

ABSTRACT

CLARK, MATTHEW RANDALL. Using Numerical Comparison Problems to Promote Middle-School Students' Understanding of Ratio as an Intensive Quantity. (Under the direction of Sarah B. Berenson and Eric N. Wiebe.)

The purpose of this qualitative study is to investigate middle-school students' understanding of their notation for ratios and to determine, through semi-structured task-based interviews, possibilities for using numerical comparison problems to promote their growth in understanding. The issue of concern is that students use fractional representations of ratios as a convenient notation for solving missing-value problems, but when they use this notation to solve numerical comparison problems, they are unable to interpret and compare the ratios as intensive quantities.

Patterns are reported for students' notation, their problem-solving strategies, their expressions of extensive and intensive quantities, and their use of contextual elements from the problem. A model of ratios and fractions based on Venn diagrams and a general model of ratios and other number-type domains provide the framework for charting students' activities and explanations.

Conjectures based on the data include an association between crossmultiplication and decontextualization and a hierarchy of number types that illustrates students' relative ability to interpret the number types as intensive quantities. Conclusions from the study include recommendations for using numerical comparison problems to give students at different sublevels of quantitative reasoning a stronger conceptual foundation for ratio-related topics. The study demonstrates that students in middle school can make progress in the short term at solving numerical comparison problems using comparisons based on

both extensive and intensive quantities, but in response to this short-term intervention, the students demonstrated limited transfer of knowledge across problems.

USING NUMERICAL COMPARISON PROBLEMS TO PROMOTE
MIDDLE-SCHOOL STUDENTS' UNDERSTANDING OF RATIO AS AN
INTENSIVE QUANTITY

by
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BIOGRAPHY

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INTRODUCTION

Evidence from international studies suggests that although students in the United States perform well in mathematics through the fourth grade, their performance by the eighth grade slips to below the international average (Sowder et al., 1998). In the intervening middle grades, rational-number concepts and operations are the dominant topics in mathematics courses (Conference Board of the Mathematical Sciences [CBMS], 2001, chap. 8). As Confrey (1998) stated, by “not mastering the concepts of fractions, ratio and proportion, decimals, and percents, many students are at risk in an ‘algebra for all’ middle school movement of becoming part of the failure statistics” (p. 39).

A major concern of researchers who focus on middle-school mathematics education is that students’ understanding is often too syntactical in nature and limited in terms of the underlying concepts (Case, 1988). A suspected cause is that students are taught formal notations and strategies before they have sufficient opportunity to construct a conceptual foundation for using them (Cramer et al., 1993; Kaput & West, 1994; Lo & Watanabe, 1997). One notation that students learn in middle school is writing ratios as fractions, and fractional notation for ratios is most often presented in textbooks as the preferred notation (Clark, Berenson, & Cavey, in press). However, Clark and Berenson (2002) found that at least on a particular problem, of a type that Lamon (1993) refers to as an associated-sets problem, middle-school students who used non-fractional notation performed significantly better than those who used fractional notation.

The research topic for this study is the nature of middle-school students’ understanding of ratio, with a focus on their understanding of fractional representations

of ratios, and how they use this notation in their problem-solving activities. The instructional goal is helping students develop an understanding of fractional representations of ratios and a flexibility with these representations that enables them to solve a variety of unfamiliar problems.

The Research Domain

In this section, I identify the overall context of the mathematical subject matter by beginning with multiplication-related topics and narrowing the focus to proportional reasoning and then to ratios.

Multiplicative Conceptual Field

Vergnaud (1994) rejected the notion that reasoning related to ratios, fractions, multiplication, division, and proportions could be condensed into a framework based on information processing or linguistics. Instead, he offered the theory of conceptual fields, emphasizing knowledge acquisition as dependent on situations and problems and stressing the “local features” (p. 42) of that acquired knowledge. The multiplicative conceptual field (MCF), according to Vergnaud (pp. 57-58), comprises a set, or “bulk,” of situations for which a student uses multiplication, division, or a combination of the two; a set of schemes, which he defines as “invariant organizations of behavior for well-defined classes of problems,” that students use in these situations; a set of concepts and theorems that students use to plan and analyze their activity; and a set of formulations and symbolizations.

The practical implication of this theory is that researchers and teachers have to organize and experiment with situations that promote both the short-term development of

new competencies and conceptions related to MCF topics and the long-term development of a foundation for future learning (Vergnaud, 1988). Researchers who use an MCF-based framework recommend that instruction be oriented to giving students the opportunity to develop concepts for MCF topics “not in isolation but in concert with each other over long periods of time through experience with a large number of situations” (Lo & Watanabe, 1997, p. 217). Vergnaud’s ideas apply to my proposed study because, even though my focus is on students’ understanding of ratios and fractional representations of ratios, their understanding of multiplication, division, and decimals can facilitate or inhibit their progress at interpreting fractional representations of ratios.

Multiplicative reasoning, according to Singh (2000), is the “entry point to the world of ratio and proportion” (p. 273). It is this world of ratio and proportion where I focus my attention for the proposed study.

Proportional Reasoning

Proportional reasoning, “at the heart of middle grades mathematics” (Ben-Chaim, Fey, Fitzgerald, Benedetto, & Miller, 1998, p. 249), is a type of reasoning that students should develop in the fifth through eighth grades (CBMS, 2001; National Council of Teachers of Mathematics [NCTM], 2000). Proportional reasoning, as defined by Karplus, Pulos, and Stage (1983), is “reasoning in a system of two variables between which there exists a linear functional relationship” (p. 219). Its pivotal role in middle school was highlighted by Lesh, Post, and Behr (1988) in an often-quoted passage in which they called proportional reasoning the “capstone” of elementary-school arithmetic and the “cornerstone of all that is to follow” (p. 94).

As Tourniaire and Pulos reported in 1985, proportional reasoning became the subject of intense research efforts after Inhelder and Piaget (1958) theorized that proportional reasoning is a distinguishing feature of the formal operational stage of development. Since the Tourniaire and Pulos summary of research to date, proportional reasoning has remained a research emphasis due to the continuing problems that middle-school and even high-school students have with ratio and proportion (Carraher, 1996; CBMS, 2001; Sowder et al., 1998). According to Post, Cramer, Behr, Lesh, and Harel (1993), one major problem with instruction on proportions is that the crossmultiplication algorithm is taught too soon and as a result drives a wedge between operations and mathematical meaning. However, researchers (Ben-Chaim et al., 1998; Lamon, 1993, 2001; Singh, 2000) have demonstrated that middle-school students can develop mathematical meaning for ratio and proportion given appropriate instruction and time.

Clark et al. (in press) found that middle-school textbooks are remarkably similar in presenting fractions as the preferred notation for ratios, a convenient notation for applying the crossmultiplication algorithm to missing-value problems. But what is the nature of students' understanding of this notation? Von Glasersfeld's (1996) remarks seem appropriate for this topic: "In school...mathematical symbols are often treated as though they were self-sufficient and no concepts and mental operations had to accompany them" (p. 312). He concludes, "When students are only trained to manipulate marks on paper, it is small wonder that few of them ever come to understand the meaning of what they are doing and why they should do it." Investigating middle-school students' work on a problem for which the students needed to interpret ratios, Clark and Berenson

(2002) found that those who represented the ratios as fractions were not as successful as those who did not.

Ratios

Thompson (1994) defines *ratio* as “the result of comparing two quantities multiplicatively” (p. 190). It is this definition that best matches my broad use of the term for this study. A ratio, according to Thompson, can be a multiplicative comparison of two complete collections of objects or can be a comparison of one of the quantities in terms of one item of the other collection, what is often called a *unit rate*. He points out that although many attempts to distinguish *rate* and *ratio* as terms seem to be valid, there is no conventional distinction between these terms, and he argues that the inconsistencies in terminology, both in the classroom and in the mathematics-education research literature, have created a great deal of confusion. Often the distinction between the terms is based on the nature of or the relationship between the units of the quantities in the ratio, but Thompson emphasizes the mental operations, which are the same regardless of the measure spaces, and concludes that a precise classification scheme is not important. In this study, I use “ratio” to refer to a wide variety of multiplicative relationships and avoid using “rate” as a stand-alone term; however, I do use “unit rate” because it refers to a specific form of a ratio that is associated with a problem-solving strategy.

Rejecting the common curricular sequence of the four basic operations followed by fractions followed by ratio and proportion, many researchers (Confrey, 1994; Kaput & West, 1994; Lo & Watanabe, 1995) recommend that initial instruction in ratio and proportion, with a basis in students’ informal strategies, begin as early as the third grade. Sophian and Wood (1997) claimed that even younger students are able to progress in

their understanding of ratios, given an appropriate activity. According to Lo and Watanabe (1997), “If the goal for teaching ratio and proportion is more than introducing the cross-multiplication algorithm, we do not see any reason to wait to introduce ratios and proportions until after instruction on fractions, as is typically done” (p. 234).

A glaring omission in middle-school mathematics is the lack of opportunity for students to build connections between ratios and fractions (Carraher, 1996; Sowder et al., 1998). A conjecture by Clark et al. (in press) is that an in-depth understanding of fractional representations of ratios depends on a conceptual convergence of ratio and fraction to form this connection, which seems unlikely if fractions are treated as a prerequisite topic and instruction in ratio is based entirely on fractional notation. Kaput and West’s (1994) conclusion expresses the need for further research on students’ understanding of ratios:

The rush to put formal computational tools in students’ hands before they understand their quantitative foundations is one major factor in the widespread incompetence and alienation from mathematics among students across the nation. Although the tradition of teaching ratio reasoning in the formal style is very long-lived, we should not assume that it should be venerated, or continued. The preponderance of data indicates that it does not work and may in fact do actual harm. (pp. 284-285)

The purpose of the proposed study is to examine students’ understanding of ratio as they are about to begin Algebra I, after they have experienced classroom instruction in ratio and proportion and at a time when they need a sufficient understanding of linear relationships to be able to represent them with tables, graphs, words, and symbols

(NCTM, 2000). Students' understanding of ratio will serve as the foundation for learning about rate of change and linear equations in Algebra I (NCTM). Many researchers (Kaput & West, 1994; Lamon, 1993; Singh, 2000) examined sixth graders' understanding of ratio and proportion at the time in school when students are traditionally introduced to these topics, but as research suggests, the present curriculum is failing at teaching ratio-related concepts to middle-school students (Sowder et al., 1998). Gathering more information from seventh and eighth graders could help researchers and teachers learn more about what students need in earlier grades to develop a deeper understanding of ratios.

Classifications

In this section I define and contrast pairs of terms that I use throughout the dissertation.

Conceptual Versus Procedural Understanding

Using Stieg Mellin-Olson's terms, Skemp (1976) distinguished *instrumental understanding*, knowing "rules without reason" (p. 20), and *relational understanding*, which includes understanding why something is done. A similar classification is that of Hiebert and Lefevre (1986), who defined *conceptual knowledge* as a web of knowledge "rich in relationships" (p. 3), or what Eisenhart et al. (1993) call knowledge about underlying mathematical structures, and *procedural knowledge* as the knowledge about the language, symbols, and algorithms. Hiebert and Lefevre created two distinct subcategories of procedural knowledge—knowledge of the formal language and symbols of mathematics and knowledge of the algorithms and rules to complete mathematical

tasks. A major problem with middle-school students' mathematical knowledge, according to Wearne and Hiebert (1988), is that it "appears to be more procedural than conceptual, procedural in the sense that the syntax of the system, rather than the underlying concepts, guides students' responses" (p. 220).

However, as Silver (1986) argued, the primary focus in mathematics education should be on the complex interactions between elements of conceptual and procedural knowledge, not simply on the classifications of such knowledge. The relationship between the development of conceptual and procedural knowledge continues to be a major research focus in mathematics education (Rittle-Johnson, Kalchman, Czarnocha, & Baker, 2002). One aspect of the proposed study is to research the use of a particular problem type in promoting the conceptual understanding of ratios in fractional form and promoting students' connecting of the concepts to procedures they have previously learned.

Additive Versus Multiplicative Reasoning

Progression from a reliance solely on additive strategies to the ability to discriminate between situations for which an additive strategy is appropriate and those for which a multiplicative strategy is appropriate is a "hallmark" (CBMS, 2001) of students' development in fifth through eighth grades. Because a common problem is that students use additive reasoning in situations that require multiplicative reasoning, experts (Lamon, 1993; Sowder et al., 1998) suggest that teachers design activities that create conflict between students' additive and multiplicative strategies in an attempt to promote the transition from using only additive reasoning to using both appropriately. An example of a problem that can be solved with both additive and multiplicative strategies is the Tree

problem that the students in this study answered on their test (see Problem 9 in the Appendix).

Many researchers (Clark & Kamii, 1996; Confrey, 1994; Schwartz, 1988; Steffe, 1994) warn against the heavy emphasis of teaching multiplication based on the idea of repeated addition. According to Confrey, models of repeated addition do not adequately represent certain actions of children and therefore may be inconsistent with their informal knowledge related to MCF concepts.

Internal Versus External Representations

When I discuss fractions as representations of ratios I am referring to external representations, specifically the way that students represent ratios in written form. However, internal representations—how students represent ratios mentally—although not directly observable, is an even more important issue in the proposed study because of its impact on students’ problem-solving abilities and strategies. As Hiebert and Carpenter (1992) assume, there is a relationship between students’ internal and external representations and those internal relationships can be connected in useful ways that help students reason about problems. In addition, they assume that “the nature of external mathematical relationships influences the nature of internal mathematical relationships” and conversely that “the way in which a student deals with or generates an external representation reveals something of how the student has represented that information internally” (p. 66).

Extensive Versus Intensive Quantities

An *intensive quantity*, according to Schwartz (1988), is a “generalization of the notion of an attribute density” (p. 43), or in other words, a relational quantity, often

expressed with “per,” that does not depend directly on a simple measurement or count—what Schwartz calls an *extensive quantity*. As Singh (2000) described them, intensive quantities are not usually measures or counts but are “generated through the act of division” (p. 275). Schwartz gives the example of 5 fish as an extensive quantity and “5 fish for every 2 bicycles” (p. 44) as an intensive quantity. He stated that intensive quantities can be difficult for students to represent because the intensive quantity is invariant while the related extensive quantities are unspecified.

Schwartz (1988) explained the pedagogical challenge inherent in situations that involve an intensive quantity: As students multiply or divide they generate new quantities whose referent is different than those of the two original quantities. Using the intensive quantity of five pieces of candy per bag and the extensive quantity of six bags, for example, a student can multiply to obtain a product of 30 but must realize that the 30 now refers to pieces of candy, a different unit than those associated with the two numbers that the student multiplied. As Schwartz stated, “Multiplication and division are referent transforming compositions of quantity, and addition and subtraction are referent preserving” (pp. 47-48). He argued that teaching multiplication and division as repeated addition and subtraction, respectively, is a flawed strategy because it ignores this feature of multiplication and division—that they are referent transforming, not referent preserving.

One topic of interest in this study is whether students are able to construct the appropriate composite unit when computing, interpreting, and comparing intensive quantities. For example, when given a number of hours that it took a worker to fertilize a certain number of acres of grass, how do students describe the quotient, and are they able

to distinguish between hours per acre and acres per hour? I use the term *composite unit* to refer to the unit of an intensive quantity formed by the ratio relationship of two extensive quantities and the term *label* to refer to students' written expression of units, both for intensive and extensive quantities. My use of "composite unit" should not be confused with Steffe's (1994) use of this term by which he refers to a number composed of other numbers that a child can use as a unit without reconstructing it.

The Research Focus

In this section, I introduce the problem type that I have used in my research and list my research questions for the dissertation.

The Mathematical Problems

Within semi-structured one-on-one interviews with middle-school students, the teaching technique is primarily a problem-solving approach (Polya, 1948), which was recommended by Behr, Harel, Post, and Lesh (1992) to give students an "experiential base for internalizing the unit-conversion principles they will apply to the concepts of fraction, rational number, rate ratio, and proportion" (p. 306). Advances in problem solving are characterized by an "increase in flexibility" (Carpenter, 1986, p. 115), made possible by a stronger conceptual base, more efficient procedures, and the connections between conceptual and procedural knowledge. By presenting a series of problems to the students, I hope to encourage such an increase in flexibility as they interpret their notation, make connections between ratios and fractions, and revise their procedures.

Numerical comparison problems (Ben-Chaim et al., 1998) are problems for which the student is given complete information for two or more ratios, and to answer the

question, the student must compare those ratios, working with numerical values to make the comparison but without having to give a numerical answer. For example, the problem could specify the times and distances for two sprinters and ask which ran faster.

Providing various contexts for such problems is important because seventh and eighth graders tend to use different reasoning processes to compare ratios in a word problem than they do when comparing fractions in no particular context (Heller, Post, Behr, & Lesh, 1990).

According to Cobb, Yackel, and Wood (1995), “A situation in which a student’s current computational procedures prove inadequate can give rise to a problem, the resolution of which involves the construction of conceptual knowledge” (p. 18). On this basis, I chose numerical comparison problems as the focus of activity for the students in the study for two reasons: because in previous articles (Clark & Berenson, 2002; Clark et al., in press) I have explored and documented the inadequacy of students’ procedural knowledge and operations for solving this type of problem, and because the instructional goal embedded in the proposed study is to promote the conceptual understanding of ratio as an intensive quantity.

The Research Questions

In the proposed study, my goal is, through the interview and data-analysis process, to better understand the students’ understanding (Hunting, 1997) of ratios and to describe their learning experiences. Hiebert and Lefevre (1986) define a special type of conceptual knowledge that students develop through the “linking process...between two pieces of information that already have been stored in memory” (p. 4). I refer to this as a *retroconnective learning process*, the realization of a connection that creates a new

understanding and supports a broader understanding of concepts previously learned. In this case, the new understanding is of fractional representations of ratios, which requires a connection between the concepts of ratio and fraction.

Specifically, my research questions are as follows:

1. What is the nature of students' understanding of ratios, especially fractional representations of ratios, and what meaning, if any, do they associate with ratios in this form?
2. What characteristics of the numerical comparison problems—for example, whether variables are discrete or continuous and whether the smaller or larger number in a ratio is given first—influence students' notation, their strategy, and their likelihood of success?
3. Are students able to compute intensive quantities and construct the appropriate composite unit, and if they are not able to compare ratios based on intensive quantities, what strategies do students use to solve numerical comparison problems?
4. How do students grow in their understanding of ratios and intensive quantities during the problem-solving sessions, and are students successful at retroconnective learning, making connections between ratios and fractions to develop an understanding of ratio as an intensive quantity?

LITERATURE REVIEW

In this chapter, I provide a review of the literature, first in the context of stating my conceptual framework for this study and then for various MCF-related topics that are related to it.

Conceptual Framework

A “skeletal structure of justification,” according to Eisenhart (1991), a conceptual framework enables a researcher to blend ideas from many sources in stating the underlying assumptions for a study. She argues that conceptual frameworks are better suited than other types of frameworks, such as theoretical and practical frameworks, for educational research because researchers can focus on relevant aspects of a theory for a particular study and because a conceptual framework is less likely to bind them to a path that produces a theory-driven conclusion.

I have developed a conceptual framework for my proposed study, and in this section, I explain the following components and how they fit together to support my research:

- the emergent perspective (Cobb & Yackel, 1996)
- Lamon’s (1993) classification scheme for proportional reasoning
- a model of ratios and fractions, proposed by Clark et al. (in press), based on Venn diagrams.

The Emergent Perspective

According to Wood, Cobb, and Yackel (1995), “It is useful to see mathematics as both cognitive activity constrained by social and cultural processes and a sociocultural phenomenon that is constituted by a community of actively cognizing individuals” (p. 402). This point of view, which blends elements of psychological constructivism with elements of sociocultural theories, is often referred to as social constructivism, or the *emergent perspective* (Cobb & Yackel, 1996). From this perspective, mathematical meaning emerges from individual interpretation and taken-as-shared meaning.

In the framework of the emergent perspective, three pairs of categories are reflexively linked across the social and psychological dimensions. Classroom social norms enable and constrain how individuals change their beliefs about their role, others’ roles, and the nature of mathematical activity in the classroom, which, in turn, contributes to the evolving social norms of the classroom (Cobb, 2000). Sociomathematical norms, or a “criteria of values with regard to mathematical activities” (Voight, 1995), contribute to the classroom culture in which individual learning takes place and enable and constrain students’ changing mathematical beliefs and values, which, in turn, contribute to the evolving sociomathematical norms. Cobb gives what counts as an insightful solution, what characteristics are required for a solution to be considered different from another, and what counts as an acceptable mathematical explanation as examples of sociomathematical norms.

The third pair of categories—classroom mathematical practices, in the social dimension, and the individual’s mathematical conceptions, in the psychological dimension—is the pair of primary interest for this study. Classroom mathematical

practices are the communal aspect of the classroom that corresponds to individual's mathematical conceptions and activity (Cobb & Yackel, 1996). Explaining the reflexive relationship between the individual and communal, Cobb and Yackel stated, "Students actively contribute to the evolution of classroom mathematical practices as they reorganize their individual mathematical activities and, conversely, that these reorganizations are enabled and constrained by the students' participation in the mathematical practices" (p. 180).

The use of the emergent perspective as a research framework is usually associated with research in a classroom, where the observer is looking for evidence of individuals' learning within the context of the classroom culture and the progress of a class as a whole, composed of those individuals. For my research, with data collected through individual interviews, the psychological dimension is in the foreground and the social dimension, which is not directly observable, is in the background and only accessible through the participants' comments. As Cobb and Bauersfeld (1995) explained, "When the focus is on the individual, the social fades into the background, and vice versa" (p. 8).

In this study, I can only make inferences and conjectures about the classroom communities in which the students have participated. However, I believe that it is important to attempt to account for classroom influences, especially when analyzing students' mathematical beliefs and conceptions about notation, which are strongly influenced by classroom mathematical practices. Because each individual in my study comes from a different set of classroom experiences, I do not have sufficient data to consider a primary social analysis. However, each individual contributes to variability in

two ways, by her individual characteristics and by the characteristics of the mathematical communities that have influenced them.

Categories of Reasoning about Ratio and Proportion Problems

For my analysis, I use Lamon's (1993) standard for whether a student reasons proportionally: Proportional reasoning occurs when a student can "demonstrate understanding of the equivalence of appropriate scalar ratios and the invariance of the function ratio between two measure spaces" (p. 45). In this same paper, Lamon categorized students' strategies by semantic types of ratio and proportion problems. Table 1 shows the categories she used.

Table 1
Strategies for Solving Ratio and Proportion Problems (adapted from Lamon, 1993, p. 46)

Strategy	Characteristic
Nonconstructive strategies	
Avoidance	No serious attempt
Nonproportional visual or additive	Purely visual judgment (“It looks like...”)
	Responses without reasons
	Trial and error
	Incorrect strategy based on sums or differences
Nonproportional pattern building	Reliance on a pattern without any accompanying numerical relationship
Constructive strategies	
Preproportional reasoning	An intuitive sense-making activity, based on a picture, chart, or model, with some relative thinking
Qualitative proportional reasoning	The use of ratio and relative thinking with some understanding of numerical relationships in a qualitative sense
Quantitative proportional reasoning	The use of symbols to represent the relationships in the problem and full understanding of those relationships in a quantitative sense

One of the problems I used with the students in my study was one that Lamon (1993) used in her study with the sixth graders. She found that qualitative proportional reasoning was by far the most common strategy for this problem. One question of interest

is the following: After seventh grade, have students progressed to using quantitative proportional reasoning in their solutions of this problem?

The Overlapping Model of Ratios and Fractions

One potential connection that is the focus of the proposed study is the connection between the fractional representation of a ratio and the fraction as a rational number. A serious deficiency in middle-school mathematics is the lack of opportunity for students to make connections between ratios and fractions (Carraher, 1996; Sowder et al., 1998). From our work with teachers and with middle-school students, Clark et al. (in press) proposed a series of models based on Venn diagrams for representing possible interpretations of the relationship between ratios and fractions.

In Model 1, all ratios are fractions. Clark et al. (in press) represent Model 1 with the set notation $R \subset F$, which means that the set of ratios is contained in the set of fractions, but to be precise they note that $F \not\subset R$ because in Model 1 the sets are not identical. Figure 1 shows the Venn diagram for Model 1.

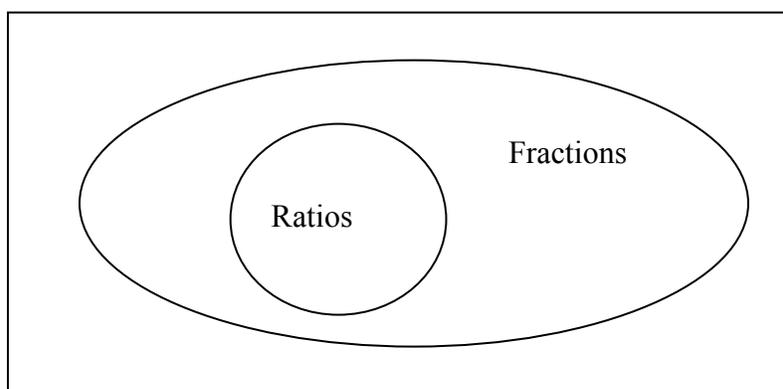


Figure 1. Ratios as a Subset of Fractions

According to Begle (1975), “A ratio is nothing but a special case of a fraction” (p. 255). Many mathematics-education textbooks published in the 1960s and 1970s use *fraction* as the shortened form of *fractional number*, synonymous with *rational number*

(Grossnickle & Reckzeh, 1973, p. 259). Some authors (for example, Brumfiel, Eicholz, Shanks, & O’Daffer, 1963, p. 145) make the distinction of *fraction* as the numeral (written notation) and *rational number* as the concept; others (for example, Willerding, 1966, p. 180) use *fraction* to refer to both the concept and the symbol.

The most common interpretation of Model 1 is that the set of all fractions is the set of all rational-number expressions, of which ratios compose a subset. But such a perspective would exclude from that subset commonly accepted ratios that are irrational, such as the “golden ratio,” the ratio of the hypotenuse to leg length of an isosceles right triangle, the ratio of the circumference of a circle to its diameter, and odds ratios in famous probability problems that involve e (Grinstead & Snell, 1997, pp. 85-86).

In Model 2, all fractions are ratios. With set notation, Clark et al. (in press) represent Model 2 by $F \subset R$ and for clarification add that $R \not\subset F$. Figure 2 shows the Venn diagram for Model 2.

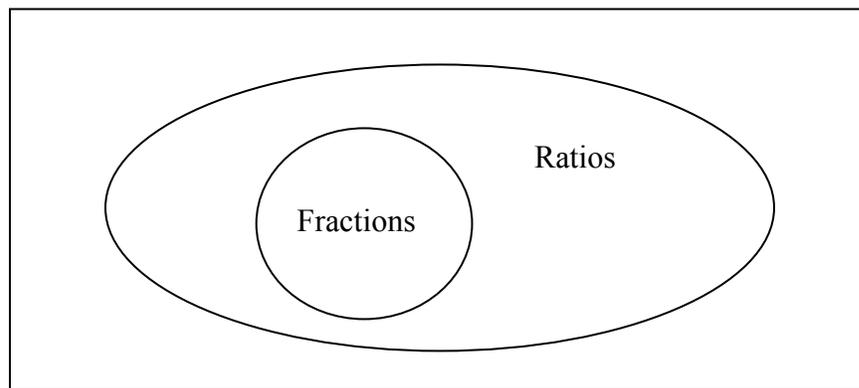


Figure 2. Fractions as a Subset of Ratios

“All fractions are ratios,” according to Van de Walle (1994, p. 275), but “it is not correct to say that all ratios are fractions.” In their activities with pre-service and in-service teachers, Clark et al. reported that Model 2 was the most popular choice. In individual written responses about the models, two thirds of pre-service teachers enrolled

in their first methods course selected Model 2 as best representing their understanding of ratios and fractions; however, their understanding of fractions, in most cases, was limited to part-whole comparisons, a common but incomplete interpretation (CBMS, 2001).

Model 2 was either the most popular choice or tied for first among experienced teachers, most of whom had a broader understanding of fractions and interpreted Model 2 as a ratio as a relative comparison, in a multiplicative sense, and a fraction as one notation for expressing that comparison.

In Model 3, ratios and fractions are separate, without any elements in common. In set notation, Model 3 is represented as $R \cap F = \emptyset$ (Clark et al., in press). Figure 3 shows the Venn diagram for Model 3.

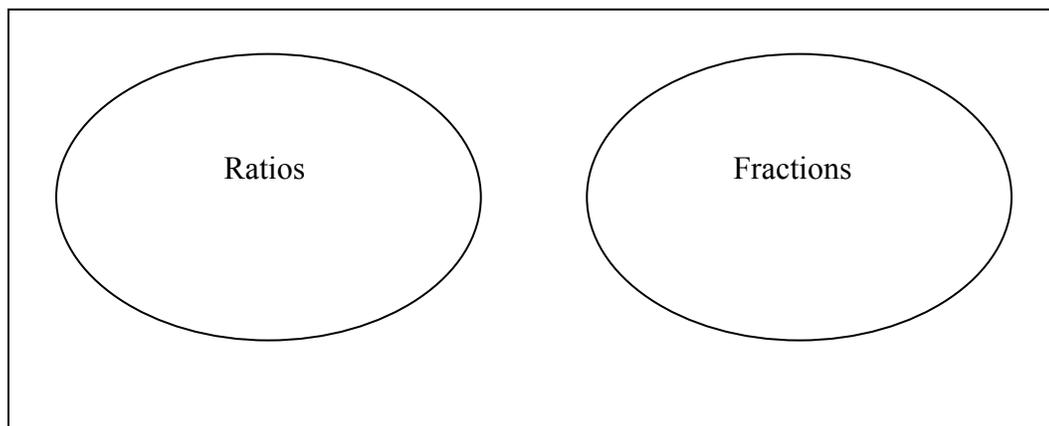


Figure 3. Ratios and Fractions as Distinct Sets

A common distinction is that a fraction represents a part of a whole, such as in Johnson's (1988) example of "one of two pieces" (p. 79), and a ratio represents "an amount compared to another amount" such as in his example of "one piece compared to two pieces." Two of the nine pre-service teachers mentioned by Clark et al. (in press) defined *fraction* as expressing a part-whole relationship and *ratio* as expressing a part-part relationship and chose Model 3.

In Model 4, some, but not all, ratios are fractions, and some, but not all, fractions are ratios. Clark et al. (in press) refer to this model as the “overlapping model.” Figure 4 shows the Venn diagram for Model 4.

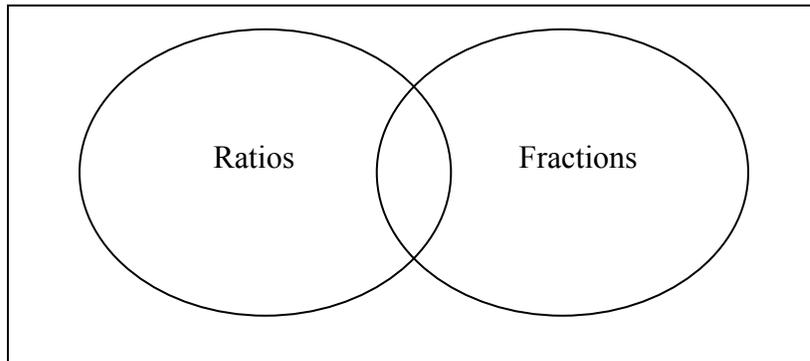


Figure 4. Ratios and Fractions as Overlapping Sets

Clark et al. reported an example of a group of teachers who defended their choice of Model 4 with the following examples: “1 cup sugar : 2 cups flour” in the ratio-only realm, “1 cup sugar / 3 cups ingredients” in the intersection, and “ $\frac{1}{2}$ cup sugar” in the fraction-only realm. Because of the three realms, Model 4 seems to be open to the widest variety of interpretation.

In Model 5, the terms have identical meanings. Using set notation, Clark et al. (in press) represent Model 5 as $R = F$. Figure 5 shows the Venn diagram for Model 5.

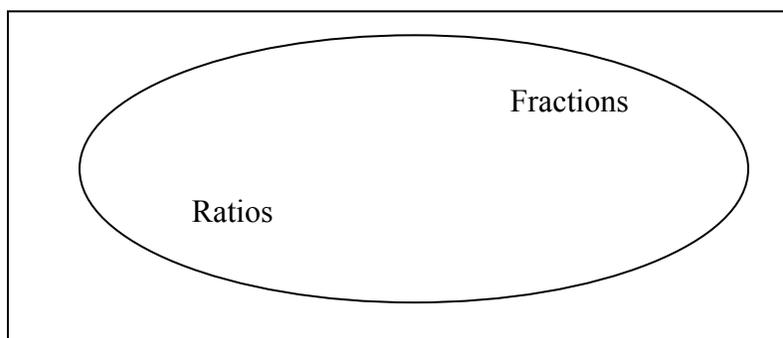


Figure 5. Ratios and Fractions as Identical Sets

Although Clark et al. reported finding no textbooks that introduce the terms as synonyms, we did find instances of the terms defined essentially the same way (for example, Washington & Triola, 1988).

Clark et al. (in press) chose the overlapping model as the one that best represents our interpretation of the relationship between ratios and fractions. One reason that the overlapping model is particularly useful for this study is that an interpretation of a fractional representation of a ratio as an intensive quantity qualifies as an understanding of the representation both as a ratio and as a fractional number. In the overlapping model, there are two possible connections for students to make across the boundaries—the connection between a non-fractional representation of a ratio (in sentence form or with the colon notation, for example) and a fractional representation of that ratio, and the connection between the fractional representation of the ratio and the fraction as a stand-alone rational number.

Kieren's (1993) theory of subconstructs of rational numbers provides a foundation for discussing how students might consider ratios and fractions relative to the overlapping model. According to Behr, Harel, Post, and Lesh (1993), when rational numbers are thought of in the context of problems, they take on "personalities" (p. 13) that are not captured by the strict mathematical definition of a rational number as an element of an infinite quotient field. Kieren identified four subconstructs of rational numbers: quotients, operators, measures, and ratios. Behr et. al (1992, 1993) also include part-whole as a separate rational-number subconstruct, and as noted by Ball (1993), Nesher has suggested probability as an additional subconstruct.

The following examples of rational-number subconstructs are taken from Lamon (2001). For the measure subconstruct, $\frac{3}{4}$ means the distance of 3 steps of the unit $\frac{1}{4}$ on a number line. For the operator subconstruct, $\frac{3}{4}$ means to multiply by 3 and divide the result by 4. For the part-whole subconstruct, $\frac{3}{4}$ means three parts out of a whole divided into 4 equal parts. For the quotient subconstruct, $\frac{3}{4}$ is associated with partitioning when four people share three of something evenly. For the ratio subconstruct, 3:4 describes a multiplicative relationship of three of something compared to four of something else.

Figure 6 shows the overlapping model of ratios and fractions, as proposed by Clark et al. (in press), with these various personalities or flavors of rational numbers placed along the Venn diagram.

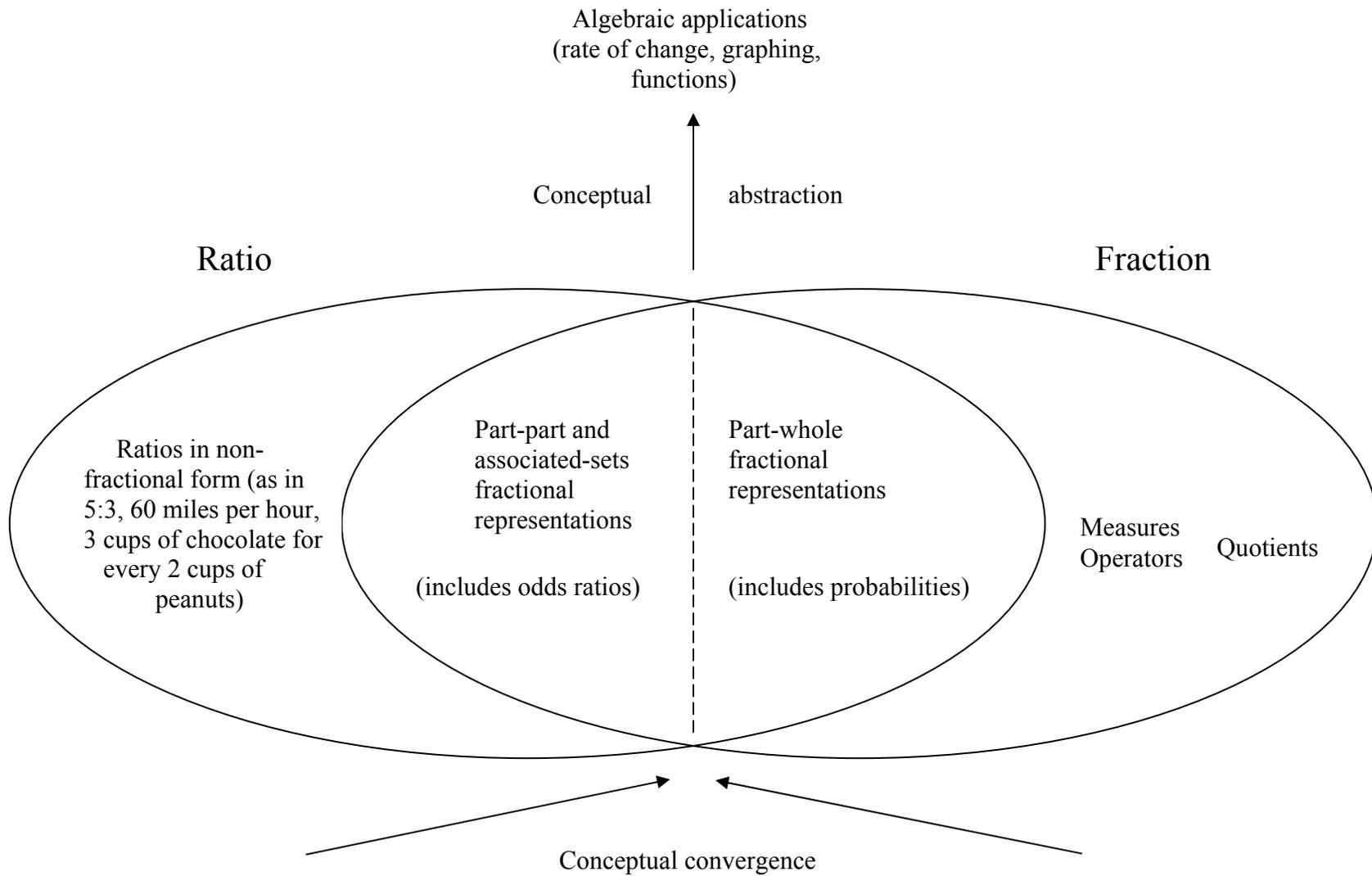


Figure 6. Subconstructs Within the Overlapping Model

In the overlapping model of ratios and fractions, the quotient subconstruct appears in the fraction-only realm, and the measure and operator subconstructs appear in the fraction-only realm as well but closer to the intersection because, depending on the context and phrasing of the problem, these two subconstructs are more likely to be interpreted as ratios. Probability is seen as a type of part-whole comparison and odds ratios as a type of part-part comparison. This view of the subconstructs within the overlapping model is at the granular level.

Pulling back several steps from that granular level provides a view of the generalized overlapping model of ratios and other number-type domains suggested by Clark et al. (in press). In our article, Clark et al. argue for a broader perspective of ratios, not one limited by their representations as fractions. Figure 7 shows how alternate number-type domains can be matched with ratio to form the same relationship of Model 4. Imagine the right side of Figure 7 as a set of lenses that can be rotated, as needed, when analyzing how a student expresses and interprets a type of number as a ratio and operates on that number.

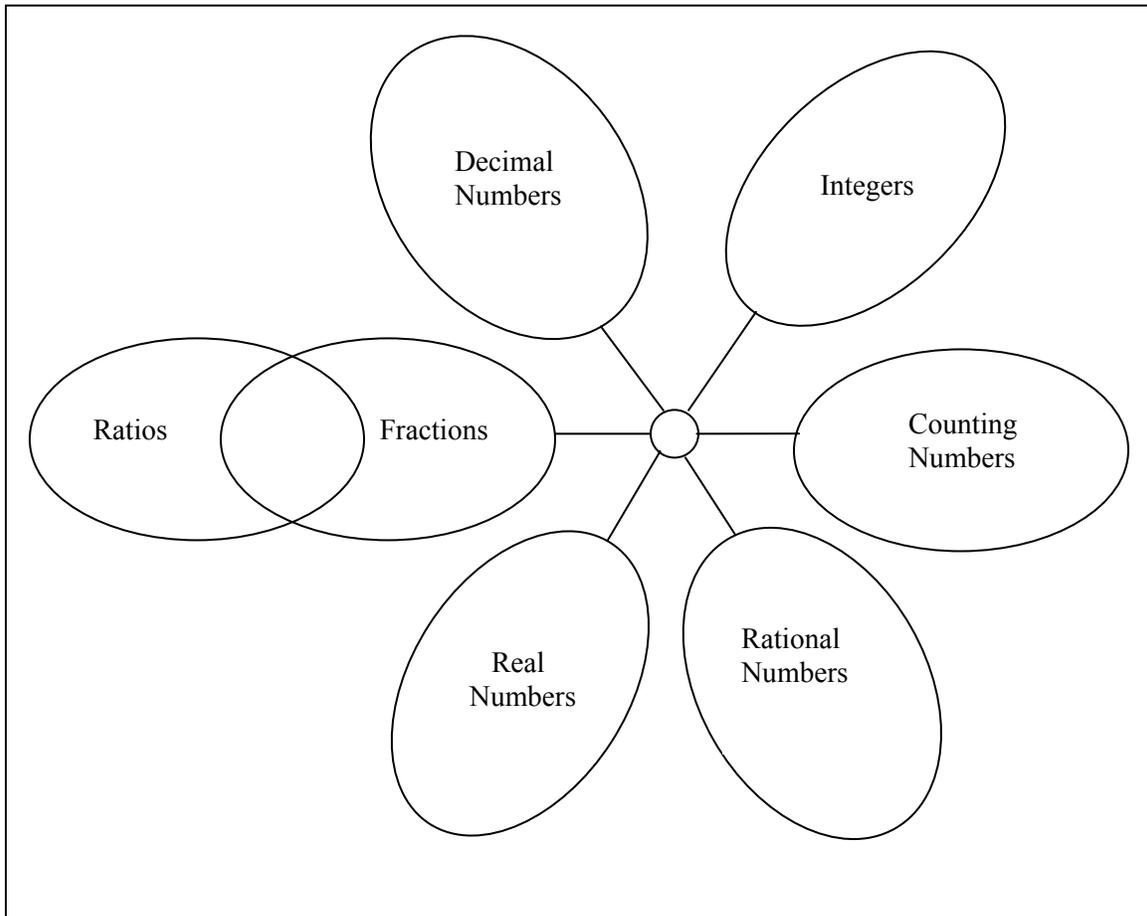


Figure 7. The Overlapping Model with Alternative Number-Type Domains

Clark and Berenson (2002) reported on the phenomenon of *decontextualization* of ratios, when students identify a ratio in a word problem, write the ratio in fractional form, and then perform operations on the fraction or on the numerator and denominator without considering the context of the problem from which those numbers came. For example, consider a numerical comparison that asks students to compare two runners—one who took 7 minutes to run 3 laps and one who took 8 minutes to run 5 laps. A potential error would be to write the ratios as $7/3$ and $8/5$, in fractional form; perform a procedure, such as crossmultiplication, long division, or conversion to fractions with a common denominator, to determine that $7/3$ is the larger; and conclude that the first runner ran

faster, without returning to the context of the problem to construct the composite unit for the intensive quantities. Figure 8 shows the overlapping model of ratios and fractions, and the path of decontextualization, as described in the example in this paragraph, is from left to right across the diagram.

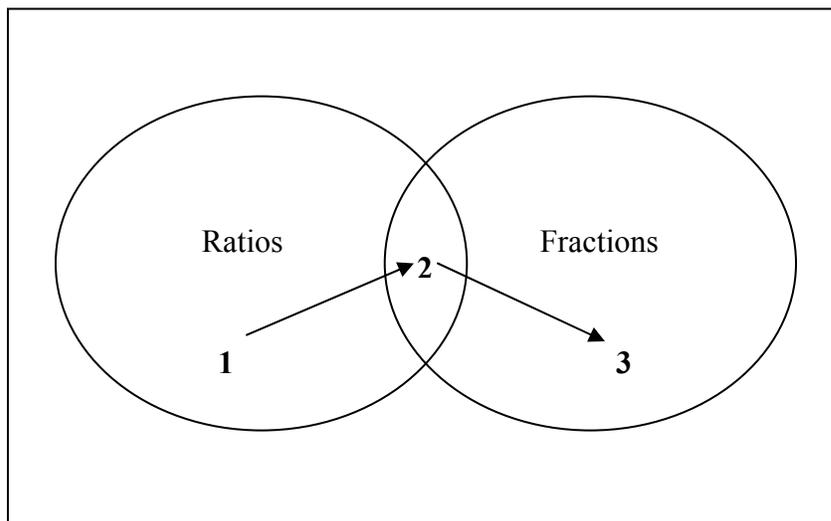


Figure 8. Path of Decontextualization across Model 4

At first, the student recognizes the multiplicative relationship between the pairs of numbers (1). Next the student writes the ratios as fractions (2). Finally, the student performs a procedure with the fractions as numbers (3), disconnected from the context of the ratio relationship.

Clark and Berenson (2002) found significant associations to provide support for a conjecture that seventh- and eighth-grade students who use non-fractional representations are more successful on numerical comparison problems than those who use fractional representations. Those who used non-fractional representations were also more likely to maintain contextual elements of the problem in their solution, such as labels or diagrams that refer to the information given in the problem. One promising sign is that the students who use fractional representations and maintain a written expression of the context in

their solution are more successful than those who use fractional representations and provide no context. Although only a result of association, this link prompted us to hypothesize that while teaching students to interpret and apply fractional representations of ratios, teachers might help students by encouraging them to write out contextual elements in their solutions.

Multiplicative Conceptual Field

In the first chapter, I defined the multiplicative conceptual field (MCF), as stated by Vergnaud (1994). The first step in using his MCF theory, according to Vergnaud, is to “account for the knowledge contained in most ordinary actions, those performed at home, at work, at school, or at play” (p. 44). Rejecting the separation between “real-life” mathematics and “school” math, Vergnaud explained that any problematic situation found in real life could potentially be used in a classroom context to add to the variety of situations that students must encounter to develop MCF concepts.

According to MCF theory, students master some classes of situations long before they master others (Vergnaud, 1994, p. 46). This is consistent with research results on proportional reasoning that have demonstrated that students are able to solve problems with integer ratios before non-integer ratios (Tourniaire & Pulos, 1985), for example.

For my study, MCF theory is relevant because of the wide range of knowledge of various topics that impact students’ progress on the activities. The problems require students to compare ratios, which they often represent as fractions, but their knowledge of decimal numbers, division, common denominators, and other MCF topics can either facilitate or inhibit their progress and are thus essential to the analytical process. As

Vergnaud (1988) recommended, it is “not wise to study the learning and teaching of fractions, ratios, and rates independently of multiplicative structures” (p. 158).

Learning with Understanding and Meaningful Learning

An early proponent of meaningful learning in the 1930s and 1940s, William Brownell suggested that teachers teach with the aim of students truly understanding the mathematical content. Concerned that teachers had misinterpreted his message, he later stressed students developing both an understanding of the material and proficiency in computation (Brownell, 1956). Brownell’s goal is often expressed today in terms of Hiebert and Lefevre’s (1986) definitions of conceptual and procedural knowledge, which I discussed in the previous chapter.

According to Ausubel (1963), meaningful learning occurs when the learner is able to “relate substantive (as opposed to verbatim) aspects of new concepts, information, or situations to relevant components of existing cognitive structure in various ways that make possible the incorporation of derivative, elaborative, correlative, supportive, qualifying, or representational relationships” (p. 22). Ausubel distinguished between factors related to the material and factors related to the learner in specifying whether learning is potentially meaningful. For learning to be potentially meaningful, the new material must be related in a nonarbitrary way to concepts previously learned, which is a characteristic primarily of the material itself, and the new material must be compatible with the existing cognitive structure of an individual learner, which Ausubel defined as “an individual’s organization, stability, and clarity of knowledge in a particular subject-matter field at any given time” (p. 26). According to Ausubel, if learning is meaningful, it

is more likely to be achieved more quickly, to be transferred to new situations, and to be retained over time.

The word “understanding” is most frequently used to describe the learner’s state when new information is appropriately connected to existing knowledge (Hiebert & Lefevre, 1986). A goal that is widely accepted by mathematics-education researchers is that students should learn mathematics with understanding (Hiebert & Carpenter, 1992). This goal is the basis for the NCTM standards (NCTM, 2000). Hiebert and Lefevre point out how closely related the terms “understanding,” “meaningful learning,” and “conceptual knowledge” are: “The heart of the process involves assimilating (Piaget, 1960) the new material into appropriate knowledge networks or structures” (p. 4). “Conceptual knowledge” refers to the substance, “meaningful learning” to the process, and “understanding” to the resulting state.

A critical feature of a network of knowledge is the connections between ideas, facts, or procedures (Hiebert & Carpenter, 1992). “Thinking mathematically involves looking for connections, and making connections builds mathematical understanding” (NCTM, 2000, p. 274). The teacher’s role, according to NCTM, should be to emphasize mathematical connections to “help students build a disposition to use connections in solving mathematical problems, rather than see mathematics as a set of disconnected, isolated concepts and skills” (p. 277). NCTM’s standards stress the alliance of factual knowledge, procedural proficiency, and conceptual understanding needed for students to become successful in their math classes—a goal that has been expressed by researchers and teachers since Brownell warned in 1956 that one type of knowledge is not sufficient.

Recent Research on Teaching MCF Topics

According to von Glasersfeld (1996), “The need for an experiential basis for the abstraction of concepts is often overlooked” (p. 312). Discussing the importance of students’ experiences with various subconstructs in their development of a complete understanding of rational numbers, Behr et. al (1993) claimed that researchers must “explore children’s ability to acquire knowledge of these personalities and determine what their informal knowledge of these personalities is” (p. 13). Informal knowledge, as described by Mack (1990), is knowledge that is applied, circumstantial, and “can be drawn upon by the student in response to problems posed in the context of real-life situations familiar to him or her” (p. 16).

In Mack’s (1990) study, she used students’ informal knowledge of fractions as a basis for teaching them how to add and subtract fractions and how to compare fractions. Another example of research of an instructional program to teach an MCF-related topic is a study by Moss and Case (1999), in which they reported success in using students’ informal knowledge of percents as a basis for teaching decimal notation. Confrey (1994) suggested that students be introduced to multiplication, division, and ratio as a trio early in elementary school and that this instruction be based on their informal knowledge of splitting activities, such as folding and sharing. Lamon (2001) found that basing rational-number instruction on children’s informal knowledge of measurement was more successful than the traditional focus on part-whole interpretation.

After basing initial instruction on students’ informal knowledge, these researchers have attempted to help students construct knowledge of more advanced concepts by basing instruction on what the students previously learned in addition to students’

informal knowledge. For example, Lachance and Confrey (1995, 2002) based their instruction of decimal notation on what the students previously learned about the trio of multiplication, division, and ratio in addition to their informal knowledge of splitting activities. In Lamon's (2001) study, after students developed an in-depth understanding of one subconstruct of rational number, such as measure or operator, she based instruction on other subconstructs on students' understanding of the initial subconstruct that they had experienced in depth.

Ben-Chaim et. al (1998) tested the effectiveness of the Connected Mathematics Project (CMP), a middle-school curriculum based on problem solving, versus a control group. In the CMP group, students were to develop their own procedures for solving proportion-related problems; in the control group, students observed the teacher solving example problems and then practiced solving similar problems using the methods they had learned from the teacher and the textbook. The researchers classified proportional-reasoning problems according to three general categories: missing-value value problems, where three pieces of information about the proportional relationship are given and the student has to compute the fourth; numerical comparison problems, which is the problem type for my study; and qualitative prediction and comparison problems, which require the students to make general comparisons and predictions not based on specific numerical values. This third category of problem is sometimes referred to as *directional questions* (Heller, Post, Behr, & Lesh, 1990) because the student is often asked if a result would be greater than, less than, or the same as a prior result. For example, if a runner runs more laps in less time than he did yesterday, the student would be able to determine that the

runner ran faster the second day, without knowing anything about the specific distance, time, or speed.

Ben-Chaim et. al (1998) found that in both the experimental and control groups students performed better on missing-value problems than on numerical-comparison problems. Overall, the students in the CMP group performed better on a written exam than those in the control group. Based on follow-up interviews, the researchers concluded that the students in the CMP were also more successful at providing explanations for their solutions. They hypothesized that the problem-solving approach of the CMP curriculum, with the emphasis on classroom discourse, might be the factor that best explains the superior performance of the CMP group.

Proportional Reasoning

In addition to Lamon's categories for reasoning strategies, I use common categories for numeric approaches that students use to solve ratio and proportion problems. These categories are important for the proposed study because the numerical comparison problems that I use are, in general, too hard for students to solve intuitively and therefore require some numerical strategy.

According to Karplus, Pulos, and Stage's (1983) classification system for numeric approaches to missing-value proportion problems, a *between* strategy involves comparing the given components of two ratios, establishing that multiplicative relationship, and applying the relationship to the other given number in one of the ratios to find a value for the other ratio; and a *within* strategy involves comparing the numbers of one ratio, establishing that multiplicative relationship, and applying the relationship to the other

ratio. Karplus, Pulos, and Stage concluded that neither of these strategies seems more natural for students and that factors such as the problem context, the numerical content of the problem, and the preceding task influence which strategy a student uses.

A problem that I wrote (Clark, 2001) and have used many times with teachers to illustrate strategies for solving missing-value problems is the following:

If you ride a bike 5 miles in 20 minutes and continue at a constant speed, how far will you travel in a total of 60 minutes?

A student applying a between strategy would see that 60 minutes is 3 times as long as 20 minutes and conclude that the missing distance, 15 miles, would also be 3 times as long as the given distance for 20 minutes. A student applying a within strategy would realize that going 5 miles in 20 minutes is a mile every 4 minutes and divide 60 minutes by 4 minutes per mile to get 15 miles. A student crossmultiplying would set up the proportion, multiply 60 by 5 and 20 by the unknown, and divide 300 by 20 to get 15.

Another common proportional-reasoning strategy, particularly among children who are just beginning to reason proportionally, is one often called *building up* (Tourniaire & Pulos, 1985), in which a student establishes a relationship between two extensive quantities and alternately iterates, or builds up, until reaching an established target. For example, if a child is asked, “If you can buy 3 cans of soda for 75 cents total, how much would 12 cans cost?” the child might build up to 12 cans, counting 3 cans for 75 cents, 6 cans for a \$1.50, 9 for \$2.25, and 12 for \$3. The building-up strategy is most commonly used by children when all the numbers are integers (Tourniaire & Pulos).

As reported by Ben-Chaim et al. (1998), a common error on numerical comparison problems is the switching of the order of units in the ratio, such as when a

student tries to work with miles per hour but actually computes hours per mile. As I began my study, I wondered if students could learn the reversibility of a rate, that it can be expressed as either amount of one unit per the other, and how this reversibility relates to fractions and their multiplicative inverses. The main reason I chose numerical comparison problems is that students must interpret and compare ratios in the context of the problem, sometimes choosing the smaller number and sometimes the larger one for their answer, depending on what the question asks; therefore, these problems cannot be solved by any procedure that students have learned to determine which fraction is the larger or smaller number.

Because of the lack of students' conceptual understanding associated with the crossmultiplication procedure for missing-value proportion problems, many researchers (for example, Post, Cramer, Behr, Lesh, & Harel, 1993) have suggested that the unit-rate method, a type of within strategy by which the student computes how many units of one variable are associated with one unit of the other variable within the given ratio and then multiplies, shows the most promise. However, Singh (2000) warned that in our rush to overturn the teaching of the crossmultiplication procedure we run the risk of replacing one procedure, crossmultiplication, that students do not understand with another procedure, the unit-rate procedure, that students do not understand either. As an example of a potential pitfall, Singh mentioned the case of a sixth-grade girl who relied so heavily on the unit-rate method that when her procedure gave a result that conflicted with her number sense she reverted to additive reasoning. Singh stressed the goal of students achieving flexibility—an ability to choose a method that is best suited to the problem—and offered the following question as an example: If 8 cups of flour are needed to make

14 donuts, how many donuts can you make with 12 cups of flour? A more efficient solution than using the unit rate of one cup for $14/8$ donuts, Singh argues, is to see 4 cups of flour, which are needed to make 7 donuts, as the unit and to iterate this unit three times. Singh concluded that the unit-rate method should not be taught until students have a good grasp of unit-coordination schemes, such as iterating the composite unit in the donut problem.

Another argument against relying too heavily on the unit-rate method was made by Christou and Philippou (2002), who found that fourth- and fifth-grade students' exclusive focus on finding an emphasized unit rate distracted them from finding the unit rate when it made more sense to reverse the roles of the variables. When given missing-value problems involving dollars and pencils, the students in their study who relied on the unit-rate method tried to compute the cost per pencil instead of figuring out how many pencils could be bought for one dollar and applying that within relationship to find the missing value. An instructional goal for this study is for students to develop this flexibility to define a ratio of one variable in terms of another and to define its multiplicative inverse as the ratio of the second variable in terms of the first and to use either ratio, depending on the situation.

Using manipulatives when solving proportion-related problems can enhance the performance of some students (Harrison, Brindley, & Bye, 1989; Tourniaire & Pulos, 1985). Tourniaire and Pulos cited a 1979 study by Kieren and Southwell in which they found that children could solve a problem using manipulatives two years before they could solve the same problem without using them. On numeric comparison problems,

manipulatives may help students experiment with groupings and various composite units to determine relationships between associated sets (Lo & Watanabe, 1997).

Ratios and Fractions

A difficulty that students have in understanding rational numbers in fractional form may be due to not seeing the fraction as representing a single number (Behr, Harel, Post, & Lesh, 1993; Post, Cramer, Behr, Lesh, & Harel, 1993). Either students see it as an expression of two numbers or not a number at all. According to Resnick and Singer (1993), one goal of teaching ratios should be students' "construction of ratios as mental entities" (p. 111)—mathematical nouns instead of simply adjectives, nouns that students will work with in the form of slope in their algebra courses.

One interesting conflict in how students may prefer to work with fractions is the apparent contradiction between students' misconception that a fraction is somehow in a better form when it represents a number between zero and one and students' avoidance of dividing a smaller by a bigger number—a tug of war between the part-whole and quotient subconstructs. Pitkethly and Hunting (1996), in their summary of research on learning fractions, cited an Italian study by Bonotto, who suggested that an overemphasis on fractions as part-whole representations interferes with the development of conceptual understanding of other types of fractional representations because knowledge of fractions becomes dependent on a visual or manipulatory context. If a fraction means some part of a whole to students, it seems natural that students might prefer writing a fraction with a larger denominator than numerator. However, Bell, Greer, Grimison, and Mangan (1989) concluded that children avoid division of a smaller by a bigger number, either because it

is impossible or because the alternative is easier and also acceptable. When students write ratios in fractional form, which number they choose to write in the numerator and which in the denominator may be influenced by one or both of these documented tendencies.

For my study, I am working under the following assumptions: that quantitative proportional reasoning is a critical skill in making the transition from middle-school mathematics to algebra courses, that students' learning is meaningful and their strategies more flexible and comprehensive if they are able to make connections between ratios and fractions, and that interpreting ratio as an intensive quantity will help them apply ratios as mathematical nouns when solving problems. I ask the following: What is the potential of using numerical comparison problems with middle-school students to promote the conceptual understanding of ratio as an intensive quantity?

METHODOLOGY

The general research methodology, which Ernest (1998) defines as the theoretical perspective that allows the specific methods, is the qualitative-research paradigm. Within this paradigm, this qualitative study meets Creswell's (1998, pp. 61-62) standard for a case study with the individual participants defined as the cases. Specifically, I refer to this study as a collective instrumental case study—"collective" because of the use of multiple cases and "instrumental" because the focus is on the mathematical topics rather than on the cases themselves. For my study, the case method is the instrument of data collection. However, the primary focus is on the problems and their role in promoting growth of understanding; therefore, it is appropriate to conduct a comparative analysis of the problems as opposed to one of the cases. As Creswell recommends, case studies should draw on multiple sources of information, which, for this study, include a written test from each student, videotaped interviews with each student, the researcher's notes from interviews, and students' written work during the interviews.

Researchers use clinical interviews for observing the mathematical activity and communication of participants and for gathering evidence upon which to base inferences about meanings, knowledge structures, cognitive processes, and changes in these over the course of the interview (Goldin, 1998). According to Singh (2000), "Interviewing as a successful tool of research must be accompanied by appropriate learning tasks" (p. 274). Although at the beginning of the first interview I did ask some general questions about the participants' school experiences with ratios, the overwhelming majority of my interaction with them was in the context of working on problems. By observing students

and asking questions, I can prompt statements in a problem-solving context about their school experiences and make inferences on that basis. Citing advice given by Cobb, Singh suggested that in interviews the students be task-involved rather than ego-involved; therefore, I wanted the majority of students' comments about their experiences to be in the context of a mathematical activity, as opposed to a measured answer to an open, general question, for which a response might be based on what the student anticipates the interviewer wants to hear or what the interviewer expects her to have studied in school. All interviews in this study were semi-structured (Bogdan & Biklen, 1998, p. 95), which means that the problems and certain important questions were arranged sequentially, the same way for every participant, but that the probes depended on a student's responses and varied from interview to interview.

The problems I used are based on an assumption of a linear relationship. They provide a realistic context and require a minimal number of assumptions to understand that context. My goal was to make sure that features of the problems and the assumption of linearity are not in conflict with real-world experience and common sense (Greer, 1993). I used numerical comparison problems as the problem domain because they are more difficult than missing-value problems (Ben-Chaim et. al, 1998) and because my focus is on conceptual understanding, as opposed to the understanding of the crossmultiplication procedure, which tends to be students' preferred method for solving missing-value problems.

Participants

The participants in my study were middle-school girls who attended the *Girls on Track* summer math program at Meredith College in June of 2002. These students were volunteer participants and attended middle schools throughout Wake County, North Carolina. In the past, the students in the program have had an average score on the state's end-of-grade mathematics exam well above the state average (Clark & Berenson, 2000) and almost all of the girls have been on the "fast track" in mathematics, as defined by taking Algebra I during or before eighth grade.

The candidates for selection were 37 of the girls who attended camp on the first day, June 20, 2002, and took a test of proportional reasoning developed by Allain (2000). Thirty-nine girls took the test, but I eliminated two from the pool of candidates because they had attended the camp the previous year and had taken the same test twice that year.

I chose a purposeful-sampling method (Bogdan & Biklen, 1998, p. 65) to include students of various ability, determined by their performance on the test of proportional reasoning and their performance on a subset of questions of particular interest. The overall test has 10 questions and a maximum score of 40, and the subset is 6 questions with a maximum score of 24. I chose the subset based on questions relevant to my study, but because I did not want to ignore the overall results of a test with a good reputation for validity in the arena of proportional reasoning, I required each participant to fit the profile of a high scorer, a medium scorer, or a low scorer on both the overall test and the subset to be selected for the study. The high scorers fall into the upper quartile, as defined by Longest's (2002) analysis of the test results from the previous year, the medium scorers fall into the middle two quartiles, and the low scorers into the bottom quartile. Table 2

shows the ten participants grouped according to their performance level on the test of proportional reasoning.

Table 2
Participants by Groups Based on Test Scores

Group	Name	Grade Entering	Race	Test Score Full/Subset
High	Marie	8	Asian	35/22
	Susan	8	Caucasian	34/24
	Sheila	8	Caucasian	33/24
	Kim	8	Caucasian	33/23
Medium	Mandy	8	Caucasian	26/17
	Angela	8	African American	25/17
	Rhonda	8	Caucasian	24/17
Low	Corina	7	African American	19/15
	Prema	8	Asian	19/14
	Melinda	7	African American	19/12

Another girl, Kelly (8, Caucasian, 27/21), was selected for the study, but she left camp before I could ask her about participating. The ten students listed in Table 2 participated in the first round of interviews. Marie and Angela were dropped from the study after the first round of interviews because Marie had already taken Algebra I and because Angela's family was preparing to move across the country before the second round of interviews was scheduled to begin. Even though Marie did not participate in the

second round, I use her work from the test and from the first interview as the benchmark for the other students because of her outstanding performance.

Phase 1: Written Assessment (June 20, 2002)

On the first day of *Girls on Track*, the students took a test of proportional reasoning in an auditorium at Meredith College. This test, developed by Allain (2000), comprises ten questions from various sources that assess a variety of skills related to proportional reasoning. The following list shows the categories of Allain's rubric, as interpreted by three experienced graders of this test, with Longest's (2002) additional category of a zero score for no response or a response that shows no mathematical effort:

- 4 correct answer with evidence of appropriate strategy
- 3 incorrect answer due to computational error but with evidence of appropriate strategy
- 2 correct answer without evidence of strategy or with evidence of inappropriate strategy
- 1 incorrect answer with evidence of inappropriate strategy
- 0 no mathematical response.

Throughout the test and in instructions before the test, the students were told to show their work and to explain how they arrived at their answers. The following questions on the test are the ones I selected for the subset of problems relevant to my study.

The Gum Problem

In terms of students' scores in the past, the gum problem is the easiest on the test (Allain, 2000). This type of problem tends to be easy for students because it involves

well-chunked measures, or two quantities that when combined produce a familiar measure that has its own identity, as defined by language and culture (Lamon, 1993). In this case, pieces of gum and number of cents are associated in the relationship that people readily identify as price. The question is as follows:

Sally bought 3 pieces of gum for 12 cents and Anna bought 5 pieces of gum for 20 cents. Who bought the cheaper gum or were they equal?

The problem could be improved by adding the phrase “a total of” before the 12 and the 20 to stress that the amount of money is the price of all the gum, not for an individual piece; however, out of almost 200 students who have taken this test in three years of the program, only a few have misinterpreted the question.

The Coffee Problem

The coffee problem, a standard missing-value problem, provided information about students’ notation and strategies for missing-value problems. The coffee problem is as follows:

To make coffee, David needs exactly 8 cups of water to make 14 small cups of coffee. How many small cups of coffee can he make with 12 cups of water?

The Pizza Problem

The pizza problem, taken from Lamon (1993), is the problem that Clark and Berenson (2002) analyzed when first considering the decontextualization phenomenon and is as follows:

There are 7 girls with 3 pizzas and 3 boys with 1 pizza. Who gets more pizza, the girls or the boys?

Lamon refers to this as a problem of associated sets because the sets, pizzas and girls and pizzas and boys, are unrelated until they are associated by a given relationship in the problem. This problem was also used by Kieren and Pirie (1991), who cited Wales as using it before them.

The Orange-Juice Problems

Taken from Noelting (1980), three questions on the test asked students to compare two mixtures of orange juice and water to determine which has the stronger orange flavor. The instructions on the test are as follows:

You and your friend are going to make orange juice for a party by mixing orange juice and water. You will be given three different situations. For each situation, you will be presented with the contents of two trays that contain various amounts of orange juice and water. The shaded box represents the orange juice and the unshaded box represents the water. **The goal for each situation is to determine which tray will create a drink that has a stronger orange taste or if the two drinks will taste the same.** Each mixture will be expressed as an ordered pair (e.g. (1, 3) with the first term corresponding to the number of glasses of orange juice and the second term to the number of glasses of water.

The required comparisons are as follows: (1,2) versus (1,5); (2,5) versus (3,8); and (2,3) versus (4,6). See the Appendix for the diagrams that accompany the notation.

Phase 2: Teaching Session (June 24-28, 2002)

During the final five days of the seven-day camp, I interviewed each participant individually once for about 45 minutes. All the interviews were conducted in the same

place, a small study room on the campus of Meredith College. The interviews were videotaped, and during the interviews the participants had access to a calculator, pencils, and paper. I also provided a different set of manipulatives for each question. Artifacts from the teaching session include the interviewer's notes, the participant's written work, and the videotape recording.

After talking with each girl for a couple of minutes about camp and her favorite activities at camp, I asked her what "ratio" means to her and how she normally writes them. Often follow-up questions included questions about what notation her teachers had used and what she had seen in her textbooks. For the rest of the session we talked about the following problems.

The Pizza and Orange-Juice Problems

I based the teaching session on the pizza problem and the orange-juice problems, which the girls had responded to on the test just a few days before. For the pizza problem, I told the girls that they could work with tower blocks, which represented the boys and girls, and circular pieces of paper, which represented the pizzas. After the initial questions about ratios, I handed each girl her test, which had already been graded but not marked on by anyone except her, and asked her to explain her solution to the pizza problem.

After our discussion of the pizza problem, I used the coin problem (see the next section) and then the orange-juice problems. Again, I asked each participant to explain her solutions on the test and to use orange and blue cubes, which represented the cups of orange juice and water, respectively, if she wanted, to help her answer the questions.

The Coin Problem

I wanted to have an easy question that I could refer to during the teaching session and a question that might prompt a discovery of the relationship between a fractional representation of a ratio and the fraction as a stand-alone number. After asking the participants about the pizza problem, I asked them to show me, using play coins, a ratio of dimes to quarters in an even exchange of money and to write the ratio. Because of only one ratio in the problem, instead of the two in the numerical comparison problems, it was my hope that students would be able to focus on the notation and interpretation of the ratio.

Attempting to minimize variation in students' knowledge based on their prior experience is crucial to being able to make valid inferences about their progress during the study (Goldin, 2000, chap. 19). At the end of the teaching session, I explained to the students that my goal was for them to learn how to interpret ratios written as fractions in terms of the units given in the problem. For those who did not make progress during the interview, I used direct instruction to tell them what I wanted them to learn so that all participants would at least be exposed to the ideas before the follow-up interview.

Phase 3: Follow-Up Interview (July 31-August 16, 2002)

I interviewed each girl once after the teaching session, with an intervening period ranging from almost five to about seven weeks. I wanted to find out if the students had changed their notation or strategy as a result of the teaching episode and also to collect more data on their responses to numerical comparison problems. These follow-up interviews were one-on-one interviews, videotaped, and about an hour long. I conducted

some of the interviews in a conference room on the N.C. State campus, some of the interviews at the participants' homes, one of the interviews in a classroom at the participant's school, and one of the interviews in a study room in the facility of an after-school program that the participant attended. Artifacts from the follow-up interviews are the interviewer's notes, the participant's written work, and the videotape recording. Participants had access to pencils, paper, and a calculator. I did not make manipulatives available for the follow-up interviews. The following problems were the focus of the follow-up interviews.

The Cookie Problem

I began the follow-up interview with the cookie problem, a warm-up problem about rewriting a cookie recipe from a book (Stewart, 1982, p. 168) to serve a large number of people at a party. The recipe, as given, calls for $\frac{1}{4}$ pound of butter, 4 eggs, $\frac{1}{4}$ teaspoon of salt, $\frac{2}{3}$ cup of sugar, a teaspoon of vanilla, and a cup of flour. After showing each participant the recipe and the photo of the cookies and explaining how to bake them, I asked her, "If I am having a party and want to make a bigger batch of cookies, using all 10 eggs in my refrigerator, could you rewrite the recipe so that my cookies taste the same as these?"

The Sidewalk Problem

The final two questions were both numerical comparison problems. The first, the sidewalk problem, is one that I wrote for these interviews, and I presented it to the participants in written form with a picture of two girls walking on a sidewalk. The problem is as follows:

Jill and Alice are walking together down the sidewalk. They notice that Jill steps on every fifth crack in the sidewalk on her seventh step and Alice steps on every third crack on her fourth step. Who has the longer stride?

The purpose of arranging the numbers this way was so that if students simply wrote $5/7$ and $3/4$, according to their order in the problem, and picked the larger fraction, they would get the correct answer. If they did not make progress in response to probes, I planned to return to this problem at the end of the interview.

The Fertilizer Problem

The final problem is another that I wrote specifically for these interviews, and it was also given to the participants in written form, but with no picture. The fertilizer problem is as follows:

In 8 hours, Tony can fertilize 3 acres. It takes Russ 12 hours to do 5 acres. Who works faster?

The purpose of arranging the numbers this way was so that if students simply wrote $8/3$ and $12/5$, according to their order in the problem, and picked the larger fraction, they would get the wrong answer. I also wanted the larger number in each pair to come first in the problem. By having the fertilizer problem set up differently than the sidewalk problem, my goal was to identify students' tendencies by looking for similarities in their notation and strategies between the two problems.

Data Analysis

In the data-analysis phase of qualitative research, the researcher codes and arranges elements of the collected data to look for patterns (Bogdan & Biklen, 1998).

Reviewing the videotapes, my fieldnotes, the girls' written work from the interviews and their work on the test of proportional reasoning, I looked for evidence of students' understanding of ratios and their understanding of fractional notation of ratios. Using the emergent perspective, I considered this evidence of the psychological dimension against the backdrop of cultural factors, both in and out of school, that have influenced their informal knowledge and their preferred external representations.

Coding is based on evidence of the following: students' notation and representations, students' connections among MCF topics, students' movement within the overlapping model of ratios and fractions, students' levels within Lamon's categories of proportional reasoning, and categories of numerical approaches (such as *between*, *within*, and *crossmultiplication*) that the girls used to solve problems. The girls' external representations and statements provide evidence for whether they maintain the context of the problem in their solution.

Summary

Using the problem domain of numerical comparison problems and the research methods of the semi-structured task-based interview, I was able to learn about how students represent ratios externally, both on paper and with manipulatives, and to collect evidence about their understanding of ratios, particularly as they relate to fractions. By experiencing their language and actions as they made progress, I was able to grow in my understanding about the potential of numerical comparison problems in promoting their conceptual understanding of fractional representations of ratios.

RESULTS

In this chapter, the results from the interviews are given, for the most part, in the sequence of the activities in the interviews. The only exception is the first section, which covers all the participants' responses to the general questions about their understanding of ratios that were asked at the beginning and the end of the first interview. In the subsequent sections, results are given for the problems that the students discussed in the interviews—the pizza problem, the coin problem, and the orange-juice problems, from the first interview, and the cookie problem, the sidewalk problem, and the fertilizer problem, from the second interview. The final major section at the end of this chapter presents the overall patterns of students' behavior for the entire study.

Subsections within each of the sections for the problems identify issues and have titles that begin with “Emerging Theme,” for issues that emerged during data analysis; “Confirming Evidence,” for evidence that supports conjectures put forth in articles mentioned in the literature review; and “Interfering Factor,” for any additional issues that seem to have inhibited one or more individuals on a particular problem. In these subsections, episodes are described, usually with quotes and often longer blocks of dialogue. Although episodes are described in detail in only one subsection for each episode, the episodes often illustrate multiple themes. For brevity, each episode is described in detail only once under one subsection heading.

In this chapter and the next, I use the following terms for the numerical representations of ratios that the students wrote. A *mixed fraction* is a number with a whole number portion and an additional amount in a fraction. For a number to be

considered a mixed fraction, the whole number portion (2, for example) must be beside but not part of the additional amount in the fraction ($3/5$, for example). In conversation, a student would say “two and three fifths,” for example. A *decimal expression* is any number represented with a decimal point (for example, 7.82). A decimal expression may or may not have digits to the left of the decimal point. *Fraction* is defined as a number in the form a/b , with a numerator, a horizontal or diagonal bar, and a denominator. A *proper fraction* is a fraction in which the denominator is greater than the numerator, and an *improper fraction* is a fraction in which the numerator is greater than the denominator.

Ratios

The first math-related questions, before the students began working on the problems, in the initial interview were designed to gather information about participants’ understanding of ratio and their classroom experiences learning about ratios. In this section, I list participants’ responses to the first question together in a block to illustrate the similarities, but keep in mind that the interviews were conducted individually. All the students remembered studying ratios in their recent math courses, and when asked what they remembered about ratios, they responded as follows.

Angela: Converting them into fractions.

Corina: Well, they look a lot like fractions.

Kim: Basically something over another number.

Mandy: Ratios are fractions and they simplify everything, like if Mark had two pieces of pie and there was a total of five, you would put that into a ratio.

Marie: It’s like something and something else. Like you put them in fractions.

Melinda: I remember that to change a fraction to a ratio there's three different ways—a fraction, a colon, or like “3 to 6.”

Prema: It's a divisibility, sets things apart. And it can be written in many ways.

Rhonda: They can be like “6 to 2” and they can be fractions too. Ratios and fractions are the same thing.

Sheila: You can do it as a fraction or you can do it with a dividing slash.

Susan: A ratio is like “2 to 5.”

When asked about fractions, Kim gave the same answer—“something over another number”—so including Kim, eight of the ten mentioned fractions immediately. Of those eight, only Melinda mentioned anything related to ratios other than fractions. None of the students mentioned multiplication, and only two, Prema and Sheila, mentioned division. When asked what she meant by “divisibility,” Prema said that when given a word problem with ratios you have to divide to get the answer, which she consistently tried with the numerical comparison problems; however, throughout both interviews, Prema demonstrated that she did not understand the difference between dividing one number by another and dividing the second by the first.

All the responses to this question were procedurally oriented, mostly focused on how ratios look when written. In our response to how ratios are most often presented in middle-school textbooks (Clark et al., in press), we concluded that ratios are defined too narrowly in terms of fractions and suggested broadening students' experience with ratios to include other number types (see Figure 7). Students' responses in this study provided

further evidence of a narrow understanding of ratios, limited by the notation that they use to write them.

Written Representations of Ratios

Each student demonstrated options for writing ratios. Table 3 shows each participant's statement of her own preference of notation for ratios, her statement about her teacher's preference of notation, and whether she gave an example of writing a ratio as a fraction, with a colon, or with the word "to" between two numbers. The only other option for notation that any of the students wrote was Prema's "2 out of 4."

Table 3
Students' Use of Notation for Ratios

Student	Notational Preference	Teacher's Preference	Fraction	Colon	"To"
Angela	No preference	No preference	Yes	Yes	No
Corina	Fraction	Fraction	Yes	No	No
Kim	Fraction	Fraction	Yes	Yes	Yes
Mandy	Fraction	Fraction	Yes	Yes	No
Marie	No preference	Did not remember	Yes	Yes	No
Melinda	Colon	No preference	Yes	Yes	Yes
Prema	Colon	Colon	Yes	Yes	No
Rhonda	No preference	No preference	Yes	Yes	No
Sheila	Fraction	No preference	Yes	Yes	No
Susan	Colon	No preference	Yes	Yes	Yes

There were no conflicts between the student's preference and her statement of her teacher's preference. All ten wrote a fraction as an acceptable form for writing ratios, and all but one, Corina, also used the colon notation. Of the students who expressed a

preference, four chose writing ratios as fractions and three chose writing ratios with a colon.

The girls who stated no preference—Angela, Marie, and Rhonda—all used fractions on the pizza problem on the test of proportional reasoning. Angela and Rhonda also used fractions on the gum, coffee, and orange-juice problems. Prema, who stated a preference for the colon notation, used fractions on the gum, coffee, and pizza problems. The other two girls who stated a preference for the colon notation—Melinda and Susan—did not use fractions or the colon notation on any of the problems on the test. The students, in general, knew the colon notation but did not use it when solving problems, either on their tests or in their work during the interviews.

Examples of Ratios

After writing their options for ratio notation, the participants were asked to state their numbers in the context of a ratio. When asked for an example of a ratio, most expressed a ratio in a phrase with “to” and two units, such as Susan’s “two cookies to five people.” Rhonda, however, could not think of any example for her “6 to 2.” Marie, the student who had already taken Algebra I, was the only participant who provided any information in her example that implied a multiplicative relationship, when she described a classroom in which “for every two boys there’s four girls.” All the other examples were phrased as two extensive quantities with no stated relationship.

The Relationship Between Ratios and Fraction

The final question of the first interview—“How are ratios and fractions related?”—was designed to gather more information about the students’ understanding of ratios and fractions but in a more direct way than at the beginning of the interview, when

I wanted to avoid influencing their problem-solving strategies by asking specifically about fractions. The following responses are presented together in a block, but remember that the participants were interviewed individually.

Corina: They have the same format, and they're basically the same thing.

Kim: How ratios and fractions can be set up. They can be set up the same way.

You can think of the line as division for both of them.

Mandy: I think they're part of everyday life. When it comes to money, you use ratios; when it comes to word problems in school and math, it all comes down to ratios. Ratios are a big part of life.

Marie: You can simplify ratios by using the fraction form...They can be the same thing. [She was then asked about differences.] Actually, I think they are the same thing because for actual numbers the numbers have to be put into a fraction.

Melinda: You really just change the fraction to a ratio or you can just keep it a fraction. They're related because you just change the format to a different form, saying it in a different way—"three to six," instead of "three sixths."

Prema: For one thing, they both have to be simplified. Another thing, they just are fractions because another way to write ratios is fractions.

Rhonda: They're basically just the same thing because you're saying "five to two" and then you're saying "five over two."

Sheila: Because in fractions, one fifth is one out of five. For ratios, it's almost the same thing—one to five.

Susan: They're like the same thing; they're just written different ways.

Those who expressed differences did so with a focus on how the notation looks or how the notation is translated into words when read. The students, in general, were not able to explain ratios well, but they were able to identify them in problems and write them, usually as fractions.

The Pizza Problem

The first problem that the participants discussed in the interview was the pizza problem. No conclusions about students' strategies in general, or even strategies of *Girls on Track* participants, can be drawn from the work of the students in this study on this problem because their written work on this problem on the test of proportional reasoning served as one component in the sampling strategy. When choosing the students whom I wanted to invite to participate in the study, I looked at this problem on the test and chose students who used a variety of methods and students with a variety of scores on this particular problem. Only four of the students in this study—Kim, Marie, Sheila, and Susan—answered the pizza problem correctly by using an appropriate strategy. Marie used ratios only in her solution; the other three divided the pizzas into slices and figured out how many slices the girls and boys would get.

Overall, the participants in this study were not as successful on the pizza problem as the sixth graders in Lamon's (1993) study, who were "not yet operating symbolically" (p. 50) and were not attempting to formalize exact numerical relationships. The students in her study were successful at reasoning informally within the context of the problem. Many of the *Girls on Track* participants, most of them more than a year older than

Lamon's students, inappropriately used formal representations that they had learned in school or used an inappropriate algorithm. Table 4 provides a summary on the students' work on the pizza problem on the test of proportional reasoning.

Table 4
Summary of Participants' Work on the Pizza Problem

Students' Method and Outcome	Students
Using fractions successfully	Marie
Using the slice strategy successfully	Kim, Sheila, Susan
Using fractions unsuccessfully	Angela, Mandy, Prema, Rhonda
No apparent strategy	Corina, Melinda

When asked about ratios in the problem, the students were able to identify 7 to 3 (the number of girls to the number of their pizzas) and 3 to 1 (the number of boys to their pizza) as the ratios. But only one student, Marie, who had already taken Algebra I, solved the problem using these exact numeric relationships. Of the five students in this study who tried to solve the problem using mathematical symbols, only she was successful. Figure 9 illustrates the path for Marie's strategy against the backdrop of the overlapping model. In the explanation that follows the figure, the numbers in parentheses match the callout numbers in the figure.

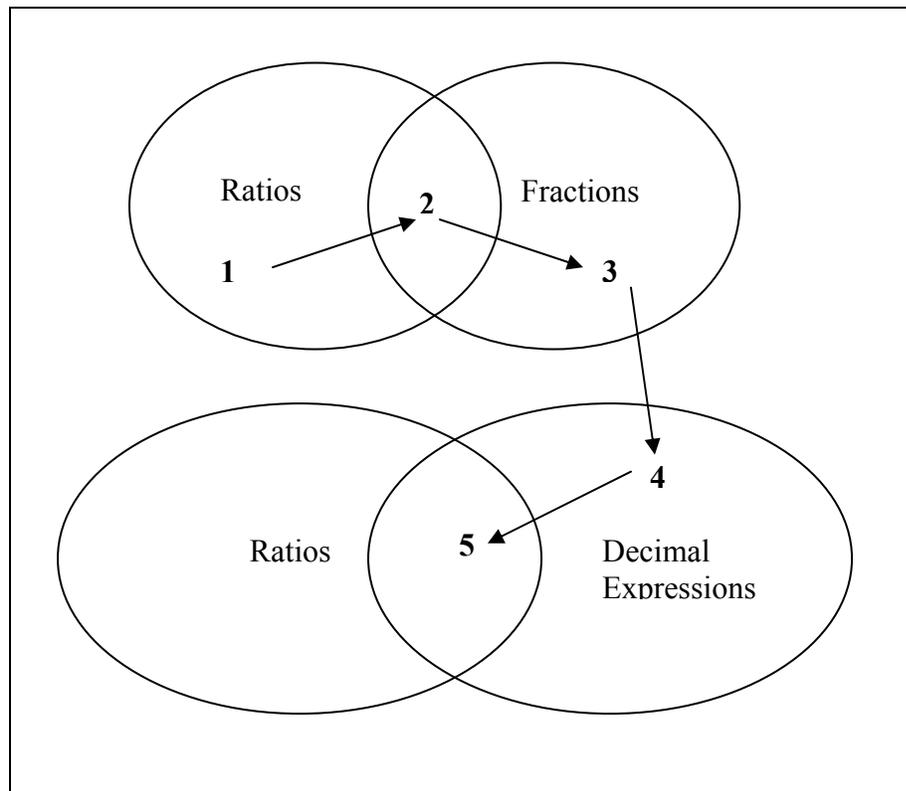


Figure 9. Marie's Path to Solving the Pizza Problem

Marie's first step (1) was recognizing the ratios and establishing the associated-sets relationships between the girls and pizzas and between the boys and pizzas. Next, she wrote these ratios as fractions (2). She then worked with fractions as numbers (3), dividing the numerators by denominators, and converted them into decimal expressions (4). On her test, she compared the decimal expressions .43 and .33, which she said represented "how much of the pizza each person got." Her choice of the girls was based on comparing the decimal expressions as numbers and interpreting them as ratios (5). Her solution is an example of solving a numerical comparison problem by interpreting the ratios as intensive quantities.

Emerging Theme: Mixed Fractions and Decimal Expressions as Ratios

The participants were successful, in general, at converting ratios as improper fractions into ratios as mixed fractions and pointing to the position of the number on a number line. Even though the students wrote the ratio of girls to pizzas as $7/3$, they said “two and a third” when discussing the ratio in the interview. None of the participants said “seven thirds” when explaining the ratio. The notation they used to write the ratio, an improper fraction, was inconsistent with how they spoke about the ratio, which was always “two and a third” or “two point three” or “two point three repeating.” This consistent conversion from their written notation of improper fractions to their spoken representations of mixed fractions or decimal expressions suggests that the students are more comfortable using mixed fractions and decimal expressions and are more likely to be able to interpret ratios in these forms in the context of the problem.

Emerging Theme: Role of Manipulatives

The pizza problem proved to be difficult for many of the students on the test, but the manipulatives of tower blocks and paper cutouts helped those students. The following exchange was with Corina, who just guessed on the test and had no apparent strategy.

Interviewer: Did you notice a ratio in the pizza problem you did last week?

Corina: No, I didn't understand what they were asking.

[The interviewer puts out the paper cutouts representing the pizzas and points to the tower blocks, the red ones representing the girls and the blue ones the boys.]

I: Do you think you could work with the pizzas and the people to solve this?

C: Yeah. Since there are three pizzas (putting two red tower blocks on each of the

girls' pizzas) I think two girls will be splitting a pizza with one girl left over. I don't know what to do with her (holding up the seventh red tower).

I: What do you think she would do at the party?

C: She could eat a little from each pizza maybe.

I: That's a good idea. We can't really split up people, but we can split up the blocks to kind of demonstrate that.

[Corina laughs and splits the seventh tower into three blocks and distributes the blocks, one for each of the girls' pizza.]

I: What about the boys?

C: I guess these three (putting three blue towers on the boys' pizza) just split one pizza.

I: So this (pointing at the blue towers) is three boys splitting one pizza?

C: Yeah.

I: And how would you describe what you've done here with the girls?

C: Well, I guess two girls would get equal amounts of the pizza and if there was any left over, this one person (picking up the single blocks that she separated from the seventh red tower) would get it.

I: We might have different toppings or something, and these two girls really like this topping, and these two this one and these two this one. And this girl is going to get a little from every pizza. So what do you think? Do you think each girl or each boy going to get more food?

C: The girls.

I: Why is that?

C: Since the boys have to split three they only get a little bit of the pizza each, but there's three pizzas for the girls and they only have to split it with two people and have just a little bit left over. So the girls will get almost a half a pizza.

I: That's a nice way of looking at it. The girls would get almost half a pizza.
What do you think about the boys?

C: They'd just get a third of a pizza each.

I: Is there a way to figure out—you said almost a half—is there a way to actually figure out how much of a pizza each girl gets?

C: Well, if you divided. I just don't know to divide by what.

This exchange lasted 3 minutes and 34 seconds. In this segment, Corina, with assistance, progressed from her nonconstructive strategy on the test to a strategy of qualitative proportional reasoning. She was able to identify the ratios and pick the girls on the basis of the smaller ratio, but she did not know how to set up the ratios as fractions and what to divide by what in each ratio.

Rhonda had written $7/3$ on her test for the ratio for the girls and had converted it to a decimal expression. Her answer, that the boys got three slices each and the girls two and a third, seemed incorrect to her immediately after putting the red towers on the pizza, as shown in the following exchange.

Rhonda: There's an uneven number on the third pizza so you have to divide it all

evenly (putting the fifth, sixth, and seventh red towers on the third girls' pizza). So the girls would get a little le-... Wait, hold on. The girls would actually get more.

Interviewer: Why?

R: Because there's three boys who have to split one pizza, and this number is how much of the pizza each girl has to divide by. So the girls would actually get more pizza. There's not exactly three on each one (pointing to the girls' pizzas), but there's exactly three on this one (pointing to the boys' pizza). A little less than three on each one (pointing to the girls' pizza), so the girls get more.

I: A little less than three what?

R: A little less than three slices. Well, a little less than three girls getting each pizza.

As soon as the manipulatives drew Rhonda's attention to the context of the numbers she realized her error on the test. A common problem throughout the interview for many of the girls was returning to slices as the unit, as Rhonda did in her slip-up at the end of this exchange and also later in the interview when she was no longer working with the manipulatives. In terms of the general overlapping model of ratios and the various number types (see Figure 7), modeling the problem with the manipulatives tends to hold the student's attention in the ratio-only realm, where she focuses on the extensive quantities and their units and the relationship between them, before she attempts to

represent that relationship as a single number. At the end of this exchange, Rhonda is making progress at expressing a comparison between the two intensive quantities.

After solving the problem with the manipulatives, Rhonda was able to return to her previous fractional representation of $7/3$ on the test and interpret it correctly.

Interviewer: You said you don't agree with your answer anymore (pointing to her test). Do you agree with this number that you came up with?

Rhonda: Yes.

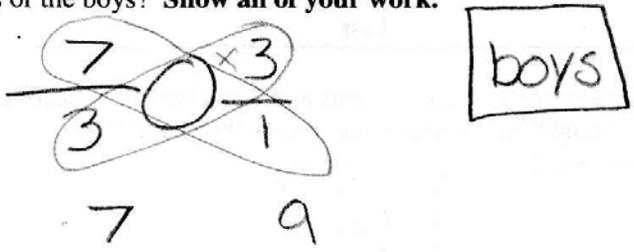
I: What does that number mean to you?

R: It means how many girls get one of the pizzas. So that would be girls. Since you can't divide girls into thirds, they have to eat from a different pizza.

Confirming Evidence: Decontextualization of Ratios

Figure 10 shows a common unsuccessful strategy used by the students at *Girls on Track*. This example is from the test of Kelly, who was chosen to participate in the study but left *Girls on Track* before the first round of interviews. Kelly's work illustrates the use of the crossmultiplication algorithm to determine the larger fraction.

There are 7 girls with 3 pizzas and 3 boys with 1 pizza. Who gets more pizza, the girls or the boys? **Show all of your work.**



7/3 1/3

7 9

boys

Figure 10. Kelly's Crossmultiplication Method

From our examination of students' written work on the test of proportional reasoning (Clark & Berenson, 2002; Clark et al., in press), we concluded that students who solved the problem by Kelly's method identified the two ratios, wrote them as fractions, crossmultiplied to compare the fractions, and chose the group of children associated with the larger number. We used the term *decontextualization* to refer to the path of students moving left to right across the overlapping model of ratios and fractions, from the ratio-only realm to the intersection to the fraction-only realm, and never associating their operations and notation with the situation described in the problem.

Students' responses confirmed the conjecture by Clark and Berenson (2002) that many students who used fractional representations of the ratios in this problem compared the fractions and chose the group of children, either the girls or the boys, associated with the larger fraction. Students, like Kelly, who wrote the ratios as $7/3$ and $3/1$ and chose the larger fraction were really writing a ratio of people to pizza and answered the question incorrectly; however, some students who have taken the test—Prema is one—wrote the ratios as $3/7$ and $1/3$ and chose the correct answer, the girls, by choosing the larger fraction. In the interview, Prema used the calculator to check her answer on the test and divided the pair of numbers in the reverse order of the way she had written the fractions on the test. This time, she chose the boys as her answer, and when I asked her how she arrived at that conclusion, she said, “When I divided 3 into 7...I got 2.3, which is for the girls, and for the boys I got 3. And 3 is more than 2.3.”

Sheila, using the slice strategy, answered the problem correctly on the test, but working with the ratios $7/3$ and $3/1$ in the interview, she changed her mind and decided that each boy got more pizza. With a common denominator, she converted $3/1$ to $9/3$ and

chose the boys because 9 is larger than 7. When asked to check her answer with the manipulatives, she returned to her slice strategy and changed her mind back to her original answer. Referring to the numbers 3 and 2.3 that she got from the fractions $3/1$ and $7/3$, she later explained why she chose the boys when working with these numbers: “Because that’s their decimal number, and I thought that whatever number was higher would be the right answer.” With this solution, Sheila followed the same path as Marie (see Figure 9) until stopping after Step 4, without interpreting the numbers as ratios. When asked about the meaning of her decimal numbers in terms of the problem, Sheila could not explain them. Her reasoning was quantitative, but quantitative only in terms the variables that she introduced into the problem, slices per girl and slices per boy.

Interfering Factor: Slices as a Unit

“What do these numbers mean in terms of the problem?” was the question that some of the students started to expect every time they wrote a fraction in their problem-solving activities during these interviews. The most common answer by far on the pizza problem was that the number $7/3$ represented the number of slices each girl would get and $3/1$ the number of slices each boy would get. Marie was the only one who was able to interpret her fractional representations of ratios. She wrote the ratios as $3/7$ and $1/3$ and correctly identified those as the part of a pizza that each girl and boy, respectively, gets.

Kim and Sheila assigned eight slices to a pizza and computed the number of slices that each girl and each boy would get. With a similar but less precise strategy, Susan divided the pizzas into seven slices each, marked three slices for each girl and for two of the three boys, and then matched the remaining boy with the last slice. Susan’s solution

was based on qualitative reasoning but was a quick and efficient way to solve this numerical comparison problem for which she did not need exact numbers.

Angela said that there was no one right answer because the answer would depend on the number of slices in the pizzas.

Perhaps the ratios of three and two and a third in this problem are close to the number of slices that these girls usually eat when sharing a pizza with friends or family members. Of course, slices is a common unit used when talking about splitting up a pizza. When trying to interpret the ratios $7/3$ and $3/1$ or $3/7$ and $1/3$, the participants often used slices as the unit.

Interfering Factor: Indivisibility of Discrete Variables

One potential conflict is that the problem requires a comparison of pizza per person but that the order of the numbers suggests writing a people-per-pizza ratio to students who automatically set up ratios according to the order of the numbers in the problem. The people-per-pizza ratios may also conflict with students' experiences, both in and out of school, thinking about ratios such as this in terms of fair sharing, with their focus on how much each person gets. Another factor in the difficulty of this problem for students who are inclined to make the comparison quantitatively is the indivisibility of the discrete variables, girls and boys. The students in this study avoided expressing a non-integer amount of a discrete variable, such as two and a third girls for every pizza. In their explanations, the participants preserved the discrete nature of the variable.

The tendency to preserve the indivisibility of the discrete variable was true even for Marie, the most advanced student, who demonstrated quantitative proportional reasoning throughout the interview. On her test, she used the ratio $3/7$ and, at the

beginning of the interview, she interpreted it correctly as the amount of pizza for each girl. However, she was unable to interpret $7/3$ in the context of the problem. After she worked the coin problem, we returned to this issue. Her short response to the following question demonstrates the tendency to express the non-integer ratio as one of a continuous variable to a discrete variable and the tendency to preserve the discrete nature of the variable—in this case, the number of girls.

Interviewer: Does that $[7/3]$ as a number mean anything to you in terms of the problem?

Marie: Maybe $7/3$ —it's two and a third... So maybe each girl got two and a thirds of a pizza. No, that wouldn't be it. Maybe there's two girls for every pizza, and then there's a little bit left over and the last girl gets a piece of every pizza.

Interfering Factor: Attempts to Solve for a Missing Value

Two of the girls, Angela and Mandy, tried to solve the pizza problem on the test by using an algorithm for solving missing-value proportion problems, even though there is no proportion given in the problem. Mandy set up two proportions—one with $7/10$ and $n/4$ for the girls and one with $3/10$ and $n/4$ for the boys. The first fraction in each pair is the number of girls or boys over the total number of children, and the second fraction's denominator of four represents the total number of pizzas. She solved for the missing values in the first and second proportion and then multiplied the results by the number of girls and boys, respectively. Her answer was based on which final product was greater. The following is her explanation from the interview.

Interviewer: Can you explain your solution to me?

Mandy: What I did was put people over total and put 7 girls out of a total of 10 people, and the variable is undefined so you put n . And there were a total of four pizzas, and then what I did was I did 4 times 7 is 28 divided by 10 is 2.8 times the 7 girls. And then I did over here the same thing, 1.2 times 3 because there are 3 boys.

I: You said that the variable is undefined. What did you mean by that?

M: To know the actual answer, like how much pizza they actually had, that's what I meant by that.

I: So if somebody said, "What is n in this case?"...

M: The undefined variable, what you have to figure out.

Mandy continued saying that the numerical results of her operations gave her the amount of pizza that each group had, until she was asked what the final products of 19.6 and 3.6 represented. She then admitted that she did not feel confident about her solution. When asked if she could use the manipulatives to help, she said, "It would probably depend on how many slices of pizza there were." When asked how she would cut the pizzas, she suggested six slices in each pizza and expressed her belief that the answer would be different if she cut the pizzas into a different number of slices. Assuming six slices per pizza, Mandy was then able to solve the problem using the slices-per-person method that Kim, Sheila, and Susan used on the test. Mandy's solution did not lead to a discussion of interpreting the fractional representations of ratios because her recognition of ratios was restricted to part-whole representations. When you write a fraction, she said,

the total always goes on the bottom, so in the pizza problem, the ratios to her were seven girls over ten people total and three boys over ten people total.

With a similar strategy to Mandy's, Angela created a variable in a proportion, crossmultiplied, and solved for the variable, which she claimed represented how much more pizza the boys had than the girls. The most plausible explanation is that both girls had worked many missing-value problems according to a procedure learned in their classes, and when they saw a problem with four values presented in two pairs, they forced the problem to fit the procedure they knew.

The Coin Problem

When the students could not progress any further on their own in interpreting the fractional representations of ratios in the pizza problem, I moved forward to the coin problem and then returned to the pizza problem. When asked to show an even exchange of dimes and quarters and to write it as a ratio, most of the girls chose five dimes and two quarters and wrote the ratio as $5/2$. Angela at first used ten dimes and four quarters and then simplified to five and two. Mandy initially insisted that the exchange could not be completed without one person receiving change

When asked to interpret the ratio in terms of the coin problem, Rhonda and Susan were able to do so immediately, but most of the girls needed a hint. I gave them a hint by arranging the coins as shown in Figure 11 and placing a pencil down the middle.

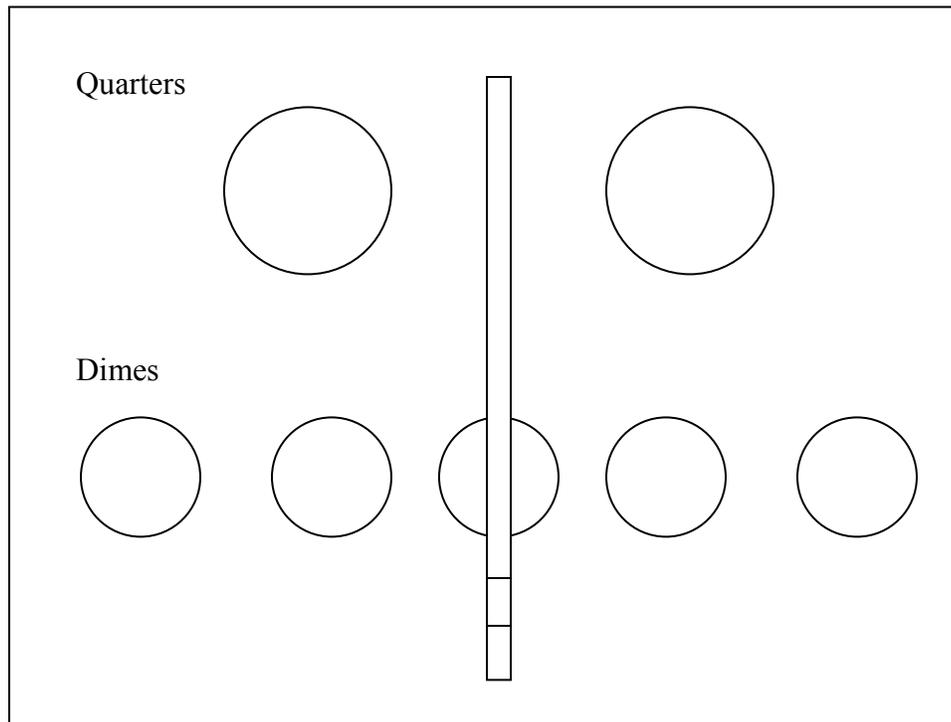


Figure 11. The Coin Arrangement and Pencil Placement

After seeing this arrangement of the coins, most of the girls, using the mixed fraction instead of the improper fraction they had written, interpreted the ratio as two and a half dimes being equivalent to one quarter.

Emerging Theme: Mimicking the Language of Intensive Quantities

After using the coin problem to provide a familiar context in which the participants could interpret a simple non-integer ratio, I realized that what many of the students were learning was to mimic the language of intensive quantities rather than learning how to coordinate two units of extensive quantities to form a third unit for an intensive quantity. With the coin problem, I wanted their interpretation of $5/2$ to serve as reference as they constructed new meanings for the ratios in the pizza problem and progressed to a level of quantitative proportional reasoning. The students seemed to

understand that I had constructed this simple and parallel context and figured out that they should mimic the relationship for the coin problem in their interpretation of the ratios in the pizza problem. For example, the following passage from the interview with Rhonda, who had already solved the pizza problem with manipulatives, illustrates how some of the participants tried to apply the language of the relationships in the coin problem without understanding the relationships in the pizza problem. Before the pencil was placed between the coins, Rhonda was able to interpret $5/2$ correctly in terms of the coin problem.

Interviewer: Does that number [$5/2$] mean anything to you in terms of the coin problem?

Rhonda: Um, two and a half... Oh yeah, if you just have one quarter, that's how many dimes you would have to have.

[The interviewer then arranges the coins as shown in Figure 11 and places the pencil down the center.]

I: Thinking about the way you were able to make sense of that, can you now make sense of this number, $7/3$?

R: If you had only one girl, they would each get 2.3 repeating pieces of pizza.

Yeah, so they'd each get 2.3 pieces of pizza.

I: What about the number 3 for the boys?

R: The number of pieces of pizza. Or wait... Yeah, three pieces of pizza. Hold on, I'm getting confused. They'd each get three pieces of pizza. Which one's right?

I: So if you look at those two numbers—you divide 7 by 3 you get 2.3 repeating

and divide 3 by 1 and you get 3—

R: So actually the boys would get more. Well, no... I don't know who'd get more.

By comparing 3 and 2.3 and choosing the boys, Rhonda returned to the strategy that she used on the test. She had solved the problem correctly with the manipulatives, but using only the ratios without the manipulatives, she was unable to construct the new unit for the intensive quantity. She learned from the coin problem that the goal was to be able to express a number of one unit in terms of another single unit, but she did not have a strategy for determining which units in the pizza problem were analogous to the dimes and quarters. In subsequent attempts, Rhonda began her attempts at interpreting $7/3$ (the number of girls per pizza) by saying, "For every *one* girl..." just as she stressed "one quarter" in her explanation of the ratio $5/2$. However, using the wrong unit as the single unit in the pizza problem, she confused herself and began to doubt the correct solution that she had developed with the manipulatives.

Throughout the conversation from the previous passage, Rhonda was working constantly with the calculator, never referring back to the manipulatives she had used earlier. She seemed to be working with the fractions and the quotients in a decontextualized manner and simply filling in the blanks with units. She was using the language of the context of the problem but was not interpreting her calculations within the context of the problem. In terms of the overlapping model of ratios and fractions, once she represented the relationship in fractions and started dividing, she moved into the realm of the number type, at first for fractions and then for decimal expressions. She

knew that her new answer conflicted with her previous answer but could not return to the intersection to resolve the conflict.

After I asked her to reconsider the manipulatives on the table, the following exchange took place.

Rhonda: It shows that there's one pizza and two and a part of a girl, but if you divide it equally, one person (touching one of the blue towers) would get three slices and one person (touching one of the red towers) would get almost two and a half slices. But the sizes of the pizza could be different.

Interviewer: We're going to assume that the pizzas are the exact same size and that the girls are going to share equally and the boys are going to share equally. So who do you think gets more pizza, a girl or a boy?

R: A boy because if you divide it out the boys get more. But if you do it with this (pointing to the manipulatives) it looks like the girls get more.

I: It seems like you're looking at our situation on the table here and coming up with one answer, but when you do the numbers you get another answer. Is that right?

R: Yeah.

I: Which one do you think is correct?

R: This one probably, when you divide it out, because it's more accurate.

Rhonda had no strategy for resolving the conflict between her two different answers. She was trying to apply the phrasing of the relationship in the coin problem to the pizza problem, but she was not able to make the conceptual connection. After I

encouraged her to write the labels “girls” and “pizzas” with her fraction $7/3$, she was finally able to make the link between the two units of the numerators of the ratios: “Since that’s how many dimes there are, that’s how many girls there are...if you had one pizza.” She then decided that each girl got more pizza than each boy.

Doing the coin problem did seem to help some of the girls interpret the ratios in the pizza problem. For example, when Sheila was asked to return to the pizza problem and interpret the numbers she had written, she said, “For one pizza, there’s 2.3 girls, which is less than 3 boys, so they don’t have to share as much as they do.” She was finally able to arrive at the same conclusion by both her methods, using numbers and using diagrams, but only after an additional four minutes thinking about the numbers. Sheila was the least talkative of all the participants; therefore, it was very difficult to make inferences about her thought processes.

Interfering Factor: Variable of Conversion

When asked to write ratios of dimes to quarters, some of the students initially wrote or said an expression, such as Mandy’s description of the fraction “50 over 50,” that indicated that they were still focused on the value in cents of both sets of coins and had not yet moved forward to thinking about the extensive quantities of dimes and quarters. In the coin problem, unlike the pizza problem, the students showed no aversion to talking about a non-integer ratio for a discrete variable. In this problem, I view the number of dimes as a discrete variable because, in a physical sense, a portion of a dime does not retain the relevant characteristics of a whole dime. In other words, a dime split in half is not worth half a dime. But cents, the variable to which the participants easily converted, can be interpreted as a continuous variable. Because they could easily convert

to cents, they had no trouble discussing a portion of a dime in theory. When discussing the number of girls sharing a pizza, the participants had trouble interpreting and discussing two and a third girls; in the coin problem they had no trouble with two and a half dimes.

One major distinction between the pizza problem and the coin problem is that the pizza problem is an associated-sets problem, for which the ratio is defined by the relationship stated in the problem, and the coin problem involves a ratio that is culturally based and consistent. This difference is illustrated in the difference between how Susan phrased the ratio relationships. When asked to interpret the ratio in the coin problem, she said, “Two and a half dimes equals one quarter,” even before I arranged the coins and placed the pencil down the middle, but when she was finally able to interpret the ratio in the pizza problem, she said, “It’s the number of girls at each pizza.” For associated sets such as girls and pizza, there is no variable of conversion that enables the problem solver to establish equivalence, but on the other hand, there is no variable of conversion that distracts students from the given extensive quantities.

The Orange-Juice Problems

The last problem set on the protocol for the first interview was the group of orange-juice problems from the test of proportional reasoning, for which the students had to compare two mixtures of orange juice and determine which has the stronger orange taste.

Notation for Ratios

On the test of proportional reasoning, the amounts of orange juice and water for each problem are given in an ordered pair, such as (2,5) and (3,8). One of the students, Susan, used this notation in her solution and demonstrated quantitative proportional reasoning by finding a common amount of water for two mixtures equivalent to the original two mixtures and comparing the new mixtures. For example, she chose (2,5) as having a stronger orange taste than (3,8) because (16,40), which would taste the same as (2,5), would have a stronger orange taste than (15,40), which is equivalent to (3,8).

Table 5 shows the notation and strategy used by each student on the orange-juice problems on the test and whether she was successful at solving them. With the additional evidence provided by the participants in the interviews, I was able to determine each student's strategy and to determine whether a student had simply guessed correctly or had chosen the correct answer based on appropriate reasoning.

Table 5
Summary of Participants' Performance on the Orange-Juice Problems

Participant	Notation	Strategy	Overall Success
Angela	Fractions	Visual, based on her drawings	No
Corina	None	None apparent	Yes, by guessing
Kim	Fractions	Common denominator	Yes
Mandy	None	Visual, based on the pictures given on the test	No
Marie	None	Long division	Yes
Melinda	None	None apparent	Yes, by guessing
Prema	None	Visual, based on the pictures given on the test	No
Rhonda	Fractions	Common denominator	Yes
Sheila	Fractions	Common denominator	Yes
Susan	Ordered Pairs	Equivalent ratios	Yes

Emerging Theme: Mixed Fractions as Ratios

When planning the activities before the interviews, I considered the most advanced level of understanding of ratios as intensive quantities that the students could express would be an interpretation of a pair of ratios as intensive quantities and a successful comparison of them in the context of the problem and then an interpretation of the multiplicative inverses and a successful comparison of them, arriving at the same conclusion from both comparisons. Expressing this understanding, students should be able to coordinate the flipping of the fractions with the reversal of the units that compose the intensive quantity. But while working with students, I realized that students could

often interpret some pairs of ratios and compare them within the context of the problem without being able to interpret the multiplicative inverses of those ratios.

Corina had spent almost 15 minutes working on the comparison of the (2,5) and (3,8) trays, during which she continued to say that the two mixtures would taste the same, before having a breakthrough. During this time she was basing her answer on visual evidence from a drawing she made and on a hybrid additive-multiplicative strategy of subtracting the corresponding numbers of the trays and working with a ratio of three glasses of water to one glass of orange juice. She had written the ratios for the trays as $\frac{2}{5}$ and $\frac{3}{8}$ and had successfully converted those fractions into decimal expressions using the calculator. But she continued to insist that the two mixtures would taste the same; she made no connection between the difference between the decimal expressions and the difference in how the mixtures would taste. Seven minutes into our work on this problem, I asked her if she could figure out how much of a glass of orange juice to add to only one glass of water to make each mixture. I thought that because of her successful conversions from fractions to decimal expressions she might be able to make the connection between the amount of a glass of orange juice for one cup of water and the original ratios of $\frac{2}{5}$ and $\frac{3}{8}$ that she wrote. This strategy was unsuccessful. After this I told her that the ratios of $\frac{2}{5}$ and $\frac{3}{8}$ in this problem can be interpreted as the amounts of a glass of orange juice needed for one cup of water. She nodded and said “Oh!” several times, but when I asked her for her answer, she still claimed that the two mixtures would taste the same. For another five minutes she continued using her previous strategy in response to my questions, until the following exchange took place.

Corina: I just realized that I was wrong because since there are five cups of water

here and five divided by two is two and a half, so one cup of orange juice couldn't equal three cups.

Interviewer: OK, so one cup of orange juice equals what for Tray A?

C: Two and a half cups of water.

I: What about over here, Tray B?

C: One of the cups equals 0.375 cups—well, 0.375 cups of orange juice. One cup of orange juice equals two and something cups of water.

I: It sounds like what you're figuring out is the ratio of cups of water to cups of orange.

C: Uh-huh.

I: What is that for Tray A?

C: For Tray A, it's two and a half cups of water.

I: And Tray B?

C: It's two and...I don't know how to figure that out.

Corina at this point knew that she could solve the problem by determining the amount of water per cup of juice for each mixture and she knew that division was the appropriate operation, but she still could not formulate the mathematical expression, with the right numbers and operation, to find what she needed. At this point, she returned to the diagram on the test and shaded the blocks that represent the glasses of water and drew arcs to match them with the blocks that represent the glasses of orange juice, as shown in Figure 12. The first orange-juice block is matched with the two water blocks with

diagonal stripes, the second orange-juice block with the two water blocks shaded in, and the third orange-juice block with the two water blocks with horizontal stripes.

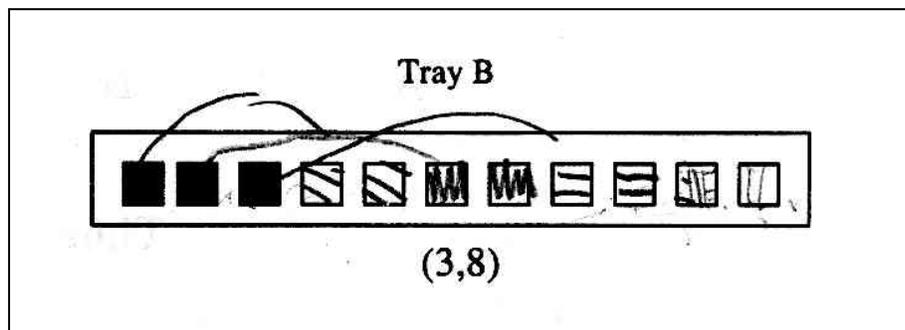


Figure 12. Corina's Matching Strategy to Compute the Ratio

She then divided the remaining two blocks of water into thirds and assigned two of the six thirds to each glass of orange juice. She explained this as follows.

Corina: So I guess each cup [of orange juice] would get a third of each of them.

They each get a third of that cup, and they each get a third of that cup too.

So each one gets two and two thirds of a cup.

Interviewer: Each cup of orange gets two and two thirds cups of water?

C: Yeah.

I: So which tray do you think is stronger?

C: This one (pointing to Tray A on the test).

I: Tray A?

C: Uh-huh.

I: Because?

C: Oh yeah, because if you only put two and a half cups of water that's less water than two and two thirds. Well, whichever cups of water per one cup of orange juice is less, then that one's stronger.

She was able to interpret the mixed fractions within the context of the problem and choose the correct tray, which had the smaller ratio associated with it. Therefore, she avoided the pitfall of “larger number is correct answer” that was common in the pizza problem. She did not realize that dividing eight by three and five by two would give her these numbers, but she was able to accomplish the division with the diagrams. She was able to solve the problem in 3 minutes by looking at the ratio of cups of water to cups of orange juice after making no progress for 15 minutes when working with the ratios of cups of orange juice to cups of water. The ratios greater than one were easier for her to visualize with the diagrams and easier for her to interpret, even though she had to match the smaller number with the correct answer.

Marie used long division to get what she called a measure of concentration for each mixture—a concentration of .4 for Tray A and .375 for Tray B. When asked if she could express the ratios as fractions, she said “two fifths and three eighths,” but when asked to interpret $2/5$, she said, “For every one cup of orange juice there’s two and a half cups of water.” So instead of interpreting the ratio as a proper fraction, she interpreted the inverse with the units reversed and stated the ratio as a mixed fraction. When I asked, “What about for every water?” she said, “And then for every water there’s...” and paused for 29 seconds before asking to use the calculator. So even the most advanced student in the group was not able to immediately identify the meaning of the ratio $2/5$ in this problem, even though she had used this ratio in her solution on the test. When asked to interpret $3/8$, Marie said, “Well, there are three orange juices and eight waters, and then you could divide three into eight. And that’s how much...If I divide three into eight, I

would get two and two thirds, and that's how many waters each glass of orange juice would have." Again, she reversed the units, interpreted the inverse, and stated the ratio as a mixed fraction. The pause in this last statement seems important because she started to interpret the ratio but then stated it as a mixed fraction before giving her interpretation. This time, she also stated the inverse relationship without being prompted: "And the other way around...each water would get three eighths of an orange-juice glass."

The students' work on this orange-juice problem provided further evidence of their preference to state ratios in mixed fractions, as they did on the pizza problem. Overall, the students demonstrated that the way they write the ratios depends on the order of the numbers in the problem. Therefore, on the pizza problem, most students worked with ratios greater than one ($7/3$ and $3/1$), and on the orange-juice problems, the students worked with ratios less than one because the number of glasses of orange juice is given before the number of glasses of water. From Corina's work, it is apparent that she was able to understand the ratios greater than one but not the ratios less than one, which is related to students' preference of stating ratios as mixed fractions in all three of the problems discussed in the first interview.

Emerging Theme: Decimal Expressions as Ratios

Melinda, whose subset score on the test was the lowest of all the participants, answered the orange-juice questions on the test correctly but provided no work or explanation as evidence of proportional reasoning. In the interview, she demonstrated proportional reasoning by saying that the mixtures of (2,3) and (4,6) would taste the same because the first mixture is just the second one "split in half" and is therefore just "a smaller version." However, when asked to compare the (2,5) and (3,8) mixtures, she

focused on the differences between the extensive quantities between the mixtures and based her explanation on visual evidence.

When asked if she could use ratios to help, she stated the ratios as two fifths and three eighths, but using the notation for long division, she wrote five divided by two and eight divided by three and then used the calculator to get the quotients, which she wrote as 2.5 and 2.7, respectively. When asked which tray would have the stronger taste, she initially said Tray B but quickly changed her answer to Tray A. Unable to put her explanation into words, she became frustrated, sighing heavily, shaking the pencil and tapping it against the table, and pressing her hand against her forehead. But she stuck with Tray A as her answer. My guess is that she instinctively chose the larger number and then caught herself because she did have some idea of what 2.5 and 2.7 mean in terms of the problem. Therefore, this problem seems to have potential breaking the reaction that the correct answer should be the one associated with the larger number. She did not state the ratios as mixed fractions because she used the calculator to divide. Independent of the preference to state the ratios as mixed fractions, there also seems to be a pattern of students being able to interpret ratios greater than one more easily than ratios less than one.

Interfering Factor: Part-Part Versus Part-Whole Representations

The students who used fractions in their solutions wrote the fractions in the form a/b where a is the number of glasses of orange juice and b is the number of glasses of water. However, there was one exception, Rhonda, who wrote her fractions as part-whole representations, where b is the number of total glasses of liquid. When asked about the (2,5) and (3,8) mixtures in the interview, she stated the ratios as two to five and three to

eight and checked her answer by dividing two by five and three by eight on the calculator. When asked about dividing by five on the calculator but writing seven in the denominator on the test, she explained the difference in how she chose the denominator for each of the fractions but could not interpret either $2/7$ or $2/5$ in terms of the problem. Her explanations were solely in extensive terms: To her, $2/7$ meant that “out of seven cups you only have two cups that are orange” and $2/5$ meant “there’s two cups of orange juice and there’s five cups of water.”

Later, after a discussion about the meaning of these two fractions in terms of the problem, I asked her to describe the ratio of the (2,5) mixture to a friend who liked the taste so that her friend could make any amount of that beverage. The following exchange took place.

Rhonda: I could tell them that there are two cups to five water—two cups of orange juice.

Interviewer: Two cups of orange juice to five cups of water?

R: Yeah.

I: What if they really liked fractions, could you explain it to them?

R: OK, I could do two fifths of the water that you add—well, that’s just for one cup—so two fifths of the liquid you add is going to be orange juice.

I: Could you say that last part one more time?

R: Two fifths of the liquid that you’re going to add is going to be orange juice and the rest is going to be water.

I: OK, so how many fifths will be water?

R: Three.

I: So do you think two fifths of the pitcher being orange and three fifths of the pitcher being water will taste the same as this?

R: Yes.

During this segment she was looking at the part-part fractional representation but interpreting it as a part-whole representation. When she tried to verify this interpretation by using the calculator, she became confused and returned to the drawing on the test. She eventually explained the ratio as two and a half cups of water for every cup of orange juice, just as Corina and Marie did.

The Cookie Problem

The second interview began with the cookie problem, which I designed mostly as a warm-up problem, thinking that it might help the girls get in the proper frame of mind after not having done any math for a few weeks. I refer to the cookie problem as a *multiple-missing-value problem* because the students have to fill in the amounts for multiple ingredients in the new cookie recipe, for which ten eggs is the only given amount. The underlying concept of scaling is the same for the cookie problem as for the coffee problem that the students answered on the test of proportional reasoning. For the coffee problem, the scale factor is 1.5 because the number of cups of water increases from 8 to 12, and for the cookie problem, the scale factor is 2.5 because the number of eggs increases from 4 to 10. The difference between the problems is that for the coffee problem the students only had to fill in one missing value as opposed to five missing values (only three of which are unique) on the cookie problem.

Table 6 compares the students' strategies on the coffee problem, a standard missing-value problem, and the cookie problem, a multiple-missing-value problem. The evidence for the coffee problem came from the test of proportional reasoning only, and the evidence for the cookie problem came from the second interview only.

Table 6
Comparison of Strategies on Missing-Value Problems

Participant	Strategy on Coffee Problem	Strategy on Cookie Problem	Overall Success on Coffee Problem	Overall Success on Cookie Problem
Corina	No attempt	Building up	No	Yes
Kim	Crossmultiplication	Building up	Yes	Yes
Mandy	Within	Building up	Yes	Yes
Melinda	Additive	Estimating	No	No
Prema	Crossmultiplication	Hybrid Additive-Multiplicative	No	No
Rhonda	Within	Building up	Yes	Yes
Sheila	Crossmultiplication	Building up	Yes	No
Susan	Within	Between	Yes	Yes

Kim and Rhonda both made small computational errors on the coffee problem on their test and did not get the answer exactly correct, but they both used and carried out a successful strategy. Therefore, for the purpose of discussing strategies on missing-value problems, their work is counted as successful.

Prema began the cookie problem by expressing a strategy that Lamon (1993) calls *random operations*. She looked at the difference of six eggs between the two recipes and said, “Maybe you have to add or multiply by six, and I’m going to go with multiply.”

Only one student, Susan, used the scale factor to write the new recipe. Using the between strategy, she immediately divided ten by four and applied the ratio to all the amounts in the original recipe. Building up was by far the most common strategy. The students did the problem in batches of cookies, thinking about the original recipe with four eggs, another batch with four eggs, and a half batch with two eggs. With the other ingredients, they used a base ratio of the amount of that ingredient to four eggs and “built up” by saying, for example, that a quarter pound of butter goes with the first four eggs, a quarter pound with the second four eggs, and then another eighth of a pound to go with the final two eggs.

What was surprising was that, with the exception of Susan, the students did not use two and a half as a multiplier, even when the transformation was from one teaspoon or one cup to two and a half. When the original amount was one unit, the students stuck with the primitive step-by-step building-up process. For example, after building up all the other ingredients, Rhonda explained her reasoning for needing two and a half cups of flour for the new recipe: “I did one plus one is two because you add those four plus four [eggs], but then you have to add half of four which is two to get ten and then you do half of one is a half.”

Another surprise was that the difficulty of this problem for the students did not seem to be directly correlated with the difficulty of other problems. For example, Corina and Mandy took a long time on all the problems in the first interview and struggled, but

they did this problem quickly. Rhonda also solved the problem quickly. However, Kim and Sheila, two of the students in the High group from their test results, took much longer to do the problem. Sheila and I spent 19 minutes on this problem, and she was only able to solve it after being asked leading questions. One plausible explanation for why this problem broke the pattern of success is that students' solutions to this problem are the most likely of all the problems used in this study to be influenced by their personal experiences. For example, Mandy mentioned that she loves to cook and knows how to cut recipes in half and that she enjoys baking.

Sheila, whose score was in the High group on the test of proportional reasoning, began the cookie problem with an additive strategy but then realized that adding six units to all the ingredients would mean going from a recipe with a quarter pound of butter to a recipe with six and a quarter pounds of butter, which seemed too much. She then doubled the recipe, after which she explained, "If you use multiplication it makes everything the same, but two times bigger." When asked again about a recipe with ten eggs, she said nothing and wrote nothing for almost two minutes, until the following exchange took place.

Interviewer: What are you thinking about now?

Sheila: I'm trying to think—because there's two [eggs] left to get to ten and two is half of what we multiplied it by, but four plus four is eight. And then you take half of these and then add it.

I: Half of what?

S: Half of the eight, which would be four, so it would be...

[She paused for over a minute, not writing or saying anything.]

I: Do you think we could write out a recipe so that we would use exactly ten eggs and still make the cookies taste the same?

S: Some way.

I: Well, first of all, when you came up with all this—one half pound of butter and eight eggs and all that—do you think that will make the cookies taste the same?

S: I think so.

I: So what was your strategy for going from four eggs to eight eggs?

S: I knew that eight is a multiple of four and ten isn't, so eight was the closest.

I: So what would your strategy be if we wanted to make a recipe with 12 eggs?

S: Four times three.

I: OK, so what about ten eggs?

S: Two and a half.

Sheila then used the calculator and the scale factor of two and a half to determine the amount of each ingredient in the new recipe. After she finished, she stated her new between strategy: "If we use 10 eggs, then it's 4 times 2.5 equals 10. So that's the ten, and then we need to do everything else by 2.5." At first, it seems that she had the right idea about how to build up but then became confused about whether the final step in the build up was half of the original recipe or half of the doubled recipe.

The most important result from the cookie problem is that no student used crossmultiplication to solve it. The most efficient solution is to use the between strategy, which Susan used successfully. It is surprising that students who are about to enter

Algebra I are still relying on building up to solve problems like this. Because of Sheila's progress, it seems that multiple-missing-value problems may have great potential in helping students who know how to use the crossmultiplication algorithm but do not understand it conceptually make the transition from building up to applying a between strategy.

The Sidewalk Problem

The first of the two numerical comparison problems in the second interview was the sidewalk problem, which asked the students to determine which of two girls has the longer stride—Alice, who covers the distance between three cracks in the sidewalk in four steps, and Jill, who covers the distance between five cracks in seven steps. Table 7 shows the participants' initial answers and strategies before the detailed questions about their work began.

Table 7
Summary of Participants' Strategies on the Sidewalk Problem

Participant	Initial answer	Initial strategy
Corina	Alice	Additive
Kim	Alice	Comparison over common number of steps
Mandy	Not enough information given in the problem	Additive
Melinda	Alice	Writing the ratios as fractions and choosing the larger number
Prema	Jill	Comparing extensive quantities
Rhonda	Jill	Visual only
Sheila	Alice	Writing the ratios as fractions and finding a common denominator
Susan	Alice	Writing the ratios as fractions and choosing the larger number

Emerging Theme: Common Reference Value for Comparing

In her solution, Kim did not express the ratios as single numbers but did base her diagram on the numerical relationships so that she could make a direct quantitative comparison. She extended the pattern for Jill's seven steps for three iterations and Alice's pattern of four steps for five iterations, giving her a way to compare each girl's number of steps when they both reach the fifteenth crack. To reach that point, it takes Jill 21 steps, but it only takes Alice 20. When asked if she could also solve the problem using only ratios, Kim wrote the ratios as $5/7$ and $3/4$, crossmultiplied, and determined that $3/4$ is larger. But she could not explain how this supported her answer of Alice. Figure 13 shows Kim's work on the sidewalk problem.

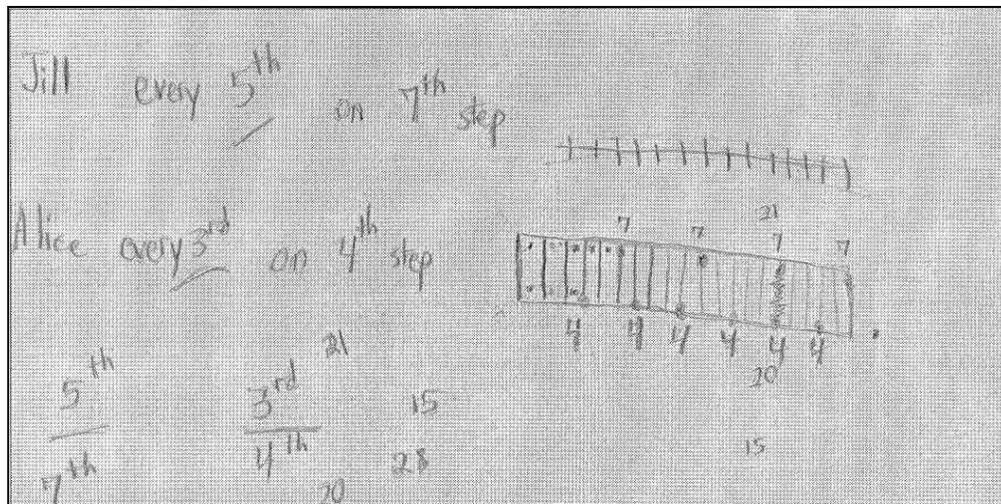


Figure 13. Kim's Extended Sidewalk and Crossmultiplication

Sheila solved the problem with the same general strategy but using fractions instead of a diagram. She wrote the ratios as $\frac{5}{7}$ and $\frac{3}{4}$, found a common denominator, and converted the numbers to $\frac{20}{28}$ and $\frac{21}{28}$. Her focus was on the extensive quantities in the new fractions, but the common denominator gave her a way to make the comparison without needing to understand the numbers as intensive quantities. When asked what the numbers mean in terms of the problem, her explanation was limited to the extensive quantities represented by the numbers in the new fractions. But Sheila's understanding of the extensive quantities in this case was sufficient for developing a solution in the context of the problem and based on her conceptual knowledge. Sheila chose Alice because, "In 28 steps, she stepped one block further." Sheila's solution is an important example of how students can develop a solution based on quantitative reasoning and on mathematical symbols, but without an understanding of the intensive quantities involved. She wrote the ratios, found a common multiple for two parallel elements across the ratios, converted the ratios to equivalent ratios with a number in common, and interpreted and compared them within the context of the problem. Sheila

was not thinking that Alice's step covers $21/28$ of a distance of a block compared to $20/28$ of a block that Jill steps.

In terms of the overlapping model of ratios and fractions, this example shows how students can move from right to left, from using their learned procedures with fractions to a ratio-based interpretation of those fractions, without necessarily understanding the numbers as intensive quantities. Sheila's path is illustrated later (see Figure 14) in contrast to Susan's path on this problem.

Confirming Evidence: Decontextualization

When I asked her how she would approach the problem, Susan said, "Put it into fractions and see which one is larger." This is the most succinct explanation of the strategy that we inferred (Clark & Berenson, 2002) from students' written work on the pizza problem. Melinda expressed the same strategy: "You do five over seven and Alice has three over four, and you see which one is bigger." Melinda then drew two pie charts, one with five of seven slices shaded and one with three of four, and decided that the numbers are about the same.

Rhonda's initial strategy was based on her diagram and was purely visual. She answered Jill but provided no rationale. After I asked her, "Is there any way to think about it in terms of the numbers?" she wrote the ratios as $5/7$ and $3/4$ and drew pie charts, the same way that Melinda did. After I reminded her of the calculator, she used it to divide and changed her answer to Alice "because .75 is bigger than .71." When asked what .75 means in terms of the problem, she answered, "Some kind of measurement maybe—feet or something."

I used the following line of questioning to encourage Susan to explore the meaning of her fractional representations of the ratios in terms of the problem.

Interviewer: Why did you choose to write the fractions in this particular way?

Susan: Because five is the first number and seven is the second.

I: What if I said, “They noticed that on Jill’s seventh step she steps on the fifth crack.” What if I turned it around?

S: I’d probably write it as seven over five, but then you have to make it into a proper fraction.

I: Do you prefer working with fractions when the denominator is bigger?

S: Yeah.

I: If I turn the sentences around, would it still be the same question?

[Susan writes $7/5$ and $4/3$, changes them to mixed fractions, and then converts the mixed fractions based on the common denominator of 15.]

S: No, it wouldn’t be the same.

I: What did you get when you wrote them in the reverse order?

S: That they’re different.

I: When you wrote them like this (pointing to the first pair she wrote), which number was bigger?

S: Alice’s.

I: When you wrote them like this (pointing to the second pair she wrote), which number was bigger?

S: Jill’s.

I: Do you still agree with what you said originally?

S: No, guess not.

At this point, Susan's tone of voice and body language suggested that my causing this state of disequilibrium was not going to prompt her to be motivated to try to resolve the conflict between the answers. In an attempt to encourage her to work more on the problem and talk more, I told her that she had been right all along, that Alice was the correct answer, and that I had been trying to "bug" her to get her to explain more. She did start talking more and offered an explanation to resolve the conflict: The correct way, as she understood it, to write fractions was that the smaller number should be on top; therefore, when she wrote the fractions the correct way and compared them, Alice's number was larger. Susan's faith in this strategy did not seem to be firm, but she stayed with it as a possible explanation for why her first comparison gave her the correct answer and the second comparison did not.

Sheila and Susan each wrote the fractions $5/7$ and $3/4$ to represent the ratios in this problem, but they compared the fractions using different methods—finding a common denominator in the case of Sheila and crossmultiplying in the case of Susan. Figure 14 shows their paths against the backdrop of the overlapping model of ratios and fractions.

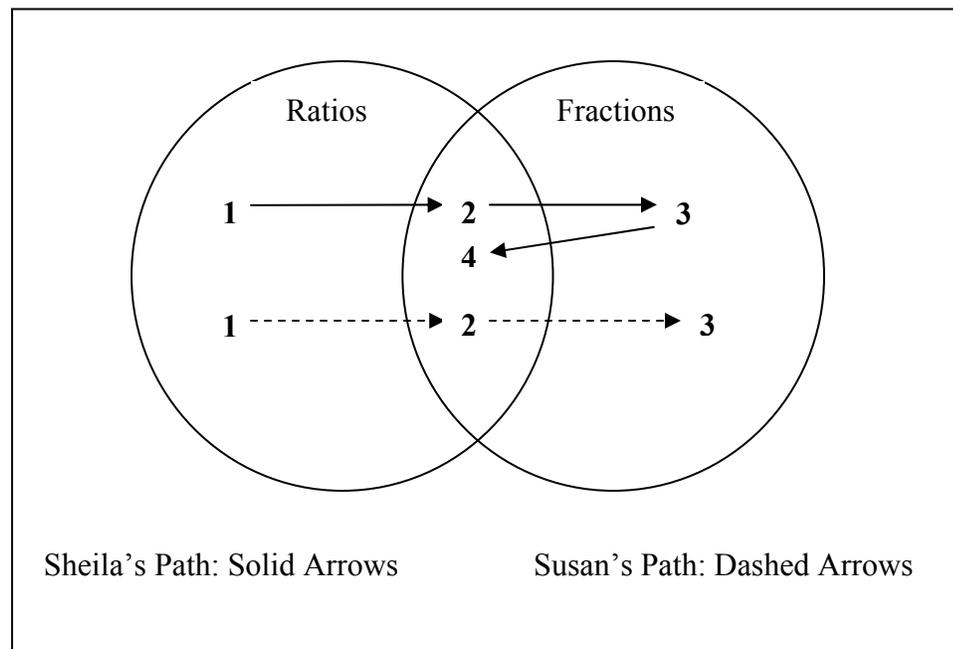


Figure 14. A Comparison of Sheila's and Susan's Paths for the Sidewalk Problem

Each of them recognized the ratios (1) and wrote the relationships as fractions (2). When Sheila compared the fractions in terms of their numerical values (3), she found the common denominator of 28 and wrote the equivalent fractions $20/28$ and $21/28$. Susan, however, crossmultiplied and chose the larger number without interpreting her work in terms of the problem. Her path ends at this point.

Sheila demonstrated right-to-left movement back to the intersection by interpreting her new fractions in terms of 20 and 21 blocks in the sidewalk for 28 steps for each girl (4). She interpreted these fractions as ratios, but only in extensive terms. Sheila's path shows an example of how students can return to the intersection of the model but without necessarily interpreting the ratios as intensive quantities. She seems to be gradually making a connection, but for her, the concepts have not converged at the point where she understands the fractions simultaneously as ratios and as rational

numbers. There is still this toggling between viewing $20/28$ and $21/28$ as ratios of blocks to steps and understanding them as numbers.

In general, students in this study who compared fractions on the basis of a common denominator were more likely than those who compared fractions by crossmultiplication to make this right-to-left movement back to the intersection. It seems that when students crossmultiply to compare fractions to help them solve a numerical comparison problem, they abandon the context of the problem completely. When a student compares the fractions by finding a common denominator, she is more likely to preserve the units in the numerators and denominators. However, when crossmultiplying numerators and denominators, attention to units disappears because the crossproducts have no units. The most likely explanation for the preservation of contextual elements of the problem for those who find a common denominator is that their new fractions match their old fractions in terms of the units in the numerators and denominators and they are able to recall these units easily, use the denominator and its unit as the common reference point, and compare the numerators in the context of the problem.

Prema's solution was similar to Susan's, but with a bias in favor of improper fractions. Prema compared the decimal-expression equivalents for Jill's ratio $7/5$ and Alice's ratio of $4/3$ and chose Jill for her answer. After encouraged to investigate and compare the inverses, Prema insisted that her original fractional representations were the correct way to write the ratios for this problem but could not explain why. As in the case with Susan, Prema was attempting to explain why one comparison was valid and the other was not. The difference between the two students is that Susan based her answer on

the comparison of the less-than-one ratios and said that comparing the greater-than-one ratios would be incorrect, and Prema did the opposite.

In terms of her ability to make transitions between numbers in different forms and in terms of her overall sense for ratios, Susan was overall the strongest of the eight students who participated in all three phases of the study. Of these eight, she had the most sophisticated and direct solutions for the orange-juice problems and the cookie problem, and her solution for the pizza problem was the most creative and efficient in the sense that she drew just enough, without computing a numerical answer, to make her decision that each girl got more pizza than each boy. But on these two more difficult numerical comparison problems during the second interview, her numerical strategies failed and she became frustrated. She showed no evidence of progress during the second interview.

Interfering Factor: Fence Versus Fenceposts

In some of the diagrams that the girls drew, they counted one too few blocks in the sidewalk for the given number of steps. For example, when Prema drew arcs on her diagram to represent the distance covered by Alice in four steps, she marked a distance of three blocks with her first arc, two blocks with the second arc, two blocks with the third arc, and three blocks with her fourth arc. This conflict is commonly referred to as the problem of the fence and the fenceposts because instead of counting the number of the units of interest (such as the length of a section of fence) the student will count the number of dividers between those units (the fenceposts). In some cases, Prema started her three count on the initial crack in the segment and therefore only covered the length of two blocks instead of three with her arc.

A related factor that caused some confusion was that one of the units of distance, a block in the sidewalk, was not mentioned by name in the problem. So the students used “cracks” when they talked about that distance. This, in general, did not seem to be a major obstacle; however, the focus on the dividers—in this case, the cracks—in the wording of the problem could contribute to students focusing on the dividers rather than the distance in their diagrams or in their counting. One change I would make in the wording of the problem would be mentioning the number of blocks in the sidewalk as the distance covered in a certain number of steps.

As with all the numerical comparison problems, students had a difficult time constructing the unit for the intensive quantity. For example, Corina at first used the unit of feet for the ratios in the sidewalk problem, but then she changed her mind to yards because less than a foot seemed too short of a distance for a person’s step. This strategy of assigning a unit is similar to the assignment of slices in the pizza problem. In the pizza problem, the students knew they needed a unit of measure for an amount of pizza, and “slices” made sense for the numbers 2.3 and 3. Corina knew that she needed a unit for a measure of distance and then picked the unit of distance that made sense with the numbers she had computed. In both cases, students introduced units that were not given in the problem.

The Fertilizer Problem

The final problem was the fertilizer problem, which asked the students to determine which of two workers can fertilize grass faster—Tony, who can fertilize 3 acres in 8 hours, and Russ, who can fertilize 5 acres in 12 hours. Table 8 shows students’ initial answer and strategy before being asked detailed questions about their work.

Table 8
Summary of Participants' Work on the Fertilizer Problem

Participant	Initial answer	Initial strategy
Corina	Russ	Dividing and comparing the unit rates
Kim	Russ	Writing the ratios as fractions and finding a common denominator
Mandy	Russ	Writing the ratios as fractions and finding a common denominator
Melinda	About the same	Comparing extensive quantities
Rhonda	Russ	Writing the ratios as fractions and choosing the larger number
Sheila	Tony	Writing the ratios as fractions and choosing the larger number
Susan	Tony	Writing the ratios as fractions and choosing the larger number

Prema and I spent more than an hour discussing the cookie problem and the sidewalk problem. Her lack of understanding of ratios and division inhibited her progress. She was cooperative, enthusiastic, and talkative, but also overconfident, not willing to challenge her assumptions. She made no progress during the second interview. I spent some extra time with her trying to help her with division, which I thought would be more helpful to her than giving her another problem that she would not be able to do. We ran out of time, and I did not ask her to work the fertilizer problem.

Emerging Theme: Common Reference Value for Comparison

Kim solved the fertilizer problem by finding a common reference point for her comparison, just as she did in her solution to the sidewalk problem. However, this time she used fractions only and did not rely on a diagram. (No student tried to draw anything

for this problem, probably because the two units of area and time are difficult to represent in a drawing.) With the fractions $\frac{8}{3}$ and $\frac{12}{5}$, Kim found the least common multiple of the denominators and converted the fractions to $\frac{40}{15}$ and $\frac{36}{15}$, respectively. She chose Russ “because it would take him 36 hours to do 15 acres while it takes Tony 40 hours to do 15.” But her interpretation of the fractional representations of ratios remained in extensive terms only. After reading the problem, she wrote “Faster = T or R” below her fractions, and it is clear from her work that she initially circled “T” for Tony and then erased it, perhaps because her initial reaction was to choose the larger number before realizing what the 40 and 36 represented.

Kim’s strategy and response fits the pattern that students who compare fractional representations of ratios by finding a common denominator are more likely to maintain the context of the problem in their solution than the students who compare fractions by crossmultiplying. Those who crossmultiply operate strictly in the fraction-only realm of the overlapping model and make their final choice based on the results of the algorithm with perhaps a selection criterion influenced by a key word from the problem. The students who compare the fractions by finding a common denominator are more likely to interpret their new fractions in terms of the problem, as Kim did, even though they may not understand what their original fractions mean as intensive quantities.

Emerging Theme: Mixed Fractions as Ratios

Corina was the weakest of the students in terms of her knowledge of procedures frequently taught in middle school. She did not have any strategy for solving the coffee problem on the test—the straightforward missing-value problem for which the overwhelming majority of the students set up a proportion and crossmultiplied. Corina

described her frustration with procedurally-focused mathematics instruction this way: “I never follow my book. Every chapter in my book tells me how to do something new. I really never understand it, so I just figure out my own crazy way to do it.” Corina never called upon a procedure that she had learned to solve anything. She approached each problem as a new experience, unlike some of the others who, after reading a problem, immediately started searching for a method to apply.

After reading the fertilizer problem, she said, “Well, I’m thinking about dividing; I just don’t know why. I’m trying to think of a reason to and it’s just not coming to me, so I’m just going to divide.” Figure 15 shows Corina’s work on the fertilizer problem. She wrote “T” for Tony and “R” for Russ. She began with the improper fractions at the top as she was collecting information from the problem. She wrote what I am calling *mixed fractions within a unit rate* after she divided each of the pairs of numbers. She did the division in her head and did not touch the calculator while working.

The image shows handwritten mathematical work. At the top, there are two columns. The left column is headed 'T' and contains the fraction $\frac{8}{3}$. The right column is headed 'R' and contains the fraction $\frac{12}{5}$. Below these, there are two mixed fractions separated by a vertical line. The left mixed fraction is $2\frac{2}{3}$ with a 'T' above the fraction part, followed by 'hrs.' and a horizontal line below it, and '1 acre' written below the line. The right mixed fraction is $2\frac{2}{5}$ with an 'R' above the fraction part, followed by a horizontal line below it, and '1 acre' written below the line.

Figure 15. Corina’s Mixed Fractions Within a Unit Rate

Kim used the same notation during the first interview for the pizza problem when she converted seven girls over three pizzas, all in a fraction, to two and a third girls over one pizza. Corina's explanation for her solution to the fertilizer problem is as follows.

Corina: Well, I know that eight divided by three is two and two thirds. I don't know two and two thirds of what exactly. Oh yeah! Two and two thirds hours for one acre.

Interviewer: How did you get that?

C: Well, I just divided. I just did that (writing the division operation in symbols) like that.

I: You said it was two and two-thirds hours for one acre. How were you able to keep that straight?

C: Well, the eight hours is on the top, so I put the two and two-thirds hours on the top. And then the acres is on the bottom, so I put the acres on the bottom.

I: OK.

C: It takes two and two-fifths hours to do one acre (writing the mixed fraction within a unit rate for Russ), I think, so since this number of hours is more than that one, it takes him less time to do one acre than it does Tony. So Russ works faster.

She confirmed her answer, and my asking her which number is larger did not rattle her confidence. Her use of mixed fractions as ratios and her ability to interpret them in terms of this problem are similar to her work on the orange-juice problem during the first interview. Corina and Melinda were a grade behind the other students in the study,

and the second interview with Corina took place three weeks before her twelfth birthday. For an eleven-year-old who had just completed the sixth grade, her solutions were outstanding. And her solutions were her own, influenced little by rules and routines learned at school. Therefore, her comfort with using mixed fractions as ratios and her ability to solve numerical comparison problems with two continuous variables by her own methods suggests that problems that fit this description could be an excellent starting point for students exploring intensive quantities. Starting with integers and moving to simple and then more complex mixed fractions seems like a natural progression.

Corina was the only participant who consistently used labels in her work. She always had the information she needed on her paper. She did not get confused about which numbers were associated with which characters in the problem and did not have to refer back to the problem the way that many of the girls did.

Emerging Theme: Decimal Expressions as Ratios

On this problem, students were more likely to be able to interpret decimal expressions of ratios as intensive quantities than fractions as intensive quantities. Rhonda wrote the ratios for Tony and Russ as $3/8$ and $5/12$, respectively, and the following exchange took place.

Rhonda: So it's probably Russ who works faster.

Interviewer: How did you get that?

R: Well, I drew a fraction. I don't know what it means and I don't know why I drew it, but it kind of seemed that it would be the answer. He gets like a higher number than he does.

I: OK, so the number for Russ when you did it as a fraction was greater than the

number for Tony.

R: Yeah.

I: I noticed you wrote it as 5 over 12 and 3 over 8. Why did you choose to do it that way?

R: Well, I guess you could do it as eight thirds and twelve fifths, but then it would be like... Well, you could do it that way too.

I: Try it.

[She works the problem the different way using the calculator to compute the quotients for the multiplicative inverses.]

R: Oh, totally different. That means then Tony—that's probably the better answer.

I: The second one?

R: Yeah.

I: Why do you think that?

R: Because it says in eight hours he can fertilize three acres, so out of the eight hours that he has he fertilizes three acres, not out of the three acres he fertilizes...

I: You originally said Russ. What do you think now?

R: It's actually pretty close, but Tony works faster.

I: OK, you wrote down here 2.6 and 2.4. Any idea of what that means in terms of the problem?

R: Maybe that's how much it takes them to do in one hour.

I: That's how much they can do in an hour?

R: I don't know. [She works on the calculator and checks the numbers on her

paper.] Maybe it is. I'm not sure if that's how much—well, that's probably not how many acres because it says in eight hours he can do three acres. Then that can't be how many acres he does in one hour.

I: How did you know that?

R: Because it's almost three. Even if you add like two hours it would be more than three. So maybe that's how many hours it takes to do one acre.

I: How did you get that?

R: That's kind of (working on the calculator)...yeah. Oh yeah, it will work that way.

I: How did you come to that conclusion?

R: Since the first one didn't work, I (laughing)...Well, it must take—he can probably fertilize 2.6 acres in one hour. Or—2.6 acres in—or 2.6 hours to do one acre.

I: 2.6 hours to do one acre?

R: Yeah.

I: Did you check yourself to see if that was right?

R: Yeah, because it was actually 2.6 repeating. So 2.6666. It was 7.9999.

I: How did you check yourself on the calculator?

R: I did 2.6666 times 3 for 3 acres, and that equals 7.9999.

I: So if a friend of yours comes along and doesn't know how to do the problem and says, "What do these numbers mean?" how would you explain it to her?

R: This number is how many hours it takes Tony to fertilize one acre. This is how

many acres—how many hours it takes Russ to fertilize one.

I: So who do you think works faster?

R: Tony.

I: Why is that?

R: Because it takes him 2.6 hours... Oh, Russ! Yeah, Russ because that's hours.

So Russ is faster.

I: What was it about this that made you realize that?

R: 2.4 hours is less than 2.6 hours.

I asked Rhonda to return to her original expressions for the ratios, $\frac{3}{8}$ and $\frac{5}{12}$, which she had converted to decimal numbers, and try to interpret them. At that moment, a voice over the intercom system called her to the school's office. I paused the videotape, and when she returned, we continued.

Interviewer: When we left off, I had asked you about these numbers.

Rhonda: Yeah, that's how many acres he'll do in an hour. So let's see (checking herself on the calculator). Yep.

Interviewer: How did you check yourself on that?

R: I did .375 times 8 hours for .375 acres and 8 hours. Oh, five.

I: How would you interpret the .416?

R: .416 acres Russ can do in one hour.

I: Just from looking at those numbers, who works faster?

R: Russ.

I: Why is that?

R: Because he fertilizes more acres in an hour.

Rhonda seemed very confident during this portion of the interview, answering quickly and at times leaning back and smiling. At first, her attempts to construct the composite unit were based on trial and error. She considered acres per hour, which seems to be the easiest of the intensive quantities in these problems for students to conceptualize, but she looked at the extensive quantities given in the problem and realized that neither Tony or Russ could be fertilizing between two and three acres per hour. On her own, she determined a strategy to check herself—that the number of hours for one acre multiplied by the number of acres should be the total number of hours, and that the number of acres in one hour multiplied by the number of hours should be the total number of acres. This was remarkable progress for someone who only seven minutes earlier in the interview was not even using the numbers in the sidewalk problem and was guessing that the decimal expressions meant how many feet each girl walked.

But later when asked to interpret $\frac{3}{8}$ in terms of the problem, Rhonda said, “He can fertilize three acres out of eight hours.” Using “out of” indicates that she was still trying to interpret fractional representations of associated-sets ratios in terms of part-whole relationships, just as she did in the first interview with the orange-juice problems. After interpreting the decimal expressions as intensive quantities, she continued to interpret the fractions as two extensive quantities. Returning to the sidewalk problem, I asked her about $\frac{5}{7}$ and $\frac{3}{4}$, and she also explained those fractions in extensive terms only. Rhonda first constructed her unit of the intensive quantity with decimal expressions greater than one and then applied that knowledge to decimal expressions less than one,

but she never interpreted the fractions, in either the sidewalk or fertilizer problems, as intensive measures.

Confirming Evidence: Well-Chunked Measures

Lamon (1993) uses the term *well-chunked measure* to refer to an intensive quantity that is culturally defined, pre-packaged as a single unit by language. Because students experience speed, for example, in the chunked form of distance and time combined, Lamon suggests that students have an easier time conceptualizing it. Acres per hour is a work rate, which, although not a well-chunked measure such as speed or price, is certainly a familiar concept to students and therefore “better chunked” than any other intensive quantity in the numerical comparison problems used in this study.

Because of Sheila’s family’s summer vacation, we were not able to schedule the second interview until the evening after the second day of school in August. A discussion from her science class that day had a major impact on her interpretation of ratios in the fertilizer problem. At first, she wrote the ratios as $8/3$ and $12/5$, used crossmultiplication to determine the larger number, and chose Tony for her answer based on $8/3$ being greater than $12/5$. Then I used the same line of questioning that I used with Susan in our discussion of the sidewalk problem.

Interviewer: Why did you write that as 8 over 3 as opposed to say 3 over 8?

Sheila: In 8 hours he covers 3 acres; in 12 hours he does 5.

I: What if I’d written, “Tony can fertilize 3 acres in 8 hours?”

[She sets up the fractions $3/8$ and $5/12$, crossmultiplies, and gets the opposite result, with the crossproducts of 40 and 36 reversed from before.]

S: Now it’s different.

I: Now the number for Russ is bigger, right?

S: Yeah.

I: So do you think your answer should be consistent no matter how I write the problem?

S: I'm not sure.

She later verified that she had been choosing the numbers for the numerators and denominators based on their given order in the problem. After the previous exchange, she paused for 34 seconds and changed her answer to Russ, explaining that the second way of writing the fractions was correct because the unit of time should go in the denominator. However, she changed her mind again because after she settled on the fractions with the unit of time in the denominator as the appropriate notation, she decided that “faster” implied that the choice of the smaller number would give her the correct answer. This, of course, was a mistake because she was looking at numbers that represented area per unit time instead of time per unit area.

At this point, she was still focused on the size of the numbers and determining a rule for choosing one or the other; she was not trying to construct units for the numbers or make sense of them in the context of the problem. She had mentioned that her decision to use the fractions with the measure of time in the denominators was based on a discussion about speed from her science class that day. She said that when writing a speed the time goes on the bottom. She was stuck, not able to make sense of the fractions she had written. Her breakthrough occurred when I asked her for an example of speed

and she was able to generalize from distance per unit time to other measures per unit time.

Sheila gave an example of three meters per second. Because of her focus on choosing the smaller number when comparing the fractions she had written, I asked her, “So what’s faster, three meters per second or four meters per second?” She immediately said three and then paused before correcting herself. I believe that this episode caused her to focus more on the meaning and interpretation of numbers and less on constructing a rule that would give her the correct answer because she immediately interpreted the decimal expression for $5/12$: “So this would be .416 acres per hour.” She wrote “ a/h ” after the decimal expressions .375 and .416 and chose Russ as her answer because the .416 for him referred to the greater amount of land, not the greater amount of time. My line of questioning created disequilibrium, but Sheila, unlike Susan, was motivated to work to resolve it and not so easily upset. Reflecting on her experience from science class helped her make sense of how an extensive quantity combined with a unit of time can form a chunked measure.

There are two important pieces of evidence from this episode. One contributes to my conjecture that students are better able to interpret decimal expressions than fractions in terms of an intensive quantity. In many cases, one could argue that students convert fractions to decimals to be able to compare them easily, but in this case, Sheila already knew from her crossmultiplication that $5/12$ is greater than $3/8$. She chose, however, to represent the acres per hour as decimal expressions. Research suggests that younger students often do not interpret a fraction as a single number because they focus on the numerators and denominators individually. In my study, the students did understand that

a fraction represents a single number, but they did seem to have a more difficult time interpreting fractions as an intensive quantity because of the presence of the two extensive quantities that compose it. Fractional representations in the form a/b are associated with students' exclusive focus on the extensive quantities.

The second piece of evidence supports Lamon's theory of well-chunked measures. Sheila was able to generalize a speed relationship to make sense of area per unit time. Her solution of computing and comparing the numbers of acres per hour was quantitative and done in the context of the problem. However, when I next asked her to return to her original fractional representations of $8/3$ and $12/5$ and interpret them, she said, "Eight hours in three acres, and this doesn't really make sense." Without the time measure in the second position, she was not able to interpret these fractional representations of ratios in intensive terms.

Mandy also was able to interpret the decimal expressions for acres per hour but was never able to interpret the inverse ratios, either as fractions or decimal expressions. Mandy had trouble resolving the conflicting answers from the two sets of ratios, but then she settled on her answer from when she put the numbers of hours in the denominators because she remembered her teacher telling her that "time is always on the bottom."

As the dialogue in the previous section demonstrates, Rhonda was able to correct herself after initially choosing Tony because she realized that the smaller number of hours per acre would give her the correct answer. However, when we returned to the sidewalk problem, she computed steps per blocks and chose Jill based on the larger number, not realizing that the girl with the smaller number for steps per block would give her the correct answer. In the fertilizer problem, the presence of time variable most likely

helped her construct her rule for choosing the correct answer. She was not able to combine the two extensive quantities in the sidewalk form and conceptualize the intensive quantity.

Interfering Factor: Attempts to Solve for a Missing Value

Mandy set up the problem as a missing-value problem, as she did with the pizza problem on the test, but this time she stumbled into a meaningful solution even though the way she set the problem up was based on a flawed strategy. She set up two proportions with $n/20$ equal to both $3/8$ and $5/12$. The choice of 20 was due to it being the total number of hours given for both workers. She was again setting up a fraction as a part-whole representation. At first when she solved for the missing value, she struggled to identify the unit. She guessed that seven and a half would be the minutes per acre and then guessed acres per hour. She then said that what she had computed was seven and a half acres for 20 hours for Tony and eight and a third acres for 20 hours for Russ, and so she chose Russ because he does more acres in the same amount of time. Using 20 was an odd choice for a common denominator and finding a common denominator might not have been her intention, but she did solve the problem this way and managed to make sense out of the “missing values” that she computed. When asked about her choice of 20 for the denominator, she replied, “Because you have to have the total number of hours between both of them.” She did not understand why her choice of 20 worked; it worked because any common number of hours would work. In the sense that Mandy used found a common reference value and compared the other extensive quantities, her solution was similar to Sheila’s solution to the sidewalk problem.

After applying this method to the sidewalk problem, Mandy returned to the fertilizer problem but this time wrote $8/3$, with the number of hours in the numerator. She interpreted the decimal expressions for $8/3$ and $12/5$ as the number of acres per hour. Therefore, it became obvious that Mandy did not have a sufficient understanding of the relationships to be able to construct the units for the intensive quantities. She said that what she had computed earlier was the number of acres for 20 hours and that her new numbers represented the number of acres in one hour. She did not see the contradiction in getting a different answer from the unit rates and from the rates of 20.

Interfering Factor: Improper Fractions

After deciding that the comparison in the sidewalk problem should be based on the proper fractions, Susan wrote the ratios for the fertilizer problem as $3/8$ and $5/12$, the first time that she wrote the first number given in each pair in the denominator. Her explanation was that improper fractions are considered incorrect unless “they tell you to write it that way.” This was also the first time that she wrote labels—“a” for acres and “h” for hours—with her fractions, perhaps because it was the first time she was not basing the position of the numbers in her fractions on the order given in the problem and therefore needed a shorthand to remind herself.

She chose Tony as her answer because the decimal expression equivalent to the fraction for him, $.375$, is less than the decimal expression equivalent to the fraction for Russ, $.416$ (with the 6 repeating). To her, “faster” implied choosing the smaller number, but without identifying the intensive measure, she actually chose the one who did fewer acres per hour than the one who worked faster. When asked if she could reverse the order of the numerators and denominators, she computed the decimal equivalents and chose

Russ because the number was smaller for him. She had no strategy for resolving the difference between the answers. Her method, which she had formalized from resolving her conflicts on both the sidewalk and fertilizer problems, was to write the smaller numbers in the numerators, compare the fractions, and choose the bigger or smaller number, depending on the key word, such as “faster” or “longer,” given in the problem. Her choice of the bigger or smaller number was not based on an understanding of the intensive quantities. Instead, her choice was based on the sound of the key word. To her, “longer” implied choose the bigger; “faster” implied choose the smaller.

Susan seemed frustrated that she could not get the right answer by simply pressing the right buttons for an operation on the calculator. She made no progress during the second interview at constructing the composite unit for an intensive quantity. When asked for an interpretation of the ratio .375, which she got from her fraction of 3 acres over 8 hours, she said, “Maybe it’s how much fertilizer they need.” After working on the fertilizer problem, Susan claimed that it was more difficult than the sidewalk problem because the bigger number was given first in each pair in the problem.

Overall Patterns

The purpose of this section is to give a brief summary of the themes mentioned in the analysis of the problems in this chapter and to report the overall results not specific to one problem. The patterns from the analysis will serve as the basis for the conclusions and recommendations in the next chapter. In Table 9, characteristics of student behavior are listed for three general categories—notation, strategy, and context.

Table 9
Overall Characteristics of Student Behavior

Category	Characteristic	Exceptions or Comments
Notation	Showing the colon notation as an example of ratio but never using it in problem solving	Corina was the only one not to give an example of it
	Writing fractional representations of ratios according to the order of the information in the problem, with the first number as the numerator	Mandy and Sheila put the measure of time in the denominator, Susan avoided improper fractions
	Writing a mixed fraction within a unit rate to help make the transition to an interpretation of the intensive quantity	Only Corina and Kim used this notation
	Not using labels for units with numerators and denominators	Corina was the only one to use labels on her own
Strategy	Building up to solve the problem with multiple missing values	Susan used a between strategy
	Using a common reference point to convert and compare extensive quantities of ratios	Kim and Sheila used this consistently, Susan did on the orange-juice problems, Mandy did on the fertilizer problem
Context	Ability to interpret ratios as mixed fractions or decimal expressions more easily in the context of the problem than ratios written as proper or improper fractions	None
	Comparing fractions via a common denominator instead of crossmultiplication increasing the likelihood of in-context interpretation	None

In the following sections, I include brief summary comments about various issues raised in this chapter.

Notation for Ratios

In general, the participants wrote ratios as fractions, with the numerator of the first of the pair of numbers given in the problem. The problem that was the exception to this rule was the fertilizer problem, for which some of the participants reversed the order of the numbers. On this problem, Mandy, Rhonda, and Sheila wrote the fractional representations of ratios with the first number listed going in the denominator because of their preference of writing the number for the measure of time in the denominator.

The students also matched the numerator with the “of” variable and the denominator with the “to” variable when asked to write a ratio of dimes to quarters. In general, they wrote $5/2$ as the ratio of dimes to quarters and $2/5$ as the ratio of quarters to dimes.

When someone is solving a problem, one possibility for choosing the numerators and denominators of ratios in fractional form is to write them according to what the problem is asking. For example, in the pizza problem, someone using this method of writing ratios would write the numbers for the pizzas in the numerators and the numbers for the children in the denominators because the question is phrased as “who gets more pizza,” which implies that the resulting ratios should be two measures of pizza per person that can be compared. In the orange-juice problems, the question asks which drink has the “stronger orange taste,” which might imply that the resulting ratios should express two measures of orange concentration that can be compared. There is some evidence that Marie may have been influenced by the way the questions are phrased on the test when she set up her fractions. Because she did not participate in the second round of

interviews, there is a limited amount of data for her. There is no evidence that any of the other participants were influenced by this method.

Levels of Proportional Reasoning

Students in Lamon's 1993 study were not using mathematical symbols to solve the problems; therefore, the students' work gave her better evidence about the students' reasoning than the work of the students in this study provided. Except for Corina and Melinda, the students in this study had completed seventh grade. The students in this study did use diagrams and creative methods in some cases, but mostly their work was based on mathematical symbols and notation that they had learned in school. Even though most attained the level of quantitative proportional reasoning for at least for some of the problems, they often did not reason proportionally when they solved the problems. They operated on numbers consistently with quantitative proportional reasoning, but they were doing so detached from the context of the problem and therefore only demonstrating quantitative proportional reasoning about the numbers they had written instead of reasoning about the actual problem.

For those who reasoned quantitatively and did so in a meaningful way within the context of the problem, I noticed three possible sublevels for Lamon's category of quantitative proportional reasoning. One is quantitative reasoning about a variable of conversion, as in Kim and Sheila's slice strategy for the pizza problem, for which they created a new variable, slices, and determined the number of slices for the girls and the boys. The second sublevel, which is more sophisticated, is to use the given extensive quantities to find a common reference value for parallel elements across the ratios. Kim and Sheila used this strategy for the sidewalk problem. The third and most sophisticated

of the quantitative–reasoning strategies is to construct the unit of the intensive quantity and to compare the ratios using the intensive quantities. Corina did this when she compared the number of hours per acres for Tony and Russ and chose Russ because he did an acre in a shorter amount of time.

Interpretations and Expressions for Forms of Ratios

Although student-specific and problem-specific factors can greatly influence students' abilities to interpret and express ratios, there are some general patterns from the study to report. Students were better able to interpret and express ratios greater than one as intensive quantities than ratios between zero and one. Mixed fractions were the form that the students preferred when speaking. They were better able to interpret and express ratios in the form of decimal expressions as intensive quantities than ratios written as fractions.

The Role of Manipulatives

Many of the students relied on diagrams for the pizza and sidewalk problems. Manipulatives played a role only in the pizza and coin problems. The manipulatives for the pizza problem were especially helpful to those students who had no strategy and did not know how to begin the problem. The manipulatives also helped those who wrote ratios as decontextualized fractions and compared the fractions without units to make their choice. The use of manipulatives forces the student to focus on the units in the problem and increases the chances of a student developing a solution in the context of the problem.

Multiplicative Conceptual Field

As I stated in the introduction and literature review, it is impossible to evaluate students' knowledge of ratios independent from other concepts in the multiplicative conceptual field. Lack of both conceptual and procedural knowledge about division proved to be the greatest interfering factor overall for students as they worked on the numerical comparison problems. Prema was the student who had the most difficulty with division, and it proved to be a roadblock to making progress on understanding ratios. Given a fraction, she did not know which number to divide by which, and she did not consistently match the order of division with her spoken language, such as "divided by" and "goes into."

The students' lack of number sense about fractions and division was surprising to me. Even Kim, who was probably the strongest student overall who participated in all three phases of the study, made a mistake once by reversing the order of the numerator and denominator when dividing. I thought that students would be able to catch a mistake by noticing that a quotient greater than one should come from an improper fraction and a quotient less than one should come from a proper fraction, but this was not an obvious rule of thumb to them. This lack of number sense can have a major impact on their inability to match ratios with intensive quantities in the numerical comparison problems because the strategy of matching the ratio that is greater than one with the intensive quantity that expresses the number for the smaller amount per the unit of the larger amount is not a conceptual shortcut that they have developed.

Knowledge Transfer Across Problems

Although many of the students demonstrated an advance in their level of reasoning in the first interview, I noticed no evidence that any of them were able to apply what they learned to any of the activities in the second interview. During the second round of interviews, no student referred to any of the problems or activities from the test of proportional reasoning, their first interview, or *Girls on Track* camp.

Knowledge transfer across problems was limited, at best. Students were often able to interpret a ratio as an intensive quantity in the fertilizer problem, for example, and then unable to interpret the ratios in the sidewalk problem as intensive quantities. The exception was the coin problem, which was designed as a simple problem from which students could develop a conceptual understanding about fractional representations of ratios that they could then apply to the pizza problem. However, some of the students only learned the language for expressing intensive quantities instead of learning about the relationship between the extensive quantities. For example, Rhonda tried to mimic “for every one quarter, there are two and a half dimes” in the pizza problem but could not coordinate the units appropriately.

CONCLUSIONS AND RECOMMENDATIONS

Recent studies (Lachance & Confrey, 2002; Lo & Watanabe, 1997; Singh, 2000) have focused on fifth- and sixth-grade students' understanding of ratio-related topics, during the years when students traditionally learn these topics in school. The goal for this study was to investigate students' understanding of ratio after the traditional school years of instruction on ratio and proportion, as the students prepare to enter Algebra I.

In general, the students who participated in this study demonstrated their procedural skills with fractions. They were able to convert improper fractions to mixed fractions, convert fractions to decimal expressions, set up missing-value problems with fractions and solve them by crossmultiplying or using a within strategy, and compare fractions by crossmultiplying. However, their ability to work with ratios was deficient in several respects. With only one exception, the students had to “build up,” or use repeated addition, to solve the multiple-missing-values problem. With only one exception, the students did not use labels to identify units in ratios. And students frequently worked with ratios in a decontextualized manner, not making sense of their operations and numerical results in terms of the problem.

Students used four numeric methods of comparing the values of ratios. They were most successful at interpreting the ratio as an intensive quantity when they could convert improper fractions to mixed fractions and could immediately determine the larger number by looking at the mixed fractions. For example, Corina used this method in the interviews to solve the orange-juice and fertilizer problems. The most likely explanation for her success at comparing the numbers is that she could compare the fraction portion of the

mixed fraction to one half and because both mixed fractions were never in the same interval (for example, between two and two and a half or between two and a half and three), she could identify the larger of the two ratios without needing to find a common denominator or convert them into decimal expressions.

The second method of comparing numbers was conversion to decimal expressions, as used by Rhonda on the fertilizer problem. Rhonda advanced quickly on this problem and solved it using intensive quantities, following the same path across the general overlapping model that Marie followed when solving the pizza problem (see Figure 9). The third method was finding a common denominator, as Sheia did on the sidewalk problem, which increased the likelihood of a student moving from the fraction-only realm back to the intersection of the overlapping model to compare the numerators and denominators in extensive terms (see Figure 14). The fourth method was crossmultiplication, as used by Kelly on the pizza problem (see Figure 10). Those who used crossmultiplication were most likely to make their comparison in terms of the numbers only, removed from the context of the problem, and simply pick the larger number or pick the larger or smaller number on the basis of a key-word association, such as “more” or “faster,” without recognizing the composite unit for the ratios.

Responses to Research Questions

In this section, I present a summary of the results in terms of the research questions listed at the end of the introduction.

Understanding of Ratios

What is the nature of students' understanding of ratios, especially fractional representations of ratios, and what meaning, if any, do they associate with ratios in this form?

With only one exception, the students demonstrated the notation for writing ratios as fractions and the notation for writing ratios with a colon. They use fractional representations of ratios but do not understand their connection to fractions, and they do understand how to represent ratios with the colon notation but do not use it when solving problems. Their explanations of ratios were only in terms of extensive quantities, which is what the colon notation can be used to represent. Therefore, it seems that the colon notation would be ideal for students who are still developing their multiplicative reasoning with extensive quantities. None of the students were consistently able to interpret fractional representations of ratios. Given a items of Object A and b items of Object B, the students write a/b and interpret the fraction only in terms of a separate numerator and denominator, not a unified number within the context of the ratio relationship. Because the students are fluent at using fractions, they understand that a/b is a number and can work with it as a number, but to them the notation a/b has two separate identities—one as the ratio subconstruct of rational numbers and one as the quotient subconstruct. Because of Melinda and Rhonda's attempts to compare numbers by shading pie charts, it does seem that students understand the connection between quotients and part-whole representations. But the ratio-as-fraction notation is a disconnected mathematical entity. They can toggle between seeing a/b as a number and seeing it as a ratio of two extensive quantities but make no connection between the two.

The students did associate ratios with multiplication and division through their problem-solving activities, but they rarely expressed this connection when discussing ratios in general. Overall, their understanding of ratios is confined by the fractional notation they use to represent them—a notation for which they have limited connections to other subconstructs of rational numbers.

In terms of the mathematical practices of the classrooms in which the students have participated, it seems that representing ratios as fractions and with the colon notation are both common, but in problem-solving activities, it seems likely that students have experienced instruction with primarily or exclusively fractional notation.

Characteristics of the Problems

What characteristics of the numerical comparison problems—for example, whether variables are discrete or continuous and whether the smaller or larger number in a ratio is given first—influence students' notation, their strategy, and their likelihood of success?

Because the problem-specific factors were not isolated and tested individually, I can offer only my impressions from the interviews and base these conclusions on the data that was presented and summarized in the previous chapter. When working the pizza problem, the students avoided expressing the ratio of girls to pizza as an exact number. Instead, they preferred to explain the sharing of the pizzas in terms of how the girls might arrange themselves at the party. They did not avoid expressing the ratio of two and a half dimes to one quarter in the coin problem, probably because of the close connection to the continuous variable of cents, on which they based the ratio.

The pizza problem provided one easy integer ratio, three boys to one pizza, for the participants to use in their comparison. While working other problems that required more complicated comparisons, the students often referred to integer ratios to help them make sense of numbers. For example, Corina made her breakthrough on the orange-juice problem when she finally realized that the ratio of cups of water to cups of orange juice would be less than three. Sheila was able to use integer ratios in the cookie problem to guide herself in determining the scale factor for converting the recipe. Using integer ratios as a guide, students can determine whether they can use qualitative reasoning or need to compute exact numbers for ratios. Susan was able to solve the pizza problem, for example, by giving each girl three slices and saw that the boys could not get as much as three slices each. She solved the problem using a convenient integer ratio as a comparison and did not need to be more precise.

The presence of a unit of time in the fertilizer problem made the problem easier for some students. Once they were able to conceptualize the ratio as a work rate it was easier for them to compare acres per hours for the two workers. However, there seems to be two competing factors in the fertilizer problem: One is that acres per hour is the more natural intensive quantity, and the other is that the hours per acre is the ratio greater than one, which, in general, is easier for students to conceptualize.

Overall, the easiest non-integer ratios for students to understand and use as intensive quantities were ratios of continuous variables (at least in the first position), ratios greater than one, and ratios that they could express and compare easily as mixed fractions. With ratios that fit this description, Corina was able to invent her own notation and express intensive quantities in a variety of ways. Problems with ratios such as these

could help students construct their own unit-rate strategies and help teachers avoid the pitfalls of teaching the unit-rate solution as a prescribed procedure.

Strategies for Solving Problems

Are students able to compute intensive quantities and construct the appropriate composite unit, and if they are not able to compare ratios based on intensive quantities, what strategies do students use to solve numerical comparison problems?

In general, students associated division with fractional representations of ratios but were unable to make sense of the quotients they computed. After making progress during the interviews, Corina and Rhonda were able to make the leap to solving problems by constructing the composite unit and comparing the ratios as intensive quantities. However, other solutions based on conversions to a familiar variable and based on the comparison of extensive quantities with a common reference point suggest ways that younger students can develop a better and broader understanding of ratios without having to solve these problems using intensive quantities. Table 10 later in this chapter lists the strategies and examples.

Growth in Making Connections

How do students grow in their understanding of ratios and intensive quantities during the problem-solving sessions, and are students successful at retroconnective learning, making connections between ratios and fractions to develop an understanding of ratio as an intensive quantity?

Students made progress in response to a variety of situations: when their computational strategies gave them contradicting answers for the same problem with the information given in reverse order, when they used manipulatives to model the situations

described in the problem, and when they were prompted to attach meaning to the numbers they used in their notation and numbers they generated with their procedures. When students were encouraged to keep track of the units and interpret numbers within the context of the problem they often examined their own work and evaluated what they had done in terms of the context of the problem. This search for meaning for the numbers was what sometimes led them to their expressions of intensive quantities.

There were no episodes that provided any evidence of retroconnective learning in terms of an “ah-ha” moment at which the student expressed the connection between ratios written as fractions and the numerical value of those fractions and then advanced to using that conceptual knowledge in her problem-solving activities. One of my predictions when I designed the study was that the students with the high scores on the test of proportional reasoning, who, I assumed, would have a solid conceptual and procedural foundation, would be likely to construct meaning for fractional representations of ratios and make the connections among the topics they had previously experienced as “distinct and disconnected ‘pockets’ of mathematical understandings” (Lachance & Confrey, 2002, p. 507). However, there was no evidence of this happening. The two students who made the most progress at interpreting ratios as intensive quantities were Rhonda and Corina, whose scores placed them in the Middle and Low groups, respectively. Both of them performed much better in the interviews than they did on the test. What they had in common was a lack of rigidity in their problem-solving approach and a willingness to try new approaches. Both were also strong procedurally with division and fractions.

The difficulty of students to realize the connection, formalize this knowledge, and apply it to new situations suggests that teaching for procedural knowledge first and then

asking students to make connections later is likely to be unsuccessful. The results of this study provide further evidence for emphasizing conceptual knowledge of ratios from the beginning of instruction and providing a variety of problems and contexts for students to construct their knowledge over a long period. Of course, a greater emphasis on problem solving and conceptual knowledge does not imply that syntactical knowledge is unimportant. As demonstrated by Prema's troubles throughout the study, an understanding of the basics of multiplication and division are needed to be able to make progress.

Conjectures

Three conjectures based on the overall results of this study are important in formulating an instructional strategy for ratios and their relationship to fractions. The first conjecture is how students may progress through increasingly sophisticated sublevels of quantitative proportional reasoning. The second conjecture explains students' work in terms of the overlapping model of ratios and fractions. The third conjecture is how they may progress in their ability to interpret and use a variety of representations of ratios.

Levels of Quantitative Reasoning

Unlike the students in Lamon's (1993) study, the students in this study frequently reasoned quantitatively in making their choice for the numerical comparison problems. Often students' numerical procedures were decontextualized and therefore not considered as reasoning about ratios at all. In most cases, students in this study used multiplication or division and based their solutions on a comparison of exact numbers. Their skills using fractions, multiplication, and division gave them the tools to solve more difficult

problems, but their use of these tools put them at risk of decontextualizing their solution. The students in Lamon's study reasoned about the pizza problem in context but did so in a qualitative sense, without the use of precise relationships. How can teachers help students continue to focus on context as they acquire these numerical skills?

Because of the various quantitative strategies used by students in this study, who were older than the students in Lamon's study, I have broken her level of quantitative reasoning into three sublevels that are appropriate for describing students' work on the numerical comparison problems. Table 10 shows the three sublevels of quantitative proportional reasoning in increasing order of sophistication in terms of efficiency and in terms of how students coordinated the variables that compose the ratios.

Table 10
Sublevels for Quantitative Proportional Reasoning

Sublevel	Description	Examples
Variable Conversion	Converting one or more extensive quantities to a new variable and computing a unit rate	Kim and Sheila's slice strategy on the pizza problem
Common Reference Value	Finding a common multiple for parallel elements across ratios and creating equivalent ratios with a common reference value	Kim and Sheila's solutions to the sidewalk problem, Kim and Susan's solutions to the orange-juice problems
Intensive Interpretation	Using fractions or decimal expressions to represent and compare intensive quantities	Marie's solution to the pizza problem, Corina and Rhonda's solutions to the fertilizer problem

The first conjecture is that teachers and curriculum developers could structure problems according to these sublevels to help students to move from qualitative solutions

to increasingly more sophisticated quantitative solutions and eventually quantitative solutions based on intensive quantities. An example is given in the “Recommendations” section at the end of the chapter.

Decontextualization

Extracting numbers from a problem, choosing an operation based on a key word or phrase (Garofalo, 1992), and computing an answer is a common method for solving word problems. When asked what their numeric expressions meant in terms of the problem, the students in this study frequently chose a unit that fit the problem and fit the number but had nothing to do with the solution. For example, slices made sense in the pizza problem and two and a third slices sounded about right for sharing a pizza.

In the “rush to put formal computational tools in students’ hands” (Kaput & West, 1994, p. 284), teachers and textbook authors are encouraging students to represent ratios as fractions, which is the most convenient notation for solving missing-value problems. But it is the notation for which students are least likely to maintain the context of the problem in their solution. The second conjecture is that the early use of fractional representations of ratios speeds students along the arc of decontextualization from thinking about ratios to representing them with fractions to manipulating fractions that are no longer connected to the ratios from which they came. Along the arc the students pass through one-way doors, never to return to the context of the problem. Using manipulatives, finding common reference values for extensive quantities, and using labels with numbers all show promise for helping students develop flexibility at moving between representations for ratios and interpreting them in terms of the context of the problem.

Hierarchy of Representations

In our expanded model (Clark et al., in press) of how ratios and various number types can be related (see Figure 8), we have tried to broaden the view of ratios in school mathematics, from their narrow association with fractions to their potential representations as various number types. Students in this study expressed ratios as a variety of number types, and these number types are included in Figure 16 in the expanded model.

4. Decimal expressions between zero and one
5. Improper fractions
6. Proper fractions

Which of these number types is “dialed in” as the oval on the right side of the model affects the thickness of the barrier for students in their ability to move right to left from the number type to the intersection. In this study, students who worked with proper or improper fractions remained in the number-type domain, unable to move back to the intersection. The exception was those who compared the fractions by finding a common denominator and then interpreted the new numerators and denominators in the context of the problem. In most cases, conversion from improper fractions to mixed fractions or decimal expressions greater than one or conversion from proper fractions to decimal expressions less than one increased the likelihood of interpreting the ratio as an intensive quantity. This “changing the channel” from one number type to another is how Marie interpreted the ratios in the pizza problem (see Figure 9). Of course, there are many potential factors related to the variables and the student that can impact the ranking as it relates to a specific problem.

From our review of textbooks (Clark et al., in press), we concluded that the notation for ratios emphasized and encouraged by authors is writing ratios as fractions. From this study, it is clear that the participants’ notation of choice was fractions. However fractions fall at the bottom of the hierarchy in terms of students’ ability to interpret and use these representations as intensive quantities. The notation recommended by textbook authors does not match the representations for ratios that middle-school students are likely to understand as intensive quantities.

Extended Paths in the Overlapping Model

Teachers and researchers can use the overlapping model of ratios and fractions and the generalized model of ratios and various number-type domains to map students' movement in a way that helps them analyze students' progress. Figure 17 shows several paths that students in this study followed while working on the numerical comparison problems and shows how more advanced students might extend their path.

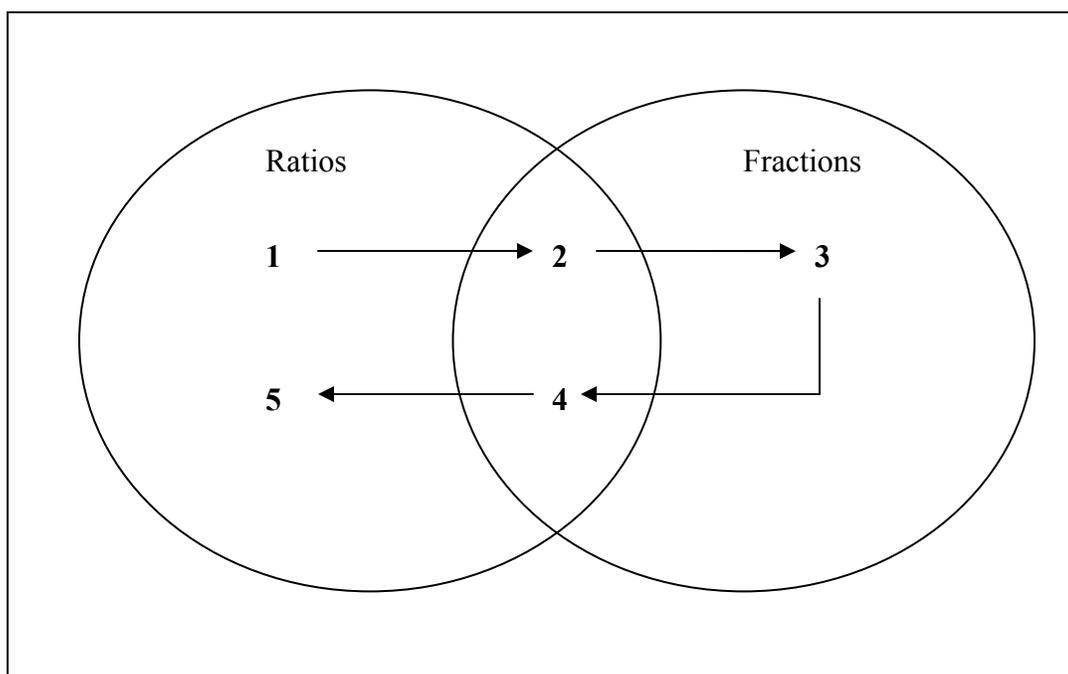


Figure 17. Paths Within the Overlapping Model

In her solution to the sidewalk problem, Kim based her initial solution on a diagram (see Figure 13) and found a common reference point for the two walkers. She used ratios in this solution but did not express the ratio as a fraction or as any other number type. Her solution was creative and precise but inefficient. Her path ends in the ratio-only realm of the overlapping model (1). Melinda often wrote fractional representations of ratios but then had trouble making the comparison. A student who

writes fractional representations of ratios but then does not work with them as numbers would have a path that ends in the intersection (2).

The most common path for students on the pizza problem was the path of decontextualization that ends in the fraction-only realm (3). The students who followed this path compared the numerical value of the fractions but never returned to interpreting the fractions as ratios.

In the path that returns to the intersection (4), there is an important distinction between those students who make their comparison based only on extensive quantities, as Sheila did on the sidewalk problem (see Figure 14), and those who interpret the fractions as intensive quantities. An efficient path for students who use the crossmultiplication algorithm to compare fractions would be to determine which fraction is larger, construct the composite unit for the intensive quantity, and interpret the fractions in this context to decide whether to make the choice associated with the larger or smaller number.

A possible extension is returning to the ratio-only realm (5) to state the ratio in terms of a common reference point. For example, a student who understands that $\frac{7}{3}$ girls sharing one pizza means that each girl gets more pizza than each boy might prefer to state the ratio in terms of integers. The student might choose to compare a party with 21 girls and a party with 21 boys and preserve the ratios by ordering nine pizzas for the girls and seven for the boys. By connecting an interpretation of fractional representations of ratios to representations such as 21:9 and 21:7 in the ratio-only realm, a student would complete the cycle back to ratios.

As I stated in the introduction, one goal of mine is to help students develop flexibility in moving between ratios and fractions, and the number of times a student

crosses a barrier in the overlapping model is one measure of this flexibility. When solving complex problems, a student may move across the model more than once and into various number-type domains. Extensions to numerical comparison problems can be designed to encourage students to continue moving back and forth as they solidify their conceptual base for ratios.

General Comments

In this section, I provide some general comments about the limitations of the study, the problems that were used in the interviews, and the students who participated.

Limitations of the Study

Because of targeting students during the summer before entering Algebra I, I had a limited amount of time to conduct the study and was able to schedule only two rounds of interviews. In all, ten students were interviewed during the first round and eight in the second round. The number of numerical comparison problems used in the study was limited by the number of rounds of interviews and the lengths of the interviews.

The sample of students was limited to female students because of drawing the sample from the pool of *Girls on Track* participants in 2002. The sample was also limited in terms of geography, all the girls attending schools in Wake County, North Carolina. However, of the eight students who completed all three phases, one had recently moved from the upper Midwest and another from the Northeast. Each of them, like all the participants, had her own set of unique abilities and obstacles, but neither provided any evidence that her school experiences with ratios and fractions had been substantially different from the others.

Characteristics of students' classroom experiences certainly have a major impact on the students' problem-solving activities, and another limitation of the study is that these characteristics were only accessible through students' comments and actions during the interviews and through their written work on the test. There was no direct view of their social participation in the classroom. A study of one class of learners working on numerical comparison problems could provide different types of evidence to enhance what has been presented here. An examination of small-group interactions could provide guidance for teachers in how to use numerical comparison problems in the classroom. However, the purpose of this study was to investigate the understanding and growth of students from different learning environments, not to focus on a group of students from one classroom.

Comments About the Problems in the Study

The consensus from researchers is that students develop their proportional-reasoning skills over a long period and in response to a variety of contexts. The problem focus for this study is numerical comparison problems, and within that type of problem, a variety of situations and variables were used. One reason that a variety of problems is so important is that each problem presents a new set of potential pitfalls, as was demonstrated in the "Interfering Factor" sections in the previous chapter. Each problem in this study has strengths and weaknesses, and teachers must make choices of which problems to use according to how these strengths and weaknesses match with the needs of their students.

The discrete variables of girls and boys in the pizza problem was a factor in the students avoiding expressing the exact numeric ratios of children to pizzas. The

advantage of this problem is that students can solve it easily using diagrams or manipulatives, which was suggested by Lamon (1993) for helping students develop proportional reasoning. Therefore, it seems appropriate for helping students advance from nonconstructive strategies or preproportional reasoning to qualitative proportional reasoning. The pizza problem could also be helpful for students who are making the transition to quantitative reasoning and can solve it by converting numbers of pizza to numbers of slices and computing the numbers of slices per person for both groups. In small groups, students could work the problem this way using different numbers for slices per pizza and realize that the way the pizzas are cut affects the number of slices per person but not the answer of who gets more pizza. However, the pizza problem does not seem suited for students who are ready to explore interpreting ratios as intensive quantities. In my study, students were motivated, in general, to try to make sense of ratios as intensive quantities when they could not simply make a qualitative judgment.

One advantage of the orange-juice problem is that teachers could use it to contrast part-whole and part-part fractional representations for students who are already using both. Students who think about word problems as mostly superfluous information from which they have to extract numbers and choose an operation before computing an answer need more practice at interpreting and labeling those numbers. The most valuable lesson, I believe, that the participants learned while working with me was that they were expected to attach some meaning to the numbers they wrote. A (2,3) mixture, for example, could be represented as $\frac{2}{3}$, $\frac{3}{2}$, $\frac{2}{5}$, or $\frac{3}{5}$, depending on what the student wants to represent. Is it the ratio of cups of orange juice to cups of water, the ratio of cups of water to cups of orange juice, the portion of the total volume that is orange juice, or the

portion of the total volume that is water? The orange-juice problem with mixtures that are similar but not the same in concentration seem suited well for students who are prepared to make the transition from qualitative to quantitative proportional reasoning. For example, after realizing that two of the mixtures were similar but not the same, Corina drew a diagram that helped her conceptualize the number of cups of water for each cup of orange juice, and she compared the two mixtures using these ratios. Another advantage of the orange-juice problems is that the intensive quantities of cups of water per cup of orange juice and cups of orange juice per cup of water can both be used to answer the question. In this study, students expressed the ratio as cups of water per cup of orange juice because of their tendency to use ratios greater than one. However, teachers could vary the amounts so that some solutions contain more orange juice than water.

The sidewalk problem was easy for most students to visualize and to represent with a diagram. The two variables in the problem are both distances, which are easy types of units for students to compare, but the difficulty for them was that the variables were hidden and therefore tough to verbalize. The two distances are the distance between steps and the distance between cracks in the sidewalk, but those measures were not obvious from the problem. The sidewalk problem seems to be a good fit with students who need to develop strategies using a common reference value to compare extensive quantities.

Perhaps the best of all the numerical comparison problems for helping students to learn to interpret ratios as intensive quantities is the fertilizer problem. Acres and hours are both continuous variables and both describe familiar measurements (area and time). Acres per hour and hours per acre are both composite units that students can conceptualize. The time variable in the denominator makes the acres per hour easier for

the students to interpret, but the larger numbers in this problem are the number of hours. So there are two competing factors involved—the tendency to express the ratio as greater than one and the tendency to put the time variable in the denominator. I suspect that the biggest weakness of the fertilizer problem would be the difficulty of representing area and time with manipulatives. Therefore, the fertilizer problem seems appropriate for helping students advance from a strategy of finding a common reference value to a strategy of comparing intensive quantities.

Singh's (2000) concern is that the unit-rate method could become the new algorithm that replaces crossmultiplication, which would do little to encourage students' flexibility in problem solving. In her work on the orange-juice and fertilizer problems, Corina demonstrated how students can develop the unit-rate method on their own in a way that is meaningful to them. Corina did this first with her diagram in the orange-juice problem and then with her mixed fractions within unit rates in the fertilizer problems. In her work on the fertilizer problem, Rhonda demonstrated how students can use their knowledge of multiplication and division to verify that their unit rates are correct.

Comments About the Students

There were many characteristics of individual participants that inhibited their progress. Because of Prema's lack of understanding about division she would divide numbers in the reverse order and not be able to match the quotients with the fractions. Mandy's consistent use of part-whole fractions with her rigid rule for composing the whole and the way that she tried to turn every problem into a missing-value problem distracted her from focusing on the ratio relationships in the numerical comparison problems. Some of her fundamental beliefs about mathematics also limited her. For

example, she did not see a contradiction between different answers; instead, she wanted to work the problem three ways and choose her answer according to which answer she got two out of three times. Therefore, she was not motivated to resolve a conflict by analyzing her work and trying to interpret the numbers. She wanted instead to come up with a new procedure to work the problem a different way.

Susan's frustration at obtaining different answers to the most difficult problems depending on the order of the given information caused a major roadblock for her. In contrast, Sheila and Rhonda were highly motivated to resolve conflicts. Kim's outstanding problem-solving skills were in a way a roadblock in this context to developing an in-depth understanding of ratio as an intensive quantity because she was able to solve the problems in creative ways that did not require her to construct composite units for her ratios.

Corina and Melinda, the two students who were one grade behind the others, were different from the rest of the participants and in some ways the opposites of each other. Melinda could repeat facts and methods that she had learned in school but had not progressed entirely from additive to multiplicative reasoning. Corina, on the other hand, did not know any methods from school and instead relied on inventing her own. Corina's firm conceptual understanding of ratios gave her the foundation she needed to make progress during the interviews and eventually solve problems by interpreting ratios in intensive terms and comparing them within the context of the problem.

The cookie problem was the only problem for which out-of-school experiences seems to have had a major impact. Students who mentioned baking at home had a much easier time than those who did not. The lack of association between students'

performance on this problem and many of the other problems illustrates the importance of teachers using a variety of situations in problems and the importance of not assessing students' ability solely on the basis of their performance on one problem. Because ratios are so common in everyday activities, teachers can design numerical comparison problems that match students' interests.

Recommendations

In this section, I provide a few comments about possibilities for further research and then conclude with recommendations for instruction.

Recommendations for Further Research

The motivation for this qualitative study came from the results of a small quantitative study (Clark & Berenson, 2002), which demonstrated that those students who used fractions to solve the pizza problem were less likely to maintain the context of the problem in their solution than those who did not use fractions and that students who used fractions scored lower on the problem than those who did not. Because of these results, students' use of fractional representations of ratios to solve numerical comparison problems became an issue of concern. Now that the qualitative study is complete, some of the conjectures based on this study could be tested with quantitative methods. One factor that could be isolated is the given order of the numbers in a problem. My hypothesis is that when neither of the intensive quantities is more familiar or easily understood than the other, students would be more successful at solving numerical comparison problems when the larger number in each ratio is given first in the problem. I also suspect that their level of success would be greater when the second number given is

five or less because students would be likely to express the ratios as improper fractions and convert them to mixed fractions, which the students in this study preferred to interpret and compare. According to the hierarchy, the difference would still exist for ratios with numbers larger than five, but because students would be more likely to convert the improper fractions to decimal expressions instead of mixed fractions, the difference would not be as great. In other words, the gap in the hierarchy between mixed fractions and proper fractions is greater than decimal expressions larger than one and decimal expressions between zero and one, and I believe that this difference could be detected in a quantitative study.

Another avenue for a follow-up quantitative study is a further exploration of students' strategies for solving multiple-missing-value problems. Using the ordered-pair notation, I can represent the number of eggs in the original cookie recipe and the number of eggs in the new recipe as $(4,10)$. Would students still rely on building up for problems such as $(4,9)$ or $(4,11)$? Would they be more likely to use the scale factor if having to build up further, to $(4,17)$ for example? Or would more difficult numbers prompt them to break the problem up into several simple missing-value problems?

A possibility for a follow-up qualitative study is conducting a teaching experiment in one or more classrooms with the same focus and instructional objectives as in this study. The time for instruction could be much longer than in this study, and the analysis could include the social dimension within the emergent perspective and how the classroom social norms, sociomathematical norms, and classroom mathematical practices enable and constrain the progress of individuals.

Recommendations for Instruction

In Lamon's (1993) investigation of problems of different semantic types, she demonstrated that the pizza problem is a relatively easy problem for students who have had no formal instruction on symbolic representations for ratios and proportions. The sixth graders in her study were able to use qualitative reasoning in their strategies to solve the pizza problem correctly. One reason that the students were successful is that the comparison can be made easily without needing to express ratios in precise numeric terms, as Susan demonstrated on her test of proportional reasoning. A reasonable approach for helping students develop quantitative reasoning skills is to allow them to explore similar numerical comparison problems in earlier grades and develop their qualitative reasoning skills by using manipulatives, diagrams, and their own descriptions of the comparison. Small-group collaboration could also give students opportunities to share solutions and move toward quantitative solutions. For example, in Corina's solution to the most difficult orange-juice problem, she recognized that both ratios of cups of water to cups of orange juice would be between two and three; therefore, she had motivation to explore the ratios in more precise terms with diagrams and eventually used quantitative reasoning to compare the solution with two and a half cups of water per cup of orange juice to the solution with two and two thirds.

The NCTM standards (2000) recommend a greater emphasis on writing in math classrooms. Teachers should require written explanations because they encourage students to reflect on their solutions and help teachers assess students' work. A written explanation of a solution to the orange-juice problem such as this would help teachers know if students were solving problems meaningfully or simply manipulating numbers:

“The Tray A mixture has a stronger orange taste because it has two and a half glasses of water for every glass of orange juice compared to the Tray B mixture, which has more water—two and two thirds glasses of water for every glass of orange juice.”

When trying to understand how students are thinking about these problems, I often do my best to remember what troubles I had and what techniques helped me when I was learning math. I remember going through a phase—it might have been in the fourth grade—of doing conversion problems. Sometimes they were simple one-step conversion problems, but the ones I enjoyed the most required multiple steps and involved large numbers, such as, “How many seconds are in a year?” Part of the intrigue was estimating and then finding how close I was. Problems such as this one require several steps of conversion to intermediate units, such as minutes and hours and days. I understood the relationships between the units in a conversion, but I remember pausing as I chose my operation, multiplication or division. For example, when converting between centimeters and inches, I could recall the magic factor of 2.54, but sometimes what to do with that ratio required a review of what I knew about the units and about their connection to the operations. An inch is longer than a centimeter, I would tell myself, so that magic number, which is greater than one, must be the number of centimeters in an inch. So if I am converting from inches to centimeters I need more centimeters than inches to cover the same distance; therefore, multiplying, not dividing, is the appropriate operation. Students need more opportunities to develop this type of number sense.

Perhaps one reason that ratios greater than one are easier for students to conceptualize and apply is that ratios are usually expressed this way in terms of conversions. Students learn the number of centimeters in an inch and pounds in a

kilogram, which are ratios of an integer and a decimal portion. Using ratios in this form seems like a natural progression from using the simple integer ratios, such as feet per yard and quarts per gallon, which children use in the simplest of conversions.

The most basic way that this number sense can apply to numerical comparison problems is how students can construct a mental reminder to guide them throughout their work, such as a reminder that the length of a block in the sidewalk is longer than either of the girl's step and therefore the number of steps will always be larger than the number of blocks. This reminder can be used as a guide for when to multiply and when to divide, the same way that I used my reminder about inches and centimeters.

An example of lack of number sense creating a roadblock was evident in Prema's difficulty with the sidewalk problem. Prema and I spent five and a half minutes discussing whether Jill's stride is less than, equal to, or greater than the length of a block in the sidewalk. I stressed that Jill covers five blocks in seven steps, that all the steps cover the same distance, and that all the blocks are the same length; I pointed to Prema's own diagram; and I rephrased the question several times. But she insisted throughout that Jill's step is longer than the length of a block in the sidewalk.

Students need more practice with ratios in extensive terms. If fractional representations of ratios offer nothing more than a notation for extensive quantities, the obvious alternative is to use the colon notation, which students are learning but not using. With the colon notation, students may be more willing to write labels. For example, Sheila's solution to the sidewalk problem could be presented as shown in Figure 17.

For Jill	5 blocks : 7 steps	→	20 blocks : 28 steps
For Alice	3 blocks : 4 steps		21 blocks : 28 steps

Figure 18. Solution to the Sidewalk Problem with Colon Notation and a Common Reference Value

Students need to explore quantitative methods with the extensive quantities, and formulating two solutions by finding a common reference value for both pairs of variables could help students maintain the context of the problem in their solution and break the “larger number is correct answer” response. Sheila and Kim demonstrated that this extrapolation of ratios to a common multiple of parallel measures encourages students to keep track of units. This type of reasoning seems less likely if students use fractional representations because of their tendency to omit contextual elements.

The sidewalk problem, for example, can also be solved by finding a common reference value for the blocks variable, as shown in Figure 18. This is, with the colon notation, what Kim did in her diagram.

For Jill	5 blocks : 7 steps	→	15 blocks : 21 steps
For Alice	3 blocks : 4 steps		15 blocks : 20 steps

Figure 19. Alternative Solution with a Common Reference Value for the Other Variable

Alice walking one block further in the same number of steps and Jill taking an extra step for the same number of blocks are two ways of making the comparison. At this sublevel of quantitative proportional reasoning students can practice using labels and

expressing their solutions in the context of the problem. Solutions like these would give students more experience in the ratio-only realm of the overlapping model of ratios and fractions, where they can experience ratios conceptually, not ratios only in terms of how they look like fractions. When students are ready to interpret ratios as a compact number, mixed fractions may be a better choice than the typical a/b expressions.

When students are ready to interpret fractional representations of ratios as intensive quantities, a productive line of questioning that teachers can use with students who are having trouble is to ask them to examine a pair of ratios and examine their multiplicative inverses. The contradiction for those who rely simply on methods for comparing fractions can provide the motivation for constructing conceptual knowledge.

From my work on this project and from other experiences with younger students, I have become convinced that Confrey's long-term approach of a ratio-centered curriculum (Lachance & Confrey, 2002) that begins as early as third grade does provide students the needed opportunity to make connections among ratio-related mathematical topics. As Mandy said at the end of the first interview, "Ratios are a big part of life." And so they should become a bigger part of school mathematics.

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APPENDIX

Protocol for the Initial Interview

Participant:

Team:

Date:

Group:

To prepare: Have camera set up; calculator, pencil, and scratch paper at participant's place at table; participant's PR test, PR score sheet, interview schedule, clock, and copy of protocol at my place; materials for tasks on table (blocks and construction-paper circles for pizza problem, cubes for OJ problems, money tray with fake nickels and quarters, number line)

Begin with questions about how camp is going, favorite camp activity, and last math course.

1. Ask what she remembers about studying ratios in school.
2. Ask about how she normally writes ratios. Ask about alternative notations and teachers' notations and notations in textbooks.
3. Ask her to explain her solution to the pizza problem. Ask about the meaning of the notation in terms of the problem.
4. Ask her to use manipulatives to solve the pizza problem.
5. Ask her about the position of any fractions on the number line.
6. Ask her to show a ratio for an even exchange of dimes and quarters. Use the pencil as a hint.
7. Ask about the meaning of $5/2$ in the coin problem and return to the fractional representations of ratios in the pizza problem. Ask her again about the meaning of those.
8. Ask her to explain her solution to the orange-juice problems.
9. *How do you think ratios and fractions are related?*

Protocol for the Follow-Up Interview

Participant:

Date:

To prepare: Have camera set up; calculator, pencil, and scratch paper at participant's place at table; participant's PR test, clock, copy of protocol, notes about first interview at my place; materials for tasks on table (graph paper and rod)

1. General questions: Summer vacation, *Girls on Track*, further math study?
Remind them to talk out loud while working the problems.
2. Present the cookie problem. Ask what ratios she sees and see if she expresses them with more words.
3. Present the sidewalk problem.
4. Present the fertilizer problem. Ask about graphing and slope, if enough time.
5. *Any more thoughts about how ratios and fractions are related?*

**PARENT/STUDENT PERMISSION FOR
GIRLS ON TRACK PARTICIPANTS**

Dear Parent/Student:

One of Wake County Public Schools' major aims is to provide a challenging and meaningful program of mathematics at the middle grades level. As part of this effort, the school system has partnered with NC State University, Meredith College, and the Department of Public Instruction in *Girls on Track*, a program targeting middle grades girls in algebra.

The purpose of this letter is to request permission for your child's participation in the evaluation and research aspect of *Girls on Track*. Evaluation of the project is a requirement of the National Science Foundation, which is funding the program; therefore, it is important to gather data that will contribute to that process. We would like your permission to:

- (1) Have your child complete a brief survey form relating to attitudes about math, technology, and careers
- (2) Have your child complete a Proportional Reasoning Test, and
- (3) Have your child provide answers to a brief videotaped interview during the course of the *Girls on Track* Summer Camp (the tapes will be kept secure and confidential and used only for purposes of curriculum improvement).
- (4) Have your child (along with other participants) videotaped during Camp activities (the tapes will be kept secure and confidential and used only for purposes of curriculum improvement, evaluation of the project, and research into girls' thinking in the context of solving problems).
- (5) Access your child's End-of-Course test scores in math, technology test scores, math grades, and a record of courses attempted and completed.

The time required will be approximately one-half hour for each of the pre/post attitude surveys and the Proportional Reasoning Test. They will be administered as part of the summer program. The videotaped interviews will take no longer than half an hour as well.

There are no risks associated with this study. All information will be kept confidential. No child will be identified in any summary of the research.

If you and your child agree to take part in the study now and you change your mind, you can withdraw your permission at any time. Your child will not be penalized in any way if you decide to withdraw.

The results of the study will be used to help the school system make wise decisions about the math courses offered to middle grades students, in particular the courses that result in a stronger program for girls. The research plan has been approved by the North Carolina State University Institutional Review Board and the Wake County Public School System's Department of Evaluation and Research.

We hope that you and your child will decide to participate in this project. This study has been approved by NCSU and school district review committees. You may contact Dr. Sarah Berenson at NCSU at 919-515-2013 if you have any questions or concerns about this project. If you feel you have not been treated according to the descriptions in this letter, or if you or your child's rights have been violated during the course of the project, you may contact Dr. Matthew T. Zingraff, Chair of the NCSU IRB for the Use of Human Subjects in Research Committee, Box 8101, NCSU, Raleigh, NC 27695-8101

NC STATE UNIVERSITY

IRB Approval Letter

Sponsored Programs and
Regulatory Compliance
Campus Box 7514
1 Leazar Hall
Raleigh, NC 27695-7514

919.515.7200
919.515.7721 (fax)

From: Debra A. Paxton, IRB Administrator
North Carolina State University
Institutional Review Board

Date: February 6, 2002

Project Title: Girls on Track: Increasing Middle Grades Girls' Interest in Math Related Careers by Engaging them in Computer-based Mathematical Explorations of Urban Problems in their Communities

IRB#: 1479c

Dear Dr. Berenson:

The continuation request for the project listed above has been approved in accordance with policy under 45 CFR 46, and is approved for one year (through February 4, 2003). If your study lasts beyond that time, including data analysis, you must apply for continuing approval before the listed expiration date.

NOTE:

1. This committee complies with requirements found in Title 45 part 46 of The Code of Federal Regulations. For NCSU projects the Assurance Number is: M1263; the IRB Number is: 01XM.
2. Review de novo of this proposal is necessary if any significant alterations/additions are made.

Sincerely,

Matt Zingraff, Ph.D.

Letter Sent Home with Participants to Arrange the First Interview

Dear parent(s):

Every year at *Girls on Track* one of the staff members interviews about 10 to 12 of the campers about their math courses in school. The responses are anonymous, and the information we gather helps us design activities for the camp.

We would like to interview your daughter, _____, next week, either before or after regular camp hours one day. Would you be able to bring her to camp early one day or pick her up late? The suggested time for the interview is circled below. If this time is OK, please initial the time and have your daughter return this page to me. If this time is not OK, please ask her to let me know if another one of these times would be better, or you can leave me a message at 919-461-0515. Thank you.

Matthew R. Clark

Monday:	7:30-8:00 AM	12:00-12:30 PM
Tuesday:	7:30-8:00 AM	12:00-12:30 PM
Wednesday:	7:30-8:00 AM	12:00-12:30 PM
Thursday:	7:30-8:00 AM	12:00-12:30 PM

Example of Letter Mailed to Participants' Parents to Arrange the Second Interview

Dear Mr. And Ms. Smith:

We hope that your daughter, Ann, enjoyed *Girls on Track* last month. The teachers and those of us at N.C. State who organize the program had a positive experience and enjoyed working with the girls. Every year during *Girls on Track* we interview some of the girls to find out about the math courses they have taken. This information helps us design activities for the camp. Ann was one of the students I interviewed this year.

This year we decided to go into more detail during the interviews about how students solve math problems, and because the interviews were so valuable, both to us and to the girls who participated, I am trying to schedule one follow-up interview with eight of these girls, including Ann. We hope that the follow-up interviews will help us learn more about students' understanding of topics that are critical to making a successful transition from middle-school math into their algebra courses. Because of the progress that the girls made during the first round of interviews, we believe that the follow-up interview will provide them with an excellent opportunity to review several math topics before beginning the new school year.

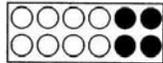
I will be conducting the interviews during the mornings of July 31, August 1, and August 2 in Poe Hall on the N.C. State campus; enclosed is a map with brief directions. I hope to schedule each student for a one-hour slot one of these days, but because of summer vacations and other activities some people may not be able to attend at this time. I will also be available August 12-14 in the late afternoons and evenings for students who cannot come next week but would like to participate. If transportation is a problem, please let me know. Because these interviews are a high priority for us, I am willing to travel to your home, your daughter's school, or another convenient location to do the interview if you are unable to bring your daughter to N.C. State.

Please call me or send me an e-mail message (mrclark@unity.ncsu.edu) to reserve a time or to ask me anything about the project. We hope that your daughter will be able to participate. These interviews have been a great learning experience for everyone involved. Thank you for your cooperation.

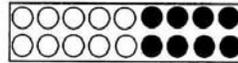
Sincerely,
Matthew R. Clark
Girls on Track Research Coordinator

3. There are 7 girls with 3 pizzas and 3 boys with 1 pizza. Who gets more pizza, the girls or the boys? Explain your thinking.

4. There are two egg cartons. The shaded circles represent brown eggs and the unshaded circles represent white eggs. The blue carton contains 8 white eggs and 4 brown eggs. The red carton contains 10 white eggs and 8 brown eggs. Which carton contains more brown eggs relative to white eggs? Explain your thinking.

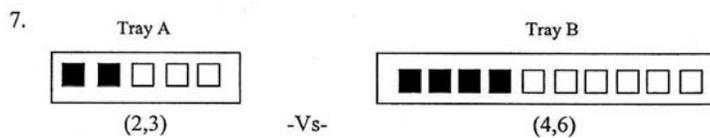
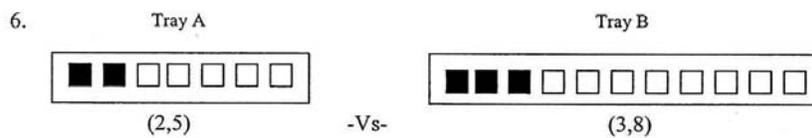
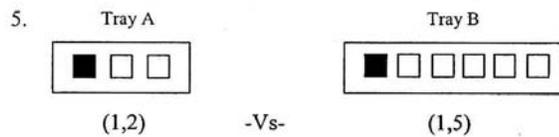


Blue Carton

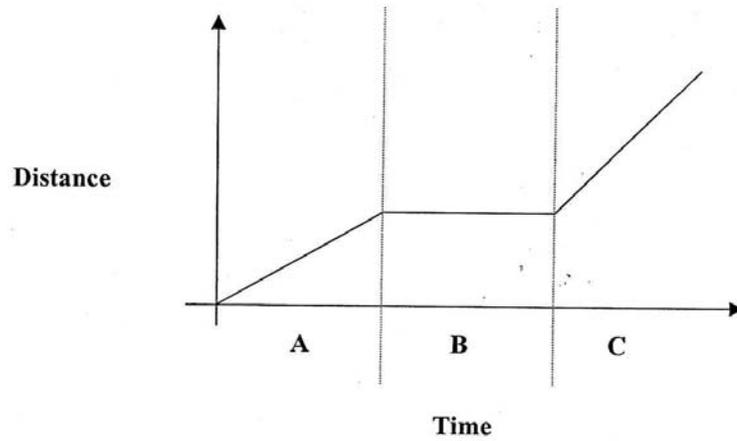


Red Carton

You and your friend are going to make orange juice for a party by mixing orange juice and water. You will be given three different situations. For each situation, you will be presented with the contents of two trays that contain various amounts of orange juice and water. The shaded box represents the orange juice and the unshaded box represents the water. **The goal for each situation is to determine which tray will create a drink that has a stronger orange taste or if the two drinks will taste the same.** Each mixture will be expressed as an ordered pair (e.g. (1, 3)) with the first term corresponding to the number of glasses of orange juice and the second term to the number of glasses of water. Show any calculations and explain your thinking.



8. Sarah took a bike ride this weekend. Below is a graph of the distance she travelled during the duration of her journey. The variable labeled **Distance** represents the distance Sarah is away from her starting point and the variable **Time** represents the amount of time that has passed since she began her journey. The graph is divided into three equal time intervals: A, B and C. What information can you deduce from the graph about how fast she was traveling during each time interval?



9. Two trees were measured five years ago. Tree A was 8 feet high and tree B was 10 feet high. Today, tree A is 14 feet high and tree B is 16 feet high. Over the last five years, which tree's height increased the most relative to its initial height? Show any calculations that lead you to your answer.
10. You are shown a flag that measures 3 feet in length and 2 feet in height. It uses 6 square feet of cloth. If you wanted to make it 3 feet longer while maintaining the same ratio of length to height, how much cloth would you need? Show your work.

	Length	Height	Area
Flag 1	3 ft	2 ft	6 ft ²
Flag 2	6 ft		