Abstract

MCLEAN, MICHAEL ALLEN. N-Symplectic Analysis of Field Theory. (Under the direction of L. K. Norris.)

Two techniques for relating n-symplectic geometry to the jet bundle formulations of classical field theory are presented.

The tangent bundle of the frame bundle of a manifold M is shown to be a principal fiber bundle over the jet bundle of the tangent bundle of M. We are able to generalize this result to symmetric and antisymmetric tensor bundles of rank p. Using this $\mathrm{GL}(m)$ gauge freedom, we interpret the standard free field Lagrangian as a symmetric type $(0, 2)$ tensor on LM.

The adapted frame bundle of an arbitrary fiber bundle π is shown to be a principal bundle over the jet bundle of π. Using this $\mathrm{GL}(m) \times \mathrm{GL}(k)$ gauge freedom we generate a modified $m + k$-symplectic geometry from a lifted Lagrangian. The modified soldering form is shown to induce the Cartan-Hamilton-Poincaré m-form on $J^1\pi$. We derive generalized Hamilton-Jacobi and Hamilton equations on $L_\pi E$, and show that the Hamilton-Jacobi and canonical equations of Carathéodory-Rund and de Donder-Weyl are obtained as special cases.

These results demonstrate that by introducing additional gauge freedom into the standard jet bundle formalism one can obtain a great of of additional geometric and algebraic structure. Such additional structure may be key in achieving a greater understanding of field theory or in attempts at quantization.

N-SYMPLECTIC ANALYSIS OF FIELD THEORY

by

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a thesis submitted to the graduate faculty of
north carolina state university
in partial fulfillment of the
requirements for the degree of
doctor of philosophy

department of mathematics

raleigh, north carolina
March 23, 2001

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Biography

Michael McLean was born on June 3, 1972 in Asheville, North Carolina. He was raised in Asheville and Raleigh, North Carolina. He graduated from The North Carolina School of Science and Mathematics in Durham, North Carolina in 1990 and went on to receive his B.S. in Physics and M.S. in Mathematics from North Carolina State University in Raleigh in 1994 and 1999 respectively. He lives in Raleigh.

Acknowledgements

I would like to credit my advisors, both official and unofficial, for the tremendous amount of wisdom, patience and time they have invested in my professional growth.

In particular, I thank Larry Norris for sharing his keen geometric insight and healthy research ethic, and Ronald Fulp for a truly mathematical view of math-physics.

I particularly appreciate the aid of Arkady Kheyfets and Erich Kaltofen in securing summer positions and the mathematics department of NCSU for their excellent teaching assistantship program.

Also, my extreme gratitude goes out to Carl Meyer for suggesting an alternate course of action many years ago.

The pursuit of a graduate degree impacts all aspects of one's life. Consequently, I would like thank Sherri for putting up with all the lost time and second-hand stress.

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Chapter 1

Introduction

The Lagrangian and Hamiltonian descriptions of classical mechanics are among the most elegant achievements of physics. Simple principles which are intuitively sensible lead to calculatable results and greater understanding. The geometric formulation of these theories extends this by clarifying certain issues and providing a solid mathematical framework.

The Lagrangian and Hamiltonian descriptions of classical field dynamics are clear generalizations of those for classical mechanics. However the development of a geometric formulation has been something of a rocky road. Much work has been done, but no approach has provided all the answers.

The geometric arenas for classical mechanics are the tangent bundle TM for Lagrangian formalism and the cotangent bundle T^*M for Hamiltonian formalism. Logically, many have tried using various generalizations or extensions of the tangent and cotangent bundle. Among the many possible generalizations that have been studied, we are mainly concerned with two:

1. The study of jet bundles

2. The study of n-symplectic geometry on frame bundles

In this work we present two different techniques for relating these two approaches. In Chapter 3, we study how n-symplectic observables on the frame bundle relate to the jets of contravariant tensor fields, and in Chapter 4 we study how n-symplectic geometry on the adapted frame bundle relates to canonical formalism on the jet and cojet bundles.

1.1 A Brief History of Generalizations

The generalizations of geometric Lagrangian and Hamiltonian formalism seem to fall into two categories. One one hand, some have tried to generalize the geometric structures of the tangent or cotangent bundles, sometimes arriving at a formalism pertinent to field theory. Another direction was to fundamentally extend the bundles themselves and then explore the natural geometry there. The former gives us the various axiomatic systems such as k-symplectic and almost k-tangent. The latter gives us such things as the jet bundle and the adapted frame bundle.

1.1.1 Cotangent-like structures

The first step in this direction of generalization was the development of symplectic geometry which is an axiomatic system based on the the essential properties of the canonical form θ on the cotangent bundle. Later, around 1960, Bruckheimer [5] introduced the notion of almost cotangent structures. These were further investigated by Clark and Goel [8] in 1974. In both cases the canonical two form became the model from which axioms were designed.

Between 1987 and 1991, several independent and closely related generalizations were developed. Polysymplectic geometry [26], almost k-tangent structures [32, 33], and k-symplectic geometry [2, 3] were based around the natural structure of the k-cotangent bundle. This bundle, which can be thought of as the fiberwise product of the cotangent bundle k times, has a k-tuple of θs with which one works. Also, the n-symplectic geometry of the frame bundle (and its $\mathbb{R}^n$-valued θ) was investigated in [39, 40, 41, 43]. While the development of k-tangent structures and k-symplectic geometry had purely geometric motivations, polysymplectic geometry was created to study field theory and n-symplectic geometry sought to generalize Hamiltonian mechanics.

1.1.2 Tangent-like structures

Around 1960, the theory of almost tangent structures was developed by Clark and Bruckheimer [6] and Eliopoulos [15] separately. Almost tangent structures are generalizations of the tangent bundle. The vector valued one-form (J), viewed as the object of central interest, was axiomatized.

Almost k-tangent structures [34, 35] arose around 1988 as a generalization of the geometry of the k-tangent bundle. This bundle is, among other interpretations, the fiberwise product of the tangent bundle with itself k times. A section of this bundle is equivalent to a k-tuple of vector fields. The central geometric object becomes a k-tuple of Js.

Another version of the tangent structure arises on the jet bundle. This is a very broad level of generalization since the idea of the jet of a section generalizes and incorporates the notions of tangent vectors, cotangent vectors, k-tangent vectors, and k-cotangent vectors. Such geometry has clear importance to field theory since one can envision any type of field as a section of a fiber bundle.

1.1.3 N-Symplectic Geometry

Although it is possible to write down axioms for n-symplectic geometry and define an abstract n-symplectic manifold, in practice one nearly always works on a frame bundle. The frame bundle of manifold has a great deal of structure. It is a principal bundle to which many other useful bundles are associated. It has both a cotangent-like structure and a tangent-like structure because of the natural correlation of frames and co-frames.

N-symplectic geometry is important because of its wealth of structure. A broad class of observables are allowed, with the symmetric ones forming a Poisson algebra and the antisymmetric ones a graded Poisson algebra [39, 40].

The powerful geometric structure of LM allows one to connect these algebras to other spaces. In particular, in [41] n-symplectic geometry is shown to be a covering theory for symplectic geometry. Additionally, the major results of chapters 3 and 4 show how n-symplectic geometry is connected with the jet formalism of field theory.

1.2 Jets of Vector and Tensor Fields

Consider a vector field as a section of the bundle $\tau : TM \to M$. In local coordinates, for a vector field $f = f^i \frac{\partial}{\partial x^i}$, the 1-jet prolongation of that section is

$$j^1_p f = d_p x^j \otimes \left(\left.\frac{\partial}{\partial x^j}\right|_{f(p)} + \frac{\partial f^i}{\partial x^j} \left.\frac{\partial}{\partial v^i}\right|_{f(p)} \right)$$

On the other hand, the Hamiltonian vector field on LM corresponding to that section is

$$X_{\hat{f}}(u) = f^i \circ \lambda(u) \left.\frac{\partial}{\partial x^i}\right|_u + \left.\frac{\partial f^i}{\partial x^j}\right|_{\lambda(u)} v^j_k \left.\frac{\partial}{\partial v^i_k}\right|_u$$

It is apparent that both contain similar information, namely the value of f and its first derivative. In fact we can demonstrate a direct and purely geometric relation.

If we start with a vector field f, we imagine obtaining its Hamiltonian vector field by the following steps:

1. generate the tensorial function $\hat{f}$ on LM from f
2. take the differential of $\hat{f}$
3. generate $X_{\hat{f}}$ from $d\hat{f}$ via the n-symplectic equation.

Not surprisingly, given the form of $X_{\hat{f}}$, each of these steps is reversible. We may easily obtain $d\hat{f}$ from $X_{\hat{f}}$, via the n-symplectic equation

$$d\hat{f}^i = -X_{\hat{f}} \lrcorner\, d\theta^i$$

It turns out (lemma 3.2.11) that for a rank-1 tensorial function

$$d\hat{f}^i(-\overset{*}{E}{}^a_a) = \hat{f}^i$$

where $\overset{*}{E}{}^a_a$ is the Euler vector field on LM. Thus we may obtain $\hat{f}$ from $d\hat{f}$. Finally we may obtain f from $\hat{f}$ by using the map $\psi : LM \times \mathbb{R}^m \to TM$ which is the standard projection used to obtain TM as an associated bundle.

$$\psi(p, \{e_i\}, \xi) = \xi^i e_i = \xi^i v_i^j(u) \left.\frac{\partial}{\partial x^j}\right|_{\lambda(u)} \qquad (u = (p, \{e_i\}))$$

It turns out that

$$\psi(u, \hat{f}(u)) = f \circ \lambda(u)$$

or alternately

$$\psi \circ (\mathrm{id}_{LM}, \hat{f}) = f \circ \lambda$$

where λ is the projection from LM to M.

Having recovered f from $X_{\hat{f}}$, we may calculate its 1-jet prolongation, which is equivalent to simply df. Let us attempt to combine this with previous steps. We start with

$$\begin{aligned} df \circ d\lambda &= d\psi \circ d(\mathrm{id}_{LM}, \hat{f}) \\ &= d\psi \circ (\mathrm{id}_{T(LM)}, d\hat{f}) \end{aligned}$$

However, we must be careful here. Strictly speaking, since $\hat{f}$ is a function from LM to $\mathbb{R}^m$, $d\hat{f}$ should be a function from $T(LM)$ to $T(\mathbb{R}^m) \cong \mathbb{R}^m \times \mathbb{R}^m$. However, it is customary with $\mathbb{R}^m$ valued functions to drop the function value from the differential, leaving $d\hat{f} : T(LM) \to \mathbb{R}^m$. Since what we *need* is the full differential, we must instead write

$$df \circ d\lambda = d\psi \circ (\mathrm{id}_{T(LM)}, (\hat{f}, d\hat{f}))$$

Now if we incorporate some of the earlier steps, we obtain

$$\begin{aligned} df \circ d\lambda &= d\psi \circ (\mathrm{id}_{T(LM)}, (-\overset{*}{E}{}^a_a \lrcorner\, d\hat{f}, d\hat{f})) \\ &= d\psi \circ \Big(\mathrm{id}_{T(LM)}, \big(-\overset{*}{E}{}^a_a \lrcorner -X_{\hat{f}} \lrcorner\, d\theta, -X_{\hat{f}} \lrcorner\, d\theta \big) \Big) \\ &= d\psi \circ \Big(\mathrm{id}_{T(LM)}, \big(X_{\hat{f}} \lrcorner\, \theta, -X_{\hat{f}} \lrcorner\, d\theta \big) \Big) \end{aligned}$$

where we have used the fact (from lemma 3.2.12) that $-\overset{*}{E}{}^a_a \lrcorner\, d\theta = \theta$. We now conclude that starting from the Hamiltonian vector field $X_{\hat{f}}$ we may define a map $\phi : T(LM) \to T(TM)$ by

$$\phi_u(Y) = d_{(u,\theta_u(X_{\hat{f}}))}\psi\big(Y, Y \lrcorner (-X_{\hat{f}}) \lrcorner\, d_u\theta\big)$$

and that this $T(TM)$ valued one-form will be equal to $df \circ d\lambda$, which is a horizontal 1-form that projects to df.

At this point something wonderful happens. So far we have been talking only of *Hamiltonian vector fields* $X_{\hat{f}}$, but the formula above can be interpreted *pointwise*. The only question is whether or not, for an arbitrary vector $X \in T_u(LM)$, the resulting ϕ_u is horizontal or not. In fact it is (this is proved in theorem 3.1.3), and so we obtain a pointwise map from $T(LM)$, the tangent bundle of the frame bundle, to $J^1\tau$, the jet bundle of the tangent bundle, which we denote (rather mundanely) by F.

$$F : T(LM) \to J^1\tau$$

Having found a map from a higher dimensional space to a lower dimensional one, the question naturally arises: "is this a fiber bundle?" It turns out the answer is yes, in fact F is a principal bundle!

On another note, the structure discovered here (the quotient of a bundle over a principal bundle) led to the development of the notion of a "weakly associated" bundle (section 2.3.4). It turns out that $J^1\tau$ can be thought of as a weakly associated bundle of LM as well.

1.2.1 Generalizing the Result

The results for the rank one case all generalize nicely to the higher order symmetric and antisymmetric cases.

In the p-th order case, we study symmetric and antisymmetric type $(p, 0)$ tensor fields on M, which we consider as sections of either $\tau^p_{(s)} : \otimes^p_{(s)} TM \to M$ or $\tau^p_{(a)} : \otimes^p_{(a)} TM \to M$. These fields induce tensorial $\otimes^p \mathbb{R}^m$-valued functions on LM, which in turn produce equivalence classes of $\otimes^p \mathbb{R}^m$-valued Hamiltonian vector fields on LM. We consider these vector fields as sections of the bundle $\bigoplus^{m^{p-1}} T(LM) \to LM$.

It turns out that we must restrict $\bigoplus^{m^{p-1}} T(LM)$ to obtain the desired results. We choose either the tensorially symmetric subbundle $\Upsilon^{p-1}_{(s)}$ or tensorially antisymmetric subbundle $\Upsilon^{p-1}_{(a)}$. With this restriction, we are able to define maps

$$F^p_{(s)} : \Upsilon^{p-1}_{(s)} \to J^1 \tau^p_{(s)}$$

and

$$F^p_{(a)} : \Upsilon^{p-1}_{(a)} \to J^1 \tau^p_{(a)}$$

which are direct generalizations of the map F described above.

1.2.2 Application: The Free Field Lagrangian

The standard free field Lagrangian on $J^1\tau$ can, when pulled up to $T(LM)$, be interpreted as the quadratic form of a symmetric $(0, 2)$ tensor on LM. This is an analogue of the fact that, in particle mechanics, the free Lagrangian can be interpreted as the quadratic form of a metric tensor.

Starting from the metric h on M, we define We define $\tilde{h}$ on LM as follows

$$\begin{aligned}
\tilde{h}(B_i, B_j) &= -\mu^2 h(\lambda_* B_i, \lambda_* B_j) \\
\tilde{h}(B_i, \overset{*}{E}{}^j_k) &= 0 \\
\tilde{h}(\overset{*}{E}{}^j_k, \overset{*}{E}{}^{\bar{j}}_{\bar{k}}) &= 2\hat{h}_{k\bar{k}}\hat{h}^{j\bar{j}} - 2\delta^j_{\bar{k}}\delta^{\bar{j}}_k
\end{aligned}$$

where $\mu^2 > 0$ is a constant, the B_i are the standard horizontal vector fields induced by the connection defined by h, and the $\overset{*}{E}{}^j_k$ are the standard vertical vector fields on LM.

The quadratic form associated with $\tilde{h}$ projects to the standard free field Lagrangian on $J^1\tau$.

$$\mathcal{L} = h^{a\bar{a}} h^{b\bar{b}} D_{[a}\psi_{b]} D_{[\bar{a}}\psi_{\bar{b}]} - \mu^2 h^{k\bar{k}} \psi_k \psi_{\bar{k}} \tag{1.1}$$

It turns out that $\tilde{h}$ is not a metric on LM. This suggests that there may be missing terms in the Lagrangian (1.1).

Chapter 2

Background Material

In this chapter we develop the basic definitions and results that are needed in the remainder of the work. These primarily concern

- Fiber bundles in general
- Specific induced bundles such as frame bundles, jet bundles, and associated bundles.
- The notion of a weakly associated bundle
- Types and categories of bundles
- N-symplectic geometry

2.1 Fiber Bundles

Definition 2.1.1. *A fiber bundle consists of a two (smooth) manifolds, E and M, and a map between them, $\pi : E \to M$, with the following properties:*

- *π is a (smooth) surjective submersion.*
- *π is locally trivial:*

  For every point $p \in M$, there exists an open neighborhood U of p and a (smooth) diffeomorphism $\mathrm{tr} : \pi^{-1}(U) \to U \times F_p$, where F_p is a (smooth) manifold and $\mathrm{pr}_1 \circ \mathrm{tr} = \pi$. Such a map is called a trivialization around p.

The manifolds E and M are called the total space and base space respectively. The map π is called the **projection**; in this work we will generally refer to fiber bundles by their projection maps. Then manifold F_p is called the **fiber**. Throughout this work, we will assume bundles of constant fiber and drop the index p.

A fiber bundle is a generalization of a product manifold. Perhaps the most common example would be the tangent bundle of a manifold. Bundles seem to be a natural extension of manifold structure.

Definition 2.1.2. *If two trivializations* tr_1 *and* tr_2 *have overlapping domains* U_1 *and* U_2*, we define their* **transition function** *to be*

$$\mathrm{tr}_1 \circ \mathrm{tr}_2^{-1} : (U_1 \cap U_2) \times F \to (U_1 \cap U_2) \times F$$

When defining a fiber bundle, it is unnecessary to give E any structure a priori, since it can be induced from the trivialization maps.

Definition 2.1.3. *Let* E *be a set,* M *and* F *manifolds, and* π *a map from* E *onto* M*. A* **trivialization atlas** *for* π *is a collection of maps,* $\mathrm{tr}_i : \pi^{-1}(U_i) \to U_i \times F$*, such that*

- $\{U_i\}$ *is an open cover of* M
- $\mathrm{pr}_1 \circ \mathrm{tr}_i = \pi|_{\pi^{-1}(U_i)}$
- $\mathrm{tr}_i \circ \mathrm{tr}_j^{-1}$ *is a diffeomorphism.*

Definition 2.1.4. *Given a trivialization atlas* $\mathcal{T}$ *for* E *and atlases* $\mathcal{A}_M$ *and* $\mathcal{A}_F$ *for* M *and* F*, we define the* **induced atlas** $\mathcal{A}_E$ *for* E *as follows.*

For each $(\mathrm{tr} : \pi^{-1}(\tilde{U}) \to M \times F) \in \mathcal{T}$*,* $(x : U \to \mathbb{R}^m) \in \mathcal{A}_M$ *and* $(y : V \to \mathbb{R}^k) \in \mathcal{A}_F$*, we construct the chart* $(x \times y) \circ \mathrm{tr} : \mathrm{tr}^{-1}(U \cap \tilde{U}) \times V \to \mathbb{R}^m \times \mathbb{R}^k$*. The set of all such charts is the induced atlas.*

In essence, each trivial subbundle $\pi^{-1}(\tilde{U})$ gets the standard product manifold atlas. The union of these atlases is an atlas for all of E. Compatibility is guaranteed because the transition functions of the trivialization are smooth. So a trivialization atlas, together with the atlases for M and F, generates an atlas for E and makes π

a fiber bundle.

Definition 2.1.5. *A* **smooth bundle morphism** *from a bundle* $\pi : E \to M$ *to another bundle* $\rho : H \to N$ *is a pair of smooth maps* $f : E \to H$ *and* $g : M \to N$ *with the property that* $\rho \circ f = g \circ \pi$*. (See Figure 2.1)*

It is worth noting that bundle morphisms preserve fibers. Specifically, the image a of fiber in the domain will lie within a single fiber in the range.

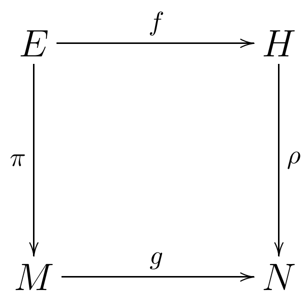

Figure 2.1: A bundle morphism

Definition 2.1.6. *Let* $\dim E = m + k$ *and* $y : U \to \mathbb{R}^{m+k}$ *be a coordinate system on* E*. Then* y *is an* **adapted coordinate system** *if*

$$\forall a, b \in U \quad \pi(a) = \pi(b) \iff \mathrm{pr}_1 \circ y(a) = \mathrm{pr}_1 \circ y(b)$$

(where $pr_1 : \mathbb{R}^{m+k} \to \mathbb{R}^m$ *is the projection.)*

Remark 2.1.7. *Adapted coordinates project to coordinates on* M*. Conversely, given coordinates* x^i *on* M*, one can construct adapted coordinates on* E *that project to* x^i *from the local product structure. Adapted coordinates will be denoted as a pair* $(x^i,\, y^A)$*.*

Definition 2.1.8. *A* **section** *of a bundle* $\pi : E \to M$ *is a map* $\phi : M \to E$ *such that* $\pi \circ \phi = \mathrm{id}_M$*. The space of sections of* π *is denoted* $\Gamma(\pi)$*.*

Definition 2.1.9. *A* **local section** *of* π *at* $p \in M$ *is a map* $\phi : U \to E$ *where* U *is an open neighborhood of* p *such that* $\pi \circ \phi = \mathrm{id}_U$*. The space of local sections of* π *at* p *is denoted* $\Gamma_p(\pi)$*.*

Sections of bundles arise many places. All vector and tensor fields on a manifold are sections of some bundle.

2.1.1 Bundle Construction Theorem

Fiber bundles are often constructed from the top down. That is, first the total space is specified, then a projection is given and whatever charts, trivializations, or other structures necessary are demonstrated. For example, one usually defines the tangent bundle by stating that

$$TM = \{(p, v) \mid p \in M, v \in T_pM\},$$

defining the projection $\tau : TM \to M$ by

$$\tau(p, v) = v,$$

giving the induced coordinate charts

$$x^i(p, v) = x^i(p) \qquad v^j(p, v) = d_px^j(v),$$

and remarking that these coordinate charts can also be considered as trivializations.

However, sometimes it is useful (or even necessary) to build a bundle from ground up. That is, we start with a base manifold and tack on the fibers in such a way as to get the desired bundle structure. The following theorem is a standard result that allows us to do this.

THEOREM 2.1.10. *Let M and F be manifolds, $\{U_\alpha \mid \alpha \in I\}$ an open covering of M, and $\tau_{\alpha\beta} : (U_\alpha \cap U_\beta) \times F \to F$ a family of smooth maps with the properties that*

1. $\tau_{\alpha\beta}(x, \tau_{\beta\gamma}(x, a)) = \tau_{\alpha\gamma}(x, a) \qquad \forall \alpha, \beta, \gamma \in I;\ x \in U_\alpha \cap U_\beta \cap U_\gamma$
2. $\tau_{\alpha\beta}(x) : F \to F$ *is a diffeomorphism* $\forall \alpha, \beta \in I; x \in U_\alpha \cap U_\beta$

Then one can define a fiber bundle $\pi : E \to M$ with trivializations $t_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ such that $t_\alpha \circ t_\beta^{-1} = \tau_{\alpha\beta}$.

Proof. We first define $S = \{(\alpha, x, a) \in I \times M \times F \mid x \in U_\alpha\}$ and an equivalence relation $\sim$ on S. We say that $(\alpha, x, a) \sim (\beta, y, b)$ iff $x = y$ and $\tau_{\alpha\beta}(x, b) = a$. The properties demanded for τ ensure that this is an equivalence relation. We denote the equivalence class of (α, x, a) by $[\alpha, x, a]$ and remark that

$$[\alpha, x, a] = [\beta, x, \tau_{\beta\alpha}(x, a)] \tag{2.1}$$

Next we define $E = S/\sim$ and denote the canonical projection by η. We define $\pi : E \to M$ by $\pi([\alpha, x, a]) = x$ which is clearly well defined. Also, we define $i_\alpha : U_\alpha \times F \to S$ by $i_\alpha(x, a) = (\alpha, x, a)$.

LEMMA 2.1.11. *$\eta \circ i_\alpha$ is 1-to-1 and onto $\pi^{-1}(U_\alpha)$*

Proof. $\eta \circ i_\alpha(x, a) = [\alpha, x, a]$

Suppose $[\alpha, x, a] = [\alpha, y, b]$, then $x = y$ and $\tau_{\alpha\alpha}(x, b) = a = b$, so $(x, a) = (y, b)$.

Suppose $[\beta, y, b] \in \pi^{-1}(U_\alpha)$, then $[\beta, y, b] = [\alpha, y, a]$ for some a so $\eta \circ i_\alpha(y, a) = [\beta, y, b]$ □

Finally we define $t_\alpha = (\eta \circ i_\alpha)^{-1} : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ and we note that

$$t_\alpha \circ t_\beta^{-1}(y, b) = t_\alpha([\beta, y, b]) = t_\alpha([\alpha, y, \tau_{\alpha\beta}(y, b)]) = (y, \tau_{\alpha\beta}(y, b)) \tag{2.2}$$

which is the promised result. □

2.2 Categories of Bundles

There are many types of fiber bundles. Often these spaces arise with additional properties. This section explores the notion that in some cases, these different types of bundles actually form categories.

A category consists of spaces and maps (or objects and arrows if you prefer). The category *Man* consists of the smooth manifolds and smooth maps between them. Fiber bundles live in a related category, called *Bun*.

For any category C, one can define the arrow category of C, denoted C^2. The objects of C^2 are the arrows of C. The arrows of C^2 are the commutative squares of C (see Figure 2.2). In other words, for two C^2 objects $f : A \to B$ and $g : \hat{A} \to \hat{B}$, an arrow in C^2 from f to g is a pair of arrows in C, $s : A \to \hat{A}$ and $t : B \to \hat{B}$, such that $g \circ s = t \circ f$.

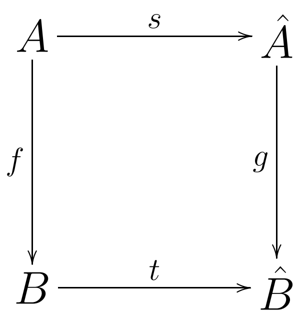

Figure 2.2: The arrows of C^2 are the commutative squares of C

Fiber bundles and bundle morphisms form a subcategory of Man^2. This is a subcategory because the objects are restricted; not all arrows in Man are fiber bundles. We will refer to the category of fiber bundles as *Bun*.

Many of the different types of bundles define subcategories of *Bun*. For example, vector bundles and their morphisms. In this section, we develop a method for generating such subcategories. In section 2.3 we use this method to determine the category of associated and weakly associated bundles.

2.2.1 Types of Bundles

Many of the standard types of bundles form categories.

Definition 2.2.1. *A* **vector bundle** *is a fiber bundle where the fiber F is a vector space and the trivialization atlas has the additional property that*

- $\mathrm{tr}_i \circ \mathrm{tr}_j^{-1} |_{\{p\}\times F}$ *is a linear isomorphism.*

Definition 2.2.2. *A* **vector bundle morphism** *is a smooth bundle morphism between two vector bundles which, when restricted to a single fiber, yields a linear map.*

Vector bundles and vector bundle morphisms form a subcategory of *Bun*.

Definition 2.2.3. *An* **affine bundle** *is a fiber bundle where the fiber F is an affine vector space and the trivialization atlas has the additional property that*

- $\text{tr}_i \circ \text{tr}_j^{-1}|_{\{p\}\times F}$ *is an affine linear isomorphism.*

DEFINITION 2.2.4. *A **affine bundle morphism** is a smooth bundle morphism between two affine bundles which, when restricted to a single fiber, yields a affine linear map.*

Affine bundles and affine bundle morphisms also form a subcategory of *Bun*.

The definition of an affine bundle given above differs from the standard definition [19, 46]. Usually an affine bundle is defined relative to a vector bundle on which it is modelled. In the following theorem we show that this vector bundle can be constructed given the structure above.

THEOREM 2.2.5. *Let (A, π, M) be a fiber bundle with an affine linear trivialization atlas. Then π is an affine bundle in the usual sense and the vector bundle on which it is modelled can be constructed.*

Proof. The idea of the proof is simple. We build the new bundle from three parts

1. The base manifold M
2. The model space of the affine fiber
3. The *linear part* of the transition functions for an affine trivialization of π

First let the typical fiber of π be F. This must of course be an affine vector space. Let F be modelled on the vector space V.

Let $\{(U_\alpha, t_\alpha) \mid \alpha \in I\}$ be the affine trivialization atlas of π. Define $T_{\alpha\beta}(x, a) = \text{pr}_2 \circ t_\alpha \circ t_\beta^{-1}(x, a)$. $T_{\alpha\beta}(x) : F \to F$ is an affine linear map; let $\tau_{\alpha\beta}(x)$ be its linear part. Construct E from M, V, and $\tau_{\alpha\beta}$. E will then be the vector bundle that π is modelled on. To show this we need to demonstrate the difference map.

If $(u, v) \in A \times_M A$, choose an $\alpha \in I$ such that $\pi(u) \in U_\alpha$ and define $\delta(u, v) = [\alpha, \pi(u), \text{pr}_2 \circ t_\alpha(u) - \text{pr}_2 \circ t_\alpha(v)]$

We must show this is independent of the choice of α. Let β also have the property that $\pi(u) \in U_\beta$. Then

$$\begin{aligned}
&\tau_{\alpha\beta}(\pi(u), \mathrm{pr}_2 \circ t_\beta(u) - \mathrm{pr}_2 \circ t_\beta(v)) \\
&= \tau_{\alpha\beta}(\pi(u))(\mathrm{pr}_2 \circ t_\beta(u) - \mathrm{pr}_2 \circ t_\beta(v)) \\
&= T_{\alpha\beta}(\pi(u))(\mathrm{pr}_2 \circ t_\beta(u)) - T_{\alpha\beta}(\pi(u))(\mathrm{pr}_2 \circ t_\beta(v)) \\
&= T_{\alpha\beta}(\pi(u), \mathrm{pr}_2 \circ t_\beta(u)) - T_{\alpha\beta}(\pi(u), \mathrm{pr}_2 \circ t_\beta(v)) \\
&= T_{\alpha\beta}(t_\beta(u)) - T_{\alpha\beta}(t_\beta(v)) \\
&= \mathrm{pr}_2 \circ t_\alpha \circ t_\beta^{-1}(t_\beta(u)) - \mathrm{pr}_2 \circ t_\alpha \circ t_\beta^{-1}(t_\beta(v)) \\
&= \mathrm{pr}_2 \circ t_\alpha(u) - \mathrm{pr}_2 \circ t_\alpha(v)
\end{aligned}$$

where we have used the fact that $\tau_{\alpha\beta}(x)$ is the linear part of $T_{\alpha\beta}(x)$, and the definition of $T_{\alpha\beta}$.

Therefore, we have that

$$(\alpha, \pi(u), \mathrm{pr}_2 \circ t_\alpha(u) - \mathrm{pr}_2 \circ t_\alpha(v)) \sim (\beta, \pi(u), \mathrm{pr}_2 \circ t_\beta(u) - \mathrm{pr}_2 \circ t_\beta(v))$$

or

$$[\alpha, \pi(u), \mathrm{pr}_2 \circ t_\alpha(u) - \mathrm{pr}_2 \circ t_\alpha(v)] = [\beta, \pi(u), \mathrm{pr}_2 \circ t_\beta(u) - \mathrm{pr}_2 \circ t_\beta(v)]$$

□

Definitions 2.2.1 (vector bundles) and 2.2.3 (affine bundles) are remarkably similar, as are the definitions of their respective morphisms. It is easy to see within these definitions a template that one could apply to other types of manifolds. This template can be stated formally in terms of categories.

DEFINITION 2.2.6. *For any subcategory C of manifolds, we define the* ***associated subcategory*** $\hat{C}$ *of fiber bundles, as follows.*

The objects of $\hat{C}$ are the fiber bundles whose fiber F is in C and whose trivialization atlas $\mathcal{T}$ has the property that

$$\mathrm{tr}_i \circ \mathrm{tr}_j^{-1}|_{\{p\}\times F} \textit{ is an isomorphism in } C \quad \forall\, \mathrm{tr}_i, \mathrm{tr}_j \in \mathcal{T}, p \in M \tag{2.3}$$

The morphisms of $\hat{C}$ are the smooth bundle morphisms that, when restricted to a single fiber, yield morphisms of C.

2.2.2 Principal Bundles

DEFINITION 2.2.7. *A **principal bundle** is a fiber bundle with a free right Lie group action such that the orbits are exactly the fibers.*

This global structure is a level beyond ordinary fiber bundles and makes principal bundles a nice place to work for several reasons.

- Local sections yield local trivializations
- Associated bundles can be generated
- Connections can be described more simply
- Canonical vertical vector fields can be defined

Principal bundles do not fit as neatly into the scheme described earlier.

We can say that a principal bundle with structure group G is a fiber bundle where the typical fiber F has a free right transitive G-action and whose trivialization atlas has the additional property that

- $\mathrm{tr}_i \circ \mathrm{tr}_j^{-1}|_{\{p\}\times F}$ is a diffeomorphism that commutes with the G-action.

The subcategory implicit here is that of manifolds with a free right transitive G-action where the morphisms satisfy the condition

$$f(u \cdot g) = f(u) \cdot g$$

This category doesn't quite give the category of principal bundles. This is partly because of the fixed G, but also because we get the wrong notion of morphisms.

2.3 Spaces of Interest

Here we briefly discuss several specific bundles and their properties in order to clarify our terminology an notation.

2.3.1 Jet Bundles

When studying the dynamics of sections of a bundle $\pi : E \to M$, one frequently encounters functions that depend on the derivatives of those sections. Lagrangians involve the (first) derivative, and any differential equation involves derivatives up to its order. The domains of such functions are jet bundles. The notation we use here follows that of [46].

DEFINITION 2.3.1. *We define two local sections at $p \in M$ to be $(1, p)$-**equivalent** if, in some adapted coordinate system,*

$$\phi(p) = \psi(p) \tag{2.4}$$

$$\left.\frac{\partial \phi^A}{\partial x^i}\right|_p = \left.\frac{\partial \psi^A}{\partial x^i}\right|_p \tag{2.5}$$

The equivalence class of ϕ at p is denoted $j^1_p\phi$.

DEFINITION 2.3.2. *The **first jet bundle** of π, $J^1\pi$, is the set of $(1, p)$ equivalence classes of local sections.*

$$J^1\pi = \left\{ j^1_p\phi \mid p \in M, \phi \in \Gamma_p(M) \right\}$$

The **projection** from $J^1\pi$ to E is $\pi_{1,0}(j^1_p\phi) = \phi(p)$.

The **projection** from $J^1\pi$ to M is $\pi_1(j^1_p\phi) = p$.

If (x^i, y^A) is an adapted coordinate system on $U \subset E$, then we induce **coordinates** on $\pi^{-1}_{1,0}(U)$ by

$$x^i(j^1_p\phi) = x^i(p)$$

$$y^A(j^1_p\phi) = y^A(p)$$

$$y^A_i(j^1_p\phi) = \left.\frac{\partial \phi^A}{\partial x^i}\right|_p$$

REMARK 2.3.3. *The second condition in the equivalence relation can be stated in* *various ways.*

$$d_p\phi = d_p\psi \tag{2.5'}$$

$$d_p\phi(T_pM) = d_p\psi(T_pM) \tag{2.5''}$$

This leads us to the three major ways of describing 1-jets:

1. Equivalence classes of sections of π.

2. Linear right-inverses to $d_u\pi$.

3. Non-vertical m-dimensional subspaces of T_uE.

$J^1\pi$ is a fiber bundle two ways:

$$\pi_{1,0} : J^1\pi \to E \quad j^1_p\phi \mapsto \phi(p)$$
$$\pi_1 : J^1\pi \to M \quad j^1_p\phi \mapsto p$$

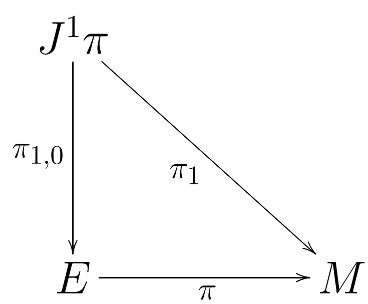

Figure 2.3: $J^1\pi$ is a fiber bundle two ways

The bundle $\pi_{1,0}$ is an affine bundle. The bundle π_1 has no particular structure in general. However we note two special cases. First, if π is a vector bundle, π_1 will inherit a vector bundle structure. Second, if π is a trivial bundle, $\pi_{1,0}$ can be given a vector bundle structure.

2.3.2 Frame Bundles

DEFINITION 2.3.4. *The* **frame bundle** *of an m-dimensional manifold M, LM, is a principal bundle that encodes M's coordinate invariance.*

$$LM = \{(p, \{e_i\}_{i=1}^m) \mid p \in M,\ \{e_i\}_{i=1}^m \text{ is a basis for } T_pM\}$$

The structure group of LM is $\mathrm{GL}(m)$.

We denote the **projection** from LM to M by λ, which is defined by $\lambda(p, \{e_i\}_{i=1}^m) = p$.

The right action of $\mathrm{GL}(m)$ on LM is given by

$$(p, \{e_i\}_{i=1}^m) \cdot g = (p, \{e_i g^i_j\}_{j=1}^m) \tag{2.6}$$

REMARK 2.3.5. *From this this point on, we will omit the braces and range in $\{e_i\}_{i=1}^m$ for convenience, instead writing just e_i with a free index.*

DEFINITION 2.3.6. *We can define two different sets of* **coordinates** *on LM, the frame coordinates*

$$x^i(p, e_k) = x^i(p) \qquad v^i_j(p, e_k) = dx^i(e_j)$$

and the co-frame coordinates

$$x^i(p, e_k) = x^i(p) \qquad \pi^i_j(p, e_k) = e^i\left(\frac{\partial}{\partial x^j}\right)$$

DEFINITION 2.3.7. *The $\mathbb{R}^n$-valued* **soldering form** *on LM is defined by*

$$\theta_u(X) = e^i(d_u\lambda(X))r_i = \theta^i_u(X)r_i$$

where $u = (e, e_i) \in LM$ and $\{r_i\}$ is the standard basis for $\mathbb{R}^n$.

In local coordinates,

$$\theta^i = \pi^i_j dx^j \tag{2.7}$$

2.3.3 Associated Bundles

THEOREM 2.3.8. *The associated bundle $(P \times F)/G$ will be of the category $\hat{C}$ induced from the category C of F iff the action of G on F consists of morphisms in C.*

Proof. Every trivialization of P induces a trivialization of $(P \times F)/G$. From the transition functions of these induced trivializations, we can see whether or not $(P \times F)/G$ is of the category $\hat{C}$.

The induced trivialization is given by

$$\mathrm{tr}_F((u,f)G) = (p, g \cdot f) \quad \text{where } (p,g) = \mathrm{tr}(u)$$

The inverse is given by

$$\begin{aligned} \mathrm{tr}_F^{-1}(p,h) &= (\mathrm{tr}^{-1}(p,g), g^{-1} \cdot h)G \quad \text{for any } g \in G \\ &= (\mathrm{tr}^{-1}(p,e), h)G \end{aligned}$$

The induced transition function is

$$\widehat{\mathrm{tr}}_F \circ \mathrm{tr}_F^{-1}(p,h) = (p, g \cdot h) \quad \text{where } (p,g) = \widehat{\mathrm{tr}} \circ \mathrm{tr}^{-1}(p,e)$$

□

So if the action of G on F consists of linear maps then $(P \times F)/G$ will be a linear bundle. An affine action would yield an affine bundle, and so forth.

2.3.4 Weakly Associated Bundles

To make an associated bundle one attaches a fiber with a group action to a principal bundle having the same group and then passes to the quotient space. In short, by going to a weakly associated bundle we are allowing the group action on the fiber to vary.

THEOREM 2.3.9. *Let $\pi_P : P \to M$ be a principal fiber bundle with group G, and let $\pi_H : H \to P$ be a fiber bundle over P with fiber F and a right G action that* *projects to the action of G on P, i.e.*

$$\pi_H(h \cdot g) = \pi_H(h) \cdot g$$

Then H/G can be given the structure of a fiber bundle over M with fiber F.

Proof. We will prove this theorem by a series of definitions which we will prove to be well-defined.

DEFINITION 2.3.10. *The projection $\rho : H/G \to M$ is given by*

$$\rho(h \cdot G) = \pi_P \circ \pi_H(h)$$

This is well defined since

$$\begin{aligned} \pi_P \circ \pi_H(h \cdot g) &= \pi_P(\pi_H(h) \cdot g) \\ &= \pi_P \circ \pi_H(h) \end{aligned}$$

DEFINITION 2.3.11. *Let tr_P be a trivialization of π_P around $m \in M$ defined over $\tilde{U} \subseteq M$. Let tr_H be a trivialization of π_H around $u_e = \mathrm{tr}_P^{-1}(m, e)$ defined over $W \subseteq P$. Let $S = \mathrm{tr}_P^{-1}(\tilde{U} \times \{e\}) \subseteq P$, $V = W \cap S$, and $U = \pi_P(V)$. Finally, define $\mathrm{tr} : \rho^{-1}(U) \to U \times F$ by*

$$\mathrm{tr}(h \cdot G) = \left(\pi_P \circ \pi_H(h), \mathrm{pr}_2 \circ \mathrm{tr}_H \left(h \cdot (\mathrm{pr}_2 \circ \mathrm{tr}_P \circ \pi_H(h))^{-1}\right)\right) \tag{2.8}$$

We need to check a few things to ensure that tr is a well-defined trivialization.

1. $\pi_P \circ \pi_H(h)$ is well-defined because it is just $\rho(h \cdot G)$ which was shown to be well-defined above.

2. We need to show that U is open in M. To do this, consider the image of V under the diffeomorphism tr_P.

$$\begin{aligned} \mathrm{tr}_P(V) &= \mathrm{tr}_P(W \cap S) \\ &= \mathrm{tr}_P(W) \cap (\tilde{U} \times \{e\}) \end{aligned}$$

Now $\tilde{U} \times \{e\}$ is a closed submanifold of $M \times G$, but given the subset topology it is homeomorphic to $\tilde{U}$ under projection. Since $\mathrm{tr}_P(W)$ is an open subset of $M \times G$, we see that $\mathrm{tr}_P(V)$ is an open subset of $\tilde{U} \times \{e\}$ in the subset topology. This then implies that $\mathrm{pr}_1 \circ \mathrm{tr}_P(V) = \pi_P(V) = U$ is an open subset of $\tilde{U}$ and hence of M.

3. We need to make sure that $\pi_H(h)$ is in the domain of tr_P for any choice of representative h.

$$\begin{aligned} h \cdot G \in \rho^{-1}(U) &\implies \pi_P \circ \pi_H(h) \in U \\ &\implies \pi_H(h) \in \pi_P^{-1}(U) \\ &\implies \pi_H(h) \in \pi_P^{-1}(\tilde{U}) = \mathrm{dom}(\mathrm{tr}_P) \end{aligned}$$

where we have used the fact that $U \subseteq \tilde{U}$, shown above.

4. We need to make sure that $h \cdot (\mathrm{pr}_2 \circ \mathrm{tr}_P \circ \pi_H(h))^{-1}$ is in the domain of tr_H. Because of the construction of U, we have that

$$\begin{aligned} U \times \{e\} &= \mathrm{tr}_P(V) \\ \mathrm{tr}_P^{-1}(U \times \{e\}) &= V \subseteq W \end{aligned}$$

So let $\mathrm{tr}_P \circ \pi_H(h) = (u, g)$, where we are assured that $u \in U$. Then let us consider the image under the trivialization map.

$$\begin{aligned} \mathrm{tr}_P \circ \pi_H\big(h \cdot (\mathrm{pr}_2 \circ \mathrm{tr}_P \circ \pi_H(h))^{-1}\big) &= \mathrm{tr}_P \circ \pi_H\big(h \cdot g^{-1}\big) \\ &= \mathrm{tr}_P(\pi_H(h) \cdot g^{-1}) \\ &= \mathrm{tr}_P(\pi_H(h)) \cdot g^{-1} \\ &= (u, g) \cdot g^{-1} \\ &= (u, e) \subseteq U \times \{e\} \end{aligned}$$

We can now conclude that

$$\begin{aligned} \pi_H\big(h \cdot (\mathrm{pr}_2 \circ \mathrm{tr}_P \circ \pi_H(h))^{-1}\big) &\subseteq \mathrm{tr}_P^{-1}(U \times \{e\}) \\ &\subseteq W \end{aligned}$$

Or

$$h \cdot (\mathrm{pr}_2 \circ \mathrm{tr}_P \circ \pi_H(h))^{-1} \subseteq \pi_H^{-1}(W) = \mathrm{dom}(\mathrm{tr}_H)$$

5. We need to make sure that tr is invertible. We do this by explicit construction of the inverse.

$$\mathrm{tr}^{-1}(m, f) = \mathrm{tr}_H^{-1}(\mathrm{tr}_P^{-1}(m, e), f) \cdot G \tag{2.9}$$

Suppose that $h = \mathrm{tr}_H^{-1}(\mathrm{tr}_P^{-1}(m, e), f)$, (a representative of $\mathrm{tr}^{-1}(m, f)$) then we have

$$\begin{aligned} \mathrm{pr}_2 \circ \mathrm{tr}_P \circ \pi_H(h) &= \mathrm{pr}_2 \circ \mathrm{tr}_P \circ \pi_H(h) \\ &= \mathrm{pr}_2 \circ \mathrm{tr}_P \circ \mathrm{tr}_P^{-1}(m, e) \\ &= e \end{aligned}$$

So we see that

$$\begin{aligned} \mathrm{tr}(h \cdot G) &= \big(\pi_P \circ \pi_H(h), \mathrm{pr}_2 \circ \mathrm{tr}_H \big(h \cdot (\mathrm{pr}_2 \circ \mathrm{tr}_P \circ \pi_H(h))^{-1}\big)\big) \\ &= \big(m, \mathrm{pr}_2 \circ \mathrm{tr}_H(h)\big) \\ &= (m, f) \end{aligned}$$

This proves that $\mathrm{tr} \circ \mathrm{tr}^{-1} = \mathrm{id}$. Next consider the reverse composition. First consider

$$\begin{aligned} \mathrm{pr}_1 \circ \mathrm{tr}_H \big(h \cdot (\mathrm{pr}_2 \circ \mathrm{tr}_P\, \pi_H(h))^{-1}\big) &= \pi_H\big(h \cdot (\mathrm{pr}_2 \circ \mathrm{tr}_P\, \pi_H(h))^{-1}\big) \\ &= \pi_H(h) \cdot (\mathrm{pr}_2 \circ \mathrm{tr}_P\, \pi_H(h))^{-1} \end{aligned}$$

Now let $\mathrm{tr}_P \circ \pi_H(h) = (m, g)$ or $\pi_H(h) = \mathrm{tr}_P^{-1}(m, g)$.

$$\begin{aligned} &= \mathrm{tr}_P^{-1}(m, g) \cdot (g)^{-1} \\ &= \mathrm{tr}_P^{-1}(m, e) \\ &= \mathrm{tr}_P^{-1}(\pi_P \circ \pi_H(h), e) \end{aligned}$$

Then the formula for the reverse composition becomes

$$\begin{aligned}
\mathrm{tr}^{-1}\circ\mathrm{tr}(h\cdot G) &= \mathrm{tr}_H^{-1}\Bigg(\mathrm{tr}_P^{-1}(\pi_P\circ\pi_H(h),e),\\
&\qquad \mathrm{pr}_2\circ\mathrm{tr}_H\left(h\cdot(\mathrm{pr}_2\circ\mathrm{tr}_P\circ\pi_H(h))^{-1}\right)\Bigg)\cdot G\\
&= \mathrm{tr}_H^{-1}\Bigg(\mathrm{pr}_1\circ\mathrm{tr}_H\left(h\cdot(\mathrm{pr}_2\circ\mathrm{tr}_P\,\pi_H(h))^{-1}\right),\\
&\qquad \mathrm{pr}_2\circ\mathrm{tr}_H\left(h\cdot(\mathrm{pr}_2\circ\mathrm{tr}_P\circ\pi_H(h))^{-1}\right)\Bigg)\cdot G\\
&= \left(h\cdot(\mathrm{pr}_2\circ\mathrm{tr}_P\,\pi_H(h))^{-1}\right)\cdot G\\
&= h\cdot G
\end{aligned}$$

6. Lastly, we need to make sure that the transition functions are smooth. First we write

$$\begin{aligned}
\mathrm{tr}(h\cdot G) &= \left(\pi_P\circ\pi_H(h),\mathrm{pr}_2\circ\mathrm{tr}_H\left(h\cdot(\mathrm{pr}_2\circ\mathrm{tr}_P\circ\pi_H(h))^{-1}\right)\right) \qquad (2.10)\\
\widetilde{\mathrm{tr}}^{-1}(m,f) &= \widetilde{\mathrm{tr}}_H^{-1}(\widetilde{\mathrm{tr}}_P^{-1}(m,e),f)\cdot G
\end{aligned}$$

Now note that if $h=\widetilde{\mathrm{tr}}_H^{-1}(\widetilde{\mathrm{tr}}_P^{-1}(m,e),f)$, we have

$$\begin{aligned}
\pi_P\circ\pi_H(h) &= m\\
\mathrm{pr}_2\circ\mathrm{tr}_P\circ\pi_H(h) &= \mathrm{pr}_2\circ\mathrm{tr}_P\circ\pi_H\circ\widetilde{\mathrm{tr}}_H^{-1}(\widetilde{\mathrm{tr}}_P^{-1}(m,e),f)\\
&= \mathrm{pr}_2\circ\mathrm{tr}_P(\widetilde{\mathrm{tr}}_P^{-1}(m,e))\\
&= \phi_P(m)
\end{aligned}$$

where we note that $\phi_P(m)=\mathrm{pr}_2\circ\mathrm{tr}_P(\widetilde{\mathrm{tr}}_P^{-1}(m,e))$ is just the transition function for the principal bundle P, thought of as a map from the base to the group. Substituting these into (2.10), we obtain

$$\mathrm{tr}\circ\widetilde{\mathrm{tr}}^{-1}(m,f)=\left(m,\mathrm{pr}_2\circ\mathrm{tr}_H\left(\widetilde{\mathrm{tr}}_H^{-1}(\widetilde{\mathrm{tr}}_P^{-1}(m,e),f)\cdot(\phi_P(m))^{-1}\right)\right) \qquad (2.11)$$

Which, although cumbersome, is clearly smooth.

This concludes the proof of Theorem 2.3.9. □

In the case of associated bundles, the properties of the group action on the fiber determine what sort of structure is inherited by the associated bundle. We will see that the same is true in the case of weakly associated bundles, at least for the structure that can be cast in terms of trivializations.

DEFINITION 2.3.12. *We say that an action of a group G on the total space of a fiber bundle $\pi : H \to M$* **preserves fibers** *if the image of each fiber under the action is also a fiber. In other words*

$$\forall p \in M \;\; \forall g \in G \;\; \exists q \in M \quad \mu_g(\pi^{-1}(p)) = \pi^{-1}(q)$$

If a group action preserves fibers then it projects to a group action on M via

$$\hat{\mu}_g(p) = \pi \circ \mu_g(\pi^{-1}(p)) \tag{2.12}$$

We can see that this action will inherit the appropriate differentiability properties. For any p and $q = \hat{\mu}_g(p)$ in M let tr_p be a trivialization around p and tr_q be a trivialization around q. Then for x in some neighborhood of p we have

$$\hat{\mu}_g(x) = \mathrm{pr}_1 \circ \mathrm{tr}_q \circ \mu_g \circ \mathrm{tr}_p^{-1}(x, f) \quad \forall f \in F \tag{2.13}$$

So in particular, if the bundle is smooth and the action μ is smooth, then $\hat{\mu}$ will be smooth also.

DEFINITION 2.3.13. *Let C be a subcategory of manifolds and let $\pi : H \to M$ be a fiber bundle in the associated subcategory $\hat{C}$ (see Definition 2.2.6). Let G act on H such that it preserves fibers. We will say that the action of G on H* **preserves fiber structure** *if for any $g \in G$ and any trivializations tr and $\widetilde{\mathrm{tr}}$ of H the map*

$$\mathrm{tr} \circ \mu_g \circ \widetilde{\mathrm{tr}}^{-1} : U \times F \to \tilde{U} \times F$$

when restricted to a single fiber, yields a morphism in C.

REMARK 2.3.14. *If an action of G on H preserves fiber structure then for any trivialization tr of H and any $g \in G$, $\mathrm{tr} \circ \mu_g$ is an admissible trivialization.*

THEOREM 2.3.15. *Let P be a principal bundle with group G. Let C be a subcategory of manifolds and let $\pi : H \to P$ be a fiber bundle in the associated subcategory $\hat{C}$. Finally, let G act on H such that*

1. *the action projects to the action on P*
2. *the action preserves fiber structure*

Then the weakly associated bundle H/G will also be of category $\hat{C}$.

Proof. To prove this, we need to show that the transition functions have the right properties. We recall the transition of two induced trivializations from equation (2.11).

$$\mathrm{tr} \circ \widetilde{\mathrm{tr}}^{-1}(m, f) = \left(m, \mathrm{pr}_2 \circ \mathrm{tr}_H \left(\widetilde{\mathrm{tr}}_H^{-1} (\widetilde{\mathrm{tr}}_P^{-1}(m, e), f) \cdot (\phi_P(m))^{-1} \right) \right)$$

or, alternately

$$\mathrm{tr} \circ \widetilde{\mathrm{tr}}^{-1}(m, f) = \left(m, \mathrm{pr}_2 \circ \mathrm{tr}_H \circ \mu_{(\phi_P(m))^{-1}} \circ \widetilde{\mathrm{tr}}_H^{-1} (\widetilde{\mathrm{tr}}_P^{-1}(m, e), f) \right)$$

We must show at $\mathrm{tr} \circ \widetilde{\mathrm{tr}}^{-1}$, when restricted to a single fiber, yields a morphism in C. Note that by restricting $\mathrm{tr} \circ \widetilde{\mathrm{tr}}^{-1}$ to $\{m\} \times F$, the factor of $\mathrm{tr}_H \circ \mu_{(\phi_P(m))^{-1}} \circ \widetilde{\mathrm{tr}}_H^{-1}$ in the formula is restricted to $\{u\} \times F$ where $u = \widetilde{\mathrm{tr}}_P^{-1}(m, e)$. And note also that $(\phi_P(m))^{-1}$ becomes a fixed group element. Hence the restriction of $\mathrm{tr}_H \circ \mu_{(\phi_P(m))^{-1}} \circ \widetilde{\mathrm{tr}}_H^{-1}$ must be a morphism in C which implies that the restriction of $\mathrm{tr} \circ \widetilde{\mathrm{tr}}^{-1}$ is also. □

2.4 N-Symplectic Geometry

N-symplectic geometry arises on the frame bundle LM when one takes the exterior derivative of the soldering form (definition 2.3.7), $d\theta$, as a vector-valued generalized symplectic form. We give a terse overview of the subject here; the curious reader is recommended to references [39, 40, 41, 42, 43] for greater detail.

2.4.1 The First Order Case

In the simplest case, we work with vector valued observables using the equation

$$d\hat{f}^i = -X_{\hat{f}} \lrcorner\, d\theta^i \tag{2.14}$$

which we call the first order n-symplectic structure equation. In this equation, $\hat{f}$ is an $\mathbb{R}^m$-valued function on LM and $X_{\hat{f}}$ is a vector field on LM. We call $X_{\hat{f}}$ the Hamiltonian vector field associated with $\hat{f}$.

Not all functions $\hat{f}$ are compatible with equation (2.14). The set of functions $\hat{f}$ for which a solution exists is denoted HF^1 and it is know [39, 40] that

$$HF^1 = T^1 \oplus C^\infty(M, \mathbb{R}^m)$$

where $C^\infty(M, \mathbb{R}^m)$ represents the lifts of smooth functions on M to LM and T^1 represents the $\mathbb{R}^m$-valued functions on LM that transform tensorially according to

$$R_g^* \hat{f} = g^{-1} \cdot \hat{f} \tag{2.15}$$

where we use the natural left action of $\mathrm{GL}(m)$ on $\mathbb{R}^m$ in the right hand side. When a solution $X_{\hat{f}}$ exists, it is unique.

For any vector field f on M, we can define its associated tensorial function $\hat{f}$ by

$$\hat{f}^i(p, e_k) = e^i(f) \tag{2.16}$$

where e^i represents the coframe of e_i. Such a function $\hat{f}$ will be an element of HF^1 and its Hamiltonian vector field is given locally by

$$f^i \frac{\partial}{\partial x^i} - \frac{\partial f^i}{\partial x^j} \pi_i^k \frac{\partial}{\partial \pi_j^k} \qquad \text{or} \qquad f^i \frac{\partial}{\partial x^i} + \frac{\partial f^i}{\partial x^j} \upsilon_k^j \frac{\partial}{\partial \upsilon_k^i} \tag{2.17}$$

where we use f^i for the components of f in the x basis.

2.4.2 The p-th Order Case

In the higher order case we allow more indices and find that unique solutions no longer exist. We isolate two cases: symmetric and anti-symmetric $\otimes^p \mathbb{R}^m$-valued observables on LM. The more general structure equation for n-symplectic geometry takes the form

$$d\hat{f}^{i_1 i_2 \cdots i_p} = -p! X_{\hat{f}}{}^{(i_1 i_2 \cdots i_{p-1}} \lrcorner\, d\theta^{i_p)} \tag{2.18}$$

or

$$d\hat{f}^{i_1 i_2 \cdots i_p} = -p! X_{\hat{f}}{}^{[i_1 i_2 \cdots i_{p-1}} \lrcorner\, d\theta^{i_p]} \tag{2.19}$$

depending on the symmetry of the observable. Here we use round brackets (parentheses) for symmetrization on indices and square brackets for antisymmetrization. In both cases the introduction of brackets prevents us from obtaining unique solutions. We note that $X_{\hat{f}}$ now represents a $\otimes^{p-1}\mathbb{R}^m$-valued vector field on LM.

The set of functions $\hat{f}^{i_1 i_2 \cdots i_p}$ for which equation (2.18) is solvable is described in [39, 40]. For our purposes, it suffices to say that it includes the symmetric tensorial functions of rank p,

$$ST^p = \left\{ \hat{f} : LM \to \otimes^p_s \mathbb{R}^m \;\middle|\; \hat{f}(u \cdot g) = g^{-1} \cdot \hat{f}(u) \;\; \forall g \in \mathrm{GL}(m) \right\} \tag{2.20}$$

Likewise, the set of functions $\hat{f}^{i_1 i_2 \cdots i_p}$ for which equation (2.19) is solvable includes the antisymmetric tensorial functions of rank p,

$$AT^p = \left\{ \hat{f} : LM \to \otimes^p_a \mathbb{R}^m \;\middle|\; \hat{f}(u \cdot g) = g^{-1} \cdot \hat{f}(u) \;\; \forall g \in \mathrm{GL}(m) \right\} \tag{2.21}$$

For any rank p contravariant tensor field f on M, we can define its associated tensorial function $\hat{f} : LM \to \otimes^p_s \mathbb{R}^m$ by

$$\hat{f}^{i_1 \cdots i_p}(p, e_k) = f(e^{i_1}, \ldots, e^{i_p}) \tag{2.22}$$

If f is a symmetric tensor field then $\hat{f} \in ST^p$ and the resulting solution space of equation (2.18) is described by

$$\frac{1}{(p-1)!} f^{j_1 \cdots j_{p-1} k} \pi^{i_1}_{j_1} \cdots \pi^{i_{p-1}}_{j_{p-1}} \frac{\partial}{\partial x^k} - \frac{1}{p!} \left(\frac{\partial f^{j_1 \cdots j_p}}{\partial x^a} \pi^{i_1}_{j_1} \cdots \pi^{i_{p-1}}_{j_{p-1}} \pi^b_{j_p} + T_a^{i_1 \cdots i_{p-1} b} \right) \frac{\partial}{\partial \pi^b_a} \quad (2.23)$$

where T satisfies the condition $T_a^{(i_1 \cdots i_{p-1} b)} = 0$ but is otherwise arbitrary. We say that $\hat{f}$ determines an equivalence class of Hamiltonian vector fields.

Likewise if f is an antisymmetric tensor field then $\hat{f} \in AT^p$ and the resulting solution space of equation (2.19) is again described by equation (2.23) but with T satisfying the alternate condition $T_a^{[i_1 \cdots i_{p-1} b]} = 0$.

Chapter 3

Jets of Tensor fields

3.1 A special case: Vector Fields

Consider a vector field as a section of the bundle $\tau : TM \to M$. In local coordinates, for a vector field $X = X^i \frac{\partial}{\partial x^i}$, the 1-jet of that section is

$$dx^k \otimes \frac{\partial}{\partial x^k} + \frac{\partial X^i}{\partial x^j} dx^j \otimes \frac{\partial}{\partial v^i} \tag{3.1}$$

On the other hand, the Hamiltonian vector field on LM corresponding to that section is

$$X^i \frac{\partial}{\partial x^i} - \frac{\partial X^i}{\partial x^j} \pi_i^k \frac{\partial}{\partial \pi_j^k} = X^i \frac{\partial}{\partial x^i} + \frac{\partial X^i}{\partial x^j} v_k^j \frac{\partial}{\partial v_k^i} \tag{3.2}$$

It is apparent that both contain similar information. In this section we will demonstrate a projection that takes Hamiltonian vector fields to the corresponding jets.

We make much use of the following function.

Definition 3.1.1. *The map $\psi : LM \times \mathbb{R}^m \to TM$ is the standard projection used to obtain TM as an associated bundle.*

$$\psi(p, \{e_i\}, \xi) = \xi^i e_i = \xi^i v_i^j(u) \left. \frac{\partial}{\partial x^j} \right|_{\lambda(u)} \tag{3.3}$$

In local coordinates, we have

$$x^i \circ \psi(u,\xi) = x^i(u,\xi) = x^i(u) \tag{3.4}$$

$$v^j \circ \psi(u,\xi) = t^i(\xi) v_i^j(u) \tag{3.5}$$

where we use t^i for the standard coordinates on $\mathbb{R}^m$.

The differential of ψ will be useful to us.

$$\begin{aligned} d\psi &: T(LM) \times T(\mathbb{R}^m) \to T(TM) \\ d_{(u,\xi)}\psi &= d_{(u,\xi)}(x^i \circ \psi) \otimes \left.\frac{\partial}{\partial x^i}\right|_{\psi(u,\xi)} + d_{(u,\xi)}(v^j \circ \psi) \otimes \left.\frac{\partial}{\partial v^j}\right|_{\psi(u,\xi)} \\ &= d_{(u,\xi)}x^i \otimes \left.\frac{\partial}{\partial x^i}\right|_{\psi(u,\xi)} + \big(t^i(\xi) d_{(u,\xi)} v_i^j + v_i^j(u) d_{(u,\xi)} t^i\big) \otimes \left.\frac{\partial}{\partial v^j}\right|_{\psi(u,\xi)} \end{aligned} \tag{3.6}$$

or in shorthand

$$d\psi = dx^i \otimes \frac{\partial}{\partial x^i} + (t^i dv_i^j + v_i^j dt^i) \otimes \frac{\partial}{\partial v^j}$$

where we use v_i^j for the frame coordinates on LM.

Definition 3.1.2. *Given any $X \in T_u(LM)$, define $\phi_X : T_u(LM) \to T_{d_u\lambda(X)}(TM)$ by*

$$\phi_X(\tilde{Y}) = d_{(u,\theta_u(X))}\psi\big(\tilde{Y}, \tilde{Y} \lrcorner (-X) \lrcorner d_u\theta\big) \tag{3.7}$$

$$= d_{(u,\theta_u(X))}\psi\big(\tilde{Y}, d_u\theta(\tilde{Y}, X)\big) \tag{3.8}$$

where $\psi : LM \times \mathbb{R}^n \to TM$ is the standard projection to the associated bundle.

Since ϕ_X is a linear map, we consider it a $T(TM)$-valued one-form at $u \in LM$.

Theorem 3.1.3. *For any $X \in T_u(LM)$, ϕ_X is a horizontal 1-form and is therefore projectible to a $T(TM)$-valued one-form $\tilde{\phi}_X$ on M (equivalent to an element of $J^1\pi$).*

Proof. Let $\tilde{Y}$ be a vertical vector at $u \in LM$, Expand $\tilde{Y}$ and X locally as

$$\tilde{Y} = \tilde{Y}_b^a \left.\frac{\partial}{\partial v_b^a}\right|_u \tag{3.9}$$

$$X = X^i \left.\frac{\partial}{\partial x^i}\right|_u + X_k^j \left.\frac{\partial}{\partial v_k^j}\right|_u \tag{3.10}$$

Using the formula $d\theta^i = -\pi^i_a \pi^b_k dv^a_b \wedge dx^k$ we find that

$$d_u\theta^i(\tilde{Y}, X) = -\pi^i_a \pi^b_k \tilde{Y}^a_b X^k - 0 \tag{3.11}$$

Substituting this into (3.7) and using the formula for $d\psi$ (3.6), we obtain

$$\phi_X(\tilde{Y}) = 0 + (\theta^i_u(X)\tilde{Y}^j_i + v^j_i d_u\theta^i(\tilde{Y}, X)) \otimes \left.\frac{\partial}{\partial v^j}\right|_{\psi(u,\theta_u(X))} \tag{3.12}$$

$$= (\pi^i_k X^k \tilde{Y}^j_i - v^j_i \pi^i_a \pi^b_k \tilde{Y}^a_b X^k) \otimes \left.\frac{\partial}{\partial v^j}\right|_{\psi(u,\theta_u(X))} \tag{3.13}$$

$$= (\pi^i_k X^k \tilde{Y}^j_i - \delta^j_a \pi^i_k \tilde{Y}^a_i X^k) \otimes \left.\frac{\partial}{\partial v^j}\right|_{\psi(u,\theta_u(X))} \tag{3.14}$$

$$= (\pi^i_k X^k \tilde{Y}^j_i - \pi^i_k \tilde{Y}^j_i X^k) \otimes \left.\frac{\partial}{\partial v^j}\right|_{\psi(u,\theta_u(X))} \tag{3.15}$$

$$= 0 \tag{3.16}$$

□

Definition 3.1.4. *For any $X \in T_u(LM)$ and any $Y \in T_{\lambda(u)}M$, define*

$$F(X)(Y) = \phi_X(\tilde{Y}) \tag{3.17}$$

where $\tilde{Y}$ is any vector at u that projects to Y.

Since $F(X)$ is a linear map from $T_{\lambda(u)}M$ to $T_{d\lambda(X)}(TM)$ that projects to the identity, it defines a 1-jet of a vector field. So, F itself is a map from the tangent bundle of the frame bundle to the 1-jet bundle of the tangent bundle.

$$F : T(LM) \to J^1\tau$$

The existence of this map is very intriguing, as it provides a connection between n-symplectic geometry on LM and the jet bundle formulations of field dynamics.

Now we will calculate a formula for F in local coordinates. Since the vertical components of $\tilde{Y}$ are irrelevant to the calculation, we will choose them to be zero.

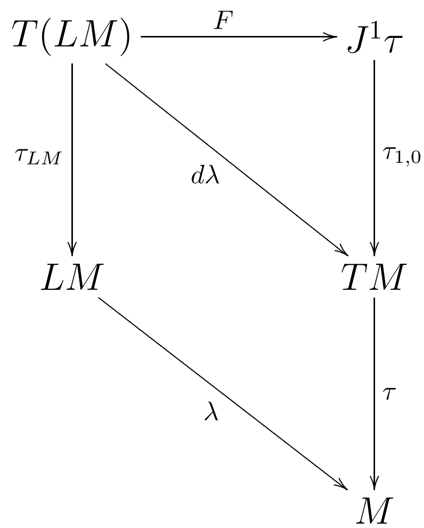

Figure 3.1: Commutative Diagram for F

$$\tilde{Y} = \tilde{Y}^i \left.\frac{\partial}{\partial x^i}\right|_u \qquad X = X^i \left.\frac{\partial}{\partial x^i}\right|_u + X_k^j \left.\frac{\partial}{\partial v_k^j}\right|_u \tag{3.18}$$

$$d_u\theta^i(\tilde{Y}, X) = 0 + \pi_a^i \pi_k^b X_b^a \tilde{Y}^k \tag{3.19}$$

From (3.17), (3.6) and (3.7) we obtain

$$F(X)(Y) = \tilde{Y}^k \left.\frac{\partial}{\partial x^k}\right|_{d_u\lambda(X)} + \left(0 + v_i^j \pi_a^i \pi_k^b X_b^a \tilde{Y}^k\right) \left.\frac{\partial}{\partial v^j}\right|_{d_u\lambda(X)} \tag{3.20}$$

$$= \tilde{Y}^k \left.\frac{\partial}{\partial x^k}\right|_{d_u\lambda(X)} + \delta_a^j \pi_k^b X_b^a \tilde{Y}^k \left.\frac{\partial}{\partial v^j}\right|_{d_u\lambda(X)} \tag{3.21}$$

$$= \tilde{Y}^k \left(\left.\frac{\partial}{\partial x^k}\right|_{d_u\lambda(X)} + \pi_k^b X_b^j \left.\frac{\partial}{\partial v^j}\right|_{d_u\lambda(X)} \right) \tag{3.22}$$

So

$$F(X) = d_{\lambda(u)}x^k \otimes \left(\left.\frac{\partial}{\partial x^k}\right|_{d_u\lambda(X)} + \pi_k^b X_b^j \left.\frac{\partial}{\partial v^j}\right|_{d_u\lambda(X)} \right) \tag{3.23}$$

From this formula it is easy to see that, in coordinates,

$$\begin{aligned} x^i \circ F &= x^i \\ v^j \circ F &= \dot{x}^j \\ v_k^j \circ F &= \pi_k^b \dot{v}_b^j \end{aligned} \tag{3.24}$$

Clearly F is a smooth function.

THEOREM 3.1.5. *Let f be a vector field on an open subset U of M (a local section of τ), $\hat{f} : \lambda^{-1}(U) \to \mathbb{R}^m$ its tensorial function on LM, and $X_{\hat{f}}$ the corresponding Hamiltonian vector field.*

$$\hat{f}^i(p, \{e^k\}) = e^i(f(p)) \qquad \textit{and} \qquad d\hat{f}^i = -X_{\hat{f}} \lrcorner\, d\theta^i \tag{3.25}$$

Then

$$F(X_{\hat{f}}(u)) = j^1_{\lambda(u)} f \tag{3.26}$$

for all $u \in \lambda^{-1}(U)$.

Proof. In local coordinates, we have

$$X_{\hat{f}}(u) = (f^k \circ \lambda)(u) \left.\frac{\partial}{\partial x^k}\right|_u + \left.\frac{\partial f^j}{\partial x^a}\right|_{\lambda(u)} v_b^a(u) \left.\frac{\partial}{\partial v_b^j}\right|_u \tag{3.27}$$

Note that $d_u\lambda(X_{\hat{f}}) = (f \circ \lambda)(u)$ ($X_{\hat{f}}$ projects to the vector field f).

So using formula (3.23),

$$F(X_{\hat{f}}) = d_{\lambda(u)}x^k \otimes \left(\left.\frac{\partial}{\partial x^k}\right|_{d_u\lambda(X_{\hat{f}})} + \pi_k^b(u) \left.\frac{\partial f^j}{\partial x^a}\right|_{\lambda(u)} v_b^a(u) \left.\frac{\partial}{\partial v^j}\right|_{d_u\lambda(X_{\hat{f}})} \right) \tag{3.28}$$

$$= d_{\lambda(u)}x^k \otimes \left(\left.\frac{\partial}{\partial x^k}\right|_{f\circ\lambda(u)} + \left.\frac{\partial f^j}{\partial x^k}\right|_{\lambda(u)} \left.\frac{\partial}{\partial v^j}\right|_{f\circ\lambda(u)} \right) \tag{3.29}$$

This is exactly $j^1_{\lambda(u)} f$. □

THEOREM 3.1.6. *$F : T(LM) \to J^1\tau$ is a fiber bundle.*

Proof. Consider the coordinate representation of F (eqn. 3.24)

$$x^i \circ F = x^i \qquad v^j \circ F = \dot{x}^j \qquad v^j_k \circ F = \pi^b_k \dot{v}^j_b$$

F is clearly a smooth function.

We will use the following induced coordinates

$$(x^i, v^j_k) \text{ and } (x^i, \pi^j_k) \qquad \text{on } \lambda^{-1}(U) \tag{3.30}$$

$$(x^i, v^j) \qquad \text{on } \tau^{-1}(U) \subset TM \tag{3.31}$$

$$(x^i, v^j_k, \dot{x}^i, \dot{v}^j_k) \text{ and } (x^i, \pi^j_k, \dot{x}^i, \dot{\pi}^j_k) \qquad \text{on } \tau^{-1}_{LM}(\lambda^{-1}(U)) \subset T(LM) \tag{3.32}$$

$$(x^i, v^j, v^j_k) \qquad \text{on } \tau_1^{-1}(U) \tag{3.33}$$

Define the functions $\tilde{v}^j_k$ on $\tau^{-1}_{TM} \circ \lambda^{-1}(U)$ by $\tilde{v}^j_k(X) = \pi^b_k \circ \tau_{LM}(X)\dot{v}^j_b(X)$. The functions $(x^i, v^j_k, \dot{x}^i, \tilde{v}^j_k)$ form local coordinates on $\tau^{-1}_{TM} \circ \lambda^{-1}(U)$. (The overlap function is $(a, b, c, d) \to (a, b, c, db^{-1})$, which is a smooth diffeomorphism). Because their domains span entire fibers, these adapted coordinates actually define a local trivialization. Hence F is a fiber bundle. □

Definition 3.1.7. *Using the right action of* GL(m) *on* LM *(denoted* $u \cdot g$ *or* $R_g(u)$*), we define a right action of* GL(m) *on* $T(LM)$ *in the obvious way.*

$$(u, X) \cdot g = (u \cdot g, R_{g*}(X)) \tag{3.34}$$

or simply,

$$X \cdot g = R_{g*}(X) \tag{3.35}$$

Let us consider how this looks in coordinates.

$$\begin{aligned} d_u R_g &= d_u(x^i \circ R_g) \otimes \left.\frac{\partial}{\partial x^i}\right|_{R_g(u)} + d_u(v^i_j \circ R_g) \otimes \left.\frac{\partial}{\partial v^i_j}\right|_{R_g(u)} \\ &= d_u x^i \otimes \left.\frac{\partial}{\partial x^i}\right|_{u \cdot g} + d_u(v^i_k g^k_j) \otimes \left.\frac{\partial}{\partial v^i_j}\right|_{u \cdot g} \\ &= d_u x^i \otimes \left.\frac{\partial}{\partial x^i}\right|_{u \cdot g} + g^k_j d_u v^i_k \otimes \left.\frac{\partial}{\partial v^i_j}\right|_{u \cdot g} \end{aligned} \tag{3.36}$$

or alternately,

$$d_u R_g = d_u x^i \otimes \left.\frac{\partial}{\partial x^i}\right|_{u\cdot g} + (g^{-1})^i_k d_u \pi^k_j \otimes \left.\frac{\partial}{\partial \pi^i_j}\right|_{u\cdot g} \tag{3.37}$$

Or, in another form,

$$\begin{aligned} x^i(X\cdot g) &= x^i(X) & \dot{x}^i(X\cdot g) &= \dot{x}^i(X) \\ v^i_j(X\cdot g) &= g^k_j v^i_k(X) & \dot{v}^i_j(X\cdot g) &= g^k_j \dot{v}^i_k(X) \end{aligned} \tag{3.38}$$

THEOREM 3.1.8. *$F : T(LM) \to J^1\tau$ is a principal bundle under the given action (def. 3.1.7).*

Proof. As we have already shown that F is a fiber bundle, what remains is to show that

1. The action is free
2. The equivalence classes are exactly the fibers.

To prove (1), suppose $(u, X)\cdot g = (u, X)$. This implies $u \cdot g = u$, which implies $g = e$, since the action on LM is free.

To prove (2), recall the coordinate form of F (eqn. 3.24).

$$x^i \circ F((u, X)\cdot g) = x^i(u\cdot g) = x^i(u) = x^i \circ F(u, X) \tag{3.39}$$

$$v^j \circ F((u, X)\cdot g) = \dot{x}^j((u, X)\cdot g) = \dot{x}^j(u, X) = v^j \circ F((u, X)) \tag{3.40}$$

$$v^j_k \circ F((u, X)\cdot g) = \pi^b_k(u\cdot g)\dot{v}^j_b((u, X)\cdot g) \tag{3.41}$$

$$= (g^{-1})^b_a \pi^a_k(u) g^c_b \dot{v}^j_c(u, X) \tag{3.42}$$

$$= \pi^a_k(u)\dot{v}^j_a(u, X) \tag{3.43}$$

$$= v^j_k \circ F((u, X)) \tag{3.44}$$

So,

$$F((u, X)\cdot g) = F(u, X) \tag{3.45}$$

This proves that the equivalence classes are subsets of the fibers. To show the reverse inclusion, suppose that $F(u, X) = F(p, Z)$.

$$\begin{aligned}
x^i \circ F(u, X) = x^i \circ F(p, Z) &\implies x^i(u) = x^i(p) \\
&\implies \exists g \;\; u \cdot g = p \qquad (3.46) \\
u \cdot g = p &\implies v^i_k(u) g^k_j = v^i_j(p) \\
&\implies \pi^b_k(u)(g^{-1})^a_b = \pi^a_k(p) \\
v^j \circ F(u, X) = v^j \circ F(p, Z) &\implies \dot{x}^j(u, X) = \dot{x}^j(p, Z) \qquad (3.47) \\
v^j_k \circ F(u, X) = v^j_k \circ F(p, Z) &\implies \pi^a_k(u)\dot{v}^j_a(u, X) = \pi^a_k(p)\dot{v}^j_a(p, Z) \\
&\implies \pi^b_k(u)\dot{v}^j_b(u, X) = \pi^b_k(u)(g^{-1})^a_b \dot{v}^j_a(p, Z) \\
&\implies g^a_b \dot{v}^j_a(u, X) = \dot{v}^j_a(p, Z) \qquad (3.48)
\end{aligned}$$

Using (3.46) and we can conclude that

$$\begin{aligned}
x^i((u, X) \cdot g) &= x^i(u \cdot g) \\
&= x^i(p) \\
&= x^i(p, Z) \\
v^i_j((u, X) \cdot g) &= v^i_j(u \cdot g) \\
&= v^i_j(p) \\
&= v^i_j(p, Z)
\end{aligned}$$

From (3.47) and (3.38) we have

$$\dot{x}^i((u, X) \cdot g) = \dot{x}^i(p, Z)$$

From (3.48) and (3.38) we have

$$\dot{v}^j_a((u, X) \cdot g) = \dot{v}^j_a(p, Z)$$

Finally, we conclude

$$(u, X) \cdot g = (p, Z)$$

This proves the reverse inclusion. □

THEOREM 3.1.9. *The bundle $J^1\tau$ is bundle isomorphic to the weakly associated bundle obtained from $T(LM)$ with the given action.*

Proof.

We recall the notion of weakly associated bundles from section 2.3.4. Since the set $T(LM)/\operatorname{GL}(m)$ is readily identified with $J^1\tau$ via the map $F/\operatorname{GL}(m)$, what remains to be shown is that the bundle structure is the same. To do this we will need to consider the trivialization maps for $\lambda : LM \to M$ and $\tau_{LM} : T(LM) \to LM$.

For any point $m_0 \in M$, let x^i be local coordinates around m_0 with domain U. We will use the trivialization of LM given by

$$\operatorname{tr}_P^{-1} : U \times \operatorname{GL}(m) \to \lambda^{-1}(U) \subseteq LM$$
$$\operatorname{tr}_P^{-1}(m, g) = (m, \{g_j^i \left.\frac{\partial}{\partial x^i}\right|_m \})$$

Secondly, we use the trivialization of $T(LM)$ given by

$$\operatorname{tr}_H^{-1} : \lambda^{-1}(U) \times (\mathbb{R}^m \times \mathbb{R}^{m^2}) \to \tau_{LM}^{-1}(\lambda^{-1}(U)) \subseteq T(LM)$$
$$\operatorname{tr}_H^{-1}(u, (A, B)) = (u, A^i \left.\frac{\partial}{\partial x^i}\right|_u + B_k^j \left.\frac{\partial}{\partial v_k^j}\right|_u)$$

where (x^i, v_k^j) represent the standard frame coordinates on LM induced by the q coordinates.

We recall that the inverse of the induced trivialization map is given by Equation (2.9).

$$\operatorname{tr}^{-1}(m, f) = \operatorname{tr}_H^{-1}(\operatorname{tr}_P^{-1}(m, e), f) \cdot G$$

Which under our identification becomes

$$\operatorname{tr}^{-1}(m, f) = F \circ \operatorname{tr}_H^{-1}(\operatorname{tr}_P^{-1}(m, e), f)$$

Now from the formula for F in equation (3.23) we see that

$$F \circ \mathrm{tr}_H^{-1}(u,(A,B)) = d_{\lambda(u)}x^k \otimes \left(\left.\frac{\partial}{\partial x^k}\right|_{A^i \frac{\partial}{\partial x^i}\big|_{\lambda(u)}} + \pi_k^b(u) B_b^j \left.\frac{\partial}{\partial v^j}\right|_{A^i \frac{\partial}{\partial x^i}\big|_{\lambda(u)}} \right)$$

Finally we obtain

$$\begin{aligned} \mathrm{tr}^{-1}(m,(A,B)) &= F \circ \mathrm{tr}_H^{-1}(\mathrm{tr}_P^{-1}(m,e),(A,B)) \\ &= d_m x^k \otimes \left(\left.\frac{\partial}{\partial x^k}\right|_{A^i \frac{\partial}{\partial x^i}\big|_m} + \delta_k^b B_b^j \left.\frac{\partial}{\partial v^j}\right|_{A^i \frac{\partial}{\partial x^i}\big|_m} \right) \\ &= d_m x^k \otimes \left(\left.\frac{\partial}{\partial x^k}\right|_{A^i \frac{\partial}{\partial x^i}\big|_m} + B_k^j \left.\frac{\partial}{\partial v^j}\right|_{A^i \frac{\partial}{\partial x^i}\big|_m} \right) \end{aligned} \tag{3.49}$$

which is exactly the expected trivialization of $J^1\tau$. Since we can cover M with such trivializations all other induced trivializations will have smooth overlap. Therefore we have proven that $J^1\tau$ is bundle isomorphic to the weakly associated bundle $T(LM)/\,\mathrm{GL}(m)$. □

THEOREM 3.1.10. *The bundle $J^1\tau$ is vector bundle isomorphic to the weakly associated bundle obtained from $T(LM)$ with the given action.*

Proof. Since the group action on $T(LM)$ is linear on fibers we conclude, by Theorem 2.3.15, that the weakly associated bundle will inherit the vector bundle structure of $T(LM)$. As such, all of the transition functions will be linear on fibers. Now we note that the induced trivialization given in equation (3.49) is the usual linear trivialization of $J^1\tau$. Hence all other induced trivializations will be admissible linear trivializations of $J^1\tau$. This demonstrates vector bundle isomorphism. □

3.2 Symmetric Tensor Fields

Consider a type $(p,0)$ tensor field as a section of the bundle $\tau^p : T^pM \to M$. In local coordinates, for a tensor field $X(u) = X^{i_1\cdots i_p}(u) \left.\frac{\partial}{\partial x^{i_1}}\right|_u \otimes \cdots \otimes \left.\frac{\partial}{\partial x^{i_p}}\right|_u$, the 1-jet prolongation of that section is

$$d_u x^j \otimes \left(\frac{\partial}{\partial x^j}\bigg|_{X(u)} + \frac{\partial X^{i_1\cdots i_p}}{\partial x^j}\bigg|_u \frac{\partial}{\partial v^{i_1\cdots i_p}}\bigg|_{X(u)} \right) \tag{3.50}$$

If X is symmetric, the equivalence class of Hamiltonian vector fields on LM corresponding to that section is represented by

$$\frac{1}{(p-1)!} X^{j_1\cdots j_{p-1}k} \pi^{i_1}_{j_1}\cdots\pi^{i_{p-1}}_{j_{p-1}} \frac{\partial}{\partial x^k} - \frac{1}{p!}\Big(\frac{\partial X^{j_1\cdots j_p}}{\partial x^a}\pi^{i_1}_{j_1}\cdots\pi^{i_{p-1}}_{j_{p-1}}\pi^b_{j_p} + T^{i_1\cdots i_{p-1}b}_a\Big)\frac{\partial}{\partial \pi^b_a} \tag{3.51}$$

where T satisfies the condition $T^{(i_1\cdots i_{p-1}b)}_a = 0$ but is otherwise arbitrary.

A similar formula applies in the anti-symmetric case. Either way, it is again apparent that both contain similar information, namely the value of X and its first derivative. It turns out that all the results of the previous section will generalize nicely to the p-th order symmetric and anti-symmetric cases. This section will demonstrate the details of this generalization.

The map ψ that was so useful in the last section generalizes as follows

Definition 3.2.1. *The map $\psi^p : LM \times (\bigotimes^p \mathbb{R}^m) \to T^pM$ is the standard projection used to obtain T^pM as an associated bundle.*

$$\psi^p(p, \{e_i\}, T) = \psi^p(u, T) \tag{3.52}$$

$$= T^{i_1\cdots i_p}\ e_{i_1}\otimes\cdots\otimes e_{i_p} \tag{3.53}$$

$$= T^{i_1\cdots i_p} v^{j_1}_{i_1}(u)\cdots v^{j_p}_{i_p}(u)\ \frac{\partial}{\partial x^{j_1}}\bigg|_{\lambda(u)}\otimes\cdots\otimes\frac{\partial}{\partial x^{j_p}}\bigg|_{\lambda(u)} \tag{3.54}$$

where the $T^{i_1\cdots i_p}$ are the components of T in the standard basis of $\bigotimes^p \mathbb{R}^m$.

In local coordinates, we have

$$x^i \circ \psi^p = x^i \tag{3.55}$$

$$v^{j_1\cdots j_p}\circ\psi^p = t^{i_1\cdots i_p} v^{j_1}_{i_1}\cdots v^{j_p}_{i_p} \tag{3.56}$$

where the $t^{i_1\cdots i_p}$ are the standard coordinates on $\bigotimes^p \mathbb{R}^m$ defined by $t^{i_1\cdots i_p}(T) = T^{i_1\cdots i_p}$.

The map ψ^p has the interesting ability to recover a p-tensor field from its tensorial function.

LEMMA 3.2.2. *Let f be a $(p,0)$ tensor field on M and $\hat{f}$ its tensorial function on LM. Then we have*

$$\psi^p(u, \hat{f}(u)) = f_{\lambda(u)}$$

or, alternately

$$\psi^p \circ (\mathrm{id}_{LM}, \hat{f}) = f \circ \lambda$$

Proof. The tensorial function of f is given by

$$\hat{f} = \hat{f}^{i_1 \cdots i_p} \; r_{i_1} \otimes \cdots \otimes r_{i_p} \tag{3.57}$$

with

$$\hat{f}^{i_1 \cdots i_p}(u) = f_{\lambda(u)}(e^{i_1}, \ldots, e^{i_p}) \tag{3.58}$$

where we have used $\{r_i\}$ for the standard basis of $\mathbb{R}^m$ and $\{e_i\}$ for the basis vectors that comprise u. In light of this and equation (3.53), it is clear that

$$\psi^p(u, \hat{f}(u)) = \psi^p(u, \hat{f}^{i_1 \cdots i_p}(u) \; r_{i_1} \otimes \cdots \otimes r_{i_p}) \tag{3.59}$$

$$= \hat{f}^{i_1 \cdots i_p}(u) \; e_{i_1} \otimes \cdots \otimes e_{i_p} \tag{3.60}$$

$$= f_{\lambda(u)}(e^{i_1}, \ldots, e^{i_p}) \; e_{i_1} \otimes \cdots \otimes e_{i_p} \tag{3.61}$$

$$= f_{\lambda(u)} \tag{3.62}$$

□

REMARK 3.2.3. *If we differentiate the result of Lemma 3.2.2, we obtain*

$$d\psi^p \circ d(\mathrm{id}_{LM}, \hat{f}) = df \circ d\lambda$$

or

$$d\psi^p \circ (\mathrm{id}_{T(LM)}, d\hat{f}) = df \circ d\lambda$$

So just as ψ^p allows us to recover the tensor from its tensorial function, the differential of ψ^p allows us to recover the differential of the tensor field, which is readily identified with the 1-jet of the tensor field. Clearly $d\psi^p$ will prove useful to us. In the following we calculate its coordinate form. Starting from the coordinate form of ψ^p given in (3.55) and (3.56), we differentiate to obtain:

$$\begin{aligned} d\psi^p &: T(LM) \times T\Big(\bigotimes^p \mathbb{R}^m\Big) \to T(T^pM) \\ d_{(u,T)}\psi^p &= d_{(u,T)}(x^i \circ \psi) \otimes \left.\frac{\partial}{\partial x^i}\right|_{\psi(u,T)} + d_{(u,T)}(v^{j_1 \cdots j_p} \circ \psi) \otimes \left.\frac{\partial}{\partial v^{j_1 \cdots j_p}}\right|_{\psi(u,T)} \\ &= d_{(u,T)}x^i \otimes \left.\frac{\partial}{\partial x^i}\right|_{\psi(u,T)} \\ &\quad + \Bigg(t^{i_1 \cdots i_p}(T) d_{(u,T)}(v^{j_1}_{i_1} \cdots v^{j_p}_{i_p}) \\ &\qquad + v^{j_1}_{i_1}(u) \cdots v^{j_p}_{i_p}(u) d_{(u,T)} t^{i_1 \cdots i_p} \Bigg) \otimes \left.\frac{\partial}{\partial v^{j_1 \cdots j_p}}\right|_{\psi(u,T)} \end{aligned} \tag{3.63}$$

or in shorthand

$$d\psi^p = dx^i \otimes \left.\frac{\partial}{\partial x^i}\right|_{\psi(u,T)} + \Big(t^{i_1 \cdots i_p} d(v^{j_1}_{i_1} \cdots v^{j_p}_{i_p}) + v^{j_1}_{i_1} \cdots v^{j_p}_{i_p} dt^{i_1 \cdots i_p} \Big) \otimes \left.\frac{\partial}{\partial v^{j_1 \cdots j_p}}\right|_{\psi(u,T)} \tag{3.64}$$

where we use $t^{i_1 \cdots i_p}$ for coordinates on $\bigotimes^p \mathbb{R}^m$, and v^j_i for the frame coordinates on LM.

3.2.1 The bundle $\bigoplus^{m^{p-1}} T(LM)$

For a rank p observable in n-symplectic geometry, we obtain an m^{p-1}-tuple of vector fields. For the moment, we consider these as sections of the bundle $\bigoplus^{m^{p-1}} T(LM)$, which is the Whitney sum of the tangent bundle m^{p-1} times.

The standard tangent bundle coordinates induced from the frame coordinates on LM are given by

$$\begin{aligned} x^i(u,X) &= x^i(u) \\ v^i_j(u,X) &= v^i_j(u) \\ \dot{x}^{i_1\cdots i_p}(u,X) &= \dot{x}^{i_p}(X^{i_1\cdots i_{p-1}}) \\ &= d_u x^{i_p}(X^{i_1\cdots i_{p-1}}) \\ &= X^{i_1\cdots i_{p-1}}\vdots^{i_p} \\ \dot{v}^{i_1\cdots i_p}_j(u,X) &= \dot{v}^{i_p}_j(X^{i_1\cdots i_{p-1}}) \\ &= d_u v^{i_p}_j(X^{i_1\cdots i_{p-1}}) \\ &= X^{i_1\cdots i_{p-1}}\vdots^{i_p}_j \end{aligned} \tag{3.65}$$

We use the unusual notation $X^{i_1\cdots i_{p-1}}\vdots^{i_p}$ and $X^{i_1\cdots i_{p-1}}\vdots^{i_p}_j$ for the local basis components of the multivectors. This notation is chosen to distinguish the global multivector indices $i_1\cdots i_{p-1}$ from the local coordinate basis indices.

Definition 3.2.4. *The following alternate set of coordinates will also prove useful. For reasons that will become apparent later, we will call these* **Lagrangian coordinates**.

$$\begin{aligned} w^i &= x^i \\ w^j_k &= v^j_k \\ w^{j_1\cdots j_p} &= (p-1)!\, v^{j_1}_{i_1}\cdots v^{j_{p-1}}_{i_{p-1}}\dot{x}^{i_1\cdots i_{p-1}j_p} \\ w^{j_1\cdots j_p}_k &= p!\, v^{j_1}_{i_1}\cdots v^{j_{p-1}}_{i_{p-1}}\pi^b_k\dot{v}^{i_1\cdots i_{p-1}j_p}_b \end{aligned} \tag{3.66}$$

Definition 3.2.5. *Define a right action of $Gl(m)$ on $\bigoplus^{m^{p-1}} T(LM)$ as follows.*

$$(u, X^{i_1\cdots i_{p-1}})\cdot g = (u\cdot g, R_{g*}(X^{j_1\cdots j_{p-1}})(g^{-1})^{i_1}_{j_1}\cdots(g^{-1})^{i_{p-1}}_{j_{p-1}}) \tag{3.67}$$

For clarification, we use the notation R_g only for the action on LM.

We recall that, in coordinates.

$$d_u R_g = d_u x^i \otimes \frac{\partial}{\partial x^i}\bigg|_{u\cdot g} + g^k_j d_u v^i_k \otimes \frac{\partial}{\partial v^i_j}\bigg|_{u\cdot g} \tag{3.68}$$

or alternately,

$$d_u R_g = d_u x^i \otimes \frac{\partial}{\partial x^i}\bigg|_{u\cdot g} + (g^{-1})^i_k d_u \pi^k_j \otimes \frac{\partial}{\partial \pi^i_j}\bigg|_{u\cdot g} \tag{3.69}$$

So we have

$$\begin{aligned} x^i(X\cdot g) &= x^i(X) \\ v^i_j(X\cdot g) &= g^k_j v^i_k(X) \\ \dot{x}^{i_1\cdots i_p}(X\cdot g) &= \dot{x}^{j_1\cdots j_{p-1}i_p}(X)(g^{-1})^{i_1}_{j_1}\cdots(g^{-1})^{i_{p-1}}_{j_{p-1}} \\ \dot{v}^{i_1\cdots i_p}_l(X\cdot g) &= g^k_l \dot{v}^{j_1\cdots j_{p-1}i_p}_k(X)(g^{-1})^{i_1}_{j_1}\cdots(g^{-1})^{i_{p-1}}_{j_{p-1}} \end{aligned} \tag{3.70}$$

Or in the w coordinates,

$$\begin{aligned} w^i(X\cdot g) &= x^i(X\cdot g) \\ &= x^i(X) \\ &= w^i(X) \\ w^j_k(X\cdot g) &= v^j_k(X\cdot g) \\ &= g^i_k v^j_i(X) \\ &= g^i_k w^j_i(X) \\ w^{j_1\cdots j_p}(X\cdot g) &= (p-1)!\, v^{j_1}_{i_1}(X\cdot g)\cdots v^{j_{p-1}}_{i_{p-1}}(X\cdot g)\dot{x}^{i_1\cdots i_{p-1}j_p}(X\cdot g) \\ &= (p-1)!\, g^{k_1}_{i_1} v^{j_1}_{k_1}(X)\cdots g^{k_{p-1}}_{i_{p-1}} v^{j_{p-1}}_{k_{p-1}}(X)\dot{x}^{l_1\cdots l_{p-1}j_p}(X)(g^{-1})^{i_1}_{l_1}\cdots(g^{-1})^{i_{p-1}}_{l_{p-1}} \\ &= (p-1)!\, v^{j_1}_{k_1}(X)\cdots v^{j_{p-1}}_{k_{p-1}}(X)\dot{x}^{k_1\cdots k_{p-1}j_p}(X) \\ &= w^{j_1\cdots j_p}(X) \end{aligned}$$

$$\begin{aligned}
w_a^{j_1\cdots j_p}(X\cdot g) &= p!\, v_{i_1}^{j_1}(X\cdot g)\cdots v_{i_{p-1}}^{j_{p-1}}(X\cdot g)\pi_a^b(X\cdot g)\dot{v}_b^{i_1\cdots i_{p-1}j_p}(X\cdot g) \\
&= p!\, g_{i_1}^{k_1} v_{k_1}^{j_1}(X)\cdots g_{i_{p-1}}^{k_{p-1}} v_{k_{p-1}}^{j_{p-1}}(X)(g^{-1})_c^b \pi_a^c(X) \\
&\quad g_b^d \dot{v}_d^{l_1\cdots l_{p-1}j_p}(X)(g^{-1})_{l_1}^{i_1}\cdots(g^{-1})_{l_{p-1}}^{i_{p-1}} \\
&= p!\, v_{k_1}^{j_1}(X)\cdots v_{k_{p-1}}^{j_{p-1}}(X)\pi_a^c(X)\dot{v}_c^{k_1\cdots k_{p-1}j_p}(X) \\
&= w_a^{j_1\cdots j_p}(X) \qquad (3.71)
\end{aligned}$$

3.2.2 The Tensorially Symmetric Subbundle $\Upsilon_{(s)}^{p-1}$

DEFINITION 3.2.6. *For any positive integer p and $X \in \bigoplus^{m^{p-1}} T(LM)$ we define the* **tensor projection** *of X to be $\mathbb{T}(X)$, where the map $\mathbb{T} : \bigoplus^{m^{p-1}} T(LM) \to T^pM$ is given by*

$$\begin{aligned}
\mathbb{T}(u,X) &= (p-1)!\,\psi^p(u,\theta_u(X)) \\
&= (p-1)!\,(X^{i_1\cdots i_{p-1}} \lrcorner\, \theta_u^{i_p})\, e_{i_1}\otimes\cdots\otimes e_{i_p}
\end{aligned}$$

Note that we are hooking X into θ and not $d\theta$. Also note the lack of any sort of symmetrization on the indices. One might be tempted to change this definition and thereby force a symmetry condition on the tensor projection. However, it turns out to be necessary to have a symmetry condition on the vector X itself, making any symmetrization in this definition redundant. We make this restriction in Definition 3.2.7, and see its necessity in Theorem 3.2.17.

Recall that $\theta^i = \pi_j^i dx^j$. If we calculate the coordinate form of $\mathbb{T}$ we find

$$\mathbb{T}(u,X) = (p-1)!\,X^{i_1\cdots i_{p-1}}\vdots^k \pi_k^{i_p}(u)\, e_{i_1}\otimes\cdots\otimes e_{i_p} \qquad (3.72)$$

$$= (p-1)!\,X^{i_1\cdots i_{p-1}}\vdots^k \pi_k^{i_p}(u)\, v_{i_1}^{j_1}(u)\cdots v_{i_p}^{j_p}(u)\, \frac{\partial}{\partial x^{j_1}}\bigg|_{\lambda(u)}\otimes\cdots\otimes\frac{\partial}{\partial x^{j_p}}\bigg|_{\lambda(u)} \qquad (3.73)$$

$$= (p-1)!\, X^{i_1\cdots i_{p-1}\vdots k}\delta_k^{j_p}\; v_{i_1}^{j_1}(u)\cdots v_{i_{p-1}}^{j_{p-1}}(u)\, \frac{\partial}{\partial x^{j_1}}\bigg|_{\lambda(u)} \otimes\cdots\otimes \frac{\partial}{\partial x^{j_p}}\bigg|_{\lambda(u)} \tag{3.74}$$

$$= (p-1)!\, X^{i_1\cdots i_{p-1}\vdots j_p}\; v_{i_1}^{j_1}(u)\cdots v_{i_{p-1}}^{j_{p-1}}(u)\, \frac{\partial}{\partial x^{j_1}}\bigg|_{\lambda(u)} \otimes\cdots\otimes \frac{\partial}{\partial x^{j_p}}\bigg|_{\lambda(u)} \tag{3.75}$$

So we see that

$$\begin{aligned} x^i \circ \mathbb{T}(u,X) &= x^i\circ\lambda(u) \\ &= x^i(u,X) \\ v^{j_1\cdots j_p}\circ\mathbb{T}(u,X) &= (p-1)!\, X^{i_1\cdots i_{p-1}\vdots j_p}\; v_{i_1}^{j_1}(u)\cdots v_{i_{p-1}}^{j_{p-1}}(u) \\ &= (p-1)!\, \dot{x}^{i_1\cdots i_{p-1}j_p}(X)\; v_{i_1}^{j_1}(u)\cdots v_{i_{p-1}}^{j_{p-1}}(u) \end{aligned} \tag{3.76}$$

Comparing (3.76) to (3.66), we see that in the w coordinates $\mathbb{T}$ appears trivial.

$$x^i\circ\mathbb{T} = w^i \qquad\qquad v^{j_1\cdots j_p}\circ\mathbb{T} = w^{j_1\cdots j_p} \tag{3.77}$$

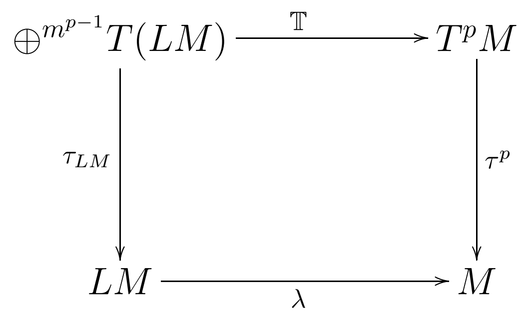

Figure 3.2: Commutative Diagram for $\mathbb{T}$

$\mathbb{T}$ is linear over the fibers of $\bigoplus^{m^{p-1}} T(LM)$. Or stated another way, the pair $(\mathbb{T},\lambda)$ is a vector bundle morphism. We note in the $p=1$ case that $\mathbb{T}$ is equal to $d\lambda$.

Definition 3.2.7. *We define the* **tensorially symmetric subbundle** *of* $\bigoplus^{m^{p-1}} T(LM)$ *to be those multivectors whose tensor projections are symmetric.*

$$\begin{aligned} \Upsilon_{(s)}^{p-1} &= \{X\in \bigoplus^{m^{p-1}} T(LM) \mid \mathbb{T}(X) \text{ is symmetric } \} \\ &= \mathbb{T}^{-1}(T_{(s)}^p(M)) \end{aligned}$$

Since $(\mathbb{T}, \lambda)$ *is a vector bundle morphism,* $\Upsilon^{p-1}_{(s)}$ *is a vector subbundle of* $\bigoplus^{m^{p-1}} T(LM)$

THEOREM 3.2.8. *The following are equivalent:*

1. $X \in \Upsilon^{p-1}_{(s)}$
2. $\theta^{i_p}(X^{i_1\cdots i_{p-1}}) = \theta^{(i_p}(X^{i_1\cdots i_{p-1})})$
3. $\pi^{i_p}_k X^{i_1\cdots i_{p-1}\vdots k} = \pi^{(i_p}_k X^{i_1\cdots i_{p-1})\vdots k}$ *in some canonical coordinate system.*
4. $X^{i_1\cdots i_{p-1}\vdots j} = X^{(i_1\cdots i_{p-1}\vdots j)}$ *in some canonical coordinate system.*

Proof.

(1) $\iff$ (2) This follows directly from the definition of $\mathbb{T}$ and the linearity of ψ^p.

(2) $\iff$ (3) The local coordinate expansion of (2) is (3). Since we are working at a single point, the two statements are equivalent.

(3) $\iff$ (4) Since the matrix (v^i_j) is nonsingular

$$\begin{aligned} \pi^{i_p}_k X^{i_1\cdots i_{p-1}\vdots k} = \pi^{(i_p}_k X^{i_1\cdots i_{p-1})\vdots k} \quad &\iff \quad v^j_{i_p}\pi^{i_p}_k X^{i_1\cdots i_{p-1}\vdots k} = v^j_{i_p}\pi^{(i_p}_k X^{i_1\cdots i_{p-1})\vdots k} \\ &\iff \quad \delta^j_k X^{i_1\cdots i_{p-1}\vdots k} = \delta^{(j}_k X^{i_1\cdots i_{p-1})\vdots k} \\ &\iff \quad X^{i_1\cdots i_{p-1}\vdots j} = X^{(i_1\cdots i_{p-1}\vdots j)} \end{aligned}$$

□

THEOREM 3.2.9. *Restriction to* $\Upsilon^{p-1}_{(s)}$ *forces the kernel of the n-symplectic equation to be vertical.*

Proof. Suppose that X satisfies

$$X^{(i_1\cdots i_{p-1}} \lrcorner\, d\theta^{i_p)} = 0$$
$$\theta^{i_p}(X^{i_1\cdots i_{p-1}}) = \theta^{(i_p}(X^{i_1\cdots i_{p-1})})$$

Then using Lemma 3.2.12, we obtain

$$\begin{aligned}0 &= \overset{*}{E}{}^a_a \lrcorner X^{(i_1\cdots i_{p-1}} \lrcorner d\theta^{i_p)} \\ &= -X^{(i_1\cdots i_{p-1}} \lrcorner \overset{*}{E}{}^{|a|}_a \lrcorner d\theta^{i_p)} \\ &= X^{(i_1\cdots i_{p-1}} \lrcorner \theta^{i_p)} \\ &= X^{i_1\cdots i_{p-1}} \lrcorner \theta^{i_p}\end{aligned}$$

which implies that X is vertical. □

THEOREM 3.2.10. *The Hamiltonian vector field corresponding to any symmetric $(p,0)$ tensor is a section of $\Upsilon^{p-1}_{(s)}$.*

Proof. If f is a symmetric $(p,0)$ tensor field on M, then the corresponding Hamiltonian vector field on LM is given by

$$\begin{aligned}X_{\hat{f}}(u) = &\frac{1}{(p-1)!} f^{j_1\cdots j_{p-1}k}\pi^{i_1}_{j_1}\cdots\pi^{i_{p-1}}_{j_{p-1}}\frac{\partial}{\partial x^k} \\ &-\frac{1}{p!}\left(\frac{\partial f^{j_1\cdots j_p}}{\partial x^a}\pi^{i_1}_{j_1}\cdots\pi^{i_{p-1}}_{j_{p-1}}\pi^b_{j_p} + T_a^{i_1\cdots i_{p-1}b}\right)\frac{\partial}{\partial \pi^b_a}\end{aligned}$$

where T satisfies the condition $T_a^{(i_1\cdots i_{p-1}b)} = 0$ but is otherwise arbitrary. According to theorem 3.2.8 we need to show that

$$f^{j_1\cdots j_{p-1}k}\pi^{i_1}_{j_1}\cdots\pi^{i_{p-1}}_{j_{p-1}}$$

is symmetric in the indices $(i_1\cdots i_{p-1}k)$. So we consider the following reduction

$$\begin{aligned}f^{l_1\cdots l_{p-1}k} = f^{(l_1\cdots l_{p-1}k)} &\iff f^{j_1 l_2\cdots l_{p-1}k}\pi^{i_1}_{j_1} = f^{j_1(l_2\cdots l_{p-1}k}\pi^{i_1)}_{j_1} \\ &\iff f^{j_1 j_2 l_3\cdots l_{p-1}k}\pi^{i_1}_{j_1}\pi^{i_2}_{j_2} = f^{j_1 j_2(l_3\cdots l_{p-1}k}\pi^{i_1}_{j_1}\pi^{i_2)}_{j_2} \\ &\cdots \\ &\iff f^{j_1\cdots j_{p-1}k}\pi^{i_1}_{j_1}\cdots\pi^{i_{p-1}}_{j_{p-1}} = f^{j_1\cdots j_{p-1}(k}\pi^{i_1}_{j_1}\cdots\pi^{i_{p-1})}_{j_{p-1}}\end{aligned}$$

where each step collapses to the previous upon multiplication by a nonsingular matrix $v^{l_z}_{i_z}$ □

It will turn out that the tensor projection of the Hamiltonian vector field of symmetric tensor field is again the tensor field. To prove this succinctly, we will need results of the following lemmas.

LEMMA 3.2.11.

$$d\hat{f}^{i_1\cdots i_p}(-\tfrac{1}{p}\overset{*}{E}{}^a_a) = \hat{f}^{i_1\cdots i_p}$$

Proof. Start with the local coordinate formulas

$$\hat{f}^{i_1\cdots i_p} = (f^{j_1\cdots j_p}\circ\lambda)\,\pi^{i_1}_{j_1}\cdots\pi^{i_p}_{j_p} \tag{3.78}$$

and

$$\overset{*}{E}{}^a_a = -\pi^a_j\frac{\partial}{\partial\pi^a_j} \tag{3.79}$$

Then

$$\begin{aligned} d\hat{f}^{i_1\cdots i_p} &= (f^{j_1\cdots j_p}\circ\lambda)\,d(\pi^{i_1}_{j_1}\cdots\pi^{i_p}_{j_p}) + \pi^{i_1}_{j_1}\cdots\pi^{i_p}_{j_p}d(f^{j_1\cdots j_p}\circ\lambda) &(3.80)\\ &= (f^{j_1\cdots j_p}\circ\lambda)\,d(\pi^{i_1}_{j_1}\cdots\pi^{i_p}_{j_p}) + \pi^{i_1}_{j_1}\cdots\pi^{i_p}_{j_p}\frac{(\partial f^{j_1\cdots j_p}\circ\lambda)}{\partial x^k}dx^k &(3.81)\end{aligned}$$

This leads to

$$\begin{aligned} d\hat{f}^{i_1\cdots i_p}(-\tfrac{1}{p}\overset{*}{E}{}^a_a) &= (f^{j_1\cdots j_p}\circ\lambda)\,d(\pi^{i_1}_{j_1}\cdots\pi^{i_p}_{j_p})(\tfrac{1}{p}\pi^i_j\frac{\partial}{\partial\pi^i_j}) &(3.82)\\ &= \frac{1}{p}(f^{j_1\cdots j_p}\circ\lambda)\sum_{k=1}^{p}\pi^{i_1}_{j_1}\cdots d\pi^{i_k}_{j_k}(\pi^i_j\frac{\partial}{\partial\pi^i_j})\cdots\pi^{i_p}_{j_p} &(3.83)\\ &= \frac{1}{p}(f^{j_1\cdots j_p}\circ\lambda)\sum_{k=1}^{p}\pi^{i_1}_{j_1}\cdots\pi^{i_k}_{j_k}\cdots\pi^{i_p}_{j_p} &(3.84)\\ &= (f^{j_1\cdots j_p}\circ\lambda)\,\pi^{i_1}_{j_1}\cdots\pi^{i_p}_{j_p} &(3.85)\\ &= \hat{f}^{i_1\cdots i_p} &(3.86)\end{aligned}$$

□

LEMMA 3.2.12.

$$-\overset{*}{E}{}^a_b \lrcorner\, d\theta^i = \delta^i_b \theta^a$$

Proof.

$$\begin{aligned}
-\overset{*}{E}{}^a_b \lrcorner\, d\theta^i &= \pi^a_j \frac{\partial}{\partial \pi^b_j} \lrcorner\, (d\pi^i_k \wedge dx^k) \\
&= \pi^a_j \delta^j_k \delta^i_b dx^k \\
&= \delta^i_b \pi^a_k dx^k \\
&= \delta^i_b \theta^a
\end{aligned}$$

□

REMARK 3.2.13. *In particular we have* $-\overset{*}{E}{}^a_a \lrcorner\, d\theta^i = \theta^i$.

THEOREM 3.2.14. *If f is a symmetric p-tensor field with tensorial function $\hat{f}$ and Hamiltonian vector field $X_{\hat{f}}$, then*

$$\mathbb{T}(u, X_{\hat{f}}) = f_{\lambda(u)}$$

Proof. By using the results of the preceding lemma and the n-symplectic equation, we obtain

$$d\hat{f}^{i_1 \cdots i_p} = -p!\, X_{\hat{f}}^{(i_1 \cdots i_{p-1}} \lrcorner\, d\theta^{i_p)} \tag{3.87}$$

$$-\frac{1}{p} \overset{*}{E}{}^a_a \lrcorner\, d\hat{f}^{i_1 \cdots i_p} = (p-1)!\, \overset{*}{E}{}^a_a \lrcorner\, X_{\hat{f}}^{(i_1 \cdots i_{p-1}} \lrcorner\, d\theta^{i_p)} \tag{3.88}$$

$$\hat{f}^{i_1 \cdots i_p} = (p-1)!\, X_{\hat{f}}^{(i_1 \cdots i_{p-1}} \lrcorner\, \theta^{i_p)} \tag{3.89}$$

$$\hat{f}^{i_1 \cdots i_p} = (p-1)!\, X_{\hat{f}}^{i_1 \cdots i_{p-1}} \lrcorner\, \theta^{i_p} \tag{3.90}$$

where we have used the result of Theorem 3.2.10 in the last step. So if we now consider the tensor projection of $X_{\hat{f}}$ we get

$$\mathbb{T}(u, X_{\hat{f}}) = (p-1)!\,\psi^p(u, \theta_u(X_{\hat{f}})) \tag{3.91}$$

$$= \psi^p(u, \hat{f}) \tag{3.92}$$

$$= f_{\lambda(u)} \tag{3.93}$$

□

And now we return to the problem of relating the Hamiltonian vector fields to 1-jets. The truly surprising thing is that we can do this pointwise, obtaining *purely geometric* results.

DEFINITION 3.2.15. *Given any $X \in \Upsilon^{p-1}_{(s)}$, define $\phi^{(s)}_X : T_u(LM) \to T_{\mathbb{T}(X)}(T^pM)$ by*

$$\phi^{(s)}_X(\tilde{Y}) = d_{(u,(p-1)!\,\theta_u(X))}\psi^p\big(\tilde{Y}, \tilde{Y} \lrcorner (-p!\,X^{(i_1\cdots i_{p-1})} \lrcorner\, d_u\theta^{i_p)}\big) \tag{3.94}$$

$$= d_{(u,(p-1)!\,\theta_u(X))}\psi^p\big(\tilde{Y}, p!\,d_u\theta^{(i_p}(\tilde{Y}, X^{i_1\cdots i_{p-1})})\big) \tag{3.95}$$

Since $\phi^{(s)}_X$ is a linear map, we consider it a $T(T^pM)$-valued one-form at $u \in LM$. We should emphasize that $\phi^{(s)}_X$ maps into $T_{\mathbb{T}(X)}(T^pM)$. This is because we take the differential of ψ^p at the point

$$\big(u, (p-1)!\,\theta^{i_p}_u(X^{i_1\cdots i_{p-1}})\big) \in LM \times \bigotimes{}^p\,\mathbb{R}^m$$

We could have defined $\phi^{(s)}_X$ for all m^p multi-vectors, but to show that $\phi^{(s)}_X$ is horizontal, we will need $X \in \Upsilon^{p-1}_{(s)}$. This is our next step, but we will need the following lemma.

LEMMA 3.2.16.

$$d(v^{(j_1}_{i_1}\cdots v^{j_p)}_{i_p})(\frac{\partial}{\partial v^a_c}) = p\,\delta^{(j_1}_a v^{j_2}_{(i_1}\cdots v^{j_p)}_{i_{p-1}}\delta^c_{i_p)}$$

Proof. We prove this by direct calculation, explicitly expanding the symmetrization and the product rule. The crucial step is the introduction of a second symmetrization, initially over $p-1$ terms, in equation (3.99).

$$
\begin{aligned}
& d(v_{i_1}^{(j_1}\cdots v_{i_p}^{j_p)})(\frac{\partial}{\partial v_c^a}) \\
&= \frac{1}{p!}\sum_{\pi\in S_p} d(v_{i_1}^{j_{\pi 1}}\cdots v_{i_p}^{j_{\pi p}})(\frac{\partial}{\partial v_c^a}) && (3.96)\\
&= \frac{1}{p!}\sum_{\pi\in S_p}\sum_{k=1}^{p} v_{i_1}^{j_{\pi 1}}\cdots dv_{i_k}^{j_{\pi k}}(\frac{\partial}{\partial v_c^a})\cdots v_{i_p}^{j_{\pi p}} && (3.97)\\
&= \frac{1}{p!}\sum_{\pi\in S_p}\sum_{k=1}^{p} v_{i_1}^{j_{\pi 1}}\cdots \delta_a^{j_{\pi k}}\delta_{i_k}^c\cdots v_{i_p}^{j_{\pi p}} && (3.98)
\end{aligned}
$$

With a slight feat of legerdemain, we introduce symmetrization over the $p-1$ terms $v_{i_{\sigma 1}}^{j_{\pi\sigma 1}}\cdots \widehat{v_{i_{\sigma k}}^{j_{\pi\sigma k}}}\cdots v_{i_{\sigma p}}^{j_{\pi\sigma p}}$, where the hat indicates that the term in left out. Since we are permuting identical factors, we are not changing the formula. By summing only over $\{\sigma\in S_p|\sigma p=k\}$ the two factors of δ remain unchanged.

$$
= \frac{1}{p!}\sum_{\pi\in S_p}\sum_{k=1}^{p}\left[\frac{1}{(p-1)!}\sum_{\substack{\sigma\in S_p\\ \sigma p=k}} v_{i_{\sigma 1}}^{j_{\pi\sigma 1}}\cdots v_{i_{\sigma(p-1)}}^{j_{\pi\sigma(p-1)}}\delta_a^{j_{\pi\sigma p}}\delta_{i_{\sigma p}}^c\right] \tag{3.99}
$$

And now the sums over σ and k collapse to a single sum.

$$
= \frac{1}{p!}\sum_{\pi\in S_p}\frac{p}{p!}\sum_{\sigma\in S_p} v_{i_{\sigma 1}}^{j_{\pi\sigma 1}}\cdots v_{i_{\sigma(p-1)}}^{j_{\pi\sigma(p-1)}}\delta_a^{j_{\pi\sigma p}}\delta_{i_{\sigma p}}^c \tag{3.100}
$$

Now we absorb the σs into the πs in the upper indices. This is allowable since π and σ are summed over all of S_p.

$$
\begin{aligned}
&= \frac{1}{p!}\sum_{\pi\in S_p}\frac{p}{p!}\sum_{\sigma\in S_p} v_{i_{\sigma 1}}^{j_{\pi 1}}\cdots v_{i_{\sigma(p-1)}}^{j_{\pi(p-1)}}\delta_a^{j_{\pi p}}\delta_{i_{\sigma p}}^c && (3.101)\\
&= p v_{(i_1}^{(j_1}\cdots v_{i_{p-1}}^{j_{p-1}}\delta_{|a|}^{j_p)}\delta_{i_p)}^c && (3.102)\\
&= p\delta_a^{(j_1} v_{(i_1}^{j_2}\cdots v_{i_{p-1}}^{j_p)}\delta_{i_p)}^c && (3.103)
\end{aligned}
$$

□

THEOREM 3.2.17. *For any $X \in \Upsilon^{p-1}_{(s)}$, $\phi^{(s)}_X$ is a horizontal 1-form and is therefore projectible to a $T(T^p_{(s)}M)$-valued one-form $\tilde{\phi}^{(s)}_X$ on M (equivalent to an element of $j^1\tau^p_{(s)}$).*

Proof. Let $\tilde{Y}$ be a vertical vector at $u \in LM$, Expand $\tilde{Y}$ and X locally as

$$\tilde{Y} = \tilde{Y}^a_b \left.\frac{\partial}{\partial v^a_b}\right|_u \tag{3.104}$$

$$X^{i_1 \cdots i_{p-1}} = X^{i_1 \cdots i_{p-1}} \vdots^{i} \left.\frac{\partial}{\partial x^i}\right|_u + X^{i_1 \cdots i_{p-1}} \vdots^{j}_{k} \left.\frac{\partial}{\partial v^j_k}\right|_u \tag{3.105}$$

Using the formula $d\theta^i = -\pi^i_a \pi^b_k dv^a_b \wedge dx^k$ we find that

$$d_u\theta^{(i_p}(\tilde{Y}, X^{i_1 \cdots i_{p-1})}) = -\tilde{Y}^a_b \pi^b_k \pi^{(i_p}_a X^{i_1 \cdots i_{p-1})} \vdots^{k} \tag{3.106}$$

Substituting this into (3.95) and using the formula for $d\psi^p$ (3.63), we obtain

$$\begin{aligned} \phi^{(s)}_X(\tilde{Y}) = 0 + \Bigg(& (p-1)!\, \theta^{i_p}_u(X^{i_1 \cdots i_{p-1}}) d(v^{j_1}_{i_1} \cdots v^{j_p}_{i_p})(\tilde{Y}) \\ & + v^{j_1}_{i_1} \cdots v^{j_p}_{i_p}\, p!\, d_u\theta^{(i_p}(\tilde{Y}, X^{i_1 \cdots i_{p-1})}) \Bigg) \otimes \left.\frac{\partial}{\partial v^{j_1 \cdots j_p}}\right|_{\mathbb{T}(X))} \end{aligned} \tag{3.107}$$

Now we introduce symmetrization on $\theta(X)$. This is valid since $X \in \Upsilon^{p-1}_{(s)}$

$$\begin{aligned} = \Bigg(& (p-1)!\, \theta^{(i_p}_u(X^{i_1 \cdots i_{p-1})}) d(v^{j_1}_{i_1} \cdots v^{j_p}_{i_p})(\tilde{Y}^a_c \frac{\partial}{\partial v^a_c}) \\ & - p!\, v^{j_1}_{i_1} \cdots v^{j_p}_{i_p} \tilde{Y}^a_b \pi^b_k \pi^{(i_p}_a X^{i_1 \cdots i_{p-1})} \vdots^{k} \Bigg) \otimes \cdots \end{aligned} \tag{3.108}$$

Next we move the symmetrization from the i indices to the j indices.

$$\begin{aligned} = \Bigg(& (p-1)!\, \pi^{i_p}_k X^{i_1 \cdots i_{p-1}} \vdots^{k} \tilde{Y}^a_c d(v^{(j_1}_{i_1} \cdots v^{j_p)}_{i_p})(\frac{\partial}{\partial v^a_c}) \\ & - p!\, v^{(j_1}_{i_1} \cdots v^{j_p)}_{i_p} \tilde{Y}^a_b \pi^b_k \pi^{i_p}_a X^{i_1 \cdots i_{p-1}} \vdots^{k} \Bigg) \otimes \cdots \end{aligned} \tag{3.109}$$

With a bit of factoring and the application of Lemma 3.2.16, the formula becomes more manageable.

$$= \pi^b_k X^{i_1 \cdots i_{p-1} \vdots k} \tilde{Y}^a_c \left((p-1)!\, \delta^{i_p}_b p \delta^{(j_1}_a v^{j_2}_{(i_1} \cdots v^{j_p)}_{i_{p-1}} \delta^c_{i_p)} - p!\, v^{(j_1}_{i_1} \cdots v^{j_p)}_{i_p} \delta^c_b \pi^{i_p}_a \right) \otimes \cdots \tag{3.110}$$

$$= p!\, \pi^b_k X^{i_1 \cdots i_{p-1} \vdots k} \tilde{Y}^a_c \left(\delta^{(j_1}_a v^{j_2}_{(i_1} \cdots v^{j_p)}_{i_{p-1}} \delta^c_{b)} - v^{(j_1}_{i_1} \cdots v^{j_{p-1}}_{i_{p-1}} \delta^{j_p)}_a \delta^c_b \right) \otimes \cdots \tag{3.111}$$

$$= p!\, \pi^b_k X^{i_1 \cdots i_{p-1} \vdots k} \tilde{Y}^a_c \left(\delta^{(j_1}_a v^{j_2}_{(i_1} \cdots v^{j_p)}_{i_{p-1}} \delta^c_{b)} - \delta^{(j_1}_a v^{j_2}_{i_1} \cdots v^{j_p)}_{i_{p-1}} \delta^c_b \right) \otimes \cdots \tag{3.112}$$

The last trick, and the one that truly requires $X \in \Upsilon^{p-1}_{(s)}$, is to remove the symmetrization on the indices $i_1 \cdots i_{p-1} b$ in the first term. We can do this because $\pi^b_k X^{i_1 \cdots i_{p-1} \vdots k}$ is symmetric in those indices (Theorem 3.2.8).

$$= p!\, \pi^b_k X^{i_1 \cdots i_{p-1} \vdots k} \tilde{Y}^a_c \left(\delta^{(j_1}_a v^{j_2}_{i_1} \cdots v^{j_p)}_{i_{p-1}} \delta^c_b - \delta^{(j_1}_a v^{j_2}_{i_1} \cdots v^{j_p)}_{i_{p-1}} \delta^c_b \right) \otimes \cdots \tag{3.113}$$

$$= 0 \tag{3.114}$$

□

Since $\phi^{(s)}_X$ is a horizontal form, we can make the following definition.

Definition 3.2.18. *For any $X \in T_u(LM)$ and any $Y \in T_{\lambda(u)}M$, define*

$$F^p_{(s)}(X)(Y) = \phi^{(s)}_X(\tilde{Y}) \tag{3.115}$$

where $\tilde{Y}$ is any vector at u that projects to Y.

Since $F^p_{(s)}(X)$ is a linear map from $T_{\lambda(u)}M$ to $T_{\mathbb{T}(X)}(T^p_{(s)}M)$ that projects to the identity, it defines a 1-jet of a symmetric tensor field at a point. So, $F^p_{(s)}$ itself maps from the tensorially symmetric m^{p-1} multi-tangent bundle of the frame bundle to the 1-jet bundle of the symmetric $(p, 0)$-tensor bundle.

$$F^p_{(s)} : \Upsilon^{p-1}_{(s)} \to J^1 \tau^p_{(s)}$$

Now we will calculate a formula for F in local coordinates. Since the vertical components of $\tilde{Y}$ are irrelevant to the calculation, we will choose them to be zero.

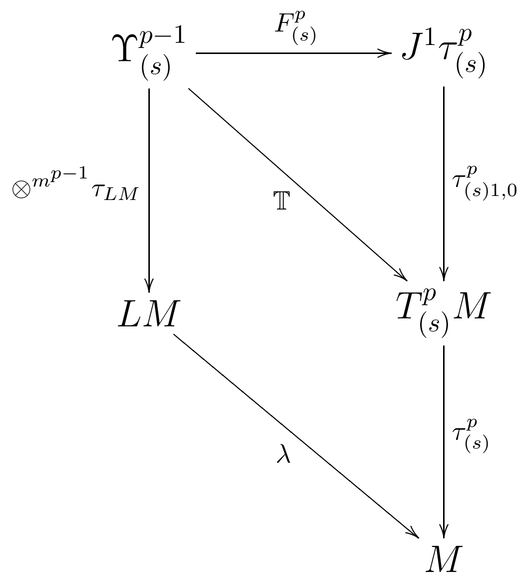

Figure 3.3: Commutative Diagram for $F^p_{(s)}$

$$\tilde{Y} = Y^i \left.\frac{\partial}{\partial x^i}\right|_u \qquad X = X^i \left.\frac{\partial}{\partial x^i}\right|_u + X^j_k \left.\frac{\partial}{\partial v^j_k}\right|_u \tag{3.116}$$

$$d_u\theta^{(i_p}(\tilde{Y}, X^{i_1\cdots i_{p-1})}) = 0 + \pi^b_k \pi^{(i_p}_a X^{i_1\cdots i_{p-1}):a}_{\phantom{i_1\cdots i_{p-1})}:b} Y^k \tag{3.117}$$

From (3.115), (3.63) and (3.95) we obtain

$$\begin{aligned} F^p_{(s)}(X)(Y) &= Y^k \left.\frac{\partial}{\partial x^k}\right|_{\mathbb{T}(X)} + \left(0 + p!\, v^{j_1}_{i_1}\cdots v^{j_p}_{i_p} \pi^b_k \pi^{(i_p}_a X^{i_1\cdots i_{p-1}):a}_{\phantom{i_1\cdots i_{p-1})}:b} Y^k\right) \left.\frac{\partial}{\partial v^{j_1\cdots j_p}}\right|_{\mathbb{T}(X)} \\ &= Y^k \left(\left.\frac{\partial}{\partial x^k}\right|_{\mathbb{T}(X)} + p!\, v^{j_1}_{i_1}\cdots v^{j_{p-1}}_{i_{p-1}} \pi^b_k \delta^{(j_p}_a X^{i_1\cdots i_{p-1}):a}_{\phantom{i_1\cdots i_{p-1})}:b} \left.\frac{\partial}{\partial v^{j_1\cdots j_p}}\right|_{\mathbb{T}(X)} \right) \\ &= Y^k \left(\left.\frac{\partial}{\partial x^k}\right|_{\mathbb{T}(X)} + p!\, v^{j_1}_{i_1}\cdots v^{j_{p-1}}_{i_{p-1}} \pi^b_k X^{(i_1\cdots i_{p-1}:j_p)}_{\phantom{(i_1\cdots i_{p-1}}:b} \left.\frac{\partial}{\partial v^{j_1\cdots j_p}}\right|_{\mathbb{T}(X)} \right) \end{aligned}$$

So

$$F^p_{(s)}(X) = d_{\lambda(u)}x^k \otimes \left(\left.\frac{\partial}{\partial x^k}\right|_{\mathbb{T}(X)} + p!\, v^{j_1}_{i_1}\cdots v^{j_{p-1}}_{i_{p-1}} \pi^b_k X^{(i_1\cdots i_{p-1}:j_p)}_{\phantom{(i_1\cdots i_{p-1}}:b} \left.\frac{\partial}{\partial v^{j_1\cdots j_p}}\right|_{\mathbb{T}(X)} \right) \tag{3.118}$$

From this formula and the formula for $\mathbb{T}$ in equation (3.76) we can see that, in coordinates,

$$\begin{aligned} x^i \circ F &= x^i \circ \mathbb{T} \\ &= x^i \qquad (3.119) \\ v^{j_1\cdots j_p} \circ F &= v^{j_1\cdots j_p} \circ \mathbb{T} \\ &= (p-1)!\, v^{j_1}_{i_1} \cdots v^{j_{p-1}}_{i_{p-1}} \dot{x}^{i_1\cdots i_{p-1} j_p} \qquad (3.120) \\ v^{j_1\cdots j_p}_k \circ F &= p!\, v^{j_1}_{i_1} \cdots v^{j_{p-1}}_{i_{p-1}} \pi^b_k \dot{v}^{(i_1\cdots i_{p-1} j_p)}_b \qquad (3.121) \end{aligned}$$

Clearly F is a smooth function.

THEOREM 3.2.19. *Let f be a symmetric $(p,0)$-tensor field on an open subset U of M (a local section of $\tau^p_{(s)}$), $\hat{f} : \lambda^{-1}(U) \to \bigoplus^p \mathbb{R}^m$ its tensorial function on LM, and $X_{\hat{f}}$ the corresponding Hamiltonian vector field.*

$$\hat{f}^{i_1\cdots i_p}(u) = f_{\lambda(u)}(e^{i_1}, \ldots, e^{i_p}) \qquad \textit{and} \qquad d\hat{f}^{i_1\cdots i_p} = -p! X^{(i_1\cdots i_{p-1}}_{\hat{f}} \lrcorner\, d\theta^{i_p)} \qquad (3.122)$$

Then

$$F(X_{\hat{f}}(u)) = j^1_p f \qquad (3.123)$$

for all $p \in U$ and $u \in \lambda^{-1}(p)$.

Proof. In local coordinates, we have [40]

$$\begin{aligned} X_{\hat{f}}(u) = &\frac{1}{(p-1)!} f^{j_1\cdots j_{p-1} k} \pi^{i_1}_{j_1} \cdots \pi^{i_{p-1}}_{j_{p-1}} \frac{\partial}{\partial x^k} \\ &- \frac{1}{p!} \left(\frac{\partial f^{j_1\cdots j_p}}{\partial x^a} \pi^{i_1}_{j_1} \cdots \pi^{i_{p-1}}_{j_{p-1}} \pi^b_{j_p} + T^{i_1\cdots i_{p-1} b}_a \right) \frac{\partial}{\partial \pi^b_a} \end{aligned}$$

where T satisfies the condition $T^{(i_1\cdots i_{p-1} b)}_a = 0$ but is otherwise arbitrary. However, we will need this vector in the v basis instead.

$$\begin{aligned} = &\frac{1}{(p-1)!} f^{j_1\cdots j_{p-1} k} \pi^{i_1}_{j_1} \cdots \pi^{i_{p-1}}_{j_{p-1}} \frac{\partial}{\partial x^k} \\ &- \frac{1}{p!} \left(\frac{\partial f^{j_1\cdots j_p}}{\partial x^a} \pi^{i_1}_{j_1} \cdots \pi^{i_{p-1}}_{j_{p-1}} \pi^b_{j_p} + T^{i_1\cdots i_{p-1} b}_a \right) (-v^a_c v^d_b) \frac{\partial}{\partial v^d_c} \end{aligned}$$

$$= \frac{1}{(p-1)!} f^{j_1\cdots j_{p-1}k} \pi^{i_1}_{j_1} \cdots \pi^{i_{p-1}}_{j_{p-1}} \frac{\partial}{\partial x^k} + \frac{1}{p!}\left(\frac{\partial f^{j_1\cdots j_p}}{\partial x^a} \pi^{i_1}_{j_1} \cdots \pi^{i_{p-1}}_{j_{p-1}} \delta^d_{j_p} v^a_c + T^{i_1\cdots i_{p-1}b}_a v^a_c v^d_b \right) \frac{\partial}{\partial v^d_c} \tag{3.124}$$

First recall that the tensor projection of $X_{\hat{f}}$ is f.

So using formula (3.118),

$$F^p_{(a)}(X) = d_{\lambda(u)} x^k \otimes \left(\left.\frac{\partial}{\partial x^k}\right|_{\mathbb{T}(X)} + p!\, v^{j_1}_{i_1} \cdots v^{j_{p-1}}_{i_{p-1}} \pi^b_k X^{(i_1\cdots i_{p-1}\,:j_p)}_{\quad :b} \left.\frac{\partial}{\partial v^{j_1\cdots j_p}}\right|_{\mathbb{T}(X)} \right) \tag{3.125}$$

$$F^p_{(a)}(X_{\hat{f}}) = d_{\lambda(u)} x^k \otimes \left(\left.\frac{\partial}{\partial x^k}\right|_{\mathbb{T}(X)} + p!\, v^{j_1}_{i_1} \cdots v^{j_{p-1}}_{i_{p-1}} \pi^b_k \frac{1}{p!} \left(\frac{\partial f^{l_1\cdots l_p}}{\partial x^a} \pi^{(i_1}_{l_1} \cdots \pi^{i_{p-1}}_{l_{p-1}} \delta^{j_p)}_{l_p} v^a_b + T^{(i_1\cdots i_{p-1}|c|}_a v^{j_p)}_c v^a_b \right) \left.\frac{\partial}{\partial v^{j_1\cdots j_p}}\right|_{\mathbb{T}(X)} \right) \tag{3.126}$$

Now $T^{(i_1\cdots i_{p-1}b)} = 0$ implies $T^{(i_1\cdots i_{p-1}|c|}_a v^{j_p)}_c = 0$ since the two expressions differ only by multiplication by the nonsingular matrix $\pi^b_{j_p}$. So we are left with

$$= d_{\lambda(u)} x^k \otimes \left(\left.\frac{\partial}{\partial x^k}\right|_{\mathbb{T}(X)} + \frac{\partial f^{l_1\cdots l_p}}{\partial x^a} \delta^{(j_1}_{l_1} \cdots \delta^{j_{p-1}}_{l_{p-1}} \delta^{j_p)}_{l_p} \delta^a_k \left.\frac{\partial}{\partial v^{j_1\cdots j_p}}\right|_{\mathbb{T}(X)} \right) \tag{3.127}$$

$$= d_{\lambda(u)} x^k \otimes \left(\left.\frac{\partial}{\partial x^k}\right|_{\mathbb{T}(X)} + \frac{\partial f^{j_1\cdots j_p}}{\partial x^k} \left.\frac{\partial}{\partial v^{j_1\cdots j_p}}\right|_{\mathbb{T}(X)} \right) \tag{3.128}$$

which is exactly $j^1_{\lambda(u)} f$. □

THEOREM 3.2.20. *$F : T(LM) \to J^1\tau$ is a fiber bundle.*

Proof. As noted earlier, F is a smooth function. The coordinate representation of F (eqn. 3.121) becomes, in the w coordinates:

$$x^i \circ F = w^i \qquad v^{j_1\cdots j_p} \circ F = w^{j_1\cdots j_p} \qquad v^{j_1\cdots j_p}_k \circ F = w^{(j_1\cdots j_p)}_k \tag{3.129}$$

Because their domains span entire fibers, these coordinates actually define a local trivialization. Since we can easily cover F with such trivializations, it is a fiber bundle. The w coordinates are then adapted coordinates. □

THEOREM 3.2.21. *$F : \Upsilon^{p-1}_{(s)} \to J^1\tau^p_{(s)}$ is a principal bundle under the action given in Definition 3.2.5.*

Proof. We have already shown that F is a fiber bundle; what remains is to show that:

1. The action is free
2. The equivalence classes are exactly the fibers.

To prove (1), suppose $(u, X) \cdot g = (u, X)$. This implies $u \cdot g = u$, which implies $g = e$, since the action on LM is free.

To prove (2), recall the w coordinate form of F from equation (3.129).

$$\begin{aligned}
x^i \circ F(X \cdot g) &= w^i(X \cdot g) \\
&= w^i(X) \\
&= x^i \circ F(X) \\
v^{j_1 \cdots j_p} \circ F(X \cdot g) &= w^{j_1 \cdots j_p}(X \cdot g) \\
&= w^{j_1 \cdots j_p}(X) \\
&= v^{j_1 \cdots j_p} \circ F(X) \\
v^{j_1 \cdots j_p}_k \circ F(X \cdot g) &= w^{(j_1 \cdots j_p)}_k(X \cdot g) \\
&= w^{(j_1 \cdots j_p)}_k(X) \\
&= v^{j_1 \cdots j_p}_k \circ F(X)
\end{aligned} \tag{3.130}$$

So,

$$F(X \cdot g) = F(X) \tag{3.131}$$

This proves that the equivalence classes are subsets of the fibers. To show the reverse inclusion, suppose that $F(u,X) = F(p,Z)$.

$$x^i \circ F(u,X) = x^i \circ F(p,Z) \implies w^i(u,X) = w^i(p,Z) \tag{3.132}$$
$$\implies x^i(u) = x^i(p)$$
$$\implies \exists g \; u \cdot g = p$$
$$u \cdot g = p \implies w^i_k(u,X) g^k_j = w^i_j(p,Z) \tag{3.133}$$
$$v^{j_1 \cdots j_p} \circ F(u,X) = v^{j_1 \cdots j_p} \circ F(p,Z) \implies w^{j_1 \cdots j_p}(u,X) = w^{j_1 \cdots j_p}(p,X) \tag{3.134}$$
$$v^{j_1 \cdots j_p}_k \circ F(u,X) = v^{j_1 \cdots j_p}_k \circ F(p,Z) \implies w^{(j_1 \cdots j_p)}_k(u,X) = w^{(j_1 \cdots j_p)}_k(p,X) \tag{3.135}$$

And using (3.132), (3.133), (3.134), (3.135), and (3.71) we see that

$$F(u,X) = F(p,Z) \implies (u,X) \cdot g = (p,Z) \tag{3.136}$$

This proves the reverse inclusion.

□

3.3 Anti-Symmetric Tensor Fields

The case of antisymmetric tensor fields progresses, expectedly, in a similar fashion. However, also expectedly, there are some calculations that acquire an additional layer of complexity.

Since the map ψ^p (Equation 3.53) is defined without reference to symmetry, we will again find it useful.

3.3.1 The Tensorially Antisymmetric Subbundle $\Upsilon^{p-1}_{(a)}$

We recall the tensor projection from Definition 3.2.6.

Definition 3.3.1. *We define the* **tensorially antisymmetric subbundle** *of* $\bigoplus^{m^{p-1}} T(LM)$ *to be those multivectors whose tensor projections are antisymmetric.*

$$\begin{aligned}\Upsilon^{p-1}_{(a)} &= \{X \in \bigoplus^{m^{p-1}} T(LM) \mid \mathbb{T}(X) \text{ is antisymmetric } \}\\ &= \mathbb{T}^{-1}(T^p_{(a)}(M))\end{aligned}$$

Since $(\mathbb{T}, \lambda)$ *is a vector bundle morphism,* $\Upsilon^{p-1}_{(a)}$ *is a vector subbundle of* $\bigoplus^{m^{p-1}} T(LM)$

Theorem 3.3.2. *The following are equivalent:*

1. $X \in \Upsilon^{p-1}_{(a)}$
2. $\theta^{i_p}(X^{i_1\cdots i_{p-1}}) = \theta^{[i_p}(X^{i_1\cdots i_{p-1}]})$
3. $\pi^{i_p}_k X^{i_1\cdots i_{p-1}\vdots k} = \pi^{[i_p}_k X^{i_1\cdots i_{p-1}]\vdots k}$ *in some canonical coordinate system.*
4. $X^{i_1\cdots i_{p-1}\vdots j} = X^{[i_1\cdots i_{p-1}\vdots j]}$ *in some canonical coordinate system.*

Proof.

(1) $\iff$ (2) This follows directly from the definition of $\mathbb{T}$ and the linearity of ψ^p.

(2) $\iff$ (3) The local coordinate expansion of (2) is (3). Since we are working at a single point, the two statements are equivalent.

(3) $\iff$ (4) Since the matrix (v^i_j) is nonsingular

$$\begin{aligned}\pi^{i_p}_k X^{i_1\cdots i_{p-1}\vdots k} = \pi^{[i_p}_k X^{i_1\cdots i_{p-1}]\vdots k} &\iff v^j_{i_p}\pi^{i_p}_k X^{i_1\cdots i_{p-1}\vdots k} = v^j_{i_p}\pi^{[i_p}_k X^{i_1\cdots i_{p-1}]\vdots k}\\ &\iff \delta^j_k X^{i_1\cdots i_{p-1}\vdots k} = \delta^{[j}_k X^{i_1\cdots i_{p-1}]\vdots k}\\ &\iff X^{i_1\cdots i_{p-1}\vdots j} = X^{[i_1\cdots i_{p-1}\vdots j]}\end{aligned}$$

□

Theorem 3.3.3. *Restriction to* $\Upsilon^{p-1}_{(a)}$ *forces the kernel of the n-symplectic equation to be vertical.*

Proof. Suppose that X satisfies

$$X^{[i_1\cdots i_{p-1}} \lrcorner\, d\theta^{i_p]} = 0$$
$$\theta^{i_p}(X^{i_1\cdots i_{p-1}}) = \theta^{[i_p}(X^{i_1\cdots i_{p-1}]})$$

Then using Lemma 3.2.12, we obtain

$$\begin{aligned} 0 &= \overset{*}{E}{}^a_a \lrcorner\, X^{[i_1\cdots i_{p-1}} \lrcorner\, d\theta^{i_p]} \\ &= -X^{[i_1\cdots i_{p-1}} \lrcorner\, \overset{*}{E}{}^{|a|}_a \lrcorner\, d\theta^{i_p]} \\ &= X^{[i_1\cdots i_{p-1}} \lrcorner\, \theta^{i_p]} \\ &= X^{i_1\cdots i_{p-1}} \lrcorner\, \theta^{i_p} \end{aligned}$$

which implies that X is vertical. □

THEOREM 3.3.4. *The Hamiltonian vector field corresponding to any antisymmetric $(p,0)$ tensor is a section of $\Upsilon^{p-1}_{(a)}$.*

Proof. If f is an antisymmetric $(p,0)$ tensor field on M, then the corresponding Hamiltonian vector field on LM is given by [40]

$$\begin{aligned} X_{\hat{f}}(u) = &\frac{1}{(p-1)!} f^{j_1\cdots j_{p-1}k} \pi^{i_1}_{j_1}\cdots\pi^{i_{p-1}}_{j_{p-1}} \frac{\partial}{\partial x^k} \\ &- \frac{1}{p!}\left(\frac{\partial f^{j_1\cdots j_p}}{\partial x^a}\pi^{i_1}_{j_1}\cdots\pi^{i_{p-1}}_{j_{p-1}}\pi^b_{j_p} + T^{i_1\cdots i_{p-1}b}_a\right)\frac{\partial}{\partial \pi^b_a} \end{aligned}$$

where T satisfies the condition $T^{[i_1\cdots i_{p-1}b]}_a = 0$ but is otherwise arbitrary. According to theorem 3.3.2 we need to show that

$$f^{j_1\cdots j_{p-1}k}\pi^{i_1}_{j_1}\cdots\pi^{i_{p-1}}_{j_{p-1}}$$

is antisymmetric in the indices $(i_1\cdots i_{p-1}k)$. We consider the following reduction

$$\begin{aligned} f^{l_1\cdots l_{p-1}k} = f^{[l_1\cdots l_{p-1}k]} &\iff f^{j_1 l_2\cdots l_{p-1}k}\pi^{i_1}_{j_1} = f^{j_1[l_2\cdots l_{p-1}k}\pi^{i_1]}_{j_1} \\ &\iff f^{j_1 j_2 l_3\cdots l_{p-1}k}\pi^{i_1}_{j_1}\pi^{i_2}_{j_2} = f^{j_1 j_2[l_3\cdots l_{p-1}k}\pi^{i_1}_{j_1}\pi^{i_2]}_{j_2} \\ &\cdots \\ &\iff f^{j_1\cdots j_{p-1}k}\pi^{i_1}_{j_1}\cdots\pi^{i_{p-1}}_{j_{p-1}} = f^{j_1\cdots j_{p-1}[k}\pi^{i_1}_{j_1}\cdots\pi^{i_{p-1}]}_{j_{p-1}} \end{aligned}$$

where each step collapses to the previous upon multiplication by a nonsingular matrix $v^{l_z}_{i_z}$ □

It will turn out that the tensor projection of the Hamiltonian vector field of an antisymmetric tensor field is again the tensor field. To prove this succinctly, we will need results of the Lemmas 3.2.11 and 3.2.12.

THEOREM 3.3.5. *If f is an antisymmetric p-tensor field with tensorial function $\hat{f}$ and Hamiltonian vector field $X_{\hat{f}}$, then*

$$\mathbb{T}(u, X_{\hat{f}}) = f_{\lambda(u)}$$

Proof. By using the results of the lemmas and the n-symplectic equation, we obtain

$$d\hat{f}^{i_1\cdots i_p} = -p!\, X_{\hat{f}}^{[i_1\cdots i_{p-1}} \lrcorner\, d\theta^{i_p]} \tag{3.137}$$
$$-\frac{1}{p}\overset{*}{E}{}^a_a \lrcorner\, d\hat{f}^{i_1\cdots i_p} = (p-1)!\, \overset{*}{E}{}^a_a \lrcorner\, X_{\hat{f}}^{[i_1\cdots i_{p-1}} \lrcorner\, d\theta^{i_p]} \tag{3.138}$$
$$\hat{f}^{i_1\cdots i_p} = (p-1)!\, X_{\hat{f}}^{[i_1\cdots i_{p-1}} \lrcorner\, \theta^{i_p]} \tag{3.139}$$
$$\hat{f}^{i_1\cdots i_p} = (p-1)!\, X_{\hat{f}}^{i_1\cdots i_{p-1}} \lrcorner\, \theta^{i_p} \tag{3.140}$$

where we have used the result of theorem 3.3.4 in the last step. So if we now consider the tensor projection of $X_{\hat{f}}$ we get

$$\mathbb{T}(u, X_{\hat{f}}) = (p-1)!\, \psi^p(u, \theta_u(X)) \tag{3.141}$$
$$= \psi^p(u, \hat{f}) \tag{3.142}$$
$$= f_{\lambda(u)} \tag{3.143}$$

□

And now we return to the problem of relating Hamiltonian vector fields to 1-jets. The truly surprising thing is that we can do this pointwise, obtaining *purely geometric* results.

DEFINITION 3.3.6. *Given any* $X \in \Upsilon^{p-1}_{(a)}$, *define* $\phi^{(a)}_X : T_u(LM) \to T_{\mathbb{T}(X)}(T^p M)$ *by*

$$\phi^{(a)}_X(\tilde{Y}) = d_{(u,(p-1)!\,\theta_u(X))}\psi^p\big(\tilde{Y}, \tilde{Y} \lrcorner (-p!\, X^{[i_1\cdots i_{p-1}}) \lrcorner d_u\theta^{i_p]}\big) \tag{3.144}$$

$$= d_{(u,(p-1)!\,\theta_u(X))}\psi^p\big(\tilde{Y}, p!\, d_u\theta^{[i_p}(\tilde{Y}, X^{i_1\cdots i_{p-1}]})\big) \tag{3.145}$$

Since $\phi^{(a)}_X$ is a linear map, we consider it a $T(T^p_{(a)}M)$-valued one-form at $u \in LM$. We should emphasize that $\phi^{(a)}_X$ maps into $T_{\mathbb{T}(X)}(T^p_{(a)}M)$. This is because we take the differential of ψ^p at the point

$$\big(u, (p-1)!\,\theta^{i_p}_u(X^{i_1\cdots i_{p-1}})\big) \in LM \times \bigotimes^p \mathbb{R}^m$$

We could have defined $\phi^{(a)}_X$ for all m^p multi-vectors, but to show that $\phi^{(a)}_X$ is horizontal, we will need $X \in \Upsilon^{p-1}_{(a)}$. This is our next step, but we will need the following lemma.

LEMMA 3.3.7.

$$d(v^{[j_1}_{i_1}\cdots v^{j_p]}_{i_p})(\frac{\partial}{\partial v^a_c}) = p\,\delta^{[j_p}_a v^{j_1}_{[i_1}\cdots v^{j_{p-1}]}_{i_{p-1}}\delta^c_{i_p]}$$

Proof. We prove this by direct calculation, explicitly expanding the antisymmetrization and the product rule. The crucial step is the introduction of a second antisymmetrization, initially over $p-1$ terms, in equation (3.149).

$$d(v^{[j_1}_{i_1}\cdots v^{j_p]}_{i_p})(\frac{\partial}{\partial v^a_c})$$

$$= \frac{1}{p!}\sum_{\pi\in S_p}(-1)^\pi d(v^{j_{\pi 1}}_{i_1}\cdots v^{j_{\pi p}}_{i_p})(\frac{\partial}{\partial v^a_c}) \tag{3.146}$$

$$= \frac{1}{p!}\sum_{\pi\in S_p}\sum_{k=1}^{p}(-1)^\pi v^{j_{\pi 1}}_{i_1}\cdots dv^{j_{\pi k}}_{i_k}(\frac{\partial}{\partial v^a_c})\cdots v^{j_{\pi p}}_{i_p} \tag{3.147}$$

$$= \frac{1}{p!}\sum_{\pi\in S_p}\sum_{k=1}^{p}(-1)^\pi v^{j_{\pi 1}}_{i_1}\cdots \delta^{j_{\pi k}}_a\delta^c_{i_k}\cdots v^{j_{\pi p}}_{i_p} \tag{3.148}$$

With a slight feat of legerdemain, we introduce symmetrization over the $p-1$ terms $v^{j_{\pi\sigma 1}}_{i_{\sigma 1}} \cdots \widehat{v^{j_{\pi\sigma k}}_{i_{\sigma k}}} \cdots v^{j_{\pi\sigma p}}_{i_{\sigma p}}$, where the hat indicates that the term in left out. Since we are permuting identical factors, we are not changing the formula. By summing only over $\{\sigma \in S_p | \sigma p = k\}$ the two factors of δ remain unchanged.

$$= \frac{1}{p!} \sum_{\pi \in S_p} \sum_{k=1}^{p} \left[\frac{1}{(p-1)!} \sum_{\substack{\sigma \in S_p \\ \sigma p = k}} (-1)^{\pi} v^{j_{\pi\sigma 1}}_{i_{\sigma 1}} \cdots v^{j_{\pi\sigma(p-1)}}_{i_{\sigma(p-1)}} \delta^{j_{\pi\sigma p}}_{a} \delta^{c}_{i_{\sigma p}} \right] \tag{3.149}$$

And now the sums over σ and k collapse to a single sum. We also replace $(-1)^{\pi}$ with $(-1)^{\pi\sigma}(-1)^{\sigma}$, which is effectively multiplying by $((-1)^{\sigma})^2 = 1$

$$= \frac{1}{p!} \sum_{\pi \in S_p} \frac{p}{p!} \sum_{\sigma \in S_p} (-1)^{\pi\sigma} (-1)^{\sigma} v^{j_{\pi\sigma 1}}_{i_{\sigma 1}} \cdots v^{j_{\pi\sigma(p-1)}}_{i_{\sigma(p-1)}} \delta^{j_{\pi\sigma p}}_{a} \delta^{c}_{i_{\sigma p}} \tag{3.150}$$

Now we absorb the σs into the πs in the upper indices. This is allowable since π and σ are summed over all of S_p. Notice how this step introduces the factor of $(-1)^{\sigma}$.

$$= \frac{1}{p!} \sum_{\pi \in S_p} \frac{p}{p!} \sum_{\sigma \in S_p} (-1)^{\pi} (-1)^{\sigma} v^{j_{\pi 1}}_{i_{\sigma 1}} \cdots v^{j_{\pi(p-1)}}_{i_{\sigma(p-1)}} \delta^{j_{\pi p}}_{a} \delta^{c}_{i_{\sigma p}} \tag{3.151}$$

$$= p v^{[j_1}_{[i_1} \cdots v^{j_{p-1}}_{i_{p-1}} \delta^{j_p]}_{|a|} \delta^{c}_{i_p]} \tag{3.152}$$

$$= p \delta^{[j_p}_{a} v^{j_1}_{[i_1} \cdots v^{j_{p-1}]}_{i_{p-1}} \delta^{c}_{i_p]} \tag{3.153}$$

□

Theorem 3.3.8. *For any $X \in \Upsilon^{p-1}_{(a)}$, $\phi^{(a)}_X$ is a horizontal 1-form and is therefore projectible to a $T(T^p_{(a)}M)$-valued one-form $\tilde{\phi}^{(a)}_X$ on M (equivalent to an element of $j^1\tau^p_{(a)}$).*

Proof. Let $\tilde{Y}$ be a vertical vector at $u \in LM$, Expand $\tilde{Y}$ and X locally as

$$\tilde{Y} = \tilde{Y}^a_b \left. \frac{\partial}{\partial v^a_b} \right|_u \tag{3.154}$$

$$X^{i_1 \cdots i_{p-1}} = X^{i_1 \cdots i_{p-1} \vdots i} \left. \frac{\partial}{\partial x^i} \right|_u + X^{i_1 \cdots i_{p-1} \vdots j}{}_{k} \left. \frac{\partial}{\partial v^j_k} \right|_u \tag{3.155}$$

Using the formula $d\theta^i = -\pi^i_a \pi^b_k dv^a_b \wedge dx^k$ we find that

$$d_u\theta^{[i_p}(\tilde{Y}, X^{i_1\cdots i_{p-1}]}) = -\tilde{Y}^a_b \pi^b_k \pi^{[i_p}_a X^{i_1\cdots i_{p-1}]}\vdots^{k} \tag{3.156}$$

Substituting this into (3.145) and using the formula for $d\psi^p$ (3.63), we obtain

$$\phi^{(a)}_X(\tilde{Y}) = 0 + \Bigg((p-1)!\,\theta^{i_p}_u(X^{i_1\cdots i_{p-1}})d(v^{j_1}_{i_1}\cdots v^{j_p}_{i_p})(\tilde{Y}) + v^{j_1}_{i_1}\cdots v^{j_p}_{i_p}\; p!\, d_u\theta^{[i_p}(\tilde{Y}, X^{i_1\cdots i_{p-1}]})\Bigg) \otimes \frac{\partial}{\partial v^{j_1\cdots j_p}}\Bigg|_{\mathbb{T}(X))} \tag{3.157}$$

Now we introduce antisymmetrization on $\theta(X)$. This is valid since $X \in \Upsilon^{p-1}_{(a)}$

$$= \Bigg((p-1)!\,\theta^{[i_p}_u(X^{i_1\cdots i_{p-1}]})d(v^{j_1}_{i_1}\cdots v^{j_p}_{i_p})(\tilde{Y}^a_c\frac{\partial}{\partial v^a_c}) - p!\, v^{j_1}_{i_1}\cdots v^{j_p}_{i_p}\tilde{Y}^a_b \pi^b_k \pi^{[i_p}_a X^{i_1\cdots i_{p-1}]}\vdots^{k}\Bigg) \otimes \cdots \tag{3.158}$$

Next we move the antisymmetrization from the i indices to the j indices.

$$= \Bigg((p-1)!\,\pi^{i_p}_k X^{i_1\cdots i_{p-1}}\vdots^{k}\tilde{Y}^a_c d(v^{[j_1}_{i_1}\cdots v^{j_p]}_{i_p})(\frac{\partial}{\partial v^a_c}) - p!\, v^{[j_1}_{i_1}\cdots v^{j_p]}_{i_p}\tilde{Y}^a_b \pi^b_k \pi^{i_p}_a X^{i_1\cdots i_{p-1}}\vdots^{k}\Bigg) \otimes \cdots \tag{3.159}$$

With a bit of factoring and the application of Lemma 3.3.7, the formula becomes more manageable.

$$= \pi^b_k X^{i_1\cdots i_{p-1}}\vdots^{k}\tilde{Y}^a_c \Bigg((p-1)!\,\delta^{i_p}_b p\delta^{[j_p}_a v^{j_1}_{[i_1}\cdots v^{j_{p-1}]}_{i_{p-1}}\delta^c_{i_p]} - p!\, v^{[j_1}_{i_1}\cdots v^{j_p]}_{i_p}\delta^c_b \pi^{i_p}_a\Bigg) \otimes \cdots \tag{3.160}$$

$$= p!\,\pi^b_k X^{i_1\cdots i_{p-1}}\vdots^{k}\tilde{Y}^a_c \Bigg(\delta^{[j_p}_a v^{j_1}_{[i_1}\cdots v^{j_{p-1}]}_{i_{p-1}}\delta^c_{b]} - v^{[j_1}_{i_1}\cdots v^{j_{p-1}}_{i_{p-1}}\delta^{j_p]}_a\delta^c_b\Bigg) \otimes \cdots \tag{3.161}$$

$$= p!\,\pi^b_k X^{i_1\cdots i_{p-1}}\vdots^{k}\tilde{Y}^a_c \Bigg(\delta^{[j_p}_a v^{j_1}_{[i_1}\cdots v^{j_{p-1}]}_{i_{p-1}}\delta^c_{b]} - \delta^{[j_p}_a v^{j_1}_{i_1}\cdots v^{j_{p-1}]}_{i_{p-1}}\delta^c_b\Bigg) \otimes \cdots \tag{3.162}$$

The last trick, and the one that truly requires $X \in \Upsilon^{p-1}_{(a)}$, is to remove the anti-symmetrization on the indices $i_1 \cdots i_{p-1} b$ in the first term. We can do this because $\pi^b_k X^{i_1 \cdots i_{p-1} \vdots k}$ is antisymmetric in those indices (Theorem 3.3.2).

$$= p!\, \pi^b_k X^{i_1 \cdots i_{p-1} \vdots k} \tilde{Y}^a_c \left(\delta^{[j_p}_a v^{j_1}_{i_1} \cdots v^{j_{p-1}]}_{i_{p-1}} \delta^c_b - \delta^{[j_p}_a v^{j_1}_{i_1} \cdots v^{j_{p-1}]}_{i_{p-1}} \delta^c_b \right) \otimes \cdots \tag{3.163}$$

$$= 0 \tag{3.164}$$

□

Since $\phi^{(a)}_X$ is a horizontal form, we can make the following definition.

Definition 3.3.9. *For any $X \in T_u(LM)$ and any $Y \in T_{\lambda(u)} M$, define*

$$F^p_{(a)}(X)(Y) = \phi^{(a)}_X(\tilde{Y}) \tag{3.165}$$

where $\tilde{Y}$ is any vector at u that projects to Y.

Since $F^p_{(a)}(X)$ is a linear map from $T_{\lambda(u)} M$ to $T_{\mathbb{T}(X)}(T^p_{(a)} M)$ that projects to the identity, it defines a 1-jet of a antisymmetric tensor field at a point. So, $F^p_{(a)}$ itself maps from the tensorially antisymmetric m^{p-1} multi-tangent bundle of the frame bundle to the 1-jet bundle of the antisymmetric $(p, 0)$-tensor bundle.

$$F^p_{(a)} : \Upsilon^{p-1}_{(a)} \to J^1 \tau^p_{(a)}$$

Now we will calculate a formula for $F^p_{(a)}$ in local coordinates. Since the vertical components of $\tilde{Y}$ are irrelevant to the calculation, we will choose them to be zero.

$$\tilde{Y} = Y^i \left. \frac{\partial}{\partial x^i} \right|_u \qquad X = X^i \left. \frac{\partial}{\partial x^i} \right|_u + X^j_k \left. \frac{\partial}{\partial v^j_k} \right|_u \tag{3.166}$$

$$d_u \theta^{[i_p}(\tilde{Y}, X^{i_1 \cdots i_{p-1}]}) = 0 + \pi^b_k \pi^{[i_p}_a X^{i_1 \cdots i_{p-1}] \vdots a}_{\ \vdots b} Y^k \tag{3.167}$$

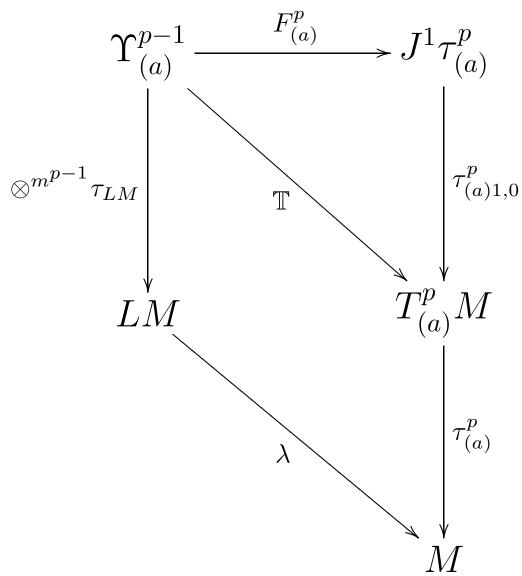

Figure 3.4: Commutative Diagram for $F^p_{(a)}$

From (3.165), (3.63) and (3.145) we obtain

$$\begin{aligned}
F^p_{(a)}(X)(Y) &= Y^k \left.\frac{\partial}{\partial x^k}\right|_{\mathbb{T}(X)} + \left(0 + p!\, v^{j_1}_{i_1}\cdots v^{j_p}_{i_p}\pi^b_k \pi^{[i_p}_a X^{i_1\cdots i_{p-1}]}{}^{:a}_{:b} Y^k\right) \left.\frac{\partial}{\partial v^{j_1\cdots j_p}}\right|_{\mathbb{T}(X)} \\
&= Y^k \left(\left.\frac{\partial}{\partial x^k}\right|_{\mathbb{T}(X)} + p!\, v^{j_1}_{i_1}\cdots v^{j_{p-1}}_{i_{p-1}}\pi^b_k \delta^{[j_p}_a X^{i_1\cdots i_{p-1}]}{}^{:a}_{:b} \left.\frac{\partial}{\partial v^{j_1\cdots j_p}}\right|_{\mathbb{T}(X)} \right) \\
&= Y^k \left(\left.\frac{\partial}{\partial x^k}\right|_{\mathbb{T}(X)} + p!\, v^{j_1}_{i_1}\cdots v^{j_{p-1}}_{i_{p-1}}\pi^b_k X^{[i_1\cdots i_{p-1}}{}^{:j_p]}_{:b} \left.\frac{\partial}{\partial v^{j_1\cdots j_p}}\right|_{\mathbb{T}(X)} \right)
\end{aligned}$$

So

$$F^p_{(a)}(X) = d_{\lambda(u)}x^k \otimes \left(\left.\frac{\partial}{\partial x^k}\right|_{\mathbb{T}(X)} + p!\, v^{j_1}_{i_1}\cdots v^{j_{p-1}}_{i_{p-1}}\pi^b_k X^{[i_1\cdots i_{p-1}}{}^{:j_p]}_{:b} \left.\frac{\partial}{\partial v^{j_1\cdots j_p}}\right|_{\mathbb{T}(X)} \right) \tag{3.168}$$

From this formula and the formula for $\mathbb{T}$ in equation (3.76) we can see that, in coordinates,

$$\begin{aligned} x^i \circ F &= x^i \circ \mathbb{T} \\ &= x^i \qquad (3.169) \\ v^{j_1\cdots j_p} \circ F &= v^{j_1\cdots j_p} \circ \mathbb{T} \\ &= (p-1)!\, v^{j_1}_{i_1} \cdots v^{j_{p-1}}_{i_{p-1}} \dot{x}^{i_1\cdots i_{p-1}j_p} \qquad (3.170) \\ v^{j_1\cdots j_p}_k \circ F &= p!\, v^{j_1}_{i_1} \cdots v^{j_{p-1}}_{i_{p-1}} \pi^b_k \dot{v}^{[i_1\cdots i_{p-1}j_p]}_b \qquad (3.171) \end{aligned}$$

Clearly F is a smooth function.

THEOREM 3.3.10. *Let f be a antisymmetric $(p,0)$-tensor field on an open subset U of M (a local section of $\tau^p_{(a)}$), $\hat{f} : \lambda^{-1}(U) \to \bigoplus^p \mathbb{R}^m$ its tensorial function on LM, and $X_{\hat{f}}$ the corresponding Hamiltonian vector field.*

$$\hat{f}^{i_1\cdots i_p}(u) = f_{\lambda(u)}(e^{i_1}, \dots, e^{i_p}) \qquad \text{and} \qquad d\hat{f}^{i_1\cdots i_p} = -p! X^{[i_1\cdots i_{p-1}}_{\hat{f}} \lrcorner\, d\theta^{i_p]} \qquad (3.172)$$

Then

$$F(X_{\hat{f}}(u)) = j^1_p f \qquad (3.173)$$

for all $p \in U$ and $u \in \lambda^{-1}(p)$.

Proof. In local coordinates, we have [40]

$$\begin{aligned} X_{\hat{f}}(u) = &\frac{1}{(p-1)!} f^{j_1\cdots j_{p-1}k} \pi^{i_1}_{j_1} \cdots \pi^{i_{p-1}}_{j_{p-1}} \frac{\partial}{\partial x^k} \\ &- \frac{1}{p!}\left(\frac{\partial f^{j_1\cdots j_p}}{\partial x^a} \pi^{i_1}_{j_1} \cdots \pi^{i_{p-1}}_{j_{p-1}} \pi^b_{j_p} + T^{i_1\cdots i_{p-1}b}_a \right) \frac{\partial}{\partial \pi^b_a} \end{aligned}$$

where T satisfies the condition $T^{[i_1\cdots i_{p-1}b]}_a = 0$ but is otherwise arbitrary. However, we will need this vector in the v basis instead.

$$\begin{aligned} = &\frac{1}{(p-1)!} f^{j_1\cdots j_{p-1}k} \pi^{i_1}_{j_1} \cdots \pi^{i_{p-1}}_{j_{p-1}} \frac{\partial}{\partial x^k} \\ &- \frac{1}{p!}\left(\frac{\partial f^{j_1\cdots j_p}}{\partial x^a} \pi^{i_1}_{j_1} \cdots \pi^{i_{p-1}}_{j_{p-1}} \pi^b_{j_p} + T^{i_1\cdots i_{p-1}b}_a \right) (-v^a_c v^d_b) \frac{\partial}{\partial v^d_c} \end{aligned}$$

$$= \frac{1}{(p-1)!} f^{j_1\cdots j_{p-1}k} \pi^{i_1}_{j_1} \cdots \pi^{i_{p-1}}_{j_{p-1}} \frac{\partial}{\partial x^k} + \frac{1}{p!}\left(\frac{\partial f^{j_1\cdots j_p}}{\partial x^a} \pi^{i_1}_{j_1} \cdots \pi^{i_{p-1}}_{j_{p-1}} \delta^d_{j_p} v^a_c + T^{i_1\cdots i_{p-1}b}_a v^a_c v^d_b \right) \frac{\partial}{\partial v^d_c} \tag{3.174}$$

First recall that the tensor projection of $X_{\hat{f}}$ is f.

So using formula (3.168),

$$F^p_{(a)}(X) = d_{\lambda(u)} x^k \otimes \left(\left.\frac{\partial}{\partial x^k}\right|_{\mathbb{T}(X)} + p!\, v^{j_1}_{i_1} \cdots v^{j_{p-1}}_{i_{p-1}} \pi^b_k X^{[i_1\cdots i_{p-1}\vdots j_p]}_{\phantom{[i_1\cdots i_{p-1}}\vdots b} \left.\frac{\partial}{\partial v^{j_1\cdots j_p}}\right|_{\mathbb{T}(X)} \right) \tag{3.175}$$

$$F^p_{(a)}(X_{\hat{f}}) = d_{\lambda(u)} x^k \otimes \left(\left.\frac{\partial}{\partial x^k}\right|_{\mathbb{T}(X)} + p!\, v^{j_1}_{i_1} \cdots v^{j_{p-1}}_{i_{p-1}} \pi^b_k \frac{1}{p!} \left(\frac{\partial f^{l_1\cdots l_p}}{\partial x^a} \pi^{[i_1}_{l_1} \cdots \pi^{i_{p-1}}_{l_{p-1}} \delta^{j_p]}_{l_p} v^a_b + T^{[i_1\cdots i_{p-1}|c|}_a v^{j_p]}_c v^a_b \right) \left.\frac{\partial}{\partial v^{j_1\cdots j_p}}\right|_{\mathbb{T}(X)} \right) \tag{3.176}$$

Now $T^{[i_1\cdots i_{p-1}b]} = 0$ implies $T^{[i_1\cdots i_{p-1}|c|}_a v^{j_p]}_c = 0$ since the two expressions differ only by multiplication by the nonsingular matrix $\pi^b_{j_p}$. So we are left with

$$= d_{\lambda(u)} x^k \otimes \left(\left.\frac{\partial}{\partial x^k}\right|_{\mathbb{T}(X)} + \frac{\partial f^{l_1\cdots l_p}}{\partial x^a} \delta^{[j_1}_{l_1} \cdots \delta^{j_{p-1}}_{l_{p-1}} \delta^{j_p]}_{l_p} \delta^a_k \left.\frac{\partial}{\partial v^{j_1\cdots j_p}}\right|_{\mathbb{T}(X)} \right) \tag{3.177}$$

$$= d_{\lambda(u)} x^k \otimes \left(\left.\frac{\partial}{\partial x^k}\right|_{\mathbb{T}(X)} + \frac{\partial f^{j_1\cdots j_p}}{\partial x^k} \left.\frac{\partial}{\partial v^{j_1\cdots j_p}}\right|_{\mathbb{T}(X)} \right) \tag{3.178}$$

This is exactly $j^1_{\lambda(u)} f$. □

THEOREM 3.3.11. *$F^p_{(a)} : \Upsilon^{p-1}_{(a)} \to J^1\tau^p_{(a)}$ is a fiber bundle.*

Proof. As noted earlier, $F^p_{(a)}$ is a smooth function. The coordinate representation of $F^p_{(a)}$ (eqn. 3.171) becomes, in the w coordinates:

$$x^i \circ F = w^i \qquad v^{j_1\cdots j_p} \circ F = w^{j_1\cdots j_p} \qquad v^{j_1\cdots j_p}_k \circ F = w^{[j_1\cdots j_p]}_k \tag{3.179}$$

Because their domains span entire fibers, these coordinates actually define a local trivialization. Since we can easily cover F with such trivializations, it is a fiber bundle. The w coordinates are then adapted coordinates. □

THEOREM 3.3.12. *$F^p_{(a)} : \Upsilon^{p-1}_{(a)} \to J^1\tau^p_{(a)}$ is a principal bundle under the action given in Definition 3.2.5.*

Proof. We have already shown that $F^p_{(a)}$ is a fiber bundle; what remains is to show that:

1. The action is free
2. The equivalence classes are exactly the fibers.

To prove (1), suppose $(u, X) \cdot g = (u, X)$. This implies $u \cdot g = u$, which implies $g = e$, since the action on LM is free.

To prove (2), recall the w coordinate form of F from equation (3.179).

$$\begin{aligned}
x^i \circ F(X \cdot g) &= w^i(X \cdot g) \\
&= w^i(X) \\
&= x^i \circ F(X) \\
v^{j_1 \cdots j_p} \circ F(X \cdot g) &= w^{j_1 \cdots j_p}(X \cdot g) \\
&= w^{j_1 \cdots j_p}(X) \\
&= v^{j_1 \cdots j_p} \circ F(X) \\
v_k^{j_1 \cdots j_p} \circ F(X \cdot g) &= w_k^{[j_1 \cdots j_p]}(X \cdot g) \\
&= w_k^{[j_1 \cdots j_p]}(X) \\
&= v_k^{j_1 \cdots j_p} \circ F(X)
\end{aligned} \tag{3.180}$$

So,

$$F(X \cdot g) = F(X) \tag{3.181}$$

This proves that the equivalence classes are subsets of the fibers. To show the reverse inclusion, suppose that $F(u,X) = F(p,Z)$.

$$x^i \circ F(u,X) = x^i \circ F(p,Z) \implies w^i(u,X) = w^i(p,Z) \tag{3.182}$$
$$\implies x^i(u) = x^i(p)$$
$$\implies \exists g\ u \cdot g = p$$
$$u \cdot g = p \implies w^i_k(u,X) g^k_j = w^i_j(p,Z) \tag{3.183}$$
$$v^{j_1 \cdots j_p} \circ F(u,X) = v^{j_1 \cdots j_p} \circ F(p,Z) \implies w^{j_1 \cdots j_p}(u,X) = w^{j_1 \cdots j_p}(p,X) \tag{3.184}$$
$$v^{j_1 \cdots j_p}_k \circ F(u,X) = v^{j_1 \cdots j_p}_k \circ F(p,Z) \implies w^{[j_1 \cdots j_p]}_k(u,X) = w^{[j_1 \cdots j_p]}_k(p,X) \tag{3.185}$$

And using (3.182), (3.183), (3.184), (3.185), and (3.71) we see that

$$F(u,X) = F(p,Z) \implies (u,X) \cdot g = (p,Z) \tag{3.186}$$

This proves the reverse inclusion. □

3.4 Application: The Free Field Lagrangian

In the following calculation, we assume that M is a Riemannian manifold with metric tensor h.

DEFINITION 3.4.1. *We define the type (0,2) symmetric tensor $\tilde{h}$ on LM as follows*

$$\begin{aligned}
\tilde{h}(B_i, B_j) &= -\mu^2 h(\lambda_* B_i, \lambda_* B_j) \\
&= -\mu^2 \hat{h}_{ij} \\
\tilde{h}(B_i, \overset{*}{E}{}^j_k) &= 0 \\
\tilde{h}(\overset{*}{E}{}^j_k, \overset{*}{E}{}^{\bar{j}}_{\bar{k}}) &= 2\hat{h}_{k\bar{k}} \hat{h}^{j\bar{j}} - 2\delta^j_{\bar{k}} \delta^{\bar{j}}_k \\
&= 2h_{a\bar{a}} h^{b\bar{b}} v^a_k v^{\bar{a}}_{\bar{k}} \pi^j_b \pi^{\bar{j}}_{\bar{b}} - 2\delta^j_{\bar{k}} \delta^{\bar{j}}_k
\end{aligned}$$

where $\mu^2 > 0$ is a constant, the B_i are the standard horizontal vector fields induced by the connection defined by h, and the $\overset{*}{E}{}^j_k$ are the standard vertical vector fields on

LM.

THEOREM 3.4.2. *The tensor* $\tilde{h}$ *is invariant under* $\mathrm{GL}(m)$*, i.e.*

$$\tilde{h}_{u\cdot g}(X\cdot g, Y\cdot g) = \tilde{h}_u(X,Y) \qquad \forall u\in LM\,\forall g\in \mathrm{GL}(m)\,\forall X,Y\in T_u(LM)$$

Proof. First let us recall that

$$d_u R_g(B_{j(u)}) = B_{k(u\cdot g)}(g^{-1})^k_j \tag{3.187}$$

$$d_u R_g(\overset{*}{E}{}^j_{i(u)}) = g^j_b \overset{*}{E}{}^b_{a(u\cdot g)}(g^{-1})^a_i \tag{3.188}$$

First we consider the horizontal diagonal term.

$$\tilde{h}_{u\cdot g}(B_{i(u)}\cdot g, B_{j(u)}\cdot g) = \tilde{h}_{u\cdot g}(d_u R_g(B_{i(u)}), d_u R_g(B_{j(u)}))$$

Since the group action moves horizontal vectors to horizontal vectors, we can say

$$\begin{aligned}\tilde{h}_{u\cdot g}(B_{i(u)}\cdot g, B_{j(u)}\cdot g) &= -\mu^2 h_{\lambda(u)}(d_{u\cdot g}\lambda(d_u R_g(B_{i(u)})), d_{u\cdot g}\lambda(d_u R_g(B_{j(u)})))\\ &= -\mu^2 h_{\lambda(u)}(d_u(\lambda\circ R_g)(B_{i(u)}), d_u(\lambda\circ R_g)(B_{j(u)}))\\ &= -\mu^2 h_{\lambda(u)}(d_u\lambda(B_{i(u)}), d_u\lambda(B_{j(u)}))\\ &= \tilde{h}_u(B_{i(u)}), B_{j(u)}))\end{aligned}$$

For the off-diagonal terms we have

$$\tilde{h}_{u\cdot g}(B_{i(u)}\cdot g, \overset{*}{E}{}^j_{k(u)}\cdot g) = 0 = \tilde{h}_u(B_{i(u)}, \overset{*}{E}{}^j_{k(u)})$$

This is true because the group action preserves the connection.

Finally, for the vertical diagonal terms we have

$$\begin{aligned}\tilde{h}_{u\cdot g}(\overset{*}{E}{}^j_{k(u)}\cdot g, \overset{*}{E}{}^{\bar{j}}_{\bar{k}(u)}\cdot g) &= \tilde{h}_{u\cdot g}(g^j_r \overset{*}{E}{}^r_{s(u\cdot g)}(g^{-1})^s_k, g^{\bar{j}}_{\bar{r}} \overset{*}{E}{}^{\bar{r}}_{\bar{s}(u\cdot g)}(g^{-1})^{\bar{s}}_{\bar{k}})\\ &= g^j_r(g^{-1})^s_k g^{\bar{j}}_{\bar{r}}(g^{-1})^{\bar{s}}_{\bar{k}}\ \tilde{h}_{u\cdot g}(\overset{*}{E}{}^r_{s(u\cdot g)}, \overset{*}{E}{}^{\bar{r}}_{\bar{s}(u\cdot g)})\\ &= g^j_r(g^{-1})^s_k g^{\bar{j}}_{\bar{r}}(g^{-1})^{\bar{s}}_{\bar{k}}\\ &\qquad (2h_{a\bar{a}}h^{b\bar{b}}v^a_{s}{}_{(u\cdot g)}v^{\bar{a}}_{\bar{s}}{}_{(u\cdot g)}\pi^r_{b}{}_{(u\cdot g)}\pi^{\bar{r}}_{\bar{b}}{}_{(u\cdot g)} - 2\delta^r_{\bar{s}}\delta^{\bar{r}}_s)\\ &= 2h_{a\bar{a}}h^{b\bar{b}}\big(g^j_r\pi^r_{b(u\cdot g)}\big)\big((g^{-1})^s_k v^a_{s(u\cdot g)}\big)\big(g^{\bar{j}}_{\bar{r}}\pi^{\bar{r}}_{\bar{b}(u\cdot g)}\big)\big((g^{-1})^{\bar{s}}_{\bar{k}}v^{\bar{a}}_{\bar{s}(u\cdot g)}\big)\\ &\qquad - 2g^j_r(g^{-1})^s_k g^{\bar{j}}_{\bar{r}}(g^{-1})^{\bar{s}}_{\bar{k}}\delta^r_{\bar{s}}\delta^{\bar{r}}_s\end{aligned}$$

$$
\begin{aligned}
&= 2h_{a\bar{a}}h^{b\bar{b}}\pi^j_b(u)v^a_k(u)\pi^{\bar{j}}_{\bar{b}}(u)v^{\bar{a}}_{\bar{k}}(u) - 2g^j_r(g^{-1})^r_{\bar{k}}g^{\bar{j}}_s(g^{-1})^s_k \\
&= 2h_{a\bar{a}}h^{b\bar{b}}v^a_k(u)v^{\bar{a}}_{\bar{k}}(u)\pi^j_b(u)\pi^{\bar{j}}_{\bar{b}}(u) - 2\delta^j_{\bar{k}}\delta^{\bar{j}}_k \\
&= \tilde{h}_u(\overset{*}{E}{}^j_k(u), \overset{*}{E}{}^{\bar{j}}_{\bar{k}}(u))
\end{aligned}
$$

□

Remark 3.4.3. *The tensor $\tilde{h}$ is, unfortunately, not a metric on LM*

We can see that this is true by demonstrating a vector X such that $\tilde{h}(X, Y) = 0$ for all Y. One such vector is

$$X = \overset{*}{E}{}^k_k = \delta^k_j \overset{*}{E}{}^j_k$$

With this choice of X, we have

$$
\begin{aligned}
\tilde{h}(X, B_i) &= \delta^k_j \tilde{h}(\overset{*}{E}{}^j_k, B_i) \\
&= 0
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{h}(X, \overset{*}{E}{}^{\bar{j}}_{\bar{k}}) &= \delta^k_j \tilde{h}(\overset{*}{E}{}^j_k, \overset{*}{E}{}^{\bar{j}}_{\bar{k}}) \\
&= 2\delta^k_j \hat{h}_{k\bar{k}}\hat{h}^{j\bar{j}} - 2\delta^k_j\delta^j_{\bar{k}}\delta^{\bar{j}}_k \\
&= 2\hat{h}_{k\bar{k}}\hat{h}^{k\bar{j}} - 2\delta^k_{\bar{k}}\delta^{\bar{j}}_k \\
&= 2\delta^{\bar{j}}_{\bar{k}} - 2\delta^{\bar{j}}_{\bar{k}} \\
&= 0
\end{aligned}
$$

Theorem 3.4.4. *The quadratic form associated with $\tilde{h}$ projects to the standard free field Lagrangian on $J^1\tau$.*

$$\mathcal{L} = h^{a\bar{a}}h^{b\bar{b}}D_{[a}\psi_{b]}D_{[\bar{a}}\psi_{\bar{b}]} - \mu^2 h^{k\bar{k}}\psi_k\psi_{\bar{k}}$$

Proof. Let ω be the induced connection on LM. Then we have

$$\omega = \omega^i_j E^j_i \tag{3.189}$$

$$\omega^i_j = \pi^i_k (dv^k_j + \Gamma^k_{ml} v^l_j dx^m) \tag{3.190}$$

Given a vector tangent to LM, $X = X^i \frac{\partial}{\partial x^i} + X^j_k \frac{\partial}{\partial v^j_k}$, we write the vertical projection

$$\begin{aligned} X_v &= \overset{*}{\omega}(X) \\ &= X^i \overset{*}{\omega}(\frac{\partial}{\partial x^i}) + X^i_j \overset{*}{\omega}(\frac{\partial}{\partial v^i_j}) \\ &= X^i \pi^k_m \Gamma^m_{il} v^l_j \overset{*}{E}{}^j_k + X^i_j \pi^k_i \overset{*}{E}{}^j_k \\ &= (X^i_j \pi^k_i + X^i \pi^k_m \Gamma^m_{il} v^l_j) \overset{*}{E}{}^j_k \\ &= \pi^k_m (X^m_j + X^i \Gamma^m_{il} v^l_j) \overset{*}{E}{}^j_k \end{aligned} \tag{3.191}$$

And the horizontal projection as

$$X_h = X^j \pi^i_j B_i \tag{3.192}$$

The quadratic form $\tilde{h}(X, X)$ splits into two terms because horizontal and vertical vectors are orthogonal with respect to $\tilde{h}$.

$$\begin{aligned} \tilde{h}(X, X) &= \tilde{h}(X_h + X_v, X_h + X_v) \\ &= \tilde{h}(X_h, X_h) + \tilde{h}(X_v, X_v) \end{aligned}$$

If we consider these terms in local coordinates, we obtain

$$\begin{aligned} \tilde{h}(X_h, X_h) &= \tilde{h}(X^j \pi^i_j B_i, X^{\bar{j}} \pi^{\bar{i}}_{\bar{j}} B_{\bar{i}}) \\ &= X^j \pi^i_j X^{\bar{j}} \pi^{\bar{i}}_{\bar{j}} \tilde{h}(B_i, B_{\bar{i}}) \\ &= -X^j \pi^i_j X^{\bar{j}} \pi^{\bar{i}}_{\bar{j}} \mu^2 h(\lambda_* B_i, \lambda_* B_{\bar{i}}) \\ &= -X^j \pi^i_j X^{\bar{j}} \pi^{\bar{i}}_{\bar{j}} \mu^2 h(v^k_i \frac{\partial}{\partial x^k}, v^{\bar{k}}_{\bar{i}} \frac{\partial}{\partial x^{\bar{k}}}) \\ &= -X^j \pi^i_j v^k_i X^{\bar{j}} \pi^{\bar{i}}_{\bar{j}} v^{\bar{k}}_{\bar{i}} \mu^2 h_{k\bar{k}} \\ &= -\mu^2 X^k X^{\bar{k}} h_{k\bar{k}} \end{aligned} \tag{3.193}$$

and

$$\begin{aligned}
\tilde{h}(X_v, X_v) &= \tilde{h}\big(\pi^k_m(X^m_j + X^i\Gamma^m_{il}v^l_j)\overset{*}{E}{}^j_k, \pi^{\bar{k}}_{\bar{m}}(X^{\bar{m}}_{\bar{j}} + X^{\bar{i}}\Gamma^{\bar{m}}_{\bar{i}\bar{l}}v^{\bar{l}}_{\bar{j}})\overset{*}{E}{}^{\bar{j}}_{\bar{k}}\big) \\
&= \pi^k_m(X^m_j + X^i\Gamma^m_{il}v^l_j)\pi^{\bar{k}}_{\bar{m}}(X^{\bar{m}}_{\bar{j}} + X^{\bar{i}}\Gamma^{\bar{m}}_{\bar{i}\bar{l}}v^{\bar{l}}_{\bar{j}})\tilde{h}(\overset{*}{E}{}^j_k, \overset{*}{E}{}^{\bar{j}}_{\bar{k}}) \\
&= \pi^k_m(X^m_j + X^i\Gamma^m_{il}v^l_j)\pi^{\bar{k}}_{\bar{m}}(X^{\bar{m}}_{\bar{j}} + X^{\bar{i}}\Gamma^{\bar{m}}_{\bar{i}\bar{l}}v^{\bar{l}}_{\bar{j}})(2h_{a\bar{a}}h^{b\bar{b}}v^a_kv^{\bar{a}}_{\bar{k}}\pi^j_b\pi^{\bar{j}}_{\bar{b}} - 2\delta^j_{\bar{k}}\delta^{\bar{j}}_k) \\
&= \pi^k_m(X^m_j + X^i\Gamma^m_{il}v^l_j)\pi^{\bar{k}}_{\bar{m}}(X^{\bar{m}}_{\bar{j}} + X^{\bar{i}}\Gamma^{\bar{m}}_{\bar{i}\bar{l}}v^{\bar{l}}_{\bar{j}})(2h_{a\bar{a}}h^{b\bar{b}}v^a_kv^{\bar{a}}_{\bar{k}}\pi^j_b\pi^{\bar{j}}_{\bar{b}}) \\
&\quad - 2\pi^k_m(X^m_j + X^i\Gamma^m_{il}v^l_j)\pi^{\bar{k}}_{\bar{m}}(X^{\bar{m}}_{\bar{j}} + X^{\bar{i}}\Gamma^{\bar{m}}_{\bar{i}\bar{l}}v^{\bar{l}}_{\bar{j}})(\delta^j_{\bar{k}}\delta^{\bar{j}}_k) \\
&= (\pi^k_mv^a_k)(X^m_j\pi^j_b + X^i\Gamma^m_{il}v^l_j\pi^j_b)(\pi^{\bar{k}}_{\bar{m}}v^{\bar{a}}_{\bar{k}})(X^{\bar{m}}_{\bar{j}}\pi^{\bar{j}}_{\bar{b}} + X^{\bar{i}}\Gamma^{\bar{m}}_{\bar{i}\bar{l}}v^{\bar{l}}_{\bar{j}}\pi^{\bar{j}}_{\bar{b}})(2h_{a\bar{a}}h^{b\bar{b}}) \\
&\quad - 2\pi^k_m(X^m_j + X^i\Gamma^m_{il}v^l_j)\pi^j_{\bar{m}}(X^{\bar{m}}_k + X^{\bar{i}}\Gamma^{\bar{m}}_{\bar{i}\bar{l}}v^{\bar{l}}_k) \\
&= \delta^a_m(X^m_j\pi^j_b + X^i\Gamma^m_{il}\delta^l_b)\delta^{\bar{a}}_{\bar{m}}(X^{\bar{m}}_{\bar{j}}\pi^{\bar{j}}_{\bar{b}} + X^{\bar{i}}\Gamma^{\bar{m}}_{\bar{i}\bar{l}}\delta^{\bar{l}}_{\bar{b}})(2h_{a\bar{a}}h^{b\bar{b}}) \\
&\quad - 2(X^m_j\pi^j_{\bar{m}} + X^i\Gamma^m_{il}v^l_j\pi^j_{\bar{m}})(X^{\bar{m}}_k\pi^k_m + X^{\bar{i}}\Gamma^{\bar{m}}_{\bar{i}\bar{l}}v^{\bar{l}}_k\pi^k_m) \\
&= (X^a_j\pi^j_b + X^i\Gamma^a_{ib})(X^{\bar{a}}_{\bar{j}}\pi^{\bar{j}}_{\bar{b}} + X^{\bar{i}}\Gamma^{\bar{a}}_{\bar{i}\bar{b}})(2h_{a\bar{a}}h^{b\bar{b}}) \\
&\quad - 2(X^m_j\pi^j_{\bar{m}} + X^i\Gamma^m_{il}\delta^l_{\bar{m}})(X^{\bar{m}}_k\pi^k_m + X^{\bar{i}}\Gamma^{\bar{m}}_{\bar{i}\bar{l}}\delta^{\bar{l}}_m) \\
&= (X^a_j\pi^j_b + X^i\Gamma^a_{ib})(X^{\bar{a}}_{\bar{j}}\pi^{\bar{j}}_{\bar{b}} + X^{\bar{i}}\Gamma^{\bar{a}}_{\bar{i}\bar{b}})(2h_{a\bar{a}}h^{b\bar{b}}) \\
&\quad - 2(X^m_j\pi^j_{\bar{m}} + X^i\Gamma^m_{i\bar{m}})(X^{\bar{m}}_k\pi^k_m + X^{\bar{i}}\Gamma^{\bar{m}}_{\bar{i}m})
\end{aligned} \tag{3.194}$$

So combining (3.193) and (3.194) we have finally that

$$\begin{aligned}
\tilde{h}(X, X) = -\mu^2X^kX^{\bar{k}}h_{k\bar{k}} &+ (X^a_j\pi^j_b + X^i\Gamma^a_{ib})(X^{\bar{a}}_{\bar{j}}\pi^{\bar{j}}_{\bar{b}} + X^{\bar{i}}\Gamma^{\bar{a}}_{\bar{i}\bar{b}})(2h_{a\bar{a}}h^{b\bar{b}}) \\
&- 2(X^m_j\pi^j_{\bar{m}} + X^i\Gamma^m_{i\bar{m}})(X^{\bar{m}}_k\pi^k_m + X^{\bar{i}}\Gamma^{\bar{m}}_{\bar{i}m})
\end{aligned} \tag{3.195}$$

Now recall from (3.24) the coordinate form of the projection,

$$\begin{aligned}
x^i \circ F &= x^i \\
v^j \circ F &= \dot{x}^j \\
v^j_k \circ F &= \pi^b_k\dot{v}^j_b
\end{aligned}$$

It is not hard to see that if the vector X projects to $j^1\psi$ we obtain

$$\begin{aligned}\tilde{h}(X,X) &= -\mu^2 h_{k\bar{k}}\psi^k\psi^{\bar{k}} + (\frac{\partial\psi^a}{\partial x^b} + \psi^i\Gamma^a_{ib})(\frac{\partial\psi^{\bar{a}}}{\partial x^{\bar{b}}} + \psi^{\bar{i}}\Gamma^{\bar{a}}_{\bar{i}\bar{b}})(2h_{a\bar{a}}h^{b\bar{b}}) \\ &\quad - 2(\frac{\partial\psi^m}{\partial x^{\bar{m}}} + \psi^i\Gamma^m_{i\bar{m}})(\frac{\partial\psi^{\bar{m}}}{\partial x^m} + \psi^{\bar{i}}\Gamma^{\bar{m}}_{\bar{i}m}) \\ &= 2h_{a\bar{a}}h^{b\bar{b}}D_{\bar{b}}\psi^a D_b\psi^{\bar{a}} - 2D_{\bar{m}}\psi^m D_m\psi^{\bar{m}} - \mu^2 h_{k\bar{k}}\psi^k\psi^{\bar{k}}\end{aligned}$$

which with a little work reduces to

$$= h^{a\bar{a}}h^{b\bar{b}}D_{[a}\psi_{b]}D_{[\bar{a}}\psi_{\bar{b}]} - \mu^2 h_{k\bar{k}}\psi^k\psi^{\bar{k}}$$

This is exactly the desired Lagrangian. □

Chapter 4

Jets of Arbitrary fields

This chapter was published as [37]. Some slight revisions have been made to blend it with the rest of the work.

4.1 Introduction

The Cartan-Hamilton-Poincaré (CHP) m-form is the central object in covariant Lagrangian field theory. The ingredients which go into the construction of this m-form are:

1. A Lagrangian $\mathrm{L} : J^1\pi \to \mathbb{R}$, on the bundle of 1-jets of sections of $\pi : E \to M$, where E is the configuration manifold of the theory,
2. A volume on the m-dimensional parameter space M,
3. The contact structure of $J^1\pi$.

It is the contact structure [46] in this mixture of ingredients that provides the geometrical foundation of the theory. In this chapter we give a new geometrical formulation of the covariant field theory on $J^1\pi$ by lifting it to the bundle of vertically adapted linear frames $L_\pi E$ of E. We will show that the full depth of Lagrangian and Hamiltonian field theory on $J^1\pi$ has a useful geometrical representation on the bundle $L_\pi E$.

In this representation the role of the contact structure of $J^1\pi$ is taken over by the canonical vector-valued soldering 1-form on $L_\pi E$. Introduction of a Lagrangian leads to the definition of a modified soldering form, and this vector-valued 1-form plays the role of the CHP-m-form. These structures pass to a certain quotient of $L_\pi E$ to give the standard structures on $J^1\pi$. The advantage gained by this reformulation is that it allows us to utilize the natural geometry that is supported on $L_\pi E$, namely n-symplectic geometry, to further develop covariant field theory.

If E is an arbitrary n-dimensional manifold, then the bundle of linear frames $\lambda : LE \to E$ supports a canonically defined $\mathbb{R}^n$-valued 1-form, the "soldering" 1-form. n-symplectic geometry on LE is the generalized symplectic geometry that emerges upon taking the soldering 1-form θ as the generalized symplectic potential. This geometry, including the notions of n-symplectic observables, the corresponding generalized Hamiltonian vector-valued vector fields, and generalized Poisson and graded Poisson brackets, has been developed in a series of papers [20, 21, 39, 40, 41, 42]. A sketch of the basic structure of the theory can be found in section 4.2, but let us point out here that in [41] it was shown that the fundamentals of the canonical symplectic geometry on the cotangent bundle T^*E can be constructed entirely in terms of the n-symplectic geometry on LE.

When E has extra structure, in particular when $\pi : E \to M$ is a fiber bundle as it is in Lagrangian field theory, then the n-symplectic geometry likewise inherits extra structure on LE. In particular, the fiber structure of $\pi : E \to M$ leads to a reduction of LE to the subbundle of vertically adapted linear frames $L_\pi E$, with a corresponding reduction in the generality of n-symplectic observables. The structure group $\mathrm{G_v}$ of $L_\pi E$ is the subgroup of $\mathrm{GL}(n)$ that is block lower triangular, corresponding to the convention that the last $k = n - m$ vectors in each linear frame are required to be vertical. Following the model construction given in [41] Lawson showed [31, 21] that the multisymplectic geometry [22] on the affine cojet bundle $J^{1*}\pi$ can also be derived directly from the n-symplectic geometry on $L_\pi E$.

Turning our attention in this chapter to the covariant field theory on $J^1\pi$, we will show that the geometrical foundations of the theory, namely the contact structure on $J^1\pi$, follows directly from the n-symplectic structure on $L_\pi E$, while the CHP-form follows from a modified soldering form. The central idea on which the analysis is based is the following theorem.

THEOREM 4.1.1. *Let $\pi : E \to M$ be an $m+k$ dimensional fiber bundle over the m-dimensional manifold M. The vertically adapted frame bundle $L_\pi E$ is a principal $H = \mathrm{GL}(m) \times \mathrm{GL}(k)$ bundle over $J^1\pi$. In particular, $J^1\pi \cong L_\pi E/H$.*

As a consequence of this theorem, which we prove in section 4.3, the canonical soldering forms on $L_\pi E$ pass to the quotient to define the contact structure of $J^1\pi$ (see section 4.5).

A simple picture of the main ideas can be sketched out as follows. Let (x^i) be local coordinates on M and let (y^A) be fiber coordinates on E, so that $(z^\alpha) = (x^i, y^A)$ are adapted local coordinates on E. With respect to such coordinates a general vertically adapted linear frame at a point in E will be of the form

$$\begin{aligned}&(e_i, e_A) = (v_i^j \tfrac{\partial}{\partial x^j} + v_i^B \tfrac{\partial}{\partial y^B}, v_A^B \tfrac{\partial}{\partial y^B})\\ &i, j = 1, \ldots, m, \quad A, B = m+1, \ldots, m+k\end{aligned}$$

The first m vectors (e_i) are non-vertical while the last k vectors (e_A) are vertical with respect to π. The matrices (v_i^j) and (v_A^B) are necessarily non-singular, while the matrix $(v_i^B) \in \mathbb{R}^{k\times m}$ is arbitrary. Hence we may take the collection $(x^i, y^A, v_i^j, v_i^B, v_A^B)$ as local coordinates on $L_\pi E$. We can represent an arbitrary adapted linear frame in terms of these local coordinates as the $(m+k)\times(m+k)$ matrix

$$\begin{pmatrix} v_i^j & 0 \\ v_i^B & v_A^B \end{pmatrix}$$

Using the notation $\pi_j^i = (v_j^i)^{-1}$ and $\pi_A^B = (v_A^B)^{-1}$, this matrix can be decomposed as follows:

$$\begin{pmatrix} v_i^j & 0 \\ v_i^B & v_A^B \end{pmatrix} = \begin{pmatrix} \delta_k^j & 0 \\ \pi_k^a v_a^B & \delta_C^B \end{pmatrix} \begin{pmatrix} v_i^k & 0 \\ 0 & v_A^C \end{pmatrix} \tag{4.1}$$

The first factor is H invariant and defines a natural projection to $J^1\pi$. We thus obtain the decomposition

$$L_\pi E = J^1\pi \times_E (LVE \times_M LM)$$

where LVE denotes the bundle of vertical frames of E.

These results suggest that it may be useful to lift the covariant Lagrangian field theory on $J^1\pi$ to $L_\pi E$. In particular on $L_\pi E$ we have available the n-symplectic geometry to use in studying the structure of field theories. We show in section 4.4 that for a lifted Lagrangian $\mathcal{L} = \rho^*(\mathrm{L})$, the n-symplectic Hamiltonian vector fields defined by vertical vector fields on E may be thought of as *variational vector fields*. If X is such a vector field then $X(\mathcal{L})$ gives the Euler-Lagrange operator to within a total divergence.

In section 4.6 we turn to the problem of constructing, on $L_\pi E$, a lifted version of the CHP m-form. We show that in fact one can use a lifted Lagrangian to define an $\mathbb{R}^n$-valued CHP-form using the canonical $\mathbb{R}^n$-valued soldering form θ. The key to the construction is to use the fundamental vertical vector fields on $L_\pi E$ together with the Lagrangian to give a global, invariant definition of the covariant momentum, which is essentially a frame bundle version of the Legendre transformation of classical theory. The result is that the $\mathbb{R}^n$-valued CHP-form is a *modified, or non-canonical soldering form* $\theta_{\mathcal{L}}$. This new vector-valued CHP-form $\theta_{\mathcal{L}}$ passes to the quotient to define the standard CHP-m-form on $J^1\pi$.

As an application of the general formalism we derive in section 4.7 a generalized Hamilton-Jacobi differential equation and generalized Hamilton equations. Under appropriate assumptions these equations reproduce the Hamilton-Jacobi equations and Hamilton equations of the de Donder-Weyl [14, 45] and Carathéodory-Rund [12, 45] theories.

We recall that there is a certain degree of arbitrariness in Rund's [45] canonical formalism for Carathéodory's theory. We find that by identifying the canonical variables introduced here with the canonical variables in Rund's formalism, the undetermined features of the Carathéodory-Rund theory can be given a natural interpretation on $L_\pi E$, namely as the variables defining linear frames for M. Looking again at the decomposition (4.1) we see now that the entries in the right-hand-factor represent a linear frame for M (the (v_i^j) factor) together with a linear frame for the fibers of E (the (v_A^B) factor). Section 4.9 contains concluding remarks together with plans for applications and extensions of the results presented in this chapter.

4.2 The Vertically Adapted Linear Frame Bundle $L_\pi E$

Let $\pi : E \to M$ be a fiber bundle where M is m-dimensional and E is $n = m + k$-dimensional. Lower case Latin indices are assumed to range over $1 \ldots m$, upper case Latin indices over $m+1 \ldots m+k$, and Greek indices over $1 \ldots m+k$. This convention will be used throughout the chapter.

An adapted frame at $e \in E$ is a frame where the last k basis vectors are vertical. Note that coordinate frames that come from adapted coordinates are adapted frames. The adapted frame bundle of π, denoted $L_\pi E$, consists of all adapted frames for E.

$$L_\pi E = \{(e, \{e_i, e_A\}) : e \in E, \{e_i, e_A\} \text{ is a basis for } T_e E, \text{ and } d_u\pi(e_A) = 0\}$$

The canonical projection, $\lambda : L_\pi E \to E$, is defined by $\lambda(e, \{e_i, e_A\}) = e$.

$L_\pi E$ is a reduced subbundle of LE, the frame bundle of E (Lawson [31]). As such it is a principal fiber bundle over E. Its structure group is $\mathrm{G_v}$, the nonsingular block lower triangular matrices.

$$\mathrm{G_v} = \left\{ \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} : A \in \mathrm{GL}(m), B \in \mathrm{GL}(k), C \in \mathbb{R}^{km} \right\}$$

$\mathrm{G_v}$ acts on $L_\pi E$ on the right by

$$(e, \{e_i, e_A\}) \cdot \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} = \{(e, \{e_i A^i_j + e_A C^A_j, e_A B^A_B\})$$

4.2.1 Coordinates

If (x^i, y^A) are adapted coordinates on an open set $U \subseteq E$, then one may induce several different coordinates on $\lambda^{-1}(U)$. First consider the coframe or n-symplectic momentum coordinates $(x^i, y^A, \pi^i_j, \pi^A_j, \pi^A_B)$ on $\lambda^{-1}(U)$ defined by

$$\begin{aligned} x^i(e,\{e_i,e_A\}) &= x^i(e) & \pi^i_j(e,\{e_i,e_A\}) &= e^i(\frac{\partial}{\partial x^j}) \\ & & \pi^A_B(e,\{e_i,e_A\}) &= e^A(\frac{\partial}{\partial y^B}) \\ y^A(e,\{e_i,e_A\}) &= y^A(e) & \pi^A_j(e,\{e_i,e_A\}) &= e^A(\frac{\partial}{\partial x^j}) \end{aligned}$$

Here (e^i, e^A) is the dual frame to (e_i, e_A). We have as is customary retained the same symbols for the induced horizontal coordinates.

Secondly consider the frame or n-symplectic velocity coordinates $(x^i, y^A, v^i_j, v^A_j, v^A_B)$ on $\lambda^{-1}(U)$ defined by

$$\begin{aligned} x^i(e,\{e_i,e_A\}) &= x^i(e) & v^i_j(e,\{e_i,e_A\}) &= e_j(x^i) \\ & & v^A_B(e,\{e_i,e_A\}) &= e_B(y^A) \\ y^A(e,\{e_i,e_A\}) &= y^A(e) & v^A_j(e,\{e_i,e_A\}) &= e_j(y^A) \end{aligned}$$

The v coordinates, viewed together as a block triangular matrix, form the inverse of the π coordinates above. The blocks have the following relations:

$$v^i_j \pi^j_k = \delta^i_k \qquad v^A_j \pi^j_k + v^A_B \pi^B_k = 0 \qquad v^A_B \pi^B_C = \delta^A_C$$

Lastly consider the following coordinates which are constructed from the previous two.

Define $(x^i, y^A, u^i_j, u^A_j, u^A_B)$ on $\lambda^{-1}(U)$ by

$$\begin{aligned} x^i(e,\{e_i,e_A\}) &= x^i(e) & u^i_j &= \pi^i_j & u^A_j &= v^A_i \pi^i_j = -v^A_B \pi^B_j \\ y^A(e,\{e_i,e_A\}) &= y^A(e) & u^A_B &= \pi^A_B \end{aligned}$$

It will turn out that the u^A_j coordinates are pull-ups of the standard jet coordinates on $J^1\pi$. As such, we will refer to these coordinates as Lagrangian coordinates.

Later in the chapter we will need the following formulas for the fundamental vertical vector fields $\overset{*}{E}{}^{\alpha}_{\beta}$ on $L_\pi E$ in Lagrangian coordinates.

$$\overset{*}{E}{}^{i}_{j} = -u^i_k \frac{\partial}{\partial u^j_k} \qquad \overset{*}{E}{}^{A}_{B} = -u^A_C \frac{\partial}{\partial u^B_C} \qquad \overset{*}{E}{}^{i}_{A} = u^i_k v^B_A \frac{\partial}{\partial u^B_k} \tag{4.2}$$

4.2.2 n-symplectic structure

n-symplectic geometry arises naturally on the frame bundle LE of any n-dimensional manifold E. LE supports a canonically defined $\mathbb{R}^n$-valued 1-form θ, the soldering 1-form, and n-symplectic geometry is the geometry on LE when one takes $d\theta$ as a vector-valued generalized symplectic form. We present here a sketch of the structure of the theory and refer the reader to the literature [20, 21, 39, 40, 41, 42] for more details. See also the works of de León, Salgado et al. [13] and Awane [1].

The intrinsic definition of the soldering 1-form θ parallels the definition of the canonical form on T^*M.

$$\theta_u(X) = e^\alpha(d_u\bar{\lambda}(X))r_\alpha = \theta^\alpha_u(X)r_\alpha \tag{4.3}$$

Here $u = (e, \{e_\alpha\}) \in LE$, $\bar{\lambda} : LE \to E$ is the canonical projection, and $\{r_\alpha\}$ is the standard basis for $\mathbb{R}^n$. In canonical coordinates,

$$\theta^\alpha = \pi^\alpha_\beta dx^\beta$$

The above formula parallels the local coordinate formula $\vartheta = p_i dq^i$ for the canonical 1-form on T^*M.

Because the soldering 1-form θ is vector-valued, the natural structure equation for n-symplectic geometry takes the generalized form

$$d\hat{f}^{\alpha_1\alpha_2\cdots\alpha_p} = -p!X_{\hat{f}}{}^{\alpha_1\alpha_2\cdots\alpha_{p-1}} \lrcorner\, d\theta^{\alpha_p} \tag{4.4}$$

Here $\hat{f} = (\hat{f}^{\alpha_1\alpha_2\cdots\alpha_p}) : LE \to \otimes^p\mathbb{R}^n$ is a vector-valued function on LE and $X_{\hat{f}} = (X_{\hat{f}}{}^{\alpha_1\alpha_2\cdots\alpha_{p-1}})$ is the corresponding set of Hamiltonian vector fields. (Each superscript α_k, $k = 1, 2, \ldots, p$, runs from 1 to n). Moreover, since the soldering form is equivariant under the free right action of the structure group $GL(n,\mathbb{R})$ on LE, the class of functions that can satisfy (4.4) is restricted. They divide naturally into vector-valued functions that map to either the symmetric tensor spaces $(\otimes_s)^p\mathbb{R}^n$ or the anti-symmetric tensor spaces $(\otimes_a)^p\mathbb{R}^n$, where $\otimes_s$ and $\otimes_a$ denote the symmetric and anti-symmetric tensor products, respectively. There is a *naturally defined Poisson bracket* for both sets of observables, and the complete set of symmetric observables is a *Poisson algebra* with respect to the bracket, while the set of anti-symmetric observables is a *graded Poisson algebra* with respect to the bracket. These brackets, when restricted to the subsets of tensorial observables, are the frame bundle versions of the Schouten-Nijenhuis brackets [42]. On $L_\pi E$ the allowable tensorial observables [31] correspond to contravariant tensor fields that are projectible to E.

As a reduced subbundle of LE, $L_\pi E$ has the n-symplectic geometry obtained by restricting the soldering form. Since this soldering form is $\mathbb{R}^{m+k}$-valued, we will denote it (θ^i, θ^A). Let $u = (e, \{e_i, e_A\})$ be a point in $L_\pi E$. If $\lambda : L_\pi E \to E$ is the canonical projection and $X \in T_u L_\pi E$, then θ defined as in (4.3) above splits naturally into the two terms

$$\theta_u(X) = \theta^i(X)r_i + \theta^A(X)r_A$$

where (r_i, r_A) is the standard basis for $\mathbb{R}^{m+k}$. In local momentum coordinates,

$$\theta^i = \pi^i_j dx^j \qquad \theta^A = \pi^A_j dx^j + \pi^A_B dy^B$$

4.3 The Relationship between $L_\pi E$ and $J^1\pi$

We will demonstrate three useful facts relating $L_\pi E$ and $J^1\pi$.

1. $J^1\pi$ is an associated bundle to $L_\pi E$ [31].

2. $L_\pi E$ is a principal fiber bundle over $J^1\pi$.

3. $L_\pi E$ is a pull-back bundle over $J^1\pi$ [31].

4.3.1 A special case

Consider the case where π is a trivial bundle. Let $M = \mathbb{R}^m$ and $E = \mathbb{R}^m \times F$ with F a k-dimensional manifold. Let $\pi : \mathbb{R}^m \times F \to \mathbb{R}^m$ be the standard projection. It is known that for this bundle each 1-jet corresponds to an m-tuple of tangent vectors to F.

$$J^1\pi \cong \mathbb{R}^m \times (TF \oplus \cdots \oplus TF)$$

It is clear that such a bundle is associated to $L_\pi E$.

Let us examine $L_\pi E$ in this case. We will make use of the other projection mapping $\bar{\pi} : \mathbb{R}^m \times F \to F$. For each frame $(u, \{e_i, e_A\})$ in $L_\pi E$, we decompose each vector into

$$e_i = (v_i, w_i) \quad e_A = (v_A, w_A)$$

where $v_i = d_u\pi(e_i)$, $w_i = d_u\bar{\pi}(e_i)$, $v_A = d_u\pi(e_A)$, and $w_A = d_u\bar{\pi}(e_A)$. Note that $v_A = 0$ by the definition of $L_\pi E$, so we have

$$e_i = (v_i, w_i) \quad e_A = (0, w_A)$$

The k vectors $\{w_A\}$ form a basis for $\mathrm{T}_{\bar{\pi}(u)}\,\mathrm{F}$, and the m vectors $\{v_i\}$ form a basis for $\mathrm{T}_{\pi(u)}\,\mathbb{R}^m$. The m vectors $\{w_i\}$ are simply an m-tuple of vectors in $\mathrm{T}_{\bar{\pi}(u)}\,\mathrm{F}$.

Decomposing all of $L_\pi E$ in this way, we obtain

$$L_\pi E \cong J^1\pi \times_E (\mathrm{L}\,\mathbb{R}^m \times \mathrm{LF})$$

This is a bundle isomorphism over $E = \mathbb{R}^m \times F$. From this decomposition, it is clear that $L_\pi E$ is a pull-back bundle over $J^1\pi$. Furthermore, the fiber is the Lie group $\mathrm{GL}(m) \times \mathrm{GL}(k)$.

4.3.2 The general case

Consider an arbitrary fiber bundle $\pi : E \to M$. In this more general setting, a 1-jet is no longer simply an m-tuple of tangent vectors. There are three major ways of describing 1-jets, each with its own charm:

1. Equivalence classes of sections of π.

2. Linear right-inverses to $d_u\pi$.

3. Non-vertical m-dimensional subspaces of T_uE.

One quick way to define the projection from $L_\pi E$ to $J^1\pi$ is to map each adapted frame to the span of its non-vertical elements.

$$(u, \{e_i, e_A\}) \mapsto (u, \mathrm{span}\{e_i\})$$

However, we will benefit from starting with $J^1\pi$ as an associated bundle.

As stated earlier, the structure group of $L_\pi E$ is $\mathrm{G_v}$, the nonsingular block lower triangular matrices. This group $\mathrm{G_v}$ can be decomposed [31] into the product of two of its subgroups, H and J, where

$$H = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A \in \mathrm{GL}(m), B \in \mathrm{GL}(k) \right\}$$

and

$$J = \left\{ \begin{pmatrix} I & 0 \\ C & I \end{pmatrix} : C \in \mathbb{R}^{km} \right\}$$

Note that J is Lie group isomorphic to the additive group $\mathbb{R}^{km}$.

We will show that $J^1\pi$ is a bundle associated to $L_\pi E$ with fiber $\mathrm{G_v}/H$. Although H is a closed Lie subgroup of $\mathrm{G_v}$ it is not normal. As such $\mathrm{G_v}/H$ does not have a natural group structure; it is a manifold with a left $\mathrm{G_v}$-action. For each coset $gH \in \mathrm{G_v}/H$, we select the unique representative in J.

$$\begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \sim \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}$$

By choosing these representatives, we identify $\mathrm{G_v}/H$ with J and hence $\mathbb{R}^{km}$. These identifications are diffeomorphisms.

Consider how the left $\mathrm{G_v}$-action looks for our selected representatives.

$$\begin{pmatrix} A & 0 \\ C & B \end{pmatrix}\begin{pmatrix} I & 0 \\ \xi & I \end{pmatrix} = \begin{pmatrix} A & 0 \\ C + B\xi & B \end{pmatrix} \sim \begin{pmatrix} I & 0 \\ CA^{-1} + B\xi A^{-1} & I \end{pmatrix}$$

So the $\mathrm{G_v}$-action appears *affine* when $\mathrm{G_v}/H$ is identified with $\mathbb{R}^{km}$. Therefore it is prudent to use this identification to define an affine structure on $\mathrm{G_v}/H$ modelled on $\mathbb{R}^{km}$. This $\mathrm{G_v}$-invariant structure will pass to the fibers of the associated bundle, making it an affine bundle.

THEOREM 4.3.1. $L_\pi E \times_{\mathrm{G_v}} (\mathrm{G_v}/H) \cong J^1\pi$

Proof. The isomorphism maps each equivalence class $[(e, \{e_i, e_A\}, \xi)]$ to the linear map $\phi : T_{\pi(e)}M \to T_eE$ defined by $\phi(\hat{e}_i) = e_i + \xi_i^A e_A$, where we use the basis $\hat{e}_i = d_e\pi(e_i)$. □

COROLLARY 4.3.2. *$L_\pi E$ is a principal fiber bundle over $J^1\pi$ with fiber H.*

Proof. This fact follows directly from proposition 5.5 in reference [30]. □

We will denote the projection from $L_\pi E$ to $J^1\pi$ by ρ. It is given by

$$\rho(e, \{e_i, e_A\}) = (e, \tau) \quad \text{where} \quad \tau(\hat{e}_i) = e_i$$

We now show that the u_j^A-coordinates defined earlier are the pull-ups of the jet coordinates. If (x^i, y^A) are adapted coordinates on an open set $U \subseteq E$ and $u = (e, \{e_i, e_A\}) \in \lambda^{-1}(U)$ then

$$\begin{aligned} y_i^A \circ \rho(u) = y_i^A(e, \tau) &= d_e y^A \circ \tau\Big(\frac{\partial}{\partial x^i}\Big|_{\pi(e)}\Big) = d_e y^A\Big(e_j \hat{e}^j\Big(\frac{\partial}{\partial x^i}\Big|_{\pi(e)}\Big)\Big) \\ &= d_e y^A(e_j) e^j\Big(\frac{\partial}{\partial x^i}\Big|_{e}\Big) = v_j^A(u)\pi_i^j(u) = u_j^A(u) \end{aligned} \tag{4.5}$$

What remains to be shown is that $L_\pi E$ is a pull-back bundle over $J^1\pi$. To see this, we will decompose each adapted frame in a manner similar to the trivial case covered earlier. We can split each adapted frame (u, e_i, e_A) into three pieces:

1. A point in LM, $(\pi(u), \tilde{e}_i)$, where $\tilde{e}_i = d_u\pi(e_i)$

2. A point in LVE, (u, e_A), where LVE is the bundle of vertical frames over E

3. A point in $J^1\pi$, (u, ϕ), where $\phi : T_{\pi(u)}M \to T_uE$ is defined by $\phi(\tilde{e}_i) = e_i$

THEOREM 4.3.3. $L_\pi E \cong J^1\pi \times_E (LVE \times_M LM)$

Proof. The isomorphism is given by $(u, e_i, e_A) \mapsto \big((u, \phi), (u, e_A), (\pi(u), \tilde{e}_i)\big)$. The inverse map is quite nice: $\big((u, \phi), (u, f_A), (p, f_i)\big) \mapsto (u, \phi(f_i), f_A)$ □

4.4 Prolongations of Vector Fields to $L_\pi E$

DEFINITION 4.4.1. *A Lagrangian on $L_\pi E$ is a function $\mathcal{L} : L_\pi E \to \mathbb{R}$. A Lagrangian on $L_\pi E$ is* lifted *if it satisfies the auxiliary conditions*

$$\overset{*}{E}{}^i_j(\mathcal{L}) = 0 \qquad \overset{*}{E}{}^A_B(\mathcal{L}) = 0$$

REMARK 4.4.2. *Using (4.2) one can show that these conditions imply that $\mathcal{L}$ is constant on the fibers of $\rho : L_\pi E \to J^1\pi$, and thus is the pull up of a function on $J^1\pi$. For the remainder of this chapter we will assume that our Lagrangians are lifted.*

In order to see the role played by the canonical n-symplectic structure on $L_\pi E$ in Lagrangian field theory, we consider a variation of a local section $\phi : M \to E$. The variation of ϕ can be defined by a vector field f on E that projects to the zero vector field on M, so that in adapted local coordinates f has the form $f = f^A\partial_A$. The associated tensorial function $\hat{f} : L_\pi E \to \mathbb{R}^{m+k}$ is given in local coordinates on $L_\pi E$ by $\hat{f} = \hat{f}^\alpha \hat{r}_\alpha$, where

$$(\hat{f}^\alpha) = (\hat{f}^i, \hat{f}^A) = (0, f^B\pi^A_B)$$

The n-symplectic Hamiltonian vector field $X_{\hat{f}}$ determined by $\hat{f}$ is the unique solution of equation (4.4) with $p = 1$. Thus $X_{\hat{f}}$ is defined by

$$d\hat{f}^\alpha = -X_{\hat{f}} \lrcorner\, d\theta^\alpha$$

and in local coordinates it has the form [39]

$$X_{\hat{f}} = f^A \partial_A - \frac{\partial f^A}{\partial x^j} \pi^B_A \frac{\partial}{\partial \pi^B_j} - \frac{\partial f^A}{\partial y^C} \pi^B_A \frac{\partial}{\partial \pi^B_C}$$

Transforming to Lagrangian coordinates we find

$$\begin{aligned} X_{\hat{f}} &= \left(f^A \partial_A + (\frac{\partial f^A}{\partial x^j} + u^B_j \frac{\partial f^A}{\partial y^B}) \frac{\partial}{\partial u^A_j} \right) - (\frac{\partial f^A}{\partial y^C} u^B_A) \frac{\partial}{\partial u^B_C} \\ &= \left(f^A \partial_A + \frac{d f^A}{d x^i} \frac{\partial}{\partial u^A_j} \right) - (\frac{\partial f^A}{\partial y^C} u^B_A) \frac{\partial}{\partial u^B_C} \end{aligned} \tag{4.6}$$

Lemma 4.4.3. *Let f be a vertical vector field on E. The projection of the associated Hamiltonian vector field $X_{\hat{f}}$ on $L_\pi E$ to $J^1\pi$ is the prolongation $j(f)$ of f to $J^1\pi$.*

Proof. The vector fields $\frac{\partial}{\partial u^B_C}$ are vertical with respect to ρ, and $\rho_*(\frac{\partial}{\partial u^A_j}) = \frac{\partial}{\partial y^A_j}$. □

This lemma shows that the Hamiltonian vector field $X_{\hat{f}}$ on $L_\pi E$ is a lift of the prolongation of f to $J^1\pi$. That $X_{\hat{f}}$ actually has the properties of the prolongation of f with respect to Lagrangians follows from the following lemma. We let

$$\mathcal{E}_A(\cdot) = \frac{\partial(\cdot)}{\partial y^A} - \frac{d}{dx^i}\left(\frac{\partial(\cdot)}{\partial u^A_i}\right)$$

denote the Euler-Lagrange operator in local coordinates on $L_\pi E$.

Lemma 4.4.4. *If $X_{\hat{f}}$ is the n-symplectic Hamiltonian vector field on $L_\pi E$ of a vertical vector field f on E, and if $\mathcal{L}$ is a lifted Lagrangian on $L_\pi E$, then*

$$X_{\hat{f}}(\mathcal{L}) = f^A \mathcal{E}_A(\mathcal{L}) + \frac{d}{dx^i}(f^A p^i_A) \tag{4.7}$$

Proof. The proof is a straightforward calculation using (4.6). □

After introducing the CHP 1-forms in the section 4.6 we will use (4.7) to lift the variational principle to $L_\pi E$.

REMARK 4.4.5. *As mentioned in section 4.2.2 there are other observables in n-symplectic geometry on $L_\pi E$ beyond those corresponding to vertical vector fields on E. In particular there is the Poisson algebra of all vertical symmetric contravariant tensor fields on E, as well as the graded Poisson algebra of all vertical antisymmetric contravariant tensor fields on E. The associated (equivalence classes of) vector-valued Hamiltonian vector fields on $L_\pi E$ also project to tensor fields on $J^1\pi$. Since these vector-valued Hamiltonian vector fields generalize the natural lift of a vector field from E to $L_\pi E$, their projections to $J^1\pi$ can be taken as the prolongation of the tensor fields on E to $J^1\pi$. These ideas will be elaborated in more detail elsewhere.*

4.5 The Contact Structure

The contact structure on $J^1\pi$ amounts to a natural splitting of the tangent and cotangent spaces to E. For every $(e,\tau) \in J^1\pi$ there is a natural splitting of T_eE and T_eE^* into horizontal and vertical subspaces. This is usually encoded via the linear projections onto the vertical and horizontal. Saunders [46] envisions the contact structure as linear endomorphisms of the pullback vector bundles $J^1\pi \times_E (TE)$ and $J^1\pi \times_E (T^*E)$. These maps can be defined invariantly as follows. For $(e,\tau) \in J^1\pi$, $X \in T_eE$, and $\omega \in T_e^*E$.

$$h(X) = \tau \circ d_e\pi(X) \qquad v(X) = X - h(X)$$
$$h^t(\omega) = \omega \circ \tau \circ d_e\pi \qquad v^t(\omega) = \omega - h^t(\omega)$$

Guillemin and Sternberg [25] prefer to think of the contact structure as TE-valued 1-forms on $J^1\pi$. To achieve this, they compose the h and v above with $d_{(e,\tau)}\pi_{1,0}$, where $\pi_{1,0} : J^1\pi \to E$.

In local coordinates, the contact structure looks like a pair of (1,1) tensor fields on E, except that they depend on jet coordinates.

$$h = dx^k \otimes (\frac{\partial}{\partial x^k} + y^A_k \frac{\partial}{\partial y^A}) \qquad v = (dy^B - y^B_j dx^j) \otimes \frac{\partial}{\partial y^B}$$

Depending on interpretation, the expressions above can be the horizontal and vertical projections for either $J^1\pi \times_E (TE)$ or $J^1\pi \times_E (T^*E)$. They can also be interpreted as TE-valued 1-forms on $J^1\pi$.

4.5.1 The Contact Structure viewed from $L_\pi E$

The contact structure arises on $J^1\pi$ because each 1-jet (e, τ) allows us to decompose T_eE into a direct sum. Similarly, the soldering form arises on $L_\pi E$ because each adapted frame $u = (e, e_i, e_A)$ allows us to represent T_eE as $\mathbb{R}^{m+k}$. *So the contact structure is analogous to the soldering form.* Recall that

$$\theta_u(X) = e^i(d_u\lambda(X))r_i + e^A(d_u\lambda(X))r_A = \theta^i_u(X)r_i + \theta^A_u(X)r_A$$

and that in local coordinates,

$$\theta^i = \pi^i_j dx^j \qquad \theta^A = \pi^A_j dx^j + \pi^A_B dy^B$$

Consider the following TE-valued one-forms on $L_\pi E$

$$\theta_h(u) = \theta^i(u) \otimes e_i \qquad \theta_v(u) = \theta^A(u) \otimes e_A$$

In local coordinates,

$$\theta_h = \pi^i_k dx^k \otimes v^l_i(\frac{\partial}{\partial x^l} + u^A_l \frac{\partial}{\partial y^A}) = dx^k \otimes (\frac{\partial}{\partial x^k} + u^A_k \frac{\partial}{\partial y^A})$$

$$\theta_v = \pi^A_B(dy^B - u^B_j dx^j) \otimes v^C_A \frac{\partial}{\partial y^C} = (dy^B - u^B_j dx^j) \otimes \frac{\partial}{\partial y^B}$$

These objects are strikingly similar to the contact structure of $J^1\pi$. In fact, they pass to the quotient to give the contact structure on $J^1\pi$. The contact structure is known to appear in "various guises" [46]; the soldering form on $L_\pi E$ is another, perhaps more potent, version.

We also remark that the contact structure falls trivially from the following theorem

THEOREM 4.5.1. *Let $\lambda : P \to E$ be a principal fiber bundle with structure group G, let $H \subseteq G$ be a closed lie subgroup, and let F be a manifold with a left G-action. Then*

$$P \times_H F \cong (P/H) \times_E (P \times_G F)$$

Proof. First note that by Proposition 5.5 in reference [30], $P/H \cong P \times_G (G/H)$ and $\rho : P \to P/H$ is a principal bundle. So $P \times_H F$ is a bundle associated to ρ. This makes sense–if F has a left G-action then it has a left H-action. The isomorphism map is

$$(p, f)H \mapsto (pH, (p, f)G)$$

It is well-defined and a smooth diffeomorphism. □

COROLLARY 4.5.2.

$$L_\pi E \times_H \mathbb{R}^{m+k} \cong J^1\pi \times_E TE$$
$$L_\pi E \times_H (\mathbb{R}^{m+k})^* \cong J^1\pi \times_E T^*E$$

The natural splitting of the fibers $\mathbb{R}^{m+k}$ and $(\mathbb{R}^{m+k})^*$ is H-invariant and passes to the quotient to form the contact structure.

4.6 The Cartan-Hamilton-Poincaré Forms

One associates [25, 22] with a given Lagrangian L on $J^1\pi$ the Cartan-Hamilton-Poincaré (CHP)-m-form θ_{L}, which one may use to re-express the action integral of the Lagrangian. This form can be defined directly [25] on $J^1\pi$, or it can be defined [22] on $J^1\pi$ as the pull back of the canonical multisymplectic form on $J^{1*}\pi$, the affine dual of $J^1\pi$. Although the CHP-form on $L_\pi E$ can be defined in terms of the n-tangent structure on $L_\pi E$, we will define this form directly in terms of invariant quantities on $L_\pi E$. We will first define CHP-1-forms, from which the CHP-m-form will be constructed.

The fundamental vertical vector fields $\overset{*}{E}{}^i_A$ are given in Lagrangian coordinates in (4.2). If $\mathcal{L} : L_\pi E \to \mathbb{R}$ is a lifted Lagrangian on $L_\pi E$, then it is the pull-up under ρ of a Lagrangian L on $J^1\pi$. Hence, since $\rho_*(\frac{\partial}{\partial u^A_i}) = \frac{\partial}{\partial y^A_i}$, we have

$$\overset{*}{E}{}^i_A(\mathcal{L}) = u^i_k v^B_A \frac{\partial \mathcal{L}}{\partial u^B_k} = u^i_k v^B_A \frac{\partial\, \mathrm{L}}{\partial y^B_k}$$

This leads us to the following definition:

Definition 4.6.1. *Let $\mathcal{L} : L_\pi E \to \mathbb{R}$ be a Lagrangian on $L_\pi E$. The* ***covariant momenta*** *of $\mathcal{L}$ are*

$$\mathcal{P}^i_A = \overset{*}{E}{}^i_A(\mathcal{L}) = (u^i_k v^B_A) p^k_B$$

where $p^k_B = \frac{\partial \mathcal{L}}{\partial u^B_k}$ denotes the ***canonical momenta*** *of the Lagrangian.*

Remark 4.6.2. *Notice that the* ***covariant momenta*** *$(\mathcal{P}^i_A)$ are globally defined tensorial objects on $L_\pi E$, while the* ***canonical momenta*** *$(p^k_B) = (\frac{\partial \mathcal{L}}{\partial u^B_k})$ are only defined locally.*

We are now in a position to give a global definition of the CHP-form on $L_\pi E$. We first define the related 1-forms.

Definition 4.6.3. *Let $\mathcal{L} : L_\pi E \to \mathbb{R}$ be a lifted Lagrangian on $L_\pi E$, and $\tau(m)$ a positive function of m, the dimension of M. The CHP-1-forms $\theta^\alpha_{\mathcal{L}}$ on $L_\pi E$ are*

$$\theta^i_{\mathcal{L}} = \tau(m)\mathcal{L}\theta^i + \overset{*}{E}{}^i_A(\mathcal{L})\theta^A \tag{4.8}$$

$$\theta^A_{\mathcal{L}} = \theta^A \tag{4.9}$$

Remark 4.6.4. *The positive function $\tau(m)$ in this definition is included to allow for various theories to occur as special cases. We will see that the choice $\tau(m) = 1$*

yields the canonical theory of Carathéodory-Rund, and $\tau(m) = \frac{1}{m}$ *yields the canonical theory of de Donder-Weyl.*

REMARK 4.6.5. *The collection of forms* $(\theta^\alpha_{\mathcal{L}}) = (\theta^i_{\mathcal{L}}, \theta^A_{\mathcal{L}})$, *where* $\theta^A_{\mathcal{L}} = \theta^A$, *is a* **modified**, *or* **non-canonical soldering form** *if* $\mathcal{L} > 0$. *This follows from the easily verifiable properties* $X \lrcorner\, \theta^\alpha_{\mathcal{L}} = 0$ *for all* $\alpha = 1, 2, \ldots, n$ *if and only if* X *is vertical with respect to* $\lambda : L_\pi E \to E$ *and* $R^*_g \theta_{\mathcal{L}} = g^{-1} \cdot \theta_{\mathcal{L}}$.

Working out the local coordinate form of the CHP-1-forms in Lagrangian coordinates we find

$$\theta^i_{\mathcal{L}} = -H^i_j dx^j + P^i_A dy^A \tag{4.10}$$

$$\theta^A_{\mathcal{L}} = P^A_j dx^j + P^A_B dy^B \tag{4.11}$$

where

$$H^i_j = u^i_k (p^k_B u^B_j - \tau(m)\mathcal{L}\delta^k_j) \tag{4.12}$$

$$P^i_B = u^i_k p^k_B \tag{4.13}$$

$$P^A_j = -u^A_B u^B_j \tag{4.14}$$

$$P^A_B = u^A_B \tag{4.15}$$

We will refer to the H^i_j as the components of the **covariant Hamiltonian**, and to the P^i_B as the components of the **covariant canonical momentum**. If we define symbols h^k_j by the formula

$$h^k_j = p^k_B u^B_j - \tau(m)\mathcal{L}\delta^k_j \tag{4.16}$$

then the covariant Hamiltonian (4.12) can be expressed as $H^i_j = u^i_k h^k_j$. Setting $\tau(m) = 1$ we find that h^i_j has the form of Carathéodory's Hamiltonian [12, 45] tensor. Similarly, setting $\tau = \frac{1}{m}$ we find that $h = h^i_i$ yields the Hamiltonian in the de Donder-Weyl theory [14, 45].

Finally we show how the CHP-1-forms can be used to construct the CHP-m-form on $J^1\pi$.

PROPOSITION 4.6.6. *Let (B_i, B_A) denote the standard horizontal vector fields of any torsion free linear connection on $\lambda : L_\pi E \to E$, and let vol denote the pull up to $L_\pi E$ of a fixed volume m-form on M. Set $vol_i = B_i \lrcorner\, vol$. Then when $\tau(m) = \frac{1}{m}$ the m-form*

$$\theta_{\mathcal{L}} := \theta^i_{\mathcal{L}} \wedge vol_i$$

passes to the quotient to define the CHP-m-form $\Theta_{\mathcal{L}}$ on $J^1\pi$.

Proof. The vector fields B_i have the local coordinate form $B_i = v^\alpha_i \frac{\partial}{\partial z^\alpha} + V$ where V is vertical with respect to $\lambda : L_\pi E \to E$. Using this and formulas (4.8) through (4.13) one can show that

$$\theta_{\mathcal{L}} = -(p^j_B u^B_j - \mathcal{L})\ vol + p^j_A dy^A \wedge (\frac{\partial}{\partial x^j} \lrcorner\, vol)$$

The right-hand-side is constant on the fibers of $\rho : L_\pi E \to J^1\pi$ and is in fact the pull-up $\rho^*(\Theta_{\mathrm{L}})$ of the CHP-m-form Θ_{L} on $J^1\pi$. □

REMARK 4.6.7. *The above geometrical construction of the CHP-m-form is analogous to the geometrical construction given by Guillemin and Sternberg [25].*

4.6.1 The Variational Principle on $L_\pi E$

We now lift the variational principle from $J^1\pi$ to $L_\pi E$. This is a simple procedure since we are using a lifted Lagrangian and only varying a section of π. Let $\phi : M \to E$ be a section of π and $j\phi$ its 1-jet prolongation to $J^1\pi$. For any section $\xi : J^1\pi \to L_\pi E$ we have that $u = \xi \circ j\phi : M \to L_\pi E$ is a section of $\pi \circ \lambda : L_\pi E \to M$.

The action integral on $J^1\pi$ lifts nicely since $\mathcal{L} = \mathrm{L} \circ \rho$.

$$\int_M j\phi^*(\mathrm{L}) vol \quad = \quad \int_M u^*(\mathcal{L}) vol$$

We recall [22] that the action integral is extremized by ϕ iff $j\phi^*(W \lrcorner\, d\Theta_{\mathrm{L}}) = 0$ for all vector fields W on $J^1\pi$. However, this condition can be weakened to $j\phi^*(j(f) \lrcorner\, d\Theta_{\mathrm{L}}) = 0$ for all vertical vector fields f on E.

Now for any such vertical vector field f, consider its $J^1\pi$ prolongation $j(f)$ and its n-symplectic Hamiltonian vector field $X_{\hat{f}}$ on $L_\pi E$ (also a kind of prolongation). From (4.4.3) we know $\rho_*(X_{\hat{f}}) = j(f)$. From proposition (4.6.6) we now have $X_{\hat{f}} \lrcorner\, d\theta_{\mathcal{L}} = X_{\hat{f}} \lrcorner\, \rho^*(d\Theta_{\mathrm{L}})$. It follows that

$$u^*(X_{\hat{f}} \lrcorner\, d\theta_{\mathcal{L}}) = j\phi^*(j(f) \lrcorner\, d\Theta_{\mathrm{L}})$$

We conclude that the action integral is extremized by ϕ iff $u^*(X_{\hat{f}} \lrcorner\, d\theta_{\mathcal{L}}) = 0$ for all vertical vector fields f on E.

4.7 The Generalized Hamilton-Jacobi Equation

As an application of our general formalism we derive the Carathéodory-Rund and de Donder-Weyl Hamilton-Jacobi equations. By analogy with the time independent Hamilton-Jacobi theory (see, for example, reference [49]) we seek Lagrangian submanifolds of $L_\pi E$. However, since the dimension of $L_\pi E$ is in general not twice the dimension of E, a new definition is needed. For our purposes here we will consider $n = m + k$ dimensional submanifolds of $L_\pi E$ that arise as sections of λ. In particular we consider sections $\sigma : E \to L_\pi E$ that satisfy

$$\sigma^*(d\theta^\alpha_{\mathcal{L}}) = 0 \tag{4.17}$$

We will refer to this equation as the generalized Hamilton-Jacobi equation.

Since $\sigma^*(d\theta^\alpha_{\mathcal{L}}) = d\left(\sigma^*(\theta^\alpha_{\mathcal{L}})\right)$ the condition (4.17) asserts that the 1-forms $\sigma^*(\theta^\alpha_{\mathcal{L}})$ are locally exact, and we express this as

$$\sigma^*(\theta^\alpha_{\mathcal{L}}) = dS^\alpha \tag{4.18}$$

in terms of $m + k$ new functions S^α defined on open subsets of E. For convenience we will denote objects on $L_\pi E$ pulled back to E using σ with an over-tilde. Thus, for example, $\tilde{H}^i_j = H^i_j \circ \sigma$ and $\tilde{P}^i_A = P^i_A \circ \sigma$. Then we get from (4.12)–(4.15) and (4.18)

$$\textbf{(a)}\quad \tilde{H}^i_j \;=\; -\frac{\partial S^i}{\partial x^j}\ , \qquad\qquad \textbf{(b)}\quad \tilde{P}^i_A = \frac{\partial S^i}{\partial y^A} \tag{4.19}$$

$$\textbf{(a)}\quad \tilde{u}^A_B \tilde{u}^B_j \;=\; -\frac{\partial S^A}{\partial x^j}\ , \qquad\qquad \textbf{(b)}\quad \tilde{u}^A_B = \frac{\partial S^A}{\partial y^B} \tag{4.20}$$

Recalling that $H^i_j = P^i_B u^B_j - \tau(m)\mathcal{L} u^i_j$ and P^i_A are functions of the coordinates x^i, y^A, u^i_j and u^A_i, equations (4.19) can be combined into the single equation

$$H^i_j(x^a, y^B, u^a_b, u^B_a, \frac{\partial S^i}{\partial y^B}) \circ \sigma = -\frac{\partial S^i}{\partial x^j} \tag{4.21}$$

Similarly combining equations (4.20) we obtain

$$\frac{dS^A}{dx^j} = 0$$

We next consider special cases of these generalized Hamilton-Jacobi equations.

4.7.1 The Theory of Carathéodory and Rund

We note from (4.12), (4.13), and (4.16) that $H^i_j = u^i_k h^k_j$ and $P^i_A = u^i_k p^k_A$, where the matrix of functions (u^i_j) is GL(m)-valued. Using the notation $P^i_j = -H^i_j$ and $\tilde{u}^i_j = u^i_j \circ \sigma$ we may rewrite (4.12) and (4.13) in the form

$$\tilde{P}^i_j = -\tilde{u}^i_k \tilde{h}^k_j\ , \qquad\qquad \tilde{P}^i_A = \tilde{u}^i_k \tilde{p}^k_A \tag{4.22}$$

If we take $t(m) = 1$ then these equations are the equations defining the canonical momenta in Rund's canonical formalism for Carathéodory's geodesic field theory (see equations (1.22), page 389 in [45], with the obvious change in notation). In this situation equation (4.21) can be identified with the Rund's Hamilton-Jacobi equation for Carathéodory's theory (see equation (3.29) on page 240 in [45]). We recall [45] that one can derive the Euler-Lagrange field equations from this Hamilton-Jacobi equation.

In (4.22) we have the result that the arbitrary non-singular matrix-valued functions $(\tilde{u}^i_j)$ that occur in Rund's canonical formalism for Carathéodory's theory have a geometrical interpretation in the present setting. Specifically they correspond to the coordinates for linear frames for M. These defining relations are derived from Rund's transversality condition, and this condition has the elegant reformulation here as the kernel of $(\theta^i_{\mathcal{L}})$.

We will say that a vector X at $e \in E$ is transverse to a solution surface through e that is defined by a given Lagrangian $\mathcal{L}$, if $X = d\lambda(\hat{X})$, where $\hat{X} \in T_u(L_\pi E)$ satisfies $\hat{X} \lrcorner \theta^i_{\mathcal{L}} = 0$, for some $u \in \lambda^{-1}(e)$. $\hat{X}$ thus satisfies the equations

$$0 = -H^i_j X^j + P^i_A X^A = u^i_k \left(-h^k_j X^j + p^k_A X^A\right)$$

$$X^j = \hat{X}(x^i) \ , \quad X^A = \hat{X}(y^A)$$

from which we infer

$$0 = -h^k_j X^j + p^k_A X^A \tag{4.23}$$

This is Rund's transversality condition for the theory of Carathéodory when we take $t(m) = 1$ (see equation (1.10), page 388 in [45]). The canonical momenta P^i_j and P^i_A are defined by Rund to be solutions of

$$0 = P^i_j X^j + P^i_A X^A \tag{4.24}$$

when (X^j, X^A) satisfy (4.23). Rund's solutions of these equations are given in (4.22). Looking at (4.22), (4.23) and (4.24) we see that the introduction of the u^i_j in (4.22) amounts to the introduction of the $\mathrm{GL}(m)$ freedom for linear frames for M.

4.7.2 de Donder-Weyl Theory

Returning to (4.21) let us reduce this equation by making several assumptions. We suppose that $\mathcal{L}$ is regular (in the usual sense on $J^1\pi$), that the section σ is such that $\tilde{u}^i_j = \delta^i_j$, and we make the choice $t(m) = \frac{1}{m}$. Now summing $i = j$ in (4.21) we obtain

$$\tilde{h}(x^i, y^B, \frac{\partial S^i}{\partial y^B}) = -\frac{\partial S^i}{\partial x^i}$$

where $\tilde{h} = \tilde{p}^i_A \tilde{u}^A_i - \tilde{\mathcal{L}}$. This equation is the Hamilton-Jacobi equation of the de Donder-Weyl theory, as presented by Rund (see equation (2.31) on page 224 in [45]). We recall [45] that one can derive in this case also the Euler-Lagrange field equations from the de Donder-Weyl Hamilton-Jacobi equation.

4.8 Hamilton's Equations

The structure of equations (4.10) - (4.13) suggests that one should be able to derive generalized Hamilton equations if the canonical momenta $p^i_A = \frac{\partial \mathcal{L}}{\partial u^A_i}$ can be introduced as part of a local coordinate system on $L_\pi E$. Part of the original philosophy used in developing n-symplectic geometry in reference [39] was to switch from scalar equations to tensor equations, motivated by the fact that the soldering 1-form is vector-valued. In particular, the basic structure equation (4.4) in n-symplectic geometry is tensor-valued. We show next that

$$u^*(\eta \lrcorner d\theta^i_{\mathcal{L}}) = 0 \tag{4.25}$$

where $u : M \to L_\pi E$ is a section of $\pi \circ \lambda$, and η is any vector field on $L_\pi E$, yields generalized canonical equations that contain known canonical equations as special cases. We consider here only $d\theta^i_{\mathcal{L}}$ since by Proposition (4.6.6) it alone is needed to construct the CHP-m-form on $J^1\pi$.

We need the following definition in order to introduce the canonical momenta as part of a coordinate system on $L_\pi E$.

Definition 4.8.1. *A Lagrangian $\mathcal{L}$ on $L_\pi E$ is* **regular** *if the $(m+k) \times (m+k)$ matrix*

$$\left(\overset{*}{E}{}^i_A \circ \overset{*}{E}{}^j_B(\mathcal{L}) \right)$$

is non-singular. Working out the terms of this matrix in Lagrangian coordinates using (4.2) we obtain

$$\overset{*}{E}{}^i_A \circ \overset{*}{E}{}^j_B(\mathcal{L}) = u^j_a u^i_b v^E_B v^D_A \left(\frac{\partial^2 \mathcal{L}}{\partial u^E_a \partial u^D_b} \right)$$

It is clear that this definition is equivalent to the standard definition of regularity on $J^1\pi$.

We now consider the transformation of coordinates from the set $(x^i, y^A, u^i_j, u^A_k, u^A_B)$ to the new set $(\bar{x}^i, \bar{y}^A, \bar{u}^i_j, p^j_A, \bar{u}^A_B)$ where

$$\bar{x}^i = x^i \ , \quad \bar{y}^A = y^A \ , \quad \bar{u}^i_j = u^i_j \ , \quad \bar{u}^A_B = u^A_B \ , \quad p^i_A = \frac{\partial \mathcal{L}}{\partial u^A_i}$$

Computing the Jacobian one finds that the new barred functions will be a proper coordinate system whenever the Lagrangian is regular. For the remainder of this section we shall assume that $\mathcal{L}$ has this property, despite the fact that many important examples (see [22]) have non-regular Lagrangians. Moreover, for simplicity we will drop the bars on the new coordinates.

In the generalized canonical equation (4.25) we now take $\eta = \frac{\partial}{\partial p^i_A}$. We find the result

$$0 = \left(\frac{\partial H^j_k}{\partial p^i_A} \circ u \right) + (u^j_i \circ u) \left(\frac{\partial (y^A \circ u)}{\partial x^k} \right)$$

Using $H^j_k = u^j_i h^i_k$ and the fact that (u^i_j) is a non-singular matrix valued function, this last equation reduces to

$$\frac{\partial h^j_k}{\partial p^i_A} \circ u = \frac{\partial (y^A \circ u)}{\partial x^k} \delta^j_i$$

This is our first set of generalized Hamilton equations. Notice that by summing $j = k$ in this equation we obtain

$$\frac{\partial h}{\partial p^i_A} \circ u = \frac{\partial (y^A \circ u)}{\partial x^i} \tag{4.26}$$

Upon setting $t(m) = \frac{1}{m}$ we obtain half of the de Donder-Weyl canonical equations. Under suitable but complicated conditions these equations, with $t(m) = 1$, will also reproduce part of Rund's canonical equations for the theory of Carathéodory.

In the generalized canonical equation (4.25) we now take $\eta = \frac{\partial}{\partial y^A}$. We find

$$0 = u^* \left(d(u^i_k p^k_A) + u^i_k \frac{\partial h^k_j}{\partial y^A} dx^j \right)$$

Using an "over bar" notation to denote objects pulled back to M by u we may write this as

$$\frac{\partial}{\partial x^j} \left(\bar{u}^i_k \bar{p}^k_A \right) = -\bar{u}^i_k \left(\frac{\partial h^k_j}{\partial y^A} \right) \circ u \tag{4.27}$$

This is our second set of generalized Hamilton's equations.

Notice that what is non-standard in (4.27) is the appearance of the derivatives of the functions $\bar{u}^i_j = u^i_j \circ u$. If, however, the section $u : M \to L_\pi E$ is such that the $\bar{u}^i_j$ are constants, then these equations reduce to

$$\frac{\partial(\bar{p}^k_A)}{\partial x^j} = -\frac{\partial h^k_j}{\partial y^A} \circ u$$

Setting $\tau(m) = \frac{1}{m}$ and summing $k = j$ in this equation we obtain

$$\frac{\partial(\bar{p}^i_A)}{\partial x^i} = -\frac{\partial h}{\partial y^A} \circ u$$

These equations, together with equations (4.26) when $\tau(m) = \frac{1}{m}$, are the complete canonical equations in the de Donder-Weyl theory.

4.9 Conclusions

In this chapter we have reformulated covariant field theory for sections of $\pi : E \to M$ on the bundle of vertically adapted linear frames $L_\pi E$. The advantage gained by this reformulation is that it allows us to utilize the natural geometry that is supported on $L_\pi E$, namely n-symplectic geometry, to further develop covariant field theory. We have concentrated on demonstrating that $L_\pi E$ with its canonical n-symplectic structure provides an appropriate arena for formulating covariant field theory, leaving aside the development of the modified n-symplectic geometry defined by the Cartan-Hamilton-Poincaré (CHP) 1-forms to future papers.

To this end we showed that covariant field theory on $J^1\pi$ lifts in a natural way to $L_\pi E$. The analysis was based on the theorem, presented in section 4.3, that $J^1\pi$ is a principal fiber bundle $\rho : L_\pi E \rightarrow J^1\pi$ over the bundle of 1-jets of sections of π. This theorem was used to show that the soldering 1-forms θ^α on $L_\pi E$ play the role of the contact structure on $J^1\pi$. The soldering 1-forms and a lifted Lagrangian $\mathcal{L} = \rho^*(\mathrm{L})$ were then used to construct modified soldering 1-forms $\theta^\alpha_{\mathcal{L}}$ on $L_\pi E$, the CHP 1-forms. These CHP 1-forms were shown to pass to the quotient to define the standard CHP m-form on $J^1\pi$. Further we used the CHP 1-forms to derive generalized Hamilton-Jacobi and generalized canonical equations on $L_\pi E$, and then showed that the Hamilton-Jacobi and canonical equations in the theories of Carathéodory-Rund and de Donder-Weyl are contained as special cases. What we did not do was develop the explicit structure of the modified n-symplectic geometry, including allowable observables, Hamiltonian vector fields, and Poisson and graded Poisson brackets, that one should be able to define using the CHP 1-forms. Nor did we develop the variational principle on $L_\pi E$ for a Lagrangian that is not lifted from $J^1\pi$. These problems we leave to future publications.

Chapter 5

Conclusions

The overall them of this work has been to introduce additional freedom into the standard formalisms for field theory, lifting them to higher dimensional spaces with additional geometric and algebraic structure. This sort of trick is a standard way of taming a difficult problem; in a sense we are creating "room to maneuver." We have presented two approaches: one for (anti)symmetric contravariant vector fields and another for arbitrary vector fields.

These two approaches share many similarities. Both cases revolve around a frame bundle, and in both cases the new projection becomes a principal bundle, hence we use the term gauge freedom. Also in both cases, we interpret the jet bundle as an associated bundle, which itself is a powerful relationship. Even the coordinate form of our projections share similarities; compare (3.24) with (4.5).

On the other hand there are significant differences. In chapter 3 we lift to the tangent bundle of the frame bundle and in chapter 4 we lift to the adapted frame bundle. The significance being that the adapted frame bundle clearly has n-symplectic structure, whereas the tangent bundle of LM is not yet known to possess such structure. On the other hand, $T(LM)$ has a vector bundle structure which $L_\pi E$ lacks. In particular this vector space structure allowed for the interpretation of the free field Lagrangian as symmetric type $(0, 2)$ tensor.

5.1 Future Work

The relationship between these two approaches needs to be explored further. Although we have the result from chapter 4 that

$$\begin{aligned} L_\tau TM &\cong J^1\tau \times_{TM} (LV(TM) \times_M LM) \\ &\cong J^1\tau \times_{TM} (LM \times_M LM) \end{aligned}$$

it is unclear how this related to $T(LM)$.

Also there is much development yet to be done in the approach of chapter 3. Dynamical equations have not been written. This will probably require lifting the soldering form to $T(LM)$, which will give it n-symplectic, or perhaps k-cotangent, structure. A Legendre transformation needs to be defined. One expects this to be a map to $T^*(LM)$, but which one is unknown.

The result for the free field Lagrangian may yet lead to a metric. It is possible that the addition of more terms will do the trick. Also, Lagrangians for higher rank observables have yet to be explored.

Additionally, there are a number of side topics that have arisen in this work. In [39, 40], a certain symmetry is demanded of Hamiltonian vector fields. In chapter 3, we were forced to demand a slightly different symmetry. The full relationship between these restrictions in not know and it is worth exploring. It would be interesting to study n-symplectic geometry using the symmetry of chapter 3 only.

The notion of a weakly associated bundle presented in chapter 2 may be new to the literature. If so, there are many details to work out regarding it. Many of the statements about associated bundles should generalize.

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