Abstract

KERMES, JEREMIAH MITCHELL. Fermat Curves on Weighted Projective Planes. (Under the direction of Amassa Fauntleroy.)

This paper takes the classical result that a homogeneous polynomial of degree $d \geq 2$ defines a curve of genus $\left(\frac{d-1}{2}\right)$ on $\mathbb{P}^2$ and generalizes it to some weighted projective planes using their structure as toric varieties. Throughout the work, focus is on a specific class of curves called Weighted Fermat Curves (WFC)'s. Part I of the paper covers the surface $\mathbb{P}(1,1,n)$ and uses two techniques to determine the genus of a WFC. The first involves Hurwitz's Theorem, while the second involves pulling the curve up to a Hirzebruch surface.

Part II of the paper extends both of these techniques to $\mathbb{P}(1,m,n)$. The second technique no longer involves Hirzebruch surfaces (since they fail to desingularize $\mathbb{P}(1,m,n)$). Instead a desingularization algorithm due to Oda [6] is used to construct a new toric variety, $\mathcal{D}(1,m,n)$. Here the Hirzebruch surface's structure as a ruled surface is replaced by using the homogeneous coordinate ring of a toric variety.
Fermat Curves on Weighted Projective Planes

by

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Dedication

This paper is dedicated to all the wonderful storytellers I’ve listened to over the years. May you continue to live, create, and share your wonderful tales with the world - and know that the warmest place by my fire is always reserved for you.
Biography

Jeremiah Kermes was born June 10, 1980 in Hartford, CT. At age four he moved to Miami, FL where he spent 4 years as a varsity swimmer at Christopher Columbus High School and became an Eagle Scout in April of 1998.

Jeremiah obtained a B.A. from DePauw University in May, 2002 double-majoring in Physics and Mathematics. While at DePauw he did research on Orthogonal Polynomials and Cryptography with Dr. Mark Kannowski and on Magic Cubes with Dr. Michelle Penner. He served as both President and V.P. of the Physics Club and worked to repair and improve telescopes at McKim Observatory in Greencastle, IN. Jeremiah’s musical life also flourished at DePauw, playing in two bands while there as well as becoming the house-bassist for both of the town’s open mic nights.

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Much thanks also goes to my parents, Kathy Hibbs and Thom Kermes, who’ve constantly encouraged me to seek out new challenges and pursue them to the end.

Thank you Dr.’s Larry Norris and Jack Silverstein (aka ”The Professors”) as well as the members of my other musical projects while in school: Shagbark and The Back Porch Band. The relaxed enjoyment of playing good music with good friends has helped keep me grounded throughout this process.

Thank you Dr.’s Tom Lada and Martin Markl, for the opportunity to experience life and mathematics outside of the U.S.

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Thank you to my room-mates Jeremy Scott, Andrew Bickle, and Peter Hansma for being there whenever I needed a beer-break.

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Preface

“Your art takes place where your understanding and your ingorance blur.”

China Miéville

“Perdido Street Station”

What is the genus of a curve on $\mathbb{P}^2$ defined by a homogeneous polynomial of degree $d$? For $d \geq 2$ the answer is well known to be $g = \binom{d-1}{2}$.

In this paper we address the same question, but posed on those weighted projective planes with at least one trivial weight, $\mathbb{P}(1,m,n)$. A significant portion of the work, however, focuses specifically on weighted Fermat curves (or WFC’s for short).

The process begins with the case where $m = 1$. Two approaches to the problem are given here. The first involves applying Hurwitz’s Theorem to a map from a WFC to $\mathbb{P}^1$. The second is to pull the WFC, $C$, up to the desingularization of $\mathbb{P}(1,1,n)$, which is the Hirzebruch surface $\mathbb{F}_n$. On this surface, the curve’s divisor class, $[C] \in \text{Pic}(X)$, can be determined. Then the adjunction formula can be used to determine $g(C)$. This technique has the added benefit that it can be applied to not only WFC’s, but also any smooth curve on $\mathbb{P}(1,1,n)$ that misses the surface’s sole singular point.

Next these two techniques are applied to the case where $m \neq 1$. In this
setting, though, the Jacobian criterion fails to show that a WFC is smooth, so an alternate proof must be obtained before Hurwitz’s theorem can be applied. Fortunately a WFC misses both of $\mathbb{P}(1, m, n)$’s singular points. Subsequently, smoothness can be proven on the surface’s desingularization, $\mathbb{D}(1, m, n)$. In order to describe this new surface, the description of a weighted projective plane as a toric variety must be utilized – in particular, an algorithm (due to Oda [6]) that gives the minimal desingularization of a toric surface.

Even more of the toric structure of $X = \mathbb{D}(1, m, n)$ gets involved when using the adjunction formula. This structure allows for computation of the surface’s canonical divisor – $K_X$, a description of the Chow group – $A_{n-1}(X)$, and a method for intersecting divisors on $X$. In addition, a toric variety’s homogeneous coordinate ring [1] is used to find $C$’s divisor class, $[C] \in A_{n-1}(X)$.

Unlike the case $m = 1$, an explicit construction of $\mathbb{D}(1, m, n)$ is unavailable. However, one is constructed for the cases $m = 2$ and $n = \pm 1 \ (mod \ m)$. At present the author is working on writing some programs in the Python programming language to compute $\mathbb{D}(1, m, n)$ explicitly for large $m, n$. 
Part I: \( \mathbb{P}(1, 1, n) \)

1 Preliminaries on Toric Varieties

Since all weighted projective spaces are toric varieties, we’ll need to go over a few preliminaries. Recall that for any algebraically closed field, \( k \), an \( n \)-dimensional algebraic torus is the multiplicative group \( (k^*)^n \).

Definition 1.0.1

A toric variety is a variety that contains an algebraic torus as a dense open subset. In addition, the group action of the torus on itself extends to an action of the torus on the entire variety.

1.1 Constructing \( X(\Delta) \) From It’s Fan

The greatest benefit in working with toric varieties is that they can all be constructed from a combinatorial structure called a fan. A fan, \( \Delta \), is a special collection of strongly convex rational polyhedral cones, \( \sigma \). For each cone we will construct an affine variety, \( U_\sigma \), that will form a cover of the variety \( X(\Delta) \).

Definition 1.1.1

Given a lattice, \( N \), a strongly convex rational polyhedral cone is a subset of \( N \otimes_\mathbb{Z} \mathbb{R} \) that contains the origin but no entire line passing through it, and is generated (over \( \mathbb{R}_+ \)) by vectors in \( N \).
In order for a collection of cones to be a fan it must satisfy certain properties that will later allow us to glue the corresponding affine cover together. In particular, if two cones intersect each other then their intersection must be a face of each. Also, any face of any cone in $\Delta$ must also be in $\Delta$.

Given a cone, $\sigma$, the construction of the affine variety $U_\sigma$ is as follows. First find the dual cone, $\sigma^\vee = \{ u \in (N \otimes_{\mathbb{Z}} \mathbb{R})^* \mid u(v) \geq 0 \ \forall \ v \in \sigma \}$. Let $\mathcal{M} = \text{Hom}(N, \mathbb{Z})$ be the dual lattice to $N$ so that $\sigma^\vee \subseteq \mathcal{M} \otimes_{\mathbb{Z}} \mathbb{R}$. The next step is to find the semi-group $S_\sigma = \sigma^\vee \cap \mathcal{M}$. Finally, this semi-group is used to construct a finitely generated $k$-algebra, $k[\sigma]$. The corresponding affine variety will then be $U_\sigma = \text{Spec}(k[\sigma])$.

In order to construct $k[\sigma]$ consider (for each $u \in S_\sigma$) the formal symbol $\chi^u$. Then let $k[\sigma]$ be the $k$-algebra generated by the $\chi^u$’s where multiplication of the $\chi$’s is determined by the semi-group operation of $S_\sigma$ (i.e. $\chi^u \cdot \chi^v = \chi^{u+v}$).

The result will be a ring of the form $k[z_0, \ldots, z_n]/I$. Here the ideal $I$ is generated by polynomials that arise from linear dependence relations in $S_\sigma$ (i.e. if $u_1 + u_2 = v_1 + v_2$ in $S_\sigma$, then $\chi^{u_1} \chi^{u_2} - \chi^{v_1} \chi^{v_2} \in I$).

Example 1.1.2 : $\mathbb{A}^2$

Starting with the lattice $N = \mathbb{Z}^2$, take $\sigma$ to be the first quadrant; i.e. the cone generated by the standard basis ($e_1$ and $e_2$). Let $e_1^\vee$ and $e_2^\vee$ be the corresponding dual basis for $\mathcal{M}$. Then $\sigma^\vee = \{ a \ e_1^\vee + b \ e_2^\vee \mid a, b \geq 0 \}$ is the first quadrant in $\mathcal{M}$. In other words $S_\sigma$ will turn out to be the free semigroup
generated by \( e_1^\vee \) and \( e_2^\vee \). Let \( X = \chi e_1^\vee \) and \( Y = \chi e_2^\vee \), so that \( k[\sigma] = k[X, Y] \).

There are no relations on \( X \) and \( Y \) since \( e_1^\vee \) and \( e_2^\vee \) are linearly independent. \( X \) and \( Y \) are all the generators of the ring because \( e_1^\vee \) and \( e_2^\vee \) form a basis for \( \mathbb{M} \) as a \( \mathbb{Z} \)-module. Thus \( U_\sigma = \text{Spec}(k[X, Y]) = \mathbb{A}^2 \) is the affine plane.

Given a fan, \( \Delta \), a scheme can be obtained by gluing together the affine schemes, \( U_\sigma = \text{Spec}(k[\sigma]) \), for all of the cones \( \sigma \) in \( \Delta \). To do this recall that the intersection of two cones in \( \Delta \) (say \( \sigma \) and \( \tau \)) must also be a cone in \( \Delta \). Note that \( \rho = \sigma \cap \tau \subset \sigma \) implies \( \sigma^\vee \subset \rho^\vee \). Subsequently \( S_\sigma \subset S_\rho \) gives an inclusion of rings \( k[\sigma] \hookrightarrow k[\rho] \) (likewise for \( \tau \)). These maps of rings yield maps on the varieties \( U_\rho \hookrightarrow U_\sigma \). The subsequent gluing uses the identification \( U_\sigma \cap U_\tau = U_\rho \) and treats the corresponding maps as inclusions.

**Example 1.1.3 :** \( \mathbb{P}^1 \)

Take \( \mathbb{N} = \mathbb{Z} \) and let \( \Delta \) consist of the cones \( \sigma_+ = \mathbb{R}_+ \), \( \sigma_- = \mathbb{R}_- \), and \( \sigma_0 = \{0\} \). If we let \( e_1^\vee \) denote the identity map on \( \mathbb{Z} \), then \( \sigma_+^\vee \) are generated by \( \pm e_1^\vee \) respectively, and \( \sigma_0^\vee = \mathbb{M} \) is the entire dual space (i.e. it’s generated by \( e_1^\vee \) and \( -e_1^\vee \)). Use \( X \) to denote \( \chi e_1^\vee \) and observe the following:

\[
\begin{align*}
  k[\sigma_+] &= k[X] & \sim U_+ & \cong k \\
  k[\sigma_-] &= k[X^{-1}] & \sim U_- & \cong k \\
  k[\sigma_0] &= k[X, X^{-1}] & \sim U_0 & \cong k^* 
\end{align*}
\]

So the resulting affine pieces are two lines that intersect in a torus. Using local coordinates \( x = X \) and \( y = X^{-1} \) on each respective line will show that
they glue together along $U_0$ using the function $y = 1/x$. Thus $X(\Delta) \cong \mathbb{P}^1$.

1.2 Complete Toric Varieties

Several properties of a toric variety will be of use to us later on. In particular, the property of completeness will be useful since weighted projective planes are all complete.

Definition 1.2.1

A fan, $\Delta$, is **complete** if and only if $\Delta$ covers $\mathbb{N} \otimes_{\mathbb{Z}} \mathbb{R}$.

When $\Delta$ is complete, we say that the corresponding variety, $X(\Delta)$ is complete as well. It turns out that a toric variety is compact if and only if it is complete. The only complete toric curve is $\mathbb{P}^1$. In the case of complete toric surfaces one does not need to specify $\Delta$ in its entirety. Rather, indicating the set of one-dimensional cones, $\Delta(1)$, will suffice. This is because the two-dimensional cones will be the regions of the plane separated by the rays generated by the given one-dimensional cones. One example of a complete toric variety is the projective plane, whose fan is obtained from the lattice $\mathbb{N} = \mathbb{Z}^2$ by letting $\Delta(1) = \left\{ \begin{bmatrix} -1 & 1 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. 
In fact there are other toric representations of this space. Take $\mathbb{N} = \mathbb{Z}^2$. Then any choice of $\Delta(1) = \{v_0, v_1, v_2\}$ that satisfies $\sum_{i=0}^{2} v_i = 0$ and $\mathbb{N} = \text{Span}_{\mathbb{Z}}(v_0, v_1, v_2)$ will yield $X(\Delta) \cong \mathbb{P}^2$.

In a similar fashion, the weighted projective plane, $\mathbb{P}(l,m,n)$ can be described as a complete toric variety by using the lattice $\mathbb{Z}^2$ and choosing $\Delta(1) = \{v_0, v_1, v_2\}$ such that $lv_0 + mv_1 + nv_2 = 0$ and $\mathbb{N} = \text{Span}_{\mathbb{Z}}(v_0, v_1, v_2)$. For general $(l,m,n)$ the second condition makes finding such $v_i$’s difficult. However, in the case where one weight is trivial (say $l = 1$) the task is much simpler. In particular one can take $v_0 = \begin{bmatrix} -m \\ -n \end{bmatrix}$, $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Because of this convenient construction we confine our consideration to $\mathbb{P}(1,m,n)$ for the duration of this paper. For convenience we let each two-dimensional cone, $\sigma_i$, be the cone generated (over $\mathbb{R}_+$) by the vectors other than $v_i$. This would yield a fan that looks like:
At this point it is beneficial to recall a detail about the construction of \( k[\sigma] \) from a cone \( \sigma \). In particular, the fact that this construction really depends on \( \sigma^\vee \) which sits in the dual space, not of the lattice \( \mathbb{N} \), but of \( \mathbb{N} \otimes_{\mathbb{Z}} \mathbb{R} \). Since this is a vector space we see that complete toric varieties are determined by the rays generated by \( \Delta(1) \), not by the actual lattice points.

Since the construction merely depends on the rays dividing \( \mathbb{N} \) one needs only consider the smallest \( v \in \mathbb{N} \) that lies in the given ray. For our case of \( \mathbb{P}(1,m,n) \) let \( a = \gcd(m,n) \) so that \( \begin{bmatrix} -m \\ -n \end{bmatrix} = a \begin{bmatrix} -m' \\ -n' \end{bmatrix} \) where \( m',n' \) are relatively prime. Then the corresponding ray is generated by \( \begin{bmatrix} -m' \\ -n' \end{bmatrix} \). This establishes an isomorphism between \( \mathbb{P}(1,m,n) \) and \( \mathbb{P}(1,m',n') \).

The condition that \( m < n \) may also be imposed since the map of lattices that switches the \( x \) and \( y \) axes establishes an isomorphism between \( \mathbb{P}(1,m,n) \) and \( \mathbb{P}(1,n,m) \). Subsequently, we have reduced to the problem of considering weighted projective planes \( \mathbb{P}(1,m,n) \) where \( m,n \) are relatively prime with \( m < n \).
2 Singular Varieties: Constructing $\mathbb{P}(1,1,n)$

In this section we treat the notion of a singular toric variety. The goal is to be able to construct $\mathbb{P}(1,1,n)$ from its fan. Since the classical description of a weighted projective plane as $\text{Proj} \, (k[x_0, x_1, x_2])$ is so well known, we will also relate this classical construction to the toric one. To begin, recall the following.

**Theorem 2.0.1** : ([4], p.29)

Let $\sigma$ be a cone of dimension $a \leq b = \text{rank}(\mathbb{N})$. Then $U_\sigma$ is non-singular if and only if $\sigma$ is generated by part of a $\mathbb{Z}$-basis for $\mathbb{N}$. In this case

$$U_\sigma \cong k^a \times (k^*)^{b-a}$$

This result hinges on the fact that the generators of $\sigma^\vee$ (as an $\mathbb{R}$-module) will not usually generate $S_\sigma$ (as a $\mathbb{Z}$-module). Once the rest of the generators of $S_\sigma$ are included there will be relations among them that generate the ideal that vanishes on the affine variety $U_\sigma$. In the case of surfaces, if the edges of $\sigma$ (say $v_1, v_2$) satisfy $\det|v_1 v_2| = 1$ (i.e. form a $\mathbb{Z}$-basis for $\mathbb{N}$) then their duals, $v_1^\vee, v_2^\vee$, will generate $S_\sigma$ with no relations among them. From here letting $X = \chi^{v_1^\vee}$ and $Y = \chi^{v_2^\vee}$ yields an affine plane $U_\sigma \cong \text{Spec} (k[X,Y]) = \mathbb{A}^2$.

Consider now the cone consisting of the origin alone (this is a cone in any fan since it is the intersection of all cones). By the theorem, the resulting space is the algebraic torus, $(k^*)^{\text{rank}(\mathbb{N})}$. To construct $U_{\{0\}}$ note that all of
\( \mathbb{M} \) vanishes on \( \{0\} \). Pick a basis, \( e_1^\vee, \ldots, e_n^\vee \) for \( \mathbb{M} \) and let \( X_i = \chi e_i^\vee \). The resulting \( k \)-algebra is \( k[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}] \) which gives \( U_0 = (k^*)^n \) as expected. In fact, this torus is the one from the definition of a toric variety.

2.1 The Toric Construction of \( \mathbb{P}(1, 1, n) \)

The affine plane, the torus, and \( \mathbb{A}^1 \times k^* \) (take \( \sigma \) to be a single vector in \( \mathbb{Z}^2 \)), are the only non-singular affine toric surfaces. For an example of a singular surface, we look to \( \mathbb{P}(1, 1, n) \). This is a complete toric surface where \( \Delta(1) \) of the fan in \( \mathbb{Z}^2 \) consists of the vectors \( v_0 = \begin{bmatrix} -1 \\ -n \end{bmatrix}, v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \) and \( v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \).

As before, let \( \sigma_i \) be the cone generated by the vectors other than \( v_i \). Since

\[
\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ 1 & -n \end{vmatrix} = 1
\]

we know that \( U_{\sigma_0} \) and \( U_{\sigma_1} \) will both be non-singular. In fact, both will be copies of \( \mathbb{A}^2 \).

Note, however, that \( \begin{vmatrix} -1 & 1 \\ -n & 0 \end{vmatrix} = n \neq 1 \), which means \( U_{\sigma_2} \) will be a singular surface. The first step in describing \( U_{\sigma_2} \) is to find the edges of the dual cone, \( \sigma_2^\vee \). Consider \( u = a e_1^\vee + b e_2^\vee \in Hom(\mathbb{N}, \mathbb{Z}) \) where \( e_1^\vee, e_2^\vee \) is dual to the standard basis for \( \mathbb{Z}^2 \). Next find \( a, b \) such that \( u \) is zero on one edge of \( \sigma_2 \) and positive on the other. Note that \( u \begin{bmatrix} 1 \\ 0 \end{bmatrix} = a \) and \( u \begin{bmatrix} -1 \\ -n \end{bmatrix} = -a - bn \).

Thus one edge satisfies \( a = 0 \) and \( -a - bn = -b n > 0 \), which means \( b < 0 \). So take \( b = -1 \) to get \( u = -e_2^\vee \). The second edge satisfies \( a > 0 \) and \( -a - bn = 0 \Rightarrow a = -b n \). Since taking \( a = 1 \) would yield the fractional
\( b = \frac{1}{n} \) we will take \( a = n \) and \( b = -1 \) to get \( u = n e_1^\vee - e_2^\vee \) for the second generator of \( \sigma_2^\vee \). This process can be repeated for the other two cones with the following edges for each \( \sigma_i^\vee \).

<table>
<thead>
<tr>
<th>( i )</th>
<th>Edge 1</th>
<th>Edge 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( e_1^\vee )</td>
<td>( e_2^\vee )</td>
</tr>
<tr>
<td>1</td>
<td>( -n e_1^\vee + e_2^\vee )</td>
<td>( -e_1^\vee )</td>
</tr>
<tr>
<td>2</td>
<td>( -e_2^\vee )</td>
<td>( n e_1^\vee - e_2^\vee )</td>
</tr>
</tbody>
</table>

Since \( U_{\sigma_2} \) is singular, \( -e_2^\vee \) and \( n e_1^\vee - e_2^\vee \) will not generate \( S_{\sigma_2} = \sigma_2^\vee \cap \mathbb{M} \) over \( \mathbb{Z} \). In order to find generators we must find the other lattice points in the parallelogram created by \( -e_2^\vee \) and \( n e_1^\vee - e_2^\vee \). To do this consider rational linear combinations \( s(-e_2^\vee) + t(n e_1^\vee - e_2^\vee) \) where \( 0 \leq s, t \leq 1 \) (and are not simultaneously 0 or 1). This is merely \( u = (t n) e_1^\vee - (s + t) e_2^\vee \). In order for \( u \) to be a lattice point we must have \( t = \frac{j}{n} \) for any \( 0 \leq j \leq n \). In regard to the \( e_2^\vee \) coefficient, we must recall that \( s \leq 1 \). This means \( (s + t) = (s + \frac{j}{n}) \) cannot be as large as 2 (that would mean either \( s > 1 \) or \( s = t = 1 \)). Since \( s, t \) cannot both be 0, the only integer left is 1. Thus, the generators for \( S_{\sigma_2} \) are \( u_j = (\frac{j}{n} n) e_1^\vee - e_2^\vee = j e_1^\vee - e_2^\vee \) where \( 0 \leq j \leq n \).

It should be noted that this is the part of the process that, if used in a situation where \( U_{\sigma} \) is non-singular (i.e. \( \det |v_1 \ v_2| = 1 \)), will simply return only the two edges of \( \sigma^\vee \) that you began with. What will happen is that the only \((s, t)\) pairs that satisfy the necessary conditions will be \((1, 0)\) and \((0, 1)\). This is because the two edges of \( \sigma^\vee \) will form the basis for \( \mathbb{Z}^2 \) that is dual to
In order to obtain the coordinate ring of $U_{\sigma_2}$ let $X = \chi^{v_1}$ and $Y = \chi^{v_2}$. This gives the generators for our $k$-algebra as $z_j = X^j Y^{-1}$. Subsequently, $k[\sigma_2] \cong k[Y^{-1}, XY^{-1}, \ldots, X^j Y^{-1}, \ldots, X^n Y^{-1}]$. As with any toric variety, there is an ideal, $I \subset k[z_0, \ldots, z_n]$, generated by relations of the form monomial = monomial that vanish on $U_{\sigma_2} \subset \mathbb{A}^{n+1}$ (so that $k[\sigma_2] \cong k[z_0, \ldots, z_n]/I$).

One method for finding some of these relations is to consider the relation $u_j = s u_0 + t u_n$ used to find $S_{\sigma}$ from $\sigma^\vee$. Clearing the common denominator, $d$, of $s$ and $t$ will yield an integral relation $d u_j = \tilde{s} u_0 + \tilde{t} u_n$ which in turn yields $z^d_j = \tilde{z}_0^\prime \tilde{z}_n^\prime$. In our case this gives relations of the form $z^n_j = z_0^{n-j} z_j^n$ for each $1 \leq j \leq n - 1$.

It should also be noted that other polynomials in $I$ can be obtained by picking any pair of generators for $S_{\sigma}$ and regarding them as a basis for the vector space $M \otimes \mathbb{Z} R$, then writing any of the other generators of $S_{\sigma}$ in terms of this basis. Using the edges of $\sigma^\vee$, as we have here, has the added benefit that the coefficients involved will turn out to be positive giving relations of the form $z^a_j = \frac{b}{c} z_0^\prime z_n^\prime$. Instead of this approach we could also use the relations based off of $u_0$ and $u_1$, namely $z^1_1 = X^j Y^{-1} = (X^j Y^{-1}) (Y^{-(j-1)}) = z_j z_0^{j-1}$ for $2 \leq j \leq n$.
2.2 The Classical Construction

For the moment we turn our attention away from the toric aspects of $\mathbb{P}(1, 1, n)$ and recall the classical construction in terms of the graded polynomial ring $k[x_0, x_1, x_2]$ where $\text{deg}(x_0) = \text{deg}(x_1) = 1$ and $\text{deg}(x_2) = n$. As with the usual projective plane $U_i$ will denote the affine region defined by $x_i \neq 0$. In particular the local rings, $k[U_i]$, will be $k$-algebras generated by forms of degree zero in $k[x_0, x_1, x_2, x_i^{-1}]$.

The cases where $i = 0, 1$ will each yield two algebraically independent forms. For example, $x_0 \neq 0$ gives $\frac{x_1}{x_0}$ and $\frac{x_2}{x_0}$. Similarly $x_1 \neq 0$ gives the forms $\frac{x_0}{x_1}$ and $\frac{x_2}{x_1}$. In both cases, there are no polynomial relations among the generators, meaning that $k[U_0]$ and $k[U_1]$ will be isomorphic to polynomial rings in two variables. Subsequently, $U_0 \cong U_1 \cong \mathbb{A}^2$.

The region $U_2$ is somewhat trickier. In order to obtain all of the generators, begin with the obvious form, $\frac{x_0^n}{x_2}$. The key in finding the rest is to include successively higher powers of $x_1$ in the numerator, while dropping the number of $x_0$’s present (so that the resulting form still has a degree of 0). One can continue this process until the other obvious generator, $\frac{x_1^n}{x_2}$, has been reached. Thus for $0 \leq j \leq n$, $k[U_2]$ is generated by

$$\tilde{z}_j = \frac{x_0^{n-j}x_1^j}{x_2}.$$ 

It should be noted here that every one of these forms (including the
ones for $U_0$ and $U_1$) can be obtained by taking products of the two forms $\frac{x_1}{x_0}, \frac{x_2}{x_0}$ and their inverses. It is this insight that allows us to construct the isomorphism between this classical construction of $\mathbb{P}(1,1,n)$, and the surface’s toric construction.

**Theorem 2.2.1**

Let $k[x_0, x_1, x_2]$ be graded as above, and let $\Delta$ be the complete fan in $\mathbb{Z}^2$ given by $\Delta(1) = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -n \end{bmatrix} \right\}$. Then $\text{Proj}(k[x_0, x_1, x_2]) \cong X(\Delta)$.

**Proof**

Not only will the two spaces be isomorphic, but the isomorphism will be very natural. In particular $U_i \cong U_{\sigma_i}$. The method of the construction is to show the three rings for the affine cover are isomorphic. Recall that from the toric construction we have $X = \chi^{\epsilon_1}$ and $Y = \chi^{\epsilon_2}$. The isomorphism on all three rings can be described by the correspondences

$$X \sim \frac{x_1}{x_0} \quad \text{and} \quad Y \sim \frac{x_2}{x_0}$$

Clearly this correspondence yeilds an isomorphism when $i = 0$ since it means $k[X,Y] \cong k[\frac{x_1}{x_0}, \frac{x_2}{x_0}]$ as desired. For $i = 1$ we must realize that

$$k[\sigma_1] = k[X^{-1}, X^{-n}Y] \cong k \left[ \left( \frac{x_1}{x_0} \right)^{-1}, \left( \frac{x_1}{x_0} \right)^{-n} \cdot \left( \frac{x_2}{x_0} \right) \right] = k \left[ \frac{x_0}{x_1}, \frac{x_2}{x_1} \right] = k[U_1]$$

All that remains is to check this in the case where $i = 2$. To do this we will demonstrate the correspondence on the respective generators of $k[U_2]$
and $k[\sigma_2]$.

$$z_j = X_j Y^{-1} \sim \left( \frac{x_1}{x_0} \right)^j \left( \frac{x_2}{x_0} \right)^{-1} = \frac{x_1 x_0^{n-j}}{x_2} = z_j$$

\[\square\]

### 2.3 The Singular Point of $\mathbb{P}(1,1,n)$

Now that we are able to go back and forth between the two constructions seamlessly, there is enough machinery to figure out exactly where $\mathbb{P}(1,1,n)$ is singular.

**Theorem 2.3.1**

*Using the homogeneous coordinates $[x_0, x_1, x_2]$, the surface $\mathbb{P}(1,1,n)$ has an isolated singular point at $[0,0,1]$.***

**Proof**

Recall that the edges of $\sigma_0$ and $\sigma_1$ respectively form a $\mathbb{Z}$-basis for $\mathbb{N}$ since

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ 1 & -n \end{vmatrix} = 1.$$  

Thus, by Theorem 2.0.1 we know the regions $U_0$ and $U_1$ are non-singular. Thus, any singular points must sit solely on $U_2$.

In order to find the singular point we will recall that $U_2 \subseteq \mathbb{A}^{n+1}$. We will use the $(0,1)$-based equations obtained earlier. Namely, for $2 \leq j \leq n$, 

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the polynomials \( f_j = z_j z_{j-1}^0 - z_j^j \) vanish on \( U_2 \). The corresponding Jacobian matrix looks like

\[
\begin{pmatrix}
  z_2 & -2z_1 & z_0 & \ldots & 0 & \ldots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
  z_j(j-1)z_{j-2}^0 & -jz_{j-1}^1 & 0 & \ldots & z_{0}^{j-1} & \ldots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
  z_n(n-1)z_{n-2}^0 & -nz_{n-1}^1 & 0 & \ldots & 0 & \ldots & z_{0}^{n-1}
\end{pmatrix}
\]

Clearly any point with \( z_0 \neq 0 \) will be non-singular. Thus the only potential singular points must have \( z_0 = 0 \). Furthermore, from the \((0, n)\)-based equations, \( z_j^n = z_{0}^{n-j} z_{n}^j \), we know that if \( z_0 = 0 \) then \( z_j = 0 \) for \( 1 \leq j \leq n - 1 \).

It turns out, however, that not all such points will be singular. In fact, consider points of the form \( P = (0, 0, \ldots, 0, z_n) \) where \( z_n \neq 0 \). We know from our construction of the isomorphism between the classical and toric constructions that this means \( z_n = \frac{x_n}{x_2} \neq 0 \). This can only occur if \( x_1 \neq 0 \).

But this means that \( P \in U_1 \), which is smooth everywhere. Subsequently, the only singular point can be \( Q = (0, \ldots, 0) \in \mathbb{A}^{n+1} \).

In order to write \( Q \) in terms of the homogeneous coordinates, note that \( 0 = z_0 = \frac{x_0^n}{x_2} \) gives \( x_0 = 0 \). Similarly \( 0 = z_n = \frac{x_n^n}{x_2} \), so that \( x_1 = 0 \). Since \( Q \in U_2 \), we know that \( x_2 \neq 0 \). Subsequently, the singular point must be \( Q = [0, 0, x_2] \sim [0, 0, 1] \), and the theorem is proven \( \square \)

It is important to note that the associations \( z_0 = \frac{x_0^n}{x_2} \) and \( z_n = \frac{x_n^n}{x_2} \) are actually enough to show that the point \([0,0,1]\) is the only part of \( \mathbb{P}(1, 1, n) \)
that is not covered by $U_0$ and $U_1$. Since both of these regions are smooth the singularity of the point is demonstrated by the fact that $U_2$ must be singular because

$$
\begin{vmatrix}
-1 & 1 \\
-n & 0
\end{vmatrix} = n \neq 1.
$$
3 Hurwitz’s Theorem

3.1 A Map To \( \mathbb{P}^1 \)

Now that we know there is an isolated singularity on \( \mathbb{P}(1, 1, n) \) it should make sense to consider curves that avoid this singular point. There are several reasons for this. One of the more obvious ones is that any singularities the curve may have will be the sole responsibility of the curve itself, making it so one doesn’t have to worry about untangling curve singularities from surface singularities. Another bonus is that such a curve admits a very natural map to the projective line. It is this property that will be taken advantage of in this section.

Theorem 3.1.1

If one removes the point \([0, 0, 1]\) from \( \mathbb{P}(1, 1, n) \), then the resulting space has a well defined map, \( \phi \), to \( \mathbb{P}^1 \) given by \([x_0, x_1, x_2] \mapsto [x_0, x_1] \).

Proof

Let \( P = [x_0, x_1, x_2] \) be a point on \( \mathbb{P}(1, 1, n) \) other than \([0, 0, 1]\). This means that at least one of \( x_0 \) or \( x_1 \) is non-zero. Subsequently \([x_0, x_1] \) will be a point on \( \mathbb{P}^1 \) (the point \([0, 0] \) does not make sense on projective space).

The next thing to check is that this map respects the equivalence relation \([x_0, x_1, x_2] \sim [\lambda x_0, \lambda x_1, \lambda^n x_2] \) for any \( \lambda \in k^* \). In particular \([\lambda x_0, \lambda x_1, \lambda^n x_2] \) gets sent to the point \([\lambda x_0, \lambda x_1] \) which is indeed equivalent to the point.
[\[x_0, x_1\]. Thus, the map \( \phi \) is well defined.

At this point we will limit our discussion to a specific collection of curves. The first thing that should be noted is that the degree of a polynomial defining a curve on \( \mathbb{P}(1,1,n) \) that does not contain the singular point, \([0,0,1]\), must be a multiple of \( n \), say \( an \). This is because such a polynomial must have a term with no \( x_0 \)'s or \( x_1 \)'s. In other words \( x_2^a \) must be a term of the homogeneous polynomial defining the curve. Having said this, we will begin our discussion of a special class of curves called weighted Fermat curves. These are the natural extension to a weighted projective plane of the infamous Fermat curve on \( \mathbb{P}^2 \) defined by the polynomial from his last theorem:

\[ x_0^a + x_1^a - x_2^a. \]

**Definition 3.1.2**

A weighted Fermat curve (or WFC) on \( \mathbb{P}(1,1,n) \) is the curve defined by the "homogeneous" polynomial

\[ F = x_0^{an} + x_1^{an} - x_2^{an} \]

The reader should note that the notion of a WFC will be reintroduced when we generalize to \( \mathbb{P}(1,m,n) \). This is because the difference in grading the rings will make a slight modification necessary in order to preserve the homogeneity of the polynomial.
Proposition 3.1.3

A WFC on $\mathbb{P}(1, 1, n)$ is smooth.

Proof

To check this first note that $F(0, 0, 1) = 1$, so that the curve does indeed miss the surface’s singular point. Not only that, but this means the curve is contained in $U_0 \cap U_1$, so that smoothness need only be checked on these two affine planes.

On $U_0$, localize $F$ at $x_0$ and use the local coordinates $X_0 = \frac{x_1}{x_0}$ and $Y_0 = \frac{x_2}{x_0}$ to obtain $F_0 = 1 + X_0^a - Y_0^a$. Since its differential is $dF_0 = anX_0^{a-1}dX_0 - aY_0^{a-1}dY_0$ the Jacobian criterion shows the only possible singular point of $C$ satisfies $X_0 = Y_0 = 0$. However, $F_0(0, 0) = 1$, so this point isn’t even on the curve. Subsequently, a WFC is smooth on $U_0$.

To handle $U_1$ use the same process. Localize $F$ at $x_1$ and use local coordinates $X_1 = \frac{x_0}{x_1}$ and $Y_1 = \frac{x_2}{x_1}$ to obtain $F_1 = X_1^a + 1 - Y_1^a$. The differential is $dF_1 = anX_1^{a-1}dX_1 - aY_1^{a-1}dY_1$. The only place this vanishes is the point given by $X_1 = Y_1 = 0$, which is not on $C$, so that the curve is smooth on $U_1$ as well.

Another happy consequence of the fact that $[0, 0, 1] \notin C$ is that the projection to $\mathbb{P}^1$, $\phi$, given in Theorem 3.1.1 is well defined when restricted to the curve. Since we now have a morphism of curves, one of which is extremely
well known, Hurwitz’s Theorem will become a very useful tool. This result
gives a relationship between the geni of two curves when there is a morphism
from one to the other. Before we can get to the statement, however, we will
need to provide a little background information on ramification of a map of
curves.

3.2 The Ramification of $\phi|_C$

**Definition 3.2.1 : Ramification**

Let $f : X \to Y$ be a finite morphism of smooth curves where $P \in X$ and
$f(P) = Q$. Consider the local parameter at $Q$, namely $t \in \mathcal{O}_Q$. Since $f$
gives a homomorphism of rings, $f^* : \mathcal{O}_Q \to \mathcal{O}_P$ we can define the ramification
index of $f$ at the ideal $P$ as

$$e_P = v_P(f^*(t))$$

where $v_P$ is the usual valuation in the local valuation ring, $\mathcal{O}_P$.

If $e_P = 1$ we say that $P$ is unramified. If $e_P > 1$ then $P$ is called a
ramification point. The picture to think of here is that the map $f : X \to Y$
depicts $X$ as a branched covering of $Y$. The branch points of this covering
are then the images of the ramification points. Since $f$ is a finite morphism,
the fibers of the map will be discrete sets. The ramification index, $e_P$, can
be thought of as measuring how much the map deviates from being a simple
$n$-sheeted covering at $P$.  

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One useful method of keeping track of all the ramifications of \( f \) at one time is use of the ramification divisor. While this divisor is usually defined in terms of the sheaf \( \Omega_{X/Y} \), there is a simpler approach available to us in characteristic zero. This is because all of our ramifications will be tame (i.e. \( e_P \) is never divisible by \( \text{char}(k) \) ). The support of this divisor will consist exactly of the ramification points of \( f \).

**Definition 3.2.2 : Ramification Divisor**

The ramification divisor associated to a map of curves \( f : X \to Y \) is the divisor on \( X \)

\[
R = \sum_{P \in X} (e_P - 1) \cdot P
\]

Now that the ramification divisor has been defined, we may state Hurwitz’s Theorem. This will allow us to relate the geni of two curves given a morphism from one to the other, in terms of the ramification divisor.

**Theorem 3.2.3 : ([5] p. 301)**

Let \( f : X \to Y \) be a finite separable morphism of curves and let \( d = \deg(f) \).

Then the geni of \( X \) and \( Y \) are related by

\[
2g(X) - 2 = d(2g(Y) - 2) + \deg(R)
\]

where \( R \) is the ramification divisor associated to \( f \).
Our use of this result arises by letting $f$ be the map $\phi : C \to \mathbb{P}^1$ that sends $[x_0, x_1, x_2]$ to $[x_0, x_1]$ where $C$ is a WFC. The key steps here will be to compute both the degree of $\phi$, and its ramification divisor on $C$.

**Lemma 3.2.4**

Let $C$ be the WFC on $\mathbb{P}(1, 1, n)$ of degree $an$. Then the degree of the map $\phi : C \to \mathbb{P}^1$ given above is $\text{deg}(\phi) = a$.

**Proof**

While the degree of a map $X \to Y$ is usually defined as the degree of the extension of function fields, $[K(X), K(Y)]$, we will consider a more geometric characterization. In particular, the degree is the number of points in a fiber of a non-branched point of the morphism (i.e. number of sheets in the branched covering).

With this in mind, consider the fiber over a point $[x_0, x_1]$ on $\mathbb{P}^1$. Since $C$ is defined by $x_0^{an} + x_1^{an} = x_2^n$ (and $k$ is algebraically closed) we see that, as long as $x_0^{an} + x_1^{an} \neq 0$ there will be a distinct possible values for $x_2$. Subsequently, each fiber will consist of a different points, giving the degree of $\phi$ to be $a$. $\square$

It should be noted here that the points on $\mathbb{P}^1$ that satisfy $x_0^{an} + x_1^{an} = 0$ are actually the branch points of $\phi$. This leads us part of the way to determining the ramification divisor of $\phi$, since the fibers over these points will also satisfy $x_2 = 0$. This means that the ramification points of $\phi$ are in one-to-one
correspondence with the branch points. Using this we can explicitly compute
the ramification divisor.

**Lemma 3.2.5**

Let $\phi$ and $C$ be as in Lemma 3.2.4. Also let $\{\alpha_j\}$ denote the distinct $an^{th}$
roots of $-1$ for $1 \leq j \leq an$. Then the ramification divisor associated to $\phi$ is
given by

$$R = \sum_{j=1}^{an} (a - 1) \cdot [1, \alpha_j, 0]$$

**Proof**

The first step is to determine the branch points $[x_0, x_1]$ of $\phi$ on $\mathbb{P}^1$, since we
know that for each one of these $[x_0, x_1, 0]$ will be the only ramification point
that maps to it. These branch points occur exactly when $x_0^{an} + x_1^{an} = 0$.
Note that there can be no solutions that have $x_0 = 0$ since this would mean
$x_1 = 0$ as well, which cannot happen on $\mathbb{P}^1$. Thus, we can work on the open
neighborhood of $\mathbb{P}^1$ where $x_0 \neq 0$.

Upon localizing to this neighborhood we find that branch points are
determined by $1 + \left(\frac{x_1}{x_0}\right)^{an} = 0$. Subsequently, branch points occur at all possible
occasions where $\frac{x_1}{x_0}$ is an $an^{th}$ root of $-1$. We will denote these $an$ distinct
roots by $\alpha_j$ where $1 \leq j \leq an$. Since $x_0 \neq 0$ we see that for any branch point
we have $[x_0, x_1] = [x_0, \alpha_jx_0] \sim [1, \alpha_j]$. Consequently there are $an$ distinct
ramification points on $C$, each one of the form $[1, \alpha_j, 0]$. 

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The final step will be to compute the ramification index at each point. Since all the ramification occurs where \( x_0 \neq 0 \) we can work on the corresponding affine regions of both \( \mathbb{P}(1, 1, n) \) and \( \mathbb{P}^1 \). Locally, the ramification points, \( P_j \), are the ideals \( \langle \frac{x_1}{x_0} - \alpha_j, \frac{x_2}{x_0} \rangle \) of the ring \( k \left[ \frac{x_1}{x_0}, \frac{x_2}{x_0} \right] / I \) where \( I = \langle 1 + \left( \frac{x_1}{x_0} \right)^a - \left( \frac{x_2}{x_0} \right)^a \rangle \). Similarly, the branch points, \( Q_j = \phi(P_j) \), are the ideals \( \langle \frac{n_1}{x_0} - \alpha_j \rangle \) of the ring \( k \left[ \frac{x_1}{x_0} \right] \).

Since our WFC is a smooth curve we know that the point \( P_j \) must be a principal ideal, so the generator must be either \( \frac{x_1}{x_0} - \alpha_j \) or \( \frac{x_2}{x_0} \). But the ideal \( I \) must vanish on \( C \), so that

\[
\left( \frac{x_2}{x_0} \right)^a = \left( \frac{x_1}{x_0} \right)^{an} + 1 = \left( \frac{x_1}{x_0} \right)^{an} - (\alpha_j)^a = \left( \frac{x_1}{x_0} - \alpha_j \right) p \left( \frac{x_1}{x_0} \right)
\]

where \( p(\alpha_j) \neq 0 \). Since \( p(\alpha_j) \neq 0 \), the polynomial \( p \) must be a unit in \( O_{P_j} \).

Subsequently, the local parameter \( \frac{x_1}{x_0} - \alpha \) is \( \left( \frac{x_2}{x_0} \right)^a \) times a unit. This demonstrates two things. The first is that the ideal \( P_j \) is actually \( \langle \frac{x_2}{x_0} \rangle \). The second is that \( v_{P_j} \left( \frac{n_1}{x_0} - \alpha_j \right) = a \). Since this is \( e_{P_j} \), the lemma is proven. \( \square \)

Now that both the degree and ramification divisor of \( \phi \) have been determined, we can finally compute the genus of a weighted Fermat curve.

**Theorem 3.2.6**

*If \( C \) is a weighted Fermat curve on \( \mathbb{P}(1, 1, n) \) whose degree is an, then its*
The genus is

\[ g(C) = \frac{(a - 1)(an - 2)}{2} \]

**Proof**

By the previous lemma, we know that \( \deg(R) = \sum_{j=1}^{an} (a - 1) = an(a - 1) \).

Recall that the genus of a projective line is 0. Since we have also shown that \( \deg(\phi) = a \), we know from Hurwitz’s Theorem that

\[ 2g(C) - 2 = a(2 \cdot 0 - 2) + an(a - 1) \]

and a little elementary algebra completes the proof. \( \square \)
4 Desingularizations and $F_n$

Earlier we saw that $\mathbb{P}(1, 1, n)$ has an isolated singularity at $[0, 0, 1]$. This raises the question as to what the minimal desingularization of this surface is. The resulting space is a well known surface called a Hirzebruch surface, denoted $F_n$. We will see that this surface can be obtained by a single blow-up of the singular point.

4.1 Constructing $F_n$

A well known construction of $F_n$ is as the $\mathbb{P}^1$ bundle over $\mathbb{P}^1$ associated to the rank 2 vector bundle $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)$. In particular, this construction has a canonical map from $\pi : F_n \rightarrow \mathbb{P}^1$. This observation is actually a demonstration of a second use for blow-ups that is not as popularized as are their uses in desingularization. Namely, that a blow-up can be used to extend a rational map of varieties into a morphism. In particular, the bundle map $\pi$ and the surface $F_n$ are a natural way of extending the rational map $\phi : \mathbb{P}(1, 1, n) \rightarrow \mathbb{P}^1$ from the last section to a smooth morphism.

Another advantage of working on $F_n$ is that many useful theorems about curves (and divisors) on surfaces enter the fray if the surface is smooth. In particular, the adjunction formula and some techniques in the intersection theory of toric varieties become tools available for our manipulation. In particular, these will allow us to extend the main result of the last section
to include all smooth curves on $\mathbb{P}(1, 1, n)$ of degree $an$ that do not contain $[0, 0, 1]$.

**Definition 4.1.1 : Hirzebruch Surface**

The $n^{th}$ Hirzebruch surface, $\mathbb{F}_n$, is the complete toric surface described by

$$\Delta(1) = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -n \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}$$

where the lattice is $\mathbb{Z}^2$ as usual.

This is the same as the set of vertices used to construct $\mathbb{P}(1, 1, n)$ but with the vector $-e_2$ thrown into the mix. This divides the plane into four cones, $\tau_i$. It is a simple thing to check that the determinant of the edges of each cone is 1 so that each affine surface will be a smooth copy of $\mathbb{A}^2$. The resulting toric variety, $\mathbb{F}_n \cong X(\Delta)$ will also be smooth.

The technique for computing the affine rings for each cone progresses in the same manner as before. On a given cone, $\tau_i$, find a $u \in M = Hom(\mathbb{N}, \mathbb{Z})$ that vanishes on one edge of the cone and is positive on the other. Since the edges of $\tau_i$ form a $\mathbb{Z}$-basis for $\mathbb{N}$, the resulting pair of $u$’s will be sufficient.
to generate $S_{\tau_i}$. Subsequently, the generators of $k[\tau_i]$ will then be $\chi^u$. A quick summary of the edges of $\tau_i$ and the resulting $k$-algebras is given here.

As before we take $e_1^\vee, e_2^\vee$ as dual to the standard basis for $\mathbb{N}$ while letting $X = \chi^{e_1^\vee}$ and $Y = \chi^{e_2^\vee}$.

<table>
<thead>
<tr>
<th>$i$</th>
<th>Edge 1</th>
<th>Edge 2</th>
<th>$k[\tau_i]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$e_2^\vee$</td>
<td>$e_1^\vee$</td>
<td>$k[Y, X]$</td>
</tr>
<tr>
<td>1</td>
<td>$-e_1^\vee$</td>
<td>$-n e_1^\vee + e_2^\vee$</td>
<td>$k[X^{-1}, X^{-n}Y]$</td>
</tr>
<tr>
<td>2</td>
<td>$ne_1^\vee - e_2^\vee$</td>
<td>$-e_1^\vee$</td>
<td>$k[X^nY^{-1}, X^{-1}]$</td>
</tr>
<tr>
<td>3</td>
<td>$e_1^\vee$</td>
<td>$-e_2^\vee$</td>
<td>$k[X, Y^{-1}]$</td>
</tr>
</tbody>
</table>

It is important to note here that as you move from cone to cone in a counterclockwise manner Edge 2 of one $\tau_i^\vee$ is the negative of Edge 1 of the next dual cone. This is because they are the $u$’s that vanish on the common edge of the cones in question. The opposite signs are there because one $u$ must be positive on one side of that ray, while the other $u$ must be positive on the opposite side of it.

This phenomenon of related edges for adjacent cones occurs in any discussion of toric surfaces and it will be used quite enthusiastically in our later analysis of $\mathbb{P}(1, m, n)$. The major detail to keep in mind is that this technique is really dealing with the generators of $\tau_i^\vee$, and not necessarily the generators of $S_{\tau_i}$ (the former is generated using $\mathbb{R}_+$ while the other is generated using $\mathbb{Z}_+$). It works nicely here because the surfaces are non-singular and the
generators of $\tau_i^\vee$ are the same as the generators of $S_{\tau_i}$. In the singular case one must still address the lattice points of $\mathbb{N}^*$ contained in the parallelogram formed by the edges of $\tau_i^\vee$.

As a matter of convenience, when using local coordinates we will use the convention that in a cone, $\tau_i$, we will let $x_i$ be the ring generator that corresponds to Edge 1 of $\tau_i^\vee$ and $y_i$ be the other ring generator. For example $x_0 = Y$ and $y_1 = X^{-n}Y$. In this notation, the above observation about adjacent cones may be more simply stated as $x_{i+1} = y_i^{-1}$ (where the addition in the subscript is considered to be in $\mathbb{Z}/\langle 4 \rangle$).

One of the main conveniences of this notation is that in trying to write down the coordinate changes from one affine surface to another, some of the work is done for you already. The remaining work is a matter of writing one of the variables as a product of powers (possibly negative) of the variables for another region. However, since each pair of variables for a region come from a $\mathbb{Z}$-basis for $\mathbb{N}^*$ this is the same thing as writing a vector in terms of this basis.

As an example we will write down coordinate changes between $U_{\tau_1}$ and $U_{\tau_3}$. For the variables $x_1 = X^{-1}$ and $y_1 = X^{-n}Y$ we consider the vectors $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -n \\ 1 \end{bmatrix}$ respectively. Similarly the variables $x_3 = X$ and $y_3 = Y^{-1}$ correspond to the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$. Solving the vector equation
\[
\begin{bmatrix}
0 \\
-1
\end{bmatrix}
= a \begin{bmatrix}
-1 \\
0
\end{bmatrix} + b \begin{bmatrix}
-n \\
1
\end{bmatrix}
\] yields a solution of \( a = n, b = -1 \). The corresponding coordinate change is \( y_3 = x_1^n y_1^{-1} \). The remaining coordinate change happens just as easily, and is seen to be \( x_3 = x_1^{-1} \). The remaining coordinate changes are summarized here.

Table 4.1.2 - Coordinate Changes on \( \mathbb{F}_n \)

<table>
<thead>
<tr>
<th>( \tau_0 )</th>
<th>( \tau_1 )</th>
<th>( \tau_2 )</th>
<th>( \tau_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_0 = -x )</td>
<td>( x_1^n y_1 )</td>
<td>( x_2^1 y_2^{-n} )</td>
<td>( y_3 )</td>
</tr>
<tr>
<td>( y_0 = -y )</td>
<td>( x_1^{-1} )</td>
<td>( y_2^{-1} )</td>
<td>( x_3 )</td>
</tr>
<tr>
<td>( \tau_1 )</td>
<td>( x_1 = y_0^{-1} )</td>
<td>( y_2 )</td>
<td>( x_2^{-1} )</td>
</tr>
<tr>
<td>( y_1 = x_0 y_0 y_1 )</td>
<td>( x_1^{-1} )</td>
<td>( x_2^{-1} )</td>
<td>( x_3^{-1} )</td>
</tr>
<tr>
<td>( \tau_2 )</td>
<td>( x_2 = x_0 y_0 y_1 )</td>
<td>( y_1^{-1} )</td>
<td>( y_2^{-1} )</td>
</tr>
<tr>
<td>( y_2 = y_0^{-1} )</td>
<td>( x_1 )</td>
<td>( y_2^{-1} )</td>
<td>( x_3 )</td>
</tr>
<tr>
<td>( \tau_3 )</td>
<td>( x_3 = y_0 )</td>
<td>( x_1^{-1} )</td>
<td>( y_2^{-1} )</td>
</tr>
<tr>
<td>( y_3 = x_0^{-1} )</td>
<td>( x_1^n y_1^{-1} )</td>
<td>( x_2 y_2^n )</td>
<td>( - )</td>
</tr>
</tbody>
</table>

One important thing to note is that some of the coordinate changes involve negative exponents. Subsequently, the corresponding maps of affine surfaces will be undefined at certain points (actually, it will be on coordinate axes when it does occur). These points are merely those points on \( U_{\tau_i} \) that are not on \( U_{\tau_j} \).

4.2 Maps of Toric Varieties

Now that we have all of our local rings and coordinate changes for \( \mathbb{F}_n \), we need to take a few moments to regard morphisms of toric varieties \( X \to Y \). In particular, \( T \)-equivariant maps that respect the action of their respective tori. Such maps arise from a map of fans underlying the two toric varieties.
A map of fans is a homomorphism of lattices with the additional property that for any cone in the fan for $X$, its image must be contained in a single cone in the fan for $Y$. This criterion ensures that these maps will be nicely defined on each affine neighborhood of the standard covering.

The method for deriving the morphism of toric varieties from a given map of fans follows much of the method used to construct the varieties themselves. Say there is a map of lattices $\psi : N_1 \to N_2$ that sends the cone $\tau$ to the cone $\sigma$. The first step is to find the corresponding map on the dual cones $\sigma^\vee \to \tau^\vee$. After restricting this map to the semi-group, $S_\sigma$, one can obtain a homomorphism of rings (usually an inclusion) $k[\sigma] \to k[\tau]$. Finally, associated to this homomorphism is a morphism on their spectra, $U_\tau \to U_\sigma$.

**Example 4.2.1 :** $\psi : \mathbb{F}_n \to \mathbb{P}(1, 1, n)$

As an example, consider the map $\psi : \mathbb{F}_n \to \mathbb{P}(1, 1, n)$ that is induced by the identity map on the lattice $\mathbb{Z}^2$. The first thing to note is that $\tau_0 = \sigma_0$ and $\tau_1 = \sigma_1$. Since we wind up getting the same rings for these cones, we see that $\psi$ will be an isomorphism of rings, and subsequently an isomorphism on the spaces $U_{\tau_0}$ and $U_{\tau_1}$.

The main difference lies in the inclusion of the cones $\tau_2$ and $\tau_3$ into the cone $\sigma_2$. The first thing to note is that the map of the dual cones will be the identity map. Subsequently, the maps on rings will be determined by mapping the generators $X, Y$ to themselves.
Recall that $k[\tau_2] \cong k[X^nY^{-1}, X^{-1}]$ and $k[\sigma_2] \cong k[Y^{-1}, XY^{-1}, \ldots, X^nY^{-1}]$. The first thing to note is that $X^jY^{-1} = (X^nY^{-1})(X^{-1})^{n-j}$. Subsequently, we see that $k[\sigma_2] \subset k[\tau_2]$, and our homomorphism is the corresponding inclusion of rings. Furthermore, writing this relationship in terms of the local coordinates for $\tau_2$, $(x_2, y_2)$, and $\sigma_2$, $(z_0, \ldots, z_n)$ we can deduce the morphism of affine schemes $U_{\tau_2} \to U_{\sigma_2}$. Namely, because $x_2 = X^nY^{-1}$, $y_2 = X^{-1}$, and $z_j = X^jY^{-1}$ the previous computation indicates that $z_j = x_2y_2^{n-j}$. This gives our map of schemes as desired.

In a similar fashion, the map $U_{\tau_3} \to U_{\sigma_2}$ may be deduced. This comes from the relationship $z_j = X^jY^{-1} = (X)^j(Y^{-1}) = x_3^jy_3$. As noted before, the maps on $U_{\tau_0}$ and $U_{\tau_1}$ are induced by the identity homomorphism on the rings $k[Y, X]$ and $k[X^{-1}, X^{-n}Y]$ respectively. Writing things a little more explicitly shows the maps to $U_{\sigma_2}$:

$$(x_2, y_2) \mapsto (x_2y_2^n, x_2y_2^{n-1}, \ldots, x_2y_2, x_2)$$

$$(x_3, y_3) \mapsto (y_3, x_3y_3, \ldots, x_3^{n-1}y_3, x_3^ny_3)$$

Now that the map has been written down locally, it is an easy task determine the fibers over any point on the map. First, since we have an isomorphism when $i = 0, 1$ we know the fibers on these regions will consist of a single point. Fibers of points on $U_2$, however, are a little trickier. Let $P = (z_0, \ldots, z_n)$ be a point on $U_2$. Upon looking at the maps we reduce to 4
cases. One where no $z_j$ is zero, points where all $z_j$ are zero except for one of $z_0$ or $z_n$, and the origin.

The first case is simple, for a fiber on $\tau_2$ recall that $z_n = x_2 y_2^n - y = x_2$ and $z_{n-1} = x_2 y_2$. Thus $\psi^{-1}(P)$ is the point $\left(z_n, \frac{z_n}{z_{n-1}}\right)$. Similarly on $\tau_3$ we can find $\psi^{-1}(P)$ to be $\left(\frac{z_n}{z_{n-1}}, \frac{z_n}{z_{n-1}}\right)$. Note that we could also have used the point $(\frac{z_n}{z_0}, z_0)$, but the first choice makes it easier to verify that $\psi^{-1}(P)$ gives the same point under a change of coordinates between $\tau_2$ and $\tau_3$. This shows that $\psi$ is one-to-one on the 2-torus defined by $x_2 \cdot y_2 \neq 0$ (which is the same as the one defined by $x_3 \cdot y_3 \neq 0$). In fact, this is the torus that gives the group action on the toric variety.

The next case is $P = (z_0, 0, \ldots, 0)$ where $z_0 \neq 0$. First, suppose that $P \in \psi(U_{\tau_2})$. Then $0 = z_n = x_2$. But then $z_0 = x_2 y_2^n = 0 \cdot y_2^n = 0$, which is a contradiction of $z_0 \neq 0$. Consequently, $P \not\in \psi(U_{\tau_2})$. For $\tau_3$, however, we know that $y_3 = z_0 \neq 0$ and then $x_3 = \frac{z_3}{z_0} = \frac{0}{z_0} = 0$. Thus, the pre-image of $P$ is the point $(0, z_0)$. $\psi$ is an isomorphism of the two 1-tori $(0, y_3)$ and $(z_0, 0, \ldots, 0)$.

The next case is similar, in that we are addressing the 1-torus $(0, \ldots, 0, z_n)$ with $z_n \neq 0$. Regarding $\tau_3$ we see that $y_3 = z_0 = 0$ gives $z_n = x_3^n y_3 = 0$, so that $P \not\in \psi(U_{\tau_3})$. As far as $\tau_2$ is concerned, we see that $x_2 = z_n \neq 0$ and $y_2 = \frac{z_n}{z_2} = \frac{0}{z_2} = 0$. Thus $\psi^{-1}(P)$ is $(z_n, 0)$, giving us an isomorphism of the 1-tori $(x_2, 0)$ and $(0, \ldots, 0, z_n)$. 

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Lastly, we consider the fiber of \( P = (0, \ldots, 0) \). First, for the fiber over \( \tau_2 \) we see that \( x_2 = z_n = 0 \). However, \( y_2 \) can be any element of the field, \( k \). Thus, \( \psi^{-1}(P) \cap U_{\tau_2} \) is the line \((0, y_2)\). Similarly, on \( \tau_3 \) we have \( y_3 = z_0 = 0 \), while \( x_3 \) is undetermined, giving a fiber of \((x_3, 0)\). However, our change of coordinates between the two neighborhoods involves \( x_3 = y_2^{-1} \). So what we really have here is a pair of lines, related by a coordinate change of \( 1/x \). In otherwords, \( \psi^{-1}(P) \cong \mathbb{P}^1 \). This is what is referred to as a blow-up. It is also important to note that the point being blown-up here is actually \([0, 0, 1]\), the singular point of \( \mathbb{P}(1, 1, n) \).

### 4.3 The Bundle Map \( \pi : \mathbb{F}_n \to \mathbb{P}^1 \)

Another map of toric varieties of great interest is the one that exhibits \( \mathbb{F}_n \) as a \( \mathbb{P}^1 \)-bundle over \( \mathbb{P}^1 \). Recall that \( \mathbb{F}_n \) comes from the lattice \( \mathbb{Z}^2 \) and \( \mathbb{P}^1 \) arises from the lattice \( \mathbb{Z} \). If we then consider the map of lattices \((x, y) \mapsto x\), we will arrive at the bundle map, \( \pi \). This will give a well defined map because the cones \( \tau_0 \) and \( \tau_3 \) are mapped completely inside of the positive \( x \)-axis, \( \sigma_+ \), while the cones \( \tau_1 \) and \( \tau_2 \) are sent inside the cone \( \sigma_- \) (i.e., the negative \( x \)-axis).

The dual maps derive from the inclusion of \( \mathbb{Z}^* \) into \((\mathbb{Z}^2)^* \) that sends \( e_1^\vee \) to itself. Subsequently, the maps of rings will be inclusions sending \( X \) to itself, and \( X^{-1} \) to \( X^{-1} \). Thus, sitting over \( \sigma_+ \) we have the inclusions \( k[X] \hookrightarrow k[Y, X] = k[\tau_0] \) and \( k[X] \hookrightarrow k[X, Y^{-1}] = k[\tau_3] \). Similarly, over \( \sigma_- \) we have the
Inclusions $k[X^{-1}] \hookrightarrow k[X^{-1}, X^{-n}Y] = k[\tau_1]$ and $k[X^{-1}] \hookrightarrow k[X^nY^{-1}, X^{-1}] = k[\tau_2]$.

Our next step will be to show that this map actually exhibits a bundle structure for $\mathbb{P}_n$. To do this, we must verify that the fibers are isomorphic to $\mathbb{P}^1$. Consider a point on $U_{\sigma_+}$ using the local coordinate $X$. For $\tau_0$ we see $X = y_0$ and $x_0$ is undetermined, so that the fiber over $X$ is the line $(x_0, X)$. Similarly, on $\tau_3$, $X = x_3$ while $y_3$ is indeterminate. Thus, our fiber is the line $(X, y_3)$. However, on the overlap of the two regions we note that the point $(x_0, y_0)$ of $U_{\tau_0}$ when written on $U_{\tau_3}$ is $(y_0, x_0^{-1})$, so that $(x_0, X) \sim (X, y_3)$. Thus, the two affine lines glue together via the relationship $x_0 = y_3^{-1}$, and our fiber is a copy of $\mathbb{P}^1$ as desired.

The construction over $\sigma_-$ is similar, and uses the local coordinate $X^{-1}$. The fiber over $\tau_1$ is the line $(X^{-1}, y_1)$ while the fiber over $\tau_2$ is the line $(x_2, X^{-1})$. Moreover the points are related by the coordinate change $x_1 = y_2$, $y_1 = x_2^{-1}$. Thus, we have two lines whose local coordinates $y_1$ and $x_2$ satisfy $x_2 = y_1^{-1}$. The result is a fiber isomorphic to $\mathbb{P}^1$.

4.4 Using Blow-Ups To Extend Rational Maps

Recall that we had a rational map (undefined at $[0,0,1]$) $\phi : \mathbb{P}(1,1,n) \to \mathbb{P}^1$ defined in terms of homogenous coordinates by $[x_0, x_1, x_2] \mapsto [x_0, x_1]$. The goal of this section is to find a space that projects onto both $\mathbb{P}(1,1,n)$ and
$\mathbb{P}^1$ so as to make these maps agree when the former is composed with $\phi$. In other words, we want to find a space on which $\phi$ may be extended to a smooth map.

We actually have already found such a space, and the maps involved. We just haven’t realized it yet. Consider the map of lattices $\mathbb{Z}^2 \to \mathbb{Z}$ from the last section (the one that sends $(x,y)$ to $x$). Note that this map does not yield a map of fans if you are considering the fan for $\mathbb{P}(1,1,n)$. This is because the image of the cone $\sigma_2$ under this map is not contained in any single cone of the fan for $\mathbb{P}^1$ (it takes both $\sigma_+$ and $\sigma_-$ to hold it).

If we were to somehow split up $\sigma_2$ into smaller cones by dropping a vertical line from the origin, we would be able to have the resulting cones contained in single cones for $\mathbb{P}^1$, giving a map to $\mathbb{P}^1$. In order to do this, we need only include the lattice point $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ in $\Delta(1)$ in our fan. This is exactly what we did in constructing $\mathbb{F}_n$.

There are some other important things to put together here. Not only does our space admit a map $\pi$ to $\mathbb{P}^1$, but it also has a projection $\psi$ onto $\mathbb{P}(1,1,n)$ that comes from the identity map on $\mathbb{Z}^2$. Also, we have seen that $\mathbb{F}_n$ is isomorphic to $\mathbb{P}(1,1,n)$ everywhere except $[0,0,1]$ and the copy of $\mathbb{P}^1$ that maps to it. Since this is the point where $\phi$ was undefined, it makes sense to check whether or not $\mathbb{F}_n$ does extend the map $\phi$. In fact, all we have left to check is that the following diagram commutes.
Lemma 4.4.1

Let $\psi : \mathbb{F}_n \to \mathbb{P}(1, 1, n)$ be induced by the identity map on $\mathbb{Z}^2$, and $\pi : \mathbb{F}_n \to \mathbb{P}^1$ be induced by the lattice homomorphism $(x, y) \mapsto x$. Furthermore, let $\phi : \mathbb{P}(1, 1, n) \to \mathbb{P}^1$ send $[x_0, x_1, x_2]$ to $[x_0, x_1]$. Then on the points where all three are defined we have

$$\pi = \phi \circ \psi$$

Proof

There are really two things to keep in mind here. The first is that we can do this all locally, and for each $\tau_i$ pick a point $(x_i, y_i)$ and find where it gets sent both ways. The second is to remember our isomorphism between the classical construction of $\mathbb{P}(1, 1, n)$ and the toric construction. In particular, the crucial correspondence is

$$X \sim \frac{x_1}{x_0} \quad \text{and} \quad Y \sim \frac{x_2}{x_0}$$

From this relationship we see that if we were to write down the map $\psi$ on $U_0$, then we have $[1, \frac{x_1}{x_0}, \frac{x_2}{x_0}] \mapsto [1, \frac{x_1}{x_0}]$. Really, this is the map sending $(X, Y)$ to $X$. This is exactly what $\pi$ looks like on $U_m$. 

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Now we consider the region $U_1$. Here the map sends $[\frac{a}{x_1}, 1, \frac{a}{x_1}]$ to $[\frac{a}{x_1}, 1]$. Using our $X, Y$ description we see this sends $(X^{-1}, X^{-n}Y)$ to $X^{-1}$. This is exactly the description of $\pi$ obtained previously.

As far as checking $U_2$ is concerned, there’s really nothing to check. This is because the only point of $U_2$ not already contained in $U_0 \cap U_1$ is $[0, 0, 1]$, the point where $\phi$ is extended by $F_n, \pi$. 

\textbf{Theorem 4.4.2}

Let $C$ be a curve of degree $an$ on $\mathbb{P}(1, 1, n)$ that does not contain the point $[0, 0, 1]$. Then:
1) $\psi^{-1}(C) \cong C$.
2) Restricting the bundle map $\pi: \mathbb{F}_n \to \mathbb{P}^1$ to $\psi^{-1}(C)$ gives a map of degree $a$.

\textit{Proof}

We have already shown that $\psi$ is an isomorphism between $\mathbb{F}_n$ and $\mathbb{P}(1, 1, n)$ away from the point $[0, 0, 1]$. Since $C$ does not contain this point, the restriction of $\psi$ to the curve $\psi^{-1}(C)$ gives the desired isomorphism of curves.

For the second statement, since $[0, 0, 1] \not\in C$ the polynomial defining $C$ must contain a term having only $x_2$ as a factor. This is because every term with $x_0$ or $x_1$ as a factor will vanish at $[0, 0, 1]$. Since the degree of the polynomial is $an$ and $\deg(x_2) = n$ this term must be a constant multiple of $x_2^a$. 

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The next step is to compute the degree of the map $\phi$ when restricted to $C$. To do this, pick a point on $\mathbb{P}^1$. Plugging the homogeneous coordinates, $[x_0, x_1]$ into the polynomial defining $C$ will then yield a polynomial of degree $a$ in the single variable $x_2$. Since $k$ is algebraically closed with characteristic 0 this will yield at most $a$ distinct values, giving a maximum of $a$ distinct elements to a fiber. In fact, one can always find a point $[x_0, x_1]$ that will give this maximum. The reader should note, that not all points of $\mathbb{P}^1$ will give this maximum (the ramification points of $\phi$ will have fewer fiber elements).

Since there are $a$ distinct elements in a general fiber of $\phi$ we see that it has a degree of $a$. In addition, since the map $\psi : \mathbb{F}_n \to \mathbb{P}(1, 1, n)$ is an isomorphism away from $[0, 0, 1]$ it must have a degree of 1. Since the previous lemma shows us that $\pi = \phi \circ \psi$, we know $\deg(\pi) = \deg(\phi) \cdot \deg(\psi) = a \cdot 1 = a$ and the theorem is proven. \hfill \square

4.5 Intersection Theory on $\mathbb{F}_n$

There are additional benefits to working on Hirzebruch surfaces beyond smoothness and having the map $\psi$ extend to a globally defined function. In particular, the Picard group of these surfaces is particularly well known. Recall that the Picard group is the group of linear equivalence classes of divisors on a space (for surfaces, think formal $\mathbb{Z}$-linear combinations of curves).
Two divisors are of particular interest here. Both have special relationships to the bundle structure of $F_n$.

The first is the zero-section of $\pi$, and is denoted $C_0$. This curve is merely a section of the bundle map, $\pi$. In particular, $C_0$ is isomorphic to $\mathbb{P}^1$. Another way to view $C_0$ is as the exceptional divisor of the blow-up $\psi$.

The other divisor is a general fiber of $\pi$. This divisor is denoted by $f$. One thing we have already demonstrated is that, like $C_0$, the fiber $f$ is isomorphic to $\mathbb{P}^1$.

Given these two divisors, we may now address the Picard group $\text{Pic}(F_n)$. The following result comes from the fact that $F_n$ is a $\mathbb{P}^1$-bundle and can be found in [5] (p. 370).

**Lemma 4.5.1**

$\text{Pic}(F_n)$ is a free abelian group generated by the divisors $C_0$ and $f$. Furthermore, the intersection numbers for divisors on $F_n$ are determined by the intersections of these curves. Namely:

i) $C_0^2 = -n$
ii) $C_0 \cdot f = 1$
iii) $f^2 = 0$

Since the proofs of this result are so readily available, we will omit them here. A little motivation, however, is in order. As a $\mathbb{P}^1$-bundle, we may think of $F_n$ as a ruled surface. The fact that $\text{Pic}(F_n)$ is generated by $C_0$ and $f$ comes from the idea of traversing a given curve a certain number of times in
the direction of each ruling. Also, since \( C_0 \) is a simple section of \( \pi \) and \( f \) is a fiber of the same map, it is clear that the two curves should intersect exactly once. Also, any two fibers are distinct, hence \( f^2 = 0 \).

From this characterization of \( Pic(\mathbb{P}_n) \) we have that any curve, \( C \), on \( \mathbb{P}_n \) is linearly equivalent to \( aC_0 + bf \). Not every such linear system \( |aC_0 + bf| \) admits an irreducible curve, however. Clearly the systems \( |C_0| \) and \( |f| \) contain irreducible curves. For the rest, we consult [3]. The result can also be found in [5] (p. 380).

**Lemma 4.5.2**

*Excluding the cases \( |C_0| \) and \( |f| \), the linear system \( |aC_0 + bf| \) contains an irreducible, non-singular curve if and only if it contains an irreducible curve. This occurs if and only if \( a > 0 \) and \( b \geq an \).*

Before we continue, a little more about the constant \( a \) needs to be said. In particular, since it is the \( C_0 \) coefficient and \( C_0 \) is a section of \( \pi \) we have generally will have \( a \) points on a curve in \( |aC_0 + bf| \) projecting down to a single point of \( \mathbb{P}^1 \). Subsequently, \( a \) is the degree of the map \( \pi \) upon restriction to a curve in \( |aC_0 + bf| \).

**Theorem 4.5.3**

*Let \( C \) be a curve of degree \( an \) on \( \mathbb{P}(1, 1, n) \) that does not contain \( [0, 0, 1] \). Then as a curve on \( \mathbb{P}_n \), \( \psi^{-1}(C) \) is linearly equivalent to the curve \( aC_0 + an f \).*
Proof

First, we know from Theorem 4.4.2 that $\psi^{-1}(C)$ is a curve on $\mathbb{F}_n$ isomorphic to itself $C$. From the second part of the same theorem we know that a polynomial of degree $an$ will make the degree of $\pi|_{\psi^{-1}(C)}$ equal to $a$. This must be the $C_0$ coefficient, so we know that $C \sim aC_0 + bf$ for some $b \geq an$.

To determine $b$ recall that the curve does not contain the point $[0,0,1]$. Thus, on $\mathbb{F}_n$ we see that $C$ cannot intersect the exceptional divisor of $\psi$. Since $C_0$ is that very divisor it is clear that $C.C_0 = 0$. Since $C \sim aC_0 + bf$ combining this with the intersection numbers of $C_0$ and $f$ yields:

$$0 = (aC_0 + bf).C_0 = a(C_0)^2 + b(C_0,f) = a(-n) + b(1)$$

which can be solved for $b = an$. \hfill \square

4.6 Adjunction And The Genus of $C$

Now that the representative of $C$ in $\text{Pic} (\mathbb{F}_n)$ has been determined we can use the intersection theory of curves on a surface to determine the curve’s genus. In particular, we use the adjunction formula. This result may be found in [5] (p. 361) but we restate it here for convenience. Recall that for any variety, $X$, the canonical divisor $K_X$ is the divisor whose corresponding line bundle $\mathcal{L}(K_X)$ is $\omega_X$, the top exterior product of the contangent bundle for $X$.
Lemma 4.6.1: Adjunction Formula

If \( C \) is a non-singular curve of genus \( g \) on a surface \( X \) with canonical divisor \( K_X \). Then the following formula holds:

\[
2g - 2 = C \cdot (C + K_X)
\]

Since we already know the linear equivalence class of \( C \) and how to compute intersection products on \( \mathbb{F}_n \), the only thing keeping us from determining \( g \) is the canonical divisor. In particular, we need to be able to write \( K_X \) in terms of the divisors \( C_0 \) and \( f \). Luckily this is a known result that may be found in [3] (p. 418) or [5] (p. 374).

Lemma 4.6.2

The canonical divisor for a Hirzebruch surface, \( \mathbb{F}_n \), is given by

\[
K_X = -2C_0 + (-2 - n)f
\]

Finally, we may put all of this together and derive a formula for the genus of most smooth curves on \( \mathbb{P}(1,1,n) \).

Theorem 4.6.3

Let \( C \) be a non-singular curve on \( \mathbb{P}(1,1,n) \) that does not contain \([0,0,1]\) and is defined by a polynomial of degree \( an \). Then the genus of \( C \) is

\[
g(C) = \frac{(a - 1)(an - 2)}{2}
\]
Proof

As before, pull $C$ up to $F_n$. The resulting curve is isomorphic to $C$ and is linearly equivalent to $aC_0 + an f$. Since $F_n$ is smooth, we can apply the adjunction formula to obtain

$$2g - 2 = (aC_0 + an f)[(a - 2)C_0 + (an - 2 - n)f]$$

which can be manipulated on the right hand side. Namely, one obtains

$$a(a - 2)(C_0)^2 + [an(a - 2) + a(an - 2 - n)](C_0.f) + an(an - 2 - n)(f)^2.$$ 

But we know that $(C_0)^2 = -n$, $C_0.f = 1$ and $(f)^2 = 0$. Plug these values into the equation, multiply everything out and collect terms to obtain

$$2g - 2 = a^2n - an - 2a.$$ 

From here some basic algebra will complete the proof. 

This completes our analysis of curves on $F(1,1,n)$ and its desingularization, $F_n$. The next step will be address a second non-trivial weight, $m$. Many of the techniques used for the one-weight problem will generalize to this case, albeit in a more complicated manner.
Part II: $\mathbb{P}(1, m, n)$

5 Some Basics on $\mathbb{P}(1, m, n)$

As we did with $\mathbb{P}(1, 1, n)$ we will construct $\mathbb{P}(1, m, n)$ using both the classical construction and the toric construction. The next step is to establish the isomorphism between the two. Once these descriptions of the affine rings have been reconciled we will determine the singular points of the surface.

5.1 The Toric Construction

Recall that $\mathbb{P}(1, m, n)$ is the complete toric variety given by $\mathbb{N} = \mathbb{Z}^2$ and

$$\Delta(1) = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -m \\ -n \end{bmatrix} \right\}.$$  

As with the one weight case, we label the cones $\sigma_i$ with $0 \leq i \leq 2$ starting with $\sigma_0$ as the first quadrant and then moving counter-clockwise around the origin.

\[
\sigma_1 \quad \sigma_0 \quad \sigma_2
\]

Figure 5.1.1 - A Fan For $\mathbb{P}(1, m, n)$
Recall from Section 1 that the construction of $X(\Delta)$ depends on $\sigma \otimes \mathbb{R}$, which allowed us to reduce to the case where $\gcd(m, n) = 1$ and $n > m$. Now let $k = \lfloor \frac{n}{m} \rfloor$ be the floor (or integer part) of $\frac{n}{m}$ and let $r$ be the residue of $n \pmod{m}$ so that $n = mk + r$. It is important to note that $\gcd(m, r) = 1$ as well. This is because if $d|m$ and $d|r$, then $d|(mk+r) = n$ so that $\gcd(m, n) = 1$ yields $d = 1$.

The first question that arises is the issue of singular cones. To find the cones that give singular affine surfaces we must consider the determinants of the edges.

$$
\begin{vmatrix}
1 & 0 \\
0 & 1 \\
\end{vmatrix} = 1, \quad \begin{vmatrix}
0 & -m \\
1 & -n \\
\end{vmatrix} = m, \quad \begin{vmatrix}
-m & 1 \\
-n & 0 \\
\end{vmatrix} = n
$$

Subsequently, we will have singular points not only on $U_{\sigma_2}$, but on $U_{\sigma_1}$ as well.

In order to determine what these singular points are we will want to find the generators of the rings $k[\sigma_1]$ and $k[\sigma_2]$. In order to find these generators we must first find the edges of $\sigma_1^\vee$. From there we can find the generators of $S_{\sigma_1}$. We will proceed with $\sigma_1$ first, then move on to $\sigma_2$.

First, let $u = ae_1^\vee + be_2^\vee$, so that $u \begin{bmatrix} 0 \\ 1 \end{bmatrix} = b$ and $u \begin{bmatrix} -m \\ -n \end{bmatrix} = -am - bn$. One edge will be determined by $b = 0$ and $-am - bn = -am > 0$.

Subsequently, $a < 0$ gives $a = -1$, which is $u = -e_1^\vee$. The second edge has $-am - bn = 0$, or $a = \frac{-bn}{m}$, as well as $b > 0$. Since $b = 1$ would give a fractional $a$ we take $b = m$ and $a = -n$, making the second edge of $\sigma_1^\vee$.

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\[ u = -ne_1^\vee + me_2^\vee. \]

The next step is to find the generators of \( S_{\sigma_1} \). Since these are the lattice points in the parallelogram generated by the two vectors we just found, we consider rational linear combinations \( s(-e_1^\vee) + t(-ne_1^\vee + me_2^\vee) \) where \( 0 \leq s, t \leq 1 \), but \( s \) and \( t \) are not simultaneously 0 or 1. Grouping the terms shows this is \(- (s + tn)e_1^\vee + (tm)e_2^\vee\). In order to get an integer for the \( e_2^\vee \) coefficient we can have \( t = \frac{j}{m} \) for \( 0 \leq j \leq m \). This means that the \( e_1^\vee \) coefficient looks like \(- (s + \frac{nj}{m})\). Since \( 0 \leq s \leq 1 \) the only possible integer this could be is \(-\lceil\frac{nj}{m}\rceil\) (here we are using the unusual convention that \( \lceil 0 \rceil = 1 \) because \( s \) and \( t \) can’t vanish simultaneously). This means that \( S_{\sigma_1} \) is generated by the lattice points \( u_j = -\lceil\frac{nj}{m}\rceil e_1^\vee + je_2^\vee \) where \( 0 \leq j \leq m \).

The last step is to construct generators for the \( k \)-algebra, \( k[\sigma_1] \). As with the one weight case, we will let \( X = \chi e_1^\vee \) and \( Y = \chi e_2^\vee \). From here we let \( z_j = \chi^{u_j} \) which gives us

\[ k[\sigma_1] = k[z_0, \ldots, z_m] \text{ where } z_j = X^{-\frac{nj}{m}} Y^j. \]

Now we move on to \( \sigma_2 \), using the same technique as above to determine the edges of \( \sigma_2^\vee \) (i.e. finding \( u \in \mathbb{M} \) that is positive on one edge of \( \sigma_2 \) and vanishes on the other edge). The resulting lattice points are then \( ne_1^\vee - me_2^\vee \) and \(-e_2^\vee\).

To find the generators for \( S_{\sigma_2} \) consider rational linear combinations \( s(ne_1^\vee - me_2^\vee) + t(-e_2^\vee) \) where \( 0 \leq s, t \leq 1 \). Arrange terms by coordinates to obtain
\((sn)e_1^\vee - (sm + t)e_2^\vee\). In order to obtain an integer for the first coordinate, \(s\) can be \(\frac{j}{n}\) for any \(0 \leq j \leq n\). Subsequently, the second coordinate is \(-\frac{mj}{n} + t\).

Again, since \(0 \leq t \leq 1\) this must be \(-\lceil\frac{mj}{n}\rceil\) (again we use the convention that \([0] = 1\) to avoid \(s\) and \(t\) vanishing simultaneously).

This process has given us generators for \(S_{\sigma_2}\) as \(v_j = je_1^\vee - \lceil\frac{mj}{n}\rceil e_2^\vee\). In keeping with the habit that \(X = \chi e_1^\vee\) and \(Y = \chi e_2^\vee\), and letting \(w_j = \chi v_j\), we find that our affine ring is

\[k[\sigma_2] = k[w_0, \ldots, w_n]\] where \(w_j = X^j Y^{-\lceil\frac{mj}{n}\rceil}\).

### 5.2 The Classical Construction

The classical construction of \(\mathbb{P}(1, m, n)\) is \(\text{Proj}(k[x_0, x_1, x_2])\) where the grading in the ring is given by letting the degree of each \(x_i\) be 1, \(m\), and \(n\) respectively. In particular, our concern is with finding forms of degree zero that arise upon localizing at each \(x_i\). For more on the construction \(\text{Proj}(R)\) for \(R\) a graded ring, see [2] (§III.2) or [5] (§2.2).

In the case \(i = 0\), for any numerator, we can put exactly that many \(x_0\) terms in the denominator in order to match the degrees. Since any \(x_0\)'s in the numerator will cancel, the numerator for any zero form are generated by \(x_1\) and \(x_2\). Subsequently, \(k[U_0]\) is generated by the forms \(\frac{x_1}{x_0}\) and \(\frac{x_2}{x_0}\).

The case where \(x_1 \neq 0\) becomes much more challenging to deal with. This is mainly because \(\text{deg}(x_1) \neq 1\), so that we can’t just pick a number of \(x_1\)'s to
put in the denominator in order to match the degrees. Instead, we will begin by considering the number of \( x_2 \)'s in the numerator, say \( j \) of them. In this case the trick is to make the denominator \( x_1^p \) where \( p \) is just large enough for our form to have a negative degree (i.e. denominator has highest degree). Since \( \deg(x_2^j) = nj \) and \( \deg(x_1^p) = mp \), we see that we must have the smallest \( p \) such that \( mp \geq nj \). Or stated differently:

\[
p \geq \frac{n j}{m} \text{ so that } p = \left\lceil \frac{n j}{m} \right\rceil.
\]

Placing this many \( x_1 \)'s in the denominator does not give us a form of degree 0, however. But we do know that the denominator has a a degree bigger than the numerator's and that \( \deg(x_0) = 1 \). This means we can multiply the previous numerator, \( x_2^j \), by the exact number of \( x_0 \)'s necessary to balance out the degrees. Since we know that \( \deg(x_1^{\left\lceil \frac{n j}{m} \right\rceil}) = m \left\lceil \frac{n j}{m} \right\rceil \) and \( \deg(x_2^j) = nj \), the number of \( x_0 \)'s needed is the difference. Subsequently, we arrive at a collection of forms

\[
\tilde{z}_j = x_0^{m \left\lceil \frac{n j}{m} \right\rceil - nj} x_2^j x_1^{\left\lceil \frac{n j}{m} \right\rceil}.
\]

These will generate all of the cases where \( x_2 \) is in the numerator. If we decide to adopt the convention used in the toric construction that \( \left\lceil 0 \right\rceil = 1 \), then the case \( j = 0 \) gives \( \frac{x_0^n}{x_1} \). This will generate anything with \( x_0 \) in the numerators. Subsequently the \( \tilde{z}_j \)'s generate all forms of degree zero, so that:

\[
k[U_1] \cong k[\tilde{z}_0, \tilde{z}_1, \ldots, \tilde{z}_m].
\]
Lastly, when \( x_2 \neq 0 \) we have the same problem as when \( x_1 \neq 0 \). Since it’s the same problem, use the same trick. First, suppose there are \( j \) \( x_1 \)’s in the numerator. Then the denominator needs at least \( p \) \( x_2 \)’s where \( mj \leq np \). Dividing both sides by \( n \) shows that \( \left\lfloor \frac{mj}{n} \right\rfloor \) will do the trick (still maintaining that \( [0] = 1 \)). Lastly, fill in the numerator with the \( x_0 \)’s needed to match degrees and we find that \( k[U_2] = k[\hat{w}_0, \hat{w}_1, \ldots, \hat{w}_n] \) where:

\[
\hat{w}_j = \frac{x_0^{\left\lfloor \frac{mj}{n} \right\rfloor - mj} x_1^j}{x_2^{\frac{mj}{n}}}. 
\]

### 5.3 The Isomorphism

We now know enough about these rings in order to see the isomorphism between this classical construction and the toric construction of \( \mathbb{P}(1, m, n) \). If you recall from section 2, that the correspondence came from relating the generators on \( U_0 \) and \( U_{\sigma_0} \) via \( X \sim \frac{x_1}{x_0} \) and \( Y \sim \frac{x_2}{x_0} \). This motivates us to again relate \( X \) with \( x_1 \) and \( Y \) with \( x_2 \) yielding the isomorphism on \( U_0 \) given by

\[
X \sim \frac{x_1}{x_0^m} \quad \text{and} \quad Y \sim \frac{x_2}{x_0^n}. 
\]

The next thing is to check that inverting one or the other of these terms gives you the cases \( i = 1, 2 \). It will suffice to establish that this is a one-to-one correspondence on the generators of the rings \( k[U_i] \) and \( k[\sigma_i] \).

On \( U_1 \) we have \( x_1 \neq 0 \) so that we may invert \( X = \frac{x_1}{x_0^m} \). This gives us the correspondence of the generators for \( k[U_1] \), \( \hat{z}_j \), with the generators for \( \sigma_1 \), \( z_j \).
In particular, our correspondence shows:

\[ z_j = X^{-\frac{\alpha_j}{m}} Y^j = \left( x_1 x_0 \right)^{-\frac{\alpha_j}{m}} \left( x_2 x_0 \right)^j = \frac{x_0^{m - n_j} x_2^j}{x_1^{\frac{\alpha_j}{m}}} = \tilde{z}_j. \]

In a similar fashion, the region where \( x_2 \neq 0 \) will have \( Y = \frac{x_2}{x_0} \) inverted.

The subsequent correspondence is then with \( \sigma_2 \), and is given by:

\[ w_j = X^j Y^{-\frac{\alpha_j}{n}} = \left( \frac{x_1}{x_0} \right)^j \left( \frac{x_2}{x_0} \right)^{-\frac{\alpha_j}{n}} = \frac{n^{m_j - m_j} x_1^j}{x_2^{\frac{\alpha_j}{n}}} = \tilde{w}_j. \]

### 5.4 The Ideals \( \mathcal{I}(U_1) \) and \( \mathcal{I}(U_2) \)

In this section we will use both the classical and toric descriptions of \( \mathbb{P}(1, m, n) \) to find out more about its singular points. In addition, we will consider some of the polynomials in the ideals \( \mathcal{I}(U_1) \subset k[z_0, \ldots, z_m] \) and \( \mathcal{I}(U_2) \subset k[w_0, \ldots, w_n] \) of functions that vanish on the affine varieties \( U_1 \) and \( U_2 \) respectively. Much of what we do in this section proceeds in a similar fashion on both \( \sigma_1 \) and \( \sigma_2 \). Because of this, we will restrict discussion to \( \sigma_1 \) for the most part, and give the results and note some key differences for the case of \( \sigma_2 \).

The geometry of the cone \( \sigma_1 \) and its relationship to the algebra of functions on \( U_1 \) can be very useful in generating polynomials in this ideal. In particular, recall that \( S_{\sigma_1} \) is generated by lattice points of \( M = \mathbb{N}^n \) labeled \( u_j \). Also, recall, that \( k[\sigma_1] \) is generated by monomials of the form \( z_j = \chi^{u_j} \). Furthermore, since \( \{u_j\} \) is the minimal set of generators for \( S_{\sigma_1} \) no two are
colinear.

A consequence of this is that any pair $u_{i_1}, u_{i_2}$ forms a basis for the vector space $\mathbb{M} \otimes \mathbb{R}$. Since they are also lattice points we know that they will form a basis upon tensoring with $\mathbb{Q}$ rather than $\mathbb{R}$. This means that any other lattice point can be written as $u_j = \frac{a_{i_1}}{b_{i_1}} u_{i_1} + \frac{a_{i_2}}{b_{i_2}} u_{i_2}$ where the coefficients are simplified fractions. Then upon clearing denominators, one finds that $b_{i_1} b_{i_2} u_j = a_{i_1} u_{i_1} + a_{i_2} u_{i_2}$, which can then be cleared of any common factors. Subsequently, for any pair $i_1 \neq i_2$ and any $j$ not equal to either, the following equation corresponds to a polynomial in $\mathcal{I}(U_1)$:

$$z_j^{b_{i_1} b_{i_2}} = z_{i_1}^{a_{i_1}} z_{i_2}^{a_{i_2}}$$

and a similar result holds for $\mathcal{I}(U_2)$.

When we say that this equation corresponds to an element of $\mathcal{I}(U_1)$ it does not mean that $z_j^{b_{i_1} b_{i_2}} - z_{i_1}^{a_{i_1}} z_{i_2}^{a_{i_2}}$ will actually be in that ideal. In fact, if the vector $u_j$ is not inside the cone generated by $u_{i_1}$ and $u_{i_2}$, then when you write $u_j$ in terms of the basis at least one of the coefficients will be negative. As a consequence, one of $a_{i_1}$ or $a_{i_2}$ will also be negative. When this happens, simply multiply the disagreeable polynomial by the smallest monomial needed to make every term have a positive exponent. The resulting polynomial will be the one in $\mathcal{I}(U_1)$.

There is one pair of lattice points one can choose that avoids having any negative exponents at all. Since $u_0$ and $u_m$ generate the cone $\sigma_1^\vee$ (over $\mathbb{R}$) they
are the edges of $\sigma_1^\vee$. Subsequently, any lattice point in $S_{\sigma_1}$ is contained in the cone they generate, which will result in having all positive exponents. Not only that, but much of the work to find these coefficients was done when we determined the $u_j$’s by writing them as $s(-e_1^\vee)+t(-ne_1^\vee+me_2^\vee) = su_0+tu_m$. Recall that we found $t = \frac{j}{m}$ and $s + \frac{n j}{m} = \lceil \frac{n j}{m} \rceil$. Next solve for $s$ and plug these values into our equation for $u_j$. Multiplying both sides by the common denominator of $m$ shows that $m u_j = (m \lceil \frac{n j}{m} \rceil - n j) u_0 + (j) u_m$. This means that for $1 \leq j \leq m - 1$ (those indices with lattice points not on the edge of $\sigma_1^\vee$) the following equation defines a polynomial in $I(U_1)$:

$$z^m_j = z^m_0 [\frac{n j}{m}]^j - n j z^m_j.$$

The process for finding the corresponding set of equations for $\sigma_2$ follows the same technique used here. The result is a set of polynomials in $k[w_0, \ldots, w_n]$ that come from equations of the form:

$$w^n_j = w^n_0 [\frac{n j}{m}]^j w^n_j$$

where $1 \leq j \leq n - 1$.

Much can be determined about $U_1$ from these equations. In particular, if $P = (z_0, \ldots, z_m)$ is a point on $U_1$ where either $z_0$ or $z_m$ vanishes, then for every $1 \leq j \leq m - 1$ we see that $z_j = 0$. In a similar fashion we see that if any $z_j$ (other than $j = 0$ or $j = m$) vanishes, then one of these two edge coordinates must vanish as well (as will all of the remaining interior
coordinates). This shows us that $P$ can only be in one of 4 forms. The first has $z_j \neq 0$ for all $j$, while another is the origin of $\mathbb{A}^m$. The other two possibilities are either $P = (z_0, 0, \ldots, 0)$ with $z_0 \neq 0$ or $P = (0, \ldots, 0, z_m)$ with $z_m \neq 0$. A similar argument may be made regarding the points of $U_2$, to arrive at the same results (i.e. some $w_j = 0$ if an only if $w_0 = 0$ or $w_n = 0$).

As was stated earlier these equations (the ones based on $u_0$ and $u_m$) are the only ones in $\mathcal{I}(U_1)$ that can be arrived at from a basis for $\mathbb{M} \otimes \mathbb{R}$ without having to rearrange terms to avoid negative exponents. Other equations are of interest to us as well. In particular we know that $u_0 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $u_1 = \begin{bmatrix} -(k + 1) \\ 1 \end{bmatrix}$ are not colinear, so we may generate a set of equations out of them (recall that we write $n$ as $mk + r$ so that $\left\lfloor \frac{n}{m} \right\rfloor = k + 1$).

The first step is to solve the vector equation $u_j = a_0u_0 + a_1u_1$ in terms of the $a_i$'s for every $2 \leq j \leq m$. In particular

$$\begin{bmatrix} -\left\lfloor \frac{n}{m} \right\rfloor \\ j \end{bmatrix} = a_0 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + a_1 \begin{bmatrix} -(k + 1) \\ 1 \end{bmatrix}$$

so that the equation for the second coordinate means $a_1 = j$. Subsequently, we see that $a_0 = \left\lfloor \frac{n}{m} \right\rfloor - j(k + 1)$. To see that this will be negative, recall that $n = mk + r$ so that $a_0 = \left\lfloor \frac{mk + r}{m} \right\rfloor - j(k + 1) = \left\lfloor \frac{kj + \frac{r}{m}}{m} \right\rfloor - j(k + 1)$. Since $kj$ is an integer, we may bring it outside of the ceiling to see that $a_0 = kj + \left\lfloor \frac{\frac{r}{m}}{m} \right\rfloor - kj - j = \left\lfloor \frac{r}{m} \right\rfloor - j$. Since $r < m$ we see that $\frac{r}{m} < j$, and the fact that $a_0 \leq 0$ follows.

In order to write this relationship from linear algebra as a polynomial
in \( \mathcal{I}(U_1) \) we must rewrite it so as to have only positive coefficients. To do this here we must bring the \( u_0 \) term to the left hand side to see that 
\[ u_j + \left( j(k + 1) - \left\lceil \frac{mj}{n} \right\rceil \right) u_0 = j u_k. \]
This yields the polynomial equation
\[ z_j^j = z_j z_k^{j(k+1)-\left\lceil \frac{mj}{n} \right\rceil} \]
where \( 2 \leq j \leq m \).

A nearly identical process may be used generate equations in \( \mathcal{I}(U_2) \). In particular, we know that \( m < n \), so that \( \left\lceil \frac{m-1}{n} \right\rceil = 1 \), which makes \( w_1 = XY^{-1} \) (and \( w_0 = Y^{-1} \) since \( [0] = 1 \)). This means that when we solve the equation 
\[ v_j = a_0 v_0 + a_1 v_1 \]
for the \( a_i \)'s we get \( a_1 = j \) and \( a_0 = \left\lceil \frac{mj}{n} \right\rceil - j \), the second of which is clearly negative. Thus we bring the \( v_0 \) term to the left hand side of the equation, and raise \( \chi \) to both sides to see that
\[ w_j^j = w_j w_0^{j\left\lceil \frac{mj}{n} \right\rceil} \]
for any \( 2 \leq j \leq n \).

### 5.5 The Singular Points of \( \mathbb{P}(1, m, n) \)

**Theorem 5.5.1**

\( \mathbb{P}(1, m, n) \) has 2 isolated singular points whose homogeneous coordinates are \([0, 0, 1]\) and \([0, 1, 0]\) respectively.

**Proof**

Since \( \mathbb{P}(1, m, n) \) is a toric variety, it is normal ([4], p. 29). Subsequently, any
singularities have codimension $\geq 2$. Since we’re on a surface, this means that any singularities are isolated points. Furthermore, the image of a singular point under the $T^2$-action on $\mathbb{P}(1, m, n)$ must be a singular point. Subsequently, they must be the unique fixed points $x_{\sigma_1}$ and $x_{\sigma_2}$ on $U_{\sigma_1}$ and $U_{\sigma_2}$ respectively.

Since both $\sigma_1$ and $\sigma_2$ span $\mathbb{N} \otimes_{\mathbb{Z}} \mathbb{R}$, they both satisfy $\sigma_i^\perp = \{0\}$. Subsequently, the maximal ideal, $\mathcal{M}_i$, of $k[\sigma_i]$ corresponding to $x_{\sigma_i}$ contains each $\chi^u$ for every non-zero $u \in S_{\sigma_i}$. In other words, $\mathcal{M}_1 = \langle \chi^{w_0}, \ldots, \chi^{u_j}, \ldots, \chi^{u_m} \rangle = \langle z_0, \ldots, z_j, \ldots, z_m \rangle$, and $\mathcal{M}_2 = \langle \chi^{v_0}, \ldots, \chi^{v_j}, \ldots, \chi^{v_n} \rangle = \langle w_0, \ldots, w_j, \ldots, w_n \rangle$.

By the Nullstellestatz, these two points are the origins of $\mathbb{A}^{m+1}$ and $\mathbb{A}^{n+1}$ respectively. On $U_1$ (where $x_1 \neq 0$) this means that $z_0 = \frac{x_0}{x_1} = 0$ (so that $x_0 = 0$), and $z_m = \frac{x_m}{x_1} = 0$ (so $x_2 = 0$). Subsequently, this point is $[0, 1, 0]$. Similarly on $U_2$, since $x_2 \neq 0$ we have $w_0 = \frac{x_0}{x_2} = 0$ and $w_n = \frac{x_n}{x_2} = 0$. Subsequently this singular point has $x_0 = x_1 = 0$, so it must be $[0, 0, 1]$.  

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6 On The Desingularization of \( \mathbb{P}(1, m, n) \)

In section 4 we studied the Hirzebruch surface \( \mathbb{F}_n \) and used several properties of its relationship to \( \mathbb{P}(1, 1, n) \) to study curves on a weighted projective plane. The main relationship used was that \( \mathbb{F}_n \) is the minimal desingularization of \( \mathbb{P}(1, 1, n) \). In this section our object of focus is the surface, \( \mathbb{D}(1, m, n) \), that is the minimal desingularization of \( \mathbb{P}(1, m, n) \). This new surface will also be a toric variety, and its corresponding fan is obtained by picking lattice points in \( \mathbb{Z}^2 \) and using the rays that pass from the origin through those lattice points to further subdivide the original cones, \( \sigma_i \).

6.1 Oda’s Algorithm

Given a cone corresponding to any singular affine surface, there are a few algorithms for generating the additional lattice points needed to construct the fan that gives the surface’s minimal desingularization. One can be found in [4], but it relies on the fact that any cone is equivalent to cones of a certain form. Rather than chasing down the lattice automorphism that puts us in that case, we will use the more general algorithm found in [6].

To begin the algorithm let \( v_1, v_2 \) be the edges of a singular cone \( \sigma \). The next step is to find a lattice point \( n_1 \) such that \( \{v_1, n_1\} \) form a \( \mathbb{Z} \)-basis for \( \mathbb{N} \) and \( v_2 \) is in the cone they generate. In particular \( n_1 \) should be chosen so that there are \( p, q \in \mathbb{Z} \) with \( 0 \leq p < q \) such that \( v_2 = p v_1 + q n_1 \). Note
that the case \( p = 0 \) gives \( v_2 = q \, n_1 \), which must have \( q = 1 \) since \( v_2 \) must be the shortest lattice point on the corresponding ray. So the case \( p = 0 \) means \( n_1 = v_2 \) which pairs with \( v_1 \) to form a \( \mathbb{Z} \)-basis which means \( \sigma \) is non-singular.

The next part of the algorithm involves a variant of the notion of a continued fraction. The fraction \( \frac{q}{q-p} \) is clearly greater than 1. Instead of the traditional continued fraction process, we will use the Hirzebruch-Jung continued fraction. The key difference is that Hirzebruch-Jung approach has one subtracting the remaining fractions from the the previous denominator, while the traditional approach involves addition. What needs to be done here is to find \( b_1, \ldots, b_s \) where each \( b_j \geq 2 \) and

\[
\frac{q}{q-p} = b_1 - 1/(b_2 - 1/(b_3 - 1/(\ldots - 1/b_s))).
\]

This is frequently denoted by \( \frac{q}{q-p} = [[b_1, \ldots, b_s]] \). We are now ready to proceed with Oda’s algorithm.


Let \( l_0 = v_1 \) and \( n_1 \) be as above. Also, let \( v_2 = p \, v_1 + q \, n_1 \) with \( 0 \leq p < q \) and \( \gcd(p,q) = 1 \) so that \( \frac{q}{q-p} = [[b_1, \ldots, b_s]] \). We can now construct two subsets of \( \mathbb{N} \), \( \{l_0, \ldots, l_{s+1}\} \) and \( \{n_1, \ldots, n_{s+1}\} \) by the following inductive description.

\[
l_{j+1} = l_j + n_{j+1} \quad \text{and} \quad n_{j+1} = (b_j - 2) \, l_{j-1} + (b_j - 1) \, n_j.
\]

If this is done, then \( l_{s+1} = v_2 \) and the toric variety obtained by subdividing
σ by all of the l_j’s is the minimal desingularization of U_σ with a projection map induced by the identity map on \( \mathbb{N} \).

Before we apply this algorithm let’s take a moment to look at the continued fraction expansion a little bit more. The first term, \( b_1 \) must be an integer larger than \( \beta_0 = \frac{q}{p-q} \). Since each \( b_j \geq 2 \), the remaining piece, \( 1/(b_2 - \ldots) \), must be between 0 and 1. Consequently, \( b_1 \) must be the smallest integer larger than \( \beta_0 \). In other words \( b_1 = \left\lceil \frac{q}{q-p} \right\rceil \).

Now let \( \beta_1 = b_2 - 1/(b_3 - \ldots) \) so that \( \beta_0 = b_1 - \frac{1}{\beta_1} \). Solving for \( \beta_1 \) gives \( \frac{1}{\beta_1} = \frac{1}{b_1 - \beta_0} \). This may also be written as \( \beta_1 = \frac{1}{|\beta_0| - \beta_0} \). Now we try to determine \( b_2 \). In order for us to do this we must subtract \( b_1 = \left\lceil \frac{q}{q-p} \right\rceil \) from both sides of the original continued fraction expansion, then multiply by \(-1\) and invert both sides so that \( b_2 - 1/(b_3 - \ldots) = \frac{1}{|\beta_0| - \beta_0} = \beta_1 \). Since the left hand terms (excluding \( b_2 \)) amount to a number less than 1 we see that \( b_2 = \left\lceil \beta_1 \right\rceil \). In fact, this entire continued fraction process may be broken into an inductive process.

**Lemma 6.1.2**

Let \( \beta_0 = \frac{q}{q-p} \) and define the \( \beta_j \)'s by

\[
\beta_{j+1} = \frac{1}{\left\lceil \beta_j \right\rceil - \beta_j}.
\]

Then the \( b_j \)'s of the Hirzebruch-Jung continued fraction are given by \( b_j = \left\lceil \beta_{j-1} \right\rceil \).
Proof

The key here is to realize that \( \beta_j \) is merely \( b_j + 1 - \frac{1}{b_j} \) from the continued fraction expansion. What this is allowing us to do is solve a series of equations of the form \( \beta_{j-1} = b_j - \frac{1}{\beta_j} \). Now we must pick \( b_j \) to be an integer larger than \( \beta_{j-1} \), say the smallest such one \( b_j = \lceil \beta_{j-1} \rceil \). This gives the desired relationship between the \( b_j \)'s and the \( \beta_j \)'s.

Now it remains to prove the inductive relationship on the \( \beta_j \)'s. We have already reduced our work to solving successive equations \( \beta_j = b_{j+1} - \frac{1}{\beta_{j+1}} \). Upon solving this for \( \beta_{j+1} \) we see that \( \beta_{j+1} = \frac{1}{b_{j+1} - \beta_j} = \frac{1}{\lceil \beta_j - \beta_j \rceil} \) and the lemma is proven.

It should also be noted that \( 0 < \lceil \beta_{j-1} \rceil - \beta_{j-1} < 1 \). This means that its reciprocal, \( \beta_j \) is strictly larger than one. Subsequently, \( b_{j+1} = \lceil \beta_j \rceil \geq 2 \). This means that every \( b_j \) is at least 2.

**Example 6.1.3**

To use Oda’s algorithm for \( \sigma_1 \) of \( P(1, m, n) \) let \( v_1 = l_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) and \( v_2 = \begin{bmatrix} -m \\ -n \end{bmatrix} \).

In order for \( n_1 \) to form a basis with \( l_0 \), it must be of the form \( \begin{bmatrix} \pm 1 \\ c \end{bmatrix} \) where \( c \) is an integer. Now we must consider \( p \) and \( q \) alongside \( n_1 \). We know
that
\[
\begin{bmatrix}
-m \\
-n
\end{bmatrix} = p \begin{bmatrix}
0 \\
1
\end{bmatrix} + q \begin{bmatrix}
\pm 1 \\
c
\end{bmatrix}
\]
where \(0 \leq p < q\). In order to have \(q > 0\), we must take a \(-1\) for the first coordinate of \(n_1\), and subsequently, \(q = m\). The second coordinate then says that \(-n = p + m c\). Solving for \(p\) and substituting \(mk + r\) for \(n\) yields \(p = m(-c - k) - r\). In order to get this in between 0 and \(q = m\) we must pick \(c = -(k + 1)\) which yields \(p = m - r\). This also shows us that \(n_1 = \begin{bmatrix}
-1 \\
-(k + 1)
\end{bmatrix} = \begin{bmatrix}
1 \\
-\left\lfloor \frac{n}{m} \right\rfloor
\end{bmatrix}\). Lastly, recall that if \(d|m\) and \(d|m-r\), then \(d|mk + r = n\). Since, \(m\) and \(n\) are relatively prime, so are \(p = m - r\) and \(q = m\).

This gives us all of our input for Oda’s algorithm in the case of \(\sigma_1\). As far as our algorithm for the continued fraction method, we now have that \(\beta_0 = \frac{m}{m-(m-r)} = \frac{m}{r}\). The inductive relationship \(\beta_{j+1} = \frac{1}{\left\lfloor \beta_j \right\rfloor - \beta_j}\) will determine the remaining \(\beta_j\)’s while the \(b_j\)’s are merely \(\left\lfloor \beta_j-1 \right\rfloor\).

This part of the algorithm, however, is very difficult to compute without knowing more about how \(m\) and \(r\) are related. However, we do have enough to start the desingularization process. Regardless of what all of the resulting vectors are, we know that the \(l_j\)’s will subdivide \(\sigma_1\) in order as you move counter-clockwise from \(v_1 = l_0 = \begin{bmatrix}
0 \\
1
\end{bmatrix}\) to \(v_2 = l_{s+1} = \begin{bmatrix}
-m \\
-n
\end{bmatrix}\).
One fact that is clear from this ordering of the $l_j$'s is that when you consider the line between $l_j$ and the origin the slope of $l_j$ is strictly greater than the slope of $l_{j-1}$. In addition to this fact, our knowledge of the algorithm allows us to determine $l_1$ right away.

**Proposition 6.1.4**

When desingularizing the cone $\sigma_1$ using Oda’s algorithm the first subdivision is through the lattice point

$$l_1 = \left[\begin{array}{c} -1 \\ -n/m \end{array}\right].$$

**Proof**

Consider what is already known from the algorithm when applied to $\sigma_1$. In particular, we already know that $l_0 = \left[\begin{array}{c} 0 \\ 1 \end{array}\right]$ and $n_1 = \left[\begin{array}{c} -1 \\ -(k+1) \end{array}\right]$. Subsequently,

$$l_1 = l_0 + n_1 = \left[\begin{array}{c} 0 \\ 1 \end{array}\right] + \left[\begin{array}{c} -1 \\ -1 \end{array}\right] = \left[\begin{array}{c} -1 \\ -k \end{array}\right].$$

Since $k = \left\lfloor n/m \right\rfloor$, the result follows. \qed
Unfortunately, any further computation will involve more knowledge of the relationship between $m$ and $r$. Since this is undetermined, we turn our attention to $\sigma_2$.

**Example 6.1.5**

*In order to use Oda’s algorithm to desingularize $U_{\sigma_2}$, let $v_1 = l_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -m \\ -n \end{bmatrix}$.*

The result of Oda’s algorithm will give a collection of lattice points in $\sigma_2$, $\{l_0, \ldots, l_{s+1}\}$, that will have $l_0 = v_1$ and $l_{s+1} = v_2$. Furthermore, any consecutive pair $l_j, l_{j+1}$ will constitute a $\mathbb{Z}$-basis for $\mathbb{N}$.

![Figure 6.1.2 - A Fan For $\pi^{-1}(U_{\sigma_2}) \subset \mathcal{D}(1, m, n)$](image)

In order to determine the rest of the input for Oda’s algorithm we first need to find $n_1 \in \mathbb{N}$ that forms a $\mathbb{Z}$-basis with $l_0$. For this to occur, $n_1$ must be of the form $\begin{bmatrix} c \\ \pm 1 \end{bmatrix}$. In order to determine $c$ and the sign of the second coordinate we try to find $0 \leq p < q$ so that $v_2 = p v_1 + q n_1$, or

$$\begin{bmatrix} -m \\ -n \end{bmatrix} = p \begin{bmatrix} 1 \\ 0 \end{bmatrix} + q \begin{bmatrix} c \\ \pm 1 \end{bmatrix}.$$
The second coordinate tells us $-n = \pm q$. Since $q > 0$ we must have $q = n$ and $n_1$ must have a second coordinate of $-1$. Plugging this back into the first coordinate gives the equation $-m = p + n \cdot c$. Solving for $p$ yields $p = -n \cdot c - m$. The key here is to find a value for $c$ that makes $0 \leq p < q = n$. A value of $c = -1$ gives $p = n - m$ which satisfies the required conditions on $p$ and $q$.

The last step for us is to translate the values for $p$ and $q$ into our inductive version of the continued fraction argument. This is done by setting $\beta_0 = \frac{q}{q-p} = \frac{n}{n-(n-m)} = \frac{n}{m}$. Recall that in the case of $\sigma_1$ we had $\beta_0 = \frac{m}{r}$, about which we knew nothing. In the current case of $\sigma_2$, however, we have $m$ and $n$, about which much more is known. In fact we know that $b_1 = \lfloor \beta_0 \rfloor = \left\lfloor \frac{n}{m} \right\rfloor = \left\lfloor \frac{mk+\tau}{m} \right\rfloor = k + 1$. This will allow us to go a little further with Oda’s algorithm than we were able to go on $\sigma_1$.

**Proposition 6.1.6**

When desingularizing $\sigma_2$ via Oda’s algorithm the first two subdivisions occur through the lattice points

$$l_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \text{ and } l_2 = \begin{bmatrix} -1 \\ \frac{n}{m} \end{bmatrix}.$$ 

**Proof**

We already know that $l_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $n_1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$. Subsequently, Oda’s algorithm gives us $l_1 = l_0 + n_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. 

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In order to proceed towards determining \( l_2 \) we must first determine \( n_2 \).

Using the inductive step of the algorithm gives \( n_2 = (b_1 - 2) l_0 + (b_1 - 1) n_1 \).

Since we know \( b_1 = \left\lceil \frac{n}{m} \right\rceil = k + 1 \) this tells us that

\[
n_2 = (k - 1) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (k) \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -k \end{bmatrix}.
\]

Subsequently, \( l_2 = l_1 + n_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ -k \end{bmatrix} = \begin{bmatrix} -1 \\ -(k + 1) \end{bmatrix} \). Since \( \left\lceil \frac{n}{m} \right\rceil = k + 1 \), the result is proven. \( \square \)

This is about as far as we can go with Oda’s algorithm for the moment.

For the interested reader the following table summarizes the algorithm and input values once applied to our weighted projective planes.

<table>
<thead>
<tr>
<th>( \beta_{j+1} )</th>
<th>( b_{j+1} )</th>
<th>( l_{j+1} )</th>
<th>( n_{j+1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{\beta_j} [-\beta_j] )</td>
<td>( \beta_j )</td>
<td>( l_j + n_{j+1} )</td>
<td>( (b_j - 2) l_{j-1} + (b_j - 1) n_j )</td>
</tr>
<tr>
<td>( \beta_0 = \frac{m}{r} )</td>
<td>( l_0 = (0, 1) )</td>
<td>( l_0 = (1, 0) )</td>
<td>( n_1 = (-1, -\left\lceil \frac{n}{m} \right\rceil) )</td>
</tr>
<tr>
<td>( \beta_0 = \frac{n}{m} )</td>
<td>( n_1 = (-1, -1) )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### 6.2 The Projection Map \( \pi : D(1, m, n) \to P(1, m, n) \)

The process of the finding the minimal desingularization of a toric surface \( X(\Delta) \) is based upon subdividing the original cones of \( \Delta \) into smaller ones, giving a new fan \( \hat{\Delta} \). Because of this we know that every cone of \( \hat{\Delta} \) is completely contained in a cone of \( \Delta \). Consequently, the identity map on the lattice (and
the corresponding inclusion of cones) gives a map of toric varieties. This map is merely the natural projection to a surface from its desingularization. In particular, this map will be an isomorphism away from the singular points of $X(\Delta)$.

### 6.3 Maps From $\mathbb{D}(1, m, n)$ To A Hirzebruch Surface

At this point we will take a short respite from trying to determine all of the subdivisions necessary to desingularize $\mathbb{P}(1, m, n)$ and take a look at what we have already figured out about the new surface. In particular we know that $\Delta(1)$ for the fan defining $\mathbb{D}(1, m, n)$ will contain the lattice points $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (from the original fan for $\mathbb{P}(1, m, n)$) as well as $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ (from desingularizing $\sigma_2$).

The key thing to notice here is that so far we have part of the fan for a Hirzebruch surface. One can consider either the lattice point $\begin{bmatrix} -1 \\ -k \end{bmatrix}$ that arose from desingularizing $\sigma_1$ or the point $\begin{bmatrix} -1 \\ -(k+1) \end{bmatrix}$ from $\sigma_2$. Either one of these will complete $\Delta(1)$ for the Hirzebruch surfaces $F_k$ and $F_{k+1}$ respectively. This means that the fan for $\mathbb{D}(1, m, n)$ is not only a refinement of the fan for $\mathbb{P}(1, m, n)$, but for the two Hirzebruch surfaces as well. Subsequently, the identity homomorphism on $\mathbb{N} = \mathbb{Z}^2$ will induce a pair of projection maps from $\mathbb{D}(1, m, n)$ to both $F_k$ and $F_{k+1}$.

This result should not be that surprising. This is because there is a well
known result about smooth complete toric surfaces in [4] (p. 43). Namely, any complete, non-singular toric surface can be obtained from \( \mathbb{P}^2 \) or some \( \mathbb{F}_n \) by successively blowing up points of these surfaces that are fixed by the action of the torus.

### 6.4 Smoothness of WFC’s on \( \mathbb{P}(1, m, n) \)

The remainder of this section will be devoted to using our knowledge of the desingularization algorithm in order to prove that weighted Fermat curves are smooth. To begin with, we need to adapt the polynomial for a Fermat curve to find one that is “homogeneous” with respect to the new grading of \( k[x_0, x_1, x_2] \).

**Definition 6.4.1**

A weighted Fermat curve on \( \mathbb{P}(1, m, n) \) of degree \( amn \) is the curve, \( C \), defined by the equation

\[
x_0^{amn} + x_1^{an} = x_2^{am}.
\]

**Theorem 6.4.2**

A weighted Fermat curve is smooth on the region \( U_0 \) given by \( x_0 \neq 0 \).

**Proof**

This neighborhood of \( \mathbb{P}(1, m, n) \) is merely an affine plane, \( U_0 = Spec(k[X, Y]) \), where the local coordinates are given by \( X = \frac{x_1}{x_0} \) and \( Y = \frac{x_2}{x_0} \). Localize the
homogeneous polynomial for $C$, $f = x_0^{amn} + x_1^{an} - x_2^{am}$, at the ideal $\langle x_0 \rangle$. In terms of the local coordinates, the resulting polynomial is

$$F = 1 + X^{am} - Y^{an}.$$ 

Now check the Jacobian of $F$ and see that $C$ is singular only at points where $am \cdot X^{am-1} = 0$ and $an \cdot Y^{an-1} = 0$. This means the only possible singular point of $C$ is the origin, which is not even on $C$. Subsequently, $C$ is smooth on $U_0$. $\square$

This means that the only possible singular points of $C$ would have to sit on the line $x_0 = 0$. At this point, it will save a lot of redundant computation to realize that any such points of $C$ must also satisfy $x_1 \neq 0$ and $x_2 \neq 0$. This is because substituting $x_0 = 0$ into the homogeneous polynomial yields $x_1^{am} = x_2^{am}$, so that $x_1 = 0$ if and only if $x_2 = 0$. However, there is no point with homogeneous coordinates $[0, 0, 0]$, so neither $x_1$ nor $x_2$ can vanish on this line.

Subsequently the only points of $C$ that intersect the line $x_0 = 0$ are contained in the region $U_1 \cap U_2$. This means that only one of the two affine regions needs to be checked for possible singular points. In particular, on the region $U_1$ the local coordinates are given by

$$z_i = \frac{\left. \frac{x_0^{m}}{x_1^{\frac{n}{m}}} \right] - \frac{n}{m} \left. x_2^{i} \right.}{x_1^{\frac{n}{m}}}$$

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for $0 \leq i \leq m$. Recall that under the correspondence $X = \frac{z_i}{x_0}$, $Y = \frac{z_i}{x_0}$, these coordinates are $z_i = X^{-\left\lceil \frac{n_i}{m} \right\rceil} Y^i$ in our toric description. In any case, the localized version of the polynomial defining $C$ then becomes $z_0^{a_n} + 1 - z_m^a$.

At this point one could proceed to use the Jacobian test for singularity by constructing the matrix of partial derivatives of not only this polynomial, but also the relations defining $U_1$ as a subvariety of $A^{m+1}$. In particular recall the $(0,1)$-based equations from section 5.4, $z_1^i = z_i z_0^{\left\lceil \frac{n_i}{m} \right\rceil} - z_0^{\left\lceil \frac{n_i}{m} \right\rceil}$. Unfortunately all this will demonstrate is that any possible singular points lie on the line $z_0 = 0$. This is a fact that was already known because $z_0 = \frac{x_0^m}{z_1}$ means this line is contained the line $x_0 = 0$. A similar attempt using the $(0,m)$-based equations will prove to be even more futile.

This futility arises mainly because we just don’t have enough equations to describe the ideal, $\mathcal{I}(U_1) \subset k[z_0, \ldots, z_m]$, that vanishes on $U_1$. It’s not the polynomial for the WFC that causes the Jacobian criterion to fail on this line, it’s our lack of generators for $\mathcal{I}(U_1)$. This occurs even though we know that the surface is still smooth on all of this line except for the point $[0, 1, 0]$. This is because we were able to use the torus action on $U_1$ to isolate the surface singularity away from the points on this line with $z_m \neq 0$. Unfortunately, we don’t have any such action on $C$ with which to use a similar trick. Another method must be used to untangle the WFC from the singularity.
In order to proceed notice that a WFC contains neither $[0,0,1]$ nor $[0,1,0]$, which are $\mathbb{P}(1,m,n)$’s sole singular points. Because of this, if we pull $C$ up to the surface’s desingularization, $\mathbb{D}(1,m,n)$, then the resulting curve will be isomorphic to $C$. This means that instead of having to prove $C$ is non-singular on $U_1$, it may be easier to check that $C$ is smooth on $\pi^{-1}(U_1)$.

Proving that $C$ is smooth on $\pi^{-1}(U_1)$ will have to wait until we know what the polynomial defining $C$ looks like on each of the affine regions covering the surface. Recall that Oda’s desingularization algorithm subdivides $\sigma_1$ by a collection of lattice points, $\{l_0, \ldots, l_{s+1}\}$, with $l_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $l_{s+1} = \begin{bmatrix} -m \\ -n \end{bmatrix}$. It also provides a collection of integers, $\{b_1, \ldots, b_s\}$, that comprise the Hirzebruch-Jung continued fraction expansion of $m/r$. In addition, each $b_j \geq 2$.

The fan for $\pi^{-1}(U_1)$ then consists of the cones $\tau_j = \langle l_{j-1}, l_j \rangle$ for $1 \leq j \leq s + 1$. For each $l_j$ pick an element of $\mathbb{M} = Hom(\mathbb{N}, \mathbb{Z})$ that vanishes on $l_j$ and is positive on on $l_{j-1}$ (for the case $j = 0$ think of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as $l_{-1}$). Let $l_j^\perp$ be the smallest such lattice point. In particular, note that $l_j^\perp(l_{j-1}) = det|l_{j-1}l_j| = 1$.

In a similar fashion $l_j^\perp(l_{j+1}) = det|l_{j+1}l_j| = -1$.

Using these dual elements we can now determine the edges of $\tau_j^\vee$. Namely, they are $-l_{j-1}^\perp$ and $l_j^\perp$. This means that the affine rings for each cone will be $A_j = k[x_j, y_j]$ where $x_j = \chi^{-l_j^\perp-1}$ and $y_j = \chi^{l_j^\perp}$. 

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For any two adjacent cones, $\tau_j$ and $\tau_{j+1}$, the ring corresponding to $U_{\tau_j} \cap U_{\tau_{j+1}}$ can be obtained by inverting the ring element in either $A_j$ or $A_{j+1}$ that corresponds to their common edge – namely, $\chi^\pm l^+_j$. The resulting isomorphism between $A_j[y_j^{-1}]$ and $A_{j+1}[x_{j+1}^{-1}]$ will produce the coordinate change between $U_{\tau_j}$ and $U_{\tau_{j+1}}$.

**Lemma 6.4.3**

The isomorphism $\phi_j : A_j[y_j^{-1}] \to A_{j+1}[x_{j+1}^{-1}]$ is given by:

i) $\phi_j(x_j) = x_{j+1} y_{j+1}^{-1}$

ii) $\phi_j(y_j) = x_{j+1}^{-1}$

**Proof**

Part ii) is trivial since $y_j = \chi^l_j = (\chi^{-l^+_j})^{-1} = x_{j+1}^{-1}$.

To prove the rest, note that part i) is equivalent to $\chi^{-l^+_j} = (\chi^{-l^+_j})^{b_j} \cdot \chi^l_{j+1}$. This is true if and only if $-l^+_j - 1 = -b_j l^+_j + l^+_{j+1}$. To prove this merely check that the form $l^+_j - 1 - 1 - b_j l^+_j$ vanishes on a basis for $\mathbb{N}$. Since $U_{\tau_j}$ is nonsingular we know that its edges, $l_{j-1}$ and $l_j$ form the requisite basis.

First note that $l^+_{j-1}(l_{j-1}) = 0$ so $[l^+_{j-1} + l^+_{j-1} - b_j l^+_j] (l_{j-1}) = l^+_{j+1}(l_{j-1}) - b_j l^+_{j+1}(l_{j-1})$. Also, we know that $l^+_j(l_{j-1}) = 1$, so this reduces to $l^+_{j+1}(l_{j-1}) - b_j$. To evaluate the first term, recall from Oda’s desingularization algorithm that $l_{j-1} + l_{j+1} = b_j l_j$. Solve for $l_{j-1}$ and substitute into our form to obtain $l^+_{j+1}(b_j l_j - l_{j+1}) - b_j$. Since $l^+_{j+1}(l_j) = 1$ and $l^+_{j+1}(l_{j+1}) = 0$, this simplifies to $b_j - b_j = 0$. 72
Now consider \( [l_{j-1}^+ + l_{j-1}^- - b_j l_j^+] (l_j) \). Use the identities \( l_{j-1}^+ (l_j) = -1 \), \( l_{j+1}^+ (l_j) = 1 \) and \( l_j^+ (l_j) = 0 \) to obtain \(-1 + 1 - b_j \cdot 0 = 0\). Subsequently, the form vanishes on a basis for \( \mathbb{N} \), and the lemma is proven. \( \square \)

Before proceeding to describe \( C \) on \( U_{\tau_j} \), a return to our previous description of the Hirzebruch-Jung continued fraction is in order. Recall that \( \beta_j = b_{j+1} - (1/b_{j+2} - (1/b_{j+3} - \ldots) \) is described by the recursion relation

\[
\beta_{j+1} = \frac{1}{[\beta_j] - \beta_j}
\]

where \( \beta_0 = \frac{m}{r} \) (the \( b_j \)'s can be recovered via \( b_j = \lceil \beta_j \rceil \)).

Now define the numbers \( r_j \) by letting \( r_{-1} = m \) and \( r_{j+1} = \frac{r_j}{b_j} \). Note that \( r_0 = r_{-1}/\beta_0 = m/(\frac{m}{r}) = r \). It will also be important to note that \( \beta_j \geq 1 \) and \( r_{-1} > 0 \) means every \( r_j \) will be positive.

**Lemma 6.4.4**

The \( r_j \)'s defined above satisfy

\[
r_{j+1} = b_{j+1} \cdot r_j - r_{j-1}.
\]

**Proof**

The equation defining the \( r_j \)'s is equivalent to \( \beta_{j+1} = \frac{r_j}{r_{j+1}} \). On the other hand \( \beta_{j+1} = 1/(\lceil \beta_j \rceil - \beta_j) \). Substituting \( b_{j+1} \) for \( \lceil \beta_j \rceil \) and \( \frac{r_{j-1}}{r_j} \) for \( \beta_j \) yields

\[
\frac{r_j}{r_{j+1}} = \frac{1}{b_{j+1} - \frac{r_{j-1}}{r_j}} = \frac{r_j}{b_{j+1} r_j - r_{j-1}}.
\]
Solving for $r_{j+1}$ will complete the proof.

One important consequence of this result is that since $r_{-1}$, $r_0$, and $b_j$ are all integers so is every $r_j$. Another key observation is how much this resembles the identity from Oda’s algorithm: $l_{j-1} + l_{j+1} = b_j \cdot l_j$. In fact there is another useful sequence, $\{t_j\}$, defined by a similar relationship. Let $t_0 = 0$ and $t_1 = 1$ and generate the remaining $t_j$’s by letting $t_{j+1} = b_j \cdot t_j - t_{j-1}$. Clearly, each $t_j$ will be an integer, but it is not immediately clear whether or not they are all positive. It turns out, however, that not only are they positive, but they’re increasing as well.

**Proposition 6.4.5**

The $t_j$’s defined above satisfy $t_{j+1} \geq t_j$.

**Proof**

The proof is by induction. First note that $t_2 = b_1 t_1 - t_0 = b_1$. Since $b_1 \geq 2$ and $t_1 = 1$ we see that $t_2 > t_1$.

Next suppose that $t_j \geq t_{j-1}$. Then $t_{j+1} = b_j t_j - t_{j-1} \geq b_j t_j - t_j = (b_j - 1) t_j$. However, since $b_j \geq 2$ it is observed that $t_{j+1} \geq (2 - 1) t_j = t_j$. \hfill \square

Now that these sequences are in hand we can determine what $C$ looks like on each region. In particular we can determine each $F_j \in A_j = k[\tau_j]$ that defines $C$ locally.

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Lemma 6.4.6

When a WFC of degree $amn$ is pulled up to $\pi^{-1}(U_1) \subset D(1,m,n)$ its restriction to $U_{\tau_j}$ is defined by the polynomial

$$F_j = x_j^{an-t_j}y_j^{an-t_j-1} - x_j^{a-r_j-1}y_j^{a-r_j-2} + 1.$$ 

Proof

The proof is by induction on $j$. In the initial case, $j = 1$, the theorem is equivalent to saying

$$F_1 = x_1^{an}y_1^{an} - x_1^{ar}y_1^{am} + 1 = x_1^{an} - x_1^{ar}y_1^{am} + 1.$$ 

Recall that $\tau_1 = \left\langle l_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, l_1 = \begin{bmatrix} -1 \\ -k \end{bmatrix} \right\rangle$, where $k = \left\lfloor \frac{n}{m} \right\rfloor$. Subsequently $\tau_1^\perp$ is generated by $-l_0^\perp = -e_1^\perp$ and $l_1^\perp = -k \cdot e_1^\perp + e_2^\perp$. Thus $A_1 = k[x_1, y_1]$ where $x_1 = \chi^{-e_1^\perp} = X^{-1}$ and $y_1 = \chi^{-ke_1^\perp + e_2^\perp} = X^{-k}Y$.

However, the polynomial that defines $C$ on $U_{\tau_1} \subset \mathbb{P}(1,m,n)$ is $z_0^a - z_m^a + 1$ where $z_0 = X^{-1}$ and $z_m = X^{-n}Y^m$. This polynomial is then $X^{-an} - X^{-an}Y^{am} + 1$. Since $n = mk + r$ this becomes

$$(X^{-1})^an - (X^{mk+r})^arY^{am} + 1 = (X^{-1})^an - (X^{-1})^ar(X^{-k}Y)^{am} + 1$$

which in turn is $x_1^{an} - x_1^{ar}y_1^{am} + 1$ as desired.

Now that an initial case is nailed down, assume that $F_j = x_j^{an-t_j}y_j^{an-t_j-1} - x_j^{a-r_j-1}y_j^{a-r_j-2} + 1$. Since $F_j$ and $F_{j+1}$ describe $C$ on their respective neighborhoods, they must agree on the overlap $U_{\tau_j} \cap U_{\tau_{j+1}}$. In other words, the isomorphism $\phi_j : A_j[y_j^{-1}] \to A_{j+1}[x_{j+1}^{-1}]$ sends $F_j$ to $F_{j+1}$. Since we have assumed the theorem to be true for $F_j$, this yields...
\[ F_{j+1} = \phi_j(F_j) \]
\[ = \phi_j(x_j^{an\cdot t_j} y_j^{an\cdot t_j-1} - x_j^{a_{r_j-1}} y_j^{a_{r_j-2}} + 1) \]
\[ = (x_j^{b_j} y_j^{1})^{an\cdot t_j} (x_j^{a_{r_j-1}})^{an\cdot t_j-1} - (x_j^{b_j} y_j^{1})^{a_{r_j-1}} (x_j^{a_{r_j-2}} + 1) \]
\[ = x_j^{an\cdot (b_j-1)} y_j^{1} - x_j^{a_{r_j-1}} y_j^{1} + 1 \]
which concludes the proof. \[ \square \]

Now that the polynomials defining \( \pi^{-1}(C) \) on each \( U_{\tau_j} \) we can prove the following.

**Theorem 6.4.7**

When pulled up to \( \mathbb{D}(1, m, n) \) a WFC is smooth on \( \pi^{-1}(U_1) \).

**Proof**

First consider the part of \( \pi^{-1}(C) \) contained in \( U_{\tau_1} \). This is defined by \( F_1 = x_1^{an} - x_1^{ar} y_1^{am} + 1 \). By the Jacobian criterion, singularities can only occur where \( \partial_{x_1} F_1 = \partial_{y_1} F_1 = 0 \). Since \( \partial_{y_1} F_1 = am x_1^{ar} y_1^{am-1} \), this means that one of \( x_1, y_1 \) must vanish for a point to be singular. However, no point with \( x_1 = 0 \) sits on this curve. Subsequently, any possible singular points satisfy \( y_1 = 0 \).

Setting \( \partial_{x_1} (F_1) = an x_1^{an-1} - ar x_1^{ar-1} y_1^{am} \) equal to zero at such a point would yield \( an x_1^{an-1} = 0 \), or \( x_1 = 0 \). Since such a point cannot lie on \( C \) the curve is smooth on \( U_{\tau_1} \).

In order to deal with the remaining coordinate charts recall that \( C \cap U_{\tau_j} \) is defined by \( F_j = x_j^{an\cdot t_j} y_j^{an\cdot t_j-1} - x_j^{a_{r_j-1}} y_j^{a_{r_j-2}} + 1 \). From this we can see that no point of \( C \) can satisfy \( x_j = 0 \) or \( y_j = 0 \). As a result, any point of
C corresponds to an ideal in $A_j[x_j^{-1}]$. More to the point, this means $C$ is contained in $\text{Spec}(A_j[x_j^{-1}] = U_{\tau j} \cap U_{\tau j-1} \subset U_{\tau j-1}$. Repeat this process $j - 1$ times and you will see that $C$ is then contained in $U_{\tau 1}$, where it is known to be smooth.

The only place that this argument fails is when an $r_j$ vanishes. The only time this can occur is when the continued fraction argument stops because $\beta_{s-1}$ is an integer, so that $b_s = \beta_{s-1}$. Since $\frac{r_j}{r_{j-1}} = \frac{1}{\beta_j} = b_j - \beta_{j-1}$ this means that once $j$ reaches $s - 1$ and $\beta_{s-1} = b_s$, then we will have $\frac{r_{s-1}}{r_{s-2}} = b_s - \beta_{s-1} = 0$. From this, not only can we deduce that $r_s = 0$, but that $r_{s-1} > 0$ as well. Also, since the $t_j$’s are increasing, none of them will be zero here. Subsequently, $C \cap U_{\tau s+1}$ is defined by $F_{s+1} = x_{s+1}^{an} t_{s+1}^{an} y_{s+1}^{an} t_s^{an} - x_{s+1}^{r_{s-1} - 1} + 1$.

First note that this curve contains no point with $x_{s+1} = 0$. Now apply the Jacobian criterion to find possible singular points. Since $\partial_{y_{s+1}}(F_{s+1}) = an \cdot t_s \cdot x_{s+1}^{an} t_{s+1}^{an} y_{s+1}^{an} t_s^{an} - x_{s+1}^{a r_{s-1} - 1}$ and $x_{s+1} \neq 0$ it holds that any potential singular point satisfies $y_{s+1} = 0$. If this is true, then $\partial_{x_{s+1}}(F_{s+1}) = an \cdot t_{s+1} x_{s+1}^{an} t_{s+1}^{an} y_{s+1}^{an} t_s^{an} - a \cdot r_{s-1} x_{s+1}^{a r_{s-1} - 1} = a \cdot r_{s-1} x_{s+1}^{a r_{s-1} - 1}$ is non-zero since $x_{s+1} \neq 0$. Subsequently, the curve must be smooth on $U_{\tau_{s+1}}$, as well as the rest of $\pi^{-1}(U_1)$, proving the theorem. \qed
Corollary 6.4.8 A weighted Fermat curve on $\mathbb{P}(1,m,n)$ of degree $amn$ is smooth.

Proof

First recall that a WFC is contained in $U_0 \cup U_1$. The curve has been shown to be smooth on $U_0$, so $U_1$ is all that remains. Since $C$ avoids the singular point of $U_1$ the curve is isomorphic to $\pi^{-1}(C) \subset \mathbb{P}(1,m,n)$. Since $\pi^{-1}(C)$ is smooth on $\pi^{-1}(U_1)$, $C$ must then be smooth on $U_1$, as well. $\square$
7 Hurwitz Revisited

In this section we generalize the approach of section 3 to WFC’s on $\mathbb{P}(1,m,n)$. This will be done by first constructing a map from our curve, $C$, to another well known curve. Viewing this map as a branched covering will allow us to apply Hurwitz’s theorem and determine the genus of $C$.

7.1 A Map of Curves from $C$

Recall that a WFC is defined by $x_0^m + x_1^m - x_2^m$, so that $[0,0,1]$ is not on $C$. In section 3 this property allowed us to use the map $\phi : C \rightarrow \mathbb{P}^1$ defined by $[x_0, x_1, x_2] \mapsto [x_0, x_1]$. We will attempt to use the same map here, but we must first make sure that it is well defined. In particular, consider another representative of the point $[x_0, x_1, x_2]$, i.e. $[\lambda x_0, \lambda^m x_1, \lambda^n x_2]$. Using our map, the image of this point is $[\lambda x_0, \lambda^m x_1]$. On $\mathbb{P}^1$, this is not equivalent to $[x_0, x_1]$. Rather, the image curve of this map must be the weighted projective line $\mathbb{P}(1,m)$ where these two points will be equivalent. As it turns out, this really doesn’t complicate things much.

Lemma 7.1.1

If $s, t$ are relatively prime then $\mathbb{P}(s, t) \cong \mathbb{P}^1$. In particular, $\mathbb{P}(1, n) \cong \mathbb{P}^1$.

Proof

Let $[x_0, x_1]$ be homogeneous coordinates for $\mathbb{P}(s, t)$ and try to construct the
standard affine cover for this curve. On the region $V_0$ defined by $x_0 \neq 0$ all forms of degree zero are generated by $x = \frac{x_1}{x_0}$, while on $V_1$ they are generated by $y = \frac{x_0}{x_1}$. This gives two copies of $\mathbb{A}^1$ that are glued together via the association $x = y^{-1}$, i.e. $\mathbb{P}^1$.\[\square\]

One way to think of this is that $\mathbb{P}(1, m)$ is really just $\mathbb{P}^1$ with homogeneous coordinates of $[x_0^m, x_1]$ rather than $[x_0, x_1]$. Another way to arrive at this conclusion is to contract $\mathbb{P}(1, m)$ as the complete toric variety given by $\mathbb{N} = \mathbb{Z}$ and $\Delta(1) = \{-m, 1\}$. Tensoring these cones with $\mathbb{R}$ gives the same cones as in the case of $\mathbb{P}^1$, giving the desired isomorphism.

Because of this isomorphism we have a well defined map of curves from our WFC to $\mathbb{P}(1, m) \cong \mathbb{P}^1$ defined by $[x_0, x_1, x_2] \mapsto [x_0, x_1]$ using homogenous coordinates on $\mathbb{P}(1, m, n)$ and $\mathbb{P}(1, m)$ respectively. Thus, when we use Hurwitz’s Theorem to determine the genus of our WFC, we know that the genus of our image curve is $g(\mathbb{P}(1, m)) = g(\mathbb{P}^1) = 0$.

It is important to remember when performing local computations for this map that we are using coordinates for $\mathbb{P}(1, m)$. This means we need to bear the grading of the ring in mind when determining things like the degree of the map and its ramification divisor (both of which we’ll need to apply Hurwitz’s Theorem).
Lemma 7.1.2

The degree of \( \phi : C \to \mathbb{P}(1, m) \) is \( am \).

Proof

In order to determine \( \deg(\phi) \) we need to find the number of elements in a generic fiber of \( \phi \). To do this consider \( V_0 = \phi|_{U_0} \) so that we have a map into the corresponding affine region of \( \mathbb{P}(1, m) \) defined by \( x_0 \neq 0 \). In otherwords, restrict to the affine line \( k[\frac{x_1}{x_0}] \cong k[X] \). Similarly, our coordinates on \( U_0 \subset \mathbb{P}(1, m, n) \) are given by \( X = \frac{x_1}{x_0} \) and \( Y = \frac{x_2}{x_0} \) so that \( k[U_0] = k[X, Y] \) as usual.

Since \( \phi \) sends \([x_0, x_1, x_2]\) to \([x_0, x_1]\), this localization sends \( (\frac{x_1}{x_0}, \frac{x_2}{x_0}) \) to \( \frac{x_1}{x_0} \), and the corresponding map of rings is the inclusion \( \phi^\#: k[X] \hookrightarrow k[X, Y] \).

Now take the polynomial defining our WFC and localize it at \( x_0 \) so that \( C \cap U_0 \) is defined by \( 1 + (\frac{x_1}{x_0})^{an} - (\frac{x_2}{x_0})^{am} \). This means that \( \phi|_{C \cap U_0} \) comes from the ring homomorphism from \( k[X] \) sending \( X \) to itself in \( k[C \cap U_0] = k[X, Y]/\langle 1 + X^{an} - Y^{am} \rangle \). Thus we see that as long as \( 1 + X^{an} \neq 0 \) we have \( am \) possible values for \( Y \), each corresponding to a distinct point in the fiber of \( \phi \) over the point \( X \in V_0 \). This gives the degree of \( \phi \) as \( am \), proving the lemma.
### 7.2 The Ramification of $\phi$

In order to proceed with determining the ramification divisor of $\phi$ we need to split the points of $C$ into two sets. In particular we first consider those points where $x_0 \neq 0$ then deal with the points where $x_0 = 0$. This splits the ramification divisor as $R = \bar{R} + R_0$ where $|\bar{R}| \subset C \cap U_0$ and the points of $|R_0|$ lie on the line $x_0 = 0$.

**Lemma 7.2.1**

In terms of the homogeneous coordinates on $\mathbb{P}(1, m, n)$

$$\bar{R} = \sum_{j=1}^{an} (am - 1) \cdot [1, \alpha_j, 0]$$

where each $\alpha_j$ is a distinct $an^{th}$ root of $-1$.

**Proof**

Since $|\bar{R}| \subset U_0$, use the local coordinates above. In this case we have already seen that branching occurs exactly when $1 + X^{an} = 0$. Subsequently each $\alpha_j$ gives a distinct branch point given by $X = \frac{x_1}{x_0} = \alpha_j$, or in terms of homogeneous coordinates on $\mathbb{P}(1, m), [1, \alpha_j]$. The fibers on $C$ over these points must satisfy $Y^{an} = 1 + X^{an} = 0$, so that $Y = \frac{x_2}{x_0} = 0 \Rightarrow x_2 = 0$. This means that our ramification points are $P_j = [1, \alpha_j, 0]$ and are in one-to-one correspondence with the branch points $\phi(P_j) = [1, \alpha_j]$.

Now it remains to find $e_{P_j}$. Working in the affine ring $k[X] = k[U_0]$ we see that the local ring $\mathcal{O}_{\mathbb{P}(1, m), \phi(P_j)}$ has maximal ideal $\langle X - \alpha_j \rangle$, so that $t = X - \alpha_j$. 

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is our local parameter for \(\mathbb{P}(1,m)\). Subsequently, \(e_{P_j} = v_{P_j}(\phi^#(X - \alpha_j))\).

Since \(\phi^#\) is the inclusion of rings \(k[X] \hookrightarrow k[X,Y]\) followed by the quotient map \(k[X,Y] \to k[X,Y]/\langle 1 + X^{an} - Y^{am} \rangle\) this is merely \(v_{P_j}(X - \alpha_j)\).

On \(U_0\) the point \(P_j\) has local coordinates \((\alpha_j,0)\) so that it corresponds to the ideal \(\langle X - \alpha_j, Y \rangle\). This means that the local parameter for \(\mathcal{O}_{C,P_j}\) is the generator for this ideal (since \(C\) is a smooth curve at \(P_j\) the ideal at that point must be principal).

At \(P_j \in C\) we know that \(Y^{am} = 1 + X^{an} = X^{an} - (-1) = (X - \alpha_j) \cdot p(X)\) where \(p(\alpha_j) \neq 0\). Subsequently \(p(X)\) is a unit in \(\mathcal{O}_{C,P_j}\) so we can call its inverse \(u(X)\). This means \(X - \alpha_j = u(X) \cdot Y^{am}\). Subsequently we see that not only is \(\langle X - \alpha_j, Y \rangle = \langle Y \rangle\), but also that \(v_{P_j}(X - \alpha_j) = v_Y(X - \alpha_j) = am\). Subsequently, we see that \(e_{P_j} = am\) for every \(1 \leq j \leq an\). Since \(\bar{R} = \sum_{j=1}^{an} (e_{P_j} - 1) \cdot P_j\), this completes the proof. \(\square\)

Now that we know both the degree of \(\phi\) and the ramification for all points where \(x_0 \neq 0\) it remains only to find what happens at those points where \(x_0 = 0\) (i.e. \(R_0\)). None of these points are on the region \(U_0\), but since they are points of \(C\) they must satisfy \(0 + x_1^{an} = x_2^{am}\). Since not all three homogeneous coordinates can vanish simultaneously we see that \(|R_0| \subset U_1 \cap U_2\). Because
of this we will use the local coordinates on \( U_1 \) determined in section 5.2:

\[
z_i = x^2 \cdot x_0 \frac{[\frac{n_i}{m}]}{x_1} - n_i
\]

where \( 0 \leq i \leq m \). Similarly the region \( V_1 = \phi(U_1) \subset \mathbb{P}(1, m) \) given by \( x_1 \neq 0 \) has a local coordinate of \( X^{-1} = \frac{x_0^m}{x_1} = z_0 \). Subsequently the map of rings \( \phi# \) is obtained by composing the inclusion \( k[z_0] \hookrightarrow k[z_0, \ldots, z_m] \) with the quotient map to \( k[\sigma_1] = k[z_0, \ldots, z_m]/\mathcal{I}(U_1) \).

The next step is to identify what points of \( U_1 \) lie on the line \( x_0 = 0 \) because \(|R_0|\) will be contained in this set. In particular any such \((z_0, \ldots, z_m) \in U_1\) must then satisfy \( z_0 = \frac{x_0^m}{x_1} = 0 \). Now recall from section 5.4 that \( \mathcal{I}(U_1) \), the ideal of those \( f \in k[z_0, \ldots, z_m] \) that vanish on \( U_1 \subset \mathbb{A}^{m+1} \), contains the polynomials

\[
z_i^m - z_0 \frac{[\frac{n_i}{m}]}{x_1} \cdot z_m^i
\]

for each \( 1 \leq i \leq m - 1 \). Substituting \( z_0 = 0 \) into each of these shows that \( z_i^m = 0 = z_i \). Subsequently, \( x_0 = 0 \) defines the line given locally by \( \{(0, \ldots, 0, z_m)|z_m \in k\} \). In particular, this is the fiber of \( \phi \) sitting over the origin in \( V_1 = \text{Spec}(k[z_0]) \).

The next step will be to find what points on \( x_0 = 0 \) lie on the WFC (since \(|R_0| \subset C\)). Since we are working on \( U_1 \) we must localize the equation defining \( C \), namely \( x_0^{nm} + x_1^m - x_2^{zm} \). Dividing by \( x_1^m \) and rewriting in terms
of the $z_i$ gives

$$f = z_0^a + 1 - z_m^a.$$ 

Subsequently $k[C \cap U_1] = k[U_1] / \langle f \rangle$. Since $z_0 = 0$ any points of $C$ on this line are defined by $z_m^a = 1$. Subsequently, we have $a$ distinct points of $C$ corresponding to each $a^{th}$ root of unity, $\beta_j$ where $1 \leq j \leq a$.

Thus we know that every possible point on $|R_0|$ must be one of the $a$ points in $C \cap \{Q \in U_1 | x_0 = 0 \}$. Explicitly these points are $Q_j = (0, \ldots, 0, \beta_j)$. Subsequently $M_{Q_j}$, the maximal ideal of $\mathcal{O}_{C,Q_j}$ is $\langle Q_j \rangle = \langle z_0, \ldots, z_m-1, z_m - \beta_j \rangle$.

Each of these points sits over $[1,0]$, which is the origin of $V_1 = \phi(U_1) = \text{Spec}(k[z_0])$ corresponding to the maximal ideal $M_{\phi(Q_j)} = \langle z_0 \rangle$.

All that remains is to compute the ramification index of $\phi$ at each $Q_j$. In particular, since $\langle z_0 \rangle$ is the local parameter for all $\phi(Q_j)$ we must determine $v_{Q_j} \left( \phi^\#(\langle z_0 \rangle) \right) = v_{Q_j}(\langle z_0 \rangle)$.

**Lemma 7.2.2**

*Let $Q_j = (0, \ldots, 0, \beta_j)$ be as above. Then for $1 \leq j \leq a$*

$$v_{Q_j}(z_0) = m.$$ 

**Proof**

First note that since $z_m(Q_j) = \beta_j \neq 0$ then $z_m \not\in M_{Q_j}$ (i.e. $z_m$ is a unit). Subsequently $v_{Q_j}(\langle z_m \rangle) = 0$. 

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Now consider equations in $I(U_1)$ from section 5.4. In particular, take the case where $i = 1$: $z_1^m = z_m z_0^{m \left[ \frac{m}{n} \right] - n}$. After substituting $mk + r$ for $n$, this simplifies to $z_1^m = z_m z_0^{m - r}$. Taking the valuations of both sides shows that $m \cdot v_{Q_j}(z_1) = v_{Q_j}(z_m) + (m - r) \cdot v_{Q_j}(z_0)$. But $v_{Q_j}(z_m) = 0$ so that this is merely $v_{Q_j}(z_1) = \left( \frac{m - r}{m} \right) \cdot v_{Q_j}(z_0)$. In particular, since $m$ and $m - r$ are relatively prime this means that $m v_{Q_j}(z_0)$.

In order to proceed we must use the fact that $C$ is smooth (in particular, $C$ is smooth at each $Q_j$). Since $C$ is a smooth curve we know the maximal ideal $\mathcal{M}_{Q_j} = \langle z_0, \ldots, z_{m-1}, z_m - \beta_j \rangle$ must be a principal ideal. In other words it must be generated by either $z_m - \beta_j$ or one of the $z_i$’s for $0 \leq i \leq m - 1$.

The first case may be ruled out by the fact that, on $C$, $z_0^m = z_m - 1 = (z_m - \beta_j) \cdot p(z_m)$ where $p(z_m)$ is a unit in $O_{C,Q_j}$. Since $z_m - \beta_j$ is a power of $z_0$ times a unit, it cannot generate $\mathcal{M}_{Q_j}$.

Because of this there is a single $0 \leq i \leq m - 1$ such that $\mathcal{M}_{Q_j} = \langle z_i \rangle$. Furthermore this monomial satisfies $v_{Q_j}(z_i) = 1$ since $C$ is smooth at $Q_j$.

Now consider the $i$th copy of our 0,1-based equations from section 5.4: $z_i^1 = z_i z_0^{i \left[ \frac{m}{n} \right] - \left[ \frac{ni}{m} \right]}$. Substituting $n = mk + r$ simplifies this to

$$z_i^1 = z_i z_0^{i - \left[ \frac{ri}{m} \right]}.$$ 

Now take valuations of both sides of this to obtain

$$i \cdot v_{Q_j}(z_1) = v_{Q_j}(z_i) + \left( i - \left[ \frac{ri}{m} \right] \right) v_{Q_j}(z_0).$$
which can be solved for $v_{Q_j}(z_i)$. Next, substitute $\left(\frac{m-r}{m}\right)v_{Q_j}(z_0)$ for $v_{Q_j}(z_1)$ to see that

$$v_{Q_j}(z_i) = \left(\left\lceil \frac{ri}{m} \right\rceil - \frac{ri}{m}\right) \cdot v_{Q_j}(z_0).$$

Since the number in parentheses is a fraction with denominator no larger than $m$ and $m|v_{Q_j}(z_0)$, the only way that we can have $v_{Q_j}(z_i) = 1$ is if the fraction is $\frac{1}{m}$ and $v_{Q_j}(z_0) = m$, proving the lemma.

Corollary 7.2.3

The ramification divisor for $\phi : C \to \mathbb{P}(1, m)$ is given by

$$R = \left(\sum_{i=1}^{an} (am-1)P_i\right) + \left(\sum_{j=1}^{a} (m-1)Q_j\right)$$

where $P_i = (\alpha_i, 0) \in U_0$ and $Q_j = (0, \ldots, 0, \beta_j) \in U_1$ correspond to an\textsuperscript{th} roots of $-1$ and an\textsuperscript{th} roots of unity respectively.

Proof

The first term is simply $\bar{R}$ which was determined in Lemma 7.2.1. The second term is $R_0$. Since we found every point of $C \cap \{x_0 = 0\} = \{Q_j\}$ is a ramification point with an index of $m$, the corollary follows.

Theorem 7.2.4

If $C$ is a WFC on $\mathbb{P}(1, m, n)$ of degree $amn$ then

$$g(C) = \frac{(am-1)(an-2) + a(m-1)}{2}.$$
Proof

From the corollary above we can compute \( \text{deg}(R) = \left(\sum_{j=1}^{a} (am - 1)\right) + \left(\sum_{j=1}^{a} (m - 1)\right) = an(am - 1) + a(m - 1) \). Also, from Lemma 7.1.2 we know that \( \text{deg}(\phi) = am \). Also, since \( \phi(C) = \mathbb{P}(1, m) \cong \mathbb{P}^1 \) the image of \( \phi \) has a genus of 0.

Now recall that Hurwitz’s Theorem on a map of curves \( f : X \to Y \) states \( 2g(X) - 2 = \text{deg}(f)(2g(Y) - 2) + \text{deg}(R) \). Applying this to the map \( \phi|_C \) yields

\[
2g(C) - 2 = am(-2) + an(am - 1) + a(m - 1)
\]

Subsequently

\[
2g(C) = am(-2) + 2 + an(am - 1) + a(m - 1)
\]
\[
= (-2)(am - 1) + an(am - 1) + a(m - 1)
\]
\[
= (am - 1)(an - 2) + a(m - 1)
\]

so that dividing both sides by 2 completes the proof. \( \square \)
8 The Intersection Approach

In this section we will use the smooth surface $\mathbb{D}(1,m,n)$ to study curves on $\mathbb{P}(1,m,n)$. Just as in Section 4, the key is to pull a curve $C \subset \mathbb{P}(1,m,n)$ up to $\mathbb{D}(1,m,n)$ via the projection map $\pi$. Once on the smooth surface the adjunction formula may be applied to $\pi^{-1}(C)$ to obtain its genus. In order to do this we will need to determine generators for Pic($\mathbb{D}(1,m,n)$) as well as their respective intersection numbers. Once this is accomplished, we need only describe the class $[\pi^{-1}(C)]$ and the canonical divisor, $[K_{\mathbb{D}(1,m,n)}]$, in terms of these generators to find the curve’s genus.

8.1 An Exact Sequence for Toric Varieties

If $X(\Delta)$ is a toric variety, then divisors on $X(\Delta)$ may be determined from the lattice points in $\Delta(1)$. In the case of surfaces the process is straightforward. Given a $v_i \in \Delta(1)$ consider the two maximal cones ($\sigma_1, \sigma_2$) that share $v_i$ as an edge. One of the rings, say $k[\sigma_1]$ will contain the monomial $\chi^{v_i}$, while the other will contain its inverse (here $v_i^\perp \in \{u \in \mathbb{M}|u(v_i) = 0\}$). We then obtain the divisor $D_i$ by considering the subvarieties determined by the vanishing of these respective monomials. Moreover these divisors are $T$-invariant and glue together to form a copy of $\mathbb{P}^1$.

The main tool that arises when considering these divisors is a commutative diagram with exact rows:
$0 \to M \to \text{Div}_T(X) \to \text{Pic}(X) \to 0$

$0 \to M \to \mathbb{Z}^{\Delta(1)} \xrightarrow{\pi} A_{n-1}(X) \to 0$

where the vertical arrows are inclusions. Here, $\text{Div}_T(X)$ is the group of Cartier divisors on $X$ invariant under the action of the torus, $T$, and $\mathbb{Z}^{\Delta(1)}$ is the free abelian group of Weil divisors generated by $D_i$ corresponding to each $v_i \in \Delta(1)$. Also, $A_{n-1}(X)$ is the Chow group of $(n-1)$-dimensional sub-varieties of $X$ modulo rational equivalence (an algebraic geometer’s analogue of the homology group $H_{n-1}(X,\mathbb{Z})$).

The key here is to use the exactness of the bottom row to determine $A_{n-1}$. In particular, the bottom left map $M \to \mathbb{Z}^{\Delta(1)}$ is the homomorphism determined by $e^\vee_j \to \left[ e^\vee_j(v_1), \ldots, e^\vee_j(v_{|\Delta(1)|}) \right]$. This will allow us to determine the image of $M$, which (due to the exactness of the sequence) is the kernel of the surjective map $\mathbb{Z}^{\Delta(1)} \to A_{n-1}(X)$. The inclusion of $\text{Pic}(X)$ into $A_{n-1}(X)$ will allow us to work in the Chow group instead of the Picard group, since we are ultimately out to find the intersection numbers of certain divisor classes.

**Example 8.1.1 : $\mathbb{F}_n$**

Recall that the fan for the complete toric surface $X=\mathbb{F}_n$ is given by:

$$\Delta(1) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -n \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}.$$
Label these lattice points as \(v_1, \ldots, v_4\), and let the corresponding divisors \(D_1, \ldots, D_4\) form a basis for \(\mathbb{Z}^4\). Then the map \(M \to \mathbb{Z}^4\) is given by \(e_1^\vee \mapsto [1, 0, -1, 0]\) and \(e_2^\vee \mapsto [0, 1, -n, -1]\). These two vectors will subsequently form a basis for the kernel of the map \(\pi : \mathbb{Z}^4 \to A_{n-1}(X)\) by the exactness of the sequence.

Next we find the orthogonal complement of \(\ker(\pi)\) in \(\mathbb{Z}^{[\Delta(1)]}\). Suppose the point \([a_1, \ldots, a_4] \in \mathbb{Z}^4\) is orthogonal to \(\ker(\pi)\). Since its dot product with the vectors \([1, 0, -1, 0]\) and \([0, 1, -n, -1]\) must vanish we see that \(a_1 = a_3\) and \(a_2 = n \cdot a_3 + a_4\). Considering the cases determined by \(a_3 = 1, a_4 = 0\) and \(a_3 = 0, a_4 = 1\) respectively gives the vectors \([1, n, 1, 0]\) and \([0, 1, 0, 1]\) as generators for the orthogonal complement of the image of \(\mathbb{M} = \ker(\pi)\).

Since our exact sequence contains \(\pi : A_{n-1}(X) \to 0\) we know that \(\pi\) is surjective, so that \(A_1(\mathbb{C}_n) = \mathbb{Z}^4/\ker(\pi) = \mathbb{Z}^2\). This is just the orthogonal complement of \(\ker(\pi)\) which we just computed a basis for. Thus take \(\pi\) to be the homomorphism \(\mathbb{Z}^4 \to \mathbb{Z}^2\) that takes each of the vectors \([1, n, 1, 0]\) and \([0, 1, 0, 1]\) to the standard basis for \(\mathbb{Z}^2\). In other words, given a vector in \(\mathbb{Z}^4\) project it onto each basis vector for the orthogonal complement (via the dot product). For example, taking a point \([a_1, \ldots, a_4] \in \mathbb{Z}^4\) and dotting with \([1, n, 1, 0]\) gives \(a_1 + n a_2 + a_3\). Similarly, dotting with \([0, 1, 0, 1]\) gives \(a_2 + a_4\). Subsequently, the image in \(A_{n-1}(X)\) of \([a_1, \ldots, a_4]\) is \([a_1 + n a_2 + a_3, a_2 + a_4] \in \mathbb{Z}^2\). In particular, the images of the divisors \(D_i\) are given by:
Table 8.1.1 - $T$-Invariant Divisors on $\mathbb{F}_n$

| $D_1$ | $\mapsto [1, 0]$ |
| $D_2$ | $\mapsto [n, 1]$ |
| $D_3$ | $\mapsto [1, 0]$ |
| $D_4$ | $\mapsto [0, 1]$ |

Here, we have $A_1(\mathbb{F}_n)$ generated by the divisors $D_3$ and $D_4$. The other divisors from the toric structure are given by $D_1 \sim D_3$ and $D_2 \sim n \cdot D_3 + D_4$.

8.2 The Homogeneous Coordinate Ring of a Toric Variety

Associated to any toric variety, $X$, there is a finitely generated $k$-algebra, $S$, that is graded by the group $A_{n-1}(X)$ that is called the variety’s *homogenous coordinate ring* [1]. Treating this algebra as an $A_{n-1}(X)$-graded polynomial ring will yield $X = \text{Proj}(S)$.

To construct $S$, for each $v_i \in \Delta(1)$, there is a corresponding generator $x_i \in S$. The grading on $S = k[x_1, \ldots, x_{\Delta(1)}]$ is then given by $\text{deg}(x_i) = [D_i]$, the class of the divisor associated to $v_i$ in $A_{n-1}(X)$. Also, the multiplication of two monomials $x_D, x_{D'}$ associated to divisors $D, D'$ respects this grading so $x_D \cdot x_{D'} = x_{D+D'}$. In order to construct the affine cover for $X$ that comes from $\text{Proj}(S)$ one associates to each cone $\sigma$ in $\Delta$ a ring generated by forms of degree 0 in varying localizations of $S$. In particular, every $v_i \in \Delta(1)$ that is not an edge of $\sigma$ allows us to invert the monomial $x_i$. 

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Example 8.2.1 : $F_n$

Let $\Delta(1)$ be as above so that $X(\Delta) = F_n$. Then the homogenous coordinate ring is $k[x_1, x_2, x_3, x_4]$. In order to determine $U_\tau_0$ (the cone in the first quadrant) we see that its edges are $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Thus we can invert the generators $x_3$ and $x_4$. In particular, arrange $x_1$'s and $x_2$'s in the numerator of a form, then use the $x_3$ and $x_4$ terms to balance out the degree, but using the grading of $A_1(\mathbb{F}_n)$. Recall from our exact sequence we have $\text{deg}(x_1) = [D_1] = [1,0]$ and $\text{deg}(x_2) = [n,1]$ while $\text{deg}(x_3) = [1,0]$ and $\text{deg}(x_4) = [0,1]$. Subsequently, all forms of degree 0 obtained by inverting $x_3, x_4$ are generated by $\frac{x_1}{x_3}$ and $\frac{x_2}{x_3^3x_4}$ giving $k[U_\tau_0] = k[\frac{x_1}{x_3}, \frac{x_2}{x_3^3x_4}]$.

It should be noted here that the affine $k$-algebras generated in this process are exactly the $k[\sigma]$ from the original toric construction of $X(\Delta)$. The forms of degree 0 here match up with the generators of $\sigma^\vee \cap M$. For example, on the cone $\tau_0$ the toric description has $k[\tau_0] = k[X,Y]$ where $X = \chi^{e_1}$ and $Y = \chi^{e_2}$. In particular, $X = 0$ corresponds to a divisor that vanishes on $e_1$ (i.e. $D(e_1) = D(v_1) = D_1$) while $Y = 0$ corresponds to one vanishing on $D_2$. Subsequently, the isomorphism between the usual toric construction and the coordinate ring approach is given by

$$X = \frac{x_1}{x_3} \quad \text{and} \quad Y = \frac{x_2}{x_3^3x_4}.$$ 

The real power of this correspondence arises when you consider a polynomial $f \in k[X,Y]$. Since $U_{\tau_0}$ is dense on the smooth variety $F_n$ we know
that the curve defined by \( f \) will extend to a unique curve \( C \) on \( \mathbb{F}_n \). Thus, one can use the isomorphism on \( k[\tau_0] \) and consider the local polynomial \( f(\frac{x_1}{x_3}, \frac{x_2}{x_3^3 x_4}) \). Once this is cleared of any denominators we will have a homogeneous polynomial in the \( A_{n-1}(X) \)-graded ring, \( S \). In particular, the degree of the homogenized polynomial will be the curve’s divisor class \([C]\). As an example, we will find the divisor class of a weighted Fermat curve.

**Lemma 8.2.2**

*The divisor class of a WFC of degree \( an \) on the smooth toric variety \( \mathbb{F}_n \) is \([C] = [an, a]\).*

**Proof**

Recall that a WFC on \( \mathbb{P}(1, 1, n) \) is defined by \( x_0^{an} + x_1^{an} - x_2^a \). Localized to the region \( \sigma_0 = \tau_0 \) we have a polynomial \( 1 + \left( \frac{x_1}{x_0} \right)^{an} - \left( \frac{x_2}{x_0} \right)^a \). Using the isomorphism with the toric construction we can define \( C \) locally by \( 1 + X^{an} - Y^a \).

Carry the isomorphism on this region upstairs to \( \mathbb{F}_n \) and use the homogeneous coordinate ring description to write \( f = 1 + \left( \frac{x_1}{x_3} \right)^{an} - \left( \frac{x_2}{x_3^3 x_4} \right)^a \) (these are different \( x_i \)'s than the ones in our original definition of a WFC). Clearing the denominators of this polynomial gives \( x_3^{an} x_4^a + x_1^{an} x_4^a - x_2^a \). Subsequently, the degree of this polynomial will be the degree of any of its monomials. In particular, since \( \text{deg}(x_3) = [1, 0] \) and \( \text{deg}(x_4) = [0, 1] \) the first term tells us that \([C] = \text{deg} (x_3^{an} x_4^a) = [an, a]\). \( \square \)
8.3 Intersecting Divisors in $A_{n-1}(X)$

Once the class of a curve, $C$, has been determined, there are only a few pieces missing before we can use the adjunction formula to determine its genus. We need to know the canonical divisor of the smooth surface $X(\Delta)$, as well as a method for computing intersection products in $A_{n-1}(X)$.

Determining the canonical divisor of a complete non-singular toric variety is well known. This is because the irreducible $T$-invariant divisors of $X$ are exactly the $D(v_i)$ from our exact sequence.

Proposition 8.3.1 ([4] p. 85)

The canonical divisor of a complete non-singular toric variety $X(\Delta)$ is

$$K = - \sum_{v_i \in \Delta(1)} D(v_i).$$

For our current example of $\mathbb{F}_n$ we must recall the classes of the divisors $D_i$ obtained from our exact sequence. Doing this gives a canonical divisor of

$$K = - \sum_{i=1}^{4} D_i = -[1, 0] - [n, 1] - [1, 0] - [0, 1] = -[n + 2, 2]$$

which may also be written in terms of the generators for $A_{n-1}(\mathbb{F}_n)$ as $K = -(n + 2)D_3 - 2D_4$.

With the canonical divisor firmly in our grasp we may proceed to finding the intersection products of divisors on $X$. The easiest way to do this is to
find a set of generators for $A_{n-1}(X)$ and determine their intersections with each other. Any other intersections may then be computed using linearity of the intersection product. In particular, one can use some collection of $D_i$’s as a generating set. Finding the intersections of one of these divisors with one another is easy, as $D_i \cdot D_j = 1$ if $v_i$ and $v_j$ have a cone for which they are both an edge. Otherwise $D_i \cdot D_j = 0$.

Finding the self-intersection numbers of these divisors requires a little more work. Since we are working on a smooth toric surface we know that it is the desingularization of some toric surface. In particular, there is some way of grouping the $v_i \in \Delta(1)$ so that $v_i = l_i$ are the lattice points from Oda’s desingularization algorithm. This means that for each $v_i$ there is an $a_i$ such that $v_{i-1} + v_{i+1} = a_i v_i$ where the self intersection number of the divisor corresponding to $v_i$ is $D_i^2 = -a_i$. This will allow us to construct a symmetric matrix whose associated bi-linear form $(A_{n-1}(X) \times A_{n-1}(X) \mapsto \mathbb{Z})$ is the intersection product.

Returning to our running example of the surface $\mathbb{P}_n$, we recall that $D_3 = [1, 0]$ and $D_4 = [0, 1]$ generate $A_1(\mathbb{P}_n)$. Since both $v_3 = \begin{bmatrix} -1 \\ -n \end{bmatrix}$ and $v_4 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ are edges of the cone $\tau_2$ we see that $D_3 \cdot D_4 = 1$. Also, we see that $v_2 + v_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} = 0 \cdot v_3$ so that $D_3^2 = 0$. In a similar fashion note that $v_3 + v_1 = \begin{bmatrix} -1 \\ -n \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -n \end{bmatrix} = n \cdot v_4$ produces $D_4^2 = -n$. 

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The subsequent intersection matrix is

\[ I = \begin{bmatrix} 0 & 1 \\ 1 & -n \end{bmatrix}. \]

None of these results for \( F_n \) should be surprising. In fact, these are all results that were stated in section 4. To see this note that matching the divisor \( D_4 \) with the zero section \( C_0 \) and \( D_3 \) with the fiber \( f \) gives the same intersection products as before. This correspondence also gives the same canonical divisor as in section 4.6.

**Theorem 8.3.2**

The genus of a WFC on \( \mathbb{P}(1, 1, n) \) of degree \( an \) is

\[ g = \frac{(a-1)(an-2)}{2}. \]

**Proof**

Once one pulls the WFC up to the smooth surface \( \mathbb{F}_n \) the resulting curve, \( C \), is isomorphic to the original curve. Furthermore, its class in \( A_1(\mathbb{F}_n) \) is \([C] = [an, a]\). In addition, this means that the divisor \([C + K] = [an - n - 2, a - 2]\). Since \( \mathbb{F}_n \) is smooth we may use the adjunction formula to determine \( C \)'s genus.

\[ 2g - 2 = C.(C + K) = [an, a] \cdot \begin{bmatrix} 0 & 1 \\ 1 & -n \end{bmatrix} \cdot \begin{bmatrix} an - n - 2 \\ a - 2 \end{bmatrix} \]

Multiplying out these matrices gives \( 2g - 2 = a^2n - an - 2a \) which may be solved for \( g \) to prove the theorem. \( \square \)
While this is a result we already knew there is some new machinery involved with this approach. Some of these tools include the use of the homogeneous coordinate ring to find the divisor class of a curve as well as the construction of $A_{n-1}(X)$ from $\Delta(1)$ using the exact sequence. The advantage to these techniques is that they will allow us to perform the same construction on any smooth toric surface. Of particular interest to us will be the application of this technique to the desingularizations of weighted projective planes, $\mathbb{D}(1,m,n)$. We will spend the next few sections working out these details for the the cases where $\Delta(1)$ is known.

8.4 The Case $m = 2$

Now that we've handled the simplest case (i.e. $m = 1$) we can try our hand at the next level of difficulty, $m = 2$.

Lemma 8.4.1

$\Delta(1)$ for $X = \mathbb{D}(1,2,2k + 1)$ is

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -k \end{bmatrix}, \begin{bmatrix} -2 \\ -(2k + 1) \end{bmatrix}, \begin{bmatrix} -1 \\ -(k + 1) \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}$$

or more concisely $\{v_1, \ldots, v_6\}$.

Proof

By Proposition 6.1.4, $\Delta(1)$ must contain $\begin{bmatrix} -1 \\ -k \end{bmatrix}$. Similarly, Proposition 6.1.6 says that $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -(k + 1) \end{bmatrix}$ must be there as well. A quick check of
the determinants on each resulting cone shows that the resulting surface is non-singular.

As before consider the image of the map \( M \rightarrow \mathbb{Z}^{\Delta(1)} = \mathbb{Z}^6 \) from our exact sequence. In particular

\[
e_1 \mapsto [1, 0, -1, -2, -1, 0],
\]
\[
e_2 \mapsto [0, 1, -k, -(2k + 1), -(k + 1), -1]
\]
generates the kernel of the map \( \pi : \mathbb{Z}^6 \rightarrow A_1(\Delta(1, 2, 2k + 1)) \). Subsequently \([a_1, \ldots, a_6]\) is in the orthogonal complement if and only if \( a_1 = a_3 + 2a_4 + a_5 \) and \( a_2 = k a_3 + (2k + 1) a_4 + (k + 1) a_5 + a_6 \). In particular, generate the orthogonal complement by setting one \( a_j = 1 \) and the remaining ones zero \((3 \leq j \leq 6)\) and solving for \( a_1, a_2 \).

Once we determine the orthogonal complement of \( \text{im}(\mathcal{M}) = \text{ker}(\pi) \) to be

\[
\begin{bmatrix}
1 \\
2k + 1 \\
k + 1 \\
k + 1 \\
0 \\
0
\end{bmatrix},
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
1 \\
1
\end{bmatrix},
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
1 \\
1
\end{bmatrix}
\]

we know that its image will generate \( A_1(X) \). As before, obtain the map \( \pi \) by projecting vectors in \( \mathbb{Z}^6 \) onto each of these. In particular we find that

\[
\pi(a_1, \ldots, a_6) = \begin{bmatrix}
a_1 + k a_2 + a_3 \\
2 a_1 + (2k + 1) a_2 + a_4 \\
a_1 + (k + 1) a_2 + a_5 \\
a_2 + a_6
\end{bmatrix}.
\]
This will allow us to determine the classes of the irreducible $T$-divisors, $D_i$. This proves simple for $3 \leq i \leq 6$, since the $D_i$'s turn out to be the standard basis for $\mathbb{Z}^4$. This leaves $D_1 = [1, 2, 1, 0]$ and $D_2 = [k, (2k + 1), (k + 1), 1]$.

Now that we know the class of $D_i$ we know $\text{deg}(x_i) = [D_i]$ in the $A_1(X)$-graded algebra, $k[x_1, \ldots, x_6]$. The main reason for doing this is to determine the 0-forms that generate the local ring for the cone, $\tau_0$, corresponding to the first quadrant. Since $v_1, v_2$ are the only $v_i \in \Delta(1)$ that border this cone we may invert $x_3, \ldots, x_6$ in $k[\tau_0]$. Subsequently, the numerator for any 0-forms are generated by $x_1$’s and $x_2$’s, yielding 0-forms

$$X = \frac{x_1}{x_3x_4^2x_5} \quad \text{and} \quad Y = \frac{x_2}{x_3^kx_4^{2k+1}x_5^{k+1}x_6}$$

that correspond to the original generators of $k[\tau_0]$ by $X = \chi^{e_1}$ and $Y = \chi^{e_2}$ respectively.

Now consider a WFC that has been pulled up to $D(1, 2, 2k + 1)$. On the region $U_{\tau_0}$ the curve $C$ is defined by the polynomial $1 + X^{a(2k+1)} - Y^{2a}$. Using the correspondence with the homogeneous coordinate ring we just found gives the local form for $C$ as

$$1 + \left( \frac{x_1}{x_3x_4^2x_5} \right)^{a(2k+1)} - \left( \frac{x_2}{x_3^kx_4^{2k+1}x_5^{k+1}x_6} \right)^{2a}.$$

In order to clear the denominators of this polynomial we must determine which term has most $x_i$'s in the denominator for each $3 \leq i \leq 6$, then multiply the entire polynomial by that many $x_i$'s (i.e. multiply everything
by the least common multiple of the denominators). For our current case we compare the number of $x_i$’s in each term

Table 8.4.1 - Comparison of Terms in a WFC

<table>
<thead>
<tr>
<th>$i$</th>
<th>$x_i$’s in $X$ term</th>
<th>$x_i$’s in $Y$ term</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$a(2k + 1)$</td>
<td>$2a(k)$</td>
</tr>
<tr>
<td>4</td>
<td>$2a(2k + 1)$</td>
<td>$2a(2k + 1)$</td>
</tr>
<tr>
<td>5</td>
<td>$a(2k + 1)$</td>
<td>$2a(k + 1)$</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>$2a$</td>
</tr>
</tbody>
</table>

to find a least common multiple of $x_3^{a(2k+1)}x_4^{2a(2k+1)}x_5^{2a(k+1)}x_6^{2a}$. Since our original polynomial contains the constant term, 1, we know that the corresponding term of our homogenized polynomial will be exactly the least common multiple just found. Now that we have a polynomial that is homogeneous in our $A_1(\mathbb{D}(1, 2, 1k + 1))$-graded ring, in order to find the divisor class of $[C]$ we need only find the degree of this polynomial. Since for $3 \leq i \leq 6$ we know that $deg(x_i) = D_i$ form the standard basis for $A_1(X) = \mathbb{Z}^4$ the easiest term to use to compute the degree would be one containing only $x_3, \ldots, x_6$. The least common multiple fits the bill and

$[C] = deg \left( x_3^{a(2k+1)}x_4^{2a(2k+1)}x_5^{2a(k+1)}x_6^{2a} \right) = [a(2k + 1), 2a(2k + 1), 2a(k + 1), 2a]$. 

Now that we know the class of a WFC the next step is to determine the canonical divisor of $\mathbb{D}(1, 2, 2k + 1)$. Using the theorem of Fulton from earlier
we see that
\[ K = -\sum_{i=1}^{6} D_i = -D_1 - D_2 - \sum_{i=3}^{6} D_i = -D_1 - D_2 - [1, 1, 1, 1]. \]

Substituting \([1, 2, 1, 0]\) for \(D_1\) and \([k, 2k+1, k+1, 1]\) for \(D_2\) gives a canonical divisor of \(K = -[k + 2, 2k + 4, k + 3, 2]\). This also allows us to compute \([C + K] = [a(2k + 1) - k - 2, 2a(2k + 1) - 2k - 4, 2a(k + 1) - k - 3, 2a - 2]\).

The next step is to compute the intersection numbers of our basis for \(A_1(X)\), i.e. \(\{D_3, \ldots, D_6\}\). Since our \(v_i\) are the edges of successive cones we see that \(D_i \cdot D_{i+1} = 1\), and all other cross products are \(D_i \cdot D_j = 0\). Lastly we need the self-intersections obtained from the identity from Oda’s algorithm \(l_{j-1} + l_{j+1} + b_j \cdot l_j = 0\) where \(b_j = D_j^2\). For example, to compute \(D_3^2\) see that \(l_2 + l_4 = \left[ \begin{array}{c} 0 \\ 1 \end{array} \right] + \left[ \begin{array}{c} -1 \\ -k \end{array} \right] = \left[ \begin{array}{c} -2 \\ -2k \end{array} \right] = 2 \cdot v_3\) which yields \(D_3^2 = -2\). Proceed in a similar fashion to obtain the intersection matrix for \(\mathcal{D}(1, 2, 2k + 1)\)

<table>
<thead>
<tr>
<th>(D_3)</th>
<th>(D_4)</th>
<th>(D_5)</th>
<th>(D_6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>-2</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-(k + 1)</td>
</tr>
</tbody>
</table>

In order to compute \([C],[C + K]\) we will split the problem up and first find \([C],[D_i]\) for \(3 \leq i \leq 6\). For example \([C],[D_3] = (a(2k + 1)D_3 + 2a(2k + 1)D_4 + 2a(k + 1)D_5 + 2aD_6).[D_3]\). Since intersection products are linear this gives \([C],[D_3] = a(2k + 1)(D_3^2) + 2a(2k + 1)(D_3.D_4) + 2a(k + 1)(D_3.D_5) + \ldots\)
2a(D_3,D_6). Now substitute the intersection numbers from the matrix to find [C].[D_3] = 0. Proceed in a similar fashion to find that [C].[D_4] = a and [C].[D_5] = [C].[D_6] = 0. The fact that we have [C].[D_i] vanishing for i = 3, 5, 6 comes as no surprise since these are the exceptional divisors of the desingularization map π : D(1, 2, 2k + 2) \to P(1, 2, 2k + 1). Since our weighted Fermat curve, C, does not contain the singular points being blown up in the map, [C] has zero intersection with the exceptional divisors that are the fibers over those points.

The next step is to find [C].[C + K] = [C].((2ak + a - k - 2)D_3 + (4ak + 2a - 2k - 4)D_4 + (2ak + 2a - k - 3)D_5 + (2a - 2)D_6). Use linearity and the intersections with [C] just found to simplify this to [C].[C + K] = (4ak + 2a - 2k - 4)(a). Subsequently, the adjunction formula yields

$$2g - 2 = 4a^2k + 2a^2 - 2ak - 4a \Rightarrow$$
$$2g = (4a^2k + 2a^2 - 4a) + (-2ak - a + 2) + a$$
$$= 2a(2ak + a - 2) - (2ak + a - 2) + a$$
$$= (2a - 1)(a(2k + 1) - 2) + a$$

which proves the following.

**Theorem 8.4.2**

The proper transform of a WFC of degree 2a(2k + 1) on D(1, 2, 2k + 1) has a genus of

$$g(C) = \frac{(2a - 1)(a(2k + 1) - 2) + a}{2}.$$ 

It should be noted that this fits the more general form $g = \frac{(am-1)(an-2)+a(m-1)}{2}$ that was found using Hurwitz’s Theorem.
8.5 The Case \( n = 1(\text{mod } m) \)

In this case we know that \( n = mk + 1 \) where \( k = \left\lfloor \frac{n}{m} \right\rfloor \).

Lemma 8.5.1

For \( 0 \leq j \leq m \) let \( v_j = \begin{bmatrix} -j \\ -(jk + 1) \end{bmatrix} \) and let \( v_{m+1} = \begin{bmatrix} -1 \\ -k \end{bmatrix} \), \( v_{m+2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), and \( v_{m+3} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \). Then \( \Delta(1) \) for \( X = \mathbb{D}(1, m, mk + 1) \) is \( \{v_0, \ldots, v_{m+3}\} \).

Proof

Checking a couple of determinants is all that is needed to verify that subdividing \( \sigma_1 \) by the lattice point \( \begin{bmatrix} -1 \\ -k \end{bmatrix} \) from Proposition 6.1.4 will desingularize the cone.

Checking the determinants of the remaining cones shows that this fan will result in a smooth surface.

\[
|v_j \ v_{j+1}| = \begin{vmatrix} -(j + 1) & -j \\ -(j + 1)(k + 1) & -(jk + 1) \end{vmatrix} = (j + 1)(jk + 1) - j(jk + k + 1) = 1
\]

Since the removal of any of the \( v_j \)'s will result in a non-smooth surface, this desingularization is minimal.

We also know \( \Delta(1) \) for \( X = \mathbb{D}(1, m, mk + 1) \). It should be noted that this ordering starts just at \( v_0 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \) and moves around \( \Delta(1) \) in a clockwise fashion (as opposed to our usual counter-clockwise orderings).
Since we have $\Delta(1)$ for our smooth surface we can use the exact sequence

$$0 \to M \to \mathbb{Z}^{m+4} \xrightarrow{\pi} A_{n-1}(\mathcal{D}(1,m,mk+1)) \to 0$$

to determine $A_1(X)$. As before, the image of $M$ is given by evaluation on the $v_j$’s and our basis for $\mathbb{Z}^{m+4}$ is $\{D_0, \ldots, D_{m+3}\}$. Subsequently

$$e_1^j \mapsto [0, -1, \ldots, -j, \ldots, -m, -1, 0, 1]$$
$$e_2^j \mapsto [-1, -(k+1), \ldots, -(jk+1), \ldots, -(mk+1), -k, 1, 0]$$

provides a basis for $\ker(\pi)$.

Let $[a_0, \ldots, a_{m+3}]$ be in the orthogonal complement of the image of $M$. Then this point must satisfy

$$a_{m+2} = a_0 + \ldots + (jk+1)a_j + \ldots + (mk+1)a_m + ka_{m+1}$$
$$a_{m+3} = a_1 + \ldots + ja_j + \ldots + ma_m + a_{m+1}.$$

To get a basis for the orthogonal complement take the cases generated by a single $a_j$ being 1 and the rest vanishing for each $0 \leq j \leq m+1$. This gives

one the basis consisting of

$$\begin{bmatrix}
1 \\
0 \\
\vdots \\
0 \\
\vdots \\
0 \\
1 \end{bmatrix}, \begin{bmatrix}
0 \\
1 \\
\vdots \\
0 \\
\vdots \\
0 \\
k+1 \end{bmatrix}, \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
\vdots \\
0 \\
1 \end{bmatrix}, \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
\vdots \\
0 \\
1 \end{bmatrix}.$$

Since $\pi$ is surjective $A_1(X) = \mathbb{Z}^{m+4}/\ker(\pi)$ is generated by this basis. Thus, we can represent $\pi$ by projecting $[a_0, \ldots, a_{m+3}]$ onto these vectors.
The resulting map sends this point to the divisor
\[
\begin{bmatrix}
a_0 + a_{m+2} \\
a_1 + (k+1)a_{m+2} + a_{m+3} \\
\vdots \\
a_j + (jk+1)a_{m+2} + ja_{m+3} \\
\vdots \\
a_m + (mk+1)a_{m+2} + ma_{m+3} \\
a_{m+1} + ka_{m+2} + a_{m+3}
\end{bmatrix}.
\]

Notice for \(0 \leq j \leq m + 1\) the divisors \(D_j = D(v_j)\) project onto our basis for \(A_1(X) = \mathbb{Z}^{m+2}\). We can also use this map to determine the divisor classes of the remaining \(D_j\)'s. The end result is

\[
[D_j] = \begin{cases} 
[0, 0, \ldots, 1, \ldots, 0, 0] & 0 \leq j \leq m + 1 \\
[1, k + 1, \ldots, ik + 1, \ldots, mk + 1, k] & j = m + 2 \\
[0, 1, \ldots, i, \ldots, m, 1] & j = m + 3.
\end{cases}
\]

Now that we know the class of each \(D_j\) we can determine the canonical divisor of \(D(1, m, mk+1)\). Recall that since our toric variety is complete and non-singular we have

\[
K = -\sum_{v_j \in \Delta(1)} D(v_j) = -\left(\sum_{j=0}^{m+1} D_j\right) - D_{m+2} - D_{m+2} = -[1, \ldots, 1, \ldots, 1] - [1, k + 1, \ldots, jk + 1, \ldots, mk + 1, k] - [0, 1, \ldots, j, \ldots, m, 1] = -[2, k + 3, \ldots, j(k+1) + 2, \ldots, m(k+1) + 2, k + 2].
\]

The next step is to use \(A_1(X)\) to construct the homogenous coordinate ring for \(D(1, m, mk+1)\) and us it to determine the divisor class of a weighted Fermat curve. For each \(v_j \in \Delta(1)\) use the monomial \(x_j\) giving one the
ring $k[x_0, \ldots, x_{m+3}]$. Now put an $A_1(X)$-grading on this ring by setting $\text{deg}(x_j) = [D_j]$.

Of particular interest are the 0-forms in this ring that arise while working on the cone $\tau_0 = \langle v_{m+2}, v_{m+3} \rangle$ (i.e. the first quadrant of $\mathbb{Z}^2$). Since the two edges are the only $v_j \in \Delta(1)$ that touch $\tau_0$ we can invert all of the monomials $x_0, \ldots, x_{m+1}$. As a consequence of this, our zero forms will be generated by having $x_{m+2}$ and $x_{m+3}$ in the numerator. Thus one obtains the two 0-forms

$$\frac{x_{m+2}}{x_0 x_1^{k+1} \cdots x_m^{k+1} x_{m+1}^{k+1}} \quad \text{and} \quad \frac{x_{m+3}}{x_1^{k} \cdots x_m^{k+1} x_{m+1}^{k}}$$

as generators for the local ring $k[\tau_0]$.

The next step is to take advantage of the fact that the affine region corresponding to the cone $\tau_0$ sits on both surfaces $\mathbb{D}(1, m, mk+1)$ and $\mathbb{P}(1, m, mk+1)$ and relate our 0-forms to the classical toric variables, $X = \chi^{e_1'}$ and $Y = \chi^{e_2'}$. On both surfaces a WFC is locally defined by $1 + X^{an} - Y^{am}$.

To relate this to the homogeneous coordinate ring note that $v_{m+3} = e_1$ and $v_{m+2} = e_2$. Thus $x_{m+3}$ can be thought of as monomial that vanishes along the divisor corresponding to $e_1$ and in a similar fashion $x_{m+2}$ vanishes on the divisor for $e_2$. Seeing this we can relate back to our toric construction via the correspondences

$$X = \chi^{e_1'} = \frac{x_{m+3}}{x_1 \cdots x_j \cdots x_m^{k} x_{m+1}^{k+1}}$$

and

$$Y = \chi^{e_2'} = \frac{x_{m+2}}{x_0 x_1^{k+1} \cdots x_j^{k+1} \cdots x_m^{k+1} x_{m+1}^{k}}.$$
When we clear the denominators of $1 + X^{an} - Y^{am}$ to obtain a homogenous polynomial, the leading term will simply be the least common multiple of the denominators. Since the polynomial will be homogeneous the degree of this common multiple will be the degree of $C$, which is exactly the curve’s class $[C] \in A_1(X)$. This reduces finding $[C]$ to the simple task of comparing the number of $x_j$’s in each denominator for $0 \leq j \leq m + 1$ (since on $U_m$ we cannot invert $x_{m+2}$ or $x_{m+3}$). This task is broken up into two cases.

In the first case we consider $x_j$ where $0 \leq j \leq m$. In this case the denominator of $X^{an} = X^{a(mk+1)}$ has $j \cdot a(mk + 1) = ajmk + aj$ of the $x_j$’s, while $Y^{am}$ has $(jk + 1)am = ajmk + am$ of them. Since $j \leq m$ we see there are more in the $Y^{am}$ term. Subsequently the $j^{th}$ component of $[C]$ is $am(jk + 1)$.

The second case is the only remaining $x_j$ in the denominator, namely $x_{m+1}$. $X^{an}$ has $1(an) = amk + a$ of them. $Y^{am}$, however, only has $k(am)$ of them. Subsequently the $m + 1^{st}$ component of $[C]$ is $an$. This finally determines the class of our weighted Fermat curve.

$$[C] = [am, am(k+1), \ldots, am(jk + 1), \ldots, amn, an]$$

Now we carry on by determining the intersection numbers on our basis $\{D_0, \ldots, D_{m+1}\}$ for $A_1(X)$. As before, the cross products $D_i.D_j$ are 1 if $v_i$ and $v_j$ are both edges of the same cone, and vanish otherwise. For the self intersections we use the identity from Oda’s desingularization algorithm.
\( v_{i-1} + v_{i+1} + (D^2_i)v_i = 0 \). For example consider the case \( 1 \leq i \leq m - 1 \). In this case
\[
v_{i-1} + v_{i+1} = \begin{bmatrix} -(i-1) \\ -(i-1)k - 1 \end{bmatrix} + \begin{bmatrix} -(i+1) \\ -(i+1)k - 1 \end{bmatrix} = \begin{bmatrix} -2i \\ -2ik - 2 \end{bmatrix} = 2v_i
\]
shows that \( D^2_i = -2 \) for \( 1 \leq i \leq m - 1 \). The process for the remaining cases works similarly and results in finding \( D^2_0 = -(k+1) \), \( D^2_m = -1 \), and \( D^2_{m+1} = -m \). The interested reader could use the same process to find the self-intersections of the remaining \( D_i \)'s (those not part of our basis) and find \( D^2_{m+2} = k \) and \( D^2_{m+3} = 0 \). Putting all of these intersection products together gives us our intersection matrix.

Table 8.5.1 - Intersection Product on \( \mathbb{D}(1,m, mk+1) \)

<table>
<thead>
<tr>
<th>( D_0 )</th>
<th>( D_1 )</th>
<th>( \ldots )</th>
<th>( D_i )</th>
<th>( \ldots )</th>
<th>( D_m )</th>
<th>( D_{m+1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D_0 )</td>
<td>( -(k+1) )</td>
<td>1</td>
<td>( \ldots )</td>
<td>0</td>
<td>( \ldots )</td>
<td>0</td>
</tr>
<tr>
<td>( D_1 )</td>
<td>1</td>
<td>( -2 )</td>
<td>( \ldots )</td>
<td>0</td>
<td>( \ldots )</td>
<td>0</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \ddots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td></td>
</tr>
<tr>
<td>( D_i )</td>
<td>0</td>
<td>0</td>
<td>( \ldots )</td>
<td>( -2 )</td>
<td>( \ldots )</td>
<td>0</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \ddots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td></td>
</tr>
<tr>
<td>( D_m )</td>
<td>0</td>
<td>0</td>
<td>( \ldots )</td>
<td>0</td>
<td>( \ldots )</td>
<td>( -1 )</td>
</tr>
<tr>
<td>( D_{m+1} )</td>
<td>0</td>
<td>0</td>
<td>( \ldots )</td>
<td>0</td>
<td>( \ldots )</td>
<td>( 1 )</td>
</tr>
</tbody>
</table>

Now that we know the intersections of our basis elements for \( A_1(X) \) we can use the bilinearity of the intersection product to intersect any pair of divisors. In particular we are ultimately after \( [C].[C + K] \) so that we may apply the adjunction formula and obtain the genus of \( C \). However, we will first find \( [C].D_i \) for \( \leq i \leq m + 1 \) to make the computation a little easier. This must be handled in a few cases: \( i = 0, 1 \leq i \leq m - 1, i = m, \) and \( i = m + 1 \).
We treat the case $1 \leq i \leq m-1$ first. To this write $[C] = [am, \ldots, am(jk+1), \ldots, amn, an]$ as $[C] = \left( \sum_{j=0}^{m} am(jk + 1)[D_j] \right) + an[D_{m+1}]$. Now use the linearity of the intersection product to write $[C],[D_i]$ as a linear combination of $D_i[D_j]'s$. Since most of these vanish we are left with $[C],[D_i] = am(k(i-1) + 1)[D_{i-1}][D_i] + am(ik + 1)[D_j]^2 + am(k(i + 1) + 1)[D_i][D_{i+1}] = (amik - amk + am)(1) + (amik + am)(-2) + (amik + amk + am)(1) = 0.

Using the same technique for $i = 0$ gives $[C],[D_0] = am[D_0]^2 + am(k + 1)[D_0].[D_1] = am(-k + 1) + am(k + 1)(1) = 0$. The same can be said of $i = m + 1$. In this case we have $[C],[D_{m+1}] = am(mk + 1)[D_m].[D_{m+1}] + a(mk + 1)[D_{m+1}]^2 = am(mk + 1)(1) + a(mk + 1)(-m) = 0.

Note that all the divisors so far have 0-intersection with $[C]$. This should be clear because all of the divisors treated so far are exceptional divisors of the map $\mathbb{D}(1, m, mk+1) \rightarrow \mathbb{P}(1, m, mk+1)$ that blows up the singular points of weighted projective plane, none of which are on $C$. The case $[C],[D_m]$ will not vanish, however, since $v_m = \left[ \begin{array}{c} -m \\ -(mk + 1) \end{array} \right]$ was already in $\Delta(1)$ for $\mathbb{P}(1, m, mk + 1)$. In particular

$[C],[D_m] = am((m - 1)k + 1)[D_{m-1}][D_m] + am(mk + 1)[D_m]^2 + a(mk + 1)[D_{m+1}.D_m] = am(mk - k + 1)(1) + am(mk + 1)(-1) + a(mk + 1)(1) = am(-k) + a(mk + 1) = a.$
Now that we have our $[C].[D_j]$’s we move one step closer to using the adjunction formula by considering the divisor

$$
[C + K] = \left( \sum_{j=0}^{m} am(jk + 1)[D_j] \right) + an[D_{m+1}]
- \left( \sum_{j=0}^{m} (j(k + 1) + 2)[D_j] \right) - (k + 2)[D_{m+1}]
= \left( \sum_{j=0}^{m} (am(jk + 1) - j(k + 1) - 2)[D_j] \right) + (an - k - 2)[D_{m+1}]
$$

**Theorem 8.5.2**

Upon pulling a WFC of degree $am(mk + 1)$ up to $\mathcal{D}(1, m, mk + 1)$ its genus is given by

$$g(C) = \frac{(am - 1)(am(mk + 1) - 2) + a(m - 1)}{2}.
$$

**Proof**

We already know $[C]$ and $[C + K]$, so all that remains is to use the adjunction formula to solve $2g - 2 = [C].[C + K]$ for the genus. Since we have written $[C + K]$ as a linear combination of the $[D_j]$’s for $0 \leq j \leq m + 1$ we can use the linearity of the intersection product to distribute $[C]$ across it. Since $[C].[D_i] = 0$ for all $i \neq m$ the only term we are left with is

$$
2g - 2 = \left( am(mk + 1) - m(k + 1) - 2 \right) [C].[D_m]
= \left( am(mk + 1) - mk - m - 2 \right) (a).
$$

Subsequently

$$
g = \frac{a^2(mk+1)-amk-am-2a+2}{2}
= \frac{a^2(mk+1)-amk-a-2am+2+am-a}{2}
= \frac{a^2(mk+1)-a(mk+1)-2(am-1)+a(m-1)}{2}
= \frac{a(mk+1)(am-1)-2(am-1)+a(m-1)}{2}
$$

and factoring out the $(am - 1)$ term completes the proof. \qed
8.6 The Case $n = -1 (mod m)$

We next turn to the case where $n = -1 (mod m)$ so that $n = mk + (m - 1)$.

**Lemma 8.6.1**

For $1 \leq j \leq m$ let $v_j = \begin{bmatrix} -j \\ -(jk + (j - 1)) \end{bmatrix}$. Also let $v_{m+1} = \begin{bmatrix} -1 \\ -(k + 1) \end{bmatrix}$, $v_{m+2} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, $v_{m+3} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $v_{m+4} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then $\Delta(1)$ for $\mathbb{D}(1, m, mk + m - 1)$ is $\{v_1, \ldots, v_{m+3}\}$.

**Proof**

Checking the determinants of the three cones that are contained in the original $\sigma_2$ (for $\mathbb{P}(1, m, n)$ will show that $v_{m+1}$ and $v_{m+2}$ (both given by Proposition 6.1.6) will suffice in desingularizing $U_2$.

For the remainder of the proof, check the determinants

$$
|v_j v_{j+1}| = \begin{vmatrix} -j \\ -(jk + j - 1) \end{vmatrix} = j(jk + k + j) - (j + 1)(jk + j - 1) = 1
$$

to see that this fan desingularizes $\sigma_1$. Also note that removing any of these gives a non-smooth surface, so this desingularization is minimal. \qed

Now consider the image in $\mathbb{Z}^{m+4}$ of $M$ from our exact sequence.

$$
e_1^\vee \mapsto (-1, -2, \ldots, -j, \ldots, -m, -1, 0, 1, 0)
$$

$$
e_2^\vee \mapsto (-k, -(2k + 1), \ldots, -[jk + (j - 1)], \ldots, -n, -(k + 1), -1, 0, 1)
$$
Subsequently a point $\sum_{v_i \in \Delta(1)} a_i D_i$ is in the orthogonal complement of the kernel of the projection $\pi : \mathbb{Z}^{m+4} \to A_1(X)$ if and only if

$$a_{m+3} = a_{m+1} + \sum_{j=1}^{m} ja_j$$

and

$$a_{m+4} = (k + 1)a_{m+1} + a_{m+2} + \sum_{j=1}^{m} (jk + j - 1)a_j.$$

As before we form a basis for this space by taking vectors that have a single non-zero entry of 1 in their $j^{th}$ coordinate for $1 \leq j \leq m + 2$. The basis we then arrive at consists of the vectors in $\mathbb{Z}^{m+4}$

$$\begin{bmatrix}
1 \\
0 \\
\vdots \\
0 \\
\vdots \\
0 \\
k \\
\end{bmatrix}, \begin{bmatrix}
0 \\
0 \\
\vdots \\
1 \\
\vdots \\
0 \\
jk + (j - 1) \\
\end{bmatrix}, \ldots, \begin{bmatrix}
0 \\
0 \\
\vdots \\
m \\
\vdots \\
0 \\
jk + (j - 1) \\
\end{bmatrix}, \ldots, \begin{bmatrix}
0 \\
0 \\
\vdots \\
k+1 \\
\vdots \\
0 \\
k+1 \\
\end{bmatrix}, \ldots, \begin{bmatrix}
0 \\
0 \\
\vdots \\
k+1 \\
\vdots \\
0 \\
k+1 \\
\end{bmatrix}$$

Now we can construct the map $\pi : \mathbb{Z}^{m+4} \to \mathbb{Z}^{m+2}$ by projecting $[a_1, \ldots, a_{m+4}]$ onto each of these vectors. The resulting homomorphism then sends this
point to the divisor class:

$$
\begin{bmatrix}
  a_1 + a_{m+3} + ka_{m+4} \\
  \vdots \\
  a_j + ja_{m+3} + (jk + j - 1)a_{m+4} \\
  \vdots \\
  a_m + ma_{m+3} + (mk + m - 1)a_{m+4} \\
  a_{m+1} + a_{m+3} + (k + 1)a_{m+4} \\
  a_{m+2} + a_{m+4}
\end{bmatrix}
$$

In particular the irreducible $T$-divisors $D_i = D(v_i)$ for each $v_i \in \Delta(1)$ are given by

$$[D_i] = \begin{cases}
[0, \ldots, 1, \ldots, 0] & 1 \leq i \leq m + 2 \\
[1, \ldots, j \ldots, m, 1, 0] & i = m + 3 \\
[k, \ldots, jk + (j - 1) \ldots, n, k + 1, 1] & i = m + 4
\end{cases}.$$

This will allow us to find the canonical divisor for $\mathbb{D}(1, m, mk + m - 1)$.

Namely

$$K = - \sum_{v_j \in \Delta(1)} D(v_j) = -[k + 2, \ldots, j(k + 2), \ldots, m(k + 2), k + 3, 2].$$

Knowing the classes of the $D_j$’s will also allow us to construct the homogeneous coordinate ring for $\mathbb{D}(1, m, mk + m - 1)$. Begin by considering the polynomial ring $k[x_1, \ldots, x_{m+4}]$. Then give it the $\mathbb{Z}^{m+2}$-grading given by $\deg(x_j) = [D_j]$ from the divisors classes above.

Now consider the affine region corresponding to the cone $\tau_0 = \langle v_{m+3}, v_{m+4} \rangle = \langle e_1, e_2 \rangle$ (the first quadrant). These two edges are the only $v_j \in \Delta(1)$ that touch $\tau_0$, so when working on this affine region we may invert $x_1, \ldots, x_{m+2}$ to construct our 0-forms in the homogeneous coordinate ring. Using the
divisor classes for our grading we find the two 0-forms \( \frac{x_{m+3}}{x_1 \ldots x_j \ldots x_{m-1} x_{m+1}} \) and \( \frac{x_{m+4}}{x_1 \ldots x_{j+k+1} \ldots x_{m} x_{m+1} x_{m+2}} \). As with the previous cases, we see the isomorphism between this construction and the usual toric construction arises from the correspondences between \( x_j \) and the monomial in \( X = \chi^{e_1} \) and \( Y = \chi^{e_2} \) that vanishes on \( D_j \). Namely

\[
X = \frac{x_{m+3}}{x_1 \ldots x_j \ldots x_{m} x_{m+1}}
\]

and

\[
Y = \frac{x_{m+4}}{x_1 \ldots x_{j+k+1} \ldots x_{m} x_{m+1} x_{m+2}}.
\]

Now let \( C \) be a WFC on \( \mathbb{P}(1, m, mk + m - 1) \) of degree \( amn \) that has been pulled up to \( \mathbb{D}(1, m, mk + m - 1) \). In particular recall that the function defining \( C \) on the local region \( U_\tau \) is given by \( 1 + X^{an} - Y^{am} \). The next step is to substitute our 0-forms for \( X \) and \( Y \) and clear any \( x_j \)'s from the denominator yielding a "homogeneous" polynomial whose degree is \([C]\).

As before, the existence of the constant term 1 in our local form means that we only need to find the least common multiple of the denominators of \( X^{an} \) and \( Y^{am} \) in order to compute \([C]\). This will be done by comparing the number of \( x_j \)'s in each denominator, and keeping track of the larger one.

For \( 1 \leq j \leq m \) our \( X^{an} \) denominator has \( x_j \) to the power \( ajn = aj(mk + m - 1) = ajmk + ajm - aj \), while in the \( Y^{am} \) term \( x_j \) occurs to the power \( am(jk + j - 1) = amjk + ajm - am \). The first number is the larger of the
two (compare the negative terms of each and use the fact that $j \leq m$) so that our $j^{th}$ coordinate in $[C]$ will be $ajn$.

Now consider the other $x_j$’s, namely $x_{m+1}$ and $x_{m+2}$. For the first, $X^an$ has only $an = amk + am - a x_{m+1}$’s while $Y^am$ has $am(k + 1) = amk + am$ of them, yielding coordinate in $[C]$ of $am(k + 1)$. For the second, realize that $X^an$ has no $x_{m+2}$’s while $Y^am$ has $am$ of them. Subsequently the class of a WFC of degree $amn$ on $\mathbb{D}(1, m, mk + m - 1)$ is

$$[C] = [an, \ldots, ajn, \ldots, amn, am(k + 1), am].$$

The next step is to construct the intersection matrix for $A_1(\mathbb{D}(1, m, mk + m - 1))$. The cross products work just like before, where $D_j, D_{j+1} = 1$ and the remaining cross terms are 0. For the self-intersections we use the identity $v_{j-1} + v_{j+1} + (D^2)v_j = 0$ from Oda’s desingularization algorithm. As an example we compute $D^2_j$ for $2 \leq j \leq m - 1$. Note that for $1 \leq j \leq m$ we have $v_j = \left[\begin{array}{c} -j \\ -(jk + j - 1) \end{array} \right]$. Subsequently for $2 \leq j \leq m - 1$

$$v_{j-1} + v_{j+1} = \left[\begin{array}{c} -(j - 1) \\ -[(j - 1)k + j - 2] \end{array} \right] + \left[\begin{array}{c} -(j + 1) \\ -[(j + 1)k + j] \end{array} \right]$$

$$= \left[\begin{array}{c} -2j \\ -[(j + 1 + j - 1)k + 2j - 2] \end{array} \right]$$

$$= 2 \left[\begin{array}{c} -j \\ -(jk + j - 1) \end{array} \right] = 2v_j$$

yielding $D^2_j = -2$. One may repeat this process for the remaining $D_j$ with $1 \leq j \leq m + 2$ (i.e. our basis for $A_1(\mathbb{D}(1, m, mk + m - 1))$). This yields
\[ D_1^2 = -2, \ D_m^2 = -1, \ D_{m+1}^2 = -m \] and \[ D_{m+2}^2 = -(k+1) \] which results in the following intersection matrix.

Table 8.6.1 - Intersection Product on \( \mathbb{D}(1, m, mk + m - 1) \)

<table>
<thead>
<tr>
<th>( D )</th>
<th>( D_1 )</th>
<th>( D_2 )</th>
<th>( \cdots )</th>
<th>( D_j )</th>
<th>( \cdots )</th>
<th>( D_m )</th>
<th>( D_{m+1} )</th>
<th>( D_{m+2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D_1 )</td>
<td>-2</td>
<td>1</td>
<td>( \cdots )</td>
<td>0</td>
<td>( \cdots )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( D_2 )</td>
<td>1</td>
<td>-2</td>
<td>( \cdots )</td>
<td>0</td>
<td>( \cdots )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
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<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( D_j )</td>
<td>0</td>
<td>0</td>
<td>( \cdots )</td>
<td>-2</td>
<td>( \cdots )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
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<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( D_m )</td>
<td>0</td>
<td>0</td>
<td>( \cdots )</td>
<td>0</td>
<td>( \cdots )</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( D_{m+1} )</td>
<td>0</td>
<td>0</td>
<td>( \cdots )</td>
<td>0</td>
<td>( \cdots )</td>
<td>1</td>
<td>( -m )</td>
<td>1</td>
</tr>
<tr>
<td>( D_{m+2} )</td>
<td>0</td>
<td>0</td>
<td>( \cdots )</td>
<td>0</td>
<td>( \cdots )</td>
<td>0</td>
<td>1</td>
<td>( -(k + 1) )</td>
</tr>
</tbody>
</table>

Now that we can intersect divisors in \( A_1(X) \) we can compute \( [C], [C + K] \) and use the adjunction formula to obtain the genus of \( C \). As before we break this process into pieces by determining \( [C], [D_j] \) for each \( [D_j] \) in our basis for \( A_1(X) \). To do this recall that \( [C] = [an, \ldots, ajn, \ldots, amn, am(k + 1), am] \).

Subsequently \( [C], [D_1] = an(D_1^2) + 2an(D_1.D_2) = (-2)an + 2an(1) = 0. \) Also, for \( 2 \leq j \leq m \) we have

\[
[C].[D_j] = a(j - 1)n(D_{j-1}.D_j) + ajn(D_j^2) + a(j + 1)n(D_j.D_{j+1})
= (ajn - an)(1) + ajn(-2) + (ajn + an)(1)
= 0.
\]

Proceeding in a similar fashion yields \( [C].[D_m] = a \) and \( [C].[D_{m+1}] = [C].[D_{m+2}] = 0. \) Since \( D_1, \ldots, D_{m+2} \) form a basis for \( A_1(X) \) we can write \( D_{m+3} \) and \( D_{m+4} \) in terms of them, then use the intersections we just found to determine \( [C].[C_{m+3}] = am \) and \( [C].[C_{m+4}] = an. \) It should be noted that the
divisors that have 0 intersection with $C$ are exactly the ones corresponding to the extra vectors $v_j \in \Delta(1)$ added to the fan for $\mathbb{P}(1, m, mk + m - 1)$ when constructing $D(1, m, mk + m - 1)$. This is what one should expect since the desingularization is a repeated blowing up of the singular points $[0, 0, 1]$ and $[0, 1, 0]$ of $\mathbb{P}(1, m, mk + m - 1)$. Since $C$ does not contain any of these points, it will have 0 intersection with their exceptional divisors, as we have seen.

Now that we know $[C].[D_j]$ for each $[D_j]$ in our basis for $A_1(X)$ we can move on to determining the genus of $C$.

**Theorem 8.6.2**

Let $C$ be a WFC on $\mathbb{P}(1, m, mk + m - 1)$ of degree $am(mk + m - 1)$. Then the curves genus is determined by

$$g(C) = \frac{(am - 1)(a(mk + m - 1) - 2) + a(m - 1)}{2}.$$ 

**Proof**

As before, pull the curve $C$ up to $D(1, m, mk + m - 1)$. Since $C$ does not contain $[0, 0, 1]$ or $[0, 1, 0]$ the resulting curve is isomorphic to $C$. Subsequently, the divisors class for $[C] \in A_1(X)$ is $[an, \ldots, ajn, \ldots, amn, am(k + 1), am]$. Since we also know $[K] = -[k + 2, \ldots, j(k + 2), \ldots, m(k + 2), k + 3, 2]$ we can determine $[C + K] = [an - k - 2, \ldots, j(an - k - 2), \ldots, m(an - k - 2), am(k +...
1) \(-k - 3, am - 2\]. Then using the adjunction formula we see that

\[
2g - 2 = [C]_*[C + K] = (an - k - 2)[C]_*[D_1] + \ldots + j(an - k - 2)[C]_*[D_j] + \ldots \\
+ m(an - k - 2)[C]_*[D_m] \\
+ (am(k + 1) - k - 3)[C]_*[D_{m+1}] \\
+ (am - 2)[C]_*[D_{m+2}] \\
= m(an - k - 2)(a)
\]

where the last line follows from the fact that the only non-zero intersection of 

\([C] \) with one of our basis elements for \(A_1(X)\) is \([C]_*[D_m] = a\). Subsequently, 

one finds that

\[
2g = am(an - k - 2) + 2 \\
= a^2mn - amk - 2am + 2 \\
= a^2mn - 2am + 2 - amk - a(m - 1) + a(m - 1) \\
= a^2mn - 2am + 2 - an + a(m - 1) \\
= an(am - 1) - 2(an - 1) + a(m - 1) \\
= (am - 1)(an - 2) + a(m - 1).
\]

Since \(n = mk + m - 1\) dividing both sides by 2 will prove the result. \(\square\)
References


