suggestions and help. He has been with me at every moment of frustration. At last I would like to extend my gratitude to my friend, Dr. Xianwen Zhou from Lehman Brothers, for his help.

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Chapter 1

Introduction

1.1 Risks, Credit Market and Credit Derivatives

1.1.1 Risk Associated with Bonds

A variety of risks are associated with bonds, such as interest rate risk, yield curve risk, liquidity risk, inflation risk, event risk, sovereign risk, and credit risk. First, the interest rate risk measures the potential effect of change in market interest rates on bond prices. Because of the time value of money (TVM), the present value (PV) of a bond is computed by bringing its cash flows back to a time point. The interest rate, a discount factor, is always negatively related to the present value of the bond. If we buy a three-year corporate bond at a fixed interest rate, we hope the market interest rate would not change too much; otherwise we will lose money if the market interest rate is higher than the interest rate at which we get paid. Next, we know long-term bonds usually have higher yields than short-term bonds. The yield curve is used to show the relationship between the bond yield and the maturity. The shape of the yield curve may be upward

sloping, downward sloping, flat or their combinations. Once the market interest rate varies, the yield changes differently with respect to different maturities. The yield curve risk estimates the potential effect of change in the shape of the yield curve on bond prices. For instance, long-term bonds are more likely to have a significant increase in the yield and hence to experience a significant decrease in their values than short-term bonds. Especially for a portfolio of bonds, the yields of individual bonds are expected to change by different values. It changes the shape of the portfolio yield curve. Another issue associated with bond investment is the liquidity. Since investors always prefer the bonds that could be easily exchanged or cashed at any time, they are willing to buy these bonds at higher prices. The liquidity risk refers to the potential cost or loss due to the lack of liquidity. For example, the prices of treasury bonds are higher and their yields are lower than corporate bonds because they can always be sold at prevailing market prices very easily and quickly.

Inflation decreases the purchasing power of money. If the inflation rate is higher than what we expected, generally the prices of goods and services will increase. The higher required yield will decrease the value of the bond issued previously. This is called inflation risk. Other risks such as event risk and sovereign risk are not coming from the financial market. The event risk describes possible substantial impact on a company's financial condition and on its asset value by some significant issues such as disasters, corporate restructurings or regulatory implementations. For example, natural disasters may make it hard for a company to meet its debt obligations. Sovereign risk is caused by relative governmental restrictions and policies on debt repayments to investors.

There are two main types of credit risks. One is the credit spread risk that measures the impact of observed credit spreads on the value of the corresponding bond. The credit spread is the premium paid to bond holders for the default risk. It is the difference

between the yield on a risky bond and the yield on a risk-free bond of the same maturity. The credit spread risk is one of the market risks. It only appears when the mark-to-market accounting policy is applied. The marking to market is the process of adding gains to or subtracting losses from the margin account daily, based on the change in closing prices from one day to the next [9]. When the credit spread gets wider, the credit risk becomes higher. The other is the credit default risk that is the possibility that a bond issuer can not make the promised payments. We will study the credit default risk in detail later.

1.1.2 Rising of Credit Market

First let us review several big credit crises happened in the past 20 years. In the late 1980s, high volume investments in junk bonds provided by the Savings & Loan (S& L) institutions made lots of bankruptcies. In the 1990s, huge "hot money" flowed into South-East Asia. Most of the money was invested in long-term securities. The sharp fall of economic growth caused severe financial crisis in the mid 1990s. In 2001, Argentina suffered a serious economic recession. Its central bank raised the interest rate of treasury bills from 9% to 14% in July 2001 in order to decrease capital outflow; however, high repayment burden forced the central bank to devalue the domestic currencies. It forced Argentina to officially default on their debt at the end of 2001. In 2001, Enron, once a utility giant, filed bankruptcy. WorldCom declared bankruptcy in 2002. Billions of dollars were lost in the collapse of these two big companies. (See [25]).

The increased awareness of credit default risk leads to the tremendous growth of credit market. Credit derivatives began trading actively in the mid 1990s. They are over-the-counter (OTC) contracts designed to meet the customer demands for a wide

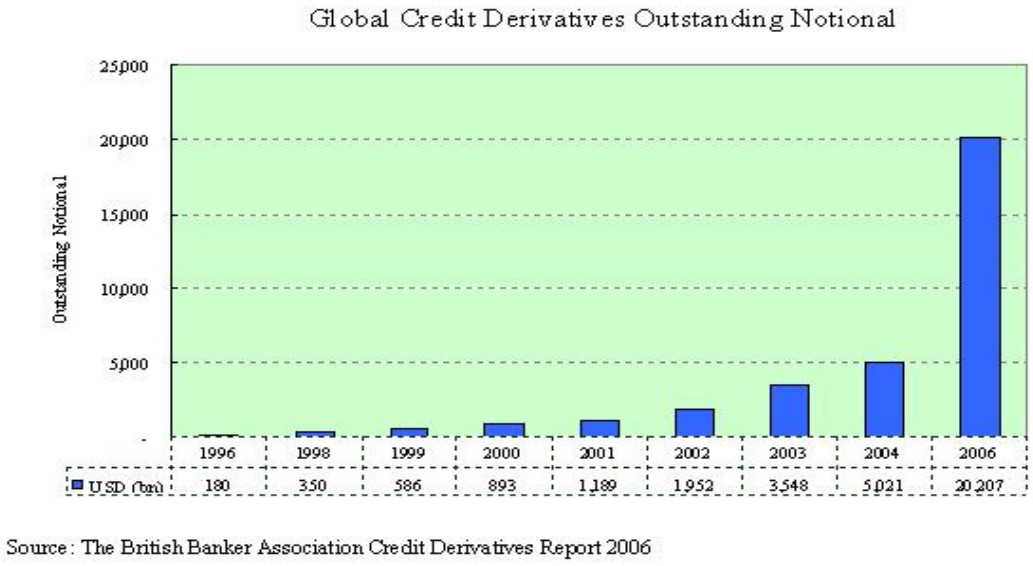

Figure 1.1: Credit Derivative Market Evolution

range of purposes. The value of a credit derivative is derived from the creditworthiness of a third-party obligation [19]. The credit derivatives market has grown very fast. The outstanding notional amount of global credit derivatives is over $20 trillion in 2006 [13]. The index and portfolio credit products have higher growth rate than the single-name credit products since 2003. Figure 1.1 demonstrates the evolution of credit derivative market [2]. Credit derivatives are used by a variety of market participants such as banks, insurance companies, hedge funds and pension funds. Figure 1.2 shows the main participants on the credit derivative market [13].

Credit derivatives are complicated financial instruments. The return on the credit derivatives may become over-expected. For example, in the past few years the returns of the asset-backed collateralized debt obligations on the subprime mortgage market

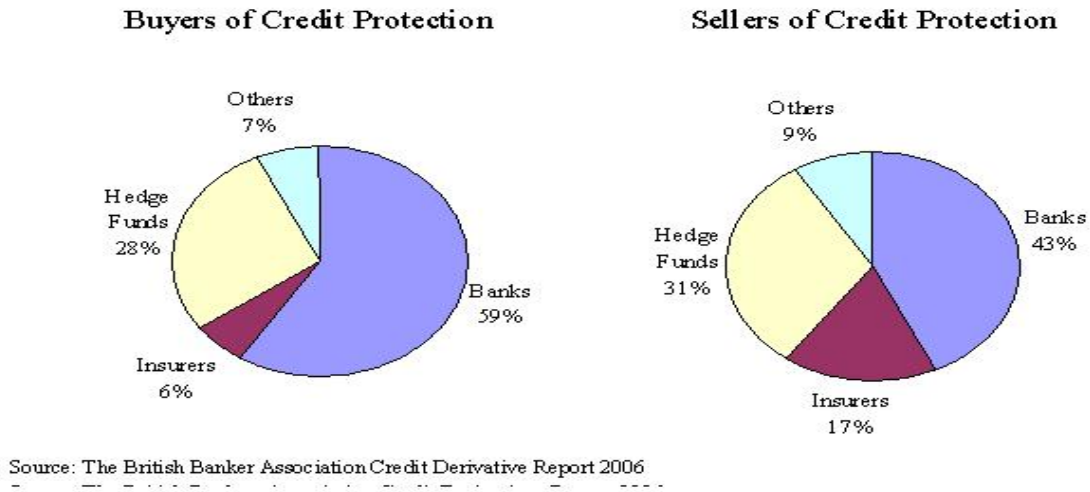

Figure 1.2: Credit Derivative Market Participants

were very high. It is estimated that 170 billion U.S. dollars have been invested on this market. However the relaxation of mortgage lending restrictions brings about severe defaults. In June 2007, two hedge funds of Bear Stearns collapsed due to the investment in the collateralized debt obligations (CDO). Standard & Poor and Moody's downgraded a number of bonds backed by subprime mortgages. The loss of CDO on the subprime market made the stock market fall sharply since July 2007. The 85-year-old Bern Stearns, the nation's fifth-largest investment bank, announced to be acquired by JPMorgan Chase & Co. on March 16, 2008. The credit crunch crushes the stability of the global financial system.

1.1.3 Credit Derivatives

Credit default swaps (CDS), collateralized debt obligations (CDO), synthetic collateralized debt obligations (synthetic CDO) and standardized synthetic CDO indexes are the main products on the credit derivative market. The credit default swap is similar as the "plain vanilla" interest rate swap that is the most common type of swap. A swap is an agreement between two companies to exchange cash flows in the future. In the interest rate swap, a company agrees to pay the cash flows equal to the interest at a predetermined fixed rate on a notional principal for a number of years. In return, it receives interest at a floating rate on the same notional principal for the same period of time. The floating rate in many interest rate swap agreements is the LIBOR (London Interbank Offer Rate). For a floating-rate bond, interest is usually set at the beginning of the period to which it will apply and is paid at the end of the period. Analogously a credit default swap is a contract where one party makes advance or periodical payments to get a contingent payment from the other party, given a predefined credit event occurs within a period of time. It is used to transfer the credit risk from one party to the other. It is easy to understand the mechanics of the CDS from Figure 1.3.

A collateralized debt obligation is a security whose underlying securities can be various types of debt obligations, such as business loans, mortgages, debt of developing countries, corporate bonds of various ratings or asset-backed securities [9]. A synthetic CDO is a credit instrument that comprises a collection of single-name CDS. The pool of CDS is cut into several slices based on their credit risk. The ordered slices are technically named as super senior tranche, second senior tranche, first senior tranche, second mezzanine tranche, first mezzanine tranche and equity tranche. Figure 1.4 illustrates the structure of a synthetic CDO. The riskiest equity tranche offers the highest return and it first

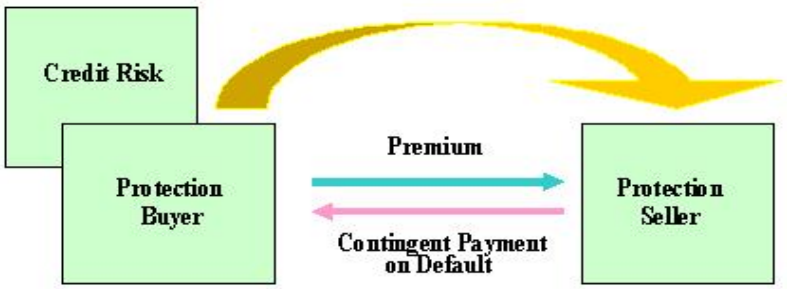

Figure 1.3: Credit Default Swap

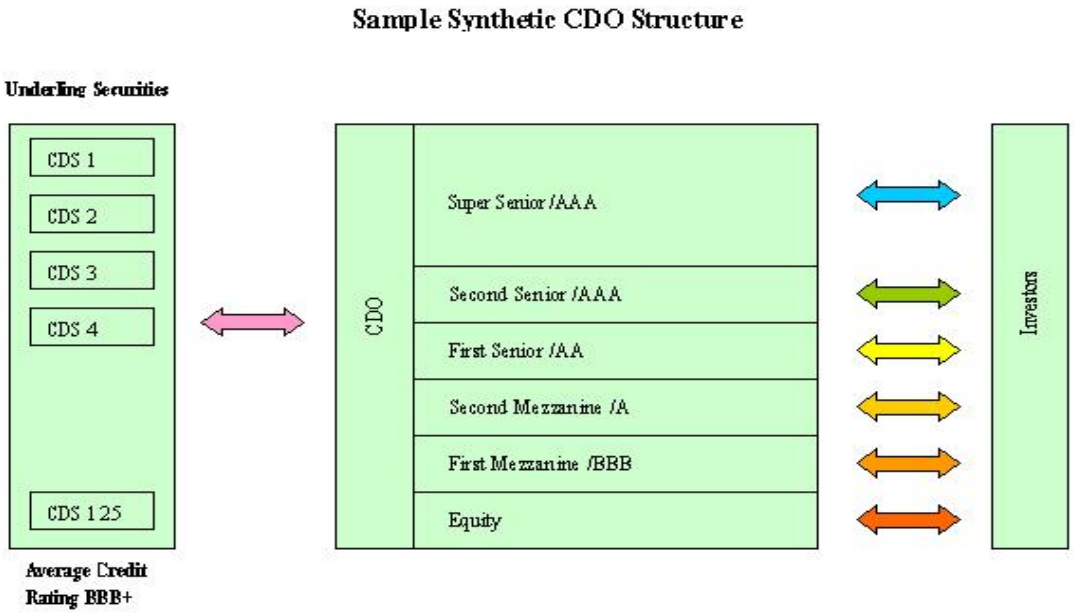

Figure 1.4: Synthetic CDO Structure

absorbs the loss up to some extent as long as any loss occurs to the credit pool. There are two families of standardized synthetic CDO indexes traded on market. One is the Dow Jones CDX Indexes that comprise the most liquid names in the North American and emerging markets. The CDX are administered by CDS Index Company (CDS IndexCo) and marketed by Markit Group. The other is the iTraxx Indexes that consist of the most liquid entities in the European and Asian markets. The iTraxx are managed by International Index Company (IIC). The market data that we use is the iTraxx Europe Series 4 (5-year). This index comprises 125 equally-weighted European names. It has five standardized tranches: $0-3\%$, $3-6\%$, $6-9\%$, $9-12\%$ and $12-22\%$.

CDO, synthetic CDO and standardized synthetic CDO indexes are all portfolio credit derivatives. The pricing is strongly dependent on the default correlation between indi-

vidual securities in a portfolio. If entities are highly correlated, they may collapse just like dominos when an economic tragedy appears. Then the return of the equity tranche will not differ too much from that of the other tranches. If the collateralized pool has low correlation, there would exist significant difference between the values of different tranches. We are interested in studying the tail dependence, the dependence of credit defaults, of a credit portfolio.

1.2 Credit Risk Modeling

The credit risk is mainly determined by the default probability of the obligor, the size of exposure to the credit default and the severity of the credit default. In order to model credit risk, we have to consider several key factors such as default probability, recovery rate upon default, risk-free interest rate and credit premium. Loss distribution plays a very important role in modeling credit risk. It is necessary for pricing credit derivatives. Also by using the loss distribution, we could calculate the expected loss to develop the financial strategies in risk management. For the example of the economic capital, the adequate capital held against risk, is generally set equal to the 1% quantile of the loss distribution [20]. Typically the loss distribution is skewed, so the central research interest is the tail of loss distribution. We have to make sure that the tail of loss distribution is not distorted by some misspecified assumptions.

Freedom of arbitrage is one of the important assumptions of the market. It means investors can not attempt to profit by exploiting price differences from identical or similar financial instruments on different markets or in different forms. Thus the present value of the expected payments made by the buyers of a CDS or a CDO (premium leg) must be equal to the present value of the expected payoff on default (loss leg).

1.2.1 Modeling Single-Name Credit Risk

The structural model and the reduced form model are the two common methods for modeling the single-name credit risk. Black & Scholes [3] and Merton [41] developed the classical structural model that economically defined default as the firm value falling below a designated level at a predefined time. It is based on the idea that the firm's liability is an option on its asset. The firm value is modeled by a geometric Brownian motion. One of the benefits of the structural model is that it is a function of observed variables. In the structural model, the default has its economic meaning. The reduced form model was developed by Jarrow, Robert & Turnbull [30] and Duffie & Singleton [16]. It uses an exogenous process of the hazard rate or the default probability density to describe the dynamics of the time to default. It models the firm default free of the economic sense.

1.2.2 Modeling Portfolio Credit Risk

For the portfolio credit risk, we have to concern ourselves with the issue of default dependence. The typical approaches include the structural model, the reduced form model, the mixture binomial model and the copula-based model. Analogous to the single-name credit risk, the structural model for the portfolio credit risk has the same definition of the default. It models the dependent defaults of the obligors through the correlation of their asset values. CreditMetrics developed by JPMorgan is based on the structural model. The reduced form model studies the default time or the survival time. It models the dependent default of the obligors through the correlation of their default times or survival times. CreditRisk+ designed by Credit Swiss is based on the reduced form model.

The mixture binomial model is based on the assumption that conditioning on some

common information, the defaults of the obligors are independent. Therefore given a homogenous credit portfolio, the conditional distribution of the defaults is simplified to a binomial distribution. Here "homogenous" means each obligor has the same default probability and the same recovery rate.

The copula is a general approach to model the dependence structure of defaults. The merit of the copula is that it constructs a joint distribution without any restriction on its marginal distributions. The latent variables could have arbitrary marginal distributions. Either the dependent firm values or the dependent default time could be used to construct the copula model. The current market standard model is the Gaussian copula. The dependence of the default events is induced by modeling a vector of latent variables that have a multivariate normal distribution.

1.3 Organization

First we introduce the structural model and the reduced form model for modeling the single-name credit risk in Chapter 2. Several common methods for modeling the portfolio credit risk are presented in Chapter 3. In Chapter 4, we study the loss distribution of the large portfolio by using the Gaussian copula. We derive several asymptotic approximations to the loss distributions of the Gaussian copula and compare each of them to the exact loss distribution by using the Hellinger distance of the probability density functions.

However, the tail dependence of the extreme events such as credit defaults can not be captured by the Gaussian copula. In Chapter 5, we develop the Student's t copula process to define an appropriate dependence structure of the defaults. The dependence of the default events is induced by modeling a vector of latent variables that have a

multivariate Student's t distribution. We study the loss distribution of the large portfolio by using the Student's t copula. We also derive several asymptotic approximations to the loss distributions of the Student's t copula and compare each of them to the exact loss distribution by using the Hellinger distance of the probability density functions.

In Chapter 6, we demonstrate the calibration algorithm for pricing the synthetic CDO. The market data of iTraxx Europe Series 4 (5-year) is investigated by using both the Gaussian copula and the Student's t copula. We say the Student's t copula works better than the Gaussian copula to explain the tail dependence of the extreme events. The parameters are estimated by using the weighted least squares of the mark to market. The base correlation curve implied from the Gaussian copula is skewed. The base correlation curve implied from the Student's t copula is also skewed unless we allow varying degrees of freedom. Since the implied base correlations of the individual tranches are not consistent, it is impossible to interpolate the base correlation for a non-standard tranche. Therefore it will be one of our interests to find a less skewed implied base correlation curve in the future study that is discussed in Chapter 7.

Chapter 2

Single-Name Credit Risk Modeling

2.1 Structural Models

Li[34] mentioned the critical thing in pricing credit derivatives is to construct a credit curve that is similar to the yield curve of fixed-income derivatives. It gives the instantaneous default probabilities of the counterparty. The structural model is a common method to get the default probabilities. It was first developed by Black & Scholes [3] and Merton [41]. This classical structural model is based on the idea that the firm's liability is a kind of option on its asset. It models the process of the firm value V_t by a geometric Brownian motion

$$dV_t = \mu V_t dt + \sigma V_t dW_t, \quad V_0 > 0,$$

where μ is called drift, σ is called volatility and W is the standard Brownian motion.

We assume the firm is financed by an equity and a zero coupon bond that has the face value K and the maturity T. The firm defaults when its asset value is falling below K at maturity. This definition of the default is closely related to the firm's capital structure.

The default probability is termed as $P[V_T < K]$. The default time τ is defined as

$$\tau = \begin{cases} T & \text{if } V_T < K \\ \infty & \text{if else.} \end{cases}$$

One of the benefits of the structural model is that it is a function of observed variables [24]. And the default has its economic meaning. Since a firm may default at any time before maturity, later Black & Cox [4] gave a more realistic definition of the default. The firm defaults once its asset value falling below some predetermined level $Z \in (0,\ V_0)$. It is called the first-passage approach. The default probability is given by $P[\underset{t \leq T}{min}\, V_t < Z]$. And the default time τ is defined as

$$\tau = inf\{t > 0 : V_t < Z\}.$$

However this modified definition of the default still dose not cover one situation under which $V_t > Z$ but $V_T < K$. The default may occur at any time before maturity when $V_t < Z < K$ or at maturity when $V_T < K$. Therefore a better definition of the default time τ is given by

$$\tau = min\{\tau_1,\ \tau_2\},$$

where τ_1 is the default time from the classical approach and τ_2 is the default time from the first-passage approach. The closed form solution of the default probability can be obtained by solving the stochastic differential equations in these three cases [24].

The credit spread is another issue that we are interested in. It is the difference between the yield on a defaultable zero coupon bond and the yield on a default free zero coupon bond. It is the compensation to the bond holders for their exposure to the default risk. If the risk-free interest rate and the credit spread are quoted on an continuous basis, the

credit spread at time t is given by

$$C_{spd}(t,T) = -\frac{1}{T-t} log\left(\frac{B_t^T}{\bar{B}_t^T}\right), \quad t < T,$$

where B_t^T is the price of a defaultable bond at time t for \$1 at maturity T, and $\bar{B}_t^T$ is the price of a default free bond at time t for \$1 at maturity T. Given a constant risk free interest rate r, we have $\bar{B}_t^T = e^{-r(T-t)}$ for \$1 at T. (See [24]).

2.2 Reduced Form Models

The time to default, τ, measures the time length from the origin to the default time. The reduced form model uses an exogenous process of the hazard rate or the default probability density to describe the dynamics of the time to default. It was first developed by Jarrow & Turnbull [30] and Duffie & Singleton [16]. The default is characterized by a point process with jump size equal to one,

$$N_t = \begin{cases} 1 & \text{if } \tau \leq t \\ 0 & \text{if else.} \end{cases}$$

The default process has an uptrend. By the Doob-Meyer decomposition theorem, $N_t = A_t^\tau + (N_t - A_t^\tau)$, where A_t^τ is an increasing process starting from zero and $N_t - A_t^\tau$ is a martingale. A_t^τ is called compensator whose properties are closely related to the default. A_t^τ is continuous if and only if τ is unpredictable [24]. In the reduced form model, A_t^τ is modeled through a non-negative process of the hazard rate λ,

$$A_t^\tau = \int_0^t \lambda(\nu) 1_{\tau > \nu} d\nu.$$

The merit of the reduced form model is that we can use any non-negative process of λ to model the default through the time to default τ. In the credit risk literature λ is often called intensity.

2.2.1 Survival Function

From the survival analysis, the cumulative distribution function and the survival function of τ are defined as

$$\begin{aligned} F(t) &= P(\tau \leq t), \quad t \geq 0 \\ S(t) &= P(\tau > t), \quad t \geq 0 \\ &= 1 - P(\tau \leq t). \end{aligned}$$

The probability density function of τ is $f(t)$. By definition, $f(t)h$ is the limiting unconditional default probability from time t to $t+h$, as h goes to zero.

$$\begin{aligned} f(t) &= F'(t) \\ &= -S'(t) \\ &= \lim_{h \to 0} \frac{P[t < \tau \leq t+h]}{h}. \end{aligned}$$

The conditional probability of τ is denoted by $P_{h|t}$. It measures the default probability from time t to $t+h$ given there is no default before and at time t. When $h = 1$, it is called marginal conditional probability [34].

$$P_{h|t} = Pr[\tau \leq t+h | \tau > t], \quad h > 0.$$

The instantaneous conditional probability is called hazard rate $\lambda(t)$. By definition, $\lambda(t)h$ is the default probability from time t to $t+h$ conditioning on no earlier default from

time zero to time t.

$$\begin{aligned}\lambda(t) &= \lim_{h\to 0} \frac{Pr[t < \tau \leq t+h | \tau > t]}{h} \\ &= \frac{f(t)}{1-F(t)} \\ &= -\frac{S'(t)}{S(t)}.\end{aligned}$$

The survival function and the conditional probability function can be expressed in terms of the hazard rate,

$$\begin{aligned}S(t) &= e^{-\int_0^t \lambda(v)\,dv} \\ P_{h|t} &= 1 - e^{-\int_t^{t+h} \lambda(v)\,dv} \\ &= 1 - e^{-\int_0^h \lambda(t+v)\,dv}.\end{aligned}$$

If the hazard rate is a constant over time, the time to default follows an exponential distribution with parameter λ. Therefore $P_{h|t} = 1 - e^{-\lambda h}$. It requires non-negative λ because $P_{h|t} \leq 1$. Li [34] developed the credit curve analogous to building the yield curve from the market price of bonds or swaps. It is used by most of the credit derivative trading desks. In the discrete case, the credit curve consists of the conditional default probabilities $P_{1|0}, P_{1|1}, \ldots, P_{1|n}$. In continuous case, it is actually the hazard function.

2.2.2 Affine Diffusion Models

In the affine diffusion model, the default intensity λ is a affine function of the state variable X_t such that $\lambda(x) = \lambda_0 + \lambda_1 x$ [15] . X_t satisfies the stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t,$$

where the coefficients are the affine functions of X_t:

$$\mu(x) = \mu_0 + \mu_1 x$$

$$\sigma(x) = \sigma_0 + \sigma_1 x$$

and W_t is the standard Brownian motion. When $\mu(x) = c(\mu - x)$ and $\sigma(x) = \sigma$, X_t follows the Vasicek process [24]

$$dX_t = c(\mu - X_t)dt + \sigma dW_t.$$

It is also called the mean-reverting process. Please refer to Duffie & Kan [15] for the closed form solution of X_t. When $\mu(x) = c(\mu - x)$ and $\sigma(x) = \sigma\sqrt{x}$, X_t follows the square-root diffusion process proposed by Cox, Ingersoll & Ross [10]

$$dX_t = c(\mu - X_t)dt + \sigma\sqrt{X_t}dW_t.$$

A further extension of the affine model is made by adding an extra item for the unexpected jumps J_t,

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + dJ_t.$$

The arrival intensity of J_t is given by $h(X_t)$ such that $h(x) = h_0 + h_1 x$. Please refer to Duffie, Filipović & Schachermayer [18] for details.

One of the benefits of the reduced form model is the parallel pricing formula for the risky bonds and the risk-free bonds. Given the constant risk-free interest rate r quoted on a continuous basis and the constant recovery rate R, the price of a risky bond with the maturity T is given by

$$\begin{aligned} B_0^T &= e^{-rT}\mathrm{E}[R1_{\tau \le T} + 1_{\tau > T}] \\ &= e^{-rT} - e^{-rT}(1 - R)P(T), \end{aligned}$$

where $P(T)$ is the risk-neutral default probability at time 0 [24]. When the recovery rate $R = 0$ and the default intensity λ is a constant, $B_0^T = e^{-(r+\lambda)T}$. Therefore the calibration of the risky bonds is just like the risk-free bonds except the discount rate is risk-adjusted.

Chapter 3

Portfolio Credit Risk Modeling

3.1 Default Correlation

Pearson's correlation coefficient is a very popular statistic to describe the linear relationship between two variables

$$\rho = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}.$$

It is estimated by the sample correlation coefficient

$$\hat{\rho} = \frac{\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{\sum_{i=1}^{n}(X_i - \bar{X})^2}\sqrt{\sum_{i=1}^{n}(Y_i - \bar{Y})^2}}.$$

The correlation coefficient has the special meaning for the bivariate normal distribution. If $(X,\ Y)$ jointly have a bivariate normal distribution $\Phi_2(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$, the conditional distribution of X given $Y = y$ is a normal with the explicit conditional mean and conditional variance

$$\mathrm{E}(X|Y = y) = \mu_1 + \frac{\rho\sigma_1(y - \mu_2)}{\sigma_2}$$

$$\mathrm{Var}(X|Y = y) = {\sigma_1}^2(1 - \rho^2).$$

It means if Y is going away from its mean by one standard deviation then X is expected to be apart from its mean by ρ standard deviation; however, the similar interpretation is not applicable to the other distributions rather than the bivariate normal. Therefore we need to consider what could be used to estimate the default correlation, say historical defaults, equity returns, asset returns or credit spreads [39]. Due to the scarcity of the observed defaults, we consider to use equity returns, asset returns or credit spreads as the input to estimate the default correlation. The distribution of asset returns or equity returns is close to a normal distribution in general. Therefore the CreditMetrics model developed by JPMorgan uses the correlation of asset returns. The credit returns, such as corporate bonds returns, CDS returns and CDO returns, generally have a "heavy tail" distribution. The typical joint distribution of a credit portfolio is more like a star than an ellipse. Figure 3.1 is the scatter plot of the daily credit spread changes of the $0-3\%$ tranche and the $3-6\%$ tranche of iTraxx Europe Series 4 (5-year) from 1/1/2006 to 8/21/2007. While Figure 3.2 shows the bivariate normal distribution with the same sample mean and the same sample variance-covariance matrix. It is an example that different distributions may give us the same Pearson's correlation coefficient. Therefore the Pearson's correlation coefficient is not very useful for modeling the portfolio credit risk. It is necessary to have an explicit dependent structure.

3.2 CreditMetrics

JPMorgan developed CreditMetrics to measure the loss distribution of a credit portfolio in 1997. It assumes the credit rating of an entity is the basic measure of its default risk. For name i, its asset value (or distance to default) X_i is treated as the underlying driving factor for the change of its credit rating. Given a credit portfolio with n securities,

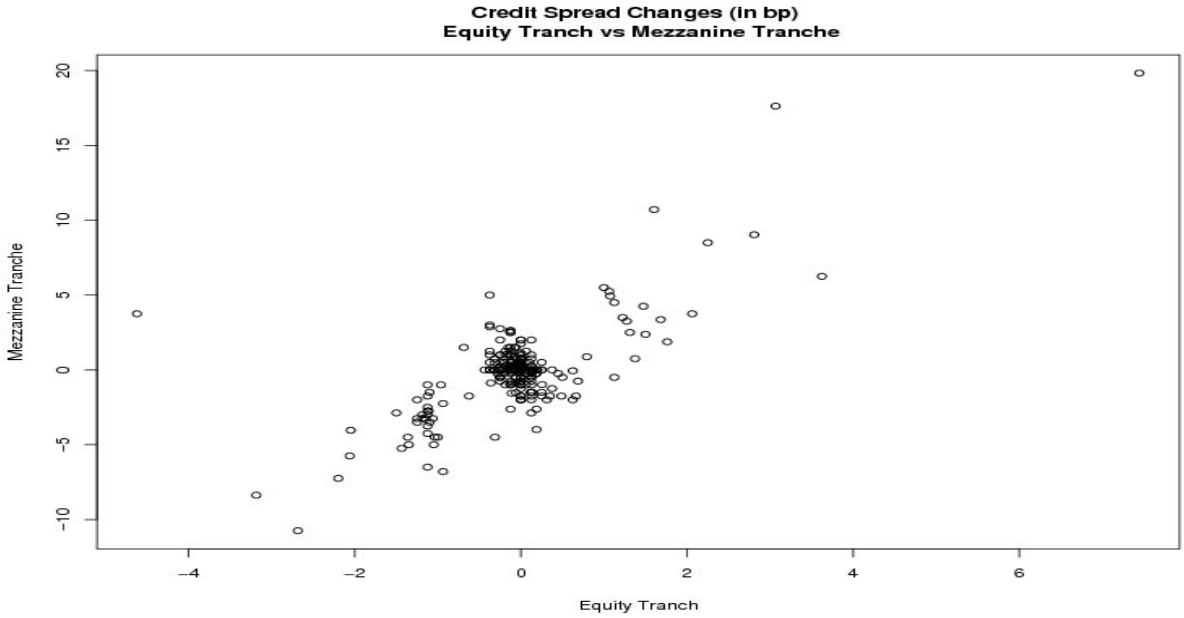

Figure 3.1: Joint Distribution of Credit Returns

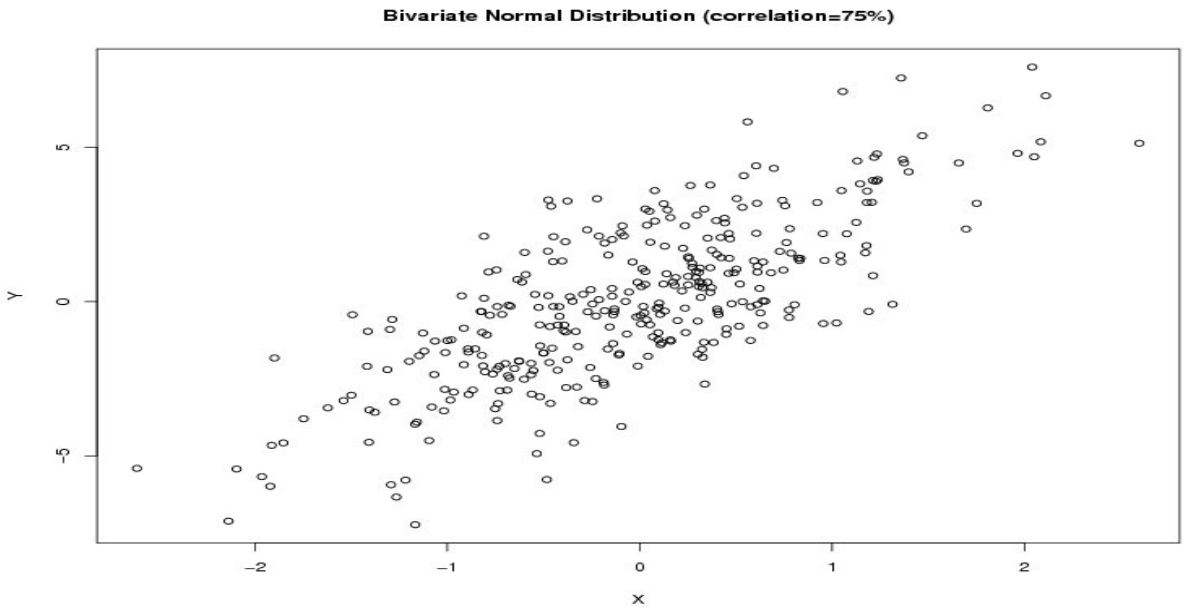

Figure 3.2: Bivariate Normal Distribution

and each security having k credit ratings respectively denoting the state of the risk-free to the state of the default, there would exist k^n situations in a one-year period. The cash flows generated by each security in each scenario have to be discounted to get its present value. The present value of the portfolio is a weighted average of individual security values for the given scenario. Finally the loss distribution of the portfolio is obtained by evaluating the present values for all the scenarios. It uses credit rating transition matrix to derive the loss distributions. CreditMetrics takes into account the obligors' correlations, the changes in credit rating and the recovery rates.

Because the asset values are not observable, they are approximated by using the stock prices. As CreditMetrics assumes that the asset values result in the change of credit rating, the probability distribution of credit rating of an obligor is defined as [20],

$$P(R_i = j) = P(S_{k-1} < X_i < S_k),$$

where R_i is the credit rating of obligor i, and $i = 1, \ldots, n$. j denotes the credit ratings such as $j = 1, \ldots, k$. S_j refers to the corresponding interval bound of the stock return for each credit rating. Another assumption of CreditMetrics is that $X_1, \ldots, X_n$ have a joint normal distribution with a covariance matrix Σ. Therefore the probability that one scenario occurs is given by,

$$P(R_1 = j_1, \ldots, R_n = j_n) = \int_{S_{i_1-1}}^{S_{i_1}} \ldots \int_{S_{i_n-1}}^{S_{i_n}} f(X_1, \ldots, X_n; \Sigma)\, dX_1 \ldots\, dX_n.$$

The covariance matrix Σ can be estimated by using the time series data of the industrial indexes and the recovery rate can be assumed to follow a beta distribution [20].

One of the drawbacks of this approach is that it requires a huge amount of input data for parameter estimation. It also takes extremely a long time for computation. Since it is a one-period model and the credit risk is modeled by using the change of credit

rating, Schönbucher and Schubert [43] showed that this approach was not accurate and not flexible enough to be used for pricing. Hull [27] pointed out that the historical data always gave much lower default probabilities than those from the market price of bonds. The reason is that the latter gives an estimate in the risk-neutral world whereas the former is an estimate in the real world. The real-world default probabilities are based on historical data whereas the risk-neutral default probabilities are based on market prices.

3.3 CreditRisk+

CreditRisk+ is designed by Credit Suisse in 1997 [12]. It does not make any assumptions about the cause of defaults; however, the economic situation in a country may make the defaults to be correlated. For example, the number of defaults increases during the economic recession. It makes the default rate increased. In order to incorporate the effect of common factors into the default rate, the default rate volatility is used and it makes the loss distribution positive skewed with a heavy tail. It is caused by the pairwise default correlation between obligors. The input of this approach includes credit exposure, obligor default rate (default probability), obligor default rate volatility (standard deviation) and recovery rate [12].

The securities contained in a credit portfolio are classified into several groups in terms of the similarity of their credit ratings, which is called sector analysis. It is assumed that each group is affected by the same systematic factors. It is also assumed that groups are independent. Within each group, it models the number of credit defaults within a one-year period via a Poisson distribution, as it is reasonable to assume the credit defaults are rare event and the time when they occur is not dependent on the current time. Therefore

the probability of the number of defaults, Y, is given by

$$P(Y = n) = \frac{\lambda^n e^{-\lambda}}{n!},$$

where λ is the expected number of defaults for a given group.

Suppose a credit portfolio contains k groups and within group j there are n_j counterparties that have the same default probability p_j over a one-year period. For small p_j, the number of defaults has a Poisson distribution with $\lambda_j = n_j p_j$. Due to the independence of groups, the number of defaults of the credit portfolio is also following a Poisson distribution with $\lambda = \sum_{j=1}^{k} n_j p_j$. The obligors' default rates could be estimated by using the observed credit spreads. Or the default statistics provided by the rating agencies could be adopted for the approximation of the obligors' default rates.

Since the exposure sizes are different for different obligors, the loss distribution of the whole portfolio is not a Poisson distribution. The loss distribution can be derived from its probability generating function. In terms of an auxiliary variable a, the probability generating function for the loss of group j is given by

$$\begin{aligned} G_j(a) &= \sum_{n=0}^{\infty} P(\text{aggregate losses} = n * L) a^{n v_j} \\ &= \sum_{n=0}^{\infty} P(\text{number of defaults} = n) a^{n v_j} \\ &= \sum_{n=0}^{\infty} \frac{e^{-\lambda_j} \lambda_j^n}{n!} a^{n v_j} \\ &= e^{-\lambda_j + \lambda_j a^{v_j}}, \end{aligned}$$

where L is the unit amount of exposure. v_j is the exposure of group j expressed as multiples of the unit. It means the loss of group j is equal to $L * v_j$. Here the exposure of group j is adjusted by the expected recovery rate [12]. Therefore the generating function

for the loss of the whole portfolio is given by

$$G(a) = \prod_{j=1}^{k} e^{\lambda_j - \lambda_j a^{v_j}}$$
$$= e^{-\sum_{j=1}^{k} \lambda_j + \sum_{j=1}^{k} \lambda_j a^{v_j}}.$$

From the probability generating function, we could obtain the loss distribution of the whole portfolio. (See [12]).

The recovery rates are treated as exogenous variables whose values are usually provided by the rating agencies such as Standard & Poor and Moody's.

An obvious downside of this approach is the assumption of the independence of the defaults. Given the firms coming from the same industry and having similar background, the defaults may be contagious to each other. Macroeconomics affect mostly all the firms, especially in the economic recession. By using this approach, the expected value and the variance corresponding to the Poisson distribution are always equal; however, from the statistics provided by Standard & Poor or Moody's, the variance is always higher than the expected value. That is to say that the Poisson distribution is not a very appropriate assumption.

3.4 Mixed Binomial Models

Lando [33] used a zero-one indicator variable to denote whether or not a firm defaults within the next period,

$$D_i = \begin{cases} 1 & \text{if firm i defaults, with probability } \tilde{p} \\ 0 & \text{if firm i does not default, with probability } 1 - \tilde{p}, \end{cases}$$

where $\tilde{p}$ could be a random variable. Given the value of $\tilde{p}$, the default events occur independently. Then the default correlation coefficient ρ of D_i and D_j is given by

$$\begin{aligned}\rho &= \frac{\mathrm{E}(D_i D_j) - \mathrm{E}(D_i)\mathrm{E}(D_j)}{\sqrt{\mathrm{Var}(D_i)\mathrm{Var}(D_j)}} \\ &= \frac{\mathrm{E}(\tilde{p}^2) - [\mathrm{E}(\tilde{p})]^2}{\mathrm{E}(\tilde{p})[1 - \mathrm{E}(\tilde{p})]}.\end{aligned}$$

The number of defaults in a credit portfolio, $D = \sum_{i=1}^{n} D_i$, conditionally follows a binomial distribution. And it has

$$\begin{aligned}\mathrm{E}(D) &= n\bar{p} \\ \mathrm{Var}(D) &= n\bar{p}(1-\bar{p}) + n(n-1)[\mathrm{E}(\tilde{p}^2) - \bar{p}^2],\end{aligned}$$

where $\bar{p} = \mathrm{E}(\tilde{p})$. If the default probability is a constant, $\tilde{p} = \bar{p}$, then the defaults happen independently. We have $\rho = 0$ and $\mathrm{Var}(D) = n\bar{p}(1-\bar{p})$. If $\tilde{p}$ is a Bernoulli random variable with $\mathrm{E}(\tilde{p}) = \bar{p}$, then the defaults are perfectly correlated. We have $\rho = 1$ and $\mathrm{Var}(D) = n^2\bar{p}(1-\bar{p})$. Therefore different distribution of the default probability $\tilde{p}$ results in different distribution of the number of defaults. For example we could assign a gamma distribution or a beta distribution to $\tilde{p}$. Increased variability of the default probability makes the higher correlation of the defaults and the long and heavy tails of the loss distribution. Please refer to Lando [33] for details.

3.4.1 Mixed Binomial Structural Models

We've already mentioned that in Merton's structural model, the definition of the default has its advantage in interpreting the economic causes. Suppose V_t^i, the asset value of firm i, follows a stochastic process in the form of

$$dV_t^i = \mu_i V_t^i dt + \sigma_i V_t^i dW_t, \quad 0 < t \leq T.$$

By Ito's lemma,

$$d(logV_t^i) = (\mu_i - \frac{1}{2}{\sigma_i}^2)dt + \sigma_i dW_t.$$

Hence

$$V_t^i = V_0^i e^{(\mu_i - \frac{1}{2}{\sigma_i}^2)t + \sigma_i W_t}.$$

We say the default occurs when

$$V_0^i e^{(\mu_i - \frac{1}{2}{\sigma_i}^2)t + \sigma_i W_t} < Z_i,$$

where Z_i is the contractual default level of firm i. Therefore the default probability is given by

$$\begin{aligned} P_{\tau_i}(t) &= P\left(V_t^i < Z_i\right) \\ P_{\tau_i}(t) &= P\left[\frac{W_t}{\sqrt{t}} < \frac{logZ_i - logV_0^i - (\mu_i - \frac{1}{2}{\sigma_i}^2)t}{\sigma_i\sqrt{t}}\right] \end{aligned}$$

where τ_i is the time to default of firm i. By letting

$$\begin{aligned} p_i &= P_{\tau_i}(t) \\ c_i &= \frac{logZ_i - logV_0^i - (\mu_i - \frac{1}{2}{\sigma_i}^2)t}{\sigma_i\sqrt{t}}, \end{aligned}$$

we have

$$p_i = \Phi(c_i)$$

$$\Phi^{-1}(p_i) = c_i.$$

For a given t, the distribution of W_t is the same as that of $\sqrt{t}\left(\beta X^G + \sqrt{1-\beta^2}X^I\right)$, where β is a non-negative constant, X^G and $X^1, \ldots, X^n$ are independent standard normal

random variables. Therefore given X^G, the conditional default probability is given by

$$\begin{aligned}
P_{\tau_i}\left(t\middle|X^G=x\right) &= P\left(V_t^i < Z_i\middle|X^G=x\right) \\
&= P\left(\frac{W_t}{\sqrt{t}} < \frac{logZ_i - logV_0^i - (\mu_i - \frac{1}{2}{\sigma_i}^2)t}{\sigma_i\sqrt{t}}\middle| X^G=x\right) \\
&= P\left(X^I < \frac{logZ_i - logV_0^i - (\mu_i - \frac{1}{2}{\sigma_i}^2)t - \sigma_i\sqrt{t}\beta x}{\sigma_i\sqrt{t(1-\beta^2)}}\right) \\
&= \Phi\left(\frac{logZ_i - logV_0^i - (\mu_i - \frac{1}{2}{\sigma_i}^2)t - \sigma_i\sqrt{t}\beta x}{\sigma_i\sqrt{t(1-\beta^2)}}\right) \\
&= \Phi\left(\frac{\sigma_i\sqrt{t}c_i - \sigma_i\sqrt{t}\beta x}{\sigma_i\sqrt{t(1-\beta^2)}}\right) \\
&= \Phi\left(\frac{\Phi^{-1}(p_i) - \beta x}{\sqrt{(1-\beta^2)}}\right).
\end{aligned}$$

This conditional default probability play a key role in the factor model proposed by Vasicek [47]. It is used to get the loss distribution of a large homogeneous portfolio.

3.4.2 Mixed Binomial Intensity Models

We introduce the general intensity affine model with exponentially distributed jumps such as

$$d\lambda_t = \alpha(\theta - \lambda_t)dt + \sigma\sqrt{\lambda_t}dW_t + dJ_t,$$

where J_t refers to the exponentially distributed jumps with mean μ [33]. It assumes τ_i is the first arrival time of a Poisson process that has the deterministic time-varying intensity λ_i. Also conditioning on the intensity processes $\lambda_1, \ldots, \lambda_n$ of n individual names, the default times $\tau_1, \ldots, \tau_n$ are independent. So the dependence of the defaults is driven by the covariance of the default intensities. One of the shortcomings of this kind of model is that the estimated correlation is much lower than the result from the empirical study. Another difficulty of this approach is to derive and analyze the dependence structure.

(See [43]). An improvement of this approach is by introducing a separate point process that describes the joint defaults. However, it is not realistic to assume common default time for several entities in a credit portfolio, such as the collateralized default obligations (CDO).

Duffie and Singleton [17] gave an example of dependent jump-intensity process. It decomposes the total jump arrival intensity into two parts: the common jump default intensity and the individual default intensity. Another example is the doubly stochastic correlated default model that gives the dependent lognormal default intensities [17]. The correlation is introduced through the error term ϵ_i.

3.5 Copula-Based Models

The copula function is a function that links univariate distributions to the full multivariate distribution. It uses the inverse-CDF algorithm. Let random variable X_i, $i = 1, \ldots, m$, has continuous CDF, $F_{X_i}(x_i)$. Define the random variables Y_i as $Y_i = F_{X_i}(x_i)$. Y_i is uniformly distributed in $[0, 1]$, that is $P(Y_i \leq y_i) = y_i$, $0 \leq y_i \leq 1$ [8]. Sklar [45] proved that for a joint distribution function with the marginal distributions, there exists a copula function such that

$$\begin{aligned} F(x_1, x_2, \ldots, x_m) &= C\left(F_{X_1}(x_1), F_{X_2}(x_2), \ldots, F_{X_m}(x_m),\ \rho\right) \\ &= C(y_1, y_2, \ldots, y_m,\ \rho), \end{aligned}$$

It is the same to say

$$C(y_1, y_2, \ldots, y_m,\ \rho) = F\left(F_{X_1}^{-1}(y_1), F_{X_2}^{-1}(y_2), \ldots, F_{X_m}^{-1}(y_m)\right),$$

where $F_{X_i}^{-1}(y_i) = inf\{x_i : F_{X_i}(x_i) \geq y_i\}$ that is referred to as the y_i-quantile of X_i. If the multivariate distribution function F is continuous, then the copula function is unique [24].

For example, the bivariate Gaussian copula gives us the cumulative distribution function of two dependent normal random variables,

$$C(y_1, y_2,\ \rho) = \Phi_2\left(\Phi^{-1}(y_1), \Phi^{-1}(y_2),\ \rho\right).$$

It helps us to have the two-name Merton's structural model such as

$$P\left[V_{T_1}^1 < K_1, V_{T_2}^2 < K_2\right] = \Phi_2\left(\frac{logL_1 - m_1T_1}{\sigma_1\sqrt{T_1}}, \frac{logL_2 - m_2T_2}{\sigma_2\sqrt{T_2}},\ \rho\right),$$

where $L_i = \dfrac{K_i}{V_0^i}$, K_i is the face value of the corporate bond issued by firm i, V_0^i is the asset value of firm i at time 0, and $m_i = \mu_i - \frac{1}{2}{\sigma_i}^2$. For the two-name first-passage approach, Zhou [52] showed the dependent structure is given by the bivariate inverse Gaussian distribution function with the correlation ρ.

Li [34] argued that the default correlation is time-dependent. He proposed the default correlation of the survival time. The survival function of each underlying credit risk can be estimated from the market data such as the risky bond prices or the swap spreads by constructing the credit curve. The credit curve gives the conditional default probabilities over a period of time. In the case of two names, the joint survival function of firm A and firm B in terms of their time to default τ_1 and τ_2 is given by

$$S_{\tau_1,\tau_2}(t_1, t_2) = P[\tau_1 > t_1, \tau_2 > t_2].$$

And the joint cumulative distribution function is given by

$$\begin{aligned} F(t_1, t_2) &= P[\tau_1 \le t_1, \tau_2 \le t_2] \\ &= 1 - S_{\tau_1}(t_1) - S_{\tau_2}(t_2) + S_{\tau_1,\tau_2}(t_1, t_2). \end{aligned}$$

The correlation coefficient in terms of the time to default τ_1 and τ_2 is given by

$$\begin{aligned} \rho &= \frac{\mathrm{Cov}(\tau_1, \tau_2)}{\sqrt{\mathrm{Var}(\tau_1)\mathrm{Var}(\tau_2)}} \\ &= \frac{\mathrm{E}(\tau_1, \tau_2) - \mathrm{E}(\tau_1)\mathrm{E}(\tau_2)}{\sqrt{\mathrm{Var}(\tau_1)\mathrm{Var}(\tau_2)}}. \end{aligned}$$

There is a relationship between the copula in terms of the survival function and the copula in terms of the distribution function. It is given by

$$\begin{aligned}\bar{C}\left(S_1(t_1), S_2(t_2)\right) &= P[\tau_1 > t_1, \tau_2 > t_2] \\ &= S_1(t_1) + S_2(t_2) + C\left(F_1(t_1), F_2(t_2)\right) - 1,\end{aligned}$$

where $S_i(t_i) = 1 - F_i(t_i)$.

The merit of the copula is that it constructs a joint distribution without any restrictions on its marginal distributions. The copula can be either in terms of the firm value or in terms of the time to default. If we model the dependent defaults through the time to default, then any non-negative process of the hazard rate can be used. We do not need any economic model of the default in terms of the firm value [24]. In the next two chapters, we will study the loss distribution of the large portfolio by using the Gaussian copula and the Student's t copula.

Chapter 4

Loss Distribution by Gaussian Copula

4.1 Introduction

A portfolio of credit securities is subject to the loss that results from the defaults of its components. Schwarz [44] mentioned the default correlation was a measure for the sensitivity of the defaults to the systematic risk factor which represents the state of economy. If two companies have a strong dependence on the same systematic risk factor then they have a higher default correlation. We are interested in studying the loss distribution of a large pool of credit securities, as it is critical in pricing the collateralized debt obligation (CDO). We study the loss distribution of a large credit portfolio, both in the homogeneous case and in the nonhomogeneous case, in terms of the default probabilities and the recovery rates of individual names. For the large pool of credit securities, we have the following assumptions:

(1) The portfolio consists of n credit securities in equal dollar amounts and with the

same term T.

(2) The correlation coefficient of any two names remains the same.

(3) The recovery rates are the same.

(4) In homogeneous case, the default probabilities of individual names are identical such that $p_i = p_j = p$.

By Merton's classical structural model, the asset value of firm i, V_t^i, is described by a stochastic process

$$dV_t^i = \mu_i V_t^i dt + \sigma_i V_t^i dW_t, \quad 0 < t \leq T,$$

where μ is called drift, σ is called volatility and W is the standard Brownian motion. The asset value has the closed-form solution by Ito's lemma,

$$V_t^i = V_0^i e^{(\mu_i - \frac{1}{2}{\sigma_i}^2)t + \sigma_i W_t}.$$

The default probability is defined by

$$\begin{aligned} P_{\tau_i}(t) &= P\left(V_t^i < Z_i\right) \\ p_i &= P\left(\frac{logV_t^i - logV_0^i - (\mu_i - \frac{1}{2}{\sigma_i}^2)t}{\sigma_i\sqrt{t}} < \frac{logZ_i - logV_0^i - (\mu_i - \frac{1}{2}{\sigma_i}^2)t}{\sigma_i\sqrt{t}}\right) \\ p_i &= P\left(U_i < c_i\right), \end{aligned}$$

where τ_i is the time to default of name i, V_t^i is the asset value of firm i, Z_i is the contractual default level of firm i, and

$$\begin{aligned} U_i &= \frac{logV_t^i - logV_0^i - (\mu_i - \frac{1}{2}{\sigma_i}^2)t}{\sigma_i\sqrt{t}} \\ c_i &= \frac{logZ_i - logV_0^i - (\mu_i - \frac{1}{2}{\sigma_i}^2)t}{\sigma_i\sqrt{t}}. \end{aligned}$$

Therefore we say firm i defaults when

$$U_i < F_{U_i}^{-1}(p_i).$$

In the case of the Gaussian copula, U_i follows the standard normal distribution. We could write U_i as

$$U_i = \beta X^G + \sqrt{1-\beta^2} X_i^I,$$

where $i = 1, \ldots, n$. X^G is the common systematic risk factor and has the standard normal distribution. $X_1^I, \ldots, X_n^I$ are the idiosyncratic risk factors and they follow the independent and identical standard normal distribution. We also assume that X^G is independent of X_i^I. β^2 refers to the pairwise correlation coefficient. We define

$$D_i = \begin{cases} 1 & \text{if firm i defaults, with probability } \tilde{p} \\ 0 & \text{if firm i does not default, with probability } 1-\tilde{p}. \end{cases}$$

Then the number of defaults in a credit portfolio is given by $D = \sum_{i=1}^{n} D_i$. The portfolio loss is $L = N_{ml} \sum_{i=1}^{n} (1 - R_i) 1_{\{\tau_i \leq T\}}$, where N_{ml} is the nominal amount of the credit portfolio and R_i is the recovery rate of name i. Please refer to the detail notations at the end of this section. If R_i is the same for all i, the distribution of the number of defaults D is closely related to the distribution of the portfolio loss L. Therefore we have the factor Gaussian copula for the pool loss L such that

$$L = L(t, U_1, \ldots, U_n)$$
$$U_i = \beta X^G + \sqrt{1-\beta^2} X_i^I.$$

This factor Gaussian copula model is the combination of the mixed binomial structure model and the copula model that we have introduced in Chapter 3. It constructs the joint distribution of individual defaults independent of their marginal distributions. It is simple and easy to implement in practice. Therefore it is the market standard model for pricing credit portfolios. Vasicek [47] proposed to get the loss distribution by using

the conditional default probability given the common systematic factor X^G. We will introduce his idea in this chapter. Vasicek also gave the limiting distribution of the proportional pool loss [47]. However, it does not work well when the correlation β^2 is very small. First we try to find a more appropriate asymptotic approximation for the loss of a homogeneous large pool of credit securities. We will compare the asymptotic approximations with the exact loss distribution by using the Hellinger distance of their probability density functions.

In the nonhomogeneous case, we derive the so-called recursive calculation to induce the exact loss distribution of a large pool of credit securities. It allows different default probabilities of individual names, while keeping the other assumptions of the homogeneous large pool as the same. We also give several asymptotic approximations for the nonhomogeneous large pool of credit securities.

Notations

Value date: t_0

Life of CDO from value date to its maturity: T

Total number of underlying credit names: n

Nominal amount of CDO: N_{ml}

Time to default of name i: τ_i

Default time of CDO: τ

Risk-neutral default probability of name i: $p_i = P(\tau_i \leq T)$

Recovery rate of name i: R_i

Correlation coefficient in the factor copulas: β^2

Portfolio loss: $L = N_{ml} \sum_{i=1}^{n} (1 - R_i) 1_{\{\tau_i \leq T\}}$

Portfolio loss distribution: $P(L \leq l)$, where $0 \leq l \leq N_{ml} \sum_{i=1}^{n} (1 - R_i)$

4.2 Model in Homogeneous Case

We study the exact loss distribution and several asymptotic approximations in the homogeneous case, including:

Exact Model: Vasicek factor model with conditional binomial distribution;

Approx2: Vasicek approximation as $n \to \infty$;

Approx3: Approximation as $\beta = 0$ and $n \to \infty$;

Approx4: Approximation as $\beta \to 0$ and $n \to \infty$;

Approx5/Approx5a/Approx5b: Alternative approximations as $\beta \to 0$ and $n \to \infty$;

Approx6/Approx6a/Approx6b: Alternative approximations as $\beta \to 0$ and $n \to \infty$ (approximation of bivariate normal);

Approx7a/Approx7b: Alternative approximations as $\beta \to 0$ and $n \to \infty$.

4.2.1 Vasicek Factor Model with Conditional Binomial Distribution

Vasicek [47] defined the pool loss as $D = \sum_{i=1}^{n} D_i$, where D_i is an indicator variable for the default of name i. Therefore the portfolio loss distribution over the period of T is given by

$$\begin{aligned} P(D = i) &= \binom{n}{i} P\left[V_T^1 < Z_1, \ldots, V_T^i < Z_i, V_T^{i+1} \geq Z_{i+1}, \ldots, V_T^n \geq Z_n\right] \\ &= \binom{n}{i} P\left[U_1 < c_1, \ldots, U_i < c_i, U_{i+1} \geq c_{i+1}, \ldots, U_n \geq c_n\right]. \end{aligned}$$

From the results in Chapter 3, given the common systematic risk factor X^G, the conditional default probability is given by

$$\begin{aligned} P_{\tau_i}\left(T\middle|X^G = x\right) &= P\left(V_T^i < Z_i\middle|X^G = x\right) \\ &= \Phi\left(\frac{\Phi^{-1}(p) - \beta x}{\sqrt{1-\beta^2}}\right), \end{aligned}$$

where p is the risk-neutral default probability of the individual names over the period of T. In the homogeneous case, we have $p_i = p_j = p$. As we assume X^G has the standard normal distribution, the unconditional probability of the pool loss in terms of D is given by

$$P(D \le k) = \sum_{i=0}^{k} \binom{n}{i} \int_{-\infty}^{+\infty} \left[\Phi\left(\frac{\Phi^{-1}(p) - \beta x}{\sqrt{1-\beta^2}}\right)\right]^i \left[1 - \Phi\left(\frac{\Phi^{-1}(p) - \beta x}{\sqrt{1-\beta^2}}\right)\right]^{n-i} d\,\Phi(x), \quad (4.1)$$

where $0 \le k \le n$. We can use the Gaussian-Hermite quadrature (GHQ) method to numerically approximate the integral that has the Gaussian kernel. The Vasicek model provides us the exact distribution of the homogeneous pool loss.

4.2.2 Vasicek Approximation as $n \to \infty$

Vasicek [48] gave the limiting distribution of the proportional pool loss $\dfrac{D}{n}$ as n goes to infinity

$$P\left(\frac{D}{n} \le \frac{k}{n}\right) \approx \Phi\left(\frac{\sqrt{1-\beta^2}\Phi^{-1}\left(\frac{k}{n}\right) - \Phi^{-1}(p)}{\beta}\right).$$

We use an alternative way to prove the Vasicek limiting distribution of the proportional pool loss.

$$\begin{aligned}
P\left(\frac{D}{n} \leq \frac{k}{n}\right) &\approx P\left[\mathrm{E}\left(\frac{D}{n}\middle| X^G\right) \leq \frac{k}{n}\right] \\
&\approx P\left[\Phi\left(\frac{\Phi^{-1}(p) - \beta X^G}{\sqrt{1-\beta^2}}\right) \leq \frac{k}{n}\right] \\
&\approx P\left[\Phi^{-1}(p) - \beta X^G \leq \sqrt{1-\beta^2}\Phi^{-1}\left(\frac{k}{n}\right)\right] \\
&\approx P\left[-\beta X^G \leq -\Phi^{-1}(p) + \sqrt{1-\beta^2}\Phi^{-1}\left(\frac{k}{n}\right)\right] \\
&\approx P\left[X^G \leq \frac{-\Phi^{-1}(p) + \sqrt{1-\beta^2}\Phi^{-1}\left(\frac{k}{n}\right)}{\beta}\right] \\
&\approx \Phi\left(\frac{\sqrt{1-\beta^2}\Phi^{-1}\left(\frac{k}{n}\right) - \Phi^{-1}(p)}{\beta}\right)
\end{aligned}$$

However, this asymptotic approximation will not work when the correlation β^2 goes to zero. One of our interests is to find a more appropriate asymptotic approximation for the loss of the homogeneous large pool of credit securities.

4.2.3 Approximation as $\beta = 0$ and $n \to \infty$

From the factor Gaussian copula model, when β equals to zero, the number of defaults D follows the binomial distribution with the parameters p (the default probability of the individual names) and n (the number of underlying credit names). As n goes to infinity, by the central limit theorem, the asymptotic distribution of the proportion pool loss $\frac{D}{n}$ is given by

$$\frac{D}{n} \overset{a}{\sim} N\left(p, \quad \tfrac{p(1-p)}{n}\right).$$

4.2.4 Approximation as $\beta \to 0$ and $n \to \infty$

As β, the square root of the correlation coefficient, is approaching zero, we let $\frac{k}{n} = p + \lambda\beta$, where $\lambda \in R$. Then

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) = \Phi\left(\frac{\sqrt{1-\beta^2}\Phi^{-1}\left(\frac{k}{n}\right) - \Phi^{-1}(p)}{\beta}\right)$$
$$P\left(\frac{D}{n} \leq p + \lambda\beta\right) = \Phi\left(\frac{\sqrt{1-\beta^2}\Phi^{-1}(p+\lambda\beta) - \Phi^{-1}(p)}{\beta}\right).$$

By taking expansion

$$\begin{aligned}\Phi^{-1}(p+\lambda\beta) &= \Phi^{-1}(p) + \left(\Phi^{-1}(p^*)\right)'\lambda\beta \\ &= \Phi^{-1}(p) + \frac{\lambda\beta}{\phi\left(\Phi^{-1}(p^*)\right)}\end{aligned}$$

where p^* is between p and $p + \lambda\beta$. And

$$\phi\left(\Phi^{-1}(p^*)\right) \to \phi\left(\Phi^{-1}(p)\right).$$

Therefore under the regularity conditions, as $\beta \to 0$

$$P\left(\frac{D}{n} \leq p + \lambda\beta\right) \to \Phi\left(\frac{\lambda}{\phi\left(\Phi^{-1}(p)\right)}\right).$$

Let $\lambda^* = \dfrac{\lambda}{\phi\left(\Phi^{-1}(p)\right)}$, then

$$P\left(\frac{D}{n} \leq p + \lambda^*\beta\phi\left(\Phi^{-1}(p)\right)\right) \to \Phi\left(\lambda^*\right)$$
$$P\left(\frac{\left(\frac{D}{n} - p\right)}{\phi\left(\Phi^{-1}(p)\right)\beta} \leq \lambda^*\right) \to \Phi\left(\lambda^*\right).$$

Hence we say as β goes to zero, the proportional pool loss $\dfrac{D}{n}$ has an asymptotic normal distribution with mean p and variance $\left[\phi\left(\Phi^{-1}(p)\right)\beta\right]^2$ such as

$$\frac{D}{n} \overset{a}{\sim} N\left(p, \quad \left[\phi\left(\Phi^{-1}(p)\right)\beta\right]^2\right).$$

4.2.5 Alternative Approximations as $\beta \to 0$ and $n \to \infty$

From Vasicek's model, we know the conditional expected value of D_i and the conditional variance of D_i.

$$\mathrm{E}\left(D_i \middle| X^G = x\right) = \Phi\left(\frac{\Phi^{-1}(p) - x\beta}{\sqrt{1-\beta^2}}\right)$$

$$\mathrm{Var}\left(D_i \middle| X^G = x\right) = \Phi\left(\frac{\Phi^{-1}(p) - x\beta}{\sqrt{1-\beta^2}}\right)\left[1 - \Phi\left(\frac{\Phi^{-1}(p) - x\beta}{\sqrt{1-\beta^2}}\right)\right],$$

where D_i is the indicator variable for the default of firm i and $i = 1, \ldots, n$.

By the law of iterated expectations, we have

$$\begin{aligned}
\mathrm{E}(D_i) &= \mathrm{E}\left[\mathrm{E}\left(D_i \middle| X^G\right)\right] \\
&= \mathrm{E}\left[\Phi\left(\frac{\Phi^{-1}(p) - X^G\beta}{\sqrt{1-\beta^2}}\right)\right] \\
&= \int_{-\infty}^{+\infty} \Phi\left(\frac{\Phi^{-1}(p) - x\beta}{\sqrt{1-\beta^2}}\right) d\Phi(x) \\
&= \Phi\left(\Phi^{-1}(p)\right) \\
&= p,
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{E}(D_i^2) &= \mathrm{E}\left[\mathrm{E}\left(D_i^2 \middle| X^G\right)\right] \\
&= \mathrm{E}\left[\Phi\left(\frac{\Phi^{-1}(p) - X^G\beta}{\sqrt{1-\beta^2}}\right)\Phi\left(\frac{\Phi^{-1}(p) - X^G\beta}{\sqrt{1-\beta^2}}\right)\right] \\
&= \int_{-\infty}^{+\infty} \Phi\left(\frac{\Phi^{-1}(p) - x\beta}{\sqrt{1-\beta^2}}\right)\Phi\left(\frac{\Phi^{-1}(p) - x\beta}{\sqrt{1-\beta^2}}\right) d\Phi(x) \\
&= \Phi\left(\Phi^{-1}(p), \Phi^{-1}(p), \beta^2\right),
\end{aligned}$$

where $\Phi_2\left(\Phi^{-1}(p), \Phi^{-1}(p), \beta^2\right)$ is the cumulative distribution function of the bivariate normal with $\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\Sigma = \begin{pmatrix} 1 & \beta^2 \\ \beta^2 & 1 \end{pmatrix}$. Please refer to Appendix $A.1$ and $A.2$ for the detail proofs. Vasicek [50] gave the same results of the expected values.

As $D = D_1 + \ldots + D_n$ by definition, we have

$$\begin{aligned} \mathrm{E}(D) &= n\mathrm{E}(D_i) \\ &= np, \end{aligned}$$

and

$$\begin{aligned} \mathrm{Var}(D) &= \mathrm{E}\left[\mathrm{Var}\left(D\middle|X^G\right)\right] + \mathrm{Var}\left[\mathrm{E}\left(D\middle|X^G\right)\right] \\ &= n\mathrm{E}\left\{\Phi\left(\frac{\Phi^{-1}(p) - X^G\beta}{\sqrt{1-\beta^2}}\right)\left[1 - \Phi\left(\frac{\Phi^{-1}(p) - X^G\beta}{\sqrt{1-\beta^2}}\right)\right]\right\} \\ &\quad + n^2\mathrm{E}\left[\Phi^2\left(\frac{\Phi^{-1}(p) - X^G\beta}{\sqrt{1-\beta^2}}\right)\right] - n^2p^2 \\ &= np - n^2p^2 + (n^2-n)\Phi_2\left(\Phi^{-1}(p), \Phi^{-1}(p), \beta^2\right) \\ &= np - n^2p^2 - np^2 + np^2 + (n^2-n)\Phi_2\left(\Phi^{-1}(p), \Phi^{-1}(p), \beta^2\right) \\ &= np(1-p) + (n^2-n)\left[\Phi_2\left(\Phi^{-1}(p), \Phi^{-1}(p), \beta^2\right) - p^2\right]. \end{aligned}$$

Thus

$$\mathrm{Var}\left(\frac{D}{n}\right) = \frac{p(1-p)}{n} + \left(1 - \frac{1}{n}\right)\left[\Phi_2\left(\Phi^{-1}(p), \Phi^{-1}(p), \beta^2\right) - p^2\right] \tag{4.2}$$

$$\overset{n\to\infty}{\approx} \frac{p(1-p)}{n} + \left[\Phi_2\left(\Phi^{-1}(p), \Phi^{-1}(p), \beta^2\right) - p^2\right]. \tag{4.3}$$

Hence

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \Phi\left(\frac{\frac{k}{n} - p}{\sqrt{\frac{p(1-p)}{n} + \left[\Phi_2\left(\Phi^{-1}(p), \Phi^{-1}(p), \beta^2\right) - p^2\right]}}\right).$$

We make some approximation of the cumulative distribution function of the bivariate

normal. For $x \in R$ and $y \in R$

$$\begin{aligned}
\Phi_2\left(x, y, \beta^2\right) &= \int_{-\infty}^{x}\int_{-\infty}^{y} \frac{1}{2\pi\sqrt{1-\beta^4}} e^{-\frac{u^2-2\beta^2uv+v^2}{2(1-\beta^4)}}\, du\, dv \\
&= \int_{-\infty}^{x}\int_{-\infty}^{y} \frac{1}{2\pi\sqrt{1-\beta^4}} e^{-\frac{u^2+v^2}{2(1-\beta^4)}} e^{-\frac{\beta^2uv}{1-\beta^4}}\, du\, dv \\
&\overset{\beta\to 0}{\approx} \int_{-\infty}^{x}\int_{-\infty}^{y} \frac{1}{2\pi\sqrt{1-\beta^4}} e^{-\frac{u^2+v^2}{2(1-\beta^4)}\left(1+\frac{\beta^2uv}{1-\beta^4}\right)}\, du\, dv \\
&\approx \int_{-\infty}^{x}\int_{-\infty}^{y} \frac{1}{2\pi} e^{-\frac{u^2+v^2}{2}}\, du\, dv + \int_{-\infty}^{x}\int_{-\infty}^{y} \frac{1}{2\pi} e^{-\frac{u^2+v^2}{2}} \beta^2 uv\, du\, dv \\
&\approx \Phi(x)\Phi(y) + \beta^2 \int_{-\infty}^{x}\int_{-\infty}^{y} \frac{1}{2\pi} e^{-\frac{u^2}{2}} d\left(-\frac{u^2}{2}\right) e^{-\frac{v^2}{2}} d\left(-\frac{v^2}{2}\right) \\
&\approx \Phi(x)\Phi(y) + \beta^2\phi(x)\phi(y).
\end{aligned}$$

Thus

$$\begin{aligned}
\mathrm{Var}(D) &= np - n^2p^2 + (n^2-n)\Phi_2\left(\Phi^{-1}(p), \Phi^{-1}(p), \beta^2\right) \\
&= np - n^2p^2 - np^2 + np^2 + (n^2-n)\Phi_2\left(\Phi^{-1}(p), \Phi^{-1}(p), \beta^2\right) \\
&= np(1-p) + (n^2-n)\left[\Phi_2\left(\Phi^{-1}(p), \Phi^{-1}(p), \beta^2\right) - p^2\right] \\
&\overset{\beta\to 0}{\approx} np(1-p) + (n^2-n)\beta^2\left[\phi\left(\Phi^{-1}(p)\right)\right]^2.
\end{aligned}$$

And

$$\mathrm{Var}\left(\frac{D}{n}\right) \approx \frac{p(1-p)}{n} + \left(1-\frac{1}{n}\right)\beta^2\left[\phi\left(\Phi^{-1}(p)\right)\right]^2 \tag{4.4}$$

$$\approx \frac{p(1-p)}{n} + \beta^2\left[\phi\left(\Phi^{-1}(p)\right)\right]^2. \tag{4.5}$$

When both n goes to infinity and β goes to zero, we say the asymptotic variance of the proportional pool loss consists of two components. One is the asymptotic variance from the binomial distribution for n approaching infinity, and the other is the asymptotic variance derived from Approx4. In the proof of Approx4, we have showed that as $\beta \to 0$,

$$P\left(\frac{D}{n} \le \frac{k}{n}\right) \approx \Phi\left(\frac{\sqrt{1-\beta^2}\Phi^{-1}\left(\frac{k}{n}\right) - \Phi^{-1}(p)}{\beta}\right)$$

implies

$$\frac{D}{n} \overset{a}{\sim} N\left(p, \quad \left[\phi\left(\Phi^{-1}(p)\right)\beta\right]^2\right).$$

By equating the variances from Equation (4.3) and the variance of Approx4, we have

$$\frac{p(1-p)}{n} + \left[\Phi_2\left(\Phi^{-1}(p), \Phi^{-1}(p), \beta^2\right) - p^2\right] = \left[\phi\left(\Phi^{-1}(p)\right)\right]^2 \beta^{*2}.$$

Then

$$\beta^* = \sqrt{\frac{p(1-p)}{n}\frac{1}{\left[\phi\left(\Phi^{-1}(p)\right)\right]^2} + \left[\Phi_2\left(\Phi^{-1}(p), \Phi^{-1}(p), \beta^2\right) - p^2\right]\frac{1}{\left[\phi\left(\Phi^{-1}(p)\right)\right]^2}}.$$

Therefore we can say that

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \Phi\left(\frac{\sqrt{1-\beta^2}\Phi^{-1}\left(\frac{k}{n}\right) - \Phi^{-1}(p)}{\beta^*}\right)$$

implies

$$\frac{D}{n} \overset{a}{\sim} N\left(p, \quad \frac{p(1-p)}{n} + \left[\Phi_2\left(\Phi^{-1}(p), \Phi^{-1}(p), \beta^2\right) - p^2\right]\right).$$

Similarly we equate the variances from Equation (4.5) and the variance of Approx4 such that

$$\frac{p(1-p)}{n} + \left[\phi\left(\Phi^{-1}(p)\right)\beta\right]^2 = \left[\phi\left(\Phi^{-1}(p)\right)\right]^2 \beta^{*2}.$$

Then

$$\beta^* = \sqrt{\frac{p(1-p)}{n}\frac{1}{\left[\phi\left(\Phi^{-1}(p)\right)\right]^2} + \beta^2}.$$

Therefore we can also say that

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \Phi\left(\frac{\sqrt{1-\beta^2}\Phi^{-1}\left(\frac{k}{n}\right) - \Phi^{-1}(p)}{\beta^*}\right)$$

implies

$$\frac{D}{n} \overset{a}{\sim} N\left(p, \quad \frac{p(1-p)}{n} + \left[\phi\left(\Phi^{-1}(p)\right)\beta\right]^2\right).$$

4.2.6 Summary

Exact Model: Vasicek factor model with conditional binomial distribution

$$P(D \leq k) = \sum_{i=0}^{k} \binom{n}{i} \int_{-\infty}^{+\infty} \left[\Phi\left(\frac{\Phi^{-1}(p) - \beta x}{\sqrt{1-\beta^2}}\right)\right]^i \times \left[1 - \Phi\left(\frac{\Phi^{-1}(p) - \beta x}{\sqrt{1-\beta^2}}\right)\right]^{n-i} d\Phi(x). \tag{4.6}$$

Approx2: Vasicek approximation as $n \to \infty$

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \Phi\left(\frac{\sqrt{1-\beta^2}\Phi^{-1}\left(\frac{k}{n}\right) - \Phi^{-1}(p)}{\beta}\right). \tag{4.7}$$

Approx3: Approximation as $\beta = 0$ and $n \to \infty$

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \Phi\left(\frac{\frac{k}{n} - p}{\sqrt{\frac{p(1-p)}{n}}}\right). \tag{4.8}$$

Approx4: Approximation as $\beta \to 0$ and $n \to \infty$

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \Phi\left(\frac{\frac{k}{n} - p}{\phi\left(\Phi^{-1}(p)\right)\beta}\right). \tag{4.9}$$

Approx5: Alternative approximation as $\beta \to 0$ and $n \to \infty$

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \Phi\left(\frac{\frac{k}{n} - p}{\sqrt{\frac{p(1-p)}{n} + \left[\Phi_2\left(\Phi^{-1}(p), \Phi^{-1}(p), \beta^2\right) - p^2\right]}}\right). \tag{4.10}$$

Approx5a: Based on Approx2 and Approx5, let

$$\beta^* = \sqrt{\frac{p(1-p)}{n}\frac{1}{\left[\phi\left(\Phi^{-1}(p)\right)\right]^2} + \left[\Phi_2\left(\Phi^{-1}(p), \Phi^{-1}(p), \beta^2\right) - p^2\right]\frac{1}{\left[\phi\left(\Phi^{-1}(p)\right)\right]^2}}, \tag{4.11}$$

then

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \Phi\left(\frac{\sqrt{1-\beta^2}\Phi^{-1}\left(\frac{k}{n}\right) - \Phi^{-1}(p)}{\beta^*}\right). \tag{4.12}$$

Approx5b: Based on Approx2 and Approx5, let

$$\beta^* = \sqrt{\frac{p(1-p)}{n}\frac{1}{\left[\phi\left(\Phi^{-1}(p)\right)\right]^2} + \left[\Phi_2\left(\Phi^{-1}(p), \Phi^{-1}(p), \beta^2\right) - p^2\right]\frac{1}{\left[\phi\left(\Phi^{-1}(p)\right)\right]^2}}, \tag{4.13}$$

then

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \Phi\left(\frac{\sqrt{1-\beta^{*2}}\Phi^{-1}\left(\frac{k}{n}\right) - \Phi^{-1}(p)}{\beta^*}\right). \tag{4.14}$$

Approx6: Alternative approximation as $\beta \to 0$ and $n \to \infty$

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \Phi\left(\frac{\frac{k}{n} - p}{\sqrt{\frac{p(1-p)}{n} + \left[\phi\left(\Phi^{-1}(p)\right)\beta\right]^2}}\right). \tag{4.15}$$

Approx6a: Based on Approx2 and Approx6, let

$$\beta^* = \sqrt{\frac{p(1-p)}{n}\frac{1}{\left[\phi\left(\Phi^{-1}(p)\right)\right]^2} + \beta^2}, \tag{4.16}$$

then

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \Phi\left(\frac{\sqrt{1-\beta^2}\Phi^{-1}\left(\frac{k}{n}\right) - \Phi^{-1}(p)}{\beta^*}\right). \tag{4.17}$$

Approx6b: Based on Approx2 and Approx6, let

$$\beta^* = \sqrt{\frac{p(1-p)}{n}\frac{1}{\left[\phi\left(\Phi^{-1}(p)\right)\right]^2} + \beta^2}, \tag{4.18}$$

then

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \Phi\left(\frac{\sqrt{1-\beta^{*2}}\Phi^{-1}\left(\frac{k}{n}\right) - \Phi^{-1}(p)}{\beta^*}\right). \tag{4.19}$$

Approx7a: Based on Approx6a, let

$$\beta^* = \sqrt{\left(\frac{p(1-p)}{n}\frac{1}{\left[\Phi_2\left(\Phi^{-1}(p), \Phi^{-1}(p), \beta^2\right) - p^2\right]} + 1\right)\beta^2}, \tag{4.20}$$

then

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \Phi\left(\frac{\sqrt{1-\beta^2}\Phi^{-1}\left(\frac{k}{n}\right) - \Phi^{-1}(p)}{\beta^*}\right). \tag{4.21}$$

Approx7b: Based on Approx 6b, let

$$\beta^* = \sqrt{\left(\frac{p(1-p)}{n}\frac{1}{\left[\Phi_2\left(\Phi^{-1}(p), \Phi^{-1}(p), \beta^2\right) - p^2\right]} + 1\right)\beta^2}, \tag{4.22}$$

then

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \Phi\left(\frac{\sqrt{1-\beta^{*2}}\Phi^{-1}\left(\frac{k}{n}\right) - \Phi^{-1}(p)}{\beta^*}\right). \tag{4.23}$$

4.2.7 Comparison of Exact Distribution and Asymptotic Approximations

The loss distribution plays a very important role for the credit analysis, as we have discussed in previous chapters. First we would like to study the properties of each approximate loss distribution function that we have derived from the Gaussian copula in the homogeneous case. We treat the Vasicek model as the exact model of the portfolio loss distribution. In order to compare each asymptotic approximation with the exact loss distribution, we use the Hellinger distance of their probability density functions. The comparison is also demonstrated by their loss distribution curves. In Chapter 6, we will evaluate the market data by using the Gaussian copula.

Hellinger Distance of Loss Distributions

If f, g are the densities of the measures μ, ν with respect to the dominating measure λ, then the Hellinger distance is defined as

$$d_H(\mu,\nu) = \left[\int (\sqrt{f} - \sqrt{g})^2 d\lambda\right]^{\frac{1}{2}}.$$

This definition is independent of the choice of the dominating measure λ. For a countable state space Ω,

$$d_H(\mu,\nu) = \left[\sum_{\omega\in\Omega} (\sqrt{\mu\omega} - \sqrt{\nu\omega})^2\right]^{\frac{1}{2}}.$$

See Diaconis and Zabell 1982 [14], Gibbs and Su 2002 [23] for details.

We use a, the continuity correction factor, to improve the approximation accuracy. Please refer to Appendix B.1.2 for the study of choosing continuity correction a. Table 4.1 and Table 4.2 give the Hellinger distance between each asymptotic approximation to the exact loss distribution at different levels of correlation coefficient β^2.

Table 4.1: Hellinger Distance of Loss Distributions at Different Level of Correlation Coefficient by Gaussian Copula–Continuity Correction $a = \frac{1}{2}$ (part1)

Approx.#	$\beta^2 = 0.0001$	$\beta^2 = 0.1$	$\beta^2 = 0.2$	$\beta^2 = 0.3$	$\beta^2 = 0.4$
2	0.431415	0.003684	0.001560	0.003024	0.003188
3	0.015676	0.031371	0.064564	0.113759	0.170324
4	0.432518	0.014435	0.021316	0.025849	0.030047
5	0.015757	0.018164	0.026936	0.032364	0.032762
5a	0.005491	0.003419	0.007334	0.008550	0.010142
5b	0.015112	0.015351	0.033407	0.051222	0.084455
6	0.015757	0.015563	0.022965	0.026745	0.029406
6a	0.005491	0.002018	0.003596	0.003873	0.003655
6b	0.015112	0.008348	0.010831	0.009896	0.008359
7a	0.005491	0.001791	0.002966	0.003520	0.003402
7b	0.015109	0.006058	0.007555	0.006875	0.005660

Table 4.2: Hellinger Distance of Loss Distributions at Different Level of Correlation Coefficient by Gaussian Copula–Continuity Correction $a = \frac{1}{2}$ (part2)

Approx.#	$\beta^2 = 0.5$	$\beta^2 = 0.6$	$\beta^2 = 0.7$	$\beta^2 = 0.8$	$\beta^2 = 0.9$
2	0.003053	0.002778	0.002278	0.001565	0.000752
3	0.229899	0.290766	0.352574	0.416232	0.485528
4	0.039512	0.055147	0.075255	0.098856	0.127281
5	0.032854	0.037844	0.049280	0.065447	0.085254
5a	0.011030	0.010960	0.015176	0.032151	0.061676
5b	NA	NA	NA	NA	NA
6	0.037650	0.052829	0.072865	0.096494	0.124912
6a	0.003417	0.003024	0.002388	0.001566	0.000706
6b	0.007190	0.006291	0.005428	0.004577	0.004418
7a	0.003197	0.002864	0.002310	0.001563	0.000739
7b	0.004733	0.003981	0.003172	0.002237	0.001255

Loss Distribution Curves

The following figures demonstrate the probability density functions and the cumulative distribution functions of Approx2, Approx5, Approx5a, Approx6, Approx6a and Approx7a compared with those of the exact distribution function. In order to improve

the visualization of the graphs, we take the natural log transformation of the probability densities. The graphs on the right hand side are the normal quantile-quantile (Q-Q) plots of the number of defaults. We can say that when the correlation coefficient β^2 is small, Approx6a and Approx7a are superior to the other approximations. They overcome the difficulty of the Vasicek approximation that has β in the denominator of the asymptotic variance. When the correlation coefficient β^2 is not small, the performances of Approx6a and Approx7a are comparable to that of the Vasicek approximation. All of them are close to the exact loss distribution. Comparatively, Approx7a is a little better than Approx6a. Please refer to Appendix B.1 for the detail of the numerical computation in R.

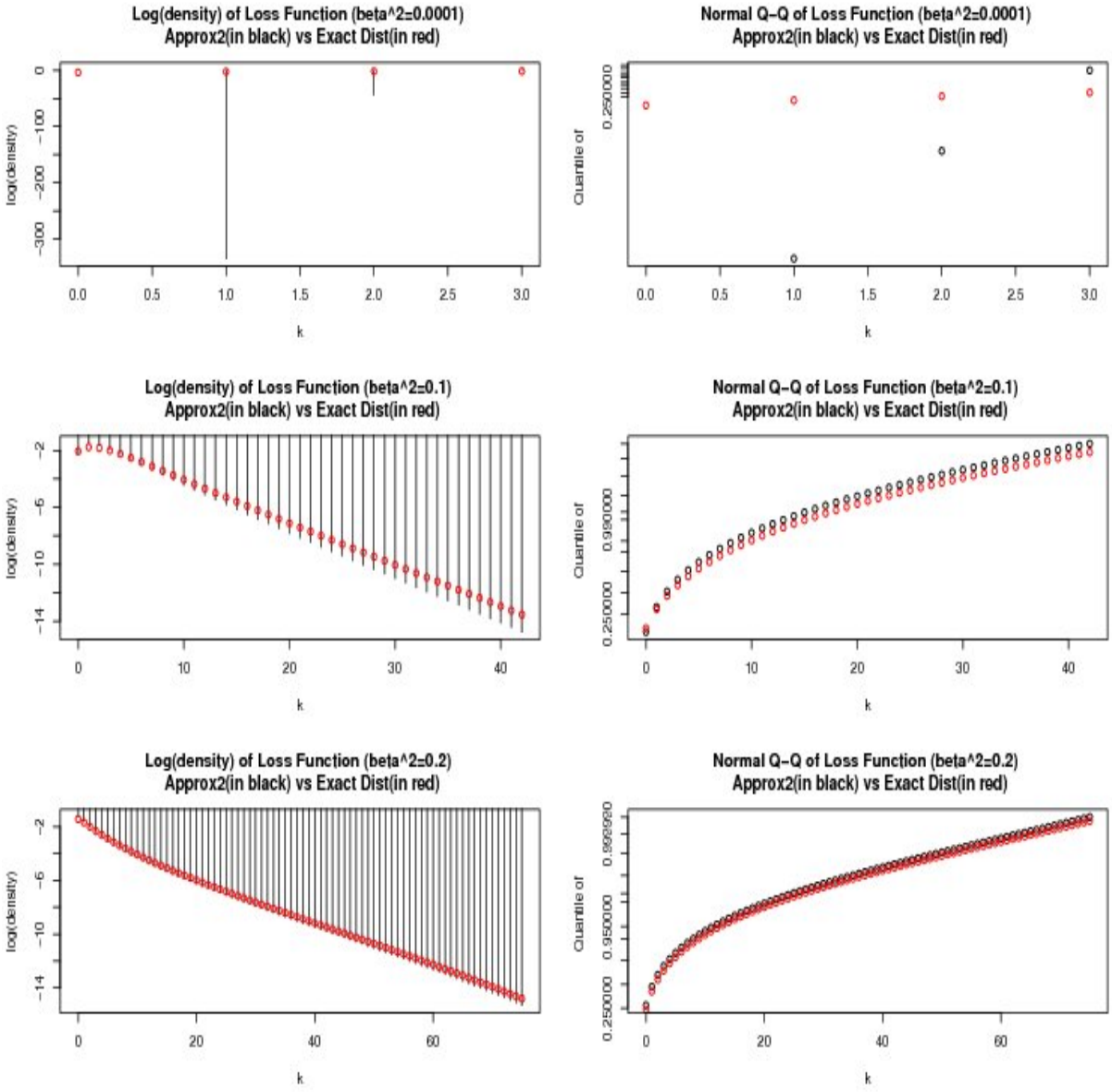

Figure 4.1: Loss Distribution by Gaussian Copula–Approx2 vs. Exact Model (part1)

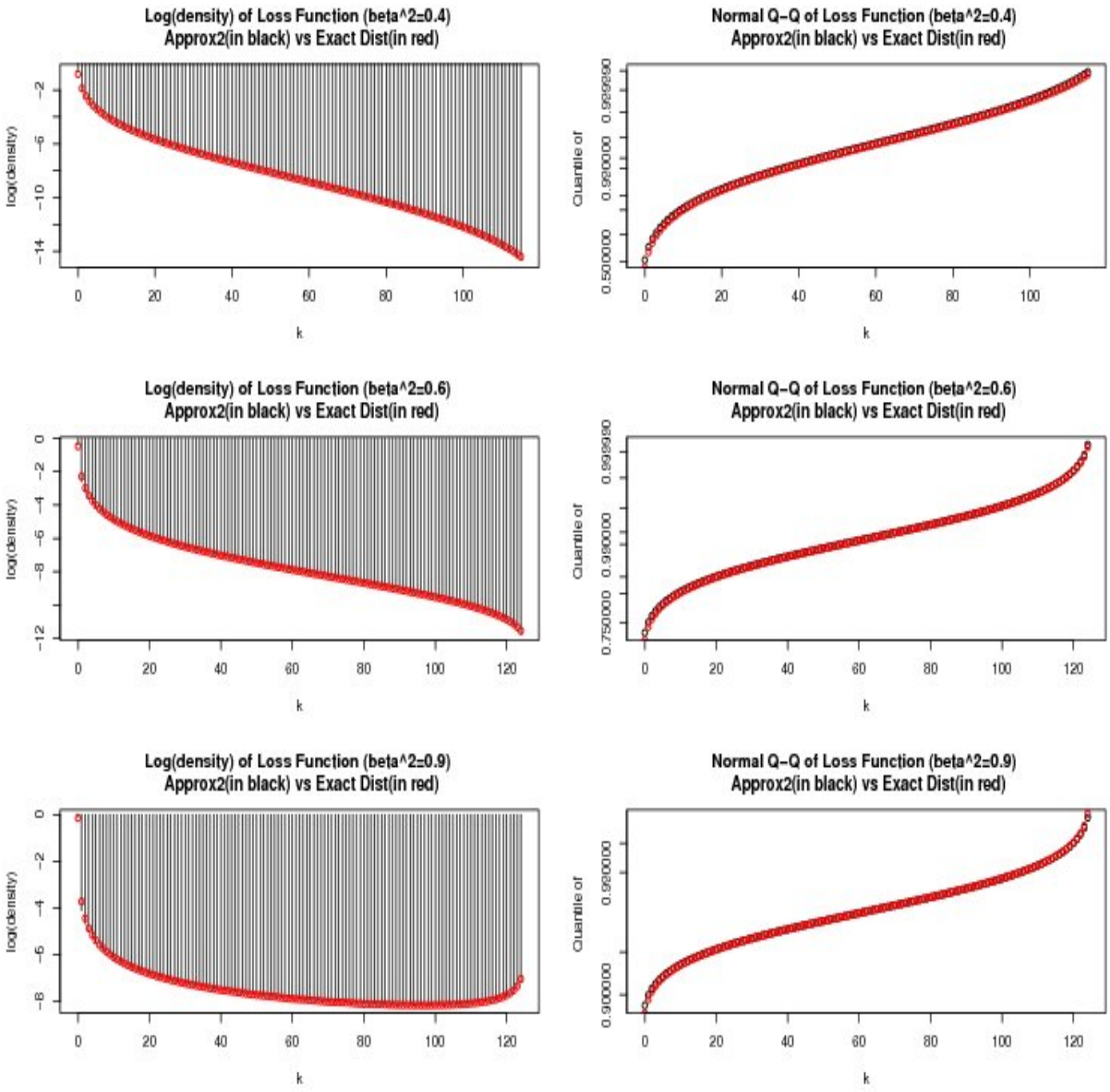

Figure 4.2: Loss Distribution by Gaussian Copula–Approx2 vs. Exact Model (part2)

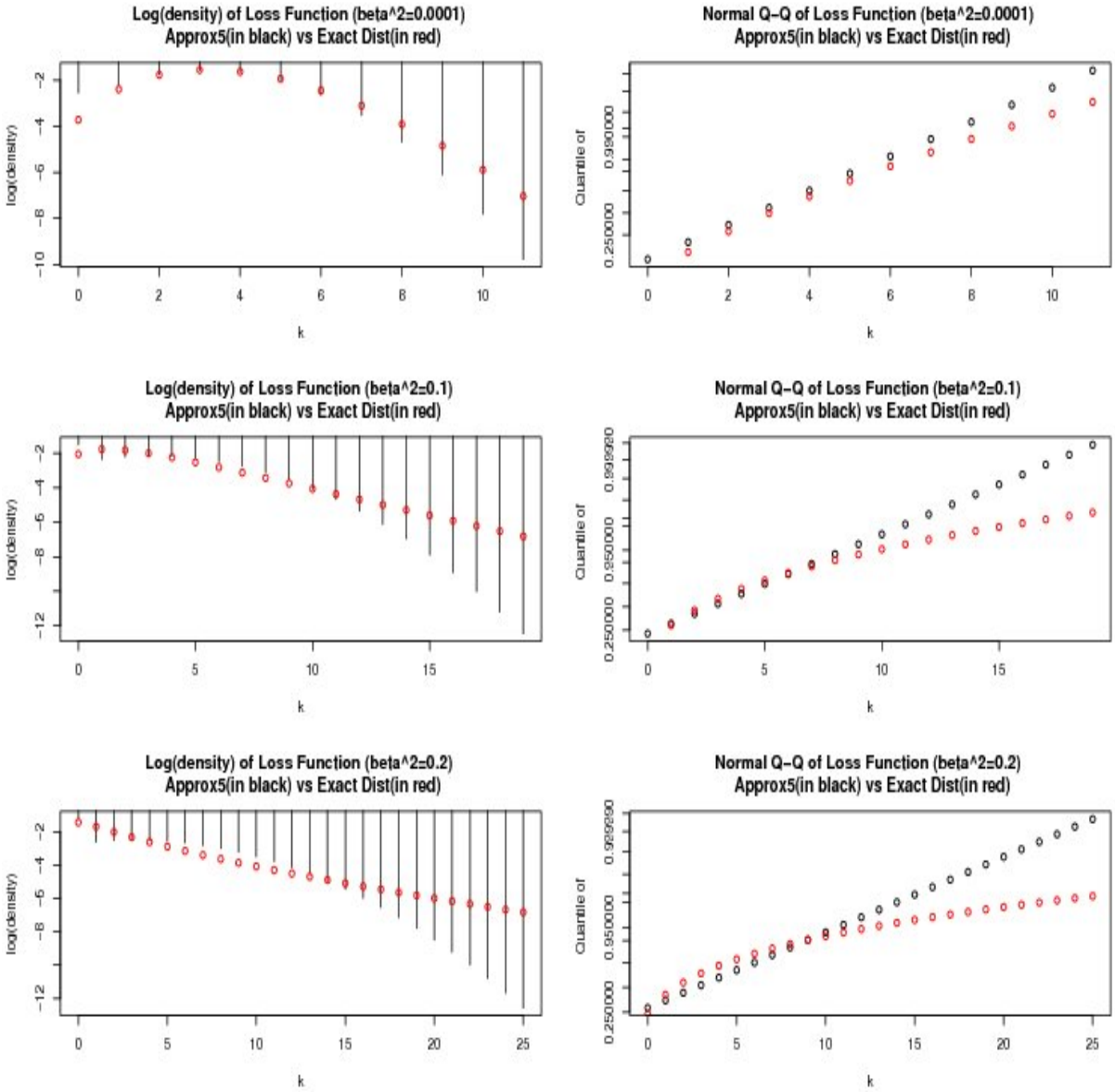

Figure 4.3: Loss Distribution by Gaussian Copula–Approx5 vs. Exact Model (part1)

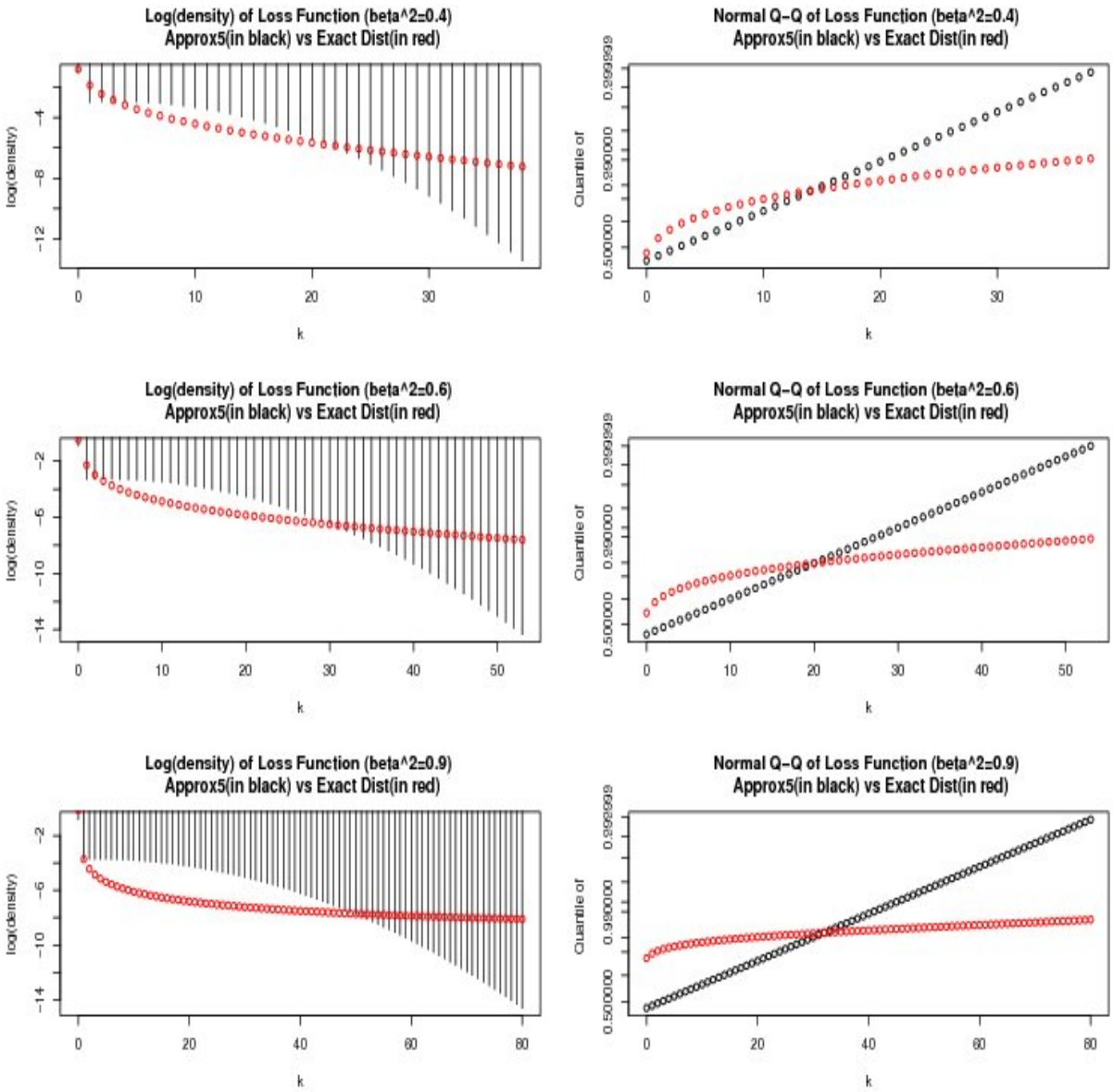

Figure 4.4: Loss Distribution by Gaussian Copula–Approx5 vs. Exact Model (part2)

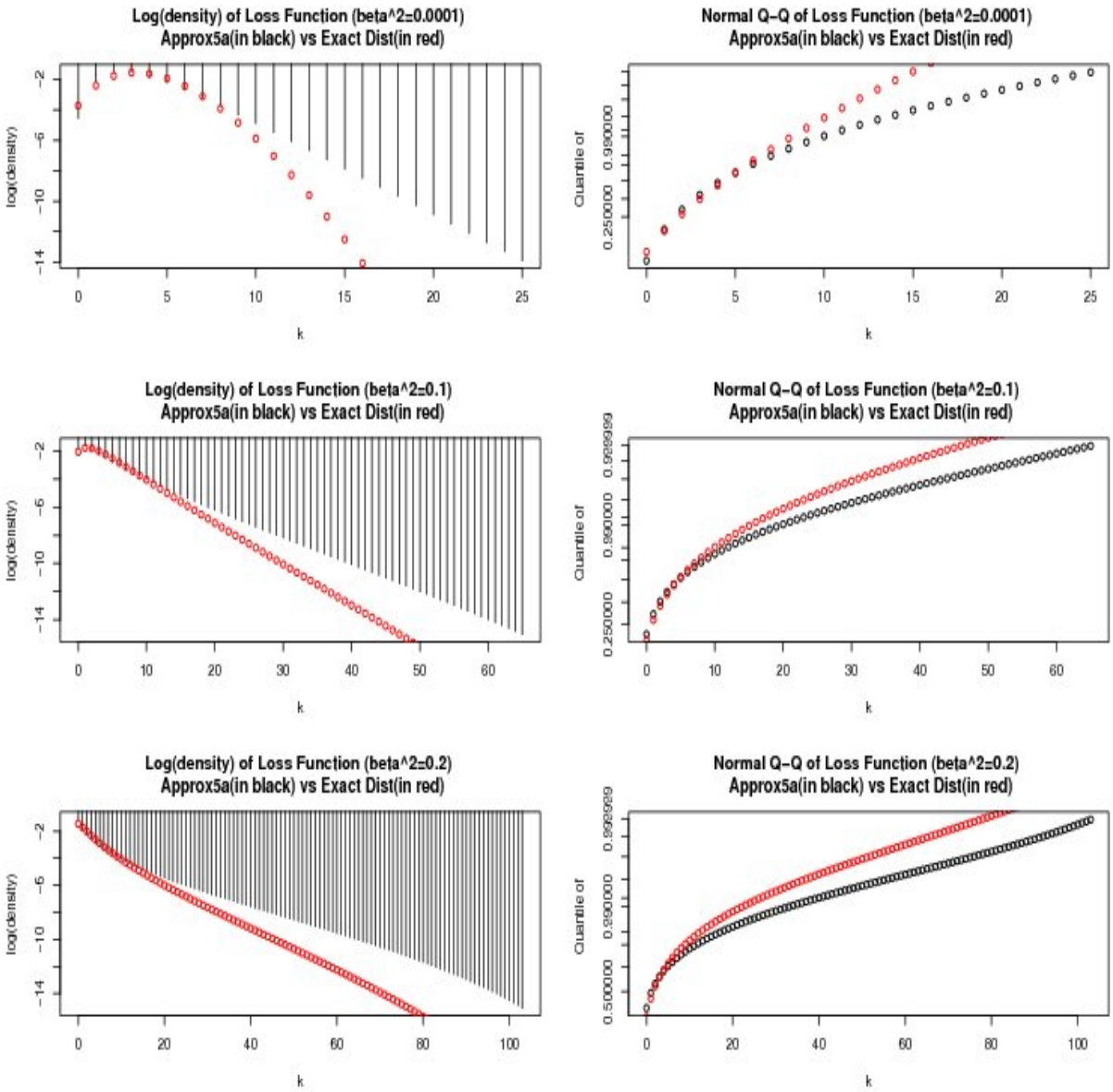

Figure 4.5: Loss Distribution by Gaussian Copula–Approx5a vs. Exact Model (part1)

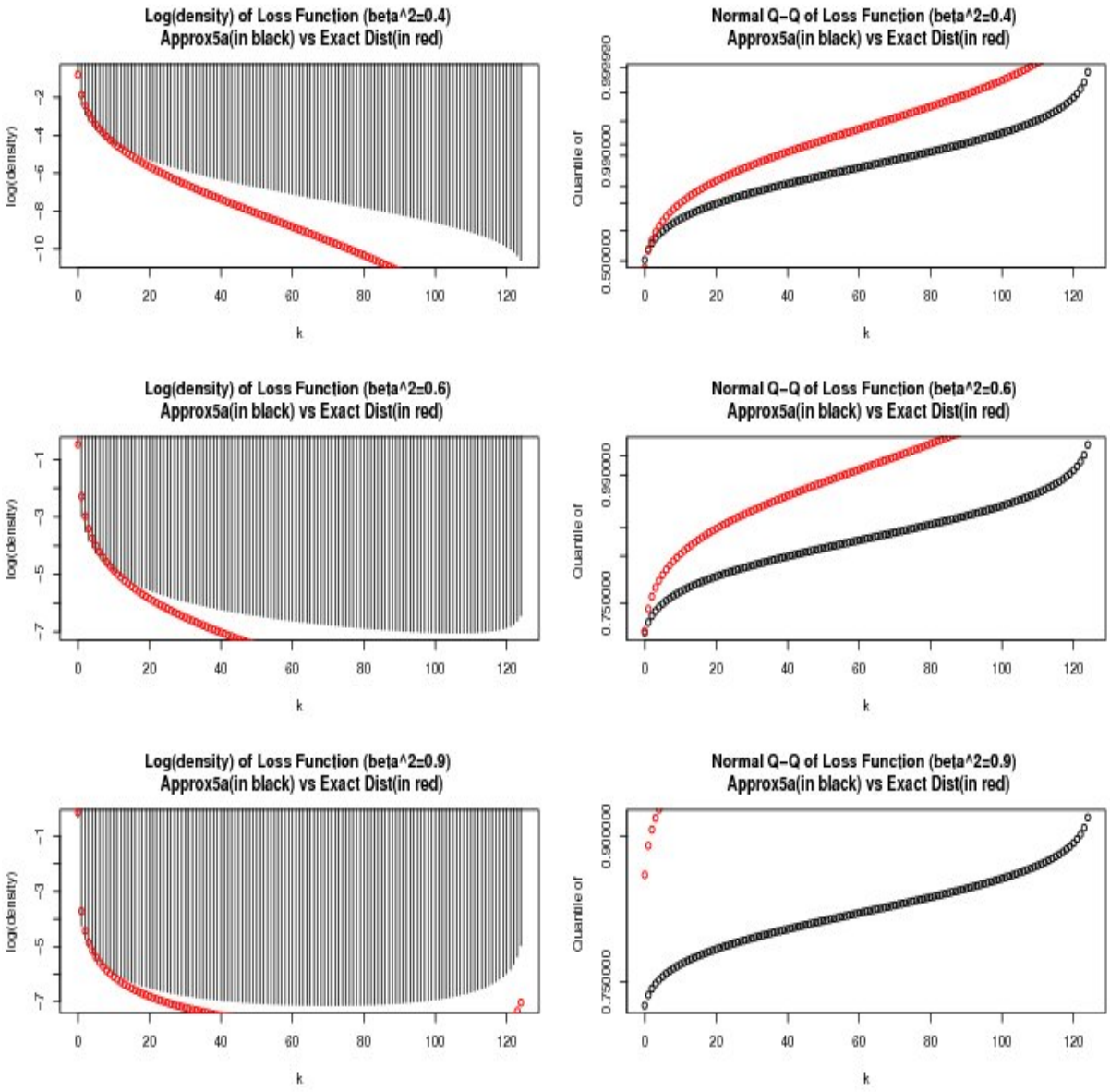

Figure 4.6: Loss Distribution by Gaussian Copula–Approx5a vs. Exact Model (part2)

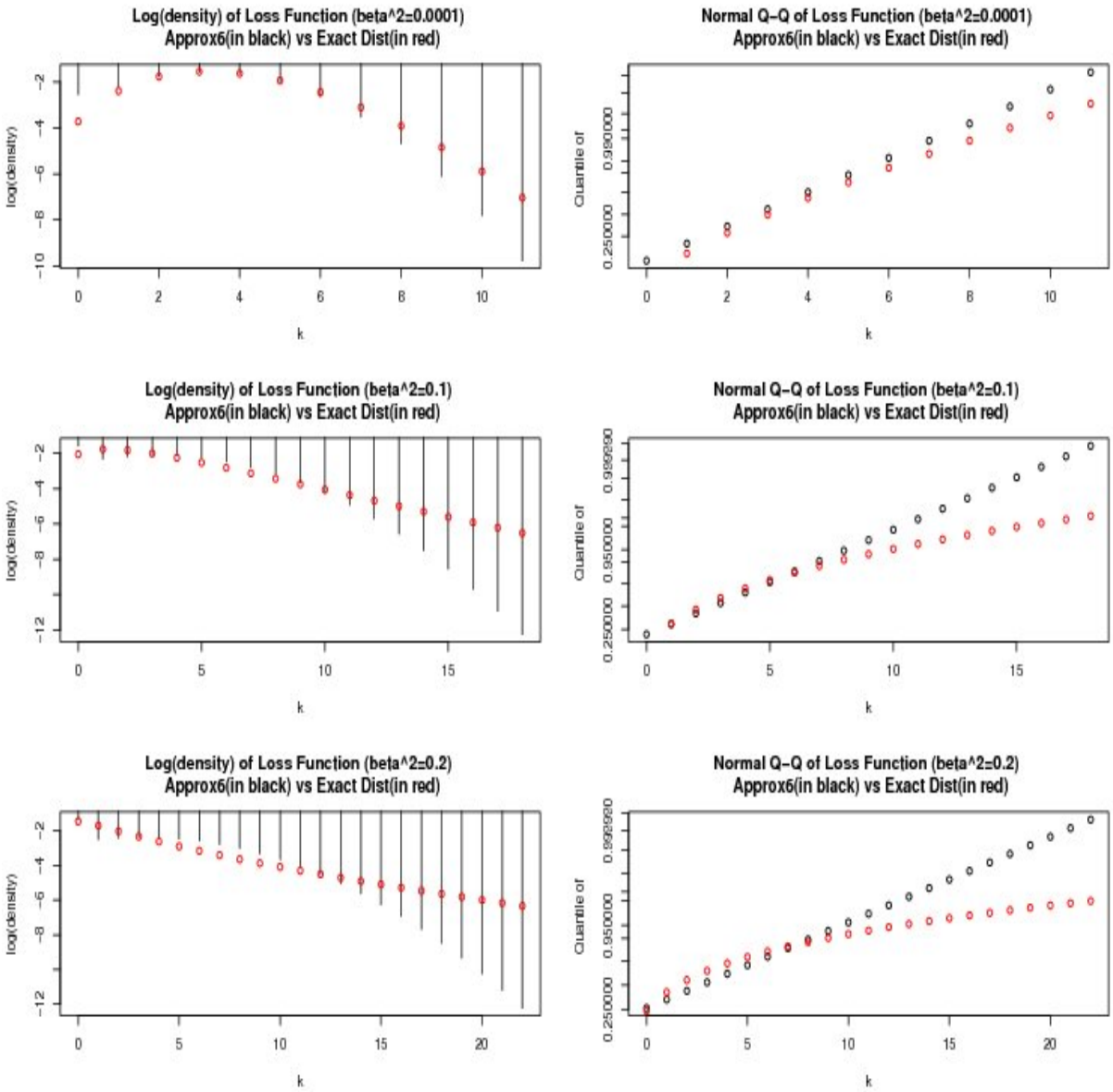

Figure 4.7: Loss Distribution by Gaussian Copula–Approx6 vs. Exact Model (part1)

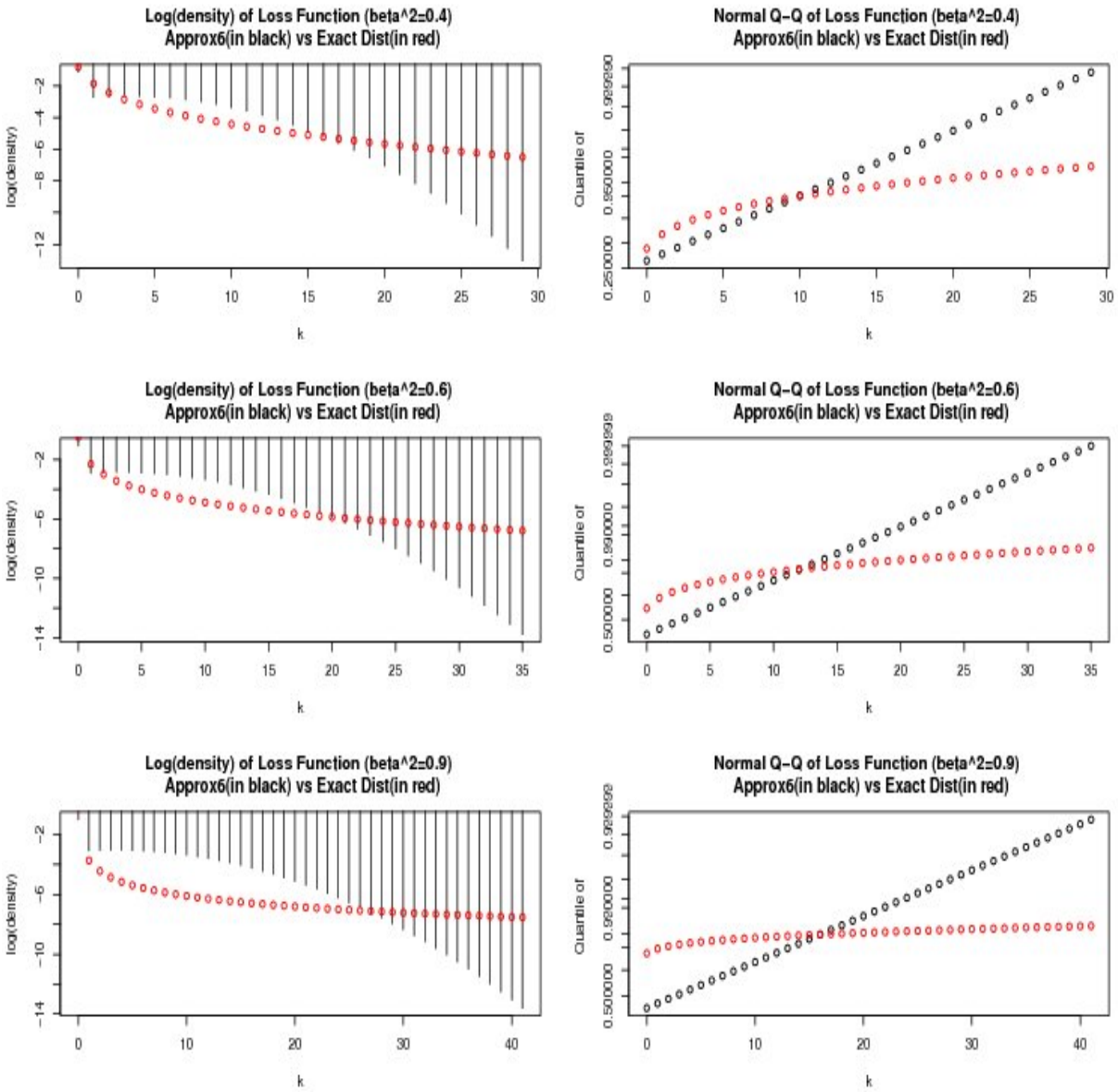

Figure 4.8: Loss Distribution by Gaussian Copula–Approx6 vs. Exact Model (part2)

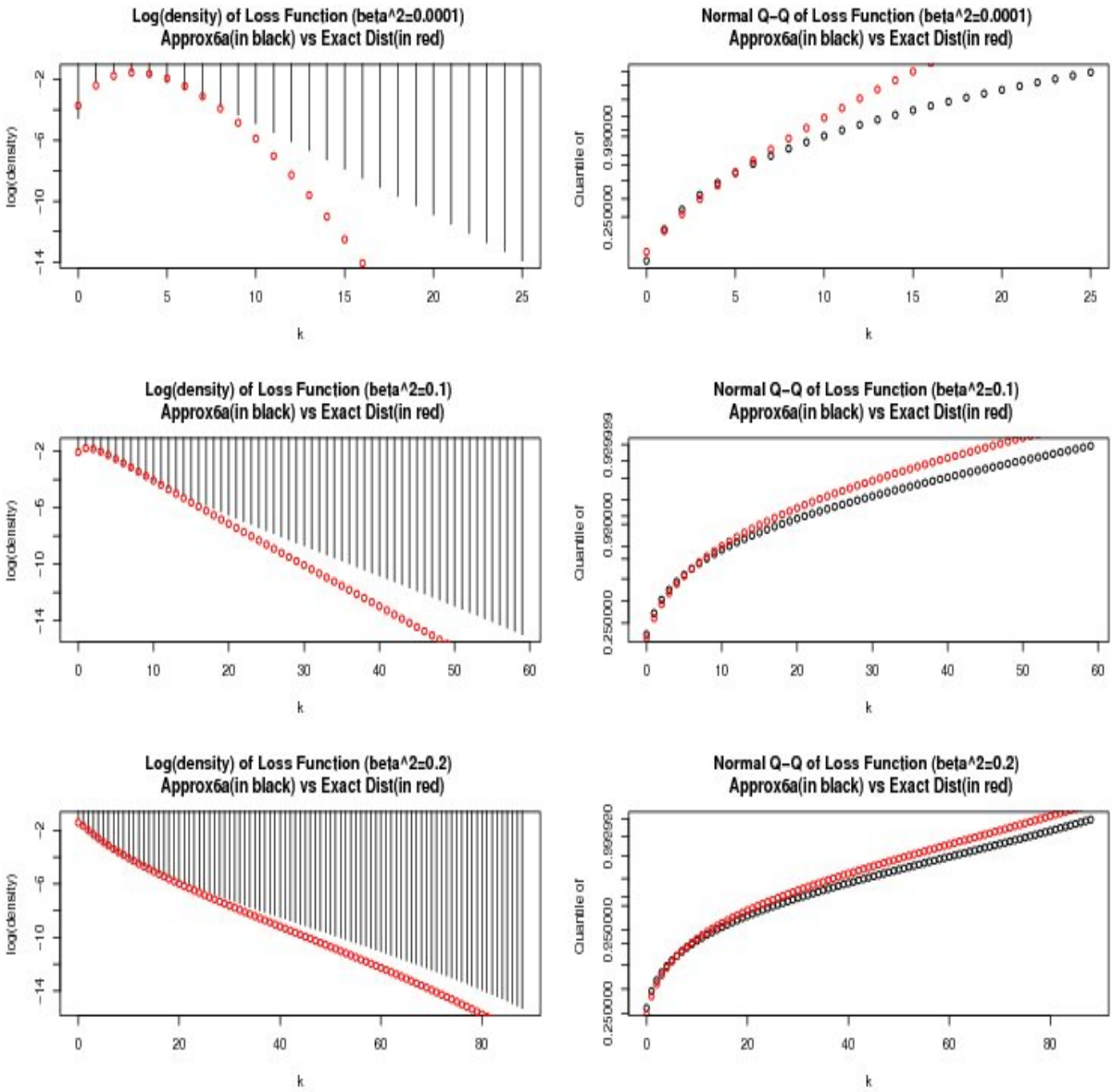

Figure 4.9: Loss Distribution by Gaussian Copula–Approx6a vs. Exact Model (part1)

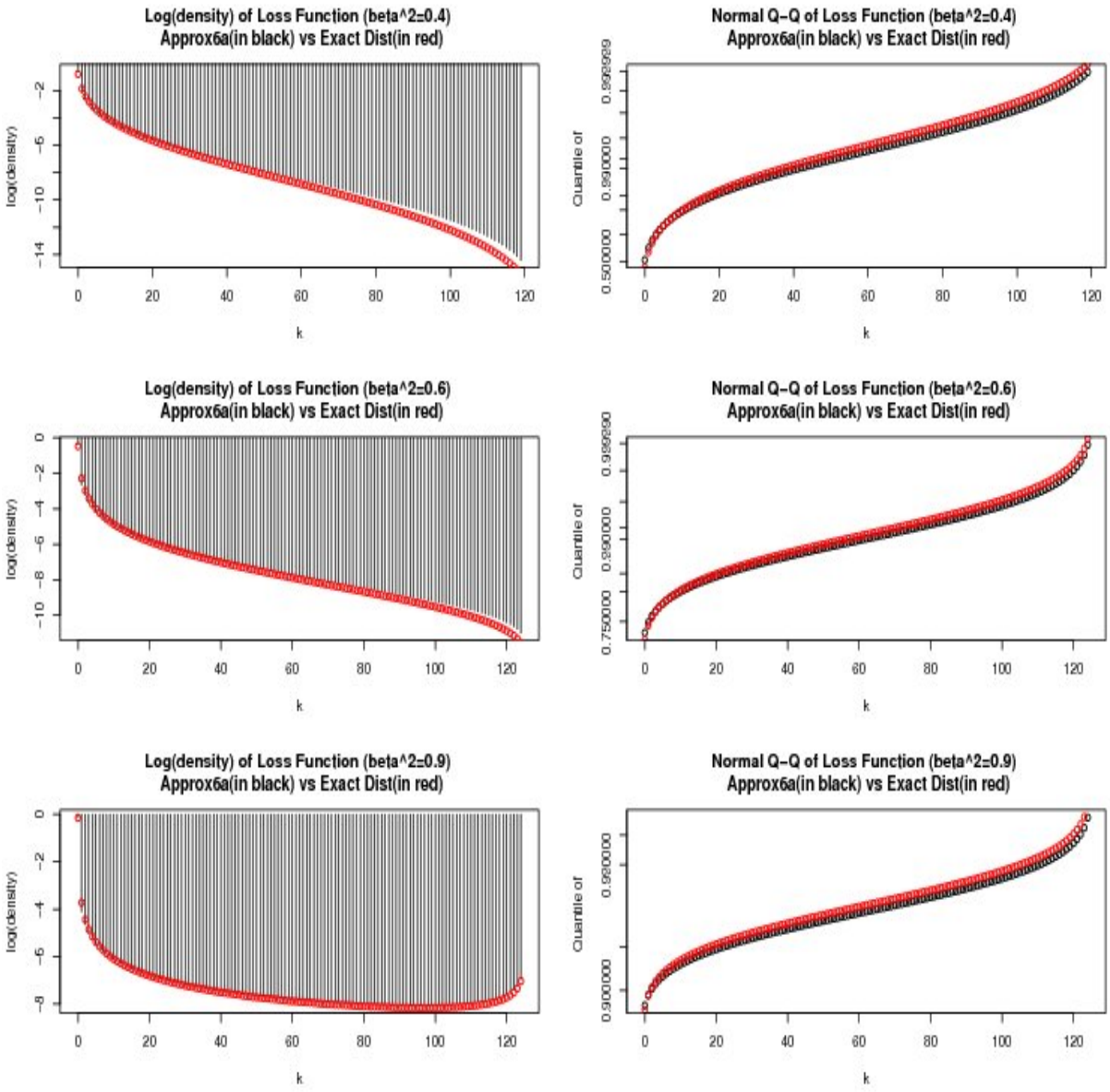

Figure 4.10: Loss Distribution by Gaussian Copula–Approx6a vs. Exact Model (part2)

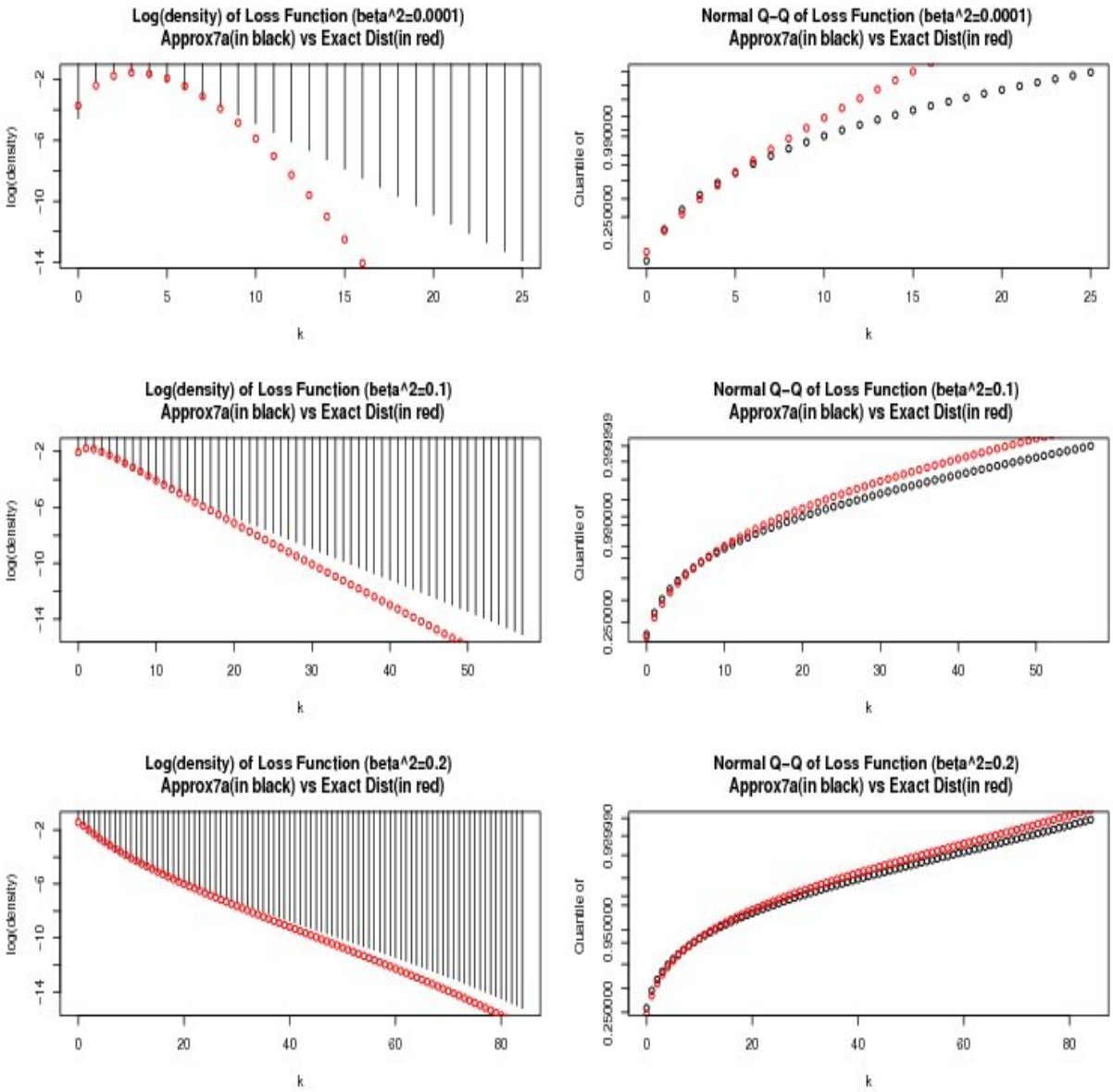

Figure 4.11: Loss Distribution by Gaussian Copula–Approx7a vs. Exact Model (part1)

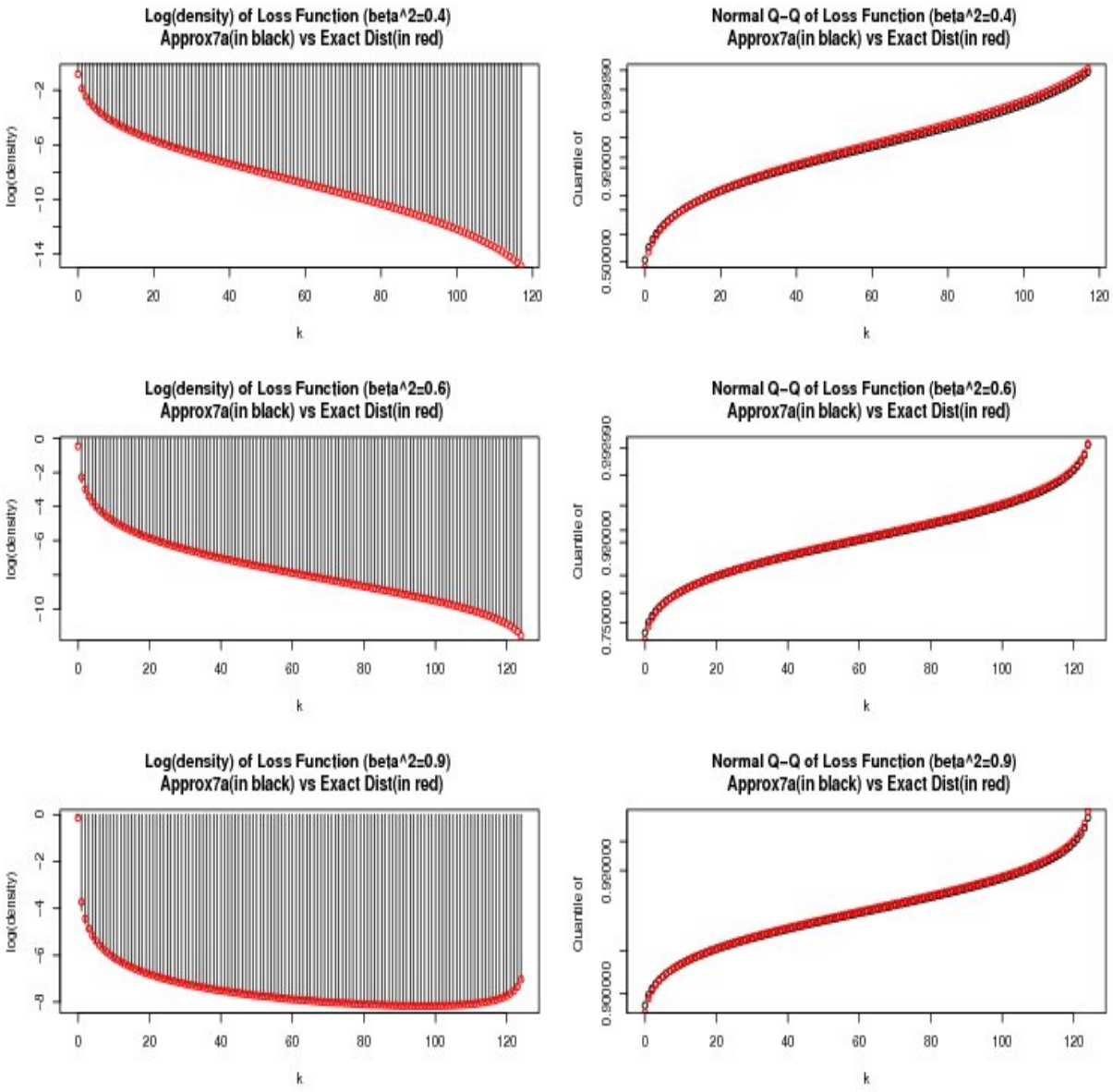

Figure 4.12: Loss Distribution by Gaussian Copula–Approx7a vs. Exact Model (part2)

4.3 Model in Nonhomogeneous Case

We study the exact loss distribution and several asymptotic approximations in the nonhomogeneous case, including:

Exact Model: Recursive Model with Conditional Binomial Distribution;

Approx2: Approximation as $n \to \infty$;

Approx3: Approximation as $\beta = 0$ and $n \to \infty$;

Approx4: Approximation as $\beta \to 0$ and $n \to \infty$;

Approx5/Approx5a/Approx5b: Alternative approximations as $\beta \to 0$ and $n \to \infty$;

Approx6/Approx6a/Approx6b: Alternative approximations as $\beta \to 0$ and $n \to \infty$ (approximation of bivariate normal);

Approx7a/Approx7b: Alternative approximations as $\beta \to 0$ and $n \to \infty$.

We derive the so-called recursive calculation that gives the exact distribution of the portfolio loss in the nonhomogeneous case. We obtain the asymptotic approximations by using similar approaches used in the homogeneous case. The format of each asymptotic approximation is almost the same in the nonhomogeneous case as that in the homogeneous case except for the replacement of p by $\Phi\left(\overline{\Phi^{-1}(p_i)}\right)$, where $\overline{\Phi^{-1}(p_i)} = \dfrac{1}{n}\sum_{i=1}^{n} \Phi^{-1}(p_i)$.

Since

$$\begin{aligned} P_{\tau_i}\left(T\middle|X^G = x\right) &= \mathrm{E}\left(D_i\middle|X^G = x\right) \\ &= \Phi\left(\frac{\Phi^{-1}(p_i) - \beta x}{\sqrt{1-\beta^2}}\right), \end{aligned}$$

therefore

$$\begin{aligned}
\mathrm{E}\left(\frac{D}{n}\middle| X^G = x\right) &= \frac{1}{n}\sum_{i=1}^{n}\mathrm{E}\left(D_i\middle| X^G = x\right) \\
&= \frac{1}{n}\sum_{i=1}^{n}\Phi\left(\frac{\Phi^{-1}(p_i) - \beta x}{\sqrt{1-\beta^2}}\right) \\
&= \frac{1}{n}\sum_{i=1}^{n}\Phi\left(\frac{\overline{\Phi^{-1}(p_i)} - \beta x}{\sqrt{1-\beta^2}}\right) \\
&\quad + \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\Phi^{-1}(p_i) - \overline{\Phi^{-1}(p_i)}}{\sqrt{1-\beta^2}}\right)^2 \Phi'\left(\frac{\Phi^{-1}(p_i^*) - \beta x}{\sqrt{1-\beta^2}}\right) \\
&\approx \frac{1}{n}\sum_{i=1}^{n}\Phi\left(\frac{\overline{\Phi^{-1}(p_i)} - \beta x}{\sqrt{1-\beta^2}}\right),
\end{aligned}$$

where $\Phi^{-1}(p_i^*)$ is a value between $\Phi^{-1}(p_i)$ and $\overline{\Phi^{-1}(p_i)}$.

4.3.1 Recursive Model with Conditional Binomial Distribution

We define an indicator variable X_i for the default of name i as

$$X_i = \begin{cases} 1 & \text{name } i \text{ defaults} \\ 0 & \text{name } i \text{ does not default.} \end{cases}$$

where $i = 1, \ldots, n$. We denote $P(X_i = 1)$ by p_i. For the purpose of the computation, we define $X_0 = 0$ and let $P(X_0 = 0) = 1$ and $P(X_0 = 1) = 0$. We also define the variable S_i as

$$S_i = X_0 + X_1 + X_2 + \ldots + X_i,$$

where $i = 1, \ldots, n$. It is easy to have the following equation

$$S_i = S_{i-1} + X_i.$$

The distribution of S_i can be derived by the following recursive rule as

$$P(S_i = s) = P(S_{i-1} = s \cap X_i = 0) + P(S_{i-1} = s - 1 \cap X_i = 1),$$

where $s = 0, \ldots, i$. And the exact loss distribution of the nonhomogeneous large pool is given by the distribution of S_n,

$$P(S_n \leq k),$$

where $k = 0, \ldots, n$.

4.3.2 Approximation as $n \to \infty$

Approx2:

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \Phi\left(\frac{\sqrt{1-\beta^2}\Phi^{-1}\left(\frac{k}{n}\right) - \overline{\Phi^{-1}(p_i)}}{\beta}\right). \tag{4.24}$$

4.3.3 Approximation as $\beta = 0$ and $n \to \infty$

Approx3:

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \Phi\left(\frac{\frac{k}{n} - \Phi\left(\overline{\Phi^{-1}(p_i)}\right)}{\sqrt{\frac{\Phi\left(\overline{\Phi^{-1}(p_i)}\right)\left[1-\Phi\left(\overline{\Phi^{-1}(p_i)}\right)\right]}{n}}}\right). \tag{4.25}$$

4.3.4 Approximation as $\beta \to 0$ and $n \to \infty$

Approx4:

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \Phi\left(\frac{\frac{k}{n} - \Phi\left(\overline{\Phi^{-1}(p_i)}\right)}{\phi\left(\overline{\Phi^{-1}(p_i)}\right)\beta}\right). \tag{4.26}$$

4.3.5 Alternative Approximations as $\beta \to 0$ and $n \to \infty$

Approx5:

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \Phi\left(\frac{\frac{k}{n} - \Phi\left(\overline{\Phi^{-1}(p_i)}\right)}{\sqrt{\frac{\Phi\left(\overline{\Phi^{-1}(p_i)}\right)\left[1-\Phi\left(\overline{\Phi^{-1}(p_i)}\right)\right]}{n} + \left[\Phi_2\left(\overline{\Phi^{-1}(p_i)}, \overline{\Phi^{-1}(p_i)}, \beta^2\right) - \Phi^2\left(\overline{\Phi^{-1}(p_i)}\right)\right]}}\right). \tag{4.27}$$

Approx5a: Based on Approx5, let

$$\beta^* = \sqrt{\frac{\Phi\left(\overline{\Phi^{-1}(p_i)}\right)\left[1-\Phi\left(\overline{\Phi^{-1}(p_i)}\right)\right]}{n}\frac{1}{\left[\phi\left(\overline{\Phi^{-1}(p_i)}\right)\right]^2} + \left[\Phi_2\left(\overline{\Phi^{-1}(p_i)},\overline{\Phi^{-1}(p_i)},\beta^2\right) - \Phi^2\left(\overline{\Phi^{-1}(p_i)}\right)\right]\frac{1}{\left[\phi\left(\overline{\Phi^{-1}(p_i)}\right)\right]^2}}, \tag{4.28}$$

then

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \Phi\left(\frac{\sqrt{1-\beta^2}\Phi^{-1}\left(\frac{k}{n}\right) - \overline{\Phi^{-1}(p_i)}}{\beta^*}\right). \tag{4.29}$$

Approx5b: Based on Approx5, let

$$\beta^* = \sqrt{\frac{\Phi\left(\overline{\Phi^{-1}(p_i)}\right)\left[1-\Phi\left(\overline{\Phi^{-1}(p_i)}\right)\right]}{n}\frac{1}{\left[\phi\left(\overline{\Phi^{-1}(p_i)}\right)\right]^2} + \left[\Phi_2\left(\overline{\Phi^{-1}(p_i)},\overline{\Phi^{-1}(p_i)},\beta^2\right) - \Phi^2\left(\overline{\Phi^{-1}(p_i)}\right)\right]\frac{1}{\left[\phi\left(\overline{\Phi^{-1}(p_i)}\right)\right]^2}}, \tag{4.30}$$

then

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \Phi\left(\frac{\sqrt{1-\beta^{*2}}\Phi^{-1}\left(\frac{k}{n}\right) - \overline{\Phi^{-1}(p_i)}}{\beta^*}\right). \tag{4.31}$$

Approx6: Alternative Approximation as $\beta \to 0$ and $n \to \infty$

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \Phi\left(\frac{\frac{k}{n} - \Phi\left(\overline{\Phi^{-1}(p_i)}\right)}{\sqrt{\frac{\Phi\left(\overline{\Phi^{-1}(p_i)}\right)\left[1-\Phi\left(\overline{\Phi^{-1}(p_i)}\right)\right]}{n} + \left[\phi\left(\overline{\Phi^{-1}(p_i)}\right)\beta\right]^2}}\right). \tag{4.32}$$

Approx6a: Based on Approx6, let

$$\beta^* = \sqrt{\frac{\Phi\left(\overline{\Phi^{-1}(p_i)}\right)\left[1-\Phi\left(\overline{\Phi^{-1}(p_i)}\right)\right]}{n}\frac{1}{\left[\phi\left(\overline{\Phi^{-1}(p_i)}\right)\right]^2} + \beta^2}, \tag{4.33}$$

then

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \Phi\left(\frac{\sqrt{1-\beta^2}\Phi^{-1}\left(\frac{k}{n}\right) - \overline{\Phi^{-1}(p_i)}}{\beta^*}\right). \tag{4.34}$$

Approx6b: Based on Approx6, let

$$\beta^* = \sqrt{\frac{\Phi\left(\overline{\Phi^{-1}(p_i)}\right)\left[1-\Phi\left(\overline{\Phi^{-1}(p_i)}\right)\right]}{n}\frac{1}{\left[\phi\left(\overline{\Phi^{-1}(p_i)}\right)\right]^2} + \beta^2}, \tag{4.35}$$

then

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \Phi\left(\frac{\sqrt{1-\beta^{*2}}\Phi^{-1}\left(\frac{k}{n}\right) - \overline{\Phi^{-1}(p_i)}}{\beta^*}\right). \tag{4.36}$$

Approx7a: Based on Approx6a, let

$$\beta^* = \sqrt{\left[\frac{\Phi\left(\overline{\Phi^{-1}(p_i)}\right)\left[1-\Phi\left(\overline{\Phi^{-1}(p_i)}\right)\right]}{n}\frac{1}{\left[\Phi_2\left(\overline{\Phi^{-1}(p_i)},\overline{\Phi^{-1}(p_i)},\beta^2\right) - \Phi^2\left(\overline{\Phi^{-1}(p_i)}\right)\right]} + 1\right]\beta^2}, \tag{4.37}$$

then

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \Phi\left(\frac{\sqrt{1-\beta^{2}}\Phi^{-1}\left(\frac{k}{n}\right) - \overline{\Phi^{-1}(p_i)}}{\beta^*}\right). \tag{4.38}$$

Approx7b: Based on Approx 6b, let

$$\beta^* = \sqrt{\left[\frac{\Phi\left(\overline{\Phi^{-1}(p_i)}\right)\left[1-\Phi\left(\overline{\Phi^{-1}(p_i)}\right)\right]}{n}\frac{1}{\left[\Phi_2\left(\overline{\Phi^{-1}(p_i)},\overline{\Phi^{-1}(p_i)},\beta^2\right) - \Phi^2\left(\overline{\Phi^{-1}(p_i)}\right)\right]} + 1\right]\beta^2}, \tag{4.39}$$

then

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \Phi\left(\frac{\sqrt{1-\beta^{*2}}\Phi^{-1}\left(\frac{k}{n}\right) - \overline{\Phi^{-1}(p_i)}}{\beta^*}\right). \tag{4.40}$$

Chapter 5

Loss Distribution by Student's t Copula

5.1 Introduction

We have the same assumptions for the large pool of credit securities as in Chapter 4,

(1) The portfolio consists of n credit securities in equal dollar amounts and with the same term T.
(2) The correlation coefficient of any two names remains the same.
(3) The recovery rates are the same.
(4) In homogeneous case, the default probabilities of individual names are identical such that $p_i = p_j = p$.

As the tail dependence of the Gaussian copula is zero, it could not explain the dependence of the extreme events such as defaults. We would like to extend the Gaussian copula by using the Student's t copula to model the portfolio loss L. Analogously the

default probability is induced through a latent variable U_i such as

$$P_{\tau_i}(T) = P(U_i < c_i)$$

$$p_i = P(U_i < c_i),$$

where τ_i is the time to default of name i and c_i is the default level corresponding to U_i. We say firm i defaults when

$$U_i < F_{U_i}^{-1}(p_i).$$

In the case of the Student's t, $U_1, \ldots, U_n$ follow the multivariate Student's t distribution with the degrees of freedom $\nu = q * T$. We let U_i be

$$U_i = \frac{\beta X^G + \sqrt{1-\beta^2} X_i^I}{\sqrt{W}},$$

where $i = 1, \ldots, n$. X^G is the common systematic risk factor and has the standard normal distribution. $X_1^I, \ldots, X_n^I$ are the idiosyncratic risk factors and they follow the independent and identical standard normal distribution. We also assume that X^G is independent of X_i^I. β^2 refers to the pairwise correlation coefficient. W has a gamma distribution with the shape parameter $\frac{\nu}{2}$ and the scale parameter $\frac{2}{\nu}$, where $\nu = q * T$ and q is a constant and T is the time interval from the value date to the maturity date. It is equivalent to say that $\frac{1}{W}$ has a inverse gamma distribution with both parameters equal to $\frac{\nu}{2}$. Please refer to Appendix $A.3$ for the proof of the statement above. W is independent of X^G and X_i^I. Please refer to $A.4$ for the proof for that U_i has a Student's t distributed with ν degrees of freedom. For the simple notation, we use ν instead of $q * T$ to denote the degrees of freedom; however q is the exact parameter in our model. We define

$$D_i = \begin{cases} 1 & \text{if firm i defaults, with probability } \tilde{p} \\ 0 & \text{if firm i does not default, with probability } 1 - \tilde{p}. \end{cases}$$

Then the number of defaults in a credit portfolio is given by $D = \sum_{i=1}^{n} D_i$. The portfolio loss is $L = N_{ml} \sum_{i=1}^{n} (1 - R_i) 1_{\{\tau_i \leq T\}}$, where N_{ml} is the nominal amount of the credit portfolio and R_i is the recovery rate of name i. If R_i is the same for all i, the distribution of the number of defaults D is closely related to the distribution of the portfolio loss L. Therefore we have the factor Student's t copula for the pool loss L such that

$$
\begin{aligned}
L &= L(T, U_1, \ldots, U_n) \\
U_i &= \frac{\beta X^G + \sqrt{1 - \beta^2} X_i^I}{\sqrt{W}} \\
W &\sim Gamma(\frac{\nu}{2}, \frac{2}{\nu}) \\
\nu &= q * T.
\end{aligned}
$$

This factor Student's t copula model is an extension of the factor Gaussian copula that we have discussed in Chapter 4. It is an attractive method for the pricing of CDO as we may get more appropriate results by having an extra parameter, the degrees of freedom. We will give the exact loss distribution of a homogeneous large pool of credit securities by using the Student's t copula. We will also derive several asymptotic approximations. We will compare these asymptotic approximations with the exact loss distribution by using the Hellinger distance of their probability density functions.

In the nonhomogeneous case, we use the recursive calculation to induce the exact loss distribution of a large pool of credit securities. It allows different default probabilities of individual names, while keeping the other assumptions of the homogeneous large pool as the same. We also give several asymptotic approximations for the nonhomogeneous large pool of credit securities.

5.2 Model in homogeneous case

We study the exact loss distribution and several asymptotic approximations in the homogeneous case, including:

Exact Model: Factor model with conditional binomial distribution;

Approx2: Approximation as $n \to \infty$;

Approx3: Approximation as $\beta = 0$ and $n \to \infty$;

Approx4: Approximation as $\beta \to 0$ and $n \to \infty$;

Approx5/Approx5a/Approx5b: Alternative approximations as $\beta \to 0$ and $n \to \infty$;

Approx6/Approx6a/Approx6b: Alternative approximations as $\beta \to 0$ and $n \to \infty$ (approximation of bivariate normal);

Approx7a/Approx7b: Alternative approximations as $\beta \to 0$ and $n \to \infty$;

Approx8: Alternative approximation as $\beta \to 0$ and $n \to \infty$.

5.2.1 Factor Model with Conditional Binomial Distribution

In the homogenous case, the distribution of the pool loss is closely related to that of the total number of default, $D = \sum_{i=1}^{n} D_i$, where D_i is an indicator variable for the default of name i. By our early discussion, we know

$$\begin{aligned} P_{\tau_i}(T) &= P\left(U_i < F_{U_i}^{-1}(p)\right) \\ &= P\left(\frac{\beta X^G + \sqrt{1-\beta^2} X_i^I}{\sqrt{W}} < t_\nu^{-1}(p)\right), \end{aligned}$$

where p is the risk-neutral default probability of the individual names over the period of T. In the homogeneous case, we have $p_i = p_j = p$. t denotes the CDF of the Student's t distribution. When W and the common systematic risk factor X^G are fixed,

the conditional default probability of any underlying name is given by

$$\begin{aligned} P_{\tau_i}\left(T\middle|W=w,X^G=x\right) &= P\left(U_i < F_{U_i}^{-1}(p_i)\middle|W=w,X^G=x\right) \\ &= P\left(\frac{\beta x+\sqrt{1-\beta^2}X_i^I}{\sqrt{w}} \le t_\nu^{-1}(p)\right) \\ &= \Phi\left(\frac{w^{\frac{1}{2}}t_\nu^{-1}(p)-\beta x}{\sqrt{1-\beta^2}}\right). \end{aligned}$$

As we assume X^G follows the standard normal distribution, the unconditional probability of the pool loss in terms of D is given by,

$$\begin{aligned} P(D\le k) = \sum_{i=0}^{k}\binom{n}{i}\int_0^{+\infty}\int_{-\infty}^{+\infty}&\left[\Phi\left(\frac{w^{\frac{1}{2}}t_\nu^{-1}(p)-\beta x}{\sqrt{1-\beta^2}}\right)\right]^k \\ &\times\left[1-\Phi\left(\frac{w^{\frac{1}{2}}t_\nu^{-1}(p)-\beta x}{\sqrt{1-\beta^2}}\right)\right]^{n-k} d\Phi(x)dF(w), \end{aligned}$$

where $0\le k\le n$ and $F(w)$ is the cumulative distribution function of W. It is gamma$(\frac{\nu}{2},\frac{2}{\nu})$.

We use the Gaussian-Hermite quadrature (GHQ) method to numerically approximate the integral with inspect to X^G and use the Gaussian-Laguerre quadrature (GLQ) method to numerically approximate the integral with inspect to W. In the homogeneous case, this factor model with conditional binomial distribution provides us the exact loss distribution of the homogeneous pool of credit securities.

5.2.2 Approximation as $n\to\infty$

Conditioning on W and X^G, X_i^I are independent and identical random variables. By the law of large number, the proportion of pool loss $\dfrac{D}{n}$ is conditionally converges to $\mathrm{E}\left(\dfrac{D}{n}\middle|W,X^G\right)$ in probability.

From our early result,

$$\mathrm{E}\left(\frac{D}{n}\middle|W,X^G\right) = \Phi\left(\frac{W^{\frac{1}{2}}t_\nu^{-1}(p)-\beta X^G}{\sqrt{1-\beta^2}}\right), \tag{5.1}$$

then

$$\begin{aligned}
P\left(\frac{D}{n} \leq \frac{k}{n}\middle| W=w\right) &\approx P\left[\mathrm{E}\left(\frac{D}{n}\middle| w, X^G\right) \leq \frac{k}{n}\middle| W=w\right] \\
&\approx P\left[\Phi\left(\frac{w^{\frac{1}{2}}t_\nu^{-1}(p)-\beta X^G}{\sqrt{1-\beta^2}}\right) \leq \frac{k}{n}\right] \\
&\approx P\left[w^{\frac{1}{2}}t_\nu^{-1}(p)-\beta X^G \leq \sqrt{1-\beta^2}\Phi^{-1}\left(\frac{k}{n}\right)\right] \\
&\approx P\left(\left[\beta X^G \leq -w^{\frac{1}{2}}t_\nu^{-1}(p)+\sqrt{1-\beta^2}\Phi^{-1}\left(\frac{k}{n}\right)\right]\right] \\
&\approx P\left[X^G \leq \frac{-w^{\frac{1}{2}}t_\nu^{-1}(p)+\sqrt{1-\beta^2}\Phi^{-1}\left(\frac{k}{n}\right)}{\beta}\right] \\
&\approx \Phi\left(\frac{\sqrt{1-\beta^2}\Phi^{-1}\left(\frac{k}{n}\right)-w^{\frac{1}{2}}t_\nu^{-1}(p)}{\beta}\right).
\end{aligned}$$

Therefore as n goes to infinity, the distribution of proportional pool loss $\dfrac{D}{n}$ is given by

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \int_0^{+\infty} \Phi\left(\frac{\sqrt{1-\beta^2}\Phi^{-1}\left(\frac{k}{n}\right)-w^{\frac{1}{2}}t_\nu^{-1}(p)}{\beta}\right) dF(w).$$

However, this asymptotic approximation will not work when the correlation β^2 goes to zero. One of our interests is to find a more appropriate asymptotic approximation for the loss of the homogeneous large pool of credit securities.

5.2.3 Approximation as $\beta = 0$ and $n \to \infty$

From the factor Student's t copula model, when β is equal to zero, given $W = w$, the portfolio loss D follows the binomial distribution with the default probability of $\Phi\left(w^{\frac{1}{2}}t_\nu^{-1}(p)\right)$ and n underlying credit names. As n goes to infinity, by the central limit theorem, the asymptotic distribution of the proportion pool loss $\dfrac{D}{n}$ is given by

$$\frac{D}{n}\middle| W=w \overset{a}{\sim} N\left(\Phi\left(w^{\frac{1}{2}}t_\nu^{-1}(p)\right), \quad \frac{\Phi\left(w^{\frac{1}{2}}t_\nu^{-1}(p)\right)\left[1-\Phi\left(w^{\frac{1}{2}}t_\nu^{-1}(p)\right)\right]}{n}\right).$$

That is the same to say

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \int_0^{+\infty} \Phi\left(\frac{\frac{k}{n} - \Phi\left(w^{\frac{1}{2}} t_\nu^{-1}(p)\right)}{\sqrt{\frac{\Phi\left(w^{\frac{1}{2}} t_\nu^{-1}(p)\right)\left[1-\Phi\left(w^{\frac{1}{2}} t_\nu^{-1}(p)\right)\right]}{n}}}\right) dF(w).$$

5.2.4 Approximation as $\beta \to 0$ and $n \to \infty$

As the square root of the correlation coefficient, β, approaches zero, we let $\frac{k}{n} = \Phi\left(w^{\frac{1}{2}} t_\nu^{-1}(p)\right) + \lambda\beta$, where $\lambda \in R$. Then from

$$P\left(\frac{D}{n} \leq \frac{k}{n}\middle| W = w\right) \approx \Phi\left(\frac{\sqrt{1-\beta^2}\Phi^{-1}\left(\frac{k}{n}\right) - w^{\frac{1}{2}} t_\nu^{-1}(p)}{\beta}\right),$$

we have

$$P\left(\frac{D}{n} \leq \Phi\left(w^{\frac{1}{2}} t_\nu^{-1}(p)\right) + \lambda\beta\middle| W = w\right) \approx \Phi\left(\frac{\sqrt{1-\beta^2}\Phi^{-1}\left[\Phi\left(w^{\frac{1}{2}} t_\nu^{-1}(p)\right) + \lambda\beta\right] - w^{\frac{1}{2}} t_\nu^{-1}(p)}{\beta}\right).$$

By taking expansion

$$\begin{aligned}\Phi^{-1}\left[\Phi\left(w^{\frac{1}{2}} t_\nu^{-1}(p)\right) + \lambda\beta\right] &= \Phi^{-1}\left[\Phi\left(w^{\frac{1}{2}} t_\nu^{-1}(p)\right)\right] \\ &\quad + \left\{\Phi^{-1}\left[\Phi\left(w^{\frac{1}{2}} t_\nu^{-1}(p)\right)\right]\right\}' \lambda\beta + o(\lambda\beta) \\ &= w^{\frac{1}{2}} t_\nu^{-1}(p) + \frac{\lambda\beta}{\phi\left(w^{\frac{1}{2}} t_\nu^{-1}(p)\right)} + o(\lambda\beta),\end{aligned}$$

Therefore under regularity conditions, as $\beta \to 0$

$$P\left(\frac{D}{n} \leq \Phi\left(w^{\frac{1}{2}} t_\nu^{-1}(p)\right) + \lambda\beta\middle| W = w\right) \to \Phi\left(\frac{\lambda}{\phi\left(w^{\frac{1}{2}} t_\nu^{-1}(p)\right)}\right).$$

Let $\lambda^* = \dfrac{\lambda}{\phi\left(w^{\frac{1}{2}} t_\nu^{-1}(p)\right)}$, then

$$P\left(\frac{D}{n} \leq \Phi\left(w^{\frac{1}{2}} t_\nu^{-1}(p)\right) + \lambda^* \beta \phi\left(w^{\frac{1}{2}} t_\nu^{-1}(p)\right) \middle| W = w\right) \to \Phi(\lambda^*)$$

$$P\left(\frac{\frac{D}{n} - \Phi\left(w^{\frac{1}{2}} t_\nu^{-1}(p)\right)}{\phi\left(w^{\frac{1}{2}} t_\nu^{-1}(p)\right)\beta} \leq \lambda^* \middle| W = w\right) \to \Phi(\lambda^*).$$

Hence we say as β goes to zero, given $W = w$, the proportional pool loss $\dfrac{D}{n}$ has a conditional asymptotic normal distribution with mean $\Phi\left(w^{\frac{1}{2}} t_\nu^{-1}(p)\right)$ and variance $\left[\phi\left(w^{\frac{1}{2}} t_\nu^{-1}(p)\right)\beta\right]^2$ such that

$$\frac{D}{n}\middle| W = w \overset{a}{\sim} N\left(\Phi\left(w^{\frac{1}{2}} t_\nu^{-1}(p)\right), \quad \left[\phi\left(w^{\frac{1}{2}} t_\nu^{-1}(p)\right)\beta\right]^2\right).$$

That is the same to say

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \int_0^{+\infty} \Phi\left(\frac{\frac{k}{n} - \Phi\left(w^{\frac{1}{2}} t_\nu^{-1}(p)\right)}{\phi\left(w^{\frac{1}{2}} t_\nu^{-1}(p)\right)\beta}\right) dF(w).$$

5.2.5 Alternative Approximations as $\beta \to 0$ and $n \to \infty$

From the factor model with conditional binomial distribution, we know the conditional expected value of D_i and the conditional variance of D_i.

$$\mathrm{E}\left(D_i \middle| W = w, X^G = x\right) = \Phi\left(\frac{w^{\frac{1}{2}} t_\nu^{-1}(p) - \beta x}{\sqrt{1-\beta^2}}\right)$$

$$\mathrm{Var}\left(D_i \middle| W = w, X^G = x\right) = \Phi\left(\frac{w^{\frac{1}{2}} t_\nu^{-1}(p) - \beta x}{\sqrt{1-\beta^2}}\right)\left[1 - \Phi\left(\frac{w^{\frac{1}{2}} t_\nu^{-1}(p) - \beta x}{\sqrt{1-\beta^2}}\right)\right],$$

where D_i is the indicator variable for the default of firm i and $i = 1, \ldots, n$.

By the law of iterated expectations, we have

$$\begin{aligned}
\mathrm{E}\left(D_i\middle|W=w\right) &= \mathrm{E}_{X^G}\left[\mathrm{E}\left(D_i\middle|W,X^G\right)\right] \\
&= \mathrm{E}_{X^G}\left[\Phi\left(\frac{w^{\frac{1}{2}}t_\nu^{-1}(p)-\beta X^G}{\sqrt{1-\beta^2}}\right)\right] \\
&= \int_{-\infty}^{+\infty}\Phi\left(\frac{w^{\frac{1}{2}}t_\nu^{-1}(p)-\beta x}{\sqrt{1-\beta^2}}\right)d\Phi(x) \\
&= \int_{-\infty}^{t_\nu^{-1}(p)}\frac{w^{\frac{1}{2}}}{\sqrt{2\pi}}e^{-\frac{w^{-1}y^2}{2}}dy \\
&= \Phi\left(w^{\frac{1}{2}}t_\nu^{-1}(p)\right),
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{E}\left(D_i^2\middle|W=w\right) &= \mathrm{E}_{X^G}\left[\mathrm{E}\left(D_i^2\middle|W,X^G\right)\right] \\
&= \mathrm{E}_{X^G}\left[\Phi\left(\frac{w^{\frac{1}{2}}t_\nu^{-1}(p)-\beta X^G}{\sqrt{1-\beta^2}}\right)\Phi\left(\frac{w^{\frac{1}{2}}t_\nu^{-1}(p)-\beta X^G}{\sqrt{1-\beta^2}}\right)\right] \\
&= \int_{-\infty}^{t_\nu^{-1}(p)}\int_{-\infty}^{t_\nu^{-1}(p)}\frac{w^{-1}}{\sqrt{2\pi}\sqrt{1-\beta^2}\sqrt{1+\beta^2}}e^{-\frac{wy_1+wy_2-2\beta^2wy_1y_2}{2(1-\beta^2)(1+\beta^2)}}dy_1dy_2 \\
&= \Phi_2\left(w^{\frac{1}{2}}t_\nu^{-1}(p),w^{\frac{1}{2}}t_\nu^{-1}(p),\beta^2\right),
\end{aligned}$$

where $\Phi_2\left(w^{\frac{1}{2}}t_\nu^{-1}(p),w^{\frac{1}{2}}t_\nu^{-1}(p),\beta^2\right)$ is a bivariate normal cumulative distribution function with

$$\mu=\begin{pmatrix}0\\0\end{pmatrix}$$

and

$$\Sigma=\begin{pmatrix}1 & \beta^2\\ \beta^2 & 1\end{pmatrix}$$

Please refer to Appendix $A.5$ and $A.6$ for the detail proofs. For the total number of

defaults $D = D_1 + \ldots + D_n$ by definition, we have

$$\begin{aligned}
\mathrm{E}(D) &= n\mathrm{E}(D_i) \\
&= n\mathrm{E}(D_i|W = w) \\
&= n\Phi\left(w^{\frac{1}{2}} t_\nu^{-1}(p)\right).
\end{aligned}$$

Let $a = \mathrm{E}(D_i|W = w) = \Phi\left(w^{\frac{1}{2}} t_\nu^{-1}(p)\right)$. Then we have $\mathrm{E}(D) = na$. And

$$\begin{aligned}
\mathrm{Var}(D|W = w) &= \mathrm{E}_{X^G}\left[\mathrm{Var}\left(D\middle|W, X^G\right)\right] + \mathrm{Var}_{X^G}\left[\mathrm{E}\left(D\middle|W, X^G\right)\right] \\
&= n\mathrm{E}_{X^G}\left\{\Phi\left(\frac{w^{\frac{1}{2}} t_\nu^{-1}(p) - \beta X^G}{\sqrt{1-\beta^2}}\right)\left[1 - \Phi\left(\frac{w^{\frac{1}{2}} t_\nu^{-1}(p) - \beta X^G}{\sqrt{1-\beta^2}}\right)\right]\right\} \\
&\quad + n^2\mathrm{E}_{X^G}\left[\Phi^2\left(\frac{w^{\frac{1}{2}} t_\nu^{-1}(p) - \beta X^G}{\sqrt{1-\beta^2}}\right)\right] - n^2\left[\Phi\left(w^{\frac{1}{2}} t_\nu^{-1}(p)\right)\right]^2 \\
&= n\Phi\left(w^{\frac{1}{2}} t_\nu^{-1}(p)\right) - n^2\left[\Phi\left(w^{\frac{1}{2}} t_\nu^{-1}(p)\right)\right]^2 \\
&\quad + (n^2 - n)\Phi_2\left(w^{\frac{1}{2}} t_\nu^{-1}(p), w^{\frac{1}{2}} t_\nu^{-1}(p), \beta^2\right) \\
&= na - n^2a^2 - na^2 + na^2 + (n^2 - n)\Phi_2\left(\Phi^{-1}(a), \Phi^{-1}(a), \beta^2\right) \\
&= na(1-a) + (n^2 - n)\left[\Phi_2\left(\Phi^{-1}(a), \Phi^{-1}(a), \beta^2\right) - a^2\right].
\end{aligned}$$

Thus

$$\mathrm{Var}\left(\frac{D}{n}\middle|W = w\right) = \frac{a(1-a)}{n} + \left(1 - \frac{1}{n}\right)\left[\Phi_2\left(\Phi^{-1}(a), \Phi^{-1}(a), \beta^2\right) - a^2\right] \tag{5.2}$$

$$\approx \frac{a(1-a)}{n} + \Phi_2\left(\Phi^{-1}(a), \Phi^{-1}(a), \beta^2\right) - a^2. \tag{5.3}$$

Hence

$$P\left(\frac{D}{n} \le \frac{k}{n}\middle|W = w\right) \approx \Phi\left(\frac{\frac{k}{n} - a}{\sqrt{\frac{a(1-a)}{n} + \Phi_2\left(\Phi^{-1}(a), \Phi^{-1}(a), \beta^2\right) - a^2}}\right).$$

We make some approximation of the cumulative distribution function of the bivariate

normal.

$$\Phi_2(x, y, \beta^2) \approx \Phi(x)\Phi(y) + \beta^2\phi(x)\phi(y)$$

$$\Phi_2\left(w^{\frac{1}{2}}t_\nu^{-1}(p), w^{\frac{1}{2}}t_\nu^{-1}(p), \beta^2\right) \approx \left[\Phi\left(w^{\frac{1}{2}}t_\nu^{-1}(p)\right)\right]^2 + \left[\phi\left(w^{\frac{1}{2}}t_\nu^{-1}(p)\right)\beta\right]^2$$

$$\approx a^2 + \left[\phi\left(\Phi^{-1}(a)\right)\beta\right]^2.$$

Thus

$$\mathrm{Var}\left(D\middle|W = w\right) = na(1-a) + (n^2 - n)\left[\Phi_2\left(\Phi^{-1}(a), \Phi^{-1}(a), \beta^2\right) - a^2\right]$$

$$\overset{\beta\to 0}{\approx} na(1-a) + (n^2-n)\left[\phi\left(\Phi^{-1}(a)\right)\beta\right]^2.$$

And

$$\mathrm{Var}\left(\frac{D}{n}\middle|W = w\right) \approx \frac{a(1-a)}{n} + \left(1 - \frac{1}{n}\right)\beta^2\left[\phi\left(\Phi^{-1}(a)\right)\right]^2 \tag{5.4}$$

$$\approx \frac{a(1-a)}{n} + \left[\phi\left(\Phi^{-1}(a)\right)\beta\right]^2. \tag{5.5}$$

When both n goes to infinity and β goes to zero, we say conditioning on W, the asymptotic variance of the proportional pool loss consists of two components. One is the asymptotic variance from the binomial distribution for n approaching infinity, and the other is the asymptotic variance derived from Approx4. In the proof of Approx4, we have showed that as $\beta \to 0$,

$$P\left(\frac{D}{n} \le \frac{k}{n}\middle|W = w\right) \approx \Phi\left(\frac{\sqrt{1-\beta^2}\Phi^{-1}\left(\frac{k}{n}\right) - w^{\frac{1}{2}}t_\nu^{-1}(p)}{\beta}\right)$$

implies

$$\frac{D}{n}\middle|W = w \overset{a}{\sim} N\left(\Phi\left(w^{\frac{1}{2}}t_\nu^{-1}(p)\right), \quad \left[\phi\left(w^{\frac{1}{2}}t_\nu^{-1}(p)\right)\beta\right]^2\right).$$

By equating the variance from Equation (5.3) and the variance of Approx4,

$$\frac{a(1-a)}{n} + \Phi_2\left(\Phi^{-1}(a), \Phi^{-1}(a), \beta^2\right) - a^2 = \left[\phi\left(w^{\frac{1}{2}}t_\nu^{-1}\right)\beta^*\right]^2$$

$$= \left[\phi\left[\Phi^{-1}(a)\right)\beta^*\right]^2.$$

Then

$$\beta^* = \sqrt{\frac{a(1-a)}{n}\frac{1}{\left[\phi\left(\Phi^{-1}(a)\right)\right]^2} + \left[\Phi_2\left(\Phi^{-1}(a), \Phi^{-1}(a), \beta^2\right) - a^2\right]\frac{1}{\left[\phi\left(\Phi^{-1}(a)\right)\right]^2}}.$$

Therefore we have

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \int_0^{+\infty} \Phi\left(\frac{\sqrt{1-\beta^2}\Phi^{-1}\left(\frac{k}{n}\right) - w^{\frac{1}{2}}t_\nu^{-1}(p)}{\beta^*}\right) dF(w).$$

By equating the variance from Equation (5.5) and the variance of Approx4,

$$\begin{aligned}\frac{a(1-a)}{n} + \left[\phi\left(\Phi^{-1}(a)\right)\beta\right]^2 &= \left[\phi\left(w^{\frac{1}{2}}t_\nu^{-1}\right)\beta^*\right]^2 \\ &= \left[\left(\phi\left(\Phi^{-1}(a)\right)\beta^*\right)\right]^2.\end{aligned}$$

Then

$$\beta^* = \sqrt{\frac{a(1-a)}{n}\frac{1}{\left[\phi\left(\Phi^{-1}(a)\right)\right]^2} + \beta^2}.$$

Therefore we have

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \int_0^{+\infty} \Phi\left(\frac{\sqrt{1-\beta^2}\Phi^{-1}\left(\frac{k}{n}\right) - w^{\frac{1}{2}}t_\nu^{-1}(p)}{\beta^*}\right) dF(w).$$

If we take the full expectation of D_i, then

$$\begin{aligned}\mathrm{E}(D_i) &= \mathrm{E}_{W,X^G}\left[\mathrm{E}\left(D_i \middle| W, X^G\right)\right] \\ &= \mathrm{E}_{W,X^G}\left[\Phi\left(\frac{W^{-\frac{1}{2}}t_\nu^{-1}(p) - \beta X^G}{\sqrt{1-\beta^2}}\right)\right] \\ &= \int_0^{+\infty}\int_{-\infty}^{+\infty} \Phi\left(\frac{w^{\frac{1}{2}}t_\nu^{-1}(p) - \beta x}{\sqrt{1-\beta^2}}\right) d\Phi(x) dF(w) \\ &= \int_{-\infty}^{t_\nu^{-1}(p)} \frac{\left(\frac{\nu}{2}\right)^{\frac{\nu}{2}}}{\sqrt{2\pi}\Gamma\left(\frac{\nu}{2}\right)} + \frac{\Gamma\left(\frac{\nu}{2}+\frac{1}{2}\right)}{\left(\frac{y^2}{2}+\frac{\nu}{2}\right)^{\frac{\nu+\frac{1}{2}}{2}}} dy \\ &= t_\nu\left(t_\nu^{-1}(p)\right) \\ &= p.\end{aligned}$$

And

$$\begin{aligned}
\mathrm{E}(D_i^2) &= \mathrm{E}_{W,X^G}\left[\mathrm{E}\left(D_i^2\middle|W,X^G\right)\right] \\
&= \mathrm{E}_{W,X^G}\left[\Phi\left(\frac{W^{\frac{1}{2}}t_\nu^{-1}(p)-\beta X^G}{\sqrt{1-\beta^2}}\right)\Phi\left(\frac{W^{\frac{1}{2}}t_\nu^{-1}(p)-\beta X^G}{\sqrt{1-\beta^2}}\right)\right] \\
&= \int_{-\infty}^{t_\nu^{-1}(p)}\int_{-\infty}^{t_\nu^{-1}(p)}\frac{\Gamma\left(\frac{\nu}{2}+1\right)}{\nu\pi\sqrt{1-\beta^4}\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{y_1^2+y_2^2}{(1-\beta^4)\nu}\right)^{-\frac{\nu}{2}-1}dy_1dy_2 \\
&= t_{2,\nu}\left(t_\nu^{-1}(p),t_\nu^{-1}(p),\beta^2\right),
\end{aligned}$$

where $t_{2,\nu}\left(t_\nu^{-1}(p),t_\nu^{-1}(p),\beta^2\right)$ is the cumulative distribution function of a bivariate Student's t with $df=\nu$,

$$\mu=\begin{pmatrix}0\\0\end{pmatrix}$$

and

$$\Sigma=\begin{pmatrix}1 & \beta^2\\ \beta^2 & 1\end{pmatrix}$$

Please refer to Appendix $A.7$ and $A.8$ for the detail proofs. Then

$$\begin{aligned}
\mathrm{Var}(D) &= \mathrm{E}_{W,X^G}\left[\mathrm{Var}\left(L\middle|W,X^G\right)\right]+\mathrm{Var}_{W,X^G}\left[\mathrm{E}\left(L\middle|W,X^G\right)\right] \\
&= n\mathrm{E}_{W,X^G}\left\{\Phi\left(\frac{W^{\frac{1}{2}}t_\nu^{-1}(p)-\beta X^G}{\sqrt{1-\beta^2}}\right)\left[1-\Phi\left(\frac{W^{\frac{1}{2}}t_\nu^{-1}(p)-\beta X^G}{\sqrt{1-\beta^2}}\right)\right]\right\} \\
&\quad + n^2\mathrm{E}_{W,X^G}\left\{\left[\Phi\left(\frac{W^{\frac{1}{2}}t_\nu^{-1}(p)-\beta X^G}{\sqrt{1-\beta^2}}\right)\right]^2\right\}-n^2p^2 \\
&= np-n^2p^2+(n^2-n)t_{2,\nu}\left(t_\nu^{-1}(p),t_\nu^{-1}(p),\beta^2\right) \\
&= np-n^2p^2-np^2+np^2+(n^2-n)t_{2,\nu}\left(t_\nu^{-1}(p),t_\nu^{-1}(p),\beta^2\right) \\
&= np(1-p)+(n^2-n)\left[t_{2,\nu}\left(t_\nu^{-1}(p),t_\nu^{-1}(p),\beta^2\right)-p^2\right].
\end{aligned}$$

And

$$\begin{aligned}
\mathrm{Var}\left(\frac{D}{n}\right) &= \frac{p(1-p)}{n}+\left(1-\frac{1}{n}\right)\left[t_{2,\nu}\left(t_\nu^{-1}(p),t_\nu^{-1}(p),\beta^2\right)-p^2\right] \\
&\approx \frac{p(1-p)}{n}+t_{2,\nu}\left(t_\nu^{-1}(p),t_\nu^{-1}(p),\beta^2\right)-p^2.
\end{aligned}$$

Hence

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \Phi\left(\frac{\frac{k}{n} - p}{\sqrt{\frac{p(1-p)}{n} + t_{2,\nu}\left(t_\nu^{-1}(p), t_\nu^{-1}(p), \beta^2\right) - p^2}}\right).$$

5.2.6 Summary

Exact Model: Factor model with conditional binomial distribution

$$\begin{aligned} P(D \leq k) = \sum_{i=0}^{k} \binom{n}{i} \int_0^{+\infty} \int_{-\infty}^{+\infty} \left[\Phi\left(\frac{w^{\frac{1}{2}} t_\nu^{-1}(p) - \beta x}{\sqrt{1-\beta^2}}\right)\right]^k \\ \times \left[1 - \Phi\left(\frac{w^{\frac{1}{2}} t_\nu^{-1}(p) - \beta x}{\sqrt{1-\beta^2}}\right)\right]^{n-k} d\Phi(x) dF(w). \end{aligned} \tag{5.6}$$

Approx2: Approximation as $n \to \infty$

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \int_0^{+\infty} \Phi\left(\frac{\sqrt{1-\beta^2}\Phi^{-1}\left(\frac{k}{n}\right) - w^{\frac{1}{2}} t_\nu^{-1}(p)}{\beta}\right) dF(w). \tag{5.7}$$

Approx3: Binomial as $\beta = 0$ and normal approximation as $n \to \infty$

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \int_0^{+\infty} \Phi\left(\frac{\frac{k}{n} - \Phi\left(w^{\frac{1}{2}} t_\nu^{-1}(p)\right)}{\sqrt{\frac{\Phi\left(w^{\frac{1}{2}} t_\nu^{-1}(p)\right)\left[1 - \Phi(w^{\frac{1}{2}} t_\nu^{-1}(p))\right]}{n}}}\right) dF(w). \tag{5.8}$$

Approx4: Approximation as $\beta \to 0$ and $n \to \infty$

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \int_0^{+\infty} \Phi\left(\frac{\frac{k}{n} - \Phi\left(w^{\frac{1}{2}} t_\nu^{-1}(p)\right)}{\phi\left(w^{\frac{1}{2}} t_\nu^{-1}(p)\right)\beta}\right) dF(w). \tag{5.9}$$

Approx5: Approximation as $\beta \to 0$ and $n \to \infty$

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \int_0^{+\infty} \Phi\left(\frac{\frac{k}{n} - a}{\sqrt{\frac{a(1-a)}{n} + \Phi_2\left(\Phi^{-1}(a), \Phi^{-1}(a), \beta^2\right) - a^2}}\right) dF(w), \tag{5.10}$$

where $a = \Phi\left(w^{\frac{1}{2}} t_\nu^{-1}(p)\right)$.

Approx5a: Based on Approx2 and Approx5

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \int_0^{+\infty} \Phi\left(\frac{\sqrt{1-\beta^2}\Phi^{-1}\left(\frac{k}{n}\right) - w^{\frac{1}{2}}t_\nu^{-1}(p)}{\beta^*}\right) dF(w), \tag{5.11}$$

Approx5b: Based on Approx2 and Approx5

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \int_0^{+\infty} \Phi\left(\frac{\sqrt{1-\beta^{*2}}\Phi^{-1}\left(\frac{k}{n}\right) - w^{\frac{1}{2}}t_\nu^{-1}(p)}{\beta^*}\right) dF(w), \tag{5.12}$$

where

$$\beta^* = \sqrt{\frac{a(1-a)}{n}\frac{1}{\left[\phi(\Phi^{-1}(a))\right]^2} + \left[\Phi_2\left(\Phi^{-1}(a), \Phi^{-1}(a), \beta^2\right) - a^2\right]\frac{1}{\left[\phi(\Phi^{-1}(a))\right]^2}}.$$

Approx6: Approximation as $\beta \to 0$ and $n \to \infty$ (with approximation of bivariate normal)

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \int_0^{+\infty} \Phi\left(\frac{\frac{k}{n} - a}{\sqrt{\frac{a(1-a)}{n} + \left[\phi(\Phi^{-1}(a)\beta)\right]^2}}\right) dF(w), \tag{5.13}$$

where $a = \Phi\left(w^{\frac{1}{2}}t_\nu^{-1}(p)\right)$.

Approx6a: Based on Approx2 and Approx6

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \int_0^{+\infty} \Phi\left(\frac{\sqrt{1-\beta^2}\Phi^{-1}\left(\frac{k}{n}\right) - w^{\frac{1}{2}}t_\nu^{-1}(p)}{\beta^{**}}\right) dF(w), \tag{5.14}$$

Approx6b: Based on Approx2 and Approx6

$$Pr\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \int_0^{+\infty} \Phi\left(\frac{\sqrt{1-\beta^{**2}}\Phi^{-1}\left(\frac{k}{n}\right) - w^{\frac{1}{2}}t_\nu^{-1}(p)}{\beta^{**}}\right) dF(w), \tag{5.15}$$

where

$$\beta^{**} = \sqrt{\frac{a(1-a)}{n}\frac{1}{\left[\phi(\Phi^{-1}(a))\right]^2} + \beta^2}.$$

Approx7a: Based on Approx6a

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \int_0^{+\infty} \Phi\left(\frac{\sqrt{1-\beta^2}\Phi^{-1}\left(\frac{k}{n}\right) - w^{\frac{1}{2}} t_\nu^{-1}(p)}{\beta^{***}}\right) dF(w), \tag{5.16}$$

Approx7b: Based on Approx6b

$$Pr\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \int_0^{+\infty} \Phi\left(\frac{\sqrt{1-\beta^{***2}}\Phi^{-1}\left(\frac{k}{n}\right) - w^{\frac{1}{2}} t_\nu^{-1}(p)}{\beta^{***}}\right) dF(w), \tag{5.17}$$

where

$$\beta^{***} = \sqrt{\left(\frac{a(1-a)}{n}\frac{1}{\left[\Phi_2\left(\Phi^{-1}(a), \Phi^{-1}(a), \beta^2\right) - a^2\right]} + 1\right)\beta^2}.$$

Approx8: Approximation as $\beta \to 0$ and $n \to \infty$

$$P\left(\frac{D}{n} \leq \frac{k}{n}\right) \approx \Phi\left(\frac{\frac{k}{n} - p}{\sqrt{\frac{p(1-p)}{n} + t_{2,\nu}\left(t_\nu^{-1}(p), t_\nu^{-1}(p), \beta^2\right) - p^2}}\right). \tag{5.18}$$

5.2.7 Comparison of Exact Distribution and Asymptotic Approximations

First we would like to study the properties of each asymptotic approximation that we have derived from the Student's t copula in the homogeneous case. We treat the factor model with conditional binomial distribution as the exact model for the portfolio loss distribution. In order to compare each asymptotic approximation with the exact loss distribution, we use the Hellinger distance of their probability density functions. The comparison is also demonstrated by their loss distribution curves. In Chapter 6, we will evaluate the market data by using the Student's t copula.

Hellinger Distance of Loss Distributions

Table 5.1 and Table 5.2 give the Hellinger distance between each asymptotic approximation to the exact loss distribution at different levels of correlation coefficient β^2. We choose $a = \frac{1}{2}$ for continuity correction.

Table 5.1: Hellinger Distance of Loss Distributions at Different Level of Correlation Coefficient by Student's t Copula–Continuity Correction $a = \frac{1}{2}$ (part1)

Approx.#	$\beta^2 = 0.0001$	$\beta^2 = 0.1$	$\beta^2 = 0.2$	$\beta^2 = 0.3$	$\beta^2 = 0.4$
2	0.098458	0.003096	0.001742	0.003062	0.003182
3	0.013683	0.021599	0.053103	0.099678	0.153056
4	0.098457	0.008843	0.017134	0.021950	0.026913
5	0.013754	0.015659	0.024266	0.029665	0.030414
5a	0.003021	0.003484	0.006987	0.008273	0.009937
5b	NA	NA	NA	NA	NA
6	0.013754	0.012940	0.019773	0.023341	0.026473
6a	0.003021	0.001973	0.003610	0.003807	0.003610
6b	NA	NA	NA	NA	NA
7a	0.003020	0.001590	0.003049	0.003499	0.003380
7b	NA	0.006533	0.007722	0.006836	0.005614
8	0.017186	0.018646	0.027625	0.032513	0.032713

Table 5.2: Hellinger Distance of Loss Distributions at Different Level of Correlation Coefficient by Student's t Copula–Continuity Correction $a = \frac{1}{2}$ (part2)

Approx.#	$\beta^2 = 0.5$	$\beta^2 = 0.6$	$\beta^2 = 0.7$	$\beta^2 = 0.8$	$\beta^2 = 0.9$
2	0.003043	0.002762	0.002259	0.001549	0.000744
3	0.209390	0.267150	0.326052	0.387042	0.453939
4	0.037314	0.053455	0.073676	0.097213	0.125572
5	0.031214	0.036995	0.048882	0.065169	0.084954
5a	0.010811	0.010805	0.015431	0.032883	0.062679
5b	NA	NA	NA	NA	NA
6	0.035571	0.051253	0.071392	0.094939	0.123272
6a	0.003382	0.002988	0.002353	0.001540	0.000693
6b	NA	NA	NA	NA	NA
7a	0.003179	0.002842	0.002288	0.001545	0.000730
7b	0.004700	0.003951	0.003143	0.002212	0.001240
8	0.032957	0.038200	0.049727	0.065824	0.085486

Loss Distribution Curves

The following figures demonstrate the probability density functions and the cumulative distribution functions of Approx2, Approx5, Approx5a, Approx6, Approx6a, Approx7a and Approx8 compared with those of the exact model. In order to improve the visualization of the graphs, we take the natural log transformation of the probability densities. The graphs on the right hand side are the normal quantile-quantile (Q-Q) plots of the number of defaults. We can say that when the correlation coefficient β^2 is small, Approx6a and Approx7a are superior to the other approximations. They overcome the difficulty of Approx2 that has β in the denominator of the conditional asymptotic variance. When the correlation coefficient β^2 is not small, the performances of Approx6a and Approx7a are comparable to that of Approx2. All of them are close to the exact loss distribution. Comparatively, Approx7a is a little better than Approx7a. Please refer to Appendix $B.2$ for the detail of the numerical computation in R.

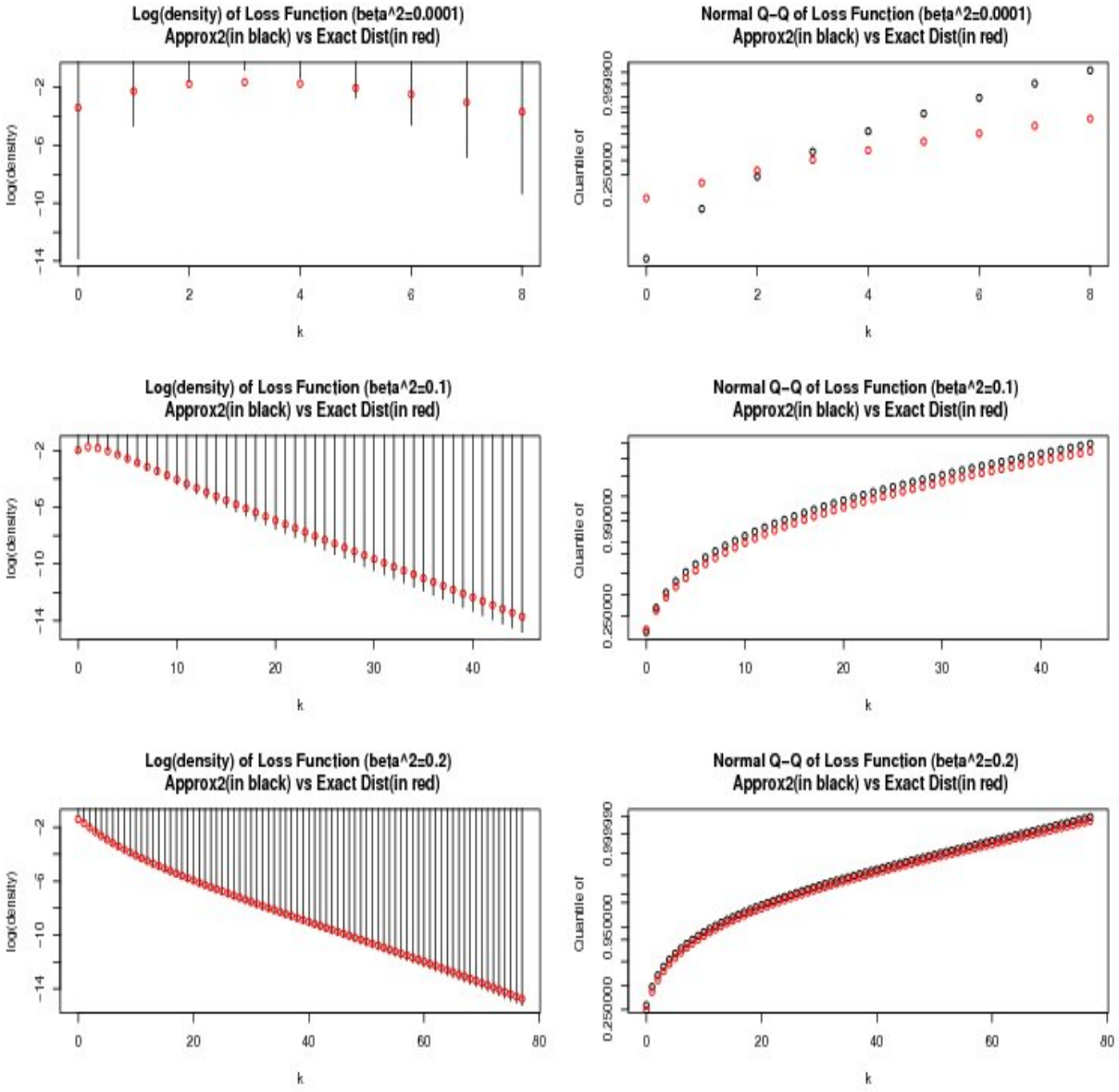

Figure 5.1: Loss Distribution by Student's t Copula–Approx2 vs. Exact Model (part1)

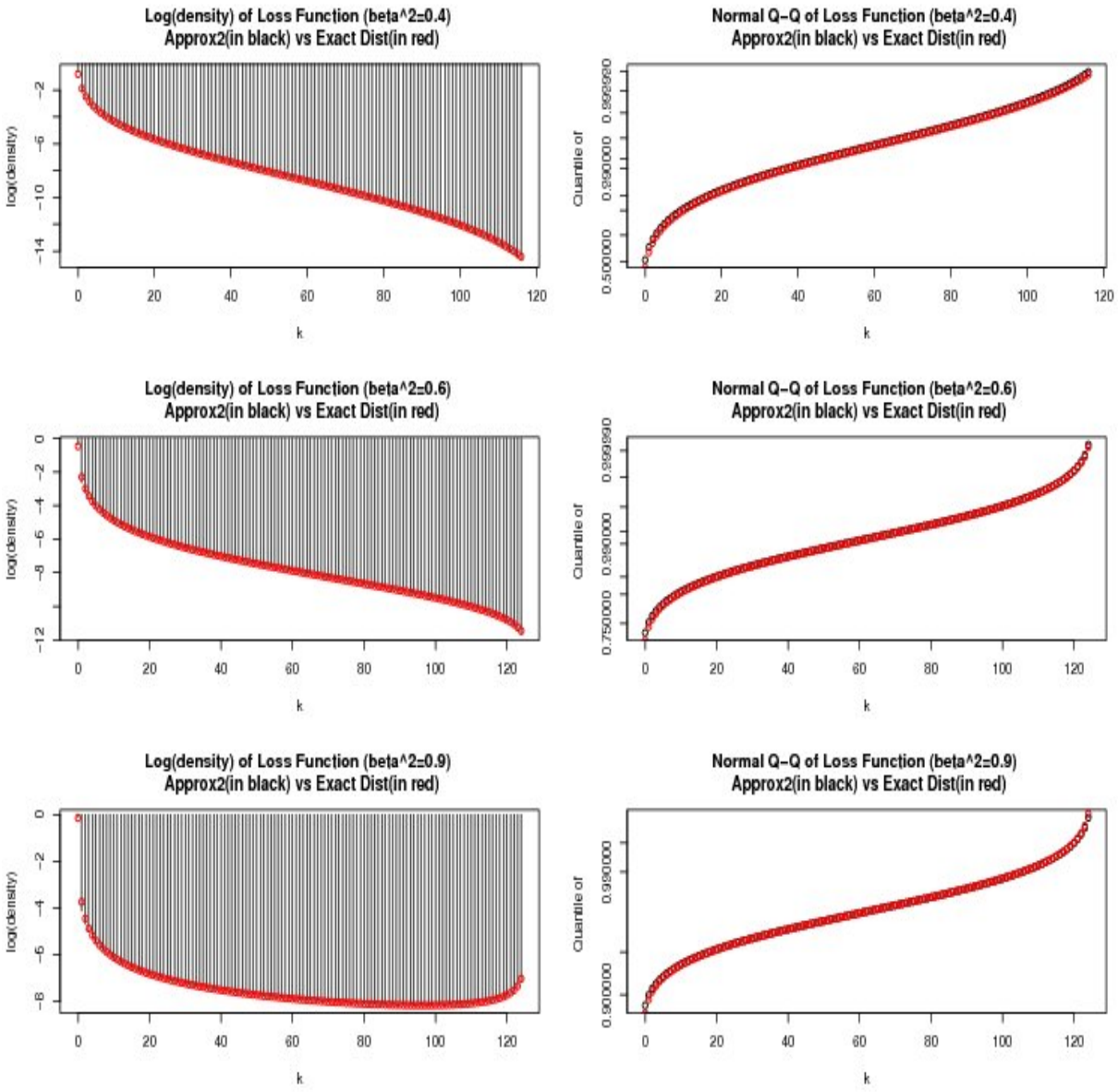

Figure 5.2: Loss Distribution by Student's t Copula–Approx2 vs. Exact Model (part2)

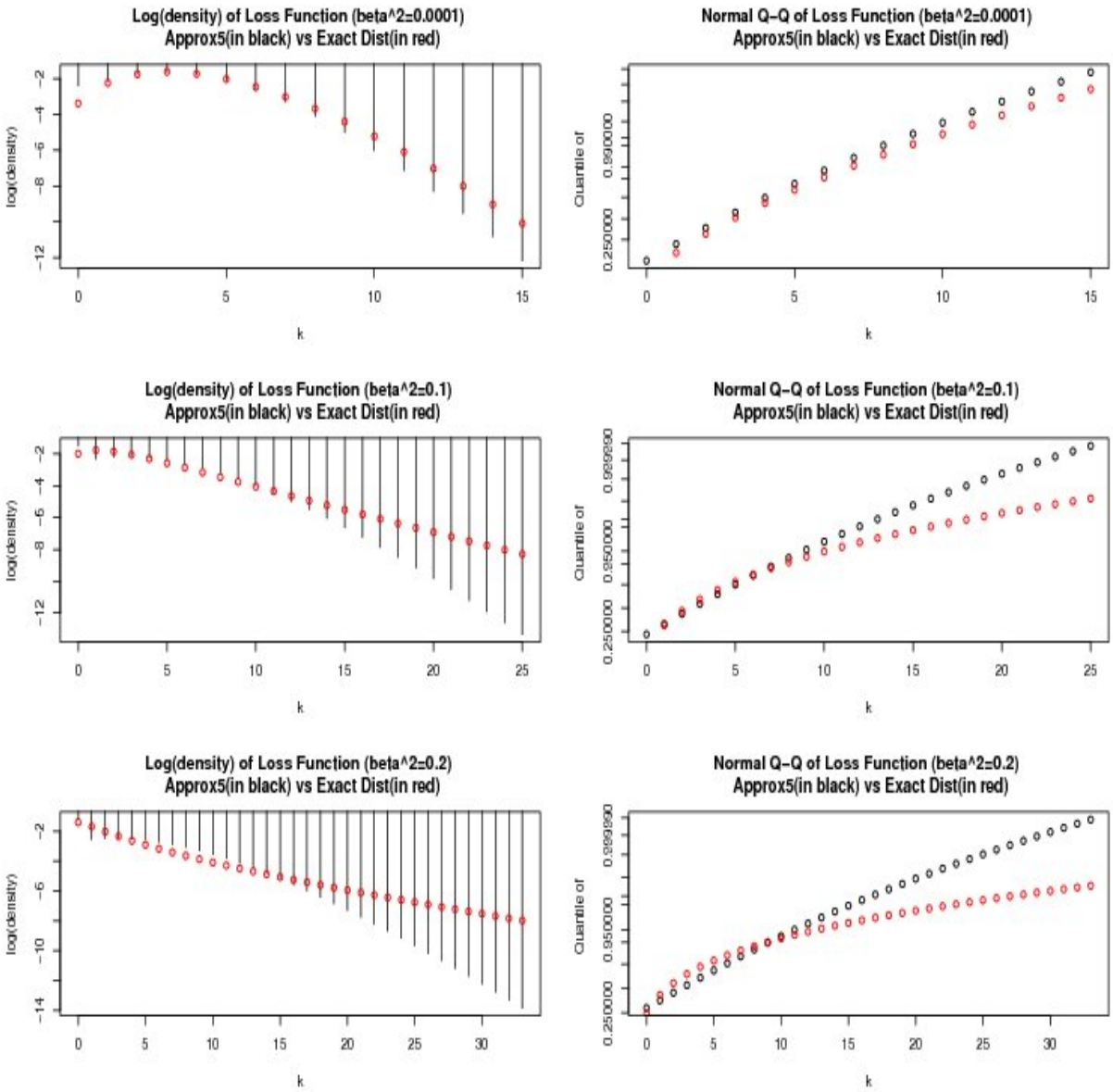

Figure 5.3: Loss Distribution by Student's t Copula–Approx5 vs. Exact Model (part1)

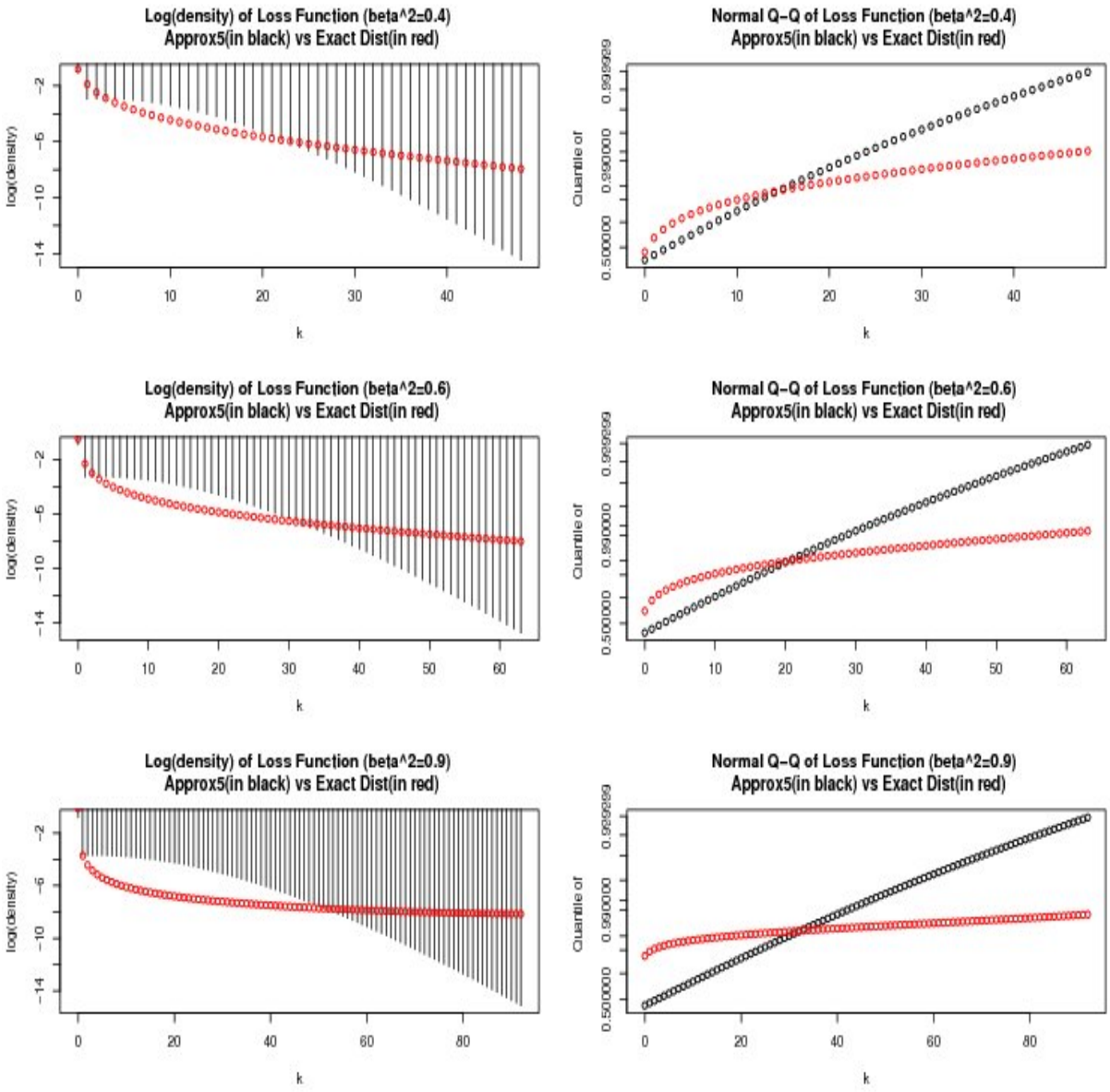

Figure 5.4: Loss Distribution by Student's t Copula–Approx5 vs. Exact Model (part2)

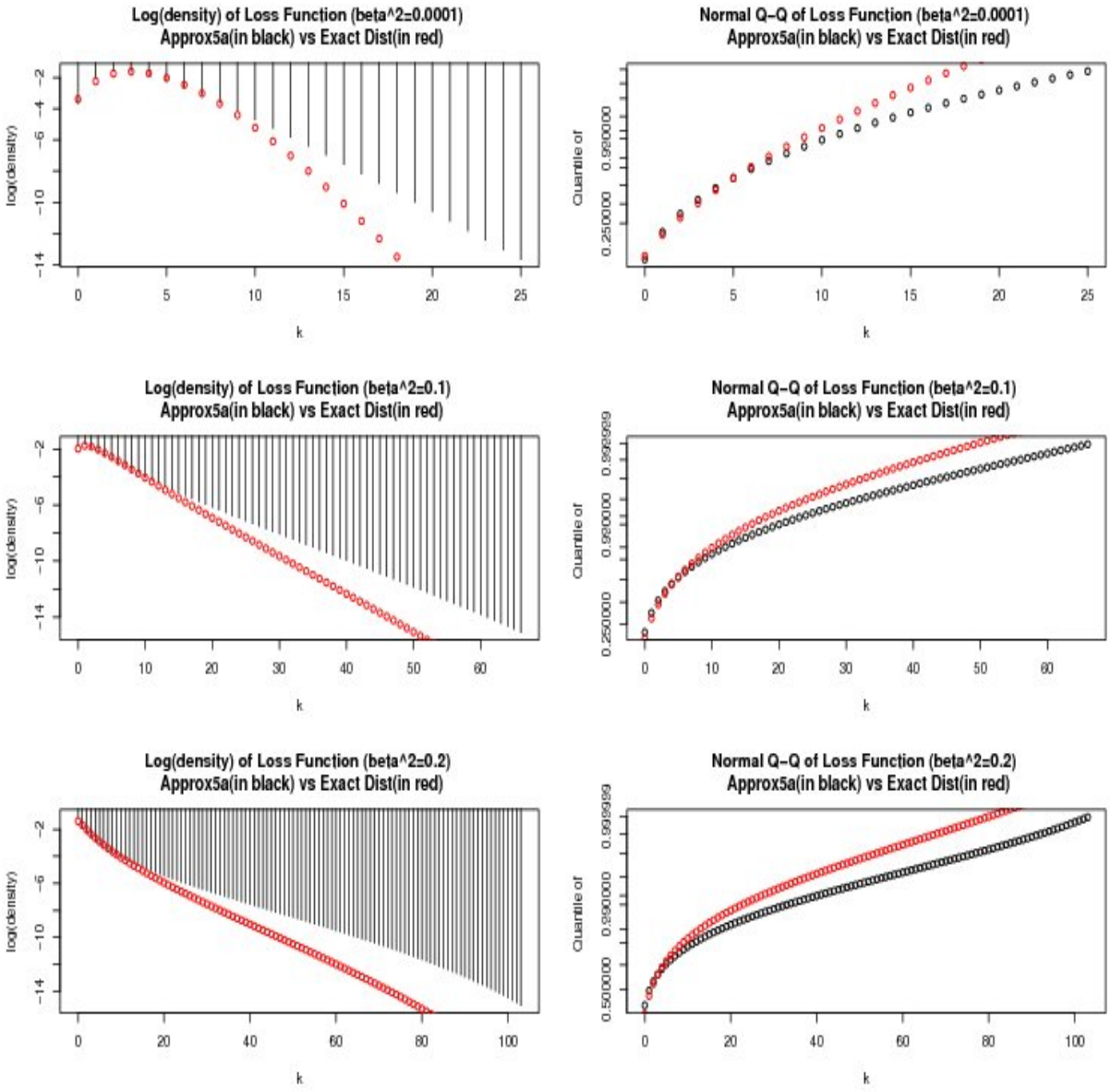

Figure 5.5: Loss Distribution by Student's t Copula–Approx5a vs. Exact Model (part1)

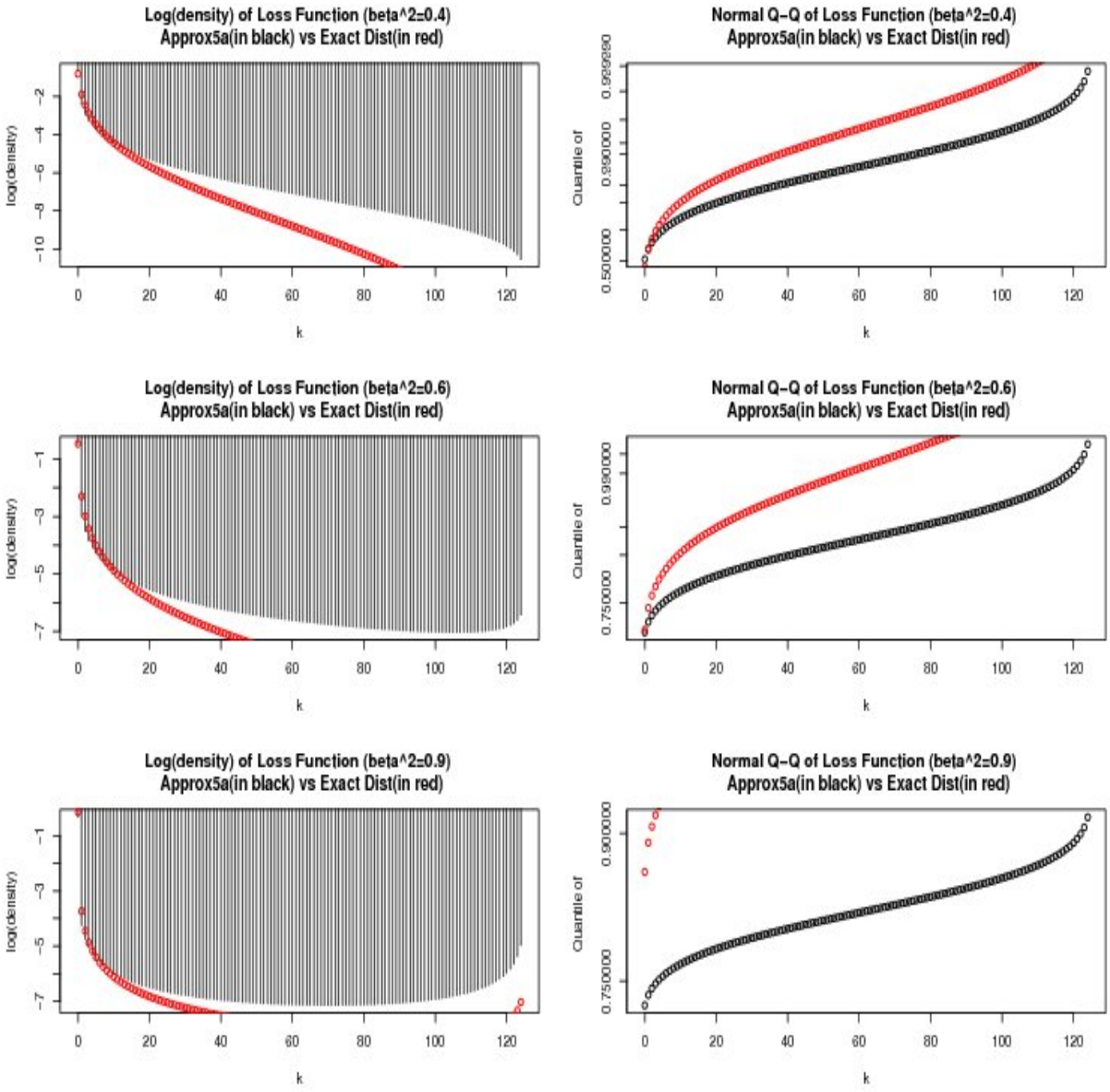

Figure 5.6: Loss Distribution by Student's t Copula–Approx5a vs. Exact Model (part2)

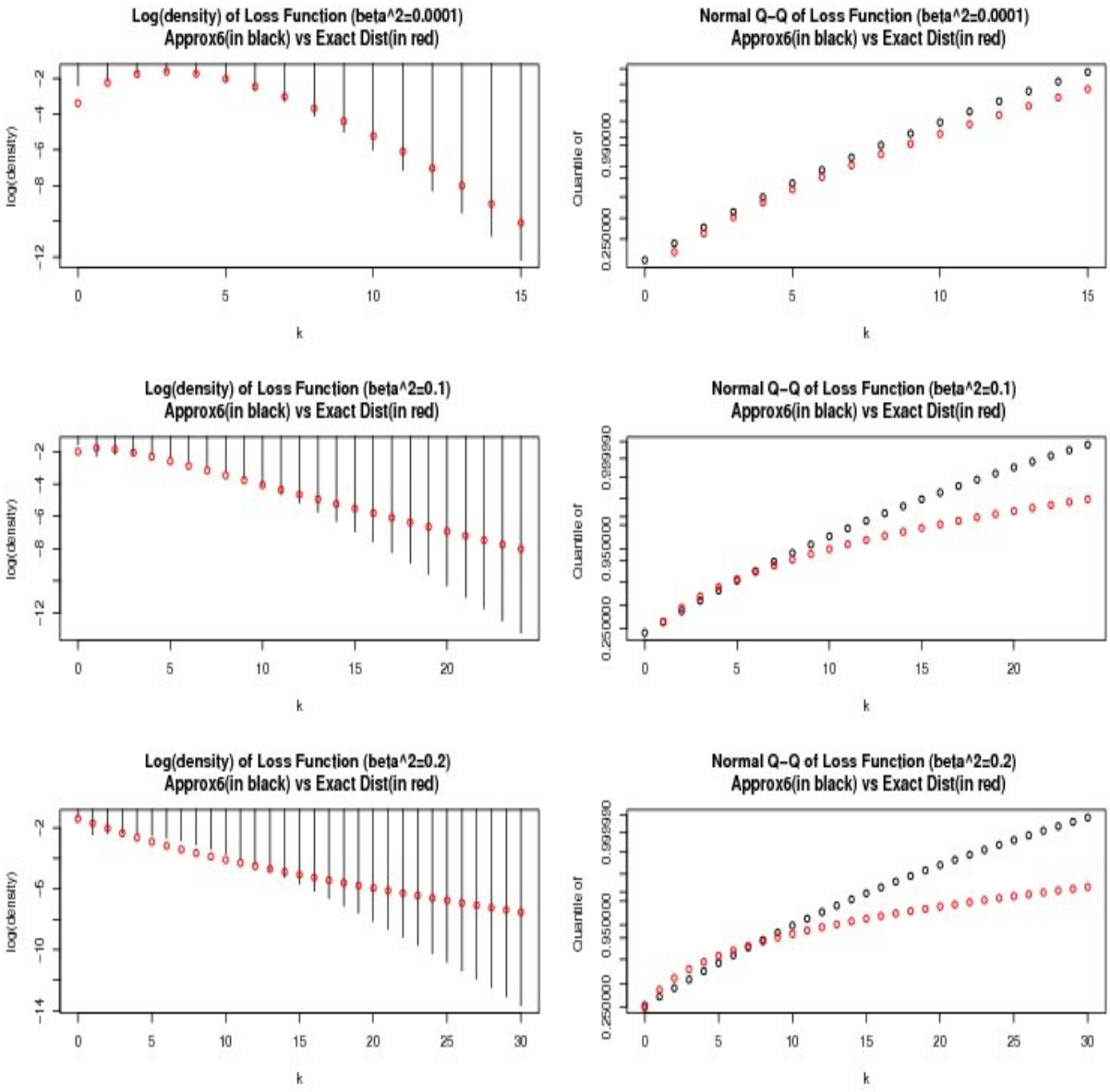

Figure 5.7: Loss Distribution by Student's t Copula–Approx6 vs. Exact Model (part1)

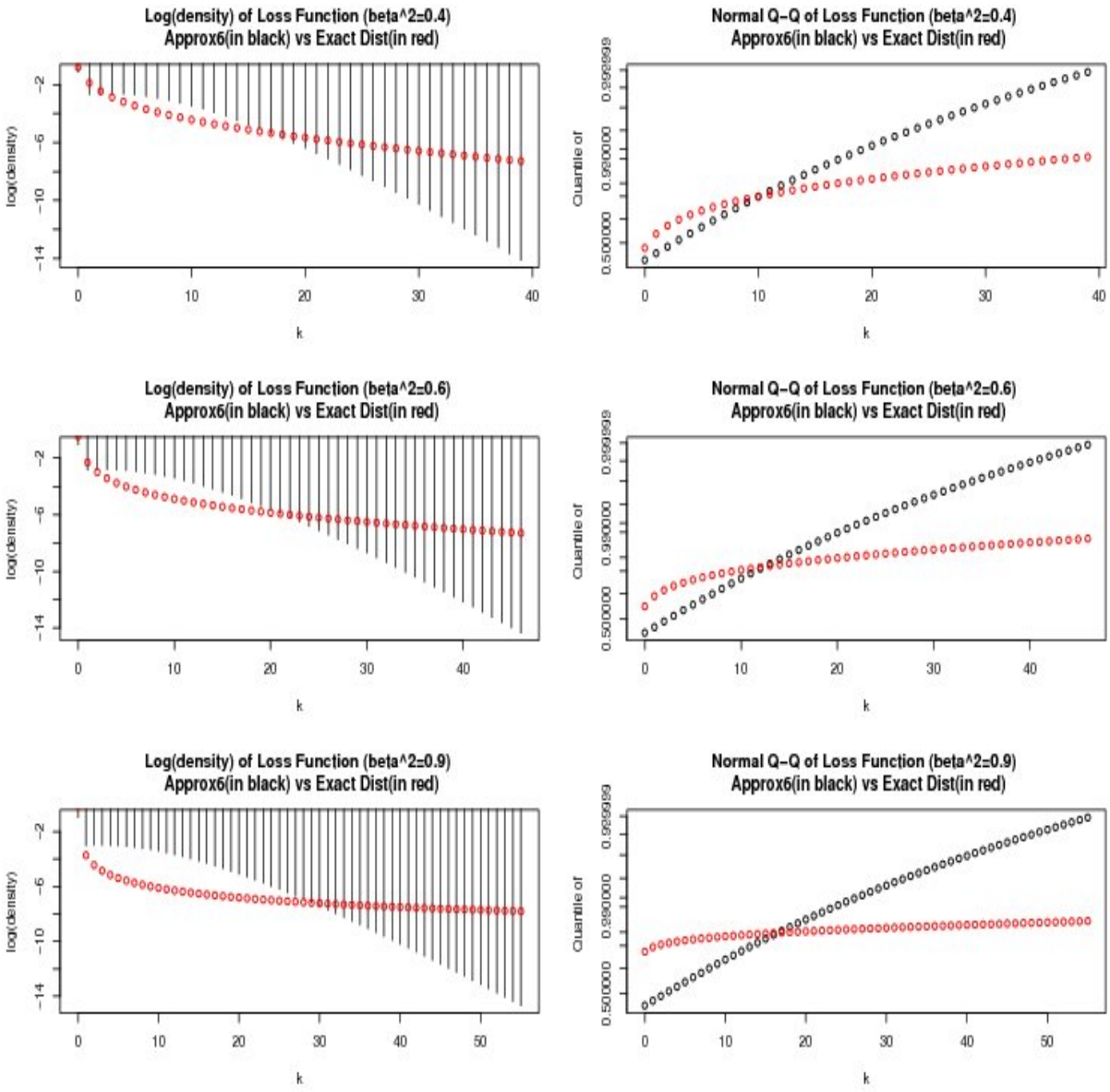

Figure 5.8: Loss Distribution by Student's t Copula–Approx6 vs. Exact Model (part2)

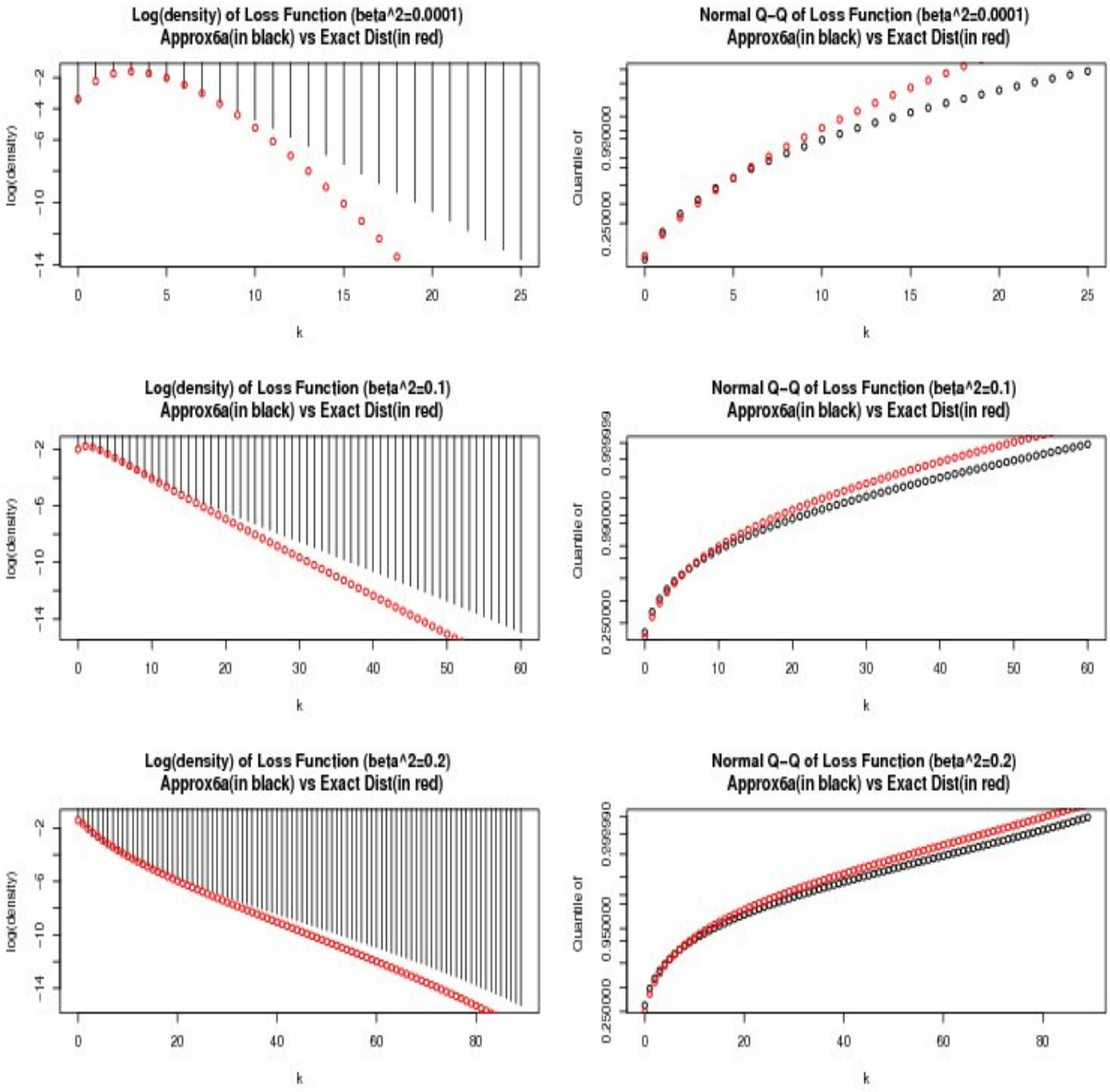

Figure 5.9: Loss Distribution by Student's t Copula–Approx6a vs. Exact Model (part1)

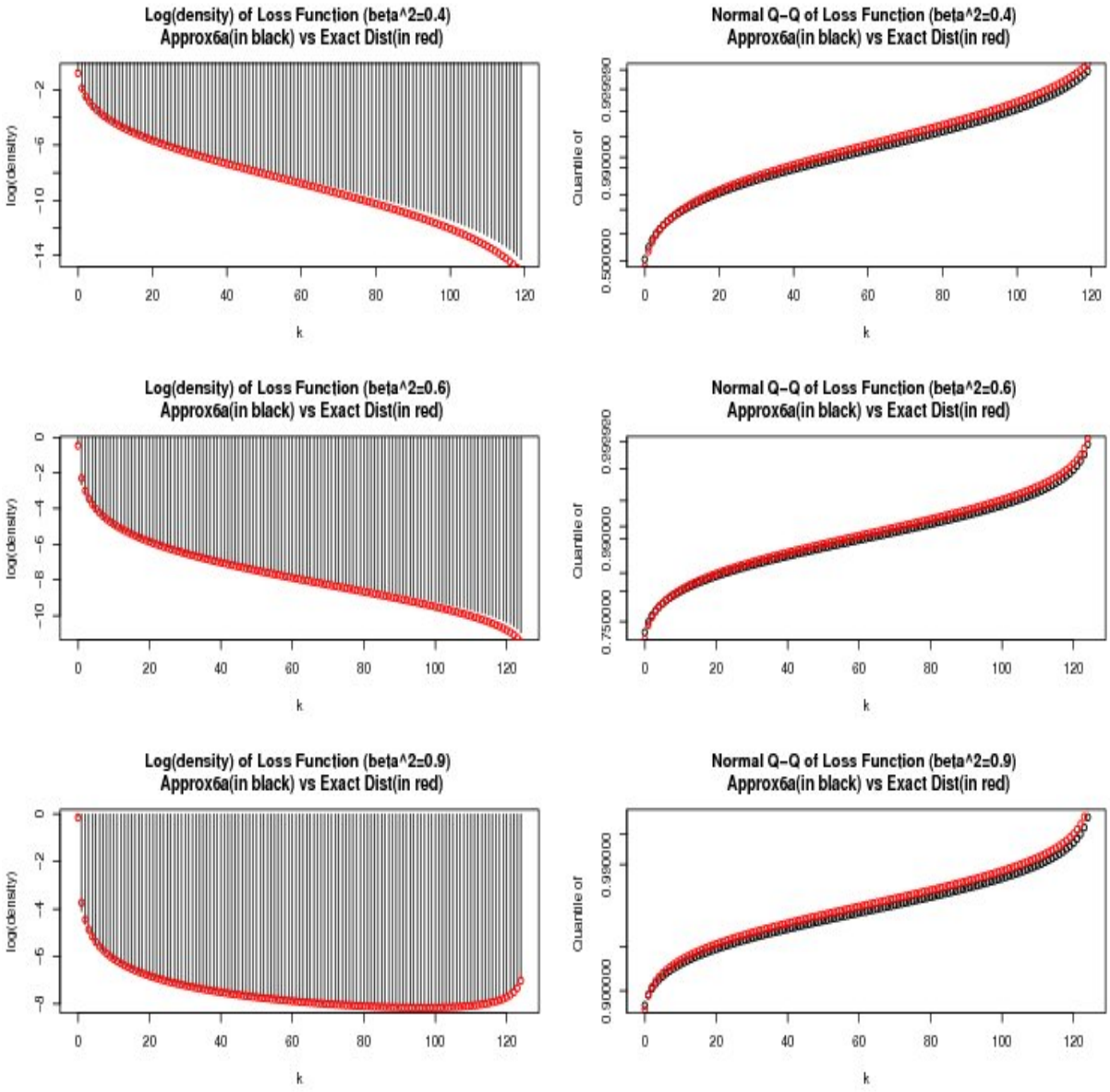

Figure 5.10: Loss Distribution by Student's t Copula–Approx6a vs. Exact Model (part2)

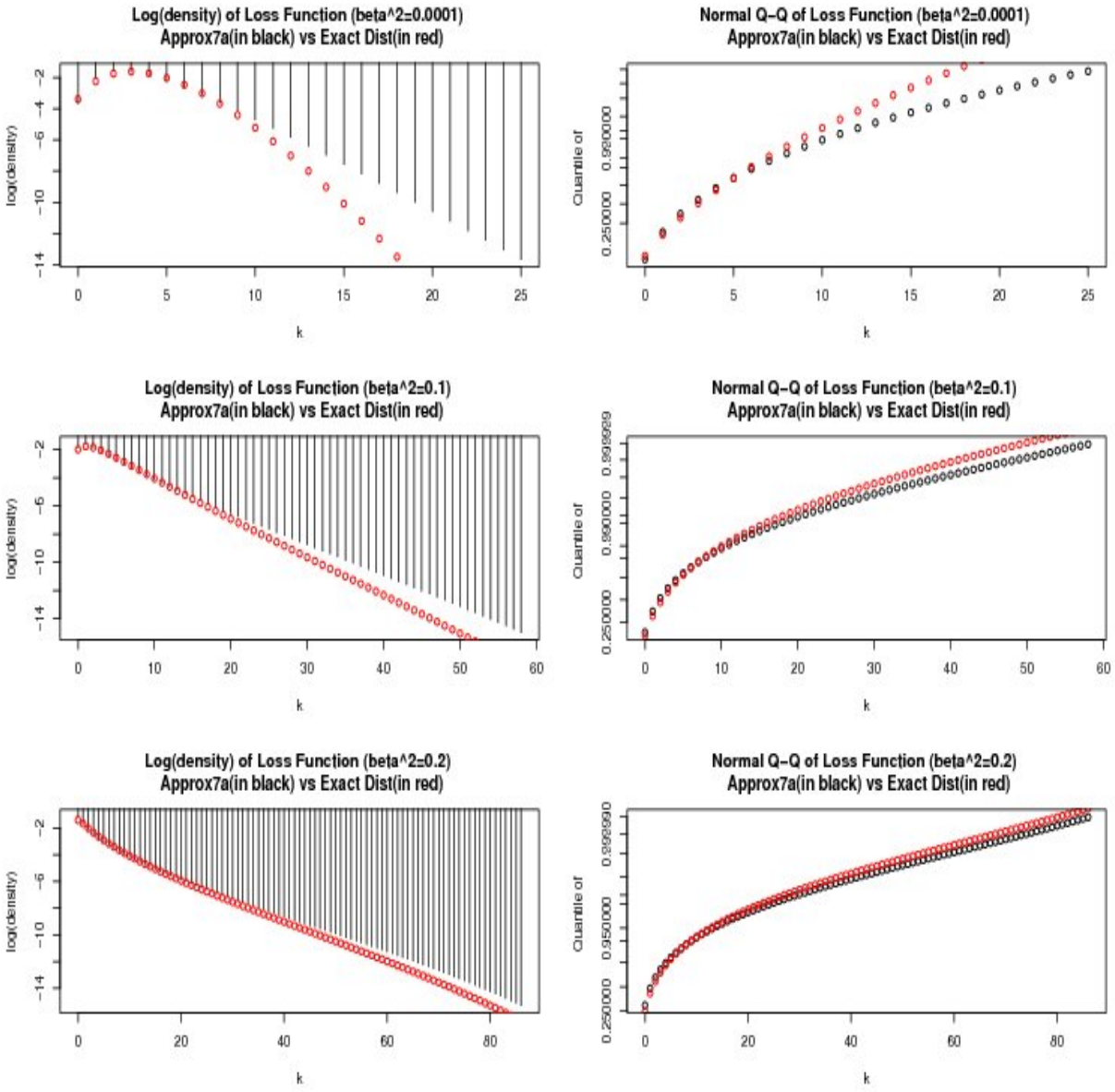

Figure 5.11: Loss Distribution by Student's t Copula–Approx7a vs. Exact Model (part1)

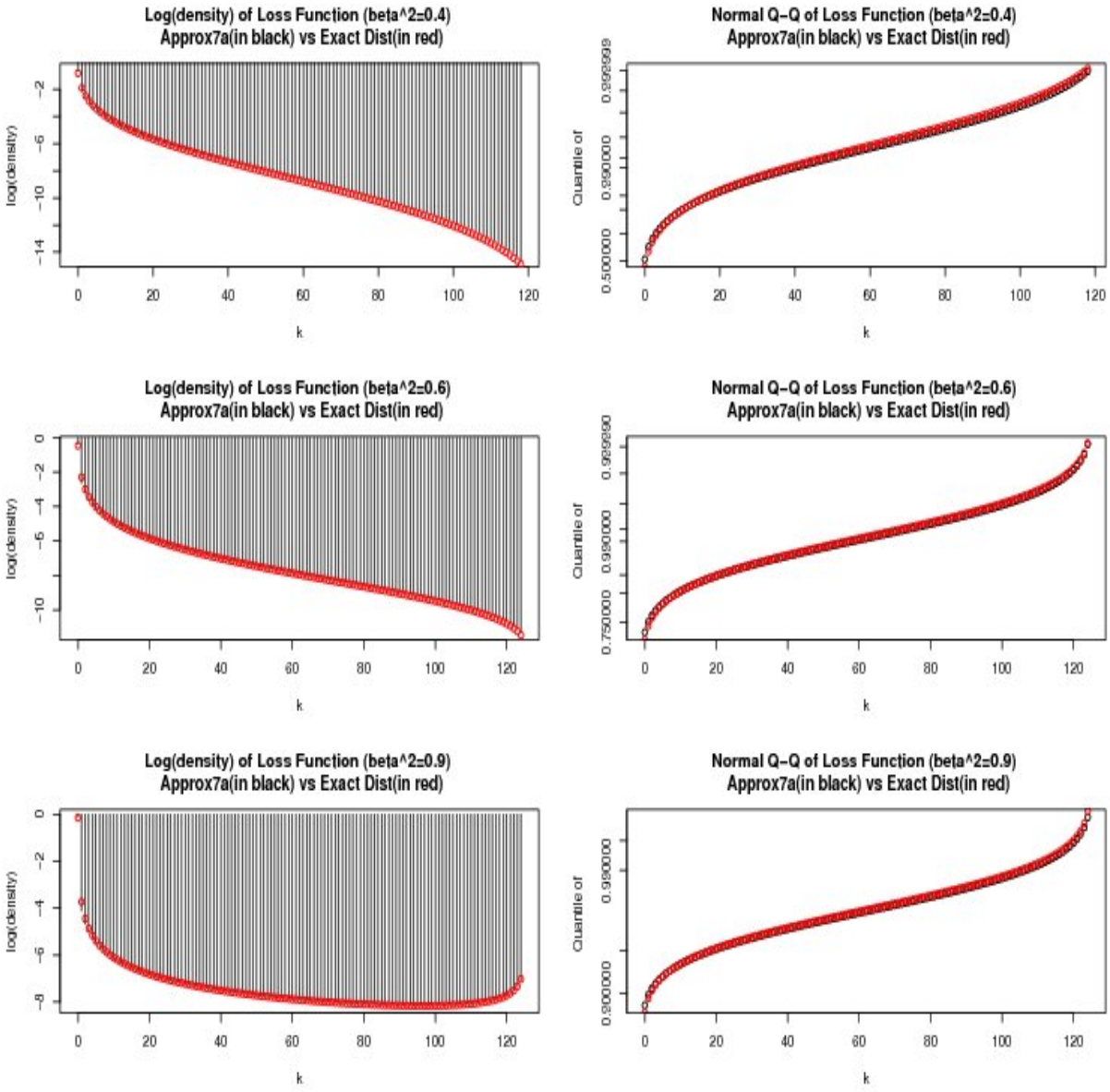

Figure 5.12: Loss Distribution by Student's t Copula–Approx7a vs. Exact Model (part2)

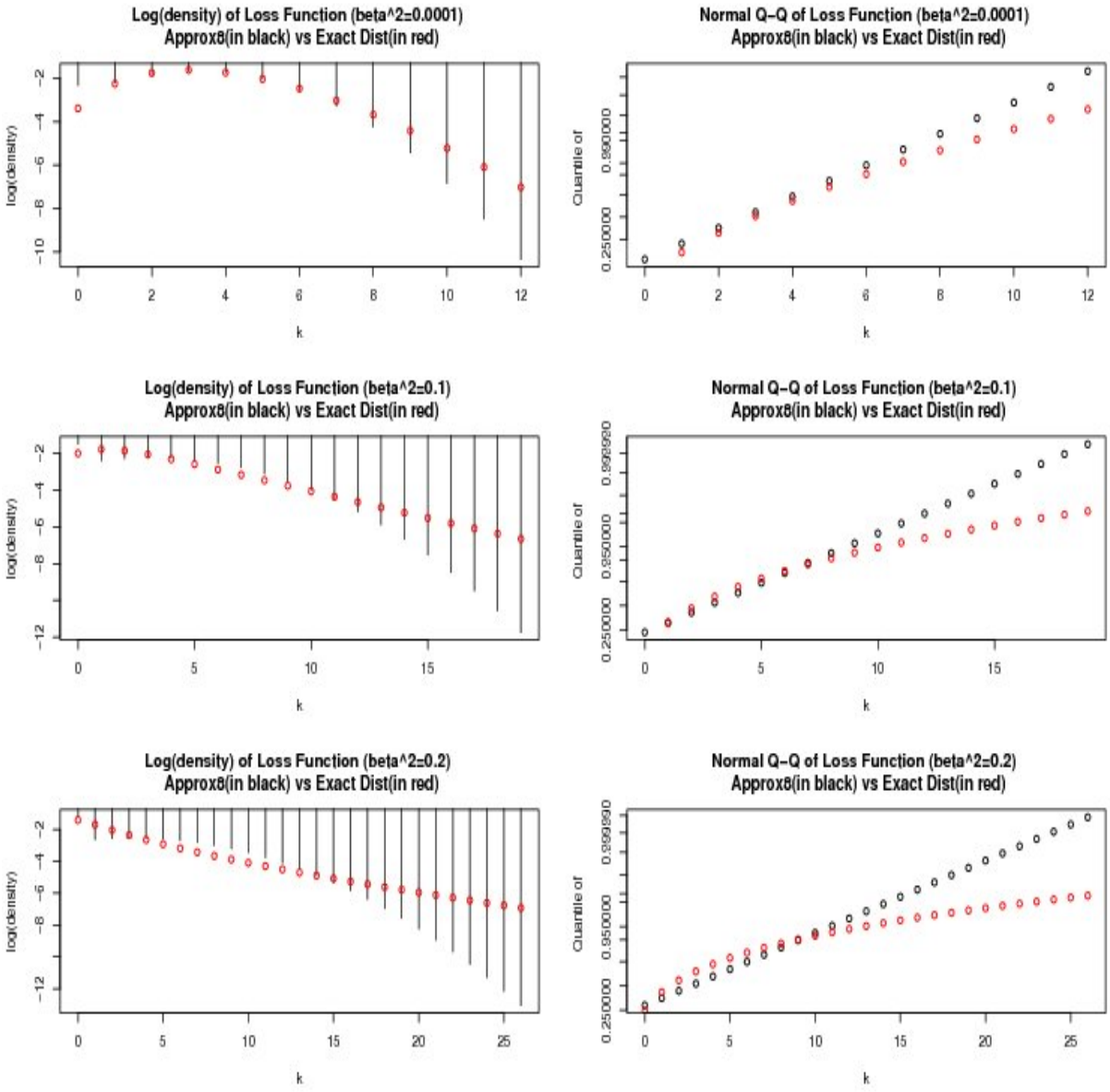

Figure 5.13: Loss Distribution by Student's t Copula–Approx8 vs. Exact Model (part1)

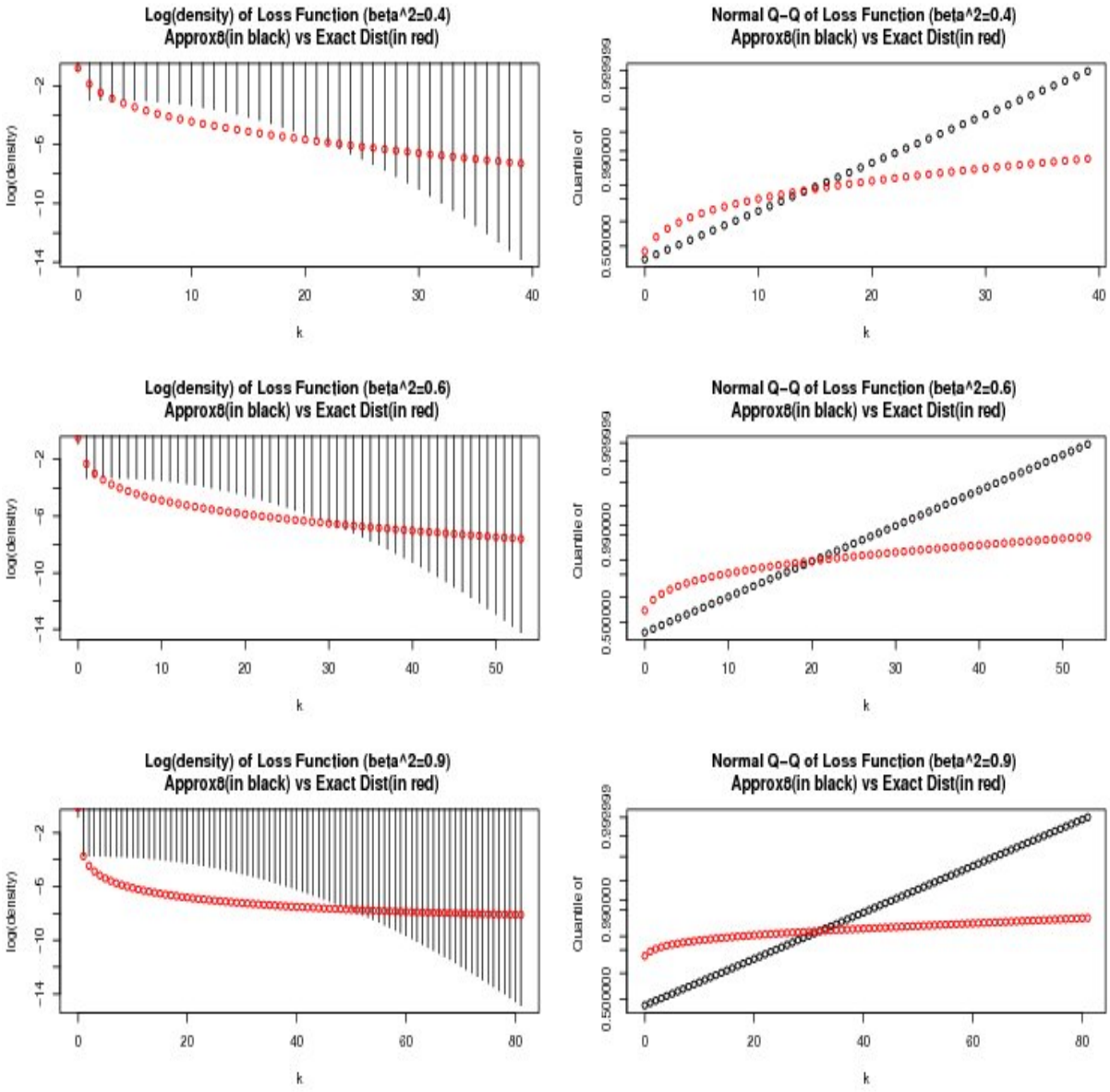

Figure 5.14: Loss Distribution by Student's t Copula–Approx8 vs. Exact Model (part2)

5.3 Model in Nonhomogeneous Case

Analogously to what we discussed in Chapter 4, we could use the recursive model that gives the exact distribution of the portfolio loss in the nonhomogeneous case. We obtain the asymptotic approximations by using the similar approaches used in the homogeneous case. The format of each asymptotic approximation is almost the same in the nonhomogeneous case as in the homogeneous case except for the replacement of p by $\Phi\left(\overline{\Phi^{-1}(p_i)}\right)$, where $\overline{\Phi^{-1}(p_i)} = \frac{1}{n}\sum_{i=1}^{n}\Phi^{-1}(p_i)$. Please refer to the proof in Chapter 4. We study the exact loss distribution and several asymptotic approximations in the nonhomogeneous case, including:

Exact Model: One factor model with conditional binomial distribution;

Approx2: Approximation as $n \to \infty$;

Approx3: Approximation as $\beta = 0$ and $n \to \infty$;

Approx4: Approximation as $\beta \to 0$ and $n \to \infty$;

Approx5/Approx5a/Approx5b: Alternative approximations as $\beta \to 0$ and $n \to \infty$;

Approx6/Approx6a/Approx6b: Alternative approximations as $\beta \to 0$ and $n \to \infty$ (approximation of bivariate normal);

Approx7a/Approx7b: Alternative approximations as $\beta \to 0$ and $n \to \infty$;

Approx8: Alternative approximation as $\beta \to 0$ and $n \to \infty$.

Since the derivation would be similar to what we have done by using the Gaussian copula method in Chapter 4, we would skip the detail derivation here.

Chapter 6

Synthetic CDO Pricing

6.1 Introduction

Pricing credit derivatives are the determination of the credit spread that is the default rate of a specific credit security [37]. For the single-name credit derivative such as the credit default swap, the credit spread is a function of the default probability and the recovery rate. It is essential to estimate the expected loss by using the derived risk-neutral default probability from the market quotes. Providing the absence of arbitrage, the present value of the expected payments made by the buyers of the credit risk protection has to be equal to the present value of the expected payoff at default.

For the portfolio credit derivative such as the CDO, its credit spread depends on the default correlation between the credit names in the pool. We've already seen that the loss distribution of a CDO is a function of the default correlation, the individual default probabilities and the recovery rates in Chapter 4 and Chapter 5. We are interested in studying the relationship between the default correlation and the tranche credit spread. The credit product used for our investigation is the iTraxx Europe, one of the stan-

dardized synthetic CDO indexes. This index comprises 125 equally-weighted European names. It has five standardized tranches: $0-3\%$, $3-6\%$, $6-9\%$, $9-12\%$ and $12-22\%$.

In order to study the credit default risk, we assume that the default is the only reason that causes the value of a corporate bond less than the value of a similar treasury bond. In the life of a CDO, the buyers of a tranche make premium payments on the predetermined dates. The payments last until the default time or its maturity date. We assume default can happen on any premium payment date, $m_1 < m_2 < \cdots < m_n$. The claim made by the tranche holders is equal to the minimum of the non-recovery portfolio loss over its lower detachment level and its upper detachment level over its lower detachment level. For example, the iTraxx has 5 tranches such as $0-3\%$, $3-6\%$, $6-9\%$, $9-12\%$ and $12-22\%$ in terms of their detachment points. If the proportional portfolio loss is 8%, the buyers of the $0-3\%$ equity tranche only lose the 3% portfolio loss. The next 3% loss would go to the buyers of the $3-6\%$ first mezzanine tranche and the remaining 2% loss will be borne by the buyers of the $6-9\%$ second mezzanine tranche. It is clear that the CDO diversifies the credit risk. It would satisfy the investors at different levels of credit risk. For easy exposition, we assume the risk-free interest rate remains constant during two payment dates and the recovery rate is equal to its expected value. According to the official documentation of the iTraxx, its recovery rate is set equal to 40% [29].

The present value of the expected total premium paid by the buyers of a specific tranche during the period from the value date to its maturity date is referred to as the premium leg of the tranche. The present value of the expected payoff on default from the tranche is called the default leg of the tranche. The freedom of arbitrage requires the equality of the premium leg and the default leg. By solving this equation, we could have the so-called fair spread that is the implied credit spread or the implied correlation for each tranche.

Next we introduce the algorithm for pricing a synthetic CDO tranche. We will estimate the base correlation by using the weighted least squares in the risk-neutral world.

6.2 Pricing Algorithm

6.2.1 Notations

Value date: t_0

Life of CDO from value date to its maturity: T

Total number of underlying credit names: n

Nominal amount of CDO: N_{ml}

Default time of name i: τ_i

Default time of CDO: τ

Risk-neutral default probability of name i: $p_i = P(\tau_i \leq T)$

Recovery rate of name i: R_i

Correlation coefficient in the factor copulas: β^2

Portfolio loss: $L = N_{ml} * \sum_{i=1}^{n}(1 - R_i)1_{\{\tau_i \leq T\}}$

Portfolio loss distribution: $P(L \leq l)$, where $0 \leq l \leq N_{ml} * \sum_{i=1}^{n}(1 - R_i)$

Loss of tranche j: L_j

Total number of premium payments: z

Premium payment date: m_k

Premium of tranche j: C_j

Present value of \$1 received at time t with certainty: $\nu(t)$

Present value of \$1 received at time t with credit risk, relative to that with certainty: $\gamma(t)$

Risk-free interest rate over m_k to m_{k+1}: $\pi(m_k)$

Default probability of tranche j at time t: $F_j(t) = P_j(\tau \leq t)$

Survival function of tranche j at time t: $S_j(t) = P_j(\tau > t)$

6.2.2 Pricing Procedure

The evaluation procedure of a synthetic CDO tranche is described by the following steps.

1. **Estimate the default probability of individual credit name from the market credit spread of a synthetic CDO.**

There are several approaches to get the default probability, for example,

(1) by the historical default probability from the rating agencies;
(2) by the structural models;
(3) by the implied approach from the market prices of the defaultable bonds or the asset swap spreads.

Li [34] suggested building the credit spread curve by using the market prices rather than the historical information of the defaults provided by the rating agencies such as Moody and Standard & Poor. Market information reflects the current market anticipation of the future that is always used by the trading desk; however, the default ratings are classified variables that are delayed information and are neglecting some information of a specific firm.

By using the survival function that we introduced in Chapter 2, the conditional default probability $P_{h|t}$ is given by

$$\begin{aligned} P_{h|t} &= P(t < \tau \leq t + h | \tau > t) \\ &= 1 - e^{-\int_t^{t+h} \lambda(v)\, dv} \\ &= 1 - e^{-\int_0^h \lambda(t+v)\, dv}, \end{aligned}$$

where $\lambda(\nu)$ is the hazard rate function. In the risk-neutral world, we assume the hazard rate is independent of the time. Then the time to default follows the exponential distribution with the parameter λ. That is $P_{h|t} = 1 - e^{-\lambda h}$. The risk-neutral default probability over the period from the value data to the maturity date is given by

$$P_{(\text{MDate}-\text{VDate})|\text{VDate}} = 1 - e^{-\lambda(\text{MDate}-\text{VDate})}.$$

The hazard rate of the synthetic CDO could be estimated from its market credit spread by [40],

$$\lambda = \frac{C_{spd}}{1 - R},$$

where C_{spd} is the composite credit spread quoted on market and R is the recovery rate. We assume the recovery rate is also independent of the time. For example, the estimated recovery rate of the 5-year iTraxx Series 4 is 40% provided by International Index Company Limited (IIC) [29].

2. **Derive the portfolio loss distribution by using a copula and then estimate the expected tranche loss.**

The portfolio loss distribution plays a very important role in pricing a synthetic CDO tranche. We need it to get the expected value of the tranche loss. The copula method

is useful for modeling a joint distribution of the defaults. It links the marginal default distributions of individual credit names to the joint default distribution of the credit portfolio. The Gaussian copulas are the current market standard models. The Student's t-copula is an alternative approach that describes the tail dependence by the degrees of freedom. We've already discussed these two methods in Chapter 4 and Chapter 5. Both the exact loss distribution and the asymptotic approximations that we derived earlier could be used to get the expected value of the tranche loss.

There are two common ways to calculate the expected tranche loss. Consider the buyers of the tranche whose detachment points are d_l and d_u ($d_l < d_u$). This specific tranche absorbs the default loss in excess of $d_l * N_{ml}$, up to and including the loss of $d_u * N_{ml} - d_l * N_{ml}$, where N_{ml} is the nominal amount of the credit portfolio. The first approach is letting the expected loss of a given tranche equal to the difference of the expected loss between two first-loss pieces, one is 0% to its upper detachment point and the other is 0% to its lower detachment point,

$$\mathrm{E}[L_j] = \mathrm{E}[L_F(d_u)] - \mathrm{E}[L_F(d_l)],$$

where L_j is the loss of the tranche j whose detachment points are d_l and d_u. L_F is the loss of the first-loss tranche whose lower detachment point is 0%. The expected loss of a first-loss tranche is given by

$$\mathrm{E}[L_F(d_u)] = \mathrm{E}[min(L,\ d_u * N_{ml})],$$

where L is the portfolio loss of the synthetic CDO. Since we've already obtained the proportional loss distribution of a synthetic CDO in Chapter 4 and Chapter 5, we could

compute the expected loss of the first-loss tranche by using

$$\mathrm{E}[L_F(d_u)] = N_{ml} * \mathrm{E}\left[min(L_P,\ d_u)\right] \tag{6.1}$$

$$= N_{ml} * \sum_{i=1}^{n} \left\{ min\left(\frac{i}{n}(1-R),\ d_u\right) P\left(L_P = \frac{i}{n}\right)\right\}, \tag{6.2}$$

where L_P is the proportional portfolio loss. The expected tranche loss is then given by the difference of the expected loss of the two first-loss tranches such as

$$\mathrm{E}[L_j] = \mathrm{E}[L_F(d_u)] - \mathrm{E}[L_F(d_l)].$$

The second approach to calculate the expected tranche loss is given by

$$\mathrm{E}[L_j] = N_{ml} * \mathrm{E}\left\{min\left[max(L_P - d_l,\ 0),\ (d_u - d_l)\right]\right\}.$$

Later from the market data evaluation, we will see these two different calibration approaches give us different implied correlations.

3. **Estimate the expected premium payments (premium leg) and the expected default loss (default leg) for each tranche.**

Once we know the expected tranche loss over the period from the value date to the maturity date, it is easy to estimate the probability of the loss for a given tranche by just using the sample proportion [40],

$$\begin{aligned}\bar{F}_j &= \frac{\mathrm{E}(L_j)}{(d_u - d_l) * N_{ml}} \\ &= 1 - \frac{(d_u - d_l) * N_{ml} - \mathrm{E}(L_j)}{(d_u - d_l) * N_{ml}} \\ &= 1 - \bar{S}_j,\end{aligned}$$

where F_j is the default probability of tranche j. And S_j is its probability of survival.

If we assume the default time τ is a continuous random variable, then the survival probability over the period from the value date to the maturity is given by

$$S\left(\text{MDate} - \text{VDate}\right) = e^{-\int_{\text{VDate}}^{\text{MDate}} \lambda(\tau) d\tau},$$

where $\lambda(\tau)$ denotes the hazard rate function. It is the risk-neutral probability measure at the value data. Since the premium of the tranche is paid quarterly, the probability of survival is given by [40]

$$S(\delta t) = \left(1 + \frac{L_r}{4}\right)^{-4*\delta t}$$

where L_r plays the role as a discount rate in the discrete framework, and $\delta t = \text{MDate} - \text{VDate}$. Here we assume L_r is a constant over the period from the value date to the maturity. We assume the default events happen on payment dates m_k. Over the period from the value date to each possible default time m_k, the probability of survival could also be obtained from the above formula. The only change is letting $\delta t = m_k - \text{VDate}$.

The present value of the expected loss (also the payoff) from tranche j, called the default leg, is given by

$$\begin{aligned}
\text{DLeg}_j &= N_{ml} * \text{E}^p \left[\int_{\text{VDate}}^{\text{MDate}} (d_u - d_l) \nu(t) dt \right] \\
&= N_{ml} * \int_{\text{VDate}}^{\text{MDate}} (d_u - d_l) \nu(t) dF_j(t) \\
&= N_{ml} * \sum_{m_k = \text{VDate}}^{\text{MDate}} (d_u - d_l) \nu(m_k) \left[F_j(m_k) - F_j(m_{k-1}) \right],
\end{aligned}$$

where p means risk-neutral probability measure, and ν is the risk-free discount factor.

The tranche buyers pay the premium until the occurrence of the default or until its maturity date, whichever is earlier. Let $m_1, m_2, \ldots, m_z$ be the series of premium payment dates. If the default happens at time m_k, the present value of the single payment is

$C_j * \nu(m_k)$. The expected value of the total payments of tranche j, called the premium leg, is given by

$$\begin{aligned}\text{PLeg}_j &= \text{E}^p\left[\int_{\text{VDate}}^{\text{MDate}} C_j\nu(t)dt\right] \\ &= \int_{\text{VDate}}^{\text{MDate}} C_j\nu(t)S_j(t)dt \\ &= \sum_{m_k=\text{VDate}}^{\text{MDate}} C_j\nu(m_k)S_j(m_k),\end{aligned}$$

where p is the risk-neutral probability measure, C_j is the premium of tranche j, and ν is the risk-free discount factor. See [40] for the detail.

4. **Calculate the implied base correlation or the implied credit spread of a tranche by letting the premium leg equal to the default leg.**

$$\text{PLeg}_j = \text{DLeg}_j$$

In practice, if we know the composite credit spread of a synthetic CDO and its correlation structure, we can calibrate the credit spreads of the individual tranches. The implied tranche credit spread is called the fair spread. In the other way around, we can get the implied correlation of each tranche from the composite spread of a synthetic CDO and the credit spreads of the individual tranches.

6.3 Market Data Investigation

6.3.1 Market Data

We investigate the market data of iTraxx that is a standardized synthetic CDO indexes. The iTraxx indexes have four maturities: 3, 5, 7 and 10 year. The five-year index

is the most liquid product on the market. A new series is issued every six month. The data that we use is the 5-year iTraxx Europe Series 4. It was issued on 3/20/2005 and will mature on 6/20/2010. It consists of 125 liquid European names. The nominal amount for a single name is EUR80,000 and the total nominal amount of the whole portfolio is EUR10,000,000. It has five standardized tranches: $0-3\%$, $3-6\%$, $6-9\%$, $9-12\%$ and $12-22\%$. For the $0-3\%$ equity tranche, the market credit spread is quoted as an upfront fee (in percentage) plus a running spread equal to 500 basis points. For the other tranches, the market credit spreads are quoted in basis points. The premiums are paid quarterly on the contracted payment dates. The time series data that we use is from 1/3/2006 to 8/21/2007. The daily market data are including:

(1) Composite credit spread of the iTraxx Europe Series 4 (5-year);
(2) Credit spread of individual tranches;
(3) Base correlation of individual tranches;
(4) 3-month LIBOR.

The first three quotes are provided by Lehman Brother. Here we thank Lehman Brother for providing us the market data for our research. The last quotes are available in [36]. Table 6.1 give us the daily market quotes of the credit spreads. Table 6.2 give us the daily market quotes of the base correlation. Table 6.3 give us the market quotes of LIBOR 3-month.

Table 6.1: Market Quotes of Credit Spread. Source: Lehman Brothers

Date of Quote	Composite Spd(bp)	$0-3\%$ Spd(%)	$3-6\%$ Spd(bp)	$6-9\%$ Spd(bp)	$9-12\%$ Spd(bp)	$12-22\%$ Spd(bp)
1/3/2006	36.92	27.50	88.50	27.13	12.50	6.20
2/17/2006	36.44	26.94	79.25	25.13	11.00	5.50
3/31/2006	30.89	20.50	7.00	12.00	7.50	3.50
7/5/2006	28.75	15.88	50.75	13.90	5.75	2.70
8/9/2006	29.53	16.44	48.00	12.00	7.50	2.88
9/6/2006	26.13	12.38	35.25	8.00	5.00	2.06
10/13/2006	25.00	12.00	33.50	8.00	4.00	1.81
11/3/2006	20.18	6.65	27.75	7.25	3.25	1.25
1/3/2007	20.75	5.63	19.00	6.25	3.00	1.13
2/22/2007	18.72	5.25	21.00	5.88	3.13	0.94
3/27/2007	18.85	3.31	21.75	5.25	2.19	0.81
4/20/2007	17.75	3.13	19.88	5.75	2.38	1.50
7/20/2007	26.25	8.26	45.08	15.06	6.78	3.79
8/16/2007	41.75	12.55	70.91	23.79	11.69	6.69

Table 6.2: Market Quotes of Base Correlation. Source: Lehman Brothers

Date of Quote	$0-3\%$ BCorr(%)	$3-6\%$ BCorr(%)	$6-9\%$ BCorr(%)	$9-12\%$ BCorr(%)	$12-22\%$ BCorr(%)
1/3/2006	12.58	24.55	33.24	40.55	58.92
2/17/2006	11.32	22.65	30.63	37.49	54.88
3/31/2006	11.05	23.29	32.12	39.19	57.80
7/5/2006	13.31	23.27	31.05	37.94	54.99
8/9/2006	11.81	20.50	26.49	30.64	41.69
9/6/2006	14.30	24.71	32.99	39.36	56.13
10/13/2006	14.09	24.67	32.84	39.49	56.63
11/3/2006	17.05	27.69	36.43	43.17	61.02
1/3/2007	16.37	27.83	35.78	42.31	59.51
2/22/2007	17.51	28.72	36.93	43.51	61.20
3/27/2007	23.87	36.73	46.20	53.83	72.54
4/20/2007	18.66	28.68	36.16	42.62	57.83
7/20/2007	27.80	40.66	49.69	57.07	74.22
8/16/2007	29.72	42.56	51.91	59.46	76.83

Table 6.3: Market Quotes of LIBOR 3-Month. Source: [36]

Date	LIBOR 3-Month
1/3/2006	4.68
2/17/2006	4.82
3/31/2006	4.99
7/5/2006	5.49
8/9/2006	5.40
9/6/2006	5.37
10/13/2006	5.37
11/3/2006	5.37
1/3/2007	5.36
2/22/2007	5.36
3/27/2007	5.36
4/20/2007	5.36
7/20/2007	5.36
8/16/2007	5.36

6.3.2 Implied Correlation

If we know the composite credit spread of a synthetic CDO and the credit spreads of the individual tranches, we can calibrate the implied correlation of the individual tranches. As we mentioned in the second step of the pricing algorithm, there are two types of implied correlations from different calculation approaches.

1. **Compound correlation.**

The compound correlation is an implied correlation with respect to the individual standardized tranches. It makes the calibrated credit spread of the standardized tranche equal to its market credit spread. If we use the following formula to caculate the expected loss of a given tranche, we will have the compound correlation.

$$\mathrm{E}[L_{T_j}] = N_{ml} * \mathrm{E}\left\{min\left[max(L_P - d_l,\ 0),\ (d_u - d_l)\right]\right\},$$

where L_j is the tranche loss, and L_P is the proportional portfolio loss. Here we use the Gaussian copula to get the proportional loss distribution. Figure 6.1 demonstrates the calibrated compound correlations. Since the investors have different preferences, the implied compound correlations of different tranches are not consistent with each other. This is the reason for the existence of correlation smile in Figure 6.1. It is not possible to use the implied compound correlations of the standard tranches to price the non-standard tranche such as $6\% - 7\%$.

2. **Base Correlation**

The base correlation is introduced by JPMorgan in 2004 [39]. It is another implied correlation with respect to the first-loss tranche, $0\% - x\%$ of a synthetic CDO. It also

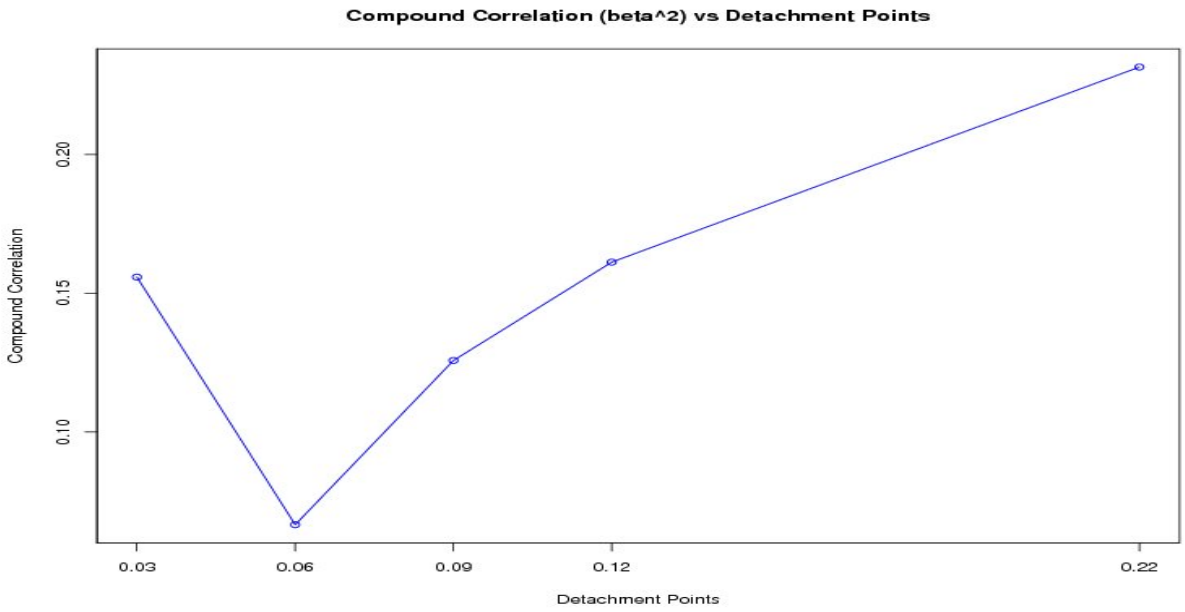

Figure 6.1: Implied Compound Correlation vs. Tranche

matches the market credit spread to the calibrated credit spread for each tranche; however, the expected loss of a given tranche is given by the difference of the expected loss of two first-loss tranches,

$$\mathrm{E}[L_j] = \mathrm{E}[L_F(d_u)] - \mathrm{E}[L_F(d_l)],$$

where L_j is the loss of tranche j whose detachment points are d_l and d_u. L_F is the loss of the first-loss tranche whose lower detachment point is 0%. Its value is given by

$$\mathrm{E}[L_F(x)] = N_{ml} * \mathrm{E}[min(L_P,\ x)],$$

where L_P is the proportional portfolio loss of the synthetic CDO, and N_{ml} is its nominal amount. Figure 6.2 shows a consistent uptrend of the base correlations calculated by using the Gaussian copula. Compared with the compound correlation, the base correlation is a better choice to interpolate the correlation for a non-standard tranche. The implied base correlation is one of our research interests.

6.3.3 Evaluation Results by Gaussian Copula

McGinty and Ahluwalia [40] compute the unconditional expected tranche loss by using the conditional proportional loss distribution, given the common systematic risk factor X^G,

$$\mathrm{E}\left[\mathrm{E}\left[L_F \middle| X^G\right]\right] \approx N_{ml} * \mathrm{E}\left\{min\left[\mathrm{E}(L_P \middle| X^G)(1-R),\ d_u\right]\right\}. \tag{6.3}$$

We compare the implied base correlations from two different approaches of computing the expected loss of the first-loss tranche. One is the McGinty and Ahluwalia's approach by Equation (6.3). The other is using the exact loss distribution from the Gaussian copula by Equation (6.2). Table 6.4 and Table 6.5 gives us the calibration results. The

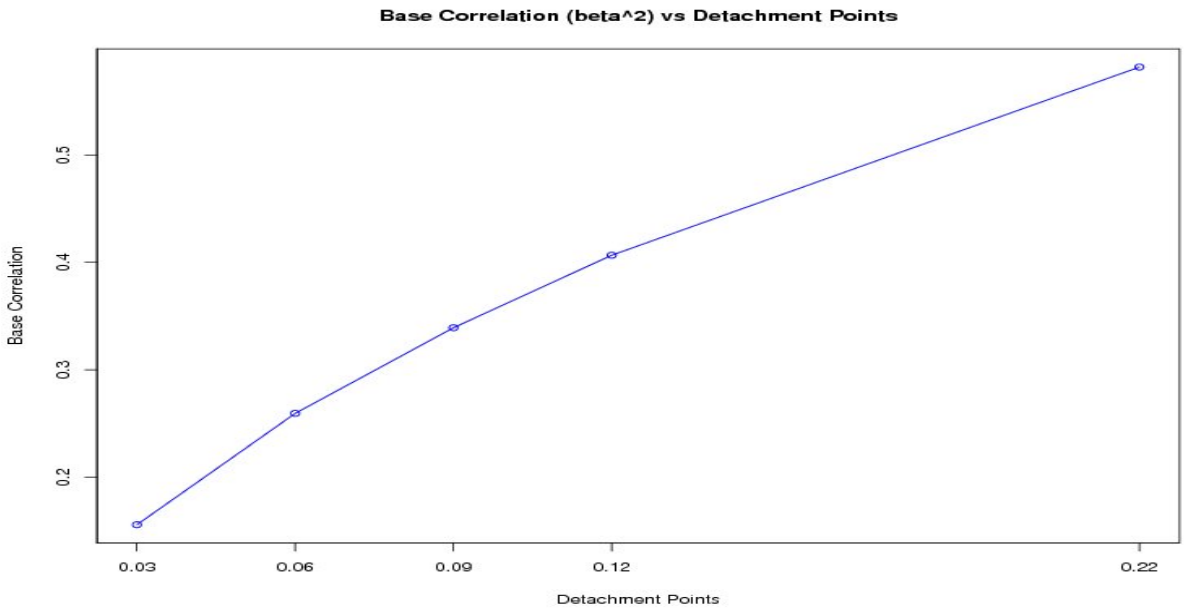

Figure 6.2: Implied Base Correlation vs. Tranche

implied base correlations by these two methods both have the consistent uptrend. As different dealers may use different calibration methods, we do not expect the implied base correlations are the same as that from Lehman Brothers.

Table 6.4: Implied Base Correlation Calibrated by Method of [40]

Date	$0-3\%$ BCorr(%)	$3-6\%$ BCorr(%)	$6-9\%$ BCorr(%)	$9-12\%$ BCorr(%)	$12-22\%$ BCorr(%)
1/3/2006	16.15	26.45	34.39	41.27	58.64
2/17/2006	15.58	25.94	33.90	40.65	58.17
3/31/2006	16.02	27.02	35.72	42.51	60.09
7/5/2006	18.89	29.07	37.21	44.24	61.85
8/9/2006	18.87	29.63	38.22	45.06	62.79
9/6/2006	20.17	31.39	40.27	47.22	64.90
10/13/2006	17.72	27.21	34.98	41.44	58.11
11/3/2006	18.07	25.46	31.68	37.06	51.83
1/3/2007	23.30	35.19	43.71	50.62	67.74
2/22/2007	15.54	20.34	23.66	25.52	25.41
3/27/2007	24.73	35.20	43.43	50.20	67.13
4/20/2007	20.49	28.58	35.03	40.70	54.73
7/20/2007	27.82	38.64	46.72	53.49	69.71
8/16/2007	40.22	54.93	64.96	72.38	88.10

Table 6.5: Implied Base Correlation Calibrated by Gaussian Copula

Date	$0-3\%$ BCorr(%)	$3-6\%$ BCorr(%)	$6-9\%$ BCorr(%)	$9-12\%$ BCorr(%)	$12-22\%$ BCorr(%)
1/3/2006	12.97	24.64	33.02	40.32	58.54
2/17/2006	12.39	24.17	32.47	39.87	57.54
3/31/2006	12.84	25.23	34.45	42.02	58.68
7/5/2006	15.82	27.34	36.15	43.69	60.12
8/9/2006	15.80	27.90	37.18	44.64	61.10
9/6/2006	17.14	29.67	39.36	46.74	63.16
10/13/2006	14.58	25.45	33.85	40.23	57.42
11/3/2006	14.92	23.63	30.23	36.01	51.89
1/3/2007	20.36	33.67	42.82	49.34	66.76
2/22/2007	12.24	18.40	22.24	24.29	25.43
3/27/2007	21.85	33.72	42.18	48.77	67.21
4/20/2007	17.43	26.84	33.73	39.97	54.09
7/20/2007	25.12	37.14	46.21	52.82	67.96
8/16/2007	37.97	54.29	63.15	70.82	91.74

6.3.4 Evaluation Results by Student's t Copula

In the Student's t copula, there are two parameters. One is β^2, the pairwise base correlation. The other is q that consists the degrees of freedom. Figure 6.3 demonstrates the lease square error that is the least square of mark to market. By using the least squares estimates, we calibrate the credit spread of the individual tranches. Table 6.6 give us the results by using the Student's t copula. The base correlation curve from the Student's t copula is also skewed unless we allow varying degrees of freedom. Figure 6.4 compares the implied base correlations from the Gaussian Copula and that from the Student's t Copula.

Table 6.6: Fair Spread Calibrated by Student's t Copula

Tranche	Par Spread Market(bp)	Par Spread t-Copula(bp)	Abs Error (bp)	Base Corr (%)	q
$0-3\%$	2694	2694	0	11.64	42.58
$3-6\%$	79.25	79.25	0	23.52	42.58
$6-9\%$	25.13	25.13	0	31.98	42.58
$9-12\%$	11.00	11.00	0	39.23	42.58
$12-22\%$	5.50	5.50	0	57.33	42.58

We study the relationship between the credit spread and the two parameters in the Student's t copula. Figure 6.5 demonstrates the sensitivity of tranche spreads to the two parameters respectively. For the equity tranche, the tranche spread is negatively related to β^2 and positively related to q. For the other four tranches, the tranche spread is positively related to β^2 and negatively related to q. In Figure 6.6, we use the lines in black to represent the equity tranche and the lines in red to represent the other tranches as a group. For larger q (larger degrees of freedom), the β^2 of the equity tranche is much different from that of the other tranches. As q is decreasing, both the black line and the

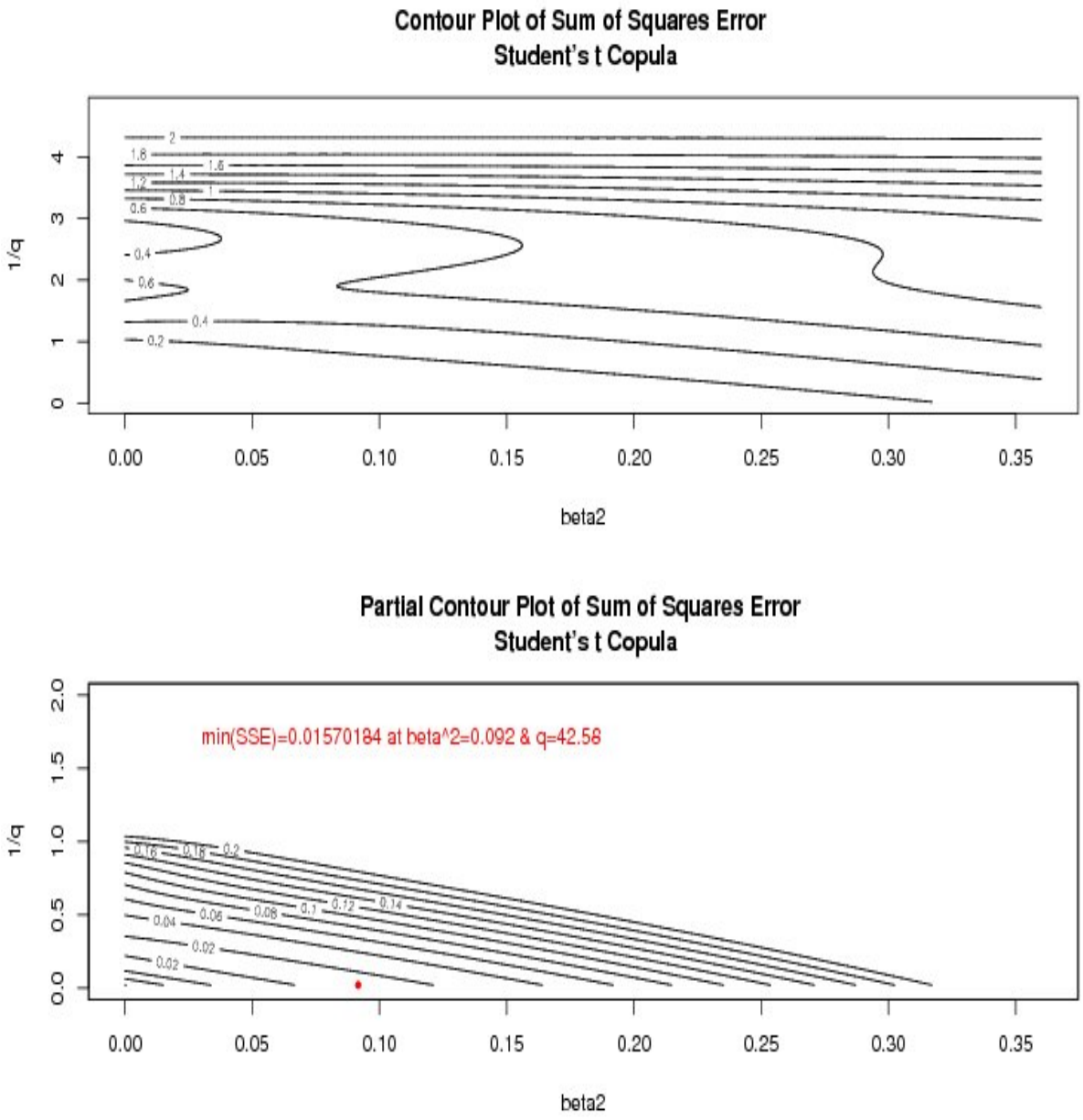

Figure 6.3: Least Square Error–Student's t Copula

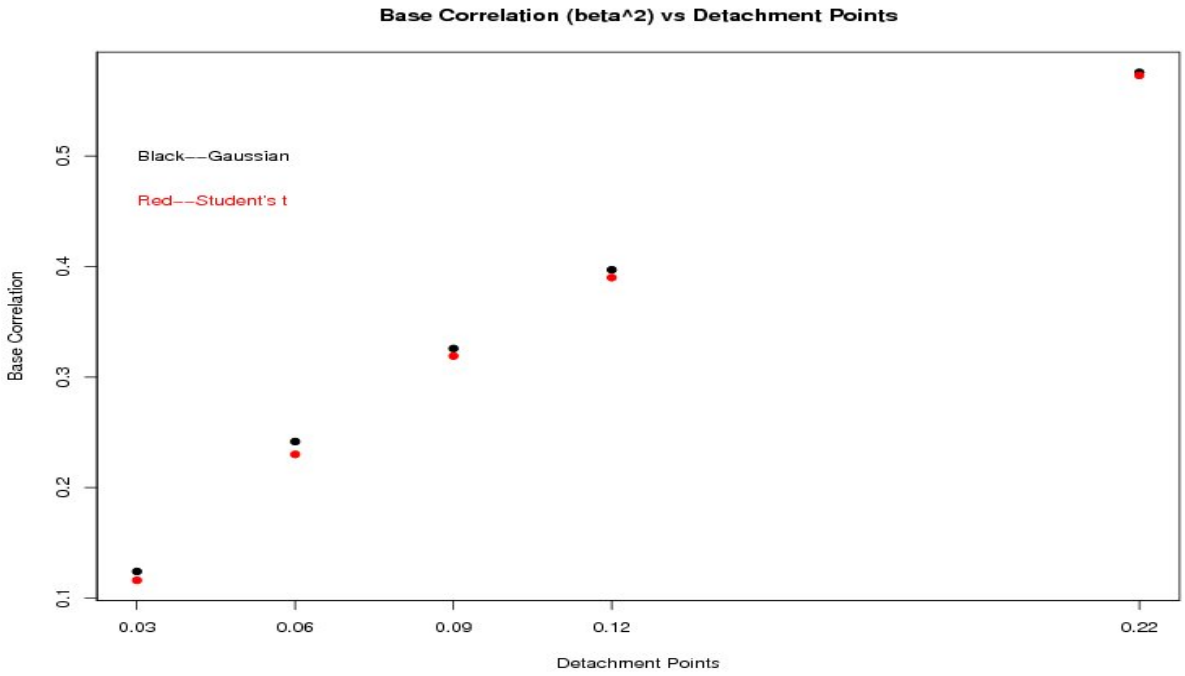

Figure 6.4: Gaussian Copula vs Student's t Copula – Implied Base Correlations

red line move to the left. It is impossible for them to intersect. That is why we can not find the same β^2 and q simultaneously for all the tranches. The question is that can we have a model that has the black line move to the right and the red line move to the left when q is decreasing. Then it is possible to find a less skewed implied base correlation curve.

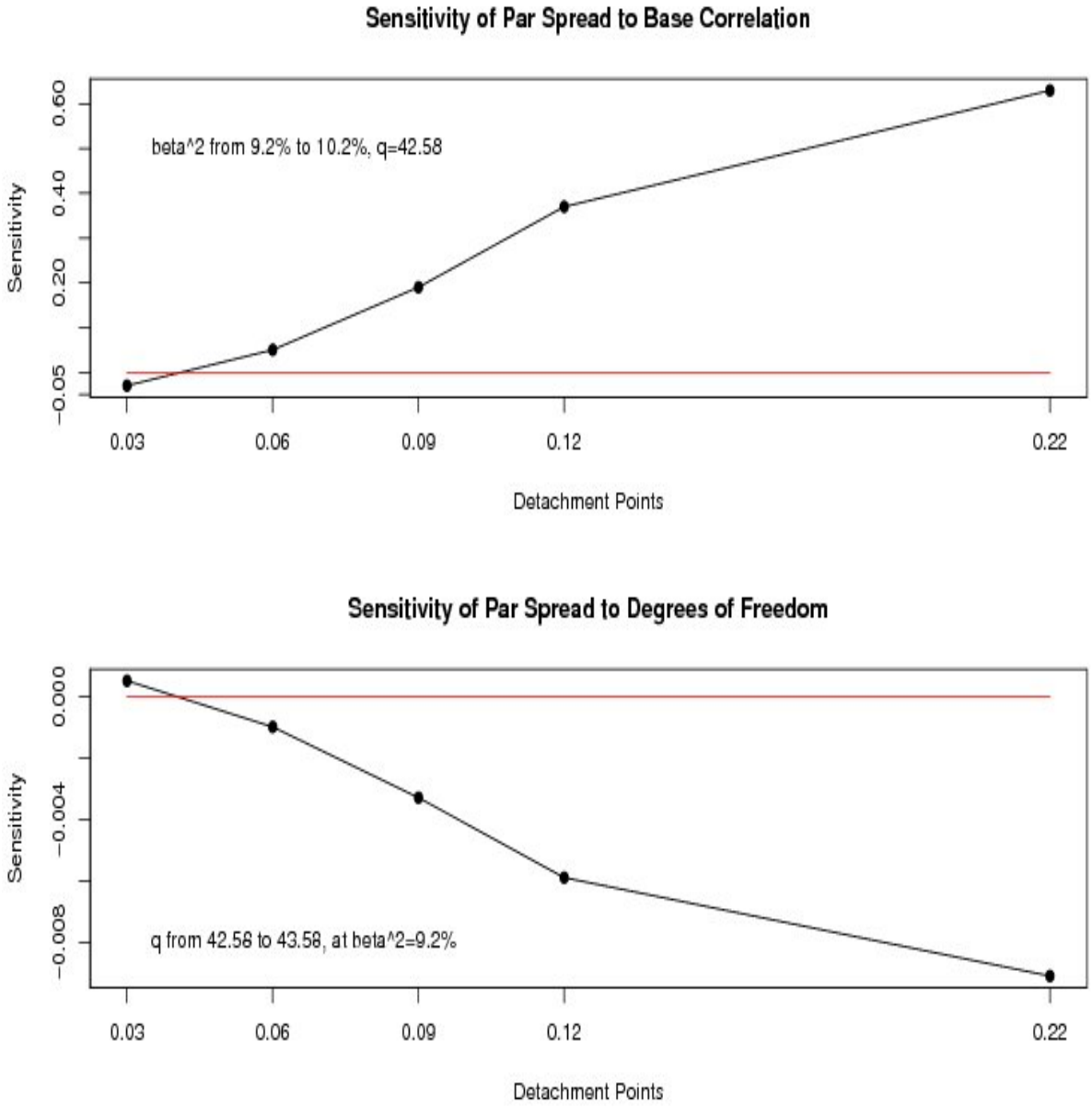

Figure 6.5: Spread Sensitivity to Parameters – Student's t Copula

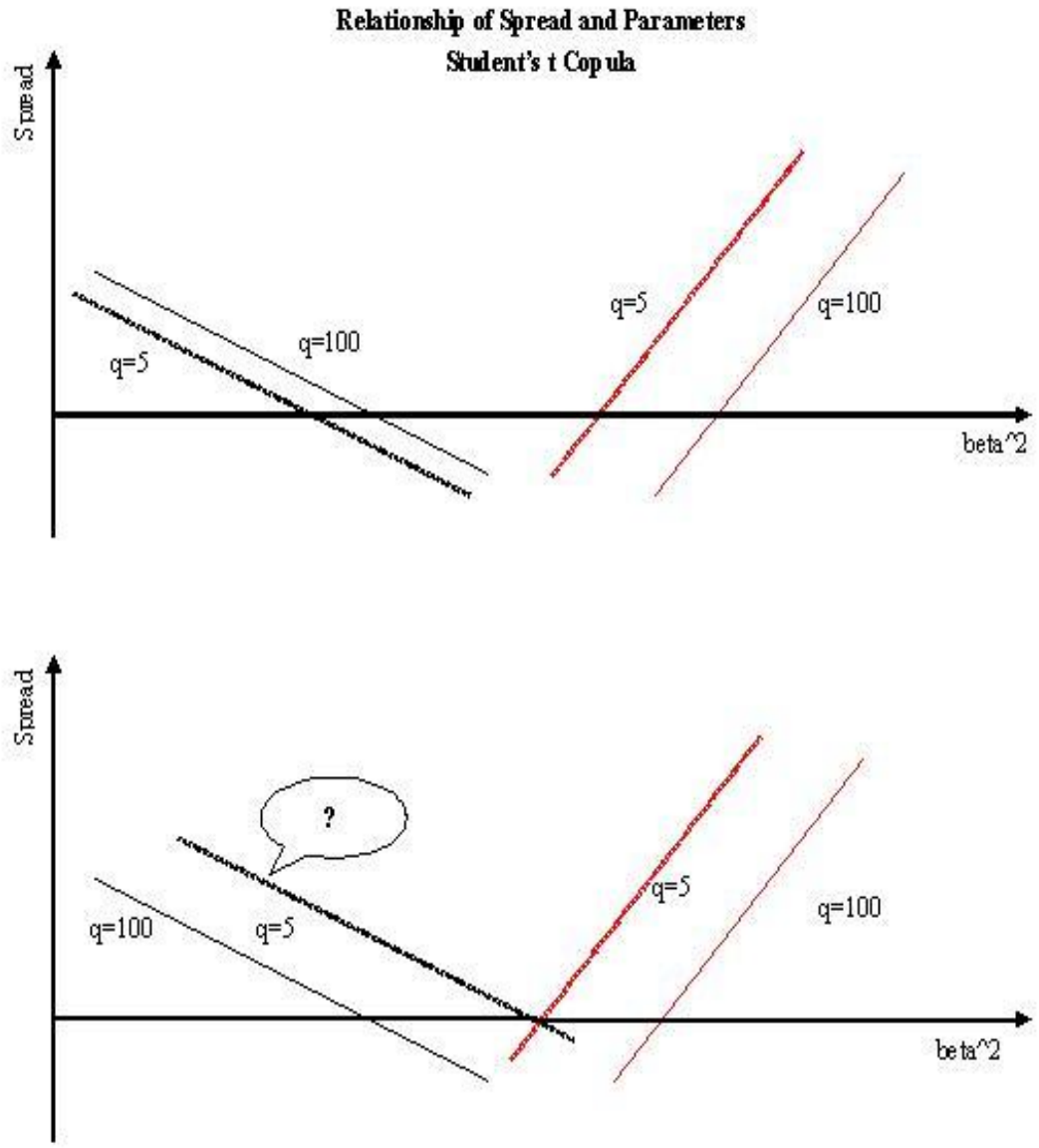

Figure 6.6: Relationship of Spread and Parameters – Student's t Copula

Chapter 7

Summary and Future Works

7.1 Dynamic Correlation

We already show the advantage of the copulas on constructing the dependence structure of the defaults. The joint distribution from a copula is not restricted on the marginal distributions. In the Gaussian copula, the dependent defaults of a credit portfolio are induced via a vector of latent variables that have a multivariate normal distribution. It is easy to understand and simple to implement. One of the asymptotic approximations that we derived in Chapter 4 works well in most of the cases. When the correlation coefficient β^2 is small, it is superior to the limiting distribution developed by Vasicek. When β^2 is not small, its performance is comparable to the Vasicek's approximation.

Since the normal distribution has zero tail dependence, the Gaussian copula could not be expect to capture the dependence between the extreme events such as credit defaults. In Chapter 5, we develop the Student's t copula process to define an appropriate dependence structure of the defaults. In the Student's t copula, the dependence among the default events is induced by modeling a vector of latent variables that have a mul-

tivariate Student's t distribution. Some of the tail dependence could be explained by an extra parameter, the degrees of freedom, in the Student's t copula. The standard Gaussian copula gives a correlation independent of the time to maturity; however, our developed Student's t copula is dependent on the time that is from the value date to the maturity date. One of the asymptotic approximations that we derived in Chapter 5 works especially well for the small correlation coefficient β^2 in the Student's t copula.

From the market data investigation in Chapter 6, we see the skewed base correlation curve implied from the Gaussian copula. The base correlation curve implied from the Student's t copula is also skewed unless we allow varying degrees of freedom. Since the implied base correlations of the individual tranches are not consistent, it is impossible to interpolate the base correlation for a non-standard tranche. Therefore it will be one of our interests to find a less skewed implied base correlation curve in the future study.

The following figures are the time series of the market data of the 5-year iTraxx Europe Series 4. It is from 1/3/2006 to 8/21/2007. The data are missing for April, May and June in 2006. The total number of the observations is 341. Figure 7.1 is the time series of the logarithm of the credit spreads by tranche. Since the credit spreads of the equity tranche is much bigger than those of the other tranches, Figure 7.2 gives us the time series of the logarithm of the credit spreads for the other tranches except the equity tranche. Figure 7.3 demonstrates the time series of the base correlations by tranche. Figure 7.4 shows the daily base correlation and the weekly, monthly and quarterly moving averages of the base correlations for the equity tranche. We can see the gradually increasing trend and the repeated variations from the time series of the market implied base correlations. In Febrary, 2007, there was a sharp upward trend and it leveled off 2 months later. There was another sharp upward trend in July, 2007. We knew in these two periods, there were two big collapses in the stock market due to the

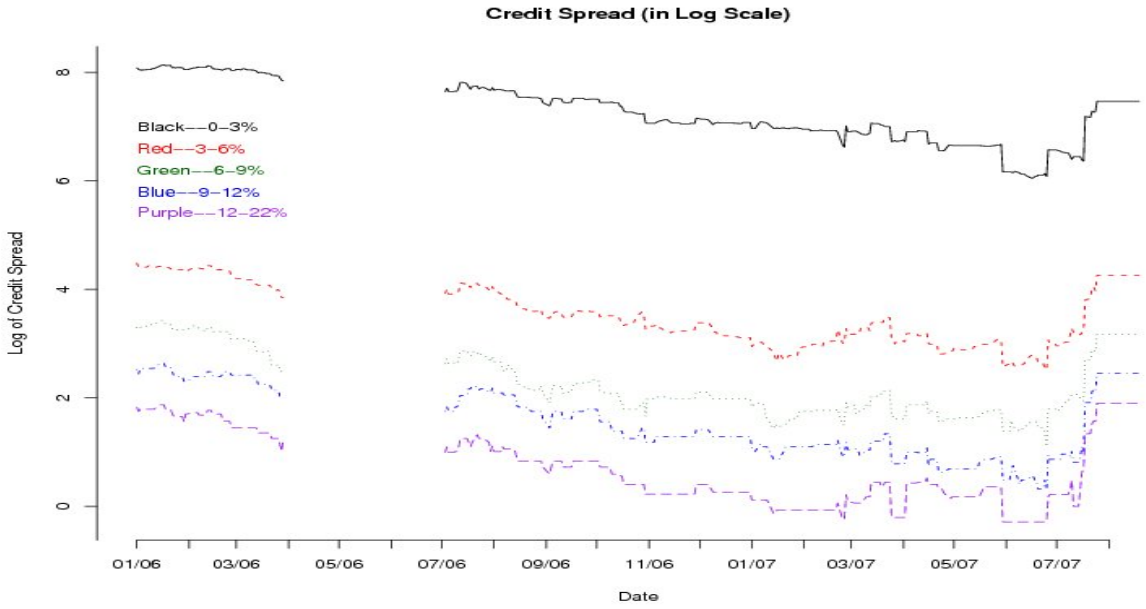

Figure 7.1: Credit Spread Time Series by Tranche

trouble coming from the subprime home loan market. We can see the highly volatile periods are clustering. This kind of problem could be handled by using the time series models such as ARCH, GARCH and stochastic volatility models [46]. It may be another future study for us.

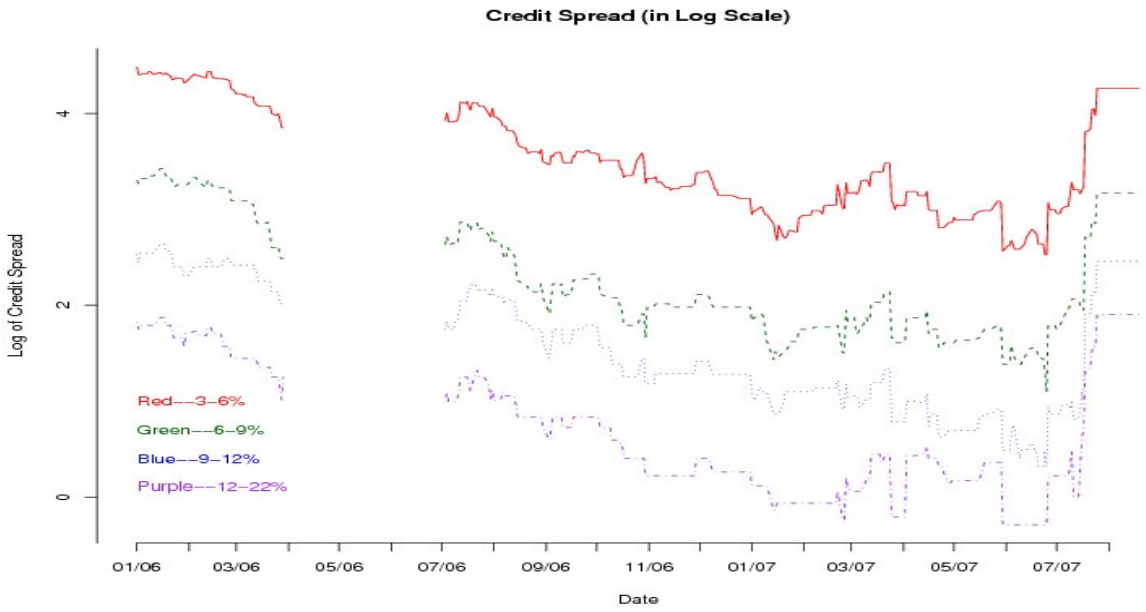

Figure 7.2: Credit Spread Time Series by Tranche (except equity tranche)

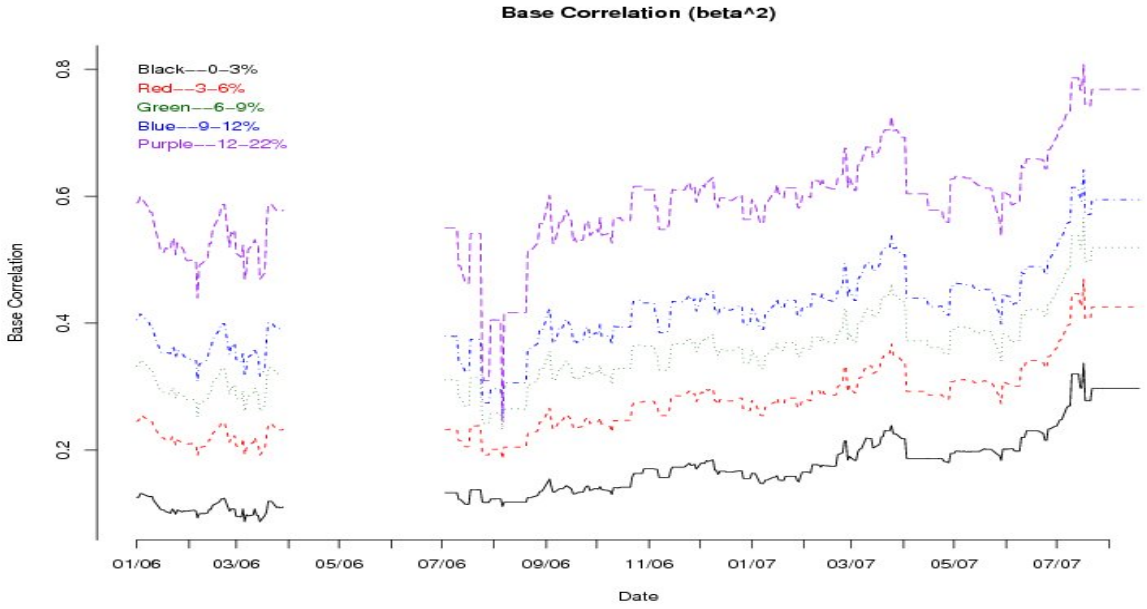

Figure 7.3: Base Correlation Time Series by Tranche

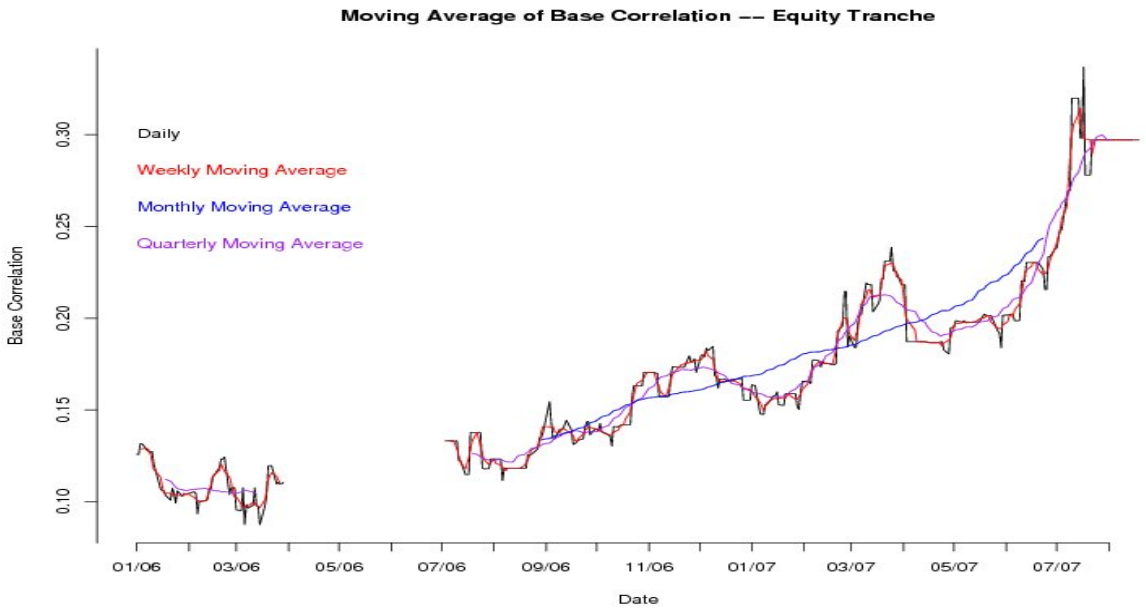

Figure 7.4: Base Correlation Time Series Filtration

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Appendix A

Proofs

1. Prove $E(D_i) = p$.

Proof:

$$
\begin{aligned}
E(D_i) &= E\left[E\left(D_i \middle| X^G = x\right)\right] \\
&= E\left[\Phi\left(\frac{\Phi^{-1}(p) - x\beta}{\sqrt{1-\beta^2}}\right)\right] \\
&= \int_{-\infty}^{+\infty} \Phi\left(\frac{\Phi^{-1}(p) - x\beta}{\sqrt{1-\beta^2}}\right) d\Phi(x) \\
&= \int_{-\infty}^{+\infty} \phi(x)\Phi\left(\frac{\Phi^{-1}(p) - x\beta}{\sqrt{1-\beta^2}}\right) dx \\
&= \int_{-\infty}^{+\infty} \phi(x) \int_{-\infty}^{\frac{\Phi^{-1}(p) - x\beta}{\sqrt{1-\beta^2}}} \phi(z) dz\, dx
\end{aligned}
$$

Let $y = z\sqrt{1-\beta^2} + \beta x$, then

$$\begin{aligned}
E(D_i) &= \int_{-\infty}^{+\infty} \phi(x) \int_{-\infty}^{\Phi^{-1}(p)} \phi\left(\frac{y-\beta x}{\sqrt{1-\beta^2}}\right) d\left(\frac{y}{1-\beta^2}\right) dx \\
&= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \int_{-\infty}^{\Phi^{-1}(p)} \frac{1}{\sqrt{2\pi}\sqrt{1-\beta^2}} e^{-\frac{(y-\beta x)^2}{2(1-\beta^2)}} \, dy \, dx \\
&= \frac{1}{2\pi\sqrt{1-\beta^2}} \int_{-\infty}^{+\infty} \int_{-\infty}^{\Phi^{-1}(p)} e^{-\frac{x^2}{2} + \frac{y^2 - 2\beta xy + \beta^2 x^2}{2(1-\beta^2)}} \, dy \, dx \\
&= \frac{1}{2\pi\sqrt{1-\beta^2}} \int_{-\infty}^{+\infty} \int_{-\infty}^{\Phi^{-1}(p)} e^{-\frac{x^2+y^2-2\beta xy}{2(1-\beta^2)}} \, dy \, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\Phi^{-1}(p)} e^{-\frac{y^2}{2}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sqrt{1-\beta^2}} e^{-\frac{1}{2}\left(\frac{x-\beta y}{\sqrt{1-\beta^2}}\right)^2} \, dx \, dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\Phi^{-1}(p)} e^{-\frac{y^2}{2}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \, dt \, dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\Phi^{-1}(p)} e^{-\frac{y^2}{2}} \, dy \\
&= \Phi\left(\Phi^{-1}(p)\right) \\
&= p.
\end{aligned}$$

Therefore $E(D_i) = p$.

2. Prove $E(D_i^2) = \Phi\left(\Phi^{-1}(p), \Phi^{-1}(p), \beta^2\right)$**.**

Proof:

$$
\begin{aligned}
E(D_i^2) &= E\left[E\left(D_i^2 \middle| X^G = x\right)\right] \\
&= E\left[\Phi\left(\frac{\Phi^{-1}(p) - x\beta}{\sqrt{1-\beta^2}}\right)\Phi\left(\frac{\Phi^{-1}(p) - x\beta}{\sqrt{1-\beta^2}}\right)\right] \\
&= \int_{-\infty}^{+\infty} \Phi\left(\frac{\Phi^{-1}(p) - x\beta}{\sqrt{1-\beta^2}}\right)\Phi\left(\frac{\Phi^{-1}(p) - x\beta}{\sqrt{1-\beta^2}}\right) d\Phi(x) \\
&= \int_{-\infty}^{+\infty} \phi(x)\Phi\left(\frac{\Phi^{-1}(p) - x\beta}{\sqrt{1-\beta^2}}\right)\Phi\left(\frac{\Phi^{-1}(p) - x\beta}{\sqrt{1-\beta^2}}\right) dx \\
&= \int_{-\infty}^{+\infty} \phi(x) \int_{-\infty}^{\frac{\Phi^{-1}(p) - x\beta}{\sqrt{1-\beta^2}}} \int_{-\infty}^{\frac{\Phi^{-1}(p) - x\beta}{\sqrt{1-\beta^2}}} \phi(z_1)\phi(z_2)\, dz_1\, dz_2\, dx.
\end{aligned}
$$

Let $y_1 = \tau_1\sqrt{1-\beta^2} + \beta x$ and $y_2 = y_2\sqrt{1-\beta^2} + \beta x$, then

$$
\begin{aligned}
E(D_i^2) &= \int_{-\infty}^{+\infty} \phi(x) \int_{-\infty}^{\Phi^{-1}(p)} \int_{-\infty}^{\Phi^{-1}(p)} \phi\left(\frac{y_1 - \beta x}{\sqrt{1-\beta^2}}\right) \phi\left(\frac{y_2 - \beta x}{\sqrt{1-\beta^2}}\right) d\left(\frac{y_1}{1-\beta^2}\right) \\
&\quad \times d\left(\frac{y_2}{1-\beta^2}\right) dx \\
&= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \int_{-\infty}^{\Phi^{-1}(p)} \int_{-\infty}^{\Phi^{-1}(p)} \frac{1}{2\pi(1-\beta^2)} e^{-\frac{(y_1-\beta x)^2}{2(1-\beta^2)}} e^{-\frac{(y_2-\beta x)^2}{2(1-\beta^2)}} \\
&\quad \times dy_1\, dy_2\, dx \\
&= \frac{1}{(2\pi)^{\frac{3}{2}}(1-\beta^2)} \int_{-\infty}^{+\infty} \int_{-\infty}^{\Phi^{-1}(p)} \int_{-\infty}^{\Phi^{-1}(p)} e^{-\frac{x^2}{2} + \frac{{y_1}^2 - 2\beta x y_1 + \beta^2 x^2 + {y_2}^2 - 2\beta x y_2 + \beta^2 x^2}{2(1-\beta^2)}} \\
&\quad \times dy_1\, dy_2\, dx \\
&= \frac{1}{(2\pi)^{\frac{3}{2}}(1-\beta^2)} \int_{-\infty}^{+\infty} \int_{-\infty}^{\Phi^{-1}(p)} \int_{-\infty}^{\Phi^{-1}(p)} e^{-\frac{x^2 + {y_1}^2 + {y_2}^2 - 2\beta x y_1 - 2\beta x y_2 + \beta^2 x^2}{2(1-\beta^2)}} \\
&\quad \times dy_1\, dy_2\, dx \\
&= \int_{-\infty}^{\Phi^{-1}(p)} \int_{-\infty}^{\Phi^{-1}(p)} \frac{1}{2\pi\sqrt{1-\beta^2}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sqrt{1-\beta^2}} \\
&\quad \times e^{-\frac{\left(x^2(1+\beta^2) + {y_1}^2 + {y_2}^2 - 2\beta x y_1 - 2\beta x y_2\right)\left(1+\beta^2\right)}{2(1-\beta^2)(1+\beta^2)}} dx\, dy_1\, dy_2 \\
&= \int_{-\infty}^{\Phi^{-1}(p)} \int_{-\infty}^{\Phi^{-1}(p)} \frac{1}{2\pi\sqrt{1-\beta^2}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sqrt{1-\beta^2}} \\
&\quad \times e^{-\frac{\left(x(1+\beta^2) - \beta(y_1+y_2)\right)^2 - \beta^2(y_1+y_2)^2 + ({y_1}^2 + {y_2}^2)(1+\beta^2)}{2(1-\beta^2)(1+\beta^2)}} dx\, dy_1\, dy_2 \\
&= \int_{-\infty}^{\Phi^{-1}(p)} \int_{-\infty}^{\Phi^{-1}(p)} \frac{1}{2\pi\sqrt{1-\beta^2}\sqrt{1+\beta^2}} e^{-\frac{{y_1}^2 + {y_2}^2 - 2\beta^2 y_1 y_2}{2(1-\beta^4)}} \\
&\quad \times \int_{-\infty}^{+\infty} \frac{\sqrt{1+\beta^2}}{\sqrt{2\pi}\sqrt{1-\beta^2}} e^{-\frac{1}{2}\frac{\left(x(1+\beta^2) - \beta(y_1+y_2)\right)^2}{(1-\beta^2)^2(1+\beta^2)^2}} dx\, dy_1\, dy_2 \\
&= \int_{-\infty}^{\Phi^{-1}(p)} \int_{-\infty}^{\Phi^{-1}(p)} \frac{1}{2\pi\sqrt{1-\beta^4}} e^{-\frac{{y_1}^2 + {y_2}^2 - 2\beta^2 y_1 y_2}{2(1-\beta^4)}} dy_1\, dy_2 \\
&= \Phi_2\left(\Phi^{-1}(p), \Phi^{-1}(p), \beta^2\right),
\end{aligned}
$$

where $\Phi_2\left(\Phi^{-1}(p), \Phi^{-1}(p), \beta^2\right)$ is the cumulative distribution function of the bivariate normal with $\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\Sigma = \begin{pmatrix} 1 & \beta^2 \\ \beta^2 & 1 \end{pmatrix}$.
Therefore $E(D_i^2) = \Phi_2\left(\Phi^{-1}(p), \Phi^{-1}(p), \beta^2\right)$.

3. Prove if $W \sim$ gamma$(\frac{\nu}{2}, \frac{2}{\nu})$ then $\frac{1}{W}$ has the inverted gamma distribution with both parameters equal to $\frac{\nu}{2}$.

Proof: Let $Y = \frac{1}{W}$, then

$$\begin{aligned} f_Y &= f_W \left| \frac{dw}{dy} \right| \\ &= \frac{1}{\Gamma\frac{\nu}{2}\frac{2}{\nu}^{\frac{\nu}{2}}} w^{\frac{\nu}{2}-1} e^{-w\frac{\nu}{2}} \left| -\frac{1}{y^2} \right| \\ &= \frac{1}{\Gamma\frac{\nu}{2}\frac{2}{\nu}^{\frac{\nu}{2}}} \frac{1}{y}^{\frac{\nu}{2}-1} e^{-\frac{\nu}{2y}} \left| -\frac{1}{y^2} \right| \\ &= \frac{\frac{\nu}{2}^{\frac{\nu}{2}}}{\Gamma\frac{\nu}{2}} y^{-\frac{\nu}{2}-1} e^{-\frac{\nu}{2y}}. \end{aligned}$$

Therefore $\frac{1}{W}$ has the inverted gamma distribution with both parameters equal to $\frac{\nu}{2}$.

4. Prove if $Y\sim$ inverted gamma$(\frac{\nu}{2},\frac{\nu}{2})$, $X\sim N(0,1)$ and Y is independent of X, then $\sqrt{Y}X$ have the Student's t distribution with ν degrees of freedom.

Proof: Let $U=\sqrt{Y}X$ and $M=Y$, then

$$\begin{aligned}
f_U &= \int_0^{+\infty} f_{U,M}\,dm \\
&= \int_0^{+\infty} f_{Y,X}\begin{vmatrix} \frac{dy}{du} & \frac{dy}{dm} \\ \frac{dx}{du} & \frac{dx}{dm} \end{vmatrix} dm \\
&= \int_0^{+\infty} \frac{\frac{\nu}{2}^{\frac{\nu}{2}}}{\Gamma\frac{\nu}{2}} y^{-\frac{\nu}{2}-1} e^{-\frac{\nu}{2y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \frac{1}{m^{\frac{1}{2}}}\,dm \\
&= \int_0^{+\infty} \frac{\frac{\nu}{2}^{\frac{\nu}{2}}}{\Gamma\frac{\nu}{2}} m^{-\frac{\nu}{2}-1-\frac{1}{2}} e^{-\frac{\nu}{2m}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2m}}\,dm \\
&= \frac{\frac{\nu}{2}^{\frac{\nu}{2}}}{\Gamma\frac{\nu}{2}\sqrt{2\pi}} \int_0^{+\infty} m^{-\frac{\nu}{2}-1-\frac{1}{2}} e^{-\frac{1}{m}(\frac{\nu}{2}+\frac{u^2}{2})}\,dm \\
&= \frac{\Gamma\frac{\nu+1}{2}}{\Gamma\frac{\nu}{2}\sqrt{\nu\pi}} \frac{1}{(1+\frac{u^2}{\nu})^{\frac{\nu+1}{2}}}.
\end{aligned}$$

Therefore $\sqrt{Y}X$ have the Student's t distribution with ν degrees of freedom.

5. Prove $E(D_i|W=w)=\Phi\left(w^{\frac{1}{2}}t_\nu^{-1}(p)\right)$.

Proof:

$$\begin{aligned}
E(D_i|W=w) &= E_{X^G}\left[E\left(D_i\middle|W,X^G\right)\right] \\
&= E_{X^G}\left[\Phi\left(\frac{w^{\frac{1}{2}}t_\nu^{-1}(p)-\beta X^G}{\sqrt{1-\beta^2}}\right)\right] \\
&= \int_{-\infty}^{+\infty} \Phi\left(\frac{w^{\frac{1}{2}}t_\nu^{-1}(p)-\beta x}{\sqrt{1-\beta^2}}\right) d\Phi(x) \\
&= \int_{-\infty}^{+\infty} \phi(x) \int_{-\infty}^{\frac{w^{\frac{1}{2}}t_\nu^{-1}(p)-\beta x}{\sqrt{1-\beta^2}}} \phi(z)dz\,dx
\end{aligned}$$

Let $y = w^{\frac{1}{2}}(z\sqrt{1-\beta^2} + \beta x)$, then

$$\begin{aligned}
E\left(D_i\middle|W=w\right) &= \int_{-\infty}^{+\infty} \phi(x) \int_{-\infty}^{t_\nu^{-1}(p)} \phi\left(\frac{w^{\frac{1}{2}}y - \beta x}{\sqrt{1-\beta^2}}\right) d\left(\frac{w^{\frac{1}{2}}y}{1-\beta^2}\right) dx \\
&= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \int_{-\infty}^{t_\nu^{-1}(p)} \frac{w^{\frac{1}{2}}}{\sqrt{2\pi}\sqrt{1-\beta^2}} e^{-\frac{(w^{\frac{1}{2}}y-\beta x)^2}{2(1-\beta^2)}}\, dy\, dx \\
&= \frac{w^{\frac{1}{2}}}{2\pi\sqrt{1-\beta^2}} \int_{-\infty}^{+\infty} \int_{-\infty}^{t_\nu^{-1}(p)} e^{-\frac{x^2}{2} + \frac{wy^2 - 2\beta w^{\frac{1}{2}}xy + \beta^2 x^2}{2(1-\beta^2)}}\, dy\, dx \\
&= \frac{w^{\frac{1}{2}}y}{2\pi\sqrt{1-\beta^2}} \int_{-\infty}^{+\infty} \int_{-\infty}^{t_\nu^{-1}(p)} e^{-\frac{x^2 + wy^2 - 2\beta w^{\frac{1}{2}}xy}{2(1-\beta^2)}}\, dy\, dx \\
&= \int_{-\infty}^{t_\nu^{-1}(p)} \frac{w^{\frac{1}{2}}}{\sqrt{2\pi}} e^{-\frac{(1-\beta^2)wy^2}{2(1-\beta^2)}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sqrt{1-\beta^2}} e^{-\frac{x - 2\beta w^{\frac{1}{2}}y}{2(1-\beta^2)}}\, dx\, dy \\
&= \int_{-\infty}^{t_\nu^{-1}(p)} \frac{w^{\frac{1}{2}}}{\sqrt{2\pi}} e^{-\frac{wy^2}{2}}\, dy \\
&= \Phi\left(w^{\frac{1}{2}} t_\nu^{-1}(p)\right)
\end{aligned}$$

Therefore $E\left(D_i\middle|W=w\right) = \Phi\left(w^{\frac{1}{2}} t_\nu^{-1}(p)\right)$.

6. Prove $E\left(D_i^2\middle|W=w\right) = \Phi_2\left(w^{\frac{1}{2}} t_\nu^{-1}(p), w^{\frac{1}{2}} t_\nu^{-1}(p), \beta^2\right)$**.**

Proof:

$$\begin{aligned}
E\left(D_i^2\middle|W=w\right) &= E_{X^G}\left[E\left(D_i^2\middle|W, X^G\right)\right] \\
&= E_{X^G}\left[\Phi\left(\frac{w^{\frac{1}{2}} t_\nu^{-1}(p) - \beta X^G}{\sqrt{1-\beta^2}}\right) \Phi\left(\frac{w^{\frac{1}{2}} t_\nu^{-1}(p) - \beta X^G}{\sqrt{1-\beta^2}}\right)\right] \\
&= \int_{-\infty}^{+\infty} \phi(x) \Phi\left(\frac{w^{\frac{1}{2}} t_\nu^{-1}(p) - \beta X^G}{\sqrt{1-\beta^2}}\right) \Phi\left(\frac{w^{\frac{1}{2}} t_\nu^{-1}(p) - \beta X^G}{\sqrt{1-\beta^2}}\right) dx \\
&= \int_{-\infty}^{+\infty} \phi(x) \int_{-\infty}^{\frac{w^{\frac{1}{2}} t_\nu^{-1}(p) - \beta x}{\sqrt{1-\beta^2}}} \int_{-\infty}^{\frac{w^{\frac{1}{2}} t_\nu^{-1}(p) - \beta x}{\sqrt{1-\beta^2}}} \phi(z_1)\phi(z_2)\, dz_1\, dz_2\, dx.
\end{aligned}$$

Let $y_1 = w^{\frac{1}{2}}(z_1\sqrt{1-\beta^2}+\beta x)$ and $y_1 = w^{\frac{1}{2}}(z_2\sqrt{1-\beta^2}+\beta x)$, then

$$\begin{aligned}
E(D_i^2|W=w) &= \int_{-\infty}^{+\infty}\phi(x)\int_{-\infty}^{t_\nu^{-1}(p)}\int_{-\infty}^{t_\nu^{-1}(p)}\phi(\frac{w^{\frac{1}{2}}y_1-\beta x}{\sqrt{1-\beta^2}})\phi(\frac{w^{\frac{1}{2}}y_2-\beta x}{\sqrt{1-\beta^2}}) \\
&\quad \times d(\frac{w^{\frac{1}{2}}y_1}{1-\beta^2})d(\frac{w^{\frac{1}{2}}y_2}{1-\beta^2})\,dx \\
&= \int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\int_{-\infty}^{t_\nu^{-1}(p)}\int_{-\infty}^{t_\nu^{-1}(p)}\frac{1}{2\pi(1-\beta^2)}e^{(-\frac{(wy_1-\beta x)^2}{2(1-\beta^2)})}e^{-\frac{(wy_2-\beta x)^2}{2(1-\beta^2)}} \\
&\quad \times\ dy_1\,dy_2\,dx \\
&= \int_{-\infty}^{t_\nu^{-1}(p)}\int_{-\infty}^{t_\nu^{-1}(p)}\frac{1}{2\pi\sqrt{1-\beta^2}}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}\frac{w}{2\pi(1-\beta^2)} \\
&\quad \times e^{-\frac{x^2(1-\beta^2)+w{y_1}^2+w{y_2}^2-2\beta w^{\frac{1}{2}}x(y_1+y_2)}{2(1-\beta^2)}}\,dx\,dy_1\,dy_2 \\
&= \int_{-\infty}^{t_\nu^{-1}(p)}\int_{-\infty}^{t_\nu^{-1}(p)}\frac{1}{2\pi\sqrt{1-\beta^2}}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}\frac{w}{2\pi(1-\beta^2)} \\
&\quad \times e^{-\frac{(x(1+\beta^2)-\beta w^{\frac{1}{2}}({y_1}^2+y_2))^2+w{y_1}^2+w{y_2}^2-2\beta^2wy_1y_2}{2(1-\beta^2)(1+\beta^2)}}\,dx\,dy_1\,dy_2 \\
&= \int_{-\infty}^{t_\nu^{-1}(p)}\int_{-\infty}^{t_\nu^{-1}(p)}\frac{w^{-1}}{2\pi\sqrt{1-\beta^2}\sqrt{1+\beta^2}}e^{-\frac{w{y_1}^2+w{y_2}^2-2\beta^2wy_1y_2}{2(1-\beta^2)(1+\beta^2)}}\,dy_1\,dy_2 \\
&= \Phi_2\left(w^{\frac{1}{2}}t_\nu^{-1}(p), w^{\frac{1}{2}}t_\nu^{-1}(p), \beta^2\right),
\end{aligned}$$

where $\Phi_2\left(w^{\frac{1}{2}}t_\nu^{-1}(p), w^{\frac{1}{2}}t_\nu^{-1}(p), \beta^2\right)$ is a bivariate normal cumulative distribution function with,

$$\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and

$$\Sigma = \begin{pmatrix} 1 & \beta^2 \\ \beta^2 & 1 \end{pmatrix}$$

Therefore $E(D_i^2|W=w) = \Phi_2\left(w^{\frac{1}{2}}t_\nu^{-1}(p), w^{\frac{1}{2}}t_\nu^{-1}(p), \beta^2\right)$.

7. Prove $E(D_i) = p$.

Proof: From the results of $A.5$

$$\begin{aligned} E(D_i) &= E\left[E\left(D_i \middle| W = w\right)\right] \\ &= \int_0^{+\infty} \Phi\left(w^{\frac{1}{2}} t_\nu^{-1}(p)\right) f_w \, dw, \end{aligned}$$

by the result of $A.4$, we have

$$\begin{aligned} E(D_i) &= t_\nu\left(t_\nu^{-1}(p)\right) \\ &= p \end{aligned}$$

Therefore $E(D_i) = p$.

8. Prove $E(D_i^2) = t_{2,\nu}\left(t_\nu^{-1}(p), t_\nu^{-1}(p), \beta^2\right)$.

Proof: From the results of $A.6$

$$\begin{aligned} E(D_i^2) &= E\left[E(D_i^2|W = w)\right] \\ &= \int_0^{+\infty} \int_{-\infty}^{t_\nu^{-1}(p)} \int_{-\infty}^{t_\nu^{-1}(p)} \frac{w^{-1}}{2\pi\sqrt{1-\beta^4}} e^{-\frac{{wy_1}^2 + {wy_2}^2 - 2\beta^2 wy_1 y_2}{2(1-\beta^4)}} f_w \, dy_1 \, dy_2 \, dw \\ &= \int_{-\infty}^{t_\nu^{-1}(p)} \int_{-\infty}^{t_\nu^{-1}(p)} \int_0^{+\infty} \frac{\frac{\nu}{2}^{\frac{\nu}{2}}}{2\pi\sqrt{1-\beta^4}\Gamma\frac{\nu}{2}} w^{(\frac{\nu}{2}+1)-1} e^{-\frac{{wy_1}^2 + {wy_2}^2 - 2\beta^2 wy_1 y_2 + \nu(1-\beta^4)}{2(1-\beta^4)}} \\ &\qquad \times \, dw \, dy_1 \, dy_2 \\ &= \int_{-\infty}^{t_\nu^{-1}(p)} \int_{-\infty}^{t_\nu^{-1}(p)} \frac{\frac{\nu}{2}^{\frac{\nu}{2}}}{2\pi\sqrt{1-\beta^4}\Gamma\frac{\nu}{2}} \frac{\Gamma(\frac{\nu}{2}+1)}{(\frac{{wy_1}^2 + {wy_2}^2 - 2\beta^2 wy_1 y_2 + \nu(1-\beta^4)}{2(1-\beta^4)})^{\frac{\nu}{2}+1}} \, dw \, dy_1 \, dy_2 \\ &= \int_{-\infty}^{t_\nu^{-1}(p)} \int_{-\infty}^{t_\nu^{-1}(p)} \frac{1}{2\pi\sqrt{1-\beta^4}} (1 + \frac{{wy_1}^2 + {wy_2}^2 - 2\beta^2 wy_1 y_2}{\nu(1-\beta^4)})^{-(\frac{\nu}{2}+1)} \, dw \, dy_1 \, dy_2 \\ &= t_{2,\nu}\left(t_\nu^{-1}(p), t_\nu^{-1}(p), \beta^2\right), \end{aligned}$$

where $t_{2,\nu}\left(t_\nu^{-1}(p), t_\nu^{-1}(p), \beta^2\right)$ is a bivariate Student's t cumulative distribution function with $df = \nu$,

$$\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and

$$\Sigma = \begin{pmatrix} 1 & \beta^2 \\ \beta^2 & 1 \end{pmatrix}$$

Therefore $E(D_i^2) = t_{2,\nu}\left(t_\nu^{-1}(p), t_\nu^{-1}(p), \beta^2\right)$.

Appendix B

Computation Algorithm

B.1 Evaluation for Gaussian Copula

Specification of Factors

To evaluate the exact loss distribution and the asymptotic approximations, we specify the values for the factors:

Total number of names: $n = 125$;

Nominal amount of name i: $N_{ml} = \text{EUR}10,000,000$;

Proportion of default exposure of name i: $A_i = 100\%$;

Recovery rate of name i: $R_i = 40\%$;

Default probability of name i: $p = 0.029$;

Correlation coefficient in factor model: $\beta^2 = 0.0001, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9$.

We would like to compare the performance of each limiting loss distribution with the exact loss distribution at ten levels of β^2.

B.1.1 Computation Algorithm

Step 1. Calculate the probability density function (PDF) and the cumulative distribution function (CDF) of the portfolio loss by using the Vasicek model and each asymptotic approximation.

We use R for computation. For the Vasicek model, we get the conditional probability of the binomial distribution in R. The Gaussian-Hermite quadrature method is applied to get the approximation of the integration with respect to the Gaussian kernel in the Vasicek model. Please refer [6], [31], [38], [50]and [51] for the numerical method of the Gaussian-Hermite quadrature in detail.

Step 2. Calculate the Hellinger distance between the Vasicek model and each asymptotic approximation.

Step 3. Repeated step 1 to step 2 at each level of β^2.

B.1.2 Continuity Correction

Since all the asymptotic approximations use the asymptotic normal distributions to approximate the default loss of the large pool of credit securities, we consider to improve the approximation accuracy by using continuity correction. Let's take the asymptotic approximation Approx3 as the example. Its continuity-corrected normal approximation of the binomial distribution is given by

$$\sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{N-i} \approx 1 - \Phi\left[(\frac{k + \frac{1}{2} + a}{n + 2a} - p) / \sqrt{\frac{p(1-p)}{n}} \right], \tag{B.1}$$

where $a \in [0, 1]$ is the continuity correction factor [21]. The value $a = 0$ is used most commonly in Equation B.1. Since we are interested in the left tail of the loss distribution, we choose $a = \frac{1}{2}$ to improve the approximation accuracy. Table B.1 and Table B.2

give the Hellinger distance between each asymptotic approximation to the exact loss distribution at different levels of correlation coefficient β^2 for $a = 0$, while Table 4.1 and Table 4.2 in Chapter 4 give the Hellinger distances for $a = \frac{1}{2}$. It is clear that $a = \frac{1}{2}$ is a better choice, since it gives more consistent results.

Table B.1: Hellinger Distance of Loss Distributions at Different Level of Correlation Coefficient by Gaussian Copula–Continuity Correction $a = 0$ (part1)

Approx.#	$\beta^2 = 0.0001$	$\beta^2 = 0.1$	$\beta^2 = 0.2$	$\beta^2 = 0.3$	$\beta^2 = 0.4$
2	0.412230	0.041199	0.013248	0.005132	0.002354
3	0.004068	0.041630	0.094085	0.156620	0.223258
4	0.410782	0.018624	0.025757	0.030521	0.036688
5	0.004105	0.017836	0.029525	0.034306	0.034786
5a	0.019730	0.010829	0.002240	0.002116	0.002823
5b	0.015740	0.006755	0.006800	0.027531	0.066632
6	0.004105	0.016850	0.026552	0.030374	0.034931
6a	0.019730	0.015827	0.006400	0.002836	0.001355
6b	0.015740	0.008298	0.003926	0.001717	0.000621
7a	0.019734	0.018553	0.007917	0.003592	0.001767
7b	0.015744	0.010644	0.004906	0.002444	0.001140

Table B.2: Hellinger Distance of Loss Distributions at Different Level of Correlation Coefficient by Gaussian Copula–Continuity Correction $a = 0$ (part2)

Approx.#	$\beta^2 = 0.5$	$\beta^2 = 0.6$	$\beta^2 = 0.7$	$\beta^2 = 0.8$	$\beta^2 = 0.9$
2	0.001128	0.000533	0.000246	0.000112	0.000051
3	0.291102	0.358979	0.426735	0.495266	0.568062
4	0.048407	0.065622	0.086791	0.111151	0.140104
5	0.035949	0.042186	0.054278	0.070605	0.090278
5a	0.003741	0.005568	0.011973	0.028279	0.057077
5b	NaN	NaN	NaN	NaN	NaN
6	0.045654	0.062592	0.083793	0.108253	0.137262
6a	0.000667	0.000337	0.000180	0.000106	0.000072
6b	0.000470	0.000564	0.000652	0.000893	0.002042
7a	0.000888	0.000442	0.000218	0.000108	0.000053
7b	0.000472	0.000167	0.000060	0.000046	0.000054

B.1.3 Effect of Number of Nodes on Integration by Gaussian-Hermite Quadrature

Nodes and Weights by Gaussian-Hermite Quadrature Method

In order to numerically approximate the exact loss distribution by using the Vasicek model, we use the Gaussian-Hermite quadrature method to approximate the integration with respect to the Gaussian kernel. Figure B.1 and Figure B.2 demonstrate the relationship between the nodes and the relative weights from the Gaussian-Hermite quadrature.

Effect of Number of Nodes on Conditional CDF

We study the relationship between the integrand and the domain of integration by using different number of nodes from the Gaussian-Hermite quadrature method. The integrand is the conditional CDF of the binomial distribution in the Vasicek model. From Figure B.3, Figure B.4 and Figure B.5 we find the small values in the domain of integration determine the result of the integration. As the value of β^2 is increasing, the number of nodes has more effect on the integrand.

Effect of Number of Nodes on PDF

We study the Hellinger distance between the probability density function of the Vasicek model approximated by using different numbers of nodes and that approximated by using 1000 nodes from the Gaussian-Hermite quadrature method. Table B.3 and Table B.4 give us the results. The first 18 rows are the Hellinger distances for the number of nodes from 10 up to 900 by using the Gaussian-Hermite quadrature method. The last row of the results is given by using the 101 nodes that make the equal subintervals from -5 to 5. It is clear that the result of 700 nodes is comparable to that of 101 nodes from

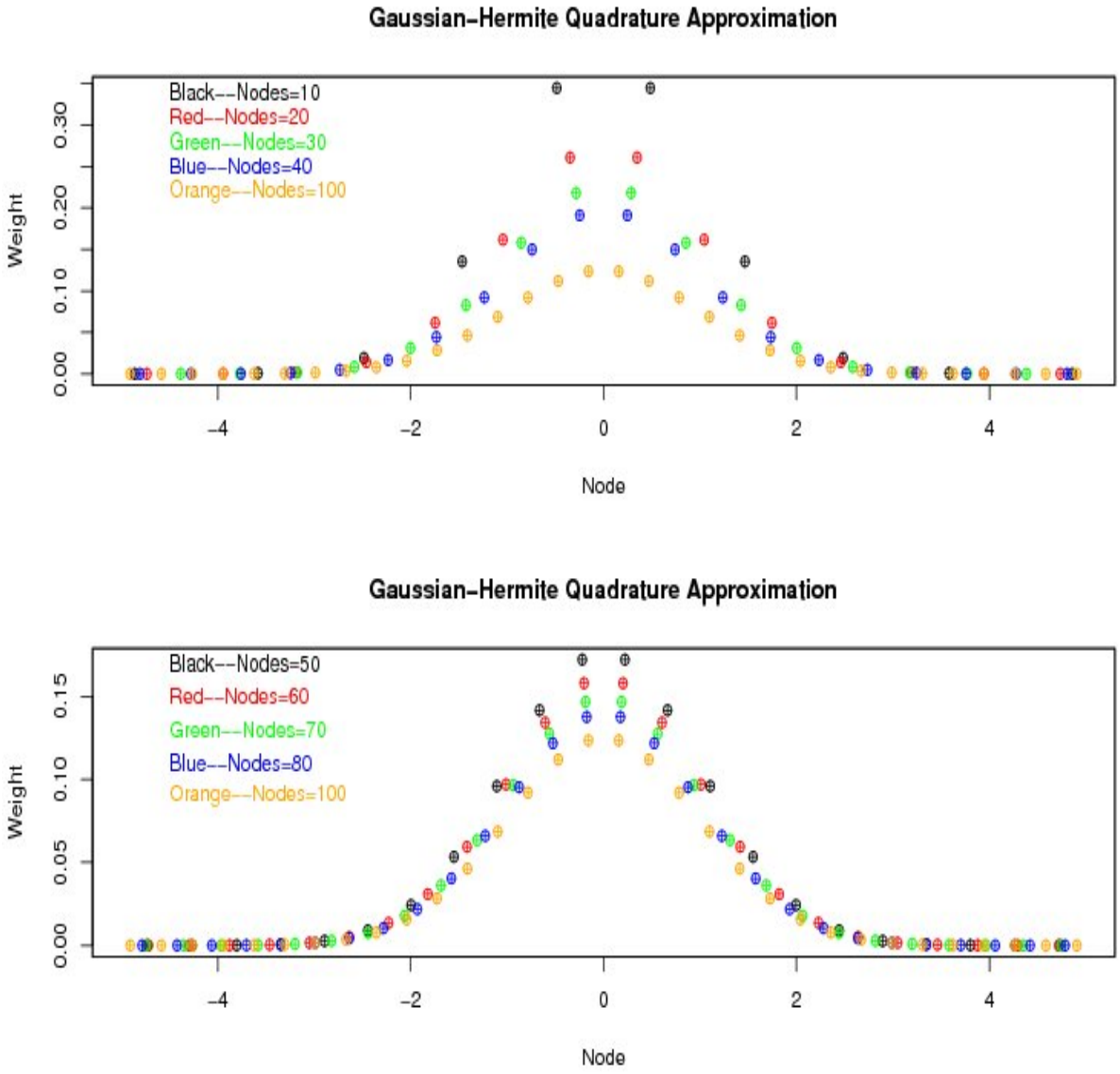

Figure B.1: Nodes vs Weights from Gaussian-Hermite Quadrature (part1)

-5 to 5. As the small values in the domain of integration have a significant effect on the result of the integration, we use 2000 nodes to approximate the loss distribution of the

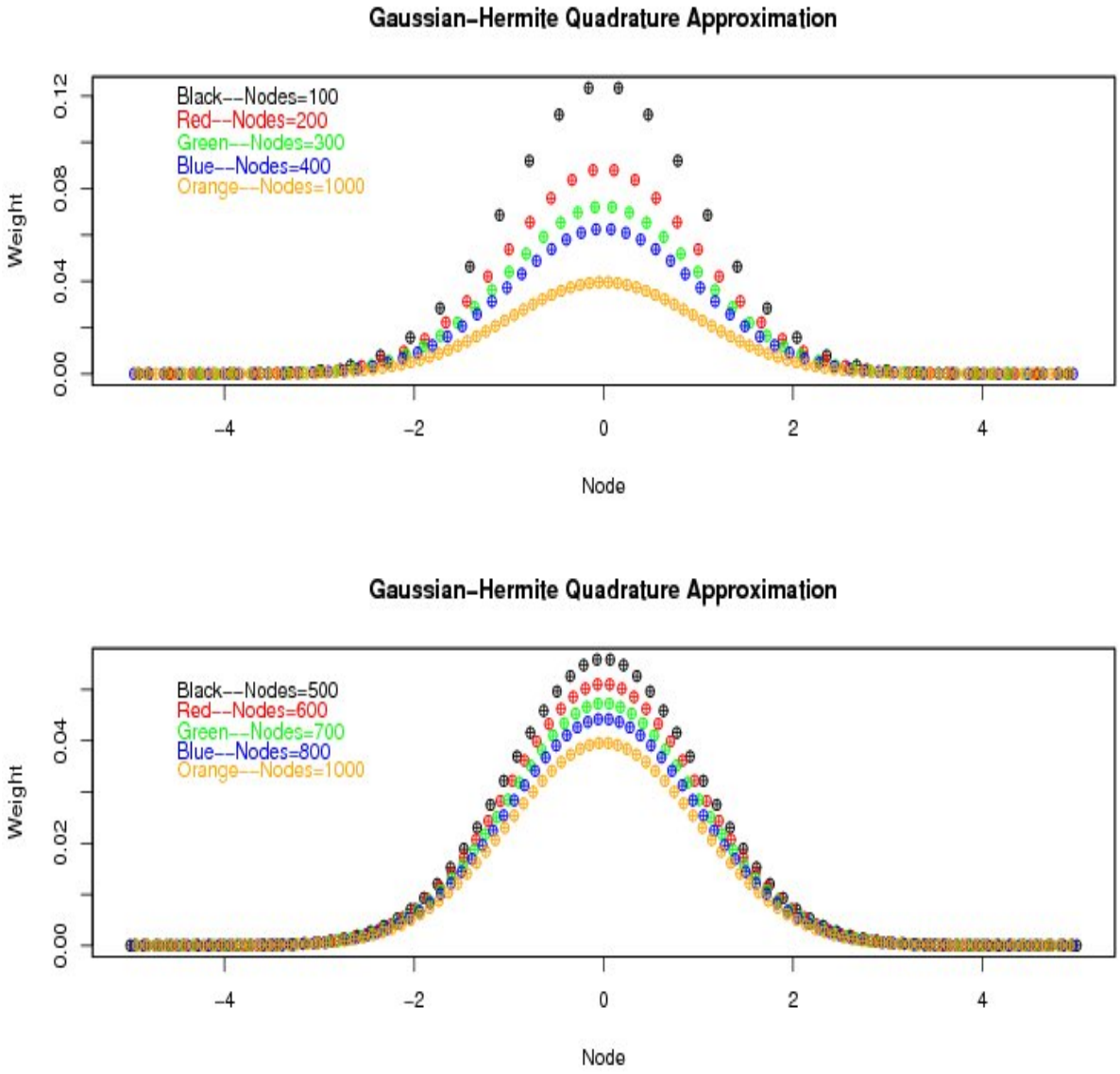

Figure B.2: Nodes vs Weights from Gaussian-Hermite Quadrature (part2)

Vasicek model.

Table B.3: Nodes Study–Hellinger Distance of Loss Distributions at Different Level of Correlation Coefficient by Gaussian Copula (part1)

# Nodes	$\beta^2 = 0.0001$	$\beta^2 = 0.1$	$\beta^2 = 0.2$	$\beta^2 = 0.3$	$\beta^2 = 0.4$
10	0.0000000	1.18e-05	0.0006605	0.0031119	0.0072447
20	0.0000000	0.0000000	0.0000270	0.0003718	0.0015016
30	0.0000000	0.0000000	0.0000018	0.0000661	0.0004326
40	0.0000000	0.0000000	0.0000002	0.0000141	0.0001464
50	0.0000000	0.0000000	0.0000000	0.0000033	0.0000543
60	0.0000000	0.0000000	0.0000000	0.0000008	0.0000214
70	0.0000000	0.0000000	0.0000000	0.0000002	0.0000087
80	0.0000000	0.0000000	0.0000000	0.0000001	0.0000037
90	0.0000000	0.0000000	0.0000000	0.0000000	0.0000016
100	0.0000000	0.0000000	0.0000000	0.0000000	0.0000007
200	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
300	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
400	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
500	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
600	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
700	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
800	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
900	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
101	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000

Effect of Number of Nodes on Conditional Expected Loss of Equity Tranche

We study the relationship between the conditional expected loss of the equity tranche and the domain of integration by using different numbers of nodes from the Gaussian-Hermite quadrature method and by using the 101 nodes that make the 100 equal subinterval from -5 to 5. Figure B.6 demonstrates their relationship at $\beta^2 = 0.14$. We say the result of the integration depends on the small values in the domain of integration.

Table B.4: Nodes Study–Hellinger Distance of Loss Distribution at Different Level of Correlation Coefficient by Gaussian Copula (part2)

# Nodes	$\beta^2 = 0.5$	$\beta^2 = 0.6$	$\beta^2 = 0.7$	$\beta^2 = 0.8$	$\beta^2 = 0.9$
10	0.0123095	0.0173454	0.0216668	0.0245927	0.0223678
20	0.0035774	0.0063130	0.0090918	0.0114057	0.0125421
30	0.0013899	0.0030634	0.0053133	0.0077000	0.0087839
40	0.0006225	0.0016795	0.0034076	0.0055720	0.0075383
50	0.0003010	0.0009720	0.0022775	0.0042393	0.0064069
60	0.0001534	0.0005893	0.0015740	0.0032303	0.0050034
70	0.0000810	0.0003684	0.0011184	0.0025793	0.0047133
80	0.0000439	0.0002360	0.0008077	0.0020582	0.0040163
90	0.0000243	0.0001539	0.0005929	0.0016620	0.0034626
100	0.0000136	0.0001019	0.0004408	0.0013598	0.0031352
200	0.0000001	0.0000024	0.0000329	0.0002396	0.0011682
300	0.0000000	0.0000001	0.0000032	0.0000547	0.0005285
400	0.0000000	0.0000000	0.0000003	0.0000138	0.0002638
500	0.0000000	0.0000000	0.0000000	0.0000036	0.0001409
600	0.0000000	0.0000000	0.0000000	0.0000010	0.0000809
700	0.0000000	0.0000000	0.0000000	0.0000003	0.0000404
800	0.0000000	0.0000000	0.0000000	0.0000001	0.0000134
900	0.0000000	0.0000000	0.0000000	0.0000000	0.0000023
101	0.0000000	0.0000000	0.0000000	0.0000000	0.0000209

Effect of Number of Nodes on Expected Loss of Equity Tranche

As the expected tranche loss is of our interest, we study the relationship between the expected loss of the equity tranche and the base correlation β^2 by using different numbers of nodes from the Gaussian-Hermite quadrature method and by using the 101 nodes that make 100 equal subinterval from -5 to 5. Figure B.7 demonstrates their relationship. We say 30 nodes are enough for evaluating the expected tranche loss. Therefore we use 30 nodes for the pricing of iTraxx in Chapter 6.

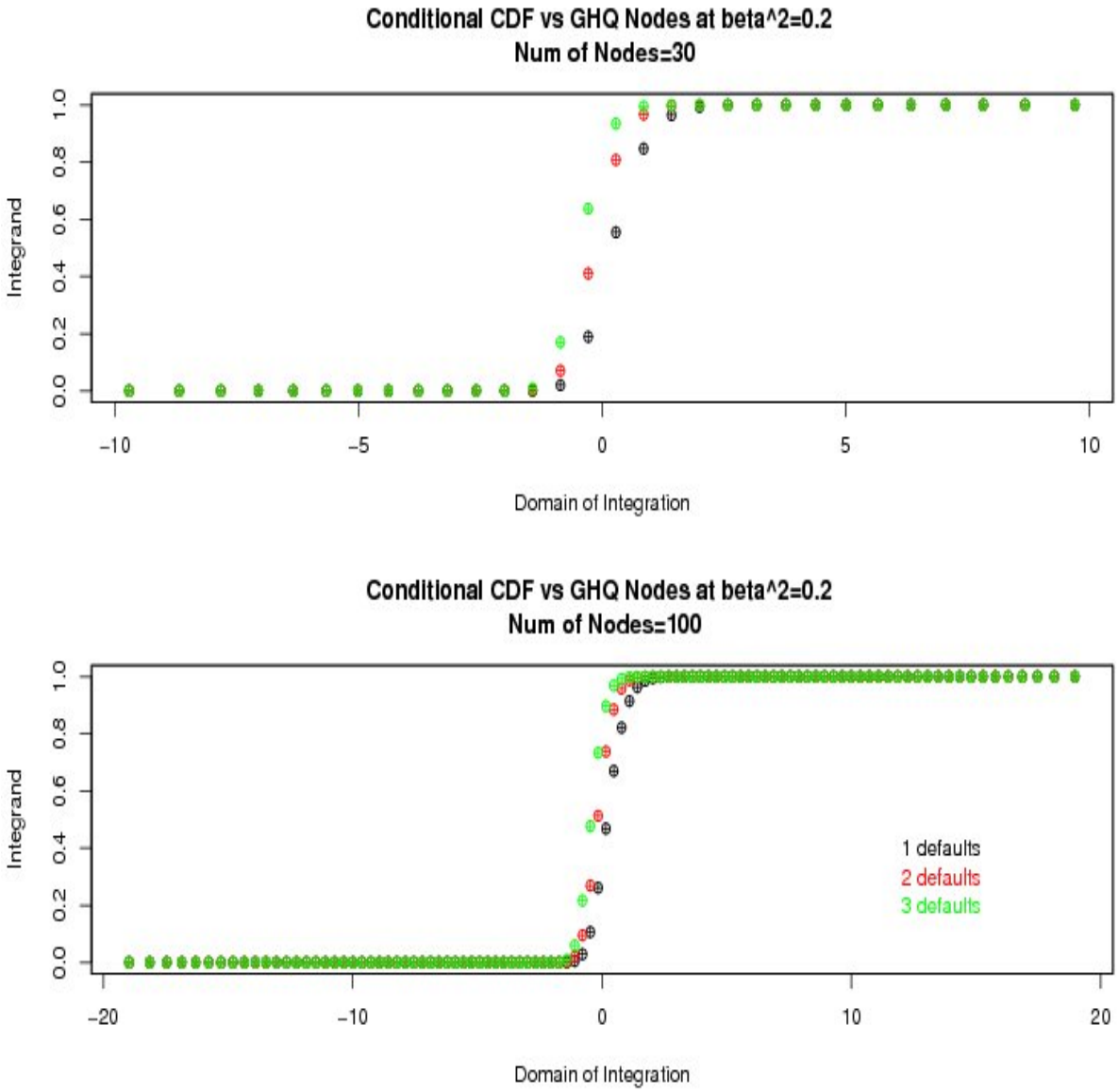

Figure B.3: Conditional Binomial CDF at $\beta^2 = 0.2$ with Different Number of Nodes from Gaussian-Hermite Quadrature

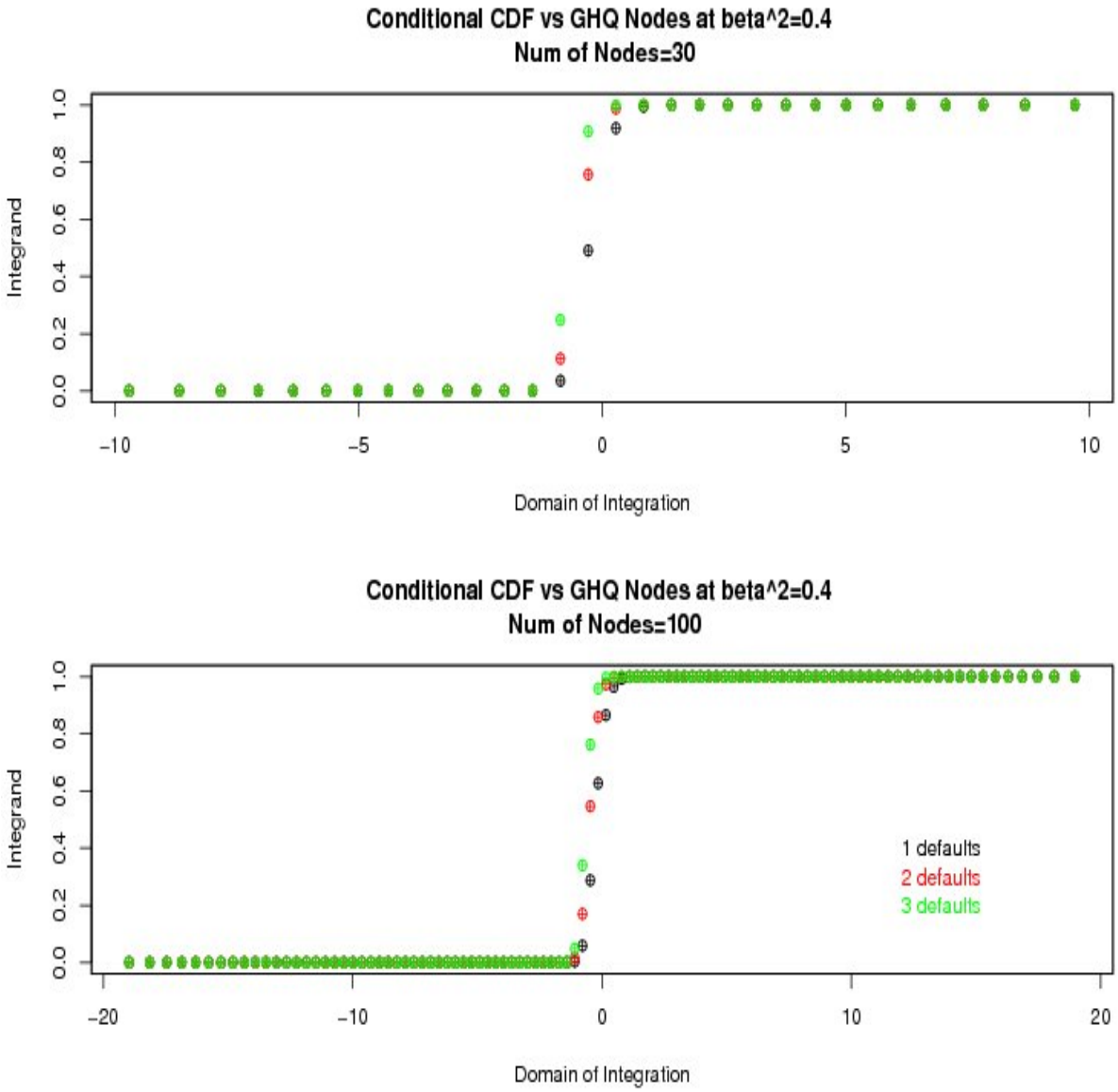

Figure B.4: Conditional Binomial CDF at $\beta^2 = 0.4$ with Different Number of Nodes from Gaussian-Hermite Quadrature

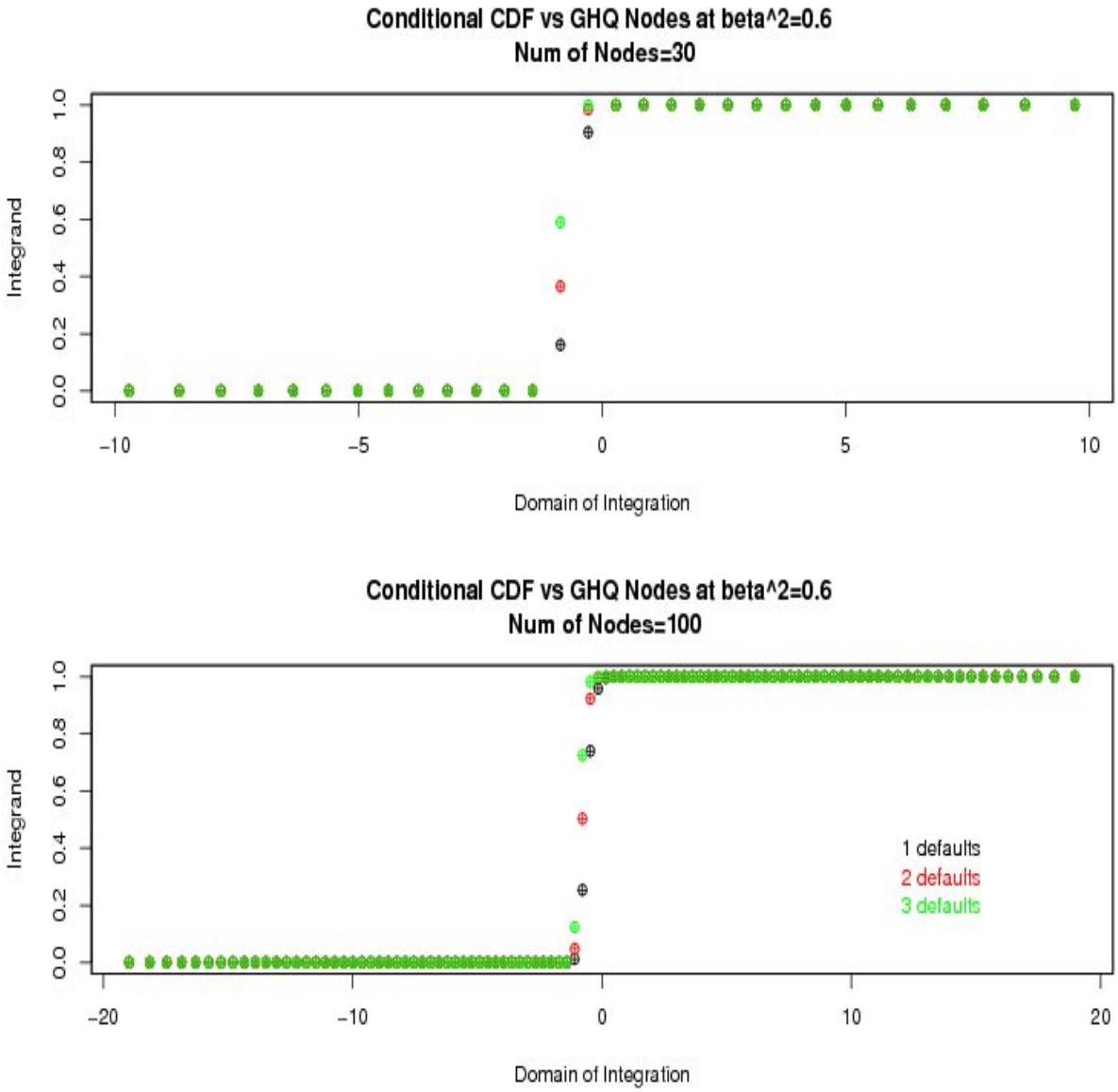

Figure B.5: Conditional Binomial CDF at $\beta^2 = 0.6$ with Different Number of Nodes from Gaussian-Hermite Quadrature

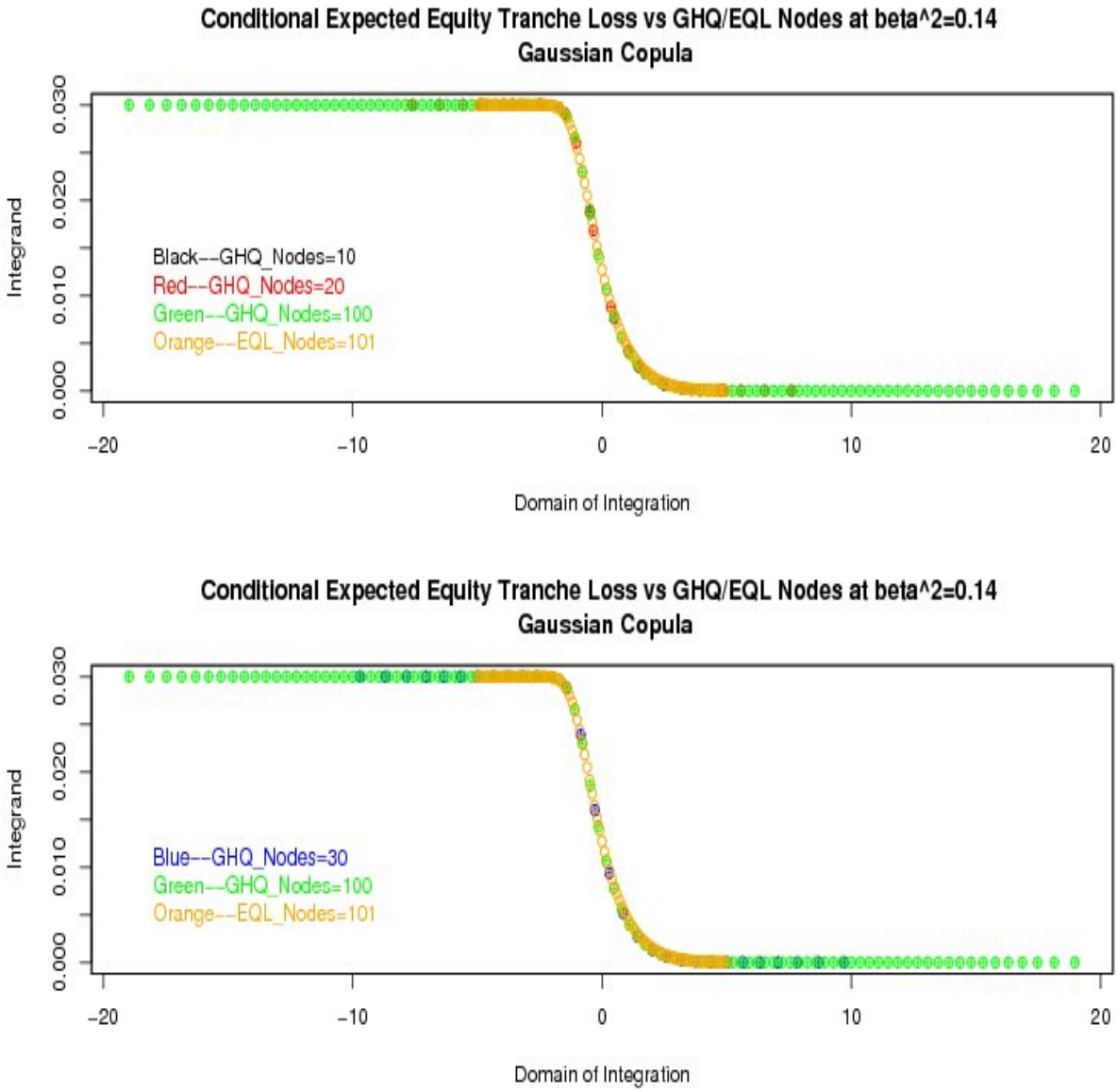

Figure B.6: Conditional Expected Loss of Equity Tranche at $\beta^2 = 0.14$ with Different Number of Nodes from Gaussian-Hermite Quadrature

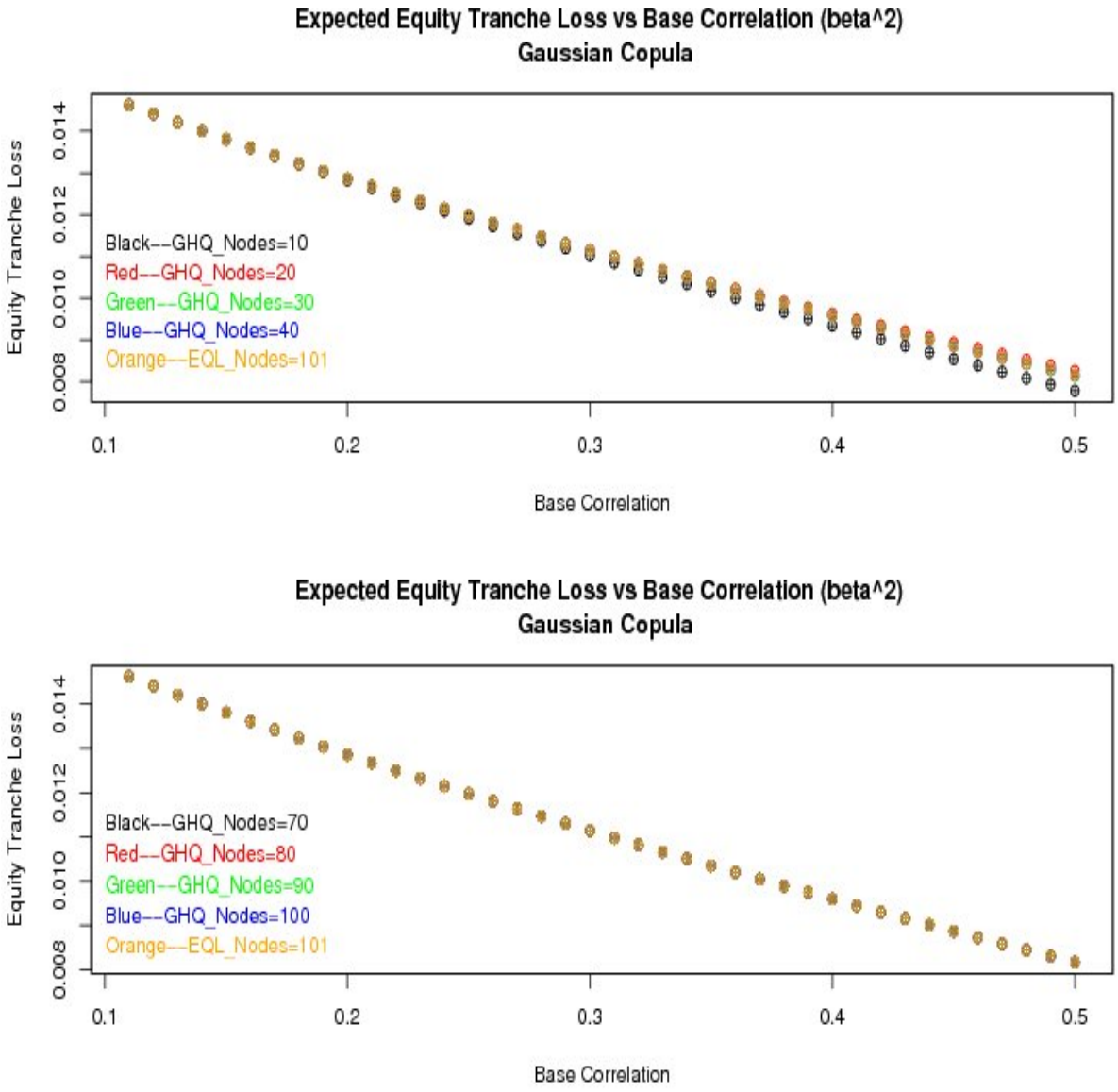

Figure B.7: Expected Loss of Equity Tranche vs β^2 with Different Number of Nodes from Gaussian-Hermite Quadrature

B.2 Evaluation for Student's t Copula

Specification of Factors

We use the same specification of the factors as the Gaussian copula.

B.2.1 Computation Algorithm

Step 1. Calculate the PDF and the CDF of the loss distributions by using the exact model and each asymptotic approximation.

For the exact model, the Gaussian-Hermite quadrature method is used for the integral with respect to the common risk factor X^G and the Gaussian-Laugerre quadrature method is used for the integral with respect to W that follows the distribution of gamma$(\frac{\nu}{2}, \frac{2}{\nu})$.

All the computation is done in R. For the asymptotic approximations, we use $a = \frac{1}{2}$ for continuity correction.

Step 2. Calculate the Hellinger distance between the exact model and each approximation.

Step 3. Repeated step 1 to step 2 at each level of β^2.

B.2.2 Effect of Number of Nodes on Integration by Gaussian-Laugerre Quadrature

Effect of Number of Nodes on PDF

We use the Gaussian-Hermite quadrature method to approximate the integral with respect to the common risk factor X^G and use the Gaussian-Laugerre quadrature method to approximate the integral with respect to W. We study the numeric approximation of the probability density function of the exact model by using different numbers of nodes

from the Gaussian quadrature methods. We calculate the Hellinger distance between the numerically approximated probability density functions by using different numbers of nodes from the Gaussian-Laugerre quadrature & the Gaussian-Hermite quadrature and that has 100 nodes from the Gaussian-Laugerre quadrature combined with the 101 nodes that make 100 equal subintervals from -5 to 5. Table B.5 and Table B.6 demonstrate the results. The first 10 rows are the results for the number of nodes from 10 (combined with 30 nodes from the Gaussian-Hermite quadrature) up to 100 (combined with 30 nodes from the Gaussian-Hermite quadrature) by using the Gaussian-Laugerre quadrature method. The next 9 rows of results are given by using the number of nodes from 10 (combined with 101 nodes making the equal subintervals from -5 to 5) up to 100 (combined with 101 nodes making the equal subintervals from -5 to 5) from the Gaussian-Laugerre quadrature. It is clear that the integral with respect to the common risk factor X^G has more effect on the result of the whole integration. As the small values in the domain of integration determine the accuracy of the integration, we use 1000 nodes for the outer integral with respect to W that has the gamma distribution and use 1000 nodes for the inner integral with respect to X^G that has the normal distribution.

Effect of Number of Nodes on Conditional Expected Loss of Equity Tranche

We study the relationship between the conditional expected value of the equity tranche loss and the domain of integration with different numbers of nodes from the Gaussian quadrature methods. Figure B.8 demonstrates their relationship at $\beta^2 = 0.14$. We say the result of the integration depends mainly on the small values in the domain of integration.

Table B.5: Nodes Study–Hellinger Distance of Loss Distribution at Different Level of Correlation Coefficient by Student's t Copula (part1)

# Nodes GLQ	# Nodes GHQ	$\beta^2 = 0.0001$	$\beta^2 = 0.1$	$\beta^2 = 0.2$	$\beta^2 = 0.3$	$\beta^2 = 0.4$
10	30	0.0000000	0.0000000	0.0000000	0.0000018	0.0000239
20	30	0.0000000	0.0000000	0.0000000	0.0000018	0.0000239
30	30	0.0000000	0.0000000	0.0000000	0.0000018	0.0000239
40	30	0.0000000	0.0000000	0.0000000	0.0000018	0.0000239
50	30	0.0000000	0.0000000	0.0000000	0.0000018	0.0000239
60	30	0.0000000	0.0000000	0.0000000	0.0000018	0.0000239
70	30	0.0000000	0.0000000	0.0000000	0.0000018	0.0000239
80	30	0.0000000	0.0000000	0.0000000	0.0000018	0.0000239
90	30	0.0000000	0.0000000	0.0000000	0.0000018	0.0000239
100	30	0.0000000	0.0000000	0.0000000	0.0000018	0.0000239
10	101	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
20	101	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
30	101	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
40	101	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
50	101	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
60	101	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
70	101	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
80	101	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
90	101	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000

Effect of Number of Nodes on Expected Loss of Equity Tranche

As the expected tranche loss is of our interest, we study the relationship between the expected loss of the equity tranche and the base correlation by using different numbers of nodes from the Gaussian quadrature methods. Figure B.9 demonstrates their relationship. We say 10 nodes of the Gaussian-Laugerre quadrature for the integral with respect to W and 30 node of the Gaussian-Hermite quadrature for the integral with respect to X^G are enough for evaluating the expected tranche loss. Therefore we use 10 by 30 nodes for the pricing of iTraxx in Chapter 6.

Table B.6: Nodes Study–Hellinger Distance of Loss Distribution at Different Level of Correlation Coefficient by Student's t Copula (part2)

# Nodes GLQ	# Nodes GHQ	$\beta^2 = 0.5$	$\beta^2 = 0.6$	$\beta^2 = 0.7$	$\beta^2 = 0.8$	$\beta^2 = 0.9$
10	30	0.0001186	0.0003489	0.0007502	0.0012948	0.0016985
20	30	0.0001186	0.0003491	0.0007510	0.0012937	0.0016836
30	30	0.0001186	0.0003491	0.0007510	0.0012937	0.0016834
40	30	0.0001186	0.0003491	0.0007510	0.0012937	0.0016834
50	30	0.0001186	0.0003491	0.0007510	0.0012937	0.0016834
60	30	0.0001186	0.0003491	0.0007510	0.0012937	0.0016834
70	30	0.0001186	0.0003491	0.0007510	0.0012937	0.0016834
80	30	0.0001186	0.0003491	0.0007510	0.0012937	0.0016834
90	30	0.0001186	0.0003491	0.0007510	0.0012937	0.0016834
100	30	0.0001186	0.0003491	0.0007510	0.0012937	0.0016834
10	101	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
20	101	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
30	101	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
40	101	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
50	101	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
60	101	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
70	101	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
80	101	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
90	101	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000

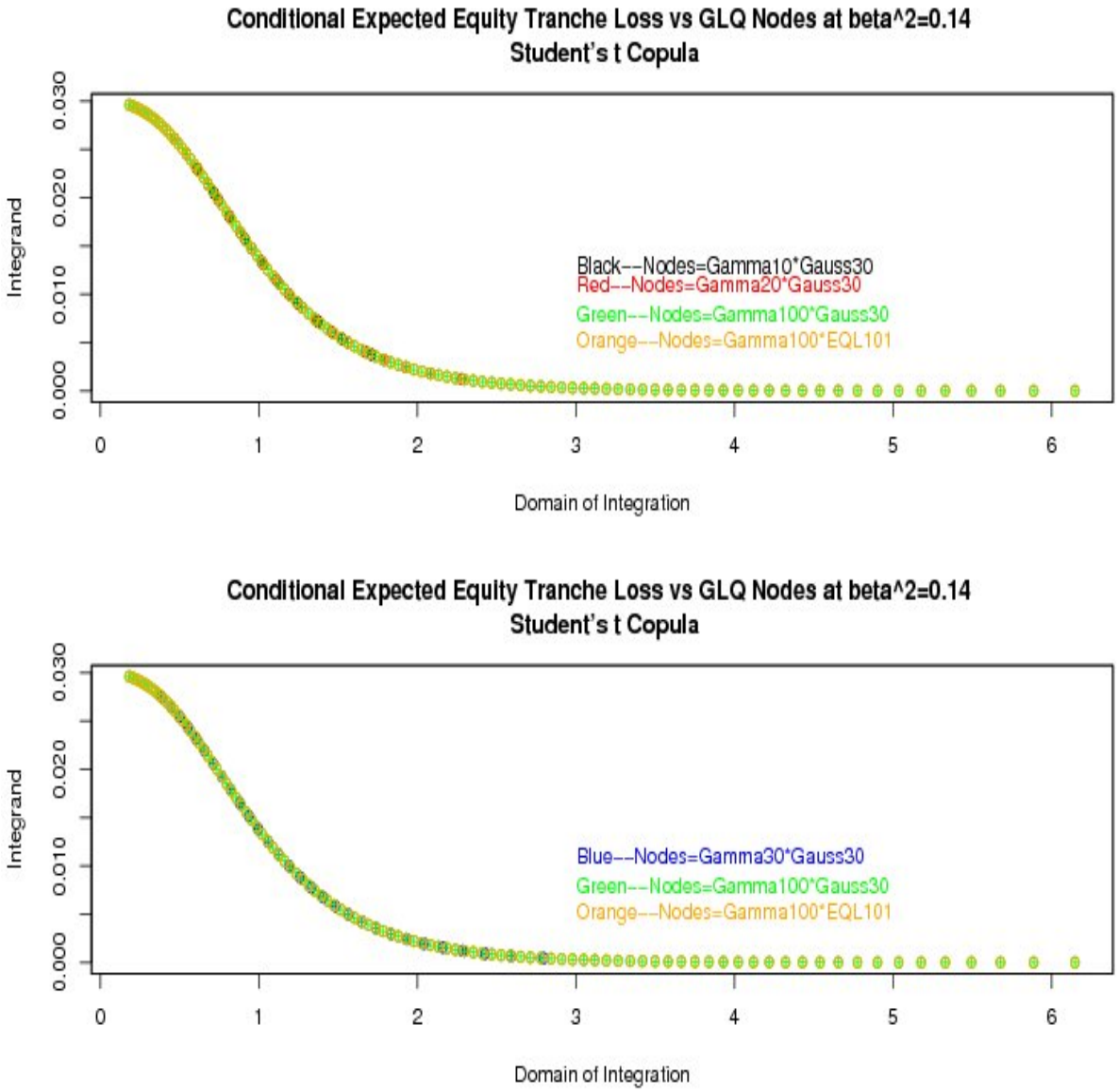

Figure B.8: Conditional Expected Loss of Equity Tranche at $\beta^2 = 0.14$ with Different Number of Nodes from Gaussian-Laguerre Quadrature

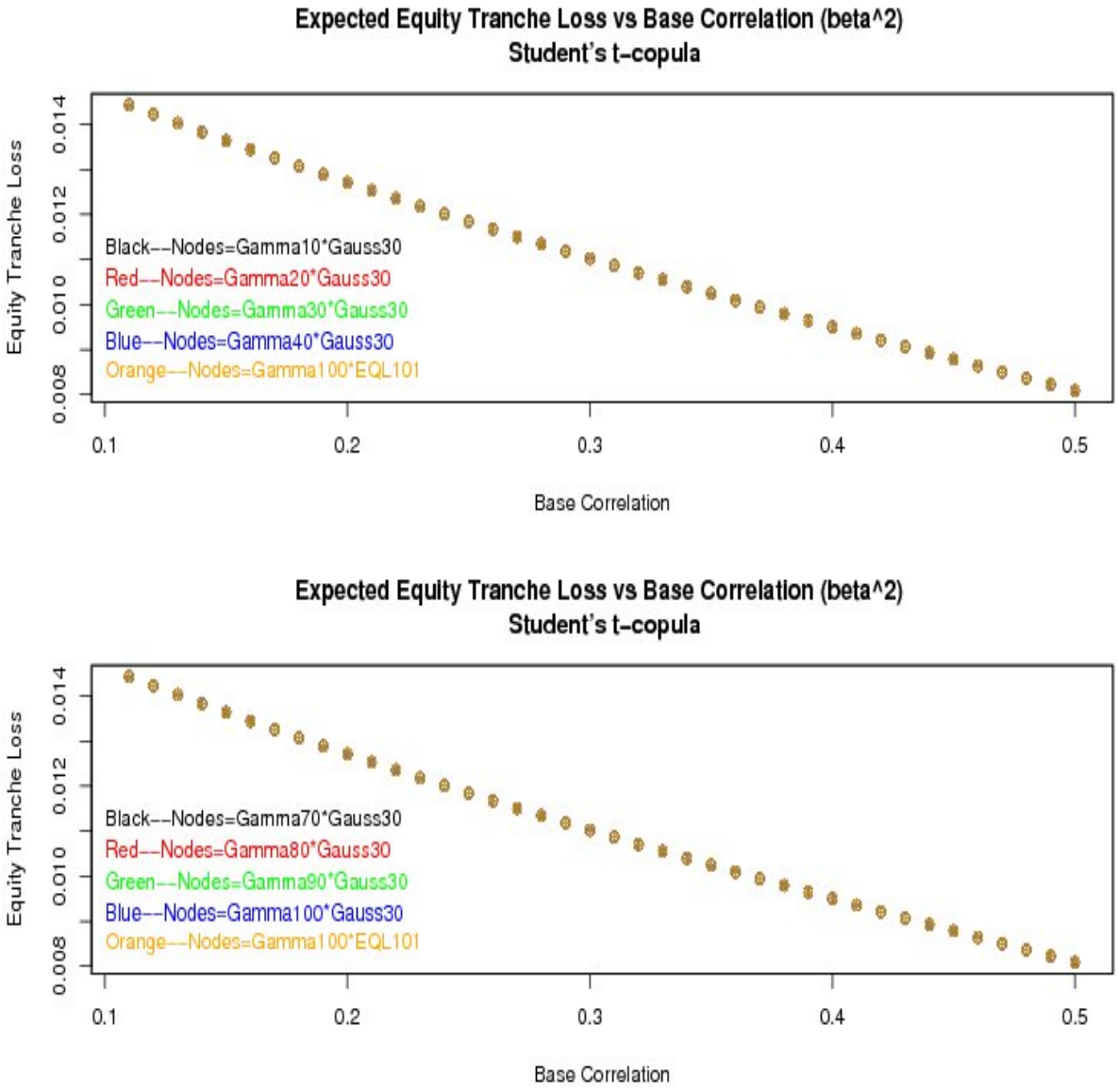

Figure B.9: Expected Loss of Equity Tranche vs β^2 with Different Number of Nodes from Gaussian-Laguerre Quadrature

Appendix C

Programs Used for Evaluation

C.1 R Function for Calibration by Gaussian Copula

```
library(statmod)

BaseCorrCal1<-function(beta1, beta2, beta3, beta4, beta5, beta6, m){

cprob1loop<-matrix(0, 126, 6)
prob1loop<-matrix(0, 126, 6)

FTL1_innerloop<-matrix(0, 126, 6)
FTL1loop<-matrix(0, 6, 1)
TL1loop<-matrix(0, 6, 1)
S1_horizonloop<-matrix(0, 6, 1)
S1_Rateloop<-matrix(0, 6, 1)

S1_DFactorloop<-matrix(0, PTimes, 6)
TL1_DFactor_deltaloop<-matrix(0, PTimes, 6)
RDuration1loop<-matrix(0, 6, 1)
PLeg1loop<-matrix(0, 6, 1)
DLeg1loop<-matrix(0, 6, 1)
FairSpd1loop<-matrix(0, 6, 1)
MTM1mloop<-matrix(0, 1, 1)
```

```
x_w<-gauss.quad.prob(30,"normal")

for (b in 1:m){
if (b==1) beta=beta1
if (b==2) beta=beta2
if (b==3) beta=beta3
if (b==4) beta=beta4
if (b==5) beta=beta5
if (b==6) beta=beta6

for (k in 1:126)
{
cncdf<-matrix(0, 30, 1)
for (i in 1:30)
{
cnpdf=pnorm(1/(sqrt(1-beta^2))*(qnorm(pdi)-beta*x_w \$ nodes[i]))
cncdf[i,1]=pbinom(k-1, 125, cnpdf)
}
cprob1loop[k, b]=x_w \$ weights \%*\% cncdf
if (k==1) prob1loop[k, b]=cprob1loop[k, b] else prob1loop[k, b]
  =cprob1loop[k, b]-cprob1loop[k-1, b]
FTL1_innerloop[k, b]=min(max((k-1)*(1-R_Indx)/n, 0), TUpper[b])
}
FTL1loop[b, 1]=t(FTL1_innerloop[ , b])\%*\% prob1loop[ , b]
if (b==1) TL1loop[b, 1]=FTL1loop[b, 1] else TL1loop[b, 1]
      =FTL1loop[b, 1]-FTL1loop[b-1, 1]
S1_horizonloop[b, 1]=1-TL1loop[b, 1]/(TUpper[b]-TLower[b])
S1_Rateloop[b, 1]=PPY*(S1_horizonloop[b, 1]^-(1/(PPY*horizon))-1)

for (TIdx in 1:PTimes)
{
if (TIdx==1) P2DatesYr[TIdx, 1]=(PDates[TIdx]-VDate)/360
else P2DatesYr[TIdx, 1]=(PDates[TIdx]-PDates[TIdx-1])/360
PDatesYr[TIdx, 1]=as.numeric(PDates[TIdx]-VDate)/360
RFDFactor[TIdx, 1]=(1+RFRate/PPY)^(-PPY*PDatesYr[TIdx, 1])
S1_DFactorloop[TIdx, b]
=(1+S1_Rateloop[b, 1]/PPY)^(-PPY*PDatesYr[TIdx, 1])
if (TIdx==1) TL1_DFactor_deltaloop[TIdx, b]=1-S1_DFactorloop[TIdx, b]
else TL1_DFactor_deltaloop[TIdx, b]
 =S1_DFactorloop[TIdx-1, b]-S1_DFactorloop[TIdx, b]
}
```

```
RFDuration=t(RFDFactor)\%*\% P2DatesYr
RDuration1loop[b, 1]=t(S1_DFactorloop[ , b]*RFDFactor)\%*\% P2DatesYr
PLeg1loop[b, 1]=t(TL1_DFactor_deltaloop[ , b])
    \%*\% RFDFactor*Ntnl*(TUpper[b]-TLower[b])
FairSpd1loop[b, 1]=PLeg1loop[b, 1]/(RDuration1loop[b, 1]
       *Ntnl*(TUpper[b]-TLower[b]))
MTM1mloop
=((FairSpd1loop[b, 1]-TSpd[b])*RDuration1loop[b, 1]-TFee[b])*100
}
}
```

C.2 R Function for Calibration by Student's t Copula

```
library(statmod)

BaseCorrCaldf<-function(x){

degreef<-x[1]   #degree of freedom
y_w<-gauss.quad.prob(10,"gamma", alpha=degreef/2, beta=2/degreef)
#For gamma integration with respect to W
x_w<-gauss.quad.prob(30,"normal")
beta1<-x[2]
beta2<-x[2]
beta3<-x[2]
beta4<-x[2]
beta5<-x[2]
beta6<-x[2]

m<-5

FTL1loop<-matrix(0, 6, 1)
TL1loop<-matrix(0, 6, 1)
S1_horizonloop<-matrix(0, 6, 1)
S1_Rateloop<-matrix(0, 6, 1)

S1_DFactorloop<-matrix(0, PTimes, 6)
TL1_DFactor_deltaloop<-matrix(0, PTimes, 6)
RDuration1loop<-matrix(0, 6, 1)
PLeg1loop<-matrix(0, 6, 1)
DLeg1loop<-matrix(0, 6, 1)
FairSpd1loop<-matrix(0, 6, 1)
MTM1mloop<-matrix(0, 6, 1)

for (b in 1:m){
if (b==1) beta=beta1
if (b==2) beta=beta2
if (b==3) beta=beta3
if (b==4) beta=beta4
if (b==5) beta=beta5
```

```
if (b==6) beta=beta6

cprob1loop<-matrix(0, 126, 6)
prob1loop<-matrix(0, 126, 6)
FTL1_innerloop<-matrix(0, 126, 6)

for (k in 1:126)

{
cncdf<-matrix(0, 10, 1) #Gamma
for (i in 1:10)
{
innercncdf<-matrix(0, 30, 1) #Gaussian
for (j in 1:30)
{
innercnpdf=pnorm(1/(sqrt(1-beta^2))*(y_w \$ nodes[i]^(1/2)
   *qt(pdi, df=degreef)-beta*x_w \$ nodes[j]))
innercncdf[j,1]=pbinom(k-1, 125, innercnpdf)
}
if (y_w \$ weights[i]==0) cncdf[i,1]=0
else cncdf[i,1]=x_w \$ weights \%*\% innercncdf
}
cprob1loop[k, b]=y_w \$ weights \%*\% cncdf
if (k==1) prob1loop[k, b]=cprob1loop[k, b]
else prob1loop[k, b]=cprob1loop[k, b]-cprob1loop[k-1, b]
FTL1_innerloop[k, b]=min((k-1)*(1-R_Indx)/n, TUpper[b])
}
FTL1loop[b, 1]=t(FTL1_innerloop[ , b]) \%*\% prob1loop[ , b]
if (b==1) TL1loop[b, 1]=FTL1loop[b, 1]
else TL1loop[b, 1]=FTL1loop[b, 1]-FTL1loop[b-1, 1]
S1_horizonloop[b, 1]=1-TL1loop[b, 1]/(TUpper[b]-TLower[b])
S1_Rateloop[b, 1]=PPY*(S1_horizonloop[b, 1]^-(1/(PPY*horizon))-1)

for (TIdx in 1:PTimes)
{
if (TIdx==1) P2DatesYr[TIdx, 1]=(PDates[TIdx]-VDate)/360
else P2DatesYr[TIdx, 1]=(PDates[TIdx]-PDates[TIdx-1])/360
PDatesYr[TIdx, 1]=as.numeric(PDates[TIdx]-VDate)/360
RFDFactor[TIdx, 1]=(1+RFRate/PPY)^(-PPY*PDatesYr[TIdx, 1])
S1_DFactorloop[TIdx, b]
=(1+S1_Rateloop[b, 1]/PPY)^(-PPY*PDatesYr[TIdx, 1])
```

```
if (TIdx==1) TL1_DFactor_deltaloop[TIdx, b]=1-S1_DFactorloop[TIdx, b]
else TL1_DFactor_deltaloop[TIdx, b]
=S1_DFactorloop[TIdx-1, b]-S1_DFactorloop[TIdx, b]
}
RFDuration=t(RFDFactor)\%*\% P2DatesYr
RDuration1loop[b, 1]=t(S1_DFactorloop[ , b]*RFDFactor)
\%*\% P2DatesYr
PLeg1loop[b, 1]=t(TL1_DFactor_deltaloop[ , b])
\%*\% RFDFactor*Ntnl*(TUpper[b]-TLower[b])
FairSpd1loop[b, 1]=PLeg1loop[b, 1]/(RDuration1loop[b, 1]
*Ntnl*(TUpper[b]-TLower[b]))
MTM1mloop[b,1]=((FairSpd1loop[b, 1]-TSpd[b])
*RDuration1loop[b, 1]-TFee[b])*100
}
MTM1m=t(MTM1mloop)\%*\% MTM1mloop
}
```