ABSTRACT

BENEDICT, BRANDY ANN. Axisymmetric Poroelastic Boundary Element Methods for Biphasic Mechanics of Articular Cartilage. (Under the direction of Professor Mansoor A. Haider.)

In this study, an axisymmetric Laplace domain boundary element method for modeling linear biphasic articular cartilage mechanics was developed. A boundary integral formulation was derived by writing the associated integral equations in terms of axisymmetric poroelastic fundamental solutions. Formulas for these fundamental solutions were derived from their three-dimensional Cartesian counterparts via transformations from Cartesian to cylindrical polar coordinates. The fundamental solutions of the poroelastic partial differential equations represent the effects at a particular boundary point of placing vector or scalar sources at all points on the boundary. In the axisymmetric formulation, these sources on the axisymmetric boundary are rotated about the z-axis, creating a ring of sources.

Axisymmetric boundary element methods were developed for solving the resulting boundary integral equations in the Laplace transform domain. The axisymmetric boundary was discretized by placing nodal points along a one-dimensional curve using three-node isoparametric quadratic boundary elements. Gaussian quadrature was employed to evaluate integrals over the boundary elements, which give rise to double integrals over strip regions on the axisymmetric surface. Weakly- and strongly-singular integrals were evaluated, separately, via specialized methods. In the case of weakly-singular integrals, transformation to local polar coordinates at the element nodes regularized the integrals. Strongly-singular integrals were evaluated using three known analytical solutions that enabled determination of unknown strongly-singular entries in terms of previously computed matrix entries.
Accuracy of the boundary element methods was demonstrated for configurations of biphasic compressive stress, pure radial stretching, and uniaxial confined compression, where analytical solutions are known. Potential use of the axisymmetric boundary element method and a Laplace inversion technique were illustrated via simulation of confined compression stress relaxation of a biphasic cartilage cell in a cylindrical sample of extracellular matrix.
Axisymmetric Poroelastic Boundary Element Methods for Biphasic Mechanics of Articular Cartilage

by

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DEDICATION

To my family, whose love, support, and cooking lessons have gotten me this far.
BIOGRAPHY

Brandy Ann Benedict was born in Orangevale, CA, on July 12, 1980. She attended Pershing Elementary School, Andrew Carnegie Middle School, Bella Vista High School, Occidental College, the University of Glasgow, and North Carolina State University. She joined the faculty of Merrimack College as Assistant Professor of Mathematics starting in the Fall of 2008.

At various points in her life, Brandy has been a theater technician, an Irish step dancer, a drummer, a Stegosaurus (only on Halloween), a sno-cone maker, a movie theater projectionist and a newspaper reporter, to name just a few. But for nearly all of her life she has been a student. This dissertation is the culmination of those years, and marks the transition from one major period of her life to another. Brandy is excited to see what lies ahead, and can’t wait to get going.
ACKNOWLEDGMENTS

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Chapter 1

Introduction and Background

Articular cartilage is the primary load-bearing soft tissue in diarthrodial joints, such as the knee, shoulder and hip. It provides a highly lubricated surface for transmission of mechanical forces as the bony ends of joints come into contact. Under mechanical loading, cartilage exhibits apparent viscoelasticity, arising mostly due to frictional drag as interstitial fluid moves past the deforming extracellular matrix in the tissue [26]. Cartilage extracellular matrix is comprised of a solid phase containing a fibrous network of type II collagen, proteoglycan macromolecules that contain fixed negative charges, and cells, called chondrocytes. The interstitial fluid phase in the extracellular matrix is comprised of water with dissolved ions and constitutes roughly 80% of the tissue mixture volume. Under conditions where ionic effects [14, 23] can be neglected, the biphasic (mixture) model [25, 26] has been shown to accurately model mechanical interactions within articular cartilage extracellular matrix.

The chondrocytes are sparsely distributed within the extracellular matrix (comprising roughly 1–10% of tissue volume) and maintain the matrix in a homeostatic balance between biosynthesis and degradation. This balance is altered in degenerative diseases such as osteoarthritis, leading to mechanical degradation and ultimately complete breakdown of the tissue. Since cartilage is avascular and aneural, the local biomechanical and biophysical environment of single cells is known to strongly influence their biosynthetic activity [17]. However, the role of specific extracellular biophysical and biomechanical variables in influencing chondrocyte activity is not well
understood, due in large part to coupled effects among these mechanisms [27]. Thus, mathematical models of the underlying biophysical and biomechanical processes serve to provide insight on factors in the local extracellular environment that are difficult or impossible to measure experimentally.

An example of one such biomechanical factor is cell volume. Under typical deformations in articular cartilage, it is known that changes are induced in the shape and volume of the cell, and in particular the cell nucleus [15]. While recent experimental techniques enable determination of in situ cell volumetric deformation via confocal microscopy [9], mathematical models are required to relate this deformation to the local extracellular forces, since these forces cannot be directly measured. Within this context, the complexity of the biphasic governing equations and associated boundary value problems makes the development of analytical solutions impractical.

Computational simulation of cartilage mechanics is most commonly performed within the framework of the domain-based finite element method. Two main approaches have included displacement-velocity-pressure penalty formulations [29, 30] and displacement-pressure formulations [2, 32]. Versatility of the FEM approach for simulating biphasic cartilage mechanics is well-established, including applications to nonlinear strain, joint contact, cell mechanics, matrix anisotropy, and simulations of cell-matrix mechanical interactions [3, 5, 13, 16, 22, 30, 31].

When the governing equations of the mathematical model are linear, an alternative numerical method called the boundary element method can be employed. Boundary element methods (BEMs) involve fundamental solutions of the governing partial differential equations and are particularly useful when quantities of interest lie on a boundary, as the method relies on discretization of only surface quantities. In the context of cartilage mechanics, previous studies have developed BEMs for simulation of single phase elastic and viscoelastic contact mechanics for single cells under micropipette aspiration [18, 19]. Recently, a three-dimensional BEM for biphasic cartilage mechanics was developed [20], based on an existing boundary integral formulation from the poroelasticity literature [4] specialized to boundary conditions that arise in biphasic cartilage mechanics. However, for geometries that are axisymmetric, this approach is inefficient and motivates the need for development of a specialized
axisymmetric formulation. Many configurations in cartilage mechanics are assumed to have axial symmetry due to the extraction of cartilage explants as cylindrical plugs [24] and the approximation of cells as spheres or ellipsoids [18, 19] (Figure 1.1).

Figure 1.1: (Left) Illustration of a uniaxial confined compression test performed on a cylindrical cartilage explant. (Right) An isolated cartilage cell (chondrocyte) with attached pericellular matrix [image courtesy of F. Guilak, Duke University Medical Center].

The primary goal of this study was to extend the three-dimensional BEM for biphasic cartilage mechanics developed in [20] for the specialized case of axisymmetric geometry. Previous formulations of axisymmetric BEMs for linear elastic materials have combined Cartesian fundamental solutions with transformations to cylindrical polar coordinates to derive axisymmetric fundamental solutions in terms of ring sources taken around the symmetry axis [6, 10, 11]. A similar approach is adopted in this study, employing the Cartesian poroelastic fundamental solutions [4, 20] to develop their axisymmetric counterparts.

The development of BEMs for biphasic cartilage mechanics relies on the mathematical correspondence to Biot’s theory of linear quasi-static poroelasticity, specialized to the case of incompressible constituents [7]. This correspondence is now briefly outlined. Under assumptions of isotropy and small strain, the governing equations for the biphasic continuum mixture model of cartilage mechanics are:

\[
\nabla \cdot \sigma^s + \Pi = 0, \quad \nabla \cdot \sigma^f - \Pi = 0, \quad (1.1)
\]

\[
\phi \nabla \cdot (\partial_t \mathbf{u}) + (1 - \phi)(\nabla \cdot \mathbf{v}) = 0, \quad (1.2)
\]
where

\begin{align}
\sigma^s &= [-\phi p + \lambda \text{tr}(e)] I + 2\mu e, \quad (1.3) \\
\sigma^f &= -(1 - \phi) p I, \quad (1.4) \\
\Pi &= \frac{(1 - \phi)^2}{k} (v - \partial_t u), \quad (1.5) \\
e &= \frac{1}{2} [\nabla u + \nabla u^T]. \quad (1.6)
\end{align}

In the above equations, the primary variables are the solid phase displacement \( u \), the velocity of the fluid phase \( v \), and the pore pressure \( p \). The momentum equations (1.1) are written in terms of the partial Cauchy stress tensors \( \sigma^s, \sigma^f \), and the fluid-solid drag \( \Pi \). The remaining quantities are the solid volume fraction \( \phi \), the hydraulic permeability \( k \), the infinitesimal strain tensor \( e \), the solid phase Poisson’s ratio \( \nu \), and the solid phase shear modulus \( \mu = 2(1 + \nu)/E \), where \( E \) is the Young’s modulus.

To illustrate the correspondence to poroelasticity theory, equations (1.1)–(1.2) can be reduced to the following \((u-p)\) model:

\begin{align}
\mu [(1 - 2\nu)^{-1} \nabla (\nabla \cdot u) + \nabla^2 u] &= \nabla p, \quad (1.7) \\
\partial_t (\nabla \cdot u) &= k \nabla^2 p. \quad (1.8)
\end{align}

From the soil consolidation literature [4, 28], the governing equations for the linear, isotropic, quasi-static poroelastic model are

\begin{align}
\nabla \cdot \sigma &= 0, \quad (1.9) \\
\partial_t \zeta + \nabla \cdot q &= 0, \quad (1.10)
\end{align}

where

\begin{align}
\sigma &= [-\alpha p + \lambda \text{tr}(e)] I + 2\mu e, \quad (1.11) \\
\frac{p}{M} &= \zeta - \alpha \text{tr}(e), \quad (1.12) \\
q &= -\kappa \nabla p. \quad (1.13)
\end{align}

In the above equations, \( \sigma \) is the total stress tensor, \( \alpha \) is the Biot effective stress coefficient, \( p \) is the pore pressure, \( \zeta \) is the variation of fluid volume per unit reference
volume, $\mathbf{q}$ is the specific discharge vector, $\kappa$ is a permeability coefficient, and $M$ is an inverse storage coefficient. Equivalence between the biphasic model (1.1)–(1.6) and the poroelastic model (1.9)–(1.13) is established by writing

$$\boldsymbol{\sigma} = \sigma^s + \sigma^f, \quad \alpha = 1, \quad \mathbf{q} = (1 - \phi)(\mathbf{v} - \partial_t \mathbf{u}), \quad M = \infty, \text{ and } \kappa = k.$$  

(1.14)

This correspondence facilitates development of BEMs for the linear biphasic model using fundamental solutions available in the poroelasticity literature. For an excellent review of poroelastic boundary integral equation formulations, see [8].

In this study, axisymmetric boundary element methods for modeling linear biphasic articular cartilage mechanics are developed in the Laplace transform domain. In Chapter 2, formulation of the axisymmetric fundamental solutions is presented based on cylindrical transformations of Cartesian fundamental solutions. In Chapter 3, detailed numerical methods for discretization and implementation of the axisymmetric boundary integral formulations presented in Chapter 2 are developed within the context of quadratic isoparametric boundary elements. Lastly, in Chapter 4, accuracy of the numerical method is evaluated and an application modeling the effect of uniaxial confined compression stress relaxation on cell deformation within the extracellular matrix is illustrated.
Chapter 2

Boundary Integral Formulation for Axisymmetric Poroelasticity

2.1 Introduction

In this chapter a boundary integral formulation for use in an axisymmetric poroelastic boundary element method is derived. The associated integral equations are written in terms of axisymmetric fundamental solutions that must be derived from their three-dimensional Cartesian counterparts. Previous three-dimensional poroelastic boundary element methods have been developed in the Laplace transform domain, and expressions for the associated Cartesian fundamental solutions are available in the literature [4]. In this study, an axisymmetric poroelastic boundary integral formulation is also developed in the Laplace transform domain.

2.2 Governing equations: Laplace transform domain

The boundary integral equations are derived from the linear, isotropic biphasic formulation of the governing equations of fluid-solid cartilage mechanics. When writ-
Using terminology of poroelasticity theory, the equations have the form:

\[ \nabla \cdot \sigma = 0, \quad (2.1) \]
\[ \partial_t \zeta + \nabla \cdot q = 0, \quad (2.2) \]

where

\[ \sigma = -pI + \lambda \text{tr}(e)I + 2\mu e, \quad (2.3) \]
\[ \zeta = \text{tr}(e), \quad (2.4) \]
\[ q = -\kappa \nabla p. \quad (2.5) \]

In (2.1)–(2.5), \( \sigma \) is the mixture stress, \( p \) is pore pressure, \( e \) is the strain tensor, \( q = q \cdot n \) is the normal flux, \( \kappa \) is the hydraulic permeability, and \( \lambda, \mu \) are Lamé coefficients for the solid phase.

The governing equations (2.1)–(2.5) are transformed to the Laplace transform domain to remove the time dependence in the equations. Nonhomogenous terms are introduced into each equation to allow for derivation of the fundamental solutions due to scalar and vector sources. The transformed set of governing equations, written in indicial summation notation, are

\[ \tilde{\sigma}_{ij,j} = -\tilde{F}_i, \quad (2.6) \]
\[ s\tilde{\zeta} + \tilde{q}_{i,i} = \tilde{\Gamma} \text{ (assuming } \zeta|_{t=0} = 0), \quad (2.7) \]

where

\[ \tilde{\sigma}_{ij} = -\tilde{p}\delta_{ij} + \lambda \tilde{e}_{aa}\delta_{ij} + 2\mu \tilde{e}_{ij}, \quad (2.8) \]
\[ \tilde{\zeta} = \tilde{e}_{aa}, \quad (2.9) \]
\[ \tilde{q}_i = -\kappa \tilde{p}_{i,i}. \quad (2.10) \]

Variables in the Laplace transform domain are denoted with a tilde, and spatial partial derivatives are denoted with a comma. In the equations above, all repeated indices are summed over the range 1-3, while all free indices represent a component equation. In the Laplace domain, the quantity \( \tilde{F}_i \) is the vector source in the \( i^{th} \) direction; the scalar source is represented by \( \tilde{\Gamma} \).
2.3 Derivation of poroelastic boundary integral equations

To derive the poroelastic boundary integral equations, the following reciprocal relation (Somigliana identity), which relates two different sets of solutions, is employed:

\[
\sigma_{ij}^{(1)} e_{ij}^{(2)} + p^{(1)} \zeta^{(2)} = \sigma_{ij}^{(2)} e_{ij}^{(1)} + p^{(2)} \zeta^{(1)}.
\]  

(2.11)

Let the solutions with a superscript “(1)” denote physical quantities, and those with a superscript “(2)” denote the fundamental solutions to the governing partial differential equations. After transforming (2.11) to the Laplace transform domain, the equivalent reciprocal relation in the Laplace domain is obtained:

\[
\tilde{\sigma}_{ij}^{(1)} \tilde{e}_{ij}^{(2)} + \tilde{p}^{(1)} \tilde{\zeta}^{(2)} = \tilde{\sigma}_{ij}^{(2)} \tilde{e}_{ij}^{(1)} + \tilde{p}^{(2)} \tilde{\zeta}^{(1)}.
\]  

(2.12)

The reciprocal relation (2.12) is integrated over the domain \( \Omega \). Due to symmetry of the stress tensor, it is straightforward to show that \( \tilde{\sigma}_{ij}^{(1)} \tilde{u}_{i,j} = \tilde{\sigma}_{ij}^{(1)} \tilde{u}_{i,j} \) (similarly, \( \tilde{\sigma}_{ij}^{(2)} \tilde{u}_{i,j} \)) so that:

\[
\int_{\Omega} \tilde{\sigma}_{ij}^{(1)} \tilde{u}_{i,j} dV + \int_{\Omega} \tilde{p}^{(1)} \tilde{\zeta}^{(2)} dV = \int_{\Omega} \tilde{\sigma}_{ij}^{(2)} \tilde{u}_{i,j} dV + \int_{\Omega} \tilde{p}^{(2)} \tilde{\zeta}^{(1)} dV.
\]  

(2.13)

Let \( \tilde{\Gamma}^{(1)} = 0 \) (giving \( s \tilde{\zeta}^{(1)} = \tilde{q}_{a,a}^{(1)} = 0 \)) and let \( \tilde{\Gamma}^{(2)} \) be the scalar source in the fundamental solution \( s \tilde{\zeta}^{(2)} + \tilde{q}_{a,a}^{(2)} = \tilde{\Gamma}^{(2)} \). Substituting these values into (2.13) yields

\[
\int_{\Omega} \tilde{\sigma}_{ij}^{(1)} \tilde{u}_{i,j} dV + \int_{\Omega} \tilde{p}^{(1)} s^{-1} (\tilde{\Gamma}^{(2)} - \tilde{q}_{a,a}^{(2)}) dV = \int_{\Omega} \tilde{\sigma}_{ij}^{(2)} \tilde{u}_{i,j} dV - \int_{\Omega} \tilde{p}^{(2)} s^{-1} \tilde{q}_{a,a}^{(1)} dV.
\]  

(2.14)

Using the divergence theorem, we can write (2.14) as

\[
\int_{\partial \Omega} \tilde{\sigma}_{ij}^{(1)} \tilde{u}_{i,j} n_j dS - \int_{\Omega} \tilde{u}_{i,j} \tilde{\sigma}_{ij}^{(1)} dV + \int_{\Omega} s^{-1} \tilde{p}^{(1)} \tilde{\Gamma}^{(2)} dV \\
- \int_{\partial \Omega} s^{-1} \tilde{p}^{(1)} \tilde{q}_{a,a} dS + \int_{\Omega} s^{-1} \tilde{p}_{a,a} \tilde{q}_{a,a}^{(2)} dV \\
= \int_{\partial \Omega} \tilde{\sigma}_{ij}^{(2)} \tilde{u}_{i,j} n_j dS + \int_{\Omega} \tilde{u}_{i,j} \tilde{F}_{i,j}^{(2)} dV \\
- \int_{\partial \Omega} s^{-1} \tilde{p}^{(2)} \tilde{q}_{a,a} dS + \int_{\Omega} s^{-1} \tilde{p}_{a,a} \tilde{q}_{a,a}^{(1)} dV.
\]  

(2.15)
Now, substitution of the following relations
\[
\delta_{ij,j}^{(1)} = 0, \quad \tilde{r}_{ij}^{(1)} = \delta_{ij}^{(1)} n_j, \quad \tilde{r}_{ij}^{(2)} = \delta_{ij}^{(2)} n_j, \quad \tilde{p}_{\alpha}^{(1)} = -k^{-1}q_{\alpha}^{(1)}, \quad \tilde{p}_{\alpha}^{(2)} = -k^{-1}q_{\alpha}^{(2)}
\]
into (2.15) gives the reduced equation
\[
\int_{\partial \Omega} \tilde{t}_{i}^{(1)} \tilde{u}_{i}^{(2)} dS + \int_{\Omega} s^{-1} \tilde{p}_{(1)}^{(1)} \tilde{F}_{i}^{(2)} dV - \int_{\partial \Omega} s^{-1} \tilde{p}_{\alpha}^{(1)} q_{\alpha}^{(2)} n_{\alpha} dS
\]
\[
= \int_{\partial \Omega} \tilde{t}_{i}^{(2)} \tilde{u}_{i}^{(1)} dS + \int_{\Omega} \tilde{u}_{i}^{(1)} \tilde{F}_{i}^{(2)} dV - \int_{\partial \Omega} s^{-1} \tilde{p}_{\alpha}^{(2)} q_{\alpha}^{(1)} n_{\alpha} dS,
\]
or, by grouping similar terms,
\[
\int_{\partial \Omega} \left[ \tilde{t}_{i}^{(1)} \tilde{u}_{i}^{(2)} - \tilde{t}_{i}^{(2)} \tilde{u}_{i}^{(1)} \right] dS - \int_{\partial \Omega} s^{-1} \left[ \tilde{p}_{\alpha}^{(1)} q_{\alpha}^{(2)} - \tilde{p}_{\alpha}^{(2)} q_{\alpha}^{(1)} \right] n_{\alpha} dS
\]
\[
+ \int_{\Omega} s^{-1} \tilde{p}_{\alpha}^{(1)} \tilde{F}_{i}^{(2)} dV - \int_{\Omega} \tilde{u}_{i}^{(1)} \tilde{F}_{i}^{(2)} dV = 0. \tag{2.16}
\]

**Impulse point force solution (vector source)** In (2.16), let the "(2)" solution correspond to placing a vector source at node \( Q \) with strength \( \alpha_j \) in the \( j^{th} \) direction; i.e.,
\[
\delta_{jk,k}^{(2)} = -\tilde{F}_{j}^{(2)} = -\alpha_k \delta_{jk}(y - x), \quad \tilde{F}_{i}^{(2)} = 0, \tag{2.17}
\]
and write \( \tilde{u}_{i}^{(2)} = \tilde{t}_{ki}^{f} \alpha_k, \tilde{r}_{i}^{(2)} = \tilde{t}_{ki}^{f} \alpha_k, \tilde{q}_{i}^{(2)} = \tilde{q}_{k}^{(2)} n_k = \tilde{q}_{k}^{f} \alpha_k, \) and \( \tilde{p}_{\alpha}^{(2)} = \tilde{p}_{k}^{f} \alpha_k. \) The "(1)" solutions then correspond to physical variables that are either prescribed or unknown. Then (2.16) can be written as
\[
\int_{\partial \Omega} \left[ \tilde{t}_{i}^{f} \tilde{u}_{i}^{f} \alpha_k - \tilde{t}_{ki}^{f} \alpha_k \tilde{u}_{i}^{f} \right] dS - \int_{\partial \Omega} s^{-1} \left[ \tilde{p}_{k}^{f} \alpha_k - \tilde{p}_{k}^{f} \alpha_k \tilde{q}_{k}^{f} \right] dS
\]
\[
- \int_{\Omega} \tilde{u}_{i} \alpha_k \delta_{ik}(y - x) dV = 0.
\]
Factoring out the \( \alpha_k \) term gives
\[
\alpha_k \left[ \int_{\partial \Omega} \left[ \tilde{t}_{i}^{f} \tilde{u}_{i}^{f} - \tilde{t}_{ki}^{f} \tilde{u}_{i}^{f} \right] dS - \int_{\partial \Omega} s^{-1} \left[ \tilde{p}_{k}^{f} \alpha_k - \tilde{p}_{k}^{f} \alpha_k \tilde{q}_{k}^{f} \right] dS - \int_{\Omega} \tilde{u}_{k}(y - x) dV \right] = 0,
\]
which yields
\[
\Theta(x) \tilde{u}_{k}(x) = \int_{\partial \Omega} \left[ \tilde{u}_{ki}^{f}(y - x) \tilde{t}_{i}(y) - \tilde{t}_{ki}^{f}(y - x) \tilde{u}_{i}(y) \right] dS(y)
\]
\[
- \int_{\partial \Omega} s^{-1} \left[ \tilde{q}_{k}^{f}(y - x) \tilde{p}(y) - \tilde{p}_{k}^{f}(y - x) \tilde{q}(y) \right] dS(y). \tag{2.18}
\]
For clarity, in (2.18) the dependence of all quantities on the spatial coordinate $x$ and location of the source $y$ is included. The quantity $\Theta(x)$ depends on the interior solid angle, e.g. $\Theta(x) = 1$ for $x$ in the interior of $\Omega$ and, in the case where $\partial\Omega$ is smooth, $\Theta(x) = 1/2$ on the boundary.

**Impulse fluid source solution (scalar source)** Alternatively, a scalar source of strength $\gamma$ can be placed at point $Q$ and the scalar and vector effects at point $P$ can be determined. In (2.16), let

$$
\tilde{\sigma}_{jk,k}^{(2)} = -\tilde{F}_j^{(2)} = 0, \quad \tilde{\Gamma}^{(2)} = \gamma \delta(y - x),
$$

and write $\tilde{u}_i^{(2)} = \tilde{u}_i^s \gamma$, $\tilde{t}_i^{(2)} = \tilde{t}_i^s \gamma$, $\tilde{q}^{(1)} = \tilde{q}_\alpha n_\alpha$, $\tilde{q}^{(2)} = \tilde{q}_\alpha^{(2)} n_\alpha = \tilde{q}_s \gamma$, and $\tilde{p}^{(2)} = \tilde{p}_s \gamma$. Then (2.16) can be written

$$
\int_{\partial\Omega} \left[ \tilde{t}_i \tilde{u}_i^s \gamma - \tilde{t}_i^s \gamma \tilde{u}_i \right] dS - \int_{\partial\Omega} s^{-1} \left[ \tilde{p}\tilde{q}_s \gamma - \tilde{p}_s \gamma \tilde{q} \right] dS + \int_{\Omega} s^{-1} \tilde{p}\gamma \delta(y - x) dV = 0.
$$

Factoring $\gamma$ out of the equation and multiplying through by $s$ gives

$$
-\Theta(x) \tilde{p}(x) = \int_{\partial\Omega} s \left[ \tilde{u}_i^s(y - x) \tilde{t}_i(y) - \tilde{t}_i^s(y - x) \tilde{u}_i(y) \right] dS(y) - \int_{\partial\Omega} \left[ \tilde{q}_s(y - x) \tilde{p}(y) - \tilde{p}_s(y - x) \tilde{q}(y) \right] dS(y).
$$

**2.3.1 Cartesian and axisymmetric boundary integral equations**

Integral equations formulated exclusively on the boundary are obtained from equations (2.18) and (2.20) by taking the limit as $x \to \partial \Omega$. In summary, the Cartesian
integral equations have the following form, for \( x \in S \) and \( j = 1, 2, 3 \), where \( S = \partial \Omega \):

\[
\Theta(x) \tilde{u}_j(x; s) = \sum_{i=1}^{3} \int_{S} \left[ \tilde{u}_{ij}^f(y - x; s) \tilde{t}_i(y; s) - \tilde{t}_{ij}^f(y - x; s) \tilde{u}_i(y; s) \right] dS(y) \\
+ \sum_{i=1}^{3} \int_{S} s^{-1} \left[ \tilde{p}_j^f(y - x; s) \tilde{q}(y; s) - \tilde{q}_j^f(y - x; s) \tilde{p}(y; s) \right] dS(y); \quad (2.21)
\]

\[
-\Theta(x) \tilde{p}(x; s) = \sum_{i=1}^{3} \int_{S} s[\tilde{u}_i^s(y - x; s) \tilde{t}_i(y; s) - \tilde{t}_i^s(y - x; s) \tilde{u}_i(y; s)] dS(y) \\
+ \sum_{i=1}^{3} \int_{S} \left[ \tilde{p}_i^s(y - x; s) \tilde{q}(y; s) - \tilde{q}_i^s(y - x; s) \tilde{p}(y; s) \right] dS(y). \quad (2.22)
\]

Recall that the primary variables are boundary values of the mixture tractions \( \tilde{t}_i \), solid displacements \( \tilde{u}_i \), the pore pressure \( \tilde{p} \), and normal flux \( \tilde{q} = \sum_{i=1}^{3} \tilde{q}_i n_i \).

To develop an axisymmetric specialization of (2.21)–(2.22), the \( i, j \) indices range over \( r \) and \( z \), giving three equations total:

\[
\Theta(x) \tilde{u}_j(x; s) = \sum_{i=r,z}^{3} \int_{S} \left[ \tilde{u}_{ij}^f(y - x; s) \tilde{t}_i(y; s) - \tilde{t}_{ij}^f(y - x; s) \tilde{u}_i(y; s) \right] dS(y) \\
+ \sum_{i=r,z}^{3} \int_{S} s^{-1} \left[ \tilde{p}_j^f(y - x; s) \tilde{q}(y; s) - \tilde{q}_j^f(y - x; s) \tilde{p}(y; s) \right] dS(y); \quad (2.23)
\]

\[
-\Theta(x) \tilde{p}(x; s) = \sum_{i=r,z}^{3} \int_{S} s[\tilde{u}_i^s(y - x; s) \tilde{t}_i(y; s) - \tilde{t}_i^s(y - x; s) \tilde{u}_i(y; s)] dS(y) \\
+ \sum_{i=r,z}^{3} \int_{S} \left[ \tilde{p}_i^s(y - x; s) \tilde{q}(y; s) - \tilde{q}_i^s(y - x; s) \tilde{p}(y; s) \right] dS(y). \quad (2.24)
\]

### 2.4 Fundamental solutions

The boundary integral equations are evaluated at a general point \( P \) on \( S \) and depend on boundary variables at all points \( Q \in S \). The Cartesian fundamental solutions are used to derive the effects of axisymmetric ring sources \( Q \) (for \( 0 \leq \theta < 2\pi \)) on vector and scalar solutions at \( P \). In the axisymmetric formulation, the plane \( \theta = 0 \) is chosen to coincide with the Cartesian plane \( x_2 = 0 \). Thus, the following coordinate representations are used for \( P \) and \( Q \) (Figure 2.1):

\[
P(x_1, x_2, x_3) = (R_P, 0, Z_P), \quad Q(y_1, y_2, y_3) = (r_Q \cos \theta, r_Q \sin \theta, z_Q) \quad (2.25)
\]
Figure 2.1: Cylindrical coordinate representation of $\mathcal{P}$ and $\mathcal{Q}$ for development of axisymmetric fundamental solutions. The fundamental solutions are constructed at point $\mathcal{P}$ for scalar and vector sources placed at point $\mathcal{Q}$.

Figure 2.2: Diagram of scalar and vector effects at $\mathcal{P}$ when a scalar source of strength $\gamma$ is placed at point $\mathcal{Q}$. Formulas for (scalar) pressure and (vector) traction at point $\mathcal{P}$ are given as examples: $\tilde{p} = \tilde{p}^s \gamma$; and $\tilde{t} = t_1^s \gamma \mathbf{E}_1 + t_2^s \gamma \mathbf{E}_2 + t_3^s \gamma \mathbf{E}_3 = \tilde{t}^s_1 \gamma \mathbf{e}_r + \tilde{t}^s_2 \gamma \mathbf{e}_z$. 
Figure 2.3: Diagram of scalar and vector effects at $P$ when a vector source of strength $\alpha$ is placed at point $Q$. Formulas for (scalar) pressure and (vector) traction at point $P$ are given as examples: 

$$\tilde{p}(P) = \tilde{p}f_1\alpha_1 + \tilde{p}f_2\alpha_2 + \tilde{p}f_3\alpha_3; \quad \tilde{t}(P) = (\tilde{t}f_{11}\alpha_1 + \tilde{t}f_{21}\alpha_2 + \tilde{t}f_{31}\alpha_3)E_1 + (\tilde{t}f_{12}\alpha_1 + \tilde{t}f_{22}\alpha_2 + \tilde{t}f_{32}\alpha_3)E_2 + (\tilde{t}f_{13}\alpha_1 + \tilde{t}f_{23}\alpha_2 + \tilde{t}f_{33}\alpha_3)E_3 = (\tilde{t}f_{rr}\alpha_r + \tilde{t}f_{rz}\alpha_z)e_r + (\tilde{t}f_{rz}\alpha_r + \tilde{t}f_{zz}\alpha_z)e_z.$$

2.4.1 Cartesian fundamental solutions

The Cartesian fundamental solutions [4] due to an impulse fluid source of strength $\gamma$, as in (2.19), are

$$\tilde{u}_i^s = \frac{r,i}{sr^2}[1 - (1 + \zeta)e^{-\zeta}], \quad (2.26)$$

$$\tilde{t}_i^s = \frac{2\mu}{sr^3}\{n_i[1 - (1 + \zeta + \zeta^2)e^{-\zeta}] + r,i r,j n_j[(3 + 3\zeta + \zeta^2) - 3]\}, \quad (2.27)$$

$$\tilde{p}^s = \frac{1}{\kappa} \frac{e^{-\zeta}}{r}, \quad (2.28)$$

$$\tilde{q}^s = \frac{r,i n_i}{r^2}(1 + \zeta)e^{-\zeta}, \quad (2.29)$$

where

$$r = ||x - y||; \quad r,i = r^{-1}(y_i - x_i); \quad \kappa = k; \quad \zeta = r\sqrt{s\frac{1}{\kappa(\lambda + 2\mu)}}, \quad \lambda = \frac{2\mu\nu}{1 - 2\nu}. \quad (2.30)$$

As illustrated in Figure 2.2,

$$\tilde{u}_i = \tilde{u}_i^s \gamma, \quad \tilde{t}_i = \tilde{t}_i^s \gamma, \quad \tilde{p} = \tilde{p}^s \gamma, \quad \tilde{q} = \tilde{q}^s \gamma. \quad (2.31)$$
The fundamental solutions due to an impulse point force of strength $\alpha_i$ in the $i^{th}$ direction, as in (2.17), are

$$\tilde{u}_{ij}^f = \frac{1}{2\mu r} \delta_{ij} + \frac{1}{2\mu} \frac{r_i r_j}{r} + \kappa \frac{1}{sr^3} \{\delta_{ij}[(1 - (1 + \zeta) e^{-\zeta}) + r_i r_j[(3 + 3\zeta + \zeta^2) e^{-\zeta} - 3]]\},$$

(2.32)

$$\tilde{t}_{ij}^f = -\frac{3}{r^2} r_i r_j r_k n_k + 2\mu \frac{\kappa}{sr^4} \{(\delta_{ij} r_k n_k + n_j r_i)[(3 + 3\zeta + \zeta^2) e^{-\zeta} - 3]
+ n_i r_j[(3 + 3\zeta + 2\zeta^2 + \zeta^3) e^{-\zeta} - 3] + r_i r_j r_k n_k[15 - (15 + 15\zeta + 6\zeta^2 + \zeta^3) e^{-\zeta}]\},$$

(2.33)

$$\tilde{p}_{ij}^f = \frac{r_i}{r^2} [1 - (1 + \zeta) e^{-\zeta}],$$

(2.34)

$$\tilde{q}_{ij}^f = \frac{1}{r^3} \{n_j[(1 + \zeta) e^{-\zeta} - 1] + r_i r_j r_k n_k[3 - (3 + 3\zeta + \zeta^2) e^{-\zeta}]\},$$

(2.35)

where, as shown in Figure 2.3,

$$\tilde{u}_r = \tilde{u}_r^f, \quad \tilde{u}_z = \tilde{u}_z^f, \quad \tilde{t}_r = \tilde{t}_r^f, \quad \tilde{t}_z = \tilde{t}_z^f, \quad \tilde{p} = \tilde{p}_r^f, \quad \tilde{q} = \tilde{q}_r^f,$$

(2.36)

(2.37)

### 2.4.2 Axisymmetric fundamental solutions

The Cartesian fundamental solutions of the previous section are now used to derive axisymmetric fundamental solutions. For a scalar source of strength $\gamma$ placed along a ring containing $Q$, the axisymmetric dependent variables are

$$\tilde{u}_r = \tilde{u}_r^s \gamma, \quad \tilde{u}_z = \tilde{u}_z^s \gamma, \quad \tilde{t}_r = \tilde{t}_r^s \gamma, \quad \tilde{t}_z = \tilde{t}_z^s \gamma, \quad \tilde{p} = \tilde{p}^s \gamma, \quad \tilde{q} = \tilde{q}^s \gamma,$$

(2.38)
where, for \( x = (R_P, 0, Z_P) \):

\[
\tilde{u}_r^s = \frac{1}{2\pi} \int_0^{2\pi} \tilde{u}_1^s(x; r_Q \cos \theta, r_Q \sin \theta, z_Q) \, d\theta,
\]
\( (2.39) \)

\[
\tilde{u}_z^s = \frac{1}{2\pi} \int_0^{2\pi} \tilde{u}_3^s(x; r_Q \cos \theta, r_Q \sin \theta, z_Q) \, d\theta,
\]
\( (2.40) \)

\[
\tilde{t}_r^s = \frac{1}{2\pi} \int_0^{2\pi} \tilde{t}_1^s(x; r_Q \cos \theta, r_Q \sin \theta, z_Q) \, d\theta,
\]
\( (2.41) \)

\[
\tilde{t}_z^s = \frac{1}{2\pi} \int_0^{2\pi} \tilde{t}_3^s(x; r_Q \cos \theta, r_Q \sin \theta, z_Q) \, d\theta,
\]
\( (2.42) \)

\[
\tilde{p}^s = \frac{1}{2\pi} \int_0^{2\pi} \tilde{p}^s(x; r_Q \cos \theta, r_Q \sin \theta, z_Q) \, d\theta,
\]
\( (2.43) \)

\[
\tilde{q}^s = \frac{1}{2\pi} \int_0^{2\pi} \tilde{q}^s(x; r_Q \cos \theta, r_Q \sin \theta, z_Q) \, d\theta.
\]
\( (2.44) \)

For a vector source of strength \( \alpha = \alpha_r e_r + \alpha_z e_z \) placed at \( Q \), the Cartesian components of \( \alpha \) at a point \( Q \) along a ring of sources are:

\[
\alpha = \alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 = \alpha_r \cos \theta E_1 + \alpha_r \sin \theta E_2 + \alpha_z E_3
\]
\( (2.45) \)

For a vector source, the axisymmetric dependent variables are written as

\[
\tilde{u}_r = \tilde{u}_r^f \alpha_r + \tilde{u}_z^f \alpha_z, \quad \tilde{u}_z = \tilde{u}_z^f \alpha_r + \tilde{u}_z^f \alpha_z, \quad \tilde{t}_r = \tilde{t}_r^f \alpha_r + \tilde{t}_z^f \alpha_z, \quad \tilde{t}_z = \tilde{t}_z^f \alpha_r + \tilde{t}_z^f \alpha_z, \quad \tilde{p} = \tilde{p}_r^f \alpha_r + \tilde{p}_z^f \alpha_z, \quad \tilde{q} = \tilde{q}_r^f \alpha_r + \tilde{q}_z^f \alpha_z.
\]
\( (2.46) \)

After some algebraic manipulation, substitution of (2.45) into (2.46)–(2.48), and (2.36)–(2.37), yields the axisymmetric fundamental solutions as integrals over \( \theta \) from
0 to $2\pi$ written in terms of the Cartesian fundamental solutions:

$$\tilde{u}_r^f = \frac{1}{2\pi} \int_0^{2\pi} [\tilde{u}_{11}^f(x; r_Q \cos \theta, r_Q \sin \theta, z_Q) \cos \theta + \tilde{u}_{21}^f(x; r_Q \cos \theta, r_Q \sin \theta, z_Q) \sin \theta] \, d\theta,$$

(2.49)

$$\tilde{u}_z^f = \frac{1}{2\pi} \int_0^{2\pi} [\tilde{u}_{13}^f(x; r_Q \cos \theta, r_Q \sin \theta, z_Q) \cos \theta + \tilde{u}_{23}^f(x; r_Q \cos \theta, r_Q \sin \theta, z_Q) \sin \theta] \, d\theta,$$

(2.50)

$$\tilde{u}_r^z = \frac{1}{2\pi} \int_0^{2\pi} \tilde{u}_{31}^f(x; r_Q \cos \theta, r_Q \sin \theta, z_Q) \, d\theta,$$

(2.51)

$$\tilde{u}_z^r = \frac{1}{2\pi} \int_0^{2\pi} \tilde{u}_{33}^f(x; r_Q \cos \theta, r_Q \sin \theta, z_Q) \, d\theta,$$

(2.52)

$$\tilde{t}_r^f = \frac{1}{2\pi} \int_0^{2\pi} [\tilde{t}_{11}^f(x; r_Q \cos \theta, r_Q \sin \theta, z_Q) \cos \theta + \tilde{t}_{21}^f(x; r_Q \cos \theta, r_Q \sin \theta, z_Q) \sin \theta] \, d\theta,$$

(2.53)

$$\tilde{t}_z^f = \frac{1}{2\pi} \int_0^{2\pi} [\tilde{t}_{13}^f(x; r_Q \cos \theta, r_Q \sin \theta, z_Q) \cos \theta + \tilde{t}_{23}^f(x; r_Q \cos \theta, r_Q \sin \theta, z_Q) \sin \theta] \, d\theta,$$

(2.54)

$$\tilde{t}_r^z = \frac{1}{2\pi} \int_0^{2\pi} \tilde{t}_{31}^f(x; r_Q \cos \theta, r_Q \sin \theta, z_Q) \, d\theta,$$

(2.55)

$$\tilde{t}_z^r = \frac{1}{2\pi} \int_0^{2\pi} \tilde{t}_{33}^f(x; r_Q \cos \theta, r_Q \sin \theta, z_Q) \, d\theta,$$

(2.56)

$$\tilde{p}_r^f = \frac{1}{2\pi} \int_0^{2\pi} [\tilde{p}_1^f(x; r_Q \cos \theta, r_Q \sin \theta, z_Q) \cos \theta + \tilde{p}_2^f(x; r_Q \cos \theta, r_Q \sin \theta, z_Q) \sin \theta] \, d\theta,$$

(2.57)

$$\tilde{p}_z^f = \frac{1}{2\pi} \int_0^{2\pi} \tilde{p}_3^f(x; r_Q \cos \theta, r_Q \sin \theta, z_Q) \, d\theta,$$

(2.58)

$$\tilde{q}_r^f = \frac{1}{2\pi} \int_0^{2\pi} [\tilde{q}_1^f(x; r_Q \cos \theta, r_Q \sin \theta, z_Q) \cos \theta + \tilde{q}_2^f(x; r_Q \cos \theta, r_Q \sin \theta, z_Q) \sin \theta] \, d\theta,$$

(2.59)

$$\tilde{q}_z^f = \frac{1}{2\pi} \int_0^{2\pi} \tilde{q}_3^f(x; r_Q \cos \theta, r_Q \sin \theta, z_Q) \, d\theta.$$

(2.60)

These fundamental solutions are employed in the axisymmetric boundary integral formulation, which for convenience is summarized in section 2.5.
2.5 Summary of axisymmetric boundary integral equations

For \( x \in S \), the following boundary integral equations hold:

\[
\Theta \tilde{u}_r(x) + \int_S \left[ \tilde{t}_{rr}(y - x)\tilde{u}_r(y) + \tilde{t}_{zr}(y - x)\tilde{u}_z(y) + s^{-1}\tilde{q}_r(y - x)\tilde{p}(y) \right] dS(y)
\]

\[
- \int_S \left[ \tilde{u}_{rr}(y - x)\tilde{t}_r(y) + \tilde{u}_{zr}(y - x)\tilde{t}_z(y) + s^{-1}\tilde{p}_r(y - x)\tilde{q}(y) \right] dS(y) = 0, \quad (2.61)
\]

\[
\Theta \tilde{u}_z(x) + \int_S \left[ \tilde{t}_{rz}(y - x)\tilde{u}_r(y) + \tilde{t}_{zz}(y - x)\tilde{u}_z(y) + s^{-1}\tilde{q}_z(y - x)\tilde{p}(y) \right] dS(y)
\]

\[
- \int_S \left[ \tilde{u}_{rz}(y - x)\tilde{t}_r(y) + \tilde{u}_{zz}(y - x)\tilde{t}_z(y) + s^{-1}\tilde{p}_z(y - x)\tilde{q}(y) \right] dS(y) = 0, \quad (2.62)
\]

\[
- \Theta \tilde{p}(x) + \int_S \left[ \tilde{s}_r(y - x)\tilde{u}_r(y) + \tilde{s}_z(y - x)\tilde{u}_z(y) + \tilde{q}(y - x)\tilde{p}(y) \right] dS(y)
\]

\[
- \int_S \left[ \tilde{s}_{rr}(y - x)\tilde{t}_r(y) + \tilde{s}_{zz}(y - x)\tilde{t}_z(y) + \tilde{p}(y - x)\tilde{q}(y) \right] dS(y) = 0, \quad (2.63)
\]

where the fundamental solutions are given in (2.39)–(2.44) and (2.49)–(2.60).
Chapter 3

Axisymmetric Boundary Element Methods

3.1 Introduction

In Chapter 2, a set of poroelastic boundary integral equations modeling the response of a biphasic material under mechanical loading was developed in the Laplace domain. The integrals depended on fundamental solutions of the poroelastic PDEs, which represent the effects at a particular boundary point of placing vector or scalar sources at all points on the boundary. In the axisymmetric BEM, these sources on the axisymmetric boundary are rotated about the $z$-axis, creating a ring of sources.

In this chapter, an axisymmetric boundary element method is presented for solving the poroelastic boundary integral equations in the Laplace domain. The axisymmetric boundary is discretized by placing nodal points along a one-dimensional curve in the plane $\theta = 0$. Every three adjacent nodes are grouped into elements, and quantities along the boundary are approximated by interpolating between nodal values in the element using quadratic shape functions. Although the method can be applied to any axisymmetric boundary, in this derivation and subsequent application, the method is specialized to a spherical surface.

Gaussian quadrature is used to integrate the fundamental solutions over the elements, which are rotated once about the $z$-axis from 0 to $2\pi$ to obtain integrals over
the entire surface. Each boundary integral represents the effect at node \( P \) due to a source placed at node \( Q \). All possible combinations of \( P \) and \( Q \) are integrated, including the case when \( P \) and \( Q \) are in the same element, leading to singular integrals as \( r \to 0 \) which must be treated specially.

Weakly-singular \((r^{-1})\) integrals are transformed to local polar coordinates, which removes the singularity by adding an ‘\( r \)’ term from the Jacobian of the coordinate transformation [4, 20]. The strongly-singular \((r^{-2})\) integrals cannot be evaluated numerically, so known solutions are substituted into the linear system of equations, allowing the unknown strongly-singular entries to be expressed in terms of matrix entries which have already been computed [6, 18, 19].

Once all matrix entries have been determined, a set of boundary conditions is prescribed at each node. A new linear system is assembled by keeping unknown nodal quantities on the left side of the equation and moving known nodal values to the right. This system is solved in the Laplace domain for multiple values of the Laplace parameter \( s \).

### 3.2 Boundary elements

The first step in the axisymmetric boundary element method is to define a mesh on the surface of the sphere. Since the boundary is symmetric about the \( z \)-axis, discretization only needs to be carried out on a one-dimensional curve that represents the boundary surface in the plane \( \theta = 0 \). The elements are effectively rotated about the \( z \)-axis to obtain influence integrals over the entire surface.

A mesh is generated by partitioning the boundary into \( M \) elements, each with three nodes (two endpoints and a midpoint), as in Figure 3.1-Left. It is noted that, apart from the first and last nodes, the endpoint nodes belong to two elements. For a curve with \( M \) elements, there are \( n = 2M + 1 \) total nodes on the boundary. To generate the nodal points, the boundary curve is parameterized in \( \theta \) counterclockwise from \(-\pi/2\) to \(\pi/2\). The global \((x, y)\) coordinates of each node are given by

\[
(x, y) = (R \cos \hat{\theta}^{k,b}, R \sin \hat{\theta}^{k,b}),
\]
where $R$ is the radius of the sphere and $\hat{\theta}^{k,b}$ is the angle $\hat{\theta}$ corresponding to node $b$ ($b = 1, 2, 3$) in element $k$ ($k = 1, \ldots, M$) on the boundary. The global elements are transformed to local isoparametric elements that are parameterized by the local coordinate $\xi \in [-1, 1]$ (Figure 3.1-Right). The global coordinates of the three local nodes $b = 1$, $b = 2$, and $b = 3$ are mapped to the local coordinate $\xi = -1$, $\xi = 0$, and $\xi = 1$, respectively.

Numerical integration over these boundary elements requires evaluation of quantities of interest at non-nodal points within the element. These quantities of interest in the $k^{th}$ boundary element are approximated via quadratic interpolation using a linear combination of nodal values $(\cdot)^{k,b}$ at node $b$ in element $k$:

$$ (\cdot)^{k}(\xi) = \sum_{b=1}^{3} (\cdot)^{k,b} N_b(\xi), \quad k = 1, \ldots, M. \quad (3.1) $$

The three quadratic shape functions $N_b(\xi)$ all have the property that they are equal to one at their corresponding node $b$ and zero at the other two nodes (Figure 3.2):

$$ N_1(\xi) = -\frac{1}{2} \xi (1 - \xi), \quad (3.2) $$

$$ N_2(\xi) = (1 - \xi)(1 + \xi), \quad (3.3) $$

$$ N_3(\xi) = \frac{1}{2} \xi (1 + \xi). \quad (3.4) $$
3.3 Discretization of boundary integral equations

The boundary integral equations (2.21)–(2.22) in axisymmetric coordinates reduce to the following three equations which must hold at each point $P$ on the boundary $S$:

\[
\int_S \left[ \tilde{t}_r \tilde{u}_r + \tilde{t}_z \tilde{u}_z + s^{-1} \tilde{q} \tilde{p} \right] dS + \Theta \tilde{u}_r - \int_S \left[ \tilde{u}_r \tilde{t}_r + \tilde{u}_z \tilde{t}_z + s^{-1} \tilde{p} \tilde{q} \right] dS = 0, \quad (3.5)
\]

\[
\int_S \left[ \tilde{t}_r \tilde{u}_r + \tilde{t}_z \tilde{u}_z + s^{-1} \tilde{q} \tilde{p} \right] dS + \Theta \tilde{u}_z - \int_S \left[ \tilde{u}_r \tilde{t}_r + \tilde{u}_z \tilde{t}_z + s^{-1} \tilde{p} \tilde{q} \right] dS = 0, \quad (3.6)
\]

\[
\int_S \left[ s \tilde{t}_r \tilde{u}_r + s \tilde{t}_z \tilde{u}_z + \tilde{q} \tilde{p} \right] dS - \Theta \tilde{p} - \int_S \left[ s \tilde{u}_r \tilde{t}_r + s \tilde{u}_z \tilde{t}_z + \tilde{p} \tilde{q} \right] dS = 0. \quad (3.7)
\]

In (3.5)–(3.7), the fundamental solutions (‘f’ and ‘s’ superscripts) depend on the distance $r$ between $P$ and $Q$ (i.e., the magnitude of $r = y - x$), while all remaining boundary quantities in the integrands (except the Laplace variable $s$) depend on $y$.

By transforming to the local coordinate $\xi \in [-1, 1]$, the boundary integrals over $S$ are partitioned into integrals over each element $k$. Within each element, boundary variables are approximated using the quadratic shape function interpolation, i.e. $(\cdot)^k(\xi) = \sum_{b=1}^{3} (\cdot)^k,b N_b(\xi)$, which is determined by the nodal values. To discretize the integral equations, we replace the continuous boundary quantities $\tilde{(\cdot)}(y)$ with interpolated boundary values in the local element. The integral equations (3.5)–(3.7) can then be written as a system of $n = 2M + 1$ linear equations, one for each node $n$. For
each element \( k = 1, \ldots, M \), the following boundary integrals arise:

\[
\begin{align*}
\sum_{b=1}^{3} \int_{-1}^{1} \left[ t_{rr} \tilde{u}_{r}^{k,b} + t_{rz} \tilde{u}_{z}^{k,b} + s^{-1} q_{r} \tilde{p}^{k,b} \right] N_{b}(\xi) J^{k}(\xi) d\xi + \Theta \tilde{u}_{r} \\
\sum_{b=1}^{3} \int_{-1}^{1} \left[ t_{rz} \tilde{u}_{r}^{k,b} + t_{zz} \tilde{u}_{z}^{k,b} + s^{-1} q_{z} \tilde{p}^{k,b} \right] N_{b}(\xi) J^{k}(\xi) d\xi + \Theta \tilde{u}_{z} \\
\sum_{b=1}^{3} \int_{-1}^{1} \left[ s t_{r}^{b} \tilde{u}_{r}^{k,b} + s t_{z}^{b} \tilde{u}_{z}^{k,b} + \tilde{q}^{k,b} \right] N_{b}(\xi) J^{k}(\xi) d\xi - \Theta \tilde{p}
\end{align*}
\]

Note that the nodal values \((\cdot)^{k,b}\) are either prescribed or unknown, depending on the particular type of boundary value problem under consideration. Thus, they are independent of the integration variable \( \xi \) and can be pulled outside the integrals in (3.8).

The fundamental solutions can be assembled into 3 × 3 matrices to write (3.8) more compactly as the system of \( n \) equations at each node \( l \):

\[
\sum_{k=1}^{M} \sum_{b=1}^{3} T^{k,b}(x^{k,b} - x^l; s) \omega^{k,b} - \sum_{k=1}^{M} \sum_{b=1}^{3} U^{k,b}(x^{k,b} - x^l; s) \tau^{k,b} = 0, \quad l = 1, \ldots, n. \quad (3.9)
\]

The 3 × 3 matrices \( T^{k,b} \) and \( U^{k,b} \) are comprised of integrals of the axisymmetric fundamental solutions over element \( k \) and around a ring of sources, i.e. \( (\xi, \theta) \in [-1,1] \times [0,2\pi] \):

\[
T^{k,b} = \int_{-1}^{1} \int_{0}^{2\pi} N_{b}(\xi) \tau^{k,b}_{*}(x^{k,b} - x^l; \xi, \theta) J_{k}(\xi) d\theta d\xi + \Theta I^{*}, \quad (3.10)
\]

\[
U^{k,b} = \int_{-1}^{1} \int_{0}^{2\pi} N_{b}(\xi) \omega^{k,b}_{*}(x^{k,b} - x^l; \xi, \theta) J_{k}(\xi) d\theta d\xi + \Theta I^{*}, \quad (3.11)
\]

where \( I^{*} \) is the 3 × 3 identity matrix with -1 in the (3,3) entry:

\[
I^{*} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{bmatrix}
\]

if \( x^l = x^{k,b} \); otherwise \( I^{*} = 0 \). \quad (3.12)
In (3.12), the coordinates of fixed point \( P \) are given by \( x^l \), and the coordinates of source point \( Q \) on the boundary are \( x^{k,b} \). The \( 3 \times 3 \) matrices \( \tau^*_k \) and \( \omega^*_k \) of axisymmetric fundamental solutions in (3.10) and (3.11) are:

\[
\tau^*_k = \begin{bmatrix}
\tilde{t}_{rr} & \tilde{t}_{rz} & s^{-1}\tilde{q}_{r}
\tilde{t}_{rz} & \tilde{t}_{zz} & s^{-1}\tilde{q}_{z}
\tilde{s}t_r & \tilde{s}t_z & s
\end{bmatrix}, \quad \omega^*_k = \begin{bmatrix}
\tilde{u}_{rr} & \tilde{u}_{rz} & s^{-1}\tilde{p}_r
\tilde{u}_{rz} & \tilde{u}_{zz} & s^{-1}\tilde{p}_z
\tilde{s}u_r & \tilde{s}u_z & s
\end{bmatrix}, \quad (3.13)
\]

where all components depend exclusively on \( r = x^{k,b} - x^l \). The nodal values of displacements, tractions, pressure and flux at the \( b^{th} \) node in the \( k^{th} \) element are denoted with a \( k,b \) superscript:

\[
\omega^{k,b} = \begin{bmatrix}
\tilde{u}^{k,b}_r \\
\tilde{u}^{k,b}_z \\
\tilde{p}^{k,b}_r
\end{bmatrix}, \quad \tau^{k,b} = \begin{bmatrix}
\tilde{t}^{k,b}_r \\
\tilde{t}^{k,b}_z \\
\tilde{q}^{k,b}
\end{bmatrix}. \quad (3.14)
\]

In the integrands in (3.10) and (3.11), \( N_b(\xi) \) is the quadratic shape function defined in (3.2)–(3.4) corresponding to node \( b \) evaluated at local coordinate \( \xi \), and \( J^k(\xi) \) is the Jacobian of the transformation in element \( k \) from global coordinates to local element coordinates. Derivatives of \( x = [r; z] \) in element \( k \) with respect to local coordinate \( \xi \) are expressed in terms of derivatives of the shape functions:

\[
r'(\xi) = \sum_{b=1}^{3} r^{k,b} N'_b(\xi); \quad z'(\xi) = \sum_{b=1}^{3} z^{k,b} N'_b(\xi).
\]

The unit outward normal vector is also required to evaluate several fundamental solutions. For the specific case of a spherical boundary, it can be obtained by simply dividing the coordinate vector by its norm. For more general boundaries, the unit outward normal is determined as the vector \( < z'(\xi), -r'(\xi) > \) divided by its norm. The Jacobian in (3.10) and (3.11) is

\[
J^k(\xi) = r(\xi) \sqrt{r'(\xi)^2 + z'(\xi)^2}, \quad (3.15)
\]

where the square root term comes from the transformation from global coordinates to local element coordinates \( \xi \in [-1, 1] \), and the \( r(\xi) \) term is due to the transformation from Cartesian to cylindrical coordinates.
3.4 Numerical integration

3.4.1 Nonsingular integrals

The large majority of integrals in (3.10) and (3.11) are nonsingular and can be evaluated using standard Gaussian quadrature:

\[ \int_{-1}^{1} f(x) \, dx \approx \sum_{i=1}^{nGQ} w_i f(x_i), \]  

where \( w_i, x_i \) are the weights and abscissas for Gaussian quadrature integration of order \( n_{GQ} \) [1]. Integrals with nonstandard limits \([a, b]\) are transformed to standard \([-1, 1]\) Gaussian quadrature limits via:

\[ \int_{a}^{b} f(t) \, dx = \frac{b-a}{2} \int_{-1}^{1} f \left( \frac{b-a}{2} x + \frac{a+b}{2} \right) \, dx \approx \frac{b-a}{2} \sum_{i=1}^{n} w_i f \left( \frac{b-a}{2} x_i + \frac{a+b}{2} \right) . \]  

3.4.2 Singular integrals

The fundamental solutions in the integrals of \( T_{k,b} \) and \( U_{k,b} \) in (3.10) and (3.11) are evaluated for all choices of \( P \) and \( Q \) along the boundary curve. When \( P \) and \( Q \) are close together, \( r \to 0 \) and the fundamental solutions become singular. For integrals evaluated when \( P \) and \( Q \) are at nodes that belong to the same element, including when \( P \) and \( Q \) are at the same node, special techniques are employed to bypass the singularities.

Weakly-singular integrals have an \( r^{-1} \) singularity, and arise in the fundamental solutions \( \tilde{u}_{rr}, \tilde{u}_{rz}, \tilde{u}_{zz}, \tilde{u}_r, \tilde{u}_z, \tilde{p}_r, \tilde{p}_z, \tilde{q}_r, \tilde{q}_z, \tilde{t}_r \) and \( \tilde{t}_z \). To remove the singularity, the axisymmetric element integral over the strip \((\xi, \theta) \in [-1, 1] \times [0, 2\pi]\) is transformed to a double integral in local polar coordinates introduced at a node in the rectangle. The Jacobian of this additional transformation appears in the integrand, effectively canceling out the weak \( r^{-1} \) singularity in the fundamental solutions.

The remaining fundamental solutions \( \tilde{t}_{rr}, \tilde{t}_{rz}, \tilde{t}_{zz}, \tilde{q}_r, \tilde{q}_z, \tilde{q}^s \) are strongly-singular, and are not evaluated directly. Their evaluation is bypassed at a later stage of the
method by substituting known solutions of the governing equations into the linear algebraic system for the nodal unknowns, thus allowing strongly-singular matrix entries to be expressed in terms of nonsingular and weakly-singular integrals that have already been computed.

3.4.3 Weakly-singular integrals

To compute the integrals that have a weak $r^{-1}$ singularity, the axisymmetric double integrals over the strip $((\xi, \theta) \in [\frac{-1}{2}, \frac{1}{2}] \times [0, 2\pi])$ (Figure 3.3) are transformed to a new local polar coordinate system. These local coordinates are introduced at points in the rectangle corresponding to the location of the nodes on $\theta = 0$, i.e. $\xi = -1, 0, 1$.

![Figure 3.3: The weakly-singular integrals are evaluated over the strip $(\xi, \theta) \in [-1, 1] \times [0, 2\pi]$, corresponding to rotating the $k^{th}$ element about the symmetry axis.](image)

When $P$ and $Q$ are in the same element and weakly-singular integrals arise, the strip $(\xi, \theta) \in [-1, 1] \times [0, 2\pi]$ is divided into two parts: a rectangular region $(\xi, \theta) \in [-1, 1] \times [\epsilon, -\epsilon]$ surrounding the boundary element over which the polar transformation is applied (Figure 3.4), and the nonsingular region of the strip $(\xi, \theta) \in [-1, 1] \times [\epsilon, 2\pi - \epsilon]$ away from the plane $\theta = 0$. Numerical Gaussian quadrature is performed over both regions, and the two results are added together to obtain integrals over the entire strip.
Figure 3.4: A nine-node biquadratic element is used for integration over the singular portion of the strip $(\xi, \theta) \in [-1, 1] \times [0, 2\pi]$. Singularities of strength $r^{-1}$ are removed via introduction of a local polar coordinate system, shown here at node $b = 2$ ($\xi = 0$).

Local polar coordinate systems and subdomains of integration

In the rectangular region it is convenient to introduce the new coordinate representation $(\eta, \xi)$, where $\xi \in [-1, 1]$ and $\eta$ is aligned in the direction of $\theta$ (Figure 3.4). For $P$ at each of the three boundary element nodes, a local polar coordinate system $(\rho, \varphi)$ is introduced at $P$, as outlined below:

Figure 3.5: The three subdomains of integration when a local polar coordinate system is introduced at $b = 1$ ($\xi = -1$).
Node 1: \((\eta, \xi) = (\rho \cos \varphi, -1 + \rho \sin \varphi)\) 

\[ s_1 = \{(\rho, \varphi) : 0 \leq \varphi \leq \tan^{-1}(2/\varepsilon); 0 \leq \rho \leq \frac{\varepsilon}{\cos \varphi}\} \] (3.19) 

\[ s_2 = \{(\rho, \varphi) : \tan^{-1}(2/\varepsilon) \leq \varphi \leq \pi - \tan^{-1}(2/\varepsilon); 0 \leq \rho \leq \frac{2}{\sin \varphi}\} \] (3.20) 

\[ s_3 = \{(\rho, \varphi) : \pi - \tan^{-1}(2/\varepsilon) \leq \varphi \leq \pi; 0 \leq \rho \leq \frac{-\varepsilon}{\cos \varphi}\} \] (3.21) 

Figure 3.6: The five subdomains of integration when a local polar coordinate system is introduced at \(b = 2\) (\(\xi = 0\)).

Node 2: \((\eta, \xi) = (\rho \cos \varphi, \rho \sin \varphi)\) 

\[ s_1 = \{(\rho, \varphi) : 0 \leq \varphi \leq \tan^{-1}(1/\varepsilon); 0 \leq \rho \leq \frac{\varepsilon}{\cos \varphi}\} \] (3.23) 

\[ s_2 = \{(\rho, \varphi) : \tan^{-1}(1/\varepsilon) \leq \varphi \leq \pi - \tan^{-1}(1/\varepsilon); 0 \leq \rho \leq \frac{1}{\sin \varphi}\} \] (3.24) 

\[ s_3 = \{(\rho, \varphi) : \pi - \tan^{-1}(1/\varepsilon) \leq \varphi \leq \pi + \tan^{-1}(1/\varepsilon); 0 \leq \rho \leq \frac{-\varepsilon}{\cos \varphi}\} \] (3.25) 

\[ s_4 = \{(\rho, \varphi) : \pi + \tan^{-1}(1/\varepsilon) \leq \varphi \leq 2\pi - \tan^{-1}(1/\varepsilon); 0 \leq \rho \leq \frac{-1}{\sin \varphi}\} \] (3.26) 

\[ s_5 = \{(\rho, \varphi) : 2\pi - \tan^{-1}(1/\varepsilon) \leq \varphi \leq 2\pi; 0 \leq \rho \leq \frac{\varepsilon}{\cos \varphi}\} \] (3.27)
Figure 3.7: The three subdomains of integration when a local polar coordinate system is introduced at $b = 3$ ($\xi = 1$).

Node 3 :  

\[(\eta, \xi) = (\rho \cos \varphi, 1 + \rho \sin \varphi)\]  \hspace{1cm} (3.28)

\[s_1 = \{(\rho, \varphi) : \pi \leq \varphi \leq \pi + \tan^{-1}(2/\varepsilon); \quad 0 \leq \rho \leq \frac{-\varepsilon}{\cos \varphi}\} \hspace{1cm} (3.29)\]

\[s_2 = \{(\rho, \varphi) : \pi + \tan^{-1}(2/\varepsilon) \leq \varphi \leq 2\pi - \tan^{-1}(2/\varepsilon); \quad 0 \leq \rho \leq \frac{-2}{\sin \varphi}\} \hspace{1cm} (3.30)\]

\[s_3 = \{(\rho, \varphi) : 2\pi - \tan^{-1}(2/\varepsilon) \leq \varphi \leq 2\pi; \quad 0 \leq \rho \leq \frac{\varepsilon}{\cos \varphi}\} \hspace{1cm} (3.31)\]

When $P$ is at any of the three boundary element nodes and $Q$ is in an element that has $P$ as a node, the appropriate local polar coordinate system is employed at that node, as defined above. The technique for evaluating the integrals in (3.10) and (3.11) using the transformations in (3.18)–(3.31) is now described.

For each of the subdomains $s_j = \{\alpha \leq \varphi \leq \beta; \quad 0 \leq \rho \leq g(\varphi)\}$, the Gaussian quadrature formula for the double polar integral over the subdomain is of the form

\[
\int_{\alpha}^{\beta} \int_{0}^{g(\varphi)} f(\eta(\rho, \varphi), \xi(\rho, \varphi)) \rho \, d\rho \, d\varphi,
\]  \hspace{1cm} (3.32)

where the limits $\alpha$, $\beta$, and $g(\varphi)$ are defined in (3.18)–(3.31). Introducing the transformations

\[
\varphi = \frac{\alpha + \beta}{2} + \frac{\beta - \alpha}{2} \tilde{\varphi} \quad \text{and} \quad \rho = \frac{g(\tilde{\varphi})}{2} (1 + \tilde{\rho})
\]  \hspace{1cm} (3.33)
gives
\[ d\varphi = \frac{\beta - \alpha}{2} d\tilde{\varphi} \quad \text{and} \quad d\rho = \frac{g(\tilde{\varphi})}{2} d\tilde{\rho}, \] (3.34)
where \( \tilde{\varphi}, \tilde{\rho} \in [-1, 1] \). Thus, the double polar integral over a general region is transformed into an iterated integral over standard Gaussian quadrature limits of \([-1, 1]\):
\[ \frac{\beta - \alpha}{4} \int_{-1}^{1} g(\tilde{\varphi}) \int_{-1}^{1} f(\eta(\tilde{\rho}, \tilde{\varphi}), \xi(\tilde{\rho}, \tilde{\varphi})) \rho \, d\tilde{\rho} \, d\tilde{\varphi}. \] (3.35)

**Biquadratic element for singular portion of strip**

For integrals over the singular portion of the strip, i.e. \((\xi, \theta) \in [-1, 1] \times [\varepsilon, -\varepsilon]\), a 9-node biquadratic rectangular element was derived. The nine shape functions \((\alpha = 1, 2, \ldots, 9)\) have the form
\[ N^\alpha(\eta, \xi) = a^\alpha_1 + a^\alpha_2 \xi + a^\alpha_3 \xi^2 + a^\alpha_4 \eta + a^\alpha_5 \eta^2 + a^\alpha_6 \eta \xi + a^\alpha_7 \eta^2 \xi + a^\alpha_8 \eta^2 \xi^2 + a^\alpha_9 \eta^2 \xi^2, \] (3.36)
where the shape function is 1 at node \(\alpha\) and 0 at all other nodes. To determine the 81 coefficients of the shape functions, the following linear system \(I = Aa\) for the \(9 \times 9\) matrix of unknown coefficients \(a^i_j\) is solved:
\[
\begin{bmatrix}
N^1(0, -1) \\
N^2(0, 0) \\
N^3(0, 1) \\
\text{diag}
\begin{bmatrix}
N^4(-\varepsilon, -1) \\
N^5(-\varepsilon, 0) \\
N^6(-\varepsilon, 1) \\
N^7(\varepsilon, -1) \\
N^8(\varepsilon, 0) \\
N^9(\varepsilon, 1)
\end{bmatrix}
\end{bmatrix}
= \begin{bmatrix}
1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 1 & -\varepsilon & \varepsilon^2 & \varepsilon & -\varepsilon & -\varepsilon & \varepsilon^2 \\
1 & 0 & 0 & -\varepsilon & \varepsilon^2 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & -\varepsilon & \varepsilon^2 & -\varepsilon & \varepsilon^2 & -\varepsilon & \varepsilon^2 \\
1 & -1 & 1 & \varepsilon & \varepsilon^2 & -\varepsilon & -\varepsilon & \varepsilon & \varepsilon^2 \\
1 & 0 & 0 & \varepsilon & \varepsilon^2 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & \varepsilon & \varepsilon^2 & \varepsilon & \varepsilon^2 & \varepsilon & \varepsilon^2
\end{bmatrix}\begin{bmatrix}
a^1_1 \\
\vdots \\
a^i_1 \\
\vdots \\
a^9_1
\end{bmatrix},
\] (3.37)
where diag indicates that the 9 scalar entries are placed along the diagonal of the \(9 \times 9\) identity matrix. The Jacobian of the transformation is the norm of the cross product of the tangent vector components:
\[ J_k(\eta, \xi) = \left| \frac{\partial \mathbf{x}}{\partial \eta} \times \frac{\partial \mathbf{x}}{\partial \xi} \right|. \] (3.38)
The derivatives are expressed in terms of derivatives of the shape functions. For the general shape function

\[ N^\alpha(\eta, \xi) = a_1^\alpha + a_2^\alpha \xi + a_3^\alpha \xi^2 + a_4^\alpha \eta + a_5^\alpha \eta \xi + a_6^\alpha \eta^2 + a_7^\alpha \xi + a_8^\alpha \eta \xi^2 + a_9^\alpha \eta^2 \xi^2; \]  \hspace{1cm} (3.39)

the derivatives are

\[ \frac{\partial N^\alpha(\eta, \xi)}{\partial \eta} = a_4^\alpha + 2a_5^\alpha + 2a_7^\alpha \eta \xi + a_8^\alpha \xi^2 + 2a_9^\alpha \eta \xi^2; \]  \hspace{1cm} (3.40)

\[ \frac{\partial N^\alpha(\eta, \xi)}{\partial \xi} = a_2^\alpha + 2a_3^\alpha \xi + a_6^\alpha \eta + a_7^\alpha \eta^2 + 2a_8^\alpha \eta \xi + 2a_9^\alpha \eta^2 \xi; \]  \hspace{1cm} (3.41)

so that

\[ \frac{\partial \mathbf{x}}{\partial \eta} = \sum_{b=1}^{9} x^{k,b} \frac{\partial N^b(\eta, \xi)}{\partial \eta} \quad \text{and} \quad \frac{\partial \mathbf{x}}{\partial \xi} = \sum_{b=1}^{9} x^{k,b} \frac{\partial N^b(\eta, \xi)}{\partial \xi}. \]  \hspace{1cm} (3.42)

Using the above formulas for the nine-node biquadratic element, the integration procedure is performed as in (3.32)--(3.35).

### 3.5 Assembly and solution of linear system

#### 3.5.1 Assembly of nodal equations

The equations in (3.9) are assembled at each node \( l \), adding together integrals in (3.10) and (3.11) for each element to which a node belongs, e.g., \( T^{k,3} + T^{k+1,1} \), in element \( k < M \). The resulting linear system for all nodal quantities can be written as

\[ \sum_{m=1}^{n} F^{lm} \omega^m - \sum_{m=1}^{n} G^{lm} \tau^m = 0, \quad l = 1, \ldots, n, \]  \hspace{1cm} (3.43)

where

\[ \omega^m = \begin{bmatrix} \tilde{u}_r^m \\ \tilde{u}_z^m \\ \tilde{p}^m \end{bmatrix}, \quad \tau^m = \begin{bmatrix} \tilde{t}_r^m \\ \tilde{t}_z^m \\ \tilde{q}^m \end{bmatrix}. \]  \hspace{1cm} (3.44)

The linear system (3.43) can be written more succinctly as the block linear system

\[ \mathbf{Fw} - \mathbf{Gt} = 0, \]  \hspace{1cm} (3.45)
where \( \mathbf{w} = (\omega^1|\ldots|\omega^n)^T \) and \( \mathbf{t} = (\tau^1|\ldots|\tau^n)^T \). Thus, \( F^{lm} \) and \( G^{lm} \) are the \( 3 \times 3 \) \((l,m)^{th}\) blocks of \( \mathbf{F} \) and \( \mathbf{G} \), respectively. All singular integrals occur exclusively in the diagonal blocks of \( \mathbf{F} \) and \( \mathbf{G} \), as indicated by the representation

\[
F^{ll} = \begin{bmatrix} S & S & W \\ S & S & W \\ W & W & S \end{bmatrix}, \quad G^{ll} = \begin{bmatrix} W & W & W \\ W & W & W \\ W & W & W \end{bmatrix}, \tag{3.46}
\]

where ‘S’ denotes a strongly-singular matrix entry and ‘W’ denotes a weakly-singular entry. This representation determines which entries require application of the weakly-singular numerical integration technique presented in the previous section. Note that all off-diagonal blocks of \( \mathbf{F} \) and \( \mathbf{G} \) are nonsingular.

### 3.5.2 Substitution of strongly singular integrals

Based on (3.46), there are five unknown strongly-singular entries in \( \mathbf{F} \) that remain to be determined. Three analytical solutions are substituted into the linear system \( \mathbf{Fw} - \mathbf{Gt} = \mathbf{0} \) to determine the unknown strongly-singular matrix entries in terms of the other entries which have been previously determined.

**Rigid-body \( z\)-displacement** The Laplace-domain solution corresponding to a rigid-body translation in the axial direction is given by [4]:

\[
\tilde{u}^l_r = 0, \quad \tilde{u}^l_z = s^{-1}, \quad \tilde{\tau}^l_r = \tilde{\tau}^l_z = 0, \quad \tilde{q}^l = \tilde{\rho}^l = 0. \tag{3.47}
\]

Substituting this solution into the linear system will allow the second column of the diagonal block of \( \mathbf{F} \) to be determined as the sum

\[
F^{ll}_{\gamma 2} = - \sum_{m=1; m \neq l}^{n} F^{lm}_{\gamma 2} \quad \text{for} \ \gamma = 1, 2, 3. \tag{3.48}
\]

Note that rigid-body displacements in non-axial directions are incompatible with an axisymmetric formulation of the method.
Compressive stress  The biphasic compressive stress solution [4]

\[
\tilde{u}_r = \tilde{u}_z = 0, \quad \tilde{t}_r = -s^{-1}n_r^l, \quad \tilde{t}_z = -s^{-1}n_z^l, \quad \tilde{q} = 0, \quad \tilde{p} = s^{-1}
\]  

(3.49)
is used to determine the third column of the diagonal blocks of \( F \). The \((\gamma, 3)\) entry is determined from the sum

\[
F_{\gamma 3}^l = -\sum_{m=1}^{n} F_{\gamma m}^{lm} - \sum_{m=1}^{n} (G_{\gamma 1}^{lm} n_r^l + G_{\gamma 2}^{lm} n_z^l) \text{ for } \gamma = 1, 2, 3.
\]  

(3.50)

Biphasic radial stretching  To determine the strongly-singular entries in the first column of \( F^l \), a biphasic time-domain solution for purely radial stretching was derived for a solution of the form

\[
u = <u(r, t), 0, 0>
\]  

(3.51)

that satisfies the biphasic governing equations in axisymmetric cylindrical coordinates. Substituting (3.51) into the cylindrical form of \( \nabla \cdot \mathbf{\sigma}_E = \nabla p \), the radial displacement must satisfy the differential equation

\[
H_A \left[ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{1}{r^2} u \right] = \frac{\partial p}{\partial r}
\]  

(3.52)

where \( H_A = \lambda + 2\mu \). The second governing equation \( \partial_t (\nabla \cdot u) = k \nabla^2 p \), written in cylindrical coordinates, becomes

\[
\frac{\partial}{\partial t} \left( \frac{\partial u}{\partial r} + \frac{u}{r} \right) = k \left[ \frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} \right].
\]  

(3.53)

Equation (3.53) reduces to

\[
\frac{\partial u}{\partial t} = k \frac{\partial p}{\partial r} + f(t).
\]  

(3.54)

Enforcing the condition that radial displacement at the axial nodes must be zero implies that \( f(t) = 0 \). Substituting \( \partial p / \partial r \) as defined in (3.54) into equation (3.52) gives a differential equation in \( u \). After some manipulation, the differential equation can be shown to have the Laplace domain representation

\[
r^2 \frac{d^2 \tilde{u}}{dr^2} + r \frac{d \tilde{u}}{dr} - \left( 1 + \frac{s}{kH_A r^2} \right) \tilde{u} = 0,
\]  

(3.55)
which has the general solution
\[
\tilde{u}(r) = c_1 I_1(r \sqrt{s/(kH_A)}) + c_K K_1(r \sqrt{s/(kH_A)}),
\] (3.56)
where \( I_n \) is a modified Bessel function of the first kind and \( K_n \) is a modified Bessel function of the second kind, which tends to infinity as \( r \to 0 \). Since \( \tilde{u} \) is zero as \( r \to 0 \) in the axisymmetric case, and the constants are arbitrary if no initial conditions are prescribed, the Laplace domain (radial) displacement solution is
\[
\tilde{u}_r(r; s) = I_1(r \sqrt{s/(kH_A)}).
\] (3.57)
In the Laplace domain, (3.54) becomes
\[
\frac{\partial p}{\partial r} = \frac{1}{k} s I_1 (r \sqrt{s/(kH_A)}),
\] (3.58)
so that \( \tilde{q} = -k n_r \partial p/\partial r \), and the mixture tractions are
\[
\tilde{t}_r(r; s) = -\tilde{p} n_r + \tilde{t}^E_r; \quad \tilde{t}_z(r; s) = -\tilde{p} n_z + \tilde{t}^E_z.
\] (3.59)
The biphasic radial stretching Laplace-domain solution is summarized as
\[
\tilde{u}_r = I_1(r \sqrt{s/(kH_A)}); \quad \tilde{u}_z = 0; \quad \tilde{p} = \sqrt{s/(kH_A)} I_0(r \sqrt{s/(kH_A)});
\] (3.60)
\[
\tilde{t}_r = -\tilde{p} n_r + \tilde{t}^E_r; \quad \tilde{t}_z = -\tilde{p} n_z + \tilde{t}^E_z; \quad \tilde{q} = -s I_1(r \sqrt{s/(kH_A)}) n_r;
\] (3.61)
where the elastic tractions are given by
\[
\tilde{t}^E_r(r; s) = C_t \frac{n_r}{r} \left[ r \sqrt{s/(kH_A)} (\nu - 1) I_0(r \sqrt{s/(kH_A)}) + (1 - 2\nu) I_1(r \sqrt{s/(kH_A)}) \right],
\] (3.62)
\[
\tilde{t}^E_z(r; s) = -C_t n_z \sqrt{s/(kH_A)} I_0(r \sqrt{s/(kH_A)}),
\] (3.63)
and \( C_t = E/[(1 + \nu)(2\nu - 1)] \).

The solutions (3.60)–(3.63) are substituted into the linear system to determine the unknown strongly-singular entries in the first column of \( F^u \).
3.5.3 Assembled system for specified boundary conditions

The matrices in the linear system \( Fw - Gt = 0 \) can be determined prior to specifying boundary conditions. A major advantage of the boundary element method is that, once an axisymmetric boundary and mesh is given, the time-consuming computation of \( F \) and \( G \) can be performed once and stored. Specifically, for a fixed set of material properties and type of boundary conditions prescribed at each node (e.g. Dirichlet, Neumann, mixed, etc.), these matrices remain the same irrespective of the actual boundary values of the prescribed quantities.

At every node, exactly three quantities must be specified: \( \tilde{u}_r \) or \( \tilde{t}_r \); \( \tilde{u}_z \) or \( \tilde{t}_z \); and \( \tilde{p} \) or \( \tilde{q} \). For a particular boundary value problem, unknown quantities are assembled on the left side of the linear system, and prescribed nodal values are moved to the right. A new linear system \( Cx = d \) results, where \( C \) and \( d \) are both formed from entries in \( F \) and \( G \).

In all axisymmetric problems, the radial displacement and radial traction at the axial nodes must be exactly zero: \( \tilde{u}_r^l = \tilde{t}_r^l = 0 \) for \( l = 1 \) and \( l = n \). Ones and zeros are placed in the corresponding entries of \( C \) and \( d \) to enforce this condition. Similarly, \( \partial p/\partial r \) at the axial nodes should be zero. Within the first and last elements, enforcement of the condition \( \partial p/\partial r = 0 \) is equivalent to the requirement that

\[
-\frac{3}{2}p_1 + 2p_2 - \frac{1}{2}p_3 = 0 \quad \text{(for } k = 1) \quad \text{and} \quad \frac{1}{2}p_1 - 2p_2 + \frac{3}{2}p_3 = 0 \quad \text{(for } k = M). \tag{3.64}
\]

The relations in (3.64) can be derived by requiring that \( \partial p/\partial \xi = 0 \) at \( \xi = -1 \) in the first element \( (k = 1) \) and \( \partial p/\partial \xi = 0 \) at \( \xi = 1 \) in the last element \( k = M \). Since \( \partial p/\partial r \) is not a quantity that appears in the method, the nodal values of \( p \) are retained as unknowns in the first and last elements, and constrained according to (3.64). The linear system \( Cx = d \) is solved in the Laplace domain for multiple values of \( s \) using the MATLAB backslash command, and the quantities at every node are reconstructed from the solution \( x \) and the prescribed values.
Chapter 4

Verification and Results

In this chapter, accuracy of the boundary element methods developed in Chapters 2 and 3 is evaluated. First, the validity of the Cartesian fundamental solutions (Chapter 2) is verified via construction of all primary variables in cylindrical coordinates. The resulting representations were substituted into the poroelastic governing equations, written in cylindrical coordinates in the Laplace transform domain, to verify that they are solutions of the equations. Second, the axisymmetric boundary element method was used to simulate configurations of biphasic compressive stress, pure radial stretching, and uniaxial confined compression. In these three cases, analytical solutions are known and were employed for verification. Lastly, potential use of the axisymmetric boundary element method is illustrated in the context of simulating confined compression stress relaxation of a biphasic cartilage cell in a cylindrical sample of extracellular matrix subjected to confined compression.

4.1 Symbolic verification of fundamental solutions

The formulation of axisymmetric fundamental solutions presented in the previous section is based on the coordinate transformation from Cartesian to cylindrical polar coordinates, along with an assumption of axial symmetry. Due to the algebraic complexity of the fundamental solutions in (2.26)–(2.29) and (2.32)–(2.35), their validity was first verified by constructing all primary variables under transformation from
Cartesian to cylindrical coordinates. These representations were then substituted into the governing equations (2.1)–(2.2), also written in cylindrical coordinates.

By use of a symbolic computation software package (Maple 11), exact satisfaction of these equations was verified. The details of this procedure, along with the associated Maple output, are presented in Appendix A. It should be noted that this procedure for symbolic verification took a few hours for some components of the governing equations. The primary purpose of conducting this verification was to circumvent potential inaccuracy in the numerical method due to incorrect mathematical representations in the fundamental solutions.

4.2 Numerical results

As outlined in Chapter 3, the axisymmetric extension of the Cartesian boundary element method requires explicit enforcement of the symmetry condition $\partial p/\partial r = 0$ along $r = 0$ (see equation (3.64)). Consequently, pressure was retained as an unknown for elements coincident with the symmetry axis $r = 0$. For a spherical cell, this corresponds to the first and last elements at the bottom and top of the sphere, respectively. In applications to biphasic cell mechanics within articular cartilage, current experimental methods allow for measurement of cell deformation along principal axes [9]. Thus, the focus in this study was simulating boundary value problems in which mixture tractions are prescribed and displacements are retained as unknowns. However, to prevent pure rigid-body translations, displacements must be prescribed for at least one point along the domain boundary. Considering these factors, displacements were prescribed in the first boundary element while mixture tractions were prescribed at all other boundary elements.

To verify accuracy of the axisymmetric boundary element method, three boundary value problems were considered: biphasic compressive stress, pure radial stretching, and uniaxial confined compression. In each of these three cases, analytical solutions are known. Note that the biphasic compressive stress and pure radial stretching solutions were employed to determine strongly-singular integrals, as described in section 3.5.2. While such usage will enhance accuracy for simulation of the corresponding
boundary value problems, exact agreement is not guaranteed due to the mixed nature of the prescribed boundary conditions. A more accurate assessment of numerical accuracy was also conducted using an analytical series solution for uniaxial confined compression stress relaxation (which was not used to compute singular integrals). Finally, this solution was also used to illustrate potential application of the axisymmetric boundary element method to modeling cell deformation in extracellular matrix subjected to uniaxial confined compression stress relaxation.

The axisymmetric BEM code was implemented in MATLAB 7.5 on a MacBookPro laptop computer (2.4 GHz Intel Core Duo processor). To the greatest extent possible, the code was designed so that the large majority of operations were vectorized. For example, fundamental interactions between source and influence points in the method are determined by $3 \times 3$ block matrices represented in (3.13), and many operations in the method were applied to columns of these blocks. All results shown employed $M = 8$ quadratic boundary elements on a one-dimensional semicircular curve (Figure 3.1) with 9-point Gaussian quadrature [1], which was sufficient for demonstrating accuracy between the numerical and analytical solutions. For this mesh resolution, the time required to compute the solution for one value of the Laplace transform variable $s$ was 21.4 seconds. In contrast, the three-dimensional Cartesian BEM [20] with 16 triangular biquadratic elements, distributed over the surface of a sphere, took 159.8 seconds to compute the solution for one value of $s$.

### 4.2.1 Biphasic compressive stress

In the Laplace transform domain, the biphasic compressive stress solution is given by

$$
\tilde{u}_r = \tilde{u}_z = 0, \quad \tilde{l}_r = -s^{-1}n^l_r, \quad \tilde{l}_z = -s^{-1}n^l_z, \quad \tilde{q}^l = 0, \quad \tilde{p}^l = s^{-1}.
$$

(4.1)

This solution represents a biphasic state with a constant, compressive pore pressure and zero solid phase displacements. Thus, all mixture stress components are also constant and arise exclusively due to the pressure effect, producing mixture tractions that are proportional to the normal direction along the boundary. A comparison of axisymmetric BEM solutions to the known solution is shown for eight values of $s$ and
demonstrates excellent agreement (Figure 4.1).

4.2.2 Radial biphasic stretching

In the Laplace transform domain, the radial biphasic stretching solution is given by

\[
\tilde{u}_r = I_1(r \sqrt{s/(kH_A)}); \quad \tilde{u}_z = 0; \quad \tilde{p} = \sqrt{s/(kH_A)} I_0(r \sqrt{s/(kH_A)});
\]

\[
\tilde{t}_r = -\tilde{p}n_r + \tilde{t}^E_r; \quad \tilde{t}_z = -\tilde{p}n_z + \tilde{t}^E_z; \quad \tilde{q} = -s I_1 r \sqrt{s/(kH_A)}; \quad (4.2)
\]

where \( H_A = \lambda + 2\mu \) and the elastic tractions are given by

\[
\tilde{t}^E_r(r; s) = C_t \frac{n_r}{r} \left[ r \sqrt{s/(kH_A)}(\nu - 1) I_0(r \sqrt{s/(kH_A)}) + (1 - 2\nu) I_1(r \sqrt{s/(kH_A)}) \right];
\]

\[
\tilde{t}^E_z(r; s) = -C_t n_z \sqrt{s/(kH_A)} I_0(r \sqrt{s/(kH_A)}).
\] (4.4)

(4.5)

and \( C_t = E/[(1 + \nu)(2\nu - 1)] \). This solution represents a time-varying biphasic state in which solid phase displacement occurs exclusively in the radial direction and, as a result, all mixture shear stresses are also zero. A comparison of axisymmetric BEM solutions to the known solution is shown for radial displacement, and radial and axial mixture tractions, demonstrating excellent agreement (Figure 4.2).
Figure 4.1: Simulations of biphasic compressive stress (4.1) shown for 8 values of \( s_i = e^{-3+(i-1)/2}, i = 1, \ldots, 8 \) with \( \nu = 0.35 \) comparing BEM solutions (circles) to the analytical solution (solid line): (Top-Left) \( \tilde{u}_r \) vs. \( r \); (Top-Right) \( \tilde{u}_z \) vs. \( r \); (Bottom-Left) \( \tilde{t}_r \) vs. \( r \); (Bottom-Right) \( \tilde{t}_z \) vs. \( r \). In all cases, the magnitude of all quantities approaches zero successively as the value of \( s \) is increased.
Figure 4.2: Simulations of radial biphasic stretching (4.2)–(4.5) shown for 8 values of $s_i = e^{-3+(i-1)/2}, \ i = 1, \ldots, 8$ with $\nu = 0.35$ comparing BEM solutions (circles) to the analytical solution (solid line): (Top-Left) $\tilde{u}_r$ vs. $r$; (Top-Right) $\tilde{t}_r$ vs. $r$; (Bottom) $\tilde{t}_z$ vs. $r$. In all cases, the magnitude of all quantities approaches zero successively as the value of $s$ is increased.
4.2.3 Uniaxial confined compression stress relaxation

In the time domain, the uniaxial confined compression stress relaxation solution is given by (see e.g. [20]):

\[
\begin{align*}
  u_z(z,t) &= \frac{\gamma z}{h} [tH(t_0 - t) + t_0 H(t - t_0)] \\
  &\quad + \frac{2\gamma h^2}{k H_A} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3 \pi^3} \left[ H(t - t_0)(e^{-a_n(t-t_0)} - e^{-a_n t}) \\
  &\quad + H(t_0 - t)(1 - e^{-a_n t}) \right] \sin \left( \frac{n\pi z}{h} \right),
\end{align*}
\]

where \( H_A = \lambda + 2\mu \) and \( a_n = \frac{n^2 \pi^2 k H_A}{h^2} \).

This solution represents a time-varying biphasic state in which solid phase displacement occurs exclusively in the axial direction (Figure 4.3). The series solution above is specific to the case of the following boundary conditions in the time domain:

\[
\begin{align*}
  u_z(h, t) &= \begin{cases} 
    -\gamma t & 0 \leq t \leq t_0 \\
    -\gamma t_0 & t > t_0
  \end{cases} \text{ for } t > 0; \quad u_z(0, t) = 0 \text{ for } t > 0. 
\end{align*}
\]

In this loading configuration, all mixture shear stresses are also zero, and axial stress profiles exhibit stress relaxation; i.e. the peak axial stress at \( z = h \) occurs when \( t = t_0 \). These boundary conditions model application of a ramp-and-hold compressive displacement to a cylindrical sample of biphasic material with a ramp time of \( t_0 \).

Using cylindrical coordinates in the Laplace transform domain, the confined com-
Figure 4.3: Diagram of confined compression test on a cylindrical biphasic sample. The sample is confined on the sides and bottom, and a porous platen on the top surface allows fluid to flow out of the sample as the plate is pushed down. To model cell deformation, forces generated by the macroscopic solution are applied as boundary conditions in a boundary value problem for a spherical cell of radius \( R_{cell} \ll h \). (Note: Diagram not to scale)

Pressure stress relaxation solution is written as:

\[
\tilde{u}_r(z, s) = 0,
\]

\[
\tilde{u}_z(z, s) = \frac{\gamma z}{h} \frac{1 - e^{-st_0}}{s^2} + \frac{2\gamma h^2}{kH_A} \sum_{n=1}^{\infty} \frac{(-1)^n a_n}{n^3 \pi^3} \frac{1 - e^{-t_0}}{s(s + a_n)} \sin\left(\frac{n\pi z}{h}\right),
\]

\[
\tilde{r}_r(z, s) = \tilde{\sigma}_{rr} n_r,
\]

\[
\tilde{r}_z(z, s) = \tilde{\sigma}_{zz} n_z,
\]

\[
\tilde{p}(z, s) = \frac{2\gamma h}{k} \sum_{n=1}^{\infty} \frac{a_n}{n^2 \pi^2} \frac{1 - e^{-st_0}}{s(s + a_n)} \left[ (-1)^n \cos\left(\frac{n\pi z}{h}\right) - 1 \right],
\]

\[
\tilde{q}_z(z, s) = -k \frac{\partial \tilde{p}}{\partial z} = -2\gamma h \sum_{n=1}^{\infty} \frac{a_n}{n^2 \pi^2} \frac{1 - e^{-st_0}}{s(s + a_n)} \left[ (-1)^{n+1} \frac{n\pi}{h} \sin\left(\frac{n\pi z}{h}\right) \right],
\]
Figure 4.4: Simulations of uniaxial confined compression stress relaxation (4.6)–(4.7) shown for 8 values of $s_i = e^{-3+(i-1)/2}$, $i = 1, \ldots, 8$ with $\nu = 0.35$ comparing BEM solutions (circles) to the analytical solution (solid line): (Top) $\tilde{u}_z$ vs. $r$; (Bottom-Left) $\tilde{t}_r$ vs. $r$; (Bottom-Right) $\tilde{t}_z$ vs. $r$. In all cases, the magnitude of all quantities approaches zero successively as the value of $s$ is increased.
where \( \tilde{q} = \tilde{q}_zn_z \), and

\[
\tilde{\sigma}_{rr}(z, s) = \frac{\lambda\gamma}{h} \frac{1 - e^{-st_0}}{s^2} + \frac{2\gamma h}{k} \sum_{n=1}^{\infty} \frac{a_n}{n^2\pi^2} \frac{1 - e^{-st_0}}{s(s + a_n)} \left[ (-1)^{n+1} \frac{2\mu}{H_A} \cos \left( \frac{n\pi z}{h} \right) + 1 \right],
\]

\( \text{(4.15)} \)

\[
\tilde{\sigma}_{zz}(z, s) = \frac{H_A\gamma}{h} \frac{1 - e^{-st_0}}{s^2} + \frac{2\gamma h}{k} \sum_{n=1}^{\infty} \frac{a_n}{n^2\pi^2} \frac{1 - e^{-st_0}}{s(s + a_n)} \left[ (-1)^{n+1} \cos \left( \frac{n\pi z}{h} \right) + 1 \right].
\]

\( \text{(4.16)} \)

For the linear biphasic equations, a natural set of characteristic scales are \( h \) as the unit of length, \( \mu \) as the unit of stress, and \( h^2/(k\mu) \) as the unit of time. Under this nondimensionalization, the mathematical character of the solution can be ascertained by solving the dimensional equations with effective parameter values of \( k = 1 \) and \( \mu = 1 \). Thus, the primary material property governing the nature of the solutions is the Poisson ratio \( \nu \). For all simulations shown, \( \gamma = 0.1 \) and \( t_0 = 0.1 \) in these nondimensional units.

A comparison of axisymmetric BEM solutions to the known solution (4.9)–(4.14) is shown for radial displacement, and radial and axial mixture tractions, and demonstrates excellent agreement (Figure 4.4).

### 4.2.4 Illustrative application: microscopic cell deformation in confined compression

To illustrate potential application of the axisymmetric BEM, biphasic deformation of a spherical cell within extracellular matrix subjected to confined compression stress relaxation (Figure 4.3) was considered. A typical depth of a cartilage layer \( (h \approx 1 \text{ mm}) \) was taken as the unit of length, the cell radius was taken as \( R_{\text{cell}} = 0.007 \), i.e. \( 7 \mu m \), and the center of the cell was located at 50\% depth within the tissue layer.

Cell deformation was idealized as a boundary value problem on the sphere, in which material properties were chosen based on values for the pericellular matrix (PCM) of articular cartilage. The PCM is a specialized zone that, in contrast to the extracellular matrix, has a higher proteoglycan concentration and exclusive presence of type-VI collagen. Consequently, its material properties are distinct from those
Figure 4.5: Time-domain confined compression solutions for cell deformation are evaluated at the three nodes shown on the boundary element mesh. Axial displacement was evaluated at the top node (I) and prescribed at the bottom node (II). Radial displacement was evaluated at the equatorial node (III).

of the extracellular matrix. Based on [22], the PCM properties for prescription of boundary conditions using (4.9) – (4.14) were chosen as

\[ \nu_{PCM} = 0.05, \quad k_{PCM} = 1, \quad \mu_{PCM} = 1, \]  
\[ \text{i.e. } E_{PCM} = 2\mu_{PCM}(1 + \nu_{PCM}) \]  
\[ E_{cell} \equiv E = \frac{1}{50} E_{PCM} \quad \text{and} \quad k_{cell} \equiv k = 1. \]  

The Poisson’s ratio for the cell \( \nu \) was varied in the range 0.35-0.45, reflecting the fact that cells are moderately incompressible [31].

**Numerical Laplace inversion**

The primary focus in this study has been the development and validation of an axisymmetric boundary element method for biphasic mechanics in the Laplace transform domain. In practical application of this method, BEM solutions are generated for multiple values of the Laplace parameter \( s \) and a numerical Laplace inversion technique is employed to reconstruct the time-domain solution. The specific number of \( s \) values required and the choice of inversion technique are dependent on the specific application of interest.
The original three-dimensional Cartesian poroelastic BEM [4] considered models of soil consolidation in which a simple exponential series approximation for a quantity of interest \( f(t) = \sum_{i=1}^{\tilde{n}} c_i e^{-\alpha_i t} \) was used. For most applications in biphasic cartilage mechanics an extended representation is required. Specifically, a quantity of interest along the boundary is given the general representation

\[
g(t) = \sum_{i=1}^{n} d_i \phi_i(t),
\]

where the basis functions \( \phi_i(t) \) are tailored to the particular boundary value problem under consideration. The basis functions are chosen so that they admit explicit analytical representations \( \tilde{\phi}_i(s) \) in the Laplace transform domain, i.e.,

\[
\tilde{g}(s) = \sum_{i=1}^{n} d_i \tilde{\phi}_i(s).
\]

The coefficients \( d_i \) are determined via least-squares optimization against BEM solutions generated for a range of Laplace parameter \( (s) \) values.

**Simulations**

For simulation of confined compression stress relaxation, the time-domain representation of displacement components

\[
g(t) = H(t_0 - t) \left[ g_0 t + \sum_{i=1}^{\tilde{n}} g_i (1 - e^{-\delta_i t}) \right] + H(t - t_0) \left[ d_0 t_0 + \sum_{i=1}^{\tilde{n}} d_i e^{-\delta_i t} \right]
\]

was used [20], where \( H(t) \) is the Heaviside step function. Continuity of \( g(t) \) at \( t = t_0 \) yields the relations \( g_0 = d_0 \) and \( g_i = e^{-\delta_i t_0}(1 - e^{-s t_0})^{-1} d_i \) for \( i = 1, \ldots, \tilde{n} \). The corresponding representation in the Laplace domain is:

\[
\tilde{g}(s) = d_0 \left[ \frac{1 - e^{-st_0}}{s^2} \right] + \sum_{i=1}^{\tilde{n}} d_i \left[ \frac{\delta_i}{e^{\delta_i t_0} - 1} \frac{1 - e^{-st_0}}{s (s + \delta_i)} \right].
\]

The following asymptotic approximation is also noted and is useful for initializing the parameter \( d_0 \) in the optimization:

\[
\tilde{g}(s) \sim \frac{d_0 t_0}{s} \quad \text{as} \quad s \to 0.
\]
Figure 4.6: Time-domain simulations of axial displacement at node I (lower curve), axial displacement at node II (middle curve) and radial displacement at node III (upper curve) for the cases $\nu = 0.35$ (Top-Left), $\nu = 0.40$ (Top-Right), and $\nu = 0.45$ (Bottom) in the case where cell and PCM properties are set equal. BEM solutions (circles) are compared to the analytical solutions (solid lines).
Figure 4.7: Time-domain simulations using PCM properties given in (4.17)–(4.18). (Left) axial displacement at node I (lower curve), axial displacement at node II (middle curve) and radial displacement at node III (upper curve) for the cases $\nu = 0.35$ (solid), $\nu = 0.40$ (dashed), and $\nu = 0.45$ (dotted). (Right) The ratio of deformed to undeformed cell volume, approximated via ellipsoids, is shown for the case $\nu = 0.35$ (solid), $\nu = 0.40$ (dashed), and $\nu = 0.45$ (dotted).
The axisymmetric BEM was run for 12 values of $s$, generated as

$$s_i = e^{-3+(i-1)/2} \text{ for } i = 1, 2, \ldots, 12,$$

and data for the optimization was generated for displacement components at the three nodes $(r, z) = (0, 0.5 + R_{cell})$ (node I), $(r, z) = (0, 0.5 - R_{cell})$ (node II) and $(r, z) = (R_{cell}, 0.5)$ (node III) (Figure 4.5). Limitation of the simulations to three principal nodes is motivated by experimental techniques for in situ measurement of cell deformation within the extracellular matrix [9]. In this technique, the cell geometry is idealized to be an axisymmetric ellipsoid and lengths of the principal axes are used to calculate changes in cell volume and surface area under deformation. Taking $\tilde{n} = 2$, equation (4.22) was curve-fit to the natural logarithm of the BEM displacement solutions by minimizing a least-squares cost function at the 12 data points using a Nelder-Mead simplex (direct search) method (‘fminsearch’, MATLAB 7.5).

To assess effectiveness of the Laplace inversion technique, the PCM properties were set equal to those of the cell in simulating time-domain displacements at nodes I-III (Figure 4.6). The unknown time-domain quantities were in good agreement with the analytical series solution. The Laplace inversion technique exhibited some sensitivity to small errors in the BEM solution as illustrated by the difference in accuracy between inversion of prescribed displacements (node II) and unknown displacements (nodes I & III).

To model cell deformation in confined compression stress relaxation, PCM properties were chosen according to (4.17)–(4.18) and time-varying cell displacement at nodes I-III was simulated (Figure 4.7-Left). Based on an approximation of cell geometry as an axisymmetric ellipsoid, the ratio between deformed and undeformed cell volume was also simulated (Figure 4.7-Right). In the case $\nu = 0.35$, changes in cell shape, including volumetric deformation and recovery, are shown in Figure 4.8. It is observed that changes in cell volume over time are highly dependent on values of $\nu$ in the range 0.35-0.45 (Figure 4.7-Right). These simulations suggest that small changes in material properties of the cell and its surrounding matrix may strongly influence volumetric deformation in the cell, which is known to contribute to regulation of
cellular metabolic activity.

Figure 4.8: Deformation of the cell boundary under uniaxial confined compression stress relaxation for $\nu = 0.35$. The undeformed spherical cell (dotted line) first undergoes compression (solid lines), followed by relaxation of cell volume at later times (dashed line). See also Figure 4.7-Right.

4.3 Discussion

The axisymmetric boundary element methods developed in this study exhibit substantial savings in computational time as compared to the previous three-dimensional BEM [20]. Specifically, computation for a single value of Laplace transform parameter $s$ was reduced by a factor of 7. It is also noted that the bottleneck portion of the overall computation is determination of the matrices $F$ and $G$ in (3.45); however, for a fixed set of material parameters, this equation holds for any class of boundary conditions. Consequently, parametric analyses for effects of loading parameters such as $t_0$ in (4.8) can be conducted in a highly efficient manner by computing $F$ and $G$ once and storing the results. In addition, the Laplace domain formulation is amenable to parallelization, in that computations for different values of $s$ can be distributed to multiple processors.

The primary limitation of boundary element methods is their restriction to linear PDEs that admit analytical fundamental solutions, in contrast to domain-based
methods, e.g. finite differences and finite elements. While use of a Laplace domain formulation avoids time-stepping, the trade-off is the necessity for numerical inversion and potential inaccuracies due to the particular inversion method chosen. In the illustration shown earlier, a relatively simple inversion technique was demonstrated. A more sophisticated extension would involve integration of the axisymmetric BEM into a numerical Laplace inversion method based on deHoog’s algorithm [12]. (It is noted that this inversion algorithm has been implemented in MATLAB [21].) Such an algorithm adaptively determines an values of the Laplace parameter at which the BEM solutions should be evaluated. Given the efficiency the axisymmetric BEM developed in this study, this suggested integrated approach has potential for improving accuracy in application of the BEM while preserving overall efficiency.
Bibliography


Appendix
Appendix A

Verification of Cylindrical Fundamental Solutions with Maple

A.1 Derivation of cylindrical solutions

The transformation between Cartesian and cylindrical coordinate bases is:

\[ E_1 = \cos \theta \mathbf{e}_r - \sin \theta \mathbf{e}_\theta; \quad E_2 = \sin \theta \mathbf{e}_r + \cos \theta \mathbf{e}_\theta; \quad E_3 = \mathbf{e}_z. \]

The cylindrical basis vector components can be written in terms of Cartesian fundamental solutions as:

\[ \mathbf{\tilde{u}} = \mathbf{\tilde{u}}_r^s \mathbf{E}_1 + \mathbf{\tilde{u}}_\theta^s \mathbf{E}_2 + \mathbf{\tilde{u}}_z^s \mathbf{E}_3 \]

\[ = \mathbf{\tilde{u}}_r^s (\cos \theta \mathbf{e}_r - \sin \theta \mathbf{e}_\theta) + \mathbf{\tilde{u}}_\theta^s (\sin \theta \mathbf{e}_r + \cos \theta \mathbf{e}_\theta) + \mathbf{\tilde{u}}_z^s \mathbf{e}_z \]

\[ = (\mathbf{\tilde{u}}_r^s \cos \theta + \mathbf{\tilde{u}}_\theta^s \sin \theta) \mathbf{e}_r + (-\mathbf{\tilde{u}}_r^s \sin \theta + \mathbf{\tilde{u}}_\theta^s \cos \theta) \mathbf{e}_\theta + \mathbf{\tilde{u}}_z^s \mathbf{e}_z. \quad (A.1) \]

Comparing (A.1) to \( \mathbf{\tilde{u}} = \mathbf{\tilde{u}}_r^s \mathbf{e}_r + \mathbf{\tilde{u}}_\theta^s \mathbf{e}_\theta + \mathbf{\tilde{u}}_z^s \mathbf{e}_z \), the cylindrical components of displacement due to a scalar source are

\[ \mathbf{\tilde{u}}_r^s = \mathbf{\tilde{u}}_r^s \cos \theta + \mathbf{\tilde{u}}_\theta^s \sin \theta; \]

\[ \mathbf{\tilde{u}}_\theta^s = -\mathbf{\tilde{u}}_r^s \sin \theta + \mathbf{\tilde{u}}_\theta^s \cos \theta; \]

\[ \mathbf{\tilde{u}}_z^s = \mathbf{\tilde{u}}_3^s. \]
Unlike the scalar source, a vector source placed at $Q$ can have different magnitudes $\alpha_i$ in each direction. The displacement due to a vector source is $\tilde{u} = \tilde{u}_{ij}^f \alpha_i E_j$, where $\tilde{u}_{ij}^f$ is the displacement in the $j^{th}$ direction due to a vector source of strength $\alpha_i$ in the $i^{th}$ direction, so that:

$$\tilde{u} = \left[ (\tilde{u}_{11} f \alpha_1 + \tilde{u}_{21} f \alpha_2 + \tilde{u}_{31} f \alpha_3) \cos \theta + (\tilde{u}_{12} f \alpha_1 + \tilde{u}_{22} f \alpha_2 + \tilde{u}_{32} f \alpha_3) \sin \theta \right] e_r$$

$$+ \left[ -(\tilde{u}_{11} f \alpha_1 + \tilde{u}_{21} f \alpha_2 + \tilde{u}_{31} f \alpha_3) \sin \theta + (\tilde{u}_{12} f \alpha_1 + \tilde{u}_{22} f \alpha_2 + \tilde{u}_{32} f \alpha_3) \cos \theta \right] e_\theta$$

$$+ (\tilde{u}_{13} f \alpha_1 + \tilde{u}_{23} f \alpha_2 + \tilde{u}_{33} f \alpha_3)e_z.$$  (A.2)

Then, since displacement can be written in cylindrical coordinates as $\tilde{u} = \tilde{u}_r e_r + \tilde{u}_\theta e_\theta + \tilde{u}_z e_z$, where

$$\tilde{u}_r = \tilde{u}_{rr} f \alpha_1 + \tilde{u}_{r\theta} f \alpha_2 + \tilde{u}_{rz} f \alpha_3,$$  (A.3)

$$\tilde{u}_\theta = \tilde{u}_{\theta r} f \alpha_1 + \tilde{u}_{\theta \theta} f \alpha_2 + \tilde{u}_{\theta z} f \alpha_3,$$  (A.4)

$$\tilde{u}_z = \tilde{u}_{z r} f \alpha_1 + \tilde{u}_{z \theta} f \alpha_2 + \tilde{u}_{zz} f \alpha_3,$$  (A.5)

the cylindrical point force fundamental solutions due to a vector source are obtained via grouping the components of $\tilde{u}$ by $\alpha_i$:

$$\tilde{u}_{rr}^f = \tilde{u}_{11}^f \alpha_1 \cos \theta + \tilde{u}_{12}^f \alpha_2 \sin \theta;$$  (A.6)

$$\tilde{u}_{r\theta}^f = \tilde{u}_{21}^f \alpha_1 \cos \theta + \tilde{u}_{22}^f \alpha_2 \sin \theta;$$  (A.7)

$$\tilde{u}_{rz}^f = \tilde{u}_{31}^f \alpha_1 \cos \theta + \tilde{u}_{32}^f \alpha_2 \sin \theta;$$  (A.8)

$$\tilde{u}_{\theta r}^f = -\tilde{u}_{11}^f \alpha_1 \sin \theta + \tilde{u}_{12}^f \alpha_2 \cos \theta;$$  (A.9)

$$\tilde{u}_{\theta \theta}^f = -\tilde{u}_{21}^f \alpha_1 \sin \theta + \tilde{u}_{22}^f \alpha_2 \cos \theta;$$  (A.10)

$$\tilde{u}_{\theta z}^f = -\tilde{u}_{31}^f \alpha_1 \sin \theta + \tilde{u}_{32}^f \alpha_2 \cos \theta;$$  (A.11)

$$\tilde{u}_{rz}^f = \tilde{u}_{13}^f;$$  (A.12)

$$\tilde{u}_{\theta z}^f = \tilde{u}_{23}^f;$$  (A.13)

$$\tilde{u}_{zz}^f = \tilde{u}_{33}^f.$$  (A.14)

Since tractions and displacements are both vector quantities, the form of the fundamental solutions for traction follows the same pattern as for displacement. For a
scalar source placed at $Q$, the corresponding cylindrical tractions are:

\[ t^s_r = t^s_1 \cos \theta + t^s_2 \sin \theta, \]
\[ t^s_\theta = -t^s_1 \sin \theta + t^s_2 \cos \theta, \]
\[ t^s_z = t^s_3, \]

and

\[ \tilde{t}^f_{rr} = \tilde{t}^f_{11} \cos \theta + \tilde{t}^f_{12} \sin \theta; \]
\[ \tilde{t}^f_{r\theta} = \tilde{t}^f_{21} \cos \theta + \tilde{t}^f_{22} \sin \theta; \]
\[ \tilde{t}^f_{rz} = \tilde{t}^f_{31} \cos \theta + \tilde{t}^f_{32} \sin \theta; \]
\[ \tilde{t}^f_{r\theta} = -\tilde{t}^f_{11} \sin \theta + \tilde{t}^f_{12} \cos \theta; \]
\[ \tilde{t}^f_{\theta\theta} = -\tilde{t}^f_{21} \sin \theta + \tilde{t}^f_{22} \cos \theta; \]
\[ \tilde{t}^f_{\theta z} = \tilde{t}^f_{31} \sin \theta + \tilde{t}^f_{32} \cos \theta; \]
\[ \tilde{t}^f_{rz} = \tilde{t}^f_{13}; \]
\[ \tilde{t}^f_{\theta z} = \tilde{t}^f_{23}; \]
\[ \tilde{t}^f_{zz} = \tilde{t}^f_{33}. \]

The scalar pore pressure at a point $P$ due to a vector source at $Q$ is given by:

\[ \tilde{p} = \tilde{p}^f_1 \alpha_1 + \tilde{p}^f_2 \alpha_2 + \tilde{p}^f_3 \alpha_3 \]
\[ = \tilde{p}^f_r \alpha_1 + \tilde{p}^f_\theta \alpha_2 + \tilde{p}^f_z \alpha_3, \]

so $\tilde{p}^f_r = \tilde{p}^f_1$, $\tilde{p}^f_\theta = \tilde{p}^f_2$, and $\tilde{p}^f_z = \tilde{p}^f_3$. When a scalar source is placed at $Q$, then $\tilde{p} = \tilde{p}^s$.

The fluid source flux is:

\[ \tilde{q} = \tilde{q}^s_1 n_1 + \tilde{q}^s_2 n_2 + \tilde{q}^s_3 n_3 + \tilde{q}^r_1 n_r + \tilde{q}^r_2 n_\theta + \tilde{q}^r_z n_z \]
\[ = (\tilde{q}^s_1 \cos \theta + \tilde{q}^s_2 \sin \theta)n_r + (\tilde{q}^s_1 \sin \theta + \tilde{q}^s_2 \cos \theta)n_\theta + \tilde{q}^s_3 n_z, \]

so $\tilde{q} = \tilde{q}^s_r n_r + \tilde{q}^s_\theta n_\theta + \tilde{q}^s_z n_z$ implies that

\[ \tilde{q}^s_r = \tilde{q}^s_1 \cos \theta + \tilde{q}^s_2 \sin \theta, \]
\[ \tilde{q}^s_\theta = -\tilde{q}^s_1 \sin \theta + \tilde{q}^s_2 \cos \theta, \]
\[ \tilde{q}^s_z = \tilde{q}^s_3. \]
where $\tilde{q}_i^s$ are the Cartesian fundamental solutions in $\tilde{q} = \tilde{q}_i^s n_i$:

$$\tilde{q}_i^s = \frac{r_i^3}{r^2} (1 - \zeta) e^{\xi}.$$  \hfill (A.34)

When a vector source is placed at $Q$, the flux can be expressed as a linear combination of the strength of the source $\alpha_i$ in each of the three directions:

$$\tilde{q} = \tilde{q}_i^f \alpha_j = \tilde{q}_1^f \alpha_1 + \tilde{q}_2^f \alpha_2 + \tilde{q}_3^f \alpha_3.$$  \hfill (A.35)

Since flux can also be written as the dot product

$$q = q \cdot n = \tilde{q}_1^f n_1 + \tilde{q}_2^f n_2 + \tilde{q}_3^f n_3,$$ \hfill (A.36)

the quantities $\tilde{q}_i^f$ can be determined by grouping the terms in (2.35) by $n_i$, and then by $\alpha_j$, to get:

$$\tilde{q}_1^f = \frac{k}{r^3} (C_1 + (r, 1)^2 C_2) \alpha_1 + \frac{k}{r^3} (r, 2 r, 1 C_2) \alpha_2 + \frac{k}{r^3} (r, 3 r, 1 C_2) \alpha_3$$ \hfill (A.37)

$$\tilde{q}_2^f = \frac{k}{r^3} (r, 1 r, 2 C_2) \alpha_1 + \frac{k}{r^3} (C_1 + (r, 2)^2 C_2) \alpha_2 + \frac{k}{r^3} (r, 3 r, 2 C_2) \alpha_3$$ \hfill (A.38)

$$\tilde{q}_3^f = \frac{k}{r^3} (r, 1 r, 3 C_2) \alpha_1 + \frac{k}{r^3} (r, 2 r, 3 C_2) \alpha_2 + \frac{k}{r^3} (C_1 + (r, 3)^2 C_2) \alpha_3$$ \hfill (A.39)

where $C_1 = (1 + \zeta) e^{-\xi} - 1$ and $C_2 = 3 - (3 + 3 \zeta + \zeta^2) e^{-\xi}$. By substituting (A.37)–(A.39) into (A.36), the flux $q$ is decomposed into terms involving $n_j$ and $\alpha_i$:

$$\tilde{q} = (\tilde{q}_1^f \cos \theta + \tilde{q}_2^f \sin \theta) n_r + (-\tilde{q}_1^f \sin \theta + \tilde{q}_2^f \cos \theta) n_\theta + (q_3^f) n_z$$

$$= \frac{k}{r^3} \left\{ \left[ (C_1 + (r, 1)^2 C_2) \cos \theta + (r, 1 r, 2 C_2) \sin \theta \right] \alpha_1 
\quad + \left[ (r, 2 r, 1 C_2) \cos \theta + (C_1 + (r, 2)^2 C_2) \sin \theta \right] \alpha_2 
\quad + \left[ (r, 3 r, 1 C_2) \cos \theta + (r, 3 r, 2 C_2) \sin \theta \right] \alpha_3 \right\} n_r 
\quad + \frac{k}{r^3} \left\{ \left[ -(C_1 + (r, 1)^2 C_2) \sin \theta + (r, 1 r, 2 C_2) \cos \theta \right] \alpha_1 
\quad + \left[ -(r, 2 r, 1 C_2) \sin \theta + (C_1 + (r, 2)^2 C_2) \cos \theta \right] \alpha_2 
\quad + \left[ -(r, 3 r, 1 C_2) \sin \theta + (r, 3 r, 2 C_2) \cos \theta \right] \alpha_3 \right\} n_\theta 
\quad + \frac{k}{r^3} \left\{ (r, 1 r, 3 C_2) \alpha_1 + \frac{k}{r^3} (r, 2 r, 3 C_2) \alpha_2 + \frac{k}{r^3} (C_1 + (r, 3)^2 C_2) \alpha_3 \right\} n_z.$$
Finally, grouping by \( \alpha_j \) in \( \tilde{q} = \tilde{q}_r' \alpha_1 + \tilde{q}_\theta' \alpha_2 + \tilde{q}_z' \alpha_3 \) produces the cylindrical components of flux:

\[
\tilde{q}_r' = \frac{k}{r^3} \left\{ n_r \left[ (C_1 + (r,1)^2 C_2) \cos \theta + (r,1 r,2) C_2 \sin \theta \right]\right.
+ n_\theta \left[ -(C_1 + (r,1)^2 C_2) \sin \theta + (r,1 r,2) C_2 \cos \theta \right]
+ n_z \left[ r,1 r,3 C_2 \right] \};
\]

(A.40)

\[
\tilde{q}_\theta' = \frac{k}{r^3} \left\{ n_r \left[ (r,2 r,1) C_2 \cos \theta + (C_1 + (r,2)^2 C_2) \sin \theta \right]\right.
+ n_\theta \left[ -(r,2 r,1) C_2 \sin \theta + (C_1 + (r,2)^2 C_2) \cos \theta \right]
+ n_z \left[ r,2 r,3 C_2 \right] \};
\]

(A.41)

\[
\tilde{q}_z' = \frac{k}{r^3} \left\{ n_r \left[ (r,3 r,1) C_2 \cos \theta + (r,3 r,2) C_2 \sin \theta \right]\right.
+ n_\theta \left[ -(r,3 r,1) C_2 \sin \theta + (r,3 r,2) C_2 \cos \theta \right]
+ n_z \left[ C_1 + (r,3)^2 C_2 \right] \}.\]

(A.42)

### A.2 Satisfaction of governing equations in cylindrical polar coordinates

To verify that the cylindrical fundamental solutions are correct, the symbolic computation software package Maple was used to symbolically verify that all field variables satisfy the stress equilibrium relation \( \nabla \cdot \sigma = 0 \) in the Laplace domain, or

\[
\nabla \cdot \sigma_E = \nabla p,
\]

(A.43)

and the continuity equation \( \partial_t \zeta + \nabla \cdot \mathbf{q} = 0 \), which can be rewritten in the Laplace transform domain as

\[
\text{str}(\varepsilon) + \nabla \cdot \mathbf{q} = 0,
\]

(A.44)

By writing the left side of (A.43) componentwise, using \( \sigma_E = \lambda \text{tr}(\varepsilon) I + 2\mu \varepsilon \), where the strain tensor \( \varepsilon \) is given by \( \varepsilon = 1/2[\nabla \mathbf{u} + (\nabla \mathbf{u})^T] \), three equations represented by (A.43) in terms of stress and its derivatives are obtained. The cylindrical fundamental solutions must satisfy the vector cylindrical stress equation (A.43), which corresponds
to the three component equations

\[
\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = \frac{\partial p}{\partial r} \tag{A.45} \\
\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2}{r} \sigma_{r\theta} = \frac{1}{r} \frac{\partial p}{\partial \theta} \\
\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{1}{r} \sigma_{rz} = \frac{\partial p}{\partial z} \tag{A.47}
\]

The scalar cylindrical continuity equation (A.44) is

\[
s \left( \frac{\partial u_r}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{1}{r} u_r + \frac{\partial u_z}{\partial z} \right) + \frac{1}{r} q_r + \frac{\partial q_r}{\partial r} + \frac{\partial q_\theta}{\partial \theta} + \frac{\partial q_z}{\partial z} = 0 \tag{A.48}
\]

where the trace of the strain tensor \(e\) is written in terms of displacement components.

Stress can be determined from derivatives of displacement, using the decomposition involving the strength of the vector source in each direction:

\[
\tilde{u}_r = \tilde{u}_{rr} \alpha_r + \tilde{u}_{r\theta} \alpha_\theta + \tilde{u}_{rz} \alpha_z \tag{A.49}
\]

\[
\tilde{u}_\theta = \tilde{u}_{r\theta} \alpha_r + \tilde{u}_{\theta\theta} \alpha_\theta + \tilde{u}_{z\theta} \alpha_z \tag{A.50}
\]

\[
\tilde{u}_z = \tilde{u}_{rz} \alpha_r + \tilde{u}_{\theta z} \alpha_\theta + \tilde{u}_{zz} \alpha_z \tag{A.51}
\]

Tractions follow the same pattern as the displacements above, while pressure and flux have similar forms:

\[
p = p_r \alpha_r + p_\theta \alpha_\theta + p_z \alpha_z \tag{A.52}
\]

\[
q = q_r \alpha_r + q_\theta \alpha_\theta + q_z \alpha_z \tag{A.53}
\]

After substituting (A.49)–(A.53) into (A.45)–(A.48) and grouping the terms by \(\alpha\), the fundamental solutions can be shown to satisfy the cylindrical governing equations if the coefficients of the three \(\alpha\) components in each equation are zero. The Maple code showing symbolic validation of the cylindrical fundamental solutions is included in the following section.

### A.3 Maple code

#### A.3.1 Verification of fluid source solutions
> point p(x1,x2,x3) is fixed; point Q(y1,y2,y3) moves
> x1:=Rp: x2:=0: x3:=Zp:  # p coord's
> y1:=rQ*cos(theta): y2:=rQ*sin(theta): y3:=zQ:  # Q coord's

> Cartesian fluid source fundamental solutions
> u1:=rd1*(1-(1+eta)*exp(-eta))/(s*r^2):
> u2:=rd2*(1-(1+eta)*exp(-eta))/(s*r^2):
> u3:=rd3*(1-(1+eta)*exp(-eta))/(s*r^2):
> q1:=-s*u1+rd1/r^2:
> q2:=-s*u2+rd2/r^2:
> q3:=-s*u3+rd3/r^2:
> p:=1/k*exp(-eta)/r:

> cylindrical transformation
> ur:=u1*cos(theta)+u2*sin(theta):
> ut:=-u1*sin(theta)+u2*cos(theta):
> uz:=u3:
> qr:=q1*cos(theta)+q2*sin(theta):
> qt:=-q1*sin(theta)+q2*cos(theta):
> qz:=q3:

> definitions
> r:=sqrt((x1-y1)^2+(x2-y2)^2+(x3-y3)^2):
> eta:=r*sqrt(s/k/(lambda+2*mu)):
> lambda:=2*mu*nu/(1-2*nu):
> rd1:=(y1-x1)/r:
> rd2:=(y2-x2)/r:
> rd3:=(y3-x3)/r:

> Verify: d(eta)/dt + div(Q) = 0
> or s*tr(E) + div(Q) = 0

> trE:=diff(ur,r)+1/r*ur+1/r*diff(ut,theta)+diff(uz,z):
> divQ:=(diff(qr,r)+qr/r+1/r*diff(qt,theta)+(diff(qz,z)):

> eq1:=s*trE+divQ:
> simplify(eq1);

> Verify: div(SigE) = grad(p)
> SigE_rr:=lambda*trE+2*mu*diff(ur,r):
> SigE_rt:=mu*(1/r*diff(ur,theta)-1/r*ut+diff(ut,r)):
> SigE_rz:=mu*(diff(ur,z)+diff(uz,r)):
A.3.2 Verification of point force solutions

\begin{verbatim}
> simplify(eq2A);
0
> simplify(eq2B);
0
> simplify(eq2C);
0
\end{verbatim}

\begin{verbatim}
x1:=Rp: x2:=0: x3:=Zp:
y1:=rQ*cos(theta): y2:=rQ*sin(theta): y3:=zQ:

# Point source fundamental solutions
U11:=1/(2*mu*r)+1/(2*mu*r)*rd1^2
    +k/(s*r^3)*((1-(1+zeta)*exp(-zeta))
    +rd1^2*((3+3*zeta+zeta^2)*exp(-zeta)-3)):
U12:=1/(2*mu*r)*rd1*rd2
    +k/(s*r^3)*((rd1*rd2*((3+3*zeta+zeta^2)*exp(-zeta)-3)):
U13:=1/(2*mu*r)*rd1*rd3
    +k/(s*r^3)*((rd1*rd3*((3+3*zeta+zeta^2)*exp(-zeta)-3)):
U21:=1/(2*mu*r)*rd2*rd1
    +k/(s*r^3)*((rd2*rd1*((3+3*zeta+zeta^2)*exp(-zeta)-3)):
U22:=1/(2*mu*r)+1/(2*mu*r)*rd2^2
    +k/(s*r^3)*((1-(1+zeta)*exp(-zeta))
    +rd2^2*((3+3*zeta+zeta^2)*exp(-zeta)-3)):
U23:=1/(2*mu*r)*rd3*rd2
    +k/(s*r^3)*((rd3*rd2*((3+3*zeta+zeta^2)*exp(-zeta)-3)):
\end{verbatim}
U31 := \frac{1}{2\mu r}rd1rd3 + \frac{k}{s r^3}(rd1rd3 ((3+3\zeta+\zeta^2)e^{-\zeta} - 3))

U32 := \frac{1}{2\mu r}rd3rd2 + \frac{k}{s r^3}(rd3rd2 ((3+3\zeta+\zeta^2)e^{-\zeta} - 3))

U33 := \frac{1}{2\mu r} + \frac{1}{2\mu r}rd3^2 + \frac{k}{s r^3}((1-(1+\zeta)e^{-\zeta})
+ rd3^2 ((3+3\zeta+\zeta^2)e^{-\zeta} - 3))

p1 := rd1/r^2(1-(1+\zeta)e^{-\zeta})

p2 := rd2/r^2(1-(1+\zeta)e^{-\zeta})

p3 := rd3/r^2(1-(1+\zeta)e^{-\zeta})

U_{rr} := U11\cos(\theta) + U21\sin(\theta)

U_{rt} := U12\cos(\theta) + U22\sin(\theta)

U_{rz} := U13\cos(\theta) + U23\sin(\theta)

U_{tr} := -U11\sin(\theta) + U21\cos(\theta)

U_{tt} := -U12\sin(\theta) + U22\cos(\theta)

U_{tz} := -U13\sin(\theta) + U23\cos(\theta)

U_{zr} := U31

U_{zt} := U32

U_{zz} := U33

qr_{\alpha1} := \frac{k}{r^3}((C1+rd1^2C2)\cos(\theta) + (rd1rd2C2)\sin(\theta))

qr_{\alpha2} := \frac{k}{r^3}((rd1rd2C2)\cos(\theta) + (C1+rd2^2C2)\sin(\theta))

qr_{\alpha3} := \frac{k}{r^3}((rd3rd1C2)\cos(\theta) + (rd3rd2C2)\sin(\theta))

qt_{\alpha1} := \frac{k}{r^3}(-(C1+rd1^2C2)\sin(\theta) + (rd1rd2C2)\cos(\theta))

qt_{\alpha2} := \frac{k}{r^3}(-(rd1rd2C2)\sin(\theta) + (C1+rd2^2C2)\cos(\theta))

qt_{\alpha3} := \frac{k}{r^3}(-(rd3rd1C2)\sin(\theta) + (rd3rd2C2)\cos(\theta))

qz_{\alpha1} := \frac{k}{r^3}(rd1rd3C2)

qz_{\alpha2} := \frac{k}{r^3}(rd2rd3C2)

qz_{\alpha3} := \frac{k}{r^3}(C1+rd3^2C2)

p_r := p1

p_t := p2

p_z := p3

# definitions
C1 := (1+\zeta)e^{-\zeta}-1
C2 := 3-(3+3\zeta+\zeta^2)e^{-\zeta}
> r := sqrt((x1-y1)^2+(x2-y2)^2+(x3-y3)^2):
> zeta := r * sqrt(s/k/(lambda+2*mu)):
> lambda := 2*mu*mu/(1-2*nu):
> rd1 := (y1-x1)/r:
> rd2 := (y2-x2)/r:
> rd3 := (y3-x3)/r:

> # Verify: d(zeta)/dt + div(Q) = 0
> # or s*tr(E) + div(Q) = 0
> trE_alpha1 := diff(Urr, rQ) + 1/rQ*diff(Utr, theta) + 1/rQ*Urr + diff(Uzr, zQ):
> trE_alpha2 := diff(Urt, rQ) + 1/rQ*diff(Utt, theta) + 1/rQ*Urt + diff(Uzt, zQ):
> trE_alpha3 := diff(Urz, rQ) + 1/rQ*diff(Utz, theta) + 1/rQ*Urz + diff(Uzz, zQ):

> divQ_alpha1 := 1/rQ*qr_alpha1 + diff(qr_alpha1, rQ) + diff(qt_alpha1, theta) + diff(qz_alpha1, zQ):
> divQ_alpha2 := 1/rQ*qr_alpha2 + diff(qr_alpha2, rQ) + diff(qt_alpha2, theta) + diff(qz_alpha2, zQ):
> divQ_alpha3 := 1/rQ*qr_alpha3 + diff(qr_alpha3, rQ) + diff(qt_alpha3, theta) + diff(qz_alpha3, zQ):

> eq1A := simplify(s*trE_alpha1 + divQ_alpha1);
> eq1A := 0
> eq1B := simplify(s*trE_alpha2 + divQ_alpha2);
> eq1B := 0
> eq1C := simplify(s*trE_alpha3 + divQ_alpha3);
> eq1C := 0

> # Verify: div(SigE) = grad(p)
> SigE_rr_alpha1 := lambda*trE_alpha1 + 2*mu*diff(Urr, rQ):
> SigE_rr_alpha2 := lambda*trE_alpha2 + 2*mu*diff(Urt, rQ):
> SigE_rr_alpha3 := lambda*trE_alpha3 + 2*mu*diff(Urz, rQ):
> SigE_rt_alpha1 := mu*(1/rQ*diff(Urr, theta) - 1/rQ*Utr + diff(Utr, rQ)):
> SigE_rt_alpha2 := mu*(1/rQ*diff(Urt, theta) - 1/rQ*Utt + diff(Utt, rQ)):
> SigE_rt_alpha3 := mu*(1/rQ*diff(Urz, theta) - 1/rQ*Utz + diff(Utz, rQ)):
> SigE_rz_alpha1 := mu*(diff(Uzr, rQ) + diff(Urr, zQ)):
> SigE_rz_alpha2 := mu*(diff(Uzt, rQ) + diff(Urt, zQ)):
> SigE_rz_alpha3 := mu*(diff(Uzz, rQ) + diff(Urz, zQ)):
> SigE_tr_alpha1 := SigE_rt_alpha1:
\begin{verbatim}
> SigE_tr_alpha2:=SigE_rt_alpha2:
> SigE_tr_alpha3:=SigE_rt_alpha3:
> SigE_tt_alpha1:=lambda*trE_alpha1+2*mu/rQ*(diff(Utr,theta)+Urr):
> SigE_tt_alpha2:=lambda*trE_alpha2+2*mu/rQ*(diff(Utt,theta)+Urt):
> SigE_tt_alpha3:=lambda*trE_alpha3+2*mu/rQ*(diff(Utz,theta)+Urz):
> SigE_tz_alpha1:=mu*(1/rQ*diff(Uzr,theta)+diff(Utr,zQ)):
> SigE_tz_alpha2:=mu*(1/rQ*diff(Uzt,theta)+diff(Utt,zQ)):
> SigE_tz_alpha3:=mu*(1/rQ*diff(Uzz,theta)+diff(Utz,zQ)):
> SigE_zr_alpha1:=SigE_rz_alpha1:
> SigE_zr_alpha2:=SigE_rz_alpha2:
> SigE_zr_alpha3:=SigE_rz_alpha3:
> SigE_zt_alpha1:=SigE_tz_alpha1:
> SigE_zt_alpha2:=SigE_tz_alpha2:
> SigE_zt_alpha3:=SigE_tz_alpha3:
> SigE_zz_alpha1:=lambda*trE_alpha1+2*mu*diff(Uzr,zQ):
> SigE_zz_alpha2:=lambda*trE_alpha2+2*mu*diff(Uzt,zQ):
> SigE_zz_alpha3:=lambda*trE_alpha3+2*mu*diff(Uzz,zQ):

> eq2A_alpha1 := simplify(diff(SigE_rr_alpha1,rQ)
   + 1/rQ*diff(SigE_rt_alpha1,theta)+diff(SigE_rz_alpha1,zQ)
   + 1/rQ*(SigE_rr_alpha1-SigE_tt_alpha1) - diff(pr,rQ));
   eq2A_alpha1 := 0
> eq2A_alpha2 := simplify(diff(SigE_rr_alpha2,rQ)
   + 1/rQ*diff(SigE_rt_alpha2,theta)+diff(SigE_rz_alpha2,zQ)
   + 1/rQ*(SigE_rr_alpha2-SigE_tt_alpha2) - diff(pt,rQ));
   eq2A_alpha2 := 0
> eq2A_alpha3 := simplify(diff(SigE_rr_alpha3,rQ)
   + 1/rQ*diff(SigE_rt_alpha3,theta)+diff(SigE_rz_alpha3,zQ)
   + 1/rQ*(SigE_rr_alpha3-SigE_tt_alpha3) - diff(pz,rQ));
   eq2A_alpha3 := 0
> eq2B_alpha1 := simplify(diff(SigE_rt_alpha1,rQ)
   +1/rQ*diff(SigE_tt_alpha1,theta)+diff(SigE_tz_alpha1,zQ)
   +2/rQ*SigE_rt_alpha1-1/rQ*diff(pr,theta));
   eq2B_alpha1 := 0
> eq2B_alpha2 := simplify(diff(SigE_rt_alpha2,rQ)
   +1/rQ*diff(SigE_tt_alpha2,theta)+diff(SigE_tz_alpha2,zQ)
   +2/rQ*SigE_rt_alpha2-1/rQ*diff(pt,theta));
   eq2B_alpha2 := 0
> eq2B_alpha3 := simplify(diff(SigE_rt_alpha3,rQ)
   +1/rQ*diff(SigE_tt_alpha3,theta)+diff(SigE_tz_alpha3,zQ)
   +2/rQ*SigE_rt_alpha3-1/rQ*diff(pz,theta));
\end{verbatim}
\[ \text{eq2B}_\alpha \text{eq3} := 0 \]

> \text{eq2C}_\alpha \text{eq1} := \text{simplify}(\text{diff}(\text{SigE}_r\text{z}_\alpha, rQ) + 1/rQ * \text{diff}(\text{SigE}_t\text{z}_\alpha, \theta) + \text{diff}(\text{SigE}_z\text{z}_\alpha, zQ) + 1/rQ * \text{SigE}_r\text{z}_\alpha - \text{diff}(pr, zQ)); \\
> \text{eq2C}_\alpha \text{eq1} := 0 \\

> \text{eq2C}_\alpha \text{eq2} := \text{simplify}(\text{diff}(\text{SigE}_r\text{z}_\alpha, rQ) + 1/rQ * \text{diff}(\text{SigE}_t\text{z}_\alpha, \theta) + \text{diff}(\text{SigE}_z\text{z}_\alpha, zQ) + 1/rQ * \text{SigE}_r\text{z}_\alpha - \text{diff}(pt, zQ)); \\
> \text{eq2C}_\alpha \text{eq2} := 0 \\

> \text{eq2C}_\alpha \text{eq3} := \text{simplify}(\text{diff}(\text{SigE}_r\text{z}_\alpha, rQ) + 1/rQ * \text{diff}(\text{SigE}_t\text{z}_\alpha, \theta) + \text{diff}(\text{SigE}_z\text{z}_\alpha, zQ) + 1/rQ * \text{SigE}_r\text{z}_\alpha - \text{diff}(pz, zQ)); \\
> \text{eq2C}_\alpha \text{eq3} := 0