Abstract

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In this thesis, we use finite semisimple Lie theory to compute the structure of a local symmetric space $\mathfrak{p}$ and its corresponding symmetric space P. Helminck classifies the local symmetric spaces over algebraically closed fields in. Here I extend these first results and write algorithms that implement combinatorial methods to compute "nice" bases, restricted root systems, multiplicities and restricted Weyl groups for these spaces.

ALGORITHMS FOR COMPUTATIONS IN LOCAL SYMMETRIC SPACES

BY

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To my daughter, Reyna Elizabeth,
the sky was hung for you, darling.

You are all the inspiration that I will ever need.

Biography

Jennifer R. Fowler was born on June 24, 1973 in Shreveport, Louisiana. She received her elementary and secondary education in the Deep East Texas town of Nacogdoches. She received her Bachelor of Science degree in Mathematics from The University of New Orleans in 1998, and her Master of Science degree in Mathematics from North Carolina State University in 2000.

In the fall of 1998, she entered the Ph.D program in Mathematics with a concentration in algebra at North Carolina State University at Raleigh. After graduation, Ms. Fowler will remain in academia as Assistant Professor of Mathematics at Lamar University in Beaumont, Texas.

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Chapter 1

The Problem

Symmetric spaces arise naturally in various fields of mathematics and physics. Traditionally the geometry and fine structure of these spaces over the real numbers were studied vigorously by Helgason [Hel78] and others. Recently symmetric spaces over arbitrary fields have proven useful as well.

As with the underlying Lie group, a lot of the structure of a symmetric space can be recovered by looking at it locally. In 1988, Helminck [Hel88] classified the local symmetric spaces over algebraically closed fields. He found 24 different cases. In this work, I use the excellent tables of Helminck to examine the structure of these spaces.

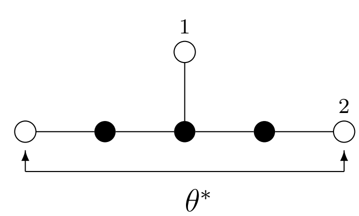
While Lie groups and their associated Lie algebras have been studied for quite some time, the advent of computer algebra made it possible to work with those groups and algebras that are just too big to deal with by hand. This prompted many mathematicians to create and implement algorithms to analyze their structure. **LiE** [vLCL92], **GAP** [GAP], **Magma** [Mag], and **Coxeter/Weyl** [Ste02] are just a few examples of excellent packages where these algorithms have been implemented.

While these packages are quite sufficient for work with Lie Groups and Lie Algebras, no such packages exist for symmetric spaces. In fact until recently no algorithms existed for any computations in symmetric spaces. In [Hel96] and [Hel00] the first algorithms for computing some of the structure of these symmetric spaces were given. In this thesis, I extend these first results and write algorithms for several other aspects of the structure of these symmetric spaces. I use combinatorial methods to compute "nice" bases, restricted root systems, multiplicities, restricted Weyl groups, etc. for these spaces.

For an arbitrary lie algebra $\mathfrak{g}$ and an involution θ, we can define the local symmetric space

$$\mathfrak{p} = \{x \in \mathfrak{g} \mid \theta(x) = -x\}.$$

Helminck borrows the Dynkin diagram from finite dimensional semisimple theory and is able to represent the action of the involution on the roots of $\mathfrak{g}$. We call the new diagram a Theta diagram.



The roots in the basis that are fixed by theta are colored in black on our Theta Diagram. The roots that are not colored project down to some root in the local symmetric space. For each case, I first recover the action of the involution on the original root system and compute the roots in the restricted root system by letting the natural projection map act on the roots of $\mathfrak{g}$. I then apply this clear-cut algorithm to each of the 24 cases.

(1) Determine all the positive roots of $\mathfrak{g}$ that project down to each root in the base for the restricted root system.

(2) Find representatives in the original Weyl group for each element in the restricted Weyl group.

(3) Use the Weyl group representatives to find a complete list of positive roots in the restricted root system.

(4) Determine all the positive roots of $\mathfrak{g}$ that project down to each root in the base for the restricted root system.

(5) Write the eigenspaces for each root in the restricted root system as the direct sum of eigenspaces from the original root system.

(6) Determine basis for $\mathfrak{p}$ by using the projection map and eigenspace decomposition.

Chapter 2

Preliminaries and Background Material

2.1 Symmetric Spaces

Let G be a reductive algebraic group over an arbitrary field k of characterisitic not 2. Let $\theta \in \mathrm{Aut}(G), \theta^2 = \mathrm{id}$.

2.1.1 Definition

Define the two sets $K = \{A \in G \mid \theta(A) = A\}$ and $P = \{A\theta(A)^{-1} \mid A \in G\}$. P is called a *reductive symmetric space*. If G is semisimple, P is called a *semisimple symmetric space*.

2.1.2 Example

Let $G = \mathrm{GL}_n(\mathbb{R})$ and $\theta(A) = (A^T)^{-1}$. To find K we need $\theta(A) = A = (A^T)^{-1}$. Taking the inverse of both sides $A^{-1} = A^T$. So K consists of the orthogonal matrices. To find P we need to calculate $A\theta(A)^{-1} = AA^T$. So P consists of symmetric matrices with positive eigenvalues.

2.2 Local Symmetric Spaces

Linear algebraic groups themselves are complicated entities. Besides having a group structure, a linear algebraic group is also an algebraic variety. A lot of the time it is easier to compute in the corresponding Lie algebra and lift it back to the group level.

In studying symmetric spaces, we have an analogous phenomenon.

2.2.1 Definition

For an algebraic group G, let $\mathfrak{g}$ denote it corresponding Lie algebra. For an involution θ in $\mathrm{Aut}(G)$, consider its differential map $\theta^* : \mathfrak{g} \to \mathfrak{g}$.

In most of the literature, we drop the * and just denote the differential by θ.

Now define

$$\mathfrak{k} = \{x \in \mathfrak{g} \mid \theta(x) = x\} = +1\text{-eigenspace of } \theta$$
$$\mathfrak{p} = \{x \in \mathfrak{g} \mid \theta(x) = -x\} = -1\text{-eigenspace of } \theta$$

$\mathfrak{p}$ is called a *local symmetric space*.

2.2.2 Some Results

Lemma 1. $\mathfrak{k} \perp \mathfrak{p}$

Proof. Since θ is a Lie algebra homomorphism it leaves the Killing form invariant. Take $x \in \mathfrak{k}$ and $y \in \mathfrak{p}$, then

$$\kappa(x, y) = \kappa(\theta(x), \theta(y)) = \kappa(x, -y) = -\kappa(x, y)$$

therefore $\kappa(x, y) = 0$ and $\mathfrak{k} \perp \mathfrak{p}$ with respect to the Killing form. □

Lemma 2. $\mathfrak{k}$ *is a subalgebra of* $\mathfrak{g}$

Proof. Take $x, y \in \mathfrak{k}$

$$\theta[x, y] = [\theta(x), \theta(y)] = [x, y]$$

therefore $[x, y] \in \mathfrak{k}$. □

$\mathfrak{p}$ is not a subalgebra, but it is a subspace.

Lemma 3. $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$

Proof. Take $x \in \mathfrak{g}$

$$x + \theta(x) \in \mathfrak{k} \text{ and } x - \theta(x) \in \mathfrak{p}$$

then $x = \frac{1}{2}(x + \theta(x)) + \frac{1}{2}(x - \theta(x))$. □

$\mathfrak{k}$ and $\mathfrak{p}$ decompose $\mathfrak{g}$ orthogonally.

2.3 Root Space Decompostion

For a vector space V, the collection of all linear functionals on V form a vector space. This vector space, V^*, is called the *dual space of* V.

Let $\mathfrak{h}$ be a toral subalgebra of the Lie algebra $\mathfrak{g}$. For α, β two functionals in $\mathfrak{h}^*$ with the positive definite, symmetric, bilinear form (α, β), we define the reflection

$$s_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha.$$

Notation 1. The quantity $\frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ is used so often in our work , that we choose to notate it $\langle \beta, \alpha \rangle$.

For $\alpha \in \mathfrak{h}^*$, let $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \quad \text{for all } h \in \mathfrak{h}\}$. Let

$$\Phi(\mathfrak{h}) = \{\alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \neq 0\}.$$

$\Phi(\mathfrak{h})$ is called the set of roots.

Remark 1. For some toral subalgebras $\mathfrak{h}$, $\Phi(\mathfrak{h})$ is a root system.

Definition 1. [Hum72] A subset $\Phi(\mathfrak{h})$ of the Euclidean space $\mathfrak{h}^*$, is called a *root system* in $\mathfrak{h}^*$ if

(1) $\Phi(\mathfrak{h})$ is finite, spans $\mathfrak{h}^*$, and does not contain 0.

(2) If $\alpha \in \Phi(\mathfrak{h})$, the only multiples of $\alpha \in \Phi(\mathfrak{h})$ are $\pm\alpha$.

(3) For all $\alpha \in \Phi(\mathfrak{h})$, s_α leaves $\Phi(\mathfrak{h})$ invariant.

(4) If $\alpha, \beta \in \Phi(\mathfrak{h})$, then $\langle \beta, \alpha \rangle \in \mathbb{Z}$

For $\mathfrak{g}$ a nonzero semisimple Lie algebra and $\mathfrak{t}$ a maximal toral subalgebra in $\mathfrak{g}$ such that $\Phi(\mathfrak{t})$ is a root system, we have the root space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha \in \Phi(\mathfrak{t})} \mathfrak{g}_\alpha.$$

This decomposition has the properties that

- $\mathfrak{t} = \mathfrak{g}_0$
- $\dim \mathfrak{g}_\alpha = 1$

We can use the above finite dimensional Lie theory to help describe the structure of $\mathfrak{k}$ and $\mathfrak{p}$.

2.3.1 Decomposition over a local Symmetric Space

Consider the set of toral subalgebras contained in $\mathfrak{p}$ and let $\mathfrak{a}$ be a maximal toral subalgebra in $\mathfrak{p}$. Note that $\mathfrak{a}$ is not neccesarily a maximal toral subalgebra in $\mathfrak{g}$.

Lemma 4. *Let $\mathfrak{t} \supseteq \mathfrak{a}$ be a maximal toral subalgebra of $\mathfrak{g}$. Then we have the following:*

(1) $\mathfrak{t}$ is θ-stable.

(2) $\mathfrak{t} = \mathfrak{t}_+ \oplus \mathfrak{t}_-$, where $\mathfrak{t}_\pm = \{x \in \mathfrak{t} \mid \theta(x) = \pm x\}$

(3) $\mathfrak{t}_- = \mathfrak{a}$ *and* $\mathfrak{t}_+ \subset \mathfrak{k}$

For a proof, see Richardson [Ric82].

Now take a maximal toral subalgebra $\mathfrak{a}$ contained in $\mathfrak{p}$ and look at the decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\lambda \in \Phi(\mathfrak{a})} \mathfrak{g}_\lambda$$

where $\mathfrak{g}_\lambda = \{x \in \mathfrak{g} \mid [t, x] = \lambda(t)x \quad \text{for all } t \in \mathfrak{a}\}$ and

$\Phi(\mathfrak{a}) = \{\lambda \in \mathfrak{a}^* \mid \lambda \neq 0 \text{ and } \mathfrak{g}_\lambda \neq 0\}$. We want to use this decomposition to find a basis for $\mathfrak{k}$ and $\mathfrak{p}$.

Definition 2. A subset Φ of the Euclidean space E is called a *reduced root system* in E if the following axioms are satisfied:

(1) Φ is finite, spans E and does not contain 0.

(2) If $\lambda \in \Phi$, the reflection s_λ leaves Φ invariant.

(3) If $\lambda_1, \lambda_2 \in \Phi$, then $\langle \lambda_1, \lambda_2 \rangle \in \mathbb{Z}$.

Lemma 5. *If* $\mathfrak{a} \subset \mathfrak{p}$ *is a maximal toral subalgebra, then* $\Phi(\mathfrak{a})$ *is a reduced root system.*

For a proof, see [Hel88].

Notation 2. To avoid confusion between roots in $\Phi(\mathfrak{t})$ and roots in $\Phi(\mathfrak{a})$, we will reserve α for roots in $\Phi(\mathfrak{t})$ and λ for roots in $\Phi(\mathfrak{a})$.

2.3.2 Nice Basis

Let $R(\mathfrak{t}) = \mathbb{Z}_{span}\{\Phi(\mathfrak{t})\}$ be the root lattice of $\Phi(\mathfrak{t})$ and E be the Euclidean space spanned by $\Phi(\mathfrak{t})$, i.e. $E = R(\mathfrak{t}) \otimes_{\mathbb{Z}} \mathbb{R}$. Since $\mathfrak{k}$ and $\mathfrak{p}$ decompose $\mathfrak{g}$ orthogonally, we are looking at the projection of E onto $\mathfrak{a}$.

The next natural question is how are the roots in $\Phi(\mathfrak{t})$ and the roots in $\Phi(\mathfrak{a})$ related?

Lemma 6. *Let $\alpha \in \Phi(\mathfrak{t})$ and $\lambda = \alpha|\mathfrak{a}$. Then $\lambda \in \Phi(\mathfrak{a}) \cup \{0\}$ and $\mathfrak{g}_\alpha \subset \mathfrak{g}_\lambda$.*

Proof. Assume $\lambda = \alpha|\mathfrak{a} \neq 0$ and let $x \in \mathfrak{g}_\alpha$ then for all $h \in \mathfrak{a} \subseteq \mathfrak{t}$ we have $[h, x] = \alpha(h)x = \lambda(h)x$ so $x \in \mathfrak{g}_\lambda$ and $\lambda \in \Phi(\mathfrak{a})$. □

Let $\Delta(\mathfrak{t})$ be a basis for $\Phi(\mathfrak{t})$ and $\Delta(\mathfrak{a})$ be a basis for $\Phi(\mathfrak{a})$. For $\lambda_i \in \Phi(\mathfrak{a})$, we have that $\mathfrak{g}_{\lambda_i} = \mathfrak{g}_{\alpha_{i_1}} \oplus \mathfrak{g}_{\alpha_{i_2}} \oplus \ldots \oplus \mathfrak{g}_{\alpha_{i_r}}$. We need to find all roots $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_r}$ in $\Phi(\mathfrak{t})$ with $\alpha_{i_j}|\mathfrak{a} = \lambda_i$ and all roots that project to zero $\alpha_{i_j}|\mathfrak{a} = 0$. Ideally we would like the natural projection map $\pi : \mathfrak{g}^* \to \mathfrak{p}^*$, defined by $\pi(\alpha) = \frac{1}{2}(\alpha - \theta^*(\alpha))$ to satisfy $\pi(\Delta(\mathfrak{t})) = \Delta(\mathfrak{a})$. This will only be the case with a special choice of basis for $\Phi(\mathfrak{t})$.

Lemma 7. *If $\alpha \in \Phi(\mathfrak{t})$ with $\alpha|\mathfrak{a} = 0$, then $\mathfrak{g}_\alpha \subset \mathfrak{g}_0$.*

Proof. If $\alpha|\mathfrak{a} = 0$ then $[t, x] = 0 \cdot x = 0$ for all $t \in \mathfrak{a}$. By definition this implies $x \in \mathfrak{g}_0$ for all $x \in \mathfrak{g}_\alpha$, or $\mathfrak{g}_\alpha \subset \mathfrak{g}_0$. □

Now denote $\Phi_0 = \{\alpha \in \Phi(\mathfrak{t}) \mid \alpha|\mathfrak{a} = 0\}$.

Lemma 8. *Let $\mathfrak{t} \supseteq \mathfrak{a}$ be a maximal toral subalgebra of $\mathfrak{g}$. Then we have the following:*

(1) *Φ_0 is a closed subset of $\Phi(\mathfrak{t})$, i.e., Φ_0 is a root system.*

(2) $\mathfrak{g}_0 = \mathfrak{t} \oplus \sum_{\alpha \in \Phi_0} \mathfrak{g}_\alpha$

(3) $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{t}_+ \oplus \sum_{\alpha \in \Phi_0} \mathfrak{g}_\alpha$

Proof. Let $\alpha_1, \alpha_2 \in \Phi_0$.

(1) $(\alpha_1 + \alpha_2)|_\mathfrak{a} = \alpha_1|_\mathfrak{a} + \alpha_2|_\mathfrak{a} = 0 + 0 = 0$. Hence $\alpha_1 + \alpha_2 \in \Phi_0$

(2) Let $x \in \mathfrak{g}_0$. Since $x \in \mathfrak{g}$ we have $x = h + \sum_{\alpha \in \Phi(\mathfrak{t})} x_\alpha$. Here $h \in \mathfrak{t}$ and $x_\alpha \in \mathfrak{g}_\alpha$.

Then for all $t \in \mathfrak{a}$ we have

$$0 = [t, x] = [t, h] + \sum_{\alpha \in \Phi(\mathfrak{t})} [t, x_\alpha] = 0 + \sum_{\alpha \in \Phi(\mathfrak{t})} \alpha(t) x_\alpha.$$

This forces either $x_\alpha = 0$ for all α or $\alpha(t) = 0$. If $x_\alpha = 0$, then $x \in \mathfrak{t}$. Otherwise, if $\alpha(t) = 0$ then for all $t \in \mathfrak{a}$ either $\alpha|\mathfrak{a} = 0$ or $\alpha \in \Phi_0$. It follows that $x = h + \sum_{\alpha \in \Phi_0} x_\alpha$.

(3) This follows directly from Lemma 4.

□

Remark 2. In the literature $\mathfrak{t}_+ \oplus \sum_{\alpha\in\Phi_0} \mathfrak{g}_\alpha$ is denoted by $\mathfrak{m}$.

Corollary 1. $\mathfrak{m} \subset \mathfrak{k}$.

Proof. If $\theta|_\mathfrak{m} \neq \mathrm{id}$ then $\mathfrak{m}$ contains a non-trivial toral subalgebra $\mathfrak{a}_1$. But then $\mathfrak{a}_1 + \mathfrak{a} \subset \mathfrak{a}$, contradicting the maximality of $\mathfrak{a}$. □

2.3.3 Root Systems and Weyl Groups

$\theta|\mathfrak{t}$ induces an involution on $\mathfrak{t}^*$ and hence on $\Phi(\mathfrak{t})$. By abuse of notation, we will denote the restricted involution also by θ. Let $X_0(\theta) = \{\chi \in R(\mathfrak{t}) \mid \theta(\chi) = \chi\}$ and $\Phi_0(\theta) = \Phi(\mathfrak{t}) \cap X_0(\theta)$.

Lemma 9. *For $X_0(\theta)$ and $\Phi_0(\theta)$ defined above*

(1) $X_0(\theta)$ and $\Phi_0(\theta)$ are θ-stable.

(2) $\Phi_0(\theta)$ is a closed subsystem of $\Phi(\mathfrak{t})$.

(3) $\Phi_0(\theta) = \Phi_0$.

Proof. Let $\chi \in X_0(\theta)$ and $\alpha_1, \alpha_2 \in \Phi_0(\theta)$

(1) Since $\chi \in X_0(\theta)$ we have $\theta(\chi) = \chi \in X_0$. The same argument holds for $\Phi_0(\theta)$.

(2) Since $\Phi_0(\theta) \subset \Phi(\mathfrak{t})$ we have $\alpha_1+\alpha_2 \in \Phi(\mathfrak{t})$. Applying θ we have $\theta(\alpha_1 + \alpha_2) = \theta(\alpha_1) + \theta(\alpha_2) = \alpha_1 + \alpha_2$. So $\alpha_1 + \alpha_2 \in \Phi(\mathfrak{t}) \cap X_0(\theta) = \Phi_0(\theta)$

(3) Since $\alpha_1 \in \Phi_0(\theta)$ we have $\theta(\alpha_1) = \alpha_1$. So $\pi(\alpha_1) = \frac{1}{2}(\alpha_1 - \theta(\alpha_1)) = 0$ or $\alpha_1|_\mathfrak{a} = 0$. Therefore $\Phi_0(\theta) \subset \Phi_0$. Likewise, $\Phi_0 \subset \Phi_0(\theta)$. So $\Phi_0 = \Phi_0(\theta)$.

□

Definition 3. For a subset $S \subset \Phi(\mathfrak{t})$, we let $W(S)$ denote the subgroup of $W(\Phi(\mathfrak{t}))$ generated by the reflections s_α with $\alpha \in S$.

Identify $W_0(\theta)$ with the subgroup $W(\Phi_0(\theta))$ of $W(\Phi(\mathfrak{t}))$. Let

$$W_1(\theta) = \{w \in W(\Phi(\mathfrak{t})) \mid w(X_0(\theta)) = X_0(\theta)\}.$$

Let $\overline{\Phi} = \pi(\Phi(\mathfrak{t}) - \Phi_0(\theta))$ denote the set of *restricted roots* of $\Phi(\mathfrak{t})$ relative to θ.

Denote $X_\theta = R(\mathfrak{t})/X_0(\theta)$ and define the projection: $\pi : X \to X_\theta$ by $\pi(\alpha) = \frac{1}{2}(\alpha - \theta(\alpha))$. All $w \in W_1(\theta)$ induce a mapping $\pi(w) \in \mathrm{Aut}(X_\theta)$ such that $\pi(w(\chi)) = \pi(w)(\pi(\chi))$. Define $\overline{W} = \{\pi(w) \mid w \in W_1(\theta)\}$.

Notation 3. Let $W(\mathfrak{a})$ denote the Weyl group of the restricted root system $\Phi(\mathfrak{a})$.

Theorem 2.1. *Richardson*

(1) $\overline{\Phi} = \Phi(\mathfrak{a})$.

(2) $W(\mathfrak{a}) = \overline{W} \cong W_1(\theta)/W_0(\theta)$.

For a proof, see [Ric82].

Definition 4. $W(\mathfrak{a})$ is called the *restricted Weyl group* with respect to the action of θ on $R(\mathfrak{t})$.

2.3.4 Theta-Ordering

Define a θ-order on $\Phi(\mathfrak{t})$ related to the projection of E on $\mathfrak{p}$ by

$$\text{if } \chi \in R(\mathfrak{t}), \chi > 0, \text{ and } \chi \notin X_0(\theta), \text{ then } \theta(\chi) < 0.$$

Geometrically this means that $\theta(\chi) > 0$ iff $\chi \perp \mathfrak{a}$.

Remark 3. A basis $\Delta(\mathfrak{t})$ for $\Phi(\mathfrak{t})$ with respect to the θ-order will be called a θ-basis of $\Phi(\mathfrak{t})$.

Theorem 2.2. $\pi(\Delta(\mathfrak{t}) - \Delta_0(\mathfrak{t})) = \Delta(\mathfrak{a})$, *where* $\Delta(\mathfrak{a})$ *is a basis for* $\Phi(\mathfrak{a})$.

Proof. For $\alpha \in \Delta(\mathfrak{t}) - \Delta_0(\theta)$, we have that $\alpha|_{\mathfrak{a}} \neq 0$. Therefore, there exists $\lambda \in \Phi(\mathfrak{a})$ such that $\alpha|_{\mathfrak{a}} = \lambda$. So $\pi(\alpha) = \lambda$. Choose a basis for $\Phi(\mathfrak{a})$ that contains λ. The map π is surjective so $\pi(\Delta(\mathfrak{t}) - \Delta_0(\theta)) = \Delta(\mathfrak{a})$.

□

Definition 5. $\theta^* = -\operatorname{id} \circ \theta \circ w_0(\theta)$

Theorem 2.3. $\theta^* = \begin{cases} \text{id} \\ \textit{Dynkin diagram automorphism of order 2} \end{cases}$

Corollary 2. *The induced involution can be recovered from the* θ*-diagram by*

$$\theta = -\operatorname{id} \circ \theta^* \circ w_0(\theta).$$

For proofs of these results, see [Hel88].

Chapter 3

Advent of the Theta Diagram

In this chapter, we mimic the work of Helminck and discover the nature of the theta diagram. For the example we choose the Lie algebra $\mathfrak{g} = A_7$ and the involution of type II from Helgason [Hel78].

3.1 Realizing $\mathfrak{p}$ and $\mathfrak{k}$

Here we have 8×8 matrices with trace 0 and the involution is $\theta(A) = J(-A)^T J^{-1}$ where

$$J = \begin{pmatrix} 0 & \mathrm{I}_4 \\ -\mathrm{I}_4 & 0 \end{pmatrix}.$$

So for a matrix $A \in \mathfrak{g}$, $\theta(A) = A$ results in the set of matrices

$$\mathfrak{k} = \left\{ \begin{pmatrix} A_1 & A_2 \\ A_3 & -A_1{}^T \end{pmatrix} \mid A_2 = A_2^T \text{ and } A_3 = A_3^T \right\}$$

and $\theta(A) = -A$ results in the set of matrices

$$\mathfrak{p} = \left\{ \begin{pmatrix} A_1 & A_2 \\ A_3 & A_1{}^T \end{pmatrix} \mid \mathrm{trace}(A_1) = 0, A_2 = -A_2^T \text{ and } A_3 = -A_3^T \right\}$$

3.2 Toral Subalgebras

The next step is to find the maximal toral subalgebra $\mathfrak{a} \subset \mathfrak{p}$ and the maximal torus $\mathfrak{t} \subset \mathfrak{g}$ such that $\mathfrak{a} \subseteq \mathfrak{t}$.

In this particular case, $\mathfrak{a}$ consists of the diagonal matrices in $\mathfrak{p}$ and $\mathfrak{t} \supset \mathfrak{a}$ is the set of diagonal matrices of trace zero in $\mathfrak{g}$. To compute the action of the involution θ on $\mathfrak{t}$ and $\mathfrak{t}^*$ it is easier to extend θ to $\mathfrak{gl}_8(k)$. Then the set of diagonal matrices

$$\mathfrak{t}_1 = \langle E_{11}, E_{22}, E_{33}, E_{44}, E_{55}, E_{66}, E_{77}, E_{88} \rangle$$

is a maximal torus of $\mathfrak{gl}_8(k)$ containing $\mathfrak{t}$. As a basis of $\mathfrak{t}_1$ we take the vectors $e_1, \ldots, e_8$ with $e_1 = E_{11}$, $e_2 = E_{22}$, $e_3 = E_{33}$, $e_4 = E_{44}$, $e_5 = E_{55}$, $e_6 = E_{66}$, $e_7 = E_{77}$ and $e_8 = E_{88}$. Then $\mathfrak{t}$ is spanned by $\{e_i - e_j \mid i, j = 1, \ldots 8 \text{ and } i \neq j\}$ and

$$\Phi(\mathfrak{t}) = \{\epsilon_i - \epsilon_j \mid i, j = 1, \ldots 8 \text{ and } i \neq j\}$$

similar as the description of the root system of type A as in [Bou81].

The involution θ acts on $\mathfrak{t}_1$ and $\mathfrak{t}$ as follows: $\theta(E_{11}) = -E_{55}$, $\theta(E_{22}) = -E_{66}$, $\theta(E_{33}) = -E_{77}$ and $\theta(E_{44}) = -E_{88}$. So $\theta(e_1) = -e_5$, $\theta(e_2) = -e_6$, $\theta(e_3) = -e_7$ and $\theta(e_4) = -e_8$. The matrix for the action on θ on this basis for $\mathfrak{t}_1$ is:

$$[\theta(\mathfrak{t}_1)] = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since we are working on the root space level, we only need to find the induced map of $\theta(A) = J(-A)^T J^{-1}$ in the dual space. As I stated in Chapter 2, we abuse notation and denote this map θ as well. From [HK61], we know the matrix of θ on the dual space is the transpose of $[\theta(\mathfrak{t}_1)]$. So in particular $\theta(\epsilon_1) = -\epsilon_5$, $\theta(\epsilon_2) = -\epsilon_6$, $\theta(\epsilon_3) = -\epsilon_7$ and $\theta(\epsilon_4) = -\epsilon_8$.

When we apply θ on the roots in $\Phi(\mathfrak{t})$ it follows that

$$\Phi_0(\theta) = \{\pm(\epsilon_1 - \epsilon_5), \pm(\epsilon_2 - \epsilon_6), \pm(\epsilon_3 - \epsilon_7), \pm(\epsilon_4 - \epsilon_8)\}.$$

As a basis we take $\Delta_0(\theta) = \{\epsilon_1 - \epsilon_5, -\epsilon_2 + \epsilon_6, \epsilon_3 - \epsilon_7, -\epsilon_4 + \epsilon_8\}$. If $\alpha = \epsilon_i - \epsilon_j \notin \Phi_0(\theta)$, then $\pi(\alpha) = \frac{1}{2}(\alpha - \theta(\alpha)) = \frac{1}{2}(\epsilon_i - \epsilon_j - \theta(\epsilon_i) + \theta(\epsilon_j))$ involves 4 of the basis vectors of $\mathfrak{t}_1$.

We choose a basis for $\mathfrak{a} \subset \mathfrak{p}$ as:

$$\begin{aligned}\mathfrak{a} =& \langle -E_{11} + E_{22} + E_{55} - E_{66}, E_{22} - E_{33} - E_{66} + E_{77}, -E_{33} + E_{44} + E_{77} - E_{88}\rangle \\ =& \langle -e_1 + e_2 + e_5 - e_6, e_2 - e_3 - e_6 + e_7, -e_3 + e_4 + e_7 - e_8\rangle.\end{aligned}$$

The corresponding basis of $\Phi(\mathfrak{a})$ is:

$$\Delta(\mathfrak{a}) = \{\frac{1}{2}(-\epsilon_1 + \epsilon_2 + \epsilon_5 - \epsilon_6), \frac{1}{2}(\epsilon_2 - \epsilon_3 - \epsilon_6 + \epsilon_7), \frac{1}{2}(-\epsilon_3 + \epsilon_4 + \epsilon_7 - \epsilon_8)\}.$$

The basis for the maximal torus $\mathfrak{t}$ that projects down to this basis of $\mathfrak{a}$ is

$$\begin{aligned}\mathfrak{t} =& \langle E_{11} - E_{55}, -E_{11} + E_{22}, -E_{22} + E_{66}, -E_{66} + E_{77}, E_{33} - E_{77}, -E_{33} + E_{44}, -E_{44} + E_{88}\rangle \\ =& \langle e_1 - e_5, -e_1 + e_2, -e_2 + e_6, -e_6 + e_7, e_3 - e_7, -e_3 + e_4, -e_4 + e_8\rangle.\end{aligned}$$

Here the associated roots in the basis of $\Phi(\mathfrak{t})$ are

$$\Delta(\mathfrak{t}) = \{\epsilon_1 - \epsilon_5, -\epsilon_1 + \epsilon_2, -\epsilon_2 + \epsilon_6, -\epsilon_6 + \epsilon_7, \epsilon_3 - \epsilon_7, -\epsilon_3 + \epsilon_4, -\epsilon_4 + \epsilon_8\}.$$

We will denote these roots by:

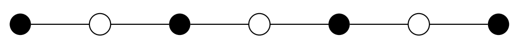
$$\Delta(\mathfrak{t}) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}.$$

This basis is a θ-basis for $\Phi(\mathfrak{t})$, since it was induced from bases on $\Phi_0(\theta)$ and $\Phi(\mathfrak{a})$.

When we let the map $\theta \in \mathrm{Aut}(\mathfrak{t}_1^*)$ act on the roots in $\Delta(\mathfrak{t})$ we get.

$$\begin{aligned}
\theta(\alpha_1) &= \alpha_1 \\
\theta(\alpha_2) &= -\alpha_1 - \alpha_2 - \alpha_3 \\
\theta(\alpha_3) &= \alpha_3 \\
\theta(\alpha_4) &= -\alpha_3 - \alpha_4 - \alpha_5 \\
\theta(\alpha_5) &= \alpha_5 \\
\theta(\alpha_6) &= -\alpha_5 - \alpha_6 - \alpha_7 \\
\theta(\alpha_7) &= \alpha_7.
\end{aligned}$$

From this information, we have the resulting θ-diagram:



Moreover using the involution and the action of the natural projection map, $\pi(\alpha) = \frac{1}{2}(\alpha - \theta(\alpha))$ the corresponding basis of $\Phi(\mathfrak{a})$ is the set of roots

$$\begin{aligned}
\lambda_1 &= \frac{1}{2}(\alpha_1 + 2\alpha_2 + \alpha_3) \\
\lambda_2 &= \frac{1}{2}(\alpha_3 + 2\alpha_4 + \alpha_5) \\
\lambda_3 &= \frac{1}{2}(\alpha_5 + 2\alpha_6 + \alpha_7).
\end{aligned}$$

Remark 4. Depending on the Lie algebra $\mathfrak{g}$ and the type of involution θ, we have cases where $\mathfrak{a}$ and $\mathfrak{t}$ consist of non-diagonal matrices.

Chapter 4

Classification of Local Symmetric Spaces

The work of Helminck [Hel88] classifies involutions over reductive algebraic groups and their associated Lie algebras. This classification allots for a finite number of symmetric and local symmetric spaces up to isomorphism.

Helminck borrows the Dynkin diagram from finite dimensional semisimple Lie theory. Here he is able to represent the action of the involution on $\Phi(\mathfrak{t})$ by using a θ-basis $\Delta(\mathfrak{t})$ and adding some additional information.

Recall from definition 5 that we defined $\theta^* = -\operatorname{id} \circ \theta \circ w_0(\theta)$. We had the result that

$$\theta^* = \begin{cases} \text{id} \\ \text{Dynkin diagram automorphism of order 2.} \end{cases}$$

Helminck represents θ^* clearly on the diagrams by using arrows to represent the Dynkin diagram automorphism.

The following tables indicate that type of involution and Lie algebra that each case resulted from, the complete θ-diagram, and the resulting base of the restricted root system $\Phi(\mathfrak{a})$.

Table 4.1: θ-diagrams

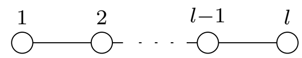
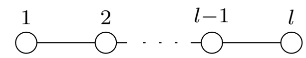
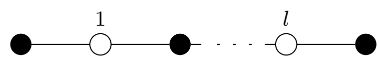
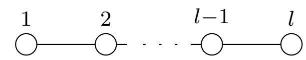
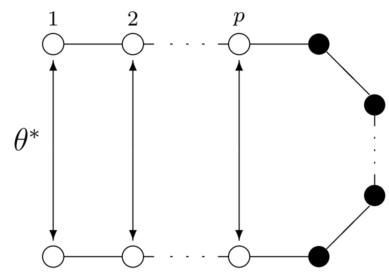
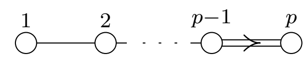
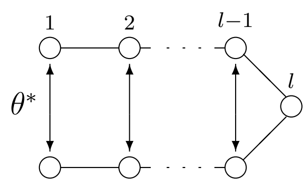
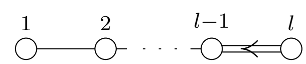
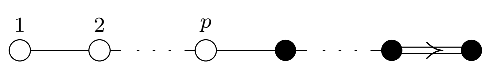
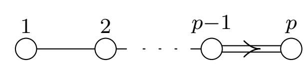
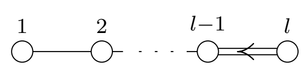
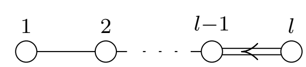
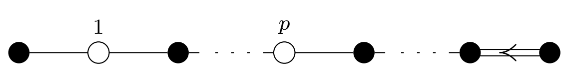
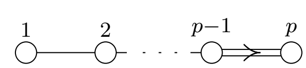
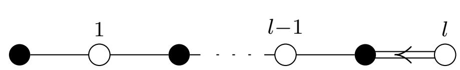
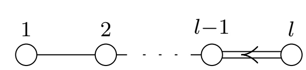
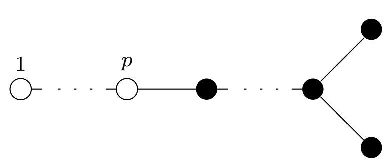
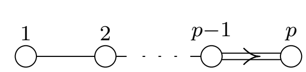
Type θ	θ-diagram	$\Delta(\mathfrak{a})$
AI		
AII		
AIII$_a$ (AIV ($p=1$)) ($1 \leq 2p \leq l$)		
AIII$_b$ ($l \geq 2$)		
BI (BII ($p=1$)) ($l \geq 2,\ 1 \leq p \leq l$)		
CI		
CII$_a$ ($l \geq 3$) ($1 \leq p \leq \frac{1}{2}(l-1)$)		
CII$_b$ ($l \geq 2$)		
DI$_a$ (DII ($p=1$)) ($l \geq 4,\ 1 \leq p \leq l-1$)		

Table 4.2: θ-diagrams (continued)

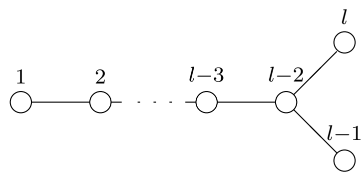
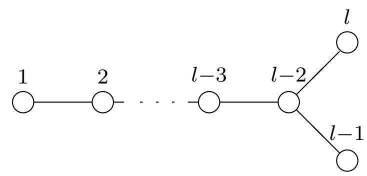
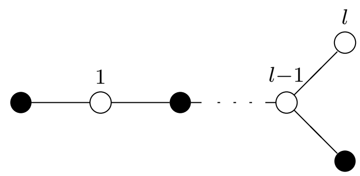
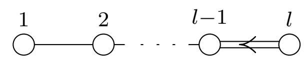
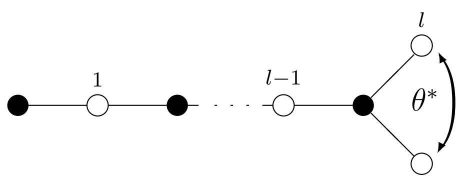
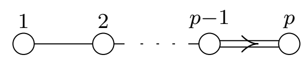
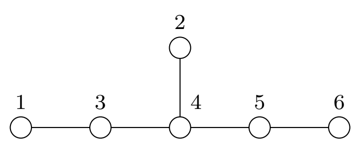
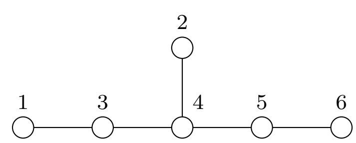
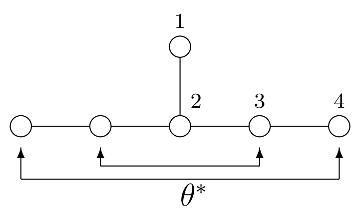
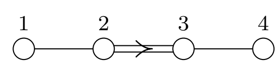
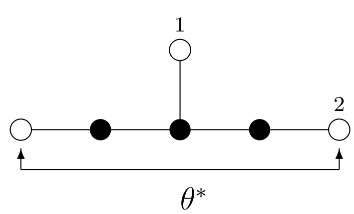
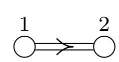
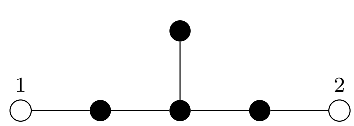
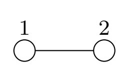
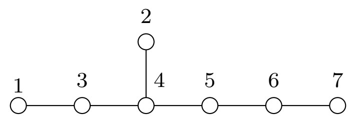
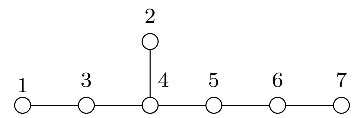
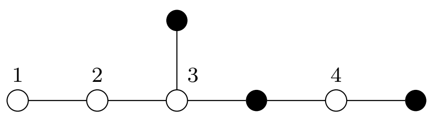
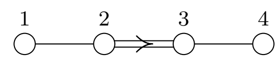
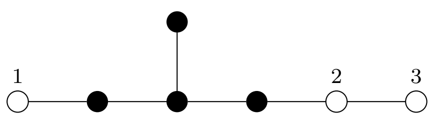
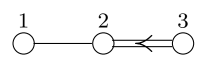
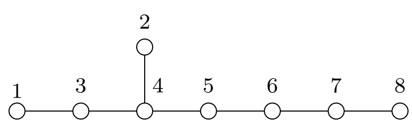
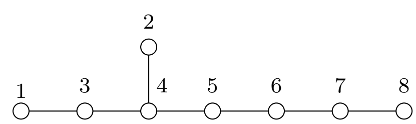
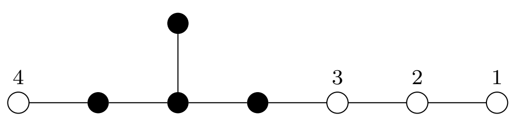
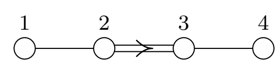
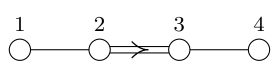
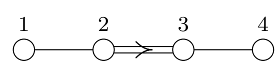
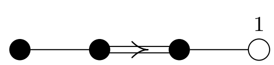
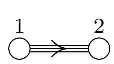
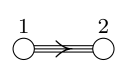
Type θ	θ-diagram	$\Delta(\mathfrak{a})$
$D\mathrm{I}_b$ ($l \geq 4$)		
$D\mathrm{III}_a$ ($l \geq 2$)		
$D\mathrm{III}_b$ ($l \geq 2$)		
$E\mathrm{I}$		
$E\mathrm{II}$		
$E\mathrm{III}$		
$E\mathrm{IV}$		

Table 4.3: θ-diagrams (continued)

Type θ	θ-diagram	$\Delta(\mathfrak{a})$
EV		
EVI		
EVII		
EVIII		
EIX		
FI		
FII		
G		

Chapter 5

The Algorithm

After recovering the action of θ on $\Phi(\mathfrak{t})$ and explicitly determining each λ, there is a beautiful six step algorithm that I can follow to determine a "nice" basis for the local symmetric space $\mathfrak{p}$.

5.1 Step 1

Definition 6. For all λ in $\Phi(\mathfrak{a})$, we define

$$\Phi(\lambda) = \{\alpha \in \Phi(\mathfrak{t}) \mid \alpha|_{\mathfrak{a}} = \lambda\}.$$

The first step in our algorithm is to compute $\Phi(\lambda)$ for all λ in $\Delta(\mathfrak{a})$. Essentially we are computing all α_i that project down to each λ or just the multiplicity of λ.

5.2 Step 2

Recall from lemma 2.1 that

$$W(\mathfrak{a}) \simeq W_1(\theta)/W_0(\theta).$$

It becomes advantageous to compute representatives w_i in $W_1(\theta)$ for each s_{λ_i} in $W(\mathfrak{a})$.

5.3 Step 3

We can compute the list of all positive roots in $\Phi(\mathfrak{a})$ by using the weyl group representatives w_i applied to all λ in $\Delta(\mathfrak{a})$.

5.4 Step 4

The next step is to compute the multiplicity of all the roots in $\Phi(\mathfrak{a})$. A nice way to do this is to note that

$$\Phi(s_{\lambda_i}(\lambda_j)) = \Phi(\lambda_j - \langle \alpha_j, \alpha_i \rangle \lambda_i) = w_i(\Phi(\lambda_j)).$$

Essentially, we use the same weyl group representatives that we used above in Step 3 to compute the list $\Phi(\mathfrak{a})^+$, but this time we apply them set-wise to the $\Phi(\lambda_i)$ from Step 1.

5.5 Step 5

We have that $\mathfrak{g}_{\lambda_i} = \mathfrak{g}_{\alpha_{i_1}} \oplus \mathfrak{g}_{\alpha_{i_2}} \oplus \ldots \oplus \mathfrak{g}_{\alpha_{i_r}}$. All we need to do to compute each $\mathfrak{g}_{\lambda_i}$ is to note that each element in $\Phi(\lambda_i)$ will be in the direct sum decomposition.

5.6 Step 6

Since $x_\alpha \in \mathfrak{g}_\alpha$ implies $x_\alpha + \theta(x_\alpha) \in \mathfrak{k}$ and $x_\alpha - \theta(x_\alpha) \in \mathfrak{p}$,
a basis for $\mathfrak{k} = \{x^i_\alpha + \theta(x^i_\alpha)\}_{\alpha \in \Phi(\mathfrak{a})^+} + \{x^i_\alpha\}_{\alpha \in \Phi_0(\theta)}$ + a basis for $\mathfrak{t}_0$ and
a basis for $\mathfrak{p} = \{x^i_\alpha - \theta(x^i_\alpha)\}_{\alpha \in \Phi(\mathfrak{a})^+}$ + a basis for $\mathfrak{a}$.

Chapter 6

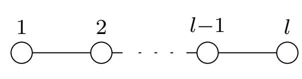
Techniques used for Computing the Bases and Weyl Groups

Before the restricted root system, $\Phi(\mathfrak{a})$, can be realized there are still some preliminary steps that must be taken. First, the longest Weyl group element $w_0(\theta)$ must be computed and consequently the action of the involution recovered. Then using the involution and the action of the natural projection map, $\pi(\alpha) = \frac{1}{2}(\alpha - \theta(\alpha))$, the roots in the basis, $\Delta(\mathfrak{a})$, can explicitly be determined.

6.1 A cases

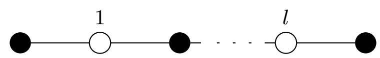
There are 4 isomorphy classes of local symmetric spaces that result from a Lie algebra of type A.

6.1.1 Type AI



When the involution is of type I, nothing is fixed by θ. Therefore, $w_0(\theta) = \mathrm{id}$ and $\theta = -\mathrm{id}$. So for all i, $\pi(\alpha_i) = \alpha_i$, i.e., every root projects down to itself and $\Delta(\mathfrak{t}) = \Delta(\mathfrak{a})$.

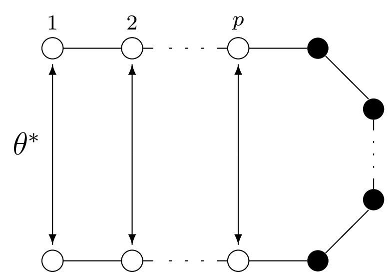
6.1.2 Type AII



The original Lie algebra is $\mathfrak{g} = A_{2l+1}$. When the involution is of type II, α_{2i-1} is fixed for $i = 1, \cdots, l+1$. Therefore, $w_0(\theta) = s_{\alpha_1} s_{\alpha_3} ... s_{\alpha_{2l+1}}$ and $\theta = -s_{\alpha_1} s_{\alpha_3} ... s_{\alpha_{2l+1}}$. So the roots α_{2i-1} project down to zero and the roots α_{2i} project down to

$$\lambda_i = \frac{1}{2}(\alpha_{2i-1} + 2\alpha_{2i} + \alpha_{2i+1}).$$

6.1.3 Type AIII$_a$



Here we start with the Lie algebra $\mathfrak{g} = A_l$. Notice that $l = 2p+m$ where $p = |\Delta(\mathfrak{a})|$ and $m = |\Delta_0(\theta)|$. The root α_{p+i} is fixed for $i = 1, \cdots, m$. Therefore,

$$w_0(\theta) = s_{\alpha_{p+1}} s_{\alpha_{p+2}} s_{\alpha_{p+1}} s_{\alpha_{p+3}} s_{\alpha_{p+2}} s_{\alpha_{p+1}} \cdots s_{\alpha_{p+m}} s_{\alpha_{p+m-1}} \cdots s_{\alpha_{p+1}}$$

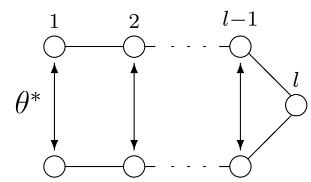
and $\theta = -w_0(\theta)\theta^*$.

For λ_i in $\Delta(\mathfrak{a})$ with $i = 1, \cdots, p-1$, notice that there are two base roots that project down to λ_i. Basically we find that

$$\pi(\alpha_i) = \pi(\alpha_{l+1-i}) = \lambda_i \text{ where } \lambda_i = \frac{1}{2}(\alpha_i + \alpha_{l+1-i}) \text{ for } i = 1, \cdots, p-1 \text{ and}$$

$$\lambda_p = \frac{1}{2}(\alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m} + \alpha_{p+m+1}).$$

6.1.4 Type AIII$_b$



Here we start with the Lie algebra $\mathfrak{g} = A_{2l-1}$ where $l = |\Delta(\mathfrak{a})|$. Since $\Delta_0(\theta) = \emptyset$, $w_0(\theta) = \text{id}$ and $\theta = -\theta^*$.

For λ_i in $\Delta(\mathfrak{a})$ with $i = 1, \cdots, l-1$, notice that there are two base roots that project down to λ_i. Basically we find that

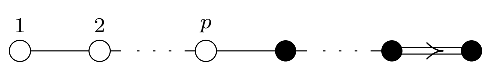
$$\pi(\alpha_i) = \pi(\alpha_{2l-i}) = \lambda_i \text{ where } \lambda_i = \frac{1}{2}(\alpha_i + \alpha_{2l-i}) \text{ for } i = 1, \cdots, l-1 \text{ and}$$

$$\lambda_l = \alpha_l.$$

6.2 B case

There is only 1 isomorphy class of local symmetric space that results from a Lie algebra of type B.

6.2.1 Type BI



Here we start with the Lie algebra $\mathfrak{g} = B_l$. Notice that $l = p+m$ where $p = |\Delta(\mathfrak{a})|$ and $m = |\Delta_0(\theta)|$. The root α_{p+i} is fixed for $i = 1, \cdots, m$. Therefore,

$$w_0(\theta) = s_{\alpha_{p+1}} s_{\alpha_{p+2}} s_{\alpha_{p+3}} \cdots s_{\alpha_{p+m-1}} s_{\alpha_{p+m}} s_{\alpha_{p+m-1}} \cdots s_{\alpha_{p+3}} s_{\alpha_{p+2}} s_{\alpha_{p+1}} s_{\alpha_{p+2}} s_{\alpha_{p+3}} \cdots$$
$$\cdots s_{\alpha_{p+m}} \cdots s_{\alpha_{p+3}} s_{\alpha_{p+2}} \cdots s_{\alpha_{p+m-2}} s_{\alpha_{p+m-1}} s_{\alpha_{p+m}}$$
$$s_{\alpha_{p+m-1}} s_{\alpha_{p+m-2}} s_{\alpha_{p+m-1}} s_{\alpha_{p+m}} s_{\alpha_{p+m-1}} s_{\alpha_{p+m}}$$

and $\theta = -w_0(\theta)$. So for $i = 1, \cdots, m$, the roots α_{p+i} project down to zero .

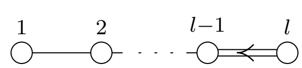
We find that

$$\pi(\alpha_i) = \lambda_i \text{ where } \lambda_i = \alpha_i \text{ for } i = 1, \cdots, p-1 \text{ and}$$
$$\lambda_p = \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m-1} + \alpha_{p+m}.$$

6.3 C cases

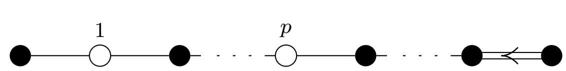
There are 3 isomorphy classes of local symmetric spaces that result from a Lie algebra of type C.

6.3.1 Type CI



When the involution is of type I, nothing is fixed by θ. Therefore, $w_0(\theta) = \mathrm{id}$ and $\theta = -\mathrm{id}$. So for all i, $\pi(\alpha_i) = \alpha_i$, i.e., every root projects down to itself and $\Delta(\mathfrak{t}) = \Delta(\mathfrak{a})$.

6.3.2 Type CII$_a$



Here we start with the Lie algebra $\mathfrak{g} = C_l$. Notice that $l = 2p+m$ where $p = |\Delta(\mathfrak{a})|$ and $|\Delta_0(\theta)| = m + p$. The root α_{2i-1} is fixed by θ for $i = 1, \cdots, p$ and α_{2p+i} is fixed for $i = 1, \cdots, m$. Therefore,

$$\begin{aligned} w_0(\theta) =& s_{\alpha_1} s_{\alpha_3} \cdots s_{\alpha_{2p-1}} s_{\alpha_{2p+1}} s_{\alpha_{2p+2}} s_{\alpha_{2p+3}} \cdots s_{\alpha_{2p+m-1}} s_{\alpha_{2p+m}} s_{\alpha_{2p+m-1}} \cdots s_{\alpha_{2p+3}} \\ & s_{\alpha_{2p+2}} s_{\alpha_{2p+1}} s_{\alpha_{2p+2}} s_{\alpha_{2p+3}} \cdots s_{\alpha_{2p+m}} \cdots s_{\alpha_{2p+3}} s_{\alpha_{2p+2}} \cdots s_{\alpha_{2p+m-2}} s_{\alpha_{2p+m-1}} s_{\alpha_{2p+m}} \\ & s_{\alpha_{2p+m-1}} s_{\alpha_{2p+m-2}} s_{\alpha_{2p+m}} s_{\alpha_{2p+m-1}} s_{\alpha_{2p+m}} \end{aligned}$$

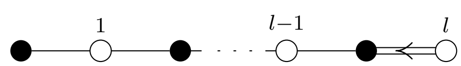
and $\theta = -w_0(\theta)$.

Basically we find that

$$\lambda_i = \pi(\alpha_{2i}) = \frac{1}{2}(\alpha_{2i-1} + 2\alpha_{2i} + \alpha_{2i+1}) \text{ for } i = 1, \cdots, p-1 \text{ and}$$

$$\lambda_p = \pi(\alpha_{2p}) = \frac{1}{2}(\alpha_{2p-1} + 2\alpha_{2p} + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}).$$

6.3.3 Type CII$_b$



Here we start with the Lie algebra $\mathfrak{g} = C_{2l}$. Notice that $|\Delta(\mathfrak{a})| = l$ and $|\Delta_0(\theta)| = l$. $w_0(\theta) = s_{\alpha_1} s_{\alpha_3} ... s_{\alpha_{2l-1}}$ and $\theta = -s_{\alpha_1} s_{\alpha_3} ... s_{\alpha_{2l-1}}$.

So

$$\pi(\alpha_{2i-1}) = 0 \text{ for } i = 1, \cdots, l,$$

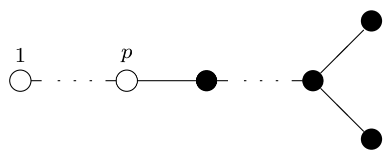
$$\pi(\alpha_{2i}) = \frac{1}{2}(\alpha_{2i-1} + 2\alpha_{2i} + \alpha_{2i+1}) = \lambda_i \text{ for } i = 1, \cdots, l-1, \text{ and}$$

$$\pi(\alpha_{2l}) = \alpha_{2l-1} + \alpha_{2l} = \lambda_l.$$

6.4 D cases

There are 4 isomorphy classes of local symmetric spaces that result from a Lie algebra of type D.

6.4.1 Type DI$_a$



Here we start with the Lie algebra $\mathfrak{g} = D_l$. Notice that $l = p+m$ where $p = |\Delta(\mathfrak{a})|$ and $|\Delta_0(\theta)| = m$.

Here

$$\begin{aligned} w_0(\theta) =& s_{\alpha_{p+1}} s_{\alpha_{p+2}} s_{\alpha_{p+1}} s_{\alpha_{p+3}} s_{\alpha_{p+2}} s_{\alpha_{p+1}} s_{\alpha_{p+4}} s_{\alpha_{p+3}} s_{\alpha_{p+2}} s_{\alpha_{p+1}} \cdots s_{\alpha_{p+m-1}} \\ & s_{\alpha_{p+m-2}} \cdots s_{\alpha_{p+1}} s_{\alpha_{p+m}} s_{\alpha_{p+m-2}} \cdots s_{\alpha_{p+1}} s_{\alpha_{p+m-1}} \cdots s_{\alpha_{p+2}} s_{\alpha_{p+m}} s_{\alpha_{p+m-2}} \\ & \cdots s_{\alpha_{p+3}} s_{\alpha_{p+m-1}} \cdots s_{\alpha_{p+4}} s_{\alpha_{p+m}} s_{\alpha_{p+m-2}} \cdots s_{\alpha_{p+5}} \cdots \end{aligned}$$

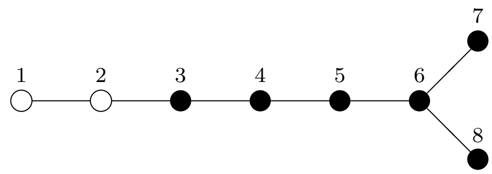
and $\theta = -w_0(\theta)$.

We find that $\lambda_i = \alpha_i$ for $i = 1, \cdots, p-1$ and

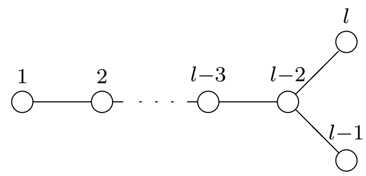
$$\lambda_p = \frac{1}{2}(2\alpha_p + 2\alpha_{p+1} + \cdots + 2\alpha_{p+m-2} + \alpha_{p+m-1} + \alpha_{p+m}).$$

Example 6.1. For the local symmetric space $\mathrm{D}_8^2(\mathrm{I}_a)$

$$\begin{aligned} w_0(\theta) = & s_{\alpha_3} s_{\alpha_4} s_{\alpha_3} s_{\alpha_5} s_{\alpha_4} s_{\alpha_3} s_{\alpha_6} s_{\alpha_5} s_{\alpha_4} s_{\alpha_3} s_{\alpha_7} s_{\alpha_6} s_{\alpha_5} s_{\alpha_4} s_{\alpha_3} \\ & s_{\alpha_8} s_{\alpha_6} s_{\alpha_5} s_{\alpha_4} s_{\alpha_3} s_{\alpha_7} s_{\alpha_6} s_{\alpha_5} s_{\alpha_4} s_{\alpha_8} s_{\alpha_6} s_{\alpha_5} s_{\alpha_7} s_{\alpha_6} s_{\alpha_8}. \end{aligned}$$

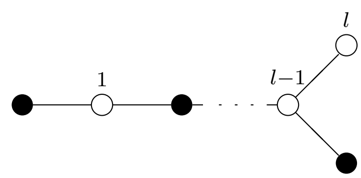


6.4.2 Type DI$_b$



Here nothing is fixed by θ. Therefore, $w_0(\theta) = \mathrm{id}$ and $\theta = -\mathrm{id}$. So for all i, $\lambda_i = \pi(\alpha_i) = \alpha_i$, i.e., every root projects down to itself and $\Delta(\mathfrak{t}) = \Delta(\mathfrak{a})$.

6.4.3 Type DIII$_a$



Here we start with the Lie algebra $\mathfrak{g} = D_{2l}$. Notice that $|\Delta(\mathfrak{a})| = l$ and $|\Delta_0(\theta)| = l$. $w_0(\theta) = s_{\alpha_1} s_{\alpha_3} ... s_{\alpha_{2l-1}}$ and $\theta = -s_{\alpha_1} s_{\alpha_3} ... s_{\alpha_{2l-1}}$.

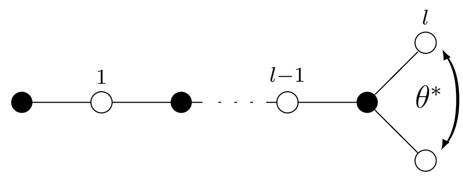
So

$$\pi(\alpha_{2i-1}) = 0 \text{ for } i = 1, \cdots, l,$$

$$\pi(\alpha_{2i}) = \frac{1}{2}(\alpha_{2i-1} + 2\alpha_{2i} + \alpha_{2i+1}) = \lambda_i \text{ for } i = 1, \cdots, l-1, \text{ and}$$

$$\pi(\alpha_{2l}) = \alpha_{2l} = \lambda_l.$$

6.4.4 Type DIII$_b$



Here we start with the Lie algebra $\mathfrak{g} = D_{2l+1}$. Notice that $|\Delta(\mathfrak{a})| = l$ and $|\Delta_0(\theta)| = l$. $w_0(\theta) = s_{\alpha_1}s_{\alpha_3}...s_{\alpha_{2l-1}}$ and $\theta = -s_{\alpha_1}s_{\alpha_3}...s_{\alpha_{2l-1}}$.

Basically we find that

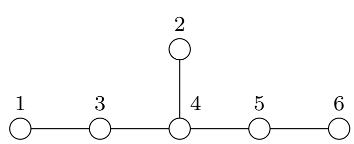
$$\pi(\alpha_{2i}) = \frac{1}{2}(\alpha_{2i-1} + 2\alpha_{2i} + \alpha_{2i+1}) = \lambda_i \text{ for } i = 1, \cdots, l-1 \text{ and}$$

$$\pi(\alpha_{2l}) = \pi(\alpha_{2l+1}) = \frac{1}{2}(\alpha_{2l-1} + \alpha_{2l} + \alpha_{2l+1}) = \lambda_l.$$

6.5 E$_6$ cases

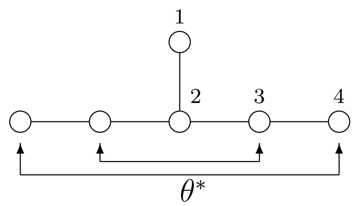
There are 4 isomorphy classes of local symmetric spaces that result from a Lie algebra of type E$_6$.

6.5.1 Type EI



Here nothing is fixed by θ. Therefore, $w_0(\theta) = \text{id}$ and $\theta = -\text{id}$. So for all i, $\pi(\alpha_i) = \alpha_i$, i.e., every root projects down to itself and $\Delta(\mathfrak{t}) = \Delta(\mathfrak{a})$.

6.5.2 Type EII



Here nothing is fixed by θ. Therefore, $w_0(\theta) = \mathrm{id}$ and $\theta = -\theta^*$.

We find that

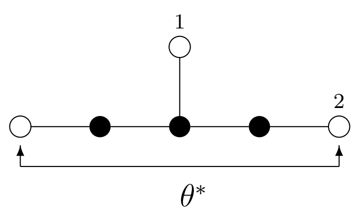
$$\pi(\alpha_2) = \alpha_2 = \lambda_1,$$

$$\pi(\alpha_4) = \alpha_4 = \lambda_2,$$

$$\pi(\alpha_3) = \pi(\alpha_5) = \frac{1}{2}(\alpha_3 + \alpha_5) = \lambda_3, \text{ and}$$

$$\pi(\alpha_1) = \pi(\alpha_6) = \frac{1}{2}(\alpha_1 + \alpha_6) = \lambda_4.$$

6.5.3 Type EIII



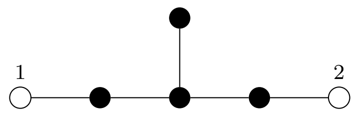
Here $\Delta_0(\theta) = \{\alpha_3, \alpha_4, \alpha_5\}$ and $w_0(\theta) = s_{\alpha_3} s_{\alpha_4} s_{\alpha_5} s_{\alpha_4} s_{\alpha_3} s_{\alpha_4}$. Also in this case θ^* is nontrivial.

We find that

$$\pi(\alpha_2) = \frac{1}{2}(2\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5) = \lambda_1, \text{ and}$$

$$\pi(\alpha_1) = \pi(\alpha_6) = \frac{1}{2}(\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6) = \lambda_2.$$

6.5.4 Type EIV



Here $\Delta_0(\theta) = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ and $w_0(\theta) = s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}$. So $\theta = -w_0(\theta)$.

We have

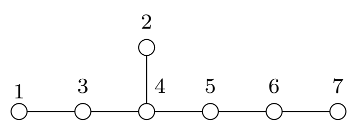
$$\pi(\alpha_1) = \frac{1}{2}(2\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5) = \lambda_1, \text{ and}$$

$$\pi(\alpha_6) = \frac{1}{2}(\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6) = \lambda_2.$$

6.6 E_7 cases

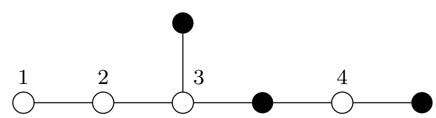
There are 3 isomorphy classes of local symmetric spaces that result from a Lie algebra of type E_7.

6.6.1 Type EV



Here nothing is fixed by θ. Therefore, $w_0(\theta) = \text{id}$ and $\theta = -\text{id}$. So for all i, $\pi(\alpha_i) = \alpha_i$, i.e., every root projects down to itself and $\Delta(\mathfrak{t}) = \Delta(\mathfrak{a})$.

6.6.2 Type EVI



Here $\Delta_0(\theta) = \{\alpha_2, \alpha_5, \alpha_7\}$ and $w_0(\theta) = s_{\alpha_2}s_{\alpha_5}s_{\alpha_7}$. So $\theta = -w_0(\theta)$.

We find that

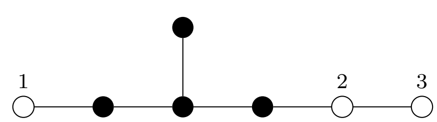
$$\pi(\alpha_1) = \alpha_1 = \lambda_1,$$

$$\pi(\alpha_3) = \alpha_3 = \lambda_2,$$

$$\pi(\alpha_4) = \frac{1}{2}(\alpha_2 + 2\alpha_4 + \alpha_5) = \lambda_3, \text{ and}$$

$$\pi(\alpha_6) = \frac{1}{2}(\alpha_5 + 2\alpha_6 + \alpha_7) = \lambda_4.$$

6.6.3 Type EVII



Here $\Delta_0(\theta) = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ and $w_0(\theta) = s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}$. So $\theta = -w_0(\theta)$.

We have

$$\pi(\alpha_1) = \frac{1}{2}(2\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5) = \lambda_1,$$

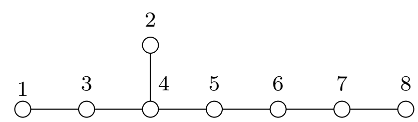
$$\pi(\alpha_6) = \frac{1}{2}(\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6) = \lambda_2, \text{ and}$$

$$\pi(\alpha_7) = \alpha_7 = \lambda_3.$$

6.7 E_8 cases

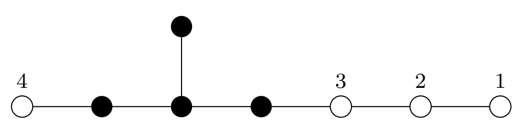
There are 2 isomorphy classes of local symmetric spaces that result from a Lie algebra of type E_8.

6.7.1 Type EVIII



Here nothing is fixed by θ. Therefore, $w_0(\theta) = \mathrm{id}$ and $\theta = -\mathrm{id}$. So for all i, $\pi(\alpha_i) = \alpha_i$, i.e., every root projects down to itself and $\Delta(\mathfrak{t}) = \Delta(\mathfrak{a})$.

6.7.2 Type EIX



Here $\Delta_0(\theta)^+ = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$.We find that

$$w_0(\theta) = s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}.$$

We have

$$\pi(\alpha_8) = \alpha_8 = \lambda_1,$$

$$\pi(\alpha_7) = \alpha_7 = \lambda_2,$$

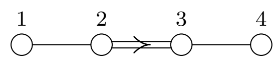
$$\pi(\alpha_6) = \frac{1}{2}(\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6) = \lambda_3, \text{ and}$$

$$\pi(\alpha_1) = \frac{1}{2}(2\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5) = \lambda_4.$$

6.8 F_4 cases

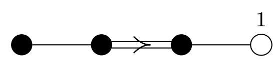
There are 2 isomorphy classes of local symmetric spaces that result from a Lie algebra of type F_4.

6.8.1 Type FI



Here nothing is fixed by θ. Therefore, $w_0(\theta) = \mathrm{id}$ and $\theta = -\mathrm{id}$. So for all i, $\pi(\alpha_i) = \alpha_i$, i.e., every root projects down to itself and $\Delta(\mathfrak{t}) = \Delta(\mathfrak{a})$.

6.8.2 Type FII



Here we start with the Lie algebra $\mathfrak{g} = F_4$. Notice that $\Delta(\mathfrak{a})$ is of type A_1 and $\Delta_0(\theta)^+ = \{\alpha_1, \alpha_2, \alpha_3\}$. $w_0(\theta) = s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}s_{\alpha_3}s_{\alpha_2}s_{\alpha_1}s_{\alpha_3}s_{\alpha_2}s_{\alpha_3}$.

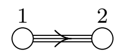
Basically we find that

$$\pi(\alpha_4) = \frac{1}{2}(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4) = \lambda_1.$$

6.9 G_2 case

There is only 1 isomorphy class of local symmetric space that results from a Lie algebra of type G_2.

6.9.1 Type G



Here nothing is fixed by θ. Therefore, $w_0(\theta) = \mathrm{id}$ and $\theta = -\mathrm{id}$. So for all i, $\pi(\alpha_i) = \alpha_i$, i.e., every root projects down to itself and $\Delta(\mathfrak{t}) = \Delta(\mathfrak{a})$.

Table 6.1: $w_0(\theta)$

Type θ	$w_0(\theta)$
AI	id
AII	$s_{\alpha_1}s_{\alpha_3}...s_{\alpha_2 l+1}$
AIII$_a$	$s_{\alpha_{p+1}}s_{\alpha_{p+2}}s_{\alpha_{p+1}}s_{\alpha_{p+3}}s_{\alpha_{p+2}}s_{\alpha_{p+1}} \cdots s_{\alpha_{p+m}}s_{\alpha_{p+m-1}} \cdots s_{\alpha_{p+1}}$
AIII$_b$	id
BI	$s_{\alpha_{p+1}}s_{\alpha_{p+2}}s_{\alpha_{p+3}} \cdots s_{\alpha_{p+m-1}}s_{\alpha_{p+m}}s_{\alpha_{p+m-1}} \cdots s_{\alpha_{p+3}}s_{\alpha_{p+2}}s_{\alpha_{p+1}}s_{\alpha_{p+2}}s_{\alpha_{p+3}} \cdots s_{\alpha_{p+m}} \cdots s_{\alpha_{p+3}}s_{\alpha_{p+2}} \cdots s_{\alpha_{p+m-2}}s_{\alpha_{p+m-1}}s_{\alpha_{p+m}} s_{\alpha_{p+m-1}}s_{\alpha_{p+m-2}}s_{\alpha_{p+m-1}}s_{\alpha_{p+m}}s_{\alpha_{p+m-1}}s_{\alpha_{p+m}}$
CI	id
CII$_a$	$s_{\alpha_1}s_{\alpha_3} \cdots s_{\alpha_{2p-1}}s_{\alpha_{2p+1}}s_{\alpha_{2p+2}}s_{\alpha_{2p+3}} \cdots s_{\alpha_{2p+m-1}}s_{\alpha_{2p+m}}s_{\alpha_{2p+m-1}} \cdots s_{\alpha_{2p+3}}s_{\alpha_{2p+2}} s_{\alpha_{2p+1}}s_{\alpha_{2p+2}}s_{\alpha_{2p+3}} \cdots s_{\alpha_{2p+m}} \cdots s_{\alpha_{2p+3}} s_{\alpha_{2p+2}} \cdots s_{\alpha_{2p+m-2}}s_{\alpha_{2p+m-1}}s_{\alpha_{2p+m}} s_{\alpha_{2p+m-1}}s_{\alpha_{2p+m-2}}s_{\alpha_{2p+m}}s_{\alpha_{2p+m-1}}s_{\alpha_{2p+m}}$
CII$_b$	$s_{\alpha_1}s_{\alpha_3}...s_{\alpha_{2l-3}}s_{\alpha_{2l-1}}$
DI$_a$	$s_{\alpha_{p+1}}s_{\alpha_{p+2}}s_{\alpha_{p+1}}s_{\alpha_{p+3}}s_{\alpha_{p+2}}s_{\alpha_{p+1}}s_{\alpha_{p+4}}s_{\alpha_{p+3}} s_{\alpha_{p+2}}s_{\alpha_{p+1}} \cdots s_{\alpha_{p+m-1}}s_{\alpha_{p+m-2}} \cdots s_{\alpha_{p+1}} s_{\alpha_{p+m}}s_{\alpha_{p+m-2}} \cdots s_{\alpha_{p+1}}s_{\alpha_{p+m-1}} \cdots s_{\alpha_{p+2}} s_{\alpha_{p+m}}s_{\alpha_{p+m-2}} \cdots s_{\alpha_{p+3}}s_{\alpha_{p+m-1}} \cdots s_{\alpha_{p+4}} s_{\alpha_{p+m}}s_{\alpha_{p+m-2}} \cdots s_{\alpha_{p+5}} \cdots$
DIII$_a$	$s_{\alpha_1}s_{\alpha_3}...s_{\alpha_{2l-1}}$
DIII$_b$	$s_{\alpha_1}s_{\alpha_3}...s_{\alpha_{2l-1}}$
EI	id
EII	id
EIII	$s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}s_{\alpha_4}s_{\alpha_3}s_{\alpha_4}$
EIV	$s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}$
EV	id
EVI	$s_{\alpha_2}s_{\alpha_5}s_{\alpha_7}$
EVII	$s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}$
EVIII	id
EIX	$s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}$
FI	id
FII	$s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}s_{\alpha_3}s_{\alpha_2}s_{\alpha_1}s_{\alpha_3}s_{\alpha_2}s_{\alpha_3}$
G	id

Table 6.2: Basis of $\Phi(\mathfrak{a})$ in terms of basis of $\Phi(\mathfrak{t})$

Type θ	λ
AI	$\lambda_i = \alpha_i \; \forall i$
AII	$\lambda_i = \frac{1}{2}(\alpha_{2i-1} + 2\alpha_{2i} + \alpha_{2i+1}) \; \forall i$
AIII$_a$	$\lambda_i = \frac{1}{2}(\alpha_i + \alpha_{l+1-i})$ for $i = 1, \cdots, p-1$ $\lambda_p = \frac{1}{2}(\alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m} + \alpha_{p+m+1})$
AIII$_b$	$\lambda_i = \frac{1}{2}(\alpha_i + \alpha_{l+1-i})$ for $i = 1, \cdots, l-1$ $\lambda_l = \alpha_l$
BI	$\lambda_i = \alpha_i$ for $i = 1, \cdots, p-1$ $\lambda_p = \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m-1} + \alpha_{p+m}$
CI	$\lambda_i = \alpha_i \; \forall i$
CII$_a$	$\lambda_i = \frac{1}{2}(\alpha_{2i-1} + 2\alpha_{2i} + \alpha_{2i+1})$ for $i = 1, \cdots, p-1$ $\lambda_p = \frac{1}{2}(\alpha_{2p-1} + 2\alpha_{2p} + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m})$
CII$_b$	$\lambda_i = \frac{1}{2}(\alpha_{2i-1} + 2\alpha_{2i} + \alpha_{2i+1})$ for $i = 1, \cdots, l-1$ $\lambda_l = \alpha_{2l-1} + \alpha_{2l}$
DI$_a$	$\lambda_i = \alpha_i$ for $i = 1 \cdots p-1$ $\lambda_p = \frac{1}{2}(2\alpha_p + 2\alpha_{p+1} + \cdots + 2\alpha_{p+m-2} + \alpha_{p+m-1} + \alpha_{p+m})$
DI$_b$	$\lambda_i = \alpha_i \; \forall i$
DIII$_a$	$\lambda_i = \frac{1}{2}(\alpha_{2i-1} + 2\alpha_{2i} + \alpha_{2i+1})$ for $i = 1, \cdots, l-1$ $\lambda_l = \alpha_{2l}$
DIII$_b$	$\lambda_i = \frac{1}{2}(\alpha_{2i-1} + 2\alpha_{2i} + \alpha_{2i+1})$ for $i = 1, \cdots, l-1$ $\lambda_l = \frac{1}{2}(\alpha_{2l-1} + \alpha_{2l} + \alpha_{2l+1})$
EI	$\lambda_i = \alpha_i \; \forall i$

Table 6.3: Basis of $\Phi(\mathfrak{a})$ in terms of basis of $\Phi(\mathfrak{t})$(continued)

Type θ	λ
EII	$\lambda_1 = \alpha_2$
	$\lambda_2 = \alpha_4$
	$\lambda_3 = \frac{1}{2}(\alpha_3 + \alpha_5)$
	$\lambda_4 = \frac{1}{2}(\alpha_1 + \alpha_6)$
EIII	$\lambda_1 = \frac{1}{2}(2\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5)$
	$\lambda_2 = \frac{1}{2}(\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)$
EIV	$\lambda_1 = \frac{1}{2}(2\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5)$
	$\lambda_2 = \frac{1}{2}(\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6)$
EV	$\lambda_i = \alpha_i \; \forall i$
EVI	$\lambda_1 = \alpha_1$
	$\lambda_2 = \alpha_3$
	$\lambda_3 = \frac{1}{2}(\alpha_2 + 2\alpha_4 + \alpha_5)$
	$\lambda_4 = \frac{1}{2}(\alpha_5 + 2\alpha_6 + \alpha_7)$
EVII	$\lambda_1 = \frac{1}{2}(2\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5)$
	$\lambda_2 = \frac{1}{2}(\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6)$
	$\lambda_3 = \alpha_7$
EVIII	$\lambda_i = \alpha_i \; \forall i$
EIX	$\lambda_1 = \alpha_8$
	$\lambda_2 = \alpha_7$
	$\lambda_3 = \frac{1}{2}(\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6)$
	$\lambda_4 = \frac{1}{2}(2\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5)$
FI	$\lambda_i = \alpha_i \; \forall i$
FII	$\lambda_1 = \frac{1}{2}(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4)$
G	$\lambda_i = \alpha_i \; \forall i$

Chapter 7

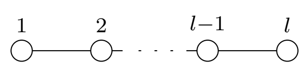
Computing Weyl Group Elements

From theorem 2.1, we have that $W(\mathfrak{a}) \simeq W_1(\theta)/W_0(\theta)$. It becomes quite adventageous in our computations to compute the representative in $W_1(\theta)$ for each s_λ, λ in $\Delta(\mathfrak{a})$.

7.1 A cases

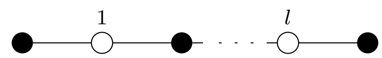
There are 4 isomorphy classes of local symmetric spaces that result from a Lie algebra of type A.

7.1.1 Type AI



When the involution is of type I, nothing is fixed by θ. In fact, every root projects down to itself, i.e., $\Phi(\mathfrak{t}) = \Phi(\mathfrak{a})$. This makes computing Weyl group representatives quite simple. For s_{λ_i}, λ_i in $\Phi(\mathfrak{a})$, the Weyl group representative w_i is only s_{α_i}.

7.1.2 Type AII

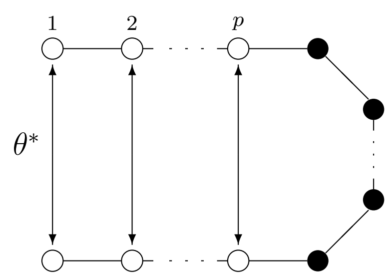


When the involution is of type II, α_{2i-1} is fixed for $i = 1, \cdots, l+1$. Recall from 6.1.2 that $\lambda_i = \frac{1}{2}(\alpha_{2i-1} + 2\alpha_{2i} + \alpha_{2i+1})$. In an essence our w_i needs to send each λ_i to its negative. Since each λ_i is a linear combination of α_{2i-1}, α_{2i}, and α_{2i+1}, it seem only reasonable that our Weyl group representative for s_{λ_i} would consist of the product of elements $s_{\alpha_{2i-1}}$, $s_{\alpha_{2i}}$, and $s_{\alpha_{2i+1}}$.

We actually arise with

$$w_i = s_{\alpha_{2i}} s_{\alpha_{2i-1}} s_{\alpha_{2i+1}} s_{\alpha_{2i}}.$$

7.1.3 Type AIII$_a$



Recall that we start with the Lie algebra $\mathfrak{g} = A_l,\ l = 2p + m$ where $p = |\Delta(\mathfrak{a})|$ and $m = |\Delta_0(\theta)|$. Here the Weyl group representatives for λ_i are quite easy when $i < p$. They are only

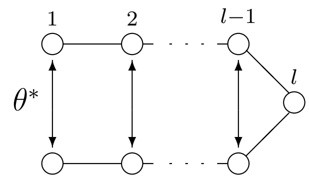
$$w_i = s_{\alpha_i} s_{\alpha_{l-i+1}}.$$

Finding the Weyl group representative for λ_p is quite a bit more challenging.

The Weyl group element for λ_p is

$$w_p = s_{\alpha_{p+m+1}} s_{\alpha_{p+m}} \cdots s_{\alpha_{p+1}} s_{\alpha_p} s_{\alpha_{p+1}} \cdots s_{\alpha_{p+m}} s_{\alpha_{p+m+1}}.$$

7.1.4 Type AIII$_b$



Here we start with the Lie algebra $\mathfrak{g} = A_{2l-1}$ where $l = |\Delta(\mathfrak{a})|$ and $\Delta_0(\theta) = \emptyset$.

The Weyl group representatives for λ_i when $i = 1, \cdots, l-1$ are

$$w_i = s_{\alpha_i} s_{\alpha_{2l-i}}.$$

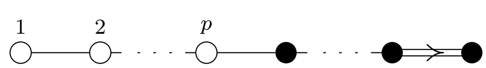
The Weyl group element for λ_l is

$$w_l = s_{\alpha_l}.$$

7.2 B case

There is only 1 isomorphy class of local symmetric space that results from a Lie algebra of type B.

7.2.1 Type BI



Here we start with the Lie algebra $\mathfrak{g} = B_l$. Recall that $l = p+m$ where $p = |\Delta(\mathfrak{a})|$ and $m = |\Delta_0(\theta)|$.

When $i = 1, \cdots, p-1$, the Weyl group representatives are only

$$w_i = s_{\alpha_i}.$$

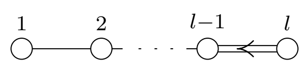
The representative for λ_p is

$$w_p = s_{\alpha_{p+m}} s_{\alpha_{p+m-1}} \cdots s_{\alpha_{p+1}} s_{\alpha_p} s_{\alpha_{p+1}} \cdots s_{\alpha_{p+m-1}} s_{\alpha_{p+m}}.$$

7.3 C cases

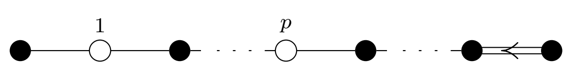
There are 3 isomorphy classes of local symmetric spaces that result from a Lie algebra of type C.

7.3.1 Type CI



When the involution is of type I, nothing is fixed by θ. In fact, every root projects down to itself, i.e., $\Phi(\mathfrak{t}) = \Phi(\mathfrak{a})$. This makes computing Weyl group representatives quite simple. For s_{λ_i}, λ_i in $\Phi(\mathfrak{a})$, the Weyl group representative w_i is only s_{α_i}.

7.3.2 Type CII$_a$



We start with the Lie algebra $\mathfrak{g} = C_l$. Recall that $l = 2p + m$ where $p = |\Delta(\mathfrak{a})|$ and $|\Delta_0(\theta)| = m + p$.

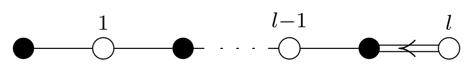
When $i = 1, \cdots, p-1$, the Weyl group representatives are only

$$w_i = s_{\alpha_{2i}} s_{\alpha_{2i+1}} s_{\alpha_{2i-1}} s_{\alpha_{2i}}.$$

The representative for λ_p is

$$w_p = s_{\alpha_{2p}} s_{\alpha_{2p+1}} \cdots s_{\alpha_{2p+m-1}} s_{\alpha_{2p+m}} s_{\alpha_{2p+m-1}} \cdots s_{\alpha_{2p}} s_{\alpha_{2p-1}} s_{\alpha_{2p}} \cdots$$
$$s_{\alpha_{2p+m-1}} s_{\alpha_{2p+m}} s_{\alpha_{2p+m-1}} \cdots s_{\alpha_{2p+1}} s_{\alpha_{2p}}.$$

7.3.3 Type CII$_b$



We begin with the Lie algebra $\mathfrak{g} = C_{2l}$. Recall that $|\Delta(\mathfrak{a})| = l$ and $|\Delta_0(\theta)| = l$.

When $i = 1, \cdots, l-1$, the Weyl group representatives are

$$w_i = s_{\alpha_{2i}} s_{\alpha_{2i-1}} s_{\alpha_{2i+1}} s_{\alpha_{2i}}.$$

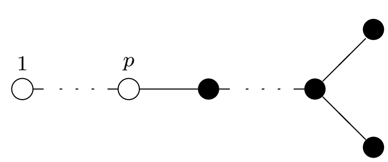
The representative for λ_l is

$$w_l = s_{\alpha_{2l}} s_{\alpha_{2l-1}} s_{\alpha_{2l}}.$$

7.4 D cases

There are 4 isomorphy classes of local symmetric spaces that result from a Lie algebra of type D.

7.4.1 Type DI$_a$



We begin with the Lie algebra $\mathfrak{g} = D_l$. Recall that $|\Delta(\mathfrak{a})| = p$ and $|\Delta_0(\theta)| = m$.

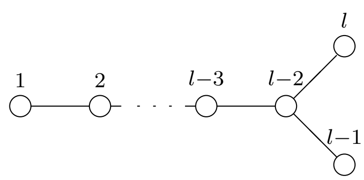
When $i = 1, \cdots, p-1$, the Weyl group representatives are

$$w_i = s_{\alpha_i}.$$

The representative for λ_p is

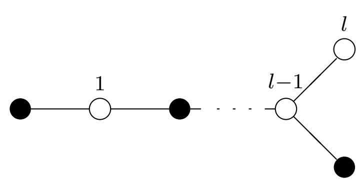
$$w_p = s_{\alpha_p} s_{\alpha_{p+1}} \cdots s_{\alpha_{p+m-2}} s_{\alpha_{p+m-1}} s_{\alpha_{p+m}} s_{\alpha_{p+m-2}} \cdots s_{\alpha_{p+1}} s_{\alpha_p}.$$

7.4.2 Type DI$_b$



When the involution is of type I, nothing is fixed by θ. In fact, every root projects down to itself, i.e., $\Phi(\mathfrak{t}) = \Phi(\mathfrak{a})$. This makes computing Weyl group representatives quite simple. For s_{λ_i}, λ_i in $\Phi(\mathfrak{a})$, the Weyl group representative w_i is only s_{α_i}.

7.4.3 Type DIII$_a$



We start with the Lie algebra $\mathfrak{g} = D_{2l}$. Recall that $|\Delta(\mathfrak{a})| = l$ and $|\Delta_0(\theta)| = l$.

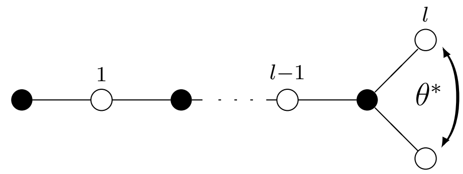
When $i = 1, \cdots, l-1$, the Weyl group representatives are

$$w_i = s_{\alpha_{2i}} s_{\alpha_{2i-1}} s_{\alpha_{2i+1}} s_{\alpha_{2i}}.$$

The representative for λ_l is

$$w_l = s_{\alpha_{2l}}.$$

7.4.4 Type DIII$_b$



We start with the Lie algebra $\mathfrak{g} = D_{2l+1}$. Recall that $|\Delta(\mathfrak{a})| = l$ and $|\Delta_0(\theta)| = l$. When $i = 1, \cdots, l-1$, the Weyl group representatives are

$$w_i = s_{\alpha_{2i}} s_{\alpha_{2i-1}} s_{\alpha_{2i+1}} s_{\alpha_{2i}}.$$

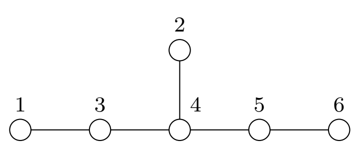
The representative for λ_l is

$$w_l = s_{\alpha_{2l+1}} s_{\alpha_{2l}} s_{\alpha_{2l-1}} s_{\alpha_{2l}} s_{\alpha_{2l+1}}.$$

7.5 E$_6$ cases

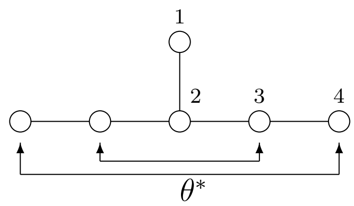
There are 4 isomorphy classes of local symmetric spaces that result from a Lie algebra of type E$_6$.

7.5.1 Type EI



When the involution is of type I, nothing is fixed by θ. In fact, every root projects down to itself, i.e., $\Phi(\mathfrak{t}) = \Phi(\mathfrak{a})$. This makes computing Weyl group representatives quite simple. For s_{λ_i}, λ_i in $\Phi(\mathfrak{a})$, the Weyl group representative w_i is only s_{α_i}.

7.5.2 Type EII



From the Lie algebra $\mathfrak{g} = E_6$, we have the restricted root system F_4.

We have the Weyl group representatives

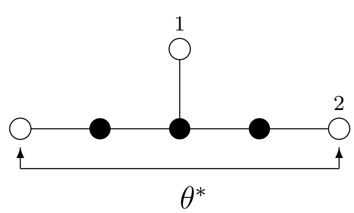
$$w_1 = s_{\alpha_2},$$

$$w_2 = s_{\alpha_4},$$

$$w_3 = s_{\alpha_3}s_{\alpha_5}, \text{ and}$$

$$w_4 = s_{\alpha_1}s_{\alpha_6}.$$

7.5.3 Type EIII



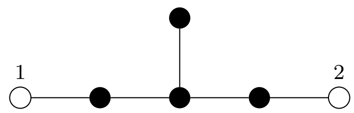
Here the resulting restricted root system is of type BC_2.

We have the Weyl group representatives

$$w_1 = s_{\alpha_2}s_{\alpha_4}s_{\alpha_5}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2} \text{ and}$$

$$w_2 = s_{\alpha_1}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}s_{\alpha_6}s_{\alpha_5}s_{\alpha_4}s_{\alpha_3}s_{\alpha_1}.$$

7.5.4 Type EIV



Here the resulting restricted root system is of type A_2.

We have the Weyl group representatives

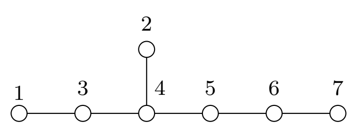
$$w_1 = s_{\alpha_1} s_{\alpha_3} s_{\alpha_4} s_{\alpha_2} s_{\alpha_5} s_{\alpha_4} s_{\alpha_3} s_{\alpha_1} \text{ and}$$

$$w_2 = s_{\alpha_6} s_{\alpha_5} s_{\alpha_4} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\alpha_5} s_{\alpha_6}.$$

7.6 E_7 cases

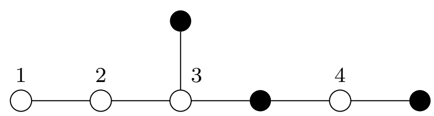
There are 3 isomorphy classes of local symmetric spaces that result from a Lie algebra of type E_7.

7.6.1 Type EV



When the involution is of type I, nothing is fixed by θ. In fact, every root projects down to itself, i.e., $\Phi(\mathfrak{t}) = \Phi(\mathfrak{a})$. This makes computing Weyl group representatives quite simple. For s_{λ_i}, λ_i in $\Phi(\mathfrak{a})$, the Weyl group representative w_i is only s_{α_i}.

7.6.2 Type EVI



From the Lie algebra $\mathfrak{g} = E_7$, we have the restricted root system F_4.

We have the Weyl group representatives

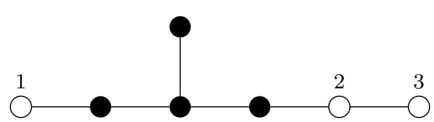
$$w_1 = s_{\alpha_1},$$

$$w_2 = s_{\alpha_3},$$

$$w_3 = s_{\alpha_4} s_{\alpha_2} s_{\alpha_5} s_{\alpha_4}, \text{ and}$$

$$w_4 = s_{\alpha_6} s_{\alpha_5} s_{\alpha_7} s_{\alpha_6}.$$

7.6.3 Type EVII



Here the resulting restricted root system is of type C_3.

We have the Weyl group representatives

$$w_1 = s_{\alpha_1} s_{\alpha_3} s_{\alpha_4} s_{\alpha_2} s_{\alpha_5} s_{\alpha_4} s_{\alpha_3} s_{\alpha_1}$$

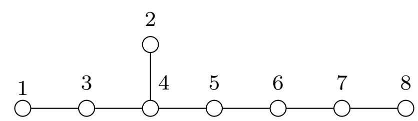
$$w_2 = s_{\alpha_6} s_{\alpha_5} s_{\alpha_4} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\alpha_5} s_{\alpha_6} \text{ and}$$

$$w_3 = s_{\alpha_7}.$$

7.7 E_8 cases

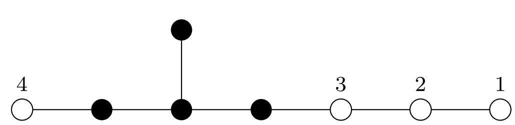
There are 2 isomorphy classes of local symmetric spaces that result from a Lie algebra of type E_8.

7.7.1 Type EVIII



When the involution is of type I, nothing is fixed by θ. In fact, every root projects down to itself, i.e., $\Phi(\mathfrak{t}) = \Phi(\mathfrak{a})$. This makes computing Weyl group representatives quite simple. For s_{λ_i}, λ_i in $\Phi(\mathfrak{a})$, the Weyl group representative w_i is only s_{α_i}.

7.7.2 Type EIX



We have the Weyl group representatives

$$w_1 = s_{\alpha_8}$$

$$w_2 = s_{\alpha_7}$$

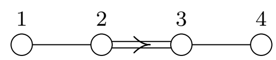
$$w_3 = s_{\alpha_6}s_{\alpha_5}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}s_{\alpha_6} \text{ and}$$

$$w_4 = s_{\alpha_1}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2}s_{\alpha_5}s_{\alpha_4}s_{\alpha_3}s_{\alpha_1}.$$

7.8 F_4 cases

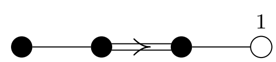
There are 2 isomorphy classes of local symmetric spaces that result from a Lie algebra of type F_4.

7.8.1 Type FI



When the involution is of type I, nothing is fixed by θ. In fact, every root projects down to itself, i.e., $\Phi(\mathfrak{t}) = \Phi(\mathfrak{a})$. This makes computing Weyl group representatives quite simple. For s_{λ_i}, λ_i in $\Phi(\mathfrak{a})$, the Weyl group representative w_i is only s_{α_i}.

7.8.2 Type FII



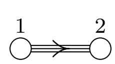
Here the Weyl group representative is

$$w_1 = s_{\alpha_4} s_{\alpha_3} s_{\alpha_2} s_{\alpha_1} s_{\alpha_3} s_{\alpha_2} s_{\alpha_4} s_{\alpha_3} s_{\alpha_4} s_{\alpha_2} s_{\alpha_3} s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4}$$

7.9 G_2 case

There is only 1 isomorphy class of local symmetric space that results from a Lie algebra of type G_2.

7.9.1 Type G



When the involution is of type I, nothing is fixed by θ. In fact, every root projects down to itself, i.e., $\Phi(\mathfrak{t}) = \Phi(\mathfrak{a})$. This makes computing Weyl group representatives quite simple. For s_{λ_i}, λ_i in $\Phi(\mathfrak{a})$, the Weyl group representative w_i is only s_{α_i}.

Table 7.1: $W(\mathfrak{a})$

Type θ	s_{λ_i}-representative
AI	$w_i = s_{\alpha_i}\ \forall i$
AII	$w_i = s_{\alpha_{2i}} s_{\alpha_{2i-1}} s_{\alpha_{2i+1}} s_{\alpha_{2i}}\ \forall i$
AIII$_a$	$w_i = s_{\alpha_i} s_{\alpha_{l+1-i}}$ for $i = 1, \cdots, p-1$ $w_p = s_{\alpha_{p+m+1}} s_{\alpha_{p+m}} \cdots s_{\alpha_{p+1}} s_{\alpha_p} s_{\alpha_{p+1}} \cdots s_{\alpha_{p+m}} s_{\alpha_{p+m+1}}$
AIII$_b$	$w_i = s_{\alpha_i} s_{\alpha_{l+1-i}}$ for $i = 1, \cdots, l-1$ $w_l = s_{\alpha_l}$
BI	$w_i = s_{\alpha_i}$ for $i = 1, \cdots, p-1$ $w_p = s_{\alpha_{p+m}} s_{\alpha_{p+m-1}} \cdots s_{\alpha_{p+1}} s_{\alpha_p} s_{\alpha_{p+1}} \cdots s_{\alpha_{p+m-1}} s_{\alpha_{p+m}}.$
CI	$w_i = s_{\alpha_i}\ \forall i$
CII$_a$	$w_i = s_{\alpha_{2i}} s_{\alpha_{2i+1}} s_{\alpha_{2i-1}} s_{\alpha_{2i}}$ for $i = 1, \cdots, p-1$ $w_p = s_{\alpha_{2p}} s_{\alpha_{2p+1}} \cdots s_{\alpha_{2p+m-1}} s_{\alpha_{2p+m}} s_{\alpha_{2p+m-1}} \cdots s_{\alpha_{2p}} s_{\alpha_{2p-1}} s_{\alpha_{2p}} \cdots$ $s_{\alpha_{2p+m-1}} s_{\alpha_{2p+m}} s_{\alpha_{2p+m-1}} \cdots s_{\alpha_{2p+1}} s_{\alpha_{2p}}$
CII$_b$	$w_i = s_{\alpha_{2i}} s_{\alpha_{2i+1}} s_{\alpha_{2i-1}} s_{\alpha_{2i}}$ for $i = 1, \cdots, p-1$ $w_l = s_{\alpha_{2l}} s_{\alpha_{2l-1}} s_{\alpha_{2l}}$
DI$_a$	$w_i = \alpha_i$ for $i = 1 \cdots p-1$ $w_p = s_{\alpha_p} s_{\alpha_{p+1}} \cdots s_{\alpha_{p+m-2}} s_{\alpha_{p+m-1}} s_{\alpha_{p+m}} s_{\alpha_{p+m-2}} \cdots s_{\alpha_{p+1}} s_{\alpha_p}$
DI$_b$	$w_i = s_{\alpha_i}\ \forall i$
DIII$_a$	$w_i = s_{\alpha_{2i}} s_{\alpha_{2i+1}} s_{\alpha_{2i-1}} s_{\alpha_{2i}}$ for $i = 1, \cdots, l-1$ $w_l = s_{\alpha_{2l}}$
DIII$_b$	$w_i = s_{\alpha_{2i}} s_{\alpha_{2i+1}} s_{\alpha_{2i-1}} s_{\alpha_{2i}}$ for $i = 1, \cdots, l-1$ $w_l = s_{\alpha_{2l+1}} s_{\alpha_{2l}} s_{\alpha_{2l-1}} s_{\alpha_{2l}} s_{\alpha_{2l+1}}$
EI	$w_i = s_{\alpha_i}\ \forall i$
EII	$w_1 = s_{\alpha_2}$ $w_2 = s_{\alpha_4}$ $w_3 = s_{\alpha_3} s_{\alpha_5}$ $w_4 = s_{\alpha_1} s_{\alpha_6}$
EIII	$w_1 = s_{\alpha_2} s_{\alpha_4} s_{\alpha_5} s_{\alpha_3} s_{\alpha_4} s_{\alpha_2}$ $w_2 = s_{\alpha_1} s_{\alpha_3} s_{\alpha_4} s_{\alpha_5} s_{\alpha_6} s_{\alpha_5} s_{\alpha_4} s_{\alpha_3} s_{\alpha_1}$
EIV	$w_1 = s_{\alpha_1} s_{\alpha_3} s_{\alpha_4} s_{\alpha_2} s_{\alpha_5} s_{\alpha_4} s_{\alpha_3} s_{\alpha_1}$ $w_2 = s_{\alpha_6} s_{\alpha_5} s_{\alpha_4} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\alpha_5} s_{\alpha_6}$
EV	$w_i = s_{\alpha_i}\ \forall i$

Table 7.2: $W(\mathfrak{a})$(continued)

Type θ	s_{λ_i}-representative
EIII	$w_1 = s_{\alpha_2}s_{\alpha_4}s_{\alpha_5}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2}$ $w_2 = s_{\alpha_1}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}s_{\alpha_6}s_{\alpha_5}s_{\alpha_4}s_{\alpha_3}s_{\alpha_1}$
EIV	$w_1 = s_{\alpha_1}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2}s_{\alpha_5}s_{\alpha_4}s_{\alpha_3}s_{\alpha_1}$ $w_2 = s_{\alpha_6}s_{\alpha_5}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}s_{\alpha_6}$
EV	$w_i = s_{\alpha_i}\ \forall i$
EVI	$w_1 = s_{\alpha_1}$ $w_2 = s_{\alpha_3}$ $w_3 = s_{\alpha_4}s_{\alpha_2}s_{\alpha_5}s_{\alpha_4}$ $w_4 = s_{\alpha_6}s_{\alpha_5}s_{\alpha_7}s_{\alpha_6}$
EVII	$w_1 = s_{\alpha_1}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2}s_{\alpha_5}s_{\alpha_4}s_{\alpha_3}s_{\alpha_1}$ $w_2 = s_{\alpha_6}s_{\alpha_5}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}s_{\alpha_6}$ $w_3 = s_{\alpha_7}$
EVIII	$w_i = s_{\alpha_i}\ \forall i$
EIX	$w_1 = s_{\alpha_8}$ $w_2 = s_{\alpha_7}$ $w_3 = s_{\alpha_6}s_{\alpha_5}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}s_{\alpha_6}$ $w_4 = s_{\alpha_1}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2}s_{\alpha_5}s_{\alpha_4}s_{\alpha_3}s_{\alpha_1}$
FI	$w_i = s_{\alpha_i}\ \forall i$
FII	$w_1 = s_{\alpha_4}s_{\alpha_3}s_{\alpha_2}s_{\alpha_1}s_{\alpha_3}s_{\alpha_2}s_{\alpha_4}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2}s_{\alpha_3}s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}$
G	$w_i = s_{\alpha_i}\ \forall i$

Chapter 8

The Structure of $\Phi(\mathfrak{a})$

Now that we have explicitly computed each λ_i in $\Delta(\mathfrak{a})$ and each Weyl group representative in $W_1(\theta)$, we would like to explore the structure of $\Phi(\mathfrak{a})$. In doing so, we will essentially complete Steps 1, 3 and 4 from our algorithm.

Recall from 5.1 that step 1 of the algorithm has us computing for all λ_i in $\Delta(\mathfrak{a})$,

$$\Phi(\lambda_i) = \{\alpha \in \Phi(\mathfrak{t}) \mid \alpha|_{\mathfrak{a}} = \lambda_i\}.$$

In finding this set, we are also computing the multiplicity of λ_i.

In step 3 we compute the list of all positive roots in $\Phi(\mathfrak{a})$ by using the Weyl group representatives w_i applied to all λ_i in $\Delta(\mathfrak{a})$.

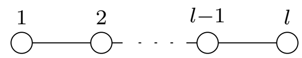
Lastly in step 4, we determine $\Phi(\lambda_i)$ for all λ_i in $\Phi(\mathfrak{a})$ by using the nice fact from 5.4 that

$$\Phi(s_{\lambda_i}(\lambda_j)) = \Phi(\lambda_j - \langle \alpha_j, \alpha_i \rangle \lambda_i) = w_i(\Phi(\lambda_j)).$$

8.1 A cases

There are 4 isomorphy classes of local symmetric spaces that result from a Lie algebra of type A.

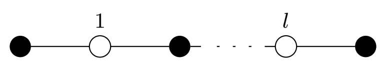
8.1.1 Type AI



We have for $i = 1, \cdots, l$ that $\Phi(\lambda_i) = \{\alpha_i\}$. So the multiplicity of each root is $m_{\lambda_i} = 1$.

Also since $w_i = s_{\alpha_i}$ for all i, we have that $\Phi(\mathfrak{a}) = \Phi(\mathfrak{t})$.

8.1.2 Type AII



Here the multiplicity of each root is $m_{\lambda_i} = 4$. We have that

$$\Phi(\lambda_i) = \{\alpha_{2i}, \alpha_{2i} + \alpha_{2i+1}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1}, \alpha_{2i-1} + \alpha_{2i}\}.$$

$\Phi(\mathfrak{a})$ is of type A_l and can be computed as in lemma 10.

Computing the $\Phi(\lambda_i)$ is fairly systematic as well.

Roots of length 2:

$$\Phi(s_{\lambda_{i+1}}\lambda_i) = \Phi(\lambda_i + \lambda_{i+1}) = w_{i+1}\Phi(\lambda_i) =$$

$\{\alpha_{2i}+\alpha_{2i+1}+\alpha_{2i+2}, \alpha_{2i}+\alpha_{2i+1}+\alpha_{2i+2}+\alpha_{2i+3}, \alpha_{2i-1}+\alpha_{2i}+\alpha_{2i+1}+\alpha_{2i+2}+\alpha_{2i+3}, \alpha_{2i-1}+\alpha_{2i}+\alpha_{2i+1}+\alpha_{2i+2}\}$

Roots of length 3:

$$\Phi(s_{\lambda_{i+2}}(\lambda_i + \lambda_{i+1})) = \Phi(\lambda_i + \lambda_{i+1} + \lambda_{i+2}) = w_{i+2}\Phi(\lambda_i + \lambda_{i+1}) =$$

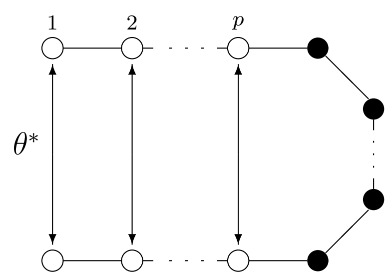
$\{\alpha_{2i}+\alpha_{2i+1}+\alpha_{2i+2}+\alpha_{2i+3}+\alpha_{2i+4}, \alpha_{2i}+\alpha_{2i+1}+\alpha_{2i+2}+\alpha_{2i+3}+\alpha_{2i+4}+\alpha_{2i+5}, \alpha_{2i-1}+\alpha_{2i}+\alpha_{2i+1}+\alpha_{2i+2}+\alpha_{2i+3}+\alpha_{2i+4}+\alpha_{2i+5}, \alpha_{2i-1}+\alpha_{2i}+\alpha_{2i+1}+\alpha_{2i+2}+\alpha_{2i+3}+\alpha_{2i+4}\}$

$$\vdots$$

Roots of length l:

$\Phi(s_{\lambda_l}(\lambda_1+\cdots+\lambda_{l-1})) = \Phi(\lambda_1+\cdots+\lambda_{l-1}+\lambda_l) = w_l\Phi(\lambda_1+\cdots+\lambda_{l-1}) = \{\alpha_2+\cdots+\alpha_{2l}, \alpha_2+\cdots+\alpha_{2l+1}, \alpha_1+\cdots+\alpha_{2l+1}, \alpha_1+\cdots+\alpha_{2l}\}$

8.1.3 Type AIII$_a$



Here the multiplicity of each λ_i is

$$\begin{aligned} m_{\lambda_i} &= 2 \text{ for } i = 1, \cdots, p-1 \\ m_{\lambda_p} &= 2(m+1) \\ m_{2\lambda_p} &= 1 \end{aligned}$$

We have that

$$\Phi(\lambda_i) = \{\alpha_i, \alpha_{l+1-i}\} \text{ for } i = 1, \cdots, p-1.$$

$\Phi(\lambda_p) = \{\alpha_p, \alpha_{p+m+1}, \alpha_p + \alpha_{p+1}, \alpha_{p+m} + \alpha_{p+m+1}, \cdots, \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m}, \alpha_{p+1} + \cdots + \alpha_{p+m+1}\}$

$$\Phi(2\lambda_p) = \{\alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m+1}\}$$

$\Phi(\mathfrak{a})$ is of type BC_p and can be computed as in lemma 13.

Computing the $\Phi(\lambda_i)$ is fairly systematic as well.

Roots of length 2:

$$\Phi(s_{\lambda_{i-1}}\lambda_i) = \Phi(\lambda_{i-1} + \lambda_i) = w_{i-1}\Phi(\lambda_i) =$$

$$\{\alpha_{i-1} + \alpha_i, \alpha_{l+1-i} + \alpha_{l+2-i}\} \text{ for } i = 2, \cdots, p-1$$

$$\Phi(s_{\lambda_{p-1}}\lambda_p) = \Phi(\lambda_{p-1} + \lambda_p) = w_{p-1}\Phi(\lambda_p) =$$

$\{\alpha_{p-1} + \alpha_p, \alpha_{p+m+1} + \alpha_{p+m+2}, \alpha_{p-1} + \alpha_p + \alpha_{p+1}, \alpha_{p+m} + \alpha_{p+m+1} + \alpha_{p+m+2}, \cdots, \alpha_{p-1} + \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m}, \alpha_{p+1} + \cdots + \alpha_{p+m+1} + \alpha_{p+m+2}\}$

$$\Phi(s_{\lambda_p}(\lambda_{p-1})) = \Phi(\lambda_{p-1} + 2\lambda_p) = w_p\Phi(\lambda_{p-1})$$

$$\{\alpha_{p-1} + \alpha_p + \cdots + \alpha_{p+m} + \alpha_{p+m+1}, \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m+1} + \alpha_{p+m+2}\}$$

$$\Phi(s_{\lambda_{p-1}}2\lambda_p) = \Phi(2\lambda_{p-1} + 2\lambda_p) = w_{p-1}\Phi(2\lambda_p) =$$

$$\{\alpha_{p-1} + \alpha_p + \cdots + \alpha_{p+m+1} + \alpha_{p+m+2}\}$$

Roots of length 3:

$$\Phi(s_{\lambda_{i-2}}s_{\lambda_{i-1}}\lambda_i) = \Phi(\lambda_{i-2}+\lambda_{i-1}+\lambda_i) = w_{i-2}w_{i-1}\Phi(\lambda_i) =$$

$$\{\alpha_{i-2}+\alpha_{i-1}+\alpha_i, \alpha_{l+1-i}+\alpha_{l+2-i}+\alpha_{l+3-i}\} \text{ for } i = 3, \cdots, p-1$$

$$\Phi(s_{\lambda_{p-2}}s_{\lambda_{p-1}}\lambda_p) = \Phi(\lambda_{p-2}+\lambda_{p-1}+\lambda_p) = w_{p-2}w_{p-1}\Phi(\lambda_p) =$$

$\{\alpha_{p-2}+\alpha_{p-1}+\alpha_p, \alpha_{p+m+1}+\alpha_{p+m+2}+\alpha_{p+m+3}, \alpha_{p-2}+\alpha_{p-1}+\alpha_p+\alpha_{p+1}, \alpha_{p+m}+\alpha_{p+m+1}+ \alpha_{p+m+2}+\alpha_{p+m+3}, \cdots, \alpha_{p-2}+\alpha_{p-1}+\alpha_p+\alpha_{p+1}+\cdots+\alpha_{p+m}, \alpha_{p+1}+\cdots+\alpha_{p+m+1}+ \alpha_{p+m+2}+\alpha_{p+m+3}\}$

$$\Phi(s_{\lambda_{p-2}}s_{\lambda_p}(\lambda_{p-1})) = \Phi(\lambda_{p-2}+\lambda_{p-1}+2\lambda_p) = w_{p-2}w_p\Phi(\lambda_{p-1})$$

$$\{\alpha_{p-2}+\alpha_{p-1}+\alpha_p+\cdots+\alpha_{p+m}+\alpha_{p+m+1}, \alpha_p+\alpha_{p+1}+\cdots+\alpha_{p+m+1}+\alpha_{p+m+2}+\alpha_{p+m+3}\}$$

$$\Phi(s_{\lambda_{p-1}}s_{\lambda_p}(\lambda_{p-2}+\lambda_{p-1})) = \Phi(\lambda_{p-2}+2\lambda_{p-1}+2\lambda_p) = w_{p-1}w_p\Phi(\lambda_{p-2}+\lambda_{p-1})$$

$\{\alpha_{p-2}+\alpha_{p-1}+\alpha_p+\cdots+\alpha_{p+m}+\alpha_{p+m+1}+\alpha_{p+m+2}, \alpha_{p-1}+\alpha_p+\alpha_{p+1}+\cdots+\alpha_{p+m+1}+ \alpha_{p+m+2}+\alpha_{p+m+3}\}$

$$\Phi(s_{\lambda_{p-2}}s_{\lambda_{p-1}}2\lambda_p) = \Phi(2\lambda_{p-2}+2\lambda_{p-1}+2\lambda_p) = w_{p-2}w_{p-1}\Phi(2\lambda_p) =$$

$$\{\alpha_{p-2}+\alpha_{p-1}+\alpha_p+\cdots+\alpha_{p+m+1}+\alpha_{p+m+2}+\alpha_{p+m+3}\}$$

$$\vdots$$

Roots of length p:

$$\Phi(s_{\lambda_1}(\lambda_2+\lambda_3+\cdots+\lambda_p)) = \Phi(\lambda_1+\lambda_2+\cdots+\lambda_p) = w_1\Phi(\lambda_2+\lambda_3+\cdots+\lambda_p)$$

$= \{\alpha_1+\alpha_2+\cdots+\alpha_p, \alpha_1+\alpha_2+\cdots+\alpha_p+\alpha_{p+1}, \cdots, \alpha_1+\alpha_2+\cdots+\alpha_p+\cdots+\alpha_{p+m}, \alpha_{p+m+1}+\alpha_{p+m+2}+\cdots+\alpha_{2p+m}, \alpha_{p+m}+\alpha_{p+m+1}+\alpha_{p+m+2}+\cdots+\alpha_{2p+m}, \cdots, \alpha_{p+1}+\cdots+\alpha_{2p+m}\}$

$$\Phi(s_{\lambda_p}(\lambda_1+\lambda_2+\cdots+\lambda_{p-1})) = \Phi(\lambda_1+\lambda_2+\cdots+\lambda_{p-1}+2\lambda_p) = w_p\Phi(\lambda_1+\lambda_2+\cdots+\lambda_{p-1})$$

$$= \{\alpha_1+\alpha_2+\cdots+\alpha_p+\cdots+\alpha_{p+m+1}, \alpha_p+\cdots+\alpha_{p+m+1}+\alpha_{p+m+2}+\cdots+\alpha_{2p+m}\}$$

$$\vdots$$

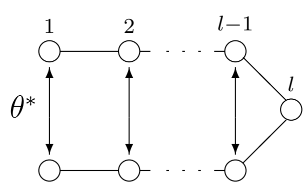
$\Phi(s_{\lambda_2}(\lambda_1+\lambda_2+2\lambda_3+\cdots+2\lambda_{p-1}+2\lambda_p)) = \Phi(\lambda_1+2\lambda_2+\cdots+2\lambda_{p-1}+2\lambda_p) = w_2\Phi(\lambda_1+\lambda_2+2\lambda_3+\cdots+2\lambda_{p-1}+2\lambda_p)$

$$= \{\alpha_1+\alpha_2+\cdots+\alpha_p+\cdots+\alpha_{2p+m-1}, \alpha_2+\cdots+\alpha_{p+m+1}+\alpha_{p+m+2}+\cdots+\alpha_{2p+m}\}$$

$\Phi(s_{\lambda_1}\cdots s_{\lambda_{p-2}}s_{\lambda_{p-1}}2\lambda_p) = \Phi(2\lambda_1+\cdots+2\lambda_{p-2}+2\lambda_{p-1}+2\lambda_p) = w_1\cdots w_{p-2}w_{p-1}\Phi(2\lambda_p) =$

$$\{\alpha_1+\cdots+\alpha_{p-2}+\alpha_{p-1}+\alpha_p+\cdots+\alpha_{p+m+1}+\alpha_{p+m+2}+\alpha_{p+m+3}+\cdots+\alpha_{2p+m}\}$$

8.1.4 Type AIII$_b$



Here the multiplicity of each λ_i is

$m_{\lambda_i} = 2$ for $i = 1, \cdots, l-1$
$m_{\lambda_l} = 1$

We have that

$$\Phi(\lambda_i) = \{\alpha_i, \alpha_{2l-i}\} \text{ for } i = 1, \cdots, l-1$$

$$\Phi(\lambda_l) = \{\alpha_l\}$$

$\Phi(\mathfrak{a})$ is of type C_l and can be computed as in lemma 12.

Now computing the $\Phi(\lambda_i)$ we have:

Roots of length 2:

$$\Phi(s_{\lambda_{i+1}}\lambda_i) = \Phi(\lambda_i + \lambda_{i+1}) = w_{i+1}\Phi(\lambda_i) =$$

$$\{\alpha_i + \alpha_{i+1}, \alpha_{2l-i-1} + \alpha_{2l-i}\} \text{ for } i = 1, \cdots, l-1$$

$$\Phi(s_{\lambda_{l-1}}\lambda_l) = \Phi(2\lambda_{l-1} + \lambda_l) = w_{l-1}\Phi(\lambda_l) =$$

$$\{\alpha_{l-1} + \alpha_l + \alpha_{l+1}\}$$

Roots of length 3:

$$\Phi(s_{\lambda_{i+2}}s_{\lambda_{i+1}}\lambda_i) = \Phi(\lambda_i + \lambda_{i+1} + \lambda_{i+2}) = w_{i+2}w_{i+1}\Phi(\lambda_i) =$$

$$\{\alpha_i + \alpha_{i+1} + \alpha_{i+2}, \alpha_{2l-i-2} + \alpha_{2l-i-1} + \alpha_{2l-i}\} \text{ for } i = 1, \cdots, l-2$$

$$\Phi(s_{\lambda_{l-1}}(\lambda_{l-2}+\lambda_{l-1}+\lambda_l)) = \Phi(\lambda_{l-2}+2\lambda_{l-1}+\lambda_l) = w_{l-1}\Phi(\lambda_{l-2}+\lambda_{l-1}+\lambda_l) =$$
$$\{\alpha_{l-2}+\alpha_{l-1}+\alpha_l+\alpha_{l+1}, \alpha_{l-1}+\alpha_l+\alpha_{l+1}+\alpha_{l+2}\}$$

$$\Phi(s_{\lambda_{l-2}}s_{\lambda_{l-1}}\lambda_l) = \Phi(2\lambda_{l-2}+2\lambda_{l-1}+\lambda_l) = w_{l-2}w_{l-1}\Phi(\lambda_l) =$$
$$\{\alpha_{l-2}+\alpha_{l-1}+\alpha_l+\alpha_{l+1}+\alpha_{l+2}\}$$

$$\vdots$$

Roots of length l:

$$\Phi(s_{\lambda_l}(\lambda_1+\lambda_2+\cdots+\lambda_{l-1})) = \Phi(\lambda_1+\lambda_2+\cdots+\lambda_l) = w_l\Phi(\lambda_1+\lambda_2+\cdots+\lambda_{l-1})$$
$$= \{\alpha_1+\alpha_2+\cdots+\alpha_l, \alpha_l+\alpha_{l+1}+\cdots+\alpha_{2l-1}\}$$

$$\Phi(s_{\lambda_{l-1}}(\lambda_1+\lambda_2+\cdots+\lambda_l)) = \Phi(\lambda_1+\lambda_2+\cdots+2\lambda_{l-1}+\lambda_l) = w_{l-1}\Phi(\lambda_1+\lambda_2+\cdots+\lambda_l)$$
$$= \{\alpha_1+\alpha_2+\cdots+\alpha_l+\alpha_{l+1}, \alpha_{l-1}+\alpha_l+\alpha_{l+1}+\cdots+\alpha_{2l-1}\}$$

$$\vdots$$

$\Phi(s_{\lambda_2}(\lambda_1 + \lambda_2 + 2\lambda_3 + \cdots + 2\lambda_{l-1} + \lambda_l)) = \Phi(\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{l-1} + \lambda_l) = w_2\Phi(\lambda_1 + \lambda_2 + 2\lambda_3 + \cdots + 2\lambda_{l-1} + \lambda_l)$

$$= \{\alpha_2 + \cdots + \alpha_l + \cdots + \alpha_{2l-1}, \alpha_1 + \alpha_2 + \cdots + \alpha_l + \cdots + \alpha_{2l-2}\}$$

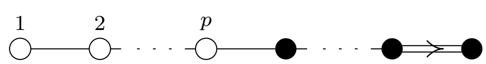
$\Phi(s_{\lambda_1}(2\lambda_2 + 2\lambda_3 + \cdots + 2\lambda_{l-1} + \lambda_l)) = \Phi(2\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{l-1} + \lambda_l) = w_1\Phi(2\lambda_2 + 2\lambda_3 + \cdots + 2\lambda_{l-1} + \lambda_l)$

$$= \{\alpha_1 + \alpha_2 + \cdots + \alpha_l + \cdots + \alpha_{2l-1}\}$$

8.2 B case

There is 1 isomorphy class of local symmetric space that results from a Lie algebra of type B.

8.2.1 Type BI



Here the multiplicity of each λ_i is

$m_{\lambda_i} = 1$ for $i = 1, \cdots, p-1$
$m_{\lambda_p} = 2m + 1$

We have that

$$\Phi(\lambda_i) = \{\alpha_i\} \text{ for } i = 1, \cdots, p-1$$

$\Phi(\lambda_p) = \{\alpha_p, \alpha_p + \alpha_{p+1}, \cdots, \alpha_p + \cdots + \alpha_{p+m}, \alpha_p + \cdots + \alpha_{p+m-1} + 2\alpha_{p+m}, \cdots, \alpha_p + 2\alpha_{p+1} \cdots + 2\alpha_{p+m-1} + 2\alpha_{p+m}\}$

$\Phi(\mathfrak{a})$ is of type B_p and can be computed as in lemma 11.

Computing the $\Phi(\lambda_i)$ is systematic.

Roots of length 2:

$$\Phi(s_{\lambda_{i-1}}\lambda_i) = \Phi(\lambda_{i-1} + \lambda_i) = w_{i-1}\Phi(\lambda_i) =$$
$$\{\alpha_{i-1} + \alpha_i\} \text{ for } i = 2, \cdots, p-1$$

$$\Phi(s_{\lambda_{p-1}}\lambda_p) = \Phi(\lambda_{p-1} + \lambda_p) = w_{p-1}\Phi(\lambda_p) =$$

$\{\alpha_{p-1} + \alpha_p, \alpha_{p-1} + \alpha_p + \alpha_{p+1}, \cdots, \alpha_{p-1} + \alpha_p + \cdots + \alpha_{p+m}, \alpha_{p-1} + \alpha_p + \cdots + \alpha_{p+m-1} + 2\alpha_{p+m}, \cdots, \alpha_{p-1} + \alpha_p + 2\alpha_{p+1} + \cdots + 2\alpha_{p+m-1} + 2\alpha_{p+m}\}$

$$\Phi(s_{\lambda_p}(\lambda_{p-1})) = \Phi(\lambda_{p-1} + 2\lambda_p) = w_p\Phi(\lambda_{p-1})$$
$$\{\alpha_{p-1} + 2\alpha_p + \cdots + 2\alpha_{p+m} + 2\alpha_{p+m+1}\}$$

Roots of length 3:

$$\Phi(s_{\lambda_{i-2}}s_{\lambda_{i-1}}\lambda_i) = \Phi(\lambda_{i-2} + \lambda_{i-1} + \lambda_i) = w_{i-2}w_{i-1}\Phi(\lambda_i) =$$
$$\{\alpha_{i-2} + \alpha_{i-1} + \alpha_i\} \text{ for } i = 3, \cdots, p-1$$

$$\Phi(s_{\lambda_{p-2}}s_{\lambda_{p-1}}\lambda_p) = \Phi(\lambda_{p-2} + \lambda_{p-1} + \lambda_p) = w_{p-2}w_{p-1}\Phi(\lambda_p) =$$

$\{\alpha_{p-2} + \alpha_{p-1} + \alpha_p, \alpha_{p-2} + \alpha_{p-1} + \alpha_p + \alpha_{p+1}, \cdots, \alpha_{p-2} + \alpha_{p-1} + \alpha_p + \cdots + \alpha_{p+m}, \alpha_{p-2} + \alpha_{p-1} + \alpha_p + \cdots + \alpha_{p+m-1} + 2\alpha_{p+m}, \cdots, \alpha_{p-2} + \alpha_{p-1} + \alpha_p + 2\alpha_{p+1} + \cdots + 2\alpha_{p+m-1} + 2\alpha_{p+m}\}$

$$\Phi(s_{\lambda_p}(\lambda_{p-2}+\lambda_{p-1})) = \Phi(\lambda_{p-2}+\lambda_{p-1}+2\lambda_p) = w_p\Phi(\lambda_{p-2}+\lambda_{p-1})$$

$$\{\alpha_{p-2}+\alpha_{p-1}+2\alpha_p+\cdots+2\alpha_{p+m-1}+2\alpha_{p+m}\}$$

$$\Phi(s_{\lambda_{p-1}}(\lambda_{p-2}+\lambda_{p-1}+2\lambda_p)) = \Phi(\lambda_{p-2}+2\lambda_{p-1}+2\lambda_p) = w_{p-1}\Phi(\lambda_{p-2}+\lambda_{p-1}+2\lambda_p)$$

$$\{\alpha_{p-2}+2\alpha_{p-1}+2\alpha_p+\cdots+2\alpha_{p+m-1}+2\alpha_{p+m}\}$$

$$\vdots$$

Roots of length p

$$\Phi(s_{\lambda_1}(\lambda_2+\lambda_3+\cdots+\lambda_p)) = \Phi(\lambda_1+\lambda_2+\cdots+\lambda_p) = w_1\Phi(\lambda_2+\lambda_3+\cdots+\lambda_p)$$

$= \{\alpha_1+\alpha_2+\cdots+\alpha_p, \alpha_1+\alpha_2+\cdots+\alpha_p+\alpha_{p+1}, \alpha_1+\alpha_2+\cdots+\alpha_p+\cdots+\alpha_{p+m}, \alpha_1+\alpha_2+\cdots+\alpha_p+\cdots+\alpha_{p+m-1}+2\alpha_{p+m}, \cdots, \alpha_1+\alpha_2+\cdots+\alpha_p+2\alpha_{p+1}+\cdots+2\alpha_{p+m-1}+2\alpha_{p+m}\}$

$$\Phi(s_{\lambda_p}(\lambda_1+\lambda_2+\cdots+\lambda_{p-1})) = \Phi(\lambda_1+\lambda_2+\cdots+\lambda_{p-1}+2\lambda_p) = w_p\Phi(\lambda_1+\lambda_2+\cdots+\lambda_{p-1})$$

$$= \{\alpha_1+\alpha_2+\cdots+2\alpha_p+\cdots+2\alpha_{p+m}\}$$

$$\vdots$$

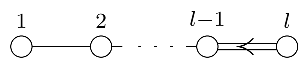
$\Phi(s_{\lambda_2}(\lambda_1+\lambda_2+2\lambda_3+\cdots+2\lambda_{p-1}+2\lambda_p)) = \Phi(\lambda_1+2\lambda_2+\cdots+2\lambda_{p-1}+2\lambda_p) = w_2\Phi(\lambda_1+\lambda_2+2\lambda_3+\cdots+2\lambda_{p-1}+2\lambda_p)$

$$= \{\alpha_1+2\alpha_2+\cdots+2\alpha_p+\cdots+2\alpha_{p+m}\}$$

8.3 C cases

There are 3 isomorphy classes of local symmetric spaces that result from a Lie algebra of type C.

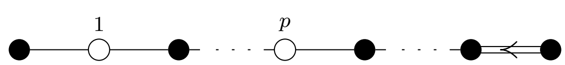
8.3.1 Type CI



For $i = 1, \cdots, l$ we have $\Phi(\lambda_i) = \{\alpha_i\}$. So the multiplicity of each root is $m_{\lambda_i} = 1$.

Also since $w_i = s_{\alpha_i}$ for all i, we have that $\Phi(\mathfrak{a}) = \Phi(\mathfrak{t})$.

8.3.2 Type CII$_a$



Here the multiplicity of each λ_i is

$$\begin{aligned} m_{\lambda_i} &= 4 \text{ for } i = 1, \cdots, p-1 \\ m_{\lambda_p} &= 4m \\ m_{2\lambda_p} &= 3 \end{aligned}$$

We have that

$$\Phi(\lambda_i) = \{\alpha_{2i}, \alpha_{2i-1} + \alpha_{2i}, \alpha_{2i} + \alpha_{2i+1}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1}\} \text{ for } i = 1, \cdots, p-1$$

$$\Phi(\lambda_p) =\{\alpha_{2p}, \alpha_{2p}+\alpha_{2p+1}, \cdots, \alpha_{2p}+\cdots+\alpha_{2p+m}, \alpha_{2p-1}+\alpha_{2p}, \alpha_{2p-1}+\alpha_{2p}+\alpha_{2p+1}, \cdots, \alpha_{2p-1}+\alpha_{2p}+\cdots+\alpha_{2p+m}, \alpha_{2p}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m}, \alpha_{2p}+\cdots+2\alpha_{2p+m-2}+2\alpha_{2p+m-1}+\alpha_{2p+m}, \cdots, \alpha_{2p}+2\alpha_{2p+1}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m}, \alpha_{2p-1}+\alpha_{2p}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m}, \alpha_{2p-1}+\alpha_{2p}+\cdots+2\alpha_{2p+m-2}+2\alpha_{2p+m-1}+\alpha_{2p+m}, \cdots, \alpha_{2p-1}+\alpha_{2p}+2\alpha_{2p+1}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m}\}$$

$\Phi(2\lambda_p) = \{2\alpha_{2p}+2\alpha_{2p+1}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m}, \alpha_{2p-1}+2\alpha_{2p}+2\alpha_{2p+1}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m}, 2\alpha_{2p-1}+2\alpha_{2p}+2\alpha_{2p+1}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m}\}$

$\Phi(\mathfrak{a})$ is of type BC_p and can be computed as in lemma 13.

Now computing the $\Phi(\lambda_i)$ we have:

Roots of length 2:

$$\Phi(s_{\lambda_{i-1}}\lambda_i) = \Phi(\lambda_{i-1}+\lambda_i) = w_{i-1}\Phi(\lambda_i) =$$

$\{\alpha_{2i-2}+\alpha_{2i-1}+\alpha_{2i}, \alpha_{2i-3}+\alpha_{2i-2}+\alpha_{2i-1}+\alpha_{2i}, \alpha_{2i-2}+\alpha_{2i-1}+\alpha_{2i}+\alpha_{2i+1}, \alpha_{2i-3}+\alpha_{2i-2}+\alpha_{2i-1}+\alpha_{2i}+\alpha_{2i+1}\}$ for $i = 2, \cdots, p-1$

$$\Phi(s_{\lambda_{p-1}}\lambda_p) = \Phi(\lambda_{p-1}+\lambda_p) = w_{p-1}\Phi(\lambda_p) =$$

$\{\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}, \alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\alpha_{2p+1}, \cdots, \alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\cdots+\alpha_{2p+m}, \alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}, \alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\alpha_{2p+1}, \cdots, \alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\cdots+\alpha_{2p+m}, \alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m}, \alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\cdots+2\alpha_{2p+m-2}+2\alpha_{2p+m-1}+\alpha_{2p+m}, \cdots, \alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+2\alpha_{2p+1}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m}, \alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m}, \alpha_{2p-3}+$

$\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\cdots+2\alpha_{2p+m-2}+2\alpha_{2p+m-1}+\alpha_{2p+m},\cdots,\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+2\alpha_{2p+1}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m}\}$

$$\Phi(s_{\lambda_p}(\lambda_{p-1}))=\Phi(\lambda_{p-1}+2\lambda_p)=w_p\Phi(\lambda_{p-1})$$

$\{\alpha_{2p-2}+\alpha_{2p-1}+2\alpha_{2p}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m},\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+2\alpha_{2p}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m},\alpha_{2p-2}+2\alpha_{2p-1}+2\alpha_{2p}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m},\alpha_{2p-3}+\alpha_{2p-2}+2\alpha_{2p-1}+2\alpha_{2p}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m}\}$

$\Phi(s_{\lambda_{p-1}}2\lambda_p)=\Phi(2\lambda_{p-1}+2\lambda_p)=w_{p-1}\Phi(2\lambda_p)=\{2\alpha_{2p-2}+2\alpha_{2p-1}+2\alpha_{2p}+2\alpha_{2p+1}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m},\alpha_{2p-3}+2\alpha_{2p-2}+2\alpha_{2p-1}+2\alpha_{2p}+2\alpha_{2p+1}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m},2\alpha_{2p-3}+2\alpha_{2p-2}+2\alpha_{2p-1}+2\alpha_{2p}+2\alpha_{2p+1}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m}\}$

Roots of length 3:

$$\Phi(s_{\lambda_{i-2}}s_{\lambda_{i-1}}\lambda_i)=\Phi(\lambda_{i-2}+\lambda_{i-1}+\lambda_i)=w_{i-2}w_{i-1}\Phi(\lambda_i)=$$

$\{\alpha_{2i-4}+\alpha_{2i-3}+\alpha_{2i-2}+\alpha_{2i-1}+\alpha_{2i},\alpha_{2i-5}+\alpha_{2i-4}+\alpha_{2i-3}+\alpha_{2i-2}+\alpha_{2i-1}+\alpha_{2i},\alpha_{2i-4}+\alpha_{2i-3}+\alpha_{2i-2}+\alpha_{2i-1}+\alpha_{2i}+\alpha_{2i+1},\alpha_{2i-5}+\alpha_{2i-4}+\alpha_{2i-3}+\alpha_{2i-2}+\alpha_{2i-1}+\alpha_{2i}+\alpha_{2i+1}\}$
for $i=3,\cdots,p-1$

$$\Phi(s_{\lambda_{p-2}}s_{\lambda_{p-1}}\lambda_p)=\Phi(\lambda_{p-2}+\lambda_{p-1}+\lambda_p)=w_{p-2}w_{p-1}\Phi(\lambda_p)=$$

$\{\alpha_{2p-4}+\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p},\alpha_{2p-4}+\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\alpha_{2p+1},\cdots,\alpha_{2p-4}+\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\cdots+\alpha_{2p+m},\alpha_{2p-5}+\alpha_{2p-4}+\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p},\alpha_{2p-5}+\alpha_{2p-4}+\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\alpha_{2p+1},\cdots,\alpha_{2p-5}+\alpha_{2p-4}+\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\cdots+\alpha_{2p+m},\alpha_{2p-4}+\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m},\alpha_{2p-4}+\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\cdots+2\alpha_{2p+m-2}+2\alpha_{2p+m-1}+\alpha_{2p+m},\cdots,\alpha_{2p-4}+\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+2\alpha_{2p+1}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m},\alpha_{2p-5}+\alpha_{2p-4}+\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m},\alpha_{2p-5}+\alpha_{2p-4}+\alpha_{2p-3}+$

$\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+\cdots+2\alpha_{2p+m-2}+2\alpha_{2p+m-1}+\alpha_{2p+m},\cdots,\alpha_{2p-5}+\alpha_{2p-4}+\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+\alpha_{2p}+2\alpha_{2p+1}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m}\}$

$$\Phi(s_{\lambda_{p-2}}s_{\lambda_p}(\lambda_{p-1}))=\Phi(\lambda_{p-2}+\lambda_{p-1}+2\lambda_p)=w_{p-2}w_p\Phi(\lambda_{p-1})$$

$\{\alpha_{2p-4}+\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+2\alpha_{2p}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m},\alpha_{2p-5}+\alpha_{2p-4}+\alpha_{2p-3}+\alpha_{2p-2}+\alpha_{2p-1}+2\alpha_{2p}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m},\alpha_{2p-4}+\alpha_{2p-3}+\alpha_{2p-2}+2\alpha_{2p-1}+2\alpha_{2p}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m},\alpha_{2p-5}+\alpha_{2p-4}+\alpha_{2p-3}+\alpha_{2p-2}+2\alpha_{2p-1}+2\alpha_{2p}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m}\}$

$$\Phi(s_{\lambda_{p-1}}s_{\lambda_p}(\lambda_{p-2}+\lambda_{p-1}))=\Phi(\lambda_{p-2}+2\lambda_{p-1}+2\lambda_p)=w_{p-1}w_p\Phi(\lambda_{p-2}+\lambda_{p-1})$$

$\{\alpha_{2p-4}+\alpha_{2p-3}+2\alpha_{2p-2}+2\alpha_{2p-1}+2\alpha_{2p}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m},\alpha_{2p-5}+\alpha_{2p-4}+\alpha_{2p-3}+2\alpha_{2p-2}+2\alpha_{2p-1}+2\alpha_{2p}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m},\alpha_{2p-4}+2\alpha_{2p-3}+2\alpha_{2p-2}+2\alpha_{2p-1}+2\alpha_{2p}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m},\alpha_{2p-5}+\alpha_{2p-4}+2\alpha_{2p-3}+2\alpha_{2p-2}+2\alpha_{2p-1}+2\alpha_{2p}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m}\}$

$$\Phi(s_{\lambda_{p-2}}s_{\lambda_{p-1}}2\lambda_p)=\Phi(2\lambda_{p-2}+2\lambda_{p-1}+2\lambda_p)=w_{p-2}w_{p-1}\Phi(2\lambda_p)=$$

$\{2\alpha_{2p-4}+2\alpha_{2p-3}+2\alpha_{2p-2}+2\alpha_{2p-1}+2\alpha_{2p}+2\alpha_{2p+1}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m},\alpha_{2p-5}+2\alpha_{2p-4}+2\alpha_{2p-3}+2\alpha_{2p-2}+2\alpha_{2p-1}+2\alpha_{2p}+2\alpha_{2p+1}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m},2\alpha_{2p-5}+2\alpha_{2p-4}+2\alpha_{2p-3}+2\alpha_{2p-2}+2\alpha_{2p-1}+2\alpha_{2p}+2\alpha_{2p+1}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m}\}$

$$\vdots$$

Roots of length p:

$\Phi(s_{\lambda_1}(\lambda_2+\lambda_3+\cdots+\lambda_p))=\Phi(\lambda_1+\lambda_2+\cdots+\lambda_p)=w_1\Phi(\lambda_2+\lambda_3+\cdots+\lambda_p)$
$\{\alpha_2+\alpha_3+\cdots+\alpha_{2p},\cdots,\alpha_2+\alpha_3+\cdots+\alpha_{2p}+\cdots+\alpha_{2p+m},\alpha_2+\alpha_3+\cdots+\alpha_{2p}+\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m},\cdots,\alpha_2+\alpha_3+\cdots+\alpha_{2p}+2\alpha_{2p+1}\cdots+2\alpha_{2p+m-1}+\alpha_{2p+m},\alpha_1+\alpha_2+$

$\alpha_3 + \cdots + \alpha_{2p}, \cdots, \alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_{2p} + \cdots + \alpha_{2p+m}, \alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_{2p} + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}, \cdots, \alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_{2p} + 2\alpha_{2p+1} + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}\}$

$\Phi(s_{\lambda_p}(\lambda_1+\lambda_2+\cdots+\lambda_{p-1})) = \Phi(\lambda_1+\lambda_2+\cdots+\lambda_{p-1}+2\lambda_p) = w_p\Phi(\lambda_1+\lambda_2+\cdots+\lambda_{p-1})$

$= \{\alpha_2 + \cdots + \alpha_{p-1} + 2\alpha_p + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}, \alpha_1 + \alpha_2 + \cdots + \alpha_{p-1} + 2\alpha_p + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}, \alpha_2 + \cdots + 2\alpha_{p-1} + 2\alpha_p + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}, \alpha_1 + \alpha_2 + \cdots + 2\alpha_{p-1} + 2\alpha_p + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}\}$

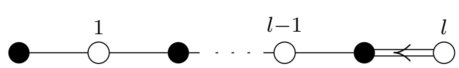
$$\vdots$$

$\Phi(s_{\lambda_2}(\lambda_1 + \lambda_2 + 2\lambda_3 + \cdots + 2\lambda_{p-1} + 2\lambda_p)) = \Phi(\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{p-1} + 2\lambda_p) = w_2\Phi(\lambda_1 + \lambda_2 + 2\lambda_3 + \cdots + 2\lambda_{p-1} + 2\lambda_p) = \{\alpha_2 + \alpha_3 + 2\alpha_4 + \cdots + 2\alpha_{p-1} + 2\alpha_p + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \cdots + 2\alpha_{p-1} + 2\alpha_p + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}, \alpha_2 + 2\alpha_3 + 2\alpha_4 + \cdots + 2\alpha_{p-1} + 2\alpha_p + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \cdots + 2\alpha_{p-1} + 2\alpha_p + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}\}$

$\Phi(s_{\lambda_1} \cdots s_{\lambda_{p-2}} s_{\lambda_{p-1}} 2\lambda_p) = \Phi(2\lambda_1+\cdots+2\lambda_{p-2}+2\lambda_{p-1}+2\lambda_p) = w_1 \cdots w_{p-2} w_{p-1} \Phi(2\lambda_p) =$

$= \{2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \cdots + 2\alpha_{p-1} + 2\alpha_p + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \cdots + 2\alpha_{p-1} + 2\alpha_p + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}, 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \cdots + 2\alpha_{p-1} + 2\alpha_p + \cdots + 2\alpha_{2p+m-1} + \alpha_{2p+m}\}$

8.3.3 Type CII$_b$



Here the multiplicity of each λ_i is

$m_{\lambda_i} = 4$ for $i = 1, \cdots, l-1$
$m_{\lambda_l} = 3$

We have that

$$\Phi(\lambda_i) = \{\alpha_{2i}, \alpha_{2i-1} + \alpha_{2i}, \alpha_{2i} + \alpha_{2i+1}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1}\} \text{ for } i = 1, \cdots, l-1$$

$$\Phi(\lambda_l) = \{\alpha_{2l}, \alpha_{2l-1} + \alpha_{2l}, 2\alpha_{2l-1} + \alpha_{2l}\}$$

$\Phi(\mathfrak{a})$ is of type C_l and can be computed as in lemma 12.

Now computing the $\Phi(\lambda_i)$ we have:

Roots of length 2:

$$\Phi(s_{\lambda_{i+1}}\lambda_i) = \Phi(\lambda_i + \lambda_{i+1}) = w_{i+1}\Phi(\lambda_i) =$$

$\{\alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2}, \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3}\}$
for $i = 1, \cdots, l-1$

$$\Phi(s_{\lambda_{l-1}}\lambda_l) = \Phi(2\lambda_{l-1} + \lambda_l) = w_{l-1}\Phi(\lambda_l) =$$

$$\{2\alpha_{2l-2} + 2\alpha_{2l-1} + \alpha_{2l}, \alpha_{2l-3} + 2\alpha_{2l-2} + 2\alpha_{2l-1} + \alpha_{2l}, 2\alpha_{2l-3} + 2\alpha_{2l-2} + 2\alpha_{2l-1} + \alpha_{2l}\}$$

Roots of length 3:

$$\Phi(s_{\lambda_{i+2}}s_{\lambda_{i+1}}\lambda_i) = \Phi(\lambda_i + \lambda_{i+1} + \lambda_{i+2}) = w_{i+2}w_{i+1}\Phi(\lambda_i) =$$

$\{\alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3} + \alpha_{2i+4}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3} + \alpha_{2i+4}, \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3} + \alpha_{2i+4} + \alpha_{2i+5}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3} + \alpha_{2i+4} + \alpha_{2i+5}\}$ for $i = 1, \cdots, l-2$

$$\Phi(s_{\lambda_{l-1}}(\lambda_{l-2} + \lambda_{l-1} + \lambda_l)) = \Phi(\lambda_{l-2} + 2\lambda_{l-1} + \lambda_l) = w_{l-1}\Phi(\lambda_{l-2} + \lambda_{l-1} + \lambda_l) =$$

$\{\alpha_{2l-4} + \alpha_{2l-3} + 2\alpha_{2l-2} + 2\alpha_{2l-1} + \alpha_{2l}, \alpha_{2l-5} + \alpha_{2l-4} + \alpha_{2l-3} + 2\alpha_{2l-2} + 2\alpha_{2l-1} + \alpha_{2l}, \alpha_{2l-4} + 2\alpha_{2l-3} + 2\alpha_{2l-2} + 2\alpha_{2l-1} + \alpha_{2l}, \alpha_{2l-5} + \alpha_{2l-4} + 2\alpha_{2l-3} + 2\alpha_{2l-2} + 2\alpha_{2l-1} + \alpha_{2l}\}$

$$\Phi(s_{\lambda_{l-2}} s_{\lambda_{l-1}} \lambda_l) = \Phi(2\lambda_{l-2} + 2\lambda_{l-1} + \lambda_l) = w_{l-2}w_{l-1}\Phi(\lambda_l) =$$

$\{2\alpha_{2l-4} + 2\alpha_{2l-3} + 2\alpha_{2l-2} + 2\alpha_{2l-1} + \alpha_{2l}, \alpha_{2l-5} + 2\alpha_{2l-4} + 2\alpha_{2l-3} + 2\alpha_{2l-2} + 2\alpha_{2l-1} + \alpha_{2l}, 2\alpha_{2l-5} + 2\alpha_{2l-4} + 2\alpha_{2l-3} + 2\alpha_{2l-2} + 2\alpha_{2l-1} + \alpha_{2l}\}$

$$\vdots$$

Roots of length l:

$$\Phi(s_{\lambda_l}(\lambda_1 + \lambda_2 + \cdots + \lambda_{l-1})) = \Phi(\lambda_1 + \lambda_2 + \cdots + \lambda_l) = w_l\Phi(\lambda_1 + \lambda_2 + \cdots + \lambda_{l-1})$$

$\{\alpha_2 + \alpha_3 + \cdots + \alpha_{2l}, \alpha_1 + \alpha_2 + \cdots + \alpha_{2l}, \alpha_2 + \alpha_3 + \cdots + 2\alpha_{2l-1} + \alpha_{2l}, \alpha_1 + \alpha_2 + \cdots + 2\alpha_{2l-1} + \alpha_{2l}\}$

$$\Phi(s_{\lambda_{l-1}}(\lambda_1 + \lambda_2 + \cdots + \lambda_l)) = \Phi(\lambda_1 + \lambda_2 + \cdots + 2\lambda_{l-1} + \lambda_l) = w_{l-1}\Phi(\lambda_1 + \lambda_2 + \cdots + \lambda_l)$$

$\{\alpha_2 + \alpha_3 + \cdots + 2\alpha_{2l-2} + 2\alpha_{2l-1} + \alpha_{2l}, \alpha_1 + \alpha_2 + \cdots + 2\alpha_{2l-2} + 2\alpha_{2l-1} + \alpha_{2l}, \alpha_2 + \alpha_3 + \cdots + 2\alpha_{2l-3} + 2\alpha_{2l-2} + 2\alpha_{2l-1} + \alpha_{2l}, \alpha_1 + \alpha_2 + \cdots + 2\alpha_{2l-3} + 2\alpha_{2l-2} + 2\alpha_{2l-1} + \alpha_{2l}\}$

$$\vdots$$

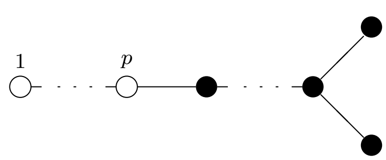
$\Phi(s_{\lambda_2}(\lambda_1 + \lambda_2 + 2\lambda_3 + \cdots + 2\lambda_{l-1} + \lambda_l)) = \Phi(\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{l-1} + \lambda_l) = w_2\Phi(\lambda_1 + \lambda_2 + 2\lambda_3 + \cdots + 2\lambda_{l-1} + \lambda_l) = \{\alpha_2 + \alpha_3 + 2\alpha_4 + \cdots + 2\alpha_{2l-1} + \alpha_{2l}, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \cdots + 2\alpha_{2l-2} + 2\alpha_{2l-1} + \alpha_{2l}, \alpha_2 + 2\alpha_3 + 2\alpha_4 + \cdots + 2\alpha_{2l-1} + \alpha_{2l}, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \cdots + 2\alpha_{2l-1} + \alpha_{2l}\}$

$\Phi(s_{\lambda_1}(2\lambda_2 + 2\lambda_3 + \cdots + 2\lambda_{l-1} + \lambda_l)) = \Phi(2\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{l-1} + \lambda_l) = w_1\Phi(2\lambda_2 + 2\lambda_3 + \cdots + 2\lambda_{l-1} + \lambda_l) = \{2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \cdots + 2\alpha_{2l-1} + \alpha_{2l}, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \cdots + 2\alpha_{2l-2} + 2\alpha_{2l-1} + \alpha_{2l}, 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \cdots + 2\alpha_{2l-1} + \alpha_{2l}\}$

8.4 D cases

There are 4 isomorphy classes of local symmetric spaces that result from a Lie algebra of type D.

8.4.1 Type DI$_a$



Here the multiplicity of each λ_i is

$m_{\lambda_i} = 1$ for $i = 1, \cdots, p-1$
$m_{\lambda_p} = 2m$

We have that

$$\Phi(\lambda_i) = \{\alpha_i\} \text{ for } i = 1, \cdots, p-1$$

$\Phi(\lambda_p) = \{\alpha_p, \alpha_p + \alpha_{p+1}, \cdots, \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m-1}, \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m-2} + \alpha_{p+m}, \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m-2} + \alpha_{p+m-1} + \alpha_{p+m}, \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m-3} + 2\alpha_{p+m-2} + \alpha_{p+m-1} + \alpha_{p+m}, \cdots, \alpha_p + 2\alpha_{p+1} + \cdots + 2\alpha_{p+m-3} + 2\alpha_{p+m-2} + \alpha_{p+m-1} + \alpha_{p+m}\}$

$\Phi(\mathfrak{a})$ is of type B_p and can be computed as in lemma 11.

Now computing the $\Phi(\lambda_i)$ we have:

Roots of length 2:

$$\Phi(s_{\lambda_{i-1}}\lambda_i) = \Phi(\lambda_{i-1} + \lambda_i) = w_{i-1}\Phi(\lambda_i) =$$

$$\{\alpha_{i-1} + \alpha_i\} \text{ for } i = 2, \cdots, p-1$$

$$\Phi(s_{\lambda_{p-1}}\lambda_p) = \Phi(\lambda_{p-1} + \lambda_p) = w_{p-1}\Phi(\lambda_p) =$$

$\{\alpha_{p-1} + \alpha_p, \alpha_{p-1} + \alpha_p + \alpha_{p+1}, \cdots, \alpha_{p-1} + \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m-1}, \alpha_{p-1} + \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m-2} + \alpha_{p+m}, \alpha_{p-1} + \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m-2} + \alpha_{p+m-1} + \alpha_{p+m}, \alpha_{p-1} + \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+m-3} + 2\alpha_{p+m-2} + \alpha_{p+m-1} + \alpha_{p+m}, \cdots, \alpha_{p-1} + \alpha_p + 2\alpha_{p+1} + \cdots + 2\alpha_{p+m-3} + 2\alpha_{p+m-2} + \alpha_{p+m-1} + \alpha_{p+m}\}$

$$\Phi(s_{\lambda_p}(\lambda_{p-1})) = \Phi(\lambda_{p-1} + 2\lambda_p) = w_p\Phi(\lambda_{p-1})$$

$$\{\alpha_{p-1} + 2\alpha_p + \cdots + 2\alpha_{p+m-2} + \alpha_{p+m-1} + \alpha_{p+m}\}$$

Roots of length 3:

$$\Phi(s_{\lambda_{i-2}}s_{\lambda_{i-1}}\lambda_i) = \Phi(\lambda_{i-2} + \lambda_{i-1} + \lambda_i) = w_{i-2}w_{i-1}\Phi(\lambda_i) =$$

$$\{\alpha_{i-2}+\alpha_{i-1}+\alpha_i\} \text{ for } i=3,\cdots,p-1$$

$$\Phi(s_{\lambda_{p-2}}s_{\lambda_{p-1}}\lambda_p)=\Phi(\lambda_{p-2}+\lambda_{p-1}+\lambda_p)=w_{p-2}w_{p-1}\Phi(\lambda_p)=$$

$\{\alpha_{p-2}+\alpha_{p-1}+\alpha_p, \alpha_{p-2}+\alpha_{p-1}+\alpha_p+\alpha_{p+1},\cdots,\alpha_{p-2}+\alpha_{p-1}+\alpha_p+\alpha_{p+1}+\cdots+\alpha_{p+m-1}, \alpha_{p-2}+\alpha_{p-1}+\alpha_p+\alpha_{p+1}+\cdots+\alpha_{p+m-2}+\alpha_{p+m}, \alpha_{p-2}+\alpha_{p-1}+\alpha_p+\alpha_{p+1}+\cdots+\alpha_{p+m-2}+\alpha_{p+m-1}+\alpha_{p+m}, \alpha_{p-2}+\alpha_{p-1}+\alpha_p+\alpha_{p+1}+\cdots+\alpha_{p+m-3}+2\alpha_{p+m-2}+\alpha_{p+m-1}+\alpha_{p+m},\cdots,\alpha_{p-2}+\alpha_{p-1}+\alpha_p+2\alpha_{p+1}+\cdots+2\alpha_{p+m-3}+2\alpha_{p+m-2}+\alpha_{p+m-1}+\alpha_{p+m}\}$

$$\Phi(s_{\lambda_p}(\lambda_{p-2}+\lambda_{p-1}))=\Phi(\lambda_{p-2}+\lambda_{p-1}+2\lambda_p)=w_p\Phi(\lambda_{p-2}+\lambda_{p-1})$$

$$\{\alpha_{p-2}+\alpha_{p-1}+2\alpha_p+\cdots+2\alpha_{p+m-2}+\alpha_{p+m-1}+\alpha_{p+m}\}$$

$$\Phi(s_{\lambda_{p-1}}(\lambda_{p-2}+\lambda_{p-1}+2\lambda_p))=\Phi(\lambda_{p-2}+2\lambda_{p-1}+2\lambda_p)=w_{p-1}\Phi(\lambda_{p-2}+\lambda_{p-1}+2\lambda_p)$$

$$\{\alpha_{p-2}+2\alpha_{p-1}+2\alpha_p+\cdots+2\alpha_{p+m-2}+\alpha_{p+m-1}+\alpha_{p+m}\}$$

$$\vdots$$

Roots of length p

$$\Phi(s_{\lambda_1}(\lambda_2+\lambda_3+\cdots+\lambda_p))=\Phi(\lambda_1+\lambda_2+\cdots+\lambda_p)=w_1\Phi(\lambda_2+\lambda_3+\cdots+\lambda_p)$$

$=\{\alpha_1+\alpha_2+\cdots+\alpha_p, \alpha_1+\alpha_2+\cdots+\alpha_p+\alpha_{p+1},\cdots,\alpha_1+\alpha_2+\cdots+\alpha_p+\cdots+\alpha_{p+m-2}, \alpha_1+\alpha_2+\cdots+\alpha_p+\cdots+\alpha_{p+m-2}+\alpha_{p+m-1}, \alpha_1+\alpha_2+\cdots+\alpha_p+\cdots+\alpha_{p+m-2}+\alpha_{p+m}, \alpha_1+\alpha_2+\cdots+\alpha_p+\cdots+\alpha_{p+m-2}+\alpha_{p+m-1}+\alpha_{p+m}, \alpha_1+\alpha_2+\cdots+\alpha_p+\cdots+2\alpha_{p+m-2}+$

$\alpha_{p+m-1}+\alpha_{p+m},\cdots,\alpha_1+\alpha_2+\cdots+\alpha_p+2\alpha_{p+1}+\cdots+2\alpha_{p+m-2}+\alpha_{p+m-1}+\alpha_{p+m}\}$

$$\Phi(s_{\lambda_p}(\lambda_1+\lambda_2+\cdots+\lambda_{p-1}))=\Phi(\lambda_1+\lambda_2+\cdots+\lambda_{p-1}+2\lambda_p)=w_p\Phi(\lambda_1+\lambda_2+\cdots+\lambda_{p-1})$$

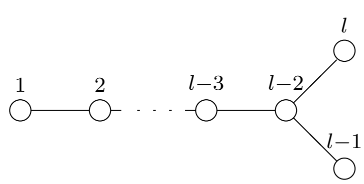
$$=\{\alpha_1+\alpha_2+\cdots+2\alpha_p+\cdots+2\alpha_{p+m-2}+\alpha_{p+m-1}+\alpha_{p+m}\}$$

$$\vdots$$

$\Phi(s_{\lambda_2}(\lambda_1+\lambda_2+2\lambda_3+\cdots+2\lambda_{p-1}+2\lambda_p))=\Phi(\lambda_1+2\lambda_2+\cdots+2\lambda_{p-1}+2\lambda_p)=w_2\Phi(\lambda_1+\lambda_2+2\lambda_3+\cdots+2\lambda_{p-1}+2\lambda_p)$

$$=\{\alpha_1+2\alpha_2+\cdots+2\alpha_p+\cdots+2\alpha_{p+m-2}+\alpha_{p+m-1}+\alpha_{p+m}\}$$

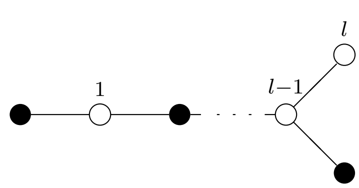
8.4.2 Type DI$_b$



For $i=1,\cdots,l$ we have $\Phi(\lambda_i)=\{\alpha_i\}$. So the multiplicity of each root is $m_{\lambda_i}=1$.

Also since $w_i=s_{\alpha_i}$ for all i, we have that $\Phi(\mathfrak{a})=\Phi(\mathfrak{t})$.

8.4.3 Type DIII$_a$



Here the multiplicity of each λ_i is

$m_{\lambda_i} = 4$ for $i = 1, \cdots, l-1$
$m_{\lambda_l} = 1$

We have that
$\Phi(\lambda_i) = \{\alpha_{2i}, \alpha_{2i-1} + \alpha_{2i}, \alpha_{2i} + \alpha_{2i+1}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1}\}$ for $i = 1, \cdots, l-1$

$$\Phi(\lambda_l) = \{\alpha_{2l}\}$$

$\Phi(\mathfrak{a})$ is of type C_l and can be computed as in lemma 12.

Now computing the $\Phi(\lambda_i)$ we have:

Roots of length 2:

$$\Phi(s_{\lambda_{i+1}}\lambda_i) = \Phi(\lambda_i + \lambda_{i+1}) = w_{i+1}\Phi(\lambda_i) =$$

$\{\alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2}, \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3}\}$ for $i = 1, \cdots, l-2$

$$\Phi(s_{\lambda_l}\lambda_{l-1}) = \Phi(\lambda_{l-1} + \lambda_l) = w_l\Phi(\lambda_{l-1}) =$$

$\{\alpha_{2l-2} + \alpha_{2l}, \alpha_{2l-3} + \alpha_{2l-2} + \alpha_{2l}, \alpha_{2l-2} + \alpha_{2l-1} + \alpha_{2l}, \alpha_{2l-3} + \alpha_{2l-2} + \alpha_{2l-1} + \alpha_{2l}\}$

$$\Phi(s_{\lambda_{l-1}}\lambda_l) = \Phi(2\lambda_{l-1} + \lambda_l) = w_{l-1}\Phi(\lambda_l) =$$

$$\{\alpha_{2l-3} + 2\alpha_{2l-2} + \alpha_{2l-1} + \alpha_{2l}\}$$

Roots of length 3:

$$\Phi(s_{\lambda_{i+2}}s_{\lambda_{i+1}}\lambda_i) = \Phi(\lambda_i + \lambda_{i+1} + \lambda_{i+2}) = w_{i+2}w_{i+1}\Phi(\lambda_i) =$$

$\{\alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3} + \alpha_{2i+4}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3} + \alpha_{2i+4}, \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3} + \alpha_{2i+4} + \alpha_{2i+5}, \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1} + \alpha_{2i+2} + \alpha_{2i+3} + \alpha_{2i+4} + \alpha_{2i+5}\}$ for $i = 1, \cdots, l-3$

$$\Phi(s_{\lambda_l}s_{\lambda_{l-1}}\lambda_{l-2}) = \Phi(\lambda_{l-2} + \lambda_{l-1} + \lambda_l) = w_l w_{l-1}\Phi(\lambda_{l-2}) =$$

$\{\alpha_{2l-4} + \alpha_{2l-3} + \alpha_{2l-2} + \alpha_{2l}, \alpha_{2l-5} + \alpha_{2l-4} + \alpha_{2l-3} + \alpha_{2l-2} + \alpha_{2l}, \alpha_{2l-4} + \alpha_{2l-3} + \alpha_{2l-2} + \alpha_{2l-1} + \alpha_{2l}, \alpha_{2l-5} + \alpha_{2l-4} + \alpha_{2l-3} + \alpha_{2l-2} + \alpha_{2l-1} + \alpha_{2l}\}$

$$\Phi(s_{\lambda_{l-1}}(\lambda_{l-2} + \lambda_{l-1} + \lambda_l)) = \Phi(\lambda_{l-2} + 2\lambda_{l-1} + \lambda_l) = w_{l-1}\Phi(\lambda_{l-2} + \lambda_{l-1} + \lambda_l) =$$

$\{\alpha_{2l-4} + \alpha_{2l-3} + 2\alpha_{2l-2} + \alpha_{2l-1} + \alpha_{2l}, \alpha_{2l-5} + \alpha_{2l-4} + \alpha_{2l-3} + 2\alpha_{2l-2} + \alpha_{2l-1} + \alpha_{2l}, \alpha_{2l-4} + 2\alpha_{2l-3} + 2\alpha_{2l-2} + \alpha_{2l-1} + \alpha_{2l}, \alpha_{2l-5} + \alpha_{2l-4} + 2\alpha_{2l-3} + 2\alpha_{2l-2} + \alpha_{2l-1} + \alpha_{2l}\}$

$$\Phi(s_{\lambda_{l-2}}s_{\lambda_{l-1}}\lambda_l) = \Phi(2\lambda_{l-2} + 2\lambda_{l-1} + \lambda_l) = w_{l-2}w_{l-1}\Phi(\lambda_l) =$$
$$\{\alpha_{2l-5} + 2\alpha_{2l-4} + 2\alpha_{2l-3} + 2\alpha_{2l-2} + \alpha_{2l-1} + \alpha_{2l}\}$$

Roots of length l:

$$\Phi(s_{\lambda_l}s_{\lambda_{l-1}}\cdots s_{\lambda_2}(\lambda_1)) = \Phi(\lambda_1 + \lambda_2 + \cdots + \lambda_{l-1} + \lambda_l) = w_l w_{l-1}\cdots w_2\Phi(\lambda_1) =$$

$\{\alpha_1 + \cdots + \alpha_{2l-2} + \alpha_{2l}, \alpha_2 + \cdots + \alpha_{2l-2} + \alpha_{2l}, \alpha_1 + \cdots + \alpha_{2l-2} + \alpha_{2l-1} + \alpha_{2l}, \alpha_2 + \cdots + \alpha_{2l-2} + \alpha_{2l-1} + \alpha_{2l}\}$

$$\Phi(s_{\lambda_{l-1}}(\lambda_1+\cdots+\lambda_{l-1}+\lambda_l)) = \Phi(\lambda_1+\cdots+\lambda_{l-2}+2\lambda_{l-1}+\lambda_l) = w_{l-1}\Phi(\lambda_1+\cdots+\lambda_{l-1}+\lambda_l) =$$

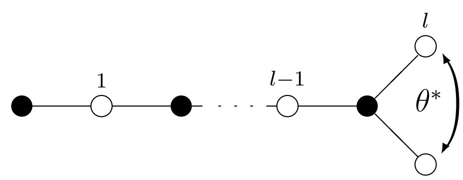
$\{\alpha_1 + \cdots + 2\alpha_{2l-2} + \alpha_{2l-1} + \alpha_{2l}, \alpha_2 + \cdots + 2\alpha_{2l-2} + \alpha_{2l-1} + \alpha_{2l}, \alpha_1 + \cdots + 2\alpha_{2l-3} + 2\alpha_{2l-2} + \alpha_{2l-1} + \alpha_{2l}, \alpha_2 + \cdots + 2\alpha_{2l-3} + 2\alpha_{2l-2} + \alpha_{2l-1} + \alpha_{2l}\}$

$$\vdots$$

$$\Phi(s_{\lambda_1}\cdots s_{\lambda_{l-2}}s_{\lambda_{l-1}}\lambda_l) = \Phi(2\lambda_1+\cdots+2\lambda_{l-1}+\lambda_l) = w_1\cdots w_{l-2}w_{l-1}\Phi(\lambda_l) =$$

$$\{\alpha_1+2\alpha_2+\cdots+2\alpha_{p+m-2}+\alpha_{p+m-1}+\alpha_{p+m}\}$$

8.4.4 Type DIII$_b$



Here the multiplicity of each λ_i is

$$m_{\lambda_i} = 4 \text{ for } i = 1,\cdots,l$$
$$m_{2\lambda_l} = 1$$

We have that

$$\Phi(\lambda_i) = \{\alpha_{2i}, \alpha_{2i-1}+\alpha_{2i}, \alpha_{2i}+\alpha_{2i+1}, \alpha_{2i-1}+\alpha_{2i}+\alpha_{2i+1}\} \text{ for } i = 1,\cdots,l-1$$

$$\Phi(\lambda_l) = \{\alpha_{2l}, \alpha_{2l+1}, \alpha_{2l-1}+\alpha_{2l}, \alpha_{2l-1}+\alpha_{2l+1}\}$$

$$\Phi(2\lambda_l) = \{\alpha_{2l-1}+\alpha_{2l}+\alpha_{2l+1}\}$$

$\Phi(\mathfrak{a})$ is of type BC_l and can be computed as in lemma 13.

Now computing the $\Phi(\lambda_i)$ we have:

Roots of length 2:

$$\Phi(s_{\lambda_{i-1}}\lambda_i) = \Phi(\lambda_{i-1} + \lambda_i) = w_{i-1}\Phi(\lambda_i) =$$

$\{\alpha_{2i-2} + \alpha_{2i-1} + \alpha_{2i}, \alpha_{2i-3} + \alpha_{2i-2} + \alpha_{2i-1} + \alpha_{2i}, \alpha_{2i-2} + \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1}, \alpha_{2i-3} + \alpha_{2i-2} + \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1}\}$ for $i = 2, \cdots, l-1$

$$\Phi(s_{\lambda_{l-1}}\lambda_l) = \Phi(\lambda_{l-1} + \lambda_l) = w_{l-1}\Phi(\lambda_l) =$$

$\{\alpha_{2l-2} + \alpha_{2l-1} + \alpha_{2l}, \alpha_{2l-2} + \alpha_{2l-1} + \alpha_{2l+1}, \alpha_{2l-3} + \alpha_{2l-2} + \alpha_{2l-1} + \alpha_{2l}, \alpha_{2l-3} + \alpha_{2l-2} + \alpha_{2l-1} + \alpha_{2l+1}\}$

$$\Phi(s_{\lambda_l}(\lambda_{l-1})) = \Phi(\lambda_{l-1} + 2\lambda_l) = w_l\Phi(\lambda_{l-1})$$

$\{\alpha_{2l-2} + \alpha_{2l-1} + \alpha_{2l} + \alpha_{2l+1}, \alpha_{2l-2} + 2\alpha_{2l-1} + \alpha_{2l} + \alpha_{2l+1}, \alpha_{2l-3} + \alpha_{2l-2} + \alpha_{2l-1} + \alpha_{2l} + \alpha_{2l+1}, \alpha_{2l-3} + \alpha_{2l-2} + 2\alpha_{2l-1} + \alpha_{2l} + \alpha_{2l+1}\}$

$$\Phi(s_{\lambda_{l-1}}2\lambda_l) = \Phi(2\lambda_{l-1} + 2\lambda_l) = w_{l-1}\Phi(2\lambda_l) =$$

$$\{\alpha_{2l-3} + 2\alpha_{2l-2} + 2\alpha_{2l-1} + \alpha_{2l} + \alpha_{2l+1}\}$$

Roots of length 3:

$$\Phi(s_{\lambda_{i-2}}s_{\lambda_{i-1}}\lambda_i) = \Phi(\lambda_{i-2} + \lambda_{i-1} + \lambda_i) = w_{i-2}w_{i-1}\Phi(\lambda_i) =$$

$\{\alpha_{2i-4} + \alpha_{2i-3} + \alpha_{2i-2} + \alpha_{2i-1} + \alpha_{2i}, \alpha_{2i-5} + \alpha_{2i-4} + \alpha_{2i-3} + \alpha_{2i-2} + \alpha_{2i-1} + \alpha_{2i}, \alpha_{2i-4} + \alpha_{2i-3} + \alpha_{2i-2} + \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1}, \alpha_{2i-5} + \alpha_{2i-4} + \alpha_{2i-3} + \alpha_{2i-2} + \alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1}\}$ for $i = 3, \cdots, l-1$

$$\Phi(s_{\lambda_{l-2}}s_{\lambda_{l-1}}\lambda_l) = \Phi(\lambda_{l-2} + \lambda_{l-1} + \lambda_l) = w_{l-2}w_{l-1}\Phi(\lambda_l) =$$

$\{\alpha_{2l-4}+\alpha_{2l-3}+\alpha_{2l-2}+\alpha_{2l-1}+\alpha_{2l}, \alpha_{2l-4}+\alpha_{2l-3}+\alpha_{2l-2}+\alpha_{2l-1}+\alpha_{2l+1}, \alpha_{2l-5}+\alpha_{2l-4}+\alpha_{2l-3}+\alpha_{2l-2}+\alpha_{2l-1}+\alpha_{2l}, \alpha_{2l-5}+\alpha_{2l-4}+\alpha_{2l-3}+\alpha_{2l-2}+\alpha_{2l-1}+\alpha_{2l+1}\}$

$$\Phi(s_{\lambda_{l-2}}s_{\lambda_l}(\lambda_{l-1})) = \Phi(\lambda_{l-2}+\lambda_{l-1}+2\lambda_l) = w_{l-2}w_l\Phi(\lambda_{l-1})$$

$\{\alpha_{2l-4}+\alpha_{2l-3}+\alpha_{2l-2}+\alpha_{2l-1}+\alpha_{2l}+\alpha_{2l+1}, \alpha_{2l-4}+\alpha_{2l-3}+\alpha_{2l-2}+2\alpha_{2l-1}+\alpha_{2l}+\alpha_{2l+1}, \alpha_{2l-5}+\alpha_{2l-4}+\alpha_{2l-3}+\alpha_{2l-2}+\alpha_{2l-1}+\alpha_{2l}+\alpha_{2l+1}, \alpha_{2l-5}+\alpha_{2l-4}+\alpha_{2l-3}+\alpha_{2l-2}+2\alpha_{2l-1}+\alpha_{2l}+\alpha_{2l+1}\}$

$$\Phi(s_{\lambda_{l-1}}s_{\lambda_l}(\lambda_{l-2}+\lambda_{l-1})) = \Phi(\lambda_{l-2}+2\lambda_{l-1}+2\lambda_l) = w_{l-1}w_l\Phi(\lambda_{l-2}+\lambda_{l-1})$$

$\{\alpha_{2l-4}+\alpha_{2l-3}+2\alpha_{2l-2}+2\alpha_{2l-1}+\alpha_{2l}+\alpha_{2l+1}, \alpha_{2l-4}+2\alpha_{2l-3}+2\alpha_{2l-2}+2\alpha_{2l-1}+\alpha_{2l}+\alpha_{2l+1}, \alpha_{2l-5}+\alpha_{2l-4}+\alpha_{2l-3}+2\alpha_{2l-2}+2\alpha_{2l-1}+\alpha_{2l}+\alpha_{2l+1}, \alpha_{2l-5}+\alpha_{2l-4}+2\alpha_{2l-3}+2\alpha_{2l-2}+2\alpha_{2l-1}+\alpha_{2l}+\alpha_{2l+1}\}$

$$\Phi(s_{\lambda_{l-2}}s_{\lambda_{l-1}}2\lambda_l) = \Phi(2\lambda_{l-2}+2\lambda_{l-1}+2\lambda_l) = w_{l-2}w_{l-1}\Phi(2\lambda_l) =$$

$$\{\alpha_{2l-5}+2\alpha_{2l-4}+2\alpha_{2l-3}+2\alpha_{2l-2}+2\alpha_{2l-1}+\alpha_{2l}+\alpha_{2l+1}\}$$

$$\vdots$$

Roots of length l:

$$\Phi(s_{\lambda_1}(\lambda_2+\lambda_3+\cdots+\lambda_l)) = \Phi(\lambda_1+\lambda_2+\cdots+\lambda_l) = w_1\Phi(\lambda_2+\lambda_3+\cdots+\lambda_l)$$

$\{\alpha_1+\cdots+\alpha_{2l-1}+\alpha_{2l}, \alpha_1+\cdots+\alpha_{2l-1}+\alpha_{2l+1}, \alpha_2+\cdots+\alpha_{2l-1}+\alpha_{2l}, \alpha_2+\cdots+\alpha_{2l-1}+\alpha_{2l+1}\}$

$$\Phi(s_{\lambda_l}(\lambda_1+\lambda_2+\cdots+\lambda_{l-1})) = \Phi(\lambda_1+\lambda_2+\cdots+\lambda_{l-1}+2\lambda_l) = w_l\Phi(\lambda_1+\lambda_2+\cdots+\lambda_{l-1})$$

$\{\alpha_1+\cdots+\alpha_{2l-1}+\alpha_{2l}+\alpha_{2l+1}, \alpha_2+\cdots+\alpha_{2l-1}+\alpha_{2l}+\alpha_{2l+1}, \alpha_1+\cdots+\alpha_{2l-2}+2\alpha_{2l-1}+\alpha_{2l}+\alpha_{2l+1}, \alpha_2+\cdots+\alpha_{2l-2}+2\alpha_{2l-1}+\alpha_{2l}+\alpha_{2l+1}\}$

$$\vdots$$

$\Phi(s_{\lambda_2}(\lambda_1 + \lambda_2 + 2\lambda_3 + \cdots + 2\lambda_{l-1} + 2\lambda_l)) = \Phi(\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{l-1} + 2\lambda_l) = w_2\Phi(\lambda_1 + \lambda_2 + 2\lambda_3 + \cdots + 2\lambda_{l-1} + 2\lambda_l) = \{\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \cdots + 2\alpha_{2l-1} + \alpha_{2l} + \alpha_{2l+1}, \alpha_2 + \alpha_3 + 2\alpha_4 + \cdots + 2\alpha_{2l-1} + \alpha_{2l} + \alpha_{2l+1}, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \cdots + 2\alpha_{2l-2} + 2\alpha_{2l-1} + \alpha_{2l} + \alpha_{2l+1}, \alpha_2 + 2\alpha_3 + 2\alpha_4 + \cdots + 2\alpha_{2l-2} + 2\alpha_{2l-1} + \alpha_{2l} + \alpha_{2l+1}\}$

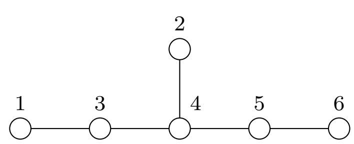
$\Phi(s_{\lambda_1} \cdots s_{\lambda_{l-2}} s_{\lambda_{l-1}} 2\lambda_l) = \Phi(2\lambda_1 + \cdots + 2\lambda_{l-2} + 2\lambda_{l-1} + 2\lambda_l) = w_1 \cdots w_{l-2} w_{l-1} \Phi(2\lambda_l) =$

$$\{\alpha_1 + 2\alpha_2 + 2\alpha_3 + \cdots + 2\alpha_{2l-2} + 2\alpha_{2l-1} + \alpha_{2l} + \alpha_{2l+1}\}$$

8.5 E_6 cases

There are 4 isomorphy classes of local symmetric spaces that result from a Lie algebra of type E_6.

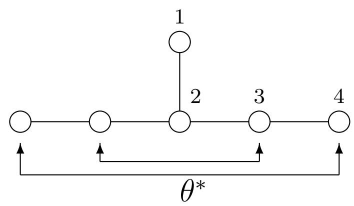
8.5.1 Type EI



For $i = 1, \cdots, 6$ we have $\Phi(\lambda_i) = \{\alpha_i\}$. So the multiplicity of each root is $m_{\lambda_i} = 1$.

Also since $w_i = s_{\alpha_i}$ for all i, we have that $\Phi(\mathfrak{a}) = \Phi(\mathfrak{t})$.

8.5.2 Type EII



Here the multiplicity of each λ_i is

$$m_{\lambda_1} = m_{\lambda_2} = 1$$
$$m_{\lambda_3} = m_{\lambda_4} = 2$$

We have that

$$\Phi(\lambda_1) = \{\alpha_2\}$$
$$\Phi(\lambda_2) = \{\alpha_4\}$$
$$\Phi(\lambda_3) = \{\alpha_3, \alpha_5\}$$
$$\Phi(\lambda_4) = \{\alpha_1, \alpha_6\}$$

$\Phi(\mathfrak{a})$ is of type F_4 and can be computed as in lemma 14.

Roots of length 2:

$$\Phi(s_{\lambda_1}\lambda_2) = \Phi(\lambda_1 + \lambda_2) = w_1\Phi(\lambda_2) =$$
$$\{\alpha_2 + \alpha_4\}$$
$$\Phi(s_{\lambda_2}\lambda_3) = \Phi(\lambda_2 + \lambda_3) = w_2\Phi(\lambda_3) =$$
$$\{\alpha_3 + \alpha_4, \alpha_4 + \alpha_5\}$$
$$\Phi(s_{\lambda_3}\lambda_4) = \Phi(\lambda_3 + \lambda_4) = w_3\Phi(\lambda_4) =$$
$$\{\alpha_1 + \alpha_3, \alpha_5 + \alpha_6\}$$

$$\Phi(s_{\lambda_3}\lambda_2) = \Phi(\lambda_2 + 2\lambda_3) = w_3\Phi(\lambda_2) =$$

$$\{\alpha_3 + \alpha_4 + \alpha_5\}$$

Roots of length 3:

$$\Phi(s_{\lambda_1}s_{\lambda_2}\lambda_3) = \Phi(\lambda_1 + \lambda_2 + \lambda_3) = w_1w_2\Phi(\lambda_3) =$$

$$\{\alpha_2 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_4 + \alpha_5\}$$

$$\Phi(s_{\lambda_2}s_{\lambda_3}\lambda_4) = \Phi(\lambda_2 + \lambda_3 + \lambda_4) = w_2w_3\Phi(\lambda_4) =$$

$$\{\alpha_1 + \alpha_3 + \alpha_4, \alpha_4 + \alpha_5 + \alpha_6\}$$

$$\Phi(s_{\lambda_1}s_{\lambda_3}\lambda_2) = \Phi(\lambda_1 + \lambda_2 + 2\lambda_3) = w_1w_3\Phi(\lambda_2) =$$

$$\{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5\}$$

$$\Phi(s_{\lambda_3}s_{\lambda_2}s_{\lambda_3}\lambda_4) = \Phi(\lambda_2 + 2\lambda_3 + \lambda_4) = w_3w_2w_3\Phi(\lambda_4) =$$

$$\{\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6\}$$

$$\Phi(s_{\lambda_2}s_{\lambda_1}s_{\lambda_3}\lambda_2) = \Phi(\lambda_1 + 2\lambda_2 + 2\lambda_3) = w_2w_1w_3\Phi(\lambda_2) =$$

$$\{\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5\}$$

$$\Phi(s_{\lambda_4}s_{\lambda_3}\lambda_2) = \Phi(\lambda_2 + 2\lambda_3 + 2\lambda_4) = w_4w_3\Phi(\lambda_2) =$$

$$\{\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6\}$$

Roots of length 4:

$$\Phi(s_{\lambda_1}s_{\lambda_2}s_{\lambda_3}\lambda_4) = \Phi(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) = w_1w_2w_3\Phi(\lambda_4) =$$

$$\{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6\}$$

$$\Phi(s_{\lambda_3}s_{\lambda_1}s_{\lambda_2}s_{\lambda_3}\lambda_4) = \Phi(\lambda_1 + \lambda_2 + 2\lambda_3 + \lambda_4) = w_3w_1w_2w_3\Phi(\lambda_4) =$$

$$\{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6\}$$

$$\Phi(s_{\lambda_2}s_{\lambda_3}s_{\lambda_1}s_{\lambda_2}s_{\lambda_3}\lambda_4) = \Phi(\lambda_1 + 2\lambda_2 + 2\lambda_3 + \lambda_4) = w_2w_3w_1w_2w_3\Phi(\lambda_4) =$$

$$\{\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5, \alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6\}$$

$$\Phi(s_{\lambda_4}s_{\lambda_1}s_{\lambda_3}\lambda_2) = \Phi(\lambda_1+\lambda_2+2\lambda_3+2\lambda_4) = w_4w_1w_3\Phi(\lambda_2) =$$

$$\{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6\}$$

$$\Phi(s_{\lambda_3}s_{\lambda_2}s_{\lambda_3}s_{\lambda_1}s_{\lambda_2}s_{\lambda_3}\lambda_4) = \Phi(\lambda_1+2\lambda_2+3\lambda_3+\lambda_4) = w_3w_2w_3w_1w_2w_3\Phi(\lambda_4) =$$

$$\{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5, \alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6\}$$

$$\Phi(s_{\lambda_4}s_{\lambda_2}s_{\lambda_1}s_{\lambda_3}\lambda_2) = \Phi(\lambda_1+2\lambda_2+2\lambda_3+2\lambda_4) = w_4w_2w_1w_3\Phi(\lambda_2) =$$

$$\{\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6\}$$

$$\Phi(s_{\lambda_4}s_{\lambda_3}s_{\lambda_2}s_{\lambda_3}s_{\lambda_1}s_{\lambda_2}s_{\lambda_3}\lambda_4) = \Phi(\lambda_1+2\lambda_2+3\lambda_3+2\lambda_4) = w_4w_3w_2w_3w_1w_2w_3\Phi(\lambda_4) =$$

$$\{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6, \alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6\}$$

$$\Phi(s_{\lambda_3}s_{\lambda_4}s_{\lambda_2}s_{\lambda_1}s_{\lambda_3}\lambda_2) = \Phi(\lambda_1+2\lambda_2+4\lambda_3+2\lambda_4) = w_3w_4w_2w_1w_3\Phi(\lambda_2) =$$

$$\{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6\}$$

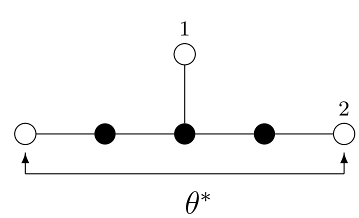
$$\Phi(s_{\lambda_2}s_{\lambda_3}s_{\lambda_4}s_{\lambda_2}s_{\lambda_1}s_{\lambda_3}\lambda_2) = \Phi(\lambda_1+3\lambda_2+4\lambda_3+2\lambda_4) = w_2w_3w_4w_2w_1w_3\Phi(\lambda_2) =$$

$$\{\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6\}$$

$$\Phi(s_{\lambda_1}s_{\lambda_2}s_{\lambda_3}s_{\lambda_4}s_{\lambda_2}s_{\lambda_1}s_{\lambda_3}\lambda_2) = \Phi(2\lambda_1+3\lambda_2+4\lambda_3+2\lambda_4) = w_1w_2w_3w_4w_2w_1w_3\Phi(\lambda_2) =$$

$$\{\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6\}$$

8.5.3 Type EIII



Here the multiplicity of each λ_i is

$$m_{\lambda_1} = 6$$
$$m_{\lambda_2} = 8$$
$$m_{2\lambda_2} = 1$$

We have that

$\Phi(\lambda_1) = \{\alpha_2, \alpha_2+\alpha_4, \alpha_2+\alpha_3+\alpha_4, \alpha_2+\alpha_4+\alpha_5, \alpha_2+\alpha_3+\alpha_4+\alpha_5, \alpha_2+\alpha_3+2\alpha_4+\alpha_5\}$

$\Phi(\lambda_2) = \{\alpha_1, \alpha_6, \alpha_1+\alpha_3, \alpha_1+\alpha_3+\alpha_4, \alpha_1+\alpha_3+\alpha_4+\alpha_5, \alpha_5+\alpha_6, \alpha_4+\alpha_5+\alpha_6, \alpha_3+\alpha_4+\alpha_5+\alpha_6\}$

$$\Phi(2\lambda_2) = \{\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6\}$$

$\Phi(\mathfrak{a})$ is of type BC_2 and can be computed as in lemma 13.

Roots of length 2:

$$\Phi(s_{\lambda_1}\lambda_2) = \Phi(\lambda_1+\lambda_2) = w_1\Phi(\lambda_2) =$$

$\{\alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_2+\alpha_4+\alpha_5+\alpha_6, \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5, \alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5, \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5, \alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6, \alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6, \alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6\}$

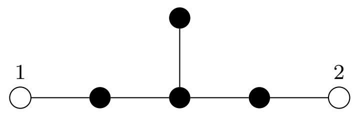
$$\Phi(s_{\lambda_2}\lambda_1) = \Phi(\lambda_1+2\lambda_2) = w_2\Phi(\lambda_1) =$$

$\{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6, \alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6, \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6, \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6, \alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6\}$

$$\Phi(s_{\lambda_1}2\lambda_2) = \Phi(2\lambda_1+2\lambda_2) = w_1\Phi(2\lambda_2) =$$

$$\{\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6\}$$

8.5.4 Type EIV



Here the multiplicity of each λ_i is

$m_{\lambda_i} = 8$

We have that

$\Phi(\lambda_1) = \{\alpha_1, \alpha_1+\alpha_3, \alpha_1+\alpha_3+\alpha_4, \alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_1+\alpha_3+\alpha_4+\alpha_5, \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5, \alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5, \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5,\}$

$\Phi(\lambda_2) = \{\alpha_6, \alpha_5+\alpha_6, \alpha_4+\alpha_5+\alpha_6, \alpha_3+\alpha_4+\alpha_5+\alpha_6, \alpha_2+\alpha_4+\alpha_5+\alpha_6, \alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6, \alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6, \alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6,\}$

$\Phi(\mathfrak{a})$ is of type A_2 and can be computed as in lemma 10.

Root of length 2:

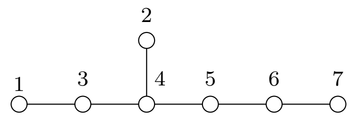
$$\Phi(s_{\lambda_1}\lambda_2) = \Phi(\lambda_1+\lambda_2) = w_1\Phi(\lambda_2) =$$

$\{\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6, \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6, \alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6, \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6, \alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6, \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6, \alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6, \alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6\}$

8.6 E_7 cases

There are 3 isomorphy classes of local symmetric spaces that result from a Lie algebra of type E_7.

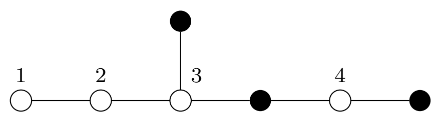
8.6.1 Type EV



For $i = 1, \cdots, 7$ we have $\Phi(\lambda_i) = \{\alpha_i\}$. So the multiplicity of each root is $m_{\lambda_i} = 1$.

Also since $w_i = s_{\alpha_i}$ for all i, we have that $\Phi(\mathfrak{a}) = \Phi(\mathfrak{t})$.

8.6.2 Type EVI



Here the multiplicity of each λ_i is

$$m_{\lambda_1} = m_{\lambda_2} = 1$$
$$m_{\lambda_3} = m_{\lambda_4} = 4$$

We have that

$$\Phi(\lambda_1) = \{\alpha_1\}$$

$$\Phi(\lambda_2) = \{\alpha_3\}$$

$$\Phi(\lambda_3) = \{\alpha_4, \alpha_2 + \alpha_4, \alpha_4 + \alpha_5, \alpha_2 + \alpha_4 + \alpha_5\}$$

$$\Phi(\lambda_4) = \{\alpha_6, \alpha_5 + \alpha_6, \alpha_6 + \alpha_7, \alpha_5 + \alpha_6 + \alpha_7\}$$

$\Phi(\mathfrak{a})$ is of type F_4 and can be computed as in lemma 14.

Roots of length 2:

$$\Phi(s_{\lambda_1}\lambda_2) = \Phi(\lambda_1 + \lambda_2) = w_1\Phi(\lambda_2) =$$

$$\{\alpha_1 + \alpha_3\}$$

$$\Phi(s_{\lambda_2}\lambda_3) = \Phi(\lambda_2 + \lambda_3) = w_2\Phi(\lambda_3) =$$

$$\{\alpha_3 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \alpha_3 + \alpha_4 + \alpha_5, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5\}$$

$$\Phi(s_{\lambda_3}\lambda_4) = \Phi(\lambda_3 + \lambda_4) = w_3\Phi(\lambda_4) =$$

$$\{\alpha_4 + \alpha_5 + \alpha_6, \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7\}$$

$$\Phi(s_{\lambda_3}\lambda_2) = \Phi(\lambda_2 + 2\lambda_3) = w_3\Phi(\lambda_2) =$$

$$\{\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5\}$$

Roots of length 3:

$$\Phi(s_{\lambda_1}s_{\lambda_2}\lambda_3) = \Phi(\lambda_1 + \lambda_2 + \lambda_3) = w_1w_2\Phi(\lambda_3) =$$

$$\{\alpha_1 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5\}$$

$$\Phi(s_{\lambda_2}s_{\lambda_3}\lambda_4) = \Phi(\lambda_2 + \lambda_3 + \lambda_4) = w_2w_3\Phi(\lambda_4) =$$

$$\{\alpha_3+\alpha_4+\alpha_5+\alpha_6, \alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6, \alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7, \alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7\}$$

$$\Phi(s_{\lambda_1}s_{\lambda_3}\lambda_2) = \Phi(\lambda_1 + \lambda_2 + 2\lambda_3) = w_1w_3\Phi(\lambda_2) =$$

$$\{\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5\}$$

$$\Phi(s_{\lambda_3}s_{\lambda_2}s_{\lambda_3}\lambda_4) = \Phi(\lambda_2 + 2\lambda_3 + \lambda_4) = w_3w_2w_3\Phi(\lambda_4) =$$

$\{\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6, \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7\}$

$$\Phi(s_{\lambda_2}s_{\lambda_1}s_{\lambda_3}\lambda_2) = \Phi(\lambda_1 + 2\lambda_2 + 2\lambda_3) = w_2w_1w_3\Phi(\lambda_2) =$$

$$\{\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5\}$$

$$\Phi(s_{\lambda_4}s_{\lambda_3}\lambda_2) = \Phi(\lambda_2 + 2\lambda_3 + 2\lambda_4) = w_4w_3\Phi(\lambda_2) =$$

$$\{\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7\}$$

Roots of length 4:

$$\Phi(s_{\lambda_1}s_{\lambda_2}s_{\lambda_3}\lambda_4) = \Phi(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) = w_1w_2w_3\Phi(\lambda_4) =$$

$\{\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6, \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6, \alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7\}$

$$\Phi(s_{\lambda_3}s_{\lambda_1}s_{\lambda_2}s_{\lambda_3}\lambda_4) = \Phi(\lambda_1 + \lambda_2 + 2\lambda_3 + \lambda_4) = w_3w_1w_2w_3\Phi(\lambda_4) =$$

$\{\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7\}$

$$\Phi(s_{\lambda_2}s_{\lambda_3}s_{\lambda_1}s_{\lambda_2}s_{\lambda_3}\lambda_4) = \Phi(\lambda_1 + 2\lambda_2 + 2\lambda_3 + \lambda_4) = w_2w_3w_1w_2w_3\Phi(\lambda_4) =$$

$\{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6, \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6, \alpha_1+\alpha_2+2\alpha_3+2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7\}$

$$\Phi(s_{\lambda_4}s_{\lambda_1}s_{\lambda_3}\lambda_2) = \Phi(\lambda_1 + \lambda_2 + 2\lambda_3 + 2\lambda_4) = w_4w_1w_3\Phi(\lambda_2) =$$

$$\{\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7\}$$

$$\Phi(s_{\lambda_3}s_{\lambda_2}s_{\lambda_3}s_{\lambda_1}s_{\lambda_2}s_{\lambda_3}\lambda_4) = \Phi(\lambda_1 + 2\lambda_2 + 3\lambda_3 + \lambda_4) = w_3w_2w_3w_1w_2w_3\Phi(\lambda_4) =$$

$\{\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7\}$

$$\Phi(s_{\lambda_4}s_{\lambda_2}s_{\lambda_1}s_{\lambda_3}\lambda_2) = \Phi(\lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4) = w_4w_2w_1w_3\Phi(\lambda_2) =$$

$$\{\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7\}$$

$$\Phi(s_{\lambda_4}s_{\lambda_3}s_{\lambda_2}s_{\lambda_3}s_{\lambda_1}s_{\lambda_2}s_{\lambda_3}\lambda_4) = \Phi(\lambda_1 + 2\lambda_2 + 3\lambda_3 + 2\lambda_4) = w_4w_3w_2w_3w_1w_2w_3\Phi(\lambda_4) =$$

$\{\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7, \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7\}$

$$\Phi(s_{\lambda_3}s_{\lambda_4}s_{\lambda_2}s_{\lambda_1}s_{\lambda_3}\lambda_2) = \Phi(\lambda_1 + 2\lambda_2 + 4\lambda_3 + 2\lambda_4) = w_3w_4w_2w_1w_3\Phi(\lambda_2) =$$

$$\{\alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7\}$$

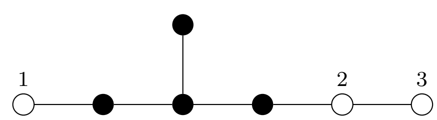
$$\Phi(s_{\lambda_2}s_{\lambda_3}s_{\lambda_4}s_{\lambda_2}s_{\lambda_1}s_{\lambda_3}\lambda_2) = \Phi(\lambda_1 + 3\lambda_2 + 4\lambda_3 + 2\lambda_4) = w_2w_3w_4w_2w_1w_3\Phi(\lambda_2) =$$

$$\{\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7\}$$

$$\Phi(s_{\lambda_1}s_{\lambda_2}s_{\lambda_3}s_{\lambda_4}s_{\lambda_2}s_{\lambda_1}s_{\lambda_3}\lambda_2) = \Phi(2\lambda_1 + 3\lambda_2 + 4\lambda_3 + 2\lambda_4) = w_1w_2w_3w_4w_2w_1w_3\Phi(\lambda_2) =$$

$$\{2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7\}$$

8.6.3 Type EVII



Here the multiplicity of each λ_i is

$m_{\lambda_1} = m_{\lambda_2} = 8$
$m_{\lambda_3} = 1$

We have that

$\Phi(\lambda_1) = \{\alpha_1, \alpha_1 + \alpha_3, \alpha_1 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5\}$

$\Phi(\lambda_2) = \{\alpha_6, \alpha_5 + \alpha_6, \alpha_4 + \alpha_5 + \alpha_6, \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6\}$

$$\Phi(\lambda_3) = \{\alpha_7\}$$

$\Phi(\mathfrak{a})$ is of type C_3 and can be computed as in lemma 12.

Roots of length 2:

$$\Phi(s_{\lambda_2}\lambda_1) = \Phi(\lambda_1 + \lambda_2) = w_2\Phi(\lambda_1) =$$

$\{\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6, \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6, \alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6, \alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6, \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6, \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6, \alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6, \alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6\}$

$$\Phi(s_{\lambda_3}\lambda_2) = \Phi(\lambda_2 + \lambda_3) = w_3\Phi(\lambda_2) =$$

$\{\alpha_6+\alpha_7, \alpha_5+\alpha_6+\alpha_7, \alpha_4+\alpha_5+\alpha_6+\alpha_7, \alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7, \alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7, \alpha_2+\alpha_4+\alpha_5+\alpha_6+\alpha_7, \alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7, \alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7\}$

$$\Phi(s_{\lambda_2}\lambda_3) = \Phi(2\lambda_2 + \lambda_3) = w_2\Phi(\lambda_3) =$$

$$\{\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7\}$$

Roots of length 3:

$$\Phi(s_{\lambda_3}s_{\lambda_2}\lambda_1) = \Phi(\lambda_1 + \lambda_2 + \lambda_3) = w_3w_2\Phi(\lambda_1) =$$

$\{\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7, \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7, \alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7, \alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7, \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7, \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7, \alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+\alpha_7, \alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+\alpha_7\}$

$$\Phi(s_{\lambda_2}(\lambda_1 + \lambda_2 + \lambda_3)) = \Phi(\lambda_1 + 2\lambda_2 + \lambda_3) = w_2\Phi(\lambda_1 + \lambda_2 + \lambda_3) =$$

$\{\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7, \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7, \alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+2\alpha_6+\alpha_7, \alpha_1+\alpha_2+2\alpha_3+3\alpha_4+3\alpha_5+2\alpha_6+\alpha_7, \alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+2\alpha_6+\alpha_7, \alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+3\alpha_5+2\alpha_6+\alpha_7, \alpha_1+2\alpha_2+2\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7, \alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7\}$

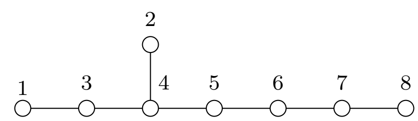
$$\Phi(s_{\lambda_1}s_{\lambda_2}\lambda_3) = \Phi(2\lambda_1 + 2\lambda_2 + \lambda_3) = w_1w_2\Phi(\lambda_3) =$$

$$\{2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7\}$$

8.7 E_8 cases

There are 2 isomorphy classes of local symmetric spaces that result from a Lie algebra of type E_8.

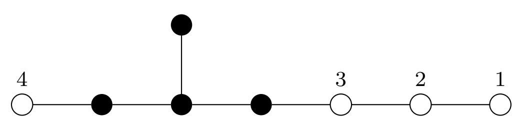
8.7.1 Type EVIII



For $i = 1, \cdots, 8$ we have $\Phi(\lambda_i) = \{\alpha_i\}$. So the multiplicity of each root is $m_{\lambda_i} = 1$.

Also since $w_i = s_{\alpha_i}$ for all i, we have that $\Phi(\mathfrak{a}) = \Phi(\mathfrak{t})$.

8.7.2 Type EIX



Here the multiplicity of each λ_i is

$m_{\lambda_1} = m_{\lambda_2} = 1$
$m_{\lambda_3} = m_{\lambda_4} = 8$

We have that

$$\Phi(\lambda_1) = \{\alpha_8\}$$
$$\Phi(\lambda_2) = \{\alpha_7\}$$

$\Phi(\lambda_3) = \{\alpha_6, \alpha_5 + \alpha_6, \alpha_4 + \alpha_5 + \alpha_6, \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6\}$

$\Phi(\lambda_4) = \{\alpha_1, \alpha_1 + \alpha_3, \alpha_1 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5\}$

$\Phi(\mathfrak{a})$ is of type F_4 and can be computed as in lemma 14.

Roots of length 2:

$$\Phi(s_{\lambda_1}\lambda_2) = \Phi(\lambda_1 + \lambda_2) = w_1\Phi(\lambda_2) =$$

$$\{\alpha_7 + \alpha_8\}$$

$$\Phi(s_{\lambda_2}\lambda_3) = \Phi(\lambda_2 + \lambda_3) = w_2\Phi(\lambda_3) =$$

$\{\alpha_6 + \alpha_7, \alpha_5 + \alpha_6 + \alpha_7, \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7\}$

$$\Phi(s_{\lambda_3}\lambda_4) = \Phi(\lambda_3 + \lambda_4) = w_3\Phi(\lambda_4) =$$

$\{\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6\}$

$$\Phi(s_{\lambda_3}\lambda_2) = \Phi(\lambda_2 + 2\lambda_3) = w_3\Phi(\lambda_2) =$$

$$\{\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7\}$$

Roots of length 3:

$$\Phi(s_{\lambda_1}s_{\lambda_2}\lambda_3) = \Phi(\lambda_1 + \lambda_2 + \lambda_3) = w_1w_2\Phi(\lambda_3) =$$

$\{\alpha_6+\alpha_7+\alpha_8, \alpha_5+\alpha_6+\alpha_7+\alpha_8, \alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8, \alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8, \alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8, \alpha_2+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8, \alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8, \alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7+\alpha_8\}$

$$\Phi(s_{\lambda_2}s_{\lambda_3}\lambda_4) = \Phi(\lambda_2+\lambda_3+\lambda_4) = w_2w_3\Phi(\lambda_4) =$$

$\{\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7, \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7, \alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7, \alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7, \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7, \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7, \alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+\alpha_7, \alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+\alpha_7\}$

$$\Phi(s_{\lambda_1}s_{\lambda_3}\lambda_2) = \Phi(\lambda_1+\lambda_2+2\lambda_3) = w_1w_3\Phi(\lambda_2) =$$

$$\{\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+\alpha_8\}$$

$$\Phi(s_{\lambda_3}s_{\lambda_2}s_{\lambda_3}\lambda_4) = \Phi(\lambda_2+2\lambda_3+\lambda_4) = w_3w_2w_3\Phi(\lambda_4) =$$

$\{\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7, \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7, \alpha_1+\alpha_2+3\alpha_3+3\alpha_4+2\alpha_5+2\alpha_6+\alpha_7, \alpha_1+\alpha_2+2\alpha_3+3\alpha_4+3\alpha_5+2\alpha_6+\alpha_7, \alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+2\alpha_6+\alpha_7, \alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+3\alpha_5+2\alpha_6+\alpha_7, \alpha_1+2\alpha_2+2\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7, \alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7\}$

$$\Phi(s_{\lambda_2}s_{\lambda_1}s_{\lambda_3}\lambda_2) = \Phi(\lambda_1+2\lambda_2+2\lambda_3) = w_2w_1w_3\Phi(\lambda_2) =$$

$$\{\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+2\alpha_7+\alpha_8\}$$

$$\Phi(s_{\lambda_4}s_{\lambda_3}\lambda_2) = \Phi(\lambda_2+2\lambda_3+2\lambda_4) = w_4w_3\Phi(\lambda_2) =$$

$$\{2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7\}$$

Roots of length 4:

$$\Phi(s_{\lambda_1}s_{\lambda_2}s_{\lambda_3}\lambda_4) = \Phi(\lambda_1+\lambda_2+\lambda_3+\lambda_4) = w_1w_2w_3\Phi(\lambda_4) =$$

$\{\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8, \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8, \alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8, \alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7+\alpha_8, \alpha_1+\alpha_2+2\alpha_3+$

$2\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8, \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7+\alpha_8, \alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+\alpha_7+\alpha_8, \alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+\alpha_7+\alpha_8\}$

$$\Phi(s_{\lambda_3}s_{\lambda_1}s_{\lambda_2}s_{\lambda_3}\lambda_4) = \Phi(\lambda_1+\lambda_2+2\lambda_3+\lambda_4) = w_3w_1w_2w_3\Phi(\lambda_4) =$$

$\{\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+\alpha_8, \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+\alpha_8, \alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+\alpha_8, \alpha_1+\alpha_2+2\alpha_3+3\alpha_4+3\alpha_5+2\alpha_6+\alpha_7+\alpha_8, \alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+\alpha_8, \alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+3\alpha_5+2\alpha_6+\alpha_7+\alpha_8, \alpha_1+2\alpha_2+2\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7+\alpha_8, \alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7+\alpha_8\}$

$$\Phi(s_{\lambda_2}s_{\lambda_3}s_{\lambda_1}s_{\lambda_2}s_{\lambda_3}\lambda_4) = \Phi(\lambda_1+2\lambda_2+2\lambda_3+\lambda_4) = w_2w_3w_1w_2w_3\Phi(\lambda_4) =$$

$\{\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+2\alpha_7+\alpha_8, \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+2\alpha_7+\alpha_8, \alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+2\alpha_6+2\alpha_7+\alpha_8, \alpha_1+\alpha_2+2\alpha_3+3\alpha_4+3\alpha_5+2\alpha_6+2\alpha_7+\alpha_8, \alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+2\alpha_6+2\alpha_7+\alpha_8, \alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+3\alpha_5+2\alpha_6+2\alpha_7+\alpha_8, \alpha_1+2\alpha_2+2\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+2\alpha_7+\alpha_8, \alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+2\alpha_7+\alpha_8\}$

$$\Phi(s_{\lambda_4}s_{\lambda_1}s_{\lambda_3}\lambda_2) = \Phi(\lambda_1+\lambda_2+2\lambda_3+2\lambda_4) = w_4w_1w_3\Phi(\lambda_2) =$$

$$\{2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7+\alpha_8\}$$

$$\Phi(s_{\lambda_3}s_{\lambda_2}s_{\lambda_3}s_{\lambda_1}s_{\lambda_2}s_{\lambda_3}\lambda_4) = \Phi(\lambda_1+2\lambda_2+3\lambda_3+\lambda_4) = w_3w_2w_3w_1w_2w_3\Phi(\lambda_4) =$$

$\{\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+3\alpha_5+3\alpha_6+2\alpha_7+\alpha_8, \alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+3\alpha_5+3\alpha_6+2\alpha_7+\alpha_8, \alpha_1+2\alpha_2+2\alpha_3+4\alpha_4+3\alpha_5+3\alpha_6+2\alpha_7+\alpha_8, \alpha_1+2\alpha_2+2\alpha_3+4\alpha_4+4\alpha_5+3\alpha_6+2\alpha_7+\alpha_8, \alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+3\alpha_6+2\alpha_7+\alpha_8, \alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+4\alpha_5+3\alpha_6+2\alpha_7+\alpha_8, \alpha_1+2\alpha_2+3\alpha_3+5\alpha_4+4\alpha_5+3\alpha_6+2\alpha_7+\alpha_8, \alpha_1+3\alpha_2+3\alpha_3+5\alpha_4+4\alpha_5+3\alpha_6+2\alpha_7+\alpha_8\}$

$$\Phi(s_{\lambda_4}s_{\lambda_2}s_{\lambda_1}s_{\lambda_3}\lambda_2) = \Phi(\lambda_1+2\lambda_2+2\lambda_3+2\lambda_4) = w_4w_2w_1w_3\Phi(\lambda_2) =$$

$$\{2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+2\alpha_7+\alpha_8\}$$

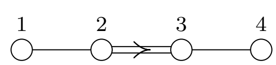
$$\Phi(s_{\lambda_4}s_{\lambda_3}s_{\lambda_2}s_{\lambda_3}s_{\lambda_1}s_{\lambda_2}s_{\lambda_3}\lambda_4) = \Phi(\lambda_1+2\lambda_2+3\lambda_3+2\lambda_4) = w_4w_3w_2w_3w_1w_2w_3\Phi(\lambda_4) =$$

$\{2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+3\alpha_6+2\alpha_7+\alpha_8, 2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+4\alpha_5+3\alpha_6+2\alpha_7+\alpha_8, 2\alpha_1+2\alpha_2+3\alpha_3+5\alpha_4+4\alpha_5+3\alpha_6+2\alpha_7+\alpha_8, 2\alpha_1+3\alpha_2+3\alpha_3+5\alpha_4+4\alpha_5+3\alpha_6+2\alpha_7+\alpha_8, 2\alpha_1+2\alpha_2+4\alpha_3+5\alpha_4+4\alpha_5+3\alpha_6+2\alpha_7+\alpha_8, 2\alpha_1+3\alpha_2+4\alpha_3+5\alpha_4+4\alpha_5+3\alpha_6+2\alpha_7+\alpha_8, 2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+4\alpha_5+3\alpha_6+2\alpha_7+\alpha_8, 2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+5\alpha_5+3\alpha_6+2\alpha_7+\alpha_8\}$

$$\Phi(s_{\lambda_3}s_{\lambda_4}s_{\lambda_2}s_{\lambda_1}s_{\lambda_3}\lambda_2) = \Phi(\lambda_1+2\lambda_2+4\lambda_3+2\lambda_4) = w_3w_4w_2w_1w_3\Phi(\lambda_2) =$$
$$\{2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+5\alpha_5+4\alpha_6+2\alpha_7+\alpha_8\}$$
$$\Phi(s_{\lambda_2}s_{\lambda_3}s_{\lambda_4}s_{\lambda_2}s_{\lambda_1}s_{\lambda_3}\lambda_2) = \Phi(\lambda_1+3\lambda_2+4\lambda_3+2\lambda_4) = w_2w_3w_4w_2w_1w_3\Phi(\lambda_2) =$$
$$\{2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+5\alpha_5+4\alpha_6+3\alpha_7+\alpha_8\}$$
$$\Phi(s_{\lambda_1}s_{\lambda_2}s_{\lambda_3}s_{\lambda_4}s_{\lambda_2}s_{\lambda_1}s_{\lambda_3}\lambda_2) = \Phi(2\lambda_1+3\lambda_2+4\lambda_3+2\lambda_4) = w_1w_2w_3w_4w_2w_1w_3\Phi(\lambda_2) =$$
$$\{2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+5\alpha_5+4\alpha_6+3\alpha_7+2\alpha_8\}$$

8.8 F cases

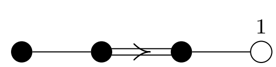
8.8.1 Type FI



For $i = 1, \cdots, 4$ we have $\Phi(\lambda_i) = \{\alpha_i\}$. So the multiplicity of each root is $m_{\lambda_i} = 1$.

Also since $w_i = s_{\alpha_i}$ for all i, we have that $\Phi(\mathfrak{a}) = \Phi(\mathfrak{t})$.

8.8.2 Type FII



Here the multiplicity of λ_i is

$$m_{\lambda_1} = 8$$
$$m_{2\lambda_1} = 7$$

We have that

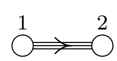
$\Phi(\lambda_1) = \{\alpha_4, \alpha_3 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \alpha_2 + 2\alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4\}$

$\Phi(2\lambda_1) = \{\alpha_2 + 2\alpha_3 + 2\alpha_4, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4, \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4, \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4, \alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4, 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4\}$

$\Phi(\mathfrak{a})$ is of type A_1 and $\Phi(\mathfrak{a})^+ = \{\lambda_1\}$.

8.9 $\mathbf{G}_2$ case

8.9.1 Type G



For $i = 1, 2$ we have $\Phi(\lambda_i) = \{\alpha_i\}$. So the multiplicity of each root is $m_{\lambda_i} = 1$.

Also since $w_i = s_{\alpha_i}$ for all i, we have that $\Phi(\mathfrak{a}) = \Phi(\mathfrak{t})$.

Chapter 9

Basis for $\mathfrak{k}$ and $\mathfrak{p}$

Now that we have computed all of the multiplicities, positive roots in $\Phi(\mathfrak{a})$, and the projection spaces, we can find a "nice" basis for both $\mathfrak{k}$ and our local symmetric space $\mathfrak{p}$.

In step 5, we found the eigenspaces for each λ_i in $\Phi(\mathfrak{a})$. From 5.5 we have that $\mathfrak{g}_{\lambda_i} = \mathfrak{g}_{\alpha_{i_1}} \oplus \mathfrak{g}_{\alpha_{i_2}} \oplus \ldots \oplus \mathfrak{g}_{\alpha_{i_r}}$. To explicitly compute each $\mathfrak{g}_{\lambda_i}$ we note that each element in $\Phi(\lambda_i)$ will be in the direct sum decomposition.

The last step in the algorithm consists of a nice way to compute the basis.

From 5.6, we have that $x_\alpha \in \mathfrak{g}_\alpha$ implies $x_\alpha + \theta(x_\alpha) \in \mathfrak{k}$ and $x_\alpha - \theta(x_\alpha) \in \mathfrak{p}$,
a basis for $\mathfrak{k} = \{x_\alpha^i + \theta(x_\alpha^i)\}_{\alpha \in \Phi(\mathfrak{a})^+} + \{x_\alpha^i\}_{\alpha \in \Phi_0(\theta)}$ + a basis for $\mathfrak{t}_0$ and
a basis for $\mathfrak{p} = \{x_\alpha^i - \theta(x_\alpha^i)\}_{\alpha \in \Phi(\mathfrak{a})^+}$ + a basis for $\mathfrak{a}$.

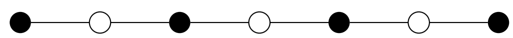
The computation depends on the original choice of basis for the Lie algebra $\mathfrak{g}$ and the choice of the θ-basis for $\Phi(\mathfrak{t})$.

To demonstrate the beauty of this algorithm, I include a complete example.

9.1 Example $A_7^3(II)$

9.1.1 Step 0:

Before we can begin implementing the algorithm, we need to recover the action of the involution on $\Phi(\mathfrak{t})$ and find the formulation for the λ in $\Phi(\mathfrak{a})$ in terms of the α in $\Phi(\mathfrak{t})$.



Here we start off with a Lie algebra of type A_7. We notice that

$$\Delta_0(\theta) = \{\alpha_1, \alpha_3, \alpha_5, \alpha_7\}.$$

We can easily calculate $w_0(\theta) = s_{\alpha_1} s_{\alpha_3} s_{\alpha_5} s_{\alpha_7}$ and recover the action of the involution, θ, on the roots $\Delta(\mathfrak{a})$ because Helminck gives us that

$$\theta = -\operatorname{id} w_0(\theta)\theta^*$$

where $w_0(\theta)$ and θ^* commute.

We have that

$$\theta(\alpha_1) = \alpha_1,$$

$$\theta(\alpha_2) = -(\alpha_1 + \alpha_2 + \alpha_3),$$

$$\theta(\alpha_3) = \alpha_3,$$

$$\theta(\alpha_4) = -(\alpha_3 + \alpha_4 + \alpha_5),$$

$$\theta(\alpha_5) = \alpha_5,$$

$$\theta(\alpha_6) = -(\alpha_5 + \alpha_6 + \alpha_7), \text{ and}$$

$$\theta(\alpha_7) = \alpha_7.$$

Now using the natural projection map $\pi(\alpha) = \frac{1}{2}(\alpha - \theta(\alpha))$, we find that

$$\pi(\alpha_2) = \frac{1}{2}(\alpha_1 + 2\alpha_2 + \alpha_3) = \lambda_1,$$

$$\pi(\alpha_4) = \frac{1}{2}(\alpha_3 + 2\alpha_4 + \alpha_5) = \lambda_2,$$

$$\pi(\alpha_6) = \frac{1}{2}(\alpha_5 + 2\alpha_6 + \alpha_7) = \lambda_3, \text{ and}$$

$$\pi(\alpha_1) = \pi(\alpha_3) = \pi(\alpha_5) = \pi(\alpha_7) = 0.$$

9.1.2 Step 1:

We defined $\Phi(\lambda_i) = \{\alpha \in \Phi(\mathfrak{t}) \mid \alpha|_{\mathfrak{a}} = \lambda_i\}$. For this example, we have

$$\Phi(\lambda_1) = \{\alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3\},$$

$$\Phi(\lambda_2) = \{\alpha_4, \alpha_3 + \alpha_4, \alpha_4 + \alpha_5, \alpha_3 + \alpha_4 + \alpha_5\}, \text{ and}$$

$$\Phi(\lambda_3) = \{\alpha_6, \alpha_5 + \alpha_6, \alpha_6 + \alpha_7, \alpha_5 + \alpha_6 + \alpha_7\}.$$

So the respective multiplicities are $m_{\lambda_1} = 4$, $m_{\lambda_2} = 4$, and $m_{\lambda_3} = 4$.

9.1.3 Step 2:

The representatives in $W_1(\theta)$ for s_{λ_1}, s_{λ_2} and s_{λ_3} are

$$w_1 = s_{\alpha_2} s_{\alpha_1} s_{\alpha_3} s_{\alpha_2}$$

$$w_2 = s_{\alpha_4} s_{\alpha_3} s_{\alpha_5} s_{\alpha_4} \text{ and}$$

$$w_3 = s_{\alpha_6} s_{\alpha_5} s_{\alpha_7} s_{\alpha_6}.$$

9.1.4 Step 3:

Next we find the list of positive roots, $\Phi(\mathfrak{a})^+$, by using s_{λ_1}, s_{λ_2}, and s_{λ_3} applied to each λ. Here we have

$$s_{\lambda_2}(\lambda_1) = \lambda_1 + \lambda_2$$

$$s_{\lambda_3}(\lambda_2) = \lambda_2 + \lambda_3$$

$$s_{\lambda_3}s_{\lambda_2}(\lambda_1) = \lambda_1 + \lambda_2 + \lambda_3$$

Therefore, $\Phi(\mathfrak{a})^+ = \{\lambda_1, \lambda_2, \lambda_3, \lambda_1 + \lambda_2, \lambda_2 + \lambda_3, \lambda_1 + \lambda_2 + \lambda_3\}$.

9.1.5 Step 4:

Now we need to compute $\Phi(\lambda_i)$ for all λ_i in $\Phi(\mathfrak{a})^+$. This is easily accomplished by looking at the previous step. We must compute

$$\Phi(\lambda_1 + \lambda_2) = \Phi(s_{\lambda_2}(\lambda_1)) = w_2(\Phi(\lambda_1)),$$

$$\Phi(\lambda_2 + \lambda_3) = \Phi(s_{\lambda_3}(\lambda_2)) = w_3(\Phi(\lambda_2)), \text{ and}$$

$$\Phi(\lambda_1 + \lambda_2 + \lambda_3) = \Phi(s_{\lambda_3}s_{\lambda_2}(\lambda_1)) = w_3w_2(\Phi(\lambda_1)),$$

We arise with

$\Phi(s_{\lambda_2}(\lambda_1)) = \Phi(\lambda_1+\lambda_2) = \{\alpha_2+\alpha_3+\alpha_4, \alpha_2+\alpha_3+\alpha_4+\alpha_5, \alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5\}$

$\Phi(s_{\lambda_3}(\lambda_2)) = \Phi(\lambda_2+\lambda_3) = \{\alpha_4+\alpha_5+\alpha_6, \alpha_3+\alpha_4+\alpha_5+\alpha_6, \alpha_4+\alpha_5+\alpha_6+\alpha_7, \alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7\}$

$\Phi(s_{\lambda_3}s_{\lambda_2}(\lambda_1)) = \Phi(\lambda_1+\lambda_2+\lambda_3) = \{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6, \alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7, \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6, \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7\}$

9.1.6 Step 5:

The eigenspaces are

$$\mathfrak{g}_{\lambda_1} = \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_2+\alpha_3} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3}$$

$$\mathfrak{g}_{\lambda_2} = \mathfrak{g}_{\alpha_4} \oplus \mathfrak{g}_{\alpha_3+\alpha_4} \oplus \mathfrak{g}_{\alpha_4+\alpha_5} \oplus \mathfrak{g}_{\alpha_3+\alpha_4+\alpha_5}$$

$$\mathfrak{g}_{\lambda_3} = \mathfrak{g}_{\alpha_6} \oplus \mathfrak{g}_{\alpha_5+\alpha_6} \oplus \mathfrak{g}_{\alpha_6+\alpha_7} \oplus \mathfrak{g}_{\alpha_5+\alpha_6+\alpha_7}$$

$$\mathfrak{g}_{\lambda_1+\lambda_2} = \mathfrak{g}_{\alpha_2+\alpha_3+\alpha_4} \oplus \mathfrak{g}_{\alpha_2+\alpha_3+\alpha_4+\alpha_5} \oplus \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \oplus \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5}$$

$$\mathfrak{g}_{\lambda_2+\lambda_3} = \mathfrak{g}_{\alpha_4+\alpha_5+\alpha_6} \oplus \mathfrak{g}_{\alpha_4+\alpha_5+\alpha_6+\alpha_7} \oplus \mathfrak{g}_{\alpha_3+\alpha_4+\alpha_5+\alpha_6} \oplus \mathfrak{g}_{\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7}$$

$$\mathfrak{g}_{\lambda_1+\lambda_2+\lambda_3} = \mathfrak{g}_{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6} \oplus \mathfrak{g}_{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7} \oplus \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6} \oplus \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7}$$

9.1.7 Step 6:

For the basis of A_7, we will choose $\{E_{ij}, i \neq j\}$ and the maximal torus

$$\begin{aligned}\mathfrak{t} =& \langle E_{11} - E_{55}, -E_{11} + E_{22}, -E_{22} + E_{66}, -E_{66} + E_{77}, E_{33} - E_{77}, -E_{33} + E_{44}, -E_{44} + E_{88}\rangle \\ =& \langle e_1 - e_5, -e_1 + e_2, -e_2 + e_6, -e_6 + e_7, e_3 - e_7, -e_3 + e_4, -e_4 + e_8\rangle.\end{aligned}$$

The corresponding θ-basis of the root space is

$$\Delta(\mathfrak{t}) = \{\epsilon_1 - \epsilon_5, -\epsilon_1 + \epsilon_2, -\epsilon_2 + \epsilon_6, -\epsilon_6 + \epsilon_7, \epsilon_3 - \epsilon_7, -\epsilon_3 + \epsilon_4, -\epsilon_4 + \epsilon_8\}.$$

We will denote these roots by:

$$\Delta(\mathfrak{t}) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}.$$

From

$$\mathfrak{g}_{\lambda_1} = \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_2+\alpha_3} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3},$$

we consider $\mathfrak{g}_{\alpha_2} = \{E_{21}\}$. We know that $E_{21} - \theta(E_{21}) \in \mathfrak{p}$ and $E_{21} + \theta(E_{21}) \in \mathfrak{k}$. Therefore $E_{21} + E_{56} \in \mathfrak{p}$ and $E_{21} - E_{56} \in \mathfrak{k}$.

Continuing on in this fashion we find that:

$$\mathfrak{p} \supset \{E_{21} + E_{56}, E_{25} - E_{16}, E_{61} - E_{52}, E_{12} + E_{65}\} \text{ and}$$

$$\mathfrak{k} \supset \{E_{21} - E_{56}, E_{25} + E_{16}, E_{61} + E_{52}, E_{12} - E_{65}\}.$$

From

$$\mathfrak{g}_{\lambda_2} = \mathfrak{g}_{\alpha_4} \oplus \mathfrak{g}_{\alpha_3+\alpha_4} \oplus \mathfrak{g}_{\alpha_4+\alpha_5} \oplus \mathfrak{g}_{\alpha_3+\alpha_4+\alpha_5},$$

we have

$$\mathfrak{p} \supset \{E_{23} + E_{76}, E_{36} - E_{27}, E_{72} - E_{63}, E_{32} + E_{67}\} \text{ and}$$

$$\mathfrak{k} \supset \{E_{23} - E_{76}, E_{36} + E_{27}, E_{72} + E_{63}, E_{32} - E_{67}\}.$$

From

$$\mathfrak{g}_{\lambda_3} = \mathfrak{g}_{\alpha_6} \oplus \mathfrak{g}_{\alpha_5+\alpha_6} \oplus \mathfrak{g}_{\alpha_6+\alpha_7} \oplus \mathfrak{g}_{\alpha_5+\alpha_6+\alpha_7},$$

we have that

$$\mathfrak{p} \supset \{E_{43} + E_{78}, E_{47} - E_{38}, E_{83} - E_{74}, E_{34} + E_{87}\} \text{ and}$$

$$\mathfrak{k} \supset \{E_{43} - E_{78}, E_{47} + E_{38}, E_{83} + E_{74}, E_{34} - E_{87}\}.$$

From

$$\mathfrak{g}_{\lambda_1+\lambda_2} = \mathfrak{g}_{\alpha_2+\alpha_3+\alpha_4} \oplus \mathfrak{g}_{\alpha_2+\alpha_3+\alpha_4+\alpha_5} \oplus \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \oplus \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5}$$

we have

$$\mathfrak{p} \supset \{E_{35} - E_{17}, E_{31} + E_{57}, E_{13} + E_{75}, E_{71} - E_{53}\} \text{ and}$$

$$\mathfrak{k} \supset \{E_{35} + E_{17}, E_{31} - E_{57}, E_{13} - E_{75}, E_{71} + E_{53}\}.$$

From

$$\mathfrak{g}_{\lambda_2+\lambda_3} = \mathfrak{g}_{\alpha_4+\alpha_5+\alpha_6} \oplus \mathfrak{g}_{\alpha_4+\alpha_5+\alpha_6+\alpha_7} \oplus \mathfrak{g}_{\alpha_3+\alpha_4+\alpha_5+\alpha_6} \oplus \mathfrak{g}_{\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7}$$

we have that

$$\mathfrak{p} \supset \{E_{46} - E_{28}, E_{42} + E_{68}, E_{24} + E_{86}, E_{82} - E_{64}\} \text{ and}$$

$$\mathfrak{k} \supset \{E_{46} + E_{28}, E_{42} - E_{68}, E_{24} - E_{86}, E_{82} + E_{64}\}.$$

From $\mathfrak{g}_{\lambda_1+\lambda_2+\lambda_3} = \mathfrak{g}_{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6} \oplus \mathfrak{g}_{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7} \oplus \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6} \oplus \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7}$ we have

$$\mathfrak{p} \supset \{E_{41} + E_{58}, E_{45} - E_{18}, E_{81} - E_{54}, E_{14} + E_{85}\} \text{ and}$$

$$\mathfrak{k} \supset \{E_{41} - E_{58}, E_{45} + E_{18}, E_{81} + E_{54}, E_{14} - E_{85}\}.$$

From $\Delta_0(\theta) = \{\alpha_1, \alpha_3, \alpha_5, \alpha_7\}$, we get

$$\mathfrak{k} \supset \{E_{15}, E_{51}, E_{26}, E_{62}, E_{37}, E_{73}, E_{48}, E_{84}\}.$$

A basis for $\mathfrak{a}$ is $\{-E_{11}+E_{22}+E_{55}-E_{66}, E_{22}-E_{33}-E_{66}+E_{77}, -E_{33}+E_{44}+E_{77}-E_{88}\}$.

A basis for $\mathfrak{t}_0$ is $\{E_{11} - E_{55}, -E_{22} + E_{66}, E_{33} - E_{77}, -E_{44} + E_{88}\}$.

Putting all this information together, we arise with "nice" bases for both $\mathfrak{k}$ and $\mathfrak{p}$.

$\mathfrak{p} = \langle -E_{11}+E_{22}+E_{55}-E_{66}, E_{22}-E_{33}-E_{66}+E_{77}, -E_{33}+E_{44}+E_{77}-E_{88}, E_{21}+E_{56}, E_{25}-E_{16}, E_{61}-E_{52}, E_{12}+E_{65}, E_{23}+E_{76}, E_{36}-E_{27}, E_{72}-E_{63}, E_{32}+E_{67}, E_{43}+E_{78}, E_{47}-E_{38}, E_{83}-E_{74}, E_{34}+E_{87}, E_{35}-E_{17}, E_{31}+E_{57}, E_{13}+E_{75}, E_{71}-E_{53}, E_{46}-E_{28}, E_{42}+E_{68}, E_{24}+E_{86}, E_{82}-E_{64}, E_{41}+E_{58}, E_{45}-E_{18}, E_{81}-E_{54}, E_{14}+E_{85}\rangle$

$\mathfrak{k} = \langle E_{11}-E_{55}, -E_{22}+E_{66}, E_{33}-E_{77}, -E_{44}+E_{88}, E_{21}-E_{56}, E_{25}+E_{16}, E_{61}+E_{52}, E_{12}-E_{65}, E_{23}-E_{76}, E_{36}+E_{27}, E_{72}+E_{63}, E_{32}-E_{67}, E_{43}-E_{78}, E_{47}+E_{38}, E_{83}+E_{74}, E_{34}-E_{87}, E_{35}+E_{17}, E_{31}-E_{57}, E_{13}-E_{75}, E_{71}+E_{53}, E_{46}+E_{28}, E_{42}-E_{68}, E_{24}-E_{86}, E_{82}+E_{64}, E_{41}-E_{58}, E_{45}+E_{18}, E_{81}+E_{54}, E_{14}-E_{85}, E_{15}, E_{51}, E_{26}, E_{62}, E_{37}, E_{73}, E_{48}, E_{84}\rangle$

Appendix A

Computing $\Phi(\mathfrak{a})^+$

There are distinct patterns used when computing the positive roots of $\Phi(\mathfrak{a})$. Here I will explicitly go through each construction that is necessary in chapter 8. In all of these constructons $|\Delta(\mathfrak{a})| = n$.

A.1 Type A construction

Remark 5. For a root system of type A_n, the positive roots are
$\Phi(\mathfrak{a})^+ = \{\lambda_1, \cdots, \lambda_n, \lambda_1+\lambda_2, \cdots, \lambda_{n-1}+\lambda_n, \lambda_1+\lambda_2+\lambda_3, \cdots, \lambda_{n-2}+\lambda_{n-1}+\lambda_n, \cdots, \lambda_1+\lambda_2+\lambda_3+\cdots+\lambda_{n-1}, \lambda_2+\lambda_3+\lambda_4+\cdots+\lambda_n, \lambda_1+\lambda_2+\lambda_3+\lambda_4+\cdots+\lambda_n\}$.

Lemma 10. *A root system of type A_n can be found by working down the root string in this manner:*

Roots of length 2:

$$s_{\lambda_2}(\lambda_1) = \lambda_1 + \lambda_2$$

$$s_{\lambda_3}(\lambda_2) = \lambda_2 + \lambda_3$$

$$\vdots$$

$$s_{\lambda_n}(\lambda_{n-1}) = \lambda_{n-1} + \lambda_n$$

Roots of length 3:

$$s_{\lambda_3}(\lambda_1 + \lambda_2) = \lambda_1 + \lambda_2 + \lambda_3$$

$$s_{\lambda_4}(\lambda_2 + \lambda_3) = \lambda_2 + \lambda_3 + \lambda_4$$

$$\vdots$$

$$s_{\lambda_n}(\lambda_{n-2} + \lambda_{n-1}) = \lambda_{n-2} + \lambda_{n-1} + \lambda_n$$

$$\vdots$$

Roots of length n-1:

$$s_{\lambda_{n-1}}(\lambda_1 + \lambda_2 + \cdots + \lambda_{n-2}) = \lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_{n-1}$$

$$s_{\lambda_n}(\lambda_2 + \lambda_3 + \cdots + \lambda_{n-1}) = \lambda_2 + \lambda_3 + \lambda_4 + \cdots + \lambda_n$$

Root of length n:

$$s_{\lambda_n}(\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_{n-1}) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \cdots + \lambda_n$$

A.2 Type B construction

Remark 6. For a root system of type B_n, the positive roots are

$\Phi(\mathfrak{a})^+ = \{\lambda_1, \cdots, \lambda_n, \lambda_1 + \lambda_2, \cdots, \lambda_{n-1} + \lambda_n, \cdots, \lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_{n-1}, \lambda_2 + \lambda_3 + \lambda_4 + \cdots + \lambda_n, \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \cdots + \lambda_n, \lambda_{n-1} + 2\lambda_n, \lambda_{n-2} + \lambda_{n-1} + 2\lambda_n, \lambda_{n-2} +$

$2\lambda_{n-1} + 2\lambda_n, \cdots, \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \cdots + 2\lambda_n, \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \cdots + 2\lambda_{n-1} + 2\lambda_n, \cdots, \lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4 + \cdots + 2\lambda_n\}$.

Lemma 11. *A root system of type B_n can be found by by working down the root string in this manner:*

Roots of length 2:

$$s_{\lambda_1}(\lambda_2) = \lambda_1 + \lambda_2$$
$$s_{\lambda_2}(\lambda_3) = \lambda_2 + \lambda_3$$
$$\vdots$$
$$s_{\lambda_{n-1}}(\lambda_n) = \lambda_{n-1} + \lambda_n$$
$$s_{\lambda_n}(\lambda_{n-1}) = \lambda_{n-1} + 2\lambda_n$$

Roots of length 3:

$$s_{\lambda_1}(\lambda_2 + \lambda_3) = \lambda_1 + \lambda_2 + \lambda_3$$
$$s_{\lambda_2}(\lambda_3 + \lambda_4) = \lambda_2 + \lambda_3 + \lambda_4$$
$$\vdots$$
$$s_{\lambda_{n-2}}(\lambda_{n-1} + \lambda_n) = \lambda_{n-2} + \lambda_{n-1} + \lambda_n$$
$$s_{\lambda_n}(\lambda_{n-2} + \lambda_{n-1}) = \lambda_{n-2} + \lambda_{n-1} + 2\lambda_n$$
$$s_{\lambda_{n-1}}s_{\lambda_n}(\lambda_{n-2} + \lambda_{n-1}) = \lambda_{n-2} + 2\lambda_{n-1} + 2\lambda_n$$

$$\vdots$$

Roots of length n-1:

$$s_{\lambda_1}(\lambda_2 + \cdots + \lambda_{n-2} + \lambda_{n-1}) = \lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_{n-1}$$
$$s_{\lambda_2}(\lambda_3 + \cdots + \lambda_{n-1} + \lambda_n) = \lambda_2 + \lambda_3 + \lambda_4 + \cdots + \lambda_n$$

$$s_{\lambda_n}(\lambda_2 + \cdots + \lambda_{n-2} + \lambda_{n-1}) = \lambda_2 + \cdots + \lambda_{n-2} + \lambda_{n-1} + 2\lambda_n$$

$$s_{\lambda_{n-1}} s_{\lambda_n}(\lambda_2 + \cdots + \lambda_{n-2} + \lambda_{n-1}) = \lambda_2 + \cdots + \lambda_{n-2} + 2\lambda_{n-1} + 2\lambda_n$$

$$\vdots$$

$$s_{\lambda_3} \cdots s_{\lambda_{n-1}} s_{\lambda_n}(\lambda_2 + \cdots + \lambda_{n-2} + \lambda_{n-1}) = \lambda_2 + 2\lambda_3 + \cdots + 2\lambda_{n-2} + 2\lambda_{n-1} + 2\lambda_n$$

Root of length n:

$$s_{\lambda_1}(\lambda_2 + \lambda_3 + \cdots + \lambda_{n-1} + \lambda_n) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \cdots + \lambda_n$$

$$s_{\lambda_n}(\lambda_1 + \cdots + \lambda_{n-2} + \lambda_{n-1}) = \lambda_1 + \cdots + \lambda_{n-2} + \lambda_{n-1} + 2\lambda_n$$

$$s_{\lambda_{n-1}} s_{\lambda_n}(\lambda_1 + \cdots + \lambda_{n-2} + \lambda_{n-1}) = \lambda_1 + \cdots + \lambda_{n-2} + 2\lambda_{n-1} + 2\lambda_n$$

$$\vdots$$

$$s_{\lambda_2} \cdots s_{\lambda_{n-1}} s_{\lambda_n}(\lambda_1 + \cdots + \lambda_{n-2} + \lambda_{n-1}) = \lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{n-2} + 2\lambda_{n-1} + 2\lambda_n$$

A.3 Type C construction

Remark 7. For a root system of type C_n, the positive roots are
$\Phi(\mathfrak{a})^+ = \{\lambda_1, \cdots, \lambda_n, \lambda_1 + \lambda_2, \cdots, \lambda_{n-1} + \lambda_n, \cdots, \lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_{n-1}, \lambda_2 + \lambda_3 + \lambda_4 + \cdots + \lambda_n, \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \cdots + \lambda_n, 2\lambda_{n-1} + \lambda_n, \lambda_{n-2} + 2\lambda_{n-1} + \lambda_n, 2\lambda_{n-2} + 2\lambda_{n-1} + \lambda_n, \cdots, \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \cdots + 2\lambda_{n-1} + \lambda_n, \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \cdots + 2\lambda_{n-2} + 2\lambda_{n-1} + \lambda_n, \cdots, 2\lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4 + \cdots + 2\lambda_{n-1} + \lambda_n\}$.

Lemma 12. *A root system of type C_n can be found by by working down the root string in this manner:*

Roots of length 2:

$$s_{\lambda_2}(\lambda_1) = \lambda_1 + \lambda_2$$
$$s_{\lambda_3}(\lambda_2) = \lambda_2 + \lambda_3$$
$$\vdots$$
$$s_{\lambda_n}(\lambda_{n-1}) = \lambda_{n-1} + \lambda_n$$
$$s_{\lambda_{n-1}}(\lambda_n) = 2\lambda_{n-1} + \lambda_n$$

Roots of length 3:

$$s_{\lambda_3}(\lambda_1 + \lambda_2) = \lambda_1 + \lambda_2 + \lambda_3$$
$$s_{\lambda_4}(\lambda_2 + \lambda_3) = \lambda_2 + \lambda_3 + \lambda_4$$
$$\vdots$$
$$s_{\lambda_n}(\lambda_{n-2} + \lambda_{n-1}) = \lambda_{n-2} + \lambda_{n-1} + \lambda_n$$
$$s_{\lambda_{n-1}}(\lambda_{n-2} + \lambda_{n-1} + \lambda_n) = \lambda_{n-2} + 2\lambda_{n-1} + \lambda_n$$
$$s_{\lambda_{n-2}}(2\lambda_{n-1} + \lambda_n) = 2\lambda_{n-2} + 2\lambda_{n-1} + \lambda_n$$

$$\vdots$$

Roots of length n-1:

$$s_{\lambda_{n-1}}(\lambda_1 + \lambda_2 + \cdots + \lambda_{n-2}) = \lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_{n-1}$$
$$s_{\lambda_n}(\lambda_2 + \lambda_3 + \cdots + \lambda_{n-1}) = \lambda_2 + \lambda_3 + \lambda_4 + \cdots + \lambda_n$$

$$s_{\lambda_{n-1}}(\lambda_2 + \cdots + \lambda_{n-2} + \lambda_{n-1} + \lambda_n) = \lambda_2 + \cdots + \lambda_{n-2} + 2\lambda_{n-1} + \lambda_n$$

$$s_{\lambda_{n-2}} s_{\lambda_{n-1}}(\lambda_2 + \cdots + \lambda_{n-2} + \lambda_{n-1}) = \lambda_2 + \cdots + 2\lambda_{n-2} + 2\lambda_{n-1} + \lambda_n$$

$$\vdots$$

$$s_{\lambda_3} \cdots s_{\lambda_{n-2}} s_{\lambda_{n-1}}(\lambda_2 + \cdots + \lambda_{n-2} + \lambda_{n-1}) = \lambda_2 + 2\lambda_3 + \cdots + 2\lambda_{n-2} + 2\lambda_{n-1} + \lambda_n$$

$$s_{\lambda_2} \cdots s_{\lambda_{n-1}}(\lambda_n) = 2\lambda_2 + 2\lambda_3 \cdots + 2\lambda_{n-2} + 2\lambda_{n-1} + \lambda_n$$

Root of length n:

$$s_{\lambda_n}(\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_{n-1}) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \cdots + \lambda_n$$

$$s_{\lambda_{n-1}}(\lambda_1 + \cdots + \lambda_{n-2} + \lambda_{n-1} + \lambda_n) = \lambda_1 + \cdots + \lambda_{n-2} + 2\lambda_{n-1} + \lambda_n$$

$$s_{\lambda_{n-2}} s_{\lambda_{n-1}}(\lambda_1 + \cdots + \lambda_{n-1} + \lambda_n) = \lambda_1 + \cdots + 2\lambda_{n-2} + 2\lambda_{n-1} + \lambda_n$$

$$\vdots$$

$$s_{\lambda_2} \cdots s_{\lambda_{n-2}} s_{\lambda_{n-1}}(\lambda_1 + \cdots + \lambda_{n-2} + \lambda_{n-1}) = \lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{n-2} + 2\lambda_{n-1} + \lambda_n$$

$$s_{\lambda_1} \cdots s_{\lambda_{n-1}}(\lambda_n) = 2\lambda_1 + 2\lambda_1 \cdots + 2\lambda_{n-2} + 2\lambda_{n-1} + \lambda_n$$

A.4 Type BC construction

Remark 8. For a root system of type BC_n, the positive roots are $\Phi(\mathfrak{a})^+ = \{\lambda_1, \cdots, \lambda_n, \lambda_1 + \lambda_2, \cdots, \lambda_{n-1} + \lambda_n, \cdots, \lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_{n-1}, \lambda_2 + \lambda_3 + \lambda_4 + \cdots + \lambda_n, \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \cdots + \lambda_n, \lambda_{n-1} + 2\lambda_n, \lambda_{n-2} + \lambda_{n-1} + 2\lambda_n, \lambda_{n-2} + 2\lambda_{n-1} + 2\lambda_n, \cdots, \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \cdots + 2\lambda_n, \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \cdots + 2\lambda_{n-1} + 2\lambda_n, \cdots, \lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4 + \cdots + 2\lambda_n, 2\lambda_n, 2\lambda_{n-1} + 2\lambda_n, \cdots, 2\lambda_1 + \cdots + 2\lambda_n\}$.

Lemma 13. *A root system of type BC_n can be found by working down the root string in this manner:*

Roots of length 2:

$$s_{\lambda_1}(\lambda_2) = \lambda_1 + \lambda_2$$

$$s_{\lambda_2}(\lambda_3) = \lambda_2 + \lambda_3$$

$$\vdots$$

$$s_{\lambda_{n-1}}(\lambda_n) = \lambda_{n-1} + \lambda_n$$

$$s_{\lambda_n}(\lambda_{n-1}) = \lambda_{n-1} + 2\lambda_n$$

$$s_{\lambda_{n-1}}(2\lambda_n) = 2\lambda_{n-1} + 2\lambda_n$$

Roots of length 3:

$$s_{\lambda_1}(\lambda_2 + \lambda_3) = \lambda_1 + \lambda_2 + \lambda_3$$

$$s_{\lambda_2}(\lambda_3 + \lambda_4) = \lambda_2 + \lambda_3 + \lambda_4$$

$$\vdots$$

$$s_{\lambda_{n-2}}(\lambda_{n-1} + \lambda_n) = \lambda_{n-2} + \lambda_{n-1} + \lambda_n$$

$$s_{\lambda_n}(\lambda_{n-2} + \lambda_{n-1}) = \lambda_{n-2} + \lambda_{n-1} + 2\lambda_n$$

$$s_{\lambda_{n-1}} s_{\lambda_n}(\lambda_{n-2} + \lambda_{n-1}) = \lambda_{n-2} + 2\lambda_{n-1} + 2\lambda_n$$

$$s_{\lambda_{n-2}}(2\lambda_{n-1} + 2\lambda_n) = 2\lambda_{n-2} + 2\lambda_{n-1} + 2\lambda_n$$

$$\vdots$$

Roots of length n-1:

$$s_{\lambda_1}(\lambda_2 + \cdots + \lambda_{n-2} + \lambda_{n-1}) = \lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_{n-1}$$

$$s_{\lambda_2}(\lambda_3 + \cdots + \lambda_{n-1} + \lambda_n) = \lambda_2 + \lambda_3 + \lambda_4 + \cdots + \lambda_n$$

$$s_{\lambda_n}(\lambda_2 + \cdots + \lambda_{n-2} + \lambda_{n-1}) = \lambda_2 + \cdots + \lambda_{n-2} + \lambda_{n-1} + 2\lambda_n$$

$$s_{\lambda_{n-1}} s_{\lambda_n}(\lambda_2 + \cdots + \lambda_{n-2} + \lambda_{n-1}) = \lambda_2 + \cdots + \lambda_{n-2} + 2\lambda_{n-1} + 2\lambda_n$$

$$\vdots$$

$$s_{\lambda_2} \cdots s_{\lambda_{n-1}} s_{\lambda_n}(\lambda_2 + \cdots + \lambda_{n-2} + \lambda_{n-1}) = \lambda_2 + 2\lambda_3 + \cdots + 2\lambda_{n-2} + 2\lambda_{n-1} + 2\lambda_n$$

$$s_{\lambda_2}(2\lambda_3 + \cdots + 2\lambda_{n-1} + 2\lambda_{n-}) = 2\lambda_2 + \cdots + 2\lambda_{n-1} + 2\lambda_n$$

Root of length n:

$$s_{\lambda_1}(\lambda_2 + \lambda_3 + \cdots + \lambda_{n-1} + \lambda_n) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \cdots + \lambda_n$$

$$s_{\lambda_n}(\lambda_1 + \cdots + \lambda_{n-2} + \lambda_{n-1}) = \lambda_1 + \cdots + \lambda_{n-2} + \lambda_{n-1} + 2\lambda_n$$

$$s_{\lambda_{n-1}} s_{\lambda_n}(\lambda_1 + \cdots + \lambda_{n-2} + \lambda_{n-1}) = \lambda_1 + \cdots + \lambda_{n-2} + 2\lambda_{n-1} + 2\lambda_n$$

$$\vdots$$

$$s_{\lambda_2} \cdots s_{\lambda_{n-1}} s_{\lambda_n}(\lambda_1 + \cdots + \lambda_{n-2} + \lambda_{n-1}) = \lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{n-2} + 2\lambda_{n-1} + 2\lambda_n$$

$$s_{\lambda_1}(2\lambda_2 + \cdots + 2\lambda_{n-1} + 2\lambda_{n-}) = 2\lambda_1 + \cdots + 2\lambda_{n-1} + 2\lambda_n$$

A.5 Type F construction

Remark 9. For a root system of type F_4. the positive roots are $\Phi(\mathfrak{a})^+ = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_1+\lambda_2, \lambda_2+\lambda_3, \lambda_3+\lambda_4, \lambda_2+2\lambda_3, \lambda_1+\lambda_2+\lambda_3, \lambda_2+\lambda_3+\lambda_4, \lambda_1+\lambda_2+2\lambda_3, \lambda_2+2\lambda_3+\lambda_4, \lambda_1+2\lambda_2+2\lambda_3, \lambda_2+2\lambda_3+2\lambda_4, \lambda_1+\lambda_2+\lambda_3+\lambda_4, \lambda_1+\lambda_2+2\lambda_3+\lambda_4, \lambda_1+2\lambda_2+2\lambda_3+\lambda_4, \lambda_1+\lambda_2+2\lambda_3+2\lambda_4, \lambda_1+2\lambda_2+3\lambda_3+\lambda_4, \lambda_1+2\lambda_2+2\lambda_3+2\lambda_4, \lambda_1+2\lambda_2+3\lambda_3+2\lambda_4, \lambda_1+2\lambda_2+4\lambda_3+2\lambda_4, \lambda_1+3\lambda_2+4\lambda_3+2\lambda_4, 2\lambda_1+3\lambda_2+4\lambda_3+2\lambda_4\}$.

Lemma 14. *A root system of type F_4 can be found by working down the root string in this manner:*

Roots of length 2:

$$s_{\lambda_1}(\lambda_2) = \lambda_1 + \lambda_2$$

$$s_{\lambda_2}(\lambda_3) = \lambda_2 + \lambda_3$$

$$s_{\lambda_3}(\lambda_4) = \lambda_3 + \lambda_4$$

$$s_{\lambda_3}(\lambda_2) = \lambda_2 + 2\lambda_3$$

Roots of length 3:

$$s_{\lambda_1}s_{\lambda_2}(\lambda_3) = \lambda_1 + \lambda_2 + \lambda_3$$

$$s_{\lambda_2}s_{\lambda_3}(\lambda_4) = \lambda_2 + \lambda_3 + \lambda_4$$

$$s_{\lambda_1}s_{\lambda_3}(\lambda_2) = \lambda_1 + \lambda_2 + 2\lambda_3$$

$$s_{\lambda_3}s_{\lambda_2}s_{\lambda_3}(\lambda_4) = \lambda_2 + 2\lambda_3 + \lambda_4$$

$$s_{\lambda_2}s_{\lambda_1}s_{\lambda_3}(\lambda_2) = \lambda_1 + 2\lambda_2 + 2\lambda_3$$

$$s_{\lambda_4}s_{\lambda_3}(\lambda_2) = \lambda_2 + 2\lambda_3 + 2\lambda_4$$

Roots of length 4:

$$s_{\lambda_1}s_{\lambda_2}s_{\lambda_3}(\lambda_4) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$$

$$s_{\lambda_3}s_{\lambda_1}s_{\lambda_2}s_{\lambda_3}(\lambda_4) = \lambda_1 + \lambda_2 + 2\lambda_3 + \lambda_4$$

$$s_{\lambda_2}s_{\lambda_3}s_{\lambda_1}s_{\lambda_2}s_{\lambda_3}(\lambda_4) = \lambda_1 + 2\lambda_2 + 2\lambda_3 + \lambda_4$$

$$s_{\lambda_4}s_{\lambda_1}s_{\lambda_3}(\lambda_2) = \lambda_1 + \lambda_2 + 2\lambda_3 + 2\lambda_4$$

$$s_{\lambda_3}s_{\lambda_2}s_{\lambda_3}s_{\lambda_1}s_{\lambda_2}s_{\lambda_3}(\lambda_4) = \lambda_1 + 2\lambda_2 + 3\lambda_3 + \lambda_4$$

$$s_{\lambda_4}s_{\lambda_2}s_{\lambda_1}s_{\lambda_3}(\lambda_2) = \lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4$$

$$s_{\lambda_4}s_{\lambda_3}s_{\lambda_2}s_{\lambda_3}s_{\lambda_1}s_{\lambda_2}s_{\lambda_3}(\lambda_4) = \lambda_1 + 2\lambda_2 + 3\lambda_3 + 2\lambda_4$$

$$s_{\lambda_3}s_{\lambda_4}s_{\lambda_2}s_{\lambda_1}s_{\lambda_3}(\lambda_2) = \lambda_1 + 2\lambda_2 + 4\lambda_3 + 2\lambda_4$$

$$s_{\lambda_2}s_{\lambda_3}s_{\lambda_4}s_{\lambda_2}s_{\lambda_1}s_{\lambda_3}(\lambda_2) = \lambda_1 + 3\lambda_2 + 4\lambda_3 + 2\lambda_4$$

$$s_{\lambda_1}s_{\lambda_2}s_{\lambda_3}s_{\lambda_4}s_{\lambda_2}s_{\lambda_1}s_{\lambda_3}(\lambda_2) = 2\lambda_1 + 3\lambda_2 + 4\lambda_3 + 2\lambda_4$$

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