

## ABSTRACT

LI, PINGKE. Fuzzy Relational Equations: Resolution and Optimization. (Under the direction of Professor Shu-Cherng Fang).

Fuzzy relational equations play an important role as a platform in various applications of fuzzy sets and systems. The resolution and optimization of fuzzy relational equations are of our particular interests from both of the theoretical and applicational viewpoints. In this dissertation, fuzzy relational equations are treated in a unified framework and classified according to different aspects of their composite operations.

For a given finite system of fuzzy relational equations with a specific composite operation, the consistency of the system can be verified in polynomial time by constructing a potential maximum/minimum solution and characteristic matrix. The solution set of a consistent system can be characterized by a unique maximum solution and finitely many minimal solutions, or dually, by a unique minimum solution and finitely many maximal solutions. The determination of all minimal/maximal solutions is closely related to the detection of all irredundant coverings of a set covering problem defined by the characteristic matrix, which may involve additional constraints. In particular, for fuzzy relational equations with  $\text{sup-}\mathcal{T}$  composition where  $\mathcal{T}$  is a continuous triangular norm, the existence of the additional constraints depends on whether  $\mathcal{T}$  is Archimedean or not.

Fuzzy relational equation constrained optimization problems are investigated as well in this dissertation. It is shown that the problem of minimizing an objective function subject to a system of fuzzy relational equations can be reduced in general to a 0-1 mixed integer programming problem. If the objective function is linear, or more generally, separable and monotone in each variable, then it can be further reduced to a set covering problem. Moreover, when the objective function is linear fractional, it can be reduced to a 0-1 linear fractional optimization problem and then solved via parameterization methods. However, if the objective function is max-separable with continuous monotone or unimodal components, then the problem can be solved efficiently, and its optimal solution set can be well characterized.

# Fuzzy Relational Equations: Resolution and Optimization

by  
Pingke Li

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APPROVED BY:

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Dr. Shu-Cherng Fang  
Chair of Advisory Committee

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Dr. Yahya Fathi

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Dr. Simon M. Hsiang

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Dr. James R. Wilson

## DEDICATION

I dedicate this dissertation to the memory of my mother, Huali Li, and to my father, Wenlong Zheng.

## BIOGRAPHY

Pingke Li was born on September 6, 1979, in Zhoushan, Zhejiang, China. He received his B.S. degree in Control Science from Fudan University in 2002 and his M.S. degree in Mathematics from Tsinghua University in 2005. In August 2005, he joined the Ph.D. program in Industrial Engineering of the North Carolina State University.

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# Chapter 1

## Introduction

The aim of this dissertation is to advance our theoretical understanding of fuzzy relational equations which are typically defined on the unit interval of real numbers equipped with some generalized logical operations. We provide a unified framework to characterize fuzzy relational equations of various types as well as the associated optimization problems.

### 1.1 Motivation

The study of fuzzy relational equations was initiated by Sanchez [399, 400, 401] in the 1970s in his pioneering work on the applications of fuzzy relations in approximate reasoning and decision making, while the notion of fuzzy relations was coined by Zadeh in his seminal paper [508] on fuzzy sets in 1965. As of today, fuzzy relational equations have played an important role as a uniform platform in a wide range of applications of fuzzy logic, particularly, in fuzzy control and knowledge engineering. The study of fuzzy relational equations is without doubt one of the most appealing subjects in the theory of fuzzy logic, from both of the mathematical and systems modeling viewpoints.

In the generic form, fuzzy relational equations are nonlinear equations in which the pairs of input and output data are connected by fuzzy relations along with some specific compositional rules of inference. In the context of rule-based systems which aim to make computers reason like human beings, each fuzzy relational equation may stand for a single rule representing some expert knowledge of a problem domain. In this perspective, solving a system of fuzzy relational equations can be viewed as encoding a set of available rules by

means of fuzzy relations, which form a knowledge base.

Among various types of fuzzy relational equations, finite systems of fuzzy relational equations with sup- $\mathcal{T}$  composition are the most fundamental and have been extensively investigated. Such a system, in most situations, can be formulated as

$$\sup_{j \in N} \mathcal{T}(a_{ij}, x_j) = b_i, \quad \text{for every } i \in M, \quad (1.1)$$

where  $M = \{1, 2, \dots, m\}$  and  $N = \{1, 2, \dots, n\}$  are two index sets, and  $a_{ij}, x_j, b_i$  are real numbers in the unit interval for every  $i \in M$  and  $j \in N$ . The compositional rule of inference applied in this case is known as the sup- $\mathcal{T}$  composition, more accurately, max- $\mathcal{T}$  composition for finite systems, where the *supremum* operation is interpreted as logical “OR” and the binary operator  $\mathcal{T} : [0, 1]^2 \rightarrow [0, 1]$  is a given triangular norm and interpreted as logical “AND”. A triangular norm, t-norm for short, is a binary operator  $\mathcal{T}$  on the unit interval which is commutative, associative, nondecreasing, and satisfies the boundary condition  $\mathcal{T}(1, x) = x$  for every  $x \in [0, 1]$ . The most frequently used t-norm in applications of fuzzy relational equations is the *minimum* operator  $\mathcal{T}_M$ , i.e.,  $\mathcal{T}_M(x, y) = \min(x, y)$ . Another two important triangular norms are the *product* operator  $\mathcal{T}_P = x \cdot y$  and the Łukasiewicz t-norm  $\mathcal{T}_L = \max(x + y - 1, 0)$ .

A system of fuzzy relational equations with sup- $\mathcal{T}$  composition is also referred to as a system of sup- $\mathcal{T}$  equations for short. Usually, the t-norm  $\mathcal{T}$  is required to be continuous, i.e., continuous as a function of two arguments. The *minimum* operator  $\mathcal{T}_M$ , *product* operator  $\mathcal{T}_P$ , and Łukasiewicz t-norm  $\mathcal{T}_L$  are all continuous. Actually, as will be introduced in Chapter 2, they are the most important prototypes of continuous t-norms. The study of sup- $\mathcal{T}$  equations with  $\mathcal{T}$  being a continuous t-norm can be reduced, in some sense, to the study of sup- $\mathcal{T}_M$  equations, sup- $\mathcal{T}_P$  equations, and sup- $\mathcal{T}_L$  equations, respectively. Besides, approaches to handling sup- $\mathcal{T}$  equations can be naturally extended to fuzzy relational equations with a general composite operation, for instance, sup- $\mathcal{C}$  equations with  $\mathcal{C}$  being a continuous conjunctor, as long as they preserve analogous structures.

Denote  $A = (a_{ij})_{m \times n} \in [0, 1]^{m \times n}$ ,  $\mathbf{x} = (x_j)_{n \times 1} \in [0, 1]^n$  and  $\mathbf{b} = (b_i)_{m \times 1} \in [0, 1]^m$ . A finite system of sup- $\mathcal{T}$  equations can be represented in its matrix form as

$$A \circ \mathbf{x} = \mathbf{b}, \quad (1.2)$$

where “ $\circ$ ” stands for the specific sup- $\mathcal{T}$  composition which is clear from the context. The resolution of a system  $A \circ \mathbf{x} = \mathbf{b}$  is to determine all vectors  $\mathbf{x} \in [0, 1]^n$  such that  $A \circ \mathbf{x} = \mathbf{b}$

for a given coefficient matrix  $A \in [0, 1]^{m \times n}$  and a right-hand side vector  $\mathbf{b} \in [0, 1]^m$ , i.e., to identify its solution set

$$S(A, \mathbf{b}) = \{\mathbf{x} \in [0, 1]^n \mid A \circ \mathbf{x} = \mathbf{b}\}. \quad (1.3)$$

It is well-known in the literature that for a finite system of sup- $\mathcal{T}$  equations  $A \circ \mathbf{x} = \mathbf{b}$  with  $\mathcal{T}$  being a continuous t-norm, its solution set, when it is nonempty, can be characterized by one unique maximum solution and a finite number of minimal solutions, i.e.,

$$S(A, \mathbf{b}) = \bigcup_{\check{\mathbf{x}} \in \check{S}(A, \mathbf{b})} \{\mathbf{x} \in [0, 1]^n \mid \check{\mathbf{x}} \leq \mathbf{x} \leq \hat{\mathbf{x}}\}, \quad (1.4)$$

where  $\check{S}(A, \mathbf{b})$  is the set of all minimal solutions of  $A \circ \mathbf{x} = \mathbf{b}$  and  $\hat{\mathbf{x}}$  is the maximum solution. The maximum solution can be constructed in polynomial time, while the determination of all minimal solutions is a difficult problem. Although various algorithms have been proposed during the last three decades for detecting all minimal solutions, this problem has not yet been fully resolved. Most recently, it was pointed out by Elbassioni [134] that the detection of all minimal solutions is related to the dualization problem of a monotone Boolean function, of which the exact time complexity is an unresolved problem. The best known result is attributed to Fredman and Khachiyan [143]. Besides, the influence of different t-norms on the structure of solution sets has not been fully explored.

Since solutions to a system of fuzzy relational equations may not be unique in general, solutions with some particular features are usually desired, for instance, Sanchez [401] suggested that the study of fuzzy relational equations can be extended to the study of the degree of fuzziness of the solutions. This immediately leads to the problem of optimizing an objective function (or multiple objective functions) subject to a system of fuzzy relational equations. For the case of sup- $\mathcal{T}$  equations, the associated optimization problem may be formulated as

$$\begin{aligned} \min \quad & z = f(\mathbf{x}) \\ \text{s.t.} \quad & \\ & A \circ \mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \in [0, 1]^n, \end{aligned} \quad (1.5)$$

where  $f : [0, 1]^n \rightarrow \mathbb{R}$  is a real-valued function, and  $A \circ \mathbf{x} = \mathbf{b}$  is a system of sup- $\mathcal{T}$  equations such that  $S(A, \mathbf{b}) \neq \emptyset$ . Solving such an optimization problem is in general not easy since the solution set  $S(A, \mathbf{b})$  is usually not convex. Actually, it is known that the resulting

optimization problem is NP-hard even if the objective function is linear. The study of this type of optimization problems has just begun.

It has been widely recognized that there are many cross connections between mathematical logic and integer programming, although these two subjects are usually covered in separate texts. Fuzzy logic, as a generalization of classical Boolean logic and the mathematical foundation for approximate reasoning, has been extensively studied since its inception. However, there is little work on discussing the connections between fuzzy logic and integer programming. Since fuzzy relational equations serve as a basic modeling tool for the applications of fuzzy logic, the study of fuzzy relational equations as well as the associated optimization problems, from a mathematical programming perspective, would certainly advance the development of fuzzy logic and provide a quantitative approach to fuzzy logical inference. Although some work has been done along this direction, a systematic treatment is absent from the known literature.

It should be noted that fuzzy relational equations, although originated from the applications of fuzzy logic, can be viewed as equations over a specific residuated lattice, and hence, can be studied in a fairly independent manner not being linked to any particular application of fuzzy logic. Resolution of fuzzy relational equations from this perspective has been comprehensively discussed by De Baets [80] and Li and Fang [270].

## 1.2 Connection to Approximate Reasoning

The human reasoning process can handle vague, ambiguous, and imprecise information in an appropriate manner that can rarely be precisely formulated. Fuzzy logic provides a framework of approximate reasoning that resembles human decision making with its ability to work on approximate data and find precise solutions. One of the most successful areas of fuzzy logic is the application of fuzzy rule-based systems. As illustrated in Figure 1.1, a fuzzy rule-based system contains a fuzzification interface, a fuzzy rule base (knowledge base), an inference engine, and a defuzzification interface. It implements a nonlinear mapping from its input space to output space through a number of fuzzy rules together with an inference procedure. The fuzzy rule base and the inference engine are the heart of a fuzzy rule-based system. The fuzzy rule base consists of necessary knowledge of a specific problem domain and is typically characterized by a list of fuzzy if-then rules in which the antecedents and

consequents may involve linguistic terms. By suitable rules of inference and composition, the inference engine evaluates the input for each rule, and then determines what piece of knowledge to use next.

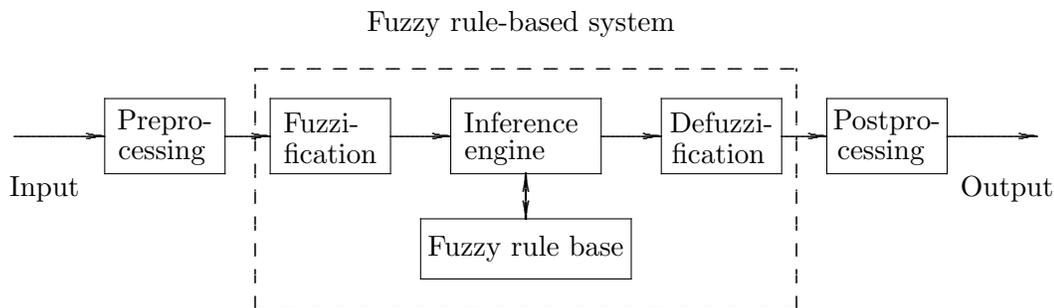


Figure 1.1: A block diagram of a fuzzy rule-based system

Modeling a fuzzy rule-based system is, in its very essence, to identify an appropriate collection of fuzzy if-then rules and incorporate them into a specific inference scheme. According to the inference schemes applied, the vast majority of existing fuzzy rule-based systems can be roughly categorized into Mamdani-Assilian models and Takagi-Sugeno models, which differ fundamentally in the formulation of if-then rules, and thus differ in the composition and defuzzification procedures accordingly. A Mamdani-Assilian model, also referred to as a linguistic fuzzy model, uses if-then rules with both fuzzy antecedents and consequents, and is essentially a qualitative expression of the concerned system. The transformation of the fuzzy if-then rules into a system of fuzzy relational equations has been a core idea in the procedure for developing a Mamdani-Assilian model. A Takagi-Sugeno model uses if-then rules with fuzzy antecedents and functional consequents, and is essentially an integrated qualitative and quantitative expression of the concerned system. For the fuzzy if-then rules of a Takagi-Sugeno model, the functional consequents are often available objective knowledge of the system while the fuzzy antecedents can be determined by minimizing the quadratic error in approximation.

A typical fuzzy if-then rule for a Mamdani-Assilian model is of the form:

“if  $x$  is  $P$ , then  $y$  is  $Q$ ,”

where variables  $x$  and  $y$  are in the universes of discourse  $\mathbf{X}$  and  $\mathbf{Y}$ , respectively, and both of

the antecedent  $P$  and consequent  $Q$  are imprecise predicates, usually expressed by linguistic terms, on  $\mathbf{X}$  and  $\mathbf{Y}$ , respectively. For instance, in an inventory control system, a fuzzy if-then rule could be

“if inventory is high, then production should be slowed.”

The knowledge base of such a fuzzy rule-based system is represented by a finite list of fuzzy if-then rules:

if  $x$  is  $A_i$ , then  $y$  is  $B_i$ ,  $i = 1, 2, \dots, m$ ,

and realized by a finite family  $\{(A_i, B_i)\}_{1 \leq i \leq m}$  of fuzzy input-output data pairs in an input space  $\mathbf{X}$  and an output space  $\mathbf{Y}$ . These fuzzy if-then rules describe a fuzzy mapping  $\Phi_R$  from the input space  $\mathbf{X}$  to the output space  $\mathbf{Y}$ , and the data are supposed to characterize the fuzzy mapping  $\Phi_R$  sufficiently well. Hence, leaving aside the appropriateness of the rule list, a major problem in building up such a knowledge base for a fuzzy rule-based system is to construct the mapping  $\Phi_R$  from the data such that

$$\Phi_R(A_i) = B_i, \quad i = 1, 2, \dots, m. \quad (1.6)$$

Following the idea of the *compositional rule of inference* introduced by Zadeh [509], the fuzzy mapping  $\Phi_R$  can be represented by a fuzzy relation  $R$  which connects fuzzy input  $A$  and fuzzy output  $B$  via

$$B = A \circ R, \quad (1.7)$$

with “ $\circ$ ” standing for a specific sup- $\mathcal{T}$  composite operation. Consequently, the construction of the fuzzy mapping  $\Phi_R$  to realize the rule list is equivalent to finding a solution to the corresponding system of sup- $\mathcal{T}$  equations:

$$A_i \circ R = B_i, \quad i = 1, 2, \dots, m, \quad (1.8)$$

where each equation  $A_i \circ R = B_i$  represents the generalized inference rule of *modus ponens*:

$$\frac{\begin{array}{l} \text{if } x \text{ is } A_i, \text{ then } y \text{ is } B_i \\ x \text{ is } A_i \end{array}}{y \text{ is } B_i.}$$

Note that, as illustrated by Dubois and Prade [130], fuzzy if-then rules may be formulated into different forms. The representation of fuzzy if-then rules is usually not unique, which results in fuzzy relational equations with various composite operations in the realization of fuzzy if-then rules. Moreover, as argued by Trillas *et al.* [437], the interpretation of fuzzy if-then rules depends on application context, and hence, there is no universal representation method.

The design of the fuzzy rule base, as well as its realization, is a key issue for the good performance of a fuzzy rule-based system. When the input/output dimensions are large, the design of an appropriate fuzzy rule base is a complicated task and could be time consuming.

### 1.3 Connection to Max-Plus Algebra

Since the late 1950s, it has been discovered independently by a number of researchers that a wide class of problems in operations research and computer science can be expressed in an attractive formulation using the framework of semirings, among which the max-plus algebra is of remarkable significance. A semiring consists of a basic set endowed with two associative binary operations which, with some additional requirements, resemble the roles of addition and multiplication in classical linear algebra, respectively. The unit interval  $[0, 1]$  endowed with the maximum operation as addition and a t-norm as multiplication is a specific semiring, and hence, also referred to as the fuzzy algebra in the context of fuzzy logic. The max-plus algebra is the set of extended real numbers  $\mathbb{R} \cup \{-\infty\}$  endowed with the maximum operation as addition and the conventional addition as multiplication. A significant effort has been developed to build up a theory of max-plus algebra analogous to that of classical linear algebra, i.e., to study systems of linear equations, eigenvalues, rank, etc. The comprehensive guides on the algebraic properties of the max-plus algebra and its applications can be found in Cuninghame-Green [65, 67], Zimmermann [527], Baccelli *et al.* [11], Gaubert [145], De Schutter [88], and Gondran and Minoux [169].

So far, the theory of max-plus algebra has offered an adequate language to model a variety of problems originated from areas of automata, discrete event systems, communication networks, project management, machine scheduling, transportation, etc. As an illustration, the following example, adapted from that in Butkovič [36], shows that how the max-plus algebra formulation arises in real-life problems.

Suppose that a factory manufactures products  $P_1, P_2, \dots, P_m$ , each of which is made out of some of  $n$  components. There are  $n$  machines available in the factory while each machine can prepare only one fixed component to be used in several products. It is known that machine  $M_j$  requires  $a_{ij}$  time units to prepare the component independently for product  $P_i$  with the convention that  $a_{ij} = -\infty$  if there is no need of the component made on machine  $M_j$  to produce  $P_i$ . Denote by  $x_1, x_2, \dots, x_n$  the starting times of the work on machines  $M_1, M_2, \dots, M_n$ , respectively. The components necessary for product  $P_i$  will be ready at time  $\max_{j \in N}(a_{ij} + x_j)$  for every  $i \in M$  where  $M = \{1, 2, \dots, m\}$  and  $N = \{1, 2, \dots, n\}$ . On the other hand, by technological reasons, the further processing of the components can start only at some specific moments  $b_1, b_2, \dots, b_m$ . In order to minimize the processing inventory during the preparation stage, the problem is to find the solutions to the following system of max-plus equations

$$\max_{j \in N}(a_{ij} + x_j) = b_i, \quad \text{for every } i \in M, \quad (1.9)$$

which can be represented as  $A \circ \mathbf{x} = \mathbf{b}$  in matrix form. It is clear that by starting early enough, all the components can be prepared before any given target times. Consequently, a way of finishing the preparation of all the components exactly on time can be obtained by artificially imposing delays. This would mean, however, the machines are not utilized in an optimal manner. Hence, obtaining the exact solutions to the system of max-plus equations  $A \circ \mathbf{x} = \mathbf{b}$  is of importance for optimal production planning. However, in case the system  $A \circ \mathbf{x} = \mathbf{b}$  has no solution, an approximate solution of particular properties would be desired, which leads to a global optimization problem:

$$\begin{aligned} \min \quad & \rho(A \circ \mathbf{x} - \mathbf{b}) \\ \text{s.t.} \quad & \mathbf{x} \in [-\infty, +\infty)^n, \end{aligned} \quad (1.10)$$

where  $\rho : [-\infty, +\infty)^n \rightarrow [0, +\infty]$  is a specific index function of performance.

From the perspective of semirings, see e.g., Hebisch and Weinert [203] and Golan [163], both of the fuzzy algebra and the max-plus algebra are specific commutative idempotent semirings. Consequently, sup- $\mathcal{T}$  equations and max-plus equations are linear equations over their respective underlying semiring structures, and hence, have many common features, e.g., solvability criteria and structures of solution sets. Methods developed for solving sup- $\mathcal{T}$  equations may be used for solving max-plus equations, and vice versa. However, as indicated

in Chapter 6, obtaining approximate solutions to a system of sup- $\mathcal{T}$  equations, as well as max-plus equations, is in general a hard problem except for the cases that use the Chebyshev distance as the performance index. The problem of finding Chebyshev approximate solutions for max-plus equations has been discussed by Cuninghame-Green [65, 67] and Cechlárová and Cuninghame-Green [49].

Another related example is the so-called max-separable optimization problem proposed by Zimmermann [520, 521] when solving some facility location problems. For instance, suppose that an emergency service system, consisting  $n$  service centers  $S_1, S_2, \dots, S_n$ , should be built to serve  $m$  customers located at  $Y_1, Y_2, \dots, Y_m$ . These service centers should be placed on  $n$  given segments  $A_1B_1, A_2B_2, \dots, A_nB_n$ , and as close as possible to the recommended locations  $C_1, C_2, \dots, C_n$ , respectively. Assume that the distances  $|A_jB_j|$ ,  $|A_jC_j|$ ,  $|Y_iA_j|$  and  $|Y_iB_j|$  are all known and denoted by  $d_j$ ,  $c_j$ ,  $a_{ij}$  and  $b_{ij}$ , respectively, for every  $i \in M$  and  $j \in N$  where  $M = \{1, 2, \dots, m\}$  and  $N = \{1, 2, \dots, n\}$ . If we require that each customer can be served always by at least one center within a certain distance, then we obtain a system of inequalities

$$\min_{j \in N} \min(a_{ij} + x_j, b_{ij} + d_j - x_j) \leq r_i, \quad \text{for every } i \in M, \quad (1.11)$$

where  $x_j = |A_jS_j|$  for every  $j \in N$  and  $r_i$  is the specified distance threshold for every  $i \in M$ . Consequently, we have to solve the following optimization problem:

$$\begin{aligned} & \min \max_{j \in N} |x_j - c_j| \\ & \text{s.t.} \\ & \quad \min_{j \in N} \min(a_{ij} + x_j, b_{ij} + d_j - x_j) \leq r_i, \quad \text{for every } i \in M, \\ & \quad 0 \leq x_j \leq d_j, \quad \text{for every } j \in N. \end{aligned} \quad (1.12)$$

This problem, as indicated by Hudec [209], is NP-hard in general. However, under some additional assumptions, Tharwat and Zimmermann [432] showed that this problem can be solved efficiently by modifying the method proposed in Zimmermann [523]. Note that the method presented in Theorems 5.4.1 and 5.4.5 coincides, in its very essence, with that presented in Zimmermann [523] and Tharwat and Zimmermann [432]. See Li and Fang [272] for details.

More discussion on the connection between the fuzzy algebra and the max-plus algebra can be found in Gondran and Minoux [167, 168], Cuninghame-Green [66], Butkovič [36, 37], Cohen *et al.* [58], Mohri [310], Rudeanu and Vaida [398], and references therein.

## 1.4 Contributions

This dissertation studies finite systems of fuzzy relational equations over the unit interval from the perspective of mathematical programming. The major contributions of this dissertation are:

- It shows that a system of fuzzy relational equations can be described by an interval-valued matrix, referred to as its characteristic matrix, and hence, can be expressed in terms of 0-1 mixed integer programming formulations. This provides a new and alternative characterization of a system of fuzzy relational equations. Moreover, the relation between minimal solutions to a system of sup- $\mathcal{T}$  equations and irredundant coverings of its characteristic matrix is fully characterized according to the well-known representation theorem of continuous triangular norms. It shows that a minimal solution to a system of sup- $\mathcal{T}$  equations corresponds to an irredundant covering of its simplified characteristic matrix and vice versa, as long as the involved continuous triangular norm  $\mathcal{T}$  is Archimedean. When  $\mathcal{T}$  is non-Archimedean, it corresponds to a constrained irredundant covering of the augmented characteristic matrix but not vice versa. Part of these results have been published in Li and Fang [268] and Li *et al.* [275]. Note that some results in Section 4.2.1 and Section 4.3.1 are generalizations of the results in Peeva [361, 366], Markovskii [296, 297], and Peeva and Kyosev [369, 370], where only the sup- $\mathcal{T}_P$  equations or sup- $\mathcal{T}_M$  equations involved. The results on sup- $\mathcal{T}$  equations with  $\mathcal{T}$  being a continuous Archimedean t-norm observed by Stamou and Tzafestas [420] are generalized in Section 4.2.1 as well.
- It shows that the problem of minimizing an objective function subject to a system of fuzzy relational equations can be reduced in general to a 0-1 mixed integer programming problem in polynomial time. In particular, if the objective function is linear, or more generally, separable and monotone in each argument, then it can be further reduced to a set covering problem. Besides, when the objective function is linear fractional, it can be reduced to a 0-1 linear fractional problem and then solved via parameterization methods. Part of these results have been published in Li and Fang [268, 271] and Li *et al.* [275].
- It shows that the problem of minimizing a max-separable function with continuous

monotone or unimodal components subject to a system of fuzzy relational equations can be solved efficiently. Furthermore, the set of optimal solutions of such a problem can be well characterized. Part of these results have been published in Li and Fang [272]. Note that some results presented in Section 5.4 coincide, in the very essence, with those in Zimmermann [523] and Tharwat and Zimmermann [432].

## 1.5 Organization

The rest of this dissertation is organized as follows. Chapter 2 introduces the generalized logical operators and their properties with an emphasis on continuous triangular norms. We also investigate elementary fuzzy relational equations induced by generalized logical operators. Chapter 3 classifies fuzzy relational equations into different categories according to the composite operations used, and discusses their solvability criteria and structures of solution sets. Chapter 4 focuses on sup- $\mathcal{T}$  equations and reveals the underlying connections between fuzzy relational equations and integer programming formulations. Chapter 5 illustrates that how the sup- $\mathcal{T}$  equation constrained optimization problems can be reformulated into 0-1 mixed integer programming problems. In particular, we show that the reformulation can be further simplified when the objective function is linear or linear fractional. The max-separable optimization problem is discussed as well. Chapter 6 presents some concluding remarks and interesting topics for further research.

The bibliographical notes at the end of each chapter are intended to provide the reader with references to the works in which important contributions were first made, with information on alternative treatments of topics covered in the text, and with suggestions for further reading.

The reference list at the end of the dissertation comprises mainly the works cited in the text and notes, and covers virtually most relevant books and significant papers on fuzzy relational equations published prior to 2009.

## 1.6 Bibliographical Notes

The history of fuzzy logic began with the simple idea conceived by Lotfi Zadeh in the early 1960s that many classes of objects encountered in the real world do not have

precisely defined criteria of membership. Abandoning the principle of excluded middle, Zadeh introduced in 1965 in his seminal paper [508] the notion of *fuzzy sets* to represent such classes of objects. Although the term *fuzzy* often carries a pejorative connotation, the mathematics of fuzzy set theory and fuzzy logic is precise. It provides formalized tools for dealing with imprecise information and modeling complex systems in a human style of reasoning. The mathematical foundation of fuzzy logic has been discussed by Gottwald [174, 178], Hájek [193], Novák *et al.* [327], Gerla [153], and Bělohlávek [17]. General discussion on fuzzy logic can be found in Wang *et al.* [458], Zadeh [510, 511, 512], and references therein.

Zadeh [509] introduced the notion of *principle of incompatibility* by which it means that “as the complexity of a system increases, our ability to make precise and yet significant statements about its behavior diminishes until a threshold is reached beyond which precision and significance (or relevance) become almost mutually exclusive characteristics.” By this principle, Zadeh [509] argued that the conventional quantitative techniques of system analysis are intrinsically unsuited for dealing with humanistic systems or systems of comparable complexity. Instead, Zadeh [509] outlined an alternative approach, now commonly known as fuzzy systems modeling, which is based on the premise that the key elements in human thinking are linguistic fuzzy values rather than crisp numbers. This proposal led to the development of a new class of systems based on fuzzy logic.

Mamdani and Assilian [295] implemented a linguistic fuzzy model for the first time in the form of fuzzy controllers. Linguistic fuzzy models were also investigated by King and Mamdani [237], Mamdani [294], Tong [434, 435], Mizumoto *et al.* [306, 307, 308], Procyk and Mamdani [382], Larsen [259], Umbers and King [439], Czogała and Pedrycz [70], Mizumoto and Zimmermann [309], Pedrycz [346], Buckley [29, 30], Sugeno and Yasukawa [425], Kosko [256], Ying [501], Ying and Chen [506], Castro and Delgado [43], Zeng and Singh [514], Ding *et al* [90], and many other researchers.

Some theoretical issues on the connections between fuzzy if-then rules of linguistic fuzzy models and fuzzy relational equations were discussed by Novák and Lehmke [326], Perfilieva [372], Perfiliva and Novák [375], and Gottwald [180].

Since the linguistic fuzzy models do not express knowledge in an explicit form, some

available objective knowledge can hardly be incorporated into the linguistic fuzzy models if it cannot be properly represented in linguistic terms.

Takagi and Sugeno [427, 428] and Sugeno and Kang [423, 424] proposed a new type of fuzzy rule-based systems, now commonly referred to as Takagi-Sugeno models or Takagi-Sugeno-Kang models, with a hybrid qualitative/quantitative formulation of if-then rules. The fundamental conception of Takagi-Sugeno models is to decompose a complex system in a fuzzy manner into simpler subsystems. See also Buckley [31], Ying [502, 503], Ying *et al.* [507], and references therein.

General discussion on fuzzy rule-based systems can be found in Pedrycz [349], Kandel [222], Yager and Filev [495], Geyer-Schulz [154], Ruan and Kerre [392], Ying [505], Piegat [379], and Pedrycz and Gomide [359].

The study of relational equations can be traced back to the work of Löwenheim [284]. Equations defined on Boolean lattices have been exclusively investigated by Rudeanu [393, 394, 395, 396] in the 1970s. A comprehensive overview of equations on Boolean lattices can be found in Rudeanu [397].

Shortly after Zadeh [508] proposed the concepts of fuzzy sets and fuzzy relations in 1965, Goguen [162] indicated that fuzzy sets, as well as fuzzy relations, can be considered in a general mathematical framework of lattice. So far, two approaches of mathematical treatment have been developed when concerning the solvability of fuzzy relational equations. The first approach follows the idea of Goguen [162] and concerns the restriction on the structure of the lattice that results in fuzzy relational equations on various lattices. The second approach is developed for fuzzy relational equations on a given lattice with different types of composite operations.

From a lattice-theoretic perspective, fuzzy relational equations can be well defined over a residuated lattice. Although equations defined on such algebraic structures may have nothing to do with the literal interpretation of “fuzzy relational equations,” this was a way how they got into existence. De Baets [80] provided an extensive investigation in a unified framework on the analytical methods for solving different types of fuzzy relational equations on various lattices. A comprehensive survey of classification and solvability for fuzzy relational equations on residuated lattices can be found in Li and Fang [270].

The pioneering work on the application of max-plus algebra includes Shimbel [416], Kleene [240], Roy [391], Cuninghame-Green [62, 63], Giffler [159, 160], Peteanu [378], Carré [42], Miniéka and Shier [301], Rinnooy Kan [387], Wongseelashote [479], Zimmermann [519], and Rote [390]. In the context of max-plus algebra, the symbols  $\oplus$  and  $\otimes$  are commonly used to denote the maximum operation and the conventional addition, respectively.

As of today, fuzzy relational equations have been widely applied in artificial intelligence, information processing, pattern analysis and classification, system control, decision making and management, etc. See, e.g., Sugeno [422], Pedrycz [349, 354], Kerre and Nachtegael [228], Szczepaniak *et al.* [426], Mordeson and Malik [311], Peeva and Kyosev [369], Kahraman [221], and Kerre [227].

In the context of fuzzy logic, fuzzy relational equations were first formulated and investigated by Sanchez [399, 400, 401] in 1970s with an emphasis on their applications in medical diagnosis. Zadeh and Desoer [513] showed that the study of relations is equivalent to the general study of systems since a system can be viewed as relations between an input space and an output space, which can be possibly modeled into a system of fuzzy relations. Investigating the behavior of such systems requires a powerful theoretical tool to deal with fuzzy relations, typically methods for formulating and solving systems of fuzzy relational equations.

The solvability criteria of  $\text{sup-}\mathcal{T}_M$  equations were established by Sanchez [400] and then extended to  $\text{sup-}\mathcal{T}$  equations by Pedrycz [343, 347] and Miyakoshi and Shimbo [303]. The structure of the solution set of  $\text{sup-}\mathcal{T}_M$  equations was characterized by Sanchez [401] and generalized to  $\text{sup-}\mathcal{T}$  equations by Di Nola *et al.* [99, 107]. Properties of  $\text{sup-}\mathcal{T}$  equations were also discussed by Bour and Lamotte [24], Di Nola *et al.* [103], Gottwald [174, 177], Klir and Yuan [248], and Li and Fang [268].

The detection of all minimal solutions of a system of  $\text{sup-}\mathcal{T}$  equations is still an ongoing research problem, although many algorithms have been proposed in the literature. Since the number of minimal solutions could be exponentially large with respect to the size of input data, the efficiency of an algorithm for generating all minimal solutions ought to be measured in terms of the combined size of input and output data. Elbassioni [134] indicated that this

problem is closely related to the dualization problem of a monotone Boolean function, of which the exact time complexity remains an open problem. The theoretically best known algorithm for solving the dualization problem of a monotone Boolean function is due to Fredman and Khachiyan [143], which runs in incremental quasi-polynomial time. See also Khachiyan *et al.* [229, 230], Eiter *et al.* [133], and references therein for detailed discussion.

The theory and applications of fuzzy relational equations developed up to 1989 were well documented by Di Nola *et al.* in the first monograph [116] on this issue. Some overviews and surveys can also be found in Di Nola *et al.* [107, 106], Perfilieva [371], Gottwald [174, 177, 179], Klir and Yuan [248], Pedrycz [354, 357], and Peeva and Kyosev [369].

The problem of minimizing a linear objective function subject to a system of sup- $\mathcal{T}_M$  equations was first investigated by Fang and Li [136] and later by Wu *et al.* [487] and Wu and Guu [485]. It was shown that such an optimization problem can be decomposed into two subproblems, one of which can be solved analytically while the other can be polynomially reduced to a 0-1 integer programming problem. This observation holds true for sup- $\mathcal{T}$  equations as well as equations with some other composite operations. A thorough investigation of sup- $\mathcal{T}$  equation constrained linear optimization problems is provided by Li and Fang [268]. See Section 5.5 for more details.

It is apparent nowadays that there is a remarkable parallel between mathematical logic and 0-1 integer programming. The interface between logical inference and mathematical programming, or quantitative methods of any sort, actually has a short history. Dantzig [71] showed in 1963 how logical propositions could be expressed using bivalent variables together with algebraic operators so that symbolic constraints can be modeled by integer programming formulations. It was this connection that was used by Karp [223] in 1972 to prove the NP-completeness of integer programming. Most of the pioneering work on this subject was done by Granot and Hammer [183], Williams [473, 474, 475, 476], Blair *et al.* [19], Jeroslow [219, 220], and Hooker [207]. Comprehensive discussions on logic/optimization interface can be found in Chandru and Hooker [53] and Williams [477].

## Chapter 2

# Generalized Logical Operators

Fuzzy logic is usually considered as an extension of the classical Boolean logic. However, the generalization of classical logical operators is not unique in fuzzy logic. It is implicitly accepted from the very beginning that, for each particular problem, there associates some generalized logical operators that are better suited than others. Consequently, the study of generalized logical operators becomes one of the most active research topics of fuzzy logic. This chapter presents an overview of various types of generalized logical operators and their roles in construction and resolution of fuzzy relational equations. The material presented is essentially from Klement *et al.* [243] and Alsina *et al.* [5].

### 2.1 Generalized Logical Operators

**Definition 2.1.1.** A conjunctor  $\mathcal{C} : [0, 1]^2 \rightarrow [0, 1]$  is a binary operator on  $[0, 1]$  such that  $\mathcal{C}(0, 1) = \mathcal{C}(1, 0) = 0$ ,  $\mathcal{C}(1, 1) = 1$ , and

$$\mathcal{C}(x_1, y_1) \leq \mathcal{C}(x_2, y_2), \quad \text{whenever } x_1 \leq x_2 \text{ and } y_1 \leq y_2. \quad (2.1)$$

A border conjunctor is a conjunctor that satisfies the neutrality principle, i.e.,  $\mathcal{C}(1, y) = y$  for every  $y \in [0, 1]$ .

**Definition 2.1.2.** A disjunctive operator  $\mathcal{D} : [0, 1]^2 \rightarrow [0, 1]$  is a binary operator on  $[0, 1]$  such that  $\mathcal{D}(0, 1) = \mathcal{D}(1, 0) = 1$ ,  $\mathcal{D}(0, 0) = 0$ , and

$$\mathcal{D}(x_1, y_1) \leq \mathcal{D}(x_2, y_2), \quad \text{whenever } x_1 \leq x_2 \text{ and } y_1 \leq y_2. \quad (2.2)$$

A border disjunctive is a disjunctive that satisfies the neutrality principle, i.e.,  $\mathcal{D}(0, y) = y$  for every  $y \in [0, 1]$ .

Note that for any conjunctive  $\mathcal{C}$ , it holds that

$$\mathcal{C}(x, 0) = \mathcal{C}(0, y) = 0, \quad \text{for every } x, y \in [0, 1], \quad (2.3)$$

while for any disjunctive  $\mathcal{D}$ , it holds that

$$\mathcal{D}(x, 1) = \mathcal{D}(1, y) = 1, \quad \text{for every } x, y \in [0, 1]. \quad (2.4)$$

In other words, 0 is a *zero element* of a conjunctive while 1 is a *zero element* of a disjunctive.

**Definition 2.1.3.** A triangular seminorm  $\mathcal{C}$  (*t-seminorm for short*) is a conjunctive that satisfies the boundary condition  $\mathcal{C}(x, 1) = \mathcal{C}(1, x) = x$  for every  $x \in [0, 1]$ . A triangular semiconorm  $\mathcal{D}$  (*t-semiconorm for short*) is a disjunctive that satisfies the boundary condition  $\mathcal{D}(x, 0) = \mathcal{D}(0, x) = x$  for every  $x \in [0, 1]$ .

**Definition 2.1.4.** An implicative  $\mathcal{I} : [0, 1]^2 \rightarrow [0, 1]$  is a binary operator on  $[0, 1]$  such that  $\mathcal{I}(0, 0) = \mathcal{I}(1, 1) = 1$ ,  $\mathcal{I}(1, 0) = 0$ , and

$$\mathcal{I}(x_1, y) \geq \mathcal{I}(x_2, y), \quad \text{whenever } x_1 \leq x_2; \quad (2.5)$$

$$\mathcal{I}(x, y_1) \leq \mathcal{I}(x, y_2), \quad \text{whenever } y_1 \leq y_2. \quad (2.6)$$

A border implicative is an implicative that satisfies the neutrality principle, i.e.,  $\mathcal{I}(1, y) = y$  for every  $y \in [0, 1]$ .

**Definition 2.1.5.** A coimplicative  $\mathcal{J} : [0, 1]^2 \rightarrow [0, 1]$  is a binary operator on  $[0, 1]$  such that  $\mathcal{J}(0, 0) = \mathcal{J}(1, 1) = 0$ ,  $\mathcal{J}(0, 1) = 1$ , and

$$\mathcal{J}(x_1, y) \geq \mathcal{J}(x_2, y), \quad \text{whenever } x_1 \leq x_2; \quad (2.7)$$

$$\mathcal{J}(x, y_1) \leq \mathcal{J}(x, y_2), \quad \text{whenever } y_1 \leq y_2. \quad (2.8)$$

A border coimplicative is a coimplicative that satisfies the neutrality principle, i.e.,  $\mathcal{J}(0, y) = y$  for every  $y \in [0, 1]$ .

Note that for any implicative  $\mathcal{I}$ , it holds that

$$\mathcal{I}(x, 1) = \mathcal{I}(0, y) = 1, \quad \text{for every } x, y \in [0, 1], \quad (2.9)$$

while for any coimplicator  $\mathcal{J}$ , it holds that

$$\mathcal{J}(x, 0) = \mathcal{J}(1, y) = 0, \quad \text{for every } x, y \in [0, 1]. \quad (2.10)$$

These boundary properties are also referred to as the absorption principles.

**Definition 2.1.6.** *A negator  $\mathcal{N} : [0, 1] \rightarrow [0, 1]$  is a decreasing unary operator on  $[0, 1]$  such that  $\mathcal{N}(0) = 1$  and  $\mathcal{N}(1) = 0$ . A negator  $\mathcal{N}$  is called strict if it is bijective. A strict negator  $\mathcal{N}$  is called strong if it is involutive, i.e.,  $\mathcal{N}(\mathcal{N}(x)) = x$  for every  $x \in [0, 1]$ .*

A negator  $\mathcal{N}$  defined on the unit interval is strict if and only if it is strictly decreasing and continuous. Moreover, if  $\mathcal{N}$  is a strong negator, then  $\mathcal{N}^{-1}(x) = \mathcal{N}(x)$  for every  $x \in [0, 1]$ , where  $\mathcal{N}^{-1}$  is the inverse operator of  $\mathcal{N}$ . A full characterization of strong negators was provided by Trillas [436].

**Theorem 2.1.7.** *The unary operator  $\mathcal{N} : [0, 1] \rightarrow [0, 1]$  is a strong negator if and only if there is a monotone bijection  $\psi : [0, 1] \rightarrow [0, 1]$  such that, for every  $x \in [0, 1]$ ,*

$$\mathcal{N}(x) = \psi^{-1}(1 - \psi(x)). \quad (2.11)$$

In particular, the negator  $\mathcal{N}_s$  with  $\mathcal{N}_s(x) = 1 - x$  is called the standard negator or the Lukasiewicz negator, which is also the most important and the most widely used one in practice. As a consequence of Theorem 2.1.7, each strong negator  $\mathcal{N}$  is a  $\psi$ -transformation of the standard negator  $\mathcal{N}_s$ , i.e.,

$$\mathcal{N}(x) = \psi^{-1}(\mathcal{N}_s(\psi(x))), \quad (2.12)$$

where  $\psi : [0, 1] \rightarrow [0, 1]$  is a monotone bijection on  $[0, 1]$ . Besides, any implicator  $\mathcal{I}$  induces a negator  $\mathcal{N}_{\mathcal{I}}(x) = \mathcal{I}(x, 0)$ , and any coimplicator  $\mathcal{J}$  also induces a negator  $\mathcal{N}_{\mathcal{J}}(x) = \mathcal{J}(x, 1)$ . Conversely, an implicator can be defined by a conjunctor  $\mathcal{C}$  and a negator  $\mathcal{N}$  via

$$\mathcal{I}_{\mathcal{C}, \mathcal{N}}(x, y) = \mathcal{N}(\mathcal{C}(x, \mathcal{N}(y))), \quad (2.13)$$

while a coimplicator can be defined by a disjunctive  $\mathcal{D}$  and a negator  $\mathcal{N}$  via

$$\mathcal{J}_{\mathcal{D}, \mathcal{N}}(x, y) = \mathcal{N}(\mathcal{D}(x, \mathcal{N}(y))). \quad (2.14)$$

Similarly, a conjunctor can be defined by an implicator  $\mathcal{I}$  and a negator  $\mathcal{N}$  via

$$\mathcal{C}_{\mathcal{I}, \mathcal{N}}(x, y) = \mathcal{N}(\mathcal{I}(x, \mathcal{N}(y))), \quad (2.15)$$

while a disjunctive can be defined by a coimplicator  $\mathcal{J}$  and a negator  $\mathcal{N}$  via

$$\mathcal{D}_{\mathcal{J},\mathcal{N}}(x, y) = \mathcal{N}(\mathcal{J}(x, \mathcal{N}(y))). \quad (2.16)$$

Furthermore, with a strong negator  $\mathcal{N}$ , a conjunctive  $\mathcal{C}$  can be induced by a disjunctive  $\mathcal{D}$  and vice versa, i.e.,

$$\mathcal{C}(x, y) = \mathcal{N}(\mathcal{D}(\mathcal{N}(x), \mathcal{N}(y))), \quad (2.17)$$

$$\mathcal{D}(x, y) = \mathcal{N}(\mathcal{C}(\mathcal{N}(x), \mathcal{N}(y))); \quad (2.18)$$

and an implicator  $\mathcal{I}$  can be induced by a coimplicator  $\mathcal{J}$  and vice versa, i.e.,

$$\mathcal{I}(x, y) = \mathcal{N}(\mathcal{J}(\mathcal{N}(x), \mathcal{N}(y))), \quad (2.19)$$

$$\mathcal{J}(x, y) = \mathcal{N}(\mathcal{I}(\mathcal{N}(x), \mathcal{N}(y))). \quad (2.20)$$

In this sense, a triple  $(\mathcal{C}, \mathcal{D}, \mathcal{N})$ , as well as  $(\mathcal{I}, \mathcal{J}, \mathcal{N})$ , forms a De Morgan triple where the conjunctive  $\mathcal{C}$  and the disjunctive  $\mathcal{D}$  are dual to each other with respect to the strong negator  $\mathcal{N}$ . Furthermore, a pair of De Morgan triples  $(\mathcal{C}, \mathcal{D}, \mathcal{N})$  and  $(\mathcal{I}, \mathcal{J}, \mathcal{N})$  are said to be isomorphic if  $\mathcal{I} = \mathcal{I}_{\mathcal{C},\mathcal{N}}$  or equivalently,  $\mathcal{C} = \mathcal{C}_{\mathcal{I},\mathcal{N}}$ . For any De Morgan triple  $(\mathcal{C}, \mathcal{D}, \mathcal{N})$ , its isomorphic De Morgan triple  $(\mathcal{I}, \mathcal{J}, \mathcal{N})$  is uniquely determined and vice versa. It is clear that for a pair of isomorphic De Morgan triples  $(\mathcal{C}, \mathcal{D}, \mathcal{N})$  and  $(\mathcal{I}, \mathcal{J}, \mathcal{N})$ , the conjunctive  $\mathcal{C}$  is a border conjunctive if and only if the implicator  $\mathcal{I}$  is a border implicator, or equivalently, if and only if the disjunctive  $\mathcal{D}$  and the coimplicator  $\mathcal{J}$  are a border disjunctive and a border coimplicator, respectively. The relations between these generalized logical operators are illustrated in Figure 2.1.

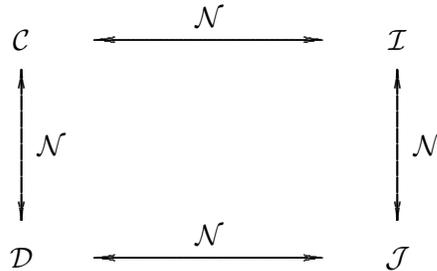


Figure 2.1: The relationship between generalized logical operators

Besides, two residual operators  $\mathcal{I}_{\mathcal{C}}$  and  $\mathcal{J}_{\mathcal{C}}$  can be defined with respect to a given border conjunctive  $\mathcal{C}$ .

**Definition 2.1.8.** *The residual implicator  $\mathcal{I}_{\mathcal{C}}$  and its accompanied residual operator  $\mathcal{J}_{\mathcal{C}}$  with respect to a border conjunctive  $\mathcal{C}$  are defined, respectively, as*

$$\mathcal{I}_{\mathcal{C}}(x, y) = \sup\{z \in [0, 1] \mid \mathcal{C}(x, z) \leq y\}, \quad (2.21)$$

$$\mathcal{J}_{\mathcal{C}}(x, y) = \inf\{z \in [0, 1] \mid \mathcal{C}(x, z) \geq y\}. \quad (2.22)$$

It can be verified that the residual implicator  $\mathcal{I}_{\mathcal{C}}$  is indeed a border implicator while the operator  $\mathcal{J}_{\mathcal{C}}$  has no particular logical interpretation. Note that the residual operators  $\mathcal{I}_{\mathcal{C}}$  and  $\mathcal{J}_{\mathcal{C}}$  can actually be defined for any conjunctive  $\mathcal{C}$ . However,  $\mathcal{I}_{\mathcal{C}}$  may fail to be an implicator when  $\mathcal{C}$  is not a border conjunctive. Moreover, the definitions of these two residual operators are not unique when the conjunctive  $\mathcal{C}$  is not commutative. The left residual operators with respect to a conjunctive  $\mathcal{C}$  can be defined accordingly.

Dually, the residual operators  $\mathcal{I}_{\mathcal{D}}$  and  $\mathcal{J}_{\mathcal{D}}$  can be defined with respect to a given border disjunctive  $\mathcal{D}$ .

**Definition 2.1.9.** *The residual coimplicator  $\mathcal{J}_{\mathcal{D}}$  and its accompanied residual operator  $\mathcal{I}_{\mathcal{D}}$  with respect to a border disjunctive  $\mathcal{D}$  are defined, respectively, as*

$$\mathcal{J}_{\mathcal{D}}(x, y) = \inf\{z \in [0, 1] \mid \mathcal{D}(x, z) \geq y\}, \quad (2.23)$$

$$\mathcal{I}_{\mathcal{D}}(x, y) = \sup\{z \in [0, 1] \mid \mathcal{D}(x, z) \leq y\}. \quad (2.24)$$

It can be verified that the residual coimplicator  $\mathcal{J}_{\mathcal{D}}$  is a border coimplicator while the operator  $\mathcal{I}_{\mathcal{D}}$  has no particular logical interpretation. With the infix notations,  $\mathcal{I}_{\mathcal{C}}(x, y)$  and  $\mathcal{J}_{\mathcal{C}}(x, y)$  are denoted by  $x\varphi_{\mathcal{C}}y$  and  $x\sigma_{\mathcal{C}}y$ , respectively, while  $\mathcal{I}_{\mathcal{D}}(x, y)$  and  $\mathcal{J}_{\mathcal{D}}(x, y)$  are denoted by  $x\beta_{\mathcal{D}}y$  and  $x\omega_{\mathcal{D}}y$ , respectively.

Analogously, residual operators can be defined with respect to border implicators and border coimplicators, respectively.

**Definition 2.1.10.** *The residual conjunctive  $\mathcal{C}_{\mathcal{I}}$  and its accompanied residual operator  $\mathcal{D}_{\mathcal{I}}$  with respect to a border implicator  $\mathcal{I}$  are defined, respectively, as*

$$\mathcal{C}_{\mathcal{I}}(x, y) = \inf\{z \in [0, 1] \mid \mathcal{I}(x, z) \geq y\}, \quad (2.25)$$

$$\mathcal{D}_{\mathcal{I}}(x, y) = \sup\{z \in [0, 1] \mid \mathcal{I}(x, z) \leq y\}. \quad (2.26)$$

**Definition 2.1.11.** *The residual disjunctive operator  $\mathcal{D}_{\mathcal{J}}$  and its accompanied residual operator  $\mathcal{C}_{\mathcal{J}}$  with respect to a border coimplicator  $\mathcal{J}$  are defined, respectively, as*

$$\mathcal{D}_{\mathcal{J}}(x, y) = \sup\{z \in [0, 1] \mid \mathcal{J}(x, z) \leq y\}, \quad (2.27)$$

$$\mathcal{C}_{\mathcal{J}}(x, y) = \inf\{z \in [0, 1] \mid \mathcal{J}(x, z) \geq y\}. \quad (2.28)$$

The residual operator  $\mathcal{C}_{\mathcal{I}}$  is always a border conjunctive operator while the residual operator  $\mathcal{D}_{\mathcal{J}}$  is always a border disjunctive operator. The operators  $\mathcal{D}_{\mathcal{I}}$  and  $\mathcal{C}_{\mathcal{J}}$  have no particular logical interpretation. With the infix notations,  $\mathcal{C}_{\mathcal{I}}(x, y)$  and  $\mathcal{D}_{\mathcal{I}}(x, y)$  are denoted by  $x\gamma_I y$  and  $x\delta_I y$ , respectively, while  $\mathcal{D}_{\mathcal{J}}(x, y)$  and  $\mathcal{C}_{\mathcal{J}}(x, y)$  are denoted by  $x\delta_J y$  and  $x\gamma_J y$ , respectively.

As has been shown, the residual operators of border conjunctive operators and border disjunctive operators lead to border implicators and border coimplicators, respectively, and vice versa. Moreover, they form pairs of adjoint operators, respectively, under additional *residuation conditions*.

**Definition 2.1.12.** *A function  $f : [0, 1] \rightarrow [0, 1]$  is said to be lower semicontinuous if  $\{x \in [0, 1] \mid f(x) > \alpha\}$  is an open set for every  $\alpha \in [0, 1]$ , and upper semicontinuous if  $\{x \in [0, 1] \mid f(x) < \alpha\}$  is an open set for every  $\alpha \in [0, 1]$ .*

**Definition 2.1.13.** *A function  $f : [0, 1] \rightarrow [0, 1]$  is said to be left-continuous if  $\lim_{x \rightarrow x_0^-} f(x) = f(x_0)$  for every  $x_0 \in (0, 1]$ , and right-continuous if  $\lim_{x \rightarrow x_0^+} f(x) = f(x_0)$  for every  $x_0 \in [0, 1)$ . A function  $f : [0, 1] \rightarrow [0, 1]$  is said to be continuous if  $\lim_{x \rightarrow x_0} f(x) = f(x_0)$  for every  $x_0 \in [0, 1]$ .*

**Theorem 2.1.14.** *A function  $f : [0, 1] \rightarrow [0, 1]$  is continuous if and only if it is both left- and right-continuous, or both lower and upper semicontinuous.*

**Theorem 2.1.15.** *An increasing function  $f : [0, 1] \rightarrow [0, 1]$  is lower semicontinuous if and only if it is left-continuous, and upper semicontinuous if and only if it is right continuous.*

Note that the left- and right-continuity mean exactly the interchangeability of the supremum operation and the infimum operation, respectively, with an increasing function.

**Theorem 2.1.16.** *Let  $\mathcal{C} : [0, 1]^2 \rightarrow [0, 1]$  be a border conjunctive operator and  $\mathcal{I}_{\mathcal{C}}$  its residual implicator. The following statements are equivalent:*

- (i) *The conjunctive operator  $\mathcal{C}$  and the implicator  $\mathcal{I}_{\mathcal{C}}$  form a pair of adjoint residual operators, i.e., for any  $a, b \in [0, 1]$ ,*

$$\mathcal{C}(a, x) \leq b \quad \text{if and only if} \quad x \leq \mathcal{I}_{\mathcal{C}}(a, b). \quad (2.29)$$

(ii) The second partial mapping of  $\mathcal{C}$ , i.e.,  $\mathcal{C}(a, \cdot)$ , is left-continuous for every  $a \in [0, 1]$ .

(iii) The second partial mapping of  $\mathcal{I}_{\mathcal{C}}$ , i.e.,  $\mathcal{I}_{\mathcal{C}}(a, \cdot)$ , is right-continuous for every  $a \in [0, 1]$ .

**Theorem 2.1.17.** *If  $\mathcal{C} : [0, 1]^2 \rightarrow [0, 1]$  is a border conjunctor with its second partial mapping  $\mathcal{C}(a, \cdot)$  being left-continuous for every  $a \in [0, 1]$ , then  $\mathcal{C}_{\mathcal{I}_{\mathcal{C}}} = \mathcal{C}$ .*

*Proof.* It follows Theorem 2.1.16 that for any  $x, y \in [0, 1]$ ,

$$\begin{aligned} \mathcal{C}_{\mathcal{I}_{\mathcal{C}}}(x, y) &= \inf\{z \in [0, 1] \mid \mathcal{I}_{\mathcal{C}}(x, z) \geq y\} \\ &= \inf\{z \in [0, 1] \mid \mathcal{C}(x, y) \leq z\} \\ &= \mathcal{C}(x, y). \end{aligned} \quad \square$$

**Theorem 2.1.18.** *Let  $\mathcal{D} : [0, 1]^2 \rightarrow [0, 1]$  be a border disjunctive and  $\mathcal{J}_{\mathcal{D}}$  its residual coimplicator. The following statements are equivalent:*

(i) *The coimplicator  $\mathcal{J}_{\mathcal{D}}$  and the disjunctive  $\mathcal{D}$  form a pair of adjoint residual operators, i.e., for any  $a, b \in [0, 1]$ ,*

$$\mathcal{J}_{\mathcal{D}}(a, x) \leq b \quad \text{if and only if} \quad x \leq \mathcal{D}(a, b). \quad (2.30)$$

(ii) *The second partial mapping of  $\mathcal{J}_{\mathcal{D}}$ , i.e.,  $\mathcal{J}_{\mathcal{D}}(a, \cdot)$ , is left-continuous for every  $a \in [0, 1]$ .*

(iii) *The second partial mapping of  $\mathcal{D}$ , i.e.,  $\mathcal{D}(a, \cdot)$ , is right-continuous for every  $a \in [0, 1]$ .*

**Theorem 2.1.19.** *If  $\mathcal{D} : [0, 1]^2 \rightarrow [0, 1]$  is a border disjunctive with its second partial mapping  $\mathcal{D}(a, \cdot)$  being right-continuous for every  $a \in [0, 1]$ , then  $\mathcal{D}_{\mathcal{J}_{\mathcal{D}}} = \mathcal{D}$ .*

*Proof.* It follows Theorem 2.1.18 that for any  $x, y \in [0, 1]$ ,

$$\begin{aligned} \mathcal{D}_{\mathcal{J}_{\mathcal{D}}}(x, y) &= \sup\{z \in [0, 1] \mid \mathcal{J}_{\mathcal{D}}(x, z) \leq y\} \\ &= \sup\{z \in [0, 1] \mid \mathcal{D}(x, y) \geq z\} \\ &= \mathcal{D}(x, y). \end{aligned} \quad \square$$

Accordingly, the De Morgan triples  $(\mathcal{C}, \mathcal{D}, \mathcal{N})$  and  $(\mathcal{I}_{\mathcal{C}}, \mathcal{J}_{\mathcal{D}}, \mathcal{N})$  adjoin to each other, when the border conjunctor  $\mathcal{C}$  is left-continuous in its second argument, or equivalently, the border disjunctive  $\mathcal{D}$  is right-continuous in its second argument. These relations are illustrated in Figure 2.2, where  $\mathcal{N}$  is a strong negator,  $\mathcal{R}$  is the associated residual operation.

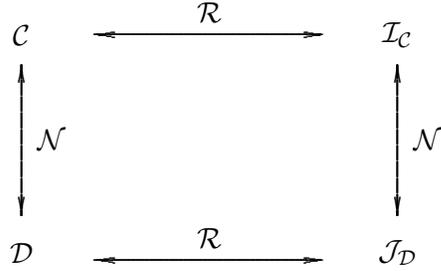


Figure 2.2: The relationship between pairwise adjoint operators

**Definition 2.1.20.** Let  $\mathcal{C} : [0, 1]^2 \rightarrow [0, 1]$  be a conjunctor. An element  $a \in (0, 1]$  is said to be a left zero divisor of  $\mathcal{C}$  if there exists some  $b \in (0, 1]$  such that  $\mathcal{C}(a, b) = 0$ . Analogously, an element  $b \in (0, 1]$  is said to be a right zero divisor of  $\mathcal{C}$  if there exists some  $a \in (0, 1]$  such that  $\mathcal{C}(a, b) = 0$ . An element that is both a left and a right zero divisor is simply called a zero divisor.

The left and right zero divisors are the same when the conjunctor is commutative. A nonzero element is called regular if it is neither a left nor a right zero divisor of the conjunctor.

**Definition 2.1.21.** Let  $\mathcal{D} : [0, 1]^2 \rightarrow [0, 1]$  be a disjunctive. An element  $a \in [0, 1)$  is said to be a left zero divisor of  $\mathcal{D}$  if there exists some  $b \in [0, 1)$  such that  $\mathcal{D}(a, b) = 1$ . Analogously, an element  $b \in [0, 1)$  is said to be a right zero divisor of  $\mathcal{D}$  if there exists some  $a \in [0, 1)$  such that  $\mathcal{D}(a, b) = 1$ . An element that is both a left and a right zero divisor is simply called a zero divisor.

Note that 1 is the “zero element” with respect to a disjunctive by Definition 2.1.2. Analogous concepts can be defined as well for implicators and coimplicators according to their respective absorption principles.

**Definition 2.1.22.** Let  $\mathcal{I} : [0, 1]^2 \rightarrow [0, 1]$  be an implicator. An element  $a \in (0, 1]$  is said to be a left zero divisor of  $\mathcal{I}$  if there exists some  $b \in [0, 1)$  such that  $\mathcal{I}(a, b) = 1$ . An element  $b \in [0, 1)$  is said to be a right zero divisor of  $\mathcal{I}$  if there exists some  $a \in (0, 1]$  such that  $\mathcal{I}(a, b) = 1$ .

**Definition 2.1.23.** Let  $\mathcal{J} : [0, 1]^2 \rightarrow [0, 1]$  be a coimplicator. An element  $a \in [0, 1)$  is said to be a left zero divisor of  $\mathcal{J}$  if there exists some  $b \in (0, 1]$  such that  $\mathcal{J}(a, b) = 0$ . An element  $b \in (0, 1]$  is said to be a right zero divisor of  $\mathcal{J}$  if there exists some  $a \in [0, 1)$  such that  $\mathcal{J}(a, b) = 0$ .

**Definition 2.1.24.** Let  $\mathcal{C} : [0, 1]^2 \rightarrow [0, 1]$  be a conjunctor.

(i) The conjunctor  $\mathcal{C}$  satisfies the right cancelation law if

$$\mathcal{C}(x, y) = \mathcal{C}(x, z) \quad \text{implies} \quad x = 0 \quad \text{or} \quad y = z. \quad (2.31)$$

(ii) The conjunctor  $\mathcal{C}$  satisfies the right conditional cancelation law if

$$\mathcal{C}(x, y) = \mathcal{C}(x, z) > 0 \quad \text{implies} \quad y = z. \quad (2.32)$$

**Theorem 2.1.25.** Let  $\mathcal{C} : [0, 1]^2 \rightarrow [0, 1]$  be a conjunctor.

(i) The conjunctor  $\mathcal{C}$  satisfies the right cancelation law if and only if the second partial mapping  $\mathcal{C}(x, \cdot)$  is strictly increasing for every  $x \in (0, 1]$ , i.e.,

$$\mathcal{C}(x, y) < \mathcal{C}(x, z) \quad \text{whenever} \quad x > 0 \quad \text{and} \quad y < z. \quad (2.33)$$

(ii) The conjunctor  $\mathcal{C}$  satisfies the right conditional cancelation law if and only if the second partial mapping  $\mathcal{C}(x, \cdot)$  is conditionally strictly increasing for every  $x \in (0, 1]$ , i.e.,

$$\mathcal{C}(x, y) < \mathcal{C}(x, z) \quad \text{whenever} \quad \mathcal{C}(x, y) > 0 \quad \text{and} \quad y < z. \quad (2.34)$$

*Proof.* If the function  $\mathcal{C}(x, \cdot)$  is strictly increasing for every  $x \in (0, 1]$ , then the conjunctor  $\mathcal{C}$  satisfies the right cancelation law. Conversely, the right cancelation law and the monotonicity of the conjunctor  $\mathcal{C}$  imply that the second partial mapping  $\mathcal{C}(x, \cdot)$  is strictly increasing for every  $x \in (0, 1]$ . The second assertion is an immediate consequence of the previous one.  $\square$

If a conjunctor  $\mathcal{C}$  satisfies the right cancelation law, it obviously fulfills the right conditional cancelation law, but not conversely. Furthermore, it follows from Theorem 2.1.25 that a conjunctor  $\mathcal{C}$  has no right zero divisors when it satisfies the right cancelation law.

When a conjunctor  $\mathcal{C}$  only satisfies the right conditional cancelation law, it necessarily has right zero divisors. Analogously, the left cancelation law and the left conditional law can be defined for conjunctors. For commutative conjunctors, the terms cancelation law and conditional cancelation law may be used, respectively.

The dual versions of Definition 2.1.24 and Theorem 2.1.25 for disjunctors can be derived without any difficulty.

**Definition 2.1.26.** *Let  $\mathcal{D} : [0, 1]^2 \rightarrow [0, 1]$  be a disjunctor.*

(i) *The disjunctor  $\mathcal{D}$  satisfies the right cancelation law if*

$$\mathcal{D}(x, y) = \mathcal{D}(x, z) \quad \text{implies} \quad x = 1 \quad \text{or} \quad y = z. \quad (2.35)$$

(ii) *The disjunctor  $\mathcal{D}$  satisfies the right conditional cancelation law if*

$$\mathcal{D}(x, y) = \mathcal{D}(x, z) > 0 \quad \text{implies} \quad y = z. \quad (2.36)$$

**Theorem 2.1.27.** *Let  $\mathcal{D} : [0, 1]^2 \rightarrow [0, 1]$  be a disjunctor.*

(i) *The disjunctor  $\mathcal{D}$  satisfies the right cancelation law if and only if the second partial mapping  $\mathcal{D}(x, \cdot)$  is strictly increasing for every  $x \in (0, 1]$ , i.e.,*

$$\mathcal{D}(x, y) < \mathcal{D}(x, z) \quad \text{whenever} \quad x < 1 \quad \text{and} \quad y < z. \quad (2.37)$$

(ii) *The disjunctor  $\mathcal{D}$  satisfies the right conditional cancelation law if and only if the second partial mapping  $\mathcal{D}(x, \cdot)$  is conditionally strictly increasing for every  $x \in (0, 1]$ , i.e.,*

$$\mathcal{D}(x, y) < \mathcal{D}(x, z) \quad \text{whenever} \quad \mathcal{D}(x, y) < 1 \quad \text{and} \quad y < z. \quad (2.38)$$

Analogously, the right cancelation laws and the right conditional cancelation laws of implicators and coimplicators can be defined in a similar manner as those of conjunctors and disjunctors, respectively. However, the left cancelation laws and the left conditional cancelation laws of implicators and coimplicators should be defined in a dual manner.

**Definition 2.1.28.** *Let  $\mathcal{I} : [0, 1]^2 \rightarrow [0, 1]$  be an impicator.*

(i) *The impicator  $\mathcal{I}$  satisfies the left cancelation law if*

$$\mathcal{I}(x, z) = \mathcal{I}(y, z) \quad \text{implies} \quad z = 1 \quad \text{or} \quad x = y. \quad (2.39)$$

(ii) The implicator  $\mathcal{I}$  satisfies the left conditional cancelation law if

$$\mathcal{I}(x, z) = \mathcal{I}(y, z) < 1 \quad \text{implies} \quad x = y. \quad (2.40)$$

**Definition 2.1.29.** Let  $\mathcal{J} : [0, 1]^2 \rightarrow [0, 1]$  be a coimplicator.

(i) The coimplicator  $\mathcal{J}$  satisfies the left cancelation law if

$$\mathcal{J}(x, z) = \mathcal{J}(y, z) \quad \text{implies} \quad z = 0 \quad \text{or} \quad x = y. \quad (2.41)$$

(ii) The coimplicator  $\mathcal{J}$  satisfies the left conditional cancelation law if

$$\mathcal{J}(x, z) = \mathcal{J}(y, z) > 0 \quad \text{implies} \quad x = y. \quad (2.42)$$

Analogous to Theorems 2.1.25 and 2.1.27, the left cancelation laws and the left conditional cancelation laws of implicators and coimplicators require that the implicators and coimplicators be strictly decreasing in their first arguments within proper domains, respectively.

For a pair of isomorphic De Morgan triples  $(\mathcal{C}, \mathcal{D}, \mathcal{N})$  and  $(\mathcal{I}, \mathcal{J}, \mathcal{N})$ , the conjunctive  $\mathcal{C}$  satisfies the left (right) cancelation law or conditional cancelation law if and only if the implicator  $\mathcal{I}$  satisfies the left (right) cancelation law or conditional cancelation law, or equivalently, if and only if the disjunctive  $\mathcal{D}$  and the coimplicator  $\mathcal{J}$  satisfy the left (right) cancelation law or conditional cancelation law, respectively. Similar results can be readily derived for the left-continuity (right-continuity) and the existence of zero divisors on the pair of isomorphic De Morgan triples  $(\mathcal{C}, \mathcal{D}, \mathcal{N})$  and  $(\mathcal{I}, \mathcal{J}, \mathcal{N})$ . These relations show that there is no essential difference between these binary generalized logical operators from an algebraic point of view regardless of their specific interpretations in the general theory of logic. Consequently, it suffices to focus on some specific class of these binary generalized logical operators in the case that only algebraic properties of them are concerned, for instance, the resolution and optimization of a system of fuzzy relational equations.

## 2.2 Triangular Norms and Conorms

Among all conjunctors and disjunctors, those commutative and associative border conjunctors and disjunctors, referred to as triangular norms and conorms, respectively, are of the most importance from different aspects.

**Definition 2.2.1.** A triangular norm (*t-norm* for short) is a binary operator  $\mathcal{T} : [0, 1]^2 \rightarrow [0, 1]$  such that for every  $x, y, z \in [0, 1]$  the following four axioms are satisfied:

- (T1)  $\mathcal{T}(x, y) = \mathcal{T}(y, x)$ . (commutativity)
- (T2)  $\mathcal{T}(x, \mathcal{T}(y, z)) = \mathcal{T}(\mathcal{T}(x, y), z)$ . (associativity)
- (T3)  $\mathcal{T}(x, y) \leq \mathcal{T}(x, z)$ , whenever  $y \leq z$ . (monotonicity)
- (T4)  $\mathcal{T}(x, 1) = x$ . (boundary condition)

**Definition 2.2.2.** A triangular conorm (*t-conorm* or *s-norm* for short) is a binary operator  $\mathcal{S} : [0, 1]^2 \rightarrow [0, 1]$  such that for every  $x, y, z \in [0, 1]$  the following four axioms are satisfied:

- (S1)  $\mathcal{S}(x, y) = \mathcal{S}(y, x)$ . (commutativity)
- (S2)  $\mathcal{S}(x, \mathcal{S}(y, z)) = \mathcal{S}(\mathcal{S}(x, y), z)$ . (associativity)
- (S3)  $\mathcal{S}(x, y) \leq \mathcal{S}(x, z)$ , whenever  $y \leq z$ . (monotonicity)
- (S4)  $\mathcal{S}(x, 0) = x$ . (boundary condition)

The infix notations like  $x \wedge_t y$  and  $x \vee_t y$  are usually used in the literature instead of the prefix notations  $\mathcal{T}(x, y)$  and  $\mathcal{S}(x, y)$ , respectively, while  $x \wedge y$  and  $x \vee y$  typically stand for the *minimum* operator  $\mathcal{T}_M(x, y)$  and the *maximum* operator  $\mathcal{S}_M(x, y)$ , respectively.

There exist uncountably many t-norms among which four basic t-norms are remarkable from different points of view, and are defined by, respectively,

$$\begin{aligned} \mathcal{T}_M(x, y) &= \min(x, y), && \text{(minimum, Gödel t-norm, Zadeh t-norm)} \\ \mathcal{T}_P(x, y) &= x \cdot y, && \text{(probabilistic product, Goguen t-norm)} \\ \mathcal{T}_L(x, y) &= \max(x + y - 1, 0), && \text{(bounded difference, Łukasiewicz t-norm)} \\ \mathcal{T}_D(x, y) &= \begin{cases} 0, & \text{if } (x, y) \in [0, 1]^2 \\ \min(x, y), & \text{otherwise.} \end{cases} && \text{(drastic product)} \end{aligned}$$

Their dual t-conorms with respect to the standard negator  $\mathcal{N}_s$  are given by, respectively,

$$\begin{aligned} \mathcal{S}_M(x, y) &= \max(x, y), && \text{(maximum, Gödel t-conorm, Zadeh t-conorm)} \\ \mathcal{S}_P(x, y) &= x + y - x \cdot y, && \text{(probabilistic sum, Goguen t-conorm)} \\ \mathcal{S}_L(x, y) &= \min(x + y, 1), && \text{(bounded sum, Łukasiewicz t-conorm)} \\ \mathcal{S}_D(x, y) &= \begin{cases} 1, & \text{if } (x, y) \in (0, 1]^2 \\ \max(x, y), & \text{otherwise.} \end{cases} && \text{(drastic sum)} \end{aligned}$$

A t-norm  $\mathcal{T}$ , as well as a t-conorm  $\mathcal{S}$ , is said to be continuous if it is continuous as a real function with two arguments, i.e., for every  $(x_0, y_0) \in [0, 1]^2$ , it holds that

$$\lim_{(x,y) \rightarrow (x_0,y_0)} \mathcal{T}(x, y) = \mathcal{T}(x_0, y_0).$$

Due to the monotonicity and commutativity properties of t-norms, a t-norm  $\mathcal{T}$  is continuous if and only if it is continuous in each argument, and hence, if and only if its partial mapping  $\mathcal{T}(a, \cdot)$  is continuous for every  $a \in [0, 1]$ . Consequently, a t-norm  $\mathcal{T}$  is said to be left-continuous (right-continuous) if and only if its partial mapping  $\mathcal{T}(a, \cdot)$  is left-continuous (right-continuous) for every  $a \in [0, 1]$ . A t-norm  $\mathcal{T}$  is continuous if and only if it is both left- and right-continuous. Moreover, Theorem 2.1.15 allows us to speak about left-continuous and right-continuous t-norms instead of lower and upper semicontinuous t-norms, respectively. The *minimum* operator  $\mathcal{T}_M$ , *product* operator  $\mathcal{T}_P$ , and Łukasiewicz t-norm  $\mathcal{T}_L$  are all continuous, while the *drastic product* operator  $\mathcal{T}_D$  is right-continuous but not left-continuous. The results for t-conorms can be derived dually.

**Theorem 2.2.3.** *A t-norm  $\mathcal{T}$  is left-continuous (right-continuous) if and only if its dual t-conorm  $\mathcal{S}$  with respect to the standard negator  $\mathcal{N}_s$  is right-continuous (left-continuous).*

The *maximum* operator  $\mathcal{S}_M$ , *probabilistic sum* operator  $\mathcal{S}_P$ , and Łukasiewicz t-conorm  $\mathcal{S}_L$  are all continuous, while the *drastic sum* operator  $\mathcal{S}_D$  is left-continuous but not right-continuous.

As a consequence of Theorems 2.1.17 and 2.1.19, the relations between a left-continuous t-norm and its dual t-conorm and residual operators, respectively, can be illustrated as in Figure 2.3. The residual implicator  $\mathcal{I}_{\mathcal{T}}$  is also called an R-implicator for short. However, only those with respect to left-continuous t-norms are of particular interest, in which case, the t-norm and its residual implicator form a pair of adjoint residual operators.

The residual operators  $\mathcal{I}_{\mathcal{T}}$  and  $\mathcal{J}_{\mathcal{T}}$  of the three most important continuous t-norms are listed in Table 2.1.

Moreover, with a De Morgan triple  $(\mathcal{T}, \mathcal{S}, \mathcal{N})$ , the implicator  $\mathcal{I}_{\mathcal{T}, \mathcal{N}}$  is defined by

$$\mathcal{I}_{\mathcal{T}, \mathcal{N}}(x, y) = \mathcal{N}(\mathcal{T}(x, \mathcal{N}(y))) = \mathcal{S}(\mathcal{N}(x), y), \quad (2.43)$$

which is also called a strong implicator, or in short form, an S-implicator. Analogously, the coimplicator  $\mathcal{J}_{\mathcal{S}, \mathcal{N}}$  is defined by

$$\mathcal{J}_{\mathcal{S}, \mathcal{N}}(x, y) = \mathcal{N}(\mathcal{S}(x, \mathcal{N}(y))) = \mathcal{T}(\mathcal{N}(x), y), \quad (2.44)$$

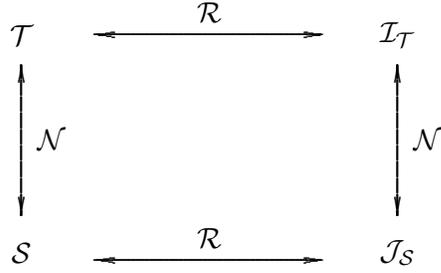


Figure 2.3: The relationship between t-norms and t-conorms and their residual operators

Table 2.1: Residual operators of the Gödel, Goguen and Łukasiewicz t-norms

$\mathcal{T}$	$\mathcal{I}_{\mathcal{T}}(x, y)$	$\mathcal{J}_{\mathcal{T}}(x, y)$
$\mathcal{T}_M$	$\begin{cases} 1, & \text{if } x \leq y \\ y, & \text{otherwise.} \end{cases}$	$\begin{cases} 1, & \text{if } x < y \\ y, & \text{otherwise.} \end{cases}$
$\mathcal{T}_P$	$\begin{cases} 1, & \text{if } x \leq y \\ y/x, & \text{otherwise.} \end{cases}$	$\begin{cases} 1, & \text{if } x < y \\ y/x, & \text{if } 0 < y \leq x \\ 0, & \text{otherwise.} \end{cases}$
$\mathcal{T}_L$	$\min(1 - x + y, 1)$	$\begin{cases} 1, & \text{if } x < y \\ 1 - x + y, & \text{if } 0 < y \leq x \\ 0, & \text{otherwise.} \end{cases}$

which is also called a strong coimplicator, an S-coimplicator for short.

In addition, the induced implicator  $\mathcal{I}_{\mathcal{T}, \mathcal{N}}$  and coimplicator  $\mathcal{J}_{\mathcal{S}, \mathcal{N}}$  may reconstruct the strong negator  $\mathcal{N}$ , respectively, i.e.,

$$\mathcal{N}(x) = \mathcal{I}_{\mathcal{T}, \mathcal{N}}(x, 0) = \mathcal{J}_{\mathcal{S}, \mathcal{N}}(x, 1). \quad (2.45)$$

This assertion holds for t-seminorms and t-semiconorms, however, not for general conjunctors and disjunctors.

The strong implicators induced by the standard negator  $\mathcal{N}_s$ , and the *minimum* operator

$\mathcal{T}_M$ , the *product* operator  $\mathcal{T}_P$  and the Łukasiewicz t-norm  $\mathcal{T}_L$ , respectively, are listed below:

$$\mathcal{I}_{\mathcal{T}_M, \mathcal{N}_s}(x, y) = \max(1 - x, y), \quad \text{the Kleene-Dienes implicator;}$$

$$\mathcal{I}_{\mathcal{T}_P, \mathcal{N}_s}(x, y) = 1 - x + x \cdot y, \quad \text{the Reichenbach implicator;}$$

$$\mathcal{I}_{\mathcal{T}_L, \mathcal{N}_s}(x, y) = \min(1 - x + y, 1), \quad \text{the Łukasiewicz implicator.}$$

Note that the residual implicator  $\mathcal{I}_{\mathcal{T}_L}$  of the Łukasiewicz t-norm  $\mathcal{T}_L$  coincides with the Łukasiewicz implicator  $\mathcal{I}_{\mathcal{T}_L, \mathcal{N}_s}$ . Consequently, the standard negator  $\mathcal{N}_s$  can be reconstructed by the Łukasiewicz implicator  $\mathcal{I}_{\mathcal{T}_L, \mathcal{N}_s}$  as well as the residual implicator  $\mathcal{I}_{\mathcal{T}_L}$ . The coimplicators induced by the standard negator  $\mathcal{N}_s$  and the corresponding t-conorms can be derived dually.

Another type of implicators, called quantum logic implicators (briefly QL-implicators) in the literature, can be defined via a t-norm  $\mathcal{T}$ , a t-conorm  $\mathcal{S}$ , and a strong negator  $\mathcal{N}$ , typically a De Morgan triple  $(\mathcal{T}, \mathcal{S}, \mathcal{N})$ , by

$$\mathcal{I}_{\mathcal{T}, \mathcal{S}, \mathcal{N}}(x, y) = \mathcal{S}(\mathcal{N}(x), \mathcal{T}(x, y)). \quad (2.46)$$

In general, a QL-implicator  $\mathcal{I}_{\mathcal{T}, \mathcal{S}, \mathcal{N}}$  is not well defined since the monotonicity in its first argument cannot be guaranteed. Some counterexamples have been constructed in Dubois *et al.* [124] for different classes of t-norms. A necessary condition can be obtained quickly by checking the boundary condition

$$\mathcal{I}_{\mathcal{T}, \mathcal{S}, \mathcal{N}}(x, 1) = \mathcal{S}(\mathcal{N}(x), x) = 1, \quad \text{for every } x \in [0, 1]. \quad (2.47)$$

Fodor [141] proposed some conditions under which  $\mathcal{I}_{\mathcal{T}, \mathcal{S}, \mathcal{N}}$  becomes an implicator. In this case,  $\mathcal{I}_{\mathcal{T}, \mathcal{S}, \mathcal{N}}$  is a border implicator and the induced negator coincides with the strong negator  $\mathcal{N}$ .

R-implicators, S-implicators, and QL-implicators, all of which are border implicators, are three fundamental classes of implicators used in fuzzy inference systems due to their sound logic foundation and good approximation capability. They will be further discussed in Section 2.5.

Since t-norms and t-conorms are simply functions from the unit square into the unit interval, an order relation can be defined among t-norms as well as t-conorms by pointwise comparison. A t-norm  $\mathcal{T}_1$  is said to be weaker than a t-norm  $\mathcal{T}_2$ , which is denoted by

$\mathcal{T}_1 \leq \mathcal{T}_2$ , if the inequality  $\mathcal{T}_1(x, y) \leq \mathcal{T}_2(x, y)$  holds for every  $(x, y) \in [0, 1]^2$ . Equivalently,  $\mathcal{T}_2$  is said to be stronger than  $\mathcal{T}_1$  if  $\mathcal{T}_1 \leq \mathcal{T}_2$ . Moreover, the notation  $\mathcal{T}_1 < \mathcal{T}_2$  is used whenever  $\mathcal{T}_1 \leq \mathcal{T}_2$  and  $\mathcal{T}_1 \neq \mathcal{T}_2$ . It is well-known that the *drastic product* operator  $\mathcal{T}_D$  is the weakest and the *minimum* operator  $\mathcal{T}_M$  is the strongest t-norm, i.e., for any t-norm  $\mathcal{T}$ , it holds that

$$\mathcal{T}_D \leq \mathcal{T} \leq \mathcal{T}_M.$$

Besides, it is clear that

$$\mathcal{T}_D < \mathcal{T}_L < \mathcal{T}_P < \mathcal{T}_M.$$

Dually, the order relation reverses for t-conorms. The *maximum* operator  $\mathcal{S}_M$  is the weakest and the *drastic sum* operator  $\mathcal{S}_D$  is the strongest t-conorm, i.e., for any t-conorm  $\mathcal{S}$ , it holds that

$$\mathcal{S}_M \leq \mathcal{S} \leq \mathcal{S}_D.$$

Besides, it holds that

$$\mathcal{S}_M < \mathcal{S}_P < \mathcal{S}_L < \mathcal{S}_D.$$

The associativity property of t-norms allows us to uniquely extend each t-norm  $\mathcal{T}$  in a recursive manner to an  $n$ -ary operator, i.e.,

$$\mathcal{T}(x_1, \dots, x_n) = \mathcal{T}(\mathcal{T}(x_1, \dots, x_{n-1}), x_n)$$

for each  $n$ -tuple  $(x_1, \dots, x_n) \in [0, 1]^n$  with the integer  $n \geq 3$ . In particular,  $\mathcal{T}(\overbrace{x, \dots, x}^n)$  is denoted as  $x_{\mathcal{T}}^{(n)}$  for each  $x \in [0, 1]$  and called the  $n$ th power of  $x$  with respect to  $\mathcal{T}$ , with the convention that  $x_{\mathcal{T}}^{(0)} = 1$  and  $x_{\mathcal{T}}^{(1)} = x$ . The  $n$ -ary extensions for the four basic t-norms are as follows:

$$\mathcal{T}_M(x_1, x_2, \dots, x_n) = \min(x_1, x_2, \dots, x_n),$$

$$\mathcal{T}_P(x_1, x_2, \dots, x_n) = x_1 \cdot x_2 \cdots x_n,$$

$$\mathcal{T}_L(x_1, x_2, \dots, x_n) = \max\left(\sum_{i=1}^n x_i - (n-1), 0\right),$$

$$\mathcal{T}_D(x_1, x_2, \dots, x_n) = \begin{cases} x_i, & \text{if } x_j = 1 \text{ for all } j \neq i, \\ 0, & \text{otherwise.} \end{cases}$$

Analogously, each t-conorm  $\mathcal{S}$  can be uniquely extended to an  $n$ -ary operator, i.e.,

$$\mathcal{S}(x_1, \dots, x_n) = \mathcal{S}(\mathcal{S}(x_1, \dots, x_{n-1}), x_n)$$

for each  $n$ -tuple  $(x_1, \dots, x_n) \in [0, 1]^n$  with the integer  $n \geq 3$ . In particular,  $\mathcal{S}(\overbrace{x, \dots, x}^n)$  is denoted as  $x_{\mathcal{S}}^{(n)}$  for each  $x \in [0, 1]$  and called the  $n$ th power of  $x$  with respect to  $\mathcal{S}$ , with the convention that  $x_{\mathcal{S}}^{(0)} = 0$  and  $x_{\mathcal{S}}^{(1)} = x$ . The  $n$ -ary extensions for the four basic t-conorms are as follows:

$$\begin{aligned} \mathcal{S}_M(x_1, x_2, \dots, x_n) &= \max(x_1, x_2, \dots, x_n), \\ \mathcal{S}_P(x_1, x_2, \dots, x_n) &= 1 - (1 - x_1)(1 - x_2) \cdots (1 - x_n), \\ \mathcal{S}_L(x_1, x_2, \dots, x_n) &= \min\left(\sum_{i=1}^n x_i, 1\right), \\ \mathcal{S}_D(x_1, x_2, \dots, x_n) &= \begin{cases} x_i, & \text{if } x_j = 0 \text{ for all } j \neq i, \\ 1, & \text{otherwise.} \end{cases} \end{aligned}$$

To better understand t-norms and t-conorms, some additional algebraic properties of t-norms and t-conorms are necessary to be presented, most of which are well-known from the general theories of semigroups and lattices because t-norms and t-conorms can be viewed as special classes of commutative semigroups, respectively.

**Definition 2.2.4.** *Let  $\mathcal{T} : [0, 1]^2 \rightarrow [0, 1]$  be a t-norm.*

- (i) *An element  $a \in [0, 1]$  is called an idempotent element of  $\mathcal{T}$  if  $\mathcal{T}(a, a) = a$ .*
- (ii) *An element  $a \in (0, 1)$  is called a nilpotent element of  $\mathcal{T}$  if there exists some positive integer  $n$  such that  $a_{\mathcal{T}}^{(n)} = 0$ .*
- (iii) *An element  $a \in (0, 1)$  is called a zero divisor of  $\mathcal{T}$  if there exists some  $b \in (0, 1)$  such that  $\mathcal{T}(a, b) = 0$ .*

Note that no element of  $(0, 1)$  can be both idempotent and nilpotent. The numbers 0 and 1 are idempotent elements for each t-norm  $\mathcal{T}$ , and hence, called trivial idempotent elements of  $\mathcal{T}$ , while each idempotent element in  $(0, 1)$  is called a nontrivial idempotent element of  $\mathcal{T}$ . Moreover, the idempotent elements of t-norms can be well characterized in the following way:

**Theorem 2.2.5.**

- (i) An element  $a \in [0, 1]$  is an idempotent of a t-norm  $\mathcal{T}$  if and only if  $\mathcal{T}(a, x) = \min(a, x)$  for every  $x \in [a, 1]$ .
- (ii) An element  $a \in [0, 1]$  is an idempotent of a continuous t-norm  $\mathcal{T}$  if and only if  $\mathcal{T}(a, x) = \min(a, x)$  for every  $x \in [0, 1]$ .

The *product* operator  $\mathcal{T}_P$ , Lukasiewicz t-norm  $\mathcal{T}_L$ , and *drastic product* operator  $\mathcal{T}_D$  possess only trivial idempotent elements. The set of idempotent elements of the *minimum* operator  $\mathcal{T}_M$  is  $[0, 1]$ . Actually,  $\mathcal{T}_M$  is the only t-norm with this property. Furthermore, the *minimum* operator  $\mathcal{T}_M$  and the *product* operator  $\mathcal{T}_P$  have neither nilpotent elements nor zero divisors, while each  $a \in (0, 1)$  is both a nilpotent element and a zero divisor of the Lukasiewicz t-norm  $\mathcal{T}_L$  as well as the *drastic product* operator  $\mathcal{T}_D$ .

Generally, each nilpotent element of a t-norm  $\mathcal{T}$  is also a zero divisor of  $\mathcal{T}$  but not vice versa, which means that the set of nilpotent elements is a subset of the set of zero divisors. However, the existence of zero divisors is equivalent to the existence of nilpotent elements for each t-norm, i.e., a t-norm  $\mathcal{T}$  has zero divisors if and only if it has nilpotent elements. Moreover, if  $a \in (0, 1)$  is a zero divisor of a t-norm  $\mathcal{T}$ , then each number  $b \in (0, a)$  is also a zero divisor of  $\mathcal{T}$  due to the monotonicity property of  $\mathcal{T}$ . This also holds for nilpotent elements.

**Definition 2.2.6.**

- (i) A t-norm  $\mathcal{T}$  is called *strictly monotone* if  $\mathcal{T}(x, y) < \mathcal{T}(x, z)$  whenever  $x > 0$  and  $y < z$ .
- (ii) A t-norm  $\mathcal{T}$  is called *strict* if it is continuous and strictly monotone.
- (iii) A t-norm  $\mathcal{T}$  is called *nilpotent* if it is continuous and each  $x \in (0, 1)$  is a nilpotent element of  $\mathcal{T}$ .
- (iv) A t-norm  $\mathcal{T}$  is called *Archimedean* if for each  $(x, y) \in (0, 1)^2$  there exists a positive integer  $n$  such that  $x_{\mathcal{T}}^{(n)} < y$ .

**Theorem 2.2.7.** A continuous t-norm  $\mathcal{T}$  is Archimedean if and only if  $\mathcal{T}(x, x) < x$  for every  $x \in (0, 1)$ .

It is evident that an Archimedean t-norm has only trivial idempotent elements. It was shown by Kolesárová [250] that the left-continuity of an Archimedean t-norm implies its continuity. Furthermore, Theorem 2.2.7 indicates that each strict or nilpotent t-norm is Archimedean. Moreover, it turns out that a continuous Archimedean t-norm is either strict or nilpotent. Clearly, the *product* operator  $\mathcal{T}_P$  is a strict t-norm and the Łukasiewicz t-norm  $\mathcal{T}_L$  is a nilpotent t-norm. The *minimum* operator  $\mathcal{T}_M$  is a continuous but not Archimedean t-norm. The *drastic product* operator  $\mathcal{T}_D$  is a noncontinuous Archimedean t-norm for which each  $x \in (0, 1)$  is a nilpotent element. Besides, as a consequence of Theorem 2.1.25, each strict t-norm satisfies the cancelation law while each nilpotent t-norm satisfies the conditional cancelation law. By Theorem 2.2.5, no continuous non-Archimedean t-norm satisfies the conditional cancelation law.

Dually, these properties can be defined for t-conorms by interchanging the roles of 0 and 1.

**Definition 2.2.8.** *Let  $\mathcal{S} : [0, 1]^2 \rightarrow [0, 1]$  be a t-conorm.*

- (i) *An element  $a \in [0, 1]$  is called an idempotent element of  $\mathcal{S}$  if  $\mathcal{S}(a, a) = a$ .*
- (ii) *An element  $a \in (0, 1)$  is called a nilpotent element of  $\mathcal{S}$  if there exists some positive integer  $n$  such that  $a_{\mathcal{S}}^{(n)} = 1$ .*
- (iii) *An element  $a \in (0, 1)$  is called a zero divisor of  $\mathcal{S}$  if there exists some  $b \in (0, 1)$  such that  $\mathcal{S}(a, b) = 1$ .*

**Theorem 2.2.9.**

- (i) *An element  $a \in [0, 1]$  is an idempotent of a t-conorm  $\mathcal{S}$  if and only if  $\mathcal{S}(a, x) = \max(a, x)$  for every  $x \in [a, 1]$ .*
- (ii) *An element  $a \in [0, 1]$  is an idempotent of a continuous t-conorm  $\mathcal{S}$  if and only if  $\mathcal{S}(a, x) = \max(a, x)$  for every  $x \in [0, 1]$ .*

The numbers 0 and 1 are idempotent elements for each t-conorm  $\mathcal{S}$ , and hence, called trivial idempotent elements of  $\mathcal{S}$ , while each idempotent element in  $(0, 1)$  is called a non-trivial idempotent element of  $\mathcal{S}$ . The *probabilistic sum* operator  $\mathcal{S}_P$ , Łukasiewicz t-conorm  $\mathcal{S}_L$ , and *drastic sum* operator  $\mathcal{S}_D$  possess only trivial idempotent elements. The *maximum*

operator  $\mathcal{S}_M$  is the only t-conorm such that its set of idempotent elements is  $[0, 1]$ . Furthermore, the *maximum* operator  $\mathcal{S}_M$  and the *probabilistic sum* operator  $\mathcal{S}_P$  have neither nilpotent elements nor zero divisors, while each  $a \in (0, 1)$  is both a nilpotent element and a zero divisor of the Łukasiewicz t-conorm  $\mathcal{S}_L$  as well as the *drastic sum* operator  $\mathcal{S}_D$ .

**Definition 2.2.10.**

- (i) A t-conorm  $\mathcal{S}$  is called *strictly monotone* if  $\mathcal{S}(x, y) < \mathcal{S}(x, z)$  whenever  $x < 1$  and  $y < z$ .
- (ii) A t-conorm  $\mathcal{S}$  is called *strict* if it is continuous and strictly monotone.
- (iii) A t-conorm  $\mathcal{S}$  is called *nilpotent* if it is continuous and each  $x \in (0, 1)$  is a nilpotent element of  $\mathcal{S}$ .
- (iv) A t-conorm  $\mathcal{S}$  is called *Archimedean* if for each  $(x, y) \in (0, 1)^2$  there exists a positive integer  $n$  such that  $x_{\mathcal{S}}^{(n)} > y$ .

**Theorem 2.2.11.** A continuous t-conorm  $\mathcal{S}$  is Archimedean if and only if  $\mathcal{S}(x, x) > x$  for every  $x \in (0, 1)$ .

It is evident that a t-conorm  $\mathcal{S}$  is Archimedean if and only if its dual t-norm with respect to the standard negator  $\mathcal{N}_s$  is Archimedean. A continuous Archimedean t-conorm is either strict or nilpotent. Clearly, the *probabilistic sum* operator  $\mathcal{S}_P$  is a strict t-conorm and the Łukasiewicz t-conorm  $\mathcal{S}_L$  is a nilpotent t-conorm. The *maximum* operator  $\mathcal{S}_M$  is a continuous but not Archimedean t-conorm. The *drastic sum* operator  $\mathcal{S}_D$  is a noncontinuous Archimedean t-conorm for which each  $x \in (0, 1)$  is a nilpotent element.

## 2.3 Construction of Triangular Norms and Conorms

From an algebraic point of view, t-norms, as well as t-conorms, are a special class of commutative semigroups, and hence, can be constructed via various methods developed in semigroup theory.

It is straightforward that, given a t-norm  $\mathcal{T}$ , any strictly increasing bijection  $\psi : [0, 1] \rightarrow [0, 1]$  defines a t-norm by

$$\mathcal{T}_{\psi}(x, y) = \psi^{-1}(\mathcal{T}(\psi(x), \psi(y))). \quad (2.48)$$

Furthermore,  $\mathcal{T}$  and  $\mathcal{T}_\psi$  are isomorphic in the sense that

$$\psi(\mathcal{T}_\psi(x, y)) = \mathcal{T}(\psi(x), \psi(y)) \quad (2.49)$$

for every  $(x, y) \in [0, 1]^2$ . Dually, given a t-conorm  $\mathcal{S}$ , any strictly increasing bijection  $\psi : [0, 1] \rightarrow [0, 1]$  defines a t-conorm as well by

$$\mathcal{S}_\psi(x, y) = \psi^{-1}(\mathcal{S}(\psi(x), \psi(y))), \quad (2.50)$$

and  $\mathcal{S}$  and  $\mathcal{S}_\psi$  are isomorphic in the sense that

$$\psi(\mathcal{S}_\psi(x, y)) = \mathcal{S}(\psi(x), \psi(y)) \quad (2.51)$$

for every  $(x, y) \in [0, 1]^2$ . Clearly,  $\mathcal{T}$  and  $\mathcal{T}_\psi$ , as well as  $\mathcal{S}$  and  $\mathcal{S}_\psi$ , share many common structural features, e.g., continuity, Archimedean property, and the existence of idempotent and nilpotent elements as well as the existence of zero divisors. The only invariants under arbitrary strictly increasing bijections are the two extremal t-norms,  $\mathcal{T}_M$  and  $\mathcal{T}_D$ , and two extremal t-conorms,  $\mathcal{S}_M$  and  $\mathcal{S}_D$ , respectively.

A more general method to construct t-norms and t-conorms involves the pseudoinverses of monotone functions.

**Definition 2.3.1.** *Let  $f : [a, b] \rightarrow [c, d]$  be a monotone function where  $[a, b]$  and  $[c, d]$  are two closed subintervals of the extended real line  $[-\infty, +\infty]$ . The pseudoinverse  $f^{(-1)} : [c, d] \rightarrow [a, b]$  is defined as*

$$f^{(-1)}(y) = \sup \{x \in [a, b] \mid (f(x) - y)(f(b) - f(a)) < 0\}.$$

**Corollary 2.3.2.** *Let  $f : [a, b] \rightarrow [c, d]$  be a monotone function where  $[a, b]$  and  $[c, d]$  are two closed subintervals of the extended real line  $[-\infty, +\infty]$ . The pseudoinverse  $f^{(-1)} : [c, d] \rightarrow [a, b]$  can be determined according to the following three cases:*

- (i) *If  $f(a) < f(b)$ , then  $f^{(-1)}(y) = \sup \{x \in [a, b] \mid f(x) < y\}$ .*
- (ii) *If  $f(a) > f(b)$ , then  $f^{(-1)}(y) = \sup \{x \in [a, b] \mid f(x) > y\}$ .*
- (iii) *If  $f(a) = f(b)$ , then  $f^{(-1)}(y) = a$ .*

The pseudoinverse of  $f$  coincides with the inverse of  $f$  if and only if  $f$  is a bijection. For more details on pseudoinverses of monotone functions, the reader may refer to Klement *et al.* [242]. With the aid of pseudoinverses, some t-norms and t-conorms can be constructed via appropriate monotone functions.

**Theorem 2.3.3.** *Let  $t : [0, 1] \rightarrow [0, +\infty]$  be a strictly decreasing function with  $t(1) = 0$ , such that  $t$  is right-continuous at 0, and*

$$t(x) + t(y) \in \text{Ran}(t) \cup [t(0), +\infty] \quad (2.52)$$

*for every  $(x, y) \in [0, 1]^2$ , where  $\text{Ran}(t) = \{t(x) \mid x \in [0, 1]\}$  is the range of  $t$ . The function  $\mathcal{T} : [0, 1]^2 \rightarrow [0, 1]$  defined by*

$$\mathcal{T}(x, y) = t^{(-1)}(t(x) + t(y)) \quad (2.53)$$

*is a t-norm.*

In Theorem 2.3.3, the function  $t : [0, 1] \rightarrow [0, +\infty]$ , which induces a t-norm  $\mathcal{T}$ , is called an additive generator of the t-norm  $\mathcal{T}$ . The multiplication of an additive generator of a t-norm  $\mathcal{T}$  by a positive constant remains to be an additive generator of  $\mathcal{T}$ . Moreover, there is a strong connection between the continuity of a t-norm and the continuity of its additive generators.

**Theorem 2.3.4.** *Let  $\mathcal{T}$  be a t-norm with an additive generator  $t : [0, 1] \rightarrow [0, +\infty]$ . The following statements are equivalent:*

- (i)  $\mathcal{T}$  is continuous.
- (ii)  $\mathcal{T}$  is left-continuous at the point  $(1, 1)$ .
- (iii)  $t$  is continuous.
- (iv)  $t$  is left-continuous at 1.

Triangular norms constructed by means of additive generators are always Archimedean. The converse, however, is not true, i.e., an Archimedean t-norm, which is necessarily not continuous, may have no additive generators. However, a t-norm is continuous Archimedean if and only if it has a continuous additive generator. Moreover, the value of the continuous additive generator at the point 0 determines whether the induced t-norm is strict or nilpotent.

**Theorem 2.3.5.** *A continuous t-norm  $\mathcal{T}$  is Archimedean if and only if it has a continuous additive generator  $t : [0, 1] \rightarrow [0, +\infty]$ , which is uniquely determined up to a positive multiplicative constant. Moreover, the value  $t(0)$  determines whether the t-norm  $\mathcal{T}$  is strict or nilpotent as follows:*

(i)  $\mathcal{T}$  is strict if and only if  $t(0) = +\infty$ ;

(ii)  $\mathcal{T}$  is nilpotent if and only if  $t(0) < +\infty$ .

The functions  $t(x) = -\ln x$  and  $t(x) = 1 - x$  are the additive generators of the *product* operator  $\mathcal{T}_P$  and the Łukasiewicz t-norm  $\mathcal{T}_L$ , respectively, while the *drastic product* operator  $\mathcal{T}_D$  can be induced by

$$t(x) = \begin{cases} 2 - x, & x \in [0, 1), \\ 0, & x = 1. \end{cases} \quad (2.54)$$

No additive generator exists for any t-norm with a nontrivial idempotent element, in particular, the *minimum* operator  $\mathcal{T}_M$ .

A similar method to construct t-norms is via the so-called multiplicative generators, which are completely dual to additive generators in some sense.

Let  $\mathcal{T}$  be a t-norm with an additive generator  $t$ . The t-norm  $\mathcal{T}$  can also be induced via

$$\mathcal{T}(x, y) = \theta^{(-1)}(\theta(x) \cdot \theta(y)) \quad (2.55)$$

where the function  $\theta(x) = e^{-t(x)}$  is called a multiplicative generator of  $\mathcal{T}$ . Any additive generator  $t : [0, 1] \rightarrow [0, +\infty]$  defines a multiplicative generator  $\theta : [0, 1] \rightarrow [0, 1]$  with  $\theta(1) = 1$ , which is right-continuous at 0, and

$$\theta(x) \cdot \theta(y) \in \text{Ran}(\theta) \cup [0, \theta(0)] \quad (2.56)$$

for every  $(x, y) \in [0, 1]^2$ . Conversely, such a function  $\theta(x)$  defines an additive generator  $t(x) = -\ln \theta(x)$ , and hence, induces a t-norm.

**Theorem 2.3.6.** *Let  $\mathcal{T}$  be a t-norm with an additive generator  $t : [0, 1] \rightarrow [0, +\infty]$ . The dual t-conorm  $\mathcal{S}$  with respect to the standard negator  $\mathcal{N}_s$  can be induced via*

$$\mathcal{S}(x, y) = s^{(-1)}(s(x) + s(y)), \quad (2.57)$$

where  $s(x) = t(1 - x)$ .

Subsequently, in analogy to Theorem 2.3.3, t-conorms can be constructed via their additive generators.

**Theorem 2.3.7.** *Let  $s : [0, 1] \rightarrow [0, +\infty]$  be a strictly increasing function with  $s(0) = 0$ , such that  $s$  is left-continuous at 1, and*

$$s(x) + s(y) \in \text{Ran}(s) \cup [s(0), +\infty] \quad (2.58)$$

for every  $(x, y) \in [0, 1]^2$ , where  $\text{Ran}(s) = \{s(x) \mid x \in [0, 1]\}$  is the range of  $s$ . The function  $\mathcal{S} : [0, 1]^2 \rightarrow [0, 1]$  defined by

$$\mathcal{S}(x, y) = s^{(-1)}(s(x) + s(y)) \quad (2.59)$$

is a t-conorm.

Similarly, if  $\mathcal{S}$  is a t-conorm with an additive generator  $s$ , the t-conorm  $\mathcal{S}$  can be induced via

$$\mathcal{S}(x, y) = \xi^{(-1)}(\xi(x) \cdot \xi(y)) \quad (2.60)$$

where the function  $\xi(x) = e^{-s(x)}$  is called a multiplicative generator of  $\mathcal{S}$ . Any additive generator  $s : [0, 1] \rightarrow [0, +\infty]$  defines a multiplicative generator  $\xi : [0, 1] \rightarrow [0, 1]$  with  $\xi(0) = 1$ , which is left-continuous at 1, and

$$\xi(x) \cdot \xi(y) \in \text{Ran}(\xi) \cup [0, \xi(1)] \quad (2.61)$$

for every  $(x, y) \in [0, 1]^2$ . Conversely, such a function  $\xi(x)$  defines an additive generator  $s(x) = -\ln \xi(x)$ , and hence, induces a t-conorm.

The functions  $s(x) = -\ln(1 - x)$  and  $s(x) = x$  are the additive generators of the *probabilistic sum* operator  $\mathcal{S}_P$  and the Łukasiewicz t-conorm  $\mathcal{S}_L$ , respectively, while the *drastic sum* operator  $\mathcal{S}_D$  can be induced by

$$s(x) = \begin{cases} 1 + x, & x \in (0, 1], \\ 0, & x = 0. \end{cases} \quad (2.62)$$

No additive generator exists for any t-conorm with a nontrivial idempotent element, in particular, the *maximum* operator  $\mathcal{S}_M$ .

The relationship between the additive and multiplicative generators of t-norms and t-conorms is illustrated in Figure 2.4. Note that a function can be an additive generator of

a t-norm and a multiplicative generator of a t-conorm simultaneously, or a multiplicative generator of a t-norm and an additive generator of a t-conorm simultaneously. For instance, the function  $f(x) = 1 - x$  is an additive generator of the Łukasiewicz t-norm  $\mathcal{T}_L$  and a multiplicative generator of the *probabilistic sum* operator  $\mathcal{S}_P$ , while the function  $g(x) = x$  is an additive generator of the Łukasiewicz t-conorm  $\mathcal{S}_L$  and a multiplicative generator of the *product* operator  $\mathcal{T}_P$ .

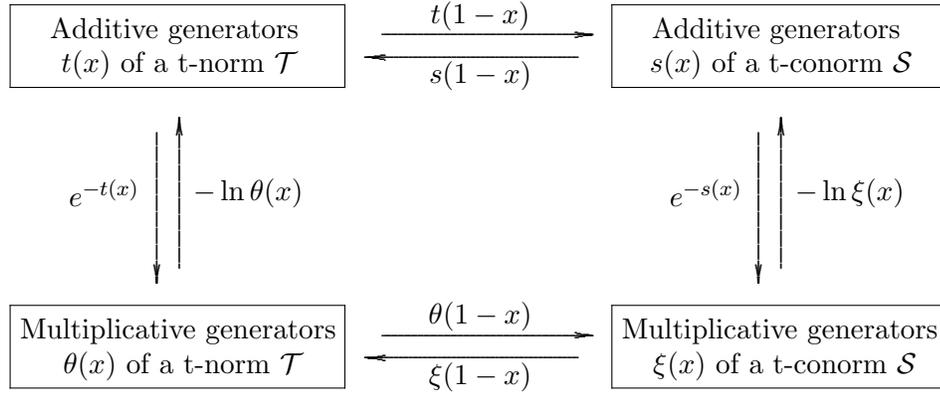


Figure 2.4: The relationship between additive and multiplicative generators

Another method to construct t-norms is by means of the ordinal sum of a family of t-norms.

**Theorem 2.3.8.** *Let  $\{\mathcal{T}_\alpha\}_{\alpha \in A}$  be a family of t-norms and  $\{(a_\alpha, b_\alpha)\}_{\alpha \in A}$  be a family of nonempty, pairwise disjoint open subintervals of  $(0, 1)$ , where the index set  $A$  is finite or countably infinite. The function  $\mathcal{T} : [0, 1]^2 \rightarrow [0, 1]$  defined by*

$$\mathcal{T}(x, y) = \begin{cases} a_\alpha + (b_\alpha - a_\alpha) \cdot \mathcal{T}_\alpha \left( \frac{x - a_\alpha}{b_\alpha - a_\alpha}, \frac{y - a_\alpha}{b_\alpha - a_\alpha} \right), & \text{if } (x, y) \in [a_\alpha, b_\alpha]^2, \\ \min(x, y), & \text{otherwise,} \end{cases} \quad (2.63)$$

*is a t-norm and called the ordinal sum of the summands  $\{(a_\alpha, b_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in A}$ .*

The t-norm determined by the ordinal sum of the summands  $\{(a_\alpha, b_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in A}$  is usually denoted by  $\mathcal{T} = (\langle a_\alpha, b_\alpha, \mathcal{T}_\alpha \rangle)_{\alpha \in A}$ . Note that the index set  $A$  may be empty, in which case the ordinal sum equals to the *minimum* operator  $\mathcal{T}_M$ . A summand  $\mathcal{T}_\alpha = \mathcal{T}_M$  in the representation of an ordinal sum can be omitted, and each t-norm  $\mathcal{T}$  can be viewed as

a trivial ordinal sum, i.e.,  $\mathcal{T} = (\langle 0, 1, \mathcal{T} \rangle)$ . If  $\mathcal{T} = (\langle a_\alpha, b_\alpha, \mathcal{T}_\alpha \rangle)_{\alpha \in A}$  is an ordinal sum of t-norms, the endpoints  $a_\alpha$  and  $b_\alpha$  are both the idempotent elements of  $\mathcal{T}$  for each  $\alpha \in A$ . Therefore, if a t-norm  $\mathcal{T}$  is a nontrivial ordinal sum  $(\langle a_\alpha, b_\alpha, \mathcal{T}_\alpha \rangle)_{\alpha \in A}$ , where  $A \neq \emptyset$  and  $(a_\alpha, b_\alpha) \subset (0, 1)$  for each  $\alpha \in A$ , then  $\mathcal{T}$  is not Archimedean. Moreover, the continuity of an ordinal sum of t-norms is equivalent to the continuity of all its summands.

**Theorem 2.3.9.** *Let  $\mathcal{T} = (\langle a_\alpha, b_\alpha, \mathcal{T}_\alpha \rangle)_{\alpha \in A}$  be an ordinal sum of t-norms with  $A \neq \emptyset$ . The t-norm  $\mathcal{T}$  is continuous if and only if  $\mathcal{T}_\alpha$  is continuous for every  $\alpha \in A$ .*

The construction of t-conorms by means of the ordinal sum of a family of t-conorms can be derived dually.

**Theorem 2.3.10.** *Let  $\{\mathcal{S}_\alpha\}_{\alpha \in A}$  be a family of t-conorms and  $\{(a_\alpha, b_\alpha)\}_{\alpha \in A}$  be a family of nonempty, pairwise disjoint open subintervals of  $(0, 1)$ , where the index set  $A$  is finite or countably infinite. The function  $\mathcal{S} : [0, 1]^2 \rightarrow [0, 1]$  defined by*

$$\mathcal{S}(x, y) = \begin{cases} a_\alpha + (b_\alpha - a_\alpha) \cdot \mathcal{S}_\alpha \left( \frac{x - a_\alpha}{b_\alpha - a_\alpha}, \frac{y - a_\alpha}{b_\alpha - a_\alpha} \right), & \text{if } (x, y) \in [a_\alpha, b_\alpha]^2, \\ \max(x, y), & \text{otherwise,} \end{cases} \quad (2.64)$$

*is a t-conorm and called the ordinal sum of the summands  $\{(a_\alpha, b_\alpha, \mathcal{S}_\alpha)\}_{\alpha \in A}$ .*

The t-conorm determined by the ordinal sum of the summands  $\{(a_\alpha, b_\alpha, \mathcal{S}_\alpha)\}_{\alpha \in A}$  is usually denoted by  $\mathcal{S} = (\langle a_\alpha, b_\alpha, \mathcal{S}_\alpha \rangle)_{\alpha \in A}$ . Note that for an ordinal sum of t-norms  $(\langle a_\alpha, b_\alpha, \mathcal{T}_\alpha \rangle)_{\alpha \in A}$ , its dual t-conorm with respect to the standard negator  $\mathcal{N}_s$  is an ordinal sum of t-conorms, i.e.,  $(\langle 1 - b_\alpha, 1 - a_\alpha, \mathcal{S}_\alpha \rangle)_{\alpha \in A}$ , where each t-conorm  $\mathcal{S}_\alpha$  is the dual of the t-norm  $\mathcal{T}_\alpha$  with respect to the standard negator  $\mathcal{N}_s$ . However, for each  $\alpha \in A$ , the dual pair  $\mathcal{T}_\alpha$  and  $\mathcal{S}_\alpha$  in general act on different intervals.

## 2.4 Representation of Continuous Triangular Norms

There is no universal representation theorem so far for all t-norms, which is actually related to the solution of the still unsolved general associativity equation. See, for instance, Alsina *et al.* [5]. However, all continuous t-norms can be well characterized and classified. The representation theorems of continuous t-norms by means of additive generators and ordinal sums of Archimedean summands were first developed by Ling [278] in the framework

of triangular norms, although they can also be derived from the results in Mostert and Shields [312] in the framework of semigroups.

As an immediate consequence of Theorem 2.3.5, a continuous Archimedean t-norm  $\mathcal{T}$  is either strict or nilpotent, which is fully determined by the value  $t(0)$  of its continuous additive generator  $t : [0, 1] \rightarrow [0, +\infty]$ . Moreover, each continuous Archimedean t-norm is isomorphic either to the *product* operator  $\mathcal{T}_P$  or to the Lukasiewicz t-norm  $\mathcal{T}_L$ . This fact makes  $\mathcal{T}_P$  and  $\mathcal{T}_L$  the most important prototypes of strict and nilpotent t-norms, respectively. Unfortunately, no such characterization exists for continuous non-Archimedean t-norms.

**Theorem 2.4.1.**

- (i) Let  $\mathcal{T}$  be a strict t-norm with an additive generator  $t : [0, 1] \rightarrow [0, +\infty]$ . The t-norm  $\mathcal{T}$  is isomorphic to the product operator  $\mathcal{T}_P$ , and

$$\mathcal{T}(x, y) = \psi^{-1}(\mathcal{T}_P(\psi(x), \psi(y))) \quad (2.65)$$

for every  $(x, y) \in [0, 1]^2$  where  $\psi(x) = e^{-t(x)}$ .

- (ii) Let  $\mathcal{T}$  be a nilpotent t-norm with an additive generator  $t : [0, 1] \rightarrow [0, +\infty]$ . The t-norm  $\mathcal{T}$  is isomorphic to the Lukasiewicz t-norm  $\mathcal{T}_L$ , and

$$\mathcal{T}(x, y) = \psi^{-1}(\mathcal{T}_L(\psi(x), \psi(y))) \quad (2.66)$$

for every  $(x, y) \in [0, 1]^2$  where  $\psi(x) = 1 - t(x)/t(0)$ .

**Theorem 2.4.2.** A t-norm  $\mathcal{T}$  is continuous if and only if it is an ordinal sum of continuous Archimedean t-norms  $(\langle a_\alpha, b_\alpha, \mathcal{T}_\alpha \rangle)_{\alpha \in A}$  where the summands  $\{\langle a_\alpha, b_\alpha, \mathcal{T}_\alpha \rangle\}_{\alpha \in A}$  are uniquely determined.

A remarkable result following Theorems 2.4.1 and 2.4.2 is the important classification of continuous t-norms.

**Theorem 2.4.3.** A t-norm  $\mathcal{T}$  is continuous if and only if exactly one of the following statements is true:

- (i)  $\mathcal{T} = \mathcal{T}_M$ .

(ii)  $\mathcal{T}$  is strict.

(iii)  $\mathcal{T}$  is nilpotent.

(iv)  $\mathcal{T}$  is a nontrivial ordinal sum of continuous Archimedean t-norms.

Consequently, a continuous non-Archimedean t-norm, which necessarily has nontrivial idempotent elements, is either the *minimum* operator  $\mathcal{T}_M$  or a nontrivial ordinal sum of continuous Archimedean t-norms.

A further classification of continuous t-norms arises from the distinction between t-norms with zero divisors and t-norms without zero divisors. A continuous Archimedean t-norm with zero divisors must be a nilpotent t-norm, and hence, isomorphic to the Łukasiewicz t-norm  $\mathcal{T}_L$ . For a continuous non-Archimedean t-norm  $\mathcal{T}$  with zero divisors, it must contain a nilpotent summand  $\langle 0, b_1, \mathcal{T}_1 \rangle$ , where  $b_1 \in (0, 1)$  is the smallest nontrivial idempotent element of  $\mathcal{T}$  and  $\mathcal{T}_1$  is a nilpotent t-norm. The converse is also true, i.e., an ordinal sum of continuous Archimedean t-norms, containing a nilpotent summand  $\langle 0, b_1, \mathcal{T}_1 \rangle$ , must have zero divisors. An interesting family of such t-norms is the ordinal sums  $\mathcal{T}_\lambda^{MT} = (\langle 0, \lambda, \mathcal{T}_L \rangle)$  for all  $\lambda \in (0, 1)$ , which are also known as the Mayor-Torrens t-norms [298]. The *minimum* operator  $\mathcal{T}_M$  is without doubt a typical but not unique representative of continuous non-Archimedean t-norms without zero divisors. Another example is the family of Dubois-Prade t-norms,  $\mathcal{T}_\lambda^{DY} = (\langle 0, \lambda, \mathcal{T}_P \rangle)$  for all  $\lambda \in (0, 1)$ , which was introduced by Dubois and Prade [126]. An illustration of the classification of continuous t-norms is shown in Figure 2.5. The reader may refer to Klement *et al.* [243] for the classification of general t-norms.

Dually, a continuous Archimedean t-conorm is either strict and isomorphic to the *probabilistic sum* operator  $\mathcal{S}_P$ , or nilpotent and isomorphic to the Łukasiewicz t-conorm  $\mathcal{S}_L$ . A continuous non-Archimedean t-conorm is either the *maximum* operator  $\mathcal{S}_M$  or a nontrivial ordinal sum of continuous Archimedean t-conorms.

**Theorem 2.4.4.** *Let  $\mathcal{T}$  be a continuous Archimedean t-norm and  $t : [0, 1] \rightarrow [0, +\infty]$  its*

	Without zero divisors	With zero divisors
Archimedean	Strict t-norms $\mathcal{T}_P$	Nilpotent t-norms $\mathcal{T}_L$
Non-Archimedean	$\mathcal{T}_M$ $\mathcal{T}_\lambda^{DP}, \lambda \in (0, 1)$	$\mathcal{T}_\lambda^{MT}, \lambda \in (0, 1)$

Figure 2.5: Continuous t-norms with typical representatives of each class

additive generator. The residual operators  $\mathcal{I}_\mathcal{T}$  and  $\mathcal{J}_\mathcal{T}$  can be represented, respectively, by

$$\mathcal{I}_\mathcal{T}(x, y) = t^{(-1)}(\max(t(y) - t(x), 0)), \quad (2.67)$$

$$\mathcal{J}_\mathcal{T}(x, y) = \begin{cases} t^{(-1)}(\max(t(y) - t(x), 0)), & \text{if } y \neq 0, \\ 0, & \text{if } y = 0. \end{cases} \quad (2.68)$$

**Theorem 2.4.5.** Let  $\mathcal{T} = (\langle a_\alpha, b_\alpha, \mathcal{T}_\alpha \rangle)_{\alpha \in A}$  be an ordinal sum of continuous Archimedean t-norms. The residual operators  $\mathcal{I}_\mathcal{T}$  and  $\mathcal{J}_\mathcal{T}$  can be obtained, respectively, by

$$\mathcal{I}_\mathcal{T}(x, y) = \begin{cases} 1, & \text{if } x \leq y, \\ a_\alpha + (b_\alpha - a_\alpha) \cdot \mathcal{I}_{\mathcal{T}_\alpha} \left( \frac{x - a_\alpha}{b_\alpha - a_\alpha}, \frac{y - a_\alpha}{b_\alpha - a_\alpha} \right), & \text{if } a_\alpha < y < x \leq b_\alpha, \\ y, & \text{otherwise.} \end{cases} \quad (2.69)$$

$$\mathcal{J}_\mathcal{T}(x, y) = \begin{cases} 1, & \text{if } x < y, \\ a_\alpha + (b_\alpha - a_\alpha) \cdot \mathcal{J}_{\mathcal{T}_\alpha} \left( \frac{x - a_\alpha}{b_\alpha - a_\alpha}, \frac{y - a_\alpha}{b_\alpha - a_\alpha} \right), & \text{if } a_\alpha < y \leq x \leq b_\alpha, \\ y, & \text{otherwise.} \end{cases} \quad (2.70)$$

**Theorem 2.4.6.** Let  $\mathcal{S}$  be a continuous Archimedean t-conorm and  $s : [0, 1] \rightarrow [0, +\infty]$  its

additive generator. The residual operators  $\mathcal{J}_S$  and  $\mathcal{I}_S$  can be represented, respectively, by

$$\mathcal{J}_S(x, y) = s^{(-1)}(\max(s(y) - s(x), 0)), \quad (2.71)$$

$$\mathcal{I}_S(x, y) = \begin{cases} s^{(-1)}(\max(s(y) - s(x), 0)), & \text{if } y \neq 1, \\ 1, & \text{if } y = 1. \end{cases} \quad (2.72)$$

**Theorem 2.4.7.** Let  $S = (\langle a_\alpha, b_\alpha, \mathcal{S}_\alpha \rangle)_{\alpha \in A}$  be an ordinal sum of continuous Archimedean  $t$ -conorms. The residual operators  $\mathcal{J}_S$  and  $\mathcal{I}_S$  can be obtained, respectively, by

$$\mathcal{J}_S(x, y) = \begin{cases} 0, & \text{if } x \geq y, \\ a_\alpha + (b_\alpha - a_\alpha) \cdot \mathcal{J}_{\mathcal{S}_\alpha} \left( \frac{x - a_\alpha}{b_\alpha - a_\alpha}, \frac{y - a_\alpha}{b_\alpha - a_\alpha} \right), & \text{if } a_\alpha \leq x < y < b_\alpha, \\ y, & \text{otherwise.} \end{cases} \quad (2.73)$$

$$\mathcal{I}_S(x, y) = \begin{cases} 0, & \text{if } x > y, \\ a_\alpha + (b_\alpha - a_\alpha) \cdot \mathcal{I}_{\mathcal{S}_\alpha} \left( \frac{x - a_\alpha}{b_\alpha - a_\alpha}, \frac{y - a_\alpha}{b_\alpha - a_\alpha} \right), & \text{if } a_\alpha \leq x \leq y < b_\alpha, \\ y, & \text{otherwise.} \end{cases} \quad (2.74)$$

It should be noted that the practical calculations of the residual operators  $\mathcal{I}_T$  and  $\mathcal{J}_T$  with respect to a given continuous  $t$ -norm  $T$  may be not easy although they are theoretically well defined, since the evaluation of the value  $T(x, y)$  for any pair of  $(x, y) \in [0, 1]^2$  may require much computational time when the  $t$ -norm  $T$  has a complicated structure. Moreover, when performing calculations via additive generators, one has to be very cautious about the behavior of the additive generators near 0 since additive generators of a strict  $t$ -norm have an asymptote at this point. Some scaling methods should be incorporated for the purpose of numerical stability. For strict  $t$ -conorms, the behavior of their additive generators near 1 should be concerned. In this text, all complexity issues concerned do not take into account the complexity of evaluating the values of these operations.

## 2.5 Model Implicators and Model Coimplicators

Among all implicators and coimplicators, model implicators and model coimplicators are of particular interest since they are essentially at a same level as  $t$ -norms for conjunctors and  $t$ -conorms for disjunctors, respectively.

**Definition 2.5.1.** An implicator  $\mathcal{I} : [0, 1]^2 \rightarrow [0, 1]$  is called *contrapositive with respect to a negator  $\mathcal{N}$*  if

$$\mathcal{I}(x, y) = \mathcal{I}(\mathcal{N}(y), \mathcal{N}(x)), \quad \text{for every } (x, y) \in [0, 1]^2. \quad (2.75)$$

**Theorem 2.5.2.** If a border implicator  $\mathcal{I}$  is contrapositive with respect to a negator  $\mathcal{N}$ , then  $\mathcal{N}$  is a strong negator and  $\mathcal{N} = \mathcal{N}_{\mathcal{I}}$ .

Theorem 2.5.2 was proposed by De Baets [76] and shows that a border implicator  $\mathcal{I}$  can only be contrapositive with respect to its induced negator  $\mathcal{N}_{\mathcal{I}}$ , and hence, the specification “with respect to a negator  $\mathcal{N}$ ” can be omitted in the definition of contraposition for border implicators. Furthermore, when the border implicator  $\mathcal{I}$  is contrapositive, the induced conjunctive  $\mathcal{C}_{\mathcal{I}, \mathcal{N}_{\mathcal{I}}}$  is a commutative t-seminorm and vice versa.

**Theorem 2.5.3.** A border implicator  $\mathcal{I}$  is contrapositive if and only if there exists a commutative t-seminorm  $\mathcal{C}$  such that  $\mathcal{I} = \mathcal{I}_{\mathcal{C}, \mathcal{N}_{\mathcal{I}}}$  with  $\mathcal{N}_{\mathcal{I}}$  being a strong negator.

*Proof.* If the border implicator  $\mathcal{I}$  is contrapositive, the induced negator  $\mathcal{N}_{\mathcal{I}}$  is a strong negator, and hence, the induced border conjunctive  $\mathcal{C}_{\mathcal{I}, \mathcal{N}_{\mathcal{I}}}$  is commutative. Consequently, the border conjunctive  $\mathcal{C}_{\mathcal{I}, \mathcal{N}_{\mathcal{I}}}$  satisfies the boundary condition  $\mathcal{C}(x, 1) = \mathcal{C}(1, x) = x$  for every  $x \in [0, 1]$ , and hence, is a t-seminorm. It is clear that the contrapositive border implicator  $\mathcal{I}$  can be induced by the commutative t-seminorm  $\mathcal{C}_{\mathcal{I}, \mathcal{N}_{\mathcal{I}}}$  and the strong negator  $\mathcal{N}_{\mathcal{I}}$ . Conversely, for any given commutative t-seminorm  $\mathcal{C}$ , any strong negator  $\mathcal{N}$  coincides with the induced negator  $\mathcal{N}_{\mathcal{I}}$  where  $\mathcal{I} = \mathcal{I}_{\mathcal{C}, \mathcal{N}}$ . Furthermore, the commutativity of  $\mathcal{C}$  implies that  $\mathcal{I}_{\mathcal{C}, \mathcal{N}}$  is contrapositive, i.e.,

$$\mathcal{I}_{\mathcal{C}, \mathcal{N}}(x, y) = \mathcal{N}(\mathcal{C}(x, \mathcal{N}(y))) = \mathcal{N}(\mathcal{C}(\mathcal{N}(y), x)) = \mathcal{I}_{\mathcal{C}, \mathcal{N}}(\mathcal{N}(y), \mathcal{N}(x)). \quad \square$$

**Definition 2.5.4.** A border implicator  $\mathcal{I}$  is said to be a *model implicator* if it is contrapositive and satisfies the exchange principle:

$$\mathcal{I}(x, \mathcal{I}(y, z)) = \mathcal{I}(y, \mathcal{I}(x, z)), \quad \text{for every } (x, y, z) \in [0, 1]^3. \quad (2.76)$$

The exchange principle for model implicators is the analogous version of associativity for t-norms, which, together with the contraposition property, leads to the following characterization of model implicators.

**Theorem 2.5.5.** A border implicator  $\mathcal{I}$  is a model implicator if and only if there exists a t-norm  $\mathcal{T}$  such that  $\mathcal{I} = \mathcal{I}_{\mathcal{T}, \mathcal{N}_{\mathcal{I}}}$  with  $\mathcal{N}_{\mathcal{I}}$  being a strong negator.

*Proof.* If  $\mathcal{I}$  is a model implicator, the induced negator  $\mathcal{N}_{\mathcal{I}}$  is a strong negator and the induced border conjunctor  $\mathcal{C}_{\mathcal{I},\mathcal{N}_{\mathcal{I}}}$  is a commutative t-seminorm. It suffices to show that  $\mathcal{C}_{\mathcal{I},\mathcal{N}_{\mathcal{I}}}$  is associative, and hence, is a t-norm. For every  $x, y, z \in [0, 1]$ , the contraposition of  $\mathcal{I}$  and the involution of  $\mathcal{N}_{\mathcal{I}}$  imply that

$$\begin{aligned}
\mathcal{C}_{\mathcal{I},\mathcal{N}_{\mathcal{I}}}(x, \mathcal{C}_{\mathcal{I},\mathcal{N}_{\mathcal{I}}}(y, z)) &= \mathcal{N}_{\mathcal{I}}(\mathcal{I}(x, \mathcal{N}_{\mathcal{I}}(\mathcal{C}_{\mathcal{I},\mathcal{N}_{\mathcal{I}}}(y, z)))) \\
&= \mathcal{N}_{\mathcal{I}}(\mathcal{I}(x, \mathcal{I}(y, \mathcal{N}_{\mathcal{I}}(z)))) \\
&= \mathcal{N}_{\mathcal{I}}(\mathcal{I}(x, \mathcal{I}(z, \mathcal{N}_{\mathcal{I}}(y)))) \\
&= \mathcal{N}_{\mathcal{I}}(\mathcal{I}(z, \mathcal{I}(x, \mathcal{N}_{\mathcal{I}}(y)))) \\
&= \mathcal{N}_{\mathcal{I}}(\mathcal{I}(\mathcal{N}_{\mathcal{I}}(\mathcal{I}(x, \mathcal{N}_{\mathcal{I}}(y))), \mathcal{N}_{\mathcal{I}}(z))) \\
&= \mathcal{C}_{\mathcal{I},\mathcal{N}_{\mathcal{I}}}(\mathcal{C}_{\mathcal{I},\mathcal{N}_{\mathcal{I}}}(x, y), z).
\end{aligned}$$

Conversely, for any given t-norm  $\mathcal{T}$ , any strong negator  $\mathcal{N}$  coincides with the induced negator  $\mathcal{N}_{\mathcal{I}}$  where  $\mathcal{I} = \mathcal{I}_{\mathcal{T},\mathcal{N}}$ . Moreover, by Theorem 2.5.3, the commutativity of  $\mathcal{T}$  implies that  $\mathcal{I}_{\mathcal{T},\mathcal{N}}$  is contrapositive. Furthermore, for every  $x, y, z \in [0, 1]$ , the associativity of  $\mathcal{T}$  and the involution of  $\mathcal{N}$  imply that

$$\begin{aligned}
\mathcal{I}_{\mathcal{T},\mathcal{N}}(x, \mathcal{I}_{\mathcal{T},\mathcal{N}}(y, z)) &= \mathcal{N}(\mathcal{T}(x, \mathcal{N}(\mathcal{I}_{\mathcal{T},\mathcal{N}}(y, z)))) \\
&= \mathcal{N}(\mathcal{T}(x, \mathcal{T}(y, \mathcal{N}(z)))) \\
&= \mathcal{N}(\mathcal{T}(x, \mathcal{T}(\mathcal{N}(z), y))) \\
&= \mathcal{N}(\mathcal{T}(y, \mathcal{T}(x, \mathcal{N}(z)))) \\
&= \mathcal{I}_{\mathcal{T},\mathcal{N}}(y, \mathcal{I}_{\mathcal{T},\mathcal{N}}(x, z)).
\end{aligned}$$

Therefore,  $\mathcal{I}_{\mathcal{T},\mathcal{N}}$  satisfies the exchange principle, and hence, is a model implicator.  $\square$

The concept of contraposition and the exchange principle can be defined dually for coimplicators. The corresponding characterization can be also derived dually.

**Definition 2.5.6.** A coimplicator  $\mathcal{J} : [0, 1]^2 \rightarrow [0, 1]$  is called *contrapositive with respect to a negator  $\mathcal{N}$*  if

$$\mathcal{J}(x, y) = \mathcal{J}(\mathcal{N}(y), \mathcal{N}(x)), \quad \text{for every } (x, y) \in [0, 1]^2. \quad (2.77)$$

**Theorem 2.5.7.** If a border coimplicator  $\mathcal{J}$  is contrapositive with respect to a negator  $\mathcal{N}$ , then  $\mathcal{N}$  is a strong negator and  $\mathcal{N} = \mathcal{N}_{\mathcal{J}}$ .

Theorem 2.5.7 was proposed by De Baets [78] and shows that a border coimplicator  $\mathcal{J}$  can only be contrapositive with respect to its induced negator  $\mathcal{N}_{\mathcal{J}}$ , and hence, the specification “with respect to a negator  $\mathcal{N}$ ” can be omitted in the definition of contraposition for border coimplicators. Furthermore, when the border coimplicator  $\mathcal{J}$  is contrapositive, the induced conjunctive  $\mathcal{D}_{\mathcal{J},\mathcal{N}_{\mathcal{J}}}$  is a commutative t-semiconorm and vice versa.

**Theorem 2.5.8.** *A border coimplicator  $\mathcal{J}$  is contrapositive if and only if there exists a commutative t-semiconorm  $\mathcal{D}$  such that  $\mathcal{J} = \mathcal{J}_{\mathcal{D},\mathcal{N}_{\mathcal{J}}}$  with  $\mathcal{N}_{\mathcal{J}}$  being a strong negator.*

**Definition 2.5.9.** *A border coimplicator  $\mathcal{J}$  is said to be a model coimplicator if it is contrapositive and satisfies the exchange principle:*

$$\mathcal{J}(x, \mathcal{J}(y, z)) = \mathcal{J}(y, \mathcal{J}(x, z)), \quad \text{for every } (x, y, z) \in [0, 1]^3. \quad (2.78)$$

**Theorem 2.5.10.** *A border coimplicator  $\mathcal{J}$  is a model coimplicator if and only if there exists a t-conorm  $\mathcal{S}$  such that  $\mathcal{J} = \mathcal{J}_{\mathcal{S},\mathcal{N}_{\mathcal{J}}}$  with  $\mathcal{N}_{\mathcal{J}}$  being a strong negator.*

Now it is clear that any De Morgan triple  $(\mathcal{T}, \mathcal{S}, \mathcal{N})$  defines a unique isomorphic De Morgan triple  $(\mathcal{I}, \mathcal{J}, \mathcal{N})$ , which contains a model implicator  $\mathcal{I}$  and its dual model coimplicator  $\mathcal{J}$ . Conversely, such a De Morgan triple  $(\mathcal{I}, \mathcal{J}, \mathcal{N})$ , with the strong negator  $\mathcal{N}$  automatically determined by  $\mathcal{N}_{\mathcal{I}}$ , defines a unique isomorphic De Morgan triple  $(\mathcal{T}, \mathcal{S}, \mathcal{N})$ . These relations are illustrated in Figure 2.6. Furthermore, the characterization of continuous t-norms and t-conorms automatically applies to continuous model implicators and coimplicators.

$$\begin{array}{ccc} \mathcal{T} & \xleftrightarrow{\mathcal{N}} & \mathcal{I}_{\mathcal{T},\mathcal{N}} \\ \uparrow \mathcal{N} & & \uparrow \mathcal{N} \\ \mathcal{S} & \xleftrightarrow{\mathcal{N}} & \mathcal{J}_{\mathcal{S},\mathcal{N}} \end{array}$$

Figure 2.6: The relationship between  $(\mathcal{T}, \mathcal{S}, \mathcal{N})$  and  $(\mathcal{I}_{\mathcal{T},\mathcal{N}}, \mathcal{J}_{\mathcal{S},\mathcal{N}}, \mathcal{N})$

## 2.6 Elementary Fuzzy Relational Equations

Let  $\mathcal{O} : [0, 1]^2 \rightarrow [0, 1]$  be a binary operator which is increasing in its second argument. If the partial mapping  $\mathcal{O}(a, \cdot)$  is lower semicontinuous, or equivalently left-continuous, for every  $a \in [0, 1]$ , then it holds that

$$\mathcal{O}(a, \sup_{x \in X} x) = \sup_{x \in X} \mathcal{O}(a, x) \quad (2.79)$$

for any nonempty set  $X \subseteq [0, 1]$ . Similarly, if  $\mathcal{O}(a, \cdot)$  is upper semicontinuous, or equivalently right-continuous, for every  $a \in [0, 1]$ , then it holds that

$$\mathcal{O}(a, \inf_{x \in X} x) = \inf_{x \in X} \mathcal{O}(a, x) \quad (2.80)$$

for any nonempty set  $X \subseteq [0, 1]$ . These properties, together with Theorems 2.1.16 and 2.1.18, lead to the following important and fundamental results on the resolution of elementary fuzzy relational equations.

### 2.6.1 The Equations $\mathcal{C}(a, x) = b$ and $\mathcal{D}(a, x) = b$

**Theorem 2.6.1.** *Let  $\mathcal{C}$  be a border conjunctor with its second partial mapping  $\mathcal{C}(a, \cdot)$  being left-continuous for every  $a \in [0, 1]$ , and  $\mathcal{I}_{\mathcal{C}}$  its associated residual implicator. The equation  $\mathcal{C}(a, x) = b$  has a solution for given  $a, b \in [0, 1]$  if and only if  $\mathcal{C}(a, \mathcal{I}_{\mathcal{C}}(a, b)) = b$ . Furthermore, if  $\mathcal{C}(a, \cdot)$  is continuous for every  $a \in [0, 1]$ , then the solution set of  $\mathcal{C}(a, x) = b$ , when it is nonempty, is the closed interval  $[\mathcal{J}_{\mathcal{C}}(a, b), \mathcal{I}_{\mathcal{C}}(a, b)]$ .*

**Theorem 2.6.2.** *Let  $\mathcal{C}$  be a border conjunctor with its second partial mapping  $\mathcal{C}(a, \cdot)$  being continuous for every  $a \in [0, 1]$ , and satisfy the right conditional cancelation law. The equation  $\mathcal{C}(a, x) = b$  with  $b \in (0, 1]$  has a unique and nonzero solution whenever  $\mathcal{C}(a, \mathcal{I}_{\mathcal{C}}(a, b)) = b$ .*

Note that the exception of Theorem 2.6.2 occurs for the equation  $\mathcal{C}(a, x) = 0$  of which the solution set depends on whether the conjunctor  $\mathcal{C}$  has right zero divisors or not. Moreover, when  $\mathcal{C}$  is a continuous t-norm, or more generally, a continuous t-seminorm, the requirement  $\mathcal{C}(a, 1) = \mathcal{C}(1, a) = a$  for every  $a \in [0, 1]$  implies that the equation  $\mathcal{C}(a, x) = b$  has a solution for given  $a, b \in [0, 1]$  if and only if  $b \leq a$ .

**Corollary 2.6.3.** *Let  $\mathcal{T}$  be a continuous t-norm, and  $\mathcal{I}_{\mathcal{T}}$  and  $\mathcal{J}_{\mathcal{T}}$  its associated residual operators. The equation  $\mathcal{T}(a, x) = b$  has a solution for given  $a, b \in [0, 1]$  if and only if  $b \leq a$ , in which case the solution set of  $\mathcal{T}(a, x) = b$  is the closed interval  $[\mathcal{J}_{\mathcal{T}}(a, b), \mathcal{I}_{\mathcal{T}}(a, b)]$ .*

Clearly, given a strictly increasing bijection  $\psi : [0, 1] \rightarrow [0, 1]$ , the equations  $\mathcal{T}_\psi(a, x) = b$  and  $\mathcal{T}(\psi(a), x) = \psi(b)$  are isomorphic in the sense that  $\mathcal{T}_\psi(a, x) = b$  implies  $\mathcal{T}(\psi(a), \psi(x)) = \psi(b)$  and vice versa. Furthermore, the fact that each continuous Archimedean t-norm satisfies the conditional cancelation law immediately leads to the following fundamental property of continuous Archimedean t-norms which plays an important role in the resolution of fuzzy relational equations.

**Corollary 2.6.4.** *Let  $\mathcal{T}$  be a continuous Archimedean t-norm. The equation  $\mathcal{T}(a, x) = b$  has a unique and nonzero solution when  $b \leq a$  and  $a, b \in (0, 1]$ .*

According to Corollary 2.6.4, the equation  $\mathcal{T}(a, x) = b$  have multiple solutions only when  $b = 0$  for a nilpotent t-norm, or  $a = b = 0$  for a strict t-norm. For a continuous non-Archimedean t-norm  $\mathcal{T}$ , the situation turns out to be a little bit complicated, especially when  $\mathcal{T}$  is an ordinal sum of continuous Archimedean t-norms, and involves nilpotent summands.

**Corollary 2.6.5.** *Let  $\mathcal{T}$  be a continuous non-Archimedean t-norm. The equation  $\mathcal{T}(a, x) = b$  with  $0 < b \leq a \leq 1$  has multiple solutions only when  $b = a < 1$  or  $b = a_\alpha \in (0, 1)$  in case  $\mathcal{T}$  involves a nilpotent summand  $\langle a_\alpha, b_\alpha, \mathcal{T}_\alpha \rangle$  and  $a \in (a_\alpha, b_\alpha)$ .*

Corollary 2.6.5 is a direct consequence of Theorem 2.4.5 and Corollary 2.6.3. Similarly, the behavior of the equation  $\mathcal{T}(a, x) = 0$  depends on whether the t-norm  $\mathcal{T}$  has zero divisors or not. As an illustration of Corollary 2.6.5, assume that  $\mathcal{T}_1 = (\langle 0.4, 0.8, \mathcal{T}_P \rangle)$  and  $\mathcal{T}_2 = (\langle 0.4, 0.8, \mathcal{T}_L \rangle)$ , the functions  $\mathcal{T}_1(0.6, x)$  and  $\mathcal{T}_2(0.6, x)$  are both piecewise linear as illustrated in Figure 2.7. It is clear that the equations  $\mathcal{T}_1(0.6, x) = 0.6$ ,  $\mathcal{T}_2(0.6, x) = 0.6$ , and  $\mathcal{T}_2(0.6, x) = 0.4$  have multiple solutions, respectively.

Note that a conjunctor  $\mathcal{C}$  may not be commutative, and hence, the equations  $\mathcal{C}(a, x) = b$  and  $\mathcal{C}(x, a) = b$  are not equivalent in general. However, it suffices to consider the equation  $\mathcal{C}(a, x) = b$  since any conjunctor  $\mathcal{C}$  also defines a conjunctor  $\mathcal{C}^T$  via  $\mathcal{C}^T(x, y) = \mathcal{C}(y, x)$  for every  $(x, y) \in [0, 1]^2$ . Similar observation holds for the equations  $\mathcal{D}(a, x) = b$  and  $\mathcal{D}(x, a) = b$  since any disjunctive  $\mathcal{D}$  defines a disjunctive  $\mathcal{D}^T$  via  $\mathcal{D}^T(x, y) = \mathcal{D}(y, x)$  for every  $(x, y) \in [0, 1]^2$ .

**Theorem 2.6.6.** *Let  $\mathcal{D}$  be a border disjunctive with its second partial mapping  $\mathcal{D}(a, \cdot)$  being right-continuous for every  $a \in [0, 1]$ , and  $\mathcal{J}_\mathcal{D}$  its associated residual coimplicator. The equation  $\mathcal{D}(a, x) = b$  has a solution for given  $a, b \in [0, 1]$  if and only if  $\mathcal{D}(a, \mathcal{J}_\mathcal{D}(a, b)) = b$ .*

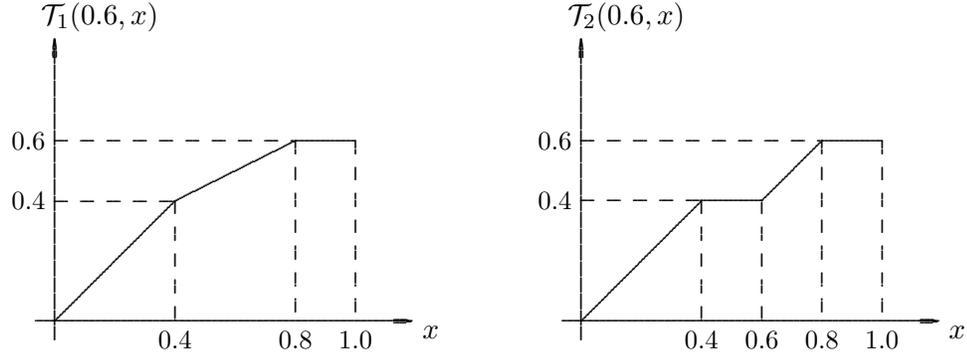


Figure 2.7: Illustrations of functions  $\mathcal{T}_1(0.6, x)$  and  $\mathcal{T}_2(0.6, x)$

Furthermore, if  $\mathcal{D}(a, \cdot)$  is continuous for every  $a \in [0, 1]$ , then the solution set of  $\mathcal{D}(a, x) = b$ , when it is nonempty, is the closed interval  $[\mathcal{J}_{\mathcal{D}}(a, b), \mathcal{I}_{\mathcal{D}}(a, b)]$ .

**Theorem 2.6.7.** Let  $\mathcal{D}$  be a border disjunctive with its second partial mapping  $\mathcal{D}(a, \cdot)$  being continuous for every  $a \in [0, 1]$ , and satisfy the right conditional cancellation law. The equation  $\mathcal{D}(a, x) = b$  with  $b \in [0, 1)$  has a unique and non-one solution whenever  $\mathcal{D}(a, \mathcal{J}_{\mathcal{D}}(a, b)) = b$ .

The exception of Theorem 2.6.7 occurs for the equation  $\mathcal{D}(a, x) = 1$  of which the solution set depends on whether the disjunctive  $\mathcal{D}$  has right zero divisors or not. Moreover, when the disjunctive  $\mathcal{D}$  is a continuous t-conorm, or more generally, a continuous t-semiconorm, the solvability of the equation  $\mathcal{D}(a, x) = b$  is operator independent.

**Corollary 2.6.8.** Let  $\mathcal{S}$  be a continuous t-conorm, and  $\mathcal{J}_{\mathcal{S}}$  and  $\mathcal{I}_{\mathcal{S}}$  its associated residual operators. The equation  $\mathcal{S}(a, x) = b$  has a solution for given  $a, b \in [0, 1]$  if and only if  $b \geq a$ , in which case the solution set of  $\mathcal{S}(a, x) = b$  is the closed interval  $[\mathcal{J}_{\mathcal{S}}(a, b), \mathcal{I}_{\mathcal{S}}(a, b)]$ .

**Corollary 2.6.9.** Let  $\mathcal{S}$  be a continuous Archimedean t-conorm. The equation  $\mathcal{S}(a, x) = b$  has a unique and non-one solution when  $b \geq a$  and  $a, b \in [0, 1)$ .

**Corollary 2.6.10.** Let  $\mathcal{S}$  be a continuous non-Archimedean t-conorm. The equation  $\mathcal{S}(a, x) = b$  with  $0 \leq a \leq b < 1$  has multiple solutions only when  $b = a > 0$  or  $b = b_{\alpha} \in (0, 1)$  in case  $\mathcal{S}$  involves a nilpotent summand  $\langle a_{\alpha}, b_{\alpha}, \mathcal{S}_{\alpha} \rangle$  and  $a \in (a_{\alpha}, b_{\alpha})$ .

Corollary 2.6.10 is a direct consequence of Theorem 2.4.7 and Corollary 2.6.8. The behavior of the equation  $\mathcal{S}(a, x) = 1$  depends on whether  $\mathcal{S}$  has zero divisors or not. Figure 2.8 illustrates that the equations  $\mathcal{S}_1(0.4, x) = 0.4$ ,  $\mathcal{S}_2(0.4, x) = 0.4$ , and  $\mathcal{S}_2(0.4, x) = 0.6$  have multiple solutions, respectively, where  $\mathcal{S}_1 = (\langle 0.2, 0.6, \mathcal{S}_P \rangle)$  and  $\mathcal{S}_2 = (\langle 0.2, 0.6, \mathcal{S}_L \rangle)$ .

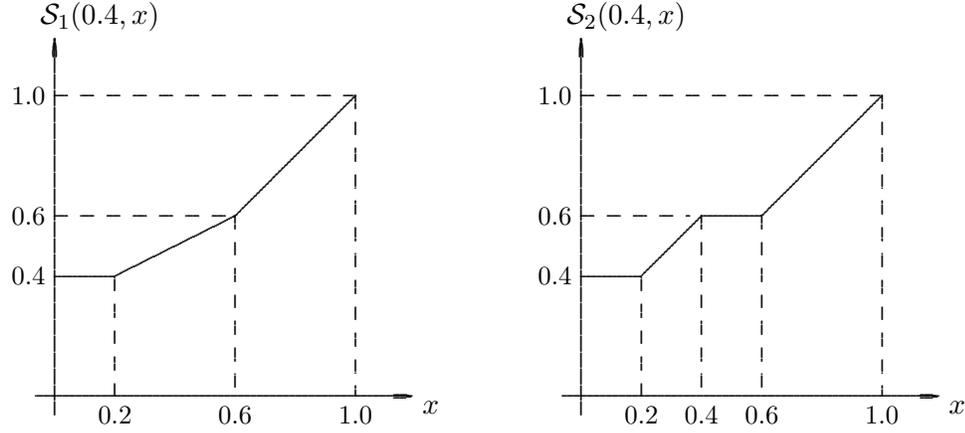


Figure 2.8: Illustrations of functions  $\mathcal{S}_1(0.4, x)$  and  $\mathcal{S}_2(0.4, x)$

### 2.6.2 The Equations $\mathcal{I}(a, x) = b$ and $\mathcal{J}(a, x) = b$

Theoretically, for a pair of isomorphic De Morgan triples  $(\mathcal{C}, \mathcal{D}, \mathcal{N})$  and  $(\mathcal{I}, \mathcal{J}, \mathcal{N})$ , it holds that

$$\mathcal{I}(x, y) = \mathcal{I}_{\mathcal{C}, \mathcal{N}}(x, y) = \mathcal{N}(\mathcal{C}(x, \mathcal{N}(y))) = \mathcal{D}(\mathcal{N}(x), y), \quad (2.81)$$

$$\mathcal{J}(x, y) = \mathcal{J}_{\mathcal{D}, \mathcal{N}}(x, y) = \mathcal{N}(\mathcal{D}(x, \mathcal{N}(y))) = \mathcal{C}(\mathcal{N}(x), y). \quad (2.82)$$

Consequently, the equations  $\mathcal{I}(a, x) = b$  and  $\mathcal{J}(a, x) = b$  can be converted to  $\mathcal{D}(\mathcal{N}(a), x) = b$  and  $\mathcal{C}(\mathcal{N}(a), x) = b$ , respectively, while the equations  $\mathcal{I}(x, a) = b$  and  $\mathcal{J}(x, a) = b$  can be converted to  $\mathcal{C}(x, \mathcal{N}(a)) = \mathcal{N}(b)$  and  $\mathcal{D}(x, \mathcal{N}(a)) = \mathcal{N}(b)$ , respectively. Therefore, these types of equations can be properly dealt with by the methods for equations of the types  $\mathcal{C}(a, x) = b$  and  $\mathcal{D}(a, x) = b$ . However, they can also be handled independently.

**Theorem 2.6.11.** *Let  $\mathcal{I}$  be a border implicator with its second partial mapping  $\mathcal{I}(a, \cdot)$  being right-continuous for every  $a \in [0, 1]$ , and  $\mathcal{C}_{\mathcal{I}}$  its associated residual conjunctive. The equation  $\mathcal{I}(a, x) = b$  has a solution for given  $a, b \in [0, 1]$  if and only if  $\mathcal{I}(a, \mathcal{C}_{\mathcal{I}}(a, b)) = b$ . Furthermore,*

if  $\mathcal{I}(a, \cdot)$  is continuous for every  $a \in [0, 1]$ , then the solution set of  $\mathcal{I}(a, x) = b$ , when it is nonempty, is the closed interval  $[\mathcal{C}_{\mathcal{I}}(a, b), \mathcal{D}_{\mathcal{I}}(a, b)]$ .

**Theorem 2.6.12.** *Let  $\mathcal{I}$  be a border implicator with its second partial mapping  $\mathcal{I}(a, \cdot)$  being continuous for every  $a \in [0, 1]$ , and satisfy the right conditional cancelation law. The equation  $\mathcal{I}(a, x) = b$  with  $b \in [0, 1)$  has a unique and non-one solution whenever  $\mathcal{I}(a, \mathcal{C}_{\mathcal{I}}(a, b)) = b$ .*

**Corollary 2.6.13.** *Let  $\mathcal{I}$  be a continuous model implicator, and  $\mathcal{C}_{\mathcal{I}}$  and  $\mathcal{D}_{\mathcal{I}}$  its associated residual operators. The implicator  $\mathcal{I}$  can be represented by  $\mathcal{I}_{\mathcal{T}, \mathcal{N}}$  where  $\mathcal{T} = \mathcal{C}_{\mathcal{I}, \mathcal{N}_{\mathcal{I}}}$  is a continuous t-norm and  $\mathcal{N} = \mathcal{N}_{\mathcal{I}}$  is a strong negator. The equation  $\mathcal{I}(a, x) = b$  has a solution for given  $a, b \in [0, 1]$  if and only if  $\mathcal{N}(a) \leq b$ , in which case the solution set of  $\mathcal{I}(a, x) = b$  is the closed interval  $[\mathcal{C}_{\mathcal{I}}(a, b), \mathcal{D}_{\mathcal{I}}(a, b)]$  with  $\mathcal{C}_{\mathcal{I}}(a, b) = \mathcal{N}(\mathcal{I}_{\mathcal{T}}(a, \mathcal{N}(b)))$  and  $\mathcal{D}_{\mathcal{I}}(a, b) = \mathcal{N}(\mathcal{J}_{\mathcal{T}}(a, \mathcal{N}(b)))$ , respectively.*

**Theorem 2.6.14.** *Let  $\mathcal{J}$  be a border coimplicator with its second partial mapping  $\mathcal{J}(a, \cdot)$  being left-continuous for every  $a \in [0, 1]$ , and  $\mathcal{D}_{\mathcal{J}}$  its associated residual disjunctive. The equation  $\mathcal{J}(a, x) = b$  has a solution for given  $a, b \in [0, 1]$  if and only if  $\mathcal{J}(a, \mathcal{D}_{\mathcal{J}}(a, b)) = b$ . Furthermore, if  $\mathcal{J}(a, \cdot)$  is continuous for every  $a \in [0, 1]$ , then the solution set of  $\mathcal{J}(a, x) = b$ , when it is nonempty, is the closed interval  $[\mathcal{C}_{\mathcal{J}}(a, b), \mathcal{D}_{\mathcal{J}}(a, b)]$ .*

**Theorem 2.6.15.** *Let  $\mathcal{J}$  be a border coimplicator with its second partial mapping  $\mathcal{J}(a, \cdot)$  being continuous for every  $a \in [0, 1]$ , and satisfy the right conditional cancelation law. The equation  $\mathcal{J}(a, x) = b$  with  $b \in [0, 1)$  has a unique and nonzero solution whenever  $\mathcal{J}(a, \mathcal{D}_{\mathcal{J}}(a, b)) = b$ .*

**Corollary 2.6.16.** *Let  $\mathcal{J}$  be a continuous model coimplicator, and  $\mathcal{D}_{\mathcal{J}}$  and  $\mathcal{C}_{\mathcal{J}}$  its associated residual operators. The coimplicator  $\mathcal{J}$  can be represented by  $\mathcal{J}_{\mathcal{S}, \mathcal{N}}$  where  $\mathcal{S} = \mathcal{D}_{\mathcal{J}, \mathcal{N}_{\mathcal{J}}}$  is a continuous t-conorm and  $\mathcal{N} = \mathcal{N}_{\mathcal{J}}$  is a strong negator. The equation  $\mathcal{J}(a, x) = b$  has a solution for given  $a, b \in [0, 1]$  if and only if  $\mathcal{N}(a) \geq b$ , in which case the solution set of  $\mathcal{J}(a, x) = b$  is the closed interval  $[\mathcal{C}_{\mathcal{J}}(a, b), \mathcal{D}_{\mathcal{J}}(a, b)]$  with  $\mathcal{C}_{\mathcal{J}}(a, b) = \mathcal{N}(\mathcal{I}_{\mathcal{S}}(a, \mathcal{N}(b)))$  and  $\mathcal{D}_{\mathcal{J}}(a, b) = \mathcal{N}(\mathcal{J}_{\mathcal{S}}(a, \mathcal{N}(b)))$ , respectively.*

Corollaries 2.6.4 and 2.6.5 are based on the classification and representation of continuous t-norms, while Corollaries 2.6.9 and 2.6.10 are based on the classification and representation of continuous t-conorms. Their analogous versions for continuous model implicators and coimplicators can be derived without any difficulty.

### 2.6.3 The Equations $\mathcal{I}(x, a) = b$ and $\mathcal{J}(x, a) = b$

Although it has been indicated that the equations  $\mathcal{I}(x, a) = b$  and  $\mathcal{J}(x, a) = b$  can be converted to  $\mathcal{C}(x, \mathcal{N}(a)) = \mathcal{N}(b)$  and  $\mathcal{D}(x, \mathcal{N}(a)) = \mathcal{N}(b)$ , respectively, they can be treated in their own frameworks of residuation.

**Definition 2.6.17.** *The left residual operators  $\mathcal{I}_{\mathcal{I}}$  and  $\mathcal{J}_{\mathcal{I}}$  with respect to a border implicator  $\mathcal{I}$  are defined, respectively, as*

$$\mathcal{I}_{\mathcal{I}}(x, y) = \sup\{z \in [0, 1] \mid \mathcal{I}(z, y) \geq x\}, \quad (2.83)$$

$$\mathcal{J}_{\mathcal{I}}(x, y) = \inf\{z \in [0, 1] \mid \mathcal{I}(z, y) \leq x\}. \quad (2.84)$$

The left residual operators  $\mathcal{I}_{\mathcal{I}}$  is in general not an implicator but it satisfies all the conditions of an implicator except for the behavior at the point  $(1, 0)$ . However, the condition  $\mathcal{I}_{\mathcal{I}}(1, 0) = 0$  can be guaranteed for many important classes of implicators, for instance, model implicators and R-implicators with respect to left continuous t-norms. It will be shown in Theorem 2.7.3 and Corollary 2.7.4 that the left residual operator  $\mathcal{I}_{\mathcal{I}}$  is an implicator if and only if the implicator  $\mathcal{I}$  satisfies the condition  $\mathcal{N}_{\mathcal{I}}(x) = \mathcal{I}(x, 0) < 1$  for every  $x \in (0, 1]$ .

**Theorem 2.6.18.** *Let  $\mathcal{I}$  be a border implicator with its first partial mapping  $\mathcal{I}(\cdot, a)$  being left-continuous for every  $a \in [0, 1]$ , and  $\mathcal{I}_{\mathcal{I}}$  its associated left residual operator which is not necessarily an implicator. The equation  $\mathcal{I}(x, a) = b$  has a solution for given  $a, b \in [0, 1]$  if and only if  $\mathcal{I}(\mathcal{I}_{\mathcal{I}}(b, a), a) = b$ . Furthermore, if  $\mathcal{I}(\cdot, a)$  is continuous for every  $a \in [0, 1]$ , then the solution set of  $\mathcal{I}(x, a) = b$ , when it is nonempty, is the closed interval  $[\mathcal{J}_{\mathcal{I}}(b, a), \mathcal{I}_{\mathcal{I}}(b, a)]$ .*

**Theorem 2.6.19.** *Let  $\mathcal{I}$  be a border implicator with its first partial mapping  $\mathcal{I}(\cdot, a)$  being continuous for every  $a \in [0, 1]$ , and satisfy the left conditional cancelation law. The equation  $\mathcal{I}(x, a) = b$  with  $b \in [0, 1)$  has a unique and nonzero solution whenever  $\mathcal{I}(\mathcal{I}_{\mathcal{I}}(b, a), a) = b$ .*

**Definition 2.6.20.** *The left residual operators  $\mathcal{J}_{\mathcal{J}}$  and  $\mathcal{I}_{\mathcal{J}}$  with respect to a border coimplicator  $\mathcal{J}$  are defined, respectively, as*

$$\mathcal{J}_{\mathcal{J}}(x, y) = \inf\{z \in [0, 1] \mid \mathcal{J}(z, y) \leq x\}, \quad (2.85)$$

$$\mathcal{I}_{\mathcal{J}}(x, y) = \sup\{z \in [0, 1] \mid \mathcal{J}(z, y) \geq x\}. \quad (2.86)$$

Similarly, the left residual operators  $\mathcal{J}_{\mathcal{J}}$  satisfies all the conditions of an coimplicator except for the behavior at the point  $(0, 1)$ . It will be shown in Theorem 2.7.3 and Corollary 2.7.4 that the left residual operator  $\mathcal{J}_{\mathcal{J}}$  is a coimplicator if and only if the coimplicator  $\mathcal{J}$  satisfies the condition  $\mathcal{N}_{\mathcal{J}}(x) = \mathcal{J}(x, 1) > 0$  for every  $x \in [0, 1)$ .

**Theorem 2.6.21.** *Let  $\mathcal{J}$  be a border coimplicator with its first partial mapping  $\mathcal{J}(\cdot, a)$  being right-continuous for every  $a \in [0, 1]$ , and  $\mathcal{J}_{\mathcal{J}}$  its associated left residual operator which is not necessarily a coimplicator. The equation  $\mathcal{J}(x, a) = b$  has a solution for given  $a, b \in [0, 1]$  if and only if  $\mathcal{J}(\mathcal{J}_{\mathcal{J}}(b, a), a) = b$ . Furthermore, if  $\mathcal{J}(\cdot, a)$  is continuous for every  $a \in [0, 1]$ , then the solution set of  $\mathcal{J}(x, a) = b$ , when it is nonempty, is the closed interval  $[\mathcal{J}_{\mathcal{J}}(b, a), \mathcal{I}_{\mathcal{J}}(b, a)]$ .*

**Theorem 2.6.22.** *Let  $\mathcal{J}$  be a border coimplicator with its first partial mapping  $\mathcal{J}(\cdot, a)$  being continuous for every  $a \in [0, 1]$ , and satisfy the left conditional cancelation law. The equation  $\mathcal{J}(x, a) = b$  with  $b \in (0, 1]$  has a unique and non-one solution whenever  $\mathcal{J}(\mathcal{J}_{\mathcal{J}}(b, a), a) = b$ .*

## 2.7 Adjoint Operators and Fodor's Closure Theorem

It is evident that residual operators play an important role in the resolution of fuzzy relational equations formed by the generalized logical operators. Actually, residual operators can be defined for a larger class of binary operators.

**Definition 2.7.1.** *Let  $\mathcal{O} : [0, 1]^2 \rightarrow [0, 1]$  be a binary operator. Two residual operators  $\mathcal{R}_{\mathcal{O}}$  and  $\mathcal{L}_{\mathcal{O}}$  with respect to  $\mathcal{O}$  are defined, respectively, by*

$$\mathcal{R}_{\mathcal{O}}(x, y) = \sup\{z \in [0, 1] \mid \mathcal{O}(x, z) \leq y\}, \quad (2.87)$$

$$\mathcal{L}_{\mathcal{O}}(x, y) = \inf\{z \in [0, 1] \mid \mathcal{O}(x, z) \geq y\}. \quad (2.88)$$

A binary operator  $\mathcal{O}$  and its residual operator  $\mathcal{R}_{\mathcal{O}}$  are called a pair of adjoint operators if, for every  $(a, b) \in [0, 1]^2$ ,

$$\mathcal{O}(a, x) \leq b \quad \text{if and only if} \quad x \leq \mathcal{R}_{\mathcal{O}}(a, b). \quad (2.89)$$

Denote

$$\mathcal{F}_{\mathcal{R}} = \{\mathcal{O} : [0, 1]^2 \rightarrow [0, 1] \mid \mathcal{O} \text{ and } \mathcal{R}_{\mathcal{O}} \text{ form a pair of adjoint operators}\}. \quad (2.90)$$

The operators in the class  $\mathcal{F}_{\mathcal{R}}$  was fully characterized by Fodor [138] via the following theorem, which is a consequence of the theory of Galois-connections.

**Theorem 2.7.2.** *Let  $\mathcal{O} : [0, 1]^2 \rightarrow [0, 1]$  be a binary operator and  $\mathcal{R}_{\mathcal{O}}$  its residual operator. The following statements are equivalent:*

- (i) *The operators  $\mathcal{O}$  and  $\mathcal{R}_{\mathcal{O}}$  form a pair of adjoint operators.*
- (ii) *The mapping  $\mathcal{O}(a, \cdot)$  is increasing, left-continuous, and satisfies  $\mathcal{O}(a, 0) = 0$  for every  $a \in [0, 1]$ .*
- (iii) *The mapping  $\mathcal{R}_{\mathcal{O}}(a, \cdot)$  is increasing, right-continuous, and satisfies  $\mathcal{R}_{\mathcal{O}}(a, 1) = 1$  for every  $a \in [0, 1]$ .*

When an operator  $\mathcal{O}$  and its residual operator  $\mathcal{R}_{\mathcal{O}}$  form a pair of adjoint operators, i.e.,  $\mathcal{O} \in \mathcal{F}_{\mathcal{R}}$ , it holds that

$$\mathcal{O}(x, y) = \inf\{z \in [0, 1] \mid \mathcal{R}_{\mathcal{O}}(x, z) \geq y\}, \quad (2.91)$$

which means that the operator  $\mathcal{O}$  can be reconstructed by the proper residual operation with respect to its residual operator  $\mathcal{R}_{\mathcal{O}}$ . It has been shown in Theorems 2.1.16 and 2.1.18 that conjunctors and coimplicators with left continuous second partial mappings all belong to the class  $\mathcal{F}_{\mathcal{R}}$ . Note that for an operator in the class  $\mathcal{F}_{\mathcal{R}}$ , the monotonicity in its first argument is not required. However, to obtain a residual implicator or a residual disjunctive, the monotonicity in the first argument is necessary.

**Theorem 2.7.3.** *Let  $\mathcal{O} \in \mathcal{F}_{\mathcal{R}}$  and  $\mathcal{R}_{\mathcal{O}}$  its residual operator, such that  $\mathcal{O}$  and  $\mathcal{R}_{\mathcal{O}}$  form a pair of adjoint operators. Then*

- (i) *the mapping  $\mathcal{O}(\cdot, a)$  is increasing for every  $a \in [0, 1]$  if and only if the mapping  $\mathcal{R}_{\mathcal{O}}(\cdot, a)$  is decreasing for every  $a \in [0, 1]$ ;*
- (ii)  *$\mathcal{O}(0, 1) = 0$  if and only if  $\mathcal{R}_{\mathcal{O}}(0, 1) = 1$ ;*
- (iii)  *$\mathcal{O}(1, y) > 0$  for every  $y \in (0, 1]$  if and only if  $\mathcal{R}_{\mathcal{O}}(1, 0) = 0$ .*

**Corollary 2.7.4.** *Let  $\mathcal{O} \in \mathcal{F}_{\mathcal{R}}$  and  $\mathcal{R}_{\mathcal{O}}$  its residual operator such that  $\mathcal{O}$  and  $\mathcal{R}_{\mathcal{O}}$  form a pair of adjoint operators. The operator  $\mathcal{R}_{\mathcal{O}}$  is an implicator if and only if  $\mathcal{O}$  is increasing in its first argument, and satisfies  $\mathcal{O}(0, 1) = 0$  and  $\mathcal{O}(1, y) > 0$  for every  $y \in (0, 1]$ .*

By Corollary 2.7.4, it is clear that a conjunctor  $\mathcal{C}$  is neither sufficient nor necessary to obtain a residual implicator  $\mathcal{I}_{\mathcal{C}}$ , since the condition  $\mathcal{C}(1, 1) = 1$  is not required while the condition  $\mathcal{C}(1, y) > 0$  for every  $y \in (0, 1]$  may not be satisfied. Furthermore, the residual operator  $\mathcal{I}_{\mathcal{C}}$  for a border conjunctor is always a border implicator, since  $\mathcal{C}(1, y) = y$  for every  $y \in [0, 1]$ , which is a necessary condition as well to obtain a border residual implicator  $\mathcal{I}_{\mathcal{C}}$ .

**Theorem 2.7.5.** *Let  $\mathcal{O} \in \mathcal{F}_{\mathcal{R}}$  and  $\mathcal{R}_{\mathcal{O}}$  its residual operator such that  $\mathcal{O}$  and  $\mathcal{R}_{\mathcal{O}}$  form a pair of adjoint operators. The operator  $\mathcal{O}$  satisfies  $\mathcal{O}(1, y) = y$  for every  $y \in [0, 1]$  if and only if its residual operator  $\mathcal{R}_{\mathcal{O}}$  satisfies  $\mathcal{R}_{\mathcal{O}}(1, y) = y$  for every  $y \in [0, 1]$ .*

**Corollary 2.7.6.** *Let  $\mathcal{O} \in \mathcal{F}_{\mathcal{R}}$  and  $\mathcal{R}_{\mathcal{O}}$  its residual operator such that  $\mathcal{O}$  and  $\mathcal{R}_{\mathcal{O}}$  form a pair of adjoint operators. The operator  $\mathcal{R}_{\mathcal{O}}$  is a border implicator if and only if the operator  $\mathcal{O}$  is a border conjunctor.*

It has been mentioned that for a conjunctor  $\mathcal{C}$ , the condition  $\mathcal{C}(1, y) = y$  for every  $y \in [0, 1]$  is also referred to as the neutrality principle. However, the element 0 plays the role of a neutral element for coimplicators and disjunctors. Moreover, notice that conjunctors and implicators, as well as coimplicators and disjunctors or even more generally, the operators in the class  $\mathcal{F}_{\mathcal{R}}$  and their residual operators, always come in isomorphic pairs with respect to a strong negator. The dual versions of Theorems 2.7.3 and 2.7.5 can be readily derived and consequently, the dual version of Corollary 2.7.6 can be obtained for a border coimplicator and its border residual disjunctive.

**Corollary 2.7.7.** *Let  $\mathcal{O} \in \mathcal{F}_{\mathcal{R}}$  and  $\mathcal{R}_{\mathcal{O}}$  its residual operator such that  $\mathcal{O}$  and  $\mathcal{R}_{\mathcal{O}}$  form a pair of adjoint operators. The operator  $\mathcal{R}_{\mathcal{O}}$  is a border disjunctive if and only if the operator  $\mathcal{O}$  is a border coimplicator.*

According to the definition of the class  $\mathcal{F}_{\mathcal{R}}$ , one immediately has the following results for the resolution of fuzzy relational equations formed by an operator  $\mathcal{O}$  in  $\mathcal{F}_{\mathcal{R}}$  or its residual operator  $\mathcal{R}_{\mathcal{O}}$ .

**Theorem 2.7.8.** *Let  $\mathcal{O} \in \mathcal{F}_{\mathcal{R}}$  and  $\mathcal{R}_{\mathcal{O}}$  its residual operator such that  $\mathcal{O}$  and  $\mathcal{R}_{\mathcal{O}}$  form a pair of adjoint operators. The equation  $\mathcal{O}(a, x) = b$  has a solution for given  $a, b \in [0, 1]$  if and only if  $\mathcal{O}(a, \mathcal{R}_{\mathcal{O}}(a, b)) = b$ . On the other hand, the equation  $\mathcal{R}_{\mathcal{O}}(a, x) = b$  has a solution for given  $a, b \in [0, 1]$  if and only if  $\mathcal{R}_{\mathcal{O}}(a, \mathcal{O}(a, b)) = b$ .*

For an operator  $\mathcal{O} \in \mathcal{F}_{\mathcal{R}}$  that is continuous in its second argument, the solution set of the equations  $\mathcal{O}(a, x) = b$ , when it is nonempty, is the closed interval  $[\mathcal{L}_{\mathcal{O}}(a, b), \mathcal{R}_{\mathcal{O}}(a, b)]$ . The situation for the equation  $\mathcal{R}_{\mathcal{O}}(a, x) = b$  is very similar, of which  $\mathcal{I}(a, x) = b$  and  $\mathcal{D}(a, x) = b$  are two special scenarios. When considering the uniqueness of the solution, the right cancelation law and conditional cancelation law can be defined accordingly as in Definitions 2.1.24 and 2.1.29, and similar results can be obtained. Note that there may be no neutral element for a general operator in the class  $\mathcal{F}_{\mathcal{R}}$  as well as its residual operator. Moreover, there is no further classification theorem so far for operators in the class  $\mathcal{F}_{\mathcal{R}}$  which may cast some light on the uniqueness of the solution of the equation  $\mathcal{O}(a, x) = b$ . However, the classification and representation theorems for continuous t-norms and t-conorms can be fully utilized when considering the equations formed by a t-norm/t-conorm or a model implicator/coimplicator.

Operators that do not belong to  $\mathcal{F}_{\mathcal{R}}$  are also used in some applications of fuzzy relational equations, of which compensatory operators are of a particular type. A compensatory operator  $\mathcal{O} : [0, 1]^2 \rightarrow [0, 1]$  is a convex combination of a t-norm  $\mathcal{T}$  and a t-conorm  $\mathcal{S}$ , i.e.,

$$\mathcal{O}(x, y) = \lambda \mathcal{T}(x, y) + (1 - \lambda) \mathcal{S}(x, y), \quad \text{for every } (x, y) \in [0, 1]^2, \quad (2.92)$$

where  $\lambda \in [0, 1]$ . Typically, the t-norm  $\mathcal{T}$  and the t-conorm  $\mathcal{S}$  are selected to be dual to each other with respect to the standard negator  $\mathcal{N}_s$ . In general, the t-norm  $\mathcal{T}$  and the t-conorm  $\mathcal{S}$  can be selected independently. A compensatory operator may be neither an operator belonging to  $\mathcal{F}_{\mathcal{R}}$  nor a residual operator with respect to an operator in  $\mathcal{F}_{\mathcal{R}}$  unless  $\lambda = 1$  or  $\lambda = 0$ , in which cases it degenerates to a t-norm or a t-conorm, respectively. Furthermore, the definition of compensatory operators can be further generalized to convex combinations of a border conjunctive operator  $\mathcal{C}$  and a border disjunctive operator  $\mathcal{D}$ . A well-known compensatory operator is the arithmetic average, which is the average of the *minimum* operator  $\mathcal{T}_M$  and the *maximum* operator  $\mathcal{S}_M$ , i.e.,

$$\mathcal{O}_{av}(x, y) = \frac{1}{2} \min(x, y) + \frac{1}{2} \max(x, y) = \frac{1}{2}(x + y), \quad \text{for every } (x, y) \in [0, 1]^2. \quad (2.93)$$

When considering the equation  $\mathcal{O}(a, x) = b$  where  $\mathcal{O}$  is a compensatory operator with its second partial mapping  $\mathcal{O}(a, \cdot)$  being continuous for every  $a \in [0, 1]$ , its solvability can no longer be detected by checking whether  $\mathcal{R}_{\mathcal{O}}(a, b)$  is a solution or not, although the residual operator  $\mathcal{R}_{\mathcal{O}}$  still can be properly defined. The failure is due to the violation of the boundary

conditions, since  $\mathcal{O}(a, 0) = (1 - \lambda)a = 0$  for every  $a \in [0, 1]$ , or  $\mathcal{O}(a, 1) = \lambda a + (1 - \lambda) = 1$  for every  $a \in [0, 1]$ , cannot be guaranteed except for the degenerate cases. However, the solution set of the equation  $\mathcal{O}(a, x) = b$ , when it is nonempty, is still a closed interval determined by  $[\mathcal{L}_{\mathcal{O}}(a, b), \mathcal{R}_{\mathcal{O}}(a, b)]$  as in the case that  $\mathcal{O} \in \mathcal{F}_{\mathcal{R}}$ . Moreover, due to the continuity and monotonicity properties of the second partial mapping  $\mathcal{O}(a, \cdot)$  for every  $a \in [0, 1]$ , the equation  $\mathcal{O}(a, x) = b$  has a solution for given  $a, b \in [0, 1]$  if and only if  $\mathcal{O}(a, 0) \leq b \leq \mathcal{O}(a, 1)$ , i.e.,  $(1 - \lambda)a \leq b \leq \lambda a + (1 - \lambda)$ .

**Theorem 2.7.9.** *Let  $\mathcal{O} : [0, 1]^2 \rightarrow [0, 1]$  be a binary operator with its second partial mapping  $\mathcal{O}(a, \cdot)$  being increasing and continuous for every  $a \in [0, 1]$ . The equation  $\mathcal{O}(a, x) = b$  has a solution for given  $a, b \in [0, 1]$  if and only if  $\mathcal{O}(a, 0) \leq b \leq \mathcal{O}(a, 1)$ , in which case the solution set is given by the closed interval  $[\mathcal{L}_{\mathcal{O}}(a, b), \mathcal{R}_{\mathcal{O}}(a, b)]$ , where  $\mathcal{L}_{\mathcal{O}}$  and  $\mathcal{R}_{\mathcal{O}}$  are two residual operators associated with the operator  $\mathcal{O}$ .*

When the second partial mapping  $\mathcal{O}(a, \cdot)$  of a binary operator  $\mathcal{O}$  on the unit interval is decreasing and continuous for every  $a \in [0, 1]$ , the equation  $\mathcal{O}(a, x) = b$  can be handled in a similar manner except that the corresponding residual operators should be defined in a dual manner as in Definition 2.6.20. It will be shown in Chapter 4 that the monotonicity and continuity properties of the binary operator  $\mathcal{O}$  in one of its arguments is sufficient to properly define a system of fuzzy relational equations, which can be fully characterized by a generalized matrix.

## 2.8 Bibliographical Notes

In some other conceptual frames, typically in Boolean algebra, the operator *negation* is also called *complement* and is equivalent to the notation of an inverse element.

A short proof of Theorem 2.1.7 can be found in Nguyen and Walker [319]. Furthermore, Theorem 2.1.7 can be immediately generalized to an arbitrary closed subinterval of  $[-\infty, +\infty]$ . See, e.g., Klement *et al.* [243] for details.

The negator  $\mathcal{N} : [0, 1] \rightarrow [0, 1]$  given by  $\mathcal{N}(x) = 1 - x^2$  is strict, but not strong. The

Gödel negator  $\mathcal{N}_G : [0, 1] \rightarrow [0, 1]$  given by

$$\mathcal{N}_G(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{otherwise,} \end{cases}$$

is not strict and, subsequently, not strong. The standard negator  $\mathcal{N}_s$  was used, for instance, by Schweizer and Sklar [406, 407] when introducing t-conorms as duals of t-norms in the framework of probabilistic metric spaces, and by Zadeh [508] when modeling the complement of a fuzzy set.

As the dual operators of implicators, coimplicators are less discussed in the literature. However, the coimplicators also play an important role in classical logic as well as in fuzzy logic. See, for instance, De Baets [78], Oh and Kandel [330, 331], and Wolter [478].

The t-norms and t-conorms were initiated by Menger [299] and have been widely used in the investigations of probabilistic metric spaces, see e.g., Schweizer and Sklar [407, 409]. They were introduced in the framework of multivalued logic and fuzzy logic for the first time by Alsina *et al.* [6] and Prade [380] and considered as suitable candidates for conjunctions and disjunctions, respectively. De Cooman and Kerre [85] also pointed out that the t-norms and t-conorms can be defined on bounded lattices. The reader may refer to the monograph [243] by Klement, Mesiar and Pap, as well as the position papers [244, 245, 246] by the same authors, and Alsina *et al.* [5] for a detailed and rather complete overview of triangular norms. An up-to-date overview of triangular norms and related operators, as well as their applications, can be found in the book [241] edited by Klement and Mesiar.

It should be noted that in Schweizer and Sklar [409], the t-norms  $\mathcal{T}_M$ ,  $\mathcal{T}_P$ ,  $\mathcal{T}_L$  and  $\mathcal{T}_D$  were denoted as  $M$ ,  $II$ ,  $W$  and  $Z$ , respectively, while the t-conorms  $\mathcal{S}_M$ ,  $\mathcal{S}_P$ ,  $\mathcal{S}_L$  and  $\mathcal{S}_D$  were denoted as  $M^*$ ,  $II^*$ ,  $W^*$  and  $Z^*$ , respectively.

Ling [278] showed that any continuous Archimedean t-norm  $\mathcal{T}$  admits the representation

$$\mathcal{T}(x, y) = t^{(-1)}(t(x) + t(y))$$

for every  $(x, y) \in [0, 1]^2$  where  $t : [0, 1] \rightarrow [0, +\infty]$  is continuous and strictly decreasing with  $t(1) = 0$ , and  $t^{(-1)}$  is its pseudoinverse. Following the results of Mostert and

Shields [312], Ling [278] also showed that each continuous t-norm is an ordinal sum of continuous Archimedean t-norms and thereby provided a complete characterization of continuous t-norms.

In the lattice theoretic framework, the definition of a residual implicator is associated with the so called commutative integral residuated  $l$ -monoid, see e.g., Gierz *et al.* [158]. In the literature of fuzzy relational equations, the residual implicators are also known as  $\varphi$ -operators which play an important role in the resolution of fuzzy relational equations. The  $\varphi$ -operators were introduced in Pedrycz [341, 347] by a different approach to describe the solutions of sup- $\mathcal{T}$  equations. They are essentially the generalization of the  $\alpha$ -operation defined by Sanchez [400, 401], which is a special  $\varphi$ -operator with respect to the Gödel t-norm  $\mathcal{T}_M$ . The connection between a  $\varphi$ -operator and its corresponding t-norm has been characterized in full generality by Gottwald [171, 173]. The residual operator  $\mathcal{J}_{\mathcal{T}}$  was also discussed in Di Nola *et al.* [116] with a slightly different definition. In the context of fuzzy logic, the residual implicator  $\mathcal{I}_{\mathcal{T}}(x, y)$  is usually denoted as  $x \rightarrow_t y$ . For more details about the residual implicators and coimplicators, the reader may refer to De Baets [78], Demirli and De Baets [86], Gottwald [174, 177, 178], and Klement *et al.* [243].

Dubois and Prade [127] and Fodor [138, 139] discussed some methods to generate implicators from general conjunctors. More discussion on generalized implicators and their applications can be found in Oh and Bandler [329], De Baets [75, 78], Demirli and De Baets [86], Wang [452], and Li *et al.* [276, 277].

Methods to construct t-norms usually originate from semigroup theory. See, e.g., Clifford [57], Schweizer and Skar [408], Ling [278], Frank [142], Krause [257], Drossos and Navara [123], and Jenei [217, 218].

The restriction of additive generators of t-norms can be further released. So far, it is still an open question to characterize the class of all monotone functions  $f : [0, 1] \rightarrow [0, +\infty]$  such that  $\mathcal{T}(x, y) = f^{(-1)}(f(x) + f(y))$  defines a t-norm. See, e.g., Vicenik [441, 442, 443].

Note that it was sometimes erroneously claimed that the *minimum*  $\mathcal{T}_M$  is the only continuous t-norm which is not Archimedean. See, e.g., Stamou and Tzafestas [420] and Shieh [414].

The solution set of the equation  $\mathcal{T}(a, x) = b$  with a continuous t-norm  $\mathcal{T}$  was characterized by Klement *et al.* [243] under the name of the preimage.

## Chapter 3

# Classification and Resolution of Fuzzy Relational Equations

Based on Li and Fang [270] and references therein, this chapter classifies fuzzy relational equations into different categories according to the composite operations. We also discuss the solvability and structure of the solution set of fuzzy relational equations. Although most results are available in the literature, they are extended to the most general situations and presented in a unified framework in this chapter.

### 3.1 Classification of Fuzzy Relational Equations

According to the type of composite operations, fuzzy relational equations can be roughly characterized into two basic categories, namely,  $\text{sup-}\mathcal{C}/\text{inf-}\mathcal{D}$  equations and  $\text{inf-}\mathcal{I}/\text{sup-}\mathcal{J}$  equations. More general but less encountered forms of fuzzy relational equations include  $\text{sup-}\mathcal{D}/\text{inf-}\mathcal{C}$  equations and  $\text{sup-}\mathcal{O}/\text{inf-}\mathcal{O}$  equations, where  $\mathcal{O} : [0, 1]^2 \rightarrow [0, 1]$  is some specifically defined binary operator, say, for instance, a compensatory operator.

#### 3.1.1 $\text{Sup-}\mathcal{C}$ Equations and $\text{Inf-}\mathcal{D}$ Equations

Fuzzy relational equations with  $\text{sup-}\mathcal{C}$  composition are associated with a border conjunctive  $\mathcal{C}$  and of the form

$$\sup_{j \in N} \mathcal{C}(a_{ij}, x_j) = b_i, \quad \text{for every } i \in M, \quad (3.1)$$

or equivalently,

$$\bigvee_{j \in N} (a_{ij} \wedge_c x_j) = b_i, \quad \text{for every } i \in M. \quad (3.2)$$

Fuzzy relational equations with  $\text{inf-}\mathcal{D}$  composition are associated with a border disjunctive  $\mathcal{D}$  and of the form

$$\text{inf}_{j \in N} \mathcal{D}(a_{ij}, x_j) = b_i, \quad \text{for every } i \in M, \quad (3.3)$$

or equivalently,

$$\bigwedge_{j \in N} (a_{ij} \vee_d x_j) = b_i, \quad \text{for every } i \in M. \quad (3.4)$$

Within a De Morgan triple  $(\mathcal{C}, \mathcal{D}, \mathcal{N})$ , any system of  $\text{sup-}\mathcal{C}$  equations defines a dual system of  $\text{inf-}\mathcal{D}$  equations and vice versa.

A system of  $\text{sup-}\mathcal{C}/\text{inf-}\mathcal{D}$  equations of the above form can be written in its matrix form as  $A \circ \mathbf{x} = \mathbf{b}$  where “ $\circ$ ” stands for  $\text{sup-}\mathcal{C}/\text{inf-}\mathcal{D}$  composition and is clear in the context. A system of  $\text{sup-}\mathcal{C}/\text{inf-}\mathcal{D}$  equations of the form  $\mathbf{x}^T \circ A^T = \mathbf{b}^T$  can be converted into the form of  $A \circ \mathbf{x} = \mathbf{b}$  by necessarily modifying the involved conjunctive  $\mathcal{C}$  or disjunctive  $\mathcal{D}$  when it is not commutative. Furthermore, a system of  $\text{sup-}\mathcal{C}$  matrix equations  $A \circ X = B$  with  $A \in [0, 1]^{m \times n}$ ,  $X \in [0, 1]^{m \times p}$  and  $B \in [0, 1]^{m \times p}$  can be either reduced to  $p$  separate systems of  $\text{sup-}\mathcal{C}$  equations or reformulated into one system of  $\text{sup-}\mathcal{C}$  equations as

$$\begin{pmatrix} A & & 0 \\ & \ddots & \\ 0 & & A \end{pmatrix} \circ \begin{pmatrix} X(1) \\ \vdots \\ X(p) \end{pmatrix} = \begin{pmatrix} B(1) \\ \vdots \\ B(p) \end{pmatrix}, \quad (3.5)$$

where  $X(k)$  and  $B(k)$ ,  $k = 1, 2, \dots, p$ , are the  $k$ th column of  $X$  and  $B$ , respectively. Similarly, a system of  $\text{inf-}\mathcal{D}$  matrix equations  $A \circ X = B$  can be either decomposed into  $p$  separate systems of  $\text{inf-}\mathcal{D}$  equations or reformulated as

$$\begin{pmatrix} A & & 1 \\ & \ddots & \\ 1 & & A \end{pmatrix} \circ \begin{pmatrix} X(1) \\ \vdots \\ X(p) \end{pmatrix} = \begin{pmatrix} B(1) \\ \vdots \\ B(p) \end{pmatrix}. \quad (3.6)$$

### 3.1.2 Inf- $\mathcal{I}$ Equations and Sup- $\mathcal{J}$ Equations

Fuzzy relational equations with inf- $\mathcal{I}$  composition are associated with a border implicator  $\mathcal{I}$  and of the form

$$\inf_{j \in N} \mathcal{I}(a_{ij}, x_j) = b_i, \quad \text{for every } i \in M, \quad (3.7)$$

or equivalently,

$$\bigwedge_{j \in N} (a_{ij} \varphi x_j) = b_i, \quad \text{for every } i \in M. \quad (3.8)$$

Since an implicator  $\mathcal{I}$  is not commutative, fuzzy relational equations of the form

$$\inf_{j \in N} \mathcal{I}(x_j, a_{ij}) = b_i, \quad \text{for every } i \in M, \quad (3.9)$$

represent a different type of inf- $\mathcal{I}$  equations. They are referred to as right and left inf- $\mathcal{I}$  equations, respectively, when a distinction is necessary.

Similarly, right sup- $\mathcal{J}$  equations, associated with a border coimplicator  $\mathcal{J}$ , are of the form

$$\sup_{j \in N} \mathcal{J}(a_{ij}, x_j) = b_i, \quad \text{for every } i \in M, \quad (3.10)$$

while left sup- $\mathcal{J}$  equations are of the form

$$\sup_{j \in N} \mathcal{J}(x_j, a_{ij}) = b_i, \quad \text{for every } i \in M. \quad (3.11)$$

In the infix notations, they are denoted, respectively, as

$$\bigvee_{j \in N} (a_{ij} \beta x_j) = b_i, \quad \text{for every } i \in M, \quad (3.12)$$

and

$$\bigvee_{j \in N} (x_j \beta a_{ij}) = b_i, \quad \text{for every } i \in M. \quad (3.13)$$

Within a De Morgan triple  $(\mathcal{I}, \mathcal{J}, \mathcal{N})$ , any system of inf- $\mathcal{I}$  equations defines a dual system of sup- $\mathcal{J}$  equations and vice versa.

Since implicators and coimplicators are decreasing in their first arguments while increasing in their second arguments, right and left inf- $\mathcal{I}$  equations, as well as right and left sup- $\mathcal{J}$  equations, should be treated separately. Besides, although inf- $\mathcal{I}$ /sup- $\mathcal{J}$  equations can be converted into sup- $\mathcal{C}$ /inf- $\mathcal{D}$  equations, they can always be handled independently.

Furthermore, it is clear that a system of right inf- $\mathcal{I}$  matrix equations  $A \circ X = B$  with  $A \in [0, 1]^{m \times n}$ ,  $X \in [0, 1]^{m \times p}$  and  $B \in [0, 1]^{m \times p}$  can be either reduced to  $p$  separate systems of right inf- $\mathcal{I}$  equations or reformulated into one system of right inf- $\mathcal{I}$  equations as

$$\begin{pmatrix} A & & 0 \\ & \ddots & \\ 0 & & A \end{pmatrix} \circ \begin{pmatrix} X(1) \\ \vdots \\ X(p) \end{pmatrix} = \begin{pmatrix} B(1) \\ \vdots \\ B(p) \end{pmatrix}, \quad (3.14)$$

where  $X(k)$  and  $B(k)$ ,  $k = 1, 2, \dots, p$ , are the  $k$ th column of  $X$  and  $B$ , respectively. Similarly, a system of left inf- $\mathcal{I}$  matrix equations  $X^T \circ A^T = B^T$  can be either decomposed into  $p$  separate systems of left inf- $\mathcal{I}$  equations or reformulated as

$$\begin{pmatrix} X(1) \\ \vdots \\ X(p) \end{pmatrix}^T \circ \begin{pmatrix} A^T & & 1 \\ & \ddots & \\ 1 & & A^T \end{pmatrix} = \begin{pmatrix} B(1) \\ \vdots \\ B(p) \end{pmatrix}^T. \quad (3.15)$$

Moreover, a system of right sup- $\mathcal{J}$  matrix equations  $A \circ X = B$  can be either decomposed into  $p$  separate systems of sup- $\mathcal{J}$  equations or reformulated as

$$\begin{pmatrix} A & & 1 \\ & \ddots & \\ 1 & & A \end{pmatrix} \circ \begin{pmatrix} X(1) \\ \vdots \\ X(p) \end{pmatrix} = \begin{pmatrix} B(1) \\ \vdots \\ B(p) \end{pmatrix}. \quad (3.16)$$

A system of left sup- $\mathcal{J}$  matrix equations  $X^T \circ A^T = B^T$  can be either decomposed into  $p$  separate systems of left sup- $\mathcal{J}$  equations or reformulated as

$$\begin{pmatrix} X(1) \\ \vdots \\ X(p) \end{pmatrix}^T \circ \begin{pmatrix} A^T & & 0 \\ & \ddots & \\ 0 & & A^T \end{pmatrix} = \begin{pmatrix} B(1) \\ \vdots \\ B(p) \end{pmatrix}^T. \quad (3.17)$$

### 3.1.3 Fuzzy Relational Equations in General Forms

In general, a system of fuzzy relational equations  $A \circ \mathbf{x} = \mathbf{b}$  with sup- $\mathcal{O}$  or inf- $\mathcal{O}$  composition can be well defined with respect to a binary operator  $\mathcal{O} : [0, 1]^2 \rightarrow [0, 1]$ . A system of right sup- $\mathcal{O}$  equations is of the form

$$\sup_{j \in N} \mathcal{O}(a_{ij}, x_j) = b_i, \quad \text{for every } i \in M, \quad (3.18)$$

while a system of right inf- $\mathcal{O}$  equations is of the form

$$\inf_{j \in N} \mathcal{O}(a_{ij}, x_j) = b_i, \quad \text{for every } i \in M. \quad (3.19)$$

Typically, the composite operator  $\mathcal{O}$  is required to be monotone in its second argument. Sup- $\mathcal{D}$ /inf- $\mathcal{C}$  equations, as well as sup- $\mathcal{I}$ /inf- $\mathcal{J}$  equations, are of such general types of fuzzy relational equations.

It should be noticed that a system of sup- $\mathcal{D}$ /inf- $\mathcal{C}$  matrix equations  $A \circ X = B$  cannot be reformulated into one system of sup- $\mathcal{D}$ /inf- $\mathcal{C}$  equations since there is no proper neutrality principle with respect to sup- $\mathcal{D}$  and inf- $\mathcal{C}$  composite operations. This observation holds for sup- $\mathcal{I}$ /inf- $\mathcal{J}$  matrix equations as well as for sup- $\mathcal{O}$ /inf- $\mathcal{O}$  matrix equations with  $\mathcal{O}$  being a compensatory operator.

Generally, fuzzy relational equations can be constructed with a more complicated composite operation, for instance,  $\mathcal{S}$ - $\mathcal{T}$  or  $\mathcal{T}$ - $\mathcal{S}$  composition, with  $\mathcal{T}$  and  $\mathcal{S}$  being a t-norm and t-conorm, respectively. However, the resolution of fuzzy relational equations of such general types still remains wide open, and are beyond the scope of this dissertation.

## 3.2 Solvability and Solution Sets

Fuzzy relational equations, formulated with different composite operations, share some common features in the solvability criteria and the structures of their solution sets. It suffices to consider fuzzy relational equations of the form  $A \circ \mathbf{x} = \mathbf{b}$ , although fuzzy relational equations of the form  $\mathbf{x}^T \circ A^T = \mathbf{b}^T$  should be discussed as well in a similar manner when an implicator or a coimplicator is involved in the composition.

Given a system of sup- $\mathcal{O}$ /inf- $\mathcal{O}$  equations  $A \circ \mathbf{x} = \mathbf{b}$ , with  $\mathcal{O} : [0, 1]^2 \rightarrow [0, 1]$  being a binary operator and monotone in its second argument, its solution set is denoted by  $S(A, \mathbf{b}) = \{\mathbf{x} \in [0, 1]^n \mid A \circ \mathbf{x} = \mathbf{b}\}$ . The system  $A \circ \mathbf{x} = \mathbf{b}$  is called consistent if  $S(A, \mathbf{b}) \neq \emptyset$ . Otherwise, it is inconsistent. The solution set  $S(A, \mathbf{b})$  is order convex, i.e., any vector  $\mathbf{x} \in [0, 1]^n$  with  $\mathbf{x}^1 \leq \mathbf{x} \leq \mathbf{x}^2$  is also in  $S(A, \mathbf{b})$  whenever  $\mathbf{x}^1, \mathbf{x}^2 \in S(A, \mathbf{b})$ . Consequently, the attention can be focused on the extremal solutions as defined below:

**Definition 3.2.1.** *Let  $A \circ \mathbf{x} = \mathbf{b}$  be a system of fuzzy relational equations with sup- $\mathcal{O}$  or inf- $\mathcal{O}$  composition where  $\mathcal{O} : [0, 1]^2 \rightarrow [0, 1]$  is monotone in its second argument. A solution  $\check{\mathbf{x}} \in S(A, \mathbf{b})$  is called a minimal or lower solution if for any  $\mathbf{x} \in S(A, \mathbf{b})$ , the relation  $\mathbf{x} \leq \check{\mathbf{x}}$*

implies  $\mathbf{x} = \tilde{\mathbf{x}}$ , while a solution  $\tilde{\mathbf{x}} \in S(A, \mathbf{b})$  is called a minimum solution if  $\tilde{\mathbf{x}} \leq \mathbf{x}$  for all  $\mathbf{x} \in S(A, \mathbf{b})$ . Dually, a solution  $\hat{\mathbf{x}} \in S(A, \mathbf{b})$  is called a maximal solution if for any  $\mathbf{x} \in S(A, \mathbf{b})$ , the relation  $\hat{\mathbf{x}} \leq \mathbf{x}$  implies  $\mathbf{x} = \hat{\mathbf{x}}$ , while a solution  $\hat{\mathbf{x}} \in S(A, \mathbf{b})$  is called a maximum solution if  $\mathbf{x} \leq \hat{\mathbf{x}}$  for all  $\mathbf{x} \in S(A, \mathbf{b})$ .

With these notions, maximal interval solutions can be defined for  $A \circ \mathbf{x} = \mathbf{b}$  which indicate how far each component of the solution can be expanded when the input  $A$  and the output  $\mathbf{b}$  are fixed.

**Definition 3.2.2.** A generalized vector  $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)^T$ , with  $\tilde{x}_j = [\underline{x}_j, \bar{x}_j]$  for every  $j \in N$ , is called an interval solution to  $A \circ \mathbf{x} = \mathbf{b}$  if any vector  $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ , such that  $x_j \in [\underline{x}_j, \bar{x}_j]$  for every  $j \in N$ , is a solution to  $A \circ \mathbf{x} = \mathbf{b}$ . An interval solution  $\tilde{\mathbf{x}}$  is called a maximal interval solution if its components are determined by a minimal solution from the left and by a maximal solution from the right.

**Theorem 3.2.3.** Let  $A \circ \mathbf{x} = \mathbf{b}$  be a system of sup- $\mathcal{O}$  equations where  $\mathcal{O} : [0, 1]^2 \rightarrow [0, 1]$  is monotone in its second argument. A vector  $\mathbf{x} \in [0, 1]^n$  is a solution to  $A \circ \mathbf{x} = \mathbf{b}$  if and only if  $\mathcal{O}(a_{ij}, x_j) \leq b_i$  for every  $i \in M$  and  $j \in N$ , and there exists an index  $j_i \in N$  for each  $i \in M$  such that  $\mathcal{O}(a_{ij_i}, x_{j_i}) = b_i$ .

*Proof.* The system of sup- $\mathcal{O}$  equations  $A \circ \mathbf{x} = \mathbf{b}$  can be written as

$$\sup_{j \in N} \mathcal{O}(a_{ij}, x_j) \leq b_i, \quad \text{and} \quad \sup_{j \in N} \mathcal{O}(a_{ij}, x_j) \geq b_i, \quad \text{for every } i \in M. \quad (3.20)$$

The inequalities

$$\sup_{j \in N} \mathcal{O}(a_{ij}, x_j) \leq b_i, \quad \text{for every } i \in M, \quad (3.21)$$

hold if and only if  $\mathcal{O}(a_{ij}, x_j) \leq b_i$  for every  $i \in M$  and  $j \in N$ , while the inequalities

$$\sup_{j \in N} \mathcal{O}(a_{ij}, x_j) \geq b_i, \quad \text{for every } i \in M, \quad (3.22)$$

hold if and only if there exists  $j_i \in N$  for each  $i \in M$  such that  $\mathcal{O}(a_{ij_i}, x_{j_i}) \geq b_i$ , and hence,  $\mathcal{O}(a_{ij_i}, x_{j_i}) = b_i$ .  $\square$

Theorem 3.2.3 holds in a dual manner for fuzzy relational equations with inf- $\mathcal{O}$  composition.

**Theorem 3.2.4.** *Let  $A \circ \mathbf{x} = \mathbf{b}$  be a system of inf- $\mathcal{O}$  equations where  $\mathcal{O} : [0, 1]^2 \rightarrow [0, 1]$  is monotone in its second argument. A vector  $\mathbf{x} \in [0, 1]^n$  is a solution to  $A \circ \mathbf{x} = \mathbf{b}$  if and only if  $\mathcal{O}(a_{ij}, x_j) \geq b_i$  for every  $i \in M$  and  $j \in N$ , and there exists an index  $j_i \in N$  for each  $i \in M$  such that  $\mathcal{O}(a_{ij_i}, x_{j_i}) = b_i$ .*

*Proof.* The system of inf- $\mathcal{O}$  equations  $A \circ \mathbf{x} = \mathbf{b}$  can be written as

$$\inf_{j \in N} \mathcal{O}(a_{ij}, x_j) \geq b_i, \quad \text{and} \quad \inf_{j \in N} \mathcal{O}(a_{ij}, x_j) \leq b_i, \quad \text{for every } i \in M. \quad (3.23)$$

The inequalities

$$\inf_{j \in N} \mathcal{O}(a_{ij}, x_j) \geq b_i, \quad \text{for every } i \in M, \quad (3.24)$$

hold if and only if  $\mathcal{O}(a_{ij}, x_j) \geq b_i$  for every  $i \in M$  and  $j \in N$ , while the inequalities

$$\inf_{j \in N} \mathcal{O}(a_{ij}, x_j) \leq b_i, \quad \text{for every } i \in M, \quad (3.25)$$

hold if and only if there exists  $j_i \in N$  for each  $i \in M$  such that  $\mathcal{O}(a_{ij_i}, x_{j_i}) \leq b_i$ , and hence,  $\mathcal{O}(a_{ij_i}, x_{j_i}) = b_i$ .  $\square$

According to Theorems 3.2.3 and 3.2.4, the resolution of a system of sup- $\mathcal{O}$ /inf- $\mathcal{O}$  equations essentially relies on the resolution of elementary fuzzy relational equations. As long as the solution sets of those involved elementary equations can be well characterized, the solution set of the system of sup- $\mathcal{O}$ /inf- $\mathcal{O}$  equations can be well characterized.

### 3.2.1 Solvability of Sup- $\mathcal{C}$ /Inf- $\mathcal{D}$ Equations

Let  $A \circ \mathbf{x} = \mathbf{b}$  be a system of sup- $\mathcal{C}$  equations where  $\mathcal{C}$  is a border conjunctive and left-continuous in its second argument. According to Theorem 3.2.3,

$$b_i \leq \sup_{j \in N} \mathcal{C}(a_{ij}, 1), \quad \text{for every } i \in M, \quad (3.26)$$

is a necessary condition for the existence of a solution to  $A \circ \mathbf{x} = \mathbf{b}$ . It becomes operator independent when the involved border conjunctive  $\mathcal{C}$  is a t-seminorm. On the other hand, it is generally not a sufficient condition unless  $\mathcal{C}$  is continuous in its second argument and the system  $A \circ \mathbf{x} = \mathbf{b}$  involves only one equation. However, Theorems 2.1.16 and 2.6.1 imply a direct way to solve the system of sup- $\mathcal{C}$  equations  $A \circ \mathbf{x} = \mathbf{b}$ , which has been well-known since the very beginning of the investigation of fuzzy relational equations.

**Theorem 3.2.5.** *Let  $A \circ \mathbf{x} = \mathbf{b}$  be a system of sup- $\mathcal{C}$  equations where  $\mathcal{C}$  is a border conjunctor and left-continuous in its second argument. The system is consistent if and only if the vector  $A^T \varphi_{\mathcal{C}} \mathbf{b}$  with its components defined by*

$$(A^T \varphi_{\mathcal{C}} \mathbf{b})_j = \inf_{i \in M} \mathcal{I}_{\mathcal{C}}(a_{ij}, b_i) = \bigwedge_{i \in M} (a_{ij} \varphi_{\mathcal{C}} b_i), \quad \text{for every } j \in N, \quad (3.27)$$

*is a solution to  $A \circ \mathbf{x} = \mathbf{b}$ . Moreover, if the system is consistent, then  $A^T \varphi_{\mathcal{C}} \mathbf{b}$  is the maximum solution in the solution set  $S(A, \mathbf{b})$ .*

By Theorem 3.2.5, the consistency of a system of sup- $\mathcal{C}$  equations  $A \circ \mathbf{x} = \mathbf{b}$  can be verified by constructing and checking a potential maximum solution in a time complexity of  $O(mn)$ . In the context of fuzzy logic, the transposed matrix  $A^T$  is also denoted by  $A^{-1}$  as the inverse relation of  $A$ , and hence, the potential maximum solution  $A^T \varphi_{\mathcal{C}} \mathbf{b}$ , also known as the principle solution, is denoted by  $A^{-1} \varphi_{\mathcal{C}} \mathbf{b}$  as well.

Furthermore, if the border conjunctor  $\mathcal{C}$  is also right-continuous in its second argument, and hence, continuous in its second argument, it follows from Theorems 2.6.1 and 3.2.3 that the solution set of each single equation in a consistent system  $A \circ \mathbf{x} = \mathbf{b}$  is determined a maximum solution  $(a_{i1} \varphi_{\mathcal{C}} b_i, a_{i2} \varphi_{\mathcal{C}} b_i, \dots, a_{in} \varphi_{\mathcal{C}} b_i)^T$  and a finite number of minimal solutions  $\check{S}_i$ , where

$$\check{S}_i = \{\check{\mathbf{x}}^k \mid b_i \leq \mathcal{C}(a_{ik}, 1), k \in N\}, \quad \text{for every } i \in M, \quad (3.28)$$

with the vector  $\check{\mathbf{x}}^k \in \check{S}_i$  being defined as

$$\check{x}_j^k = \begin{cases} \mathcal{I}_{\mathcal{C}}(a_{ik}, b_i), & \text{if } j = k, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for every } j \in N. \quad (3.29)$$

This particular structure is called a finitely generated root system by De Baets [74, 80]. The solution set  $S(A, \mathbf{b})$  of the system  $A \circ \mathbf{x} = \mathbf{b}$  is therefore the intersection of these root systems, and remains to be a finitely generated root system. In other words, the solution set  $S(A, \mathbf{b})$ , when it is nonempty, can be well characterized and determined by a unique maximum solution and a finite number of minimal solutions, i.e.,

$$S(A, \mathbf{b}) = \bigcup_{\check{\mathbf{x}} \in \check{S}(A, \mathbf{b})} \{\mathbf{x} \in [0, 1]^n \mid \check{\mathbf{x}} \leq \mathbf{x} \leq \hat{\mathbf{x}}\}, \quad (3.30)$$

where  $\check{S}(A, \mathbf{b})$  is the set of all minimal solutions of  $A \circ \mathbf{x} = \mathbf{b}$  and  $\hat{\mathbf{x}} = A^T \varphi_{\mathcal{C}} \mathbf{b}$  is the maximum solution.

However, the minimal solution of a single equation may not necessarily be a minimal solution of the system, and hence, the detection of the set  $\check{S}(A, \mathbf{b})$  becomes a rather complicated but very interesting issue for investigation, although it is sufficient to know the maximum solution of  $A \circ \mathbf{x} = \mathbf{b}$  in most practical considerations. As indicated by De Baets [74], any algorithm designed in this manner for determining the minimal solutions can always be applied independent of the conjunctor involved. Of course, this does not mean that none of the algorithms could be more efficient for some specific cases. Actually, it will be shown in Chapter 4 that sup- $\mathcal{C}$  equations with a border conjunctor  $\mathcal{C}$  satisfying the right conditional cancelation law are somehow easier to deal with, of which continuous Archimedean t-norms are a special subclass. However, the exact computational complexity of obtaining all minimal solutions of a consistent system of sup- $\mathcal{C}$  equations remains an open problem.

A system of sup- $\mathcal{C}$  equations  $A \circ \mathbf{x} = \mathbf{b}$ , with the border conjunctor  $\mathcal{C}$  being continuous in its second argument, is called homogeneous if  $\mathbf{b} = (0, 0, \dots, 0)^T$ . Otherwise it is called non-homogeneous. Solving a homogeneous system of sup- $\mathcal{C}$  equations is trivial. It has a unique minimal solution  $\check{\mathbf{x}} = (0, 0, \dots, 0)^T$  and a maximum solution  $\hat{\mathbf{x}}$  with  $\hat{x}_j = \inf_{i \in M} \mathcal{I}_{\mathcal{C}}(a_{ij}, 0)$  for every  $j \in N$ .

The solvability of a system of inf- $\mathcal{D}$  equations and the characterization of its solution set can be readily established when  $\mathcal{D}$  is a border disjunctive. Since inf- $\mathcal{D}$  equations can be viewed as the dual of sup- $\mathcal{C}$  equations, any proposition, which is true for sup- $\mathcal{C}$  equations, is also valid in its dual form for inf- $\mathcal{D}$  equations.

Let  $A \circ \mathbf{x} = \mathbf{b}$  be a system of inf- $\mathcal{D}$  equations where  $\mathcal{D}$  is a border disjunctive and right-continuous in its second argument. According to Theorem 3.2.4,

$$b_i \geq \inf_{j \in N} \mathcal{D}(a_{ij}, 0), \quad \text{for every } i \in M, \quad (3.31)$$

is a necessary condition for the existence of a solution to  $A \circ \mathbf{x} = \mathbf{b}$ . It becomes operator independent when the involved border disjunctive  $\mathcal{D}$  is a t-semiconorm. Moreover, by Theorems 2.1.18 and 2.6.6, the consistency of a system of inf- $\mathcal{D}$  equations  $A \circ \mathbf{x} = \mathbf{b}$  can be verified by constructing and checking a potential minimum solution in a time complexity of  $O(mn)$ .

**Theorem 3.2.6.** *Let  $A \circ \mathbf{x} = \mathbf{b}$  be a system of inf- $\mathcal{D}$  equations where  $\mathcal{D}$  is a border disjunctive and right-continuous in its second argument. The system is consistent if and only if the*

vector  $A^T \beta_d \mathbf{b}$  with its components defined by

$$(A^T \beta_d \mathbf{b})_j = \sup_{i \in M} \mathcal{J}_{\mathcal{D}}(a_{ij}, b_i) = \bigvee_{i \in M} (a_{ij} \beta_d b_i), \quad \text{for every } j \in N, \quad (3.32)$$

is a solution to  $A \circ \mathbf{x} = \mathbf{b}$ . Moreover, if the system is consistent, then  $A^T \beta_d \mathbf{b}$  is the minimum solution in the solution set  $S(A, \mathbf{b})$ .

Furthermore, if the border disjunctive  $\mathcal{D}$  is also left-continuous in its second argument, and hence, continuous in its second argument, then the solution set  $S(A, \mathbf{b})$ , when it is nonempty, can be well characterized and determined by a unique minimum solution and a finite number of maximal solutions, i.e.,

$$S(A, \mathbf{b}) = \bigcup_{\hat{\mathbf{x}} \in \hat{S}(A, \mathbf{b})} \{ \mathbf{x} \in [0, 1]^n \mid \check{\mathbf{x}} \leq \mathbf{x} \leq \hat{\mathbf{x}} \}, \quad (3.33)$$

where  $\hat{S}(A, \mathbf{b})$  is the set of all maximal solutions of  $A \circ \mathbf{x} = \mathbf{b}$  and  $\check{\mathbf{x}} = A^T \beta_d \mathbf{b}$  is the minimum solution. This particular structure is called a finitely generated crown system by De Baets [74, 80].

A system of inf- $\mathcal{D}$  equations  $A \circ \mathbf{x} = \mathbf{b}$ , with the border disjunctive  $\mathcal{D}$  being continuous in its second argument, is called homogeneous if  $\mathbf{b} = (1, 1, \dots, 1)^T$ . Otherwise it is called non-homogeneous. Solving a homogeneous system of inf- $\mathcal{D}$  equations is trivial. It has a unique maximal solution  $\hat{\mathbf{x}} = (1, 1, \dots, 1)^T$  and a minimum solution  $\check{\mathbf{x}}$  with  $\check{x}_j = \sup_{i \in M} \mathcal{J}_{\mathcal{D}}(a_{ij}, 1)$  for every  $j \in N$ .

### 3.2.2 Solvability of Inf- $\mathcal{I}$ /Sup- $\mathcal{J}$ Equations

Systems of right and left inf- $\mathcal{I}$ /sup- $\mathcal{J}$  equations can be dealt with in their own frameworks of residuation although they can be converted into systems of sup- $\mathcal{C}$ /inf- $\mathcal{D}$  equations, respectively.

Due to the monotonicity property of the second partial mappings of an implicator  $\mathcal{I}$ ,

$$b_i \geq \inf_{j \in N} \mathcal{I}(a_{ij}, 0) = \inf_{j \in N} \mathcal{N}_{\mathcal{I}}(a_{ij}), \quad \text{for every } i \in M, \quad (3.34)$$

is always a necessary condition for the existence of a solution to a system of right inf- $\mathcal{I}$  equations  $A \circ \mathbf{x} = \mathbf{b}$  with  $\mathcal{I}$  being a border implicator. Moreover, if  $\mathcal{I}$  is right-continuous in its second argument, by Theorems 2.1.16 and 2.6.11, the consistency of a system of right

inf- $\mathcal{I}$  equations can be verified by constructing and checking a potential minimum solution in a time complexity of  $O(mn)$ .

**Theorem 3.2.7.** *Let  $A \circ \mathbf{x} = \mathbf{b}$  be a system of right inf- $\mathcal{I}$  equations where  $\mathcal{I}$  is a border implicator and right-continuous in its second argument. The system is consistent if and only if the vector  $A^T \gamma_I \mathbf{b}$  with its components defined by*

$$(A^T \gamma_I \mathbf{b})_j = \sup_{i \in M} \mathcal{C}_{\mathcal{I}}(a_{ij}, b_i) = \bigvee_{i \in M} (a_{ij} \gamma_I b_i), \quad \text{for every } j \in N, \quad (3.35)$$

*is a solution to  $A \circ \mathbf{x} = \mathbf{b}$ . Moreover, if the system is consistent, then  $A^T \gamma_I \mathbf{b}$  is the minimum solution in the solution set  $S(A, \mathbf{b})$ .*

Furthermore, if the border implicator  $\mathcal{I}$  is also left-continuous in its second argument, and hence, continuous in its second argument, the solution set of each single equation in a consistent system of right inf- $\mathcal{I}$  equations  $A \circ \mathbf{x} = \mathbf{b}$  is a finitely generated crown system with the maximal solutions given by the set

$$\hat{S}_i = \{\hat{\mathbf{x}}^k \mid \mathcal{N}_{\mathcal{I}}(a_{ik}) \leq b_i, k \in N\}, \quad \text{for every } i \in M, \quad (3.36)$$

where the vector  $\hat{\mathbf{x}}^k \in \hat{S}_i$  is defined by

$$\hat{x}_j^k = \begin{cases} \mathcal{D}_{\mathcal{I}}(a_{ik}, b_i), & \text{if } j = k, \\ 1, & \text{otherwise,} \end{cases} \quad \text{for every } j \in N. \quad (3.37)$$

The solution set  $S(A, \mathbf{b})$  of the system  $A \circ \mathbf{x} = \mathbf{b}$  is therefore the intersection of these crown systems which remains as a finitely generated crown system. In other words, the solution set  $S(A, \mathbf{b})$ , when it is nonempty, can be well characterized and determined by a unique minimum solution and a finite number of maximal solutions, i.e.,

$$S(A, \mathbf{b}) = \bigcup_{\hat{\mathbf{x}} \in \hat{S}(A, \mathbf{b})} \{\mathbf{x} \in [0, 1]^n \mid \check{\mathbf{x}} \leq \mathbf{x} \leq \hat{\mathbf{x}}\}, \quad (3.38)$$

where  $\hat{S}(A, \mathbf{b})$  is the set of all maximal solutions of  $A \circ \mathbf{x} = \mathbf{b}$  and  $\check{\mathbf{x}} = A^T \gamma_I \mathbf{b}$  is the minimum solution.

The solvability of a system of right sup- $\mathcal{J}$  equations and the characterization of its solution set can be readily established when  $\mathcal{J}$  is a border coimplicator. Since right sup- $\mathcal{J}$

equations can be viewed as the dual of right inf- $\mathcal{I}$  equations, any proposition, which is true for right inf- $\mathcal{I}$  equations, is also valid in its dual form for right sup- $\mathcal{J}$  equations.

Let  $A \circ \mathbf{x} = \mathbf{b}$  be a system of right sup- $\mathcal{J}$  equations where  $\mathcal{J}$  is a border coimplicator and left-continuous in its second argument. According to Theorem 3.2.3, in any case,

$$b_i \leq \sup_{j \in N} \mathcal{J}(a_{ij}, 1) = \sup_{j \in N} \mathcal{N}_{\mathcal{J}}(a_{ij}), \quad \text{for every } i \in M, \quad (3.39)$$

is a necessary condition for the existence of a solution to  $A \circ \mathbf{x} = \mathbf{b}$ . Moreover, by Theorems 2.1.18 and 2.6.14, the consistency of a system of right sup- $\mathcal{J}$  equations  $A \circ \mathbf{x} = \mathbf{b}$  can be verified by constructing and checking a potential maximum solution in a time complexity of  $O(mn)$ .

**Theorem 3.2.8.** *Let  $A \circ \mathbf{x} = \mathbf{b}$  be a system of right sup- $\mathcal{J}$  equations where  $\mathcal{J}$  is a border coimplicator and left-continuous in its second argument. The system is consistent if and only if the vector  $A^T \delta_{\mathcal{J}} \mathbf{b}$  with its components defined by*

$$(A^T \delta_{\mathcal{J}} \mathbf{b})_j = \inf_{i \in M} \mathcal{D}_{\mathcal{J}}(a_{ij}, b_i) = \bigwedge_{i \in M} (a_{ij} \delta_{\mathcal{J}} b_i), \quad \text{for every } j \in N, \quad (3.40)$$

*is a solution to  $A \circ \mathbf{x} = \mathbf{b}$ . Moreover, if the system is consistent, then  $A^T \delta_{\mathcal{J}} \mathbf{b}$  is the maximum solution in the solution set  $S(A, \mathbf{b})$ .*

Furthermore, if the border coimplicator  $\mathcal{J}$  is also right-continuous in its second argument, and hence, continuous in its second argument, then the solution set  $S(A, \mathbf{b})$ , when it is nonempty, is a finitely generated root system determined by a unique maximum solution and a finite number of minimal solutions, i.e.,

$$S(A, \mathbf{b}) = \bigcup_{\tilde{\mathbf{x}} \in \check{S}(A, \mathbf{b})} \{ \mathbf{x} \in [0, 1]^n \mid \tilde{\mathbf{x}} \leq \mathbf{x} \leq \hat{\mathbf{x}} \}, \quad (3.41)$$

where  $\check{S}(A, \mathbf{b})$  is the set of all minimal solutions of  $A \circ \mathbf{x} = \mathbf{b}$  and  $\hat{\mathbf{x}} = A^T \delta_{\mathcal{J}} \mathbf{b}$  is the maximum solution.

Systems of left inf- $\mathcal{I}$ /sup- $\mathcal{J}$  equations can be dealt with in similar manners utilizing their residual operators defined in Definitions 2.6.17 and 2.6.20, respectively.

**Theorem 3.2.9.** *Let  $\mathbf{x}^T \circ A^T = \mathbf{b}^T$  be a system of left inf- $\mathcal{I}$  equations where  $\mathcal{I}$  is an implicator and left-continuous in its first argument. The system is consistent if and only*

if the potential maximum solution, whose components are defined by  $\inf_{i \in M} \mathcal{I}_{\mathcal{I}}(b_i, a_{ij})$  for every  $j \in N$ , is a solution to  $\mathbf{x}^T \circ A^T = \mathbf{b}^T$ , in which case the potential maximum solution is a maximum solution. Furthermore, if  $\mathcal{I}$  is also right-continuous in its first argument, then the solution set  $S(A, \mathbf{b})$ , when it is nonempty, is a finitely generated root system.

**Theorem 3.2.10.** *Let  $\mathbf{x}^T \circ A^T = \mathbf{b}^T$  be a system of left sup- $\mathcal{J}$  equations where  $\mathcal{J}$  is a coimplicator and right-continuous in its first argument. The system is consistent if and only if the potential minimum solution, whose components are defined by  $\sup_{i \in M} \mathcal{J}_{\mathcal{J}}(b_i, a_{ij})$  for every  $j \in N$ , is a solution to  $\mathbf{x}^T \circ A^T = \mathbf{b}^T$ , in which case the potential minimum solution is a minimum solution. Furthermore, if  $\mathcal{J}$  is also left-continuous in its first argument, then the solution set  $S(A, \mathbf{b})$ , when it is nonempty, is a finitely generated crown system.*

### 3.2.3 Solvability of General Sup- $\mathcal{O}$ /Inf- $\mathcal{O}$ Equations

By Theorems 3.2.3 and 3.2.4, a system of sup- $\mathcal{O}$ /inf- $\mathcal{O}$  equations  $A \circ \mathbf{x} = \mathbf{b}$  can be handled in a similar manner when the general binary operator  $\mathcal{O} : [0, 1]^2 \rightarrow [0, 1]$  is continuous and monotone in its second argument, in which case the solution set  $S(A, \mathbf{b})$ , when it is nonempty, is either a finitely generated root system or a finitely generated crown system.

It is clear that when  $\mathcal{O}$  is continuous and monotone in its second argument, the solution sets of the inequalities  $\mathcal{O}(a, x) \leq b$  and  $\mathcal{O}(a, x) \geq b$  and the solution set of the equation  $\mathcal{O}(a, x) = b$ , when nonempty, are all closed intervals for any given  $a, b \in [0, 1]$ . Consequently, by Theorem 2.7.8, the solution set of a system of sup- $\mathcal{O}$ /inf- $\mathcal{O}$  equations can be well characterized.

**Theorem 3.2.11.** *Let  $A \circ \mathbf{x} = \mathbf{b}$  be a system of right sup- $\mathcal{O}$  equations where  $\mathcal{O} \in \mathcal{F}_{\mathcal{R}}$ . The system is consistent if and only if the potential maximum solution, whose components are defined by  $\inf_{i \in M} \mathcal{R}_{\mathcal{O}}(a_{ij}, b_i)$  for every  $j \in N$ , is a solution to  $A \circ \mathbf{x} = \mathbf{b}$ , in which case the potential maximum solution is a maximum solution. Furthermore, if  $\mathcal{O}$  is also right-continuous in its second argument, then the solution set  $S(A, \mathbf{b})$ , when it is nonempty, is a finitely generated root system.*

**Theorem 3.2.12.** *Let  $A \circ \mathbf{x} = \mathbf{b}$  be a system of right inf- $\mathcal{O}$  equations where  $\mathcal{R}_{\mathcal{O}} \in \mathcal{F}_{\mathcal{R}}$ . The system is consistent if and only if the potential minimum solution, whose components are defined by  $\sup_{i \in M} \mathcal{R}_{\mathcal{O}}(a_{ij}, b_i)$  for every  $j \in N$ , is a solution to  $A \circ \mathbf{x} = \mathbf{b}$ , in which case the potential minimum solution is a minimum solution. Furthermore, if  $\mathcal{O}$  is also*

left-continuous in its second argument, then the solution set  $S(A, \mathbf{b})$ , when it is nonempty, is a finitely generated crown system.

However, if  $\mathcal{O}$  is continuous and increasing in its second argument, but  $\mathcal{O} \notin \mathcal{F}_{\mathcal{R}}$ , for instance,  $\mathcal{O}$  is a compensatory operator, then the solution set  $S(A, \mathbf{b})$  of a system of right sup- $\mathcal{O}$  equations  $A \circ \mathbf{x} = \mathbf{b}$ , when it is nonempty, remains a finitely generated root system. Yet, its maximum solution cannot be constructed as in the case that  $\mathcal{O}$  belongs to  $\mathcal{F}_{\mathcal{R}}$ . Analogously, for a system of right inf- $\mathcal{O}$  equations  $A \circ \mathbf{x} = \mathbf{b}$ , its solution set, when it is nonempty, remains a finitely generated crown system as long as  $\mathcal{O}$  is continuous and increasing in its second argument.

Moreover, for systems of fuzzy relational equations formulated by a binary operator which is continuous and decreasing in its second argument, everything would work in a dual manner.

### 3.3 Bibliographical Notes

Among all types of sup- $\mathcal{T}$  equations, sup- $\mathcal{T}_M$  equations are of the most importance and were first investigated by Sanchez [399, 400]. As indicated by Zimmermann [517], the sup- $\mathcal{T}_M$  composition is commonly used when a system requires conservative solutions in the sense that the goodness of one value cannot compensate the badness of another value. However, it has been reported by Oden [328], Thole *et al.* [433], Zimmermann and Zysno [518], and Dubois and Prade [128] that the *minimum* operator  $\mathcal{T}_M$  may not be the best operator for composition and the *product* operator  $\mathcal{T}_P$  would be preferred in some situations when the values of a solution vector are allowed to compensate for each other. The Łukasiewicz t-norm  $\mathcal{T}_L$  turns out to be a special candidate for composition when the general form of the law of noncontradiction is concerned. Moreover, Bellman and Zadeh [16] stated that the appropriate composite operator strongly depends on the application context and has no universal definition in the situations where the intersection connector acts interactively. Gupta and Qi [191] studied the performance of the fuzzy logic controllers with various combinations of t-norms and t-conorms implemented and concluded that the performance very much depends on the choice of the composite operators. Van de Walle *et al.* [440] and De Baets *et al.* [83] discussed the requirements of choosing a suitable t-norm in modeling a fuzzy preference structure. Di Martino *et al.* [89] and Loia and Sessa [283] reported that the

Lukasiewicz t-norm  $\mathcal{T}_L$  would be preferred in image processing. Some outlines for selecting an appropriate t-norm has been provided by Yager [494]. So far, the applications and implementations of fuzzy relational equations are developed mainly for sup- $\mathcal{T}_M$  equations. However, they can be extended under some conditions to fuzzy relational equations defined on more general structures or with general composite operations.

The result of Theorem 3.2.5 was first stated by Sanchez [400] for sup- $\mathcal{T}_M$  equations on the unit interval. It is valid for the general sup- $\mathcal{T}$  equations defined on a complete lattice as long as the t-norm  $\mathcal{T}$  is a supmorphism. See Gottwald [173, 174] and De Baets [80] for the details. The solution set was first described by Sanchez [401] for sup- $\mathcal{T}_M$  equations on the unit interval and then extended by Di Nola [94] to the case of complete chains. Some generalized results can be found in Zimmermann [519], Di Nola *et al.* [99], Pedrycz [343], Drewniak [119], Pedrycz [347], Wu [480], Turunen [438], Zhao [516], Cheng and Peng [56], Di Nola and Lettieri [97], De Baets and Kerre [82], and De Baets [73, 79]. Some recent developments can be found in Wang [466, 467], Wang and Xiong [468], and Han *et al.* [198].

Some results on the sensitivity analysis of sup- $\mathcal{T}$  equations can be found in Miyakoshi and Shimbo [305], Hsu and Wang [208], Wang and Hsu [455, 456], Peeva [361], Tang [430], and Peeva and Kyosev [369].

Interval-valued sup- $\mathcal{T}$  equations were investigated by Wagenknecht and Hartmann [447, 448, 450], Wang and Chang [454], Li and Fang [263], Wang *et al.* [463, 465], and Li and Fang [269]. Interval-valued max-min/max-plus equations on  $\mathbb{R} \cup \{-\infty\}$  were also investigated by Cechlárová [47], Cechlárová and Cuninghame-Green [48], Myšková [313, 314], and Zimmermann [524].

The solvability of a system of inf- $\mathcal{S}$  equations with inf- $\mathcal{S}_M$  composition is also attributed to Sanchez [400]. Their application in fuzzy inference was demonstrated in Kundu [258]. The general inf- $\mathcal{S}$  equations were discussed in Pedrycz [344, 347] as well as in Cuninghame-Green [65], Di Nola *et al.* [116], Klir and Yuan [248], and Peeva [364].

The essential result of Theorem 3.2.7 was first stated by Miyakoshi and Shimbo [303] for the inf- $\mathcal{I}$  equations on the real unit interval with  $\mathcal{I}$  being a residual implicator of a left-continuous t-norm. Gottwald and Pedrycz [182] and Gottwald [175] studied the same situation and claimed that a single inf- $\mathcal{I}$  equation always has a solution provided that  $\mathcal{I}$  is

the residual impicator of a strictly monotonic t-norm, which, however, is false as indicated by De Baets [80]. A general framework of the solvability of  $\text{inf-}\mathcal{I}$  equations with  $\mathcal{I}$  being a border impicator was provided by De Baets [80]. Note that the Boolean type impicators (R-impicators, S-impicators and QL-impicators) considered in Luo and Li [288] and Luo *et al.* [290] are all border impicators and therefore are within the framework of De Baets [80]. Some results can also be found in Di Nola *et al.* [102], Bour *et al.* [23], De Nola *et al.* [116], Nosková [324], and Perfiliva and Nosková [374].

In the context of fuzzy rule modeling,  $\text{inf-}\mathcal{I}$  equations correspond to the sub/super direct image equations and the sub/super composition equations.  $\text{Inf-}\mathcal{I}$  equations are not very popular in applications and therefore only some basic properties regarding the solvability are known.  $\text{Sup-}\mathcal{J}$  equations are even less known and investigated in the literature although the similar properties can be derived dually from those known for  $\text{inf-}\mathcal{I}$  equations. However,  $\text{sup-}\mathcal{J}$  equations also play an interesting role in fuzzy inference as indicated by Oh and Kandel [330, 331]. In addition, Pedrycz [354], Peeva [363], and Peeva and Kyosev [369] discussed some properties related to  $\text{sup-}\mathcal{J}$  equations where  $\mathcal{J}$  is the residual coimplicator induced by a t-conorm.

## Chapter 4

# Characterization of Sup- $\mathcal{T}$ Equations

According to the classification of fuzzy relational equations, it suffices to focus on sup- $\mathcal{C}$  equations with  $\mathcal{C}$  being a border conjunctor and continuous in its second argument. This chapter shows that systems of fuzzy relational equations with different composite operations only differ in the way to generate their characteristic matrices. The classification and representation theorems of continuous t-norms make sup- $\mathcal{T}$  equations the most important prototypes of fuzzy relational equations. Methods based on the characteristic matrices of systems of sup- $\mathcal{T}$  equations can be extended to general types of fuzzy relational equations without much difficulty.

The material presented in this chapter is essentially based on Li and Fang [268] and Li *et al.* [275], which contains the original contribution of this dissertation. Some results in Section 4.2.1 and Section 4.3.1 are generalizations of the results of Markovskii [296, 297], Peeva [361, 366], and Peeva and Kyosev [369, 370], in which only sup- $\mathcal{T}_P$  equations or sup- $\mathcal{T}_M$  equations were presented. Some results on sup- $\mathcal{T}$  equations with  $\mathcal{T}$  being a continuous Archimedean t-norm observed by Stamou and Tzafestas [420] are generalized in Section 4.2.1 as well.

## 4.1 Characteristic Matrices of Sup- $\mathcal{T}$ Equations

A system of sup- $\mathcal{T}$  equations  $A \circ \mathbf{x} = \mathbf{b}$ , with  $\mathcal{T}$  being a continuous triangular norm, is said to be in the normal form if its right-hand side elements are arranged in a decreasing order, i.e.,

$$b_1 \geq b_2 \geq \cdots \geq b_m. \quad (4.1)$$

Any system of sup- $\mathcal{T}$  equations can be converted into its normal form in polynomial time. Systems of sup- $\mathcal{T}$  equations with a same normal form have same solutions and therefore are equivalent. Typically for sup- $\mathcal{T}_M$  equations, solving a system of sup- $\mathcal{T}$  equations in its normal form will possibly offer some convenience in theoretical analysis and computation.

Furthermore, one can assume that  $b_i > 0$  for all  $i \in M$  in a system of sup- $\mathcal{T}$  equations  $A \circ \mathbf{x} = \mathbf{b}$  as long as the t-norm  $\mathcal{T}$  has no zero divisors. Otherwise, denote  $M_0 = \{i \in M \mid b_i = 0\}$ . Any solution to  $A \circ \mathbf{x} = \mathbf{b}$  must have  $x_j = 0$  for every  $j \in N_0$ , where  $N_0 = \{j \in N \mid a_{ij} > 0, i \in M_0\}$ . Therefore it is possible to delete the equations with indices from  $M_0$  and the columns of the matrix  $A$  with indices from  $N_0$ . Each solution to the reduced system can be reconstructed as a solution to the original system by setting  $x_j = 0$  for every  $j \in N_0$ . Moreover, it will be shown later that the existence of zero divisors of the t-norm  $\mathcal{T}$  actually does not make much trouble. The equations with zero right-hand side in a system of sup- $\mathcal{T}$  equations can always be eliminated after the characteristic matrix of the system has been obtained. However, the corresponding column reduction cannot be performed any more when the involved t-norm  $\mathcal{T}$  has zero divisors.

Given a system of sup- $\mathcal{T}$  equations  $A \circ \mathbf{x} = \mathbf{b}$  with  $\mathcal{T}$  being a continuous t-norm, a variable  $x_j$ , where  $j \in N$ , is called essential if  $b_j \leq a_{ij}$  holds for some  $i \in M$ , and nonessential otherwise. By Corollary 2.6.3, nonessential variables have no influence on the consistency of the system. Hence, the presence of essential variables is a necessary condition for  $S(A, \mathbf{b}) \neq \emptyset$ . Moreover, all the nonessential variables can be excluded to simplify a system since they will assume the value 1 in the maximum solution and 0 in any minimal solution.

With the potential maximum solution  $\hat{\mathbf{x}}$ , the characteristic matrix  $\tilde{Q} = (\tilde{q}_{ij})_{m \times n}$  of the

system  $A \circ \mathbf{x} = \mathbf{b}$  is defined as

$$\tilde{q}_{ij} = \begin{cases} [\mathcal{J}_{\mathcal{T}}(a_{ij}, b_i), \hat{x}_j], & \text{if } \mathcal{T}(a_{ij}, \hat{x}_j) = b_i, \\ \emptyset, & \text{otherwise,} \end{cases} \quad (4.2)$$

and can be obtained in a time complexity of  $O(mn)$ .

By Corollary 2.6.3 and Theorem 3.2.5, each element  $\tilde{q}_{ij}$  of the characteristic matrix  $\tilde{Q}$  indicates all the possible values for the variable  $x_j$  to satisfy the  $i$ th equation without violating other equations from the upper side. The column corresponding to a nonessential variable contains only empty elements, while the column corresponding to an essential variable contains at least one nonempty element. The system  $A \circ \mathbf{x} = \mathbf{b}$  is consistent if and only if each row of  $\tilde{Q}$  contains at least one nonempty element. Furthermore, any solution to the system  $A \circ \mathbf{x} = \mathbf{b}$  can be verified via the characteristic matrix  $\tilde{Q}$ .

**Theorem 4.1.1.** *Let  $A \circ \mathbf{x} = \mathbf{b}$  be a system of sup- $\mathcal{T}$  equations with  $\mathcal{T}$  being a continuous t-norm. Given its potential maximum solution  $\hat{\mathbf{x}}$  and its characteristic matrix  $\tilde{Q}$ , a vector  $\mathbf{x} \in [0, 1]^n$  is a solution to  $A \circ \mathbf{x} = \mathbf{b}$  if and only if  $\mathbf{x} \leq \hat{\mathbf{x}}$  and the induced binary matrix  $Q_x = (q'_{ij})_{m \times n}$  has no zero rows where*

$$q'_{ij} = \begin{cases} 1, & \text{if } x_j \in \tilde{q}_{ij}, \\ 0, & \text{otherwise.} \end{cases} \quad (4.3)$$

*Proof.* If  $\mathbf{x} \in S(A, \mathbf{b}) \neq \emptyset$ , then  $\mathbf{x} \leq \hat{\mathbf{x}}$ . Moreover, by Theorem 3.2.3, there exists an index  $j_i \in N$  for each  $i \in M$  such that  $\mathcal{T}(a_{ij_i}, x_{j_i}) = b_i$ , and hence,  $x_{j_i} \in \tilde{q}_{ij_i}$  and  $q'_{ij_i} = 1$ . Therefore, the induced binary matrix  $Q_x$  has no zero rows. Conversely, if  $\mathbf{x} \leq \hat{\mathbf{x}}$  and  $Q_x$  has no zero rows, then  $\mathcal{T}(a_{ij}, x_j) \leq b_i$  for every  $i \in M$  and  $j \in N$ . Moreover, by the definition of  $Q_x$ , there exists an index  $j_i \in N$  for each  $i \in M$  such that  $\mathcal{T}(a_{ij_i}, x_{j_i}) = b_i$ , and hence,  $\mathbf{x} \in S(A, \mathbf{b})$ .  $\square$

Since  $\mathcal{J}_{\mathcal{T}}(a, 0) = 0$  for every  $a \in [0, 1]$ , we see that  $\tilde{q}_{ij} \subseteq [\mathcal{J}_{\mathcal{T}}(a_{ij}, 0), \hat{x}_j]$  holds for every  $i \in M$  and  $j \in N$  in the characteristic matrix  $\tilde{Q}$  of a system of sup- $\mathcal{T}$  equations  $A \circ \mathbf{x} = \mathbf{b}$ . Hence, the induced matrix  $Q_x$  of any vector  $\mathbf{x} \in [0, 1]^n$  with  $\mathbf{x} \leq \hat{\mathbf{x}}$  has no zero rows if and only if its submatrix, with the rows corresponding to the index set  $M_0 = \{i \in M \mid b_i = 0\}$  being removed, has no zero rows. Consequently, in case the t-norm  $\mathcal{T}$  has zero divisors and the index set  $M_0$  is not empty, the corresponding equations can be removed after the

characteristic matrix  $\tilde{Q}$  has been obtained. However, the corresponding column reduction cannot be performed in this case since the equation in the form of  $\mathcal{T}(a, x) = 0$  with  $a \in (0, 1]$  is not trivial due to the existence of zero divisors. Of course, this reduction can be performed before calculating  $\tilde{Q}$  if the t-norm  $\mathcal{T}$  has no zero divisors. A variable is called pseudoessential if its corresponding column in  $\tilde{Q}$  contains only empty elements after removing the rows corresponding to  $M_0$ . Clearly, the role of pseudoessential variables is trivial.

Therefore, without loss of generality, one may always assume that  $b_i > 0$  for every  $i \in M$  in a system of sup- $\mathcal{T}$  equations  $A \circ \mathbf{x} = \mathbf{b}$ , if a resolution method based on the characteristic matrix is applied.

## 4.2 Archimedean Property and Irredundant Coverings

The Archimedean property of t-norms shows its importance in calculating the characteristic matrix  $\tilde{Q}$  of a system of sup- $\mathcal{T}$  equations  $A \circ \mathbf{x} = \mathbf{b}$ , because each strict t-norm satisfies the cancelation law and each nilpotent t-norm satisfies the conditional cancelation law. Under the assumption that  $M_0 = \{i \in M \mid b_i = 0\} = \emptyset$ , Corollary 2.6.4 indicates that for a continuous Archimedean t-norm  $\mathcal{T}$ , either strict or nilpotent, the nonempty elements in  $\tilde{Q}$  are always singletons with their values determined by the potential maximum solution. Therefore, in this case,  $\tilde{Q}$  can be further simplified as a binary matrix  $Q = (q_{ij})_{m \times n}$  with

$$q_{ij} = \begin{cases} 1, & \text{if } \tilde{q}_{ij} \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases} \quad (4.4)$$

Moreover, it turns out that the determination of  $\check{S}(A, \mathbf{b})$  is closely related to the problem of finding all irredundant coverings of the simplified characteristic matrix  $Q$ .

### 4.2.1 Minimal Solutions and Irredundant Coverings

**Definition 4.2.1.** Let  $Q = (q_{ij})_{m \times n} \in \{0, 1\}^{m \times n}$  be a binary matrix. A column  $j$  is said to cover a row  $i$  if  $q_{ij} = 1$ . A set of nonzero columns  $P$  forms a covering of  $Q$  if each row of  $Q$  is covered by some column in  $P$ . A column  $j$  in a covering  $P$  is called redundant if the set of columns  $P \setminus \{j\}$  remains a covering of  $Q$ . A covering  $P$  is irredundant if it has no redundant columns. The set of all coverings of  $Q$  is denoted by  $P(Q)$ , while the set of all irredundant coverings of  $Q$  is denoted by  $\check{P}(Q)$ .

It is well-known that the set of all coverings  $P(Q)$  of a binary matrix  $Q$  can be well represented by the feasible solution set of a *set covering problem*, i.e.,  $\{\mathbf{u} \in \{0, 1\}^n \mid Q\mathbf{u} \geq \mathbf{e}\}$  where  $\mathbf{e} = (1, 1, \dots, 1)^T \in \{0, 1\}^m$ , while the irredundant coverings of  $Q$  correspond to the minimal elements in  $\{\mathbf{u} \in \{0, 1\}^n \mid Q\mathbf{u} \geq \mathbf{e}\}$ . The problem of finding all irredundant coverings of  $Q$  is also referred to as *hypergraph transversal problem*, for which the currently known best algorithm runs in incremental quasi-polynomial time.

**Theorem 4.2.2.** *Let  $A \circ \mathbf{x} = \mathbf{b}$  be a system of sup- $\mathcal{T}$  equations with  $\mathcal{T}$  being a continuous Archimedean  $t$ -norm. Given its potential maximum solution  $\hat{\mathbf{x}}$  and its simplified characteristic matrix  $Q$ , a vector  $\mathbf{x} \in [0, 1]^n$  with  $\mathbf{x} \leq \hat{\mathbf{x}}$  is a solution to  $A \circ \mathbf{x} = \mathbf{b}$  if and only if there exists a binary vector  $\mathbf{u} \in \{0, 1\}^n$  such that  $Q\mathbf{u} \geq \mathbf{e}$  and  $\check{V}\mathbf{u} \leq \mathbf{x}$ , where  $\check{V} = \text{diag}(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n) \in [0, 1]^{n \times n}$ .*

*Proof.* If  $\mathbf{x} \in S(A, \mathbf{b}) \neq \emptyset$ , then denote  $\mathbf{u}^b = (u_1^b, u_2^b, \dots, u_n^b)^T \in \{0, 1\}^n$  with

$$u_j^b = \begin{cases} 1, & \text{if } x_j = \hat{x}_j, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for every } j \in N. \quad (4.5)$$

Hence,  $\hat{x}_j u_j^b \leq x_j$  for every  $j \in N$ , i.e.,  $\check{V}\mathbf{u}^b \leq \mathbf{x}$ . Moreover, according to Theorem 4.1.1,  $Q\mathbf{u}^b \geq \mathbf{e}$  whenever  $\mathbf{x} \in S(A, \mathbf{b})$ .

Conversely, for a vector  $\mathbf{x} \in [0, 1]^n$  with  $\mathbf{x} \leq \hat{\mathbf{x}}$ , if there exists  $\mathbf{u}^b \in \{0, 1\}^n$  such that  $Q\mathbf{u}^b \geq \mathbf{e}$  and  $\check{V}\mathbf{u}^b \leq \mathbf{x}$ , then the vector  $\mathbf{x}^b = \check{V}\mathbf{u}^b \in [0, 1]^n$  is a solution to  $A \circ \mathbf{x} = \mathbf{b}$  according to Theorem 4.1.1. Moreover, it is clear that  $\mathbf{x}^b \leq \mathbf{x} \leq \hat{\mathbf{x}}$ , and hence,  $\mathbf{x} \in S(A, \mathbf{b}) \neq \emptyset$ .  $\square$

**Theorem 4.2.3.** *Let  $A \circ \mathbf{x} = \mathbf{b}$  be a consistent system of sup- $\mathcal{T}$  equations with  $\mathcal{T}$  being a continuous Archimedean  $t$ -norm. The set of all minimal solutions  $\check{S}(A, \mathbf{b})$  of the system  $A \circ \mathbf{x} = \mathbf{b}$  is one-to-one corresponding to the set of all irredundant coverings  $\check{P}(Q)$  of its simplified characteristic matrix  $Q$ .*

*Proof.* If  $\check{\mathbf{x}} \in \check{S}(A, \mathbf{b})$ , then denote  $\check{\mathbf{u}} = (\check{u}_1, \check{u}_2, \dots, \check{u}_n)^T \in \{0, 1\}^n$  with

$$\check{u}_j = \begin{cases} 1, & \text{if } \check{x}_j = \hat{x}_j, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for every } j \in N. \quad (4.6)$$

By Theorem 4.2.2,  $Q\check{\mathbf{u}} \geq \mathbf{e}$  and  $\check{V}\check{\mathbf{u}} \leq \check{\mathbf{x}}$ , which implies  $\check{V}\check{\mathbf{u}} = \check{\mathbf{x}}$  since  $\check{\mathbf{x}} \in \check{S}(A, \mathbf{b})$ . If there exists  $\mathbf{u}^b \in \{0, 1\}^n$  such that  $Q\mathbf{u}^b \geq \mathbf{e}$  and  $\mathbf{u}^b < \check{\mathbf{u}}$ , then  $\mathbf{x}^b = \check{V}\mathbf{u}^b < \check{V}\check{\mathbf{u}} = \check{\mathbf{x}}$ ,

which leads to a contradiction. Therefore,  $\tilde{\mathbf{u}}$  must correspond to an irredundant covering of  $Q$ . Moreover, if there are two minimal solutions  $\tilde{\mathbf{x}}^1$  and  $\tilde{\mathbf{x}}^2$  corresponding to the same irredundant covering of  $Q$ , then  $\tilde{\mathbf{x}}^1 \wedge \tilde{\mathbf{x}}^2 \leq \tilde{\mathbf{x}}^1$  and  $\tilde{\mathbf{x}}^1 \wedge \tilde{\mathbf{x}}^2 \leq \tilde{\mathbf{x}}^2$ , and by Theorem 4.2.2,  $\tilde{\mathbf{x}}^1 \wedge \tilde{\mathbf{x}}^2 \in S(A, \mathbf{b})$ , which leads to a contradiction.

Conversely, if  $\tilde{\mathbf{u}} \in \{0, 1\}^n$  corresponds to an irredundant covering of  $Q$ , then by Theorem 4.2.2, the vector  $\tilde{\mathbf{x}} = \tilde{V}\tilde{\mathbf{u}}$  is a solution to  $A \circ \mathbf{x} = \mathbf{b}$ . It is clear that this mapping is injective. Moreover, if there exists  $\mathbf{x}^b \in S(A, \mathbf{b})$  such that  $\mathbf{x}^b < \tilde{\mathbf{x}}$ , then the vector  $\mathbf{u}^b = (u_1^b, u_2^b, \dots, u_n^b)^T \in \{0, 1\}^n$  with

$$u_j^b = \begin{cases} 1, & \text{if } x_j^b = \hat{x}_j, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for every } j \in N, \quad (4.7)$$

satisfies  $Q\mathbf{u}^b \geq \mathbf{e}$  and  $\mathbf{u}^b < \tilde{\mathbf{u}}$ , which leads to a contradiction. Therefore,  $\tilde{\mathbf{x}}$  must be a minimal solution to  $A \circ \mathbf{x} = \mathbf{b}$ .

Hence, there is a one-to-one correspondence between  $\check{S}(A, \mathbf{b})$  and  $\check{P}(Q)$ .  $\square$

According to Theorems 4.2.2 and 4.2.3, for a system of sup- $\mathcal{T}$  equations  $A \circ \mathbf{x} = \mathbf{b}$  with  $\mathcal{T}$  being a continuous Archimedean t-norm, its solution set  $S(A, \mathbf{b})$  can be characterized in an alternative manner by a linear system of inequalities with 0-1 mixed integer variables. The determination of the set of minimal solutions  $\check{S}(A, \mathbf{b})$  is equivalent to obtaining all irredundant coverings of its simplified characteristic matrix  $Q$ , which is known to be a hard problem and ongoing research topic. Actually, it is NP-hard even for finding some specific covering, e.g., the minimum weighted covering, for a binary matrix.

Due to the cross connections between mathematical logic and integer programming, there do exist some methods to represent all irredundant coverings of a binary matrix. In the framework of propositional calculus, a binary vector  $\mathbf{u}$  can be viewed as a group of atomic propositions, and hence, each inequality in the system  $Q\mathbf{u} \geq \mathbf{e}$  is equivalent to a disjunctive clause. Consequently, the system  $Q\mathbf{u} \geq \mathbf{e}$  can be viewed as the conjunction of these disjunctive clauses, which defines a truth function  $F_Q$  in its conjunctive normal form (CNF), while the irredundant coverings of  $Q$  correspond to the irreducible conjunctive terms of  $F_Q$  in its disjunctive normal form (DNF), i.e., the prime implicants of  $F_Q$ . In other words, once the simplified characteristic matrix  $Q$  is obtained for the system  $A \circ \mathbf{x} = \mathbf{b}$ , the

truth function  $F_Q$  can be represented in its conjunctive normal form as

$$F_Q = \bigwedge_{i \in M} \bigvee_{j \in C_i} u_j, \quad (4.8)$$

where  $C_i = \{j \in N \mid q_{ij} = 1\}$  for every  $i \in M$ . The truth function  $F_Q$  can also be represented in its disjunctive normal form, i.e.,

$$F_Q = \bigvee_{s \in S} \bigwedge_{j \in C'_s} u_j \quad (4.9)$$

where  $S$  is an index set determined posteriorly as well as  $C'_s \subseteq N$  for every  $s \in S$ . The index sets  $S$  and  $C'_s \subseteq N$  for every  $s \in S$  are determined in the process of transforming  $F_Q$  from its conjunctive normal form to its disjunctive normal form and there is no analytical way to prespecify them before the transformation is performed. If the obtained disjunctive normal form of  $F_Q$  is irreducible, i.e., no further simplification can be performed, then for each  $s \in S$ , the irreducible conjunctive term  $\bigwedge_{j \in C'_s} u_j$  forms an irredundant covering of  $Q$ , and hence, defines a minimal solution  $\tilde{\mathbf{x}}^s = (\tilde{x}_1^s, \tilde{x}_2^s, \dots, \tilde{x}_n^s)^T$  to  $A \circ \mathbf{x} = \mathbf{b}$  with

$$\tilde{x}_j^s = \begin{cases} \hat{x}_j, & \text{if } j \in C'_s, \\ 0, & \text{otherwise.} \end{cases} \quad (4.10)$$

Unfortunately, it is known that transforming  $F_Q$  from its conjunctive normal form into its irreducible disjunctive normal form may lead to an exponential explosion of the expression in some cases, i.e., the cardinality of  $S$  may increase exponentially as the input size grows.

**Example 4.2.4.** Consider a system of sup- $\mathcal{T}_L$  equations  $A \circ \mathbf{x} = \mathbf{b}$  with

$$A = \begin{pmatrix} 1 & 1 & 0.5 & 1 & 0.6 & 0.8 \\ 0.9 & 0.2 & 1 & 0.8 & 0.4 & 1 \\ 0.7 & 0.8 & 0.3 & 0.6 & 0.8 & 0.5 \\ 0.2 & 0.7 & 0.9 & 0.3 & 0.5 & 0.5 \\ 0.3 & 0.5 & 0.7 & 0.1 & 0.3 & 0.4 \\ 0 & 0.1 & 0.4 & 0 & 0.1 & 0.3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 0.8 \\ 0.6 \\ 0.4 \\ 0.2 \\ 0 \end{pmatrix}.$$

It can be verified that the system is consistent and the maximum solution is

$$\hat{\mathbf{x}} = (0.9, 0.7, 0.5, 1, 0.8, 0.7)^T.$$

Hence, the characteristic matrix  $\tilde{Q}$  can be obtained as

$$\tilde{Q} = \begin{pmatrix} \emptyset & \emptyset & \emptyset & 1 & \emptyset & \emptyset \\ 0.9 & \emptyset & \emptyset & 1 & \emptyset & \emptyset \\ 0.9 & \emptyset & \emptyset & 1 & 0.8 & \emptyset \\ \emptyset & 0.7 & 0.5 & \emptyset & \emptyset & \emptyset \\ 0.9 & 0.7 & 0.5 & \emptyset & \emptyset & \emptyset \\ [0, 0.9] & [0, 0.7] & [0, 0.5] & [0, 1] & [0, 0.8] & [0, 0.7] \end{pmatrix}.$$

Note that the last row of  $\tilde{Q}$  can be removed and the characteristic matrix can be further simplified as

$$Q = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

It is clear that  $x_6$  is pseudoessential and can be excluded in further analysis. The associated truth function  $F_Q$  can be expressed in its irreducible disjunctive normal form as

$$\begin{aligned} F_Q &= (u_4) \wedge (u_1 \vee u_4) \wedge (u_1 \vee u_4 \vee u_5) \wedge (u_2 \vee u_3) \wedge (u_1 \vee u_2 \vee u_3) \\ &= (u_2 \wedge u_4) \vee (u_3 \wedge u_4). \end{aligned}$$

Therefore, according to Theorem 4.2.3, the concerned system  $A \circ \mathbf{x} = \mathbf{b}$  has two minimal solutions, i.e.,

$$\tilde{\mathbf{x}}^1 = \begin{pmatrix} 0 \\ 0.7 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \tilde{\mathbf{x}}^2 = \begin{pmatrix} 0 \\ 0 \\ 0.5 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$



## 4.2.2 Simplification of Set Covering Problems

Taking the advantage of well-developed theory of integer and combinatorial optimization, some methods can be applied to reduce the complexity of an instance of the *set covering problem*. The most frequently used technique is to remove the redundant rows.

**Definition 4.2.6.** Let  $Q = (q_{ij})_{m \times n} \in \{0, 1\}^{m \times n}$  be a binary matrix.

- (i) A row  $i$  is said to be redundant if there is a set of rows such that any covering of these rows also covers row  $i$ .
- (ii) A row  $i_1$  is said to dominate a row  $i_2$  if any covering of row  $i_2$  also covers row  $i_1$ , or equivalently, if  $q_{i_2j} = 1$  implies  $q_{i_1j} = 1$  for every  $j \in N$ .

**Theorem 4.2.7.** Let  $Q = (q_{ij})_{m \times n} \in \{0, 1\}^{m \times n}$  be a binary matrix. A row of  $Q$  is redundant if and only if it dominates some other row.

Redundant rows can be removed from a binary matrix  $Q$  without changing the set of all irredundant coverings  $\check{P}(Q)$ . Some columns may contain only zero elements after removing the redundant rows. The variables corresponding to such type of columns are called semiessential since they also assume the value 0 in each minimal solution. Besides, each of the remaining nonzero columns belongs to at least one of the irredundant coverings of  $Q$ .

The elimination of redundant rows has to be applied in a careful way since the identification of all redundant rows could be time consuming for large scale instances. However, it may considerably reduce the size of a specific instance of concern.

**Definition 4.2.8.** Let  $Q = (q_{ij})_{m \times n} \in \{0, 1\}^{m \times n}$  be a binary matrix. The kernel  $\text{Ker}(Q)$  is the set of columns that belong to each covering of  $Q$ .

A column is in  $\text{Ker}(Q)$  if and only if there exists a row of  $Q$  that is covered only by that column. The variables corresponding to  $\text{Ker}(Q)$  are called superessential. Consequently,  $\text{Ker}(Q)$  and the rows covered by  $\text{Ker}(Q)$  can be removed as well to reduce the size of  $Q$ . Moreover, the following results are straightforward due to Theorem 4.2.3.

**Theorem 4.2.9.** Let  $A \circ \mathbf{x} = \mathbf{b}$  be a system of sup- $\mathcal{T}$  equations with  $\mathcal{T}$  being a continuous Archimedean  $t$ -norm and  $Q$  its simplified characteristic matrix. The system has a unique

solution, i.e.,  $|S(A, b)| = 1$ , if and only if all the variables are superessential, while the system has a unique minimal solution, i.e.,  $|\check{S}(A, b)| = 1$ , if and only if  $\text{Ker}(Q)$  forms a covering of  $Q$ , i.e.,  $\text{Ker}(Q) \in \check{P}(Q)$ .

In Example 4.2.4, rows 2, 3 and 5 of  $Q$  are redundant, while column 4 is in  $\text{Ker}(Q)$ . Hence the reduced matrix can be obtained as

$$Q_D = \begin{pmatrix} u_1 & u_2 & u_3 & u_4 & u_5 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \implies \begin{pmatrix} u_2 & u_3 \\ 1 & 1 \end{pmatrix}.$$

Note that  $x_4$  is superessential, while  $x_1$  and  $x_5$  are semiessential. Two irredundant coverings  $\check{\mathbf{u}}^1 = (0, 1, 0, 1, 0, 0)^T$  and  $\check{\mathbf{u}}^2 = (0, 0, 1, 1, 0, 0)^T$  can be read from  $Q_D$  which, by Theorem 4.2.3, correspond to the minimal solutions  $\check{\mathbf{x}}^1 = (0, 0.7, 0, 1, 0, 0)^T$  and  $\check{\mathbf{x}}^2 = (0, 0, 0.5, 1, 0, 0)^T$ , respectively.

### 4.3 Non-Archimedean Property and Constrained Irredundant Coverings

Consider a system of sup- $\mathcal{T}$  equations  $A \circ \mathbf{x} = \mathbf{b}$  with  $\mathcal{T}$  being a continuous non-Archimedean t-norm. Without loss of generality, one may always assume that  $b_i > 0$  for all  $i \in M$ , and the variables  $x_j$  for all  $j \in N$  are essential.

In this case, some nonempty elements of the characteristic matrix  $\tilde{Q}$  may not be singletons. Consequently, more efforts are required to figure out all minimal solutions of  $A \circ \mathbf{x} = \mathbf{b}$ . However, if all the nonempty elements of  $\tilde{Q}$  happen to be singletons, the exactly same procedure, as introduced in Section 4.2, can be followed to obtain the solution set  $S(A, \mathbf{b})$  of  $A \circ \mathbf{x} = \mathbf{b}$ . Such a system is hence called “simple.” By Corollary 2.6.5, a system  $A \circ \mathbf{x} = \mathbf{b}$  is simple if  $b_i \neq a_{ij}$  for every  $i \in M$  and  $j \in N$ , and also  $b_i \neq a_\alpha$  in case the t-norm  $\mathcal{T}$  involves a nilpotent summand  $\langle a_\alpha, b_\alpha, \mathcal{T}_\alpha \rangle$  and  $a_{ij} \in [a_\alpha, b_\alpha]$  for some  $j \in N$ . A straightforward check of these sufficient conditions can be implemented in most cases with a time complexity of  $O(mn)$ . However, particular efforts may be required when the t-norm  $\mathcal{T}$  involves countably infinite nilpotent summands, for instance,  $\mathcal{T} = (\langle 1/2^{k+1}, 1/2^k, \mathcal{T}_L \rangle)_{k=0}^\infty$ .

### 4.3.1 Minimal Solutions and Fuzzy Truth Functions

When a system of sup- $\mathcal{T}$  equations  $A \circ \mathbf{x} = \mathbf{b}$  is not simple, a variable is called multiessential if its corresponding column in the characteristic matrix  $\tilde{Q}$  contains a nonempty element that is not a singleton. A multiessential variable  $x_j$  may assume a value other than  $\hat{x}_j$  and 0 in a minimal solution. Actually, by Corollary 2.6.3, the values that  $x_j$  may assume in a minimal solution are determined by the set  $\{\mathcal{J}_{\mathcal{T}}(a_{ij}, b_i) \mid \mathcal{T}(a_{ij}, \hat{x}_j) = b_i, i \in M\}$ . Moreover, for different values of  $x_j$ , the number of equations satisfied by  $x_j$  may be different. Therefore, a minimal solution is determined by the combination of the essential variables as well as the values assumed by those multiessential variables. Peeva [361] and Peeva and Kyosev [369] proposed a method to obtain all minimal solutions for a system of sup- $\mathcal{T}_M$  equations, which essentially relies on the characteristic matrix and Theorem 4.1.1, and hence, independent of the t-norm involved. Consequently, this method can be extended to general sup- $\mathcal{T}$  equations with  $\mathcal{T}$  being a continuous non-Archimedean t-norm. However, it should be noted that this method can be traced back to Cheng and Peng [56] in which it was developed in a more general framework. The notations used below for the resolution of sup- $\mathcal{T}$  equations are attributed to Peeva [361] and Peeva and Kyosev [369].

Given a system of sup- $\mathcal{T}$  equations  $A \circ \mathbf{x} = \mathbf{b}$  with  $\mathcal{T}$  being a continuous non-Archimedean t-norm, let  $\hat{\mathbf{x}}$  be its potential maximum solution and  $\tilde{Q}$  its characteristic matrix. Denote  $r_j$  the number of different values in  $\{\mathcal{J}_{\mathcal{T}}(a_{ij}, b_i) \mid \mathcal{T}(a_{ij}, \hat{x}_j) = b_i, i \in M\}$  for each  $j \in N$ , and denote  $r = \sum_{j \in N} r_j$ . Clearly,  $r_j = 1$  if  $x_j$  is essential but not multiessential, and  $r_j \geq 1$  if  $x_j$  is multiessential. Denote  $K_j = \{1, 2, \dots, r_j\}$ , and  $\check{v}_{jk}$ , for  $k \in K_j$ , the different values in  $\{\mathcal{J}_{\mathcal{T}}(a_{ij}, b_i) \mid \mathcal{T}(a_{ij}, \hat{x}_j) = b_i, i \in M\}$  for every  $j \in N$ . By the definition of  $\tilde{Q}$ , the value of  $\mathcal{J}_{\mathcal{T}}(a_{ij}, b_i)$  indicates the minimum value that  $x_j$  may assume to satisfy the  $i$ th equation if  $\tilde{q}_{ij} \neq \emptyset$ , while the vector  $\check{\mathbf{v}}^j = (\check{v}_{j1}, \check{v}_{j2}, \dots, \check{v}_{jr_j})^T$  contains all possible values that  $x_j$  may assume in a minimal solution to satisfy some equation in the system. Hence,

$$\bigvee_{j \in C_i} \left\langle \frac{\mathcal{J}_{\mathcal{T}}(a_{ij}, b_i)}{x_j} \right\rangle, \quad C_i = \{j \in N \mid \tilde{q}_{ij} \neq \emptyset\}, \quad (4.11)$$

indicates all the possible ways to satisfy the  $i$ th equation with the minimum values of the involved variables, and consequently,

$$F_{\tilde{Q}} = \bigwedge_{i \in M} \bigvee_{j \in C_i} \left\langle \frac{\mathcal{J}_{\mathcal{T}}(a_{ij}, b_i)}{x_j} \right\rangle \quad (4.12)$$

provides all lower bound information of the values that the variables may assume in a minimal solution to the system  $A \circ \mathbf{x} = \mathbf{b}$ . The function  $F_{\tilde{Q}}$  can be regarded as a fuzzy truth function. It would reduce to a classical truth function when all the lower bound values are unique for each variable, i.e.,  $r_j = 1$  for every  $j \in N$ . The function  $F_{\tilde{Q}}$  can be viewed as in its fuzzy version of conjunctive normal form and needs to be transformed into its fuzzy version of irreducible disjunctive normal form in order to identify all minimal solutions, i.e.,

$$F_{\tilde{Q}} = \bigwedge_{i \in M} \bigvee_{j \in C_i} \left\langle \frac{\mathcal{J}_{\mathcal{T}}(a_{ij}, b_i)}{x_j} \right\rangle = \bigvee_{s \in S} \bigwedge_{j \in C'_s} \left\langle \frac{\check{v}_{jk^s}}{x_j} \right\rangle, \quad (4.13)$$

where  $S$  is an index set determined posteriorly as well as  $C'_s \subseteq N$  for every  $s \in S$ . If no further simplification can be performed, then for each  $s \in S$ , the term  $\bigwedge_{j \in C'_s} \left\langle \frac{\check{v}_{jk^s}}{x_j} \right\rangle$  forms an irreducible fuzzy conjunctive term, with  $\check{v}_{jk^s}$  being the value of  $x_j$  in the term, and hence, defines a minimal solution  $\check{\mathbf{x}}^s = (\check{x}_1^s, \check{x}_2^s, \dots, \check{x}_n^s)^T$  to  $A \circ \mathbf{x} = \mathbf{b}$  with

$$\check{x}_j^s = \begin{cases} \check{v}_{jk^s}, & \text{if } j \in C'_s, \\ 0, & \text{otherwise.} \end{cases} \quad (4.14)$$

The rules for transforming  $F_{\tilde{Q}}$  from its fuzzy conjunctive normal form into its irreducible fuzzy disjunctive normal form are analogous to those in classical propositional calculus, and summarized below:

**Rule 1: commutativity.**

$$\left\langle \frac{\check{v}_{j_1 k_1}}{x_{j_1}} \right\rangle \left\langle \frac{\check{v}_{j_2 k_2}}{x_{j_2}} \right\rangle = \left\langle \frac{\check{v}_{j_2 k_2}}{x_{j_2}} \right\rangle \left\langle \frac{\check{v}_{j_1 k_1}}{x_{j_1}} \right\rangle, \quad j_1 \neq j_2, \quad k_1 \in K_{j_1}, \quad k_2 \in K_{j_2}. \quad (4.15)$$

**Rule 2: absorption for conjunction.**

$$\left\langle \frac{\check{v}_{j_1 k_1}}{x_{j_1}} \right\rangle \left\langle \frac{\check{v}_{j_1 k_2}}{x_{j_1}} \right\rangle = \left\langle \frac{\check{v}_{j_1 k_1} \vee \check{v}_{j_1 k_2}}{x_{j_1}} \right\rangle, \quad k_1, k_2 \in K_{j_1}. \quad (4.16)$$

**Rule 3: distributivity for disjunction.**

$$\left\langle \frac{\check{v}_{j_1 k_1}}{x_{j_1}} \right\rangle \left( \left\langle \frac{\check{v}_{j_1 k_2}}{x_{j_2}} \right\rangle \vee \left\langle \frac{\check{v}_{j_3 k_3}}{x_{j_3}} \right\rangle \right) = \left\langle \frac{\check{v}_{j_1 k_1}}{x_{j_1}} \right\rangle \left\langle \frac{\check{v}_{j_2 k_2}}{x_{j_2}} \right\rangle \vee \left\langle \frac{\check{v}_{j_1 k_1}}{x_{j_1}} \right\rangle \left\langle \frac{\check{v}_{j_3 k_3}}{x_{j_3}} \right\rangle, \quad (4.17)$$

$$k_1 \in K_{j_1}, k_2 \in K_{j_2}, k_3 \in K_{j_3}.$$

**Rule 4: absorption for disjunction.**

$$\left\langle \frac{\check{v}_{j_1 k_1}}{x_{j_1}} \right\rangle \dots \left\langle \frac{\check{v}_{j_p k_p}}{x_{j_p}} \right\rangle \vee \left\langle \frac{\check{v}_{j_1 k'_1}}{x_{j_1}} \right\rangle \dots \left\langle \frac{\check{v}_{j_p k'_p}}{x_{j_p}} \right\rangle \left\langle \frac{\check{v}_{j_{p+1} k'_{p+1}}}{x_{j_{p+1}}} \right\rangle \dots \left\langle \frac{\check{v}_{j_{p+q} k'_{p+q}}}{x_{j_{p+q}}} \right\rangle$$

$$= \left\langle \frac{\check{v}_{j_1 k_1}}{x_{j_1}} \right\rangle \dots \left\langle \frac{\check{v}_{j_p k_p}}{x_{j_p}} \right\rangle, \quad \text{if } \check{v}_{j_1 k_1} \leq \check{v}_{j_1 k'_1}, \dots, \check{v}_{j_p k_p} \leq \check{v}_{j_p k'_p}. \quad (4.18)$$

**Example 4.3.1.** Consider a system of sup- $\mathcal{T}$  equations  $A \circ \mathbf{x} = \mathbf{b}$  with  $\mathcal{T} = (\langle 0.4, 0.8, \mathcal{T}_L \rangle)$  and

$$A = \begin{pmatrix} 0.8 & 1 & 0.9 & 1 & 1 \\ 0.9 & 0.7 & 0.8 & 0.8 & 1 \\ 0.4 & 0.4 & 0.6 & 0.5 & 0.2 \\ 0.6 & 0.3 & 0.1 & 0.4 & 0.9 \\ 0.2 & 0 & 0.2 & 0.1 & 0.8 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 0.8 \\ 0.6 \\ 0.4 \\ 0.2 \end{pmatrix}.$$

It can be verified that the system is consistent and the maximum solution is

$$\hat{\mathbf{x}} = (0.6, 1, 1, 1, 0.2)^T.$$

Hence, the characteristic matrix  $\tilde{Q}$  can be obtained as

$$\tilde{Q} = \begin{pmatrix} \emptyset & 1 & \emptyset & 1 & \emptyset \\ \emptyset & \emptyset & [0.8, 1] & [0.8, 1] & \emptyset \\ \emptyset & \emptyset & [0.8, 1] & \emptyset & \emptyset \\ [0.4, 0.6] & \emptyset & \emptyset & [0.4, 1] & \emptyset \\ [0.2, 0.6] & \emptyset & [0.2, 1] & \emptyset & 0.2 \end{pmatrix}.$$

The system is therefore not simple. The associated fuzzy truth function  $F_{\tilde{Q}}$  can be expressed in its irreducible disjunctive normal form as

$$\begin{aligned} F_{\tilde{Q}} &= \left( \left\langle \frac{1}{x_2} \right\rangle \vee \left\langle \frac{1}{x_4} \right\rangle \right) \wedge \left( \left\langle \frac{0.8}{x_3} \right\rangle \vee \left\langle \frac{0.8}{x_4} \right\rangle \right) \wedge \left( \left\langle \frac{0.8}{x_3} \right\rangle \right) \wedge \\ &\quad \left( \left\langle \frac{0.4}{x_1} \right\rangle \vee \left\langle \frac{0.4}{x_4} \right\rangle \right) \wedge \left( \left\langle \frac{0.2}{x_1} \right\rangle \vee \left\langle \frac{0.2}{x_3} \right\rangle \vee \left\langle \frac{0.2}{x_5} \right\rangle \right) \\ &= \left( \left\langle \frac{0.4}{x_1} \right\rangle \left\langle \frac{1}{x_2} \right\rangle \left\langle \frac{0.8}{x_3} \right\rangle \right) \vee \left( \left\langle \frac{1}{x_2} \right\rangle \left\langle \frac{0.8}{x_3} \right\rangle \left\langle \frac{0.4}{x_4} \right\rangle \right) \vee \left( \left\langle \frac{0.8}{x_3} \right\rangle \left\langle \frac{1}{x_4} \right\rangle \right). \end{aligned}$$

Therefore, the concerned system  $A \circ \mathbf{x} = \mathbf{b}$  has three minimal solutions, i.e.,

$$\tilde{\mathbf{x}}^1 = \begin{pmatrix} 0.4 \\ 1 \\ 0.8 \\ 0 \\ 0 \end{pmatrix}, \quad \tilde{\mathbf{x}}^2 = \begin{pmatrix} 0 \\ 1 \\ 0.8 \\ 0.4 \\ 0 \end{pmatrix}, \quad \tilde{\mathbf{x}}^3 = \begin{pmatrix} 0 \\ 0 \\ 0.8 \\ 1 \\ 0 \end{pmatrix}.$$

Hence, the three maximal interval solutions are

$$\tilde{\mathbf{x}}_{max}^1 = \begin{pmatrix} [0.4, 0.6] \\ 1 \\ [0.8, 1] \\ [0, 1] \\ [0, 0.4] \end{pmatrix}, \quad \tilde{\mathbf{x}}_{max}^2 = \begin{pmatrix} [0, 0.6] \\ 1 \\ [0.8, 1] \\ [0.4, 1] \\ [0, 0.4] \end{pmatrix}, \quad \tilde{\mathbf{x}}_{max}^3 = \begin{pmatrix} [0, 0.6] \\ [0, 1] \\ [0.8, 1] \\ 1 \\ [0, 0.4] \end{pmatrix}.$$

and the union of them forms the solution set  $S(A, \mathbf{b})$ .

### 4.3.2 Simplification of Characteristic Matrices

Analogously, the size of a characteristic matrix  $\tilde{Q}$  may be reduced by removing the redundant rows.

**Definition 4.3.2.** Let  $\tilde{Q} = (\tilde{q}_{ij})_{m \times n}$  be the characteristic matrix of a system of sup- $\mathcal{T}$  equations  $A \circ \mathbf{x} = \mathbf{b}$ .

- (i) A row  $i$  of  $\tilde{Q}$  is said to be redundant if there exists a set of rows of  $\tilde{Q}$  such that, for all  $\mathbf{x} \leq \hat{\mathbf{x}}$ , the corresponding rows of the induced matrix  $Q_{\mathbf{x}}$  have no zero rows implies the row  $i$  of  $Q_{\mathbf{x}}$  is also a nonzero row.
- (ii) A row  $i_1$  is said to dominate a row  $i_2$  in  $\tilde{Q}$  if, for all  $\mathbf{x} \leq \hat{\mathbf{x}}$ , row  $i_1$  dominates row  $i_2$  in the induced matrix  $Q_{\mathbf{x}}$ , or equivalently,  $\tilde{q}_{i_2j} \neq \emptyset$  implies  $\tilde{q}_{i_2j} \subseteq \tilde{q}_{i_1j}$  for every  $j \in N$ .
- (iii) A column  $j$  of  $\tilde{Q}$  is said in the kernel  $\text{Ker}(\tilde{Q})$  if there exists a row  $i$  such that  $\tilde{q}_{ij}$  is the unique nonempty element of row  $i$ .

When a continuous Archimedean t-norm is involved for composition, the concepts of redundant rows, dominant rows and the kernel are equivalent to those in Definitions 4.2.6 and 4.2.8. By Theorem 4.2.7, it is straightforward that a row of  $\tilde{Q}$  is redundant if and only if it dominates some other row. Moreover, for a system of sup- $\mathcal{T}_M$  equations  $A \circ \mathbf{x} = \mathbf{b}$ , if a row  $i_1$  dominates a row  $i_2$ , then it always holds that  $b_{i_1} \leq b_{i_2}$  since  $\mathcal{J}_{\mathcal{T}_M}(a, b) = b$  whenever  $0 \leq b \leq a \leq 1$ . Hence, the normal form of sup- $\mathcal{T}_M$  equations provides some convenience in detecting the redundant rows. However, as shown in Example 4.3.1, this is not valid in general cases.

Some columns of  $\tilde{Q}$  may contain only empty elements after removing the redundant rows. The variables corresponding to such type of columns are called semiessential and assume the value 0 in each minimal solution. The variables corresponding to the columns in  $\text{Ker}(\tilde{Q})$  are called superessential. However,  $\text{Ker}(\tilde{Q})$  and the corresponding rows can not be removed in general since the superessential variables may assume different nonzero values in minimal solutions. Furthermore, the condition that all the variables are superessential can only guarantee that the uniqueness of the minimal solution. Some discussion on the unique solvability of sup- $\mathcal{T}_M$  equations can be found in Sessa [410], Lettieri and Liguori [260, 261], Di Nola and Sessa [111], Cechlárová [44, 45], Li [265], and Gavalec [150]. Most recently, Li and Fang [274] provided a necessary and sufficient condition for the unique solvability of a system of sup- $\mathcal{T}$  equations based on the characteristic matrix.

In Example 4.3.1, row 2 and row 5 of  $\tilde{Q}$  are redundant, while column 3 is in  $\text{Ker}(\tilde{Q})$ . Hence the reduced matrix can be obtained as

$$\tilde{Q}_D = \begin{matrix} & x_1 & x_2 & x_3 & x_4 & x_5 \\ \begin{pmatrix} \emptyset & 1 & \emptyset & 1 & \emptyset \\ \emptyset & \emptyset & [0.8, 1] & \emptyset & \emptyset \\ [0.4, 0.6] & \emptyset & \emptyset & [0.4, 0.6] & \emptyset \end{pmatrix} \end{matrix}.$$

Note that  $x_3$  is superessential while  $x_5$  is semiessential. The same result of minimal solutions can be obtained from  $\tilde{Q}_D$ . In this example the superessential variable  $x_3$  assumes the same value in each minimal solution.

**Example 4.3.3.** Consider a system of  $\text{sup-}\mathcal{T}_M$  equations  $A \circ \mathbf{x} = \mathbf{b}$  with

$$A = \begin{pmatrix} 1 & 0.4 & 0.6 & 1 \\ 0.6 & 1 & 0.8 & 0.9 \\ 0.6 & 0.8 & 0.5 & 0.4 \\ 0.3 & 1 & 0.2 & 0.4 \\ 0.1 & 0.8 & 0.1 & 0.2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 0.8 \\ 0.6 \\ 0.4 \\ 0.2 \end{pmatrix}.$$

It can be verified that the system is consistent and the maximum solution is

$$\hat{\mathbf{x}} = (1, 0.2, 1, 0.8)^T.$$

Hence, the characteristic matrix  $\tilde{Q}$  can be obtained as

$$\tilde{Q} = \begin{pmatrix} 1 & \emptyset & \emptyset & \emptyset \\ \emptyset & \emptyset & [0.8, 1] & 0.8 \\ [0.6, 1] & \emptyset & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset & [0.4, 0.8] \\ \emptyset & 0.2 & \emptyset & [0.2, 0.8] \end{pmatrix}.$$

Notice that row 3 dominates row 1 and row 5 dominates row 4. Columns 1 and 4 are in  $\text{Ker}(\tilde{Q})$ . The reduced matrix can be obtained as

$$\tilde{Q}_D = \begin{matrix} & x_1 & x_2 & x_3 & x_4 \\ \begin{pmatrix} 1 & \emptyset & \emptyset & \emptyset \\ \emptyset & \emptyset & [0.8, 1] & 0.8 \\ \emptyset & \emptyset & \emptyset & [0.4, 0.8] \end{pmatrix} \end{matrix}.$$

The variables  $x_1$  and  $x_4$  are superessential, while  $x_2$  is semiessential. The system has two minimal solutions, i.e.,

$$\check{\mathbf{x}}^1 = \begin{pmatrix} 1 \\ 0 \\ 0.8 \\ 0.4 \end{pmatrix}, \quad \check{\mathbf{x}}^2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0.8 \end{pmatrix}.$$

Note that the superessential variable  $x_4$  assumes different values in the two minimal solutions.

### 4.3.3 Minimal Solutions and Constrained Irredundant Coverings

An alternative method to deal with multiessential variables is to represent each of them by a set of binary variables.

Given a system of sup- $\mathcal{T}$  equations  $A \circ \mathbf{x} = \mathbf{b}$  with  $\mathcal{T}$  being a continuous non-Archimedean t-norm, let  $\hat{\mathbf{x}}$  be its potential maximum solution and  $\tilde{Q}$  its characteristic matrix. Recall that  $\check{v}_{jk}$ , for  $k \in K_j$ , are the different values in  $\{\mathcal{J}_{\mathcal{T}}(a_{ij}, b_i) \mid \mathcal{T}(a_{ij}, \hat{x}_j) = b_i, i \in M\}$  for every  $j \in N$ . Denote  $\check{\mathbf{v}}^j = (\check{v}_{j1}, \dots, \check{v}_{jr_j})^T$  for every  $j \in N$ , and  $\check{\mathbf{v}} = (\check{v}_{11}, \dots, \check{v}_{1r_1}, \dots, \check{v}_{n1}, \dots, \check{v}_{nr_n})^T \in [0, 1]^r$ . Consequently, let

$$x_j = \sum_{k \in K_j} \check{v}_{jk} u_{jk}, \quad \text{for every } j \in N, \quad (4.19)$$

where  $u_{jk} \in \{0, 1\}$  for every  $k \in K_j$  and  $j \in N$ . To make this expression meaningful, at most one of  $u_{jk}$ , where  $k \in K_j$ , can be 1, i.e.,  $\sum_{k \in K_j} u_{jk} \leq 1$ , for every  $j \in N$ . These restrictions are called the innervariable incompatibility constraints and can be represented by  $G\mathbf{u} \leq \mathbf{e}^n$ , where  $\mathbf{e}^n = (1, 1, \dots, 1)^T \in \{0, 1\}^n$ ,  $\mathbf{u} = (u_{11}, \dots, u_{1r_1}, \dots, u_{n1}, \dots, u_{nr_n})^T \in \{0, 1\}^r$ , and  $G = (g_{jk})_{n \times r}$  with

$$g_{jk} = \begin{cases} 1, & \text{if } \sum_{s=1}^{j-1} r_s < k \leq \sum_{s=1}^j r_s, \\ 0, & \text{otherwise.} \end{cases} \quad (4.20)$$

Actually, the incompatibility constraints with  $r_j = 1$  are redundant, and hence, can be removed. In case  $r_j = 1$  for every  $j \in N$ , all the incompatibility constraints are redundant and no additional difficulties will be imposed. In this case, the values of the nonzero elements in a minimal solution are uniquely determined although they may be different from those in the maximum solution.

As a consequence of this transformation, the characteristic matrix  $\tilde{Q}$  can be transformed into its augmented characteristic matrix  $Q = (q_{ik})_{m \times r} \in \{0, 1\}^{m \times r}$  where

$$q_{ik} = \begin{cases} 1, & \text{if } \sum_{s=1}^{j-1} r_s < k \leq \sum_{s=1}^j r_s \text{ and } \check{v}_k \in \tilde{q}_{ij} \text{ for some } j \in N, \\ 0, & \text{otherwise.} \end{cases} \quad (4.21)$$

**Definition 4.3.4.** Let  $Q = (q_{ik})_{m \times r} \in \{0, 1\}^{m \times r}$  and  $G = (g_{jk})_{n \times r} \in \{0, 1\}^{n \times r}$  be two binary matrices. A column  $k$  of  $Q$  is said to cover a row  $i$  of  $Q$  if  $q_{ik} = 1$ . A set of nonzero

columns  $P$  forms a  $G$ -covering of  $Q$  if each row of  $Q$  is covered by some column in  $P$ , i.e.,  $Q\mathbf{u}^p \geq \mathbf{e}^m$ , and also satisfies  $G\mathbf{u}^p \leq \mathbf{e}^n$ , where  $\mathbf{u}^p = (u_k^p)_{r \times 1}$  and

$$u_k^p = \begin{cases} 1, & \text{if } k \in P, \\ 0, & \text{otherwise.} \end{cases} \quad (4.22)$$

A column  $k$  in a  $G$ -covering  $P$  is called *redundant* if the set of columns  $P \setminus \{k\}$  remains a  $G$ -covering of  $Q$ . A  $G$ -covering  $P$  is *irredundant* if  $P$  has no redundant columns. The set of all  $G$ -coverings of  $Q$  is denoted by  $P_G(Q)$ , while the set of all irredundant  $G$ -coverings of  $Q$  is denoted by  $\check{P}_G(Q)$ .

**Theorem 4.3.5.** *Let  $A \circ \mathbf{x} = \mathbf{b}$  be a system of sup- $\mathcal{T}$  equations with  $\mathcal{T}$  being a continuous non-Archimedean  $t$ -norm. Given its potential maximum solution  $\hat{\mathbf{x}}$ , the augmented characteristic matrix  $Q$  and the coefficient matrix of the innervariable incompatibility constraints  $G$ , a vector  $\mathbf{x} \in [0, 1]^n$  with  $\mathbf{x} \leq \hat{\mathbf{x}}$  is a solution to  $A \circ \mathbf{x} = \mathbf{b}$  if and only if there exists a binary vector  $\mathbf{u} \in \{0, 1\}^r$  such that  $Q\mathbf{u} \geq \mathbf{e}^m$ ,  $G\mathbf{u} \leq \mathbf{e}^n$ , and  $\check{V}\mathbf{u} \leq \mathbf{x}$ , where  $\check{V} = (\text{diag}(\check{v}^1, \check{v}^2, \dots, \check{v}^n))^T \in [0, 1]^{n \times r}$ .*

*Proof.* If  $\mathbf{x} \in S(A, \mathbf{b}) \neq \emptyset$ , then denote  $\mathbf{u}^b = (u_{11}^b, \dots, u_{1r_1}^b, \dots, u_{n1}^b, \dots, u_{nr_n}^b)^T \in \{0, 1\}^r$  with

$$u_{jk}^b = \begin{cases} 1, & \text{if } k = \text{argmax}\{\check{v}_{jk} \mid \check{v}_{jk} \leq x_j\}, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for every } j \in N. \quad (4.23)$$

Since for each  $j \in N$ , the values  $\check{v}_{jk}$  for all  $k \in K_j$  are different from each other, hence,  $\sum_{k \in K_j} u_{jk}^b \leq 1$  and  $\sum_{k \in K_j} \check{v}_{jk} u_{jk}^b \leq x_j$ , i.e.,  $G\mathbf{u}^b \leq \mathbf{e}^n$  and  $\check{V}\mathbf{u}^b \leq \mathbf{x}$ . Moreover, according to Theorem 4.1.1,  $Q\mathbf{u}^b \geq \mathbf{e}^m$  whenever  $\mathbf{x} \in S(A, \mathbf{b})$ .

Conversely, for a vector  $\mathbf{x} \in [0, 1]^n$  with  $\mathbf{x} \leq \hat{\mathbf{x}}$ , if there exists  $\mathbf{u}^b \in \{0, 1\}^r$  such that  $Q\mathbf{u}^b \geq \mathbf{e}^m$ ,  $G\mathbf{u}^b \leq \mathbf{e}^n$ , and  $\check{V}\mathbf{u}^b \leq \mathbf{x}$ , then the vector  $\mathbf{x}^b = \check{V}\mathbf{u}^b \in [0, 1]^n$  satisfies  $\mathbf{x}^b \leq \mathbf{x} \leq \hat{\mathbf{x}}$ . According to Theorem 4.1.1,  $\mathbf{x}^b$  is a solution to  $A \circ \mathbf{x} = \mathbf{b}$ , and hence,  $\mathbf{x} \in S(A, \mathbf{b}) \neq \emptyset$ .  $\square$

**Theorem 4.3.6.** *Let  $A \circ \mathbf{x} = \mathbf{b}$  be a consistent system of sup- $\mathcal{T}$  equations with  $\mathcal{T}$  being a continuous non-Archimedean  $t$ -norm. Each minimal solution to  $A \circ \mathbf{x} = \mathbf{b}$  corresponds to an irredundant  $G$ -covering of  $Q$ , where  $Q$  and  $G$  are the augmented characteristic matrix and the coefficient matrix of the innervariable incompatibility constraints, respectively.*

*Proof.* If  $\tilde{\mathbf{x}} \in \check{S}(A, \mathbf{b})$ , then denote  $\tilde{\mathbf{u}} = (\tilde{u}_{11}, \dots, \tilde{u}_{1r_1}, \dots, \tilde{u}_{n1}, \dots, \tilde{u}_{nr_n})^T \in \{0, 1\}^r$  with

$$\tilde{u}_{jk} = \begin{cases} 1, & \text{if } k = \operatorname{argmax}\{\check{v}_{jk} \mid \check{v}_{jk} \leq \tilde{x}_j\}, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for every } j \in N. \quad (4.24)$$

By Theorem 4.3.5,  $Q\tilde{\mathbf{u}} \geq \mathbf{e}^m$ ,  $G\tilde{\mathbf{u}} \geq \mathbf{e}^n$ , and  $\check{V}\tilde{\mathbf{u}} \leq \tilde{\mathbf{x}}$ , which imply  $\check{V}\tilde{\mathbf{u}} = \tilde{\mathbf{x}}$  since  $\tilde{\mathbf{x}} \in \check{S}(A, \mathbf{b})$ . If there exists  $\mathbf{u}^b \in \{0, 1\}^r$  such that  $Q\mathbf{u}^b \geq \mathbf{e}^m$ ,  $G\mathbf{u}^b \leq \mathbf{e}^n$ , and  $\mathbf{u}^b < \tilde{\mathbf{u}}$ , then  $\mathbf{x}^b = \check{V}\mathbf{u}^b < \check{V}\tilde{\mathbf{u}} = \tilde{\mathbf{x}}$ , which leads to a contradiction. Therefore,  $\tilde{\mathbf{u}}$  must correspond to an irredundant  $G$ -covering of  $Q$ . Moreover, if there are two minimal solutions  $\tilde{\mathbf{x}}^1$  and  $\tilde{\mathbf{x}}^2$  corresponding to a same irredundant  $G$ -covering of  $Q$ , then  $\tilde{\mathbf{x}}^1 \wedge \tilde{\mathbf{x}}^2 \leq \tilde{\mathbf{x}}^1$  and  $\tilde{\mathbf{x}}^1 \wedge \tilde{\mathbf{x}}^2 \leq \tilde{\mathbf{x}}^2$ , and by Theorem 4.3.5,  $\tilde{\mathbf{x}}^1 \wedge \tilde{\mathbf{x}}^2 \in S(A, \mathbf{b})$ , which leads to a contradiction. Hence, each minimal solution to  $A \circ \mathbf{x} = \mathbf{b}$  corresponds to an irredundant  $G$ -covering of  $Q$ ,  $\square$

Due to the existence of multiessential variables, an irredundant  $G$ -covering of the augmented characteristic matrix  $Q$  may fail to induce a minimal solution to  $A \circ \mathbf{x} = \mathbf{b}$ . As shown in Example 4.3.7, in order to construct a minimal solution to  $A \circ \mathbf{x} = \mathbf{b}$  via an irredundant  $G$ -covering of  $Q$ , some columns corresponding to different variables cannot be chosen simultaneously.

**Example 4.3.7.** Consider a system of  $\sup\text{-}\mathcal{T}_M$  equations  $A \circ \mathbf{x} = \mathbf{b}$  with

$$A = \begin{pmatrix} 0.8 & 0 & 0.8 \\ 0.6 & 0.6 & 0 \\ 0 & 0.4 & 0.2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0.8 \\ 0.6 \\ 0.4 \end{pmatrix}.$$

This example is adapted from the one in Markovskii [297]. The system has a maximum solution  $\hat{\mathbf{x}} = (1, 1, 1)^T$ . The characteristic matrix  $\tilde{Q}$  can be obtained as

$$\tilde{Q} = \begin{pmatrix} [0.8, 1] & \emptyset & [0.8, 1] \\ [0.6, 1] & [0.6, 1] & \emptyset \\ \emptyset & [0.4, 1] & \emptyset \end{pmatrix}.$$

Therefore, the concerned system has three minimal solutions, i.e.,

$$\tilde{\mathbf{x}}^1 = \begin{pmatrix} 0.8 \\ 0.4 \\ 0 \end{pmatrix}, \quad \tilde{\mathbf{x}}^2 = \begin{pmatrix} 0.6 \\ 0.4 \\ 0.8 \end{pmatrix}, \quad \tilde{\mathbf{x}}^3 = \begin{pmatrix} 0 \\ 0.6 \\ 0.8 \end{pmatrix}.$$

Let  $\mathbf{u} = (u_{11}, u_{12}, u_{21}, u_{22}, u_{31})^T \in \{0, 1\}^5$  and

$$\check{\mathbf{v}} = (\check{v}_{11}, \check{v}_{12}, \check{v}_{21}, \check{v}_{22}, \check{v}_{31})^T = (0.8, 0.6, 0.6, 0.4, 0.8)^T.$$

The augmented characteristic matrix  $Q$  can be obtained as

$$Q = \begin{pmatrix} u_{11} & u_{12} & \vdots & u_{21} & u_{22} & \vdots & u_{31} \\ 1 & 0 & \vdots & 0 & 0 & \vdots & 1 \\ 1 & 1 & \vdots & 1 & 0 & \vdots & 0 \\ 0 & 0 & \vdots & 1 & 1 & \vdots & 0 \end{pmatrix}.$$

The coefficient matrix  $G$  of the innervariable incompatibility constraints is

$$G = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

Note that the constraint for  $u_{31}$  is redundant and has been omitted. In this case, five irredundant  $G$ -coverings can be obtained, i.e.,

$$\check{P}_G^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \check{P}_G^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad \check{P}_G^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \check{P}_G^4 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \check{P}_G^5 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

However, only the first three irredundant  $G$ -coverings induce a minimal solution, respectively. In  $\check{P}_G^4$ , the variables  $u_{11}$  and  $u_{21}$  are actually incompatible to induce a minimal solution since the columns corresponding to  $u_{11}$  and  $u_{22}$  cover the same rows as those corresponding to  $u_{11}$  and  $u_{21}$ , but with a smaller value  $\check{v}_{22}$  for  $x_2$  in the induced solution to  $A \circ \mathbf{x} = \mathbf{b}$ . The similar situation occurs in  $\check{P}_G^5$ , where  $u_{12}$  and  $u_{21}$  can be replaced by  $u_{21}$  and  $u_{22}$  in order to induce a minimal solution. Furthermore, it is also possible that a group of variables are incompatible but pairwise compatible for inducing a minimal solution. This phenomenon is called the intervariable incompatibility, and essentially is a reflection of the rule of absorption for disjunction introduced in Section 4.3.1 for converting a fuzzy truth function in conjunctive normal form to its irreducible disjunctive normal form.

It is possible but not worthwhile to detect all intervariable incompatibility constraints for a system of sup- $\mathcal{T}$  equations since they do not offer any computational convenience in the determination of all minimal solutions. Moreover, the innervariable incompatibility constraints, which are easy to define, are usually sufficient when a specific minimal solution of the system is concerned. Besides, from an algorithmic point of view, it can be verified in polynomial time whether or not an irredundant  $G$ -covering induces a minimal solution. Hence, all minimal solutions of a system of sup- $\mathcal{T}$  equations can be detected by obtaining the irredundant  $G$ -coverings of its augmented characteristic matrix.

#### 4.4 Boolean Solutions of Sup- $\mathcal{T}$ Equations

In some applications, Boolean solutions of a consistent system of sup- $\mathcal{T}$  equations  $A \circ \mathbf{x} = \mathbf{b}$  are of particular interest. A Boolean solution to  $A \circ \mathbf{x} = \mathbf{b}$  is a vector  $\mathbf{x} \in S(A, \mathbf{b})$  such that  $x_j = 1$  or  $x_j = 0$  for every  $j \in N$ , i.e.,  $\mathbf{x} \in S(A, \mathbf{b}) \cap \{0, 1\}^n$ . The set of all Boolean solutions of  $A \circ \mathbf{x} = \mathbf{b}$  is denoted by  $S^b(A, \mathbf{b})$ .

Let  $A \circ \mathbf{x} = \mathbf{b}$  be a consistent system of sup- $\mathcal{T}$  equations with  $\mathcal{T}$  being a continuous  $t$ -norm. Its maximum solution  $\hat{\mathbf{x}}$  induces a Boolean vector  $\hat{\mathbf{x}}^b = (\hat{x}_1^b, \hat{x}_2^b, \dots, \hat{x}_n^b)^n \in \{0, 1\}^n$  with

$$\hat{x}_j^b = \begin{cases} 1, & \text{if } \hat{x}_j = 1, \\ 0, & \text{otherwise.} \end{cases} \quad (4.25)$$

Since  $\hat{\mathbf{x}}$  is the maximum solution in  $S(A, \mathbf{b})$ , the binary vector  $\hat{\mathbf{x}}^b$  induced by  $\hat{\mathbf{x}}$  must be a potential maximum Boolean solution to the system  $A \circ \mathbf{x} = \mathbf{b}$ .

**Theorem 4.4.1.** *Let  $A \circ \mathbf{x} = \mathbf{b}$  be a consistent system of sup- $\mathcal{T}$  equations with  $\mathcal{T}$  being a continuous  $t$ -norm. The set of Boolean solutions  $S^b(A, \mathbf{b})$  is nonempty if and only if the binary vector  $\hat{\mathbf{x}}^b$  induced by the maximum solution  $\hat{\mathbf{x}}$  is a solution to  $A \circ \mathbf{x} = \mathbf{b}$ , in which case  $\hat{\mathbf{x}}^b$  is also a maximum Boolean solution.*

*Proof.* If the binary vector  $\hat{\mathbf{x}}^b$  is a solution to  $A \circ \mathbf{x} = \mathbf{b}$ , then  $S^b(A, \mathbf{b}) = \emptyset$ . Moreover,  $\mathbf{x} \leq \hat{\mathbf{x}}$  for any  $\mathbf{x} \in S^b(A, \mathbf{b})$ , and hence,  $\mathbf{x} \leq \hat{\mathbf{x}}^b \leq \hat{\mathbf{x}}$  according to the construction of  $\hat{\mathbf{x}}^b$ , i.e.,  $\hat{\mathbf{x}}^b$  is the maximum Boolean solution in  $S^b(A, \mathbf{b})$ . Conversely, if  $S^b(A, \mathbf{b}) \neq \emptyset$ , then  $\mathbf{x} \leq \hat{\mathbf{x}}$ , and hence,  $\mathbf{x} \leq \hat{\mathbf{x}}^b \leq \hat{\mathbf{x}}$  for any  $\mathbf{x} \in S^b(A, \mathbf{b})$ . Therefore,  $\hat{\mathbf{x}}^b \in S^b(A, \mathbf{b})$  and also  $\hat{\mathbf{x}}^b$  is the maximum solution in  $S^b(A, \mathbf{b})$ .  $\square$

By Theorem 4.4.1, the existence of a Boolean solution to  $A \circ \mathbf{x} = \mathbf{b}$  can be verified by constructing and checking the potential maximum Boolean solution in a time complexity of  $O(mn)$ . Moreover, since  $S(A, \mathbf{b})$  is a finitely generated root system, the set of Boolean solutions  $S^b(A, \mathbf{b}) = S(A, \mathbf{b}) \cap \{0, 1\}^n$ , when it is nonempty, is also a finitely generated root system, i.e.,

$$S^b(A, \mathbf{b}) = \bigcup_{\hat{\mathbf{x}}^b \in \check{S}^b(A, \mathbf{b})} \{\mathbf{x} \in \{0, 1\}^n \mid \hat{\mathbf{x}}^b \leq \mathbf{x} \leq \hat{\mathbf{x}}^b\}, \quad (4.26)$$

where  $\check{S}^b(A, \mathbf{b})$  is the set of all minimal Boolean solutions of  $A \circ \mathbf{x} = \mathbf{b}$ , and  $\hat{\mathbf{x}}^b$  is the maximum Boolean solution induced by the maximum solution  $\hat{\mathbf{x}}$ . However,  $\check{S}^b(A, \mathbf{b})$  cannot be identified by checking the binary vectors induced by the minimal solutions in  $\check{S}(A, \mathbf{b})$ , but can be characterized by the characteristic matrix  $\tilde{Q}$  of  $A \circ \mathbf{x} = \mathbf{b}$  with necessary modifications.

**Theorem 4.4.2.** *Let  $A \circ \mathbf{x} = \mathbf{b}$  be a consistent system of sup- $\mathcal{T}$  equations with  $\mathcal{T}$  being a continuous t-norm and  $\tilde{Q}$  its characteristic matrix. The system  $A \circ \mathbf{x} = \mathbf{b}$  has a Boolean solution if and only if the binary matrix  $Q^b$  induced by  $\tilde{Q}$  has no zero rows, where  $Q^b = (q_{ij}^b)_{m \times n} \in \{0, 1\}^{m \times n}$  with*

$$q_{ij}^b = \begin{cases} 1, & \text{if } 1 \in \tilde{q}_{ij}, \\ 0, & \text{otherwise.} \end{cases} \quad (4.27)$$

Moreover, if  $S^b(A, \mathbf{b}) \neq \emptyset$ , there exists a one-to-one correspondence between the set of all minimal Boolean solutions  $\check{S}^b(A, \mathbf{b})$  and the set of all irredundant coverings  $\check{P}(Q^b)$  of the induced binary matrix  $Q^b$ .

*Proof.* By Theorem 4.1.1, a binary vector corresponds to a covering of  $Q^b$  if and only if it is a Boolean solution to  $A \circ \mathbf{x} = \mathbf{b}$ . Consequently,  $S^b(A, \mathbf{b}) \neq \emptyset$  if and only if  $Q^b$  has no zero rows. Moreover, when  $S^b(A, \mathbf{b}) \neq \emptyset$ , by Theorem 4.2.3, each minimal Boolean solution corresponds to an irredundant covering of  $Q^b$  and vice versa.  $\square$

According to Theorem 4.4.2, the determination of  $\check{S}^b(A, \mathbf{b})$  is equivalent to obtaining all irredundant coverings of the binary matrix  $Q^b$ . The number of minimal Boolean solutions is usually far less than the number of minimal solutions for a system of sup- $\mathcal{T}$  equations although, in general, it can be exponentially large with respect to the size of the input data.

**Example 4.4.3.** Consider the following system of sup- $\mathcal{T}$  equations  $A \circ \mathbf{x} = \mathbf{b}$  with  $\mathcal{T} = ((0.4, 0.8, \mathcal{T}_L))$  and

$$A = \begin{pmatrix} 0.8 & 1 & 0.9 & 1 & 1 \\ 0.9 & 0.7 & 0.8 & 0.8 & 1 \\ 0.4 & 0.4 & 0.6 & 0.5 & 0.2 \\ 0.6 & 0.3 & 0.1 & 0.4 & 0.9 \\ 0.2 & 0 & 0.2 & 0.1 & 0.8 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 0.8 \\ 0.6 \\ 0.4 \\ 0.2 \end{pmatrix}.$$

As has been shown in Example 4.3.1, the system has a maximum solution

$$\hat{\mathbf{x}} = (0.6, 1, 1, 1, 0.2)^T,$$

and its characteristic matrix is

$$\tilde{Q} = \begin{pmatrix} \emptyset & 1 & \emptyset & 1 & \emptyset \\ \emptyset & \emptyset & [0.8, 1] & [0.8, 1] & \emptyset \\ \emptyset & \emptyset & [0.8, 1] & \emptyset & \emptyset \\ [0.4, 0.6] & \emptyset & \emptyset & [0.4, 1] & \emptyset \\ [0.2, 0.6] & \emptyset & [0.2, 1] & \emptyset & 0.2 \end{pmatrix}.$$

Therefore, the binary vector induced by  $\hat{\mathbf{x}}$  is

$$\hat{\mathbf{x}}^b = (0, 1, 1, 1, 0)^T,$$

and the binary matrix induced by  $\tilde{Q}$  is

$$Q^b = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

By Theorems 4.4.1 and 4.4.2,  $S^b(A, \mathbf{b}) \neq \emptyset$  and  $\hat{\mathbf{x}}^b$  is the maximum Boolean solution. Moreover, columns 3 and 4 form the unique irredundant covering of  $Q^b$ , and hence,

$$\check{\mathbf{x}}^b = (0, 0, 1, 1, 0)^T$$

is the unique minimal Boolean solution.

## 4.5 Bibliographical Notes

The notion of normal form of a system of fuzzy relational equations was introduced in Czogała *et al.* [69] and Miyakoshi and Shimbo [304].

Similar types of characteristic matrices of sup- $\mathcal{T}$  equations have been introduced in the literature under various names, e.g., the index sets by Miyakoshi and Shimbo [304], Fang and Li [136], and Wu and Guu [486], the covering matrix by Cheng and Peng [56], the help matrix by Peeva [361, 366] and Peeva and Kyosev [369, 370], the solution-base matrix by Chen and Wang [54, 55], the matrix pattern by Luoh *et al.* [291, 292], the covering table by Markovskii [296, 297] and etc.

The close relation between minimal solutions of a system of sup- $\mathcal{T}$  equations and some *set covering problems* has been noticed and described from various aspects since the structure of the complete solution set was fully understood. See, for instance, Prévot [381], Czogała *et al.* [69], Higashi and Klir [205], Wang *et al.* [460], Miyakoshi and Shimbo [304], Peeva [361, 366], Wang and Hsu [456], Céchlárová [45], Luoh *et al.* [291, 292], Peeva and Kyosev [369, 370], Markovskii [296, 297], and Chen and Wang [54, 55]. However, with a small portion for sup- $\mathcal{T}_P$  equations, most of these results were developed for sup- $\mathcal{T}_M$  equations. More discussions on the minimal solutions may be found in Xu [492], Xu *et al.* [493], Di Nola [93], Peeva [360], Pappis and Sugeno [339], Bour and Lamotte [24], Han and Sekiguchi [199], Li [267], Arnould and Tano [7, 8, 9], Klir and Yuan [248], Bourke and Fisher [28], Loetamonphong and Fang [280], Shieh [415], Wu and Guu [486], and Yeh [500].

The unique solvability of a system of sup- $\mathcal{T}$  equations, as well as the existence of the minimum solution, was investigated by Di Nola and Sessa [108, 111], Sessa [410, 412], Lettieri and Liguori [260, 261], Ceclárová [44, 45], Li [265], Gavalec [150] and Gavalec and Plávka [152]. Various estimates of the number of the minimal solutions can be found in

Czogala *et al.* [69], Wang *et al.* [460], Shi [413], Peeva [361, 366], and Peeva and Kyosev [369, 370].

The resolution of sup- $\mathcal{T}$  equations is also related to the notions of linear independence and regularity. For a given matrix  $A = (a_{ij})_{m \times n} \in [0, 1]^{m \times n}$ , denote  $\mathbf{a}^j$ , for  $j \in N$ , its  $j$ th column. A vector  $\mathbf{b} \in [0, 1]^m$  is said to be a linear combination of  $\{\mathbf{a}^j \mid j \in N\}$  with respect to sup- $\mathcal{T}$  composition if there exists a vector  $\mathbf{x} \in [0, 1]^n$  such that  $A \circ \mathbf{x} = \mathbf{b}$ , where  $\mathcal{T}$  is a given continuous t-norm. In this context, the matrix  $A$  is said to have linearly independent columns if none of its columns can be expressed as a linear combination of the others with respect to sup- $\mathcal{T}$  composition. Moreover, in the case that  $A$  is a square matrix,  $A$  is called regular if  $A$  has linearly independent columns. This definition was first introduced by Cuninghame-Green [65] in the framework of max-algebra and can be extended to more general situations. To test whether or not a matrix  $A$  has linearly independent columns, it suffices to solve  $n$  systems of sup- $\mathcal{T}$  equations with a different column of  $A$  playing the role of the right-hand side vector each time. However, this task can be performed in a batch mode according to Theorem 3.2.5. For a given matrix  $A \in [0, 1]^{m \times n}$ , calculate  $A^T \varphi_t A$  and replace the entries on its main diagonal by 0. Denote the resulting matrix by  $\hat{X}$ . It is clear that  $A$  has linearly independent columns if and only if none of the columns of  $A$  is identical to the corresponding columns of the matrix  $A \circ \hat{X}$ . This elegant method was originally proposed by Cuninghame-Green [65] under the name  $\mathcal{A}$ -method, which was extended to the case of sup- $\mathcal{T}_M$  composition by Cuninghame-Green and Cechlárová [68].

The notion of linear independence typically leads to a theory of rank and dimension from a viewpoint of generalized vector spaces. However, as indicated by Cuninghame-Green [65] and Cechlárová and Plávka [52], some pathological phenomena may occur with the current definition of independence. For instance, one may construct a matrix with arbitrarily large number of linearly independent columns as well as a matrix with maximal submatrices containing different numbers of linearly independent columns. An alternative definition of independence has been suggested by Cuninghame-Green [65]. With a given continuous t-norm  $\mathcal{T}$ , a matrix  $A \in [0, 1]^{m \times n}$  is said to have strongly linearly independent columns if the system of sup- $\mathcal{T}$  equations  $A \circ \mathbf{x} = \mathbf{b}$  has a unique solution, i.e.,  $|S(A, \mathbf{b})| = 1$ , for some  $\mathbf{b} \in [0, 1]^m$ . In the case that  $A$  is a square matrix,  $A$  is called strongly regular if  $A$  has strongly linearly independent columns. Detailed discussions on this issue can be found in

Zimmermann [519], Gondran and Minoux [166], Kim and Roush [236], Kim [235], Butkovič and Hevery [39], Butkovič *et al.* [38], Cechlárová [44, 45], Butkovič [36, 37], Gavalec [150, 151], and Gavalec and Plávka [152].

The notion of computational complexity was introduced by Hartmanis and Stearns [202]. Edmonds [131, 132] argued that polynomial-time gives a good formalization of efficient computation and gave an informal description of nondeterministic polynomial-time, which set the stage for the  $P = NP$  question, the most famous problem in theoretical computer science. Cook [59, 60] and Karp [223] showed a large number of combinatorial and logical problems were NP-complete, i.e., as hard as any problem computable in nondeterministic polynomial time. The notation  $g(h) = O(h)$  means that  $g(h) \leq ch$  for some constant  $c$ . For more details on the theory of computational complexity, the reader may refer to Aho *et al.* [4], Garey and Johnson [144], Papadimitriou and Steiglitz [336], and Arora and Barak [10].

The *set covering problem* is known to be one of Karp's 21 NP-complete problems and has been extensively studied. See, for instance, Balas and Padberg [13], Caprara *et al.* [41] and Golubic and Hartman [165]. The relation between a system of sup- $\mathcal{T}_P$  equations and the *set covering problem* was presented by Markovskii [296, 297], which can be extended without difficulty to the case of continuous Archimedean t-norms. Chen and Wang [54] provided a proof by transforming polynomially the *minimum covering problem* into the problem of solving a system of sup- $\mathcal{T}_M$  equations to show that determining all minimal solutions of a system of sup- $\mathcal{T}_M$  is NP-hard.

The resolution technique via the conversion of a truth function was proposed by Markovskii [296, 297] and Peeva and Kyosev [370] for sup- $\mathcal{T}_P$  equations which, as has been shown, is valid for continuous Archimedean t-norms. The similar observation was also presented by Stamou and Tzafestas [420]. For more details on converting a truth function in CNF to its DNF, the reader may refer to Quine [385, 386], Kim [235], Wegener [472], and Milterson *et al.* [300].

The relation between the Archimedean property and the minimal solutions has been observed for a long time when solving sup- $\mathcal{T}_P$  equations. See, for instance, Loetamonphong and Fang [280], Markovskii [296, 297], and Peeva and Kyosev [370], as well as Wu and Guu [483] for strict t-norms. A version for continuous Archimedean t-norms is due to

Stamou and Tzafestas [420] and restated by Wu *et al.* [488] and Wu and Guu [486].

Ghodosian and Khorram [156] and Abbasi Molai and Khorram [2] discussed a class of sup- $\mathcal{O}_\lambda$  equations and its associated linear optimization problem where  $\mathcal{O}_\lambda$  is a convex combination of the *minimum* operator  $\mathcal{T}_M$  and the *average* operator  $\mathcal{O}_{av}$ , i.e., for every  $(x, y) \in [0, 1]^2$ ,

$$\mathcal{O}_\lambda(x, y) = \lambda\mathcal{T}_M(x, y) + (1 - \lambda)\mathcal{O}_{av}(x, y), \quad \lambda \in [0, 1].$$

It should be noted that this type of operators is nothing but compensatory operators defined by the *minimum* operator  $\mathcal{T}_M$  and the *maximum* operator  $\mathcal{S}_M$  since for each  $\lambda \in [0, 1]$ ,

$$\begin{aligned} \mathcal{O}_\lambda(x, y) &= \lambda\mathcal{T}_M(x, y) + (1 - \lambda)\mathcal{O}_{av}(x, y) \\ &= \lambda\mathcal{T}_M(x, y) + \frac{1 - \lambda}{2}(\mathcal{T}_M(x, y) + \mathcal{S}_M(x, y)) \\ &= \frac{1 + \lambda}{2}\mathcal{T}_M(x, y) + \frac{1 - \lambda}{2}\mathcal{S}_M(x, y). \end{aligned}$$

Boolean solutions of sup- $\mathcal{T}$  equations were considered in Di Nola and Ventre [117], Sessa [411], and Di Nola *et al.* [103].

## Chapter 5

# Sup- $\mathcal{T}$ Equation Constrained Optimization

This chapter is devoted to sup- $\mathcal{T}$  equation constrained optimization problems. The material presented in this chapter is essentially based on Li and Fang [268, 271, 272] and Li *et al.* [275]. The results in this chapter are original, and can be extended to more general situations.

A sup- $\mathcal{T}$  equation constrained optimization problem can be formulated as

$$\begin{aligned}
 & \min z = f(\mathbf{x}) \\
 \text{(OP-T)} \quad & \text{s.t.} \\
 & A \circ \mathbf{x} = \mathbf{b}, \\
 & \mathbf{x} \in [0, 1]^n,
 \end{aligned} \tag{5.1}$$

where  $f : [0, 1]^n \rightarrow \mathbb{R}$  is a real-valued function, and  $A \circ \mathbf{x} = \mathbf{b}$  is a system of sup- $\mathcal{T}$  equations with  $\mathcal{T}$  being a continuous t-norm. It is in general a nonconvex optimization problem even if the objective function is convex. Since the feasible set  $S(A, \mathbf{b})$ , when it is nonempty, is determined by one maximum solution and a finite number of minimal solutions, problem (OP-T) can be transformed into the following form:

$$\begin{aligned}
& \min z = f(\mathbf{y}^k) \\
& \text{s.t.} \\
\text{(GDO)} \quad & \mathbf{y}^k \in \operatorname{argmin}\{f(\mathbf{x}) \mid \check{\mathbf{x}}^k \leq \mathbf{x} \leq \hat{\mathbf{x}}\}, \\
& \check{\mathbf{x}}^k \in \check{S}(A, \mathbf{b}).
\end{aligned} \tag{5.2}$$

Problem (GDO) is also known as a generalized disjunctive optimization problem. However, when the set  $\check{S}(A, \mathbf{b})$  is hard to be identified or its cardinality  $|\check{S}(A, \mathbf{b})|$  is large, it could be very inefficient to obtain an optimal solution to problem (OP-T) via this transformation. Therefore, a systematic investigation on sup- $\mathcal{T}$  equation constrained optimization is required.

## 5.1 Sup- $\mathcal{T}$ Equations with a Linear Objective Function

When problem (OP-T) has a linear objective function, it is of the form

$$\begin{aligned}
& \min z = \mathbf{c}^T \mathbf{x} \\
& \text{s.t.} \\
\text{(LO-T)} \quad & A \circ \mathbf{x} = \mathbf{b}, \\
& \mathbf{x} \in [0, 1]^n,
\end{aligned} \tag{5.3}$$

where  $\mathbf{c} = (c_1, c_2, \dots, c_n)^T \in \mathbb{R}^n$  is the weight (or cost) vector, and  $c_j$  represents the weight associated with the variable  $x_j$  for each  $j \in N$ .

It turns out that a sup- $\mathcal{T}$  equation constrained linear optimization problem can be polynomially reduced to a *set covering problem*, and hence, is NP-hard.

**Theorem 5.1.1.** *Let  $A \circ \mathbf{x} = \mathbf{b}$  be a consistent system of sup- $\mathcal{T}$  equations with  $\mathcal{T}$  being a continuous  $t$ -norm. The maximum solution  $\hat{\mathbf{x}}$  is an optimal solution that minimizes the objective function  $z = \mathbf{c}^T \mathbf{x}$  over  $S(A, \mathbf{b})$  if  $c_j \leq 0$  for all  $j \in N$ . One of the minimal solutions is an optimal solution that minimizes the objective function  $z = \mathbf{c}^T \mathbf{x}$  over  $S(A, \mathbf{b})$  if  $c_j \geq 0$  for all  $j \in N$ .*

*Proof.* If  $c_j \leq 0$  for all  $j \in N$ , then  $\mathbf{c}^T \hat{\mathbf{x}} \leq \mathbf{c}^T \mathbf{x}$  for all  $\mathbf{x} \in S(A, \mathbf{b}) \neq \emptyset$  since  $\mathbf{0} \leq \mathbf{x} \leq \hat{\mathbf{x}}$  for all  $\mathbf{x} \in S(A, \mathbf{b})$ . Therefore,  $\hat{\mathbf{x}}$  is an optimal solution that minimizes the function  $\mathbf{c}^T \mathbf{x}$  over  $S(A, \mathbf{b})$ .

On the other hand, if  $c_j \geq 0$  for all  $j \in N$ , then there exists  $\tilde{\mathbf{x}} \in \check{S}(A, \mathbf{b})$  for any  $\mathbf{x} \in S(A, \mathbf{b}) \neq \emptyset$ , such that  $\tilde{\mathbf{x}} \leq \mathbf{x} \leq \hat{\mathbf{x}}$ , and hence,  $\mathbf{c}^T \tilde{\mathbf{x}} \leq \mathbf{c}^T \mathbf{x} \leq \mathbf{c}^T \hat{\mathbf{x}}$ . Therefore, any minimal solution  $\tilde{\mathbf{x}}^* \in \operatorname{argmin}\{\mathbf{c}^T \tilde{\mathbf{x}} \mid \tilde{\mathbf{x}} \in \check{S}(A, \mathbf{b})\}$  is an optimal solution that minimizes the function  $\mathbf{c}^T \mathbf{x}$  over  $S(A, \mathbf{b})$ .  $\square$

With the aid of Theorem 5.1.1, any given weight vector  $\mathbf{c} = (c_1, c_2, \dots, c_n)^T \in \mathbb{R}^n$  can be separated into two parts, i.e.,  $\mathbf{c}^+ = (c_1^+, c_2^+, \dots, c_n^+)^T$  and  $\mathbf{c}^- = (c_1^-, c_2^-, \dots, c_n^-)^T$  such that, for every  $j \in N$ ,

$$c_j^+ = \begin{cases} c_j, & \text{if } c_j > 0, \\ 0, & \text{if } c_j \leq 0, \end{cases} \quad \text{and} \quad c_j^- = \begin{cases} 0, & \text{if } c_j > 0, \\ c_j, & \text{if } c_j \leq 0. \end{cases} \quad (5.4)$$

Hence,  $\mathbf{c} = \mathbf{c}^+ + \mathbf{c}^-$  with  $\mathbf{c}^+ \geq \mathbf{0}$  and  $\mathbf{c}^- \leq \mathbf{0}$ . Two subproblems can be defined, respectively, as

$$\begin{aligned} \min \quad & z^+ = \sum_{j \in N} c_j^+ x_j \\ \text{s.t.} \quad & \\ & A \circ \mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \in [0, 1]^n, \end{aligned} \quad (5.5)$$

and

$$\begin{aligned} \min \quad & z^- = \sum_{j \in N} c_j^- x_j \\ \text{s.t.} \quad & \\ & A \circ \mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \in [0, 1]^n. \end{aligned} \quad (5.6)$$

**Theorem 5.1.2.** *Let  $A \circ \mathbf{x} = \mathbf{b}$  be a consistent system of sup- $\mathcal{T}$  equations with  $\mathcal{T}$  being a continuous  $t$ -norm. For any weight vector  $\mathbf{c} = (c_1, c_2, \dots, c_n)^T \in \mathbb{R}^n$ , the vector  $\mathbf{x}^* = (x_1^*, x_2^*, \dots, x_n^*)^T$  with*

$$x_j^* = \begin{cases} x_j^{+*}, & \text{if } c_j > 0, \\ \hat{x}_j, & \text{if } c_j \leq 0, \end{cases} \quad (5.7)$$

*is an optimal solution that minimizes the objective function  $\mathbf{c}^T \mathbf{x}$  over  $S(A, \mathbf{b})$ , where  $\mathbf{x}^{+*} = (x_1^{+*}, x_2^{+*}, \dots, x_n^{+*})^T$  is a solution that minimizes  $\sum_{j \in N} c_j^+ x_j$  over  $S(A, \mathbf{b})$ , and  $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n)^T$  is the maximum solution that minimizes  $\sum_{j \in N} c_j^- x_j$  over  $S(A, \mathbf{b})$ .*

*Proof.* Since  $\mathbf{x}^{+*} \leq \mathbf{x}^* \leq \hat{\mathbf{x}}$ , it is clear that  $\mathbf{x}^* \in S(A, \mathbf{b}) \neq \emptyset$ . Moreover, for any  $\mathbf{x} \in S(A, \mathbf{b}) \neq \emptyset$ , it holds that

$$\begin{aligned} \mathbf{c}^T \mathbf{x} &= (\mathbf{c}^+ + \mathbf{c}^-)^T \mathbf{x} = (\mathbf{c}^+)^T \mathbf{x} + (\mathbf{c}^-)^T \mathbf{x} \\ &\geq (\mathbf{c}^+)^T \mathbf{x}^{+*} + (\mathbf{c}^-)^T \mathbf{x}^{-*} = (\mathbf{c}^+)^T \mathbf{x}^* + (\mathbf{c}^-)^T \mathbf{x}^* \\ &= \mathbf{c}^T \mathbf{x}^*. \end{aligned}$$

Hence,  $\mathbf{x}^*$  is an optimal solution that minimizes the function  $\mathbf{c}^T \mathbf{x}$  over  $S(A, \mathbf{b})$ .  $\square$

According to Theorems 5.1.1 and 5.1.2, for an arbitrary weight vector, the problem of minimizing a linear function subject to a system of sup- $\mathcal{T}$  equations can be decomposed into two subproblems, one of which can be solved analytically while the other is not easy to handle. The subproblem with nonnegative weight vector is inevitably an NP-hard problem since the classical *set covering problem* can be regarded as a special scenario of this problem. On the other hand, this problem can be polynomially reduced to a *set covering problem* or a *constrained set covering problem*, where the existence of additional constraints depends on whether the involved continuous triangular norm is Archimedean or not.

**Theorem 5.1.3.** *Let  $A \circ \mathbf{x} = \mathbf{b}$  be a consistent system of sup- $\mathcal{T}$  equations with  $\mathcal{T}$  being a continuous Archimedean  $t$ -norm. Denote  $\hat{\mathbf{x}}$  its maximum solution and  $Q$  its associated simplified characteristic matrix. For any given weight vector  $\mathbf{c} = (c_1, c_2, \dots, c_n)^T$ , the following problem*

$$\begin{aligned} \min z_x &= \sum_{j \in N} c_j x_j \\ \text{s.t.} & \\ (LO-Ar) \quad & A \circ \mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \in [0, 1]^n, \end{aligned} \tag{5.8}$$

*is equivalent to the set covering problem*

$$\begin{aligned} \min z_u &= \sum_{j \in N} (c_j \hat{x}_j) u_j \\ \text{s.t.} & \\ (SCP) \quad & Q\mathbf{u} \geq \mathbf{e}, \\ & \mathbf{u} \in \{0, 1\}^n, \end{aligned} \tag{5.9}$$

*in the sense that any optimal solution  $\mathbf{u}^* = (u_1^*, u_2^*, \dots, u_n^*)^T$  to problem (SCP) defines an optimal solution  $\mathbf{x}^* = (\hat{x}_1 u_1^*, \hat{x}_2 u_2^*, \dots, \hat{x}_n u_n^*)^T$  to problem (LO-Ar), and any optimal*

solution  $\mathbf{x}^* = (x_1^*, x_2^*, \dots, x_n^*)^T$  to problem (LO-Ar) defines an optimal solution  $\mathbf{u}^* = (\hat{f}(x_1^*), \hat{f}(x_2^*), \dots, \hat{f}(x_n^*))^T$  to problem (SCP) where

$$\hat{f}(x_j^*) = \begin{cases} 1, & \text{if } x_j^* = \hat{x}_j, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for every } j \in N. \quad (5.10)$$

Besides, the optimal values of the two problems are equal, i.e.,  $z_x^* = z_u^*$ .

*Proof.* It follows from Theorems 4.2.2 and 5.1.2.  $\square$

**Theorem 5.1.4.** Let  $A \circ \mathbf{x} = \mathbf{b}$  be a consistent system of sup- $\mathcal{T}$  equations with  $\mathcal{T}$  being a continuous non-Archimedean  $t$ -norm. Denote  $\hat{\mathbf{x}}$  its maximum solution,  $Q$  its augmented characteristic matrix, and  $G$  the associated coefficient matrix of the innervariable incompatibility constraints. For any given weight vector  $\mathbf{c} = (c_1, c_2, \dots, c_n)^T$ , the following problem

$$\begin{aligned} (LO-nAr) \quad & \min z_x^+ = \sum_{j \in N} c_j^+ x_j \\ & \text{s.t.} \\ & A \circ \mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \in [0, 1]^n, \end{aligned} \quad (5.11)$$

is equivalent to the constrained set covering problem

$$\begin{aligned} (CSCP) \quad & \min z_u^+ = \sum_{j \in N} \sum_{k \in K_j} (c_j^+ \check{v}_{jk}) u_{jk} \\ & \text{s.t.} \\ & Q\mathbf{u} \geq \mathbf{e}^m, \\ & G\mathbf{u} \leq \mathbf{e}^n, \\ & \mathbf{u} \in \{0, 1\}^r, \end{aligned} \quad (5.12)$$

in the sense that any optimal solution  $\mathbf{u}^* = (u_{11}^*, \dots, u_{1r_1}^*, \dots, u_{n1}^*, \dots, u_{nr_n}^*)^T$  to problem (CSCP) defines an optimal solution

$$\mathbf{x}^{+*} = \left( \sum_{k \in K_1} \check{v}_{1k} u_{1k}^*, \sum_{k \in K_2} \check{v}_{2k} u_{2k}^*, \dots, \sum_{k \in K_n} \check{v}_{nk} u_{nk}^* \right)^T \quad (5.13)$$

to problem (LO-nAr), and any optimal solution  $\mathbf{x}^{+*} = (x_1^{+*}, x_2^{+*}, \dots, x_n^{+*})^T$  to problem (LO-nAr) defines an optimal solution  $\mathbf{u}^* = (\hat{f}'(x_1^{+*}), \hat{f}'(x_2^{+*}), \dots, \hat{f}'(x_n^{+*}))^T$  to problem (CSCP)

where  $\hat{f}'(x_j^{+*}) = (u_{j1}^*, u_{j2}^*, \dots, u_{jr_j}^*)^T$  and

$$u_{jk}^* = \begin{cases} 1, & \text{if } k = \operatorname{argmax}\{\check{v}_{jk} \mid \check{v}_{jk} \leq x_j^{+*}\}, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for every } j \in N. \quad (5.14)$$

Besides, the optimal values of the two problems are equal, i.e.,  $z_x^{+*} = z_u^{+*}$ .

*Proof.* It follows from Theorems 4.3.5 and 5.1.2. □

Once an optimal solution  $\mathbf{x}^{+*}$  to problem (LO-nAr) is obtained for any given weight vector  $\mathbf{c}$ , an optimal solution that minimizes the objective function  $\mathbf{c}^T \mathbf{x}$  over  $S(A, \mathbf{b})$  can be obtained according to Theorem 5.1.2. Actually, if there exists an index  $j \in N$  such that  $c_j < 0$ , then the variables  $u_{jk}$  for all  $k \in K_j$  can be aggregated as a single variable with the weight  $c_j \hat{x}_j$  in the *constrained set covering problem*. With this modification, the optimal solution can be constructed directly after solving the *constrained set covering problem*.

Moreover, note that in problem (CSCP), the weight  $c_j^+ \check{v}_{jk}$  is nonnegative for each  $k \in K_j$  and  $j \in N$ , and hence, the constraint  $G\mathbf{u} \leq \mathbf{e}^n$  can be further removed. Consequently, problem (CSCP) becomes a *set covering problem*, of which, each optimal solution will define an optimal solution to problem (LO-nAr) after taking some necessary modifications.

By Theorems 5.1.3 and 5.1.4, the problem of minimizing a linear function subject to a system of sup- $\mathcal{T}$  equations, with  $\mathcal{T}$  being a continuous t-norm, can be polynomially reduced to a *set covering problem*, which is known to be NP-hard. This reduction can be completed in a time complexity of  $O(mn)$  if the t-norm  $\mathcal{T}$  is Archimedean, or in a time complexity of  $O(m^2n)$  if it is non-Archimedean. Furthermore, taking the advantage of well developed techniques and clarity of exposition in the theory of integer programming, some methods may be applied to efficiently solve sup- $\mathcal{T}$  equation constrained linear optimization problems.

**Example 5.1.5.** *Reconsider the system of sup- $\mathcal{T}_M$  equations in Example 4.3.7 with an*

objective function  $z_x = -2x_1 + 2x_2 + x_3$ , i.e.,

$$\begin{aligned} \min \quad & z_x = -2x_1 + 2x_2 + x_3 \\ \text{s.t.} \quad & \begin{pmatrix} 0.8 & 0 & 0.8 \\ 0.6 & 0.6 & 0 \\ 0 & 0.4 & 0.2 \end{pmatrix} \circ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.6 \\ 0.4 \end{pmatrix}, \\ & x_1, x_2, x_3 \in [0, 1]. \end{aligned}$$

As shown in Example 4.3.7, the system of sup- $\mathcal{T}_M$  equations has the maximum solution  $\hat{\mathbf{x}} = (1, 1, 1)^T$  and three minimal solutions,  $\check{\mathbf{x}}^1 = (0.8, 0.4, 0)^T$ ,  $\check{\mathbf{x}}^2 = (0.6, 0.4, 0.8)^T$  and  $\check{\mathbf{x}}^3 = (0, 0.6, 0.8)^T$ . The associated characteristic matrix is

$$\tilde{Q} = \begin{pmatrix} [0.8, 1] & \emptyset & [0.8, 1] \\ [0.6, 1] & [0.6, 1] & \emptyset \\ \emptyset & [0.4, 1] & \emptyset \end{pmatrix}.$$

By evaluating the maximum solution and the three minimal solutions, the optimal solution of this problem can be constructed as  $\mathbf{x}^* = (1, 0.4, 0)^T$  according to Theorem 5.1.2.

On the other hand, in this case,  $\mathbf{c}^+ = (0, 2, 1)^T$  and

$$\check{\mathbf{v}} = (\check{v}_{11}, \check{v}_{12}, \check{v}_{21}, \check{v}_{22}, \check{v}_{31})^T = (0.8, 0.6, 0.6, 0.4, 0.8)^T. \quad (5.15)$$

Hence, the corresponding *constrained set covering problem* can be constructed as

$$\begin{aligned} \min \quad & z_u^+ = 0u_{11} + 0u_{12} + 1.2u_{21} + 0.8u_{22} + 0.8u_{31} \\ \text{s.t.} \quad & \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \\ u_{31} \end{pmatrix} \geq \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \\ & u_{11}, u_{12}, u_{21}, u_{22}, u_{31} \in \{0, 1\}. \end{aligned} \quad (5.16)$$

An optimal solution can be obtained as  $\mathbf{u}^* = (1, 0, 0, 1, 0)^T$ , which corresponds to the minimal solution  $\hat{\mathbf{x}}^1 = (0.8, 0.4, 0)^T$ . Hence, by Theorem 5.1.2, an optimal solution to the original problem can be constructed as  $\mathbf{x}^* = (1, 0.4, 0)^T$  with the optimal value  $z^* = -1.2$ .

Moreover, since  $c_1 < 0$  in this case, the variables  $u_{11}$  and  $u_{12}$  can be aggregated as  $u_1$  with a weight  $c_1 \hat{x}_1 = -2$ . The innervariable incompatibility constraint for  $u_{21}$  and  $u_{22}$  can be discarded. Consequently, the modified *set covering problem* becomes

$$\begin{aligned} \min \quad & z_u = -2u_1 + 1.2u_{21} + 0.8u_{22} + 0.8u_{31} \\ \text{s.t.} \quad & \\ & \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_{21} \\ u_{22} \\ u_{31} \end{pmatrix} \geq \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \\ & u_1, u_{21}, u_{22}, u_{31} \in \{0, 1\}, \end{aligned} \tag{5.17}$$

with an optimal solution  $\mathbf{u}^* = (1, 0, 1, 0)^T$ . An optimal solution to the original problem can be constructed directly as  $\mathbf{x}^* = (1, 0.4, 0)^T$ , which is the same as the one obtained by combining the maximum solution and the optimal solution to the subproblem with the nonnegative weight vector.

The procedure for solving sup- $\mathcal{T}$  equation constrained linear optimization problems can be directly extended to the case where the objective function is separable and monotone in each variable, i.e.,  $z_x = \sum_{j \in N} f_j(x_j)$  with  $f_j : [0, 1] \rightarrow \mathbb{R}$  being a monotone function for every  $j \in N$ . Without loss of generality, we may assume that  $f_j(0) = 0$  for every  $j \in N$ . In case the involved t-norm  $\mathcal{T}$  is a continuous Archimedean t-norm, minimizing an objective function  $z_x = \sum_{j \in N} f_j(x_j)$  subject to a consistent system of sup- $\mathcal{T}$  equations  $A \circ \mathbf{x} = \mathbf{b}$  is equivalent to the *set covering problem*

$$\begin{aligned} \min \quad & z_u = \sum_{j \in N} f_j(\hat{x}_j)u_j \\ \text{s.t.} \quad & \\ & Q\mathbf{u} \geq \mathbf{e}, \\ & \mathbf{u} \in \{0, 1\}^n, \end{aligned} \tag{5.18}$$

where  $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n)^T$  is the maximum solution, and  $Q$  is the associated simplified characteristic matrix. When  $\mathcal{T}$  is a continuous non-Archimedean t-norm, this problem is

equivalent to the *set covering problem*

$$\begin{aligned}
\min \quad & z_u = \sum_{j \in N^-} f_j(\hat{x}_j)u_j + \sum_{j \in N^+} \sum_{k \in K_j} f_j(\check{v}_{jk})u_{jk} \\
\text{s.t.} \quad & \\
& Q' \mathbf{u} \geq \mathbf{e}^m, \\
& \mathbf{u} \in \{0, 1\}^{r'},
\end{aligned} \tag{5.19}$$

where  $\mathbf{e}^m = (1, 1, \dots, 1)^T \in \{0, 1\}^m$ ,  $N^- = \{j \in N \mid f_j \text{ is a decreasing function}\}$ ,  $N^+ = N \setminus N^-$ , and  $Q' \in \{0, 1\}^{m \times r'}$  is the associated augmented characteristic matrix with some necessary modifications due to the aggregation of the variables corresponding to  $N^-$ .

**Example 5.1.6.** *Reconsider the system of sup- $\mathcal{T}_M$  equations in Example 4.3.7 with an objective function  $z_x = x_1(x_1 - 4) + (x_2)^2 + (x_3)^2$ , i.e.,*

$$\begin{aligned}
\min \quad & z_x = x_1(x_1 - 4) + (x_2)^2 + (x_3)^2 \\
\text{s.t.} \quad & \\
& \begin{pmatrix} 0.8 & 0 & 0.8 \\ 0.6 & 0.6 & 0 \\ 0 & 0.4 & 0.2 \end{pmatrix} \circ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.6 \\ 0.4 \end{pmatrix}, \\
& x_1, x_2, x_3 \in [0, 1].
\end{aligned}$$

As shown in Example 4.3.7, the system of sup- $\mathcal{T}_M$  equations has the maximum solution  $\hat{\mathbf{x}} = (1, 1, 1)^T$ . Its associated characteristic matrix is

$$\tilde{Q} = \begin{pmatrix} [0.8, 1] & \emptyset & [0.8, 1] \\ [0.6, 1] & [0.6, 1] & \emptyset \\ \emptyset & [0.4, 1] & \emptyset \end{pmatrix}.$$

In this case,  $N^- = \{1\}$ ,  $N^+ = \{2, 3\}$ , and

$$\check{\mathbf{v}} = (\check{v}_{11}, \check{v}_{12}, \check{v}_{21}, \check{v}_{22}, \check{v}_{31})^T = (0.8, 0.6, 0.6, 0.4, 0.8)^T.$$

Hence, the corresponding variables  $u_{11}$  and  $u_{12}$  can be aggregated as  $u_1$  with a weight

$\hat{x}_1(\hat{x}_1 - 4) = -3$ . The equivalent *set covering problem* is

$$\begin{aligned}
 \min \quad & z_u = -3u_1 + 0.36u_{21} + 0.16u_{22} + 0.64u_{31} \\
 \text{s.t.} \quad & \\
 & \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_{21} \\ u_{22} \\ u_{31} \end{pmatrix} \geq \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \\
 & u_1, u_{21}, u_{22}, u_{31} \in \{0, 1\},
 \end{aligned} \tag{5.20}$$

with an optimal solution  $\mathbf{u}^* = (1, 0, 1, 0)^T$ . Hence, an optimal solution to the original problem is  $\mathbf{x}^* = (1, 0.4, 0)^T$ . The optimal objective value is  $z_x^* = z_u^* = -2.84$ .

## 5.2 Sup- $\mathcal{T}$ Equations with a Linear Fractional Objective Function

The reduction techniques introduced in Section 5.1 can be performed in a similar manner when minimizing a linear fractional objective function subject to a consistent system of sup- $\mathcal{T}$  equations, i.e.,

$$\begin{aligned}
 \min \quad & z_x = \frac{\mathbf{c}^T \mathbf{x} + c_0}{\mathbf{d}^T \mathbf{x} + d_0} \\
 \text{(LFO-T)} \quad & \text{s.t.} \\
 & A \circ \mathbf{x} = \mathbf{b}, \\
 & \mathbf{x} \in [0, 1]^n,
 \end{aligned} \tag{5.21}$$

where  $\mathbf{c} = (c_1, c_2, \dots, c_n)^T$  and  $\mathbf{d} = (d_1, d_2, \dots, d_n)^T$  are two vectors,  $c_0$  and  $d_0$  are two scalars. It is customary to assume that the denominator is positive over the feasible domain, i.e.,  $\mathbf{d}^T \mathbf{x} + d_0 > 0$  for every  $\mathbf{x} \in S(A, \mathbf{b})$ .

### 5.2.1 Reduction to 0-1 Linear Fractional Optimization Problems

With the aid of Theorems 4.2.2 and 4.3.5, problem (LFO-T) can be reduced in polynomial time to a 0-1 linear fractional optimization problem which can be solved iteratively via parameterization methods.

Let  $A \circ \mathbf{x} = \mathbf{b}$  be a consistent system of sup- $\mathcal{T}$  equations with  $\mathcal{T}$  being a continuous t-norm, and  $\hat{\mathbf{x}}$  its maximum solution. For each  $j \in N$ , denote  $r_j$  the number of different values in  $\{\mathcal{J}_{\mathcal{T}}(a_{ij}, b_i) \mid \mathcal{T}(a_{ij}, \hat{x}_j) = b_i, i \in M\}$ ,  $K_j = \{1, 2, \dots, r_j\}$ , and denote  $\check{v}_{jk}$  for all  $k \in K_j$  the different values in  $\{\mathcal{J}_{\mathcal{T}}(a_{ij}, b_i) \mid \mathcal{T}(a_{ij}, \hat{x}_j) = b_i, i \in M\}$ . The following theorem reveals the relation between these values and optimal solutions to problem (LFO-T).

**Theorem 5.2.1.** *Let  $A \circ \mathbf{x} = \mathbf{b}$  be a consistent system of sup- $\mathcal{T}$  equations with  $\mathcal{T}$  being a continuous t-norm, and  $\hat{\mathbf{x}}$  its maximum solution. There exists an optimal solution  $\mathbf{x}^* = (x_1^*, x_2^*, \dots, x_n^*)^T$  to problem (LFO-T) such that  $x_j^* \in \{\check{v}_{jk} \mid k \in K_j\} \cup \{\hat{x}_j, 0\}$  for every  $j \in N$ .*

*Proof.* Suppose that  $\mathbf{x}^*$  is an optimal solution to problem (LFO-T) and there exists an index  $j_0 \in N$  such that  $x_{j_0}^* \notin \{\check{v}_{jk} \mid k \in K_j\} \cup \{\hat{x}_j, 0\}$ . Since the function

$$z(x) = \frac{c_{j_0}x + \sum_{j \neq j_0} c_j x_j^* + c_0}{d_{j_0}x + \sum_{j \neq j_0} d_j x_j^* + d_0} \quad (5.22)$$

is monotone on  $[0, 1]$ , the value of  $x_{j_0}^*$  can be either increased or decreased accordingly to a value in  $\{\check{v}_{jk} \mid k \in K_j\} \cup \{\hat{x}_j, 0\}$  without increasing the objective value. Therefore, such an optimal solution  $\mathbf{x}^*$  must exist with  $x_j^* \in \{\check{v}_{jk} \mid k \in K_j\} \cup \{\hat{x}_j, 0\}$  for every  $j \in N$ .  $\square$

**Corollary 5.2.2.** *Let  $A \circ \mathbf{x} = \mathbf{b}$  be a consistent system of sup- $\mathcal{T}$  equations with  $\mathcal{T}$  being a continuous Archimedean t-norm, and  $\hat{\mathbf{x}}$  its maximum solution. There exists an optimal solution  $\mathbf{x}^* = (x_1^*, x_2^*, \dots, x_n^*)^T$  to problem (LFO-T) such that either  $x_j^* = \hat{x}_j$  or  $x_j^* = 0$  for every  $j \in N$ .*

**Theorem 5.2.3.** *Let  $A \circ \mathbf{x} = \mathbf{b}$  be a consistent system of sup- $\mathcal{T}$  equations with  $\mathcal{T}$  being a continuous Archimedean t-norm, and  $Q$  its associated simplified characteristic matrix. The following problem*

$$\begin{aligned} \min z_x &= \frac{\mathbf{c}^T \mathbf{x} + c_0}{\mathbf{d}^T \mathbf{x} + d_0} \\ \text{(LFO-Ar)} \quad &\text{s.t.} \\ &A \circ \mathbf{x} = \mathbf{b}, \\ &\mathbf{x} \in [0, 1]^n, \end{aligned} \quad (5.23)$$

is equivalent to the 0-1 linear fractional optimization problem

$$\begin{aligned}
 \min z_u &= \frac{\hat{\mathbf{c}}^T \mathbf{u} + c_0}{\hat{\mathbf{d}}^T \mathbf{u} + d_0} \\
 \text{s.t.} & \\
 (0-1 \text{ LFO-I}) \quad & \mathbf{Q}\mathbf{u} \geq \mathbf{e}, \\
 & \mathbf{u} \in \{0, 1\}^n,
 \end{aligned} \tag{5.24}$$

where  $\hat{\mathbf{c}} = (c_1\hat{x}_1, c_2\hat{x}_2, \dots, c_n\hat{x}_n)^T$  and  $\hat{\mathbf{d}} = (d_1\hat{x}_1, d_2\hat{x}_2, \dots, d_n\hat{x}_n)^T$ , in the sense that any optimal solution  $\mathbf{u}^* = (u_1^*, u_2^*, \dots, u_n^*)^T$  to problem (0-1 LFO-I) defines an optimal solution  $\mathbf{x}^* = (\hat{x}_1 u_1^*, \hat{x}_2 u_2^*, \dots, \hat{x}_n u_n^*)^T$  to problem (LFO-Ar), and any optimal solution  $\mathbf{x}^* = (x_1^*, x_2^*, \dots, x_n^*)^T$  to problem (LFO-Ar) defines an optimal solution  $\mathbf{u}^* = (\hat{u}(x_1^*), \hat{u}(x_2^*), \dots, \hat{u}(x_n^*))^T$  to problem (0-1 LFO-I) where

$$\hat{u}(x_j^*) = \begin{cases} 1, & \text{if } x_j^* = \hat{x}_j, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for every } j \in N. \tag{5.25}$$

Besides, the optimal values of the two problems are equal, i.e.,  $z_x^* = z_u^*$ .

*Proof.* Let  $\mathbf{u}^*$  be an optimal solution to problem (0-1 LFO-I). By Theorem 4.2.2,  $\mathbf{x}^* = (\hat{x}_1 u_1^*, \hat{x}_2 u_2^*, \dots, \hat{x}_n u_n^*)^T$  is a feasible solution to problem (LFO-Ar) and  $z_x(\mathbf{x}^*) = z_u(\mathbf{u}^*)$ . If there exists  $\mathbf{x}^b \in S(A, \mathbf{b})$  such that  $z_x(\mathbf{x}^b) < z_x(\mathbf{x}^*)$ , then the vector  $\mathbf{x}^b$  can be further modified according to Corollary 5.2.2 without violating the inequality  $z_x(\mathbf{x}^b) < z_x(\mathbf{x}^*)$  such that either  $x_j^b = \hat{x}_j$  or  $x_j^b = 0$  for every  $j \in N$ . Consequently, the induced binary vector  $\mathbf{u}^b = (\hat{u}(x_1^b), \hat{u}(x_2^b), \dots, \hat{u}(x_n^b))^T$  is a feasible solution to problem (0-1 LFO-I) and  $z_u(\mathbf{u}^b) = z_x(\mathbf{x}^b) < z_x(\mathbf{x}^*) = z_u(\mathbf{u}^*)$ , which leads to a contradiction. Therefore,  $\mathbf{x}^* = (\hat{x}_1 u_1^*, \hat{x}_2 u_2^*, \dots, \hat{x}_n u_n^*)^T$  is an optimal solution to problem (LFO-Ar) when  $\mathbf{u}^*$  is an optimal solution to problem (0-1 LFO-I).

Conversely, let  $\mathbf{x}^*$  be an optimal solution to problem (LFO-Ar) such that either  $x_j^* = \hat{x}_j$  or  $x_j^* = 0$  for every  $j \in N$ . By Theorem 4.2.2, the binary vector  $\mathbf{u}^* = (\hat{u}(x_1^*), \hat{u}(x_2^*), \dots, \hat{u}(x_n^*))^T$  is a feasible solution to problem (0-1 LFO-I) and  $z_u(\mathbf{u}^*) = z_x(\mathbf{x}^*)$ . If there exists  $\mathbf{u}^b \in \{\mathbf{u} \in \{0, 1\}^n \mid \mathbf{Q}\mathbf{u} \geq \mathbf{e}\}$  such that  $z_u(\mathbf{u}^b) < z_u(\mathbf{u}^*)$ , then the induced vector  $\mathbf{x}^b = (\hat{x}_1 u_1^b, \hat{x}_2 u_2^b, \dots, \hat{x}_n u_n^b)^T$  is a feasible solution to problem (LFO-Ar) by Theorem 4.2.2 and  $z_x(\mathbf{x}^b) = z_u(\mathbf{u}^b) < z_u(\mathbf{u}^*) = z_x(\mathbf{x}^*)$ , which leads to a contradiction. Therefore,  $\mathbf{u}^* =$

$(\hat{u}(x_1^*), \hat{u}(x_2^*), \dots, \hat{u}(x_n^*))^T$  is an optimal solution to problem (0-1 LFO-I) when  $\mathbf{x}^*$  is an optimal solution to problem (LFO-Ar).  $\square$

When  $\mathcal{T}$  is a continuous non-Archimedean t-norm, problem (LFO-T) can be reduced to a 0-1 integer optimization problem in a similar manner as in Theorem 5.1.4. However, by Theorem 5.2.1, the values  $\hat{x}_j$  for all  $j \in N$  should be taken into consideration simultaneously.

Let  $A \circ \mathbf{x} = \mathbf{b}$  be a consistent system of sup- $\mathcal{T}$  equations with  $\mathcal{T}$  being a continuous non-Archimedean t-norm. Denote  $r'_j$  the number of different values in  $\{\mathcal{J}_{\mathcal{T}}(a_{ij}, b_i) \mid \mathcal{T}(a_{ij}, \hat{x}_j) = b_i, i \in M\} \cup \{\hat{x}_j\}$  for every  $j \in N$ , and  $r' = \sum_{j \in N} r'_j$ . Subsequently, for each  $j \in N$ , denote  $K'_j = \{1, 2, \dots, r'_j\}$ , and  $\check{v}'_{jk}$  for all  $k \in K'_j$  the different values in  $\{\mathcal{J}_{\mathcal{T}}(a_{ij}, b_i) \mid \mathcal{T}(a_{ij}, \hat{x}_j) = b_i, i \in M\} \cup \{\hat{x}_j\}$ . Let  $\check{\mathbf{v}}' = (\check{v}'_{11}, \dots, \check{v}'_{1r'_1}, \dots, \check{v}'_{n1}, \dots, \check{v}'_{nr'_n})^T \in [0, 1]^{r'}$  and

$$x_j = \sum_{k \in K'_j} \check{v}'_{jk} u_{jk}, \quad \text{for every } j \in N, \quad (5.26)$$

where  $u_{jk} \in \{0, 1\}$  for every  $k \in K'_j$  and  $j \in N$ . Accordingly, the innervariable incompatibility constraints should be modified as  $\sum_{k \in K'_j} u_{jk} \leq 1$  for every  $j \in N$ , and can be represented by  $G' \mathbf{u} \leq \mathbf{e}^n$ , where  $\mathbf{e}^n = (1, 1, \dots, 1)^T \in \{0, 1\}^n$ ,  $\mathbf{u} = (u_{11}, \dots, u_{1r'_1}, \dots, u_{n1}, \dots, u_{nr'_n})^T \in \{0, 1\}^{r'}$  and  $G' = (g'_{jk})_{n \times r'}$  with

$$g'_{jk} = \begin{cases} 1, & \text{if } \sum_{s=1}^{j-1} r'_s < k \leq \sum_{s=1}^j r'_s, \\ 0, & \text{otherwise.} \end{cases} \quad (5.27)$$

Note that the incompatibility constraints with  $r'_j = 1$  are redundant, and hence, can be removed. No additional difficulties will be imposed when  $r'_j = 1$  for every  $j \in N$ , or in other words, when the system  $A \circ \mathbf{x} = \mathbf{b}$  is simple.

Consequently, the augmented characteristic matrix of the characteristic matrix  $\tilde{Q}$  can be modified as  $Q' = (q'_{ik})_{m \times r'} \in \{0, 1\}^{m \times r'}$  where

$$q'_{ik} = \begin{cases} 1, & \text{if } \sum_{s=1}^{j-1} r'_s < k \leq \sum_{s=1}^j r'_s \text{ and } \check{v}'_k \in \tilde{q}_{ij} \text{ for some } j \in N, \\ 0, & \text{otherwise.} \end{cases} \quad (5.28)$$

Compared with the original augmented characteristic matrix, the modified augmented characteristic matrix  $Q'$  simply duplicates the column corresponding to the value  $\max\{\check{v}'_{jk} \mid k \in$

$K_j\}$  for each  $j \in N$  whenever  $\max\{\check{v}_{jk} \mid k \in K_j\} \neq \hat{x}_j$ . Therefore, Theorem 4.3.5 remains valid with the matrices  $Q$  and  $G$  replaced by  $Q'$  and  $G'$ , respectively. Due to Theorems 5.1.1 and 5.1.2, this modification is not necessary when dealing with the sup- $\mathcal{T}$  equation constrained optimization problem with a linear objective function or a separable and monotone objective function.

**Theorem 5.2.4.** *Let  $A \circ \mathbf{x} = \mathbf{b}$  be a consistent system of sup- $\mathcal{T}$  equations with  $\mathcal{T}$  being a continuous non-Archimedean  $t$ -norm,  $Q'$  its modified augmented characteristic matrix, and  $G'$  the associated coefficient matrix of the innervariable incompatibility constraints. The following problem*

$$\begin{aligned}
 & \min z_x = \frac{\mathbf{c}^T \mathbf{x} + c_0}{\mathbf{d}^T \mathbf{x} + d_0} \\
 \text{(LFO-nAr)} \quad & \text{s.t.} \\
 & A \circ \mathbf{x} = \mathbf{b}, \\
 & \mathbf{x} \in [0, 1]^n,
 \end{aligned} \tag{5.29}$$

is equivalent to the 0-1 linear fractional optimization problem

$$\begin{aligned}
 & \min z_u = \frac{\sum_{j \in N} \sum_{k \in K'_j} (c_j \check{v}'_{jk}) u_{jk} + c_0}{\sum_{j \in N} \sum_{k \in K'_j} (d_j \check{v}'_{jk}) u_{jk} + d_0} \\
 \text{(0-1 LFO-II)} \quad & \text{s.t.} \\
 & Q' \mathbf{u} \geq \mathbf{e}^m, \\
 & G' \mathbf{u} \leq \mathbf{e}^n, \\
 & \mathbf{u} \in \{0, 1\}^{r'},
 \end{aligned} \tag{5.30}$$

in the sense that any optimal solution  $\mathbf{u}^* = (u_{11}^*, \dots, u_{1r'_1}^*, \dots, u_{n1}^*, \dots, u_{nr'_n}^*)^T$  to problem (0-1 LFO-II) defines an optimal solution

$$\mathbf{x}^* = \left( \sum_{k \in K'_1} \check{v}'_{1k} u_{1k}^*, \sum_{k \in K'_2} \check{v}'_{2k} u_{2k}^*, \dots, \sum_{k \in K'_n} \check{v}'_{nk} u_{nk}^* \right)^T \tag{5.31}$$

to problem (LO-nAr), and any optimal solution  $\mathbf{x}^* = (x_1^*, x_2^*, \dots, x_n^*)^T$  to problem (LFO-nAr) defines an optimal solution  $\mathbf{u}^* = (\hat{f}'(x_1^*), \hat{f}'(x_2^*), \dots, \hat{f}'(x_n^*))^T$  to problem (0-1 LFO-II)

where  $\hat{f}'(x_j^*) = (u_{j1}^*, u_{j2}^*, \dots, u_{jr'_j}^*)^T$  and

$$u_{jk}^* = \begin{cases} 1, & \text{if } k = \operatorname{argmax}\{\check{v}'_{jk} \mid \check{v}'_{jk} \leq x_j^*\}, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for every } j \in N. \quad (5.32)$$

Besides, the optimal values of the two problems are equal, i.e.,  $z_x^* = z_u^*$ .

*Proof.* Note that there exists an optimal solution to problem (LFO-nAr) that assumes the form indicated in Theorem 5.2.1. In addition, Theorem 4.3.5 remains valid with respect to the modified augmented characteristic matrix  $Q'$  and the coefficient matrix of the modified innervariable incompatibility constraints  $G'$ . Hence, a similar proof can be drawn as that for Theorem 5.2.3.  $\square$

## 5.2.2 Dinkelbach's Algorithm

The early investigation of the 0-1 linear fractional optimization problem is due to Hammer and Rudeanu [195]. Jagannathan [214] provided a theoretical foundation on the relationship between nonlinear fractional optimization and nonlinear parametric optimization. Subsequently, Dinkelbach [91] developed an iterative method which can be extended to more general situations. In this particular context, Jagannathan's theorem may be stated as follows:

**Theorem 5.2.5.** *The binary vector  $\mathbf{u}^*$  is an optimal solution to problem (0-1 LFO-II) if and only if  $\mathbf{u}^*$  is an optimal solution to the following parameterized constrained set covering problem:*

$$\begin{aligned} \min \quad z_\lambda &= \left( \sum_{j \in N} \sum_{k \in K'_j} (c_j \check{v}'_{jk}) u_{jk} + c_0 \right) - \lambda \left( \sum_{j \in N} \sum_{k \in K'_j} (d_j \check{v}'_{jk}) u_{jk} + d_0 \right) \\ \text{s.t.} \quad & \\ \text{(CSCP-}\lambda\text{)} \quad & Q' \mathbf{u} \geq \mathbf{e}^m, \\ & G' \mathbf{u} \leq \mathbf{e}^n, \\ & \mathbf{u} \in \{0, 1\}^{r'}, \end{aligned} \quad (5.33)$$

with the parameter  $\lambda = z_u(\mathbf{u}^*)$ .

Theorem 5.2.5 seems a little awkward in some sense. However, it has been shown that the optimal objective value of problem (CSCP- $\lambda$ )  $z_\lambda^* = 0$  if and only if the parameter  $\lambda$  is

selected as the optimal objective value of problem (0-1 LFO-II), i.e.,  $\lambda = z_u^*$ . Therefore, the optimal solution to problem (0-1 LFO-II) can be approached by iteratively solving a sequence of (CSCP- $\lambda$ ) problems. Dinkelbach's Algorithm is one of such general purpose methods, which may be described in this context as follows:

**Dinkelbach's Algorithm:**

**Step 1:** Let  $\mathbf{u}^0$  be a feasible solution to problem (0-1 LFO-II) and  $\lambda^1 = z_u(\mathbf{u}^0)$ . Set  $k = 1$  and go to Step 2.

**Step 2:** Solve the *constrained set covering problem* (CSCP- $\lambda$ ) with the parameter  $\lambda^k$ . Denote the optimal solution by  $\mathbf{u}^k$  and the optimal objective value by  $z_\lambda^k$ .

**Step 3:** If  $z_\lambda^k = 0$ , then  $\mathbf{u}^k$  is an optimal solution to problem (0-1 LFO-II) and stop. Otherwise, set  $\lambda^{k+1} = z_u(\mathbf{u}^k)$ ,  $k = k + 1$ , and go to Step 2.

Dinkelbach's algorithm terminates in a finite number of iterations and guarantees the global optimum in this case since the value  $z_\lambda^k$  strictly decreases/increases in each iteration. Moreover, Dinkelbach's Algorithm also applies to problem (0-1 LFO-I) since it is simply a special scenario of problem (0-1 LFO-II).

**Example 5.2.6.** Consider the following linear fractional optimization problem subject to a system of sup- $\mathcal{T}_P$  equations, which was originally presented by Wu et al. [488]:

$$\min z_x = \frac{x_1 + x_2 - 2x_3 + 5x_4 + x_5 - x_6 + x_7 - 3x_8 + 2x_9 + 5}{x_1 + 2x_2 - x_3 - x_4 - x_5 - 3x_6 + 2x_7 + x_8 + x_9 + 3}$$

s.t.

$$\begin{pmatrix} 0.6 & 0.5 & 0.84 & 0.1 & 0.7 & 0.6 & 0.4 & 0.56 & 0.2 \\ 0.4 & 0.45 & 0.96 & 0.5 & 0.8 & 0.75 & 0.5 & 0.64 & 0.5 \\ 0.6 & 0.2 & 1.0 & 0.625 & 0.8 & 0.1 & 0.4 & 0.4 & 0.5 \\ 0.8 & 0.5 & 0.7 & 0.7 & 0.8 & 0.875 & 0.65 & 0.3 & 0.4 \\ 0.9 & 0.75 & 0.5 & 0.4 & 0.8 & 0.2 & 0.875 & 0.5 & 0.7 \\ 0.5 & 0.7 & 0.7 & 0.8 & 0.5 & 1.0 & 0.7 & 0.2 & 0.65 \\ 0.2 & 0.8 & 0.4 & 0.7 & 0.5 & 0.8 & 1.0 & 0.96 & 0.8 \\ 0.8 & 1.0 & 0.7 & 0.4 & 0.8 & 0.3 & 0.4 & 0.7 & 0.2 \end{pmatrix} \circ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \end{pmatrix} = \begin{pmatrix} 0.42 \\ 0.48 \\ 0.5 \\ 0.56 \\ 0.63 \\ 0.64 \\ 0.72 \\ 0.84 \end{pmatrix},$$

$$x_1, x_2, \dots, x_9 \in [0, 1].$$

It can be verified that

$$\hat{\mathbf{x}} = (0.7, 0.84, 0.5, 0.8, 0.6, 0.64, 0.72, 0.75, 0.9)^T.$$

is the maximum solution. Subsequently, this problem can be converted into a 0-1 linear fractional optimization problem as following:

$$\min z_u = \frac{0.7u_1 + 0.84u_2 - u_3 + 4u_4 + 0.6u_5 - 0.64u_6 + 0.72u_7 - 2.25u_8 + 1.8u_9 + 5}{0.7u_1 + 1.68u_2 - 0.5u_3 - 0.8u_4 - 0.6u_5 - 1.92u_6 + 1.44u_7 + 0.75u_8 + 0.9u_9 + 3}$$

s.t.

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{pmatrix} \geq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

$$u_1, u_2, \dots, u_9 \in \{0, 1\}.$$

Set the initial solution  $\mathbf{u}^0 = (1, 1, \dots, 1)^T$ . Consequently,  $\lambda^1 = 2.10$  and  $z_\lambda^1 = -7.54$  with  $\mathbf{u}^1 = (1, 1, 1, 0, 0, 1, 1, 1, 1)^T$ . The Dinkelbach's algorithm terminates after 3 iterations and yields an optimal solution  $\mathbf{u}^* = (0, 1, 1, 0, 0, 1, 1, 1, 0)^T$  with the objective value  $z_u^* = 0.6$ . An optimal solution to the original problem can be obtained as

$$\mathbf{x}^* = (0, 0.84, 0.5, 0, 0, 0.64, 0.72, 0.75, 0)^T$$

with the objective value  $z_x^* = 0.6$ . The iterations of the algorithm are illustrated in Figure 5.1.

**Example 5.2.7.** Consider the following linear fractional optimization problem subject to a

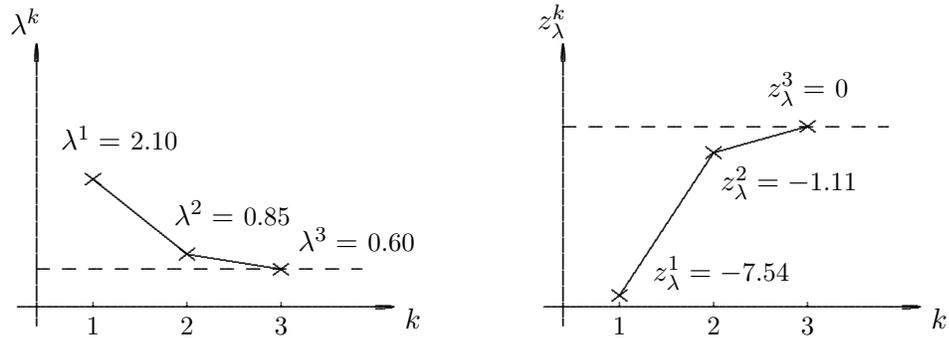


Figure 5.1: An illustration of Dinkelbach's algorithm for Example 5.2.6

system of sup- $\mathcal{T}_M$  equations:

$$\min z_x = \frac{x_1 + x_2 - 2x_3 + 5}{x_1 + 2x_2 - x_3 + 3}$$

s.t.

$$\begin{pmatrix} 1 & 0.6 & 0.8 \\ 0.8 & 0.6 & 0.5 \\ 0.4 & 0.2 & 0.4 \end{pmatrix} \circ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.6 \\ 0.4 \end{pmatrix},$$

$$x_1, x_2, x_3 \in [0, 1].$$

It can be verified that

$$\hat{x} = (0.6, 1, 1)^T$$

is the maximum solution, and the associated characteristic matrix is

$$\tilde{Q} = \begin{pmatrix} \emptyset & \emptyset & [0.8, 1] \\ 0.6 & [0.6, 1] & \emptyset \\ [0.4, 0.6] & \emptyset & [0.4, 1] \end{pmatrix}.$$

Denote

$$\check{v}' = (\check{v}'_{11}, \check{v}'_{12}, \check{v}'_{21}, \check{v}'_{22}, \check{v}'_{31}, \check{v}'_{32}, \check{v}'_{33})^T = (0.4, 0.6, 0.6, 1, 0.4, 0.8, 1)^T.$$

The concerned optimization problem can be converted into a 0-1 linear fractional optimiza-

tion problem as following:

$$\begin{aligned} \min z_u &= \frac{0.4u_{11} + 0.6u_{12} + 0.6u_{21} + u_{22} - 0.8u_{31} - 1.6u_{32} - 2u_{33} + 5}{0.4u_{11} + 0.6u_{12} + 1.2u_{21} + 2u_{22} - 0.4u_{31} - 0.6u_{32} - u_{33} + 3} \\ \text{s.t.} \quad & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \\ u_{31} \\ u_{32} \\ u_{33} \end{pmatrix} \geq \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}, \\ & u_1, u_2, \dots, u_7 \in \{0, 1\}. \end{aligned}$$

Set the initial solution  $\mathbf{u}^0 = (1, 1, \dots, 1)^T$ . Consequently,  $\lambda^1 = 0.62$  and  $z_\lambda^1 = 1.52$  with  $\mathbf{u}^1 = (0, 0, 0, 1, 0, 0, 1)^T$ . The Dinkelbach's algorithm terminates after 2 iterations and yields an optimal solution  $\mathbf{u}^* = (1, 0, 0, 1, 0, 1, 0)^T$  with the objective value  $z_u^* = 1$ . An optimal solution of the original problem can be obtained as

$$\mathbf{x}^* = (0.4, 1, 0.8)^T$$

with the objective value  $z_x^* = 1$ . The iterations of the algorithm are illustrated in Figure 5.2.

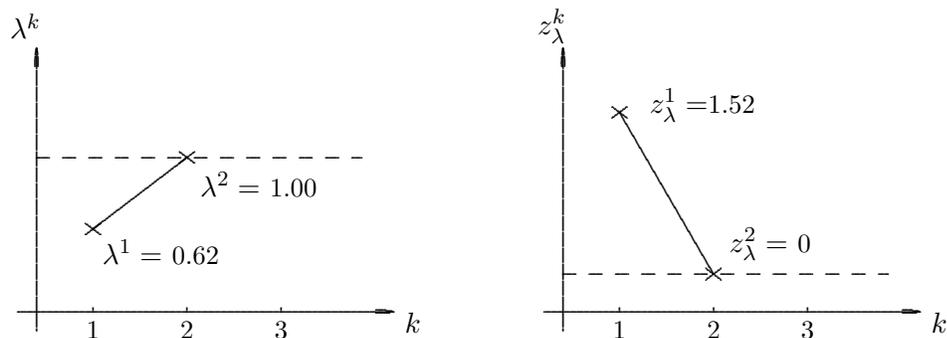


Figure 5.2: An illustration of Dinkelbach's algorithm for Example 5.2.7

Note that  $\mathbf{u}^0 = (1, 1, \dots, 1)^T$  can always be set as the initial solution as long as the feasible domain  $S(A, \mathbf{b})$  is nonempty. However, it may not necessarily serve as a good

initial solution of the optimization problem. Some heuristic methods may be applied to start Dinkelbach's algorithm. Besides, some preprocessing techniques may be applied to reduce the size of the problem. Notice that, other than Dinkelbach's algorithm, various methods have been proposed to solve the 0-1 linear fractional optimization problem and hence can be applied to solve the fuzzy relational equation constrained linear fractional optimization problem. These techniques are subject to further investigation.

Note that Theorem 5.2.1 plays an critical role in solving sup- $\mathcal{T}$  equation constrained linear fractional optimization problems. The validity of Theorem 5.2.1 essentially depends on the fact that a linear fractional function is monotone in each variable separately. Therefore, the reduction techniques introduced in this section can be applied on the following sup- $\mathcal{T}$  equation constrained optimization problem

$$\begin{aligned}
 & \min \quad z_x = f(\mathbf{x}) \\
 & \text{s.t.} \\
 \text{(MNLO-T)} \quad & A \circ \mathbf{x} = \mathbf{b}, \\
 & \mathbf{x} \in [0, 1]^n,
 \end{aligned} \tag{5.34}$$

where the objective function  $f(\mathbf{x})$  is monotone in each variable separately. Linear functions and linear fractional functions are scenarios of such type of functions. In this case, as long as  $S(A, \mathbf{b}) \neq \emptyset$ , there exists an optimal solution  $\mathbf{x}^* = (x_1^*, x_2^*, \dots, x_n^*)^T$  to problem (MNLO-T) such that  $x_j^* \in \{\check{v}_{jk} \mid k \in K_j\} \cup \{\hat{x}_j, 0\}$  for every  $j \in N$ . In case  $\mathcal{T}$  is a continuous Archimedean t-norm, it holds that either  $x_j^* = \hat{x}_j$  or  $x_j^* = 0$  for every  $j \in N$ .

Consequently, problem (MNLO-T) can be converted in polynomial time to a general 0-1 integer optimization problem in a similar manner as described in Theorems 5.2.3 and 5.2.4. In particular, when  $\mathcal{T}$  is a continuous Archimedean t-norm, problem (MNLO-T) is equivalent to the following optimization problem:

$$\begin{aligned}
 & \min \quad z_u = \hat{f}(\mathbf{u}) \\
 & \text{s.t.} \\
 \text{(0-1 MNLO-I)} \quad & Q\mathbf{u} \geq \mathbf{e}, \\
 & \mathbf{u} \in \{0, 1\}^n,
 \end{aligned} \tag{5.35}$$

where  $Q$  is the simplified characteristic matrix of the system  $A \circ \mathbf{x} = \mathbf{b}$  and  $\hat{f}(\mathbf{u}) = f(\hat{x}_1 u_1, \dots, \hat{x}_n u_n)$  for all  $\mathbf{u} \in \{0, 1\}^n$ . In general, problem (MNLO-T) is equivalent to

the following optimization problem:

$$\begin{aligned}
& \min \quad z_u = \hat{f}'(\mathbf{u}) \\
& \text{s.t.} \\
(0-1 \text{ MNLO-II}) \quad & Q'\mathbf{u} \geq \mathbf{e}^m, \\
& G'\mathbf{u} \leq \mathbf{e}^n, \\
& \mathbf{u} \in \{0, 1\}^{r'},
\end{aligned} \tag{5.36}$$

where  $Q'$  and  $G'$  are the modified augmented characteristic matrix and the modified coefficient matrix of innervariable incompatibility constraints, respectively, and for every  $\mathbf{u} = (u_{11}, \dots, u_{1r'_1}, \dots, u_{n1}, \dots, u_{nr'_n}) \in \{0, 1\}^{r'}$ ,

$$\hat{f}'(\mathbf{u}) = f\left(\sum_{k \in K'_1} \check{v}'_{1k} u_{1k}, \dots, \sum_{k \in K'_n} \check{v}'_{nk} u_{nk}\right). \tag{5.37}$$

Moreover, if  $f(\mathbf{x})$  is separable and monotone in each variable, e.g., a linear function, the function  $\hat{f}'(\mathbf{u})$ , as well as  $\hat{f}(\mathbf{u})$  in case  $\mathcal{T}$  is Archimedean, can be further simplified as shown in Section 5.1.

Note that if the objective function  $f(\mathbf{x}) : [0, 1]^n \rightarrow \mathbb{R}$  in problem (MNLO-T) fails to be monotone in each variable separately, for instance, a quadratic function, the reduction to a 0-1 integer optimization problem could be invalid. However, as shown in Section 5.3, such an optimization problem can always be reduced into a general 0-1 mixed integer optimization problem.

### 5.3 Sup- $\mathcal{T}$ Equations with a General Nonlinear Objective Function

Consider the following general sup- $\mathcal{T}$  equation constrained optimization problem:

$$\begin{aligned}
& \min \quad z = f(\mathbf{x}) \\
& \text{s.t.} \\
(OP-T) \quad & A \circ \mathbf{x} = \mathbf{b}, \\
& \mathbf{x} \in [0, 1]^n,
\end{aligned} \tag{5.38}$$

where  $f : [0, 1]^n \rightarrow \mathbb{R}$  is a real-valued function, and  $A \circ \mathbf{x} = \mathbf{b}$  is a consistent system of sup- $\mathcal{T}$  equations with  $\mathcal{T}$  being a continuous t-norm. Let  $\hat{\mathbf{x}}$  be the maximum solution,  $Q$  the

augmented characteristic matrix, and  $G$  the associated coefficient matrix of innervariable incompatibility constraints. For each  $j \in N$ , denote  $r_j$  the number of different values in  $\{\mathcal{J}_{\mathcal{T}}(a_{ij}, b_i) \mid \mathcal{T}(a_{ij}, \hat{x}_j) = b_i, i \in M\}$ ,  $K_j = \{1, 2, \dots, r_j\}$ , and denote  $\check{v}_{jk}$  for all  $k \in K_j$  the different values in  $\{\mathcal{J}_{\mathcal{T}}(a_{ij}, b_i) \mid \mathcal{T}(a_{ij}, \hat{x}_j) = b_i, i \in M\}$ . Let  $\check{\mathbf{v}}^j = (\check{v}_{j1}, \check{v}_{j2}, \dots, \check{v}_{jr_j})^T$  for every  $j \in N$  and  $\check{V} = (\text{diag}(\check{\mathbf{v}}^1, \check{\mathbf{v}}^2, \dots, \check{\mathbf{v}}^n))^T$ . The vector  $\check{\mathbf{v}}^j$  contains all possible values that  $x_j$  may assume in a minimal solution to  $A \circ \mathbf{x} = \mathbf{b}$  such that  $\mathcal{T}(a_{ij}, x_j) = b_i$  for some  $i \in M$ . Unfortunately, in an optimal solution to problem (OP-T),  $x_j$  may assume a value other than  $\{\check{v}_{jk} \mid k \in K_j\} \cup \{\hat{x}_j, 0\}$ . However, with the aid of Theorem 4.3.5, problem (OP-T) still can be reduced to the following 0-1 mixed integer programming problem:

$$\begin{aligned}
& \min \quad z_u = f(\mathbf{x}) \\
& \text{s.t.} \\
(0-1 \text{ MIO-II}) \quad & Q\mathbf{u} \geq \mathbf{e}^m, \\
& G\mathbf{u} \leq \mathbf{e}^n, \\
& \check{V}\mathbf{u} \leq \mathbf{x} \leq \hat{\mathbf{x}}, \\
& \mathbf{u} \in \{0, 1\}^r.
\end{aligned} \tag{5.39}$$

Note that in problem (0-1 MIO-II), the binary vector  $\mathbf{u}$  serves merely as a control vector and does not appear in the objective function  $f(\mathbf{x})$ . Moreover, the constraint  $\sum_{k \in K_j} q_{jk} u_{jk} \leq 1$  is redundant whenever  $r_j = 1$ , and hence, can be removed. Therefore, in case  $\mathcal{T}$  is a continuous Archimedean t-norm, problem (OP-T) can be reduced to the following 0-1 mixed integer programming problem:

$$\begin{aligned}
& \min \quad z_u = f(\mathbf{x}) \\
& \text{s.t.} \\
(0-1 \text{ MIO-I}) \quad & Q\mathbf{u} \geq \mathbf{e}^m, \\
& \check{V}\mathbf{u} \leq \mathbf{x} \leq \hat{\mathbf{x}}, \\
& \mathbf{u} \in \{0, 1\}^n,
\end{aligned} \tag{5.40}$$

which contains  $n$  binary variables and  $n$  continuous variables.

**Example 5.3.1.** *Reconsider the system of sup- $\mathcal{T}_M$  equations in Example 4.3.7 with an*

objective function  $z_x = (2x_1 + x_2)^2 + (x_2 - 2x_3)^2$ , i.e.,

$$\begin{aligned} \min \quad & z_x = (2x_1 + x_2)^2 + (x_2 - 2x_3)^2 \\ \text{s.t.} \quad & \begin{pmatrix} 0.8 & 0 & 0.8 \\ 0.6 & 0.6 & 0 \\ 0 & 0.4 & 0.2 \end{pmatrix} \circ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.6 \\ 0.4 \end{pmatrix}, \\ & x_1, x_2, x_3 \in [0, 1]. \end{aligned}$$

As shown in Example 4.3.7, the system of sup- $\mathcal{T}_M$  equations has an maximum solution  $\hat{\mathbf{x}} = (1, 1, 1)^T$ . The associated characteristic matrix is

$$\tilde{Q} = \begin{pmatrix} [0.8, 1] & \emptyset & [0.8, 1] \\ [0.6, 1] & [0.6, 1] & \emptyset \\ \emptyset & [0.4, 1] & \emptyset \end{pmatrix}.$$

Hence, the augmented characteristic matrix  $Q$  and the coefficient matrix of innervariable incompatibility constraints  $G$  are, respectively,

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that the 3rd row of  $G$  is redundant and can be removed. Moreover,  $\check{\mathbf{v}}^1 = (0.8, 0.6)^T$ ,  $\check{\mathbf{v}}^2 = (0.6, 0.4)^T$ ,  $\check{\mathbf{v}}^3 = 0.8$  and

$$\check{V} = \begin{pmatrix} 0.8 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 0.4 & 0 \\ 0 & 0 & 0 & 0 & 0.8 \end{pmatrix}.$$

Denote  $\mathbf{u} = (u_{11}, u_{12}, u_{21}, u_{22}, u_{31})^T$ . The concerned optimization problem is equivalent to

the following 0-1 mixed integer programming problem:

$$\begin{aligned}
 & \min (2x_1 + x_2)^2 + (x_2 - 2x_3)^2 \\
 & \text{s.t.} \\
 & \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ -0.8 & -0.6 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -0.6 & -0.4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -0.8 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \\ u_{31} \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \geq \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \\
 & u_{11}, u_{12}, u_{21}, u_{22}, u_{31} \in \{0, 1\}, x_1, x_2, x_3 \in [0, 1].
 \end{aligned}$$

Using a commercial solver, e.g., CPLEX, we can find an optimal solution

$$(\mathbf{u}^*; \mathbf{x}^*) = (0, 0, 1, 0, 1, 0, 0.8, 0.8)^T$$

with an objective value of 1.28. Hence  $\mathbf{x}^* = (0, 0.8, 0.8)^T$  is an optimal solution to the original nonlinear optimization problem subject to a system of sup- $\mathcal{T}_M$  equations.

## 5.4 Sup- $\mathcal{T}$ Equations with a Max-Separable Objective Function

Although it is NP-hard in general, problem (OP-T) can be solved efficiently when the objective function is max-separable with continuous monotone or unimodal components. A function  $f : [0, 1]^n \rightarrow \mathbb{R}$  is said to be max-separable if it is of the form  $f(\mathbf{x}) = \bigvee_{j \in N} f_j(x_j)$  with the component  $f_j : [0, 1] \rightarrow \mathbb{R}$  being a real-valued function of a single variable for every  $j \in N$ . In this context, we assume that all components in a max-separable function are continuous and can be evaluated efficiently. Problem (OP-T) with a max-separable objective function is called the max-separable optimization problem by Zimmermann [520, 523]. It turns out that the set of optimal solutions of such a problem remains a finitely generated root system when all the components in the objective function are monotone or unimodal. Note

that the method presented in the subsequent subsections to obtain one optimal solution to the max-separable optimization problem coincides, in its very essence, with that proposed by Zimmermann [520, 523] and Tharwat and Zimmermann [432].

#### 5.4.1 Max-Separable Functions with Monotone Components

Consider the following max-separable optimization problem:

$$\begin{aligned}
 \text{(MSO)} \quad & \min \quad z = \bigvee_{j \in N} f_j(x_j) \\
 & \text{s.t.} \\
 & A \circ \mathbf{x} = \mathbf{b}, \\
 & \mathbf{x} \in [0, 1]^n,
 \end{aligned} \tag{5.41}$$

where  $f_j : [0, 1] \rightarrow \mathbb{R}$  is a continuous monotone cost function associated with variable  $x_j$  for every  $j \in N$ , and  $A \circ \mathbf{x} = \mathbf{b}$  is a consistent system of sup- $\mathcal{T}$  equations with  $\mathcal{T}$  being a continuous t-norm.

Let  $\hat{\mathbf{x}}$  be the maximum solution to  $A \circ \mathbf{x} = \mathbf{b}$ , and  $\tilde{Q}$  its characteristic matrix. For each  $i \in M$ , the index set  $N_i = \{j \in N \mid \tilde{q}_{ij} \neq \emptyset\}$  indicates all the variables that are capable to satisfy the  $i$ th equation without violating other equations on the upper side. In order to minimize the objective function  $\bigvee_{j \in N} f_j(x_j)$  over  $S(A, \mathbf{b})$ , each equation should be satisfied with a possible minimum cost, which can be performed on the intervals  $\{\tilde{q}_{ij} \mid j \in N_i\}$  for every  $i \in M$ . This procedure leads to an optimal solution to problem (MSO) since all the components in the max-separable function act in a noninteractive manner.

Consequently, for problem (MSO) with  $f_j$  being increasing for every  $j \in N$ , denote

$$N_i^* = \operatorname{argmin} \{f_j(\mathcal{J}_{\mathcal{T}}(a_{ij}, b_i)) \mid j \in N_i\}, \quad \text{for every } i \in M. \tag{5.42}$$

The  $i$ th equation of  $A \circ \mathbf{x} = \mathbf{b}$  can be satisfied by fixing the value of  $x_{j^*}$  for every  $j^* \in N_i^*$ , with a possible minimum cost  $f_{j^*}(\mathcal{J}_{\mathcal{T}}(a_{ij^*}, b_i))$ .

Note that if there are several equations which should be satisfied by a common variable, say  $x_j$ , then they can be satisfied simultaneously by setting  $x_j$  the maximum over the values of  $\mathcal{J}_{\mathcal{T}}(a_{ij}, b_i)$  corresponding to these equations. If  $A \circ \mathbf{x} = \mathbf{b}$  is “simple,” the aforementioned operation is not necessary since in this case the value of variable  $x_j$  is uniquely determined by  $\hat{x}_j$  whenever  $j \in N_i^*$  for some  $i \in M$ . These observations lead to the following theorem for the max-separable optimization problem.

**Theorem 5.4.1.** *Let  $A \circ \mathbf{x} = \mathbf{b}$  be a consistent system of sup- $\mathcal{T}$  equations with  $\mathcal{T}$  being a continuous t-norm, and  $\tilde{Q}$  its characteristic matrix. For a max-separable function  $\bigvee_{j \in N} f_j(x_j)$  such that  $f_j : [0, 1] \rightarrow \mathbb{R}$  is continuous and increasing for every  $j \in N$ , denote  $X = (x_{ij})_{m \times n}$  with*

$$x_{ij} = \begin{cases} \mathcal{J}_{\mathcal{T}}(a_{ij}, b_i), & \text{if } j \in N_i^*, \\ 0, & \text{otherwise.} \end{cases} \quad (5.43)$$

The vector  $\mathbf{x}^* = (x_1^*, x_2^*, \dots, x_n^*)^T$  with  $x_j^* = \max_{i \in M} x_{ij}$  for every  $j \in N$  is an optimal solution that minimizes the objective function  $\bigvee_{j \in N} f_j(x_j)$  over  $S(A, \mathbf{b})$ .

*Proof.* Following the construction of  $\mathbf{x}^*$ , for each  $i \in M$ , there exists an index  $j_i \in N$  such that  $x_{j_i}^* \in \tilde{q}_{ij_i} \neq \emptyset$ , and hence,  $\mathbf{x}^* \in S(A, \mathbf{b})$  by Theorem 4.1.1. It is also clear that  $\bigvee_{j \in N} f_j(x_j^*) = f_k(x_k^*)$  holds for some index  $k \in N$ . Without loss of generality, we may assume that  $M_k^* = \{i \in M \mid k \in N_i^*\} \neq \emptyset$ . Otherwise, we have  $x_k^* = 0$  and the problem becomes trivial. Therefore, for any  $\mathbf{x} \in S(A, \mathbf{b})$ , if  $x_k \geq x_k^*$ , then

$$\bigvee_{j \in N} f_j(x_j^*) = f_k(x_k^*) \leq f_k(x_k) \leq \bigvee_{j \in N} f_j(x_j). \quad (5.44)$$

Otherwise, we have  $x_k < x_k^*$  and  $x_k \notin [x_k^*, \hat{x}_k]$ . Since  $M_k^* \neq \emptyset$ , there exists an index  $p \in M$  such that  $x_k^* = \max\{\mathcal{J}_{\mathcal{T}}(a_{ik}, b_i) \mid i \in M_k^*\} = \mathcal{J}_{\mathcal{T}}(a_{pk}, b_p)$  and  $f_k(x_k^*) = \min\{f_j(\mathcal{J}_{\mathcal{T}}(a_{pj}, b_p)) \mid j \in N_p\}$ . It is clear that the  $p$ th equations cannot be satisfied by the value  $x_k$  if  $x_k < x_k^*$ . As a consequence, there must exist another index  $k' \in N$  such that  $x_{k'} \in [\mathcal{J}_{\mathcal{T}}(a_{pk'}, b_p), \hat{x}_{k'}]$  which means  $f_{k'}(x_{k'}) \geq f_k(x_k^*)$ , and hence,

$$\bigvee_{j \in N} f_j(x_j^*) = f_k(x_k^*) \leq f_{k'}(x_{k'}) \leq \bigvee_{j \in N} f_j(x_j). \quad (5.45)$$

Therefore, for any  $\mathbf{x} \in S(A, \mathbf{b})$ , we have  $\bigvee_{j \in N} f_j(x_j^*) \leq \bigvee_{j \in N} f_j(x_j)$ , i.e.,  $\mathbf{x}^*$  is an optimal solution that minimizes the objective function  $\bigvee_{j \in N} f_j(x_j)$  over  $S(A, \mathbf{b})$ .  $\square$

By Theorem 5.4.1, an optimal solution to problem (MSO) can be constructed in a time complexity of  $O(mn)$  when  $f_j$  is continuous and increasing for every  $j \in N$ . However, problem (MSO) may have multiple optimal solutions to be found. Denote

$$S^*(A, \mathbf{b}; f) = \operatorname{argmin}\{f(\mathbf{x}) \mid \mathbf{x} \in S(A, \mathbf{b})\} \quad (5.46)$$

the set of optimal solutions to problem (MSO) with  $f(\mathbf{x}) = \bigvee_{j \in N} f_j(x_j)$ . Now suppose that  $\mathbf{x}^*$  is an optimal solution to problem (MSO) with an optimal objective value  $z^* =$

$\bigvee_{j \in N} f_j(x_j^*)$ . We define  $\hat{\mathbf{x}}^* = (\hat{x}_1^*, \hat{x}_2^*, \dots, \hat{x}_n^*)^T$  with

$$\hat{x}_j^* = \sup\{x \in [0, 1] \mid x \leq \hat{x}_j, f_j(x) \leq z^*\} \quad (5.47)$$

$$= \hat{x}_j \wedge (\sup\{x \in [0, 1] \mid f_j(x) \leq z^*\}), \quad \text{for every } j \in N. \quad (5.48)$$

Since  $\mathbf{x}^* \leq \hat{\mathbf{x}}^* \leq \hat{\mathbf{x}}$  and

$$z^* = \bigvee_{j \in N} f_j(x_j^*) \leq \bigvee_{j \in N} f_j(\hat{x}_j^*) \leq z^*, \quad (5.49)$$

we know that  $\hat{\mathbf{x}}^*$  is also an optimal solution to problem (MSO). In fact,  $\hat{\mathbf{x}}^*$  is the maximum optimal solution to problem (MSO) in this case, i.e.,  $\mathbf{x} \leq \hat{\mathbf{x}}^*$  for every  $\mathbf{x} \in S^*(A, \mathbf{b}; f)$ . On the other hand, any  $\mathbf{x} \in S(A, \mathbf{b})$  with  $\mathbf{x} \leq \hat{\mathbf{x}}^*$  is an optimal solution due to the monotonicity of the objective function. Therefore, a vector  $\mathbf{x}$  is an optimal solution to problem (MSO) if and only if  $\mathbf{x} \in S(A, \mathbf{b})$  and  $\mathbf{x} \leq \hat{\mathbf{x}}^*$ , i.e.,

$$S^*(A, \mathbf{b}; f) = S(A, \mathbf{b}) \cap \{\mathbf{x} \in [0, 1]^n \mid \mathbf{x} \leq \hat{\mathbf{x}}^*\}, \quad (5.50)$$

which is a finitely generated root system with  $\hat{\mathbf{x}}^*$  being the maximum optimal solution. Consequently, the set  $\check{S}^*(A, \mathbf{b}; f)$  of all minimal optimal solutions is the set of minimal solutions which are upper bounded by  $\hat{\mathbf{x}}^*$ , i.e.,

$$\check{S}^*(A, \mathbf{b}; f) = \check{S}(A, \mathbf{b}) \cap \{\mathbf{x} \in [0, 1]^n \mid \mathbf{x} \leq \hat{\mathbf{x}}^*\}. \quad (5.51)$$

The information of  $\check{S}^*(A, \mathbf{b}; f)$  can be characterized by a generalized matrix  $\tilde{Q}^* = (\tilde{q}_{ij}^*)_{m \times n}$  where

$$\tilde{q}_{ij}^* = \begin{cases} [\mathcal{J}_{\mathcal{T}}(a_{ij}, b_i), \hat{x}_j^*], & \text{if } \hat{x}_j^* \in \tilde{q}_{ij}, \\ \emptyset, & \text{otherwise.} \end{cases} \quad (5.52)$$

Note that  $\tilde{Q}^*$  is a modification of the characteristic matrix  $\tilde{Q}$  by updating the upper bound information.

On the other hand, for problem (MSO) with  $f_j$  being decreasing for every  $j \in N$ , the maximum solution  $\hat{\mathbf{x}} = A^T \varphi_t \mathbf{b}$  must be the maximum optimal solution. Moreover, denote  $\check{\mathbf{x}}^* = (\check{x}_1^*, \check{x}_2^*, \dots, \check{x}_n^*)^T$  with

$$\check{x}_j^* = \inf\{x \in [0, 1] \mid f_j(x) \leq z^*\}, \quad \text{for every } j \in N, \quad (5.53)$$

where  $z^* = \bigvee_{j \in N} f_j(\hat{x}_j)$  is the optimal objective value. Then  $\check{\mathbf{x}}^*$  serves as a lower bound of all optimal solutions, but itself may not necessarily be feasible. In this case, a vector  $\mathbf{x}$  is an optimal solution to problem (MSO) if and only if  $\mathbf{x} \in S(A, \mathbf{b})$  and  $\check{\mathbf{x}}^* \leq \mathbf{x}$ , i.e.,

$$S^*(A, \mathbf{b}; f) = S(A, \mathbf{b}) \cap \{\mathbf{x} \in [0, 1]^n \mid \check{\mathbf{x}}^* \leq \mathbf{x}\}, \quad (5.54)$$

which is a finitely generated root system with  $\hat{\mathbf{x}}$  being the maximum optimal solution. Consequently, the set  $\check{S}^*(A, \mathbf{b}; f)$  of all minimal optimal solutions is determined by

$$\check{S}^*(A, \mathbf{b}; f) = \{\mathbf{x} \vee \check{\mathbf{x}}^* \mid \mathbf{x} \in \check{S}(A, \mathbf{b})\}. \quad (5.55)$$

In case  $\check{\mathbf{x}}^* \in S(A, \mathbf{b})$ , we have  $S^*(A, \mathbf{b}; f) = \{\mathbf{x} \in [0, 1]^n \mid \check{\mathbf{x}}^* \leq \mathbf{x} \leq \hat{\mathbf{x}}\}$ , i.e.,  $\check{\mathbf{x}}^*$  is the unique minimal optimal solution.

For problem (MSO) of a general case, the objective function  $z = \bigvee_{j \in N} f_j(x_j)$ , with  $f_j$  being monotone, either increasing or decreasing, for every  $j \in N$ , can be represented as

$$z = \max \left\{ \bigvee_{j \in N^+} f_j(x_j), \bigvee_{j \in N^-} f_j(x_j) \right\}, \quad (5.56)$$

where  $N^+ = \{j \in N \mid f_j \text{ is increasing on } [0, 1]\}$  and  $N^- = \{j \in N \mid f_j \text{ is decreasing on } [0, 1]\}$  such that  $N^+ \cup N^- = N$  and  $N^+ \cap N^- = \emptyset$ . Note that the constant components in the objective function, if exist, are included in  $N^-$  for convenience. Subsequently, problem (MSO) can be split into two subproblems, i.e.,

$$\begin{aligned} \min \quad & z^+ = \bigvee_{j \in N^+} f_j(x_j) \\ \text{s.t.} \quad & \\ & A \circ \mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \in [0, 1]^n, \end{aligned} \quad (5.57)$$

and

$$\begin{aligned} \min \quad & z^- = \bigvee_{j \in N^-} f_j(x_j) \\ \text{s.t.} \quad & \\ & A \circ \mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \in [0, 1]^n, \end{aligned} \quad (5.58)$$

both of which can be solved in polynomial time. To write the subproblems in the standard form of problem (MSO), we can assume that the missing components in the objective

functions are the constant term  $-\infty$ . The optimal solution to the original problem can then be obtained by combining the optimal solutions of two subproblems.

**Theorem 5.4.2.** *Let  $A \circ \mathbf{x} = \mathbf{b}$  be a consistent system of sup- $\mathcal{T}$  equations with  $\mathcal{T}$  being a continuous t-norm. For a max-separable function  $z = \bigvee_{j \in N} f_j(x_j)$  with  $f_j$  being continuous and monotone for each  $j \in N$ , the vector  $\mathbf{x}^* = (x_1^*, x_2^*, \dots, x_n^*)^T$  with*

$$x_j^* = \begin{cases} x_j^{+*}, & \text{if } j \in N^+, \\ \hat{x}_j, & \text{if } j \in N^-, \end{cases} \quad (5.59)$$

*is an optimal solution that minimizes  $z = \bigvee_{j \in N} f_j(x_j)$  over  $S(A, \mathbf{b})$ , where  $\mathbf{x}^{+*} = (x_1^{+*}, x_2^{+*}, \dots, x_n^{+*})^T$  is a solution that minimizes  $z^+ = \bigvee_{j \in N^+} f_j(x_j)$  over  $S(A, \mathbf{b})$ , and  $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n)^T$  is the maximum solution that minimizes  $z^- = \bigvee_{j \in N^-} f_j(x_j)$  over  $S(A, \mathbf{b})$ .*

*Proof.* Since  $\mathbf{x}^{+*} \leq \mathbf{x}^* \leq \hat{\mathbf{x}}$ , we know that  $\mathbf{x}^* \in S(A, \mathbf{b}) \neq \emptyset$ . Moreover, for any  $\mathbf{x} \in S(A, \mathbf{b})$ , we have

$$z(\mathbf{x}) = \bigvee_{j \in N} f_j(x_j) = z^+(\mathbf{x}) \vee z^-(\mathbf{x}) \geq z^+(\mathbf{x}^{+*}) \vee z^-(\mathbf{x}^{-*}) = z(\mathbf{x}^*). \quad (5.60)$$

Hence,  $\mathbf{x}^*$  is an optimal solution that minimizes  $\bigvee_{j \in N} f_j(x_j)$  over  $S(A, \mathbf{b})$ .  $\square$

By Theorem 5.4.2, we know that an optimal solution to problem (MSO) can be obtained in a time complexity of  $O(mn)$ . Moreover, denote  $\hat{\mathbf{x}}^* = (\hat{x}_1^*, \hat{x}_2^*, \dots, \hat{x}_n^*)^T$  with

$$\hat{x}_j^* = \begin{cases} \sup\{x \in [0, 1] \mid x \leq \hat{x}_j, f_j(x) \leq z^*\}, & \text{if } j \in N^+, \\ \hat{x}_j, & \text{if } j \in N^-, \end{cases} \quad (5.61)$$

and  $\check{\mathbf{x}}^* = (\check{x}_1^*, \check{x}_2^*, \dots, \check{x}_n^*)^T$  with

$$\check{x}_j^* = \begin{cases} 0, & \text{if } j \in N^+, \\ \inf\{x \in [0, 1] \mid f_j(x) \leq z^*\}, & \text{if } j \in N^-, \end{cases} \quad (5.62)$$

where  $z^* = z^{+*} \vee z^{-*}$  is the optimal objective value with  $z^{+*}$  and  $z^{-*}$  being the optimal objective values of the two subproblems, respectively. In this case, a vector  $\mathbf{x}$  is an optimal solution to problem (MSO) if and only if  $\mathbf{x} \in S(A, \mathbf{b})$  and  $\check{\mathbf{x}}^* \leq \mathbf{x} \leq \hat{\mathbf{x}}^*$ , i.e.,

$$S^*(A, \mathbf{b}; f) = S(A, \mathbf{b}) \cap \{\mathbf{x} \in [0, 1]^n \mid \check{\mathbf{x}}^* \leq \mathbf{x} \leq \hat{\mathbf{x}}^*\}, \quad (5.63)$$

which is a finitely generated root system with  $\hat{\mathbf{x}}^*$  being the maximum optimal solution. Consequently, the set  $\check{S}^*(A, \mathbf{b}; f)$  of all minimal optimal solutions is determined by

$$\check{S}^*(A, \mathbf{b}; f) = \{\mathbf{x} \vee \check{\mathbf{x}}^* \mid \mathbf{x} \vee \check{\mathbf{x}}^* \leq \hat{\mathbf{x}}^*, \mathbf{x} \in \check{S}(A, \mathbf{b})\}. \quad (5.64)$$

In case  $\check{\mathbf{x}}^* \in S(A, \mathbf{b})$ , we have  $S^*(A, \mathbf{b}; f) = \{\mathbf{x} \in [0, 1]^n \mid \check{\mathbf{x}}^* \leq \mathbf{x} \leq \hat{\mathbf{x}}^*\}$ , i.e.,  $\check{\mathbf{x}}^*$  is the unique minimal optimal solution.

**Example 5.4.3.** Consider the following max-separable optimization problem subject to a system of sup- $\mathcal{T}_M$  equations:

$$\min \quad z = (1 \wedge (x_1)^2) \vee (0.8 \wedge x_2) \vee (2 \wedge (x_3)^{-2}) \vee (3 \wedge (x_4)^{-1})$$

s.t.

$$\begin{pmatrix} 0.8 & 0.9 & 0.3 & 1 \\ 0.2 & 0.4 & 0.9 & 0.5 \\ 0.8 & 0.4 & 0.1 & 0.1 \\ 0 & 0.2 & 0.2 & 0.1 \end{pmatrix} \circ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.6 \\ 0.4 \\ 0.2 \end{pmatrix},$$

$$x_1, x_2, x_3, x_4 \in [0, 1].$$

The system of equations is consistent with the maximum solution

$$\hat{\mathbf{x}} = (0.4, 0.8, 0.6, 0.8)^T.$$

Its characteristic matrix becomes

$$\tilde{Q} = \begin{pmatrix} \emptyset & 0.8 & \emptyset & 0.8 \\ \emptyset & \emptyset & 0.6 & \emptyset \\ 0.4 & [0.4, 0.8] & \emptyset & \emptyset \\ \emptyset & [0.2, 0.8] & [0.2, 0.6] & \emptyset \end{pmatrix}.$$

Subsequently, three minimal solutions can be obtained as

$$\check{\mathbf{x}}^1 = \begin{pmatrix} 0 \\ 0.8 \\ 0.6 \\ 0 \end{pmatrix}, \quad \check{\mathbf{x}}^2 = \begin{pmatrix} 0 \\ 0.4 \\ 0.6 \\ 0.8 \end{pmatrix}, \quad \check{\mathbf{x}}^3 = \begin{pmatrix} 0.4 \\ 0 \\ 0.6 \\ 0.8 \end{pmatrix}.$$

In this case,  $N^+ = \{1, 2\}$  and  $N^- = \{3, 4\}$ . To minimize the function  $z^+ = (1 \wedge (x_1)^2) \vee (0.8 \wedge x_2)$ , the matrix  $X^+$  is calculated as

$$X^+ = \begin{pmatrix} 0 & 0 & 0 & 0.8 \\ 0 & 0 & 0.6 & 0 \\ 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \end{pmatrix}$$

which offers an optimal solution  $\mathbf{x}^{+*} = (0.4, 0, 0.6, 0.8)^T$ . Combining  $\hat{\mathbf{x}}$  and  $\mathbf{x}^{+*}$ , we see an optimal solution to the original problem is

$$\mathbf{x}^* = (0.4, 0, 0.6, 0.8)^T$$

with the optimal objective value  $z^* = 2$ . Subsequently, since  $x_1^* = \hat{x}_1$  and  $\hat{x}_2 \wedge (\sup\{x_2 \in [0, 1] \mid 0.8 \wedge x_2 \leq 2\}) = 0.8$ , the maximum optimal solution is

$$\hat{\mathbf{x}}^* = (0.4, 0.8, 0.6, 0.8)^T.$$

Moreover, since  $\inf\{x_3 \in [0, 1] \mid 2 \wedge (x_3)^{-2} \leq 2\} = 0$  and  $\inf\{x_4 \in [0, 1] \mid 3 \wedge (x_4)^{-1} \leq 2\} = 0.5$ , the lower bound vector is

$$\check{\mathbf{x}}^* = (0, 0, 0, 0.5),$$

which, however, is not a feasible solution. Hence, by combining the lower bound vector  $\check{\mathbf{x}}^*$  and the minimal solutions that are upper bounded by  $\hat{\mathbf{x}}^*$ , three minimal optimal solutions can be identified, i.e.,

$$\check{\mathbf{x}}^{*1} = \begin{pmatrix} 0 \\ 0.8 \\ 0.6 \\ 0.5 \end{pmatrix}, \quad \check{\mathbf{x}}^{*2} = \begin{pmatrix} 0 \\ 0.4 \\ 0.6 \\ 0.8 \end{pmatrix}, \quad \check{\mathbf{x}}^{*3} = \begin{pmatrix} 0.4 \\ 0 \\ 0.6 \\ 0.8 \end{pmatrix}.$$

#### 5.4.2 Max-Separable Functions with Unimodal Components

From Theorems 5.4.1 and 5.4.2, we see that the monotonicity of all components of the max-separable function plays a crucial role in solving problem (MSO). Based on Theorem 4.1.1 and the special structure of the characteristic matrix, solving problem (MSO)

essentially relies on solving the subproblems  $\min\{f_j(x) \mid x \in \tilde{q}_{ij} \neq \emptyset\}$  and  $\min\{f_j(x) \mid x \in [0, \hat{x}_j]\}$  for every  $i \in M$  and  $j \in N$ . Therefore, as long as these subproblems can be solved in an efficient manner, the problem of minimizing a max-separable function subject to a system  $A \circ \mathbf{x} = \mathbf{b}$  can be solved efficiently in an analogous manner. One of such scenarios is minimizing a max-separable functions with continuous unimodal components over  $S(A, \mathbf{b})$ .

**Definition 5.4.4.** *A continuous function  $f : [0, 1] \rightarrow \mathbb{R}$  is said to be unimodal if there exists a number  $c \in [0, 1]$  such that  $f$  is decreasing on  $[0, c]$  and increasing on  $[c, 1]$ .*

There are several different definitions of a unimodal function in the literature. In this context, the unimodal function is defined to encompass the monotone function, either increasing or decreasing. The problem of minimizing a unimodal function over a closed interval can be solved efficiently via line search methods.

Consider the following max-separable optimization problem:

$$\begin{aligned}
 \text{(MSO)} \quad & \min \quad z = \bigvee_{j \in N} f_j(x_j) \\
 & \text{s.t.} \\
 & A \circ \mathbf{x} = \mathbf{b}, \\
 & \mathbf{x} \in [0, 1]^n,
 \end{aligned} \tag{5.65}$$

where  $f_j : [0, 1] \rightarrow \mathbb{R}$  is a continuous unimodal cost function associated with variable  $x_j$  for every  $j \in N$ , and  $A \circ \mathbf{x} = \mathbf{b}$  is a consistent system of sup- $\mathcal{T}$  equations with  $\mathcal{T}$  being a continuous t-norm.

Let  $\hat{\mathbf{x}}$  be the maximum solution to  $A \circ \mathbf{x} = \mathbf{b}$ , and  $\tilde{Q}$  its characteristic matrix. Analogously, denote

$$N_i^* = \operatorname{argmin} \{ \min\{f_j(x) \mid x \in \tilde{q}_{ij}\} \mid j \in N_i \}, \quad \text{for every } i \in M, \tag{5.66}$$

where  $N_i = \{j \in N \mid \tilde{q}_{ij} \neq \emptyset\}$  indicates all the variables that are capable to satisfy the  $i$ th equation without violating other equations on the upper side. The  $i$ th equation of  $A \circ \mathbf{x} = \mathbf{b}$  can be satisfied by fixing the value of  $x_{j^*}$  for every  $j^* \in N_i^*$ , with a possible minimum cost  $\min\{f_{j^*}(x) \mid x \in \tilde{q}_{ij^*}\}$ . Similarly, if there are several equations which should be satisfied by a common variable, say  $x_j$ , then they can be satisfied simultaneously by setting  $x_j$  the maximum over the values of  $\operatorname{argmin}\{f_j(x) \mid x \in \tilde{q}_{ij}\}$  corresponding to these equations. Such an operation is not necessary when  $A \circ \mathbf{x} = \mathbf{b}$  is “simple.” Note that the

set  $\operatorname{argmin}\{f_j(x) \mid x \in \tilde{q}_{ij}\}$  may not necessarily be a singleton. Consequently, an analogous version of Theorem 5.4.1 can be developed to solve problem (MSO) with all components in the objective function being continuous and unimodal.

**Theorem 5.4.5.** *Let  $A \circ \mathbf{x} = \mathbf{b}$  be a consistent system of sup- $\mathcal{T}$  equations with  $\mathcal{T}$  being a continuous t-norm, and  $\tilde{Q}$  its characteristic matrix. For a max-separable function  $\bigvee_{j \in N} f_j(x_j)$  such that  $f_j : [0, 1] \rightarrow \mathbb{R}$  is continuous and unimodal for every  $j \in N$ , denote  $X = (x_{ij})_{m \times n}$  with*

$$x_{ij} \in \begin{cases} \operatorname{argmin}\{f_j(x) \mid x \in \tilde{q}_{ij}\}, & \text{if } j \in N_i^*, \\ \operatorname{argmin}\{f_j(x) \mid x \in [0, \hat{x}_j]\}, & \text{otherwise.} \end{cases} \quad (5.67)$$

The vector  $\mathbf{x}^* = (x_1^*, x_2^*, \dots, x_n^*)^T$  with  $x_j^* = \max_{i \in M} x_{ij}$  for every  $j \in N$  is an optimal solution that minimizes the objective function  $\bigvee_{j \in N} f_j(x_j)$  over  $S(A, \mathbf{b})$ .

*Proof.* Following the construction of  $\mathbf{x}^*$ , for each  $i \in M$ , there exists an index  $j_i \in N$  such that  $x_{j_i}^* \in \tilde{q}_{ij_i} \neq \emptyset$ , and hence,  $\mathbf{x}^* \in S(A, \mathbf{b})$  by Theorem 4.1.1. It is also clear that  $\bigvee_{j \in N} f_j(x_j^*) = f_k(x_k^*)$  holds for some index  $k \in N$ . Without loss of generality, we may assume that  $M_k^* = \{i \in M \mid k \in N_i^*\} \neq \emptyset$ . Otherwise, we have  $x_k^* \in \operatorname{argmin}\{f_k(x) \mid x \in [0, \hat{x}_k]\}$  and the problem becomes trivial. Therefore, for any  $\mathbf{x} \in S(A, \mathbf{b})$ , if  $f_k(x_k) \geq f_k(x_k^*)$ , then

$$\bigvee_{j \in N} f_j(x_j^*) = f_k(x_k^*) \leq f_k(x_k) \leq \bigvee_{j \in N} f_j(x_j). \quad (5.68)$$

Otherwise, we have  $f_k(x_k) < f_k(x_k^*)$ . Since  $M_k^* \neq \emptyset$ , there exists an index  $p \in M$  such that  $x_k^* \in \operatorname{argmin}\{f_k(x) \mid x \in \tilde{q}_{pj}\}$  and  $f_k(x_k^*) = \min\{\min\{f_j(x) \mid x \in \tilde{q}_{pj}\} \mid j \in N_p\}$ . Since  $f_k(x_k) < f_k(x_k^*)$ , the  $p$ th equations cannot be satisfied by the value  $x_k$ . As a consequence, there must exist another index  $k' \in N$  such that  $x_{k'} \in \tilde{q}_{pk'}$  which means  $f_{k'}(x_{k'}) \geq f_k(x_k^*)$ , and hence,

$$\bigvee_{j \in N} f_j(x_j^*) = f_k(x_k^*) \leq f_{k'}(x_{k'}) \leq \bigvee_{j \in N} f_j(x_j).$$

Therefore, for any  $\mathbf{x} \in S(A, \mathbf{b})$ , we have  $\bigvee_{j \in N} f_j(x_j^*) \leq \bigvee_{j \in N} f_j(x_j)$ , i.e.,  $\mathbf{x}^*$  is an optimal solution that minimizes the objective function  $\bigvee_{j \in N} f_j(x_j)$  over  $S(A, \mathbf{b})$ .  $\square$

By Theorem 5.4.5, an optimal solution to problem (MSO) can be constructed after solving the problem of minimizing a unimodal function over a closed interval at most  $mn + n$  times when  $f_j$  is continuous and unimodal for every  $j \in N$ . Now suppose that  $\mathbf{x}^*$  is an

optimal solution to problem (MSO) with an optimal objective value  $z^* = \bigvee_{j \in N} f_j(x_j^*)$ . We define  $\hat{\mathbf{x}}^* = (\hat{x}_1^*, \hat{x}_2^*, \dots, \hat{x}_n^*)^T$  with

$$\hat{x}_j^* = \sup\{x \in [0, 1] \mid x \leq \hat{x}_j, f_j(x) \leq z^*\} \quad (5.69)$$

$$= \hat{x}_j \wedge (\sup\{x \in [0, 1] \mid f_j(x) \leq z^*\}), \quad \text{for every } j \in N, \quad (5.70)$$

and  $\check{\mathbf{x}}^* = (\check{x}_1^*, \check{x}_2^*, \dots, \check{x}_n^*)^T$  with

$$\check{x}_j^* = \inf\{x \in [0, 1] \mid f_j(x) \leq z^*\}, \quad \text{for every } j \in N. \quad (5.71)$$

Consequently,  $\hat{\mathbf{x}}^*$  is the maximum optimal solution while  $\check{\mathbf{x}}^*$  serves as a lower bound of all optimal solutions. In this case, a vector  $\mathbf{x}$  is an optimal solution to problem (MSO) if and only if  $\mathbf{x} \in S(A, \mathbf{b})$  and  $\check{\mathbf{x}}^* \leq \mathbf{x} \leq \hat{\mathbf{x}}^*$ , i.e.,

$$S^*(A, \mathbf{b}; f) = S(A, \mathbf{b}) \cap \{\mathbf{x} \in [0, 1]^n \mid \check{\mathbf{x}}^* \leq \mathbf{x} \leq \hat{\mathbf{x}}^*\}, \quad (5.72)$$

which is a finitely generated root system with  $\hat{\mathbf{x}}^*$  being the maximum optimal solution. The set  $\check{S}^*(A, \mathbf{b}; f)$  of all minimal optimal solutions is determined by

$$\check{S}^*(A, \mathbf{b}; f) = \{\mathbf{x} \vee \check{\mathbf{x}}^* \mid \mathbf{x} \vee \check{\mathbf{x}}^* \leq \hat{\mathbf{x}}^*, \mathbf{x} \in \check{S}(A, \mathbf{b})\}. \quad (5.73)$$

In case  $\check{\mathbf{x}}^* \in S(A, \mathbf{b})$ , we have  $S^*(A, \mathbf{b}; f) = \{\mathbf{x} \in [0, 1]^n \mid \check{\mathbf{x}}^* \leq \mathbf{x} \leq \hat{\mathbf{x}}^*\}$ , i.e.,  $\check{\mathbf{x}}^*$  is the unique minimal optimal solution.

**Example 5.4.6.** *Reconsider the system of sup- $\mathcal{T}_M$  equations in Example 5.4.3 with an objective function  $z = |x_1 - 0.5| \vee |x_2 - 0.5| \vee |x_3 - 0.5| \vee |x_4 - 0.5|$ , i.e.,*

$$\min \quad z = |x_1 - 0.5| \vee |x_2 - 0.5| \vee |x_3 - 0.5| \vee |x_4 - 0.5|$$

s.t.

$$\begin{pmatrix} 0.8 & 0.9 & 0.3 & 1 \\ 0.2 & 0.4 & 0.9 & 0.5 \\ 0.8 & 0.4 & 0.1 & 0.1 \\ 0 & 0.2 & 0.2 & 0.1 \end{pmatrix} \circ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.6 \\ 0.4 \\ 0.2 \end{pmatrix},$$

$$x_1, x_2, x_3, x_4 \in [0, 1].$$

By Theorem 5.4.5, the matrix  $X$  is calculated as

$$X = \begin{pmatrix} 0.4 & 0.8 & 0.5 & 0.8 \\ 0.4 & 0.5 & 0.6 & 0.5 \\ 0.4 & 0.5 & 0.6 & 0.5 \\ 0.4 & 0.5 & 0.5 & 0.5 \end{pmatrix}$$

which offers an optimal solution

$$\mathbf{x}^* = (0.4, 0.8, 0.6, 0.8)^T$$

with an optimal objective value  $z^* = 0.3$ . It happens to be the maximum solution  $\hat{\mathbf{x}}$ , and hence, is the maximum optimal solution  $\hat{\mathbf{x}}^*$ . Furthermore, the lower bound vector  $\check{\mathbf{x}}^*$  is calculated as

$$\check{\mathbf{x}}^* = (0.2, 0.2, 0.2, 0.2)^T,$$

which is not a feasible solution. Consequently, by combining  $\check{\mathbf{x}}^*$  and the minimal solutions, three minimal optimal solutions can be identified, as

$$\check{\mathbf{x}}^{*1} = \begin{pmatrix} 0.2 \\ 0.8 \\ 0.6 \\ 0.2 \end{pmatrix}, \quad \check{\mathbf{x}}^{*2} = \begin{pmatrix} 0.2 \\ 0.4 \\ 0.6 \\ 0.8 \end{pmatrix}, \quad \check{\mathbf{x}}^{*3} = \begin{pmatrix} 0.4 \\ 0.2 \\ 0.6 \\ 0.8 \end{pmatrix}.$$

## 5.5 Bibliographical Notes

Fang and Li [136] investigated the problem of minimizing a linear objective function subject to a consistent system of sup- $\mathcal{T}_M$  equations and obtained Theorems 5.1.1 and 5.1.2 for sup- $\mathcal{T}_M$  equations, which are valid for continuous t-norms, as well as general composite operators, since they only depend on the structure of the solution set  $S(A, b)$ . With Theorems 5.1.1 and 5.1.2, a sup- $\mathcal{T}_M$  equation constrained linear optimization problem was converted into a 0-1 integer programming problem by Fang and Li [136] and solved by branch-and-bound method with jump-tracking technique. The branch-and-bound method was improved by Wu *et al.* [487] and Wu and Guu [485] by providing the proper upper bounds for the branch-and-bound procedure.

Following the idea of Fang and Li [136], the linear optimization problem subject to a system of sup- $\mathcal{T}_P$  equations was discussed by Loetamonphong and Fang [281], Guu and Wu [192], Ghodousian and Khorram [155], and Qu and Wang [384]. Khorram and Ghodousian [232] and Wu [481] considered this problem under the max-average composition. However, it was pointed out by Zimmermann [525] that one of the algorithms proposed by Khorram and Ghodousian [232] may not lead to the optimal solution in some cases. Similar deficiencies were detected in Ghodousian and Khorram [155, 156] and Khorram *et al.* [233], and amended by Abbasi Molai and Khorram [1, 2] in which sup- $\mathcal{O}$  equation constrained linear optimization problems were investigated under various composite operations. Some other generalizations on this issue can be found in Wu and Guu [483], Pandey [335], Guo and Xia [190], Abbasi Molai and Khorram [3], and Ghodousian and Khorram [157]. Note that the algorithm proposed by Pandey [335] only works for strict triangular norms but not for general continuous triangular norms as claimed.

A thorough investigation of sup- $\mathcal{T}$  equation constrained linear optimization was provided by Li and Fang [268].

Wu *et al.* [488, 489] explored the linear fractional programming problem subject to a system of sup- $\mathcal{T}$  equations via the decomposition method where  $\mathcal{T}$  is a continuous Archimedean triangular norm. Note that the preprocessing procedure proposed in Wu *et al.* [488] is not exactly correct. Li and Fang [271] showed that such a problem can be reduced to a 0-1 linear fractional programming problem in polynomial time.

Yang and Cao [498, 499] considered the sup- $\mathcal{T}_M$  equation constrained optimization problem with a posynomial objective function. Shivanian and Khorram [417] discussed the sup- $\mathcal{T}_P$  inequality constrained optimization problem with a monomial objective function. As indicated by Li and Fang [272], the problems considered in Yang and Cao [499] and Shivanian and Khorram [417] can be solved in polynomial time.

Lu and Fang [285] designed a genetic algorithm to solve nonlinear optimization problems subject to a system of sup- $\mathcal{T}_M$  equations. The similar method was used by Khorram and Hassanzadeh [231] for sup- $\mathcal{O}_{av}$  equation constrained nonlinear optimization problems.

The multi-objective optimization problem was discussed in Wang [453], Loetamonphong *et al.* [282], and Wu and Guu [484].

The 0-1 linear fractional optimization has been investigated for over 40 years. Readers may refer to Stancu-Minasian [421] and Tawarmalani *et al.* [431] for details. A general discussion on Dinkelbach's Algorithm can be found in Schaible [405], Ródenas *et al.* [388], and references therein.

The problem (0-1 MNLO-II) can be viewed as a *constrained set covering problem* with a general cost function  $\hat{f}'(u)$  as well as a so-called pseudo-Boolean optimization problem with a special type of constraints. Some early developments on pseudo-Boolean optimization are due to Hammer and Rudeanu [194, 195]. See Boros and Hammer [22] for a state-of-the-art survey on pseudo-Boolean optimization.

The so-called latticized linear programming problem was investigated by Wang and Zhang [461], Wang *et al.* [462], and Peeva [362]. Its generalized version was considered by Yang and Cao [496, 497] and Wu [482]. The study of the max-separable optimization problem might be traced back to Zimmermann [526, 527] where it was called the extremal optimization problem and defined on  $\mathbb{R} \cup \{-\infty\}$ . The max-separable optimization problem with monotone components in the objective function was investigated by Zimmermann [520, 523, 524] and Li and Fang [272]. The max-separable optimization problem with unimodal components in the objective function was discussed by Tharwat and Zimmermann [432] in a slightly general situation. Note that the idea presented in Theorems 5.4.1, 5.4.2 and 5.4.5 to obtain one optimal solution to problem (MSO) coincides, in its very essence, with that of Zimmermann [520, 523] and Tharwat and Zimmermann [432]. See Li and Fang [272] for a detailed discussion. Besides, it is clear that the problem of maximizing a min-separable function can be solved efficiently in an analogous manner when each component is continuous monotone or unimodal.

## Chapter 6

# Conclusions and Further Research

This dissertation is devoted to the theoretical study of fuzzy relational equations and associated optimization problems. In this final chapter, we will review the research contributions of this dissertation, as well as discuss directions for future research.

### 6.1 Concluding Remarks

The resolution of fuzzy relational equations mainly concerns the following fundamental questions:

- (i) Is a given system of fuzzy relational equations consistent?
- (ii) If a given system of fuzzy relational equations is consistent, then how to characterize its solution set?
- (iii) If a given system of fuzzy relational equations is inconsistent, then how to resolve the inconsistency?

In the past three decades, much interest has been drawn to the first two issues, i.e., solvability criteria and characterization of solution sets, while relatively little work has been done on the third issue.

After the pioneering work of Sanchez, fuzzy relational equations, on various algebraic structures and with various composite operations, have been extensively investigated. The main attention is paid to the requirement of the underlying lattice and the composite operation such that the solution set of a consistent system of fuzzy relational equations preserves

the structure of a finitely generated root/crown system. The most essential and up-to-date results on this subject have been summarized in De Baets [80] and Li and Fang [270] for the case of general lattices, as well as in Chapters 2 and 3 of this dissertation for the case of the unit interval. It is shown that, from a lattice-theoretical point of view, there is no essential difference between  $\text{sup-}\mathcal{C}/\text{inf-}\mathcal{D}$  and  $\text{inf-}\mathcal{I}/\text{sup-}\mathcal{J}$  equations. Hence, fuzzy relational equations may be studied separately without being restricted to their particular interpretations in fuzzy logic. From this perspective, the very essence of fuzzy relational equations is somehow veiled by their own name.

Among various types of fuzzy relational equations,  $\text{sup-}\mathcal{T}$  equations with  $\mathcal{T}$  being a continuous t-norm are of the most importance. It is well-known that the solution set of a consistent system of  $\text{sup-}\mathcal{T}$  equations is characterized by a maximum solution and a finite number of minimal solutions. The maximum solution can be obtained in polynomial time, while the detection of all minimal solutions is a difficult problem. In Chapter 4, it is shown that a system of  $\text{sup-}\mathcal{T}$  equations can be characterized by an interval-valued matrix, and hence, reformulated as a system of 0-1 mixed integer inequalities. This provides an alternative characterization of  $\text{sup-}\mathcal{T}$  equations from the perspective of mathematical programming. It is this characterization that plays a significant role when tackling optimization problems that are constrained by  $\text{sup-}\mathcal{T}$  equations. Moreover, by fully utilizing the characterization of continuous t-norm, it is shown clearly that the watershed in the resolution of  $\text{sup-}\mathcal{T}$  equations is between the Archimedean property and the non-Archimedean property of the involved t-norm.

The detection of all minimal solutions of a system of  $\text{sup-}\mathcal{T}$  equations is without doubt one of the challenging issues in the study of fuzzy relational equations. Many algorithms have been proposed in the literature, most of which are for  $\text{sup-}\mathcal{T}_M$  equations and a few are for  $\text{sup-}\mathcal{T}_P$  equations. In Chapter 4, as well as in Li and Fang [268], it is indicated that, from an algorithmic viewpoint, the detection of all minimal solutions of a system of  $\text{sup-}\mathcal{T}$  equations is equivalent to the detection of all irredundant coverings of a set covering problem with or without some additional constraints. Consequently, this issue can be studied without restricting in the framework of fuzzy relational equations. Besides, Elbassioni [134] pointed out that this issue, particularly for  $\text{sup-}\mathcal{T}_M$  inequalities and  $\text{sup-}\mathcal{T}_P$  inequalities, is related to the dualization problem of a monotone Boolean function, of which the exact time complexity is a long unresolved question and the best known algorithm runs in incremental quasi-

polynomial time.

Fuzzy relational equation constrained optimization problems arise naturally when solutions of particular properties are desired. Such optimization problems in general can be reformulated into 0-1 mixed integer programming problems, and are inevitably NP-hard. However, this does not mean that there is no efficient algorithm for some particular classes of such optimization problems. Properties of objective functions are key to further simplification. It is shown in Chapter 5, as well as in Li and Fang [268, 271], that the problem of minimizing an objective function subject to a consistent system of sup- $\mathcal{T}$  equations can be reduced in polynomial time to a 0-1 integer programming problem, whenever the objective function is linear, linear fractional, or more generally, monotone in each argument separately. Moreover, as indicated in Section 5.4 and Li and Fang [272], if the objective function is max-separable with continuous monotone or unimodal components or monomial, then the corresponding optimization problem can be solved in polynomial time. Therefore, it would be an interesting topic to explore more nontrivial classes of objective functions such that the corresponding sup- $\mathcal{T}$  equation constrained optimization problems can be solved efficiently.

Note that the kernel and novel idea in this dissertation is to characterize a system of sup- $\mathcal{T}$  equations via an interval-valued matrix, which works as well for fuzzy relational equations with more general composite operations. Nevertheless, there are still many open problems related to fuzzy relational equations. Some interesting topics for further research are presented in the next section.

## 6.2 Future Research

Although the consistency of a system of sup- $\mathcal{T}$  equations can be readily verified and its solution set can be well characterized, all these investigations are of significant value only when the system under consideration is consistent. However, due to the inaccurate or deficient data used in the numerical modeling procedures, it is very often that the obtained system is inconsistent. Besides, it has been observed for a long period of time that the consistency of a system of sup- $\mathcal{T}$  equations is very sensitive to the data, i.e., small perturbations in the data could lead to an inconsistent system. For an illustration, consider the

following system of sup- $\mathcal{T}_M$  equations

$$\begin{pmatrix} 0.8 & 0.6 \\ 0.4 & 0.5 \end{pmatrix} \circ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.4 \end{pmatrix}.$$

It is consistent and has a maximum solution  $\hat{\mathbf{x}} = (1, 0.4)^T$ . However, the system of sup- $\mathcal{T}_M$  equations

$$\begin{pmatrix} 0.8 & 0.6 \\ 0.4 & 0.5 \end{pmatrix} \circ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.4 - \epsilon \end{pmatrix}$$

is inconsistent for any  $\epsilon$  with  $0 < \epsilon \leq 0.4$ . Moreover, the corresponding potential maximum solution becomes  $\hat{\mathbf{x}}_\epsilon = (0.4 - \epsilon, 0.4 - \epsilon)^T$ . Since

$$\begin{pmatrix} 0.8 & 0.6 \\ 0.4 & 0.5 \end{pmatrix} \circ \begin{pmatrix} 0.4 - \epsilon \\ 0.4 - \epsilon \end{pmatrix} = \begin{pmatrix} 0.4 - \epsilon \\ 0.4 - \epsilon \end{pmatrix},$$

the first component of the output vector deviates far away from the target value of 0.8. Hence, the potential maximum solution  $\hat{\mathbf{x}}_\epsilon = (0.4 - \epsilon, 0.4 - \epsilon)^T$  can not be a good approximate solution.

As a consequence, new approaches should be developed for handling inconsistent systems of fuzzy relational equations.

### 6.2.1 Approximate Solutions

A common idea to resolve the inconsistency in a system of fuzzy relational equations is to reformulate the system as a global optimization problem. For a system of sup- $\mathcal{T}$  equations  $A \circ \mathbf{x} = \mathbf{b}$ , it can be written as

$$\begin{aligned} \min \quad & \|A \circ \mathbf{x} - \mathbf{b}\|_p \\ \text{s.t.} \quad & \\ & \mathbf{x} \in [0, 1]^n, \end{aligned} \tag{6.1}$$

where  $\|\cdot\|_p$  denotes the  $L^p$  norm on the space of  $\mathbb{R}^n$ . It is clear that  $A \circ \mathbf{x} = \mathbf{b}$  is consistent if and only if there exists a vector  $\mathbf{x} \in [0, 1]^n$  such that  $\|A \circ \mathbf{x} - \mathbf{b}\|_p = 0$ . If  $A \circ \mathbf{x} = \mathbf{b}$  is inconsistent, the vector  $\mathbf{x}^*$  that minimizes  $\|A \circ \mathbf{x} - \mathbf{b}\|_p$  may be viewed as an  $L^p$ -approximate solution to  $A \circ \mathbf{x} = \mathbf{b}$ . The problem of obtaining an  $L^p$ -approximate solution is equivalent to

modifying the right-hand side vector as slightly as possible such that the resulting system becomes consistent, i.e.,

$$\begin{aligned} \min \quad & \|\mathbf{b}' - \mathbf{b}\|_p \\ \text{s.t.} \quad & S(A, \mathbf{b}') \neq \emptyset, \\ & \mathbf{b}' \in [0, 1]^m. \end{aligned} \tag{6.2}$$

In a more general framework, we may modify the coefficient matrix and the right-hand side vector simultaneously to obtain a consistent system, i.e.,

$$\begin{aligned} \min \quad & \|(A', \mathbf{b}') - (A, \mathbf{b})\|_p \\ \text{s.t.} \quad & S(A', \mathbf{b}') \neq \emptyset, \\ & A' \in [0, 1]^{m \times n}, \mathbf{b}' \in [0, 1]^m. \end{aligned} \tag{6.3}$$

As of today, this type of problem remains wide open except for the cases of  $L^\infty$ -norm, which have been solved by Li and Fang [269, 273] using a polynomial-time algorithm.

### 6.2.2 Maximal Consistent Subsystems

Another approach to handling an inconsistent system of fuzzy relational equations is to find a consistent subsystem containing as many equations as possible, i.e., to find a maximal consistent subsystem. The complementary problem of obtaining a maximal consistent subsystem is to remove as few equations as possible so that the resulting system is consistent. These two versions of the problem are equivalent as far as optimal solutions are concerned. In the context of rule-based systems, finding a maximal consistent subsystem amounts to identifying the maximal number of consistent rules constructed from the data.

Let  $A \circ \mathbf{x} = \mathbf{b}$  be an inconsistent system of sup- $\mathcal{T}$  equations with  $\mathcal{T}$  being a continuous t-norm. An inconsistent subsystem of  $A \circ \mathbf{x} = \mathbf{b}$  is called irreducible if each of its proper subsystems is consistent. Denote  $\mathfrak{I}(A, \mathbf{b})$  the set of all irreducible inconsistent subsystems of  $A \circ \mathbf{x} = \mathbf{b}$ . For each irreducible inconsistent subsystems  $I(A, \mathbf{b}) \in \mathfrak{I}(A, \mathbf{b})$ , let  $M_I \subseteq M = \{1, 2, \dots, m\}$  denote the corresponding index set of equations contained in  $I(A, \mathbf{b})$ . In order to achieve consistency, at least one equation should be removed from each irreducible inconsistent subsystem. Consequently, the complementary problem of finding a maximal

consistent subsystem can be formulated as

$$\begin{aligned}
& \min \sum_{i \in M} y_i \\
& \text{s.t.} \\
& \quad \sum_{i \in M_I} y_i \geq 1 \text{ for every } I(A, \mathbf{b}) \in \mathcal{I}(A, \mathbf{b}), \\
& \quad (y_1, y_2, \dots, y_m)^T \in \{0, 1\}^m,
\end{aligned} \tag{6.4}$$

where  $y_i$  with  $i \in M$  is a binary variable associated with the  $i$ th equation of  $A \circ \mathbf{x} = \mathbf{b}$  such that  $y_i = 1$  stands for removing the  $i$ th equation of  $A \circ \mathbf{x} = \mathbf{b}$ . It is clear that, for each optimal solution  $(y_1^*, y_2^*, \dots, y_m^*)^T$ , the equations corresponding to  $\{i \in M \mid y_i^* = 0\}$  form a maximal consistent subsystem of  $A \circ \mathbf{x} = \mathbf{b}$ , and vice versa.

As shown by Cechlárová and Diko [50], the problem of finding a maximal consistent subsystem is an NP-hard problem. Actually, the identification of  $\mathcal{I}(A, \mathbf{b})$  itself could be a hard problem. So far, little work has been done along this direction.

Note that finding approximate solutions and maximal consistent subsystems for an inconsistent system of fuzzy relational equations can hardly be approached by the methods presented in this dissertation, and may require new methodologies.

### 6.3 Bibliographical Notes

The investigation of approximate solutions to fuzzy relational equations was initiated by Pedrycz [345, 348], and further pursued by Wu [480], Di Nola *et al.* [116], and Klir and Yuan [247]. The notions of solvability degrees were investigated by Gottwald [173], Gottwald and Pedrycz [181], Gottwald [175, 176], and Perfiliva and Gottwald [373], that measure to what extent the system of fuzzy relational equations is consistent

Resolving the inconsistency by modifying input data was investigated by Pedrycz [350], Cuninghame-Green and Cechlárová [68], and Cechlárová [46] for  $\text{sup-}\mathcal{T}_M$  equations, and by Li and Fang [269, 273] for general  $\text{sup-}\mathcal{T}$  equations.

Cechlárová and Diko [50] showed that finding a maximal consistent subsystem for a system of  $\text{sup-}\mathcal{T}_M$  equations is NP-hard even if all the equations are defined over  $\{0, 1\}$ .

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