ABSTRACT

NA, SUNGSOO. A Heuristic Approach to a Portfolio Optimization Model with Nonlinear Transaction Costs. (Under the direction of Dr. Richard H. Bernhard and Dr. Tao Pang).

After the seminal paper of Markowitz, we have been witnesses to a great evolution with respect to the traditional mean-variance (MV) model. With all its merits, however, some of the main downsides of the MV model and its extended or modified models have been recognized: the computational complexity; the inability to incorporate practical considerations such as taxes and transaction costs; and the investment decision being at exactly one point in time for a single-period horizon. It therefore appears desirable to have an alternative method that can deal with highly demanding real-world portfolio problems considering more complex scenarios and settings.

In this thesis we extend the Markowitz MV model to a rebalancing portfolio optimization problem incorporating realistic considerations such as transaction costs and a risk-free asset with short-selling allowed, and we apply the Tabu Search (TS) heuristic to solve practical portfolio problems. First of all, we propose a bi-objective portfolio optimization model which we expect to yield a portfolio equilibrium by combining the following two objectives: maximize the portfolio's expected return and minimize its risk. For realistic portfolio problems we consider the multi-objective portfolio optimization models incorporating a risk-free asset and its short-selling and nonlinear transaction costs based on a single-period and a rebalancing portfolio optimization problem.

For the single-period portfolio problem, we propose an adaptive version of the TS heuristic. We define a feasible portfolio as the solution representation by means of a vector indicating the amount of money invested in each asset. For the initial solution, we randomly generate portfolios by considering the problem size and the purpose of diversified creation. From the initial solution, we obtain the final solution by iteratively searching with the neighborhood and tabu structure. The neighborhood of the current portfolio is generated by increasing and/or decreasing the adjacent pairwise risky assets with a variation factor. The tabu size is determined by the problem size, and the TS algorithm terminates after some number of iterations without an improvement in the objective function value.

For our primary purpose, we extend the single-period model to a rebalancing problem which also considers nonlinear transaction costs and a risk-free asset. We assume

that the time point for rebalancing the portfolio is exactly at the midpoint of the entire time horizon of the single-period model. In the rebalancing portfolio problem, since we consider nonlinear transaction costs for risky assets' transactions, a myopic policy which tries to optimize each time period independently is not optimal for the problem because portfolio decisions do affect each other time period. The final objective value at the end of the time period is affected by the portfolio decision at the beginning of the planning horizon because the final result comes from the portfolio decision at the time point of rebalancing the portfolio, which is affected by the portfolio decision at the beginning of the time period. Therefore, we have proposed an advanced, adaptive TS algorithm having an evolutionary neighborhood structure, and we have solved the rebalancing portfolio problem with an iterative folding back procedure in the decision tree structure. For computational studies we consider a risk-free asset and the number of risky assets to be 5, 10, 12, and 15 for both the single-period and rebalancing portfolio problems.

A Heuristic Approach to a Portfolio Optimization Model with Nonlinear Transaction Costs

by

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Biography

Sungsoo Na was born in Seoul, Republic of Korea on August 24, 1977 to Jong-Doo Na and Sook-Jin Lee. He is the second child of these two outstanding people, and he was raised with his older brother, Eui-Soo Na. He graduated from Kyung Bock High School in February 1996. Upon graduating from high school, Sungsoo chose to attend Myoung-Ji University, Republic of Korea, to pursue a degree in Industrial Engineering. Half way through his undergraduate studies, he joined the Korean Army and completed twenty-six months of military service. After he was released from the army, he returned to school in 2000. In August 2002, he completed his Bachelor's degree with a major in Industrial Engineering.

Sungsoo chose to continue his education at Yonsei University, Republic of Korea, in pursuit of a Master of Science degree in Industrial and Systems Engineering, immediately following the completion of his undergraduate degree. He received a M.S. degree in Industrial and Systems Engineering under the direction of Dr. Chang-Ouk Kim in August 2004. Upon his acceptance to graduate school at North Carolina State University he moved to Raleigh, North Carolina and began his Ph.D. studies in the fall of 2004. He received a Ph.D. degree in Industrial and Systems Engineering under the direction of Dr. Richard H. Bernhard and Dr. Tao Pang in May 2008.

Sungsoo's professional affiliations include active involvement with the Korea Institute of Industrial Engineering (KIIE), the Korean Financial Engineering Society (KOFES), the International Association of Financial Engineers (IAFE), and the Statistical and Applied Mathematical Sciences Institute (SAMSI). Through his SAMSI affiliation, Sungsoo participated in the working group of the SAMSI program on Financial Mathematics, Statistics, and Econometrics. His other interests include playing golf and tennis and spending time with friends and family. He hopes to continue sharing his passion for and love of learning with others.

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Finally, I would like to acknowledge my family most of all. To my parents, Jong-Doo Na and Sook-Jin Lee: Thank you for always believing in me, supporting all of my dreams, inspiring me to believe in myself, showing me unconditional love and understanding, and showing me that you are always there for me. To my older brother and his wife, Eui-Soo Na and Kyoung-Ah Jung, who support me in all things: Thank you for your continuous love and support and for always believing in me and encouraging me in all of life's pursuits.

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Chapter 1

Introduction

1.1 Motivation

Portfolio management involves formulating, modifying, and implementing an investment strategy for financial assets in the light of investment objectives or preferences (e.g., expected return vs. risk level). An investor is assumed to be faced with a choice from among an enormous number of financial assets for constructing and managing the portfolio. Each financial asset has risk and expected return, and clearly every individual asset contributes to both risk and the expected return of the overall portfolio. Therefore, the investor must accept some amount of risk in order to obtain at least a desired expected return. In other words, risk taken on by the investor is the price paid for the opportunity for a desired level of expected return.

But that raises two questions. How should risk be measured? And what is the relationship between the risk and the expected return? A formal model for the questions was devised in 1952, a feat for which Harry Markowitz eventually won the Nobel Prize for economics in 1990.

Over a half century has elapsed since Harry Markowitz [30] established the theoretical framework for portfolio management by stating a parametric optimization model called the *mean-variance* (MV) model. According to the MV model, the portfolio selection problem can be formulated as an optimization problem over real-valued variables with a

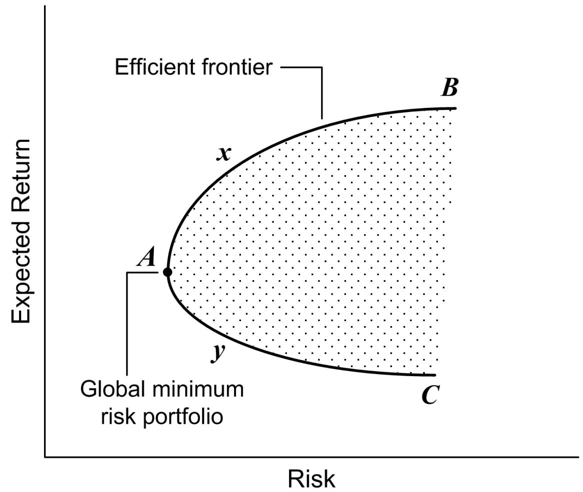


Figure 1.1: The attainable set and the efficient frontier of portfolios

quadratic objective function and linear constraints. The MV model has become a universally understood technique within the investment world for generating the trade-off of changes in risk, measured by the variance of the future asset returns, for changes in expected return called the *efficient frontier*, which is defined as one that has the smallest portfolio risk for a given level of expected return or the largest expected return for a given level of risk. Figure 1.1 illustrates the opportunities available from a given set of assets. A large number of possible portfolios exist when we assume that varying percentages of an investor's wealth can be invested in each of the assets under consideration. However, it is not necessary to evaluate all of the possible portfolios illustrated in Figure 1.1 because, fortunately, investors should be interested only in that subset of the available portfolios known as the efficient frontier.

Given the minimum-risk portfolios, we can plot the minimum-risk frontier as shown in Figure 1.1.[1] Point A represents the global minimum-risk portfolio, because no other minimum-risk portfolio has a smaller risk. The bottom segment of the minimum-risk frontier, AC, is dominated by portfolios on the upper segment, AB. For instance, since portfolio x has a larger return than portfolio y for the same level of risk, investors would want to own portfolio x instead of y. The segment of the minimum-risk frontier above the global minimum-risk portfolio, AB, offers the best risk-return combinations available to investors.

[1] Its mathematical formulation is shown in Chapter 2.

This segment is referred to as the efficient frontier of portfolios, and it can be found by the *quadratic programming* (QP) model.

After the seminal paper of Markowitz [30], we have been witnesses to a great evolution with respect to the traditional MV model. Tobin [59] removed the condition that all assets must be risky, and Black [5] removed the assumption that any asset's weight must be a non-negative real number. Black found an exact analytic solution for this modification representing *short sales*.[2] However, the computational problems related to the solution of the QP model have been pointed out gradually. In order to overcome the weakness of the QP model, Sharpe [49, 50] and Stone [55] proposed different linearizations of the objective function. More recently, Konno and Yamazaki [26] proposed a linear programming model called a *mean-absolute deviation* (MAD) model.

With all its merits, some of the main downsides of the MV model and its extended or modified models have been recognized: the computational complexity; the inability to incorporate practical considerations such as taxes and transaction costs; and the investment decision being at exactly one point in time for a single-period horizon. The QP model incorporating realistic considerations cannot be applied to find an optimal or a near-optimal portfolio. Nonlinear mixed integer programming can be of interest if, for instance, nonlinear transaction costs are considered. Even though different integer programming models can be an alternative for solving practical portfolio problems, not only do they still have a drawback of high computational requirements, but it is also almost impossible to solve a rebalancing portfolio model.

It has therefore appeared desirable to have an alternative method that can deal with highly demanding real-world portfolio problems considering more complex scenarios and settings. *Heuristic*[3] optimization techniques can be the only way out, and in this thesis we apply one of the techniques to solve practical portfolio problems. The heuristic optimization techniques do not derive the solution analytically but by iteratively searching and testing improved or modified solutions until some convergence criteria are met. The heuristic optimization techniques are attractive, as they are independent of the objective function and the structure of the model and its constraints while also being a general and

[2] Selling a security that the seller does not own but is committed to purchasing eventually. It is used to capitalize on an expected decline in the security's price. Short-selling and shorting are synonyms.

[3] Etymologically the word *heuristic* comes from the Greek work *heuriskein*, to discover; the Greek mathematician and inventor Archimedes (287-212 B.C.) is known for the famous *Heureka!* when he discovered a method for determining the purity of gold.

robust method applied to the large size of practical portfolio problems. The most attractive aspect of the heuristic optimization techniques is computational efficiency. Although for larger problem spaces the number of iterations will usually increase, the resulting increase in computational costs is usually lower than it would be for traditional optimization methods. Therefore, the computational complexity of the heuristic optimization techniques is comparatively low; even for NP-complete[4] problems, many heuristic optimization techniques have at most polynomial complexity. Hence, the heuristic optimization techniques are relatively easy to implement and have computational attractiveness, so they are well suited for empirical and computational studies. Maringer [29] studies the traditional portfolio models and their extensions with heuristic methods, and he provides empirical results. The results from the empirical studies presented in his book not only provide new insights to the discussed problems but also demonstrate that heuristics are capable of answering computationally demanding problems reliably and efficiently.

1.2 Objectives

The primary objective of this work is to extend the Markowitz MV model to a rebalancing portfolio model incorporating practical considerations (esp., transaction costs[5]) and to investigate its solution by a selected heuristic optimization method: *Tabu*[6] *Search* (TS) heuristic proposed by Glover [18] in 1986.

For this purpose, first of all, we modify the MV model. The portfolio model is formulated as a bi-criteria optimization problem: the objective function is to maximize the expected terminal wealth and minimize the average variance of the amount of wealth invested simultaneously.

Next, we propose an advanced TS heuristic: a broader set of possible neighborhood relations and search techniques; a deeper analysis on the effects of the parameter settings

[4]In computational complexity theory, the NP-complete is a subset of a nondeterministic polynomial (NP). A problem is called NP if its solution can be guessed and verified in polynomial time; nondeterministic means that no particular rule is followed to make the guess. If a problem is NP and all other NP problems are polynomial-time reducible to it, the problem is NP-complete. The knapsack problem, traveling salesman problem, and graph coloring problem are well-known NP-complete problems.

[5]Considering transaction costs is realistic, since investors are subject to brokerage fees and bid-ask spreads, which generate proportional transaction costs of trading, as well as costs of gathering and processing information, which generate fixed transaction costs.

[6]Also sometimes spelled as "Taboo".

than classical local search methods; and adaptive evolution schemes for the different time-period (single-period and rebalancing portfolio models).

Finally, the suggested TS heuristic approach is applied to empirical studies incorporating a short-selling allowance of a risk-free asset and nonlinear transaction costs. We consider a risk-free asset and 5, 10, 12, and 15 risky assets for the computational studies.

Due to restrictions in traditional optimization methods, many of the previously discussed portfolio problems had to rely on simplifying assumptions. However, we propose an alternative route and a new optimization method that are capable of dealing with heretofore unanswerable problems. Models and problems can be investigated that allow for more complexity and are therefore closer to reality than those approachable with traditional methods. That, of course, eventually also contributes to a better understanding of financial markets.

1.3 Organization of the Thesis

The dissertation is organized as follows. In **Chapter 2**, first of all, we review a fundamental framework for the portfolio return and risk, and then we examine a Markowitz-inspired mean-variance analysis. Next, we present the fundamental concepts of the selected heuristic method, tabu search, with its basic elements. Additionally, we review some discrete-time portfolio optimization models extended and modified from the Markowitz MV model. This chapter particularly provides a review of the portfolio optimization model considering real-world circumstances and practical considerations such as transaction costs. The last part of the chapter reviews papers on realistic portfolio selection models with heuristic optimization techniques.

In **Chapter 3**, we propose a bi-criteria objective function for a portfolio optimization model. Next, the single-period portfolio optimization model incorporating a short-selling allowance of the risk-free asset and nonlinear transaction costs is treated. This chapter also provides a TS heuristic approach to a single-period portfolio optimization model, and discusses the experiments and the results.

From the mathematical model of the single-period model in Chapter 3, we formulate a rebalancing portfolio optimization model in **Chapter 4**. The TS heuristic approach and computational results are provided and discussed. Also, the results are compared with those of the single-period portfolio optimization model of Chapter 3.

In addition to the numerical experiments with randomly generated data in Chapters 3 and 4, in **Chapter 5**, we apply the proposed portfolio optimization model and TS heuristic to the U.S. stock market.

Finally, in **Chapter 6**, conclusions, citation of original contributions, limitations, and suggestions for future work are presented.

1.4 Mathematical Notation

We shall use the following notation throughout this thesis. These symbols apply to all chapters. Additionally, more notation for the single-period and rebalancing models is defined in Sections 1.4.1 and 1.4.2, respectively.[7]

$\mathbb{R}$	Set of real numbers.
$\mathbb{R}^+ = \{x \in \mathbb{R} \vert x \geq 0\}$	Set of nonnegative real numbers.
$\mathbb{J} = \{1, \ldots, N\}$	Set of N risky assets.
$\mathbb{1}_{\{condition\}}$	Indicator function for the truth of condition.
i, j	Index representing assets, $i = j = 0$ for a risk-free asset and $i = j = 1, 2, \ldots, N$ for risky assets.
t	Index representing the time period.
T	Index representing the final time point of the portfolio planning horizon.
w_i	Proportional wealth balance (asset weight) of asset i.
R_i	Actual rate of return of asset i.
$R_{\mathcal{P}}$	Actual return of portfolio.
r_i	Expected rate of return of asset i.
$r_{\mathcal{P}}$	Expected nominal return of portfolio.
u_i	Amount of money invested in asset i.
$u_{\mathcal{P}}$	Expected real return of portfolio.
$\mathrm{Var}_{\mathcal{P}}$	Variance (risk) of portfolio. Equivalently, $\sigma^2_{\mathcal{P}}$.

[7] For better legibility, the expected value operator $E(\cdot)$ will be dropped.

$\mathrm{Cov}\big(R_i, R_j\big)$	Co-variation between returns on assets i and j. Equivalently, $\sigma_{i,j}$.
$\mathrm{Corr}\big(R_i, R_j\big)$	Correlation coefficient between returns on assets i and j. Equivalently, $\rho_{i,j}$.
$\mathrm{SD}_{\mathcal{P}}$	Standard deviation (volatility) of portfolio. Equivalently, $\sigma_{\mathcal{P}}$.
R_i^t	Actual rate of return of asset i during time period $[t-1,\ t]$.
$\bar{r}_i$	Average rate of return of asset i.
σ_i^2	Variance of asset i.
σ_i	Standard deviation (volatility) of asset i.
l	Short-selling limitation (credit balance) of risk-free asset.
c_b	Proportional transaction cost for buying risky asset.
c_s	Proportional transaction cost for selling risky asset.
f_b	Fixed transaction cost for buying risky asset.
f_s	Fixed transaction cost for selling risky asset.

1.4.1 Single-period model

$u_{\mathcal{P}}^*$	Desired level of expected return (target expected return) of portfolio.
W	Initial wealth (endowment) of investor.
$\sigma_{\mathcal{P}}^{2^*}$	Desired level of variance (target risk) of portfolio.
$\sigma_{i,j}$	Covariance between returns on assets i and j.
$\rho_{i,j}$	Correlation coefficient between returns on assets i and j.
α	Control parameter of risk level.
ϕ_i	Transaction costs of risky asset i.
$\phi_{\mathcal{P}\mathcal{I}}$	Transaction costs of portfolio at the beginning of the period.
$\phi_{\mathcal{P}\mathcal{F}}$	Transaction costs of portfolio at the end of the period.

1.4.2 Rebalancing model

$u_{\mathcal{P}}^t$	Expected portfolio return during time period $[t-1,\ t]$.
$\mathrm{Var}_{\mathcal{P}}^t$	Portfolio risk during time period $[t-1,\ t]$.
r_i^t	Expected rate of return of asset i during time period $[t-1,\ t]$.
x_i^t	Amount of money of asset i at time t before rebalancing.

W_t	Portfolio wealth balance at time t before rebalancing. Therefore, $W_t = \sum_{i=1}^{N} x_i^t$.
b_i^t	Amount of money of asset i for buying at time t for rebalancing.
s_i^t	Amount of money of asset i for selling at time t for rebalancing.
u_i^t	Amount of money invested in asset i at time t for rebalancing. Therefore, $u_i^t = x_i^t + b_i^t - s_i^t$.
$\sigma_{i,j}^t$	Covariance between returns on assets i and j during time period $[t-1,\ t]$.
α_t	Control parameter of risk level during time period $[t-1,\ t]$.
ϕ_i^t	Transaction costs of risky asset i at time point t.
$\phi_{\mathcal{P}}^t$	Transaction costs of portfolio at time point t.
$\phi_{\mathcal{P}}^T$	Transaction costs of portfolio at the end of the period.
a^+	$\max(a, 0)$.
a^-	$\min(a, 0)$.

Chapter 2

Literature Review

2.1 Introduction

A vast literature related to portfolio selection has been published since the seminal work of Harry Markowitz [30] in 1952. Portfolio optimization problems can be classified by the time point of the investment decision: (i) single-period versus rebalancing or multi-period models, (ii) discrete-time versus continuous-time models. Continuous-time portfolio problems have been studied by Merton [35], Duffie and Richardson [15], and Zhou [66]. In this thesis, we first consider a discrete-time single-period model, and then extend to a discrete-time rebalancing model incorporating a shorting allowance for a risk-free asset and nonlinear transaction costs.

In this chapter, we review the Markowitz MV model and other popularly cited, topic-related portfolio optimization studies. Additionally, we review a general description of the tabu search (TS) heuristic. This chapter is organized as follows. In Section 2.2, we first of all review a fundamental framework for the portfolio return and risk, and then we examine a Markowitz-inspired mean-variance analysis with three different categories. In Section 2.3, we discuss discrete-time portfolio optimization models extended and modified from the Markowitz MV model. In Section 2.4, we review papers considering real-world circumstances and practical considerations such as taxes, transaction costs, cardinality,[1]

[1] Cardinality constraints limit a portfolio to have a specific number of assets.

and restrictions on transaction volume. Studies about transaction costs in particular are discussed. After that, in Section 2.5, we describe a selected heuristic optimization method, TS, with its basic elements, and then we review papers on portfolio selection models with heuristic optimization techniques. The summary of this chapter is provided in Section 2.6.

2.2 Markowitz Mean-Variance Model

Characterization of a portfolio in terms of the expected return and risk is a rather more complicated matter than is the case for a single asset, particularly with regards to risk. In particular, the risk for individual asset returns has two components – systematic and nonsystematic risks. *Systematic risk* is market risk that cannot be diversified away because it is the risk of movement in the overall market or in the relevant market segment. Interest rate, recessions, and wars are examples of systematic risks. On the other hand, *unsystematic risk*, also known as specific or diversifiable risk, is specific to individual assets and can be diversified away by combining the assets in a diversified portfolio as shown in Figure 2.1. It represents the component of an asset's return that is not correlated with general market moves.

The difficulty of characterizing the portfolio in terms of the expected return and risk stems from the fact that asset returns are correlated, and for this reason we need to formally introduce the *correlation coefficient* to quantify precisely the correlation among

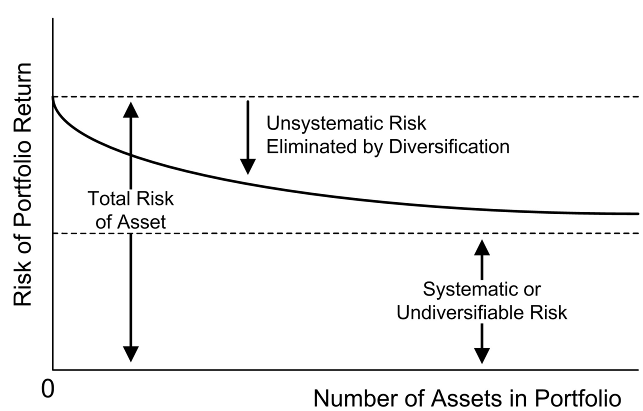


Figure 2.1: Asset risk with two components: systematic and unsystematic risks

the returns on portfolio assets.

Harry Markowitz established a framework for describing a portfolio in terms of the mean on assets' returns, the variance of their returns and the correlation between the returns on assets. This approach is known as *mean-variance analysis*, and it can be classified in three categories: (i) the minimum-risk portfolio model, (ii) the minimum-risk portfolio model for a given return level, (iii) the maximum-return portfolio model for a given risk level, and they are reviewed in Section 2.2.2.

2.2.1 Portfolio return and risk

Portfolio return is fairly easy to compute: it is the weighted sum of the returns on the constituent assets of the portfolio. The portfolio return $R_{\mathcal{P}}$ given N assets with portfolio weights (proportional wealth balances) w_i and assets' returns R_i can thus be expressed as

$$R_{\mathcal{P}} = \sum_{i=1}^{N} w_i \cdot R_i. \tag{2.1}$$

The return on each asset in the portfolio is thus simply multiplied by the proportion or weight with which that particular asset is held in the portfolio.

Precisely speaking, the returns are regarded and treated as random variables and are described with the normal distribution, the expected value (mean) of the asset returns r_i and its variance σ_i^2 (or standard deviation σ_i).[2] With the expected value of the asset returns r_i and the asset weights w_i, the expected nominal return of the portfolio is expressed as

$$r_{\mathcal{P}} = \sum_{i=1}^{N} w_i \cdot r_i. \tag{2.2}$$

Additionally, in this thesis, we use the amount of money invested in each asset u_i instead of proportional wealth balance (asset weight) of each asset w_i for mathematical formulations and empirical experiments because our portfolio problems dealt with in later chapters contain transaction costs, and especially it is easy to express and compute fixed transaction

[2] σ_i is usually referred to as *volatility* in the finance literature.

costs by using amounts of money. Therefore, the expected real return of the portfolio can be reexpressed as

$$u_{\mathcal{P}} = \sum_{i=1}^{N} u_i(1 + r_i). \tag{2.3}$$

We characterize the *risk* of a single asset as the variation over time of its returns relative to the mean of those returns, and this is quantified as *volatility*. The computations required for the risk of a single asset are relatively simple and straightforward. However, when we consider the portfolio risk of more than one asset, calculating the portfolio risk becomes a bit more complicated since then risk of the portfolio is not just a simple (weighted) sum of its parts. That is, the nature of the portfolio can yield aggregate portfolio characteristics that differ significantly from what one would expect, given individual assets in the portfolio relative to one another – their *co-variation*.

Table 2.1 illustrates N assets' returns (random variables) observed during a finite time period $[0, T]$. Assets' returns in each time period can be represented by daily, weekly or monthly data. Based on finite sample points shown in Table 2.1, the estimated co-variation $\hat{\sigma}_{A,B}$ between the returns on two assets A and B can be described mathematically by the term *covariance* as follows:

Table 2.1: Finite sample points of N assets' returns

Asset	Sample Points of Returns					Mean
	1	2	3	$\cdots$	T	
1	R_1^1	R_1^2	R_1^3	$\cdots$	R_1^T	$\overline{r}_1 = \frac{1}{T}\sum_{t=1}^{T} R_1^t$
2	R_2^1	R_2^2	R_2^3	$\cdots$	R_2^T	$\overline{r}_2 = \frac{1}{T}\sum_{t=1}^{T} R_2^t$
3	R_3^1	R_3^2	R_3^3	$\cdots$	R_3^T	$\overline{r}_3 = \frac{1}{T}\sum_{t=1}^{T} R_3^t$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
N	R_N^1	R_N^2	R_N^3	$\cdots$	R_N^T	$\overline{r}_N = \frac{1}{T}\sum_{t=1}^{T} R_N^t$

$$\widehat{\mathrm{Cov}}\big(R_A, R_B\big) = \frac{1}{T-1}\sum_{t=1}^{T}\Big(R_A^t - \overline{r}_A\Big)\Big(R_B^t - \overline{r}_B\Big).^3 \tag{2.4}$$

The covariance of asset returns is an absolute measure of co-variation between the assets. This means that it does not consider the magnitude of the deviations from the means. We can therefore usefully standardize the covariance term by adjusting for the magnitude of deviations from the respective means of the assets. This magnitude is quantified by the standard deviation of returns on the assets, so to standardize the covariance term we simply divide it by the multiple of standard deviations of the assets, which yields the *correlation coefficient*. Correlation is a measure of the statistical relationship between two comparable time series. For financial investors, these series can be two stocks, two commodities, a stock and an index or even a stock and a commodity. The relationship, which can be causal, complementary, parallel or reciprocal, is stated as the correlation coefficient and always reflects the simultaneous change in value of the pairs of numerical values over time. The correlation coefficient $\rho_{A,B}$ between the returns on asset A and B is thus expressed as

$$\widehat{\mathrm{Corr}}(R_A, R_B) = \frac{\frac{1}{T-1}\sum_{t=1}^{T}\Big(R_A^t - \overline{r}_A\Big)\Big(R_B^t - \overline{r}_B\Big)}{\sqrt{\frac{1}{T-1}\sum_{t=1}^{T}\Big(R_A^t - \overline{r}_A\Big)^2} \cdot \sqrt{\frac{1}{T-1}\sum_{t=1}^{T}\Big(R_B^t - \overline{r}_B\Big)^2}} \tag{2.5a}$$

$$= \frac{\widehat{\mathrm{Cov}}\big(R_A, R_B\big)}{\sqrt{\widehat{\mathrm{Var}}\big(R_A\big)} \cdot \sqrt{\widehat{\mathrm{Var}}\big(R_B\big)}} \tag{2.5b}$$

$$= \frac{\hat{\sigma}_{A,B}}{\hat{\sigma}_A \cdot \hat{\sigma}_B}. \tag{2.5c}$$

The covariance of the asset return with itself is equal to its variance. Examining (2.5a) – (2.5c) we see that this implies that the correlation coefficient always lies between -1 and $+1$, as a positive or negative value that the assets of a pair relate to each other. A negative reading suggests that one asset of the pair consistently moves up while the other moves

[3] $\sigma_{i,j}$ denotes the covariance between the returns of asset i and j with $\sigma_{i,i} = \sigma_i^2$ and $\sigma_{i,j} = \sigma_{j,i}$. And, the $(T-1)$ term in the denominator of the unbiased covariance formula is referred to as degree of freedom.

down. Conversely, a positive reading suggests that there is a tendency for the pair move together in the same direction. If the correlation coefficient is zero, it means the two assets have no correlation, indicating that their statistical relationship is completely random.

When dealing with risk in a portfolio context, the most important aspect to consider is the way in which the values in the portfolio move relative to each other. We therefore need to quantify how and to what extent the movement of each asset relates to the movement of every other asset in the portfolio.

The method for calculating risk in the form of the volatility for a portfolio involves both the variances of returns and the covariance between returns. We first consider a portfolio of three assets A, B, and C, and then extend to a general expression for risk of a portfolio of N assets. The variance of the three assets' portfolio, with the amount of money invested in each asset at time t equal to u_A^t, u_B^t, and u_C^t, respectively is defined as follows:

$$\begin{aligned}
\widehat{\mathrm{Var}}_{\mathcal{P}} &= \frac{1}{T-1}\sum_{t=1}^{T}\Big[\Big(u_A^t\cdot R_A^t + u_B^t\cdot R_B^t + u_C^t\cdot R_C^t\Big) \\
&\quad -\Big(u_A^t\cdot \overline{r}_A^t + u_B^t\cdot \overline{r}_B^t + u_C^t\cdot \overline{r}_C^t\Big)\Big]^2 \qquad (2.6a) \\
&= \frac{1}{T-1}\sum_{t=1}^{T}\Big[\Big(u_A^t\cdot R_A^t - u_A^t\cdot \overline{r}_A^t\Big) + \Big(u_B^t\cdot R_B^t - u_B^t\cdot \overline{r}_B^t\Big) \\
&\quad +\Big(u_C^t\cdot R_C^t - u_C^t\cdot \overline{r}_C^t\Big)\Big]^2 \qquad (2.6b) \\
&= \frac{1}{T-1}\sum_{t=1}^{T}\Big[\Big(u_A^t\Big)^2\Big(R_A^t-\overline{r}_A\Big)^2 + \Big(u_B^t\Big)^2\Big(R_B^t-\overline{r}_B\Big)^2 + \Big(u_C^t\Big)^2\Big(R_C^t-\overline{r}_C\Big)^2 \\
&\quad +2u_A^t u_B^t\Big(R_A^t-\overline{r}_A\Big)\Big(R_B^t-\overline{r}_B\Big) \\
&\quad +2u_B^t u_C^t\Big(R_B^t-\overline{r}_B\Big)\Big(R_C^t-\overline{r}_C\Big) \\
&\quad +2u_C^t u_A^t\Big(R_C^t-\overline{r}_C\Big)\Big(R_A^t-\overline{r}_A\Big)\Big] \qquad (2.6c) \\
&= \frac{1}{T-1}\sum_{t=1}^{T}\Big[\Big(u_A^t\Big)^2\widehat{\mathrm{Var}}\big(R_A\big) + \Big(u_B^t\Big)^2\widehat{\mathrm{Var}}\big(R_B\big) + \Big(u_C^t\Big)^2\widehat{\mathrm{Var}}\big(R_C\big) \\
&\quad +2u_A^t u_B^t\widehat{\mathrm{Cov}}\big(R_A,R_B\big) + 2u_B^t u_C^t\widehat{\mathrm{Cov}}\big(R_B,R_C\big) + 2u_C^t u_A^t\widehat{\mathrm{Cov}}\big(R_C,R_A\big)\Big]. \qquad (2.6d)
\end{aligned}$$

Ignoring individual time periods, and assuming that all asset variances have been calculated based on the portfolio planning horizon, (2.6d) can be generalized to yield an expression for the risk of the N-asset portfolio.

$$\mathrm{Var}_{\mathcal{P}} = \sigma_{\mathcal{P}}^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} u_i u_j \mathrm{Cov}(R_i, R_j) \tag{2.7a}$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N} u_i u_j \sigma_{i,j} \tag{2.7b}$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N} u_i u_j \sigma_i \sigma_j \rho_{i,j} \tag{2.7c}$$

which means that the portfolio volatility can be expressed as

$$\mathrm{SD}_{\mathcal{P}} = \sigma_{\mathcal{P}} = \sqrt{\mathrm{Var}_{\mathcal{P}}} \tag{2.8a}$$

$$= \sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N} u_i u_j \mathrm{Cov}(R_i, R_j)} \tag{2.8b}$$

$$= \sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N} u_i u_j \sigma_{i,j}} \tag{2.8c}$$

$$= \sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N} u_i u_j \sigma_i \sigma_j \rho_{i,j}}. \tag{2.8d}$$

The total risk of the portfolio as measured by its variance expressed in (2.7) consists of $\sum_{i \neq j} u_i u_j \sigma_{i,j}$, the systematic risk associated with the correlations between the returns on the assets in the portfolio, and $\sum_{i=1}^{N} u_i^2 \sigma_i^2$, the unsystematic or specific risk associated with the individual asset's variance alone.

2.2.2 Mean-variance model

In the Markowitz-inspired mean-variance framework [30, 31], optimization and optimal portfolios are closely linked to *efficiency*. In a portfolio context we can define efficiency as the maximum attainable return for a given level of risk, or alternatively, the minimum attainable risk for a given level of the return. An optimal portfolio is therefore a portfolio that is mean-variance efficient. In other words, in efficient portfolios we can obtain no higher return and no lower risk without paying a price in terms of either higher risk or lower return respectively.

In a portfolio when the price of one asset changes, the other assets will tend to move either by more or by less, up or down. This is the essential context for which assets in a portfolio are not perfectly correlated. So, it is possible to diminish the total amount of risk or to increase the total expected return in the portfolio by choosing to include different amount of different assets with different return, risk, and correlation characteristics. This process is known as mean-variance analysis for a portfolio optimization model, and it can be classified in the following three categories.

- minimum-risk portfolio model
- minimum-risk portfolio model for a given return level
- maximum-return portfolio model for a given risk level

Minimum-risk portfolio model

The minimum-risk portfolio model[4] is the portfolio model that generates the lowest amount of risk achievable given the particular return and the risk characteristics of each asset. In other words, the minimum-risk portfolio specifies the amounts of assets that generate the lowest possible portfolio risk, without any additional constraints on the desired return level or on the maximum or minimum extent to which an asset can enter into the portfolio.

$$\min_{u_i} \quad \mathrm{Var}_{\mathcal{P}} = \sum_{i=1}^{N} \sum_{j=1}^{N} u_i u_j \sigma_{i,j} \tag{2.9a}$$

$$\text{subject to} \quad \sum_{i=1}^{N} u_i = W \tag{2.9b}$$

$$u_i \in \mathbb{R}^+, \qquad \forall \, i \in \mathbb{J} \tag{2.9c}$$

[4] We shall use the conventional term, the minimum-risk portfolio instead of the minimum-variance portfolio.

This model includes the possibility that some assets are not invested by (2.9c), that is, they are to be dropped from the portfolio altogether. Obviously, if portfolio assets contain a risk-free asset, the investor's entire wealth will be invested in the risk-free asset based on the principle of the minimum-risk portfolio model.

Minimum-risk portfolio model for a given return level

In the minimum-risk portfolio model, we simply determine the minimum attainable level of the portfolio risk, regardless of the expected returns generated by that particular portfolio composition. However, in the minimum-risk portfolio model for a given return level, we are interested in obtaining a very specific level of the expected return that lies above that provided by the minimum-risk portfolio. For the model we need one additional constraint, namely that the portfolio return must be equal to the specific level $u^*_{\mathcal{P}}$. Therefore, the mathematical formulation for the model can be developed by simply setting the (2.3) to $u^*_{\mathcal{P}}$ predetermined.

$$\min_{u_i} \quad \mathrm{Var}_{\mathcal{P}} = \sum_{i=1}^{N} \sum_{j=1}^{N} u_i u_j \sigma_{i,j} \tag{2.10a}$$

$$\text{subject to} \quad u_{\mathcal{P}} = \sum_{i=1}^{N} u_i \big(1 + r_i\big) = u^*_{\mathcal{P}} \tag{2.10b}$$

$$\sum_{i=1}^{N} u_i = W \tag{2.10c}$$

$$u_i \in \mathbb{R}^+, \qquad \forall\, i \in \mathbb{J} \tag{2.10d}$$

Maximum-return portfolio model for a given risk level

Maximization of portfolio returns is the most common perspective on investing. For some investors this is the only objective that is given attention, which is why most of these investors end up losing their investment. This is because the portfolio risk is a function of (the variation in) portfolio returns. Conversely, there can be no potential for the return without the potential for risk; quantitative maximization of the portfolio return must be carried out subject to the prespecified risk level.

$$\max_{u_i} \quad u_{\mathcal{P}} = \sum_{i=1}^{N} u_i \big(1 + r_i\big) \tag{2.11a}$$

$$\text{subject to} \quad \mathrm{Var}_{\mathcal{P}} = \sum_{i=1}^{N} \sum_{j=1}^{N} u_i u_j \sigma_{i,j} = \sigma_{\mathcal{P}}^{2^*} \tag{2.11b}$$

$$\sum_{i=1}^{N} u_i = W \tag{2.11c}$$

$$u_i \in \mathbb{R}^+, \qquad \forall\, i \in \mathbb{J} \tag{2.11d}$$

Similarly, different strategies for the given return and risk levels can be applied to the models by modifying (2.10b) and (2.11b): (i) the minimum-risk portfolio model with no smaller than a preselected return level, (ii) the maximum-return portfolio model with no greater than a preselected risk level.

$$u_{\mathcal{P}} = \sum_{i=1}^{N} u_i \big(1 + r_i\big) \geq u_{\mathcal{P}}^* \tag{2.12}$$

$$\mathrm{Var}_{\mathcal{P}} = \sum_{i=1}^{N} \sum_{j=1}^{N} u_i u_j \sigma_{i,j} \leq \sigma_{\mathcal{P}}^{2^*} \tag{2.13}$$

Efficient portfolios

Depending on the covariances between the asset returns, the portfolio with the lowest expected return is not necessarily the portfolio with the least risk. In this case, the minimum variance portfolio $\mathcal{P}_{\mathrm{MVP}}$ has the least risk, as can be seen from Figure 2.2. Therefore, searching for the portfolio structure with least risk for the portfolio's expected return of $u_{\mathcal{P}}^*$ in (2.10b) is reasonably only for $u_{\mathcal{P}}^* \geq u_{\mathrm{MVP}}$. It is also apparent that every rational investor will choose portfolios on the upper frontier of the opportunity set, represented with a bold

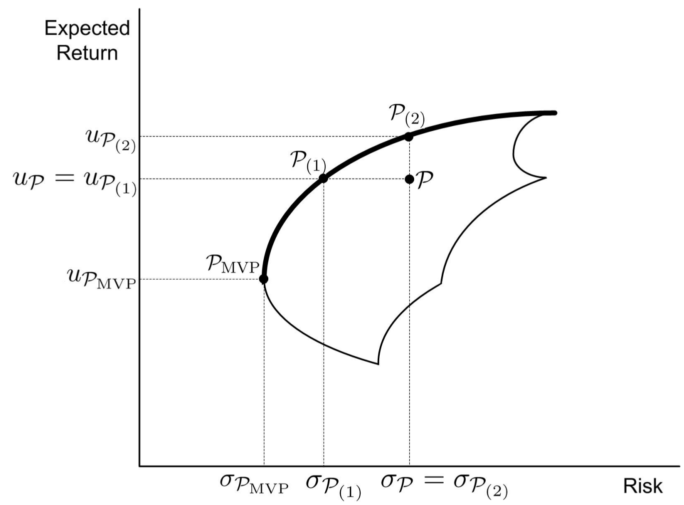


Figure 2.2: Opportunity set and efficient portfolio line in a Markowitz framework

line. That is, for any portfolio $\mathcal{P}$ which is not on the border, there exists a portfolio $\mathcal{P}_{(1)}$ with the same expected return but less risk, and a portfolio $\mathcal{P}_{(2)}$ with equal risk but higher expected return. In this case, $\mathcal{P}$ is an *inferior portfolio* whereas $\mathcal{P}_{(1)}$ and $\mathcal{P}_{(2)}$ are both *efficient portfolios*. Therefore, the upper bound of the opportunity set above the minimum variance portfolio is called the *efficient portfolio set* or *efficient portfolio line*.

In model (2.10), the desired level of the expected return (target return) of the portfolio $u^*_{\mathcal{P}}$ is chosen exogenously and might therefore well be below $u_{\mathcal{P}_{\mathrm{MVP}}}$ since the minimum variance portfolio is not known beforehand. This pitfall can be avoided by combining the constraint of the expected return (2.10b) and the corresponding risk. The original objective function (2.10a) of minimizing the risk can then be replaced with maximizing the expected return, diminished by the incurred risk,

$$\max_{u_i} \big(\lambda \cdot u_{\mathcal{P}} - (1-\lambda) \cdot \mathrm{Var}_{\mathcal{P}}\big), \tag{2.14}$$

where the trade-off between the return and risk is reflected. The efficient portfolio line can then be identified by solving this problem for different, exogenously determined values of $\lambda \in [0, 1]$: If $\lambda = 1$, the model will search for the portfolio with the highest possible expected return regardless of the variance (risk). Lower values for λ put more emphasis on the portfolio's risk and less on its expected return. With $\lambda = 0$, the minimum variance portfolio will be identified.

Similarly, in model (2.11), the original objective function (2.11a) of maximizing the expected return can be replaced with minimizing the risk, increased by the incurred expected return,

$$\min_{u_i} \big((1-\lambda)\cdot \mathrm{Var}_{\mathcal{P}} - \lambda \cdot u_{\mathcal{P}}\big). \tag{2.15}$$

We can observe that both objective functions (2.14) and (2.15) are the same, and therefore they will produce an identical solution. That is, from the definition of portfolio efficiency, minimizing portfolio risk subject to a target expected return constraint is equivalent to maximizing the expected return of the portfolio subject to a target risk constraint. This is because both problems, when solved, result in an optimal solution such that we would be unable to produce a better trade-off between the return and risk by changing the portfolio composition. In other words, we are faced with a dual problem such that we need only solve one side of it in order to obtain the result for both.

2.3 Discrete-Time Portfolio Optimization

The most important contribution of the MV approach by Markowitz [30, 31] is that it quantifies risk by using the variance, which enables investors to seek the highest expected return after specifying their acceptable risk level. In the MV model, a perfect market[5] without a risk-free asset is assumed where short sales are not allowed, but risky assets are infinitely divisible and can therefore be traded in any non-negative fraction. Tobin [59] and Black [5] overcome these assumptions in their earlier works.

First of all, Tobin [59] removes the condition that all assets must be risky. If the initial endowment is invested in a risk-free asset with risk-free rate r_0 and a portfolio of some risky assets with the expected portfolio rate $r_{\mathcal{P}}$, then the return of the resulting portfolio $\mathcal{P}^*$ is

$$r_{\mathcal{P}^*} = \mu \cdot r_0 + (1-\mu)\cdot r_{\mathcal{P}} \tag{2.16}$$

[5]The assumption includes rational investors, no transaction costs, particularly no information costs and no taxes, and equal access to market prices and information.

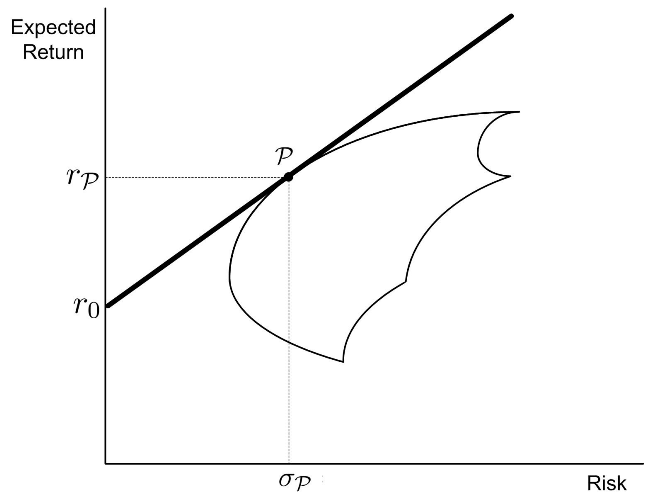


Figure 2.3: Efficient portfolio line in a Tobin framework

where μ is the proportional wealth balance (asset weight) of the risk-free asset. By assumption, obviously, the variance and covariances between the risk-free asset and risky assets are zero: $\sigma_0^2 = \sigma_{0,\mathcal{P}} = 0$. Hence, the portfolio risk is simply

$$\sigma_{\mathcal{P}^*}^2 = \mu^2\sigma_0^2 + 2\mu(1-\mu)\sigma_{0,\mathcal{P}} + (1-\mu)^2\sigma_{\mathcal{P}}^2 \tag{2.17a}$$

$$= (1-\mu)^2 \cdot \sigma_{\mathcal{P}}^2 \tag{2.17b}$$

$$\Rightarrow \sigma_{\mathcal{P}^*} = (1-\mu) \cdot \sigma_{\mathcal{P}}. \tag{2.17c}$$

Solving (2.17c) for μ and substituting the result into (2.16) reveals the linear relationship between the portfolio return and the risk in a Tobin model:

$$\mu = 1 - \frac{\sigma_{\mathcal{P}^*}}{\sigma_{\mathcal{P}}} \tag{2.18a}$$

$$r_{\mathcal{P}^*} = r_0 + (r_{\mathcal{P}} - r_0) \cdot \frac{\sigma_{\mathcal{P}^*}}{\sigma_{\mathcal{P}}}. \tag{2.18b}$$

Unlike in the Markowitz framework as can be seen from Figure 2.2, the efficient portfolio line is no longer a curve but a straight line depicted in Figure 2.3.

According to the constraints, (2.9c), (2.10d), and (2.11d) in the Markowitz MV model, any investment must be a nonnegative real number. If the non-negativity constraint is removed from Markowitz's original set of assumptions and replaced with

$$u_i \in \mathbb{R}, \qquad \forall\, i \in \mathbb{J}, \tag{2.19}$$

any asset can be invested with any real number - as long as the constraints (2.9b), (2.10c), and (2.11c) are met. Negative investments represent short sales where the investor receives today's asset price and has to pay the then current price in the future.

This modification is done by Black [5], who is therefore able to find an exact analytic solution for this simplified portfolio selection problem. With allowing of short sales, a closed-form solution[6] exists, and the efficient portfolio risk $\sigma^2_{\mathcal{P}}$ and asset weights $\boldsymbol{w}$ for a given level of portfolio return $r_{\mathcal{P}}$ can be determined by

$$\sigma^2_{\mathcal{P}} = \begin{bmatrix} r_{\mathcal{P}} & 1 \end{bmatrix} \boldsymbol{A}^{-1} \begin{bmatrix} r_{\mathcal{P}} \\ 1 \end{bmatrix} = \frac{a - 2 \cdot r_{\mathcal{P}} + c \cdot r^2_{\mathcal{P}}}{a \cdot c - b^2} \tag{2.20a}$$

$$\text{with} \quad \boldsymbol{A} = \begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} \boldsymbol{r}' \\ \mathbf{I}' \end{bmatrix} \boldsymbol{\Sigma}^{-1} \begin{bmatrix} \boldsymbol{r} & \mathbf{I} \end{bmatrix} \tag{2.20b}$$

$$\text{and} \quad \boldsymbol{w} = \boldsymbol{\Sigma}^{-1} \begin{bmatrix} \boldsymbol{r} & \mathbf{I} \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix}^{-1} \begin{bmatrix} r_{\mathcal{P}} \\ 1 \end{bmatrix} \tag{2.20c}$$

where $\boldsymbol{r} = [r_i]_{N\times 1}$ is the vector of the N assets' expected returns, $\boldsymbol{\Sigma} = [\sigma_{i,j}]_{N\times N}$ is the covariance matrix, and $\mathbf{I}$ is the unity vector. The return and risk of the minimum variance portfolio are b/c and $1/c$, respectively.

In the original MV model, it is assumed that portfolio returns follow a multivariate normal distribution. The return on a portfolio is a random variable and therefore can be completely described by the first two moments: the expected return and the variance. The portfolio risk is measured by the variance, and therefore inevitably the portfolio selection

[6] An equation is said to be a closed-form solution if it solves a given problem in terms of functions and mathematical operations from a given generally accepted set.

becomes a quadratic programming (QP) problem. For instance, the MV model becomes a mixed-integer nonlinear (quadratic) programming if it incorporates cardinality constraints as

$$\sum_{i=1}^{N} z_i = K, \qquad z_i \in [0,1] \text{ and } \forall\, i \in \mathbb{J} \tag{2.21}$$

where

$$z_i = \begin{cases} 1 & \text{if any of asset } i \text{ is invested} \\ 0 & \text{otherwise} \end{cases}$$

and K is the maximum (desired) number of asset allowed in the portfolio. Hence, the portfolio model incorporating cardinality constraints like (2.21) can be solved by using a branch-and-bound algorithm and an interior point method (e.g., see [58, 6, 4, 7]).

However, people have been more interested in other ways to measure risks,[7] and the computational problems related to the solution of the QP model have been pointed out gradually. Sharpe [49, 50] and Stone [55] propose different linearizations of the objective function.

In his earlier work, Sharpe [49] argues that the market responsiveness of a portfolio is an appropriate risk surrogate for a well-diversified portfolio. The linear programming (LP) model that results from the use of the market responsiveness as the risk measure in the objective function is

$$\max_{w_i} \sum_{i=1}^{N} w_i \big(r_i - \theta B_i\big) \tag{2.22}$$

where B_i is the market response of asset i and θ is a parameter reflecting degree of risk aversion. Also, he considers LP constraints for the imposition of an upper limit on investment[8] in each asset as

$$0 \le w_i \le p, \qquad \forall\, i \in \mathbb{J} \tag{2.23}$$

where p is the maximum fraction of the portfolio that may be invested in any one asset. This

[7] Bernstein [3] gives a good historical survey of risk in his novel.

[8] It is also called a ceiling constraint. On the contrary, a constraint for a lower limit is called a floor constraint.

linearized portfolio model can be treated as a knapsack problem,[9] and hence the problem can be computationally very simple. However, its defects appear to be that it ignores independent variability and is dependent on a single-index model.

A few years later Sharpe [50] approached the problem of capturing the essence of MV portfolio selection in an LP formulation by making a diagonalizing transformation of variables that will convert the general expression for variance and using a piece-wise linear approximation for each of the terms in the diagonalized expression for the variance. This approach possesses the advantage of not being dependent on a single-index model. Because it is based on diagonalizing the covariance matrix, it is inherently limited to the MV framework and therefore to the limitation of variance as a risk measure.

A different work for the linearized portfolio model is done by Stone [55]. He presents an alternative to capture the essence of portfolio selection as a LP model incorporating a ceiling constraint which is the same as (2.23). Stone first of all generalizes the MV criterion to differentiate between market and nonmarket risk and presents an LP approximation to the parametric QP implied by the MV criterion. Although his MV model is in the spirit of the Sharpe [49] model, the extension is the inclusion of independent variability as well as market variability.

In addition, Stone introduces a mean-variance-skewness (MVS) criterion and shows how the cubic program implied by this criterion can be approximated by a LP. By adding both market and independent skewness to the MVS model, which is still dependent on the single-index model, a new dimension to risk has been included in the problem.

More recently, Konno and Yamazaki [26] propose a LP model using mean-absolute deviation (MAD) as the risk function

$$E\left[\left|\sum_{i=1}^{N} R_i u_i - E\left[\sum_{i=1}^{N} R_i u_i\right]\right|\right] \tag{2.24}$$

where R_i is a random variable representing the rate of return (per period) of the asset i. This model is not based upon any probabilistic assumption about the rates of return, and in the case where the rates are multivariate normally distributed, it is shown to be equivalent

[9] The knapsack problem is a problem in combinatorial optimization. It derives its name from the maximization problem of the best choice of essentials that can fit into one bag to be carried on a trip. Given a set of items, each with a cost and a value, determine the number of each item to include in a collection so that the total cost is less than a given limit and the total value is as large as possible.

to the Markowitz MV model.

They define $r_{i,t}$ as the realization of random variable R_i during period t ($t = 1, \ldots, T$) which they assume to be available through the historical data or from some future projection. They also assume that the expected value of the random variable can be approximated by the sample average as

$$r_i = E\left[R_i\right] \approx \frac{1}{T}\sum_{t=1}^{T} r_{i,t}, \qquad \forall\, i \in \mathbb{J}. \tag{2.25}$$

Then they approximate the MAD risk function (2.24) as follows:

$$E\left[\left|\sum_{i=1}^{N} R_i u_i - E\left[\sum_{i=1}^{N} R_i u_i\right]\right|\right] \approx \frac{1}{T}\sum_{t=1}^{T}\left|\sum_{i=1}^{N}(r_{i,t} - r_i)u_i\right|. \tag{2.26}$$

By substituting

$$a_{i,t} = r_{i,t} - r_i, \qquad \forall\, i \in \mathbb{J} \quad \text{and} \quad t = 1, \ldots, T, \tag{2.27}$$

they propose an alternative MAD risk minimization model:

$$\min \quad \frac{1}{T}\sum_{t=1}^{T}\left|\sum_{i=1}^{N} a_{i,t} u_i\right| \tag{2.28a}$$

$$\text{subject to} \quad \sum_{i=1}^{N} r_i u_i \geq \rho W \tag{2.28b}$$

$$\sum_{i=1}^{N} u_i = W \tag{2.28c}$$

$$0 \leq w_i \leq p_i, \qquad \forall\, i \in \mathbb{J} \tag{2.28d}$$

where ρ is a parameter representing the minimum rate of return required by an investor,[10] and p_i is the maximum amount of money[11] which can be invested in asset i.

[10] Thus, ρW can be equivalent to $u^*_{\mathcal{P}}$ of (2.12).

[11] They incorporate a ceiling constraint for an individual asset, while Sharpe [49, 50] and Stone [55] use the same level for the entire asset as (2.23).

Finally, the MAD risk minimization model (2.28) is equivalent to the following LP model:

$$\min \quad \frac{1}{T}\sum_{t=1}^{T} y_t \tag{2.29a}$$

$$\text{subject to} \quad y_t + \sum_{i=1}^{N} a_{i,t} u_i \geq 0 \tag{2.29b}$$

$$y_t - \sum_{i=1}^{N} a_{i,t} u_i \geq 0 \tag{2.29c}$$

$$\sum_{i=1}^{N} r_i u_i \geq \rho W \tag{2.29d}$$

$$\sum_{i=1}^{N} u_i = W \tag{2.29e}$$

$$0 \leq w_i \leq p_i, \qquad \forall\, i \in \mathbb{J} \quad \text{and} \quad t = 1, \ldots, T. \tag{2.29f}$$

With the LP model (2.29), they do not have to compute the covariance matrix to set up the model. Also, it is easy to update the model when new data are added.

However, development of the MAD model does not ask for any specific type of return distributions, facilitating its application to portfolio optimization for mortgage-backed securities (MBSs)[12] and other investment situations where the distribution of rate of return is known not to be symmetric [65]. It turns out that the third moment (skewness) plays an important role if the distribution of the rate of return of assets is asymmetric around the mean. In particular, an investor would prefer a portfolio with a larger third moment if the mean and variance are the same. Konno, Shirakawa, and Yamazaki [23] extend the MAD approach to include skewness in the objective function under possible asymmetry of returns. For quadratic risk, Konno and Suzuki [24] consider the MV model including skewness.

Simaan [51] compares the MV model with the MAD model. He shows that ignoring the covariance matrix results in greater estimation of the risk that outweighs the benefits of the MAD model proposed by Konno and Yamazaki [26]. In both models, estimation error is

[12] Securities backed by a pool of mortgage loans.

more severe in small samples and for investors with high risk tolerance. Also, he claims that if a joint normal distribution for asset returns exhibits nontrivial skewness structure, then neither the MV model nor the MAD model is consistent with expected utility maximization modeled in his paper as an objective function.

2.4 Portfolio Optimization Model with Transaction Costs

In general, transaction costs such as bid-ask spreads,[13] brokerage commissions, market impact costs, and transaction taxes, can be classified into two types: proportional and fixed costs. Proportional costs[14] depend upon the transaction volume. Investors pay costs proportional to the traded volume when they buy or sell assets. On the other hand, investors incur fixed costs[15] regardless of the transaction volume. They pay some fixed costs when they buy or sell assets. Figure 2.4 illustrates an instance of a transaction cost function consisting of proportional and fixed costs for an asset. The example shown in Figure 2.4 has different costs scheme for buying and selling the asset.

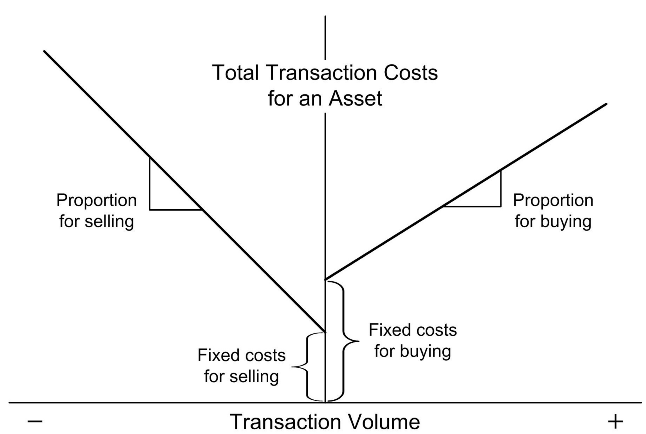


Figure 2.4: Transaction costs function for an asset: proportional and fixed costs

[13]The bid-ask spread (also known as bid-offer spread) for assets is the difference between the price available for an immediate sale (bid) and an immediate purchase (ask). The difference (spread) is kept as profit by the broker or specialist handling the transaction. Hence, in this thesis, we can consider the bid-ask spread as a type of transaction costs under the assumption of no bid-ask spreads for all assets considered.

[14]See, e.g., Pogue [42].

[15]See, e.g., Brennan [8] and Patel and Subrahmanyam [40].

Solving portfolio optimization problems incorporating fixed transaction costs is more difficult than when considering proportional transaction costs. Also, there is scarcely any literature considering proportional and fixed transaction costs simultaneously because it is hard to solve and get empirical results with existing mathematical programming techniques. In this section, we review some papers that have studied a complex structure of transaction costs in their portfolio models.

Yoshimoto [64] considers proportional transaction costs assumed to be a V-shaped function of a difference between a given existing balance w_i' and a new balance w_i of asset i.

$$\phi_i = c_i \left| w_i - w_i' \right|, \qquad \forall\, i \in \mathbb{J} \tag{2.30}$$

Accounting for the transaction costs (2.30), he constructs the following nonlinear programming model for the portfolio problem having maximization of the quadratic expected utility function:

$$\begin{aligned}
\max \quad & \sum_{i=1}^{N} \Big(w_i r_i - \phi_i \Big) - \lambda \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{i,j} && \text{(2.31a)}\\
\text{subject to} \quad & \phi_i = c_i (d_i^+ + d_i^-) && \text{(2.31b)}\\
& w_i - w_i' = d_i^+ - d_i^- && \text{(2.31c)}\\
& \sum_{i=1}^{N} w_i = 1 && \text{(2.31d)}\\
& d_i^+ \cdot d_i^- = 0 && \text{(2.31e)}\\
& d_i^+,\ d_i^- \in \mathbb{R}^+, \qquad \forall\, i \in \mathbb{J}. && \text{(2.31f)}
\end{aligned}$$

He analyzes an effect of the transaction costs on the derived portfolio problem and shows that ignoring the transaction costs results in inefficient portfolios.

Perold [41] and Mulvey [38] approximate a transaction cost function by a piecewise linear convex function. However, the results are not valid for the nonconvex shape for the actual transaction cost function since it has in most cases more than two parameters such as a cost function illustrated in Busetti's thesis [9]. Konno and Wijayanayake [25] use a branch-and-bound algorithm to solve the MAD model [26] incorporating a concave cost function approximated by linear segments. Additionally, Li, Wang, and Deng [27] introduce a linear approximation of the utility function on the return and the variance of the portfolio and are able to develop a linear programming algorithm based on the QP solution technique for this situation. They consider a V-shaped function for proportional transaction costs like (2.30) and provide numerical results with three risky assets. Their work is extended by Xia, Wang, and Deng [62] to include a risk-free asset which allows short sales.

More recently, Lynch and Balduzzi [28] examine the multi-period portfolio decisions of a long-lived investor with constant relative risk aversion (CRRA)[16] in the presence of transaction costs. They provide the impact of both proportional and fixed transaction costs on the portfolio decisions. A particular focus is how portfolio choice and rebalancing behavior when transaction costs are nonzero are affected by return predictability.[17] They use dynamic programming and consider only two assets, a risk-free and a risky asset. On the other hand, Gulpinar, Settergren, and Rustem [21] consider the multi-period extension of the MV optimization problem based on a scenario tree for a possible realization of the returns defined as non-anticipative[18] stochastic events. The set of scenarios corresponds to the set of leaves of the scenario tree and nodes of the tree at each level correspond to possible realizations of the stochastic events. They incorporate proportional transaction costs into the problem and develop a quadratic stochastic programming model.

Based on the portfolio choice in an intertemporal setting,[19] some researchers model transaction costs as proportional to the change in the holding of the risky asset, or as a fixed fraction of portfolio value. Constantinides [13] finds that proportional transaction costs affect portfolio choice since the optimal policy is a no-trade region with return to the closer boundary when rebalancing. Two risky assets with perfectly correlated rates of return and the same variance of their rate of return are considered in his study. Davis

[16] This is a description of risk aversion which holds that investors allocate the same percentage amount to risky assets as their wealth changes. Pratt [43] examines various types of risk aversions.

[17] Campbell [10] and Fama and French [17] find that the dividend yield, the term premium and the one-month T-bill rate all forecast future U.S. stock returns.

[18] A decision at a given stage does not depend on the future realization of the random events.

[19] Early works by Mossin [37], Merton [33, 34], Samuelson [44, 45], and Fama [16] were the first to address the intertemporal portfolio choice problem. After that, Merton [36] develops an intertemporal asset pricing model in which the changes in the investment opportunity set affect future asset returns, which in turn affect consumption.

and Norman [14] consider the similar problem with a bank account paying a fixed rate of interest and a stock whose price is a log-normal diffusion as investment instruments and are able to solve it exactly, without imposing restriction on the investor's consumptions at a fixed rate from the bank account.

Balduzzi and Lynch [2] extend their works based on the intertemporal setting in several ways. Their discrete-time setting solves the optimal consumption-investment problem of a finitely-lived agent facing asset-return predictability and transaction costs, who possibly ignores some aspects of the economy. The solution technique allows the simultaneous presence of both proportional and fixed transaction costs for a risky asset as follows:

$$\phi = c\left|w - w'\right| + f\mathbb{1}_{\{w-w' \neq 0\}} \tag{2.32}$$

where $(w - w')$ is a difference between asset allocations for the riskt asset.

2.5 Heuristic Optimization and Portfolio Management

Throughout the previous sections, we have reviewed the traditional MV model and its extensions. With all its attractivenesses, however, some of the main downsides of the MV model and its extended or modified models have been recognized: the computational complexity; the inability to incorporate practical considerations such as taxes and transaction costs; and the investment decision being at exactly one time point in time for a single-period horizon.

It has therefore appeared desirable to have an alternative method that can deal with highly demanding real-world portfolio problems considering more complex scenarios and settings. Heuristic optimization techniques can be the only way out, and in this thesis we apply the tabu search (TS) heuristic to solve practical portfolio problems. The heuristic methods are attractive, as they are independent of the objective function and the structure of the model and its constraints while also being a general and robust method applied to the large size of practical portfolio problems. The most attractive aspect of the heuristic optimization techniques is computational efficiency. Although for larger problem spaces the number of iterations will usually increase, the resulting increase in computational costs is usually lower than it would be for traditional optimization methods. Therefore, many researchers have considered heuristic methods to solve real-world portfolio problems, and in this section, we review some previous portfolio management problems with heuristic optimization techniques.

2.5.1 Tabu search heuristic[20]

In the last 25 years, a new kind of approximate algorithm has emerged which tries to combine basic heuristic methods in higher level frameworks aimed at efficiently and effectively exploring a search space. These methods are nowadays commonly called *metaheuristics*.[21] It is just in the last few years that some researchers in the field tried to propose a definition of the metaheuristic. In the following we quote two of them: "A metaheuristic is formally defined as an iterative generation process which guides a subordinate heuristic by combining intelligently different concepts for exploring and exploiting the search space; learning strategies are used to structure information in order to find efficiently near-optimal solutions." by Osman and Laporte [39]. "A metaheuristic is a set of concepts that can be used to define heuristic methods that can be applied to a wide set of different problems. In other words, a metaheuristic can be seen as a general algorithmic framework which can be applied to different optimization problems with relatively few modifications to make them adapted to a specific problem. Examples of metaheuristics include simulated annealing (SA), tabu search (TS), iterated local search (ILS), evolutionary algorithms (EC), and ant colony optimization (ACO)." as described on the Metaheuristics Network Website 2007 [1]. In short, metaheuristics are generally applied to problems for which there is no satisfactory problem-specific algorithm or heuristic, or when it is not practical to implement such a method. Most commonly used metaheuristics are targeted to combinatorial optimization problems, but of course they can handle any problem that can be recast in that form.

Tabu[22] *Search* (TS) which we apply to solve practical portfolio problems in this thesis is one of the metaheuristics that guide a local search procedure to explore the solu-

[20] Hertz, Taillard, and Werra [22] provide a tutorial note for the TS heuristic, and some parts of this section is written based on their tutorial note.

[21] The term *metaheuristic* was first introduced in Glover [18].

[22] The word *tabu* (or *taboo*) comes from Tongan, a language of Polynesia, where it was used by the aborigines of Tonga island to indicate things that cannot be touched because they are sacred. According to Webster's Dictionary, the word now also means "a prohibition imposed by social custom as a protective measure" or of something "banned as constituting a risk."

tion space beyond local optimality. It was first presented in its present form by Glover [18]. Additional efforts at formalization are reported in [19, 20]. Many computational experiments have shown that TS has now become an established optimization technique which can compete with almost all known techniques and which – by its flexibility – can beat many classical procedures. These successes have made TS extremely popular among those interested in finding good solutions to the large combinatorial problems such as the travelling salesman problem (TSP)[23] and the vehicle routing problem (VRP).[24]

The TS algorithm is an iterative improvement method designed to avoid terminating prematurely at a local optimum. The basic mechanism of the TS incorporates several common elements, which include the construction and selection of initial solutions, an iterative procedure through the neighborhood generation and selection, enhancement strategies such as intensification and diversification, and termination criteria of the algorithm. The general procedure of the TS can be stated as follows:

Step 1. Select an initial feasible solution and set the selected initial solution to the current solution and the overall best solution.

Step 2. Generate neighborhood solutions from the current solution, where each neighborhood solution satisfies one of the following conditions: (a) one of the tabu conditions is violated (so that the solution is not tabu); or (b) at least one of the aspiration conditions holds (so that the tabu conditions are overridden by at least one of the aspiration conditions).

Step 3. Evaluate the neighborhood solutions with respect to the performance measure and select the neighborhood-best solution out of the neighborhood solutions. Let this neighborhood-best solution be the current solution.

Step 4. Update the overall best solution by the current solution if the current solution is better than the overall best solution.

Step 5. If one of the termination criteria is satisfied, then stop. Otherwise, update the

[23]It is a problem in discrete or combinatorial optimization and asks for the shortest route to visit a collection of cities and return to the starting point. A detailed treatment and the growth of the TSP as a topic of study can be found in Alexander Schrijver's paper [48].

[24]It is a combinatorial optimization problem seeking to service a number of customers with a fleet of vehicles. Often the context is that of delivering goods located at a central depot to customers who have placed orders for such goods. Implicit is the goal of minimizing the cost of distributing the goods. Many methods have been developed for searching for good solutions to the problem since Clark and Wright [12] first solved the problem with a heuristic.

tabu list storing the attribute of move and the aspiration conditions and return to Step 2.

The TS algorithm starts with a initial solution, and the quality of the initial solution is critical since the following intermediate solutions are neighborhood solutions of the previous iteration. Therefore, a systematic way to construct the initial solution results in the higher possibility to reach the better quality of the solution with smaller iterations. Each time a neighborhood is generated and a new current solution is selected, we call the change from a current solution to a better solution a *move*. Notice that the move is defined by the mechanism that generates neighborhoods and by the rule for selecting in the neighborhood. In the TS, the custom is to select the best value of the objective function in the neighborhood; we call this the neighborhood-best solution.

When a local optimum is encountered, the TS algorithm accepts a new solution even if its solution is worse than that of the current solution. In this case, however, the procedure could cycle indefinitely. To avoid the cycling, moving to neighbor solutions that have recently been examined is forbidden. These recently visited neighbor solutions are kept in the tabu list for a certain number of iterations. The tabu list is an ordered queue containing forbidden moves: whenever the move is made, it is stored to the end of the tabu list and the first element from the list is removed. The aspiration condition is used to override the tabu status of a move if this move leads to a solution better than the best overall solution found by the search so far. Intensification strategies can be prudently applied to a search in a promising region of the solution space, so that the moves to the local optimum are intensified. On the other hand, diversification can be used to extend the search space to less explored regions by getting the moves out of the local optimum.

The TS algorithm must have at least one termination criterion. The most commonly used termination criteria in TS are:

- if every neighborhood solution belongs to the tabu list and does not meet the aspiration level;
- after a fixed number of iterations (or a fixed amount of computer time);
- after some number of iterations without an improvement in the objective function value (the criterion used in most implementations); or
- when the objective reaches a prespecified threshold value.

In applying the TS to solve a specific problem, the neighborhood structure, tabu list, and search strategy must be determined in terms of characteristics of the problem. The components of the TS for the proposed portfolio problems in this research are discussed in Chapters 3 and 4.

2.5.2 Portfolio problems with heuristic optimization

The usual (theoretical) frameworks for portfolio optimization problems have excluded realistic considerations for the way to make the problem approachable. The Markowitz MV model and its extensions in most cases cannot be applied to find an optimal or a near-optimal portfolio because of the computational complexity when we consider complex scenarios and settings such as transaction costs, short-selling assets, rebalancing the portfolio, and so forth. However, with the advent of a new type of optimization and search technique, heuristic optimization, more complex scenarios and settings can be investigated, and unrealistic assumptions are no longer necessary.

Chang, Meade, Beasley, and Sharaiha [11] present three heuristic algorithms based upon genetic algorithms (GA), tabu search (TS), and simulated annealing (SA) for finding the cardinality efficient frontier. They extend the standard MV model to include cardinality constraints as presented in (2.21). They also consider floor and ceiling constraints for N risky assets as

$$q_i z_i \leq w_i \leq p_i z_i, , \qquad \forall\ i \in \mathbb{J} \tag{2.33}$$

where

$$z_i = \begin{cases} 1 & \text{if any of asset } i \text{ is invested} \\ 0 & \text{otherwise} \end{cases}$$

and q_i and p_i are the minimum and maximum proportions, respectively that must be held of asset i if any of asset i is invested. For computational experiments, they consider five data sets, Hang Seng (Hong Kong), DAX 100 (Germany), FTSE 100 (UK), S&P 100 (USA), and Nikkei 225 (Japan), from March 1992 to September 1997 for the stocks in these indices. They compare GA, TS, and SA heuristics in terms of the median percentage error, the mean percentage error, and the total computer time. It can be seen from results of the median percentage error and the mean percentage error that over the five test data sets no one of the heuristic algorithms is uniformly dominant. For the total computer time, however, the TS heuristic dominates the GA and SA heuristics in most experimental cases.

Similarly, Schaerf [47] considers the constraints on the cardinality of the portfolio and on the quantities of individual assets as presented in (2.21) and (2.33). The portfolio problems are modeled based on the MV model and solved by three local search techniques, hill climbing (HC), SA, and TS. He experiments for his techniques on five instances taken from real stock markets.[25] He considers the percent average of the difference between benchmark instances (unconstrained problems) and each heuristic algorithm, and his computational results show that TS works much better than the others, HC and SA. Additionally, he provides experimental results that the effect of the cardinality constraint decreases the percent average quite steeply with increasing values of the cardinality variables. The other results, instead, show how the quality of the portfolio decreases while increasing the minimum quantity (variables for floor constraints). However, he does not show the results for different values of ceiling constraints because the constraints on maximum quantity are less important from a practical point of view.

Some researchers incorporate transaction costs into portfolio selection problems and solve by using heuristic techniques. First of all, Busetti [9] examines floor and ceiling constraints and cardinality constraints as well as nonlinear transaction costs including a substantial illiquidity premium for a single-period MV model. He solves the single-period portfolio problem by using two heuristic methods, GA and TS. The algorithms are applied to a large 100-stock[26] portfolio. Based on the experimental results, he claims: (i) both floor and ceiling constraints have a substantially negative impact on portfolio performance, (ii) the optimal portfolio with cardinality constraints often contains a large number of stocks with very low weightings, (iii) nonlinear transaction costs which are comparable to forecast returns in magnitude will tend to diversify portfolios materially; the effect of these costs on portfolio risk is ambiguous, depending on the degree of diversification required for cost reduction.

Additionally, Xia, Liu, Wang, and Lai [61] develop a new model for portfolio selection in which the expected returns of assets are considered as variables rather than as the arithmetic means of assets. By considering an order of the expected returns of assets

[25]Like Chang, Meade, Beasley, and Sharaiha [11], he considers five data sets from Hang Seng (Hong Kong), DAX 100 (Germany), FTSE 100 (UK), S&P 100 (USA), and Nikkei 225 (Japan). However, he does not specify the time period taken.

[26]He uses 100-stock data from JSE (South Africa).

and the change ranges of the expected returns of assets as constraints, they propose the following portfolio model incorporating proportional transaction costs:

$$\max \quad \lambda \sum_{i=1}^{N} w_i r_i - (1-\lambda) \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{i,j} - \sum_{i=1}^{N} c \left| w_i - w_i' \right| \tag{2.34a}$$

$$\text{subject to} \quad \sum_{i=1}^{N} w_i = 1 \tag{2.34b}$$

$$r_i \geq r_{i+1}, \qquad \forall\, i \in \mathbb{J} - \{N\} \tag{2.34c}$$

$$a_i \leq r_i \leq b_i, \qquad \forall\, i \in \mathbb{J} \tag{2.34d}$$

$$w_i \in \mathbb{R}^+, \qquad \forall\, i \in \mathbb{J}. \tag{2.34e}$$

where λ is a risk-aversion coefficient and (a_i, b_i) is the range in which the expected return of asset i can vary. They give the order of the expected returns of assets for (2.34c) and determine the change ranges of the expected returns of assets for (2.34d) by considering arithmetic mean, historical return tendency, and forecast of the future returns of assets. A genetic algorithm is designed to solve the proposed portfolio problem, and their numerical results show that the performance of their new model is better than that of the classical MV model when the arithmetic means are taken as the expected returns of assets in the MV model.

Based on the early study of a LP model,[27] Speranza [54] develops a single-period portfolio model incorporating fixed and proportional transaction costs as well as cardinality and floor constraints. For computational experiments, he develops a heuristic algorithm and uses stock data from the Milan stock market (Italy). First of all, to obtain the optimal solution of the model, he uses a routine for a mixed integer linear programming model, based on a branch-and-bound algorithm. However, if the number of stocks considered is > 20, the

[27] Speranza [53] shows that taking as the risk function a linear combination of the mean semi-absolute deviations, i.e., mean deviations below and above the portfolio rate of return, a model equivalent to the MAD model is obtained, whenever the sum of the coefficients of the linear combination is positive. Then in turn this model is equivalent to the Markowitz model, if the rates of return are normally distributed. Moreover, he shows that, through a suitable selection of the coefficients of the combination, 1 and 0 for the deviations below and above the average respectively, it is possible to substantially reduce the number of the constraints with respect to the MAD model and the solution time.

solution cannot be obtained within a reasonable computation time. Hence, lower than 20 stocks are considered, and he shows that the heuristic algorithm has shorter computational times, and the error of the heuristic algorithm is lower than 4.1%. Especially, the results show that the error decreases when the capital (wealth for investment) invested increases.

2.6 Summary

In this chapter, we have reviewed some topic-related portfolio optimization studies. First of all, we looked at several papers considering discrete-time portfolio optimization problems based on the MV model. Since the traditional MV model and its extensions adopt the variance as the measure of portfolio risk, the portfolio optimization problem becomes the QP problem. In order to overcome a weakness of the QP models, some researchers proposed different linearizations to measure the portfolio risk. Sharpe [49, 50] and Stone [55] propose different linearizations of the objective function. More recently, Konno and Yamazaki [26] propose a linear programming model using mean-absolute deviation (MAD) as the risk function. Their MAD model is broadly considered by some other researchers and is extended to other types of problems.

For realistic portfolio models, transaction costs and other realistic considerations have been incorporated into portfolio selection problems. Especially, transaction costs can be classified into two types, proportional and fixed costs. In general, solving portfolio optimization problems incorporating fixed transaction costs is more difficult than when considering proportional transaction costs, and thus there is scarcely any literature considering proportional and fixed transaction costs simultaneously because it is hard to solve and get empirical results with existing mathematical programming techniques. Yoshimoto [64] and some other researchers consider V-shaped proportional transaction costs and solve the portfolio selection problems by using a modified QP technique. Yoshimoto [64] shows that ignoring the transaction costs results in inefficient portfolios.

With respect to computability, the traditional MV model and its extensions have to rely on rather strict assumptions that are not always able to depict real market situations. In such situations, classical optimization methods fail to work efficiently and heuristic optimization techniques can be the only way out. They are relatively easy to implement and computationally attractive. Especially, we have reviewed a selected heuristic optimization method, TS heuristic, with its basic elements.

With heuristic algorithms, portfolio models including transaction costs and other realistic considerations can be examined. However, considering nonlinear transaction costs for a rebalancing portfolio optimization model is still hard to solve, and therefore the problem is in need of an advanced heuristic algorithm. In this thesis, we propose an adaptive, advanced TS heuristic for rebalancing portfolio optimization model incorporating nonlinear transaction costs and shorting a risk-free asset. An optimization model, the methodology of the TS algorithm for our model and empirical studies are presented in the remaining chapters.

Chapter 3

Single-Period Portfolio Optimization Model

3.1 Introduction

The MV analysis introduced in Section 2.2 has become not only the basis for theoretical models on portfolio optimization but also a major guideline for institutional portfolio management. Direct application, however, of the theoretical results to practical problems is not always possible for several reasons. First and foremost, the underlying assumptions and constraints are mostly chosen in a way to make the models solvable yet often at the cost of strong simplifications not matching real market situations. Without these simplifications and stylized facts, however, the capacities of traditional methods are quickly exceeded. For example, if we incorporate practical considerations such as short-selling of a risk-free asset and nonlinear transaction costs, we will be faced with a drawback of high computational requirements with the traditional MV approach. Therefore, we propose a new method with a TS heuristic to overcome these restrictions and make the basic formulation more accommodating to the real world's investment mechanisms.

The rest of this chapter is outlined as follows. In Section 3.2, we first propose a quadratic utility function for a multi-objective portfolio optimization model. Next, we introduce a risk-free asset and its short-selling and nonlinear transaction costs of risky assets. Then, we incorporate them into the proposed multi-objective model. An advanced version of the TS heuristic for the single-period portfolio optimization model is proposed in Section 3.3. Section 3.4 describes the numerical results for the single-period model solved by the TS algorithm and presents the results from the qualitative analysis. The last section provides a summary of this chapter.

3.2 Optimization Model

3.2.1 Multi-objective model

The Markowitz MV model is based on mean-variance portfolio selection, where the average and the variability of portfolio returns are determined in terms of the mean and the variance of the corresponding investments. In particular, two criteria or objectives have played major roles in building portfolio models these are: the maximization of the returns as reflected in the mean return and the minimization of risks as measured by the variance of the return.

Obviously, every investor would expect to maximize the portfolio return and, simultaneously, to minimize its risk. Based on (2.10a) and (2.11a), the bi-objective portfolio model expressing the idealistic expectation can be formulated as follows:

$$\max_{u_i} \quad u_{\mathcal{P}} = \sum_{i=1}^{N} u_i \big(1 + r_i\big) \tag{3.1a}$$

$$\min_{u_i} \quad \mathrm{Var}_{\mathcal{P}} = \sum_{i=1}^{N} \sum_{j=1}^{N} u_i u_j \sigma_{i,j} \tag{3.1b}$$

$$\text{subject to} \quad \sum_{i=1}^{N} u_i = W \tag{3.1c}$$

$$u_i \in \mathbb{R}^+, \qquad \forall\, i \in \mathbb{J}. \tag{3.1d}$$

However, many complex decision problems such as portfolio optimization involve multiple conflicting objectives. For portfolio optimization problems, pursuing the objectives

(3.1a) and (3.1b), simultaneously, is very possibly unacceptable in practice in terms of a trade-off between the expected return and risk. A portfolio investor would expect that more emphasis would be given to reducing risk when it is high than when it is low, and more attention would be paid to the return when it is low. This kind of rationality argues for a *state-dependent* trade-off between the mean and the variance, or, in other words, variable weights for the two criteria in the objective function.

Hence, in this framework, it is a bi-criteria (bi-objective) portfolio selection problem seeking to maximize its return and, simultaneously, to minimize its risk. For our portfolio problems we propose a quadratic utility function for the bi-criteria portfolio selection model which we can expect to yield a portfolio equilibrium by combining the two objectives, (3.1a) and (3.1b), and introducing a control parameter α for the risk level as follows:

$$\max_{u_i} \quad u_{\mathcal{P}} - \alpha \mathrm{Var}_{\mathcal{P}} \tag{3.2a}$$

$$\text{subject to} \quad u_{\mathcal{P}} = \sum_{i=1}^{N} u_i \big(1 + r_i\big) \tag{3.2b}$$

$$\mathrm{Var}_{\mathcal{P}} = \sum_{i=1}^{N} \sum_{j=1}^{N} u_i u_j \sigma_{i,j} \tag{3.2c}$$

$$\sum_{i=1}^{N} u_i = W \tag{3.2d}$$

$$u_i \in \mathbb{R}^+, \qquad \forall\, i \in \mathbb{J}. \tag{3.2e}$$

3.2.2 Constrained model

In our portfolio optimization model, we consider a risk-free asset (as considered by Tobin [59] and Davis and Norman [14]) as well as N risky assets. The risk-free asset can be considered as a bank account with the interest rate $r_0 > 0$.[1] In order to invest more money than

[1] In real market situations, it is not always true that $r_0 < r_i$. However, in this thesis, we assume that $r_0 \leq r_i$ for $i = 1, \ldots, N$.

our initial wealth, we can borrow money from a bank with the interest rate r_0. This is *short-selling*[2] of the risk-free asset. However, there is a limitation for shorting the risk-free asset. Therefore, we cannot exceed the credit balance decided by a proportional parameter l against the initial wealth W.

$$u_0 \geq -lW, \qquad u_0 \in \mathbb{R} \tag{3.3}$$

If there is short-selling of the risk-free asset for portfolio investment, the amount of short-selling and its interest clear up at the end of the period. Otherwise, it will produce risk-free gains $u_0(1+r_0)$. The expected return of the portfolio $u_{\mathcal{P}}$ in (3.2b) can be modified by incorporating the risk-free asset as

$$u_{\mathcal{P}} = \sum_{i=0}^{N} u_i \big(1 + r_i\big). \tag{3.4}$$

The main concern that leads to our study is about the transaction costs, an important factor considered by investors in financial markets. An investor must pay transaction costs when buying or selling the risky assets. We consider two types of transaction costs, proportional and fixed costs, such as bid-ask spread, brokerage commissions, market impact costs, and transaction taxes, and a transaction cost function adopted in our model follows Figure 2.4 in Section 2.4.

If an investor faces proportional costs of c_b and fixed costs of f_b, investing into asset i comes with transaction costs ϕ_i of

$$\phi_i = c_b u_i + f_b \mathbb{1}_{\{u_i \neq 0\}}. \tag{3.5}$$

The total payments associated with the purchase of N risky assets are $\phi_{\mathcal{P}\mathcal{I}}$. Most of the studies about portfolio transaction costs do not consider the costs at the end of the period. However, it is reasonable and realistic that there are transaction costs of $\phi_{\mathcal{P}\mathcal{F}}$ at the end of the period in order to realize the investor's wealth from the asset investment.

[2]Since we consider the risk-free asset as the bank account, shorting the risk-free asset represents the credit balance of the bank account.

$$
\begin{cases}
\phi_{\mathcal{PI}} = \displaystyle\sum_{i=1}^{N} \phi_i \\[2ex]
\qquad = \displaystyle\sum_{i=1}^{N} \Big[c_b u_i + f_b \mathbb{1}_{\{u_i \neq 0\}} \Big]. & \text{(3.6a)} \\[2ex]
\phi_{\mathcal{PF}} = \displaystyle\sum_{i=1}^{N} \Big[c_s u_i (1 + r_i) + f_s \mathbb{1}_{\{u_i(1+r_i) \neq 0\}} \Big]. & \text{(3.6b)}
\end{cases}
$$

We assume that initial wealth W is the only source for the portfolio investment except for short-selling of the risk-free asset. That is, there are no exogenous cash injections, and therefore the amount of risky assets bought and their transaction costs and the transaction volume of the risk-free asset have to be equal to initial wealth W. This is reflected by the following *self-financing condition*:

$$
u_0 + \sum_{i=1}^{N} \Big[u_i + c_b u_i + f_b \mathbb{1}_{\{u_i \neq 0\}} \Big] = W. \tag{3.7}
$$

Equivalently,

$$
\sum_{i=0}^{N} u_i + \phi_{\mathcal{PI}} = W. \tag{3.8}
$$

With introduction of the risk-free asset and its short-selling and incorporating proportional and fixed transaction costs for risky assets, we propose the mathematical formulation for the portfolio optimization model as follows:

$$\max_{u_i} \quad u_{\mathcal{P}} - \alpha \mathrm{Var}_{\mathcal{P}} - \phi_{\mathcal{PF}} \tag{3.9a}$$

$$\text{subject to} \quad u_{\mathcal{P}} = \sum_{i=0}^{N} u_i \big(1 + r_i\big) \tag{3.9b}$$

$$\mathrm{Var}_{\mathcal{P}} = \sum_{i=1}^{N} \sum_{j=1}^{N} u_i u_j \sigma_{i,j} \tag{3.9c}$$

$$\phi_{\mathcal{PI}} = \sum_{i=1}^{N} \Big[c_b u_i + f_b \mathbb{1}_{\{u_i \neq 0\}} \Big] \tag{3.9d}$$

$$\phi_{\mathcal{PF}} = \sum_{i=1}^{N} \Big[c_s u_i (1 + r_i) + f_s \mathbb{1}_{\{u_i(1+r_i) \neq 0\}} \Big] \tag{3.9e}$$

$$\sum_{i=0}^{N} u_i + \phi_{\mathcal{PI}} = W \tag{3.9f}$$

$$u_0 \geq -lW \tag{3.9g}$$

$$u_0 \in \mathbb{R} \tag{3.9h}$$

$$u_i \in \mathbb{R}^+, \qquad \forall\, i \in \mathbb{J}. \tag{3.9i}$$

3.3 Heuristic Optimization Approach

The portfolio optimization problem presented in Section 4.2 will be approached with a TS heuristic, a method successfully applied to portfolio optimization in Chang, Meade, Beasley, and Sharaiha [11], Schaerf [47], and Busetti [9]. Especially, for our portfolio optimization problem proposed in (4.9), we develop an adaptive version of a TS heuristic which enables the problem to be solved efficiently.

3.3.1 Solution representation

A feasible portfolio for the single-period model can be represented by means of a vector $U = (u_0, u_1, \ldots, u_N)$ in such a way that each value indicates the amount of money invested in each asset. We assume that all assets can be traded fractionally, and thus each value can be a real number. Precisely, the risk-free investment, u_0, belongs to a set of the real numbers, and the risky investments, $u_1, \ldots, u_N$, have the nonnegative real numbers as represented in (3.9h) and (3.9i).

3.3.2 Initial solution

Arguably, initial solutions of an iterative method have high impact on the final results. Yang and Nygard [63] used a heuristically selected initial population in a genetic algorithm for a time constrained travelling salesman problem. Their experiments show that this type of initialization speeds up the search process, because it saves time by finding a feasible initial solution.

For the selection of an initial solution, we randomly generate portfolios by a predetermined number. The number of the portfolios is determined by problem size (the number of assets considered).

For the purpose of diversified creation of the portfolios, we consider the following three types of the portfolios in terms of the shorting amount of the risk-free asset, and they are illustrated in Figure 3.1 reflecting the constraints from (3.9f) to (3.9i).

- $u_0 \in \mathbb{R}^+$ (no short-selling of the risk-free asset)
- $u_0 \cong -0.5lW$ (moderate short-selling of the risk-free asset)
- $u_0 \cong -lW$ (short-selling of the risk-free asset to the limit of the credit balance)

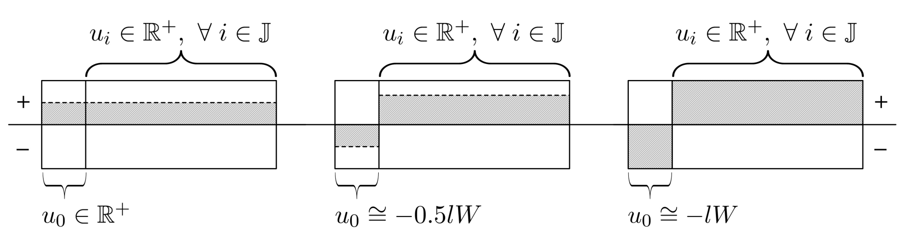


Figure 3.1: Three types of randomly generated portfolios

In addition to the randomly generated portfolios, we consider another portfolio generated by the investment of the initial wealth in the only risk-free asset. This portfolio is always feasible and the risk-free investment that surely produces $W(1+r_0)$ at the end of the period, and it can be the *lower bound* of our problem.

In TS application, the algorithm starts with the initial portfolio having the maximum value of the objective function and being valid with respect to the constraints.

3.3.3 Search space and neighborhood structure

TS is an extension of classical *Local Search* (LS) methods. In fact, basic TS can be seen simply as the combination of LS with short-term memories. It follows that the two basic elements of any TS heuristic are the definitions of its *search space* and its *neighborhood structure*.

The search space of a TS heuristic is simply the space of all possible solutions that can be considered during the search. In our portfolio optimization problem, the search space could simply be the set of feasible portfolios, where each asset invested in the search space corresponds to a set of portfolios satisfying all the specified constraints.

Closely linked to the definition of the search space is that of the neighborhood structure. At each iteration of TS, the local transformations can be applied to the current solution, denoted by S, in order to define a set of neighboring solutions in the search space, denoted by $N(S)$ (the neighborhood of S). Formally, $N(S)$ is a subset of the search space defined by:

$$N(S) = \{\text{solutions obtained by applying a single local transformation to } S\}\,. \tag{3.10}$$

In our portfolio optimization model, we consider four different types of the local transformation for the neighborhood. The neighborhood of the current portfolio $\mathcal{P}$ is generated by increasing and/or decreasing the adjacent pairwise risky assets with a variation factor $\mathit{\Delta}$: (i) increasing-increasing (APII), (ii) decreasing-decreasing (APDD), (iii) increasing-decreasing (APID), (iv) decreasing-increasing (APDI). Therefore, we redefine the neighborhood structure of (3.10) as follows:

$$N(\mathcal{P}) = \{\text{portfolios obtained by applying APII, APDD, APID, APDI to } \mathcal{P}\}\,. \tag{3.11}$$

During the neighborhood generation, the state of the risk-free asset is automatically changed by the self-financing constraint (3.9f), and thus, we only consider the adjacent pairwise risky assets. If we consider a risk-free asset and N risky assets, we produce a total of $4(N-1)$ neighborhood portfolios by applying APII, APDD, APID, and APDI. For instance, if we consider a risk-free asset and 5 risky assets ($N = 5$) with 10% variation ($\Delta = 10\%$), a total of 16 neighborhood portfolios will be produced.

However, some portfolios of $N(\mathcal{P})$ may not be satisfied with the self-financing constraint (3.9f) and/or the short-selling constraint (3.9g), and thus, the neighborhood portfolios $N(\mathcal{P})$ may include infeasible portfolio(s). Therefore, the total number of portfolios considered as the neighborhood may be less than or equal to $4(N-1)$.

During the TS iterations, the proportion of any risky asset in the portfolio may fall to a very small value. It can cause unnecessary iterations without an improvement of the objective function. Therefore, if any risky asset value in any state of the portfolio is less than the fixed transaction costs, we set the value of the risky asset to zero as follows:

$$\text{If } u_i < \max(f_b, f_s) \text{ then set } u_i = 0, \qquad \forall\, i \in \mathbb{J}. \tag{3.12}$$

On the contrary, if the state of any risky asset reaches zero, it cannot be increased again with APII, APID, or APDI. Therefore, if any risky asset value in any state of the portfolio is equal to zero and it has to be increased by APII, APID, or APDI, we set the value of the risky asset to the fixed transaction costs as follows:

$$\begin{aligned} &\text{If } u_i = 0 \text{ and asset } i \text{ has to be increased by APII, APID, or APDI} \\ &\text{then } u_i = \max(f_b, f_s), \qquad \forall\, i \in \mathbb{J}. \end{aligned} \tag{3.13}$$

3.3.4 Tabu list and termination criterion

The main feature of the TS algorithm is that it always moves to the best available neighborhood solution point even if it is worse than the current solution point. Tabus are used to prevent cycling when moving away from local optima through non-improving moves. The key realization here is that when this situation occurs, something needs to be done to prevent the search from tracking back its steps to where it came from. This is achieved by declaring tabu (disallowing) moves that reverse the effect of recent moves.

Tabus can be represented by solution points, solution attributes, or move attributes, and they are stored in a *short-term memory* of the search. This is referred to as the *tabu list* and usually only a fixed and fairly limited quantity of information is recorded. This quantity of information is the tabu size, and the choice of the tabu size is critical. If its length is too small, cycling may occur in the search process, while if too large, appealing moves may be forbidden and lead to the exploration of lower quality solutions, producing a larger number of iterations to find the desired solution. Standard tabu lists are usually implemented as circular lists of fixed length. It has been shown, however, that fixed-length tabus cannot always prevent cycling, and varying the tabu size during the search has been proposed by Glover [19, 20], Skorin-Kapov [52], and Taillard [56, 57].

In our problem, for example, we define tabus in a way where adjacent pairwise risky assets u_i and u_{i+1} are increased by applying APII, and record this in the short-term memory as the triplet $(u_i, u_{i+1}, \text{APII})$. If the tabu list is full and a new tabu is introduced, the oldest tabu will be deleted from the list.

The search algorithm could go on forever, unless the optimal value of the problem at hand is known beforehand. In practice, the search has to be stopped at some points. In our portfolio optimization problem, the TS algorithm terminates after some numbers of iterations without an improvement in the objective function value.

3.4 Computational Study

We apply the TS algorithm to scenarios involving 5, 10, 12, and 15 risky assets, respectively. That is, we consider 5, 10, 12, and 15 risky assets and a risk-free asset for numerical experiments. Assets' expected returns and covariance matrix are randomly generated, and the generation process and produced data are shown in Appendix A.

As we mentioned in Section 3.3.2, the initial solution is critical to speed up the search process. We set the size of the initial set of portfolios in terms of problem size as shown in Table 3.1. For example, in the case of 5 risky assets, we randomly generate 1,000 portfolios for the three types of the portfolios as illustrated in Figure 3.1, respectively, and consider another portfolio generated by the investment of the initial wealth in the only risk-free asset as the lower bound. Therefore, we evaluate a total of $(3 \times 1,000 + 1 = 3,001)$ portfolios to start the TS algorithm. For the neighborhood of the current portfolio $\mathcal{P}$, we set the value of the variation factor $\mathit{\Delta}$ to 10%. To use the TS algorithm, we have to decide the length of the tabu list. For the experiments, we fix the lengths in accordance with the problem size. We experiment with lengths between 1 and the maximum tabu size shown in Table 3.1. The TS algorithm terminates after 100 iterations without an improvement in the objective function value.

Since we set the credit balance to 50%, we can short the risk-free asset at most 50% of the initial wealth. That is, for instance, we can invest in risky assets and pay their transaction costs with the wealth of 15,000. The risk level α can be modified by risk preferences. If an investor tends to be risk-seeking, he or she will diminish the value of α. On the other hand, if the investor tends to be risk-avoiding, he or she will increase the value of α. We set the value to 1 for our experiments. Table 3.1 and 3.2 represent summaries for the parameters needed for the TS algorithm and for settings of the computational experiments, respectively.

Table 3.1: Parameters needed for TS algorithm

Parameter	Risky Assets			
	5	10	12	15
Initial Portfolios	3,001	6,001	15,001	30,001
Maximum Tabu Size	5	10	15	20
Variation for Neighborhood	10%	10%	10%	10%
Iterations for Termination	100	100	100	100

Table 3.2: Parameters needed for computational experiments

Parameter	Notation	Value
Initial Wealth	W	10,000.00
Credit Balance	l	50%
Proportional Costs	c_b, c_s	1.5%
Fixed Costs	f_b, f_c	10.00
Risk Level	α	1.00

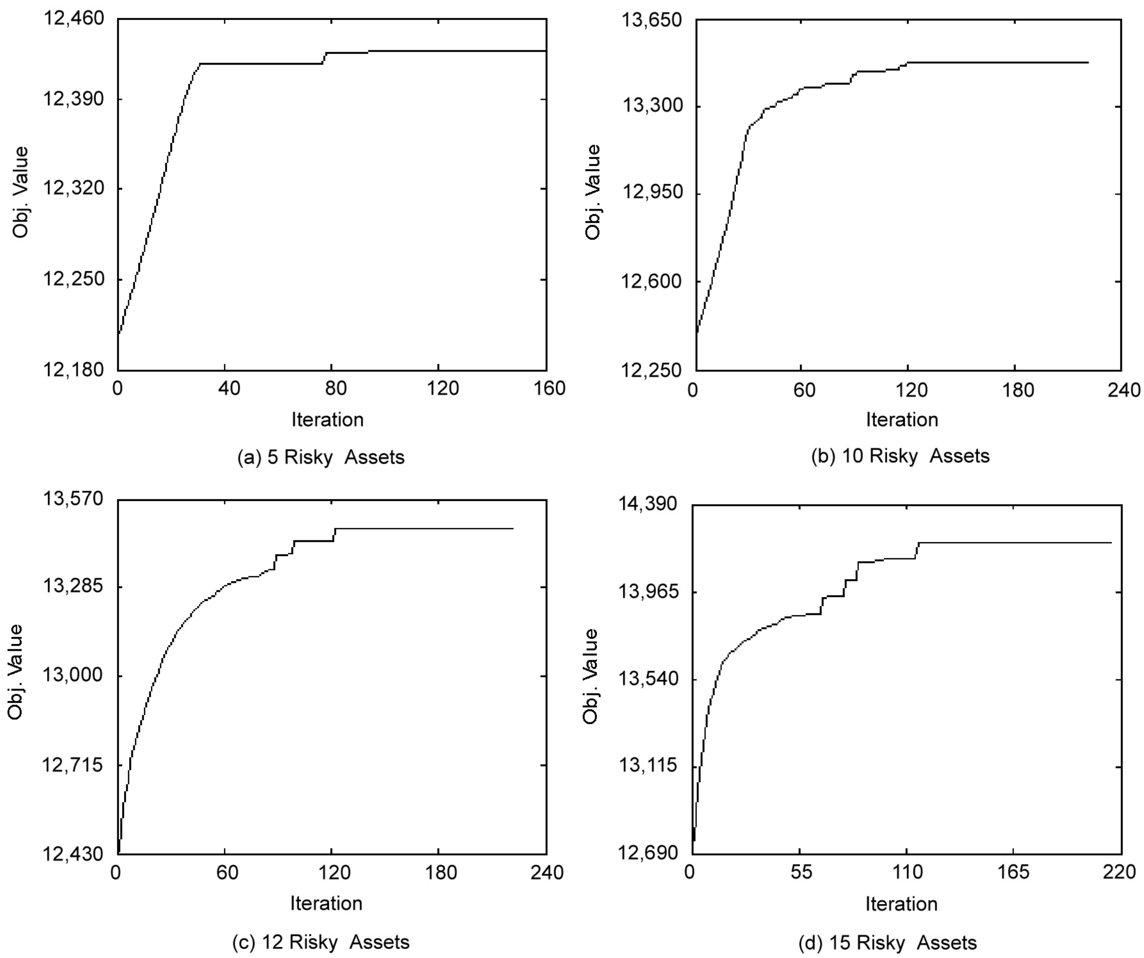


Figure 3.2: Convergence paths of the best portfolios by TS algorithm

Figure 3.2 illustrates convergence paths of the best objective values among the experimental repetitions for each problem case by TS algorithm. We can first observe that each path converges on the best objective value of each problem case. The best objective values of four problem cases are represented from Table 3.3 to 3.6. As shown in Table 3.1, we set the TS algorithm to terminate after 100 iterations without an improvement in the objective function value. We observe this in Figure 3.2.

The following four tables represent portfolio compositions for each problem case and their objective values consisting of expected return and risk. In each table, asset 0 represents the amount of money invested in the risk-free asset, and other numbers in the asset column stand for investment in risky assets.

In each table, the second column shows the lower bound for each problem case. That is, we invest our wealth in the only risk-free asset, and obviously we make the riskless return. A feasible initial portfolio represented in the third column is obtained by the selection method depicted in Section 3.3.2. If the portfolio achieved as lower bound dominates all other randomly generated portfolios for the purpose of the initial solution, it would be the lower bound and initial portfolio simultaneously. The TS iterations start with this initial portfolio, and finally converge to the best portfolio represented in the fourth column of each table.

In problem cases of 10, 12, and 15 risky assets, we short risk-free assets for the best portfolio as shown in the second row of Tables 3.4, 3.5, and 3.6. In particular, the risk-free asset is shorted up to near the credit limitation in those cases. Since the credit limitation for our investment is 50% of the initial wealth, the risk-free asset can be shorted up to 5,000, and in the cases of 10, 12, and 15 risky assets they are shorted by 4,997.49, 4,994.93, and 4,999.68, respectively.

Table 3.3: Portfolio returns and risks by the composition of the case with 5 risky assets

Asset	Lower Bound	Initial Portfolio	Best Portfolio
0	10,000.00	4,894.73	1,469.23
1	0.00	1,372.46	1,872.91
2	0.00	1,324.91	2,242.98
3	0.00	1,229.72	1,995.78
4	0.00	82.71	0.00
5	0.00	970.76	2,253.61
Expected Return	11,449.00	12,588.26	13,525.34
Expected Risk	0.00	372.77	1,085.50
Obj. Value	11,449.00	12,215.49	12,439.84

For all the problem cases, the numerical results show that we do not need to consider all risky assets as investment targets. We invest 4, 8, 8, and 14 risky assets for the problem cases of 5, 10, 12, and 15, respectively. Based on these numerical results, we conclude that our selection algorithm can exclude unnecessary assets although there is not a cardinality constraint for the purpose of portfolio diversification.

Table 3.4: Portfolio returns and risks by the composition of the case with 10 risky assets

Asset	Lower Bound	Initial Portfolio	Best Portfolio
0	$10,000.00$	$2,723.93$	$-4,997.49$
1	0.00	$1,460.65$	$2,788.82$
2	0.00	$1,325.32$	$3,368.00$
3	0.00	949.31	$1,813.12$
4	0.00	38.44	16.55
5	0.00	$1,025.24$	$1,997.89$
6	0.00	238.85	325.95
7	0.00	861.60	$3,599.13$
8	0.00	390.90	0.00
9	0.00	140.66	787.57
10	0.00	635.10	0.00
Expected Return	$11,449.00$	$12,689.10$	$14,326.13$
Expected Risk	0.00	265.51	855.36
Obj. Value	$11,449.00$	$12,423.59$	$13,470.77$

Table 3.5: Portfolio returns and risks by the composition of the case with 12 risky assets

Asset	Lower Bound	Initial Portfolio	Best Portfolio
0	$10,000.00$	957.78	$-4,994.93$
1	0.00	$1,217.21$	$3,034.57$
2	0.00	$1,138.18$	$3,388.18$
3	0.00	728.05	$1,826.54$
4	0.00	0.00	48.32
5	0.00	$1,150.85$	$1,907.04$
6	0.00	789.27	222.52
7	0.00	679.72	$3,505.41$
8	0.00	882.28	0.00
9	0.00	813.20	761.94
10	0.00	191.35	0.00
11	0.00	630.92	0.00
12	0.00	561.45	0.00
Expected Return	$11,449.00$	$12,871.75$	$14,338.55$
Expected Risk	0.00	439.94	861.90
Obj. Value	$11,449.00$	$12,431.81$	$13,476.64$

Table 3.6: Portfolio returns and risks by the composition of the case with 15 risky assets

Asset	Lower Bound	Initial Portfolio	Best Portfolio
0	$10,000.00$	$1,616.03$	$-4,999.68$
1	0.00	369.07	369.07
2	0.00	$1,219.79$	$1,341.76$
3	0.00	$1,182.11$	$1,713.43$
4	0.00	792.33	$1,103.19$
5	0.00	$1,127.25$	$2,923.80$
6	0.00	295.63	113.39
7	0.00	196.03	508.45
8	0.00	75.00	14.54
9	0.00	338.08	413.69
10	0.00	154.81	327.20
11	0.00	227.50	0.00
12	0.00	589.55	$1,529.14$
13	0.00	825.46	$3,134.70$
14	0.00	587.37	$1,040.56$
15	0.00	132.30	107.16
Expected Return	$11,449.00$	$12,829.01$	$14,511.15$
Expected Risk	0.00	136.37	257.40
Obj. Value	$11,449.00$	$12,692.64$	$14,253.75$

By comparing the objective value of the best portfolio between 4 problem cases, the objective value of the 15 risky assets case dominates all the other cases. Similarly, the 12 risky assets case dominates 5 and 10 risky assets cases, and the 10 risky assets case has the higher best objective value than that of the 5 risky assets case. These results are caused by the structure of the randomly generated expected returns and covariances shown in Appendix A. That is, for example, assets considered in the 15 risky assets case are constructed by combining 13 assets' data of the 12 risky assets case indexed by 0 to 12 in Tables A.1 and A.2 and 3 more assets' data indexed by 13 to 15 in the tables. Therefore, the best objective value of one problem case must be greater than or equal to those of the other problem cases having a smaller number of considered assets in the portfolio, and our numerical results are consistent with this conjecture.

Each experiment is repeated a total of 2 times for each tabu size shown in the fourth row of Table 3.1. That is, for the case of 5 risky assets, the experiment is repeated $(2 \times 5 = 10)$ times since the maximum tabu size is 5. Hence, for the cases of 10, 12, and 15 risky assets, the experiments are repeated 20, 30, and 40 times, respectively. It takes 0.5 to 2.5 seconds for the computation time of each experiment.[3] Every solution for the 5 risky assets case is obtained within 0.5 seconds, and at most 2.5 seconds are taken to obtain a solution of the 15 risky assets case.

With the best solution among the repetitions, the average error of the tabu search algorithm is evaluated as

$$\frac{Z^* - Z^A}{Z^*}, \tag{3.14}$$

where Z^* is the best value of the objective function, and Z^A is the average objective value obtained by the other repetitions. As shown in the second column of Table 3.7, the average error for every problem case is less than 0.3%, and the error tends to increase proportional to the number of assets in the portfolio.

Additionally, the absolute difference between the average objective value and the objective value obtained by each repetition is evaluated as

[3] For the experiment throughout this thesis, we use a personal computer on a single 1.40GHz processor and 512MB of RAM under Windows XP, and the proposed algorithm is coded in Matlab.

$$\frac{1}{J}\sum_{j=1}^{J}\left|\frac{Z_j - Z^A}{Z^A}\right|, \tag{3.15}$$

where Z_j is the objective value obtained by each repetition, and J is the total number of repetitions. The third column in Table 3.7 shows the absolute difference values, and for every problem case the value is less than 0.1%. Table 3.8 shows the percent improvement of the best objective value against the lower bound of the risk-free investment and the initial solution.

Table 3.7: Average percent errors and differences by the TS heuristic

Risky Assets	Error (%)	Difference (%)	Best Obj. Value	Avg. Obj. Value
5	0.01705	0.00440	12,439.84	12,437.72
10	0.15453	0.09570	13,470.77	13,449.95
12	0.18182	0.09531	13,476.64	13,452.14
15	0.21623	0.09736	14,253.75	14,222.93

Table 3.8: Percent improvements of the best portfolio in objective function

Risky Assets	vs. Lower Bound (%)	vs. Initial Portfolio (%)
5	8.65	1.84
10	17.66	8.43
12	17.71	8.40
15	24.50	12.30

3.5 Summary

In this chapter, first of all, we have proposed the multi-objective portfolio optimization model incorporating the risk-free asset and its short-selling and nonlinear transaction costs based on the single-period MV model. The risk-free asset incorporated into the bi-criteria portfolio selection model has a return that is certain. That is, its return has zero volatility. The covariance of the risk-free asset's return with any risky asset's return is thus zero. The presence of the risk-free asset in a portfolio implies lending or borrowing money at the risk-free rate: lending means the investment in the portfolio, whereas borrowing means a short-selling. In this study we allow the short-selling of the risk-free asset within the credit balance. For the realistic purpose we have considered transaction costs when buying or selling the risky asset. Two types of transaction costs are considered: proportional and fixed costs, and thus they are formulated with a nonlinear model.

Our multi-objective portfolio optimization problem based on the single-period model can be approached with the adaptive version of the TS heuristic. We have defined the feasible portfolio as the solution representation by means of a vector indicating the amount of money invested in each asset. For the initial solution, we randomly generate portfolios by considering the problem size and the purpose of diversified creation. From the initial solution, we obtain the final solution by iteratively searching with the neighborhood and tabu structure. The neighborhood of the current portfolio is generated by increasing and/or decreasing the adjacent pairwise risky assets with a variation factor. The tabu size is determined by the problem size, and the TS algorithm terminates after some numbers of iterations without an improvement in the objective function value.

For computational studies, we have applied the TS algorithm to multiple risky assets of 5, 10, 12, and 15, respectively. That is, we have considered 5, 10, 12, and 15 risky assets and a risk-free asset for numerical experiments. We have obtained the best portfolio compositions and their objective values within reasonable times. Each experiment is repeated a total of 2 times for each tabu size, and based on the experimental repetitions of the four problem cases, we have observed the average errors and the absolute differences by the TS heuristic reported in the single-period model. The average error for every problem case is less than 0.3%, and the results show that the absolute difference is less than 0.1%.

Chapter 4

Rebalancing Portfolio Optimization Model

4.1 Introduction

In Chapter 3, we propose the multi-objective portfolio optimization model incorporating the risk-free asset and its short-selling and nonlinear transaction costs based on the single-period MV model. The single-period model is approached with the adaptive version of a TS heuristic, and computational studies are performed with consideration of multiple risky assets and a risk-free asset. We have obtained the best portfolio compositions and their objective values within reasonable times for that model.

However, a portfolio may need to be rebalanced periodically simply as updated risk and return information is generated with the passage of time. Furthermore, any alteration to the set of investment decisions would necessitate this type of a rebalancing decision. Applying heuristic optimization techniques, some researchers have solved portfolio problems incorporating realistic situations based on the single-period model as reviewed in Section 2.5.2. However, considering real-world situations such as transaction costs and short-selling of the risk-free asset for a rebalancing portfolio optimization model is still hard to solve, and therefore the problem is in need of an advanced heuristic algorithm. In this chapter, we consider an extension of the single-period portfolio optimization problem in which nonlinear transaction costs and short-selling of the risk-free asset are allowed to rebalance an investment portfolio. To solve the considered rebalancing portfolio problem, we propose an advanced, adaptive TS algorithm, and then we perform computational studies.

We assume that the time point for rebalancing the portfolio is exactly at the midpoint of the entire time horizon of the single-period model. Therefore, we can simply say that the rebalancing model allows investors to have one more transaction chance. In terms of asset returns and covariances between risky assets, we prescribe the relationship between the single-period model and the rebalancing model as represented in Appendix A.

In the rebalancing portfolio optimization model, if we do not consider transaction costs for risky assets' transactions, then the myopic policy which tries to optimize each time period independently is always optimal for the problem because portfolio decisions do not affect decisions in each other time period. However, in our rebalancing model incorporating nonlinear transaction costs, the final objective value at the end of the time period is affected by the portfolio decision at the beginning of the planning horizon because the final result comes from the portfolio decision at the time point of rebalancing the portfolio, which is affected by the portfolio decision at the beginning of the time period. Therefore, we propose an advanced, adaptive TS algorithm having an evolutionary neighborhood structure, and solve the rebalancing portfolio problem with an iterative folding back procedure in decision tree structure.

The remainder of this chapter is organized as follows. In Section 4.2, we propose the mathematical formulation for the rebalancing portfolio optimization model. The formulation incorporates nonlinear transaction costs and a risk-free asset based on the multi-objective model proposed in the single-period model of Chapter 3. An advanced, adaptive TS algorithm and the iterative folding back procedure in decision tree structure for the rebalancing portfolio problem are proposed in Section 4.3. In Section 4.4, we describe the numerical results for the rebalancing model solved by a TS algorithm and present the results from the quantitative analysis. The last section provides a summary of this chapter.

4.2 Optimization Model

For the rebalancing portfolio model, investors can consider two types of rebalancing portfolio schemes as shown in Figure 4.1. The first case is to construct the initial portfolio with an initial wealth as depicted in Figure 4.1(a), and the second case is to construct the initial portfolio with an existing portfolio as depicted in Figure 4.1(b). However, in general, both cases are identical since investors can consider a portfolio as the existing portfolio which has no investment in risky assets and the risky asset balance with the initial wealth, which is a risk-free portfolio. In the mathematical formulation for our rebalancing portfolio optimization model, we use the general time period, for example, $[t-1,\ t]$, and the time T stands for the finalization time point depicted by t_2 and t_{k+1} of Figure 4.1. In this thesis, for numerical experiments and the mathematical optimization model, we apply the first rebalancing portfolio scheme whose initial portfolio is constructed with the given initial wealth W_{t_0}.

In our rebalancing portfolio optimization model, we consider a risk-free asset with the interest rate $r_0^t > 0$ and N risky assets with expected returns of r_i^t during time period $[t-1,\ t]$. For the rebalancing portfolio model, portfolio return during time period $[t-1,\ t]$ can be formulated as follows:

$$u_{\mathcal{P}}^t = \sum_{i=0}^{N} \left(1 + r_i^t\right)\left(x_i^{t-1} + b_i^{t-1} - s_i^{t-1}\right)^+ + \left(\mathbb{1}_{\{t=T\}} + r_0^t\right)\left(x_0^{t-1} + b_0^{t-1} - s_0^{t-1}\right)^- \quad (4.1a)$$

$$= \sum_{i=0}^{N} \left(1 + r_i^t\right)\left(u_i^{t-1}\right)^+ + \left(\mathbb{1}_{\{t=T\}} + r_0^t\right)\left(u_0^{t-1}\right)^-. \quad (4.1b)$$

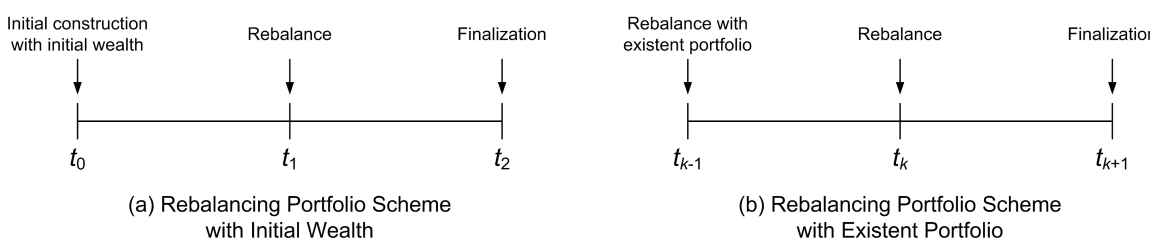


Figure 4.1: Rebalancing portfolio schemes

Obviously, b_i^{t-1} and s_i^{t-1} cannot have some values simultaneously since no investors will buy and sell the same asset at the same time period, and s_i^{t-1} cannot be positive if x_i^{t-1} is zero since short-selling is allowed only for the risk-free asset in our problem. If the risk-free asset is shorted during the planning horizon, only interest for shorting the risk-free asset is paid at the intermediate time period, and the amount of short-selling of the risk-free asset and its interest clear up at the planning horizon as formulated in each second term of the right-hand-side of the formulation (5.1) with the indicator functions. Since one of our objectives in the rebalancing portfolio model is to maximize the final wealth at the end of the planning horizon, we can set T to t_2 for mathematical formulations; hence we maximize $u_{\mathcal{P}}^{t_2}$ of the objective function.

Similarly, with formulation (2.7b) in the single-period portfolio model, portfolio risk which is measured by asset weights and covariances between assets during time period $[t-1,\ t]$ can be expressed as

$$\mathrm{Var}_{\mathcal{P}}^{t} = \sum_{i=1}^{N}\sum_{j=1}^{N} u_i^{t-1} u_j^{t-1} \sigma_{i,j}^{t}. \tag{4.2}$$

Our objective for the rebalancing portfolio model is to minimize portfolio risk as well as to maximize portfolio return as in the bi-objective model stated in Section 3.2.1. For portfolio risk in the objective function, precisely, we minimize the total portfolio risk incurred throughout the entire time periods, and hence the total portfolio risk is computed as the sum of each portfolio risk at each time period. That is, portfolio risk of our rebalancing problem $\mathrm{Var}_{\mathcal{RP}}$ based on Figure 4.1(a) can be represented as

$$\mathrm{Var}_{\mathcal{RP}} = \alpha_{t_1} \mathrm{Var}_{\mathcal{P}}^{t_1} + \alpha_{t_2} \mathrm{Var}_{\mathcal{P}}^{t_2}, \tag{4.3}$$

and in general it can be rewritten as follows:

$$\mathrm{Var}_{\mathcal{RP}} = \sum_{t=1}^{T} \alpha_t \mathrm{Var}_{\mathcal{P}}^{t}. \tag{4.4}$$

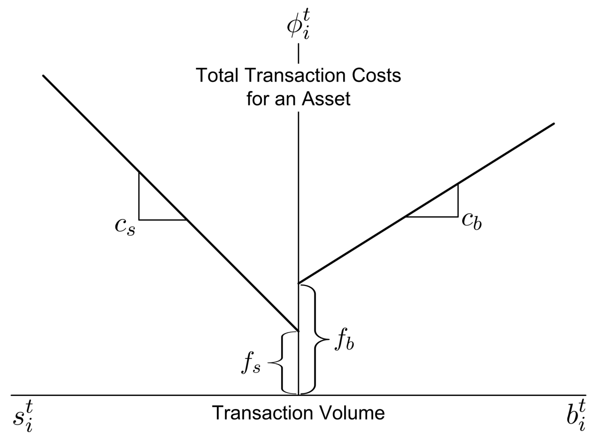


Figure 4.2: Fixed and proportional transaction costs function at time t

We can freely short-sell the risk-free asset during the portfolio planning horizon within a credit limitation decided by the proportional parameter l against the portfolio wealth balance W_t at each time period. For instance, with our notation, W_{t_0} and W_{t_2} are the initial and terminal wealth of the portfolio, respectively.

$$x_0^t + b_0^t - s_0^t = u_0^t \geq -lW_t, \qquad t = t_0,\ t_1. \tag{4.5}$$

Whenever the investor buys or sells risky assets, he or she pays proportional transaction costs in terms of transaction volume and fixed transaction costs regardless of transaction volume. Buying or selling of risky asset i at time period t comes with transaction costs ϕ_i^t of

$$\phi_i^t = c_b b_i^t + c_s s_i^t + f_b \mathbb{1}_{\{b_i^t \neq 0\}} + f_s \mathbb{1}_{\{s_i^t \neq 0\}}, \tag{4.6}$$

which is illustrated in Figure 4.2. The total payments for transaction costs at time period t associated with trading of N risky assets are denoted as $\phi_{\mathcal{P}}^t$. As we considered the realization of the wealth in (3.6b) of the single-period problem, it is also reasonable and realistic that there are transaction costs of $\phi_{\mathcal{P}}^T$ at the end of the planning horizon in order to realize the investor's wealth from the asset investment, and hence transaction costs functions, $\phi_{\mathcal{P}}^t$ and $\phi_{\mathcal{P}}^T$, are formulated as follows:

$$\begin{cases} \phi_{\mathcal{P}}^t = \sum_{i=1}^{N} \phi_i^t \\ \quad = \sum_{i=1}^{N} \left[c_b b_i^t + c_s s_i^t + f_b \mathbb{1}_{\{b_i^t \neq 0\}} + f_s \mathbb{1}_{\{s_i^t \neq 0\}} \right], & (4.7a) \\ \phi_{\mathcal{P}}^T = \sum_{i=1}^{N} \left[c_s u_i^{T-1} (1 + r_i^{T-1}) + f_s \mathbb{1}_{\{u_i^{T-1}(1+r_i^{T-1}) \neq 0\}} \right] \\ \quad = \sum_{i=1}^{N} \left[c_s x_i^T + f_s \mathbb{1}_{\{x_i^T \neq 0\}} \right]. & (4.7b) \end{cases}$$

It is assumed that our model allows money to be added to the portfolio only in the initial time period with W_{t_0}. In other words, the initial wealth W_{t_0} is the only source for the portfolio investment except for short-selling of the risk-free asset. With the portfolio scheme in Figure 4.1(a), the self-financing condition for initial construction can be represented as follows:

$$\sum_{i=1}^{N} \left[b_i^{t_0} + c_b b_i^{t_0} + f_b \mathbb{1}_{\{b_i^{t_0} \neq 0\}} \right] = W_{t_0} + s_0^{t_0} - b_0^{t_0}. \tag{4.8}$$

Equivalently,

$$\sum_{i=0}^{N} u_i^{t_0} + \phi_{\mathcal{P}}^{t_0} = W_{t_0}. \tag{4.9}$$

For the intermediate time period, there is no exogenous cash injections, and therefore the amount of assets bought and transaction costs of risky assets and the amount of assets sold and transaction costs of risky assets have to be equal. This is reflected by the following self-financing condition at the intermediate time point t.

$$b_0^t + \sum_{i=1}^{N} \left[b_i^t + c_b b_i^t + f_b \mathbb{1}_{\{b_i^t \neq 0\}} \right] = s_0^t + \sum_{i=1}^{N} \left[s_i^t - c_s s_i^t - f_s \mathbb{1}_{\{s_i^t \neq 0\}} \right]. \tag{4.10}$$

Equivalently,

$$b_0^t - s_0^t + \sum_{i=1}^{N} \left(b_i^t - s_i^t + \phi_{\mathcal{P}}^t \right) = 0. \tag{4.11}$$

No investors will buy and sell the same asset at the same time period. Therefore, the following conditions at time period t have to be satisfied by tacit understanding for self-financing conditions.

$$b_0^t \cdot s_0^t = 0 \quad \text{and} \quad b_i^t \cdot s_i^t = 0. \tag{4.12}$$

The index representing the intermediate time period t in constraints (4.10) to (4.12) can be set to t_1, t_{k-1}, or t_k in accordance with our rebalancing portfolio scheme illustrated in Figure 4.1.

Based on the rebalancing portfolio scheme shown in Figure 4.1(a), we propose the mathematical formulation for the rebalancing portfolio optimization model as follows:

$$\max_{\mathbf{b},\mathbf{s}} \quad u_{\mathcal{P}}^{t_2} - \mathrm{Var}_{\mathcal{RP}} - \phi_{\mathcal{P}}^{t_2} \tag{4.13a}$$

$$\text{subject to} \quad u_{\mathcal{P}}^{t_2} = \sum_{i=0}^{N} \left(1 + r_i^{t_2}\right)\left(u_i^{t_1}\right)^{+} + \left(1 + r_0^{t_2}\right)\left(u_0^{t_1}\right)^{-} \tag{4.13b}$$

$$\mathrm{Var}_{\mathcal{RP}} = \alpha_{t_1} \sum_{i=1}^{N}\sum_{j=1}^{N} u_i^{t_0} u_j^{t_0} \sigma_{i,j}^{t_1} + \alpha_{t_2} \sum_{i=1}^{N}\sum_{j=1}^{N} u_i^{t_1} u_j^{t_1} \sigma_{i,j}^{t_2} \tag{4.13c}$$

$$\phi_{\mathcal{P}}^{t} = \sum_{i=1}^{N} \left[c_b b_i^t + c_s s_i^t + f_b \mathbb{1}_{\{b_i^t \neq 0\}} + f_s \mathbb{1}_{\{s_i^t \neq 0\}}\right] \tag{4.13d}$$

$$\phi_{\mathcal{P}}^{t_2} = \sum_{i=1}^{N} \left[c_s x_i^{t_2} + f_s \mathbb{1}_{\{x_i^{t_2} \neq 0\}}\right] \tag{4.13e}$$

$$\sum_{i=0}^{N} u_i^{t_0} + \phi_{\mathcal{P}}^{t_0} = W_{t_0} \tag{4.13f}$$

$$b_0^{t_1} - s_0^{t_1} + \sum_{i=1}^{N} \left(b_i^{t_1} - s_i^{t_1} + \phi_{\mathcal{P}}^{t_1}\right) = 0 \tag{4.13g}$$

$$u_0^t \geq -lW_t \tag{4.13h}$$

$$u_i^t = x_i^t + b_i^t - s_i^t \tag{4.13i}$$

$$b_0^t \cdot s_0^t = 0 \tag{4.13j}$$

$$b_i^t \cdot s_i^t = 0 \tag{4.13k}$$

$$u_0^t \in \mathbb{R} \tag{4.13l}$$

$$b_0^t,\ s_0^t,\ b_i^t,\ s_i^t,\ u_i^t \in \mathbb{R}^{+}, \qquad t = t_0,\ t_1 \quad \text{and} \quad \forall\, i \in \mathbb{J}. \tag{4.13m}$$

4.3 Adaptive Heuristic Optimization Approach

The rebalancing portfolio optimization model proposed in Section 4.2 will be approached with an adaptive, advanced version of the TS heuristic. Since we solve the rebalancing portfolio problem based on the time-period scheme shown in Figure 4.1(a), the problem can be considered as a two-stage decision problem, and thus it is solved with a *folding back* procedure in a decision tree. That is, we operate two TS algorithms for time periods, $[t_0,\ t_1]$ and $[t_1,\ t_2]$, and evaluate portfolio selections via a decision tree by means of the folding back technique illustrated in Figure 4.3. Sarin and Wakker [46] demonstrate two minimal requirements of decision tree analysis, the folding back procedure and the interchangeability of consecutive event nodes. They illustrate two minimal elements of decision tree analysis by using a simple example of an ice cream vendor problem.

In this section, we present TS components for the rebalancing portfolio problem. In particular, the iterative folding back procedure combined with the TS algorithm is represented in Section 4.3.4.

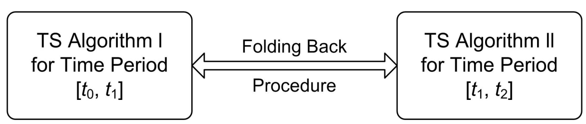


Figure 4.3: Two-stage portfolio decision by two TS algorithms and folding back procedure

4.3.1 Solution representation

The solution for the rebalancing portfolio model can be represented by two feasible portfolios for time periods, t_0 and t_1. The solution is expressed by means of a vector U consisting of two subsets in such a way that each subset represents a feasible portfolio at each time period, and each value of the subsets indicates the amount of money invested in each asset at each time period.

$$U = \left\{ \left(u_0^{t_0}, u_1^{t_0}, \ldots, u_N^{t_0} \right), \left(u_0^{t_1}, u_1^{t_1}, \ldots, u_N^{t_1} \right) \right\} \tag{4.14}$$

As in solutions for the single-period model, we assume that all assets can be traded fractionally, and thus each value can be a real number. Precisely, the risk-free investments, $u_0^{t_0}$ and $u_0^{t_1}$, belong to a set of the real numbers, and the risky investments, $u_1^{t_0}, \ldots, u_N^{t_0}$ and $u_1^{t_1}, \ldots, u_N^{t_1}$, have the nonnegative real numbers as represented in (4.13l) and (4.13m).

4.3.2 Initial solution

For the selection of an initial solution (portfolio), we randomly generate portfolios with a predetermined number. The number of the portfolios is determined by problem size (the number of assets considered).

For the purpose of diversified creation of the portfolios, as in the single-period problem in Section 3.3.2, we consider the following three types of portfolios in terms of the shorting amount of the risk-free asset at the beginning of a time period.

- $u_0^{t_0} \in \mathbb{R}^+$ (no short-selling of the risk-free asset)
- $u_0^{t_0} \cong -0.5lW_{t_0}$ (moderate short-selling of the risk-free asset)
- $u_0^{t_0} \cong -lW_{t_0}$ (short-selling of the risk-free asset to the limit of the credit balance)

In addition to the randomly generated portfolios, we consider another portfolio generated by the investment of the initial wealth W_{t_0} in the only risk-free asset. This portfolio is always feasible and the risk-free investment surely produces $W_{t_0}(1+r_0^{t_1})$ during time period $[t_0,\ t_1]$.

Since we consider the rebalancing portfolio problem, the initial solution has to consist of two portfolios at the beginning and intermediate time periods. If we randomly generate portfolios by using the above mentioned diversified creation rules for the time period $[t_0,\ t_1]$, portfolios for time period $[t_1,\ t_2]$ can be automatically created with the assets' expected returns during time period $[t_0,\ t_1]$.

$$
\begin{aligned}
U_{initial} &= \left\{ \left(u_0^{t_0}, u_1^{t_0}, \ldots, u_N^{t_0}\right), \left(u_0^{t_1}, u_1^{t_1}, \ldots, u_N^{t_1}\right) \right\} \\
&= \left\{ \left(u_0^{t_0}, u_1^{t_0}, \ldots, u_N^{t_0}\right), \left((1+r_0^{t_1})u_0^{t_0}, (1+r_1^{t_1})u_1^{t_0}, \ldots, (1+r_N^{t_1})u_N^{t_0}\right) \right\}.
\end{aligned}
\tag{4.15}
$$

Also, the lower bound can be represented as follows:

$$
\begin{aligned}
U_{LB} &= \Big\{ \Big(u_0^{t_0}, u_1^{t_0}, \ldots, u_N^{t_0} \Big), \Big(u_0^{t_1}, u_1^{t_1}, \ldots, u_N^{t_1} \Big) \Big\} \\
&= \Big\{ \Big(W_{t_0}, 0, \ldots, 0 \Big), \Big((1 + r_0^{t_1}) W_{t_0}, 0, \ldots, 0 \Big) \Big\} .
\end{aligned} \tag{4.16}
$$

We can say that portfolios randomly generated as (4.15) are always feasible due to the following relationship with assuming $r_0^t \leq r_i^t$ in our portfolio models.

$$
\begin{aligned}
&\text{If} && U_{initial}^{t_0} = \Big(u_0^{t_0}, u_1^{t_0}, \ldots, u_N^{t_0} \Big) \text{ is feasible,} \\
&\text{then} && U_{initial}^{t_1} = \Big(u_0^{t_1}, u_1^{t_1}, \ldots, u_N^{t_1} \Big) \\
&&& \qquad = \Big\{ \Big((1 + r_0^{t_1}) u_0^{t_0}, (1 + r_1^{t_1}) u_1^{t_0}, \ldots, (1 + r_N^{t_1}) u_N^{t_0} \Big) \Big\} \\
&&& \text{is always feasible} \\
&\text{because} && r_0^{t_1} \leq r_i^{t_1}, \quad \forall\, i \in \mathbb{J}, \\
&\text{and thus} && u_0^{t_1} = \big(1 + r_0^{t_1} \big) u_0^{t_0} \geq -l W_{t_1}.
\end{aligned}
$$

In the TS application, the algorithm starts with the initial portfolio having the maximum value of the objective function (4.13a) and is valid with respect to the constraints (4.13b) – (4.13m).

4.3.3 Neighborhood structure and tabu list

In our rebalancing portfolio optimization model, we consider four different types of the neighborhood for time periods, t_0 and t_1. With the current portfolio $\mathcal{P}$, the neighborhood is generated by increasing and/or decreasing the adjacent pairwise risky assets with a variation factor $\varDelta$: (i) increasing-increasing (APII), (ii) decreasing-decreasing (APDD), (iii) increasing-decreasing (APID), (iv) decreasing-increasing (APDI). Therefore, we redefine the neighborhood for each time period as follows:

$$
\begin{aligned}
N(\mathcal{P}_t) = \{\text{portfolios obtained by applying APII, APDD, APID, APDI to } \mathcal{P}_t\}, \\
t = t_0,\ t_1.
\end{aligned} \tag{4.17}
$$

During the neighborhood generation, the state of the risk-free asset is automatically changed by the self-financing conditions, (4.13f) and (4.13g), and thus, we only consider the adjacent pairwise risky assets.

For an advanced neighborhood structure of our rebalancing portfolio optimization model, we propose a two-step neighborhood generation with multiple variation factors. In the first step, we generate the neighborhood with $\mathit{\Delta} = 20\%$ for time period t_0 and $\mathit{\Delta} = 20,\ 40,\ 60,\ 80,\ 100\%$ for time period t_1. Therefore, for the first step, the neighborhood structure (4.17) can be restated as follows:

$$\begin{aligned} N1(\mathcal{P}_k^t) = \{&\text{portfolios obtained by applying APII, APDD, APID, APDI} \\ &\text{to } \mathcal{P}_t \text{ with } \mathit{\Delta} = k\%\}, \\ &t = t_0,\ t_1 \text{ and } k = 20 \text{ for } t = t_0,\ k = 20,\ 40,\ 60,\ 80,\ 100 \text{ for } t = t_1. \end{aligned} \tag{4.18}$$

After generating the neighborhood $N1(\mathcal{P}_k^t)$ in the first step, we pick up the best feasible neighborhood, denoted by $BN(\mathcal{P}_t)$, and regenerate the neighborhood with $\mathit{\Delta} = 10\%$ for the second step as follows:

$$\begin{aligned} N2(\mathcal{P}_{10}^t) = \{&\text{portfolios obtained by applying APII, APDD, APID, APDI} \\ &\text{to } BN(\mathcal{P}_t) \text{ with } \mathit{\Delta} = 10\%\}, \qquad t = t_0,\ t_1. \end{aligned} \tag{4.19}$$

If we consider a risk-free asset and N risky assets, through the two-step neighborhood generation, we produce a total of $8(N-1)$ and $24(N-1)$ neighborhood portfolios for time periods, t_0 and t_1, respectively, as represented in Table 4.1. For example, if we consider a risk-free asset and 10 risky assets (i.e., $N = 10$), $4(N-1) \cdot 5 = 180$ and $4(N-1) = 36$ neighborhood portfolios will be generated in the first and second steps, respectively, and thus a total 216 neighborhood portfolios will be produced. In Table 4.11 of Section 4.4, we will show the efficiency of the two-step neighborhood structure by comparing with numerical results of an ordinary neighborhood structure which adopts variation factor $\mathit{\Delta} = 10\%$.

During the neighborhood generation, however, some portfolios of $N1(\mathcal{P})$ or $N2(\mathcal{P})$ may violate the constraints (4.13f), (4.13g), and/or (4.13h), and thus, the neighborhood portfolios, $N1(\mathcal{P})$ or $N2(\mathcal{P})$, may include infeasible portfolio(s). Therefore, the total number of portfolios considered as the neighborhood is less than or equal to $8(N-1)$ and $24(N-1)$ for $N1(\mathcal{P})$ and $N2(\mathcal{P})$, respectively.

Table 4.1: Number of portfolios generated by the two-step neighborhood

	Time Period	
	t_0	t_1
First Step	$4(N-1)$ by $\mathit{\Delta} = 20\%$	$4(N-1) \cdot 5$ by $\mathit{\Delta} = 20,\ 40,\ 60,\ 80,\ 100\%$
Second Step	$4(N-1)$ by $\mathit{\Delta} = 10\%$	$4(N-1)$ by $\mathit{\Delta} = 10\%$
Total	$8(N-1)$	$24(N-1)$

During the TS iterations, the state of any risky asset may fall to a very small value. That can cause unnecessary iterations without the improvement of the objective function. Therefore, if any risky asset value in any state of the portfolio is less than the fixed transaction costs, we set the value of the risky asset to zero as follows:

$$\text{If } u_i^t < \max(f_b, f_s), \text{ then set } u_i^t = 0, \qquad t = t_0,\ t_1 \quad \text{and} \quad \forall\, i \in \mathbb{J}. \tag{4.20}$$

On the contrary, if the state of any risky asset reaches zero, it cannot be increased again with APII, APID or APDI. Therefore, if any risky asset value in any state of the portfolio is equal to zero, and it has to be increased by APII, APID or APDI, we set the value of the risky asset to the fixed transaction costs as follows:

$$\begin{aligned} &\text{If } u_i^t = 0 \text{ and asset } i \text{ has to be increased by APII, APID, or APDI,} \\ &\text{then } u_i^t = \max(f_b, f_s), \qquad t = t_0,\ t_1 \quad \text{and} \quad \forall\, i \in \mathbb{J}. \end{aligned} \tag{4.21}$$

For the rebalancing portfolio optimization model and its two-step neighborhood structure, we define tabus in such a way that adjacent pairwise risky assets, u_i^t and u_{i+1}^t, are increased (decreased) and/or increased (decreased) by applying APII, APDD, APID, and APDI with the variation factor $\mathit{\Delta}$, and for each time period we maintain two tabus in terms of the two-step neighborhood structure. Each tabu is a subset of a tabu expression denoted by TB_{t_0} and TB_{t_1} for time periods, t_0 and t_1, respectively, which can be represented as follows:

$$
\begin{cases}
TB_{t_0} = \left\{(u_i, u_{i+1}, \mathrm{AP}_1^{t_0}),\ (u_j, u_{j+1}, \mathrm{AP}_2^{t_0})\right\}, \\
\qquad i = j = 1, \ldots, N-1, \\
\qquad \mathrm{AP}_1^{t_0} = \text{APII, APDD, APID, APDI with } \mathit{\Delta} = 20\% \text{ and} \\
\qquad \mathrm{AP}_2^{t_0} = \text{APII, APDD, APID, APDI with } \mathit{\Delta} = 10\%. \\
TB_{t_1} = \left\{(u_i, u_{i+1}, \mathrm{AP}_1^{t_1}),\ (u_j, u_{j+1}, \mathrm{AP}_2^{t_1})\right\}, \\
\qquad i = j = 1, \ldots, N-1, \\
\qquad \mathrm{AP}_1^{t_1} = \text{APII, APDD, APID, APDI with } \mathit{\Delta} = 20,\ 40,\ 60,\ 80,\ 100\% \text{ and} \\
\qquad \mathrm{AP}_2^{t_1} = \text{APII, APDD, APID, APDI with } \mathit{\Delta} = 10\%.
\end{cases}
\begin{matrix} (4.22\text{a}) \\ \\ \\ \\ (4.22\text{b}) \end{matrix}
$$

Therefore, the possible numbers of tabus, TB_{t_0} and TB_{t_1}, are $16(N-1)$ and $80(N-1)$, respectively, and Table 4.2 shows the total number of tabus via the two-step neighborhood structure. For example, if we consider 5 risky assets, we have 64 and 320 different possible tabus for time periods, t_0 and t_1, respectively.

We operate two types of tabu lists, TB_{t_0} and TB_{t_1}, as formulated in (4.22a) and (4.22b), and if each tabu list is completed by its size and a new tabu is introduced, the oldest tabu will be deleted from the list.

Table 4.2: Number of tabus via two-step neighborhood structure

Time Period	First Step		Second Step		Total
t_0	$4(N-1)$ by $\mathit{\Delta} = 20\%$	$\times$	$4(N-1)$ by $\mathit{\Delta} = 10\%$	$=$	$16(N-1)$
t_1	$4(N-1)\cdot 5$ by $\mathit{\Delta} = 20,\ 40,\ 60,\ 80,\ 100\%$	$\times$	$4(N-1)$ by $\mathit{\Delta} = 10\%$	$=$	$80(N-1)$

4.3.4 Iterative folding back procedure in decision tree structure

In our rebalancing portfolio optimization model, we make portfolio decisions at two time points, t_0 and t_1. $D(\mathcal{P}_{t_0})$ and $D(\mathcal{P}_{t_1})$ denote portfolio decisions at time periods, t_0 and t_1, respectively. Based on our rebalancing model, if we do not consider transaction costs for risky assets' transactions, a myopic policy is always optimal for the problem because portfolio decisions do not affect each other in different time periods. However, in our rebalancing model incorporating realistic considerations such as transaction costs, the final objective value at time t_2 is affected by the portfolio decision at the beginning of the planning horizon because the final result comes from the portfolio decision at time point t_1 which is affected by the portfolio decision at time period t_0.

With the initial portfolios $U^{t_0}_{initial}$ and $U^{t_1}_{initial}$ of Section 4.3.2, we start the iterative TS procedure based on the folding back procedure in decision tree structure. First of all, we update the portfolio decision, $D(\mathcal{P}_{t_1})$, by applying the two-step neighborhood generation of (4.18) and (4.19) to $U^{t_1}_{initial}$. This iterative procedure will be stopped after some number of iterations without an improvement in the objective function value. Next, we generate the neighborhood of the initial portfolio, $U^{t_0}_{initial}$, at the beginning of the time periods by applying the first step neighborhood generation of (4.18). A total of $4(N-1)$ neighborhood portfolios at time period t_0 can be observed at time period t_1 with expected returns for assets considered. Then, we rebalance $4(N-1)$ portfolios, and decide the best portfolio decisions so far for two time points, t_0 and t_1. With the best portfolio $BN(\mathcal{P}_{t_0})$ at t_0, we regenerate neighborhood portfolios by applying the second step neighborhood generation of (4.19), and then do the same iterative TS procedure for rebalancing the portfolio decision at time period t_1. Like termination of the TS algorithm at time period t_1, the iterative procedure at the beginning of the time periods will also be stopped after some number of iterations without an improvement in the objective function value, and this will be the termination criterion for our rebalancing portfolio model.

The iterative folding back procedure in decision tree structure is briefly outlined as follows:

Step 1. From the initial solution $U^{t_1}_{initial}$, find the best portfolio at time t_1 by a two-step neighborhood generation.

Step 2. Generate neighborhood portfolios at time t_0 by the first step neighborhood generation: $\mathcal{P}_{t_0}$ and $N1(\mathcal{P}_1^{t_0})$ to $N1(\mathcal{P}_{4(N-1)}^{t_0})$ in Figure 4.4(a).

Step 3. Compute portfolios prior to rebalancing at time t_1 with expected returns of assets during $[t_0,\ t_1]$, and generate their neighborhood portfolios by the first step neighborhood generation: $\mathcal{P}_1^{t_1}$ to $\mathcal{P}_{4(N-1)}^{t_1}$ and their neighborhood portfolios $N1(\mathcal{P}_1^{t_1})$ to $N1(\mathcal{P}_{20(N-1)}^{t_1})$ in Figure 4.4(a).

Step 4. Select each of the best portfolios among $20(N-1)$ neighborhood portfolios of the first step at time t_1, and regenerate their neighborhood portfolios by the second step neighborhood generation: $BN(\mathcal{P}_1^{t_1})$ to $BN(\mathcal{P}_{4(N-1)}^{t_1})$ and $N2(\mathcal{P}_1^{t_1})$ to $N2(\mathcal{P}_{20(N-1)}^{t_1})$ in Figure 4.4(b).

Step 5. Select the best portfolios for time period t_0 and t_1 after evaluating the second step's neighborhood portfolios at time t_1.

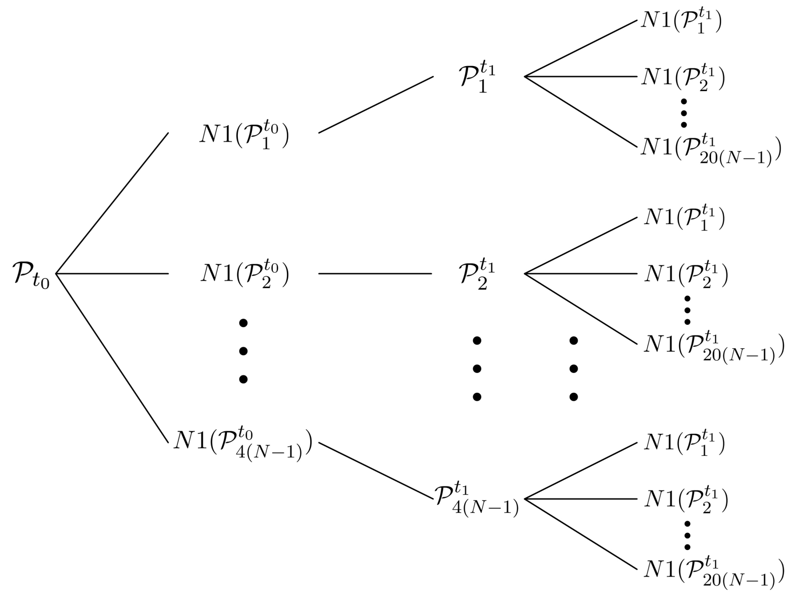
Step 6. Generate neighborhood portfolios of the current best portfolio at time t_0 by the second step neighborhood generation: $BN(\mathcal{P}_{t_0}$ and $N2(\mathcal{P}_1^{t_0})$ to $N2(\mathcal{P}_{20(N-1)}^{t_0})$ in Figure 4.4(c).

Step 7. With the neighborhood portfolios in Step 6, do Steps 3, 4 and 5: Figures 4.4(c) and (d).

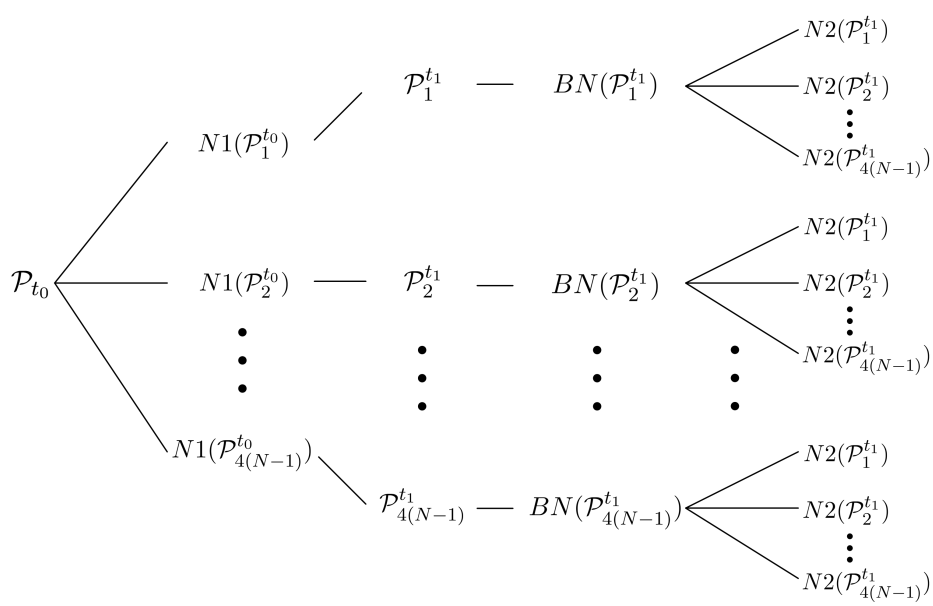
Step 8. Go to Step 2 with the current portfolio information.

The above mentioned steps for the iterative folding back procedure with the decision trees of Figure 4.4(a) to (d) illustrate one cycle of a search procedure, and the steps will be repeated by the TS iteration and termination criteria of the algorithm.

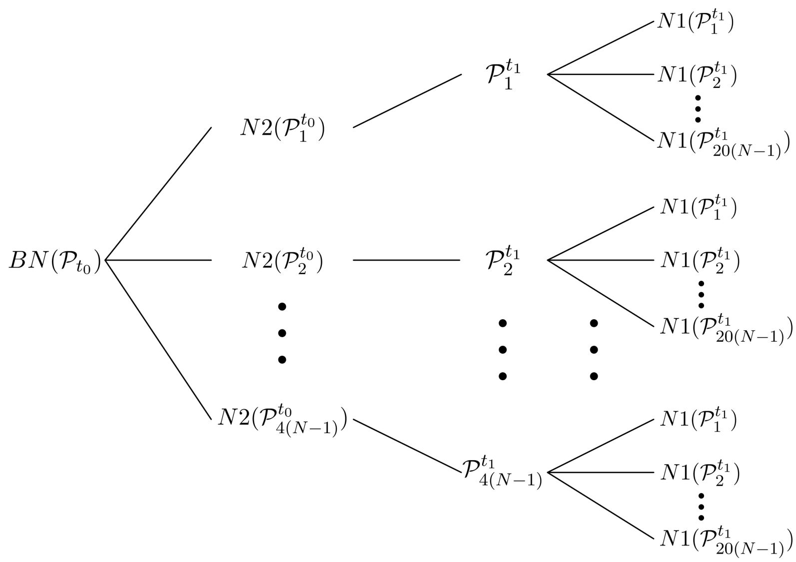
Figure 4.4: Decision trees for the iterative folding back procedure



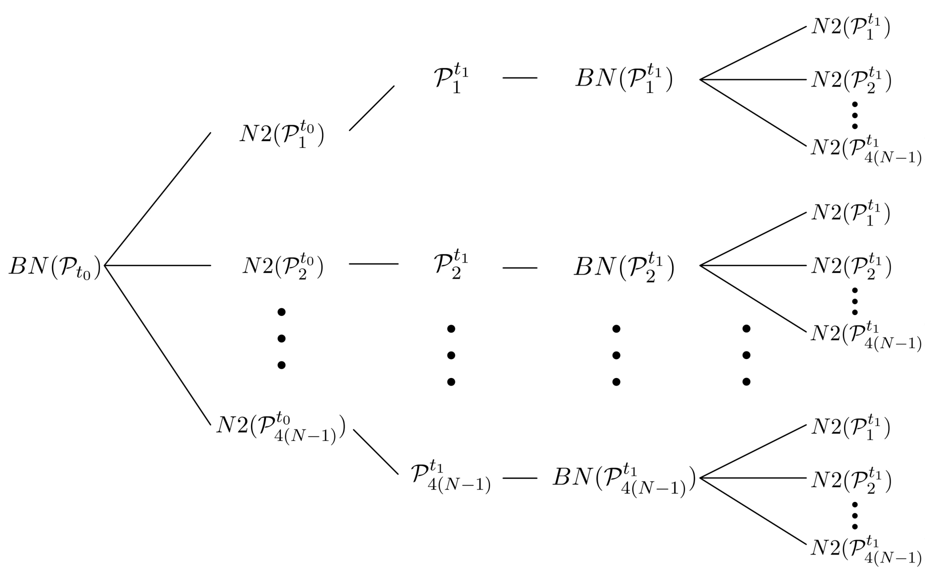
(a) Decision tree with the first steps of neighborhood generation for time periods t_0 and t_1, respectively



(b) Decision tree with the first and second steps of neighborhood generation for time periods t_0 and t_1, respectively



(c) Decision tree with the second and first steps of neighborhood generation for time periods t_0 and t_1, respectively



(d) Decision tree with the second steps of neighborhood generation for time periods t_0 and t_1, respectively

4.4 Computational Study

For numerical experiments of the rebalancing portfolio problem, we consider the number of risky assets to be 5, 10, 12, and 15 and also with a risk-free asset. Assets' expected returns and covariance matrix are randomly generated for each time period, and the generation process and produced data are shown in Appendix A.

In the rebalancing model, every solution is represented as consisting of two subsets in such a way that each subset represents a portfolio at each time period as shown by the solution representation in (4.14). The quality of the initial solution is critical to speed up the search process, and, by considering the problem size, we set the size of the initial solutions representing pairs of portfolios to 3,001, 6,001, 15,001, and 30,001 for 5, 10, 12, and 15 risky assets, respectively as shown in the third row of Table 4.3. Then, we evaluate randomly generated solutions of (4.15) and a lower bound of (4.16) in order to decide the initial best portfolio. As illustrated in the fifth row of Table 4.3, we adopt a variation factor of 10% to 100% for the two-step neighborhood generation, and we operate two TS lists for both time periods, t_0 and t_1. We decide the length of the tabu list for each time period in terms of possible numbers of tabus via the two-step neighborhood structure. We experiment with each tabu length of t_0 and t_1 between 1 and the maximum tabu size shown in the fourth row of Table 4.3. For each time period, the TS algorithms terminate after 100 iterations without an improvement in the objective function value. With the iterative folding back procedure in Section 4.3.4, the entire algorithm for rebalancing the model terminates after 100 iterations without an improvement in the objective function value at time period t_0.

Since we set the credit balance to 50%, we can short the risk-free asset at most 50% of the initial wealth W_{t_0} at the beginning of time period t_0 and of the portfolio wealth balance W_{t_1} at time period t_1. Risk levels α_{t_1} and α_{t_2} can be modified by risk preferences for each of time period and/or whole time periods. Tables 4.3 and 4.4 represent summaries for the parameters needed for the TS algorithm and for the setting of computational experiments, respectively.

Table 4.3: Parameters needed for TS algorithm

Parameter	Risky Assets			
	5	10	12	15
Initial Pair of Portfolios	3,001	6,001	15,001	30,001
Maximum Tabu Size	(10, 20)	(20, 40)	(30, 60)	(50, 100)
Variation for Neighborhood	10% – 100%	10% – 100%	10% – 100%	10% – 100%
Iterations for Termination	(100, 100)	(100, 100)	(100, 100)	(100, 100)

Table 4.4: Parameters needed for computational experiments

Parameter	Notation	Value
Initial Wealth	W_{t_0}	10,000.00
Credit Balance	l	50%
Proportional Costs	c_b, c_s	1.5%
Fixed Costs	f_b, f_c	10.00
Risk Levels	$\alpha_{t_1}, \alpha_{t_2}$	1.00

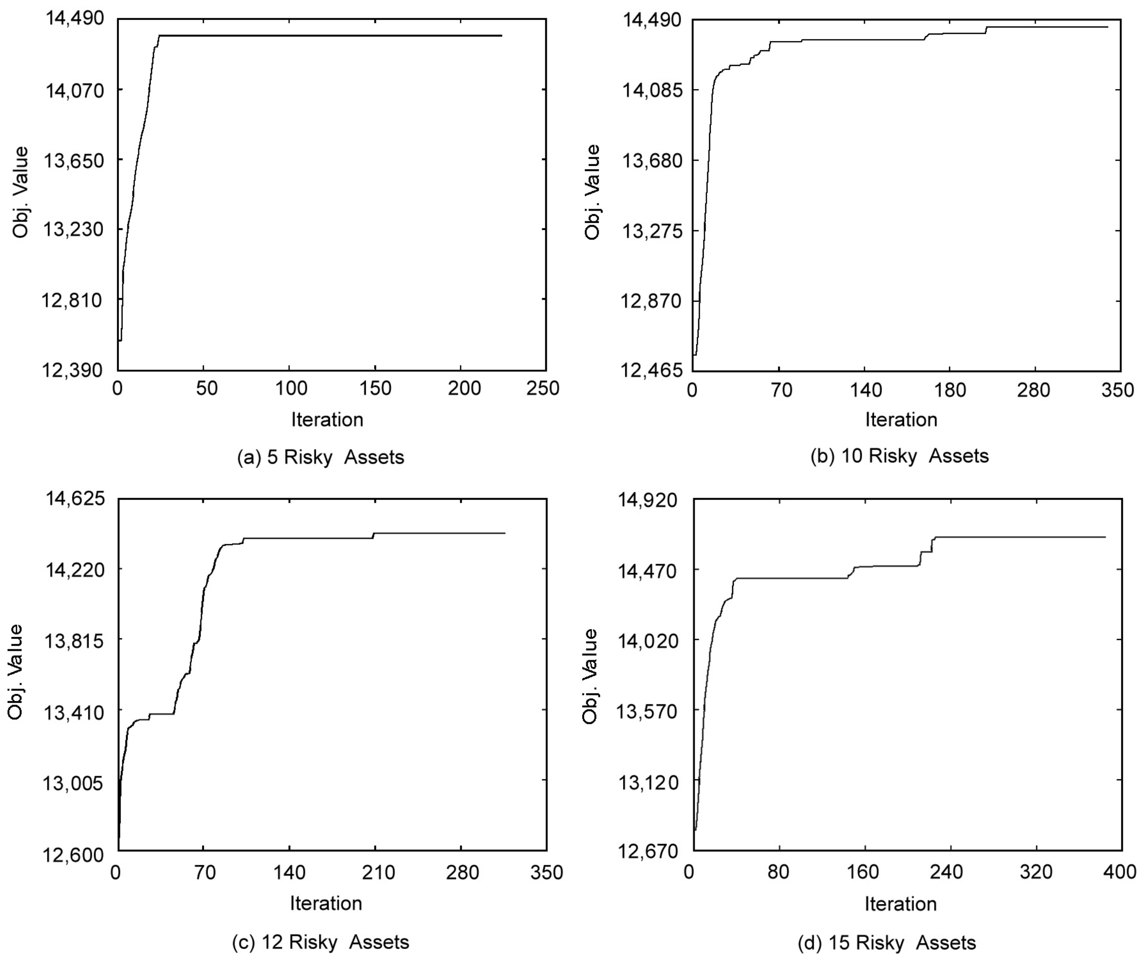


Figure 4.5: Convergence paths of the best pair of portfolios by TS algorithm

Figure 4.5 illustrates convergence paths of the best objective values for each problem case by the iterative folding back procedure with the two-stage TS algorithm. We can first observe that each path converges to the best objective value of each problem case. The best objective values of four problem cases are represented from Table 6.5 to 6.8. As shown in Table 4.3, we set that TS algorithms terminate after 100 iterations without an improvement in the objective function value, and especially, the entire algorithm for the rebalancing model terminates after 100 iterations without an improvement in the objective function value at time period t_0. We can observe this in Figure 4.5.

The following four tables represent portfolio compositions for each problem case and their objective values consisting of expected return and risk. In each table, asset 0 represents the amount of money invested in the risk-free asset, and other numbers in the asset column stand for risky assets.

In each table, we show the portfolio compositions for lower bound, initial solution and the best solution. Each of them has two columns consisting of a portfolio at time t_0 and a rebalancing portfolio at time t_1. In the tables, the second and third columns represent the lower bound of each problem case. That is, we invest our wealth in the only risk-free asset at the beginning of the time period, and maintain it until the end of the time period. Therefore, we obviously make the riskless return. A feasible initial solution represented in the forth and fifth columns is obtained by the selection method depicted in Section 4.3.2. If the solution achieved as lower bound dominates all other randomly generated solutions for the purpose of the initial solution, it would be the lower bound and initial solution simultaneously. The TS iterations start with this initial solution and finally converge to the best solution represented in the sixth and seventh columns of each table.

In all problem cases, we short risk-free assets for the best portfolio as shown in the second row of each table. In particular, the risk-free asset of every problem case is shorted

Table 4.5: Portfolio returns and risks by the composition of the case with 5 risky assets

Asset	Lower Bound		Initial Solution		Best Solution	
0	10,000.00	10,700.00	4,500.89	4,815.95	−4,993.58	−6,131.25
1	0.00	0.00	886.21	1,099.81	7,243.07	1,438.21
2	0.00	0.00	1,193.20	1,326.36	1,160.62	5,812.91
3	0.00	0.00	1,242.16	1,464.79	1,669.83	5,545.04
4	0.00	0.00	1,075.93	1,245.36	7.53	13.94
5	0.00	0.00	1,020.35	1,255.48	4,690.94	5,771.90
Expected Return	11,449.00		12,716.44		15,908.62	
Expected Risk	0.00		187.10		1,566.63	
Obj. Value	11,449.00		12,529.34		14,342.00	

up to near the credit limitation. Since the credit limitation for our investment is 50% of the wealth balance, the risk-free asset can be shorted up to 5,000 at the beginning of the time period.

For the problem cases of 10, 12, and 15 risky assets, some risky assets are invested during the time period $[t_0, t_1]$, but not invested during the time period $[t_1, t_2]$. In particular, for example, four risky assets represented by indices 1, 6, 9, and 10 are not invested during the time period $[t_1, t_2]$, and we can explain these results by the reason of a drastic decrease of the assets' return as shown in Table A.1. The results also show that the proposed advanced, adaptive TS algorithm has a consistent robustness regardless of the problem size. Based on these numerical results, we conclude that our selection algorithm excludes unnecessary assets although there is not a cardinality constraint for the purpose of portfolio diversifica-

Table 4.6: Portfolio returns and risks by the composition of the case with 10 risky assets

Asset	Lower Bound		Initial Solution		Best Solution	
0	$10,000.00$	$10,700.00$	$1,294.50$	$1,385.10$	$-4,943.00$	$-6,200.20$
1	0.00	0.00	$1,232.10$	$1,529.10$	$5,297.80$	0.00
2	0.00	0.00	$1,013.80$	$1,126.90$	495.56	$4,406.90$
3	0.00	0.00	$1,250.70$	$1,474.80$	$1,121.60$	$5,290.40$
4	0.00	0.00	948.36	$1,097.70$	30.67	35.50
5	0.00	0.00	707.30	870.29	$3,091.20$	$3,803.50$
6	0.00	0.00	972.29	$1,173.70$	352.26	0.00
7	0.00	0.00	486.59	560.63	949.89	$2,188.90$
8	0.00	0.00	764.58	820.65	287.75	$2,470.80$
9	0.00	0.00	504.81	614.79	$2,625.70$	0.00
10	0.00	0.00	597.83	687.53	371.28	0.00
Expected Return	$11,449.00$		$12,813.00$		$15,173.00$	
Expected Risk	0.00		214.58		755.96	
Obj. Value	$11,449.00$		$12,598.42$		$14,417.04$	

tion. These results are very similar to those of the single-period portfolio problem. Additionally, in the problem cases of 12 and 15 risky assets, as shown in Tables 4.4 and 4.4, not all risky assets are invested through the entire planning horizon. Two and four risky assets are excluded for the problem cases of 12 and 15 risky assets, respectively.

For the problem cases of 5, 10, 12, and 15 risky assets, the experiments are repeated 10, 20, 30, and 50 times, respectively. These numbers of repetitions are decided by the length of the tabu list for the first-time period as shown in the fourth row of Table 4.3. It takes 1.5 minutes to an hour for the computations of each experiment. Every solution for the 5 risky assets case is obtained within 1.5 minutes, and at most an hour is taken to obtain a

Table 4.7: Portfolio returns and risks by the composition of the case with 12 risky assets

Asset	Lower Bound		Initial Solution		Best Solution	
0	10,000.00	10,700.00	1,381.19	1,477.88	−4,984.18	−6,222.93
1	0.00	0.00	1,450.33	1,799.90	4,860.74	0.00
2	0.00	0.00	1,211.33	1,346.52	219.68	3,907.19
3	0.00	0.00	1,230.35	1,450.87	1,194.59	5,634.80
4	0.00	0.00	126.19	146.07	50.32	58.25
5	0.00	0.00	1,095.75	1,348.25	3,489.01	4,293.00
6	0.00	0.00	219.86	265.41	389.49	0.00
7	0.00	0.00	688.20	792.92	894.92	2,060.99
8	0.00	0.00	292.48	313.93	133.43	2,291.47
9	0.00	0.00	487.68	593.93	3,004.71	0.00
10	0.00	0.00	508.41	584.69	427.85	0.00
11	0.00	0.00	461.11	544.47	0.00	0.00
12	0.00	0.00	601.51	729.36	0.00	0.00
Expected Return	11,449.00		12,883.83		15,219.34	
Expected Risk	0.00		149.44		791.78	
Obj. Value	11,449.00		12,734.39		14,427.56	

Table 4.8: Portfolio returns and risks by the composition of the case with 15 risky assets

Asset	Lower Bound		Initial Solution		Best Solution	
0	10,000.00	10,700.00	311.64	333.46	−4,921.60	−6,141.30
1	0.00	0.00	1,431.35	1,776.34	1,224.32	0.00
2	0.00	0.00	1,049.10	1,166.19	561.55	2,496.88
3	0.00	0.00	1,131.13	1,333.87	1,954.59	2,304.92
4	0.00	0.00	583.19	675.03	447.89	518.42
5	0.00	0.00	737.25	907.14	2,219.77	5,462.57
6	0.00	0.00	777.70	938.82	1,310.97	0.00
7	0.00	0.00	613.03	706.32	610.97	1,407.87
8	0.00	0.00	468.26	502.59	326.01	699.83
9	0.00	0.00	167.49	203.97	1,577.23	0.00
10	0.00	0.00	214.83	247.06	0.00	0.00
11	0.00	0.00	530.74	626.69	0.00	0.00
12	0.00	0.00	438.49	531.69	1,508.33	1,828.92
13	0.00	0.00	663.07	819.51	2,851.09	3,523.74
14	0.00	0.00	0.00	0.00	0.00	0.00
15	0.00	0.00	601.62	662.82	0.00	0.00
Expected Return	11,449.00		12,929.53		14,898.86	
Expected Risk	0.00		107.59		211.81	
Obj. Value	11,449.00		12,821.94		14,687.05	

solution for the 15 risky assets case.

With the best solution among the repetitions, the average error of the tabu search algorithm is evaluated by using (3.14). As shown in the second column of Table 4.9, the average error for every problem case is less than 0.8%. Additionally, the absolute difference between the average objective value and the objective value obtained by each repetition is evaluated by using (3.15). The third column in Table 4.9 shows the absolute difference values, and for every problem case that value is less than 0.3%.

Table 4.10 shows the percent improvement of the best solution against the lower bound of the risk-free investment and the initial solution. For instance, in the case of the 15 risky assets, the objective value of the best solution dominates that of the lower bound and the initial solution by 28.28% and 14.55%, respectively.

Table 4.9: Average percent errors and differences by the TS heuristic

Risky Assets	Error (%)	Difference (%)	Best Obj. Value	Avg. Obj. Value
5	0.05699	0.03098	14,342.00	14,333.83
10	0.67783	0.21942	14,417.04	14,319.32
12	0.49485	0.25493	14,427.56	14,356.17
15	0.74274	0.21087	14,687.05	14,577.96

Table 4.10: Percent improvements of the best solution in objective function

Risky Assets	vs. Lower Bound (%)	vs. Initial Portfolio (%)
5	25.27	14.47
10	25.92	14.44
12	26.02	13.30
15	28.28	14.55

In Section 4.3.3, we proposed the two-step neighborhood structure for the rebalancing portfolio model. Table 4.11 shows the efficiency of the two-step neighborhood structure by comparing it with numerical results of ordinary neighborhood structure which adopts variation factor $\Delta = 10\%$ as proposed in Section 3.3.3 of the single-period model. Each value in the table represents the average objective value based on the repetitions of the experiment. We observe that results of the two-step neighborhood structure are better than those of the single-step neighborhood structure for all problem cases. Thus, we can conclude that the two-step neighborhood structure for iterative TS procedures produces better solutions in the rebalancing portfolio optimization problem.

The primary objectives of this study are to extend a single-period portfolio model to a rebalancing model and to find the effect of one more transaction of the rebalancing model. As we assumed in Section 4.1, investors have one more transaction opportunity in the rebalancing model, and the time point for rebalancing is exactly at the midpoint of the

Table 4.11: Average objective value: two-step vs. single-step neighborhood structure

Risky Assets	Two-Step	Single-Step
5	14,333.83	14,329.10
10	14,319.32	14,245.28
12	14,356.17	14,301.35
15	14,577.96	14,485.50

Table 4.12: The best objective values: single-period vs. rebalancing model

Risky Assets	Single-Period Model	Rebalancing Model	Difference (%)
5	12,439.84	14,342.00	15.29
10	13,470.77	14,417.04	7.02
12	13,476.64	14,427.56	7.06
15	14,253.75	14,687.05	3.04

entire planning horizon of the single-period model. In order to compare results between the single-period and the rebalancing models, we use the same data structure for both models and adjust randomly generated assets' return and covariance matrix considering one more transaction chance of the rebalancing model, and Appendix A shows them in detail. Table 4.12 shows the best objective values of the single-period and the rebalancing model. For all problem cases, the rebalancing model dominates the single-period model, and especially the rebalancing model with 5 risky assets improves the objective values by 12.29% in beating the transaction costs.

4.5 Summary

In this chapter, we have considered an extension of the single-period portfolio optimization problem in which nonlinear transaction costs are incurred to rebalance a portfolio as illustrated in Figure 4.2. The portfolio may need to be rebalanced periodically simply as updated risk and return information is generated with the passage of time. For the rebalancing portfolio problem, we have assumed that the time point for rebalancing the portfolio is exactly at the midpoint of the entire time horizon of the single-period model, and therefore we could simply say about the rebalancing model is such that investors have one more transaction chance.

In the mathematical formulation of the proposed rebalancing portfolio model, we have introduced two different parameters for each time period's risk level, one for each period. The parameters enable investors to control their risk preferences for each time period and/or for the whole two time periods. In particular, for the rebalancing time point, there are no exogenous cash injections, and therefore the amount of assets bought and transaction costs of risky assets and the amount of assets sold and transaction costs of risky assets have to be equal. This is reflected by the self-financing condition for the rebalancing time point.

In our rebalancing portfolio problem, since we consider nonlinear transaction costs for risky assets' transactions, myopic policies which try to optimize each time period independently are not optimal for the problem because portfolio decisions do affect each other in different time periods. The final objective value at the end of the time period is affected by the portfolio decision at the beginning of the planning horizon because the final result comes from the portfolio decision at the time point of rebalancing the portfolio which is affected by the portfolio decision at the beginning of the time period. Therefore, we have proposed an advanced, adaptive TS algorithm having an evolutionary neighborhood structure generated by increasing and/or decreasing the adjacent pairwise risky assets with variation factors over the two-step neighborhood structure, and we have solved the rebalancing portfolio problem with an iterative folding back procedure in decision tree structure.

For computational studies, we have considered a risk-free asset and scenarios involving 5, 10, 12, and 15 risky assets. We have obtained the best portfolio compositions and their objective values consisting of expected return and risk within reasonable times. Based on the repetitions of each experiment, we have observed the average errors of the TS algorithm and the absolute differences between the average objective value and the objective value obtained by each repetition. The computational results showed that the average error was less than 0.8% and the absolute difference was less than 0.3% for every problem case. In addition, we have shown the efficiency of the two-step neighborhood structure by comparing it with numerical results of an ordinary neighborhood structure which adopts variation factor $\Delta = 10\%$. We have observed that results of the two-step neighborhood structure are better than those of the single-step neighborhood structure for all problem cases. Also, in order to compare results between the single-period and rebalancing portfolio model, we have used the same data structure appropriately adjusted for both models. For all problem cases, the rebalancing model dominated the single-period model at least 3.04% and at most 15.29% of the objective value in the cases of 15 and 5 risky assets, respectively.

Chapter 5

Numerical Experiments with U.S. Stock Market

5.1 Introduction

In addition to the numerical experiments with randomly generated data, which we examined in Chapters 3 and 4, we apply the proposed portfolio optimization model and TS heuristic to the U.S. stock market. We consider a risk-free asset and 5, 10, 12, and 15 stocks traded at NYSE[1] and NASDAQ[2] as shown in Table 5.1, and historical stock prices from 2004 to 2007 are taken to calculate stocks' expected returns and covariances. The data for numerical experiments of the single-period and the rebalancing model is shown in Tables 5.2, 5.3, 5.4, and 5.5.

The entire planning horizon for the single-period model is a year, and hence each time period for the rebalancing model is 6 months since we assume that the time point for rebalancing the portfolio is exactly at the midpoint of the entire time horizon of the

[1]The New York Stock Exchange (NYSE) is a New York City-based stock exchange. It is the largest stock exchange in the world by dollar volume and, with 2,764 listed securities, has the second most securities of all stock exchanges.

[2]The National Association of Securities Dealers Automated Quotations (NASDAQ) is an American stock exchange. It is the largest electronic screen-based equity securities trading market in the United States with approximately 3,200 companies.

single-period model. We use the same computational environment and parameters as the setting for the numerical experiments in Chapters 3 and 4.

The remainder of this chapter is organized as follows. Portfolio compositions for each problem case of the single-period model and performance evaluations for the proposed TS heuristic are shown in Section 5.2. And, in Section 5.3, we examine the numerical results for comparisons between two time-period schemes as well as portfolio compositions for each problem case of the rebalancing model and performance evaluations for the proposed TS heuristic. The last section provides a summary of this chapter.

Table 5.1: A list of 15 companies for constructing a portfolio

Company (Symbol)	Industry
ExxonMobil (XOM)	Integrated Oil & Gas
Caterpillar (CAT)	Commercial Vehicles & Trucks
Hewlett-Packard (HPQ)	Diversified Computer Systems
Microsoft (MSFT)	Software
Coca-Cola (KO)	Beverages
Boeing (BA)	Aerospace & Defense
Alcoa (AA)	Aluminum
Merck (MRK)	Pharmaceuticals
United Technologies Corporation (UTX)	Aerospace, Heating/Cooling, Elevators
Procter & Gamble (PG)	Non-Durable Household Products
McDonald's (MCD)	Restaurants & Bars
Altria Group (MO)	Tobacco
Honeywell International (HON)	Aerospace/Defense Products & Services
Verizon Communications (VZ)	Telecoms
AT&T (T)	Telecoms

Table 5.2: Expected returns of 15 stocks and a risk-free return (%)

Asset	Single-Period Model	Rebalancing Model	
		$[t_0, t_1]$	$[t_1, t_2]$
Risk-Free	4.55	2.25	2.25
XOM	25.47	7.43	16.79
CAT	19.63	16.94	2.30
HPQ	37.85	15.54	19.31
MSFT	13.59	−3.59	17.82
KO	18.09	2.55	15.15
BA	22.72	17.78	4.19
AA	11.02	2.98	7.81
MRK	28.12	7.09	19.63
UTX	16.92	8.12	8.14
PG	13.48	1.53	11.77
MCD	27.78	3.41	23.57
MO	24.58	8.90	14.39
HON	23.13	5.58	16.62
VZ	10.59	−3.20	14.25
T	24.54	5.80	17.72

Table 5.3: Covariances between 15 stocks for single-period model

	XOM	CAT	HPQ	MSFT	KO	BA	AA	MRK	UTX	PG	MCD	MO	HON	VZ	T
XOM	0.01583	-0.02662	0.00397	0.00348	0.00361	0.01307	-0.02824	0.00137	-0.01022	-0.00167	-0.00984	-0.00335	-0.00203	0.00700	0.00689
CAT		0.10655	-0.03219	-0.05108	-0.00747	-0.00646	0.10556	0.00484	0.03388	-0.03608	-0.01138	-0.01401	0.02613	0.01258	0.00874
HPQ			0.02127	0.02848	-0.00165	-0.00633	-0.02322	0.00216	-0.01066	0.02125	0.02981	0.01338	-0.01072	-0.00327	0.00897
MSFT				0.03415	0.00038	-0.01379	-0.02916	0.00110	-0.00921	0.02879	0.03172	0.01496	-0.01460	-0.00871	-0.00112
KO					0.00767	-0.00273	-0.01151	0.00606	0.00221	0.00325	-0.00611	-0.00607	-0.00195	0.00867	-0.00127
BA						0.02280	-0.00311	-0.00618	-0.00841	-0.01878	-0.01897	-0.00389	0.00692	0.00106	0.00771
AA							0.09858	0.00523	0.02652	-0.02622	0.00984	-0.00411	0.02143	0.01029	0.01925
MRK								0.01416	0.00720	0.00383	0.00963	-0.00511	0.00085	0.01967	0.01553
UTX									0.01316	-0.00514	-0.00116	-0.00573	0.00553	0.00714	0.00048
PG										0.02774	0.02716	0.01101	-0.01351	-0.00561	-0.00427
MCD											0.06747	0.02273	-0.00954	-0.00090	0.02134
MO												0.01242	-0.00577	-0.01188	-0.00040
HON													0.00850	0.00599	0.00619
VZ														0.03014	0.02352
T															0.03836

Table 5.4: Covariances between 15 stocks for rebalancing model $[t_0, t_1]$

	XOM	CAT	HPQ	MSFT	KO	BA	AA	MRK	UTX	PG	MCD	MO	HON	VZ	T
XOM	0.00589	-0.00577	-0.00441	-0.00591	0.00262	0.00769	-0.00939	-0.00065	-0.00211	-0.00610	-0.01440	-0.00544	0.00145	0.00296	-0.00168
CAT		0.02652	0.00150	0.00088	0.00333	-0.01295	0.02522	0.01437	0.01168	0.00406	0.01858	-0.00225	0.00261	0.01791	0.01740
HPQ			0.00369	0.00509	-0.00276	-0.00503	0.00486	-0.00137	0.00028	0.00483	0.01018	0.00510	-0.00163	-0.00504	-0.00087
MSFT				0.00708	-0.00401	-0.00644	0.00565	-0.00257	-0.00014	0.00657	0.01338	0.00724	-0.00240	-0.00786	-0.00201
KO					0.00283	0.00190	0.00035	0.00359	0.00177	-0.00326	-0.00514	-0.00456	0.00178	0.00719	0.00370
BA						0.01145	-0.01642	-0.00441	-0.00525	-0.00748	-0.01997	-0.00515	0.00085	-0.00152	-0.00628
AA							0.02728	0.01158	0.01075	0.00826	0.02640	0.00286	0.00078	0.01125	0.01477
MRK								0.00911	0.00656	-0.00059	0.00455	-0.00438	0.00249	0.01336	0.01055
UTX									0.00518	0.00131	0.00723	-0.00154	0.00133	0.00852	0.00785
PG										0.00650	0.01451	0.00634	-0.00187	-0.00498	0.00030
MCD											0.03615	0.01168	-0.00267	-0.00277	0.00750
MO												0.00778	-0.00280	-0.01030	-0.00416
HON													0.00113	0.00474	0.00262
VZ														0.02224	0.01485
T															0.01236

Table 5.5: Covariances between 15 stocks for rebalancing model $[t_1, t_2]$

	XOM	CAT	HPQ	MSFT	KO	BA	AA	MRK	UTX	PG	MCD	MO	HON	VZ	T
XOM	0.00672	-0.01633	0.00868	0.01005	0.00003	0.00222	-0.01348	0.00196	-0.00637	0.00580	0.00860	0.00355	-0.00346	0.00256	0.00797
CAT		0.05652	-0.02466	-0.03800	-0.00799	0.00523	0.05684	-0.00738	0.01570	-0.02940	-0.02224	-0.00854	0.01708	-0.00444	-0.00683
HPQ			0.01197	0.01585	0.00171	0.00062	-0.02250	0.00309	-0.00828	0.01073	0.01139	0.00457	-0.00630	0.00293	0.00765
MSFT				0.02598	0.00643	-0.00525	-0.03958	0.00504	-0.00970	0.02102	0.01394	0.00524	-0.01216	0.00240	0.00181
KO					0.00372	-0.00498	-0.01138	0.00124	-0.00013	0.00722	0.00067	-0.00002	-0.00409	-0.00082	-0.00585
BA						0.00742	0.01073	-0.00100	-0.00196	-0.00773	0.00198	0.00122	0.00432	0.00196	0.01052
AA							0.06148	-0.00768	0.01309	-0.03352	-0.01917	-0.00699	0.01934	-0.00260	0.00193
MRK								0.00098	-0.00189	0.00407	0.00272	0.00102	-0.00236	0.00047	0.00038
UTX									0.00605	-0.00570	-0.00817	-0.00336	0.00339	-0.00240	-0.00739
PG										0.01891	0.00865	0.00299	-0.01086	0.00060	-0.00450
MCD											0.01110	0.00453	-0.00512	0.00313	0.00919
MO												0.00187	-0.00178	0.00136	0.00427
HON													0.00624	-0.00041	0.00235
VZ														0.00116	0.00435
T															0.01877

5.2 Numerical results of single-period portfolio optimization model

We apply the TS algorithm to multiple stocks of 5, 10, 12, and 15, respectively. That is, we consider 5, 10, 12, and 15 stocks and a risk-free asset for numerical experiments. Stocks' expected returns and covariance matrix are computed based on historical stock prices from 2004 to 2007 and produced data are shown in Tables 5.2 and 5.3 of Section 5.1. Also, parameters needed for the TS algorithm and for settings of the computational experiments are exactly same as those of the single-period problem illustrated in Table 3.1 and 3.2.

The following four tables represent portfolio compositions for each problem case and their objective values consisting of expected return and risk. In each table, tthe second column shows the lower bound for each problem case. That is, we invest our wealth in the only risk-free asset, and obviously we make the riskless return. A feasible initial portfolio represented in the third column is obtained by the selection method depicted in Section 3.3.2.

Table 5.6: Portfolio returns and risks by the composition of the case with 5 stocks

Stock	Lower Bound	Initial Portfolio	Best Portfolio
Risk-Free	10,000.00	4,117.63	−4,987.32
XOM	0.00	1,469.55	3,150.12
CAT	0.00	1,267.78	3,158.72
HPQ	0.00	1,101.61	6,063.62
MSFT	0.00	1,054.78	155.17
KO	0.00	852.46	2,188.95
Expected Return	10,455.00	11,232.50	13,303.85
Expected Risk	0.00	71.32	708.54
Obj. Value	10,455.00	11,161.18	12,595.31

If the portfolio achieved as lower bound dominates all other randomly generated portfolios for the purpose of the initial solution, it would be the lower bound and initial portfolio simultaneously. The TS iterations start with this initial portfolio, and finally converge to the best portfolio represented in the forth column of each table.

For all problem cases, we short risk-free assets for the best portfolio as shown in the second row of Tables 5.6, 5.7, 5.8, and 5.9. In particular, the risk-free asset is shorted up to near the credit limitation in those cases. Since the credit limitation for our investment

Table 5.7: Portfolio returns and risks by the composition of the case with 10 stocks

Stock	Lower Bound	Initial Portfolio	Best Portfolio
Risk-Free	$10,000.00$	$2,364.78$	$-4,933.91$
XOM	0.00	$1,164.38$	961.98
CAT	0.00	799.93	$2,074.11$
HPQ	0.00	$1,262.58$	$5,801.50$
MSFT	0.00	180.19	45.34
KO	0.00	775.71	$1,141.25$
BA	0.00	849.34	$2,665.59$
AA	0.00	254.20	0.00
MRK	0.00	607.72	0.00
UTX	0.00	779.34	$1,653.87$
PG	0.00	750.49	290.76
Expected Return	$10,455.00$	$11,380.16$	$13,101.82$
Expected Risk	0.00	102.31	505.86
Obj. Value	$10,455.00$	$11,277.85$	$12,595.96$

Table 5.8: Portfolio returns and risks by the composition of the case with 12 stocks

Stock	Lower Bound	Initial Portfolio	Best Portfolio
Risk-Free	$10,000.00$	$1,948.12$	$-4,994.24$
XOM	0.00	$1,463.59$	$2,852.12$
CAT	0.00	$1,239.70$	$2,320.63$
HPQ	0.00	$1,203.75$	$5,531.19$
MSFT	0.00	399.78	14.95
KO	0.00	75.21	704.49
BA	0.00	692.14	$1,348.78$
AA	0.00	0.00	0.00
MRK	0.00	717.30	0.00
UTX	0.00	698.25	$1,237.00$
PG	0.00	625.52	0.00
MCD	0.00	24.60	0.00
MO	0.00	684.67	684.67
Expected Return	$10,455.00$	$11,481.85$	$13,199.38$
Expected Risk	0.00	125.24	599.67
Obj. Value	$10,455.00$	$11,356.61$	$12,599.71$

Table 5.9: Portfolio returns and risks by the composition of the case with 15 stocks

Stock	Lower Bound	Initial Portfolio	Best Portfolio
Risk-Free	$10,000.00$	745.24	$-4,905.56$
XOM	0.00	$1,482.13$	$1,479.43$
CAT	0.00	828.02	$2,115.62$
HPQ	0.00	$1,201.12$	$5,420.71$
MSFT	0.00	103.20	0.00
KO	0.00	$1,081.21$	$1,156.47$
BA	0.00	538.29	$2,121.99$
AA	0.00	300.59	0.00
MRK	0.00	0.00	0.00
UTX	0.00	911.80	$1,298.37$
PG	0.00	715.49	0.00
MCD	0.00	359.78	0.00
MO	0.00	622.59	$1,023.72$
HON	0.00	328.71	0.00
VZ	0.00	434.51	0.00
T	0.00	72.60	0.00
Expected Return	$10,455.00$	$11,457.62$	$13,144.48$
Expected Risk	0.00	181.57	535.18
Obj. Value	$10,455.00$	$11,276.05$	$12,609.30$

is 50% of the initial wealth, the risk-free asset can be shorted up to 5,000, and in the cases of 5, 10, 12, and 15 stocks they are shorted by 4,987.32, 4,933.91, 4,994.24, and 4,999.68, respectively.

For problem cases of 10, 12, and 15 stocks, the numerical results show that we do not need to consider all stocks as investment targets. We invest 8, 8, and 7 stocks for the problem cases of 10, 12, and 15, respectively. Based on these numerical results, we conclude that our selection algorithm can exclude unnecessary assets although there is not a cardinality constraint for the purpose of portfolio diversification.

By comparing the objective value of the best portfolio between 4 problem cases, the objective value of the 15 stocks case dominates all the other cases. Similarly, the 12 stocks case dominates 5 and 10 stocks cases, and the 10 stocks case has the higher best objective value than that of the 5 stocks case. These results are caused by the structure of the expected returns and covariances shown in Section 5.1. That is, for example, stocks considered in the 15 stocks case are constructed by combining 13 stocks' data of the 12 stocks case as shown in Tables 5.2 and 5.3 and 3 more stocks' data for HON, VZ, T in the tables. Therefore, the best objective value of one problem case must be greater than or equal to those of the other problem cases having smaller considered asset in the portfolio, and our results are consistent with this conjecture by the numerical results.

Each experiment is repeated a total of 2 times for each tabu size shown in the fourth row of Table 3.1. That is, for the case of 5 stocks, the experiment is repeated $(2 \times 5 = 10)$ times since the maximum tabu size is 5. Hence, for the cases of 10, 12, and 15 stocks, the experiments are repeated 20, 30, and 40 times, respectively. It takes 0.5 to 2.5 seconds for the computation time of each experiment. Every solution for the 5 stocks case is obtained within 0.5 seconds, and at most 2.5 seconds are taken to obtain a solution of the 15 stocks case. With the best solution among the repetitions, the average error of the tabu search algorithm is evaluated by (3.14). As shown in the second column of Table 5.10, the average error for every problem case is less than 0.2%, and the error tends to increase proportional to the number of stocks in the portfolio.

Additionally, the absolute difference between the average objective value and the objective value obtained by each repetition is evaluated by (3.15). The third column in Table 5.10 shows the absolute difference values, and for every problem case the value is less than 0.1%. Table 5.11 shows the percent improvement of the best objective value against the lower bound of the risk-free investment and the initial solution.

Table 5.10: Average percent errors and differences by the TS heuristic

Stocks	Error (%)	Difference (%)	Best Obj. Value	Avg. Obj. Value
5	0.02001	0.01079	12,595.31	12,592.79
10	0.11470	0.08620	12,595.96	12,581.51
12	0.17416	0.08900	12,599.71	12,577.77
15	0.19941	0.09139	12,609.30	12,584.16

Table 5.11: Percent improvements of the best portfolio in objective function

Stocks	vs. Lower Bound (%)	vs. Initial Portfolio (%)
5	20.47	12.85
10	20.48	11.69
12	20.51	10.95
15	20.61	11.82

5.3 Numerical results of rebalancing portfolio optimization model

For numerical experiments of the rebalancing portfolio problem with real data, we consider the number of stocks to be 5, 10, 12, and 15 and a risk-free asset. Stocks' expected returns and covariance matrix are computed based on historical stock prices from 2004 to 2007 and produced data are shown in Tables 5.2, 5.4, and 5.5 of Section 5.1. And, we use same parameters with those of the rebalancing problem in Chapter 4, and tables 4.3 and 4.4 therefore represent summaries for the parameters needed for the TS algorithm and for setting of computational experiments, respectively.

The following four tables represent portfolio compositions for each problem case and their objective values consisting of expected return and risk. In each table, we show the portfolio compositions for lower bound, initial solution and the best solution. Each of them has two columns consisting of a portfolio at time t_0 and a rebalancing portfolio at time t_1. In the tables, the second and third columns represent the lower bound of each problem case. That is, we invest our wealth in the only risk-free asset at the beginning of the time period, and maintain it until the end of the time period. Therefore, we obviously

Table 5.12: Portfolio returns and risks by the composition of the case with 5 stocks

Stock	Lower Bound		Initial Solution		Best Solution	
Risk-Free	10,000.00	10,225.00	4,452.22	4,552.39	−4,924.95	−5,720.58
XOM	0.00	0.00	1,393.41	1,496.96	3,183.22	0.00
CAT	0.00	0.00	1,219.91	1,426.57	1,712.25	4,004.65
HPQ	0.00	0.00	1,229.48	1,420.49	9,135.12	10,554.32
MSFT	0.00	0.00	552.23	532.38	0.00	0.00
KO	0.00	0.00	1,021.51	1,047.58	634.37	2,602.24
Expected Return	10,455.00		11,239.78		13,511.35	
Expected Risk	0.00		38.34		267.77	
Obj. Value	10,455.00		11,201.44		13,243.58	

make the riskless return. A feasible initial solution represented in the forth and fifth columns is obtained by the selection method depicted in Section 4.3.2. If the solution achieved as lower bound dominates all other randomly generated solutions for the purpose of the initial solution, it would be the lower bound and initial solution simultaneously. The TS iterations start with this initial solution, and finally converge to the best solution represented in the sixth and seventh columns of each table.

In all problem cases, we short risk-free assets for the best portfolio as shown in the second row of each table. In particular, the risk-free asset of every problem case is shorted up to near the credit limitation. Since the credit limitation for our investment is 50% of the wealth balance, the risk-free asset can be shorted up to 5,000 at the beginning of the time period.

Table 5.13: Portfolio returns and risks by the composition of the case with 10 stocks

Stock	Lower Bound		Initial Solution		Best Solution	
Risk-Free	10,000.00	10,225.00	2,505.12	2,561.49	−4,669.53	−5,870.11
XOM	0.00	0.00	1,488.09	1,598.68	857.97	921.74
CAT	0.00	0.00	824.43	964.09	2,847.99	3,330.47
HPQ	0.00	0.00	1,186.66	1,371.01	5,247.38	6,062.59
MSFT	0.00	0.00	804.74	775.81	61.50	59.29
KO	0.00	0.00	246.88	253.18	279.17	572.59
BA	0.00	0.00	690.58	813.37	4,114.15	0.00
AA	0.00	0.00	63.90	65.81	153.09	0.00
MRK	0.00	0.00	833.99	893.15	745.59	6,387.83
UTX	0.00	0.00	854.08	923.40	57.23	61.88
PG	0.00	0.00	292.25	296.74	0.00	0.00
Expected Return	10,455.00		11,395.79		13,780.33	
Expected Risk	0.00		67.03		122.22	
Obj. Value	10,455.00		11,328.76		13,658.11	

For all problem cases, some risky assets are invested during the time period $[t_0, t_1]$, but not invested during the time period $[t_1, t_2]$. In particular, for example, BA is invested during the time period $[t_0, t_1]$, but not invested during the time period $[t_1, t_2]$, and we can explain these results by the reason of a drastic decrease of the assets' return as shown in Table 5.2. The results also show that the proposed advanced, adaptive TS algorithm has a consistent robustness regardless of the problem size. Based on these numerical results, we conclude that our selection algorithm excludes unnecessary stocks although there is not a cardinality constraint for the purpose of portfolio diversification. These results are very similar to those of the single-period portfolio problem in Section 5.2. Additionally, in all

Table 5.14: Portfolio returns and risks by the composition of the case with 12 stocks

Stock	Lower Bound		Initial Solution		Best Solution	
Risk-Free	10,000.00	10,225.00	1,186.71	1,213.41	−4,990.41	−5,562.10
XOM	0.00	0.00	769.51	826.70	163.40	175.54
CAT	0.00	0.00	499.45	584.06	1,169.33	1,367.42
HPQ	0.00	0.00	1,366.98	1,579.35	2,381.82	2,751.85
MSFT	0.00	0.00	559.67	539.56	0.00	0.00
KO	0.00	0.00	673.85	691.05	0.00	0.00
BA	0.00	0.00	1,076.32	1,267.70	6,664.30	0.00
AA	0.00	0.00	551.58	568.01	562.18	1,157.84
MRK	0.00	0.00	847.35	907.46	2,319.33	9,935.39
UTX	0.00	0.00	490.77	530.61	0.00	0.00
PG	0.00	0.00	499.96	507.63	0.00	0.00
MCD	0.00	0.00	631.32	652.82	1,094.88	1,132.17
MO	0.00	0.00	598.04	651.28	334.83	364.64
Expected Return	10,455.00		11,552.38		13,782.55	
Expected Risk	0.00		155.97		106.03	
Obj. Value	10,455.00		11,396.41		13,676.52	

Table 5.15: Portfolio returns and risks by the composition of the case with 15 stocks

Stock	Lower Bound		Initial Solution		Best Solution	
Risk-Free	10,000.00	10,225.00	160.96	164.58	−4,818.29	−5,854.46
XOM	0.00	0.00	1,247.52	1,340.23	585.14	628.63
CAT	0.00	0.00	802.83	938.83	2,268.73	2,653.07
HPQ	0.00	0.00	1,294.05	1,495.09	5,102.40	5,895.09
MSFT	0.00	0.00	170.45	164.32	0.00	0.00
KO	0.00	0.00	1,097.12	1,125.11	200.75	411.75
BA	0.00	0.00	204.65	241.04	4,275.91	0.00
AA	0.00	0.00	233.89	240.86	78.16	80.49
MRK	0.00	0.00	885.99	948.84	1,729.36	7,408.12
UTX	0.00	0.00	629.78	680.90	0.00	0.00
PG	0.00	0.00	379.86	385.69	280.03	284.33
MCD	0.00	0.00	768.01	794.17	0.00	0.00
MO	0.00	0.00	645.20	702.65	0.00	0.00
HON	0.00	0.00	507.93	536.28	0.00	0.00
VZ	0.00	0.00	97.96	94.82	0.00	0.00
T	0.00	0.00	580.61	614.26	0.00	0.00
Expected Return	10,455.00		11,694.44		13,863.37	
Expected Risk	0.00		249.50		171.26	
Obj. Value	10,455.00		11,444.94		13,692.11	

problem cases, as shown in Tables 5.12, 5.13, 5.14, and 5.15, not all stocks are invested through the entire planning horizon.

For the problem cases of 5, 10, 12, and 15 stocks, the experiments are repeated 10, 20, 30, and 50 times, respectively. These numbers of repetitions are decided by the length of the tabu list for the first-time period as shown in the fourth row of Table 4.3. It takes 1.5 minutes to an hour for the computations of each experiment. Every solution for the 5 risky assets case is obtained within 1.5 minutes, and at most an hour is taken to obtain a solution for the 15 risky assets case.

With the best solution among the repetitions, the average error of the tabu search algorithm is evaluated by using (3.14). As shown in the second column of Table 5.16, the average error for every problem case is less than 0.6%.

Table 5.16: Average percent errors and differences by the TS heuristic

Stocks	Error (%)	Difference (%)	Best Obj. Value	Avg. Obj. Value
5	0.06001	0.02189	13,243.58	13,235.63
10	0.51463	0.19973	13,658.11	13,587.82
12	0.52900	0.20039	13,676.52	13,604.17
15	0.53011	0.22000	13,692.11	13,619.53

Table 5.17: Percent improvements of the best solution in objective function

Stocks	vs. Lower Bound (%)	vs. Initial Portfolio (%)
5	26.67	18.23
10	30.64	20.56
12	30.81	20.01
15	30.96	19.63

Additionally, the absolute difference between the average objective value and the objective value obtained by each repetition is evaluated by using (3.15). The third column in Table 5.16 shows the absolute difference values, and for every problem case the value is less than 0.3%.

Table 5.17 shows the percent improvement of the best solution against the lower bound of the risk-free investment and the initial solution. For instance, in the case of the 15 stocks, the objective value of the best solution dominates that of the lower bound and the initial solution by 30.96% and 19.63%, respectively.

In Section 5.18, we proposed the two-step neighborhood structure for the rebalancing portfolio model. Table 4.11 shows the efficiency of the two-step neighborhood structure by comparing it with numerical results of ordinary neighborhood structure which adopts variation variation factor $\Delta = 10\%$ as proposed in Section 3.3.3 of the single-period model.

Table 5.18: Average objective value: two-step vs. single-step neighborhood structure

Stocks	Two-Step	Single-Step
5	13,235.63	13,211.70
10	13,587.82	13,519.00
12	13,604.17	13,571.69
15	13,619.53	13,584.51

Table 5.19: The best objective values: single-period vs. rebalancing model

Stocks	Single-Period Model	Rebalancing Model	Difference (%)
5	12,595.31	13,243.58	5.15
10	12,595.96	13,658.11	8.43
12	12,599.71	13,676.52	8.55
15	12,609.30	13,692.11	8.59

Each value in the table represents the average objective value based on the repetitions of the experiment. We observe that results of the two-step neighborhood structure are better than those of the single-step neighborhood structure for all problem cases. Thus, we can conclude that the two-step neighborhood structure for iterative TS procedures produces better solutions in the rebalancing portfolio optimization problem.

The primary objectives of this study are to extend a single-period portfolio model to a rebalancing model and to find the effect of one more transaction of the rebalancing model. As we assumed in Section 4.1, investors have one more transaction opportunity in the rebalancing model, and the time point for rebalancing is exactly at the midpoint of the entire planning horizon of the single-period model. In order to compare results between the single-period and the rebalancing model, we use the same data structure for both models and adjust real stocks' expected returns and covariance matrix considering one more transaction chance of the rebalancing model. Table 5.19 shows the best objective values of the single-period and the rebalancing model. For all problem cases, the rebalancing model dominates the single-period model, and especially the rebalancing model with 15 stocks improves the objective values by 8.59% in beating the transaction costs.

5.4 Summary

In addition to the numerical experiments with randomly generated data, which we examined in Chapters 3 and 4, in this chapter, we apply the proposed portfolio optimization model and TS heuristic to the U.S. stock market. We consider a risk-free asset and 5, 10, 12, and 15 stocks traded at NYSE and NASDAQ, and historical stock prices from 2004 to 2007 are taken to calculate stocks' expected returns and covariances.

For computational studies of the single-period model, we have applied the TS algorithm to multiple stocks of 5, 10, 12, and 15, respectively. We have obtained the best portfolio compositions and their objective values within reasonable times. Each experiment is repeated a total of 2 times for each tabu size, and based on the experimental repetitions of the four problem cases, we have observed the average errors and the absolute differences by the TS heuristic reported in the single-period model. The average error for every problem case is less than 0.2%, and the results show that the absolute difference is less than 0.1%.

For the rebalancing portfolio model, we have also considered a risk-free asset and scenarios involving 5, 10, 12, and 15 stocks. We have obtained the best portfolio compositions and their objective values consisting of expected return and risk within reasonable times. Based on the repetitions of each experiment, we have observed the average errors of the TS algorithm and the absolute differences between the average objective value and the objective value obtained by each repetition. The computational results showed that the average error was less than 0.6% and the absolute difference was less than 0.3% for every problem case. In addition, we have shown the efficiency of the two-step neighborhood structure by comparing it with numerical results of an ordinary neighborhood structure which adopts variation factor $\Delta = 10\%$. We have observed that results of the two-step neighborhood structure are better than those of the single-step neighborhood structure for all problem cases. Also, in order to compare results between the single-period and rebalancing portfolio model, we have used the same data structure appropriately adjusted for both models. For all problem cases, the rebalancing model dominated the single-period model at least 5.15% and at most 8.59% of the objective value in the cases of 5 and 15 stocks, respectively.

Chapter 6

Summarizing Remarks

6.1 Conclusion

According to the Markowitz mean-variance (MV) model [30], the portfolio selection problem can be formulated as an optimization problem over real-valued variables with a quadratic objective function and linear constraints. The MV model has become a universally understood technique within the investment world for generating the trade-off of changes in risk for changes in expected return called the efficient frontier, which is defined as one that has the smallest portfolio risk for a given level of expected return or the largest expected return for a given level of risk. The efficient frontier of portfolios can be found by solving the quadratic programming (QP) model.

After the seminal paper of Markowitz, we have been witnesses to a great evolution with respect to the traditional MV model. With all its merits, however, some of the main downsides of the MV model and its extended or modified models have been recognized: the computational complexity; the inability to incorporate practical considerations such as taxes and transaction costs; and the investment decision being at exactly one time point in time for a single-period horizon. The QP model incorporating realistic considerations cannot be applied to find an optimal or a near-optimal portfolio. Nonlinear mixed integer programming can be of interest if, for instance, nonlinear transaction costs are considered. Even though different integer programming models can be an alternative for solving practical portfolio problems, not only does it still have the drawback of high computational requirements, but also it is almost impossible to use that approach for solving a rebalancing portfolio model incorporating realistic considerations.

It therefore is desirable to have an alternative method that can deal with highly demanding real-world portfolio problems considering more complex scenarios and settings. One way is heuristic optimization. It is relatively easy to implement and computationally attractive. With various heuristic optimization techniques, portfolio models including transaction costs and other realistic considerations can be examined. However, incorporating nonlinear transaction costs for a rebalancing portfolio model is still hard to solve, and therefore the problem is in need of an advanced heuristic algorithm. In this thesis we extend the Markowitz MV model to a rebalancing portfolio optimization problem incorporating practical considerations such as nonlinear transaction costs and a risk-free asset with short-selling allowed, and propose an advanced, adaptive Tabu Search (TS) heuristic to solve the considered practical rebalancing portfolio optimization problem.

TS is a metaheuristic that guides a local search procedure to explore the solution space beyond local optimality. It is a powerful algorithmic approach that has been applied with great success to many difficult combinatorial problems such as the vehicle routing problem and the traveling salesman problem. A particularly nice feature of TS is that it can quite easily handle the dirty complicating constraints that are typically found in real-life applications. It is thus a really practical approach. The most basic steps of the development of the TS procedure are the choice of a search space and of an effective neighborhood structure. It is also extremely important to develop an effective diversification scheme in order to achieve breadth in its searching process.

Many complex decision problems such as portfolio optimization involve multiple conflicting objectives: every investor would expect to maximize the expected portfolio return and, simultaneously, to minimize its risk. It is often true that no dominant alternative will exist that is better than all other alternatives in terms of a trade-off between the expected return and risk. Therefore, in this thesis, we propose a bi-objective portfolio optimization model which we expect to yield a portfolio equilibrium by combining two objectives, and we control the risk level of the portfolio by setting a parameter in the objective function. For realistic portfolio problems, first of all, we consider the multi-objective portfolio optimization model incorporating the risk-free asset and its short-selling and nonlinear transaction costs based on the single-period MV model. The risk-free asset incorporated into the bi-criteria portfolio selection model has a return that is certain. The covariance of the risk-free asset's return with any risky asset's return is thus zero. In this study we allow the short-selling of the risk-free asset within the credit balance. Transaction costs also considered in this study, consist of proportional and fixed costs, and thus they are formulated with a nonlinear model.

For our single-period portfolio problem, we propose an adaptive version of a TS heuristic. We define the feasible portfolio as the solution representation by means of a vector indicating the amount of money invested in each asset. For the initial solution, we randomly generate portfolios by considering the problem size and the purpose of diversified creation. From the initial solution, we obtain the final solution by iteratively searching with the neighborhood and tabu structure. The neighborhood of the current portfolio is generated by increasing and/or decreasing the adjacent pairwise risky assets with a variation factor, and during the neighborhood generation, the state of the risk-free asset is automatically determined by self-financing conditions. The basic role of the tabu list is to prevent cycling, and the length of the list (tabu size) is a critical factor for quality of the solution. For our portfolio problems, the tabu size is proportionally determined by the problem size, and the TS algorithm terminates after some numbers of iterations without an improvement in the objective function value. For computational studies, we apply the TS algorithm to a risk-free asset and multiple risky assets of 5, 10, 12, and 15, respectively. We obtain the best portfolio compositions and their objective values. Based on the repetitions of each experiment, we observe the average errors of the TS algorithm and the absolute differences between the average objective value and the objective value obtained by each repetition for the TS heuristic. The average error of the proposed TS algorithm for every problem case is less than 0.3%, and the results show that the absolute difference computed by the average objective value and the objective value obtained by each repetition is less than 0.1%.

For our primary purpose, we extend the single-period model to a rebalancing problem which also considers nonlinear transaction costs and a risk-free asset and its short-selling. The portfolio may need to be rebalanced periodically simply as updated risk and return information is generated with the passage of time. For the rebalancing portfolio problem, we have assumed that the portfolio model has one intermediate, rebalancing time point, and the time point for rebalancing the portfolio is exactly at the midpoint of the entire time horizon of the single-period model, and therefore we could simply say about the rebalancing model gives investors one more transaction chance. In our rebalancing portfolio problem, since we consider nonlinear transaction costs for risky assets' transactions, a myopic policies which try to optimize each time period independently are not ordinarily optimal for the problem because portfolio decisions do affect each other time periods. The final objective value at the end of the time period is affected by the portfolio decision at the beginning of the planning horizon because the final result comes from the portfolio decision at the time point of rebalancing the portfolio, which is affected by the portfolio decision at the beginning of the time period. Therefore, we have proposed an advanced, adaptive TS algorithm having an evolutionary neighborhood structure, and have solved the rebalancing portfolio problem with an iterative folding back procedure in decision tree structure. For computational studies, as with the single-period problem, we consider a risk-free asset and risky assets of 5, 10, 12, and 15. We obtain the best portfolio compositions and their objective values consisting of weighted expected return and risk. Based on the repetitions of each experiment, we have observed the average errors of the TS algorithm and the absolute differences between the average objective value and the objective value obtained by each repetition. The computational results showed that the average error was less than 0.8% and the absolute difference was less than 0.3% for every problem case. In addition, we have shown the efficiency of the two-step neighborhood structure by comparing it with numerical results of ordinary neighborhood structure which adopts variation factor $\varDelta = 10\%$ as modeled single-period portfolio problems. We have observed that results of the two-step neighborhood structure are better than those of the single-step neighborhood structure for all problem cases. Also, in order to compare results between the single-period and rebalancing portfolio model, we have used the same data structure appropriately adjusted for both models. For all problem cases, the rebalancing model dominated the single-period model at most by 12.29% of the objective value which occurred in the case of 5 risky assets. In addition to the computational studies with randomly generated data, we also introduced numerical experiments with real data from the U.S. stock market for both the single-period and rebalancing portfolio models.

6.2 Future Research

In this thesis, we solve rebalancing portfolio problems incorporating realistic considerations such as nonlinear transaction costs and a risk-free asset with short-selling allowed, and we apply the TS heuristic to solve the practical portfolio problems. In addition to our study, there are quite a few topics that seem appropriate for future works.

First of all, we could try to design an efficient search algorithm for the TS heuristic. For our portfolio problems, it takes at most an hour to meet convergence criteria for the rebalancing problem with consideration of 15 risky assets. Although considering more than 15 assets in a portfolio is not realistically reasonable, if we consider more than 15 assets or incorporate more complicated scenarios into the portfolio model, the computation time with the current search algorithm will increase drastically.

Since Harry Markowitz [30] proposed the MV model in 1952, many researchers have studied portfolio optimization problems with consideration of diverse situations. Secondly, therefore, our model approach could be applied to more realistic and complex scenarios issued in the future as the portfolio optimization problem evolves.

Thirdly, we could apply other types of heuristic methods to solve the proposed portfolio problem. The TS is still quite attractive to solve portfolio problems in that it has the merits of viable computational time and adaptability. Chang, Meade, Beasley, and Sharaiha [11] present three heuristic algorithms to solve a problem, and they show that no one of the heuristic algorithms is uniformly dominant though the TS heuristic does dominate the other methods with regard to total computer time. However, applying other heuristic methods and comparing them could be meaningful for other computational studies.

A further potential future topic related to this study is that we may apply the proposed algorithm to advanced asset management or risk management. It is difficult to construct an optimal portfolio constituted by combining different types of assets having different return and risk forms. In the real world, many kinds of assets are traded, and some of them, especially derivatives, have really complicated return and risk structures. Under this more complex market situation, we might be able to construct a portfolio efficiently with the proposed algorithm or the advanced method appropriately modified.

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Appendix

Appendix A

Expected Returns and Covariances

For numerical experiments, we consider 5, 10, 12, and 15 risky assets and a risk-free asset. In this appendix, we first generate data for the rebalancing model presented in Chapter 4 and then produce data used for the single-period model in Chapter 3. We assume that the time point for rebalancing the portfolio is exactly at the midpoint of the entire planning horizon of the single-period model. Hence, we can simply say that the rebalancing model is such that investors have one more transaction chance with paying transaction costs at the midpoint of the entire planning horizon of the single-period model. In order to compare results between those two schemes, we must use the same data for both models and adjust the generated assets' returns and covariance matrix considering the further transaction chance of the rebalancing model. For experimental data, we generate: (i) assets' expected returns, (ii) variances, (iii) correlations between the assets, and (iv) covariances between the assets by combining (ii) and (iii).

For $t = t_1, t_2$ in the rebalancing model, we generate asset i's expected return r_i^t from uniform distributions which produce the smallest possible return 0.07 to the maximum 0.25:

$$r_i^t \sim \text{UNIF}[0.07, 0.25], \qquad \forall\, i \in \mathbb{J}. \tag{A.1}$$

Also, for asset i's variance $(\sigma_i^t)^2$, we consider the smallest possible variance 0.005 to the maximum variance 0.02 from uniform distributions:

$$(\sigma_i^t)^2 \sim \text{UNIF}[0.005, 0.02], \qquad \forall\, i \in \mathbb{J}. \tag{A.2}$$

For a fair comparison between the single-period and the rebalancing model, we generate asset i's expected return r_i and variance σ_i^2 for the single period model by considering asset i's expected returns and variances of the rebalancing model produced by (A.1) and (A.2):

$$(1+r_i) = (1+r_i^{t_1})(1+r_i^{t_2}) \quad \Leftrightarrow \quad r_i = r_i^{t_1} + r_i^{t_2} + r_i^{t_1} r_i^{t_2}, \qquad \forall\, i \in \mathbb{J}, \tag{A.3}$$

$$\sigma_i^2 = \text{Var}[(1+R_i^{t_1})(1+R_i^{t_2})] \tag{A.4a}$$

$$= \text{Var}[R_i^{t_1} + R_i^{t_2} + R_i^{t_1} R_i^{t_2}] \tag{A.4b}$$

$$= (\sigma_i^{t_1})^2(1+r_i^{t_2})^2 + (\sigma_i^{t_2})^2(1+r_i^{t_1})^2 + (\sigma_i^{t_1})^2(\sigma_i^{t_2})^2, \qquad \forall\, i \in \mathbb{J}. \tag{A.4c}$$

From among a number of methods for generating a random correlation matrix[1], we apply the following method of three steps to produce a random $N \times N$ correlation matrix. For our numerical experiments, N will be 15 since the number of risky assets considered is at most 15, and we assume that correlations between assets for the single-period and each time period of the rebalancing model are identical.

Step 1. For $j = 1, 2, \ldots, N$, generate the $N \times 1$ random vector $\boldsymbol{Z}_j$ whose components are independent standard normal random variables,

$$\boldsymbol{Z}_j = \begin{bmatrix} Z_{1,j} \\ Z_{2,j} \\ \vdots \\ Z_{N,j} \end{bmatrix} \text{ such that } \{Z_{i,j} : i = 1, 2, \ldots, N\} \overset{\text{i.i.d.}}{\sim} \text{NORM}(0,1). \tag{A.5}$$

[1]The method is brought from Wilson's lecture note [60], and other methods for generating random correlation matrices are detailed in Marsaglia and Olkin [32].

Step 2. For $j = 1, 2, \ldots, N$, project $\boldsymbol{Z}_j$ onto $C_N(1)$, the N-dimensional hypersphere of radius 1, to obtain a random vector $\boldsymbol{Y}_j$ that is uniformly distributed on $C_N(1)$,

$$\boldsymbol{Y}_j = \frac{\boldsymbol{Z}_j}{\|\boldsymbol{Z}_j\|}, \quad \text{where} \quad \|\boldsymbol{Z}_j\| \equiv \sqrt{\sum_{i=1}^{N} Z_{i,j}^2} \tag{A.6}$$

is the magnitude of the vector $\boldsymbol{Z}_j$.

Step 3. Deliver the $N \times N$ random correlation matrix $\boldsymbol{R}$ whose (i, j) entry is $\boldsymbol{Y}_i \cdot \boldsymbol{Y}_j$, the dot product of $\boldsymbol{Y}_i$ and $\boldsymbol{Y}_j$ for $i, j = 1, 2, \ldots, N$:

$$\boldsymbol{R} = \boldsymbol{Z}^{\mathrm{T}}\boldsymbol{Z}, \quad \text{where} \quad \boldsymbol{Z} \equiv [\boldsymbol{Z}_1, \boldsymbol{Z}_2, \ldots, \boldsymbol{Z}_N]\,. \tag{A.7}$$

Since $\|\boldsymbol{Y}_j\| = 1$ for $j = 1, 2, \ldots, N$ by (A.6), it follows that for $i, j = 1, 2, \ldots, N$, the cosine of the angle $\theta_{i,j}$ between the vectors $\boldsymbol{Y}_i$ and $\boldsymbol{Y}_j$ is given by

$$\cos(\theta_{i,j}) = \frac{\boldsymbol{Y}_i \cdot \boldsymbol{Y}_j}{\|\boldsymbol{Y}_i\|\,\|\boldsymbol{Y}_j\|} = \boldsymbol{Y}_i^{\mathrm{T}}\boldsymbol{Y}_j = \sum_{k=1}^{N} Y_{k,i}Y_{k,j} = \frac{\displaystyle\sum_{k=1}^{N} Z_{k,i}Z_{k,j}}{\sqrt{\left(\displaystyle\sum_{k=1}^{N} Z_{k,i}^2\right)\left(\displaystyle\sum_{k=1}^{N} Z_{k,j}^2\right)}}, \tag{A.8}$$

the (i, j) element of $\boldsymbol{R}$. Hence, the equation (A.8) represents a correlation between assets i and j, $\rho_{i,j}$. The randomly generated $\boldsymbol{R}$ is a legitimate correlation matrix since it satisfies all the requirements of a correlation matrix: it is symmetric; it has ones on the diagonal; it has off-diagonal entries that are all between -1 and $+1$; and its determinant must be nonnegative. The condition for a nonnegative determinant follows from

$$\det(\boldsymbol{R}) = \det(\boldsymbol{Z}^{\mathrm{T}}\boldsymbol{Z}) = \det(\boldsymbol{Z}^{\mathrm{T}})\det(\boldsymbol{Z}) = [\det(\boldsymbol{Z})]^2 \geq 0. \tag{A.9}$$

Therefore, for $t = t_1, t_2$ in the rebalancing model, we compute a covariance between assets i and j by the relationship in (2.5) and randomly generated values of (A.2) and (A.8):

$$\sigma_{i,j}^t = \sigma_i^t \sigma_j^t \rho_{i,j}. \tag{A.10}$$

Since we assume an identical correlation between assets i and j for the single-period and each time period of the rebalancing model, we derive a covariance between assets i and j for the single-period model as

$$\sigma_{i,j} = \sigma_i \sigma_j \rho_{i,j} \tag{A.11}$$

where $\sigma_i = \sqrt{(\sigma_i^{t_1})^2(1+r_i^{t_2})^2 + (\sigma_i^{t_2})^2(1+r_i^{t_1})^2 + (\sigma_i^{t_1})^2(\sigma_i^{t_2})^2}$ and

$\sigma_j = \sqrt{(\sigma_j^{t_1})^2(1+r_j^{t_2})^2 + (\sigma_j^{t_2})^2(1+r_j^{t_1})^2 + (\sigma_j^{t_1})^2(\sigma_j^{t_2})^2}$ by (B.4).

According to the above procedures, we produce assets' returns and covariances shown in the following four tables. In Table A.1, asset 0 represents the risk-free asset and other indexes from 1 to 15 represent 15 risky assets. For considering the case of 5, 10, 12, and 15 risky assets of numerical experiments in Chapters 3 and 4, we use the data in asset columns 0 to 5, 0 to 10, 0 to 12, and 0 to 15, respectively, and we use covariances between risky assets indexed from asset 1 and 15 in Table A.2 for the single-period model of Chapter 3 and in Tables A.3 and A.4 for the rebalancing model of Chapter 4.

Table A.1: Expected returns of 15 risky assets and a risk-free return (%)

Asset	Single-Period Model	Rebalancing Model	
		$[t_0, t_1]$	$[t_1, t_2]$
0	14.49	7.00	7.00
1	41.85	24.10	14.30
2	37.66	11.16	23.84
3	45.64	17.92	23.50
4	32.40	15.75	14.38
5	51.45	23.04	23.09
6	30.43	20.72	8.04
7	30.60	15.22	13.35
8	30.56	7.33	21.64
9	30.53	21.79	7.18
10	25.93	15.00	9.50
11	30.65	18.08	10.65
12	34.08	21.25	10.58
13	45.68	23.59	17.87
14	34.60	20.29	11.90
15	21.83	10.17	10.58

Table A.2: Covariances between 15 risky assets for single-period model

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0.02621	-0.00812	0.00994	-0.00037	0.00289	0.01033	-0.01316	-0.00829	-0.00191	-0.01273	-0.01084	-0.00382	0.00294	0.00551	0.00921
2		0.04443	-0.01242	0.01103	-0.00244	-0.00977	-0.02289	0.01379	0.01245	0.01078	0.00678	0.00009	-0.00955	-0.00872	-0.02098
3			0.03384	-0.00254	0.00209	0.00841	-0.00461	-0.01406	-0.00847	-0.00664	-0.01809	0.00414	-0.00806	-0.00344	0.00276
4				0.03783	0.00996	-0.01381	0.00379	0.01292	0.00990	0.01714	-0.00248	-0.00151	-0.02303	-0.01463	-0.01316
5					0.03014	-0.00065	-0.00158	-0.00504	-0.00804	0.00440	0.00426	-0.00623	-0.01824	-0.01555	-0.01517
6						0.02477	0.00233	-0.00108	0.00038	-0.01422	0.00448	-0.00344	0.00956	0.00835	-0.00208
7							0.04322	0.01274	-0.00409	0.00848	0.01916	0.00241	-0.00265	0.01165	0.00850
8								0.03017	0.02030	0.00829	0.02396	0.00581	-0.01188	0.00334	-0.00935
9									0.02881	0.00453	0.00859	0.00582	-0.00327	-0.00774	-0.01392
10										0.02780	0.00274	0.00613	-0.02041	-0.00872	-0.01279
11											0.04335	-0.00341	0.00128	0.00449	-0.00937
12												0.02223	-0.01675	0.00608	0.00514
13													0.04876	0.00860	0.01360
14														0.03934	0.01984
15															0.03849

Table A.3: Covariances between 15 risky assets for rebalancing model $[t_0, t_1]$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0.00523	-0.00219	0.00261	-0.00012	0.00081	0.00311	-0.00376	-0.00242	-0.00045	-0.00419	-0.00308	-0.00083	0.00073	0.00128	0.00277
2		0.01620	-0.00440	0.00472	-0.00093	-0.00398	-0.00885	0.00544	0.00397	0.00479	0.00261	0.00003	-0.00322	-0.00274	-0.00854
3			0.01168	-0.00106	0.00077	0.00333	-0.00173	-0.00540	-0.00263	-0.00287	-0.00676	0.00119	-0.00264	-0.00105	0.00109
4				0.01898	0.00445	-0.00660	0.00172	0.00598	0.00371	0.00894	-0.00112	-0.00052	-0.00911	-0.00540	-0.00628
5					0.01199	-0.00027	-0.00064	-0.00208	-0.00268	0.00204	0.00171	-0.00192	-0.00643	-0.00511	-0.00645
6						0.01128	0.00101	-0.00047	0.00014	-0.00707	0.00193	-0.00113	0.00360	0.00294	-0.00095
7							0.01769	0.00533	-0.00138	0.00400	0.00781	0.00075	-0.00095	0.00389	0.00366
8								0.01288	0.00701	0.00399	0.00996	0.00185	-0.00434	0.00114	-0.00412
9									0.00804	0.00176	0.00289	0.00150	-0.00096	-0.00213	-0.00495
10										0.01508	0.00129	0.00221	-0.00840	-0.00335	-0.00635
11											0.01757	-0.00106	0.00046	0.00149	-0.00402
12												0.00529	-0.00457	0.00155	0.00169
13													0.01522	0.00250	0.00512
14														0.01069	0.00697
15															0.01748

Table A.4: Covariances between 15 risky assets for rebalancing model $[t_1, t_2]$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0.01254	-0.00333	0.00400	-0.00013	0.00102	0.00403	-0.00540	-0.00323	-0.00089	-0.00450	-0.00448	-0.00183	0.00123	0.00256	0.00383
2		0.01564	-0.00428	0.00329	-0.00074	-0.00327	-0.00806	0.00460	0.00499	0.00327	0.00240	0.00004	-0.00343	-0.00348	-0.00748
3			0.01143	-0.00074	0.00062	0.00276	-0.00159	-0.00460	-0.00332	-0.00197	-0.00628	0.00167	-0.00284	-0.00134	0.00096
4				0.00957	0.00256	-0.00392	0.00113	0.00365	0.00336	0.00441	-0.00074	-0.00053	-0.00702	-0.00495	-0.00398
5					0.00784	-0.00019	-0.00048	-0.00145	-0.00277	0.00115	0.00130	-0.00220	-0.00564	-0.00534	-0.00465
6						0.00790	0.00078	-0.00034	0.00015	-0.00410	0.00151	-0.00135	0.00327	0.00317	-0.00071
7							0.01523	0.00426	-0.00164	0.00257	0.00680	0.00099	-0.00095	0.00465	0.00303
8								0.00954	0.00770	0.00238	0.00805	0.00226	-0.00405	0.00126	-0.00316
9									0.01313	0.00156	0.00346	0.00272	-0.00134	-0.00351	-0.00565
10										0.00726	0.00084	0.00217	-0.00632	-0.00300	-0.00393
11											0.01547	-0.00141	0.00046	0.00181	-0.00336
12												0.01068	-0.00703	0.00284	0.00214
13													0.01790	0.00350	0.00495
14														0.01780	0.00802
15															0.01390