Abstract

Bidwell, John Charles. On the periodic nature of solutions to the reciprocal delay difference equation with maximum. (Under the direction of John E. Franke.)

We prove that every positive solution of the difference equation

$$x_n = \max_{i \in [1,k]} \left\{ \frac{A_i}{x_{n-i}} \right\}$$

is eventually periodic, and that the prime period is bounded for all positive initial points. A lower bound, growing faster than polynomially, on the maximum prime period for a system of size k is given, based on a model designed to generate long periods. Conditions for systems to have unbounded preperiods are given. All cases of nonpositive systems, with either the A values and/or initial x values allowed to be negative, are analyzed. For all cases conditions are given for solutions to exist, for the system to be bounded, and for it to be eventually periodic. Finally, we examine several other difference systems, to see if the methods developed in this paper can be applied to them.

On the periodic nature of solutions to the reciprocal delay difference equation with maximum

by

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Biography

I, John Bidwell, was born in 1963, and grew up in the Santa Cruz, California area. I had an early interest in mathematics, considering topics such as number theory, fractals and summability. I gained experience in computer programming from 1981 to 83 and again from 1989 to 90. I received a bachelors in mathematics from the University of Hawaii at Hilo in 1998. In my final year there I studied dynamical systems such as the quadratic equation, $f(x) = x^2 - c$, analyzing the set of points which have nondense, nonperiodic orbits. I attended graduate school at North Carolina State University from 1998 through December 2004. My primary research at NCSU was in the topic of this thesis, while also beginning research in decidability and summability. In August 2004 I began working as a visiting instructor at the University of Toledo.

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Chapter 1

Introduction, previous results

1.1. Introduction

This thesis concerns the difference equation

$$x_n = \max_{i\in[1,k]} \left\{ \frac{A_i}{x_{n-i}} \right\} \tag{1.1}$$

with $A_k \neq 0$. For a given system, $A = (A_i)_{i=1}^k$, $A_i \in \mathbb{R}$, a vector of initial values $\vec{x} = (x_i)_{i=-k}^{-1}$, $x_i \in \mathbb{R}$ determines a sequence $(x_i)_{i=0}^{\infty}$, which is called a solution. The symbol $\vec{x}$ will be used exclusively to refer to this initial vector. Notation such as $A > 0$, where A is a vector, indicates that each component of A is positive. The system in equation (1.1) will be abbreviated **RDEM**, for Reciprocal Difference Equation with Maximum.

If one of the initial $\vec{x}$ components is zero, or if a term x_n with $n \geq 0$ is generated which equals zero, then the equation for generating the next term will involve division by zero. In such a case we say that no solution exists for that initial x vector. This system was proposed for study by Gerry Ladas [**1**] in 1996. It has potential applications in several areas, such as automated control problems, see section 2.1.

1.2. Previous results

A few partial results have been published. In [**13**] it was shown that $x_{n+1} = \max\left\{\frac{A}{x_n}, \frac{B}{x_{n-1}}\right\}$, $A, B, \vec{x} > 0$ is eventually periodic. In [**5**] Voulov showed that the slight variation $x_{n+1} = \max\left\{\frac{A}{x_n}, \frac{B}{x_{n-2}}\right\}$, $A, B, \vec{x} > 0$ is also eventually periodic. In another paper Voulov obtained a more general result, proving that systems with exactly two nonzero A components are eventually periodic. So $x_n = \max\left\{\frac{A}{x_{n-k}}, \frac{B}{x_{n-m}}\right\}$, $A, B, \vec{x} > 0$ is eventually periodic. In [**11**] Voulov gave conditions for $x_{n+1} = \max\left\{\frac{A}{x_{n-1}}, \frac{B}{x_{n-2}}, \frac{C}{x_{n-3}}\right\}$, $A, B, C, \vec{x} > 0$ to be eventually periodic, and in [**3**] he showed that $x_{n+1} = \max\left\{\frac{A}{x_{n-1}}, \frac{B}{x_{n-3}}, \frac{C}{x_{n-5}}\right\}$, $A, B, C, \vec{x} > 0$ is always eventually periodic. In [**4**] Szalkai gave a complete description of the conditions for boundedness and for the system being eventually periodic in the case that $A < 0$. This is discussed in case 5 of this paper.

1.3. Outline

In this paper we give results for the general system, with k arbitrarily large and for all possibilities of the signs of A and $\vec{x}$. This paper is divided into several sections, depending on the signs of the A and $\vec{x}$ components.

Case 1: $A \geq 0$,	$\vec{x} > 0$.
Case 2: $A \geq 0$,	exists initial $x_i > 0,\ x_j < 0$.
Case 3: $A \geq 0$,	$\vec{x} < 0$.
Case 4: $\exists A_i > 0, A_j < 0$,	$\vec{x} \in (\mathbb{R}\backslash 0)^k$.
Case 5: $A \leq 0$,	$\vec{x} \in (\mathbb{R}\backslash 0)^k$.

The first case, called the positive case, is the most important. Most previous work on this system has focussed on the positive case. It is considered the most likely to have physical interpretations and therefore applications. We begin in chapter two by considering possible applications. This is brief, since at present I know of no models of real processes that use our system. Then we begin the analysis of the positive case. First we show that the solution is bounded and persists, which was previously known [**3**]. Then a crucial transformation into a topologically conjugate system, which I call the log version, is given. The log version is

$$y_n = \max_{i \in [1,k]} \{B_i - y_{n-i}\}, \; B_i \in [-\infty, 0] \; \exists j \; s.t. \; B_j = 0. \tag{1.2}$$

We consider this as a dynamical system $f : \mathbb{R}^k \to \mathbb{R}^k$. This is useful because this dynamical system is easily shown to be nonexpansive in L_∞. As Weller has demonstrated in [**6**], bounded maps which are nonexpansive in a polyhedral norm (such as L_∞) have finite ω-limit sets. Moreover, the points in an ω-limit set are periodic with period equal to the size of the ω-limit set. Since this is such an important result, and since Weller's proof is not readily available, I have included a slightly modified version of his proof. Finally we are able to show that our system is eventually periodic, by showing that an orbit must jump onto its ω-limit set. This is because the set of distances between points in an orbit and the ω-limit set is finite.

After having shown that the positive system is eventually periodic we turn in chapter three to the question of the maximum period length for a system of size k. It is known that their is an upper limit on period lengths of bounded nonexpansive maps, depending only on k [**2**]. This limit is

$$N = \frac{k! 2^k}{(ln2)^k}. \tag{1.3}$$

It has been conjectured that the actual upper limit is just $N = 2^k$. In the same article Nussbaum finds better upper limits for classes of maps which satisfy additional properties. This is an area where similar research into our system could probably yield a better upper limit. For our system Ladas has conjectured that the upper limit of period lengths is $2k$. We show this is false by construction, giving examples of systems with periods longer than $2k$. The smallest such system has $k = 16$, and period $= 50$. There are three varieties of systems that we construct with periods longer than $2k$.

They are called the long period model, the improved model, and the unrestricted model. An example of an unrestricted model is given, but no attempt is made to determine bounds of period length as a function of k for these systems.

The long period model and the improved model are based on the idea of having independent subcycles of differing lengths, which makes the period equal to the least common multiple of these lengths. A method for constructing these long period systems is given, then the systems are analyzed to get a lower bound on maximum period length. The formulas relating k and period length involve two functions: $\mathrm{glcm}(M, N)$, the greatest least common multiple of M numbers less than or equal to N; and baselen (M), abbreviated b_M, the minimum size "base" needed to support M independent subcycles. $\mathrm{glcm}(M, N)$ is a fairly well studied function, with several known bounds, depending on the relation of M and N. But b_M is a novel function, and several methods of estimating its growth are presented. The improved model is based on a refinement which allows lower values of b_M, and thus longer periods for a similar k. We denote these improved base lengths $\hat{b}_M$. A computer program has found minimal solutions of b_M up to $M = 15$, which provides long period models up to $k = 8,000$. This program is given in appendix A. The execution time of this program grows polynomially, roughly cubic in M, so a faster computer and longer run times could easily extend these results. A more complicated program for finding $\hat{b}_M$ is given in appendix B. This program is slower, but values of $\hat{b}_M$ have also been found up to $M = 15$.

One method of estimating b_M is a probabilistic argument, which considers the number of possible tries and the probability that one try will satisfy the necessary conditions. It results in a fairly accurate estimate, slightly too high. Then a pattern is presented which is proven to satisfy the conditions, allowing a provable upper bound. It yields values for b_M which are roughly three times too high. We do not have a proven upper bound for $\hat{b}_M$, since $\hat{b}_M$ is much harder to analyze. The upper bound for b_M is used to obtain a proven lower bound on maximum period lengths, and a conjectured lower bound is given.

In chapter four we consider several other properties of the positive case. First we consider the preperiod, which is how long an initial point takes to become periodic. Systems are divided into two classes, those with a bound on preperiod for all initial points and those with unbounded preperiod. Several examples of both types are presented, along with some sufficient but not necessary conditions for unbounded preperiod. Then we consider the structure of the dynamical system from the log version. This map is piecewise linear with all partial derivatives in the set $\{-1, 0, 1\}$. One known result is that piecewise linear maps have a unique half line of translation. We also show that periodic orbits are not locally attractive.

In chapters five through eight we look at other cases. Several are very simple, with a transformation into the positive case. However, case 4 has special properties. In this case a set of necessary and sufficient conditions for the system to be bounded are given. These conditions are so strict that it is easily shown that a system which satisfies the conditions for bounded must also be eventually periodic, with a period no more than $2k$, and a preperiod no more than k.

In chapter nine we analyze a generalization of RDEM, which allows fewer than k terms in the max. For example, $x_n = \max\left\{\frac{0.6}{x_{n-2}}, \frac{-0.5}{x_{n-5}}\right\}$. In this example $k = 5$. In the positive case it is equivalent to assign all the missing A terms a value of zero, but for all the other cases the generalized system is distinct. Each case is examined, with examples and several conjectures.

In chapter ten we make a survey of several similar delay difference equations with the Max operator which have been analyzed in the literature. We check if any of these systems can be translated into a topologically conjugate system which is nonexpansive in L_∞. Most can not, but the method works for one of these systems, which implies that if it is bounded then it has a finite ω-limit set. Necessary and sufficient conditions for this system to be bounded have already been established. Using the method from chapter two it can be proven that bounded implies eventually periodic, which is a new result.

Finally in chapter eleven we list some open problems and conjectures related to this system.

1.4. Notation and Definitions

In this paper the system in equation (1.1) will be abbreviated RDEM, for the Reciprocal Difference Equation with Maximum.
If A is a set, the notation $|A|$ indicates the cardinality of A.
It will be helpful to have a precise definition of what a delay difference equation is.

Definition 1. Delay difference equation
A delay difference equation consists of a function $f : D^k \to D$ which is applied to an initial point $\vec{x} = (x_i)_{i=-k}^{-1} \in D^k$ to generate the sequence $(x_i)_{i=0}^{\infty}$ by the rule $x_n = f\left((x_i)_{i=n-k}^{n-1}\right)$.

Instead of analyzing the delay difference equation directly it is sometimes helpful to consider an equivalent dynamical system, since then the domain and range space are the same.

Definition 2. Equivalent dynamical system
For a given delay difference equation $f : D^k \to D$, with D a metric space, the equivalent dynamical system is $g : D^k \to D^k$ defined by

$$g_i\left((x_j)_{j=n}^{n+k-1}\right) = \begin{cases} x_{n+i}, & \text{if } i < k; \\ f(x) & \text{if } i = k. \end{cases}$$

Note that $g\left((x_j)_{j=n}^{n+k-1}\right) = (x_i)_{i=n+1}^{n+k}$, so the dynamic system and the difference equation both generate the same x sequence. We will frequently want to convert a dynamical system into a new one which retains some of the dynamical properties. The following definition makes this precise.

Definition 3. topologically conjugate system
We say that dynamical systems $g : E \to E$ and $f : D \to D$ are topologically conjugate if there is a continuous bijection with a continuous inverse $\phi : D \to E$, such that $\phi(f(x)) = g(\phi(x))$ for all $x \in D$.

We say that two delay difference equations are topologically conjugate if the two equivalent dynamical systems are topologically conjugate to each other. Clearly if two systems are topologically conjugate then periodic orbits get mapped into periodic orbits. We also need to define what is meant by periodic and eventually periodic for dynamic systems and delay difference equations.

Definition 4. Periodic and eventually periodic dynamic systems
For a dynamic system $f : D \to D$, $x \in D$ is periodic with period P if P is the smallest positive integer such that $x = f^P(x)$. x is eventually periodic if $\exists$ M s.t. $f^M(x) = f^{M+P}(x)$. The smallest such P is the period and the smallest such M is called the preperiod of x. If f is eventually periodic for all $x \in D$ we say that f is eventually periodic over D. If there exists a P such that $\forall x \in D$ $\exists M$ s.t. $f^M(x) = f^{M+P}(x)$ then the smallest such P is called the period of f over D. If there is an upper bound on the size of the preperiod of x for all $x \in D$ then the least upper bound is called the preperiod of f over D.

Note that a dynamic system may have a period of P even though the period of all $x \in D$ is less than P.

Definition 5. Eventually periodic delay difference equations
An initial value, $\vec{x}$ is said to be eventually periodic for the delay difference equation $f : D^k \to D$ if there exists a nonnegative integer M and a positive integer P such that $x_{M+n} = x_{M+P+n}$ $\forall$ $n \in [0, k)$. The smallest such P is the period of x and the

smallest such M is the preperiod of x. If the delay difference equation is eventually periodic for all $x \in D$ we say that it is eventually periodic over D. If there exists a P such that $\forall x \in D\ \exists M$ s.t. $x_{M+n} = x_{M+P+n}\ \forall\ n$
$in[0, k)$ then the smallest such P is called the period of f over D. If there is an upper bound on the size of the preperiod of x for all $x \in D$ then the least upper bound is called the preperiod of the delay difference equation over D.

Note that the period of a delay dynamical system for a given initial point is the same as the period of the equivalent dynamical system for that initial point.

Following Nussbaum, [**2**] we define a polyhedral norm as follows.

Definition 6. Polyhedral norm
If V is a finite dimensional vector space, a norm $\|\cdot\|$ on V is called polyhedral if the unit ball $B = \{x \in V : \|x\| \leq 1\}$ is a polyhedron and hence has only finitely many extreme points.
An equivalent definition is that a norm is polyhedral if and only if there exists finitely many linear functionals $\phi_1, \phi_2, \cdots, \phi_N$ such that

$$\|x\| = \max_{i \in [1,N]} |\phi_i(x)|.$$

Chapter 2

Case 1: $A \geq 0, \quad \vec{x} > 0$

This is the most important case, considered to have the most potential for applications. We begin this chapter with a brief discussion of possible applications. Then the analysis of the positive case begins. First we follow Voulov in translating it into a topologically conjugate system which is bounded. Then a second translation into the log version gives a dynamical system which is nonexpansive in L_∞. Then we give Weller's proof that bounded and nonexpansive implies a finite ω-limit set. Finally we use this result to prove that the positive case is eventually periodic.

2.1. Applications

The max operator appears in many models, including automated control theory. Many similar systems, such as max Lyness equations, have applications. See chapter 9 for a survey of similar systems. In [**5**] similar systems are said to have applications related to a type of differential equation with deviating arguments (DEDA). However, there does not appear to be any models that have been developed for RDEM. The positive case is the most likely to have applications, since many physical quantities only take positive values, and most applications of similar systems are for positive $\vec{x}$.

2.2. Initial translations

In the positive case solutions always exist, since $A_k > 0$, and $x_n \geq \frac{A_k}{x_{n-k}} > 0$ for all n. Also, the solution is always bounded. To show that the solution is bounded it is helpful to first transform it into a topologically conjugate system, as was done by Voulov [**3**].

$$D = \max_{i \in [1,k]} \{A_i\}, \qquad A'_i = \frac{A_i}{D}, \quad x'_i = \frac{x_i}{\sqrt{D}}. \tag{2.1}$$

The new system is $x'_n = \max_{i \in [1,k]} \left\{ \frac{A'_i}{x'_{n-i}} \right\}$. This system has $A'_i \in [0,1]$, $\exists j \in [1,k]$ $s.t.$ $A'_j = 1$, and $x_i > 0$ for all $i \geq -k$.

Lemma 1. *In case 1 the solution is bounded away from zero.*

Proof.

$$\text{Let} j \in \mathbb{N}\ s.t.\ A'_j = 1. \quad \text{Let} \;\; c = \min_{i \in [1,k]} \left\{ x'_{-i}, \frac{1}{x'_{-i}} \right\}.$$

Note $c \leq 1$. We show $x'_n \in [c, 1/c]$ $\forall\, n \geq 0$ by induction. $x'_0 \geq \frac{A'_j}{x'_{-j}} = \frac{1}{x'_{-j}} \geq c$. The maximum possible value of x'_0 is the largest possible A' value, which is 1, divided by the smallest possible x' value, which is c. So the inductive hypotheses is true for $n = 0$. Assume true up to $n = m - 1$. Then $x'_m \geq \frac{1}{x'_{m-j}} \geq c$. The maximum possible value of x'_m is again the largest possible A' value, which is 1, divided by the smallest possible x'_n value, for some $n < m$, which is c. So the maximum possible value of x'_m is $1/c$, and therefore $x'_n \in [c, 1/c]$ $\forall\, n \geq 0$. □

To show that the system is eventually periodic we make another conversion, taking the log of the A coefficients and initial x values. Since the x values are bounded away from zero the new system will also be bounded.

$$y_i = \ln x'_i \qquad B_i = \begin{cases} \ln(A'_i), & \text{if } A'_i > 0; \\ \text{`}-\infty\text{'} & \text{if } A'_i = 0. \end{cases} \tag{2.2}$$

Let $b = \max_{i\in[1,k]} |y_{-i}|$. The equivalent system is

$$y_n = \max_{i\in[1,k]}\{B_i - y_{n-i}\},\ B_i \in [-\infty, 0]\ \exists\ j\ s.t.\ B_j = 0. \tag{2.3}$$

Note $y_n \in I = [-b, b]$ for all $n \geq -k$. Now we consider the equivalent dynamical system $f : \mathbb{R}^k \to \mathbb{R}^k$, which has the property $f\left((y_i)_{i=n}^{n+k-1}\right) = (y_j)_{j=n+1}^{n+k}$. f is continuous, piecewise linear, and $\frac{\partial f_i}{\partial y_j}\Big|_y \in \{-1, 0, 1\}\ \forall\ i, j \in [1, k],\ y \in \mathbb{R}^k$. Proving that f is eventually periodic is equivalent to proving that the initial system is eventually periodic. We first observe that f is nonexpansive in the L_∞ norm.

Lemma 2. *f is nonexpansive in the L_∞ norm.*

Proof. Let $x, y \in I^k$. Let $c = \|x - y\|_\infty = \max_{i\in[1,k]} |x_i - y_i|$. Claim: $|x_{k+1} - y_{k+1}| \leq c$. Without loss of generality assume $x_{k+1} \geq y_{k+1}$. Let $j \in [1, k]\ s.t.\ x_{k+1} = B_j - x_{k+1-j}$. We know $y_{k+1} \geq B_j - y_{k+1-j}$, so $0 \leq x_{k+1} - y_{k+1} \leq B_j - x_{k+1-j} - (B_j - y_{k+1-j}) = y_{k+1-j} - x_{k+1-j} \leq c$. Thus the claim is proven. So $\|f(x) - f(y)\|_\infty = \max_{i\in[2,k+1]} |x_i - y_i| \leq c$. □

Note that f being nonexpansive implies that f^n is also nonexpansive. It is known that for D a compact set, if a function $f : D \to D$, is nonexpansive in a polyhedral norm, such as L_∞, then for any $x \in D$ the ω-limit set of x, denoted $\omega(x; f)$, is finite [**6**]. We prove this in the following section.

2.3. Compact nonexpansive maps have finite ω-limit sets.

This section comes from Weller's dissertation. Recall that a polyhedral norm can be put in the form $\|x\| = \max_{i\in[1,N]} |\phi_i(x)|$, where the ϕ_i are linear functionals on $x \in D$. Thus a polyhedral norm on D can be translated into L_∞ on $\mathbb{R}^k$ by $x_i = \phi_i x$. For the rest of this section we will assume that $x \in I^k$, and that T is bounded, $T : I^k \to I^k$, and nonexpansive in L_∞.

We need several preliminary lemmas before we can show that ω-limit sets of maps which are nonexpansive in a polyhedral norm are finite.

LEMMA 3. *For T nonexpansive $y \in \omega(x;T)$ implies $\omega(y;T) = \omega(x;T)$.*

PROOF. We show $\omega(y;T) \subset \omega(x;T)$. Let $z \in \omega(y;T)$ and consider k_i and n_i increasing sequences of positive integers such that $\lim_{i\to\infty} T^{k_i}x = y$, and $\lim_{i\to\infty} T^{n_i}y = z$. Then
$d\left(T^{n_i+k_i}x, z\right) \le d\left(T^{n_i+k_i}x, T^{n_i}y\right) + d\left(T^{n_i}y, z\right) \le d\left(T^{k_i}x, y\right) + d\left(T^{n_i}y, z\right).$
To show $\omega(x;T) \subset \omega(y;T)$ let $z \in \omega(x;T)$ and let k_i and n_i be increasing sequences of positive integers such that $\lim_{i\to\infty} T^{k_i}x \to y$ and $\lim_{i\to\infty} T^{k_i+n_i}x \to z$. Then
$d\left(T^{n_i}y, z\right) \le d\left(T^{n_i+k_i}x, T^{n_i}y\right) + d\left(T^{k_i+n_i}x, z\right) \le d\left(T^{k_i}x, y\right) + d\left(T^{k_i+n_i}x, z\right).$ ☐

For the rest of this chapter we will assume that $x \in \omega(x;T)$.

LEMMA 4. *T is an isometry on $\omega(x;T)$ for any x in the domain of T. This means that for $y, z \in \omega(x;T)$, $d(y,z) = d(Ty,Tz)$.*

PROOF. Suppose not. Then there exists $y, z \in \omega(x;T)$ and $D > 0$ such that $d(y,z) = d(Ty,Tz) + D$. By lemma 3, $z \in \omega(y;T)$, so we can find a K such that $d(T^K y, z) \le D/8$. Now since T is nonexpansive $C = \lim_{n\to\infty} d\left(T^n y, T^{n+K}y\right)$ exists, and $L > K$ can be chosen such that $C \le d\left(T^n y, T^{n+K}y\right) \le C + D/8$ for $n > L$. Now choose $n > L$ such that $d(T^n y, y) \le D/8$ and use the triangle inequality:

$$
\begin{aligned}
(2.4)\quad d(y,z) &= d(Ty,Tz) + D \\
&\ge D + d(T^{n+1}y, T^{n+K+1}y) - d(Ty, T^{n+1}y) - d(T^{n+K+1}y, T^{K+1}y) - d(T^{K+1}y, Tz) \\
&\ge D + C - D/8 - D/8 - D/8 = C + \frac{5D}{8}
\end{aligned}
$$

and also

$$
\begin{aligned}
(2.5)\quad d(y,z) &\le d(y, T^n y) + d(T^n y, T^{n+K}y) + d(T^{n+K}y, T^K y) + d(T^K y, z) \\
&\le D/8 + (C + D/8) + D/8 + D/8 = C + \frac{4D}{8}
\end{aligned}
$$

Which is a contradiction. $\square$

LEMMA 5. *For all $n \in \mathbb{N} \quad T^n$ is an isometry on $\omega(x;T)$.*

PROOF. Let $y, z \in \omega(x,T)$. By the previous lemma $d(y,z) = d(Ty,Tz)$ and $Ty, Tz \in \omega(x;T)$, so $d(y,z) = d(Ty,Tz) = d(T^2y, T^2z) = \ldots = d(T^n y, T^n z)$. $\square$

LEMMA 6. *For all $n \in \mathbb{N}, u, v \in \omega(x;T) \quad d(u, T^n u) = d(v, T^n v)$.*

PROOF. Let $u, v \in \omega(x;T)$, and fix $n \in \mathbb{N}, \epsilon > 0$. u and v may be approximated by iterations of x, since they are in its omega limit set. Let $K, M \in \mathbb{N}$ such that $d(T^K x, u) < \epsilon/5$, and $d(T^{K+M}x, v) < \epsilon/5$. By the previous lemma
$d(T^K x, T^{K+n}x) = d(T^{K+M}x, T^{K+M+n}x)$,
$d(T^n u, T^{K+n}x) = d(u, T^K x) \leq \epsilon/5$ and $d(T^n v, T^{K+M+n}x) = d(v, T^{K+M}x) \leq \epsilon/5$,
so by several applications of the triangle inequality we have

$$|d(u, T^n u) - d(T^K x, T^{K+n}x)| \leq d(u, T^K x) + d(T^n u, T^{K+n}x) \leq \frac{2\epsilon}{5}$$
$$|d(v, T^n v) - d(T^{K+M}x, T^{K+M+n}x)| \leq d(v, T^{K+M}x) + d(T^n v, T^{K+M+n}x) \leq \frac{2\epsilon}{5}$$

Hence

$$|d(u, T^n u) - d(v, T^n v)| \leq \frac{4\epsilon}{5}.$$

$\square$

Let $Y_N = \omega(x;T^N)$. So $Y_1 = \omega(x;T)$.

LEMMA 7. *Y_N is a compact subset of Y_1 and contains x.*

PROOF. Since Y_N is an ω-limit set it is closed. And Y_N is bounded since $Y_N \subset \omega(x;T)$ and we assume that $\omega(x;T)$ is bounded. We show that for all $\epsilon > 0$ there exists a $k \in \mathbb{N}$ such that $|x - T^{Nk}x| \leq \epsilon$. Choose k such that $|x - T^k x| \leq \epsilon/N$. Then $|x - T^{Nk}x| \leq |x - T^k x| + |T^k x - T^{2k}x| + \ldots + |T^{(N-1)k}x - T^{Nk}x| \leq \epsilon$. $\square$

Denote $d_N = \sup_{y,z \in Y_N} |y - z|$ for the diameter of Y_N, and
$D_N(y) = \{z \in Y_N : |y - z| = d_N\}$ for $y \in Y_N$.

LEMMA 8. *Every set* $D_N(y)$ *is a nonempty subset of* Y_N.

PROOF. Let $\epsilon > 0$ be fixed. Let $K, M \in \mathbb{N}$ such that $|T^{KN}x - T^{(K+M)N}x| \geq d_N - \epsilon$. Then $|y - T^{MN}y| = |T^{KN}x - T^{(K+M)N}x| \geq d_N - \epsilon$, and $T^{MN}y \in Y_N$, so there exists elements of Y_N whose distance from y approaches d_N. And since Y_N is closed it must therefore contain an element that is exactly d_N from y. $\square$

For $u \in I^k$ denote $I(u) = \{j \in \mathbb{N} : |u_j| = |u|\}$, the set of indices on which u achieves its norm.

LEMMA 9. *Given* $u \in I^k$ *there exists* $\epsilon > 0$ *such that* $\forall\, v \in I^k \quad |u - v| < \epsilon \Rightarrow I(v) \subset I(u)$.

PROOF. Choose $\epsilon > 0$ such that $|u_j| + 2\epsilon < |u|$ for all $j \notin I(u)$. For example $\epsilon = 1/3 \cdot \inf\{|u| - |u_j| : j \notin I(u)\}$. $\square$

LEMMA 10. *Fix* $N \in \mathbb{N}$ *and write* $D(y), Y, d$ *instead of* $D_N(y), Y_N$ *and* d_N. *Then for all* $y \in Y$ *the following hold:*
(i) $D(y)$ *is relatively open in* Y.
(ii) $\exists\, n \in \mathbb{N}$ *such that* $T^{nm}z \in D(y)\ \forall\, m \in \mathbb{N}, z \in D(y)$.

PROOF. (i) Fix $y \in Y, z \in D(y)$, so $|y - z| = d$. Let $u = z - y$. Choose $\epsilon > 0$ for u by lemma 9. Note $\epsilon < d$. Choose L a multiple of N such that $|T^L y - y| < \epsilon/2$. This is possible since $y \in \omega(y) = Y$, so there is a sequence in Y which limits on y. Each element in this sequence can be expressed as T^{L_i} for some L_i a multiple of N. We show $T^L z \in D(y)$. Note $|T^L z - z| = |T^L y - y| < \epsilon/2$. Let $j \in I(T^L z - T^L y) \subset I(z - y)$. By isometry $|T^L z - T^L y| = |z - y|$, so $|(T^L z - T^L y)_j| = d$. Suppose $(z - y)_j = -d$. Then $(z - T^L y)_j \geq -d$, so $(T^L y - y)_j \leq 0$, and similarly $(z - T^L z)_j \leq 0$. Now suppose instead that $(z - y)_j = d$. Then we get $(z - T^L z)_j \geq 0$ and $(T^L y - y)_j \geq 0$. So in either case $(z - T^L z)_j$ and $(T^L y - y)_j$ have the same sign. Also $(z - y)_j$ and $(T^L z - T^L y)_j$ must have the same sign, since $|z_j - y_j| = d$, and $|(T^L z - z)_j| = |(T^L y - y)_j| < \epsilon/2 <$

$d/2$. So we get $(z-y)_j = (T^L z - T^L y)_j = (z - y - (T^L y - y) - (z - T^L z))_j$ and hence $(T^L y - y)_j + (z - T^L z)_j = 0$. Therefore both these terms are zero, which implies that $|(y - T^L z)_j| = d$ and hence $T^L z \in D(y)$.

So, for $p \in Y$ such that $|z-p| < \epsilon/4$ we can approximate p by a sequence $T^{L_i} z$ such that $|y - T^{L_i} y| = |z - T^{L_i} z| < \epsilon/2$. Hence $|y - T^{L_i} z| = d$ and so $|y-p| = d$. This shows that a neighborhood of z is contained in $D(y)$.

(ii) $Y - D(y)$ is closed. And since $D(y)$ is compact the distance between the two sets, $\delta = \inf\{|u-v| : u \in D(y),\ v \in Y\}$ is greater than zero. Choose n a multiple of N such that $|y - T^n y| < \delta$. Then lemma 6 gives $|z - T^n z| < \delta$ for every $z \in D(y)$. So $T^n z \in D(y)$ for every $z \in D(y)$, thus $D(y)$ is invariant of $T^n z$. $\square$

Lemma 11. *Let $N \in \mathbb{N}$, $\epsilon > 0$ be given, and $y \in D_N(x)$. Then there exists an $m \in \mathbb{N}$ such that $d_m < \epsilon$, and $Y_m \subset D_N(y)$.*

Proof. Since $y \in D_N(x)$ it follows that $x \in D_N(y)$. By lemma 10 (ii) there exists N_1 such that $Y_{N_1} \subset D_N(y)$. Choose $y_1 \in D_{N_1}(x) \subset Y_{N_1}$, and apply lemma 10 again as before to find N_2. Then we have $Y_{N_2} \subset D_{N_1}(y_1) \subset Y_{N_1}$. Choose $y_2 \in D_{N_2}(x) \subset Y_{N_2}$, and continue this procedure, generating sequences y_i and N_i such that $y_i \in D_{N_i}(x) \subset Y_{N_i} \subset D_{N_j}(y_j)$, for all $j \leq i$. This implies $|y_i - y_j| = d_{N_j}$ for all $j \leq i$. By compactness the sequence (y_i) contains a Cauchy subsequence, and hence N_k exists such that $d_{N_k} < \epsilon$. Take $m = N_k$, and it is proven. $\square$

Theorem 1. *Let $C \subset E$ be a compact set, $|\cdot|$ a polyhedral norm on E and $T : C \to C$ nonexpansive in that norm. Then $\omega(x, T)$ is a finite set for all $x \in C$.*

Proof. We will prove this for $E = \mathbb{R}^k$, and $C = I^k \subset \mathbb{R}^k$, and with the norm $|\cdot| = L_\infty$. This is equivalent, as explained at the start of this section.

Choose $y_1 \in D_1(x)$, let $u_1 = y_1 - x$, and choose $\epsilon_1 > 0$ according to lemma 9, such that for all $v \in E$, $|u_1 - v| < \epsilon_1 \Longrightarrow I(v) \subset I(u_1)$. Choose N_1 by lemma 11. Then $|y - x| < \epsilon$, $\forall\, y \in Y_{N_1}$, and $Y_{N_1} \subset D_1(y_1)$. Choose $y_2 \in D_{N_1}(x) \subset Y_{N_1}$, let $u_2 = y_2 - x$, choose ϵ_2 by lemma 9, and N_2 by 11. Continuing in this manner, generate sequences $(\epsilon_i), (N_i)$, and sequences of sets such that $Y_{N_{j+1}} \subset D_{N_j}(y_j) \subset Y_{N_j}$, and a sequence of points (y_i) such that $y_{j+1} \in D_{N_j}(x) \subset Y_{N_j}$. Now consider the sets $I(y_i - x)$. Claim: there exists an $L, K \in \mathbb{N}$ such that $L < K$ and $I(y_L - x) = I(y_K - x)$. This is true by the pigeon hole principal, since possible values of $I(\cdot)$ are restricted to a set of size 2^k , which is finite. Let L, K satisfy the above property. Let $y \in Y_{N_L}$. Then $|y - x| < \epsilon_L$, hence $I(y_L - y) \subset I(y_L - x)$. For $\tilde{y} \in D_{N_L}(y_L)$ we have $|y_L - \tilde{y}| = |y_L - x|$, because $x \in Y_{N_{L+1}} \subset D_{N_L}(y_L)$. Therefore for $j \in I(y_L - y_K)$ we have $j \in I(y_L - x) = I(y_K - x)$, and furthermore, $(y_L - y_K)_j = |y_L - y_K| = |y_L - x| = (y_L - x)_j$, which implies $y_K = x$. Now $x = y_K \in D_{N_{K-1}}(x)$, so $d_{N_{K-1}} = 0$ and hence $x = T^{N_{K-1}}(x)$. $\square$

2.4. The positive case is eventually periodic.

Since the log version of the positive case is continuous (in general nonexpansive implies continuous) the result that the ω-limit set is finite implies that if $P = |\omega(x; f)|$, then all $y \in \omega(x; f)$ are periodic with period P. (Recall we use the notation $|A|$ to denote the number of elements in A). However, bounded nonexpansive maps in polyhedral norms are not always eventually periodic, since initial points may limit on the periodic ω-limit set without actually reaching it. For example, the delay difference equation $x_{n+1} = \frac{\max\{x_n, x_{n-1}\}+1}{2}$ has an equivalent dynamical system which is bounded and nonexpansive in the L_∞ norm. The ω-limit set of any initial point is $(1, 1)$, but initial points that are off this set do not reach it. We show this does not happen with our system in the following theorem.

Theorem 2. *f is eventually periodic. Moreover, the period of $y \in I^k$ is equal to $P = |\omega(y; f)|$.*

Proof. Let $\bar{y} \in I^k$, $I = [-1, 1]$. Let $P = |\omega(\bar{y}, f)|$.
Let $\dot{y} = (\dot{y}_i)_{i=1}^k = \lim_{j\to\infty} f^{Pj}(\bar{y})$. Note $f^P(\dot{y}) = \dot{y}$, and $\dot{y} \in \omega(\bar{y}; f)$. Generate $(\dot{y}_i)_{i=1}^\infty$

by applying f to the original $\dot{y}$. Set ϵ by the equation

$$\epsilon = \min \left\{ 1,\ \frac{\dot{y}_i - B_j + \dot{y}_{i-j}}{3} \ \middle|\ i \in [k+1, k+P], j \in [1,k],\ B_j - \dot{y}_{i-j} < \dot{y}_i \right\}$$

ϵ is the minimum of one third of the smallest difference between any $\dot{y}_i$ and the terms in its max equation, or 1. The min with 1 is in case the set of differences is empty. Note $\epsilon > 0$. Let $U_\epsilon = \left\{ y = (y_i)_{i=1}^k \mid \|y - \dot{y}\|_\infty < \epsilon \right\}$. Since $f^P(\bar{y})$ limits on $\dot{y}$, there exists an n such that $f^{Pn}(\bar{y}) \in U_\epsilon$. Let $n \in \mathbb{N}$ such that $f^{Pn}(\bar{y}) \in U_\epsilon$. Let $f^{Pn}(\bar{y}) = y = (y_i)_{i=1}^k$. Thus $y_i = \bar{y}_{i+Pn}$ for all $i > 0$. For $i \in [1,k]$ let $\Delta y_i = |\dot{y}_i - y_i|$. Let $D = \{0\} \cup \{\Delta y_i | i \in [1,k]\}$. Let $j \in [1,k]$ $s.t.$ $y_{k+1} = B_j - y_{k+1-j}$. Suppose $\dot{y}_{k+1} \neq B_j - \dot{y}_{k+1-j}$. Then

$0 < \epsilon \leq \frac{\dot{y}_{k+1} - B_j + \dot{y}_{k+1-j}}{3}$.

$3\epsilon \leq \dot{y}_{k+1} - B_j + \dot{y}_{k+1-j}$

$\dot{y}_{k+1-j} \geq 3\epsilon + B_j - \dot{y}_{k+1}$

$\dot{y}_{k+1-j} - y_{k+1-j} \geq 3\epsilon + (y_{k+1} - \dot{y}_{k+1})$. Which is impossible since

$\dot{y}_{k+1-j} - y_{k+1-j} \leq \epsilon$ and $\dot{y}_{k+1} - y_{k+1} \leq \epsilon$.

Therefore it must be true that $\dot{y}_{k+1} = B_j - \dot{y}_{k+1-j}$.

Therefore $y_{k+1} = B_j - y_{k+1-j} = B_j - \dot{y}_{k+1-j} \pm \Delta y_{k+1-j} = B_j - (B_j - \dot{y}_{k+1} \pm \Delta y_{k+1-j}) = \dot{y}_{k+1} \pm \Delta y_{k+1-j}$, so $|y_{k+1} - \dot{y}_{k+1}| \in D$, and by induction $|y_i - \dot{y}_i| \in D \ \forall\ i \in \mathbb{N}$. Now let $\delta = \min \{\Delta y_i \in D | \Delta y_i > 0\}$. Let $m \in \mathbb{N}$ $s.t.$ $\|f^{mP}(y) - \dot{y}\| < \delta$. Then

$\max_{i \in [1,d]} \{|y_{mP+i} - \dot{y}_i|\} < \delta$, and $\max_{i \in [1,d]} \{|y_{mP+i} - \dot{y}_i|\} \in D$, so

$\max_{i \in [1,d]} \{|y_{mP+i} - \dot{y}_i|\} = 0$, and $\dot{y} = f^{mP}(y) = f^{(n+m)P}(\bar{y})$, and therefore

$f^{(n+m+1)P}(\bar{y}) = f^{(n+m)P}(\bar{y})$, so $\bar{y}$ is eventually periodic with period $P = |\omega(\bar{y}; f)|$. $\square$

Chapter 3

Bounds on maximum period lengths

Having proven that the positive case is eventually periodic, in this chapter we establish bounds on maximum period lengths for systems of size k. The upper bound relies on a general result for upper bounds of ω-limit sets of nonexpansive maps, while the lower bound comes from a model constructed to obtain long periods, which we call the long period model.

3.1. Upper limit

It is known that for D a compact subset of $\mathbb{R}^k$ and $f : D \to D$ nonexpansive in L_∞ there exists an integer N, dependant only on k, such that $|\omega(x; f)| \leq N \ \forall \ x \in D$. Nussbaum [**2**] has established that this general upper limit is satisfied by

$$N = \frac{k!2^k}{(ln2)^k}. \tag{3.1}$$

It is conjectured that $N = 2^k$, is an upper limit, which if correct would be a sharp limit, since for example a permutation of the vertices of a unit hypercube, which can be represented as points in the set $\{(x_i)_{i=1}^k | x_i \in \{0,1\} \ \forall \ i \in [1,k]\}$, is nonexpansive in L_∞ and has period 2^k. For RDEM systems period length is equal to the size of the ω-limit set.

Definition 7. per (k) *The function* per (k) *is defined to be the least upper bound of the period lengths of all positive case systems of size k.*

LEMMA 12. $\operatorname{per}(k) \leq \frac{k!2^k}{(ln2)^k}$

PROOF. By theorem 2 RDEM systems are periodic, with period length equal to the size of the ω-limit set. So the lemma follows from Nussbaum's result in equation 3.1. $\square$

Some sharper upper bounds have been obtained for bounded nonexpansive maps which satisfy additional properties. In [**2**] it is shown that maps which satisfy

$$T(x \wedge y) = T(x) \wedge T(y), \quad \text{Where } (x \wedge y)_i = \min(x_i, y_i).$$

have a much lower upper bound. However, the positive case of RDEM does not satisfy this property. For the log version, with $T : \mathbb{R}^k \to \mathbb{R}^k$ the dynamical system, and $f : \mathbb{R}^k \to \mathbb{R}$ the difference equation we have

$$T(x)_i = \begin{cases} x_{i+1}, & \text{if } i < k; \\ f(x) = \max_{j \in [1,k]} \{B_j - x_{k+1-j}\} & \text{if } i = k. \end{cases}$$

This results in

$$T(x \wedge y)_i = \begin{cases} T(x)_i \wedge T(y)_i & \text{if } i < k; \\ \max(f(x), f(y)) & \text{if } i = k. \end{cases}$$

$$(T(x) \wedge T(y))_i = \begin{cases} T(x)_i \wedge T(y)_i & \text{if } i < k; \\ \min(f(x), f(y)) & \text{if } i = k. \end{cases}$$

So Nussbaum's results can not be applied to this system. However, it seems likely that a sharper upper bound can be obtained by using some of the special properties of this system. The long period model that we describe below is definitely not an upper bound, since we give examples of systems with longer periods for the same range of k.

3.2. A model for generating long period systems.

We analyze using the log version. Thus a system consists of $(B_i)_{i=1}^{k}$ such that $\exists\ j \in [1, k]$ such that$B_j = 0$. To aid in analysis we will construct in section 3.2.2.3 a representative initial y vector $\vec{y} = (y_i)_{i=-k}^{-1}$ having the following properties:
(i) y is periodic. In other words the preperiod of y is zero.
(ii) $y_0 = -1$.
(iii) $\|\vec{y}\|_\infty = 1$.

Note this implies $\max\{y_i | i \geq 0\} = 1$. If there is more than one B term equal to zero then the resulting pattern of -1's and 1's does not seem conducive to long periods, so we restrict our attention to systems with exactly one $j \in [1, k]$ such that $B_j = 0$. We follow Voulov [**3**] in saying that in this case B_j "dominates". Since $y_0 = -1$, we have $y_j = 1$. If there exists $y_i = -1$ for $i \in [1, 2j - 1]$, then these y positions put additional conditions on the system which seem to hinder long periods, so we will put a fourth restriction on our representative initial y vector:
(iv) $y_i > -1\ \forall\ i \in [1, 2j - 1]$.

Let $\vec{y}$ satisfy these four properties. Now since the preperiod is zero, and $y_0 = -1$, there must be a $y_i = -1$ for $i > 0$. Let i be the minimum positive integer such that $y_i = -1$. Then $y_{i-j} \geq 1$, so $y_{i-j} = 1$, and this forces $y_{i-2j} = -1$. Thus $i - 2j = 0$, and we have $y_{2nj} = -1$ and $y_{(2n+1)j} = 1$ for all $n \geq 0$. Note these are the only y positions equal to -1 or 1. The y_{nj} positions are a periodic subsequence of period length $2j$. We call this subcycle the base, and say that the base length is j. (See definition 10 below for a precise definition of subcycle.)

Definition 8. *Base, Base length: For the log version with B_j dominating we say the base length is j. For a given $\vec{y}$ with preperiod zero, and period P let i be the* *smallest nonnegative integer such that* $y_i = \min\{y_j \mid 0 \le j < P\}$*. Then the base is defined to be the y positions* y_{i+nj} *for all* $n \ge 0$*.*

Definition 9. *Generates, Generated by: We say* y_i *is generated by* y_m *if* $m < i$ *and* $y_i = B_{i-m} - y_m$*. In this case we also say that* y_m *generates* y_i*.*

Let B_j dominate. The B terms and $\vec{y}$ that we construct in section 3.2.2.3 will have the following structure. Let $i \in [1, 2j-1]$. Consider the y positions mod 2j. If y_i is generated by y_0, then we set $B_{2j-i} = B_i$ and require that y_{2j-i} is also generated by y_0. We also require that y_{2js+i} is generated from y_{2js} and y_{2js-i} is generated from y_{2js} for all nonnegative s. Then $y_i = y_{i+2j}$, and $y_{2j-i} = y_{4j-i}$, and these positions have period 2j. To get a period greater than 2j we need a set of y positions mod 2j which generate each other, which we call a subcycle.

Definition 10. *Subcycle: For a given system we say that a* $\vec{y}$ *with zero preperiod has a subcycle of* r *terms and length* q *if there exists a sequence of* r *increasing integers,* $(q_i)_{i=1}^{r}$*, with* $q_i \in [1, q-1]$ *for all* $i \in [1, r]$*, such that* $y_{q_{i+1}}$ *is generated by* y_{q_i} *for all* $1 \le i < r$*, and* y_{q+q_1} *is generated by* y_{q_r}*, and also* $y_{((p-1)q+q_i)} = y_{(pq+q_i)}$ *for all* $p \in \mathbb{N},\ i \in [1, r]$*.*
We say that a B term, B_g*, is associated with a subcycle if* $q + q_1 - q_r = g$ *or if* $\exists\, i \in [2, r]$ *s.t.*$q_i - q_{i-1} = g$*. Thus* $y_{q_i} = B_g - y_{q_{i-1}}$*.*

Note that the base fits the definition of a subcycle, but we will only use the term to refer to subcycles other than the base. A simple example of a system with one subcycle is $k = 4,\ B_3 = 0,\ B_2 = B_4 = -.4,\ B_1 = -2$. With initial vector from y_{-4} to y_{-1}, $0.6, 1, 0.6, -0.1$. There is a subcycle of two terms, $r = 2$, and length $q = 6$, with $q_1 = 1,\ q_2 = 5$. The associated B terms are B_2, B_4. The system has period 6, with solution starting from y_0: $-1, -.1, 0.6, 1, 0.6, -.3, -1, -.1, \ldots$. The following lemma makes it clear how terms in a subcycle generate each other.

LEMMA 13. *Given a system with $\vec{y}$ having a subcycle of r terms and length q, $y_{(pq+q_i)}$ is generated from $y_{(pq+q_{(i-1)})}$ and $y_{((p+1)q+q_1)}$ is generated from $y_{(pq+q_r)}$ for all $p \in \mathbb{N},\ i \in [2, r]$.*

PROOF. From the definition we have $y_{(pq+q_i)} = y_{q_i}\ \forall\ i \in [1, r],\ p \in \mathbb{N}$. Therefore for $i \in [2, r]$, and for all p, $y_{(pq+q_{(i-1)})} + y_{((p+1)q+q_i)} = y_{q_{(i-1)}} + y_{q_i} = B_{(q_i - q_{(i-1)})}$. Also $y_{((p+1)q+q_1)} + y_{(pq+q_r)} = y_{(q+q_1)} + y_{q_r} = B_{(q+q_1-q_r)}$. □

DEFINITION 11. *Group of subcycles: For a given system the term 'group of subcycles' refers to a set of all the subcycles that involve the same y positions mod twice the base length.*

Note that this definition allows for the possibility that the subcycles in a group may have different periods. For the systems that we consider this doesn't happen. If conjecture 5 is correct then such systems wouldn't have longer periods for the same k than the systems we are considering.

The idea for obtaining systems with long periods is to have two or more groups of subcycles, with the length of the subcycles in each group being a different multiple of 2j. Then the period will be equal to the least common multiple of the lengths of all the subcycles. This model is the most restrictive, requiring that there not exist any y positions from two different groups of subcycles that are separated by a B term associated with any of the subcycles. We will see later that a less restrictive requirement leads to systems with longer periods. Let M denote the number of groups of subcycles, and order the groups, with the first group of subcycles having the longest length, and define $(N_i)_{i=1}^{M}$ so that the i^{th} group of subcycles has length $2j \cdot N_i$. Set $N = N_1 = \max\{N_i \mid i \in [1, M]\}$. The period of the system is Period $=$ $\text{lcm}(N_1, N_2, ..., N_M) \cdot 2j$, and its size is at least $k \geq \frac{2jN}{r_1} + 1$, where r_1 is the number of terms in the subcycles in the first group. This lower bound for k results from subcycles in the first group having length $2jN$, and containing r_i terms, each of which can be no more than k apart. It appears that the maximal period for a given k is obtained by having each subcycle have the minimal number of terms, which is two. For the systems we consider $r_i = 2$ for all $i \leq M$, and in particular, $r_1 = 2$, which gives $k \cong jN + 1$.

Note we are assuming that the base length is large enough to support M groups of subcycles. We define the function baselen (M), abbreviated b_M, to equal the minimum base length needed to support M groups of subcycles. We want to select the $N_i \leq N$ so that the least common multiple of N_1 to N_M is maximal. Define glcm(M, N) to be the greatest least common multiple of M integers not exceeding N.

$$\text{glcm}(M, N) = \max \{\text{lcm}(N_1, N_2, ..., N_M) : N_i \leq N \; \forall \, i \in [1, M]\}.$$

For the correctly chosen initial y vector we have period lengths equal to glcm$(M, N) \cdot 2b_M$, which gives a lower bound for per k.

$$\text{per}\,(k) \geq \text{glcm}(M, N) \cdot 2b_M \quad k \cong b_M \cdot N + 1. \tag{3.2}$$

The following example demonstrates how these methods generate systems with periods longer than 2k.

3.2.1. Smallest long period model system. This system has two groups of subcycles (M=2), with $N_1 = 3,\ N_2 = 2,\ N = 3$. The minimum size base for two groups of subcycles, b_2, is 7. $k = N_1 \cdot b_2 + 1 = 22$, prime period $P = 2 \cdot 7 \cdot \text{lcm}(3, 2) = 84$. $B_i = -2$ for all i up to 22 except $B_7 = 0$ $B_{12} = B_{16} = B_{20} = B_{22} = -0.4$, $B_3 = B_5 = B_9 = B_{11} = -0.6$. An example of possible initial y values, from y_{-22} to y_{-1} is

$$\begin{array}{cccccccccccccc} . & . & . & . & . & . & . & 0.6 & 1 & 0.6 & 0.4 & -0.1 & 0.4 & 0.6 & -0.1 \\ -1 & -0.2 & 0.6 & 0.4 & -0.2 & 0.4 & 0.6 & 1 & 0.6 & 0.4 & -0.2 & 0.4 & 0.6 & -0.2 \end{array}$$

which produces the following y values, from 0 through 97:

-1	-0.3	0.6	0.4	-0.3	0.4	0.6	1	0.6	0.4	-0.2	0.4	0.6	-0.1
-1	-0.2	0.6	0.4	-0.2	0.4	0.6	1	0.6	0.4	-0.1	0.4	0.6	-0.2
-1	-0.3	0.6	0.4	-0.2	0.4	0.6	1	0.6	0.4	-0.2	0.4	0.6	-0.1
-1	-0.2	0.6	0.4	-0.3	0.4	0.6	1	0.6	0.4	-0.2	0.4	0.6	-0.2
-1	-0.3	0.6	0.4	-0.2	0.4	0.6	1	0.6	0.4	-0.1	0.4	0.6	-0.1
-1	-0.2	0.6	0.4	-0.2	0.4	0.6	1	0.6	0.4	-0.2	0.4	0.6	-0.2
-1	-0.3	0.6	0.4	-0.3	0.4	0.6	1	0.6	0.4	-0.2	0.4	0.6	-0.1

The two groups of subcycles have lengths $42 = 3 \cdot 2 \cdot 7$ and $28 = 2 \cdot 2 \cdot 7$. Each of these subcycles has length less than $2k$. However, a complete cycle requires the least common multiple of these, which is 84.

3.2.2. Constructing systems with long periods. The construction of systems like Example 3.2.1 can be described as a three stage process. After describing the three stages we will prove that such systems really do have the period lengths given in equation 3.2.

3.2.2.1. *Stage 1: A system with M subcycles.* We construct a system with M independent subcycles, supported in a pattern of size $2b_M$. Note $B_{b_M} = 0$. Each subcycle can be represented by a single number, $a_i \in [1, b_M - 1]$, with $1 \le i < M$. Each subcycle involves two y positions (mod $2b_M$), a_i and $2b_M - a_i$, and (at this stage) B terms B_{2a_i} and $B_{2b_M-2a_i}$, with the value of those B terms set to -0.4. Each subcycle looks like

$$a_i \xrightarrow{2b_M - 2a_i} 2b_M - a_i \xrightarrow{2a_i} 2b_M + a_i.$$

If we have an equation like $a_i + 2a_m = 2b_M + a_n$, with $i \neq m$ or $i \neq n$, then the y positions will interfere, and we may not have true subcycles. To avoid such interfering the following conditions will be imposed.

$$\forall\, i, m, n\ \in [1, M] \text{ the following hold} \tag{3.3}$$

$$a_i \text{ is a positive integer less than } b_M$$

$$i \neq m \longrightarrow a_i \neq a_m$$

$$\pm\, a_m + 2a_i \equiv \pm\, a_n \bmod 2b_M \longrightarrow i = m = n$$

$$\pm\, a_m + 2a_i \neq 0 \bmod 2b_M$$

$$a_m + a_i \neq b_M.$$

Values of b_M together with sets of $(a_i)_{i=1}^{M}$ which satisfy these conditions may be found through a search, such as the computer program in appendix A. In the subsection *Analyzing baselen* we give a constructive method of generating a base length together with a set of M a_i values which satisfy conditions 3.3.

In Example 3.2.1 we have $M = 2,\ b_2 = 7,\ a_1 = 1,\ a_2 = 4$. For a given $i \in [1, 2b_M - 1]$, if for all $j \in [1, M]$ we have

$$i \neq \{a_j,\ 2b_M - a_j,\ 2a_j,\ 2b_M - 2a_j\}$$

then we set $B_i = -0.6$. A possible initial y vector with this period is given in the following table. At this stage $k < 2b_M$, and the period is $2b_M$. Lemma 14 proves that the preperiod is zero, so the following values of y_n are valid for all $n \geq -k$.

Type	$r = n \bmod 2b_M$	B_r	y_n
1	0	-2	-1
2	b_M	0	$0 - (-1) = 1$
3a	a_i	-2	$-0.4 - (-0.1) = -0.3$
3b	$-a_i$	-2	$-0.4 - (-0.3) = -0.1$
4	$\pm\, 2a_i$	-0.4	$-0.4 - (-1) = 0.6$
5	All others	-0.6	$-0.6 - (-1) = 0.4$

TABLE 1. Stage 1 y values

Lemma 14. *Let M be fixed. Given a b_M, and a set $(a_i)_{i=1}^{M}$ which satisfy the conditions 3.3, the system described in table 1 with the given initial y vector has period $2b_M$, zero preperiod, and has M subcycles.*

Proof. Table 1 divides the initial y vector terms into six types. We show that this period $2b_M$ pattern is repeated by induction, assuming it is true up to y_{n-1}. Let y_n be generated by y_{n-j}. Thus $y_n = B_j - y_{n-j}$, and since all the B terms are less than or equal to zero we have $y_n \le -y_{n-j}$, $y_{n-j} \le -y_n$. Suppose y_n is type 2. Then $n \bmod 2b_M = b_M$, and $y_n \ge B_{b_M} - y_{n-b_M} = 0-1 = 1$. We conclude that $y_{n-j} \le -1$, so y_n must have been generated from type 1, and must equal 1. Similarly if y_n is of type 4 then $n = s \cdot 2b_M \pm 2a_i$ for some integers s, i, and $y_n \ge B_{2a_i} - y_{s2b_M} = -0.4-(-1) = 0.6$ or $y_n \ge B_{2b_M-2a_i} - y_{(s-1)2b_M} = -0.4-(-1) = 0.6$. In either case $y_{n-j} \le -0.6$, so y_n of type 4 must be generated from type 1, and therefore must equal 0.6. If y_n is type 5 then by the same argument it follows that y_n must be generated by type 1 and must equal 0.4.

If y_n is type 1 then $y_n = B_j - y_{n-j}$. Note B_j has the value in the table which corresponds with the type of y_{n-j}. The following max equation lists the possible types of y_{n-j} in order.

$$y_n = \max\{-2-(-1), 0-1, -2-(-.3), -2-(-.1), -.4-.6, -.6-.4\} = -1.$$

Finally, for y_n of type 3a or 3b, $y_n \ge -0.3$, so it must be generated by a $y_{n-j} \le 0.3$, which could only be type 1, 3a or 3b. All B terms of the form B_{a_i} and $B_{(2b_M-a_i)}$ have value -2, so type 1 does not generate type 3a or 3b. So a type 3a or 3b must be generated by another y term of type 3a or 3b. Let i be fixed. The type 3a y term associated with the i^{th} subcycle is $n \bmod 2b_M = a_i$, $y_n \ge B_{2a_i} - y_{n-2a_i} = -.4-(-.1) = -.3$. Knowing that y_n must be generated by a type 3a or type 3b y term, the other possible y terms that could generate y_n are $y_{n-a_i+a_j}$, $y_{n-a_i-a_j}$, where $j \ne i$. To ensure that y_n is not generated from any of these y terms it is sufficient that $B_{a_i+a_j} \le -0.6$, $B_{a_i-a_j} \le -0.6$. So these B terms can't equal 0 or -0.4. This requires $a_i + a_j \neq b_M,\ a_i + a_j \neq 2a_n,\ a_i + a_j \neq 2b_M - 2a_n,\ a_i - a_j \neq 2a_n,\ a_i - a_j \neq 2b_M - 2a_n$. These conditions must be true for all $i, j \in [1, M]$ and all $n \neq i$ or $n \neq j$. The type 3b y terms are similar. For $n \bmod 2b_M = 2b_M - a_i,\ y_n \geq B_{2b_M - 2a_i} - y_{n-2b_M+2a_i} = -.3 - (-.1) = -.1$. The other possible y terms that could generate y_n are $y_{n-2b_M+a_i+a_j},\ y_{n-2b_M+a_i-a_j}$. To ensure that y_n is not generated from any of these y terms it is sufficient that $B_{2b_M-a_i-a_j} \leq -0.6,\ B_{2b_M-a_i+a_j} \leq -0.6$. This requires $2b_M - a_i - a_j \neq b_M,\ 2b_M - a_i - a_j \neq 2a_n,\ 2b_M - a_i - a_j \neq 2b_M - 2a_n,\ 2b_M - a_i + a_j \neq 2a_n,\ 2b_M - a_i + a_j \neq 2b_M - 2a_n$. All of these requirements are satisfied for a set of $(a_i)_{i=1}^M$ and b_M which satisfy conditions 3.3. Thus for $n \bmod 2b_M = a_i$, y_n is generated from y_{n-2a_i}, and for $n \bmod 2b_M = 2b_M - a_i$, y_n is generated from $y_{(n-2b_M+2a_i)}$. So these y terms constitute a subcycle, and $B_{2a_i},\ B_{2b_M-2a_i}$ are the B terms associated with it. $\square$

3.2.2.2. *Stage 2: Expanding.* Next we expand the subcycles by selecting an N and obtaining a set $(N_i)_{i=1}^M$ of M positive integers less than or equal to N with the property that $N_1 = N$, and $\operatorname{lcm}((N_i)_{i=1}^M) = \operatorname{glcm}(M, N)$. N should be chosen so that $\operatorname{glcm}(M, N) > \operatorname{glcm}(M, N-1)$. The i^{th} subcycle becomes a group of N_i subcycles. All B terms not associated with a subcycle remain the same. The idea is to replace the two B terms associated with the i^{th} subcycle by adding a multiple of $2b_M$ to their subscripts, so that the subcycles associated with a_i have length $2b_M N_i$. Adding multiples of $2b_M$ does not affect the positions mod $2b_M$ that the y terms are generated by. The original B terms associated with each subcycle are set to -2, which ensures that they will not generate any y terms. After this expansion the period is $2b_M \operatorname{glcm}(M, N)$, while k is the highest B subscript used. To minimize k we have to break this step up into four cases, depending on the evenness of N and b_M.

Case 1: N even, b_M odd: We want $a_1 = 1$.

Setting $B_{Nb_M+2a_1} = B_{Nb_M-2a_1} = -0.4$ gives $k = Nb_M + 2$.

Case 2: N even, b_M even: We want $a_1 = 1$.

Setting $B_{Nb_M+2a_1} = B_{Nb_M-2a_1} = -0.4$ gives $k = Nb_M + 2$.

Case 3: N odd, b_M odd: We want $a_1 = \frac{b_M \pm 1}{2}$.

Setting $B_{(N-1)b_M+2a_1} = B_{(N+1)b_M-2a_1} = -0.4$ gives $k = Nb_M + 1$.

Case 4: N odd, b_M even: We want $a_1 = \frac{b_M \pm 2}{2}$.

Set $B_{(N-1)b_M+2a_1} = B_{(N+1)b_M-2a_1} = -0.4$. In this case $k = Nb_M + 2$.

In some cases there may not be any set $(a_i)_{i=1}^{M}$ of integers satisfying the conditions for M independent groups of subcycles and with a_1 the desired value. A search of M from 2 up to 15 finds that for M = 7 and M = 8 there are no sets with the needed value. For M = 7 or 8 and N odd we can use $a_i = \frac{b_M - 3}{2}$, which gives $k = Nb_M + 3$. This is perhaps an upper bound on k given M and N, but this hasn't been proven. The following lemma gives a provable upper bound on k.

LEMMA 15. *For a long period model system with M subcycles and an expansion factor of N, an upper bound on the minimum size of k is*

$$k \leq (N+1)b_M - 2M. \tag{3.4}$$

If the parity (even or odd) of N is selected to give a minimal k, then there is another upper bound,

$$k \leq (N + \frac{1}{2})b_M + M. \tag{3.5}$$

PROOF. Note the second bound is superior provided $b_M > 6M$. Consider the parity of N.

N even: The highest B subscript is either $Nb_M + 2a_1$or$(N+2)b_M - 2a_1$, with a_1 chosen to minimize k. The worst case is if the a_i are all as close to $\frac{b_M}{2}$ as possible. Since $a_i + a_j \neq b_M$, there must be an a_i at least M from $\frac{b_M}{2}$. $a_1 = \frac{b_M}{2} - M$ satisfies equation 3.4.

N odd: The highest B subscript is $(N+1)b_M - 2a_1$. This is maximized when the a_i subscripts are all as small as possible. Since the largest subscript is at least M,

$a_1 \geq M$, which satisfies equation 3.4.

If for a given M we select an even or odd N to minimize the amount $k - Nb_M$, then the worst case is if the a_i values are clustered around $\frac{b_M}{4}$ and $\frac{3b_M}{4}$. Since $a_i + a_j \neq b_M$, there must be an a_i at least $\frac{M}{2}$ from one of those values. So in the worst case we have $a_1 \bmod 2b_M = \frac{2b_M \pm b_M \pm 2M}{4}$, which results in the bound 3.5. $\square$

After the first (and longest) group of subcycles are set, the rest are simpler, since for these we don't have to worry about minimizing k.

Case N_i **even:** Set $B_{N_i b_M + 2a_i} = B_{N_i b_M - 2a_i} = -0.4$

Case N_i **odd:** Set $B_{(N_i-1)b_M + 2a_i} = B_{(N_i+1)b_M - 2a_i} = -0.4$

Lastly, we set all the other B_j terms with j in the range $2b_M < j < k$ to $B_j = -2$. Now for each i we have a group of N_i subcycles running over positions (mod $2b_M$) a_i and $2b_M - a_i$.

3.2.2.3. *Selecting an initial point.* There are of course many initial points with period $2b_M\,\mathrm{glcm}(M,N)$. The following procedure gives an initial point with this period. Set $y_j \in \vec{y}$ according to the value of j as follows, with $i \in [1, M]$.

$j \bmod 2b_M = 0$	$y_j = -1$
$j \bmod 2b_M = b_M$	$y_j = 1$
$j \bmod 2b_M = \pm\, 2a_i$	$y_j = 0.6$
$B_i = -0.6$ and $j \equiv i \mod 2b_M$	$y_j = 0.4$
$j \bmod 2b_M = a_i$	$y_j = -0.2$
$j \bmod 2b_M = -a_i$ and $j < -2b_M$	$y_j = -0.2$
$j \bmod 2b_M = -a_i$ and $j > -2b_M$	$y_j = -0.3$

The following theorem demonstrates that the expanded system really does have the stated period.

THEOREM 3. *Given a fixed M, let b_M and $(a_i)_{i=1}^M$ satisfy conditions 3.3. Select M positive integers, $(N_i)_{i=1}^M$, with $N_1 > N_j \;\forall\, j \in [2, M]$. Then the expanded system with the initial y vector described above has period $P = 2b_M\,\mathrm{lcm}\,(N_1, N_2, ..., N_M)$.*

Proof. For each $i \in [0, 2b_M - 1]$, the largest B_r such that $r \bmod 2b_M = i$ has the value given in table 1. Also the initial $\vec{y}$ is as given in table 1, except that some type 3a and 3b y terms equal -0.2. A 3a or 3b y term equal to -0.2 does not generate any other types, so lemma 14 can be applied to show that y_n satisfies table 1 for all y types except 3a or 3b. For each $i \in [1, M]$ there is exactly one term B_r such that $r \bmod 2b_M = 2a_i$, and exactly one term B_q such that $q \bmod 2b_M = 2b_M - 2a_i$. Since For each $i \in [0, 2b_M - 1]$, the largest B_r such that $r \bmod 2b_M = i$ is the same as in stage 1, we can conclude from lemma 14 that for y_n generated from y_{n-j}, if $n \bmod 2b_M = a_i$, then $(n-j) \bmod 2b_M = 2b_M - 2a_i$, and if $n \bmod 2b_M = 2b_M - a_i$, then $(n-j) \bmod 2b_M = 2a_i$. So B_r and B_q are associated with the i^{th} subcycle, which therefore has length $r+q = 2b_M N_i$. Specifically, if N_i is even, then $r = N_i b_M + 2a_i,\ q = N_i b_M - 2a_i$, while if N_i is odd, then $r = (N_i - 1)b_M + 2a_i,\ q = (N+1)b_M - 2a_i$. There are N_i subcycles in the group. All except the last subcycle in the group are given initial values in positions 3a and 3b of -0.2. These generate y terms with value -0.2, since $-0.4 - (-0.2) = -0.2$. Since only one subcycle in the group alternates between -0.3 and -0.1 the period of the group of subcycles is equal to the period of this subcycle, which is $2b_M N_i$. The period of the system using $\vec{y}$ is the least common multiple of the periods of all the subcycles, which is $2b_M \operatorname{lcm}(N_1, N_2, ..., N_M)$. □

3.2.3. Another example. $M = 4,\ b_4 = 19,\ N = 13$

$k = 19 \cdot 13 + 1 = 248$. The period is $19 \cdot 2 \cdot 13 \cdot 11 \cdot 10 \cdot 9 = 489,060$

$B_{19} = 0$. $a_1 = \frac{b_M+1}{2} = 10$. The four subcycles run over y positions (mod 38) 10 and 28, 1 and 37, 4 and 34, 11 and 27. $B_i = -0.4$ for $i = 19 \cdot 12 + 20 = 248,\ 19 \cdot 14 - 20 = 246,\ 19 \cdot 10 + 2 = 192,\ 19 \cdot 12 - 2 = 226,\ 19 \cdot 10 + 8 = 198,\ 19 \cdot 10 - 8 = 182,\ 19 \cdot 8 + 22 = 174,\ 19 \cdot 10 - 22 = 168$. $B_i = -0.6$ for $i = 3, 5, 6, 7, 9, 12, 13, 14, 15, 17, 21, 23, 24, 25, 26, 29, 31, 32, 33, 35$. All other B positions up to 248 have value -2. In this example $P > k^{2.3}$.

Tables 2, and 3 give period lengths for some systems like these, including the number of groups of subcycles in the system, the expansion term N, and the values of α, and β, which are defined as follows:

$$\text{Period} = k^{\alpha} \qquad \text{Period} = \exp\left(k^{\beta}\right).$$

K	Period	Cycles	N	α	β
22	84	2	3	1.4334	0.4816
30	168	2	4	1.5065	0.4804
36	280	2	5	1.5724	0.4825
44	420	2	6	1.5962	0.4753
50	588	2	7	1.6300	0.4736
56	1320	3	5	1.7850	0.4899
78	4620	3	7	1.9368	0.4895
90	6160	3	8	1.9392	0.4814
100	11,088	3	9	2.0224	0.4846
112	13,860	3	10	2.0211	0.4779
122	21,780	3	11	2.0792	0.4791
144	37,752	3	13	2.1206	0.4739
156	44,044	3	14	2.1175	0.4692
166	60,060	3	15	2.1524	0.4691
172	95,760	4	9	2.2282	0.4740
210	263,340	4	11	2.3342	0.4721
248	489,060	4	13	2.3761	0.4666
268	684,684	4	14	2.4033	0.4647
276	1,386,000	5	11	2.5162	0.4713
326	4,504,400	5	13	2.6475	0.4716
352	9,009,000	5	14	2.7310	0.4730
402	12,012,000	5	16	2.7185	0.4655
404	22,342,320	6	13	2.8197	0.4713
426	29,172,000	5	17	2.8390	0.4698
476	50,388,000	5	19	2.8766	0.4664
526	88,179,000	5	21	2.9200	0.4639
528	253,212,960	6	17	3.0865	0.4726
590	687,292,300	6	19	3.1893	0.4722
598	857,656,800	7	17	3.2172	0.4729
652	1,202,761,560	6	21	3.2265	0.4692
668	5,431,826,400	7	19	3.4463	0.4781
808	31,233,001,800	7	23	3.6096	0.4757
878	78,082,509,500	7	25	3.7006	0.4754
948	1.19 E11	7	27	3.7206	0.4725
1,018	2.85 E11	7	29	3.8035	0.4725
1,088	5.20 E11	7	31	3.8582	0.4712
1,122	5.94 E11	7	32	3.8603	0.4699
1,158	7.26 E11	7	33	3.8714	0.4688
1,228	1.08 E12	7	35	3.8953	0.4670
1,298	2.10 E12	7	37	3.9582	0.4667
1,332	2.35 E12	7	38	3.9594	0.4656

TABLE 2. Long period model periods for selected K=22 to 1350.

K	Period	Cycles	N	α	β
1,366	4.98 E12	8	29	4.0495	0.4675
1,460	1.19 E13	8	31	4.1317	0.4673
1,506	1.36 E13	8	32	4.1324	0.4659
1,554	1.66 E13	8	33	4.1421	0.4648
1,648	2.32 E13	8	35	4.1340	0.4626
1,712	6.87 E13	9	29	4.2793	0.4649
1,830	1.94 E14	9	31	4.3792	0.4650
1,890	2.21 E14	9	32	4.3782	0.4636
1,948	2.70 E14	9	33	4.3872	0.4625
2,002	3.21 E14	10	29	4.3941	0.4616
2,066	3.79 E14	9	35	4.3975	0.4603
2,140	2.49 E15	10	31	4.6230	0.4653
2,278	8.54 E15	10	33	4.7450	0.4660
2,388	1.11 E16	11	31	4.7501	0.4641
2,466	2.22 E16	11	32	4.8193	0.4645
2,830	6.11 E16	10	41	4.8629	0.4598
2,850	1.65 E17	11	37	4.9832	0.4626
3,038	2.12 E17	10	44	4.9754	0.4597
3,110	8.98 E17	12	37	5.1401	0.4628
3,390	4.03 E18	11	44	5.2703	0.4623
3,446	7.36 E18	12	41	5.3337	0.4630
3,614	1.02 E19	12	43	5.3426	0.4613
3,698	3.96 E19	12	44	5.4926	0.4637
3,814	4.07 E19	13	41	5.4756	0.4620
3,950	8.85 E19	12	47	5.5461	0.4621
4,000	3.50 E20	13	43	5.7036	0.4650
4,130	1.81 E21	14	43	5.8788	0.4673
4,372	2.06 E21	13	47	5.8543	0.4644
4,514	1.70 E22	14	47	6.0829	0.4677
4,706	7.44 E22	14	49	6.2275	0.4687
5,090	4.38 E23	14	53	6.3780	0.4683
5,474	7.01 E23	14	57	6.3787	0.4654
5,666	5.17 E24	14	59	6.5845	0.4676
5,858	1.85 E25	14	61	6.7065	0.4684
6,146	2.28 E25	14	64	6.6934	0.4662
6,434	7.86 E25	14	67	6.7994	0.4662
6,818	2.47 E26	14	71	6.8847	0.4653
6,834	3.68 E26	15	61	6.9277	0.4659
7,010	5.83 E26	14	73	6.9598	0.4654
7,506	2.11 E27	15	67	7.0506	0.4641
7,954	8.37 E27	15	71	7.1586	0.4636

TABLE 3. Long period model periods for selected K=1350 to 8000.

K	Period	Cycles	N	α	β
22	84	2	3	1.4334	0.4816
30	168	2	4	1.5065	0.4804
36	280	2	5	1.5724	0.4825
44	420	2	6	1.5962	0.4753
46	1080	3	5	1.8243	0.5077
64	3780	3	6	1.9807	0.5070
73	5040	3	7	1.9870	0.4995
78	9240	4	7	2.0959	0.5077
90	18,480	4	8	2.1833	0.5078
100	55,440	4	9	2.3719	0.5192
122	152,460	4	11	2.4843	0.5161
144	283,140	4	13	2.5260	0.5091
156	396,396	4	14	2.5526	0.5063
166	831,600	5	11	2.6665	0.5110
188	1,166,880	4	17	2.6678	0.5036
196	2,702,700	5	13	2.8059	0.5107
226	3,603,600	5	15	2.7852	0.5008
241	7,207,200	5	16	2.8790	0.5031
256	17,503,200	5	17	3.0076	0.5075
286	30,232,800	5	19	3.0453	0.5032
316	52,907,400	5	21	3.0898	0.5001
331	58,198,140	5	22	3.0815	0.4970
346	102,965,940	5	23	3.1558	0.4986
358	171,531,360	6	17	3.2242	0.5004
400	465,585,120	6	19	3.3312	0.4997
426	612,612,000	7	17	3.3419	0.4997
442	814,773,960	6	21	3.3685	0.4960
464	896,251,356	6	22	3.3573	0.4928
476	3.88 E9	7	19	3.5811	0.5019
548	4.259 E9	6	26	3.5159	0.4914
552	1.35 E10	8	19	3.6946	0.4989
576	2.23 E10	7	23	3.7488	0.4989
626	5.577 E10	7	25	3.8427	0.4983
668	1.035 E11	8	23	3.8994	0.4971
726	5.176 E11	8	25	4.0945	0.5002
784	5.823 E11	8	27	4.0649	0.4950
826	1.767 E12	9	25	4.1986	0.4972
842	2.412 E12	7	29	4.2328	0.4974
892	5.301 E12	9	27	4.3128	0.4972
958	2.196 E13	9	29	4.4750	0.4989
1,024	1.083 E13	9	31	4.6622	0.5014
1,057	1.238 E13	9	32	4.6602	0.4997
1,090	1.513 E13	9	33	4.6684	0.4984

TABLE 4. Improved model periods for selected K=22 to 1100.

K	Period	Cycles	N	α	β
1,156	2.118 E13	9	35	4.6772	0.4957
1,210	1.408 E14	10	31	4.9139	0.5004
1,222	6.028 E14	9	37	4.7878	0.4962
1,250	1.609 E15	10	32	4.9102	0.4986
1,354	1.826 E15	9	41	4.8734	0.4936
1,366	2.753 E15	10	35	4.9243	0.4946
1,396	6.498 E15	11	31	5.0281	0.4964
1,441	1.300 E16	11	32	5.1015	0.4969
1,600	3.452 E16	10	41	5.1615	0.4933
1,666	9.617 E16	11	37	5.2715	0.4942
1,717	1.201 E17	10	45	5.2800	0.4930
1,834	1.860 E17	10	47	5.2919	0.4901
1,846	4.381 E17	11	41	5.4012	0.4925
1,888	5.450 E17	12	37	5.4140	0.4918
1,936	1.177 E18	11	43	5.4978	0.4926
1,981	2.355 E18	11	44	5.5725	0.4933
2,092	4.469 E18	12	41	5.6159	0.4917
2,194	2.135 E19	12	43	5.7851	0.4934
2,246	2.402 E19	12	44	5.7828	0.4922
2,338	2.497 E19	13	41	5.7579	0.4898
2,386	4.270 E19	11	53	5.8119	0.4900
2,398	5.376 E19	12	47	5.8377	0.4904
2,452	2.148 E20	13	43	5.9985	0.4928
2,602	3.377 E20	12	51	6.0107	0.4903
2,680	9.013 E20	13	47	6.1126	0.4911
2,704	1.113 E21	12	53	6.1324	0.4911
2,794	3.397 E21	13	49	6.2477	0.4919
2,908	4.907 E21	13	51	6.2625	0.4904
3,009	8.095 E21	14	47	6.2983	0.4895
3,022	2.364 E22	13	53	6.4287	0.4919
3,136	5.201 E22	13	55	6.4970	0.4915
3,244	6.110 E22	15	47	6.4898	0.4898
3,364	1.809 E23	13	59	6.5944	0.4902
3,382	4.277 E23	15	49	6.6959	0.4918
3,478	4.788 E23	13	61	6.6868	0.4904
3,521	5.256 E23	14	55	6.6882	0.4899
3,649	5.892 E23	13	64	6.6730	0.4880
3,658	2.833 E24	15	53	6.8624	0.4913
3,777	3.446 E24	14	59	6.8595	0.4898
3,796	6.233 E24	15	55	6.9272	0.4908
3,905	1.236 E25	14	61	6.9863	0.4905
4,072	3.343 E25	15	59	7.0708	0.4901
4,210	2.266 E26	15	67	7.2719	0.4920

TABLE 5. Improved model periods for selected K=1150 to 4210.

3.2.4. Systems with longer periods than the long period model. We can get longer periods by relaxing the requirements for how the subcycles can interfere. In the first example, which is an example of what I will call the **unrestricted model**, the base length is five. Four y positions mod 10 (twice the base length) interact to produce a period of 50. Each of these four y positions mod 10 is generated from one of three possible y positions, so it doesn't have distinct subcycles. For this system $k = 16$. This is the shortest known system with period length greater than $2k$.
$B_5 = 0$ $B_8 = B_{12} = B_{14} = B_{16} = -.4$, with $B_i = -2$ for all other B terms less than 16. An example of initial y values, from y_{-16} to y_{-1} is

$$\begin{array}{cccccccccc} . & . & . & . & . & 0.6 & 1 & 0.6 & -0.1 & 0.6 & -0.1 \\ -1 & -0.1 & 0.6 & -0.1 & 0.6 & 1 & 0.6 & -0.3 & 0.6 & -0.3 \end{array} \tag{3.6}$$

which produces the following y values, from 0 through 60:

$$\begin{array}{cccccccccc} -1 & -0.3 & 0.6 & -0.3 & 0.6 & 1 & 0.6 & -0.1 & 0.6 & -0.1 \\ -1 & -0.1 & 0.6 & -0.1 & 0.6 & 1 & 0.6 & -0.1 & 0.6 & -0.1 \\ -1 & -0.3 & 0.6 & -0.3 & 0.6 & 1 & 0.6 & -0.3 & 0.6 & -0.1 \\ -1 & -0.1 & 0.6 & -0.1 & 0.6 & 1 & 0.6 & -0.1 & 0.6 & -0.1 \\ -1 & -0.1 & 0.6 & -0.1 & 0.6 & 1 & 0.6 & -0.3 & 0.6 & -0.3 \\ -1 & -0.3 & 0.6 & -0.3 & 0.6 & 1 & 0.6 & -0.1 & 0.6 & -0.1 \end{array}$$

In the next example, which is an example of what I will call the **improved model**, we demonstrate that the requirements for base length can be relaxed while maintaining true subcycles.

Definition 12. *$\hat{b}_M$ is the minimum positive integer such that there exists M positive integers, $(a_i)_{i=1}^M$, and also M negative numbers $(c_i)_{i=1}^M$, which satisfy conditions 3.7. Thus $\hat{b}_M$ is the minimum base length needed for M groups of subcycles in the improved model.*

$$(3.7) \qquad \forall\, i, m, n \in [1, M] \text{ the following hold}$$

$$a_i \text{ is a positive integer less than } \hat{b}_M$$

$$i \neq m \longrightarrow a_i \neq a_m$$

$$c_i < 0$$

$$\pm\, a_m + 2a_i \equiv \pm\, a_n \mod 2\hat{b}_M \longrightarrow 5c_i < c_m, \text{ and } 5c_i < c_n$$

$$\pm\, a_m + 2a_i \neq 0 \bmod 2\hat{b}_M$$

$$a_m + a_i \neq \hat{b}_M.$$

We allow $\pm\, a_m + 2a_i \equiv \pm\, a_n \mod 2\hat{b}_M$, as long as the B terms associated with a_i, which are B_{2a_i}, and $B_{(2\hat{b}_M - 2a_i)}$, are less than the B values associated with a_m and a_n. The following example shows how this works. $\hat{b}_3 = 9$. We have $a_1 = 4$ $a_2 = 1$ $a_3 = 3$, and since $1 + 2 \cdot 1 = 3$ we require that the B positions associated with a_2 be less than those with a_3. Set $N = 5$, $N_1 = 5$ $N_2 = 4$ $N_3 = 3$. Thus the period is $2 \cdot 9 \cdot \text{glcm}(3, 5) = 1080$. $B_{38} = B_{34} = -0.4$. These are the positions associated with a_2. $B_{46} = B_{44} = B_{30} = B_{24} = -0.08$. We need four more B terms to take care of the remaining positions mod 18. $B_5 = B_7 = B_{11} = B_{13} = -0.6$. Thus $k = 46$.

Initial y values, from y_{-46} to y_{-1} are

. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	0.92
1	0.92	0.4	0.92	0.4	−0.04	−0.04	0.6	−0.2
−1	−0.3	0.6	−0.04	−0.04	0.4	0.92	0.4	0.92
1	0.92	0.4	0.92	0.4	−0.04	−0.04	0.6	−0.2
−1	−0.2	0.6	−0.06	−0.06	0.4	0.92	0.4	0.92
1	0.92	0.4	0.92	0.4	−0.06	−0.06	0.6	−0.1

Thus we have three types of models for generating long periods.

1) **The long period model** – No interaction between subcycles.

2) **The improved model** – Potential interaction between subcycles, but the subcycles are maintained by putting a specified order on the values of the B terms associated with each subcycle.

3) **The unrestricted model** These systems have a base and several y positions not generated by the base, but no true subcycles. For a system with base length J, these systems obey the inequalities $\pm\, a_i + 2a_m \mod 2J \neq 0$, and $a_i + a_m \neq J$.

3.2.4.1. *The improved model.* Given a fixed M and a $\hat{b}_M$ and $(a_i)_{i=1}^M$ and $(c_i)_{i=1}^M$ that satisfy conditions 3.7, and a set of relatively prime positive integers $(N_i)_{i=1}^M$, such that $N = N_1 = \max\{N_i | i \in [1, M]\}$, and such that $\operatorname{lcm}\{N_1, N_2, ..., N_M\} = \operatorname{glcm}(M, N)$, we can construct an improved model as follows:

Scale the c_i values so that $\min\{c_i\} = -0.4$. Choose the B terms and initial $\vec{y}$ exactly as in the long period model, using $\hat{b}_M$ in place of b_M. Now if there is a B term B_r with $r \mod 2\hat{b}_M = \pm\, 2a_i$, then set $B_r := c_i$. Also modify y_j as follows:

exists i such that $j \equiv \pm\, a_i \mod 2\hat{b}_M$ **and** $j < -2\hat{b}_M$**:** $y_j := 0.5c_i$.

exists i such that $j \equiv \pm\, a_i \mod 2\hat{b}_M$ **and** $j > -2\hat{b}_M$**:** $y_j := 0.75c_i$.

The following lemma shows that these systems have M subcycles, and give a formula for k and period length.

LEMMA 16. *For a fixed M and a $\hat{b}_M$ and $(a_i)_{i=1}^M$ and $(c_i)_{i=1}^M$ that satisfy conditions 3.7, and a set of relatively prime positive integers $(N_i)_{i=1}^M$, such that $N = N_1 = \max\{N_i | i \in [1, M]\}$, and $\operatorname{lcm}\{N_1, N_2, ..., N_M\} = \operatorname{glcm}(M, N)$, there exists systems with $k \leq (N+1)\hat{b}_M - 2M$ that have period $P = 2\hat{b}_M \cdot \operatorname{glcm}(M, N)$.*

PROOF. Let the initial $\vec{y}$ be as described above. To prove that this system has M subcycles it is sufficient to show that terms from one subcycle are not generated from terms of another. Thus for any two y terms y_r and y_p, with $r > p$, if $r \mod 2\hat{b}_M = \pm\, a_m$ and $p \mod 2\hat{b}_M = \pm\, a_n$ and also $(r - p) \mod 2\hat{b}_M = \pm\, 2a_i$, and either $a_i \neq a_m$ or $a_i \neq a_n$, we need to show that $c_i - y_p < y_r$. From 3.7 we have $5c_i < c_m$, $5c_i < c_n$. Assuming that the subcycle pattern in the initial $\vec{y}$ has continued up to y_{r-1} we have

$y_r \geq 0.25c_m$, and $y_p \in \{0.25c_n,\ 0.5c_n,\ 0.75c_n\}$. So $y_r + y_p \geq 0.25c_n + 0.25c_m > 2.5c_i$. Thus y_p is not generated from y_r. So the system has M subcycles, and the period is the least common multiple of the subcycles, which is $P = 2\hat{b}_M \cdot \text{glcm}(M, N)$. Lemma 15 applies to this system, with $\hat{b}_M$ substituted for b_m. Thus equation 3.4 is an upper limit on k. □

A program for finding the improved model base lengths is given in appendix B. It is considerably slower than the program in appendix A. The following table compares $\hat{b}_M$ and b_M for M = 2 up to 15, and lists the a_i values for $\hat{b}_M$. The B terms associated with successively listed a_i terms should be in decreasing order, except for a_i terms listed together in parenthesis, which can be equal.

M	b_M	$\hat{b}_M$	a_i values
2	7	7	(1, 4)
3	11	9	8, 4, 7
4	19	11	8, 10, 5, 9
5	25	15	12, (14, 4), (5, 9)
6	31	21	18, (20, 8), 15, (19, 5)
7	35	25	16, 22, 14, (7, 24), (23, 19)
8	47	29	8, 4, 6, 26, (17, 13, 38), 27
9	59	33	6, 30, 10, 8, 32, (21, 13), 17, 31
10	69	39	(8, 36), 12, (10, 33), 38, (23, 15), 19, 37
11	77	45	(8, 12), (10, 39), 44, 18, (40, 25, 17), 21, 43
12	84	51	30, 36, 48, (16, 9, 28), (5, 38), 50, 45, 49, 29
13	93	57	(44, 48), (16, 46, 54), (8, 56), 4, (21, 35), 31, 55, 45
14	96	64	(43, 33), 38, 45, 58, 35, (60, 40), (34, 63), 54, 44, 39, 61
15	112	69	(51, 33, 43), (63, 47), 67, 57, (17, 50), (42, 40), 48, 65, (68, 20)

TABLE 6. Improved base length(M) for M=2 to 15

3.3. Analyzing the components of long period systems.

To obtain bounds on the period of these long period systems we need to analyze the functions b_M, $\text{glcm}(M, N)$ and estimate the optimal relation between M and N.

3.3.1. Analyzing baselen. As shown in lemma 14 if the conditions 3.3 are satisfied then there exists a system with base length b_M that has M subcycles. So the problem of estimating b_M reduces to finding the smallest $j = b_M$ such that there are M numbers $(a_i)_{i=1}^M$, which satisfy 3.3. We first use probability to estimate b_M, then we find a pattern which gives us a provable upper bound on b_M.

3.3.1.1. *A probabilistic estimation of baselen.* Each subcycle is defined by an integer a_i. Consider j as a candidate for b_M. We estimate the probability that $b_M = j$ by considering the probability that a set of M integers, $(a_i)_{i=1}^M$ in $[1, j-1]$ satisfies the inequalities in equations 3.3. In equations $\pm\, a_m + 2a_i \equiv \pm\, a_n \mod 2b_M \longrightarrow i = m = n$ and $\pm\, a_m + 2a_i \neq 0 \bmod 2b_M$ There are $2M + 1$ values on the right side. On the left side there are M choices for i, M choices for m, and a choice of plus or minus. This gives $2M^2$ values, in a set of size $2j$ which must avoid $2M + 1$ positions. Taking in to account the case where $i = m = n$, the probability of a randomly chosen set of $(a_i)_{i=1}^M$ integers satisfying equations 3.3 is

$$\left(1 - \frac{2M+1}{2j}\right)^{2M(M-1)} \left(1 - \frac{2M-1}{2j}\right)^{2M} = \left(\frac{2j-2M-1}{2j}\right)^{2M(M-1)} \left(\frac{2j-2M+1}{2j}\right)^{2M}.$$

There are $\binom{M}{j-1}$ possible selections of a_1 to a_M, so the probability of having no pattern of size j is

$$\left[1 - \left(\frac{2j-2M-1}{2j}\right)^{2M(M-1)} \left(\frac{2j-2M+1}{2j}\right)^{2M}\right]^{\binom{M}{j-1}}.$$

We can approximate b_M by multiplying the probability of no pattern for a sequence of integers until the probability gets below one half. Thus

$$b_M \cong \min_{i \in \mathbb{N}} \text{s.t.} \prod_{j=b_{(M-1)}}^{i} \left[1 - \left(\frac{2j-2M-1}{2j}\right)^{2M(M-1)} \left(\frac{2j-2M+1}{2j}\right)^{2M}\right]^{\binom{M}{j-1}} \leq \frac{1}{2}$$

These estimates are included in table 7. They are fairly good for low values of M, but are about 40 percent too large around $M = 15$. Because the formula for generating these estimates of b_M is so complex it does not appear to be useful in providing an equation for the rate of growth of the lower bound of the maximum period length. Also, any such bound would not be definite, since it would be based on probability.

3.3.1.2. *A provable lower bound for baselen.* We are going to investigate a specific set of a_i which satisfy equations 3.3. Let a_i be an increasing sequence. By setting $a_i \equiv 1 \mod 3 \; \forall \, i \in [1, M]$, and $b_M = 2a_M + 1$ we ensure that several conditions in 3.3 are satisfied. For example, $-a_i + 2a_n \neq -a_m$, since the left side mod 3 is 1, and the right side mod 3 is 2. Also $a_i + a_j < b_M$ for all $i, j \in [1, M]$. The conditions in 3.3 are reduced to

$$a_i + a_m = 2a_n \longrightarrow i = j = n.$$

The following lemma provides a set of integers which satisfy this condition.

LEMMA 17. *For all J there exists an increasing sequence $(c_i)_{i=1}^{J}$ of positive integers satisfying:*
i) $c_i + c_j = 2c_q \longrightarrow i = j = q.$
ii) $c_J \leq J^{\frac{\ln 3}{\ln 2}}.$

PROOF. Let $c_1 = 1, c_2 = 2, c_3 = 4, c_4 = 5$. These first values satisfy property i). Call these level one. Now let $c_{4+j} = 2c_4 - 1 + c_j$ for $j = 1$ to 4. These are level two. $c_{8+j} = 2c_8 - 1 + c_j$ for $j = 1$ to 8, level three, and in general for level N; $c_{2^N+j} = 2c_{2^N} - 1 + c_j$ for $j = 1$ to 2^N. Assume for the sake of contradiction that $\exists \; i, j, q$ such that $c_i + c_j = 2c_q$ and $i \neq j$. Let q be the smallest integer satisfying this assumption, and let $c_i < c_j$, with c_j on level N. So $j \in [2^N + 1, 2^{N+1}]$. Now if c_i is also on level N then their average, c_q, is also, since $c_i < \frac{c_i+c_j}{2} < c_j$. But then $c_{(i-2^N)} = c_i + 1 - 2c_{2^N}$, $c_{(j-2^N)} = c_j + 1 - 2c_{2^N}$, and $c_{(q-2^N)} = c_q + 1 - 2c_{2^N}$, thus $c_{(i-2^N)} + c_{(j-2^N)} = 2c_{(q-2^N)}$, so a solution exists on a lower level, which contradicts q being minimal. Therefore c_i must be on a level less than N. Then $c_i + c_j \geq 2c_{2^N} + 1$, so $c_q \geq c_{2^N} + (1/2)$, so c_q must be on level N. But $c_j \leq 2c_{2^N} - 1 + c_{2^N}$, $c_i \leq c_{2^N}$, and thus $c_q = \frac{c_i+c_j}{2} \leq 2c_{2^N} - (1/2)$. So c_q must be on a level below N, which is a contradiction. Thus the sequence satisfies property i).

To show that the sequence satisfies property ii), we first prove by induction that $2c_{2^N} - 1 = 3^N$ for all $N \geq 2$. This gives the result that for $j \in [1, 2^N]$, the j^{th} term on level N is $c_{2^N+j} = 3^N + c_j$. For $N = 2$, $2c_{2^N} - 1 = 9 = 3^2$. Assume the hypothesis is true up to N. We have $2c_{2^{N+1}} - 1 = 2c_{(2^N+2^N)} - 1 = 2(2c_{2^N} - 1 + c_{2^N}) - 1 = 6c_{2^N} - 3 = 3(c_{2^N} - 1) = 3^{N+1}$, and thus the hypothesis is proven. Now we show that the sequence satisfies property ii) by induction on levels. By direct calculation it is true for level one. Assume true up to level N. Then for $J = 2^N + j$, $j \in [1, 2^N]$, we have

$$J^{\frac{\ln 3}{\ln 2}} = (2^N + j)^{\frac{\ln 3}{\ln 2}} > (2^N)^{\frac{\ln 3}{\ln 2}} + j^{\frac{\ln 3}{\ln 2}} \geq 3^N + c_j = c_{2^N+j} = c_J.$$

□

Another sequence which satisfies properties i) can be defined by setting level one to $1, 2, 4$. The first few terms of these sequences are given below.

1	2	4	8	9	11	22	23	25	29	30	32	64	65	67	71	72	74	85	86	88
1	2	4	5	10	11	13	14	28	29	31	32	37	38	40	41	82	83	85	86	91

For some values of J the J^{th} term in the first sequence is lower than the second. Thus for $J \in \mathbb{N}$ taking the sequence with the minimum c_J value gives a slightly better bound. The graph 3.5 illustrates this.

Using this lemma we get an upper bound for b_M.

LEMMA 18. *b_M satisfies the inequality*

$$b_M \leq 6M^{\frac{\ln 3}{\ln 2}} + 3. \tag{3.8}$$

PROOF. Let $(c_i)_{i=1}^{\infty}$ be the sequence defined in lemma 17. Set $a_i = 3c_i + 1$ for all i. Fix $M \geq 2$. Note $a_M \leq 3M^{\frac{\ln 3}{\ln 2}} + 1$. Let $J = 2a_M + 1$. Claim: $b_M \leq J$. If the claim is true then the lemma is correct. Suppose $b_M = J$. Then $a_i + a_m < b_M$ for any $i, j \leq M$. By lemma 17 the (c_i) constants satisfy equation 17i), which implies that the (a_i) constants satisfy the inequalities 3.3. For example, $a_m + 2a_i \bmod 2b_M \neq a_n$, since

$a_m + 2a_i < 2b_M$, and $a_m + 2a_i \bmod 3 = 0$. Since (a_i) satisfies all these inequalities, a system with base length J can have M groups of subcycles, proving the claim. $\square$

The following table shows computed minimum values for b_M, along with the probability estimate and the set of a_i, for M ranging from 2 up to 15.

M	Prob. est.	b_M	a_i values
2	n/a	7	1, 4
3	12	11	1, 4, 5
4	18	19	10, 12, 15, 17
5	26	25	8, 9, 13, 20, 22
6	34	31	16, 18, 19, 21, 28, 29
7	44	35	3, 13, 16, 21, 24, 33, 34
8	54	47	3, 7, 22, 26, 27, 31, 38, 46
9	66	59	7, 10, 26, 29, 38, 39, 44, 55, 57
10	78	69	9, 10, 21, 26, 35, 37, 45, 50, 66, 67
11	91	77	12, 26, 32, 39, 41, 42, 53, 54, 68, 73, 75
12	105	84	8, 14, 22, 26, 32, 37, 40, 46, 49, 59, 63, 83
13	120	93	10, 23, 28, 37, 38, 47, 52, 61, 68, 77, 80, 90, 91
14	136	96	5, 8, 14, 19, 22, 31, 55, 61, 72, 73, 80, 86, 94, 95
15	153	112	4, 7, 11, 36, 43, 47, 54, 63, 70, 74, 81, 100, 106, 107, 111

TABLE 7. baselen(M) for M=2 to 15

CONJECTURE 1. *Based on table 7 I conjecture that the b_M upper bound can be reduced by a factor of 3 to*

$$b_M \leq 2M^{\frac{\ln 3}{\ln 2}} + 1. \tag{3.9}$$

See the graph 3.5.

3.3.2. Formula's and estimates of glcm(M, N)**.** In [**14**] the following lower bounds on glcm(M, N) are established.

$$\text{glcm}(M, N) > \frac{N^M}{2^M} \quad \text{provided} \quad M \leq \frac{N \ln(2)}{\ln(N)} \tag{3.10}$$

$$\text{glcm}(M, N) > \frac{N^M}{2^{N/2}} \quad \text{provided} \quad M \leq \frac{N \ln(2)}{2\ln(N)} \tag{3.11}$$

These bounds aren't very close. For example, 67 is prime, so a good choice for N. $N = 67$ satisfies equation 3.10 provided $M \leq 11.045...$, so we let $M = 11$. Then the lower bound gives $\text{glcm}(11, 67) > 5.96\text{x}10^{16}$, while the actual value is $\text{glcm}(11, 67) = 1.23\text{x}10^{19}$, over 200 times higher than the lower bound. There is also a good upper bound, based on a modulus argument.

$$\text{glcm}(M, N) \leq N \prod_{i=1}^{M-1} (N + 1 - p_i)$$

Where p_i is the i^{th} prime. Using the same example, this gives glcm $\leq 1.52\text{x}10^{19}$, much closer. From this upper bound we can derive an approximation. Assuming $N > M \ln(M)$, we have

$$\text{glcm}(M, N) \cong N^M M^{-\frac{M^2}{2N}}$$

This estimate gets slightly too high as M increases. One reason that the lower bounds in equations 3.10 and 3.11 are so low is that they have to be valid for all choices of N, such as $N = 30$. Note $\text{glcm}(5, 30) = 29 \cdot 28 \cdot 27 \cdot 25 \cdot 23 = \text{glcm}(5, 29)$. A more useful lower limit would restrict N to be a prime power. (A prime power is any prime number raised to any integer power.) I suggest the following lower bound:

Conjecture 2. *For $N > M \ln M$ and N a prime power the following lower bound holds.*

$$\text{glcm}(M, N) > N^M M^{-\frac{M^2+1}{2N}}. \tag{3.12}$$

3.3.3. The optimal range of N for a given M. Let M be fixed. By choosing an N we can calculate the size, k, of the corresponding long period system, and its period. The optimal range of N for that M is the range of N values for which systems have longer periods than any other with equal or smaller k. The following table gives the optimal range on N for $M = 2$ to 15. Optimal $\hat{N}$ is the optimal values of N for the improved $\hat{b}_M$. It is clear that N should be at least as large as p_M, the M^{th} prime, since there should be M relatively prime numbers less than or equal to N. In all cases

M	M^{th} prime	Optimal N	Optimal $\hat{N}$	$1.5M \ln M$
2	3	3 to 7	3 to 6	2.1
3	5	5 to 15	5 to 7	4.9
4	7	9 to 14	7 to 17	8.3
5	11	11 to 21	11 to 23	12.1
6	13	13 to 21	17 to 26	16.1
7	17	17 to 38	17 to 25	20.4
8	19	29 to 35	19 to 29	25.0
9	23	29 to 35	25 to 41	29.7
10	29	29 to 44	31 to 47	34.5
11	31	31 to 44	31 to 53	39.6
12	37	37 to 47	37 to 53	44.7
13	41	41 to 47	41 to 65	50.0
14	43	43 to 73(?)	43 to 61(?)	55.4
15	47	61 to ?	47 to ?	60.9

TABLE 8. Optimal range of N for M=2 to 15

except $M = 15$, the lower bound is between p_M and $p_{(M+2)}$, and in all cases except $M = 15$, $p_{(M+2)}$ is in the optimal range of N. In deriving equations which estimate period length as a function of k it is necessary to define a relationship between N and M. I have used $N = 1.5M \ln M$, which is generally in the optimal range. (See table 8). For the proven lower bound we use $M \le \frac{\ln(2)N}{\ln(N)}$, which comes from the proven bound on $\text{glcm}(M, N)$.

3.4. Calculations for lower bound of periods

Combining the information on b_M, $\text{glcm}(M, N)$, and the optimal range of N vs M, we calculate a proven lower bound, and also a conjectured lower bound and an estimate on actual period lengths.

3.4.1. Proven lower bound. We express both k and $\mathrm{per}\,(k)$ as functions of N. This defines $\mathrm{per}\,(k)$ parametrically. This parametric definition is used in the graphs which show the "proven lower bound".

$$k = (N + \frac{1}{2})b_M + M, \quad \mathrm{per}\,(k) \geq 2\,\mathrm{glcm}\,(M, N) \cdot b_M.$$

We use $M = \frac{N \ln(2)}{\ln(N)}$, which provides the lower bound on glcm(M, N) from equation 3.10, and the proven lower bound on b_M from equation 3.8, which gives

$$k \leq (6N + 3) \left(\frac{N \ln 2}{\ln N} \right)^{\frac{\ln 3}{\ln 2}} + \frac{N \ln 2}{\ln N} + 3N + 1.5 \tag{3.13}$$

Using 3.13 define $\phi(N) = k$. Then we have

$$\mathrm{per}\,(\phi(N)) \geq \left(12 \left(\frac{N \ln 2}{\ln N} \right)^{\frac{\ln 3}{\ln 2}} + 4 \right) \left(\frac{N}{2} \right)^{\frac{N \ln(2)}{\ln(N)}}. \tag{3.14}$$

3.4.2. Conjectured lower bound and estimates. Using $N = 1.5M \ln M$, and equation 3.12 for glcm, and equation 3.9 for b_M, we get a parametric definition of period length with respect to k.

$$k = 3 \ln (M) M^{\frac{\ln 6}{\ln 2}} \tag{3.15}$$

Using 3.15 define $\psi(M) = k$. then we have

$$\mathrm{per}\,(M) \geq 4(1.5)^M (\ln M)^M M^{\;M + \frac{\ln 3}{\ln 2} - \frac{M^2+1}{3M \ln M}}. \tag{3.16}$$

We also note that the assignment $\mathrm{per}\,k \geq \exp\,(k^{0.465})$ gives a good approximation of period length over the range for which we have actual data ($k = 22$ to 8000). For the improved model, using $\hat{b}_M$, $\mathrm{per}\,k \geq \exp\,(k^{0.49})$ is a good approximation. See graph 3.5.

3.5. A proof that period length grows faster than polynomially

To show that maximum period length grows faster than polynomially we use the following lemma.

Lemma 19. *Let $M > 1$ be fixed. Then*

$$\overline{\lim_{N\to\infty}} \frac{\mathrm{glcm}(M,N)}{N^M} = 1.$$

Proof. Let p_i denote the i^{th} prime. Let $M > 1$ be fixed. The numbers $1, p_1, p_2, ..., p_{M-1}$ are relatively prime. Let $p_i\#$ denote the product of the first i primes. The numbers $p_{M-1}\# + 1, p_{M-1}\# + p_1, p_{M-1}\# + p_2, \ldots, p_{M-1}\# + p_{M-1}$ are relatively prime, since the difference between any two of these numbers is less than p_{M-1}, and thus a common factor would have to consist of primes less than that. Similarly, for any positive integer t, the numbers $tp_{M-1}\# + 1, tp_{M-1}\# + p_1, \ldots, tp_{M-1}\# + p_{M-1}$ are relatively prime. Let $\epsilon \in (0,1)$ be fixed. If $N = tp_{M-1}\# + p_{M-1}$, then $\mathrm{glcm}(M,N) > (tp_{M-1})^M$. Now since

$$\lim_{t\to\infty} \frac{(tp_{M-1}\#)^M}{(tp_{M-1}\# + p_{M-1})^M} = 1$$

there exists a number L such that for any integer $t > L$, setting $N = tp_{M-1}\# + p_{M-1}$ results in $\frac{\mathrm{glcm}(M,N)}{N^M} > 1 - \epsilon$. An L that works is

$$L = \frac{(1-\epsilon)^{1/M} p_{M-1}}{p_{M-1}\# \left(1 - (1-\epsilon)^{1/M}\right)}.$$

Since there are infinitely many choices for N that get the ratio within ϵ of 1, the lemma is proved. □

Theorem 4. $\forall\, c > 0\ \exists\, k \in \mathbb{N}$ *and a system of size k with prime period greater than k^c.*

Proof. Let $c > 0$ be fixed. Let M indicate the number of independent subcycles. Set $M = [c] + 2$. Choose $N > (b_M)^{M-1}$ so that $\mathrm{glcm}(M,N) > 0.9N^M$. Then

$k \leq (N+\frac{1}{2})b_M + M < (N+1)b_M, \quad \mathrm{per}\,(k) \geq 2b_M\,\mathrm{glcm}(M,N) > 1.8b_M N^M.$

So $\mathrm{per}\,(k) > (1.8b_M)b_M^{M-1}N^{M-1} > e \cdot b_M^{M-1}N^{M-1}$.

Recall that $\frac{(N+1)^c}{N^c} < e$ for all $N, c > 1$. Thus

$k^c < (N+1)^c b_M^c < e \cdot N^c b_M^c < e \cdot N^{M-1} b_M^{M-1} < \mathrm{per}\,(k).$ □

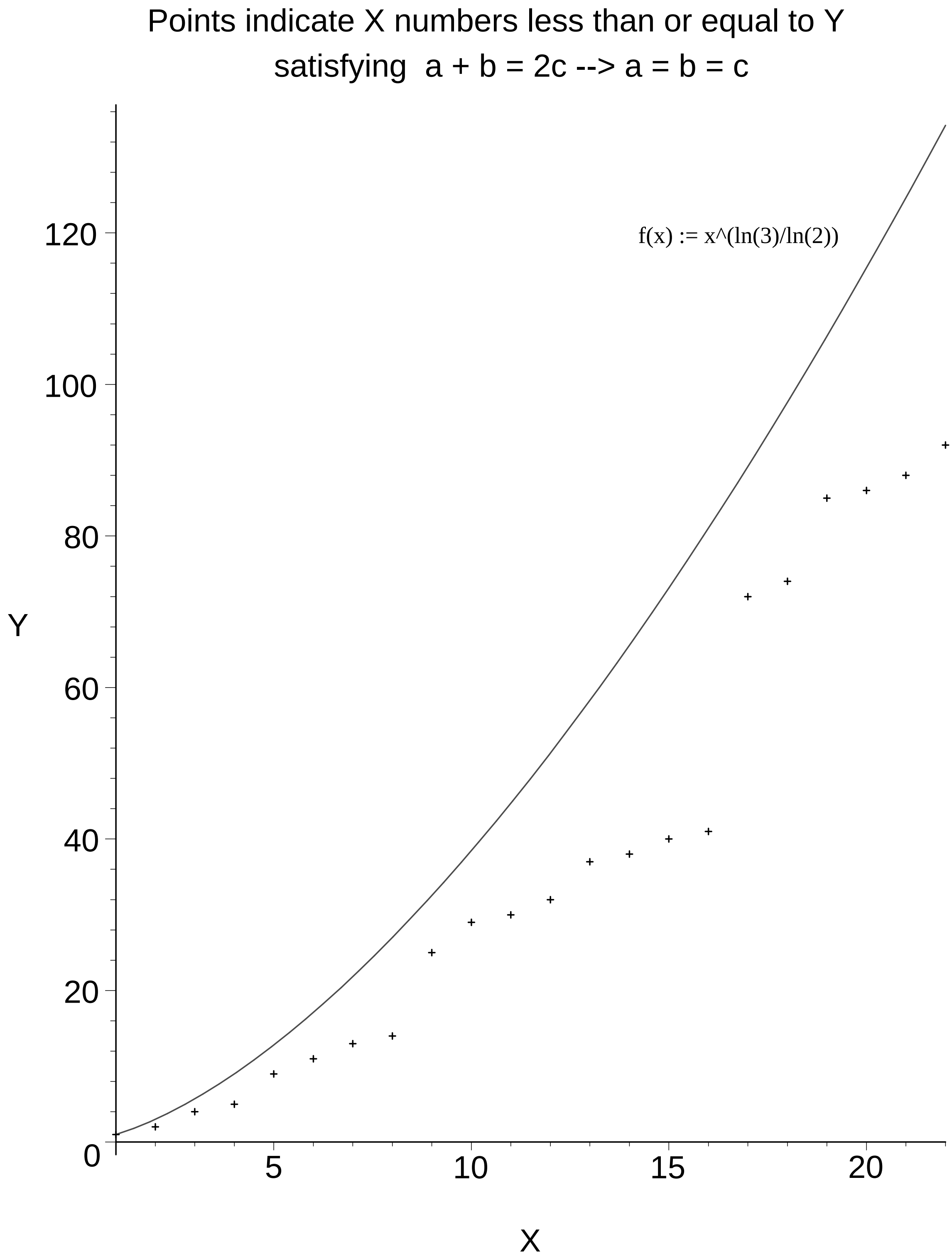

FIGURE 1. $a + b \neq 2c$

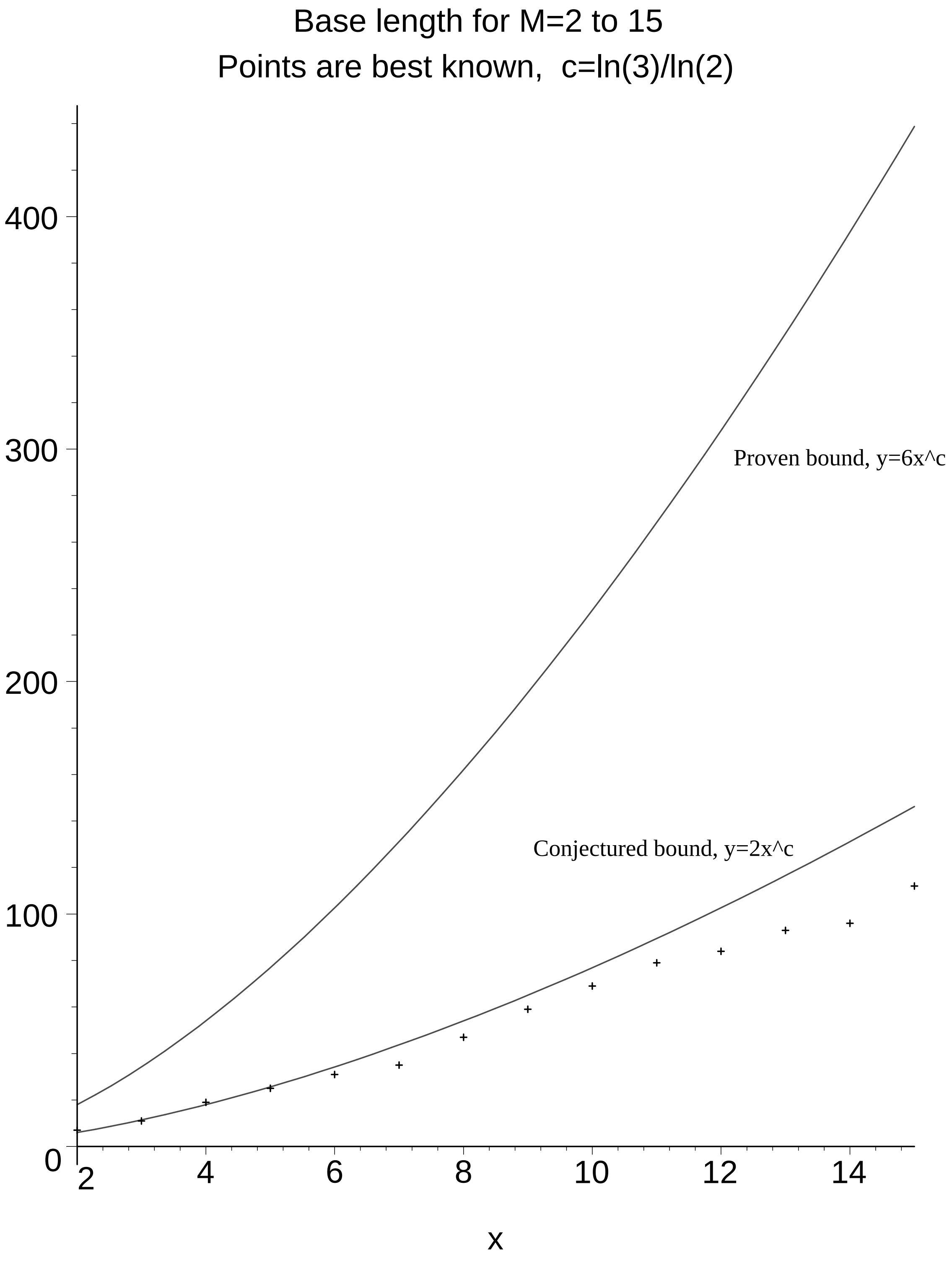

FIGURE 2. baselen(M)

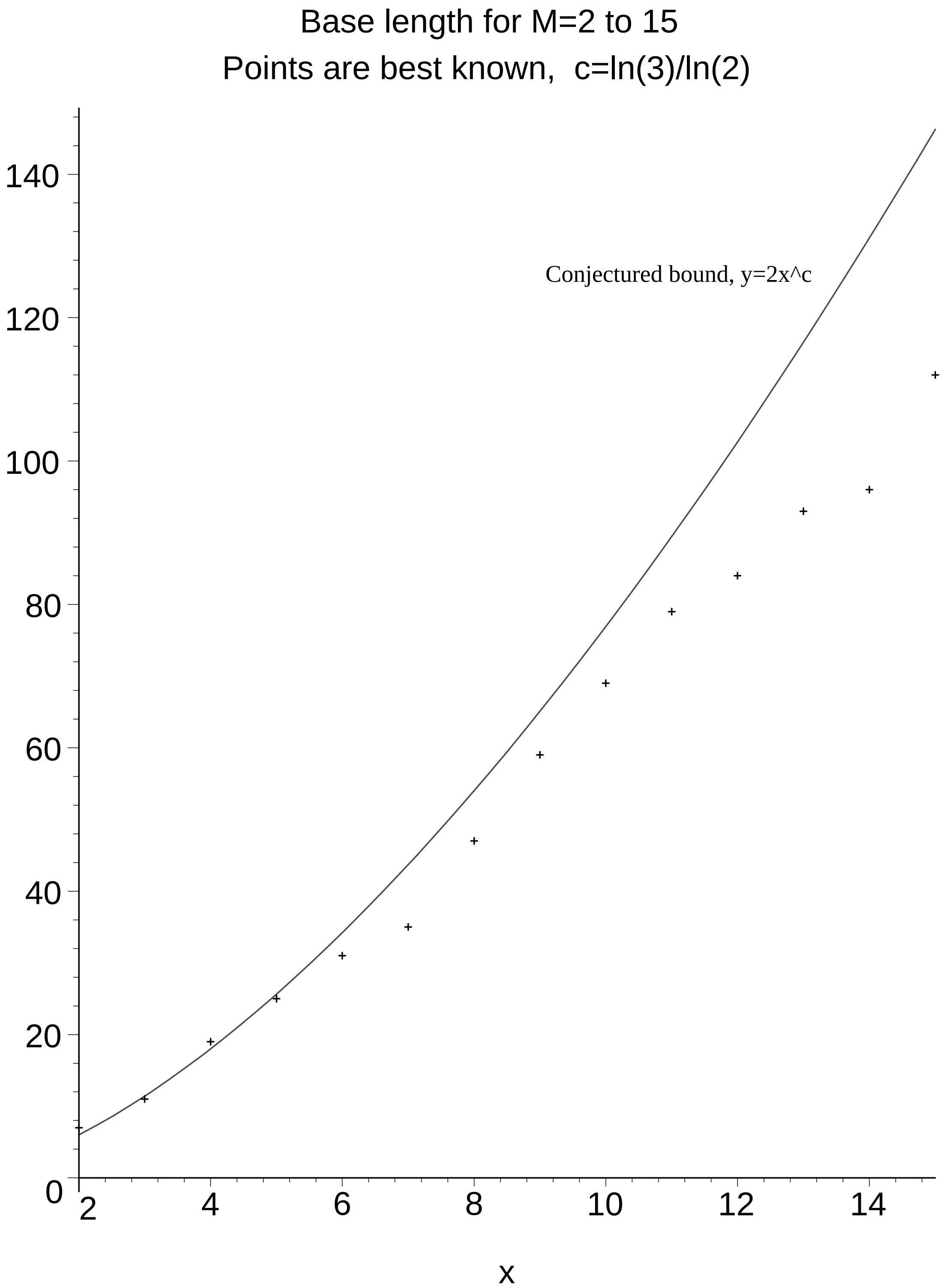

FIGURE 3. Baselen conjecture

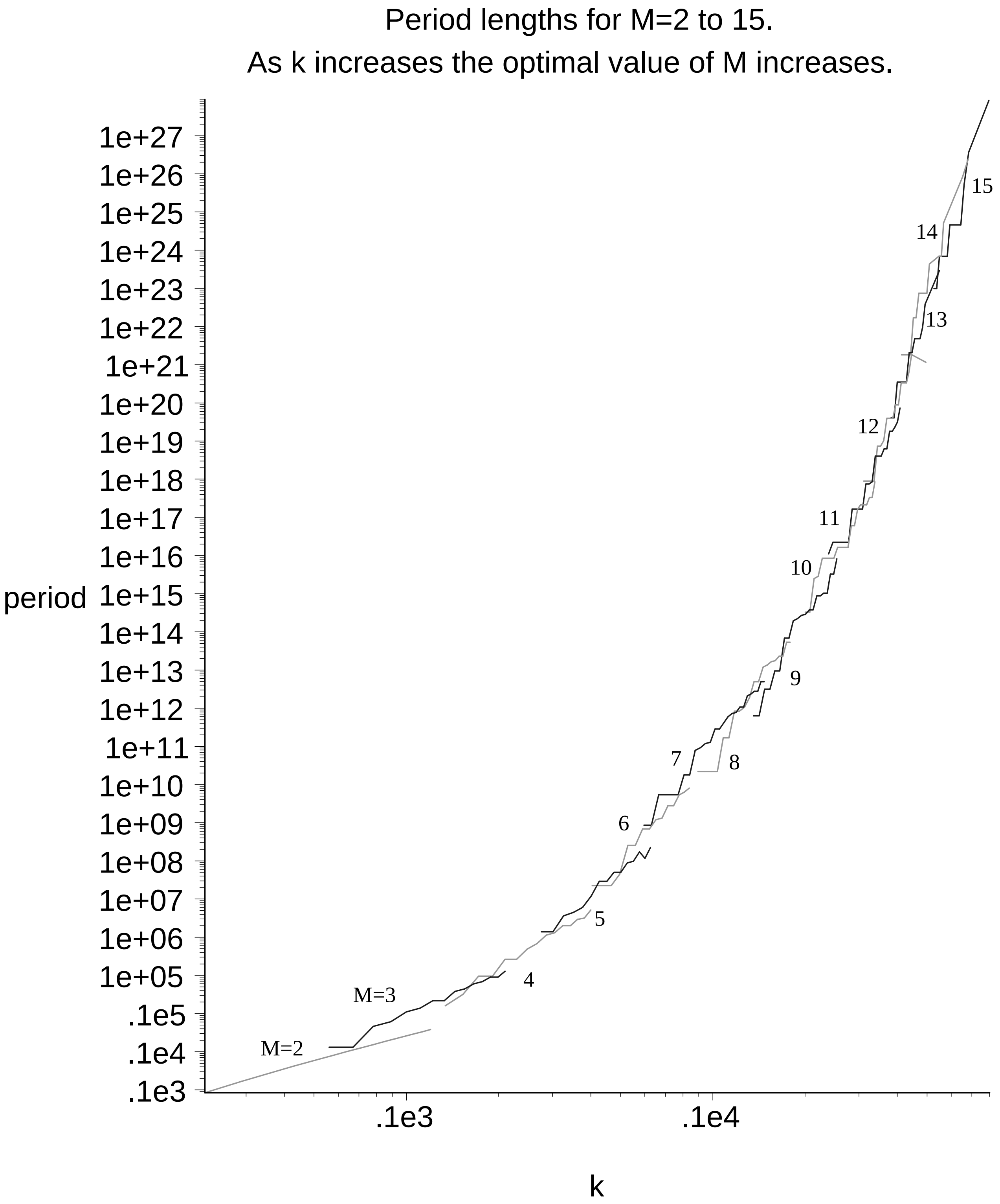

Figure 4. Period lengths for M = 2 to 15

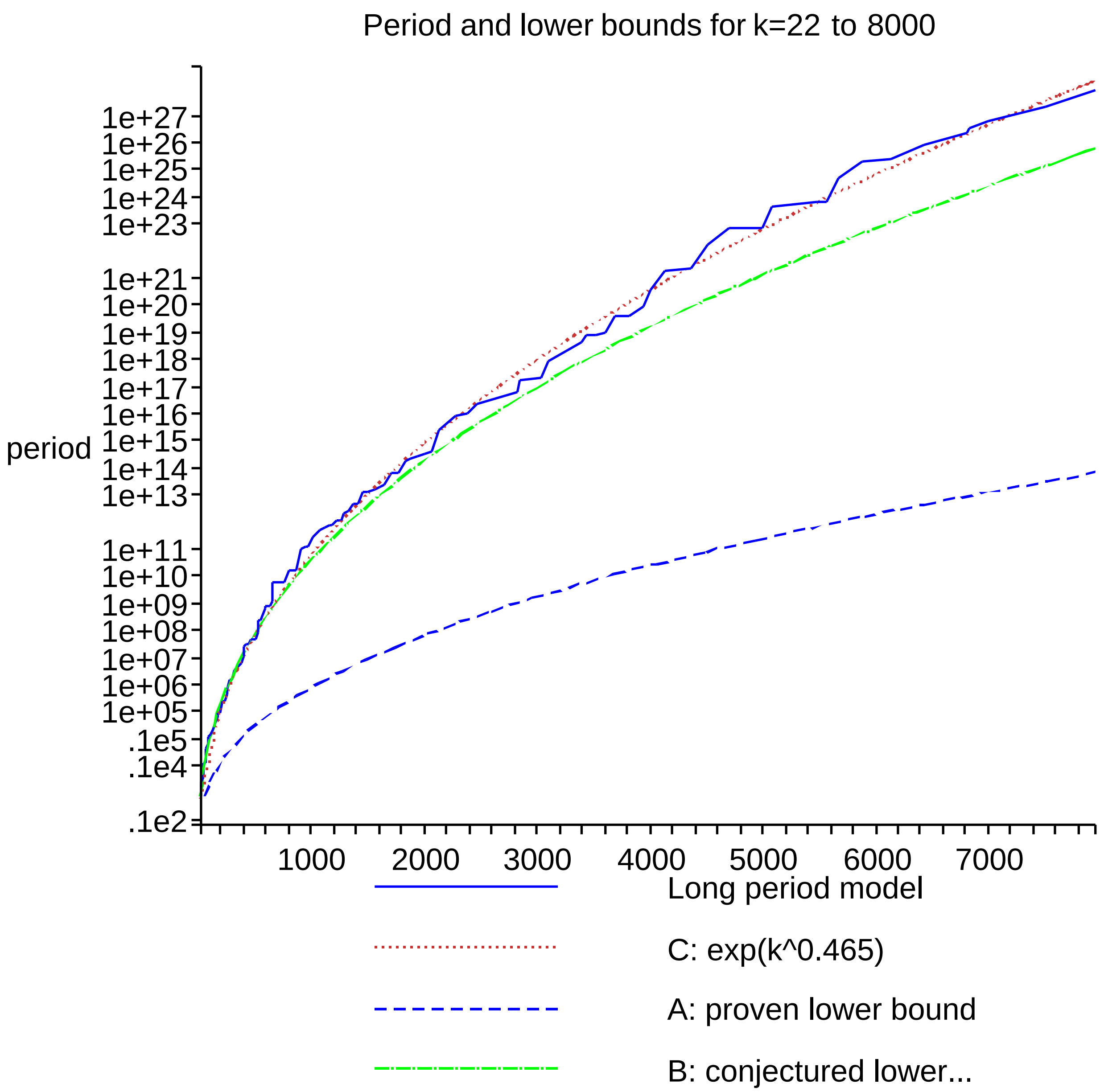

FIGURE 5. period and lower bounds for k = 22 to 8000

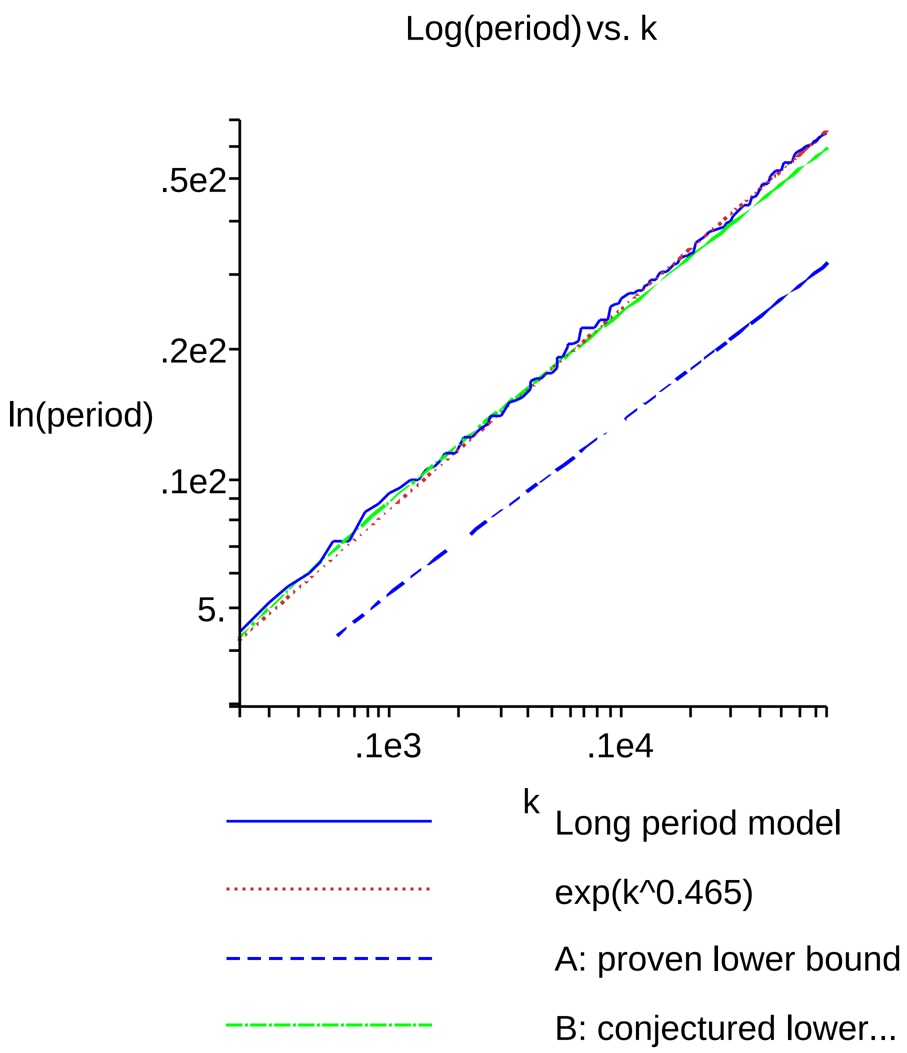

FIGURE 6. log(log(period)) versus log(k) is close to linear.

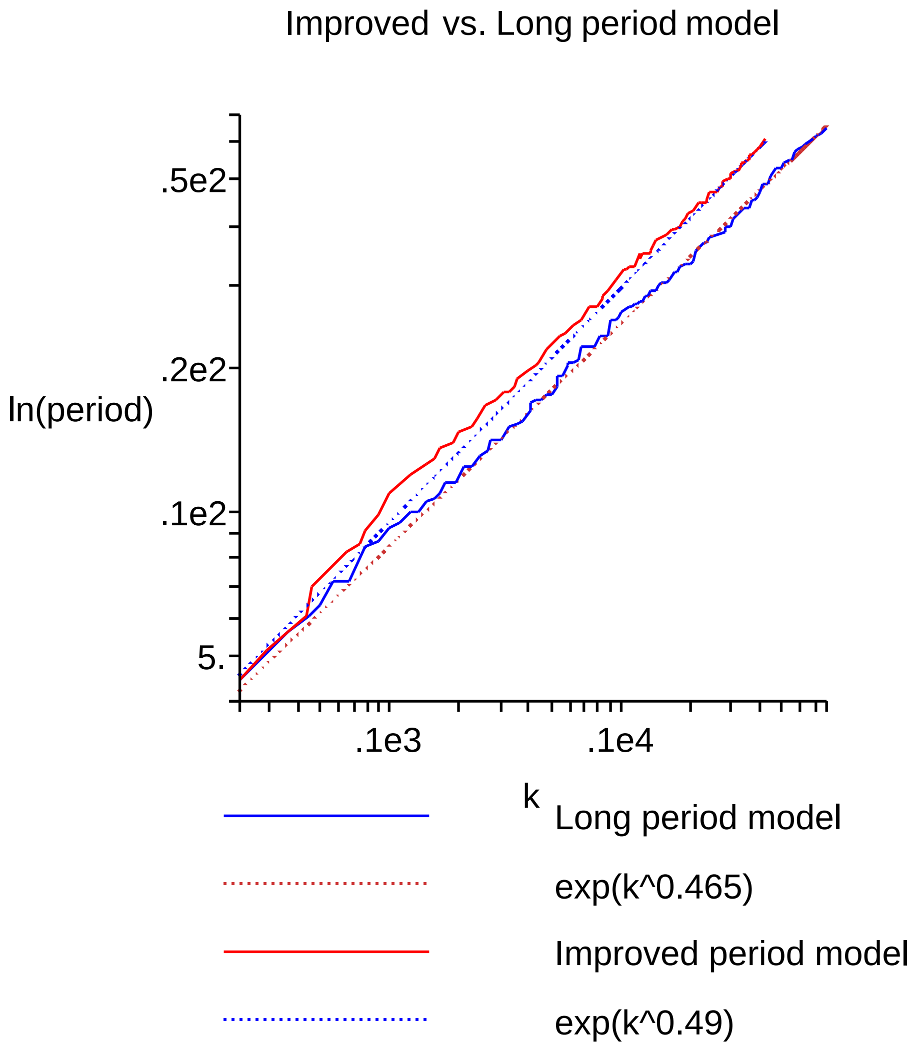

Figure 7. Improved versus Long period model

Chapter 4

Preperiod, other properties for Case 1

Having demonstrated that the positive case is eventually periodic, and having established bounds on the maximum period length, in this chapter we briefly examine some other properties of these systems. We will analyze using the log version, which we know is nonexpansive in L_∞, piecewise linear, with all partial derivatives in the set $\{-1, 0, 1\}$. First we consider the preperiod.

4.1. Conditions for unbounded preperiod

Definition 13. Unbounded preperiod:
We say that a system has unbounded preperiod if for all $N > 0$ there exists an initial positive point $\vec{y}$ with preperiod greater than N. Otherwise we say that the system has bounded preperiod.

Consider a log version system with $B_i \in -\infty \cup [-1, 0]$. Any case 1 system can be translated into a system like this. One condition that is necessary for unbounded preperiod is that there must be at least one real valued $B_i < 0$.

Lemma 20. *If there does not exist a real valued $B_i < 0$ then the system has bounded preperiod.*

Proof. Assume $B_i \in \{0, -\infty\}$ for all $i \in [1, k]$. Fix an initial vector $(y_i)_{i=-k}^{-1}$. Then $y_0 = -y_j$ for some $j \in [1, k]$. So $y_0 \in \{\pm\, y_j \mid j \in\ [-k, -1]\}$, and by induction $y_i \in \{\pm\, y_j \mid j \in\ [-k-1]\}$ for all $i \in\ \mathbb{N}$. Since the number of possible y_i values is

$2k$, the number of possible values of $(y_i)_{i=j}^{j+k}$ for any starting j is not more than $(2k)^k$. So the preperiod must be less than $(2k)^k$. $\square$

Next we show that if the set of periodic points is bounded then the system has unbounded preperiod.

LEMMA 21. *If the set of periodic points in a positive RDEM is bounded, then the system has unbounded preperiod.*

PROOF. Let $f : \mathbb{R}^k \to \mathbb{R}^k$ be the equivalent dynamical system of the log version, and suppose that M is a bound on the set of periodic points. Thus for $y \in \omega(f; y)$, $\|y\|_\infty < M$. Let $N > 0$ be fixed. Consider an initial point with $|y_i| > M + N \; \forall \; i \in [-k, -1]$. Then $|y_0| > M + N - 1$, since there exists a j such that $y_0 = B_j - y_{-j}$, and $B_j \in [-1, 0]$. Now by induction $|y_i| > M + N - i \; \forall \; i \in [1, N]$, so the preperiod of this initial point must be greater than N. $\square$

It is not clear that this lemma is very useful, since showing that the set of periodic points is bounded may be difficult. The converse of the lemma is not true, an unbounded set of periodic points does not imply a bounded preperiod, as the following example demonstrates. Let $k = 6$, $B = \{0, -\infty, 0, -\infty, 0, -1\}$. Let M be a large positive number. The initial point $y_{-6} = y_{-5} = y_{-3} = y_{-2} = y_{-1} = M, y_{-4} = -M$ generates the sequence $-M, M, M-1, M, -M, M, M-1, M, -M, ...$, which is periodic. Thus the set of periodic points is unbounded. But the initial point $y_{-6} = y_{-5} = y_{-4} = y_{-3} = y_{-2} = y_{-1} = M$ generates the sequence $-M, M, -M, M, -M, M, M-1, M, M-1, M, M-1, 1-M, M-1, 1-M, M-1, 1-M, M-1, M-2, ...$ This obeys the pattern $|y_{(11(j+1)+i)}| = |y_{(11j+i)}| - 1$ for all $i \in [0, 10]$, and all $j < M - 3$, thus the system has unbounded preperiod. This example illustrates a test for unbounded preperiod. However, failing the test does not imply a bound on the preperiod. The test consists of trying various initial points $y_i = \pm\, M$ for $i \in [-k, -1]$, with M some large unspecified number. Then for $i < j$ we have $y_i \in [-M, \frac{j+2}{3} - M] \cup [M - \frac{j+2}{3}, M]$.

Often there will be a linear translation of period p, with $|y_{n+p}| = |y_n| - c$, with $c > 0$, and for all $n < N$, where $N > 3M$. If this happens then the system has unbounded preperiod. For example, $k = 2, B_1 = 0, B_2 = -1$. Choosing the initial point $y_{-2} = y_{-1} = M$, we get the sequence $(M, M), -M, M, M-1, 1-M, M-1, M-2, ...$ which obeys the rule $|y_{n+3}| = |y_n| - 1$ for all $0 \leq n < 3M$.

It is difficult to give broad sufficient conditions for unbounded preperiod, even with just one negative real valued B_i. For example, $B_i = 0$ for all $i \in [1, 9]$ except $B_5 = -1$ has bounded preperiod, $B_i = 0$ for all $i \in [1, 9]$ except $B_4 = -1$ has unbounded preperiod. $B_i = 0$ for all $i \in [1, 9]$ except $B_4 = -1,\ B_6 = -\infty$ has bounded preperiod. Systems with more than one negative real valued B_i are even harder to classify. For example, $B_2 = B_7 = B_9 = 0, B_4 = B_6 = -1$, all other $B_i = -\infty$ has unbounded preperiod, with $|y_{n+11}| = |y_n| - 1$, while $B_3 = B_5 = B_9 = 0, B_4 = B_6 = -1$, all other $B_i = -\infty$ does not appear to have unbounded preperiod.

4.2. The unique half-line of translation.

As proven in [**10**] if f is a nonexpansive piecewise linear mapping of $\mathbb{R}^k$ into itself then there exists a unique half line that f maps onto itself such that the restriction of f thereto is a translation. This applies for any norm on $\mathbb{R}^k$. The theorem states

Theorem 5. *Let $f : \mathbb{R}^k \to \mathbb{R}^k$ be piecewise linear and nonexpansive. Then there exists a vector $\beta \in \mathbb{R}^k$, and a unique vector $\alpha \in \mathbb{R}^k$, such that $f(\beta + t\alpha) = \beta + (t+1)\alpha$ for all $t \geq 0$.*

The definition of piecewise linear is

Definition 14. Piecewise linear
We say that $f : \mathbb{R}^k \to \mathbb{R}^k$ is piecewise linear if $\mathbb{R}^k$ can be expressed as the finite union of closed polyhedral sets on each of which f is affine.

The log version of case 1 fits the definition of piecewise linear. For the system B_i $i \in [1,k]$, let $J = |\{i \in [1,k] \mid B_i \in \mathbb{R}\}|$. Then there are J closed polyhedral sets on which f is affine. Denote these sets

$$C_i = \left\{ y \in \mathbb{R}^k \mid B_i - y_{k-i} \geq B_j - y_{k-j} \; \forall \; j \in [1,k] \right\}.$$

Each set C_i consists of the points in $\mathbb{R}^k$ for which the max equation is satisfied using B_i. Note C_i is defined as the set of points which satisfy a given list of J linear inequalities, which therefore makes C_i a closed polyhedral set. For $x \in C_i$, f has the form $f(x) = Ax + b$, where $A \in \mathbb{R}^k \mathrm{x} \mathbb{R}^k$, $b \in \mathbb{R}^k$. The components of A are zero except for $A_{(j,j+1)} = 1$, and $A_{(k,i)} = -1$. $b_j = 0$ if $j \neq i$, and $b_i = B_i$. Applying this to theorem 5 we get $\beta_{j+1} + t\alpha_{j+1} = \beta_j + (t+1)\alpha_j$ for all $t \geq 0$ $j \in [1, k-1]$. This forces $\alpha_j = \alpha_1$, and $\beta_j = d + j\alpha_1$, for all $j \in [1,k]$, where d is an undetermined constant. And since $(f(x))_k = B_i - x_{k-i}$, we have $(f(\beta + t\alpha))_k = B_i - (\beta_{k-i} + t\alpha_{k-i}) = \beta_k + (t+1)\alpha_k$. Simplifying, $B_i - \beta_{k-i} - \beta k = (2t+1)\alpha_1$. This forces $\alpha = \vec{0}$, and $\beta_j = d = \frac{B_i}{2}$ for all $j \in [1,k]$.

Also, the only fixed point of any RDEM system is $\vec{0}$, since if $x = (x_i)_{i=1}^k$ is a fixed point then $(f(x))_i = x_{i+1} = x_i$ for all $i \in [1, k-1]$, so there must be a c such that $x_i = c$ for all $i \geq 1$. But then $c = (f(x))_k = -c$, so $c = 0$.

4.2.1. Other observations.

Lemma 22. *Periodic orbits of positive RDEM systems are not locally attractive.*

Proof. Let $\epsilon > 0$ be fixed. Consider the log version of a positive RDEM system of size k with terms $(B_i)_{i=1}^k$, and an initial vector $\vec{y} = (y)_{i=-k}^{-1}$. Let f be the equivalent dynamical system. Let $\dot{y} = (\dot{y})_{i=1}^k \in \omega(\vec{y}; f)$. Because of the way the y terms are generated it is true that for each $i \in [1,k]$ there are integer constants $(c_j)_{j=1}^k$ and an integer $n \in [-k,-1]$ such that $\dot{y}_i = \pm\, y_n + \sum_{t=1}^k c_t B_t$. But most real numbers can not be expressed in this manner. In particular, there exists a $\hat{y} = (\hat{y})_{i=1}^k$ such that $0 < |\hat{y}_i - \dot{y}_i| < \epsilon$ for all $i \in [1, k]$, and such that for any $i \in [1, k]$ there are no integer constants $(c_j)_{j=1}^{k}$ and $n \in [-k, -1]$ such that $\hat{y}_i = \pm\, y_n + \sum_{t=1}^{k} c_t B_t$. Let $\hat{y}$ be such a vector. The orbit of $\hat{y}$ remains within ϵ of points in $\omega(\vec{y}; f)$, and is eventually periodic, but with a different ω-limit set. This is because for $\ddot{y} \in \omega(\hat{y}; f)$ there are integer constants c_t, and n such that $\ddot{y}_i = \pm\, \hat{y}_n + \sum_{t=1}^{k}, c_t B_t$, which implies that $\ddot{y}_i$ can not be expressed as $\pm\, y_n$ plus the sum of integer multiples of the B terms, and thus $\ddot{y} \notin \omega(\vec{y}; f)$.

Therefore the log version does not have any locally attractive periodic orbits. And positive case RDEM systems are topologically conjugate to the log version, so they do not have any locally attractive periodic orbits either. □

From the proof we see the following corollary.

COROLLARY 1. *Every neighborhood of a periodic orbit of a positive RDEM system has other periodic orbits within it.*

Chapter 5

Case 2: $A \geq 0$, exists initial $x_i < 0,\ x_i > 0$.

In this chapter we begin the analysis of the non-positive cases. In this case, with A nonnegative and the initial point having both positive and negative initial values, the difference equation can not generate any negative terms. So it will either generate k positive terms in succession, thereby transposing to the positive case, or it will generate a zero, which causes division by zero, terminating the sequence. In the latter case we say that no solution exists. The following lemma can be used to determine if an initial point has a solution.

Lemma 23. *Let $j = \min(i \in \mathbb{N} \mid A_i > 0)$. A solution exists in case 2 if and only if for all i from 0 to $j-1$, There exists $t \in [1, k]$ such that $A_t > 0$ and $x_{i-t} > 0$.*

Proof. Note if all A coefficients are positive then $j = 1$ and the condition is satisfied because we specify that there is a positive initial x term, and a solution exists because $x_0 > 0$, and for all $i > 0$, $x_i \geq \frac{A_1}{x_{i-1}} > 0$. Assume there are $A_i = 0$. The condition guarantees that the first j x terms, $(x_i)_{i=0}^{j-1}$, will be positive, since for $i \in [0, j-1]$, $\exists\, t \in [1, k]$ *s.t.* $x_i \geq \frac{A_t}{x_{i-t}} > 0$. Once we have j positive terms in a row the remaining terms will all be positive, since for $i \geq j$, $x_i \geq \frac{A_j}{x_{i-j}} > 0$ by induction. Also, the condition is necessary, since if there is an $i \in [0, j-1]$ such that for all $t \in [1, k]$ with $A_t > 0$, $\quad x_{i-t} < 0$ then clearly $x_i = 0$ which results in division by zero after that term. $\square$

If a solution exists then the negative x terms play no part. After (at most) the first k terms the system is equivalent to a system starting with k positive terms, which is covered in case 1.

Chapter 6

Case 3: $A \geq 0, \quad \vec{x} < 0.$

We show that this case is topologically conjugate to case 1.

6.0.2. Case 3a: exists $A_i = 0$. There is no solution. $x_0 = 0$, so x_1 is undefined.

6.0.3. Case 3b: $A \geq 0, \quad x < 0.$. All the x terms will be negative. Taking the max of a set of negative numbers is related to taking the min of the absolute value of the numbers. We show this case is topologically conjugate to the positive case, and also how a reciprocal difference equation with minimum can be translated into a RDEM.

6.0.3.1. *Translations.* The following lemma shows that all RDEM systems have a topologically conjugate reciprocal difference equation with minimum.

Lemma 24. *For $(A_i)_{i=1}^k \in \mathbb{R}^k$, and initial vector $\vec{x} = (x_i)_{i=-k}^{-1} \in (\mathbb{R}\backslash 0)^k$, define $(x'_i)_{i=-k}^{-1}$ by $x'_i = -x_i$. Then the system 1.1 is topologically conjugate to*

$$x'_n = \min_{i \in [1,k]} \left\{ \frac{A_i}{x'_{n-i}} \right\}.$$

Proof. This is based on the fact that for real numbers $a, b, \quad a < b \rightarrow -a > -b$. Inductive hypothesis: $x'_i = -x_i$ for all $i \in \mathbb{N}$. True up to -1. Assume true up to $n-1$. Then $x'_n = \min\left\{\frac{A_i}{-x_{n-i}}\right\} = \min\left\{-\frac{A_i}{x_{n-i}}\right\} = -\max\left\{\frac{A_i}{x_{n-i}}\right\} = -x_n$, where all the min and max are for $i \in [1, k]$. $\square$

Case 1 and case 3b systems share another equivalence.

LEMMA 25. *Let $A_i > 0$ for all $i \in [1, k]$. Let $\vec{x} = (x_i)_{i=-k}^{-1}$ have the property that either $x_i < 0$ for all $i \in [-k, -1]$ or $x_i > 0$ for all $i \in [-k, -1]$. Then the system 1.1 is topologically conjugate to*

$$x_n = \frac{1}{\min_{i \in [1,k]} \left\{ \frac{x_{n-i}}{A_i} \right\}}. \tag{6.1}$$

PROOF. This uses the fact that if a, b have the same sign then $a < b \rightarrow \frac{1}{a} > \frac{1}{b}$. Let A and $\vec{x}$ satisfy the condition in the lemma. It is sufficient to show that x_0 is the same for either system. Note that the expressions $\frac{x_{0-i}}{A_i}$ have the same sign for all $i \in [1, k]$. For sets of numbers with the same sign the min of the reciprocals is equal to the reciprocal of the max. In system 6.1, $x_0 = \frac{1}{\min \left\{ \frac{x_{n-i}}{A_i} \right\}} = \max \left\{ \frac{1}{\frac{x_{n-i}}{A_i}} \right\} = \max \left\{ \frac{A_i}{x_{n-i}} \right\}$. □

Combining these equivalencies we can show that every case 3b system is topologically conjugate to a case 1 system.

LEMMA 26. *Let $A_i > 0$ for all $i \in [1, k]$. Let $\vec{x} = (x_i)_{i=-k}^{-1} < 0$. Define $B_i = \frac{1}{A_i}$, $y_{-i} = \frac{-1}{x_{-i}}$ for all $i \in [1, k]$. Then the case 3b system 1.1 is topologically conjugate to the case 1 RDEM system*

$$y_n = \max_{i \in [1,k]} \left\{ \frac{B_i}{y_{n-i}} \right\}.$$

PROOF. Inductive hypothesis: $y_i = \frac{-1}{x_i}$ for all $i \in \mathbb{N}$. True up to -1. Assume true up to $n - 1$. Then

$$y_n = \max \left\{ \frac{B_i}{y_{n-i}} \right\} = \max \left\{ \frac{\frac{1}{A_i}}{\frac{-1}{x_{n-i}}} \right\} = \max \left\{ \frac{-x_{n-i}}{A_i} \right\} =$$

$$\frac{1}{\min \left\{ \frac{A_i}{-x_{n-i}} \right\}} = \frac{-1}{\max \left\{ \frac{A_i}{x_{n-i}} \right\}} = \frac{-1}{x_n}.$$

□

Because of this topological conjugacy, all the properties of case 1 systems apply to case 3b. Since case 1 systems have solutions which are bounded, solutions for case 3b systems persist. And since solutions for case 1 systems persist, solutions for case 3b systems are bounded. Thus case 3b systems are eventually periodic, and all the long period models have equivalent case 3b systems, so the same bounds on maximum period lengths apply.

Although every case 3b system has a topologically conjugate case 1 system, the converse is not true. Consider a case 1 system with $A_i = 0$ for some i. The case 3b system would have $B_i = \frac{1}{A_i}$, division by zero. An equivalent system exists if B_i is undefined for $A_i = 0$, see section 9.0.12 for a discussion of this. If the domain of a case 1 system is restricted, so that there exists a bound $M > 1$ such that $\vec{x}$ is restricted to $\frac{1}{M} < x_j < M$ for all $j \in [-k, -1]$ then a topologically conjugate case 3b system can be constructed by setting $B_i = M^2$ for all i such that $A_i = 0$.

Chapter 7

Case 4: $\exists A_i > 0, A_j < 0, \quad \vec{x} \in (\mathbb{R}\backslash 0)^k$

This is the most significant of the non-positive cases. If there exists an $A_i = 0$ then either a solution does not exist or it transposes into the positive case. But if no $A_i = 0$ then interesting dynamics can result.

7.1. Subcase 4a: $\quad \exists A_i = 0$

Let $j = \min\{i \in \mathbb{N} | A_i > 0\}$.

Lemma 27. *A solution exists if and only if for each $i \in [0, j-1]$*

$$\exists\, t \in [i+1, k] \text{ s.t. } \frac{A_t}{x_{i-t}} > 0.$$

Proof. Note $x_n \geq 0$ for all $n \geq 0$, since for $A_i = 0$, $x_n \geq \frac{A_i}{x_{n-i}} = 0$. The condition is necessary, since if for some $i \in [0, j-1]$ there is no $t \in [i+1, k]$ such that $\frac{A_t}{x_{i-t}} > 0$ then $x_i \leq 0$ and therefore $x_i = 0$ and no solution exists. It is sufficient since it clearly implies that $x_i > 0$ for all $i \in [0, j-1]$. Now $x_j \geq \frac{A_j}{x_{j-j}}$ and by induction $x_i > 0$ for all $i \geq j$, and also $x_i > 0$ for all $i \in [0, j-1]$ by assumption. $\square$

Note this test does not require generating x_0 to x_{j-1}. If a solution exists it is always positive, since $A_j > 0$ and $x_n \geq \frac{A_j}{x_{n-j}}$. Since the solution is always positive the negative A terms play no part, so after the first k terms it is equivalent to a case 1 with $A'_i = 0\, \forall i$ *s.t.* $A_i < 0$.

7.2. Subcase 4b: $\quad A, \vec{x} \in (\mathbb{R}\backslash 0)^k, \quad \exists A_i > 0, A_j < 0,.$

In this case a solution always exists. There is a certain pattern of sign (positive or negative) needed in A, and a corresponding pattern of sign which must be in $\vec{x}$ in order for the system to generate an infinite number of negative terms. If this pattern does not exist in A or $\vec{x}$ then after the first k terms x is strictly positive, so the negative A values play no role, and by assigning zero to the negative A terms we obtain a topologically conjugate case 1 system. The following lemma gives the sign pattern needed for infinitely many negative terms.

Lemma 28. *Given $A, \vec{x} \in (\mathbb{R}\backslash 0)^k$, $(x_i)_{i=0}^{\infty}$ has infinitely many negative terms if and only if $\exists\, m \in [2, k+1],\ n \in [0, m-1]$ s.t.*

$$\operatorname{sgn}(A_i) = \begin{cases} + & \mathit{if}\ i \bmod m = 0 \\ - & \mathit{else.} \end{cases} \qquad \forall\, i \in [1, k]$$

and

$$\operatorname{sgn}(x_i) = \begin{cases} - & \mathit{if}\ i \bmod m = n \\ + & \mathit{else.} \end{cases} \qquad \forall\, i \in [n-k, -1].$$

Proof. Suppose the condition in the lemma is satisfied. Note $n < m$. For $0 \le i < n$, $A_{m-n+i} < 0$, and $x_{n-m} < 0$, so $x_i \ge \frac{A_{m-n+i}}{x_{n-m}} > 0$. Thus $x_i > 0$ for all $i \in [0, n-1]$. In order to have $x_i < 0$ we need

$$\operatorname{sgn}(A_j) \ne \operatorname{sgn}(x_{i-j})\ \ \forall\, j \in [1, k].$$

We have $\operatorname{sgn}(A_i) \ne \operatorname{sgn}(x_{n-i})\ \forall\, i \in [1, k]$, so $x_n < 0$. Also it can be verified that the next negative term generated is x_{n+m}. Thus for all $i > n-k$, $x_i < 0$ if and only if $i \bmod m = n$.

Suppose x has infinitely many negative terms. To have $x_i < 0,\ x_{i+1} < 0$ with $i \ge 0$ requires

$$sgn\,(x_{i-j}) = sgn\,(x_{i+1-j})\ \ \forall\, j \in [1, k],$$

which can only happen if x_{i-j} all have the same sign, which requires x_{i-j} to be all negative and A_i all positive, which is case 3. So x never has two negatives in a row. Hence if A_i has two positive terms in a row it can not generate negatives past x_k. Consider the first two negative terms generated, x_n and x_{n+m}. Note $m \leq k$ since otherwise we would have k consecutive positive terms after x_n, which would result in all terms after x_n being positive. Now since both x_n and x_{n+m} are negative, $\operatorname{sgn}(A_i) = \operatorname{sgn}(A_{m+i})$ for all $i \in [1, k-m]$, and also $A_i < 0$ for all $i < n$ since otherwise all the x terms would be positive. These results require the pattern of sign in A to be as described in the lemma. Also x must be as given, since otherwise x_n would not be the first negative term generated. $\square$

In the following section we show how to transform this system into a topologically conjugate nonexpansive system. However, we will analyze using the original system.

7.2.1. A nonexpansive system. To convert a system which satisfies the conditions of lemma 28 into a nonexpansive system we start by using conversions similar to those in case 1. Let n and m be as defined in lemma 28. we set

$$D = \max_{i \in [1,k]} \{|A_i|\}, \qquad A_i' = \frac{|A_i|}{D}, \qquad x_i' = \frac{|x_i|}{\sqrt{D}}. \tag{7.1}$$

$$y_i = \ln x_i' \qquad B_i = \ln (A_i').$$

The difference equation must have different rules for generating y_j, depending on the sign of x_j.

(7.2)

$$y_j = \begin{cases} \min \{B_i - y_{j-i} \mid i \in [1,k]\} & \text{if } j \bmod m = n, \\ \max \{B_i - y_{j-i} \mid i \in [1,k],\ i \equiv 0 \text{ or } i \equiv (j-n) \pmod m\} & \text{else.} \end{cases}$$

Given a set $(B_i)_{i=1}^{k}$ and a sequence $(y_i)_{i=-k}^{\infty}$ generated by it, and given the translation numbers $m,\ n,$ and D the original system can be found by

$$A_i = \begin{cases} D \cdot e^{B_i} & if\ i \bmod m = 0, \\ -D \cdot e^{B_i} & else. \end{cases} \qquad x_i = \begin{cases} -\sqrt{D} \cdot e^{y_i} & if\ i \bmod m = n, \\ \sqrt{D} \cdot e^{y_i} & else. \end{cases}$$

This system has the property that for two sequences y, y',

$$|y_n - y'_n| \le \max_{i \in [1,k]} |y_{n-i} - y'_{n-i}|.$$

So the dynamical system $f : \mathbb{R}^k \to \mathbb{R}^k$ defined by $f\,(y_i)_{i=n}^{n+k-1} = (y_i)_{i=n+1}^{n+k}$ is continuous, piece-wise linear, and nonexpansive in the sup norm. However, this system is often unbounded, which of course makes it nonperiodic.

7.2.2. Necessary and sufficient conditions for bounded. Reverting back to the original system, the following lemma gives necessary and sufficient requirements for the system to be bounded.

LEMMA 29. *Suppose that A and x have a pattern which generates infinitely many $x_i < 0$, and let n and m be as defined in lemma 28. Then the dynamical system is bounded if and only if the following condition is satisfied: For all i, j not divisible by m,*

$$A_i = A_j \ \forall \ (i+j) \bmod m = 0. \tag{7.3}$$

Moreover, this condition also implies that the solution is bounded away from zero.

PROOF. Note that equation 7.3 implies that $A_i = A_j \ \forall \ i \equiv j \mod m$, since $A_i = A_{m-(i \bmod m)} = A_j$. First we show the condition is necessary. Let i, j be integers not divisible by m, with $(i+j) \mod m = 0$, and $A_i < A_j$. Let $h = \frac{i+j}{m}$. Note $A_j < 0$. We have

$$x_{n+i} \ge \frac{A_i}{x_n} > 0, \quad x_{n+i+j} = x_{n+hm} \ge \frac{A_j}{x_{n+i}} \ge \frac{A_j}{\left(\frac{A_i}{x_n}\right)} = \frac{A_j}{A_i} \cdot x_n.$$

And $\frac{A_j}{A_i} \in (0,1)$, so $x_{n+t(hm)} \in \left[\left(\frac{A_j}{A_i}\right)^t \cdot x_n, 0\right]$, and $x_{n+t(hm)+i} \geq \frac{A_i}{x_n} \cdot \left(\frac{A_i}{A_j}\right)^t$, and $\lim_{t\to\infty} \left(\frac{A_i}{A_j}\right)^t = \infty$. So $A_i < A_j$ implies (x_i) is unbounded, and by symmetry $A_i > A_j$ also implies unbounded, therefore the condition is necessary for x to be bounded.

To show that the condition is sufficient, we prove the following claims.

Claim 1: The sequence $(x_{n+sm})_{s=0}^{\infty}$ is nondecreasing.

We have $x_{n+m} \geq \frac{A_1}{x_{n+m-1}} \geq \frac{A_1}{\frac{A_{m-1}}{x_n}} = \frac{A_1}{A_{m-1}} \cdot x_n = x_n$. The claim follows by induction.

Claim 2: If $n > 0$ then $(x_{n+sm})_{s=-1}^{\infty}$ is nondecreasing.

Assume $n > 0$. Then $x_n \geq \frac{A_1}{x_{n-1}} \geq \frac{A_1}{\frac{A_{m-1}}{x_{n-m}}} = \frac{A_1}{A_{m-1}} \cdot x_{n-m} = x_{n-m}$.

Claim 3: For all $i \in [0, m-1]$ such that $i \bmod m \neq n$, the sequence $(x_{i+sm})_{s=0}^{\infty}$ is nondecreasing.

Suppose $i < n$. Then

$$x_{i+m} \geq \frac{A_{m+i-n}}{x_n} \geq \frac{A_{m+i-n}}{\frac{A_{n-i}}{x_i}} = \frac{A_{m+i-n}}{A_{n-i}} \cdot x_i = x_i.$$

If $i > n$ then

$$x_{i+m} \geq \frac{A_{i-n}}{x_{m+n}} \geq \frac{A_{i-n}}{\frac{A_{m+n-i}}{x_i}} = \frac{A_{i-n}}{A_{m+n-i}} \cdot x_i = x_i.$$

Claim 4: $x_i \geq \sqrt{A_m}\ \forall i \geq 0$ such that $i \bmod m \neq n$. This is true since if $x_{i-m} \leq \sqrt{A_m}$ then $x_i \geq \frac{A_m}{x_{i-m}} \geq \sqrt{A_m}$, while if $x_{i-m} > \sqrt{A_m}$ then the claim is true since $(x_{i+sm})_{s=0}^{\infty}$ is nondecreasing. Note this implies that $\frac{A_m}{x_i} \leq x_i\ \forall\ i \geq 0$ such that $i \bmod m \neq n$.

Claim 5: $x_{n+(s+1)m} = x_{n+sm}$, provided ($n > 0$ and$s \geq 0$) OR ($n = 0$ and$s \geq 1$).

Assume the provision. We have shown $x_{n+m} \geq x_n$. Suppose for the sake of contradiction that $x_{n+(s+1)m} > x_{n+sm}$. Let $j \in [1,k]\ s.t. x_{n+(s+1)m} = \frac{A_j}{x_{n+(s+1)m-j}}$. Then $j \leq m$, since for $i > m$, $x_{n+(s+1)m} \neq \frac{A_i}{x_{n+(s+1)m-i}}$, because $x_{n+sm} \geq \frac{A_{i-m}}{x_{n+(s+1)m-i}}$. Also, $j \neq m$, since $x_{n+(s+1)m} \neq \frac{A_m}{x_{n+sm}}$ because $x_{n+sm} \geq \frac{A_m}{x_{n+(s-1)m}}$, and from claims 1 and 2 we have $x_{n+sm} \geq x_{n+(s-1)m}$. Therefore $j < m$. Now $x_{n+(s+1)m} = \frac{A_j}{x_{n+(s+1)m-j}} > x_{n+sm} \geq \frac{A_j}{x_{n+sm-j}}$, which implies $x_{n+(s+1)m-j} > x_{n+sm-j}$. From Claim 4 we have $x_{n+(s+1)m-j} \neq \frac{A_m}{x_{n+sm-j}} \leq x_{n+sm-j}$. Now $x_{n+(s+1)m-j}$ is positive, and $A_i < 0$ for all $i < m$, so the only possibility is $x_{n+(s+1)m-i} = \frac{A_{m-i}}{x_{n+sm}}$. But then $x_{n+(s+1)m} = \frac{A_i}{x_{n+(s+1)m-i}} = \frac{A_i}{\frac{A_{m-i}}{x_{n+sm}}} = \frac{A_i}{A_{m-i}} \cdot x_{n+sm} = x_{n+sm}$, which contradicts $x_{n+(s+1)m} > x_{n+sm}$. Thus the claim is proven.

Claim 6: $x_{n+(s+1)m+i} = x_{n+sm+i}\ \forall\ i \in [1, m-1]$, provided ($n > 0$ and$s \geq 0$) OR ($n = 0$ and$s \geq 1$).

Let s satisfy the provision. Suppose $x_{n+m+i} > x_{n+i}$. Let $j \in [1, k]$ $s.t. x_{n+(s+1)m+i} = \frac{A_j}{x_{n+(s+1)m+i-j}}$. Then $j \leq m$, since for $p > m$, $x_{n+(s+1)m+i} \neq \frac{A_p}{x_{n+(s+1)m+i-p}}$, because $x_{n+sm+i} \geq \frac{A_{p-m}}{x_{n+(s+1)m+i-p}}$. And $j \neq m$, since $\frac{A_m}{x_{sm+n+i}} \leq x_{sm+n+i}$. The only remaining possibility which give $x_{n+(s+1)m+i} > 0$ is $j = i$. But $x_{n+(s+1)m} = x_{n+sm}$ by Claim 5, so $j = i$ implies $x_{n+(s+1)m+i} = \frac{A_i}{x_{n+(s+1)m}} = \frac{A_i}{x_{n+sm}} \leq x_{n+sm+i}$. Thus the claim is proven.

The negative terms are bounded by Claim 1, and bounded away from zero by Claim 6. The positive terms are bounded by Claim 5, and bounded away from zero by Claim 3. ☐

7.2.3. Bounded implies eventually periodic. Because the requirements for the system to be bounded are so strict we can show directly that bounded implies eventually periodic.

LEMMA 30. *Let A and $\vec{x}$ have a pattern which generates infinitely many $x_i < 0$, and let n and m be as defined in lemma 28. If the system satisfies conditions 7.3, then it is eventually periodic. Moreover, the prime period $P = m$, and the preperiod is no more than m.*

PROOF. Assume A and $\vec{x}$ satisfy the given conditions. Let $i \in [1, m-1]$, $s \in \mathbb{N}$. From lemma 29 we have the following.

If $n > 0$ then $x_{n+sm} = x_n$, and $x_{n+sm+i} = x_{n+i}$. Combining these results we have $x_j = x_{j+sm}$ for all $j \geq n$, so the preperiod is no more than n and the period must be a factor of m.

If $n = 0$ then $x_{(s+1)m} = x_m$, and $x_{(s+1)m+i} = x_{m+i}$. So $x_j = x_{j+sm}$ for all $j \geq m$, and therefore the solution is periodic with period m, and a preperiod no larger than m.

Since the pattern of positive and negative x terms has prime period m, the prime period can not be less than m. $\square$

7.2.4. Unbounded solutions. It appears that all unbounded solutions of case 4b systems eventually take on a standard pattern. Suppose we have a case 4b system and an initial $\vec{x}$ that satisfies lemma 28. Let n and m be as defined in lemma 28. Let

$$c = \max\left\{ \left(\frac{A_i}{A_j}\right)^{1/h} \mid h, i, j \in \mathbb{N},\ i, j \text{ not multiples of } m,\ i + j = hm \right\}.$$

Note $c > 1$ for unbounded systems. Let h be an h which satisfies the above max. Then after an initial period it appears that the solution takes on the form described in the following conjecture.

CONJECTURE 3. *For a case 4b system and an initial $\vec{x}$ that satisfy lemma 28, let m, n, c and h be defined as above. Then there exists an integer N_0, divisible by hm, such that for all $\eta \in [0, hm-1]$, $d \in \mathbb{N}$,*

$$x_{(N_0+dhm+\eta)} = \begin{cases} x_{(N_0+\eta)} \cdot c^d & \text{if } \eta \bmod m \neq n \\ x_{(N_0+\eta)} \cdot c^{-d} & \text{if } \eta \bmod m = n. \end{cases}$$

If this conjecture is correct then after the initial N_0 terms these unbounded solutions consist of geometric subsequences $(x_{N_0+i+shm})_{s=0}^{\infty}$, where $i \in [0, hm-1]$, and where the multiple is either $1/c$ or c depending on if $i \bmod m = n$.

Chapter 8

Case 5: $A \leq 0, \quad \vec{x} \in \mathbb{R}\backslash 0$.

This case was completely solved by Szalkai, [**4**], however we will review it here to see how it compares with the other cases.

8.0.5. Case 5a: $A \leq 0,\ \exists A_j = 0,\ x \in \mathbb{R}\backslash 0$**.**

Lemma 31. *No solution exists for Case 5a RDEM systems.*

Proof. Consider a case 5a RDEM system with $A_j = 0$, and initial vector $\vec{x}$. If $x_n = 0$ for some $n < k$ then there is no solution. Suppose $x_n \neq 0$ for all $n < k$. Then $x_n \geq \frac{A_j}{x_{n-j}} = 0$, so $x_n > 0$ for all $n < k$. But then there are k consecutive positive x terms, and thus the next $x_k = 0$, so there is no solution. ☐

8.0.6. Case 5b: $A < 0, \quad x \in \mathbb{R}\backslash 0$**.** This case is similar to subcase 4b, except that it always generates infinitely many negative x terms. Let $m = k + 1$. Let

$$n = max\{0\} \cup \{i \in [1,k] | x_{i-m} < 0\}.$$

The system has a pattern of $m - 1$ positive terms followed by one negative, the negative x terms being x_{n+sm} for any $s \geq 0$. So the system is bounded if and only if it satisfies the conditions in lemma 29, and by lemma 30 bounded implies eventually periodic.

Chapter 9

Systems with fewer than k terms

In this system we consider a generalization of RDEM, allowing fewer than k terms in the max. For example, systems such as $x_n = max\left\{\frac{2}{x_{n-2}}, \frac{-1}{x_{n-5}}\right\}$. In this example $k = 5$, but there are only two terms in the max. For each of the cases of RDEM we examine the dynamics of the generalized system.

9.0.7. Formal definition. We will call these systems GRDEM, for Generalized RDEM. Let $(a_i)_{i=1}^{j}$ be an increasing sequence of positive integers with $a_j = k$. Let $A_{a_i} \in \mathbb{R}$ for all $i \in [1, j]$. Then the system

$$x_n = \max_{i\in[1,j]} \left\{ \frac{A_{a_i}}{x_{n-a_i}} \right\} \tag{9.1}$$

is a *GRDEM* system, which takes an initial vector of k terms, $\vec{x} = (x_i)_{i=-k}^{-1}$. First a general observation.

Lemma 32. *If a GRDEM system has an A term equal to zero then its solution is nonnegative.*

Proof. Let $A_i = 0$. Then $x_n \geq \frac{A_i}{x_{n-i}} = \frac{0}{x_{n-i}} = 0$. □

We divide GRDEM systems into the same cases as RDEM.

9.0.8. Case 1: $A \geq 0$, $\vec{x} > 0$. For case 1 systems it is equivalent for missing A terms to be assigned the value zero. Normally the zero coefficients aren't listed anyway, as the systems listed in the previous results section of the introduction illustrate.

9.0.9. Case 2a: $A \geq 0$, **exists** $A_i = 0$**, initial** $x_i > 0,\ x_j < 0$**.** As with RDEM systems there is either k positive terms in a row, transposing to case 1, or a solution does not exist. Lemma 23 applies to this case. As in case 1 it is equivalent to set the missing A terms to zero.

9.0.10. Case 2b: $A > 0$, **exists initial** $x_i > 0$**,** $x_j < 0$**.** In this case solutions always exist. Unlike RDEM, it is possible to have infinitely many negative values. For example, $k = 2,\ a_1 = 2,\ A_2 = 1$. An initial point such as $-1,\ 1$ generates the alternating sequence $-1,\ 1,\ -1,\ 1....$ The following conjecture proposes necessary and sufficient conditions for a solution to have infinitely many negative terms.

CONJECTURE 4. *Case 2b GRDEM systems have infinitely many negative terms if and only if* $\gcd\{a_i\} > 1$.

For solutions with infinitely many negative terms two open questions are what are the conditions for bounded solutions, and does bounded imply eventually periodic.

9.0.11. Case 3a: $A \geq 0$, **exists** $A_i = 0$, $\vec{x} < 0$**.** This case is identical to RDEM case 3a. There is no solution, $x_0 = 0$, and x_1 is undefined.

9.0.12. Case 3b: $A > 0$, $\vec{x} < 0$**.** These systems are topologically conjugate to case 1 RDEM systems, using the same translation as in chapter 6, with undefined terms in the case 3b system being assigned to zero in the case 1 system.

9.0.13. Case 4a: $\exists A_i > 0, A_j < 0, A_n = 0$. $\vec{x} \in (\mathbb{R}\backslash 0)^k$**.** This case is identical to the RDEM case 4a. Lemma 27 applies, the proof needs no modification.

9.0.14. Case 4b: $A, \vec{x} \in (\mathbb{R}\backslash 0)^k$, $\exists A_i > 0, A_j < 0$**.** If all the x terms are positive then the negative A terms play no role. So the first question is under what conditions does the solution have infinitely many negatives. It appears that lemma 28 is valid for GRDEM. Let A and $\vec{x}$ be a system and initial vector which satisfy the condition in lemma 28 for infinitely many negatives. Let m and n be as defined in that lemma.

Suppose there exists a $j \in [1, m-1]$ such that A_i is undefined for all $i \bmod m = j$. Then the sequence of positive terms $(x_{sm+n+j})_{s=0}^{\infty}$ are generated from previous terms in the sequence by the equation

$$x_{sm+n+j} = \max_{im \in [1,k]} \left\{ \frac{A_{im}}{x_{(s-i)m+n+j}} \right\}.$$

Thus the sequence $(x_{sm+n+j})_{s=0}^{\infty}$ is actually a case 1 system! For example, $k = 48$, $A_{15} = 1$, $A_{24} = A_{36} = A_{42} = A_{48} = 0.5$, $A_1 = -5$. An initial $\vec{x}$, from x_{-48} to x_{-1}

$$\begin{array}{ccccccccccccccc}
. & . & . & . & . & . & . & . & . & . & . & . & -1 & 5 & 2.5 \\
-1 & 5 & 5 & -1 & 5 & 2.5 & -1 & 5 & 0.5 & -1 & 5 & 2.5 & -1 & 5 & 0.5 \\
-1 & 5 & 0.2 & -1 & 5 & 0.5 & -1 & 5 & 2.5 & -1 & 5 & 0.5 & -1 & 5 & 2.5 \\
-1 & 5 & 5 & -1 & 5 & 2.5 & -1 & 5 & 1 & -1 & 5 & 2.5 & -1 & 5 & 1
\end{array}$$

This system satisfies the condition in lemma 28, with $m = 3$, $n = 0$. There are no A terms A_i with $i \bmod 3 = 2$, so the sequence $(x_{3s+2})_{s=0}^{\infty}$ is a case 1 system, similar to the unrestricted model example 3.6 from chapter 3. Thus case 4b GRDEM systems can have prime periods greater than $2k$. The above example has prime period 150.

9.0.15. Case 5a: $A \leq 0$, $\exists A_i = 0$, $\vec{x} \in (\mathbb{R}\backslash 0)^k$. This case is identical to RDEM case 5a. Lemma 31 applies. There is no solution, $x_n = 0$ for some $n \leq k$, and so x_{n+1} is undefined.

9.0.16. Case 5b: $A < 0$, $\vec{x} \in (\mathbb{R}\backslash 0)^k$. Solutions of these systems always have infinitely many negative terms, since if k consecutive x terms are positive then the next term must be negative. The pattern of positive and negative terms in a solution depends only on the sign pattern in the original x vector and which A terms are missing. The sign pattern in the solution is eventually periodic, but some systems have more than one possible periodic sign pattern. For example, $k = 4$, $A_2 = A_4 = -1$. The initial point $(1, 1, 1, 1)$ generates the sequence $-1, -1, 1, 1, 1, 1, -1, -1, 1, 1, 1, 1, ...$, a sign pattern with period 6, while the point $(-1, 1, 1, -1)$ generates the sequence $1, 1, -1, 1, 1, -1, 1, 1, -1...$, which is a period 3 sign pattern.

9.0.17. Conclusions. We see that all GRDEM systems that have $A_i = 0$ for some $i < k$ are equivalent to RDEM systems. Case 2b, 4b, and 5b GRDEM systems have patterns which are not seen in their RDEM counterparts.

Chapter 10

Similar systems

Many delay difference systems that have been analyzed in the literature look similar to RDEM. We would like to know if the methods used on our system will work with any of these. For each of these systems we ask the following questions:
Is the system bounded? If not what are the conditions for bounded?
If the conditions for bounded are met can the system (or just the positive case) be translated to a dynamical system which is nonexpansive in a polyhedral norm?
If so can we show that orbits must jump onto their omega limit sets?
If so, which implies that they are eventually periodic, what can we say about the maximum period lengths?

We look at various delay difference equations with maximums. The following are related to the Lyness equation. See references [**16**], [**8**]. In all cases A_n is periodic.

$$x_{n+1} = \frac{\max\{x_n, A\}}{x_n x_{n-1}} \tag{10.1}$$

$$x_{n+1} = \frac{\max\{x_n, A_n\}}{x_n x_{n-1}} \tag{10.2}$$

$$x_{n+1} = \frac{\max\{x_n, A_n\}}{x_n^2 x_{n-1}} \tag{10.3}$$

$$x_{n+1} = \frac{\max\left\{x_n^k, 1\right\}}{x_{n-1}}, n \geq 0,\ \vec{x} \text{ positive.} \tag{10.4}$$

$$x_{n+1} = \frac{\max\left\{x_n^k, 1\right\}}{x_{n-1}}, n \geq 0,\ \vec{x} \text{ positive.} \tag{10.5}$$

$$x_{n+1} = \frac{\max\{A_n x_n, B_n\}}{x_{n-1}}, A_n, B_n \text{ nonnegative with period 2, } \vec{x} \text{ positive.} \tag{10.6}$$

In each of these cases the new term, x_{n+1}, can involve a ratio of two previous terms. This feature results in the log versions being expansive, so the method used in this thesis does not appear to be applicable. Other similar difference equations are

$$x_{n+1} = \max\left\{\frac{A_n}{x_n}, \frac{B_n}{x_{n-1}}\right\},\ A_n, B_n \text{ periodic and nonnegative, } \vec{x} \text{ positive.} \tag{10.7}$$

$$x_{n+1} = \max\left\{\frac{A_n}{x_n}, \frac{B_n}{x_{n-2}}\right\},\ A_n, B_n \text{ positive with period 2, } \vec{x} \text{ positive.} \tag{10.8}$$

$$x_{n+1} = \max\left\{\frac{1}{x_n}, \frac{A_n}{x_{n-1}}\right\},\ A_n \text{ having prime period 3.} \tag{10.9}$$

In each of these cases x_{n+1} is generated by one previous term, so the log versions may be nonexpansive in L_∞.

There is an interesting dynamical system which has many applications and looks related [**2**]. Let A be an n x n matrix with the property $\min_j(a_{ij}) = 0$ for $i \in [1, n]$. Define $T : \mathbb{R}^n \to \mathbb{R}^n$ by

$$(T(x))_i = \min_j \{a_{ij} + x_j\}.$$

Note that the difference system $f : \mathbb{R}^k \to \mathbb{R}$ defined by

$$f(x) = \max_{i\in[1,k]} \{A_i - x_i\}$$

is equivalent to the log version of RDEM. To show this, let $A'_i = A_{k-i+1}$. Then $f(x) = \max_i \{A'_i - x_{k-i+1}\}$, which is the log version of RDEM with the A'_i values.

10.1. Analysis of similar systems

In 10.6 we can set $\max\{A_0, A_1, B_0, B_1\} = 1$. The log version is

$$x_{n+1} = \max\{A_n + x_n, B_n\} - x_{n-1} = \max\{A_n + x_n - x_{n-1}, B_n - x_{n-1}\}.$$

The system is expansive and appears to be unbounded. 10.1 is just a special case of 10.2 with A_n having period one. The log version is

$$x_{n+1} = \max\{0, A - x_n\} - x_{n-1}.$$

This system is expansive, so the method used in this thesis doesn't work. One method of trying to show that systems such as this are eventually periodic is to list the first few possible values in a tree structure. For example $x_0 = -x_{-2}$, or $x_0 = A - x_{-1} - x_{-2}$. If $x_0 = -x_{-2}$ then $x_1 = -x_{-1}$, or $x_1 = A + x_{-2} - x_{-1}$. If $x_0 = A - x_{-1} - x_{-2}$ then $x_1 = -x_{-1}$ or $x_1 = x_{-2}$, etcetera. If it can be shown that the set of these expressions for x terms is finite then the system is eventually periodic.

In [**16**], it is shown that for the system 10.8 bounded implies periodic. In [**8**] necessary and sufficient conditions were found for the system 10.7 to be bounded. Let p be the prime period of A_n, q the prime period of B_n. Then the system is bounded if and only if neither p nor q are multiples of three. We show that the log version of this system is nonexpansive. The log version is

$$y_{n+1} = \max\{A_n - y_n, B_n - y_{n-1}\},\ A_n, B_n \in [-d, 0] \tag{10.10}$$

$$A_n \text{ periodic with period p, } B_n \text{ periodic with period q.}$$

THEOREM 6. *Equation 10.10 is nonexpansive in L_∞.*

PROOF. Consider two initial points $\binom{y_{-1}}{y_0}$, and $\binom{x_{-1}}{x_0}$. Let $c = \|x - y\|_\infty$. We show that $|y_1 - x_1| \le c$. Without loss of generality assume that $y_1 > x_1$. $y_1 = \max\{A_0 - y_0, B_0 - y_{-1}\}$ and $x_1 = \max\{A_0 - x_0, B_0 - x_{-1}\}$, so we have

$$\begin{aligned} |y_1 - x_1| \in \{&|A_0 - y_0 - (A_0 - x_0)|, |A_0 - y_0 - (B_0 - x_{-1})|,\} \\ &\cup \{|B_0 - y_{-1} - (A_0 - x_0)|, |B_0 - y_{-1} - (B_0 - x_{-1})|\} \end{aligned} \tag{10.11}$$

There are two nontrivial cases.

Case 1: $|y_1 - x_1| = |A_0 - y_0 - B_0 + x_{-1}|$. We have $y_1 = A_0 - y_0$, $x_1 \ge A_0 - x_0$, so $0 < y_1 - x_1 = A_0 - y_0 - x_1 \le A_0 - y_0 - (A_0 - x_0) = x_0 - y_0 \le c$.

Case 2: $|y_1 - x_1| = |B_0 - y_{-1} - A_0 + x_0|$. We have $x_1 \ge B_0 - x_{-1}$, $y_1 = B_0 - y_{-1}$, so $0 < y_1 - x_1 = B_0 - y_{-1} - x_1 \le B_0 - y_{-1} - (B_0 - x_{-1}) = x_{-1} - y_{-1} \le c$. ∎

To translate the difference equation into a dynamical system we set $k = \mathrm{lcm}\,(p, q)$. Then we define $f : \mathbb{R}^{k+2} \to \mathbb{R}^{k+2}$ by $f\left((x_i)_{i=-k-1}^{0}\right) = (x_i)_{i=-1}^{k}$. A dynamical system is defined by the sequence $x, f(x), f(f(x)), \ldots$. Given the initial vector $(x_i)_{i=-k-1}^{0}$, f operates first on this vector, then on $(x_i)_{i=-1}^{k}$, then on $(x_i)_{i=k-1}^{2k}$, etcetera, each time advancing by k. This insures that the dynamical system generates the same x sequence as the difference equation. The value of the function $f\,(x_i)_{i=-1}^{k}$ is determined solely by x_{-1} and x_0, but the other values are included to make the dimension of the domain and range the same. In converting to the dynamical system, the initial value $\binom{x_{-1}}{x_0}$ can be combined with $x_i = 0\; \forall\; i \in [-k-1, -2]$.

This dynamical system is nonexpansive, so if it is also bounded it must have a finite ω-limit set. The proof of Theorem 1 can be applied to this system, showing that bounded implies eventually periodic.

The system 10.9 is a special case of 10.7, with $p = 1$, and $A_1 = 1$. The log version is

(10.12) $y_{n+1} = \max\{A - y_n, B_n - y_{n-1}\},\; A, B_n \in [-d, 0]$ B periodic with period q.

If q is not a multiple of 3, or if $A = 0$, then this system is bounded.

Lemma 33. *Equation 10.12 with $A = 0$ is bounded for any initial $\vec{y}$.*

Proof. Let $B = \max\{|y_0|, |y_{-1}|\}$. We show by induction that y_n is in the interval $I = [-B, B]$. First we show $y_1 \in I$. $y_1 \geq -y_0 \in I$, and $B_0 - y_{-1} \leq 0 - (-B) = B$. Therefore $y_1 = \max\{-y_0, B_0 - y_{-1}\} \in I$. Assume $y_i \in I$ for all $i \leq n$. Then $y_{n+1} \geq -y_n \in I$ and $B_n - y_{n-1} \leq B$, so $y_{n+1} \in I$. □

Also, this system is nonexpansive, since it is just a special case of 10.10. Therefore if it is bounded then the corresponding dynamical system has a finite ω-limit set, and using the same method as in Theorem 1 we can show that bounded implies eventually periodic.

Chapter 11

Open problems and conjectures

Here we list some unresolved problems and conjectures related to RDEM systems and generalized RDEM (GRDEM) systems.

Conjecture 1. *The minimum base length needed to support M subcycles, denoted b_M, satisfies the inequality $b_M < 2M^{\frac{\ln 3}{\ln 2}}$.*

Conjecture 2. *For $N > M \ln M$ and N a prime power the following lower bounds hold:*

$$\mathrm{glcm}(M, N) > N^M M^{-\frac{M^2+1}{2N}}.$$

$$\mathrm{glcm}(M, M^2) > M^{2M-1/2}.$$

The following conjecture asserts that unbounded solutions to case 4b systems eventually follow a specific pattern.

Conjecture 3. *For a case 4b system and an initial $\vec{x}$ that satisfy lemma 28, let m, n, c and h be defined as above. Then there exists an integer N_0, divisible by hm, such that for all $\eta \in [0, hm-1]$, $d \in \mathbb{N}$,*

$$x_{(N_0+dhm+\eta)} = \begin{cases} x_{(N_0+\eta)} \cdot c^d & if\ \eta \bmod m \neq n \\ x_{(N_0+\eta)} \cdot c^{-d} & if\ \eta \bmod m = n. \end{cases}$$

In chapter 9 we conjectured that GRDEM case 2b systems have the following property:

Conjecture 4. *Case 2b GRDEM systems have infinitely many negative terms if and only if* $\gcd\{a_i\} > 1$.

The following conjecture is relevant for systems with groups of subcycles, where each subcycle in the group has two terms. If it is false then there is an improvement on the improved long period model of chapter 3.

Conjecture 5. *Systems of the form*

$$y_n = \max_{i\in[1,d]} \left\{B_{(2i-1)} - y_{(n-(2i-1))}\right\} \quad k = 2d-1$$

which is the log version of the positive case where all the B subscripts are odd, have period length less than or equal to $2k$.

11.1. Open problems

What are the necessary and sufficient conditions for a positive case system to have an unbounded preperiod?

What is the growth rate of $\hat{b}_M$, the improved base length?

What is the growth rate of the maximum period length of the unrestricted model?

There are several open problems concerning the GRDEM systems of chapter 9.

For solutions of case 2b GRDEM systems with infinitely many negative terms two open questions are what are the conditions for bounded solutions, and does bounded imply eventually periodic. The same open problems exist for case 5b GRDEM systems.

Chapter A

Appendix: Program to find baselen(M)

Written in C.

```
// patv4.cpp
// This program looks for the smallest k which has a set of
// n numbers A[i], i=1 to n, which satisfy a set of inequalities.
// The inequalities involve triplets of the A[i], and k. For a given
// k all triplets which satisfy the inequalities are found first and
// stored in what is called the preprocessing, then the main program
// loop, which takes most of the time, checks if there is a set of n
// numbers such that every triplet is ok.

// Program asks for n, then the starting k value. The reason for a
// starting k value is to save time if the program was run before for
// the same n, and all k up to some value have been found to be no good.

// Note the c mod operator, % doesn't work like a mathematicians mod
// operator. In c (-2)%5 equals 2! For integers a%b is |a| mod b.

// #include "stdafx.h"
#include "stdlib.h"
#include "math.h"
#include "stdio.h"
#define MAXSIZE 162

short prep(short k, short s1, short s2, short s3);
short test(short b, short c);

short C[MAXSIZE][MAXSIZE-1][MAXSIZE-2], A[MAXSIZE];

int main(int argc, char* argv[])
{
        int n, k, b, i, m, j, c, ms;
        FILE *fd;
```

```
fd = fopen("pat.txt","w+");
k = 10;
b = 3;
printf("Enter n ");
i = scanf("%d",&n);
printf("Enter k ");
i = scanf("%d",&k);
printf("Enter Max k, up to %d ",MAXSIZE);
i = scanf("%d",&ms);
if (ms > MAXSIZE)
    ms = MAXSIZE;
for (;k<=ms;n++)
{
        for(;b<=n && k<=ms;k++)
        {
            b = 3;   // Preprocessing
            printf("start prep, k= %d ",k);
            for(i=3;i<k;i++)
            {
               for(j=2;j<i;j++)
               {
                   for(m=1;m<j;m++)
                   {
                       C[i][j][m] = prep(k,i,j,m);
                   }
               }
            }
            printf("done\n");
            A[1] = k-1;
            A[2] = k-2;
            c  = k-3;
            while ((b>1) && (b<=n) && (c>0))
            {
                if (test(b,c) == 0)
                {
                    A[b] = c;
                    b++;
                    c--;
                }
                else
                {
                    if((c>n-b) && (c>1))
                         c--;
                    else
                    {
                         for(b--;(A[b]<n-b) && (b>1);b--);
```

```
                                     c = A[b]-1;
                            }
                        }
                        if (b <= 2)
                        {
                            if (c>n-b+1)
                            {
                                b = 3;
                                A[2] = c;
                                c--;
                            }
                            else if (A[1]>n+1)
                            {
                                A[1]--;
                                A[2] = A[1]-1;
                                c = A[2]-1;
                                b = 3;
                            }
                            else b = 0;
                        }
                   }
                   if (b>n)
                   {
                       for (i = 1; i<=n; i++)
                       {
                       printf(" %d ",A[i]);
                       fprintf(fd," %d ",A[i]);
                       }
                       printf("\n k = %d n = %d \n",k,n);
                       fprintf(fd,"\n k = %d n = %d \n",k,n);
                   }
                }
        }
        fclose(fd);
        return 0;
}

short prep(short k, short s1, short s2, short s3)
{
        short i, m, t1, t;
        if ((2*s1 == k) || (2*s2 == k) || (2*s3 == k))
                 return 1;
        t = 2*k;
        t1 = (2*s1 + s2)%t;
        if ((t1 == s1) || (t1 == s2) || (t1 == s3) || (t1 == t-s1) ||
            (t1 == t-s2) || (t1 == t-s3) || (t1 == 0))
```

```
        return 1;
t1 = (2*s1 - s2+t)%t;
if ((t1 == s1) || (t1 == s2) || (t1 == s3) || (t1 == t-s1) ||
    (t1 == t-s2) || (t1 == t-s3) || (t1 == 0))
        return 1;
t1 = (2*s2 - s1+t)%t;
if ((t1 == s1) || (t1 == s2) || (t1 == s3) || (t1 == t-s1) ||
    (t1 == t-s2) || (t1 == t-s3) || (t1 == 0))
        return 1;
t1 = (2*s2 + s1)%t;
if ((t1 == s1) || (t1 == s2) || (t1 == s3) || (t1 == t-s1) ||
    (t1 == t-s2) || (t1 == t-s3) || (t1 == 0))
        return 1;
t1 = (2*s3 + s2)%t;
if ((t1 == s1) || (t1 == s2) || (t1 == s3) || (t1 == t-s1) ||
    (t1 == t-s2) || (t1 == t-s3) || (t1 == 0))
        return 1;
t1 = (2*s3 - s2+t)%t;
if ((t1 == s1) || (t1 == s2) || (t1 == s3) || (t1 == t-s1) ||
    (t1 == t-s2) || (t1 == t-s3) || (t1 == 0))
        return 1;
t1 = (2*s2 + s3)%t;
if ((t1 == s1) || (t1 == s2) || (t1 == s3) || (t1 == t-s1) ||
    (t1 == t-s2) || (t1 == t-s3) || (t1 == 0))
        return 1;
t1 = (2*s2 - s3+t)%t;
if ((t1 == s1) || (t1 == s2) || (t1 == s3) || (t1 == t-s1) ||
    (t1 == t-s2) || (t1 == t-s3) || (t1 == 0))
        return 1;
t1 = (2*s2 - s3+t)%t;
if ((t1 == s1) || (t1 == s2) || (t1 == s3) || (t1 == t-s1) ||
    (t1 == t-s2) || (t1 == t-s3) || (t1 == 0))
        return 1;
t1 = (2*s1 + s3)%t;
if ((t1 == s1) || (t1 == s2) || (t1 == s3) || (t1 == t-s1) ||
    (t1 == t-s2) || (t1 == t-s3) || (t1 == 0))
        return 1;
t1 = (2*s1 - s3+t)%t;
if ((t1 == s1) || (t1 == s2) || (t1 == s3) || (t1 == t-s1) ||
    (t1 == t-s2) || (t1 == t-s3) || (t1 == 0))
        return 1;
t1 = (s1 + 2*s3)%t;
if ((t1 == s1) || (t1 == s2) || (t1 == s3) || (t1 == t-s1) ||
    (t1 == t-s2) || (t1 == t-s3) || (t1 == 0))
        return 1;
t1 = (2*s3 - s1+t)%t;
```

```
        if ((t1 == s1) || (t1 == s2) || (t1 == s3) || (t1 == t-s1) ||
            (t1 == t-s2) || (t1 == t-s3) || (t1 == 0))
                 return 1;
        t1 = 3*s1%t;
        if ((t1 == s2) || (t1 == s3) || (t1 == t-s2) || (t1 == t-s3) ||
            (t1 == 0))
                 return 1;
        t1 = 3*s2%t;
        if ((t1 == s1) || (t1 == s3) || (t1 == t-s1) || (t1 == t-s3) ||
            (t1 == 0))
                 return 1;
        t1 = 3*s3%t;
        if ((t1 == s2) || (t1 == s1) || (t1 == t-s2) || (t1 == t-s1) ||
            (t1 == 0))
                 return 1;
        return 0;
}

short test(short b, short c)
{
        short i, j, s1, s2;

        for(i=2;(i<b);i++)
        {
            s2 = A[i];
            for(j=1;(j<i);j++)
            {
                s1 = A[j];
                if (C[s1][s2][c] == 1)
                     return 1;
            }
        }
        return 0;
}
```

That is the end of the program.

Chapter B

Appendix: Program to find improved baselen(m)

Written in C.

```
// patv6.c
// This program looks for the smallest k which has a set of
// n numbers A[i], i=1 to n, which satisfy a set of inequalities.
// The inequalities involve triplets of the A[i], and k. For a given
// k all triplets which satisfy the inequalities are found first and
// stored in what is called the preprocessing, then the main program
// loop, which takes most of the time, checks if there is a set of n
// numbers such that every triplet is ok.

// Program asks for M, then the starting k value. The reason for a
// starting k value is to save time if the program was run before for
// the same M, and all k up to some value have been found to be no good.

// Note the c mod operator, % doesn't work like a mathematicians mod
// operator. In c (-2)%5 equals 2! For integers a%b is |a| mod b.

// B array holds B value restrictions,
// C array holds codes. C[a][b][c] values mean the following:
// 0 means no restrictions, 19 means a,b,c not possible.
// 1 -> a<b     2 -> a<b,c   3 -> a<b<c   4 -> a<c     5 -> a<c<b
// 6 -> b<c     7 -> b<a,c   8 -> b<c<a   9 -> b<a     10 -> b<a<c
// 11 -> c<b    12 -> c<b,a  13 -> c<b<a  14 -> c<a    15 -> c<a<b
// 16 -> a,b<c  17 -> a,c<b  18 -> b,c<a.

// #include "stdafx.h"
#include "stdlib.h"
#include "math.h"
#include "stdio.h"

#define MAXSIZE 162

short prep(short k, short s1, short s2, short s3);
```

```
short test(short b, short c);

short update(short a, short b, short n);

short C[MAXSIZE][MAXSIZE-1][MAXSIZE-2], A[MAXSIZE],
       B[20][20][20];

int main(int argc, char* argv[]) {
        int n, k, b, i, m, j, c, ms, g;
        FILE *fd;

        fd = fopen("pat.txt","w+");
        k = 10;
        b = 3;
        printf("Enter starting k ");
        i = scanf("%d",&k);
        printf("Enter starting M ");
        i = scanf("%d",&n);
        printf("Enter max k, up to %d ",MAXSIZE);
        i = scanf("%d",&ms);
        if (ms > MAXSIZE)
            ms = MAXSIZE;
        for (;k<=ms;n++)
        {
                for(;b<=n && k<=ms;k++)
                {
                  b = 3;   // Preprocessing
                  printf("start prep, k= %d ",k);
                  for(i=0;i<20;i++)
                     for(j=1;j<20;j++)
                        for(m=1;m<20;m++)
                           B[i][j][m] = 0;
                  for(i=3;i<k;i++)
                     for(j=2;j<i;j++)
                        for(m=1;m<j;m++)
                           C[i][j][m] = prep(k,i,j,m);
                  printf("done\n");
                  A[1] = k-1;
                  A[2] = k-2;
                  c  = k-3;
                  while ((b>1) && (b<=n) && (c>0))
                  {
                        if (test(b,c) == 0)
                        {
                                A[b] = c;
```

```
                    b++;
                    c--;
            }
            else
            {
                    if((c>n-b) && (c>1))
                            c--;
                    else
                    {
                       for(b--;(A[b]<n-b) && (b>1);b--);
                       c = A[b]-1;
                    }
            }
            if (b <= 2)
            {
                    if (c>n-b+1)
                    {
                            b = 3;
                            A[2] = c;
                            c--;
                    }
                    else if (A[1]>n+1)
                    {
                            A[1]--;
                            A[2] = A[1]-1;
                            c = A[2]-1;
                            b = 3;
                    }
                    else b = 0;
            }
      }
      if (b>n)
      {
         for (i = 1; i<=n; i++)
         {
            printf(" %d (",A[i]);
            fprintf(fd," %d (",A[i]);
            for (j=1; j<=n; j++)
               if ((B[n][i][j] == 1) || (B[n][j][i]==2))
               {
                   printf(" %d ",A[j]);
                   fprintf(fd," %d ",A[j]);
               }
            printf(")");
            fprintf(fd,")");
         }
```

```
                    printf("\n b(M) = %d M = %d \n",k,n);
                    fprintf(fd,"\n b(M) = %d M = %d \n",k,n);
                 }
               }
        }
        fclose(fd);
        return 0;
}

short prep(short k, short s1, short s2, short s3) {
        short i, m, t1, t, v12, v23, v13;
        if ((2*s1 == k) || (2*s2 == k) || (2*s3 == k))
                return 19;
        if ((s1+s2 == k) || (s1+s3 == k) || (s2+s3 ==k))
                return 19;
        t = 2*k;
        if ((3*s1 == t) || (3*s2 == t) || (3*s3 == t))
                return 19;
        t1 = 3*s1%t;
        v12 = v13 = v23 = 0;
        if ((t1 == s2) || (t1 == t-s2))
            v12 = 1;
        if ((t1 == s3) || (t1 == t-s3))
            v13 = 1;
        t1 = 3*s2%t;
        if ((t1 == s1) || (t1 == t-s1))
            if (v12 == 1)
                return 19;
            else
               v12 = 2;
        if ((t1 == s3) || (t1 == t-s3))
            v23 = 1;
        t1 = 3*s3%t;
        if ((t1 == s2) || (t1 == t-s2))
           if (v23 == 1)
               return 19;
           else
               v23 = 2;
        if ((t1 == s1) || (t1 == t-s1))
           if (v13 == 1)
               return 19;
           else
               v13 = 2;
        t1 = (2*s1 + s2)%t;
        if ((t1 == s1) || (t1 == s2) || (t1 == t-s1) ||
                (t1 == t-s2)  || (t1 == 0))
```

```
   if (v12 == 2)
       return 19;
   else
       v12 = 1;
if ((t1 == s3) || (t1 == t-s3))
   if ((v12 == 2) || (v13 == 2))
       return 19;
   else
       v12 = v13 = 1;
t1 = (2*s1 - s2+t)%t;
if ((t1 == s1) || (t1 == s2) || (t1 == t-s1) ||
         (t1 == t-s2)  || (t1 == 0))
   if (v12 == 2)
       return 19;
   else
       v12 = 1;
if ((t1 == s3) || (t1 == t-s3))
   if ((v12 == 2) || (v13 == 2))
       return 19;
   else
       v12 = v13 = 1;
t1 = (2*s2 - s1+t)%t;
if ((t1 == s1) || (t1 == s2) || (t1 == t-s1) ||
         (t1 == t-s2)  || (t1 == 0))
   if (v12 == 1)
       return 19;
   else
       v12 = 2;
if ((t1 == s3) || (t1 == t-s3))
   if ((v12 == 1) || (v23 == 2))
       return 19;
   else
   {
       v12 = 2;
       v23 = 1;
   }
t1 = (2*s2 + s1)%t;
if ((t1 == s1) || (t1 == s2) || (t1 == t-s1) ||
         (t1 == t-s2)  || (t1 == 0))
   if (v12 == 1)
       return 19;
   else
       v12 = 2;
if ((t1 == s3) || (t1 == t-s3))
   if ((v12 == 1) || (v23 == 2))
       return 19;
```

```
   else
   {
       v12 = 2;
       v23 = 1;
   }
t1 = (2*s3 + s2)%t;
if ((t1 == s3) || (t1 == s2) || (t1 == t-s3) ||
        (t1 == t-s2)  || (t1 == 0))
   if (v23 == 1)
       return 19;
   else
       v23 = 2;
if ((t1 == s1) || (t1 == t-s1))
   if ((v13 == 1) || (v23 == 1))
       return 19;
   else
       v13 = v23 = 2;
t1 = (2*s3 - s2+t)%t;
if ((t1 == s3) || (t1 == s2) || (t1 == t-s3) ||
        (t1 == t-s2)  || (t1 == 0))
   if (v23 == 1)
       return 19;
   else
       v23 = 2;
if ((t1 == s1) || (t1 == t-s1))
   if ((v13 == 1) || (v23 == 1))
       return 19;
   else
       v13 = v23 = 2;
t1 = (2*s2 + s3)%t;
if ((t1 == s3) || (t1 == s2) || (t1 == t-s3) ||
        (t1 == t-s2)  || (t1 == 0))
   if (v23 == 2)
       return 19;
   else
       v23 = 1;
if ((t1 == s1) || (t1 == t-s1))
   if ((v12 == 1) || (v23 == 2))
       return 19;
   else
   {
       v12 = 2;
       v23 = 1;
   }
t1 = (2*s2 - s3+t)%t;
if ((t1 == s3) || (t1 == s2) || (t1 == t-s3) ||
```

```
        (t1 == t-s2)  || (t1 == 0))
   if (v23 == 2)
       return 19;
   else
       v23 = 1;
if ((t1 == s1) || (t1 == t-s1))
   if ((v12 == 1) || (v23 == 2))
       return 19;
   else
   {
       v12 = 2;
       v23 = 1;
   }
t1 = (2*s1 + s3)%t;
if ((t1 == s1) || (t1 == s3) || (t1 == t-s1) ||
        (t1 == t-s3)  || (t1 == 0))
   if (v13 == 2)
       return 19;
   else
       v13 = 1;
if ((t1 == s2) || (t1 == t-s2))
   if ((v12 == 2) || (v13 == 2))
       return 19;
   else
       v12 = v13 = 1;
t1 = (2*s1 - s3+t)%t;
if ((t1 == s1) || (t1 == s3) || (t1 == t-s1) ||
        (t1 == t-s3)  || (t1 == 0))
   if (v13 == 2)
       return 19;
   else
       v13 = 1;
if ((t1 == s2) || (t1 == t-s2))
   if ((v12 == 2) || (v13 == 2))
       return 19;
   else
       v12 = v13 = 1;
t1 = (s1 + 2*s3)%t;
if ((t1 == s1) || (t1 == s3) || (t1 == t-s1) ||
        (t1 == t-s3)  || (t1 == 0))
   if (v13 == 1)
       return 19;
   else
       v13 = 2;
if ((t1 == s2) || (t1 == t-s2))
   if ((v13 == 1) || (v23 == 1))
```

```
            return 19;
        else
            v23 = v13 = 2;
    t1 = (2*s3 - s1+t)%t;
    if ((t1 == s1) || (t1 == s3) || (t1 == t-s1) ||
            (t1 == t-s3)  || (t1 == 0))
        if (v13 == 1)
            return 19;
        else
            v13 = 2;
    if ((t1 == s2) || (t1 == t-s2))
        if ((v13 == 1) || (v23 == 1))
            return 19;
        else
            v23 = v13 = 2;
    t1 = v12 +3*v13 + 9*v23;
    switch (t1)
    {
        case 0: return 0;
        case 1: return 1;
        case 2: return 9;
        case 3: return 4;
        case 4: return 2;
        case 5: case 14: return 10;
        case 6: return 14;
        case 7: case 25: return 15;
        case 8: return 18;
        case 9: return 6;
        case 10: case 13: return 3;
        case 11: return 7;
        case 12: return 16;
        case 15: case 17: return 8;
        case 16: case 23: return 19;
        case 18: return 11;
        case 19: return 17;
        case 20: case 26: return 13;
        case 21: case 22: return 5;
        case 24: return 12;
        default: printf("LOGIC ERROR!!");
        return 19;
    }
}


short test(short b, short c) {
    short i, j, s1, s2;
```

```
for(i=1;i<=b;i++)
    for(j=1;j<=b;j++)
       B[0][i][j] = B[b-1][i][j];
for(i=2;(i<b);i++)
{
   s2 = A[i];
   for(j=1;(j<i);j++)
   {
      s1 = C[A[j]][s2][c];
      switch (s1)
      {
        case 19: return 1;
        case 0: case 1: case 9: break;
        case 2: if (update(j,i,b) == 1)
                     return 1;
                if (update(j,b,b) == 1)
                     return 1;
                break;
        case 3: if (update(j,i,b) == 1)
                     return 1;
                if (update(i,b,b) == 1)
                     return 1;
                break;
        case 4: if (update(j,b,b) == 1)
                    return 1;
                 break;
        case 5: if (update(j,b,b) == 1)
                     return 1;
                if (update(b,i,b) == 1)
                     return 1;
                break;
        case 6: if (update(i,b,b) == 1)
                    return 1;
                break;
        case 7: if (update(i,j,b) == 1)
                     return 1;
                if (update(i,b,b) == 1)
                     return 1;
                break;
        case 8: if (update(i,b,b) == 1)
                     return 1;
                if (update(b,j,b) == 1)
                     return 1;
                break;
        case 10: if (update(i,j,b) == 1)
```

```
                    return 1;
               if (update(j,b,b) == 1)
                    return 1;
               break;
       case 11: if (update(b,i,b) == 1)
                    return 1;
                break;
       case 12: if (update(b,i,b) == 1)
                    return 1;
                 if (update(b,j,b) == 1)
                    return 1;
                 break;
       case 13: if (update(b,i,b) == 1)
                    return 1;
                 if (update(i,j,b) == 1)
                    return 1;
                 break;
       case 14: if (update(b,j,b) == 1)
                    return 1;
                break;
       case 15: if (update(b,j,b) == 1)
                    return 1;
                 if (update(j,i,b) == 1)
                    return 1;
                 break;
       case 16: if (update(j,b,b) == 1)
                    return 1;
                 if (update(i,b,b) == 1)
                    return 1;
                 break;
       case 17: if (update(b,i,b) == 1)
                    return 1;
                 if (update(j,i,b) == 1)
                    return 1;
                 break;
       case 18: if (update(i,j,b) == 1)
                    return 1;
                 if (update(b,j,b) == 1)
                    return 1;
                 break;
       }
    }
}
for(i=1;i<=b;i++)
    for(j=1;j<=b;j++)
       B[b][i][j] = B[0][i][j];
```

```
        return 0;
}

short update(short a, short b, short n) {
    int i;

    if ((B[0][a][b] == 2) || (B[0][b][a] == 1))
       return 1;
    B[0][a][b] = 1;
    B[0][b][a] = 2;
    for (i=1;i<=n;i++)
    {
        if (i != a)
        {
            if ((B[0][i][a] == 1) || (B[0][a][i] == 2))
            {
                if ((B[0][i][b] == 2) || (B[0][b][i] == 1))
                   return 1;
                B[0][i][b] = 1;
                B[0][b][i] = 2;
            }
        }
        if (i != b)
        {
            if ((B[0][i][b] == 2) || (B[0][b][i] == 1))
            {
                if ((B[0][i][a] == 1) || (B[0][a][i] == 2))
                   return 1;
                B[0][i][a] = 2;
                B[0][a][i] = 1;
            }
        }
    }
    return 0;
}
```

That is the end of the program.

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