ABSTRACT

Gao, Jining The Algebraic Structure of BRST Operators. (Under the direction of Professor Ronald Fulp).

This dissertation has three related but distinct parts. In the first part of the thesis, we construct a new sequence of generators of the BRST complex and reformulate the BRST differential so that it acts on elements of the complex much like the Maurer-Cartan differential acts on left-invariant forms. Thus our BRST differential is formally analogous to the differential defined on the BRST formulation of the Chevalley-Eilenberg cochain complex of a Lie algebra. Moreover, for an important class of physical theories, we show that in fact the differential is a Chevalley-Eilenberg differential. As one of the applications of our formalism, we show that the BRST differential provides a mechanism which permits us to extend a nonintegrable system of vector fields on a manifold to an integrable system on an extended manifold.

In the second part of the thesis, we isolate a new concept which we call the chain extension of a D-algebra. We demonstrate that this idea is central to to a number of applications to algebra and physics. Chain extensions may be regarded as generalizations of ordinary algebraic extensions of Lie algebras. Applications of our theory provide a new constructive approach to BRST theories which only contains three terms; in particular, this provides a new point of view concerning consistent deformations and constrained Hamiltonian systems. Finally, we show that a similar development provides a method by which Lie algebra deformations may be encoded into the structure maps of an sh-Lie algebra with three terms.

The Algebraic Structure of BRST Operators

by

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To my parents . . .

Biography

JINING GAO was born in 1970 in the city of Shanghai in the People's Republic of China. He received his bachelors degree from the Mathematics Department of Fudan University in 1992. During the years 1992-1997, he was an instructor at the Shanghai Institute of Electric Power . He was accepted as a graduate student at the Institute of Mathematics at Fudan University in the fall of 1997 and received the Master of Science degree there in 2000.

In the fall of 2000, Jining Gao entered North Carolina State University to pursue the Ph.D degree in Mathematical Physics under the direction of Dr. Ron Fulp.

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Contents

Chapter 1

Introduction

Homological algebra has become an indispensable tool for the rigorous formulation of a wide variety of developments in theoretical physics. Applications of these techniques to physics has become so pervasive that they have gradually become identified as a new category of mathematical physics which has been called "cohomological physics". One of the fruitful branches of this theory is the "cohomology" formulation of the BRST theory of constraints. Indeed the point of BRST theory is to replace the cohomology of the reduced space of a physical theory by the cohomology of a homological resolution of the space P being constrained.

In more detail, assume that P is a symplectic manifold and that one has a system of first class constraints on P. Let Σ denote the constraint surface defined as the set of zeros of the constraints. These constraints may or may not be independent. If they are independent they are called irreducible constraints and otherwise they are reducible. The Hamiltonian vector fields of the symplectic manifold P define a possibly singular foliation of Σ and the smooth functions on Σ which are constant on the leaves of this "foliation" are said to be gauge invariant and are called the observables of the theory. There is a differential d, called the longitudinal differential, defined on a certain (dual) Chevalley-Eilenberg complex with coefficients in the algebra $C^\infty(\Sigma)$ whose cohomology in degree zero in the irreducible case is the space of observables. It is clear in the literature that the zero degree cohomology of a certain complex is the space of observables but it is not clear that the complex is a Chevalley-Eilenberg complex and that the longitudinal differential is a Maurer-Cartan differential. These facts are established here in a rigorous manner for the first time.

BRST symmetry was developed in order to replace the original gauge symmetry on the constraint surface by a symmetry on the entire phase space P in such a manner that the longitudinal differential d could be extended to a new differential S called the BRST differential to be defined on an enlarged complex in such a way that the BRST cohomology in degree zero is precisely the set of observables on Σ. The procedure is nontrivial even in the irreducible case but even more convoluted in the reducible case. An interesting question has to do with whether the BRST differential is a Maurer-Cartan differential and whether or not it is actually a Chevalley-Eilenberg differential defined on a (dual) Chevalley-Eilenberg complex as was the case for the longitudinal differential in the irreducible case.

The second chapter is mainly devoted to showing that in the case of irreducible constraints the BRST differential S is in fact a Maurer-Cartan differential and that it is a (dual) Chevalley-Eilenberg differential defined on a Chevalley-Eilenberg complex. It is also shown that in the case one has a Hamiltonian system subject to Bosonic irreducible constraints that the fact that $S^2 = 0$ implies the existence of a possibly singular "foliation" of the phase space P which agrees with the "foliation" of the constrained space defined by gauge symmetries. Generally, the BRST differential has an expansion

$$S = \delta + d + s_1 + \cdots + s_n + \cdots .$$

The Koszul Tate differential δ and the longitudinal differential d are well understood but the other terms of the expansion are less well understood. We completely characterize s_1 in the irreducible case. Finally, in this chapter we also consider systems whose constraints are reducible. In particular we introduce a new concept which we call an nth-reducible complex. This is precisely the idea needed to formulate reducible physical theories rigorously. We show that every differential on such a complex is a mildly generalized Maurer-Cartan differential. In particular we show that the BRST differential is such a generalized Maurer-Cartan differential in the reducible case.

In the third chapter we introduce the concept of a D-algebra and develop a theory of chain extensions of such algebras. These ideas do not appear in the literature and were developed in order to unify a number of seemingly disparate topics in theoretical physics such as the BRST theory of constraints, Poisson brackets of local functionals in classical and quantum field theories, and certain developments in deformation theory. When applied to the Hamiltonian theory considered in the second chapter these new techniques make it possible to show that in the case the constraints are irreducible it is sufficient to work with the first three terms of the BRST expansion δ, d, and s_1. Moreover, it is clear that this is the least number of terms required to describe the nontrivial case on which we have focussed. In this case we have acheived an optimal BRST construction. Additionally, our theory of chain extensions provides a rigorous framework in which the application of BRST theory to the theory of consistent deformations and their relations to other deformation theories are clarified. In particular, our new constructive approach to BRST theories in which the expansion of the BRST operator has only three terms provides a new point of view concerning the theory of consistent deformations.

In the last chapter, we apply the methods developed in the article [5] to certain questions concerning the deformation of Lie algebras. In order to remove obstructions to the deformation of Lie algebras, we embed the Lie algebra into an appropriate sh-Lie algebra in such a way that the obstructions vanish in the category of sh-Lie algebra deformation. As in the other developments of the third chapter we are able to show that three terms of the sh-Lie algebra are sufficient to describe the class of deformations of Lie algebras under investigation here. It is interesting that only three terms suffice as the general sh-Lie structures have infinitely many terms.

Chapter 2

Constrained Hamiltonian systems

2.1 Irreducible constraints

Let (P, ω) be a n-dimensional symplectic manifold and let $\{,\}$ be the Poisson bracket defined by ω on the algebra of smooth functions $C^{\infty}(P)$. Assume that $G_a, a = 1, \cdots, M$ are constraint functions which satisfy the condition:

$$\{G_a, G_b\} = C^c_{ab} G_c \tag{2.1}$$

where C^c_{ab} are structure functions on P and let Σ be the constraint surface which is determined by the set of zeros of G_a. When (2.1) is satisfied we say that the constraints G_a are **first class constraints.** The Hamiltonian vector fields X_a corresponding to the functions G_a are defined by $X_a(f) = \{f, G_a\}$ for $f \in C^{\infty}(P)$. The fields X_a satisfy the condition $[X_a, X_b] \approx C^c_{ab} X_c$, i.e., the equation holds only on Σ, or as we say, they hold only "on shell". Under certain conditions the fields X_a define a foliation of Σ. Functions $f \in C^{\infty}(\Sigma)$ which are constant on the leaves of the foliation are said to be "gauge invariant" and are called classical observables.

In quantum field theory, it is difficult to utilize path integrals of functionals defined on the space of observables because they are only defined on the constraint surface. To overcome this difficulty the phase space P is extended and the gauge symmetry is replaced by BRST symmetries in a such way that the path integral can be utilized on functionals defined on arbitrary functions on the extended phase space. More precisely, to achieve this, one introduces an antighost variable P_a for every constraint function G_a and a differential δ called the Koszul-Tate differential which is defined on the complex $C[P_a] \otimes C^\infty(P)$ as follows:

$$\delta P_a = -G_a \tag{2.2}$$

$$\delta f = 0 \tag{2.3}$$

where $f \in C^\infty(P)$. Additionally, new variables η^a are introduced which are in one-to-one correspondence with the space of independent gauge symmetries and another differential d called the longitudinal differential is defined on the complex $C^\infty(P) \otimes C[\eta^b]$ in a manner similar to the definition of the Chevalley-Eilenberg differential. This differential is designed to implement the gauge symmetries. One extends δ and d to all of $C[P_a] \otimes C[\eta^b]$ by requiring that $\delta(\eta^a) = 0$ and $dP_a = 0$. In some cases $\delta + d$ is a differential on the complex $C[P_a] \otimes C^\infty(P) \otimes C[\eta^b]$ whose square is zero and whose cohomology is precisely the space of classical observables. Often this fails to be true and $\delta + d$ must be extended by homological perturbation theory to obtain the (BFV version of the) BRST differential $S = \delta + d + s_1 + \cdots$ in order to obtain the observables as zero degree cohomology classes. The differential S is clearly quite different from the longitudinal differential d, but we will show that S satisfies conditions totally analogous to those characterizing d in Henneaux and Teitelboim ([9] page117-119) in the case when the constraint functions are irreducible and bosonic.

Let $\Omega = C[P_a] \bigotimes C^\infty(P) \bigotimes C[\eta^b]$ and consider $\Omega^* = \bigoplus_{p=0}^{\infty} \Omega^p$, where Ω^p is the subset of Ω having ghost number p (defined below). For simplicity, we introduce the notation $\omega^I = \eta^{b_1} \cdots \eta^{b_{p+1}} P_{a_p} \cdots P_{a_1}$, and $\Omega^1 = \{\alpha \in \Omega \mid \alpha = u_I \omega^I, u_I \in C^\infty(P)\}$ where I is the multi-index $(b_1, \cdots, b_{p+1}, a_1, \cdots, a_p)$. Obviously the elements ω^I generate all Ω^p for $p \geq 1$. For completeness and clarity, we first describe our parity conventions as follows:

$$\epsilon(P_a) = \epsilon(G_a) + 1 = \epsilon(\eta^a), \epsilon(AB) = \epsilon(A) + \epsilon(B). \tag{2.4}$$

Moreover the ghost number grading referred to above is defined as follows:

(1) the pure ghost number of each element of Ω is simply its degree as a polynomial in η^a,

(2) the anti- ghost number of each element of Ω is its degree as a polynomial in P_a,

(3) the ghost number of each element x is the number $puregh(x) - antigh(x)$. Notice that for $A, B \in \Omega$, gh(AB)=gh(A)+gh(B) and that $\epsilon(\omega^I) = 1$ whenever $\epsilon(G_a) = 0$. With these conventions, we will show that the BRST differential is essentially the Chevalley-Eilenberg differential when the constraints are bosonic and irreducible.

First, we recall how the Chevalley-Eilenberg differential is formulated in BRST notation. Let $\mathcal{G}$ be a Lie algebra spanned by a basis $\{e_i\}$ and $\mathcal{A}$ a commutative associative algebra . Let the mapping $\rho : \mathcal{G} \to End(\mathcal{A})$ be a representation of $\mathcal{G}$ with representation space $\mathcal{A}$. Introduce a ghost variable η^i for every element e_i of the basis $\{e_i\}$. Let A denote the Z-graded algebra $\mathcal{A} \otimes C[\eta^1, \eta^2, \cdots]$ with the grading defined by the ghost number.The Chevalley-Eilenberg differential $d = d_{CE}$ is defined on generators of the complex A as follows:

$$df = \rho(e_a)(f)\eta^a \tag{2.5}$$

$$d\eta^a = -\frac{1}{2} C^a_{cb} \eta^b \eta^c \tag{2.6}$$

where $f, C^c_{ab} \in \mathcal{A}$. The mapping $d = d_{CE}$ is extended to the entire graded algebra A by the Leibniz law

$$d(\alpha\beta) = (d\alpha)\beta + (-1)^{deg\alpha} \alpha(d\beta) \tag{2.7}$$

Any differential which satisfies the conditions (2.5) and (2.6) will be called a **Maurer-Cartan differential**. Moreover we will say that d is a **Chevalley-Eilenberg differential** whenever there exists a Lie algebra $\mathcal{G}$ and a representation ρ into the endomorphisms of some commutative associative algebra $\mathcal{A}$ satisfying not only (2.6) but also (2.5) and (2.7). We do not require that our Lie algebra $\mathcal{G}$ be finite dimensional but our Lie algebras are finitely generated as modules over our algebra $\mathcal{A}$.

If we choose $\mathcal{A} = C^\infty(\Sigma)$ where Σ is the constraint surface defined above and if $e_a = X_a$, then the longitudinal differential is a Chevalley-Eilenberg differential of this type. In this case the vector fields X_a must be restricted to Σ and the Lie algebra is the sub-algebra of vector fields on Σ spanned by the X_a over the algebra $C^\infty(\Sigma)$. The fact that this is a sub-Lie algebra follows from the identity $[X_a, X_b] = C^d_{ab} X_d + X_{C^d_{ab}} G_d$. The other properties follow immediately. We want to obtain an “off shell” version of this result.

Since $\epsilon(\omega^I) = 1$ and $gh(\omega^I) = 1$ we call the set of monomials ω^I multi-ghosts. Moreover, for the BRST differential S, it follows from $S\omega^I \in \Omega^2$, that

$$S\omega^K = -\frac{1}{2}C^K_{IJ}\omega^I\omega^J \tag{2.8}$$

where $C^K_{IJ} = C^K_{JI} \in C^\infty(P)$. Similarly, for $f \in \Omega^0$, one has that $Sf = (\rho_I f)\omega^I$, since $Sf \in \Omega^1$. To summarize, we have the following theorem:

Theorem 2.1 *If the constraint functions $\{G_a\}$ are irreducible and bosonic, the relevant BRST differential S defined on the complex $C[P_a] \otimes C^\infty(P) \otimes C[\eta^b]$ above is a Maurer-Cartan differential:*

$$Sf = (\rho_I f)\omega^I \tag{2.9}$$

$$S\omega^K = -\frac{1}{2}C^K_{IJ}\omega^I\omega^J. \tag{2.10}$$

Notice that even though the longitudinal differential d is not nilpotent on the space $C[P_a] \bigotimes C^\infty(P) \bigotimes C[\eta^b]$, its BRST extension S is nilpotent and so is a differential. To determine how the BRST differential S and the longitudinal differential d are related, we compare the following formulas with the formulas (2.5) and (2.6)

$$Sf = (\rho_I f)\omega^I = (X_a f)\eta^a + s_1 f + \cdots + s_n f + \cdots$$

$$S\eta^a = -\frac{1}{2}C^a_{cb}\eta^b\eta^c + s_1\eta^a + \cdots \tag{2.11}$$

Note that the terms on the right hand sides of (2.5) and (2.6) are summands of the right hand side of these equations.

We claim that the BRST differential S is essentially a Chevalley-Eilenberg differential in the case that the constraints are bosonic and irreducible. The required Lie algebra is a sub-Lie algebra of the Lie algebra $\mathcal{X}(P)$ of all vector fields on P. Because of the way S has been extended to products, each of the mappings ρ_I defined by the equation (2.9) above is a derivation of $C^\infty(P)$ and so is a vector field on P. We consider the submodule

$\mathcal{G}(\rho)$ of $\mathcal{X}(P)$ spanned by the vector fields ρ_I over $C^\infty(P)$. We eventually show that it is a sub-Lie algebra of $\mathcal{X}(P)$. Each element of $\mathcal{G}(\rho)$ clearly acts as a derivation of the algebra $\mathcal{A} = C^\infty(P)$ and therefore is in $End(\mathcal{A})$. Once we establish the fact that $\mathcal{G}(\rho)$ is a Lie algebra we will have the required data in order to show that S is a Chevalley-Eilenberg differential. First we need a lemma which is of interest in its own right.

Lemma 2.2 *Assume that the constraints are bosonic and irreducible and consider the BRST differential S on the complex $C[P_a]\bigotimes C^\infty(P)\bigotimes C[\eta^b]$. Let $\rho = \rho_I$ denote the "representation" defined by the identities in Theorem 1. Then*

$$S^2 f = \frac{1}{2}([\rho_J, \rho_I]f - C^K_{JI}(\rho_K)f)\omega^J\omega^I.$$

Moreover if $S^2 f = 0$ for all $f \in C^\infty(P)$, then

$$S^2\omega^K = -\frac{1}{6}\{[\rho_I,[\rho_J,\rho_E]]^K + [\rho_J,[\rho_E,\rho_I]]^K + [\rho_E,[\rho_I,\rho_J]]^K\}$$

Proof We prove the first identity by noticing that since S is an odd derivation, we have

$$S^2 f = S((\rho_I f)\omega^I) = S(\rho_I f)\omega^I + (\rho_I f)S\omega^I. \tag{2.12}$$

It follows from the identities of Theorem 1 that

$$\begin{aligned} S^2 f &= (\rho_J\rho_I)(f)\omega^J\omega^I + (\rho_I f)(-\frac{1}{2}C^I_{JK}\omega^J\omega^K) && (2.13)\\ &= \frac{1}{2}[(\rho_J\rho_I)f - (\rho_I\rho_J)f]\omega^J\omega^I - \frac{1}{2}C^I_{JK}\omega^J\omega^K\rho_I f && (2.14)\\ &= \frac{1}{2}[\rho_J,\rho_I]f\omega^J\omega^I - \frac{1}{2}C^K_{JI}\omega^J\omega^I(\rho_K)f && (2.15)\\ &= \frac{1}{2}([\rho_J,\rho_I]f - C^K_{JI}(\rho_K)f)\omega^J\omega^I && (2.16) \end{aligned}$$

Thus the first of the two identities is true. We now prove the second identity. Since $S\omega^K = -\frac{1}{2}C^K_{IJ}\omega^I\omega^J$, where $I = (b_1, \cdots, b_{p+1}, a_1, \cdots, a_p)$ and $J = (\tilde{b}_1, \cdots, \tilde{b}_{p+1}, \tilde{a}_1, \cdots, \tilde{a}_p)$, we have

$$
\begin{aligned}
S^2\omega^K &= -\frac{1}{2}(SC^K_{IJ})\omega^I\omega^J - \frac{1}{2}C^K_{IJ}d\omega^I\omega^J + \frac{1}{2}C^K_{IJ}\omega^I d\omega^J \\
&= -\frac{1}{2}(\rho_E C^K_{IJ})\omega^E\omega^I\omega^J - \frac{1}{2}C^K_{IJ}(-\frac{1}{2}C^I_{\tilde{K}L}\omega^{\tilde{K}}\omega^L)\omega^J \\
&\quad +\frac{1}{2}C^K_{IJ}\omega^I(-\frac{1}{2}C^J_{MN}\omega^M\omega^N) \\
&= -\frac{1}{2}\rho_E C^K_{IJ}\omega^E\omega^I\omega^J + \frac{1}{4}C^K_{IJ}C^I_{\tilde{K}L}\omega^{\tilde{K}}\omega^L\omega^J - \frac{1}{4}C^K_{IJ}C^J_{MN}\omega^I\omega^M\omega^N \\
&= -\frac{1}{2}\rho_E C^K_{IJ}\omega^E\omega^I\omega^J + \frac{1}{2}C^K_{IJ}C^I_{\tilde{K}L}\omega^{\tilde{K}}\omega^L\omega^J \\
&= -\frac{1}{6}(\rho_E C^K_{IJ} + \rho_I C^K_{JE} + \rho_J C^K_{EI})\omega^I\omega^J\omega^E \\
&\quad +\frac{1}{6}(C^K_{MI}C^M_{JE} + C^K_{MJ}C^M_{EI} + C^K_{ME}C^M_{IJ})\omega^I\omega^J\omega^E \\
&\quad +\frac{1}{6}(\rho_E C^K_{IJ} + \rho_I C^K_{JE} + \rho_J C^K_{EI})\omega^I\omega^J\omega^E \\
&\quad -\frac{1}{6}(C^K_{IM}C^M_{JE} + C^K_{JM}C^M_{EI} + C^K_{EM}C^M_{IJ})\omega^I\omega^J\omega^E \qquad (2.17)
\end{aligned}
$$

Next notice that if we assume that $S^2f = 0$ for all $f \in C^\infty(P)$, then $[\rho_J, \rho_E] = C^M_{JE}\rho_M$, $[\rho_E, \rho_I] = C^M_{EI}\rho_M$, $[\rho_I, \rho_J] = C^M_{IJ}\rho_M$, and we have

$$
\begin{aligned}
[\rho_I, [\rho_J, \rho_E]] = [\rho_I, C^M_{JE}\rho_M] &= (\rho_I C^M_{JE})\rho_M + C^M_{JE}[\rho_I, \rho_M] \qquad (2.18)\\
&= (\rho_I C^M_{JE})\rho_M + C^M_{JE}C^K_{IM}\rho_K \qquad (2.19)\\
&= (\rho_I C^K_{JE} + C^M_{JE}C^K_{IM})\rho_K. \qquad (2.20)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
[\rho_J, [\rho_E, \rho_I]] &= (\rho_J C^M_{EI})\rho_M + C^M_{EI}C^K_{JM}\rho_K \qquad (2.21)\\
&= (\rho_J C^K_{EI} + C^M_{EI}C^K_{JM})\rho_K \qquad (2.22)
\end{aligned}
$$

and

$$
\begin{aligned}
[\rho_E, [\rho_I, \rho_J]] &= (\rho_E C^M_{IJ})\rho_M + C^M_{IJ}C^K_{EM}\rho_K \qquad (2.23)\\
&= (\rho_E C^K_{IJ} + C^M_{IJ}C^K_{EM})\rho_K. \qquad (2.24)
\end{aligned}
$$

It follows from this last calculation that the negative of the sum of the K-th components of the right hand sides of the last three equations is precisely six times the right hand side of the identity for $S^2\omega^K$ (see (2.17)). The lemma follows.

Corollary 2.3 *Assume that the constraints are bosonic and irreducible and consider the BRST differential* S *on the complex* $C[P_a]\bigotimes C^\infty(P)\bigotimes C[\eta^b]$*. Since in fact* $S^2 = 0$ *we have that* $\mathcal{G}(\rho)$ *is a Lie sub-algebra of* $\mathcal{X}(P)$ *with generators the set of vector fields* $\{\rho_I\}$ *on* $C^\infty(P)$*.*

Corollary 2.4 *If the constraints of a Hamiltonian system are bosonic and irreducible, then the BRST differential is a Chevalley-Eilenberg differential on the complex* $\mathcal{A}\otimes C[\omega^I]$ *where the algebra* $\mathcal{A}$ *is the algebra of smooth functions on* P *and where the* ω^I *are called multi-ghosts instead of ghosts. Recall that the* ω^I *freely generate* $\mathcal{A}\otimes C[\omega^I]$ *as a module over* $\mathcal{A}$*.*

Proof The proof was outlined in the observations just prior to the lemma. The only gap in the argument was that we had not yet proved that $\mathcal{G}(\rho)$ is a Lie algebra which we now see is a corollary of the lemma.

Remark. The longitudinal differential d was initially defined "on shell", that is to say the underlying manifold was the constraint surface Σ. Formulated on this surface the square of d is zero and its cohomology in degree zero is the set of classical observables. The fact that d squares to zero "on shell" is related to the fact that the Hamiltonian vector fields X_a close under Lie brackets "on shell". They need not close "off shell".

In order to use the path integral formalism it is useful to extend the formalism "off shell". When the longitudinal differential d is extended "off shell" it no longer squares to zero and in fact the BRST differential was constructed to repair this defect [9].

The fact that the BRST differential squares to zero "off shell" suggests that one should be able to supplement the vector fields X_a with other fields to obtain an integrable system which "foliates" P in such a manner that the possibly singular leaves provides the "foliation" of Σ provided by the Hamiltonian vector fields. We now show that this is true.

Recall that the generators ρ_I of the Lie algebra $\mathcal{G}(\rho)$ correspond to the multi-ghosts $\omega^I = \eta^{b_1}\cdots\eta^{b_{p+1}}P_{a_p}\cdots P_{a_1}$ where I is the multi-index

$(b_1, \cdots, b_{p+1}, a_1, \cdots, a_p)$. In the case $p = 0$ it is understood that ω^I is simply η^b for some index b. Thus the equation $S(f) = (\rho_I f)\omega^I$ of Theorem 1 has the terms $(\rho_a f)\eta^a$ as certain of its summands. where d is the exterior differential and s_k is a derivation which increases the antighost degree by k. Recall that these latter terms correspond to the longitudinal differential d in the expansion

$$S = \delta + d + s_1 + \cdots + s_n + \cdots . \tag{2.25}$$

of the BRST differential. Indeed for every $f \in C^\infty(P)$,

$$Sf = (\rho_a f)\eta^a + (\rho_I f)\omega^I + \cdots \tag{2.26}$$

where antidegree$(\omega^I) \geq 1$, and we see that $\rho_a f$ is exactly the action of X_a on f. Consequently the ρ_a are simply the Hamiltonian vector fields X_a. The supplementary vector fields we require to obtain an integrable system are defined by $X_I = \rho_I$.

As an immediate consequence of these observations we have the following theorem.

Theorem 2.5 *The longitudinal differential d is defined in terms of the Hamiltonian vector fields X_i which form an open gauge algebra since $[X_i, X_j] \approx C_{ij}^k X_k$. Since $S^2 = 0$, there exists additional vector fields X_I on P such that $[X_i, X_j] = C_{ij}^k X_k + C_{ij}^I X_I$, and the fields X_i, X_I define an integrable system in the sense that they generate a subalgebra $\mathcal{G}(\rho)$ of the Lie algebra of all vector fields of P.*

We now determine further conditions imposed on the ρ_K by the fact that S is nilpotent.

Using (2.25) and the fact that $S^2 = 0$, the first three terms of the expansion of S^2 in terms of the anti-ghost degree yields :

$$\delta^2 = 0 \tag{2.27}$$

$$[\delta, d] = 0 \tag{2.28}$$

$$d^2 = -[\delta, s_1] \tag{2.29}$$

By a calculation similar to the one in the proof of the lemma, and for arbitrary $f \in C^\infty(P)$ we have

$$d^2 f = \frac{1}{2}([\rho_i, \rho_j]f - C_{ij}^k \rho_k f)\eta^i \eta^j \tag{2.30}$$

$$[\delta, s_1]f = (\delta s_1 f + s_1 \delta f) \tag{2.31}$$

$$= \delta s_1 f = \delta(\rho^c_{ab} f \eta^a \eta^b P_c) \tag{2.32}$$

where the ρ^c_{ab} are defined by the equation $s_1 f = \omega^I f = \rho^c_{ab} f \eta^a \eta^b P_c$ and the multi-index I is (abc). It follows that

$$[\delta, s_1]f = (\rho^c_{ab} f)\eta^a \eta^b \delta P_c \tag{2.33}$$

$$= -(G_c)(\rho^c_{ab} f)\eta^a \eta^b \tag{2.34}$$

Using the three identities above and comparing (2.30) with (2.34), we have

$$[X_j, X_i] = C^k_{ji} X_k + G_c \rho^c_{ij} \tag{2.35}$$

Since ω^I is a derivation for each multi-index I we see that each ρ^c_{ij} is also a derivation on $C^\infty(P)$; we distinguish it from the derivation ρ_k by referring to it as a second order derivation.

We now show how these results may be applied to Hamiltonian systems having first-class constraints restricting our remarks to the case where P is R^n for some positive integer n. We adopt the same conventions as in [9] (Page 52-53), in particular, let $G_a (a = 1, \cdots, M)$ denote the constraint functions of the system. Define vectors $X_a = X^\lambda_a \partial_\lambda$ via $X^\lambda_a = \omega^{\lambda\mu} \partial_\mu G_a$, where the matrix of components of the antisymmetric tensor $(\omega^{\lambda\mu})$ is the inverse of the matrix $\omega_{\mu\nu}$ of components of the symplectic structure ω on P. We observe that, for each $F \in C^\infty P$,

$$X_a F = X^\lambda_a \partial_\lambda F = \{F, G_a\}. \tag{2.36}$$

We know that if X^λ_a corresponds to G_a and X^λ_b corresponds to G_b then $[X_a, X_b]^\lambda$ corresponds to $[G_a, G_b]$. Moreover

$$[X_a, X_b]^\lambda = \omega^{\lambda\mu} \partial_\mu (C^c_{ab} G_c) \tag{2.37}$$

$$= C^c_{ab} X^\lambda_c + G_c \omega^{\lambda\mu} \partial_\mu C^c_{ab} \approx C^c_{ab} X^\lambda_c \tag{2.38}$$

Off the constraint surface, the second term on the right hand side of (2.38) does not vanish unless $\partial_\mu C^c_{ab} = 0$. Thus $X^\lambda_a (a = 1, \cdots, M)$ form a closed distribution only on shell $G_a = 0$.

For the remainder of this section we provide a detailed calculation which determine consistency conditions for the summand s_1 to satisfy (2.25) .

First we determine the action of s_1 on the ghosts η^α. Since the vector fields $\{X_a\}$ satisfy the Jacobi identity it follows from a computation similar to the one of Lemma (2.2) that $d^2\eta^a = 0$. Combined with the facts that $d^2 = -[\delta, s_1]$ and that $s_1\eta^\alpha$ has anti-ghost number one we have $s_1\eta^\alpha = C^{d\alpha}_{abc}\eta^a\eta^b\eta^c P_d$ Then,

$$0 = -(\delta s_1 + s_1\delta)\eta^\alpha = -\delta s_1\eta^\alpha = C^{d\alpha}_{abc}\eta^a\eta^b\eta^c G_d \tag{2.39}$$

Therefore $C^{d\alpha}_{abc}\eta^a\eta^b\eta^c G_d = 0$ and consequently we obtain the following consistency condition:

$$C^{d\alpha}_{abc} G_d = 0 \tag{2.40}$$

At this point we derive other consistency conditions which are required in order to compute $s_1 P_a$. Since d increases the ghost number by one we can write $dP_a = \eta^c C^b_{ca} P_b$ and

$$\begin{aligned} d^2 P_a = d(dP_a) &= d(\eta^c C^b_{ca} P_b) = (d\eta^c) C^b_{ca} P_b - \eta^c d(C^b_{ca} P_b) \\ &= (-\frac{1}{2} C^c_{de}\eta^d\eta^e) P_b C^b_{ca} - \eta^c(\rho_d(C^b_{ca})\eta^d P_b + C^b_{ca} dP_b) \\ &= -\frac{1}{2} C^c_{de} C^b_{ca}\eta^d\eta^e P_b - \rho_d(C^b_{ca})\eta^c\eta^d P_b - C^b_{ca}\eta^c(\eta^e C^d_{eb} P_d) \\ &= (-\frac{1}{2} C^b_{ce} C^d_{ba} - C^b_{ca} C^d_{eb} - \rho_e(C^d_{ca}))\eta^c\eta^e P_d \end{aligned} \tag{2.41}$$

Since s_1 increases the anti-ghost number by one we can write $s_1 P_a = C^{ef}_{cda}\eta^c\eta^d P_e P_f$. It follows that

$$\begin{aligned} \delta s_1 P_a &= \delta(C^{ef}_{cda}\eta^c\eta^d P_e P_f) && (2.42) \\ &= C^{ef}_{cda}\eta^c\eta^d((\delta P_e) P_f - P_e \delta P_f) && (2.43) \\ &= C^{ef}_{cda}\eta^c\eta^d(-G_e P_f + G_f P_e) && (2.44) \\ &= -C^{ef}_{cda} G_e \eta^c\eta^d P_f + C^{ef}_{cda} G_f \eta^c\eta^d P_e && (2.45) \\ &= 2C^{ef}_{cda}\eta^c\eta^d P_e G_f && (2.46) \end{aligned}$$

and

$$s_1\delta P_a = s_1(-G_a) = -s_1 G_a \tag{2.47}$$

$$= -(\rho^e_{cd} G_a))\eta^c\eta^d P_e \tag{2.48}$$

Thus

$$(\delta s^1 + s^1\delta)P_a = (2C^{ef}_{cda}G_f - \rho^e_{cd}(G_a))\eta^c\eta^d P_e \tag{2.49}$$

$$= (2C^{df}_{cea}G_f - \rho^d_{ce}(G_a))\eta^c\eta^e P_d \tag{2.50}$$

Since $d^2 = -[\delta, s^1]$,by comparing (2.41) and (2.50) we have the following consistency conditions:

$$C^{df}_{cea}G_f = \frac{1}{2}(\rho^d_{ce}(G_a) + \rho_e(C^d_{ca}) + \frac{1}{2}C^b_{ce}C^d_{ba} + C^b_{ca}C^d_{eb}). \tag{2.51}$$

Notice that in case the constraint functions are irreducible this condition is equivalent to requiring that the left hand side of (2.51) vanish "on shell".

In the last few paragraphs we have obtained consistency conditions (2.40) and (2.51) which are necessary in order to determine the action of s_1 on the generators. We have not attempted to determine the coefficients $C^{d\alpha}_{abc}$ and C^{df}_{cea} in the consistency equations since they can only guarantee the existence of perturbation terms of s_1.

2.2 Reducible Constraints

In the last section we dealt only with irreducible constraints. In this section, we will generalize some of our results to include systems of reducible constraints. To achieve that, we introduce the concept of an n-reducible complex as follows.

Definition Let $\Omega^* = \oplus_{n=0}^{\infty}\Omega^n$ be a graded algebra and assume that Ω^0 is an algebra such that Ω^* is a Ω^0-module.Let $A^p = \oplus_{n=0}^{p}\Omega^n$. If A^p is a finitely generated Ω^0-module and each component $\Omega^k (k > p)$ is generated by A^p, We call the complex Ω^* a p-reducible complex.

We are interested in investigating differentials on p-reducible complexes. First consider some examples of p-reducible complexes.

Example 1. Let R^n be n-dimensional Euclidean space and $\Omega^k(R^n)$ be the space of k-forms. Since R^n has a global coordinate chart $(x^1, \cdots, x^n)$, every k-form can be written as

$$\omega = \omega_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \tag{2.52}$$

where $\omega_{i_1 \cdots i_k}$ are smooth functions which belong to the space of 0-forms $\Omega^0(R^n)$, and where the 1-forms $dx^1, \cdots, dx^n$ generate all differential forms in $\Omega^k(R^n)$ over $\Omega^0(R^n)$ for $k \geq 1$. The complex $\Omega^*(R^n) = \oplus_{k=0}^n \Omega^k(R^n)$ is then a 1-reducible complex. The exterior differential d is defined as follows:

$$df = (\partial_i f) dx^i \tag{2.53}$$

$$d(dx^i) = 0 \tag{2.54}$$

where $f \in \Omega^0(R^n), dx^i \in \Omega^1(R^n)$. One then extends the definition above to the entire space $\Omega^*(R^n)$ via the Leibniz formula.

Example 2 (Chevalley-Eilenberg cohomology). Let a pair (d, A) be a Chevalley-Eilenberg differential with its related complex $A = \mathcal{A} \otimes C[\eta^1, \eta^2, \cdots]$ discussed earlier. Obviously, the graded algebra A is generated by ghosts $\eta^1, \eta^2, \cdots$ over the underlying algebra $\mathcal{A}$ and consequently the complex A is a 1-reducible complex.

Example 3 (Auxillary Differential Δ). Typical k-th reducible complexes arise from BRST operators in case the system is subject to $(k-1)$-reducible constraint conditions.

In BRST theory, a specific differential called the auxillary differential Δ is introduced to deal with longitudinal differentials defined on constraint surfaces subject to higher order reducibility conditions. Let Σ be a constraint surface and $\{\eta^{a_0}\}$ be the original ghosts (see [9] page 217-218).

Assume that Σ is defined by constraints which satisfy k-th order reducibility conditions. In this case the reducibility of the constraints can be written as

$$Z_{a_k}^{a_{k-1}} Z_{a_{k-1}}^{a_{k-2}} = (-1)^{\epsilon_{a_{k-2}}} C_{a_k}^{a_{k-2}, a_0} G_{a_0} \tag{2.55}$$

$$k = 1, \cdots, L \qquad a_k = 1, \cdots, m_k \tag{2.56}$$

for some m_k and appropriate functions $Z_{a_k}^{a_{k-1}}$ (see [9] page 210).

Introduce higher order ghost variables η^{a_k} along with a differential Δ as follows:

$$puregh(\eta^{a_k}) = k+1, \qquad \epsilon(\eta^{a_k}) = \epsilon_{a_k} + k + 1 \tag{2.57}$$

$$\Delta F = 0, \qquad \Delta \eta^{a_k} = \eta^{a_{k+1}} Z_{a_{k+1}}^{a_k} (-1)^{\epsilon_{a_k} + k + 1} \tag{2.58}$$

where F is an arbitary function on the constraint surface. To complete the definition of the auxillary differential Δ, one needs to introduce an auxillary grading as follows:

$$aux(z^A) = 0 = aux(\eta^{a_0}), \quad aux(\eta^{a_k}) = k \tag{2.59}$$

One then has that $aux(\Delta) = 1$ and that $puregh(A) = aux(A) + deg(A)$. The complex $\Omega = C^\infty(\Sigma) \otimes C[\eta^{a_0}, \eta^{a_1}, \cdots, \eta^{a_k}]$ is then a $(k+1)$-reducible complex.

At this point, we characterize the differential on an arbitrary n-reducible complex and thereby generalize Theorem 1. We adopt the convention of the left action for d. Obviously, a differential d on a reducible complex Ω^* is uniquely and totally determined by its values on the space $A^p = \Omega^0 \oplus \bar{A}^p$ and the Leibniz rule, where $\bar{A}^p = \oplus_{n=1}^p \Omega^n$. In order to see this in more detail we first assume that the finitely generated Ω^0 module $\bar{A}^p$ has a finite basis denoted by $(\omega_n^i)_{1 \leq n \leq p}$ where the sub-index means that $\omega_n^i \in \Omega^n$ Notice that the basis (ω_n^i) generates all the elements of every component Ω^k for $k > p$. With this notation, we define a differential d on the space A^p as follows :

$$df = (\rho_j^1 f)\omega_1^j \tag{2.60}$$

$$d\omega_m^i = -\frac{1}{2}\Sigma_{1 \leq n \leq m} C_{jkn}^i \omega_{m-n+1}^j \omega_n^k \tag{2.61}$$

where $f \in \Omega^0$ and the ρ_j^1 are derivations of Ω^0. If the complex Ω^* is a 1-reducible complex, the basis has only elements of the form ω_1^i and the formulas (2.60) and (2.61) reduce to

$$df = (\rho_j f)\omega_1^j \tag{2.62}$$

$$d\omega_1^i = -\frac{1}{2} C_{jk}^i \omega_1^j \omega_1^k \tag{2.63}$$

analogous to the formulas given by the Chevalley-Eilenberg differential.

The definition of the Koszul-Tate differential δ is then modified to reflect the reducibility conditions above. In the reducible case the longitudinal differential is not nilpotent on the space of $C^\infty(\Sigma) \otimes C[\eta^a]$, but is nilpotent on the subalgebra of longitudinal forms. In order to overcome this defect, an equivalent differential D [9] is introduced such that $H^*(D) = H^*(d)$. At this point one has "differentials" δ and D on the extended space $\Omega^* = C[P_{a_0}, P_{a_1} \cdots] \otimes C^\infty(\Sigma) \otimes C[\eta^{a_1}, \eta^{a_2}, \cdots]$ and it is possible to show that exists a BRST differential $S = \delta + D + s_1 + \cdots$ defined on the complex Ω^*. It can be seen that the longitudinal complex $C^\infty(\Sigma) \otimes C[\eta^a]$ and the BRST complex Ω^* are both n-reducible complexes for appropriate n. Therefore we obtain the following generalization of Theorem 1.

Theorem 2.6 *If the constraints are bosonic and reducible the BRST differential has a structure similar to that of the longitudinal differential.*

Chapter 3

Chain extensions of D-algebras and their applications

This chapter is organized as follows: first of all, in section 3.1 we introduce the new and fundamental concept which we call the chain extension of a D- algebra. In this section we also develop the related algebraic formalism needed for applications of these new constructs. In section 3.2.1 we show how our new constructs provide a new approach to the BRST method of encoding constraints on a symplectic manifold. In section 3.2.3 we show how our new methods apply to the homological approach to local functionals in Lagrangian field theory referred to above. In section 3.2.4 we show how the consistent deformation theory of Barnich and Henneaux relates to the chain extension formalism.

3.1 Chain extension of D-algebras

Definition Let V be a linear space over a field k and D a k-linear map such that $D^2 = 0$ on V. We call the ordered pair (V, D) a D-algebra.

First we consider some important examples of D-algebras.

Example 1:(Chain complex) Let V be a graded space such that $V = \oplus_{p=0}^{\infty} V_p$ and $(D)_p$, $p \geq 0$, a sequence of maps (boundary operators)

$$\cdots V_{p+1} \overset{D_{p+1}}{\longrightarrow} V_p \cdots \overset{D_2}{\longrightarrow} V_1 \overset{D_1}{\longrightarrow} V_0 \longrightarrow 0,$$

i.e., $D_p \circ D_{p+1} = 0$. For each p, extend D_p to the direct sum V by requiring that it be linear and that D_p restricted to V_q be zero for $q \neq p$. Define D to be the direct sum $D = D_1 + D_2 + D_3 + \cdots$. The pair (V, D) is then a D-algebra.

Example 2:(Lie algebra) Let A be a linear space over k, and let l_2 be a skew-linear map: $l_2 : A \otimes A \longrightarrow A$ which satisfies the Jacobi identity:

$$\sum_{\sigma \in unsh(2,1)} (-1)^{\sigma} l_2(l_2(x_{\sigma(1)} \otimes x_{\sigma(2)}) \otimes x_{\sigma(3)}) = 0 \tag{3.1}$$

where $x_1, x_2, x_3 \in A$. Then l_2 is a Lie bracket on A; moreover if the left side of above equation is written in an abbreviated form as

$$(l_2 l_2)(x_1 \otimes x_2 \otimes x_3),$$

then the Jacobi identity is equivalent to the statement $l_2^2 = 0$. Let $V = \oplus_{n=0}^{\infty} \otimes A$ and extend the mapping $l_2 : A \otimes A \longrightarrow A$ to a mapping $l_2 : \otimes^{2+k} A \longrightarrow \otimes^k A$ via the equation

$$\begin{aligned} & l_2(x_1 \otimes \cdots \otimes x_{2+k}) = \\ & \sum_{unsh(n,k)} (-1)^{\sigma} e(\sigma) l_2(x_{\sigma(1)} \otimes x_{\sigma(2)}) \otimes x_{\sigma(3)} \otimes \cdots \otimes x_{\sigma(2+k)}. \end{aligned} \tag{3.2}$$

Consequently we obtain an extended map $l_2 : V \to V$ such that ${l_2}^2 = 0$ and the pair (V, l_2) is a D-algebra.

Example 3: Let V be a linear space over k, $TV = \oplus_{n=0}^{\infty} (\otimes^n V)$, and D a coderivation on the cofree coalgebra TV which satisfies $D^2 = 0$. Then (TV, D) is a D-algebra. All A_∞ algebras are of this type. When V is a graded space one may obtain the class of all sh-Lie algebras by considering the cofree cocommutative coalgebra on$V$, $S^*(V)$ so that the sh-Lie structure is characterized by a coderivation D with square zero. Thus they are also included as a special type of D-algebra.

Let $(\mathcal{F}, D_{\mathcal{F}})$ be a D-algebra over a field k and $\mathcal{B}$ be an arbitrary linear space over k. Consider the ordered triple $(\mathcal{B}, \mathcal{F}, D_{\mathcal{F}})$. The primary new concept in this paper is the notion of a chain extension of such a triple.

Definition Let $(\mathcal{B}, \mathcal{F}, D_{\mathcal{F}})$ be a triple defined as above and let (X_*, δ) be a homological resolution of $\mathcal{F}$, i.e., we have a complex

$$\cdots X_{p+1} \stackrel{\delta_{p+1}}{\longrightarrow} X_p \cdots \stackrel{\delta_1}{\longrightarrow} X_1 \stackrel{\delta_0}{\longrightarrow} X_0$$

where $X_0 = \mathcal{B} \oplus \mathcal{F}, \delta_0(X_1) = \mathcal{B}$, and $H_*(X) = H_0(X) \simeq \mathcal{F}$. If there exists a sequence of maps $l_n : X_* \rightarrow X_*$ with degree $n-2$ such that $l_1 = \delta, l_2|_{\mathcal{F}} = D_{\mathcal{F}}$, and such that the summation $l = \sum_{n=0}^{\infty} l_n$ is a nilpotent operator on X_*, then we call (X_*, l) a chain extension of $\mathcal{B}$ by $(\mathcal{F}, D_{\mathcal{F}})$.

Notice that our resolution $\{X_n\}$ is defined only for $n \geq 0$, thus we do not consider augmented complexes in this paper. Although the definition above initially appears quite complicated, we will find that the notion of a chain extension is a generalization of ordinary algebraic extension theory. Indeed ordinary algebraic extensions can be described as follows. Consider a short exact sequence:

$$0 \longrightarrow C \stackrel{i}{\longrightarrow} B \stackrel{j}{\longrightarrow} A \longrightarrow 0 \tag{3.3}$$

where C, B, A are linear spaces over k and where one has a D-algebra structure on B. Furthermore assume that the restriction of D on C defines a D-algebra structure on C and induces a mapping $\bar{D} : A \rightarrow A$ such that A is a $\bar{D}$-algebra. Also notice that the chain complex $0 \longrightarrow C \stackrel{i}{\longrightarrow} B \longrightarrow 0$ provides a resolution of A for which $\delta_k = 0$ for $k > 0$, $\delta_0 = i$. If the quotient mapping $j : B \rightarrow A$ plays the role of η in the diagram below, then $S = l_1 + l_2 = \delta + D$ is a chain extension of C by A. Such D-algebra extensions of C by A occur in some applications in physics.

Sometimes a homological resolution for $\mathcal{F}$ comes with the additional structure of a contracting homotopy which we will show often guarantees the existence of a chain extension. Furthermore the chain extension in this case may be defined explicitly in terms of this homotopy. We adapt the constructions in [5] to describe such a contracting homotopy.

Note that $\mathcal{F}$ may be regarded as a differential graded vector space $\mathcal{F}_*$ with $\mathcal{F}_0 = \mathcal{F}$, and $\mathcal{F}_k = 0$ for $k > 0$. Assume that there exists a chain map $\eta : X_* \longrightarrow \mathcal{F}_*$ with homotopy inverse $\lambda : \mathcal{F}_* \longrightarrow X_*$; i.e., we have that $\eta \circ \lambda 1_{\mathcal{F}_*}$ and that $\lambda \circ \eta \sim 1_{X_*}$. Thus there is a chain homotopy $s : X_* \longrightarrow X_*$ with $\lambda \circ \eta - 1_{X_*} = l_1 \circ s + s \circ l_1$. In particular, notice that in degree zero one has that $1_{X_*} = \lambda \circ \eta - l_1 \circ s$.

We may summarize all of the above with the commutative diagram

$$
\begin{array}{ccccccc}
\cdots \longrightarrow & X_2 & \overset{s}{\underset{l_1}{\overleftarrow{\longrightarrow}}} & X_1 & \overset{s}{\underset{l_1}{\overleftarrow{\longrightarrow}}} & X_0 \\
& \lambda\uparrow\downarrow\eta & & \lambda\uparrow\downarrow\eta & & \lambda\uparrow\downarrow\eta \\
\cdots \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & H_0 = \mathcal{F}.
\end{array}
$$

Theorem 3.1 *Assume that* $(\mathcal{F}, D_{\mathcal{F}})$ *is a* D*-algebra and that the pair* $(\mathcal{F}, D_{\mathcal{F}})$ *has a homological resolution with contracting homotopy* η *as indicated above. If* $\mathcal{B} = l_1(X_1)$ *and there exists a mapping* $l_2 : X_0 \to X_0$ *which satisfies the properties:*

$$(i) \quad l_2|_{\mathcal{F}} = D_{\mathcal{F}} \tag{3.4}$$

$$(ii) \quad l_2(\mathcal{B}) \subseteq \mathcal{B} \tag{3.5}$$

$$(iii) \quad l_2^2(X_0) \subseteq \mathcal{B}. \tag{3.6}$$

then there exists a chain extension (X_*, l) *of the triple* $(\mathcal{B}, \mathcal{F}, D_{\mathcal{F}})$ *which can be explicitly constructed by induction as follows. For any* $x \in X_*$:

(1) define $l_1(x) = \delta(x)$,

(2) for deg $x > 0$, *define* $l_2(x) = (s \circ (l_2 l_1))(x)$,

(3) if deg $x = 0$, *define* $l_3(x) = (s \circ (l_2 l_2))(x)$ *and for deg* $x > 0$ *define* $l_3(x) = (s \circ (l_2 \circ l_2 + l_3 \circ l_1))(x)$,

(4) for all x *and for* $n > 3$, *define* $l_n(x) = 0$.

Using these definitions it follows that $l_2(x) = 0$ *for deg* $x > 1$ *and that* $l_3(x) = 0$ *for deg* $x > 0$.

Remark : Conditions (i) and (ii) of the theorem above were first introduced in [5] where an sh-Lie prototype of the theorem above was proved. Markl was first to notice (via a private communication) that these conditions insure that at most three of the sh-Lie structure maps are nontrivial. Barnich proved this result in [3]. Finally, Al-Ashhab [1] weakened the conditions of Markl's original statement and provided the details of his observations to the author who generalized the result to the case of D-algebras, for which we have more extensive applications.

Proof Notice first that because $\mathcal{B} = l_1(X_1)$, it follows from the commutative diagram above that $\eta(\mathcal{B}) = 0$. This fact will be used throughout the proof. Assume that there exists an operator $l_2 : X_0 \to X_0$ which satisfies the hypothesis of the theorem.

In order to prove the operator $l = l_1 + l_2 + l_3$ is nilpotent, observe that if we expand l^2 the equality $l^2 = 0$ is equivalent to the sequence of equations.

$$l_1 l_2 + l_1 l_2 = 0 \tag{3.7}$$

$${l_2}^2 + l_1 l_3 + l_3 l_1 = 0 \tag{3.8}$$

$$l_2 l_3 + l_3 l_2 = 0 \tag{3.9}$$

$${l_3}^2 = 0 \tag{3.10}$$

We give an inductive proof of these equations. For any $x \in X_0$, since $l_1 l_2(x) = l_2 l_1(x) = 0$, we have $(l_1 l_2 + l_1 l_2)(x) = 0$, and (3.7) holds in degree zero.

We now prove (3.8) in degree zero, recall that $\lambda \circ \eta - 1_{X_*} = l_1 \circ s + s \circ l_1$, and apply both sides of this equation to $({l_2}^2 + l_3 l_1)(x)$,where $x \in X_0$. Now $l_1(X_0) = 0$ and $l_2^2(X_0) \subseteq \mathcal{B}$ implies that, $(\lambda \circ \eta)({l_2}^2(x)) = 0$, consequently

$$(\lambda \circ \eta - 1_{X_*})({l_2}^2 + l_3 l_1)(x)(\lambda \circ \eta)(l_2^2(x)) - l_2^2(x) = -l_2^2(x).$$

For similar reasons, $(s \circ l_1)({l_2}^2 + l_3 l_1)(x) = 0$ and the equation above implies

$$-l_2^2(x) = (l_1 \circ s + s \circ l_1)({l_2}^2 + l_3 l_1)(x) \tag{3.11}$$

$$= (l_1 \circ s)({l_2}^2 + l_3 l_1)(x) = l_1(s(l_2^2(x))) = l_1(l_3(x)), \tag{3.12}$$

where in the last equality we used the inductive definition of l_3. This shows that $(-l_2^2)(x) = (l_1 l_3)(x)$ and since $(l_3 l_1)(x) = 0$, we have

$$({l_2}^2 + l_1 l_3 + l_3 l_1)(x) = 0 \tag{3.13}$$

Thus (3.8) holds for $x \in X_0$.

We now give an inductive proof of (3.7) in arbitary degree. Assume that (3.7) holds for any $x' \in X_*$ with $\deg x' = n$; if $\deg x = n+1$, we first prove that $(l_2 \circ l_1)(x)$ is an l_1 boundary. Since $\deg l_1(x) = n$, it follows from (3.7) that

$$l_1(l_2l_1)(x) = (l_1l_2)(l_1(x)) = (-l_2l_1)(l_1(x)) = 0 \tag{3.14}$$

Thus $(l_2l_1)(x)$ is l_1-closed. Notice that by the inductive definition l_2 preserves degree and so $\deg((l_2l_1)(x)) = n$. If $n > 0$, then the cycle $(l_2l_1)(x)$ is an l_1-boundary since $H_n(X_*) = 0$. If $n = 0$ then $\deg x = 1, l_1(x) \in \mathcal{B}$, and $(l_2l_1)(x) \in l_2(\mathcal{B}) \subseteq \mathcal{B} = l_1(X_1)$. Thus $(l_2l_1)(x)$ is also a boundary in this case. Now apply $\lambda \circ \eta - 1_{X_*} = l_1 \circ s + s \circ l_1$ to $l_2l_1(x)$; we have

$$(\lambda\eta)(l_2l_1)(x) - l_2l_1(x) = l_1s(l_2l_1(x)) + sl_1(l_2l_1(x)) \tag{3.15}$$

Since $(l_2l_1)(x)$ is a boundary and $\eta(l_2l_1)(x) = 0$,

$$l_2l_1(x) + l_1sl_2l_1(x) = 0. \tag{3.16}$$

Since $\deg x > n+1$, it follows from the inductive definition of l_2 that $s(l_2l_1)(x) = l_2(x)$ and therefore that $l_2l_1(x) + l_1l_2(x) = 0$. Thus (3.7) holds.

In order to prove (3.8) for arbitrary degree, we also need to first verify that for any $x \in X_*$, $l_2^2(x) + l_3l_1(x)$ is a boundary. Since $l_2^2(X_0) \in \mathcal{B}$, we immediately get that $l_2^2(x) = l_2^2(x) + l_3l_1(x)$ is a boundary for all $x \in X_0$. Assume that it is a boundary for every x' with $\deg x' < n+1$. Choose any x such that $\deg x = n+1$ and observe that

$$\begin{aligned} l_1(l_2^2 + l_3l_1)(x) &= (l_1l_2^2)(x) + (l_1l_3l_1)(x) \\ &= (l_1l_2^2)(x) + (l_1l_3)(l_1(x)) \end{aligned} \tag{3.17}$$

Since $\deg l_1(x) = n$, it follows from the inductive hypothesis applied to (3.8) that $l_1l_3l_1(x) = -(l_2^2 + l_3l_1)l_1(x) = -l_2^2l_1(x)$; thus $l_1(l_2^2 + l_3l_1)(x) = (l_1l_2^2)(x) - l_2^2l_1(x) = 0$ (since it follows from (2.11) that l_1 and l_2 anti-commute). It follows that $l_2^2(x) + l_3l_1(x)$ is a cycle. It also follows from the inductive definition of l_3 that it increases the degree by 1, thus deg $(l_2^2(x) + l_3l_1(x)) = n+1$. Since $H_{n+1}(X_*) = 0$, $l_2^2(x) + l_3l_1(x)$ is a boundary. Applying $\lambda \circ \eta - 1_{X_*} = l_1 \circ s + s \circ l_1$ to $l_2^2(x) + l_3l_1(x)$, we have

$$(\lambda\eta)(l_2^2 + l_3l_1)(x) - (l_2^2 + l_3l_1)(x) = l_1s(l_2^2(x) + l_3l_1(x)) + sl_1(l_2^2(x) + l_3l_1(x))$$

and $l_2^2(x)+l_3l_1(x)+l_1s(l_2^2+l_3l_1)(x)=0$. By the inductive definition of l_3, $s((l_2^2+l_3l_1)(x))=l_3(x)$, and consequently $({l_2}^2+l_3l_1+l_1l_3)(x)=0$. Thus (3.8) follows.

We now prove the two remarks in the last statement of the theorem. Assume that deg $x>1$, then deg $x\geq 2$ and deg $l_1(x)>0.$, consequently $l_2(x)=sl_2l_1(x)=s(sl_2l_1)(l_1(x))=0$. The first of the two statements follows. Now assume that deg $x\geq 1$, then using the inductive definitions of both l_2 and l_3, we have

$$\begin{aligned} l_3(x) &= s(l_2^2+l_3l_1)(x) \\ &= s((sl_2l_1)l_2+s(l_2^2+l_3l_1)l_1)(x) \\ &= s^2((l_2l_1)l_2+(l_2^2l_1+l_3{l_1}^2)) \\ &= s^2[-l_1l_2^2+l_2^2l_1(x)]=0 \end{aligned}$$

and the second statement follows. It is now easy to prove (3.9),(3.10) as follows: since $l_3(x)=0$ for any deg $x>0$, it's enough to prove (3.9),(3.10) in degree zero. Assume that $x\in X_0$, we have deg $l_3(x)$=1, thus $l_3l_3(x)=0$, and (3.10) follows.

We now prove (3.9): let $x\in X_0$, since deg $l_2(x)=0$, deg $l_3(x)=1$, and by the inductive definitions of l_2,l_3, we have

$$l_2l_3(x)+l_3l_2(x)=sl_2l_1l_3(x)+sl_2l_2l_2(x)$$
$$=sl_2(l_1l_3+l_2l_2)(x)=sl_2((l_1l_3+l_2l_2+l_3l_1)(x)=0.$$

This concludes the proof that $l^2=0$ and the fact that (X_*,l) is a chain extension.

From the theorem above, we find that a chain extension for a given triple $(\mathcal{B},\mathcal{F},D)$ is not unique,but the following theorem will tell us all chain extensions have the same homology group.

Theorem 3.2 *Let (X_*,l) be a chain extension of triple $(\mathcal{B},\mathcal{F},D)$, then $H(X_*,l)\simeq H(\mathcal{F},D)$.*

The proof of this theorem is very similar to the homological perturbation technique of the theorem on page 179-180 in [9].

3.2 Applications

3.2.1 BRST theory

As one of the applications of the theorem above, we provide a new construction for the Hamiltonian BRST operator in the irreducible case, which has the most compact form possible(three maps are enough!). In [9], a $BRST$ operator is obtained by "homological perturbation theory" but not explicitly. Using our Theorem 3, we will reconstruct a $BRST$ operator in an explicit way under some local conditions. In particular, we show that the BRST theory for the irreducible case is just a chain extension of the longitudinal differential.

Let P be a symplectic manifold, $G_a = 0,(a = 1, \cdots, n)$ a set of first class constraints defined on P, and Σ the constraint surface defined by the zeros of the functions $\{G_a\}$. We assume that Σ is a regular submanifold of P. Let N be the ideal of the algebra $C^\infty(P)$ generated by the constraint functions $G_a, a = 1, \cdots, n$ and consider the short exact sequence:

$$0 \longrightarrow N \xrightarrow{i} C^\infty(P) \xrightarrow{j} C^\infty(\Sigma) \longrightarrow 0. \tag{3.18}$$

In general, the sequence (4.16) is not split as an algebra over the complex field C, but since Σ is a regular submanifold there exists a local chart in P on which the sequence is split locally. That is to say, if P is replaced by a suitable open subset of P, there exists a splitting algebra homomorphism $\tau : C^\infty(\Sigma) \to C^\infty(P)$ such that one obtains the direct sum decomposition $C^\infty(P) = N \oplus C^\infty(\Sigma)$ as algebras.

If $[\cdot, \cdot]$ is the Poisson bracket defined by the symplectic structure on P, then we may define a mapping on $C^\infty(\Sigma) = C^\infty(P)/N$ by $g + N \to [g, G_a] + N$ for each a. This mapping is a well-defined derivation of $C^\infty(\Sigma)$. We denote its value at $f \in C^\infty(\Sigma)$ by $\partial_a f$ but sometimes by an obvious abuse of notation we denote it by $[f, G_a]$.

Let $\mathcal{F} = C^\infty(\Sigma) \otimes C(\eta^a)$ where $C(\eta^a)$ is the exterior algebra over the complex field C with generators $\{\eta^a\}$ of degree one. Physicist usually refer to the generators η^a as ghost variables.

Define the longitudinal derivative d on $\mathcal{F}$ by defining it on generators of $\mathcal{F}$ and then extending it to all of $\mathcal{F}$ as a right derivation. On generators it is defined as follows([9]):

$$df = [f, G_a]\eta^a \tag{3.19}$$

$$d\eta^a = \frac{1}{2} C^a_{cb} \eta^b \eta^c \tag{3.20}$$

where $f \in C^\infty(\Sigma)$ and C^a_{cb} are structure functions on P.

Notice the defining equations of d on generators given above is induced from the action of d on $C^\infty(P) \otimes C(\eta^a)$ given by the same equations (due to our abuse of the use of the bracket referred to above). We claim that $d^2(C^\infty(P) \otimes C(\eta^a)) \subseteq N \otimes C(\eta^a)$. The proof of this fact follows from induction on the ghost number. For the convenience of the reader we show that this is true for ghost number one. The inductive step is straightforward and is left to the reader. Let $\alpha = f_b\eta^b$ where $f_b \in C^\infty(P)$, and observe that $d^2\alpha = (d^2 f_b)\eta^b + f_b d^2\eta^b$. Now for $f \in C^\infty(P)$,

$$d^2 f = -[[f, G_a], G_b]\eta^b\eta^a - \frac{1}{2}C^a_{bc}[f, G_a]\eta^b\eta^c.$$

It follows from the Jacob identity and the anti-commutativity of the η's that

$$2[[f, G_a], G_b]\eta^b\eta^a = [f, C^c_{ab}G_c]\eta^b\eta^a.$$

Thus $d^2 f = -\frac{1}{2}[f, C^c_{ab}]G_c\eta^b\eta^a - \frac{1}{2}C^c_{ab}[f, G_c]\eta^b\eta^a - \frac{1}{2}C^a_{bc}[f, G_a]\eta^b\eta^c$ from which it follows that $d^2 f = -\frac{1}{2}[f, C^c_{ab}]G_c\eta^b\eta^a$, which is obviously in $N \otimes C(\eta^a)$.

Thus $d^2 f$ belongs to $N \otimes C(\eta^a)$ for each $f \in C^\infty(P)$ as does also $(d^2 f_b)\eta^b$ above. Recall that when the C^a_{bc} are constant one has that $d^2\eta^a = 0$. On the other hand when the structure constants are actually structure functions, then further calculations similar to those above using (3.19),(3.20) and anti-commutivity of the η's show that

$$d^2\eta^c = \frac{1}{2}\{C^c_{ab}C^b_{pq} + [C^c_{aq}, G_p]\}\eta^a\eta^p\eta^q.$$

Moreover a similar calculation shows that

$$[[G_a, G_p], G_q]\eta^a\eta^p\eta^q = -\{C^c_{ab}C^b_{pq} + [C^c_{aq}, G_p]\}G_c\eta^a\eta^p\eta^q.$$

It follows that $-2d^2\eta^c G_c = [[G_a, G_p], G_q]\eta^a\eta^p\eta^q$. The latter is zero by the Jacobi identity and the anti-commutativity of the η's. Since $d^2\eta^c G_c = 0$, it follows from the fact that the constraints are irreducible that $d^2\eta^c$ is in the ideal $N \otimes C(\eta^a)$ for each c.

It follows that $d^2\alpha \in N \otimes C(\eta^a)$ for each $\alpha \in C^\infty(P) \otimes C(\eta^a)$ of ghost degree 1. The general case now follows easily by induction on the ghost degree.

From this it follows that $(\mathcal{F}, d)$ is a D-algebra. We intend to show that the $BRST$ complex can be regarded as a chain extension of this algebra. In order to construct the homological resolution of $\mathcal{F}$, we need an antighost variable P_a for each constraint G_a and a

Kozul-Tate differential δ defined as follows:

$$\delta f = 0 \tag{3.21}$$

$$\delta P_a = -G_a \tag{3.22}$$

$$\delta \eta^a = 0 \tag{3.23}$$

where $f \in C^\infty(P)$.

Let $X_* = C(P_a) \otimes C^\infty(P) \otimes C(\eta^a)$ be the graded space where the grading is given by the antighost degree and let $\mathcal{F}_*$ be the graded space with $\mathcal{F}_0 = \mathcal{F}$ and with $\mathcal{F}_k = 0, k > 0$. If we extend the longitudinal derivative d to the anti-ghosts by requiring that $dP_a = 0$ for all a, then we can extend both d and δ to all of X_* by requiring that both of them be right derivations. We will show that (X_*, δ) is a resolution of $\mathcal{F}$.

Define $\mathcal{B}$ by $\mathcal{B} = \delta(X_1) = N \otimes C(\eta^a)$ and let $\tilde{\eta} : X_* \to \mathcal{F}_*$ be the chain map which is nontrivial only in degree zero and in that case define it to be the projection

$$X_0 \to X_0/\mathcal{B} = (C^\infty(P) \otimes C(\eta^b))/(N \otimes C(\eta^b)) = C^\infty(\Sigma) \otimes C(\eta^b) = \mathcal{F}.$$

We now construct a contracting homotopy, but to do this we need to modify the contracting homotopy formula in [9] to conform to the conventions in our paper. To make contact with [9] we adopt their notation throughout the remainder of this section. In particular we employ the formal algebraic language used by physicist in order to show that their formulation of Hamiltonian BRST theory may be reformulated in terms of chain extensions.

Since the constraints $G_a = 0$ are assumed to be independent (recall that we are working in the irreducible case), we can choose a local chart (x_i, G_a) in a neighborhood O of the manifold P. Our results are valid only on such an open set O and since all the structures of interest restrict to O we presume that $O = P$ for simplicity. Let

$$\delta = (\frac{\partial^R}{\partial P_a})G_a, \qquad \sigma = (\frac{\partial^R}{\partial G_a})P_a \tag{3.24}$$

$$\bar{N} = (\frac{\partial^R}{\partial P_a})P_a + (\frac{\partial^R}{\partial G_a})G_a \tag{3.25}$$

where $\frac{\partial^R}{\partial P_a}$ and $\frac{\partial^R}{\partial G_a}$ are right derivations on the algebra $C(P_a) \otimes C^\infty(P) \otimes C(\eta^b)$.

We will use the mapping σ to build the contracting homotopy s below. To obtain the necessary properties of s parities are important. In the calculations below we assume that the parities of $\delta, \sigma, \bar{N}$ are all zero. Moreover we require the parity of smooth functions on P be zero. The parities of ghost fields, $\varepsilon(\eta^a)$, and anti-ghost fields, $\varepsilon(P_b)$, are related to the parities of functions on P by $\varepsilon(\eta^a)\varepsilon(G_a) + 1 = \varepsilon(P_a)$. The relation $\delta\sigma + \sigma\delta = \bar{N}$ holds when evaluated at an arbitary element of $C(P_a) \otimes C^\infty(P) \otimes C(\eta^b)$ and the latter equation implies that:

$$t\frac{d}{dt}F(tP_a, tG_a, x_i, \eta^b) = (\delta\sigma + \sigma\delta)F(tP_a, tG_a, x_i, \eta^b) \tag{3.26}$$

where t is an arbitary real number. If, in addition, F is of antighost number $k > 0$, it follows from (4.19) that

$$F(P_a, G_a, x_i, \eta^b) = (\delta\sigma + \sigma\delta)(\int_0^1 \frac{F(tP_a, tG_a, x_i, \eta^b)}{t} dt). \tag{3.27}$$

If we set $\psi(F) = -\int_0^1 \frac{F(tP_a, tG_a, x_i, \eta^a)}{t} dt$ and $s = \sigma \circ \psi$, then $\delta \circ \psi = \psi \circ \delta$ and $l_1 \circ s + s \circ l_1 = -1_{X_*}$ in degree $k > 0$. Now consider the degree zero case. Recall that in the definition of chain homotopy one is required to find a homotopy inverse $\lambda : \mathcal{F}_* \to X_*$ of the chain mapping $\tilde{\eta} : X_* \to \mathcal{F}_*$. Both $\tilde{\eta}$ and λ should be nontrivial only in degree zero in which case $\tilde{\eta}$ is the projection $X_0 \to X_0/\mathcal{B} = \mathcal{F}$ defined above. Thus we need to define both λ and s in degree zero in such that $\lambda \circ \tilde{\eta} - 1_{X_*} = l_1 \circ s + s \circ l_1$.

A degree zero element of X_* is simply a function f such that $f(G_a, x_i, \eta^b) \in C^\infty(P) \otimes C(\eta^b)$. Define $sf = \int_0^1 \frac{1}{t}(\frac{\partial R}{\partial G_a}) f(tG_a, x_i, \eta^b) P_a dt$. In order to construct the map λ for the case of $antigh(f) = 0$, we first consider the map $\tilde{\lambda} : X_0 \longrightarrow X_0$ defined as follows. For any $f \in X_0$, let

$$\tilde{\lambda}(f) = f + \delta s f \tag{3.28}$$

To find the desired map the following Lemma is useful.

Proposition 3.3 $\tilde{\lambda}$ *vanishes on the subspace* $\mathcal{B}$ *and thus induces a map* $\lambda : X_0/\mathcal{B} \longrightarrow X_0$

Proof Since $f \in \mathcal{B}$, the integral $sf = \int_0^1 \frac{1}{t}(\frac{\partial^R}{\partial G_a})f(tG_a, x_i, \eta^b)P_a dt$ exists and we have:

$$\begin{aligned}
\tilde{\lambda}(f) &= f + \delta s f \\
&= f(G_a, x_i, \eta^b) - \int_0^1 \frac{1}{t}(\frac{\partial^R}{\partial G_a})f(tG_a, x_i, \eta^b)G_a dt \\
&= f(G_a, x_i, \eta^b) - \int_0^1 \frac{d}{dt} f(tG_a, x_i, \eta^b)dt \\
&= f(G_a, x_i, \eta^b) - (f(G_a, x_i, \eta^b) - f(0, x_i, \eta^b)) \\
&= f(0, x_i, \eta^b) = 0
\end{aligned}$$

where the last step follows from the fact that $f \in \mathcal{B}$.

It is now easy to verify that the maps s, λ, η satisfy the contracting homotopy formula on the space X_0.

To summarize, the contracting homotopy is obtained as follows: given any $F(P_a, G_a, x_i, \eta^b) \in C(P_a) \otimes C^\infty(P) \otimes C(\eta^b)$, $s(F) = \int_0^1 \frac{1}{t}(\frac{\partial^R}{\partial G_a})f(tG_a, x_i, \eta^b)P_a dt$ when $antigh(f) = 0$ and $s(F) = \sigma\psi(f)$ for $antigh(f) > 0$. It follows that we have a homological resolution (X_*, δ) with contracting homotopy s.

We now show that there exists maps l_1, l_2, l_3 which satisfy the definition of chain extension. To do this, we show that if $l_1 = \delta$ and $l_2 = d$ then the hypothesis of Theorem 3 is true. First observe that $X_1 = V(P_a) \otimes C^\infty(P) \otimes C(\eta^b)$ where $V(P_a)$ is the linear space spanned by the P_a's over C. Recall that $\delta(X_1) = N \otimes C(\eta^b) = \mathcal{B}$ and observe that $l_2(\mathcal{B}) = d(N \otimes C(\eta^b)) \subseteq N \otimes C(\eta^b) = \mathcal{B}$. Moreover, note that $X_0 = C^\infty(P) \otimes C(\eta^a)$ and recall that we have already shown that $d^2(C^\infty(P) \otimes C(\eta^a)) \subseteq N \otimes C(\eta^a)$ in our proof that $(\mathcal{F}, d)$ is a D-algebra. Thus $l_2^2(X_0) \subseteq N \otimes C(\eta^b) = \mathcal{B}$. It follows from Theorem 3 that there are maps l_1, l_2, l_3 which satisfy the definition of a chain extension and that they are defined inductively by the Theorem.

We can now use Theorem 3 to determine the $BRST$ operator. Let's start with the definition of l_2 on X_1; obviously knowing the values of l_2 on X_1 is enough to determine l_2 on the whole space X_* because elements of degree greater than zero are generated by X_1.

By applying formula (2) of Theorem 3, we have

$$\begin{aligned}
l_2 P_a &= dP_a = sl_2 l_1(P_a) = sl_2\delta(P_a) \\
&= sl_2(-G_a) \\
&= -s((\partial_b G_a)\eta^b) \\
&= -s([G_a, G_b]\eta^b) \\
&= -s(C_{ab}^c G_c \eta^b) \\
&= -s(C_{ab}^c) G_c \eta^b - C_{ab}^c s(G_c)\eta^b \\
&= -s(C_{ab}^c) G_c \eta^b - C_{ab}^d P_d \eta^b \qquad (3.29)
\end{aligned}$$

The next operator we need to compute is l_3, and by Theorem 3, we need only compute it in the case that $f = f(x_i, G_a, \eta^b)$ is in $C^\infty(P)\otimes C(\eta^b)$. Recall that $Y_a = \partial_a$ are the Hamiltonian vector fields of the constraints G_a and that $[\partial_a, \partial_b] = C_{ab}^c \partial_c + Y_{C_{ab}^c} G_c$. By (3) of Theorem 3, we have:

$$\begin{aligned}
l_3(f) &= s(l_2^2 + l_3 l_1)(f) \\
&= s(l_2^2)(f) \\
&= sl_2((\partial_a f)\eta^a) \\
&= s(l_2(\partial_a f)\eta^a + (\partial_a f) l_2(\eta^a)) \\
&= s(\partial_b(\partial_a f)\eta^b\eta^a + (\partial_a f)(-\frac{1}{2}C_{bc}^a \eta^b\eta^c)) \\
&= s(\frac{1}{2}([\partial_a, \partial_b] f)\eta^b\eta^a - \frac{1}{2}(\partial_c f)\eta^b\eta^{a)} \\
&= s(\frac{1}{2}([\partial_a, \partial_b] f) - C_{ba}^c \partial_c) f)\eta^b\eta^a) \\
&= s(\frac{1}{2}(G_c(Y_{C_{ba}^c} f)\eta^b\eta^a) \\
&= \frac{1}{2}s(G_c)(Y_{C_{ba}^c} f)\eta^b\eta^a + \frac{1}{2}G_c s(Y_{C_{ba}^c} f)\eta^b\eta^a \\
&= \frac{1}{2}(Y_{C_{ba}^c} f) P_c \eta^b\eta^a + \frac{1}{2}G_c s(Y_{C_{ba}^c} f)\eta^b\eta^a \\
&= \frac{1}{2}[C_{ab}^c, f] P_c \eta^b\eta^a + \frac{1}{2}G_c s(Y_{C_{ba}^c} f)\eta^b\eta^a
\end{aligned}$$

(3.30)

and $l_3(P_a) = l_3(\eta^a) = 0$

Remark : From the observations above, we see that our new development of the BRST operator has only three nonzero terms. Moreover, it is clear that this is the least number of terms required to describe the nontrivial case we have focussed on here, i.e., in this case we have acheived an optimal BRST construction. Our calculation of the $BRST$ operator above depends on the existence of an explicit contracting homotopy. However, the contracting homotopy formula required by our construction only holds locally, in other words, it depends on the choice of local coordinates. In the $BRST$ theory of Lagrangian field theory, such a choice of a local chart is referred to as a "regularity condition". General speaking, conditions of this type impose restrictions on the constraint surfaces (functions). Further details regarding regularity conditions can be found on pages 7 and 199 in [9]. Such conditions are given explicitly for field theories such as the "Klein-Gordon" and Dirac fields defined on appropriate spacetimes. We can apply the method above to examples of such field theories.

3.2.2 Local problem via chain extensions

The formulation of Lagrangian field theory in terms of jets of sections of vector bundles has proven to be quite satisfactory in the modeling of large classes of physical theories when the fields are what physicists call bosonic fields. Our purpose here is to show how the notion of chain extension may be used to reformulate "local BRST theory" in a setting involving bosonic and fermionic fields. Physicists and mathematicians often employ different notation to describe field theories involving both bosonic and fermionic fields. We will adopt a hybrid notation which permits us to deal with the issues of concern to us. For more complete details regarding the notation of this section we refer the reader to [5]. A more systematic development of the properties of infinite jet bundles may be found in [10].

Let M be a n-dimensional manifold and let $\pi : E \longrightarrow M$ be a vector bundle of fibre dimension k over M. Let $J^\infty E$ be the infinite jet bundle over π with $\pi^\infty_M : J^\infty E \longrightarrow M$. The bosonic fields under consideration will be identified with sections of π.

Fermionic fields are usually modeled by fields called ghost fields. In order to describe the ghost fields we define a finite-dimensional Grassmann algebra $\mathcal{G}$ with n linearly independent generators $\{e_a\}$. Thus a basis of $\mathcal{G}$ is the set of products $e_{a_1}e_{a_2}\cdots e_{a_k}$ where $1 \leq a_1 < a_2 < \cdots < a_k \leq n$. The linear space generated by $e_{a_1}e_{a_2}\cdots e_k$ for fixed k is denoted $\mathcal{G}_k$ and we say that elements of $\mathcal{G}_k$ have ghost number k. A single ghost field is a mapping from M into $\mathcal{G}_1$. The Lagrangian of the "fermionic sector" of a Lagrangian field theory is a function of a finite number N of such ghost fields and their derivatives. More precisely such a Lagrangian is a function of mappings from M into $\mathcal{G}_1^N$ and derivatives of such mappings. The values of these Lagrangians are in $\mathcal{G}$ itself as they are usually sums of products of the fields and their derivatives.

Similarly, to describe anti-ghost fields we choose a symmetric algebra $\mathcal{A}$ which is generated by n linearly independent elements $\{f_b\}$ of $\mathcal{A}$. As before a basis of $\mathcal{A}$ is the set of all products $f_{b_1}f_{b_2}\cdots f_{b_k}$ where $1 \leq b_1 < b_2 < \cdots < b_k \leq n$. As before $\mathcal{A}_k$ denotes the subspace generated by products $f_{b_1}f_{b_2}\cdots f_{b_k}$ and we say that elements of this subspace have anti-ghost number k.

Define a vector bundle by setting $\tilde{E}_x = E_x \oplus \mathcal{G}_1^N \oplus \mathcal{A}_1^N$ for each $x \in M$. Then $\tilde{E}$ is a vector bundle over M and the fields under consideration will be sections of this bundle.

For simplicity we will assume that the fiber bundle $E \longrightarrow M$ is trivial and has fiber the vector space V. It follows that $\tilde{\pi} : \tilde{E} \longrightarrow M$ is also trivial with fiber $V \oplus \mathcal{G}_1^N \oplus \mathcal{A}_1^N$. It follows that sections ψ of $\tilde{\pi}$ may be identified with maps from M into the fiber $V \oplus \mathcal{G}_1^N \oplus \mathcal{A}_1^N$. Moreover we choose a basis of V once for all so that mappings from M into the fiber $V \oplus \mathcal{G}_1^N \oplus \mathcal{A}_1^N$ are uniquely determined by their components. We write $\psi(x) = (\phi^i(x), \eta^a(x), P_b(x))$ to denote a typical section of $\tilde{\pi}$. Observe that the vector components of η anti-commute,

$$\eta^a\eta^b = -\eta^b\eta^a,$$

but their scalar components $\eta^{a\alpha}, \eta^{b\gamma}$ are in $C^\infty(M)$ and thus commute. If (x^μ) are the components of a local chart on M, then the infinite jet of a section ψ of $\tilde{\pi}$ may be denoted as

$$j^\infty\psi(x) = (x, \phi(x), \eta(x), P(x), \partial_I\phi(x), \partial_J\eta(x), \partial_K P(x)).$$

Here I, J, K are symmetric multi-indices where, for example, $\partial_I = \partial_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_r}$ are partial derivatives relative to the chart (x^μ). We denote coordinates on $J^\infty\tilde{E}$ as follows:

$$x^\mu(j^\infty\psi(p)) = x^\mu(p), \quad u_I^i(j^\infty\psi(p)) = \partial_I\phi^i(x)$$

$$\mu_J^{a\alpha}(j^\infty\psi(p)) = \partial_J\eta^{a\alpha}(x), \quad \nu_K^{b\beta}(j^\infty\psi(p)) = \partial_K P_{b\beta}(x).$$

One may now consider the bivariational-complex of the infinite jet bundle $J^\infty\tilde{E}$. The de Rham complex of differential forms $\Omega^*(J^\infty\tilde{E}, d)$ on $J^\infty\tilde{E}$ possesses a differential ideal, the ideal C of contact forms θ which satisfy $(j^\infty\phi)^*\theta = 0$ for all sections ϕ with compact support. This ideal is generated by the contact one-forms, which in local coordinates assume the form $\theta_I^i = du_I^i - u_{jI}^i dx^j$,$\theta_J^{a\alpha} = d\mu_J^{a\alpha} - \mu_{jJ}^{a\alpha}dx^j$, $\theta_K^{b\beta} = d\nu_K^{b\beta} - \nu_{jK}^{b\beta}dx^j$. Contact one-forms of order 0 satisfy $(j^1\phi)^*(\theta) = 0$. For example, for the *bosonic components* of a field, contact forms of *order zero* assume the form $\theta^a = du^a - u^a_j dx^i$ in local coordinates (the muti-index I is empty in this case).

Using the contact forms, we see that the complex $\Omega^*(J^\infty\tilde{E}, d)$ splits as a bicomplex $\Omega^{r,s}(J^\infty\tilde{E})$ (though the finite level complexes $\Omega^*(J^p\tilde{E})$ do not), where $\Omega^{r,s}(J^\infty\tilde{E})$ denotes the space of differential forms on $J^\infty\tilde{E}$ with r horizontal components and s vertical components. The bigrading is described by writing a differential p-form $\alpha = \alpha^{\mathbf{J}}_{IA}(\theta^A_{\mathbf{J}} \wedge dx^I)$ as an element of $\Omega^{r,s}(J^\infty\tilde{E})$, with $p = r + s$, and

$$dx^I = dx^{i_1} \wedge ... \wedge dx^{i_r} \quad \text{and} \quad \theta^A_{\mathbf{J}} = \theta^{a_1}_{J_1} \wedge ... \wedge \theta^{a_s}_{J_s}, \tag{3.31}$$

where $\theta^{a_j}_{J_j}$ denotes any one of the three types of contact forms defined above. We are interested in the complex

$$0 \to \Omega^{0,0}(J^\infty\tilde{E}) \to \Omega^{1,0}(J^\infty\tilde{E}) \to \cdots \to \Omega^{n-1,0}(J^\infty\tilde{E}) \to \Omega^{n,0}(J^\infty\tilde{E})$$

with the differential d_H defined by $d_H dx^i D_i$ where

$$D_j = \frac{\partial}{\partial x^j} + u^i_{jI}\frac{\partial}{\partial u^i_I} + \mu^{a\alpha}_{jJ}\frac{\partial}{\partial \mu^{a\alpha}_J} + \nu^{b\beta}_{jK}\frac{\partial}{\partial \nu^{b\beta}_K}.$$

Here if $\alpha = \alpha_I dx^I$ then $d_H\alpha = D_i\alpha_I(dx^i \wedge dx^I)$. Notice that this complex is exact by the algebraic Poincare lemma. The algebraic Poincare lemma is essentially the Poincare lemma for the horizontal complex on the infinite jet bundle. It is discussed and proved in [6], for example.

Definition To say that F is a **local function** (on $J^\infty\tilde{E}$) means that F is a function from $J^\infty\tilde{E}$ into $\mathbf{R}\otimes\mathcal{G}\otimes\mathcal{A}$ such that F factors through the projection of $J^\infty\tilde{E}$ onto $J^p\tilde{E}$ for some nonnegative integer p. Moreover for each section ψ of $\tilde{\pi}$, $\psi = (\phi, \eta, P)$, we require that

$$F(j^\infty\psi(x)) = F^{\mathbf{JK}}_{AB}(j^\infty\phi(x))(\partial_{J_1}\eta^{a_1})(x)\cdots(\partial_{J_r}\eta^{a_r})(x)(\partial_{K_1}P_{b_1})(x)\cdots\partial_{K_s}P_{b_s})(x).$$

where $A = \{a_1a_2\cdots a_r\}$, $B = \{b_1b_2\cdots b_s\}$ are multi-indices and $\mathbf{J} = \{J_1J_2\cdots J_r\}$, $\mathbf{K} = \{K_1K_2\cdots K_s\}$ are vectors of multi-indices. Here the set of real-valued functions $\{F^{\mathbf{JK}}_{AB}\}$ are smooth functions on the jet bundle $J^\infty E$. We denote this algebra of local functions on $J^\infty\tilde{E}$ by $Loc = Loc_{\tilde{E}}$.

Definition A **local functional**

$$\mathcal{L}[\psi] = \int_M L(x, \phi^{(p)}(x), , \partial_{j_1}\partial_{j_2}\cdots\partial_{j_s}\eta^{b_j}\partial_{i_1}\partial_{i_2}\cdots\partial_{i_r}P^{a_i})dvol_M \tag{3.32}$$

is the integral over M of a local function L evaluated on sections ψ of $\tilde{E}$ of compact support. Thus

$$\mathcal{L}(\psi) = \int_M j^\infty(\psi)^*(L)dvol_M$$

for some local function $L \in Loc$.

Remark. Notice that the integral in the last definition is the integral of a vector-valued function and the result is then a vector in the space $\mathbf{R} \otimes \mathcal{G} \otimes \mathcal{P}$. We presume that the integral respects the grading on this algebra as this is usually the case in quantum field theory in which integrals of this type regularly arise.

In physical applications, the BRST operator is often a mapping $S : Loc \longrightarrow Loc$ which commutes with the horizontal differential d_H and which satisfies the condition $S^2 = d_H\tilde{k} = \partial_i k^i$ where $\tilde{k} = k^i(-1)^{i-1}[\frac{\partial}{\partial x^i}\lrcorner\ (dx^1 \wedge dx^2 \wedge \cdots dx^n)]$ (see [6] and [9]). Thus in the Lagrangian field setting,the BRST differential is essentially a differential on a quotient space and consequently on the space $\mathcal{F}$ of local functionals. Notice that $(S, \mathcal{F})$ is a D-algebra, and if we set $\mathcal{B} = d_H(\Omega^{n-1,0})$= divergences, then we can obtain a local BRST theory via a chain extension for the triple $(\mathcal{B}, \mathcal{F}, S)$.To apply our theorem, we need to construct a homological resolution for space $\mathcal{F}$.

Obviously, $\mathcal{F} \simeq H^n(\Omega^{*,0}, d_H)$ and $\mathcal{B} = d_H(\Omega^{n-1,0})$. By the algebraic Poincare lemma [6](this is essentially the Poincare lemma for the horizontal differential), we have a commutative diagram:

$$\begin{array}{ccccccccccc}
 & & & d_H & & d_H & & d_H & & d_H & \\
\mathbf{R} & \longrightarrow & \Omega^{0,0} & \longrightarrow & \cdots & \longrightarrow & \Omega^{n-2,0} & \longrightarrow & \Omega^{n-1,0} & \longrightarrow & \Omega^{n,0} \\
\downarrow\eta & & \downarrow\eta & & & & \downarrow\eta & & \downarrow\eta & & \downarrow\eta \\
0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & H_0 = \mathcal{F}
\end{array}$$

Thus $(\Omega^{*,0}, d_H)$ provides a homological resolution for $\mathcal{F}$. It follows that if there exists an appropriate contracting homotopy then we can use Theorem 3 to construct an operator $\tilde{S} = d_H + S + l_3$ which serves in place of the BRST operator S in the usual approach to local BRST theory. The explicit expression for $\tilde{S}$ is interesting but more complicated because the contracting homotopy given by the algebraic Poincare lemma is not so simple (see [12] and [6]).

3.2.3 Consistent deformation

The purpose of this section is to reanalyze the long-standing problem of constructing consistent interactions among fields with a gauge freedom using the formalism of chain extensions. Analysis of this problem has a long history which we will not reiterate here, preferring instead to direct attention to two papers. The first of these is the paper by Berhends, Burgers, and Van Dam [2] where there is a frontal assault on the problem using direct methods and the second is the paper by Barnich and Henneaux [4] where the problem is addressed using the antibracket formalism. The latter paper reformulates the problem as a deformation problem in the sense of deformation theory [7], namely that of *deforming consistently the master equation.* They show, by using the properties of the antibracket, that there is no obstruction to constructing interactions that consistently preserve the gauge symmetries of the free theory if one allows the interactions to be non local.

It is our intent here show that the notion of chain extension clarifies the description of the algebraic aspects of the anti-bracket approach of the problem. Recall that if L is a Lagrangian with gauge degrees of freedom, then the simultaneous deformation of the Lagrangian and it's gauge transformations δ_β can be related to the deformation of the master equation.

In the Batalin-Vilkovisky formalism for gauge theories, which we consider to be irreducible, one introduces, besides the original fields ϕ^i of ghost number 0 and the ghosts C^α of ghost number 1 (related to the gauge invariance), the corresponding antifields ϕ^*_i and C^*_α of opposite Grassmann parity and ghost number -1 and -2 respectively [9]. An antibracket is defined by requiring that the fields $\phi^A \equiv (\phi^i, C^\alpha)$ and antifields ϕ^*_A be conjugate :

$$(\phi^A(x), \phi^*_B(y)) = \delta^A_B \delta(x-y) \tag{3.33}$$

Physicists (see [9]) then define the anti-bracket for arbitrary functionals A_1 and A_2 of these "extended fields" by

$$(A_1, A_2) = \int (\frac{\delta^R A_1}{\delta \phi^A} \frac{\delta^L A_2}{\delta \phi^*_A} - \frac{\delta^R A_1}{\delta \phi^*_A} \frac{\delta^L A_2}{\delta \phi^A}) d^n x \tag{3.34}$$

where the operator $\frac{\delta^R}{\delta\phi^A}$ is a functional derivative acting from the right and the other terms are defined similarly.

Notice that functionals such as A_1, A_2 above are not functions of the fields and antifields but rather are integrals of such functions. A more precise formulation of these concepts was provided in the last subsection on local functionals.

In the following paragraphs, the parity ϵ_A of a functional A is defined by requiring that $\epsilon_A = 1$ when the ghost number of A is odd and $\epsilon_A = 0$ when the ghost number of A is even. The central goal of the formalism is the construction of a proper solution to the master equation

$$(S, S) = 0, \tag{3.35}$$

where $\epsilon_S = 0$. The functional S is required to begin with the classical action S_0, to which one couples via antifields the gauge transformations with the gauge parameters replaced by the ghosts. The BRST symmetry is then canonically generated by the antibracket through the equation :

$$s = (S, \cdot). \tag{3.36}$$

The description of the anti-field formalism we have given here closely follows that of [9].

Thus we require that the components of both the fields ϕ^A and anti-fields ϕ^*_A be mappings from a given manifold M into the complex field. We then define $\mathcal{R}$ to be the algebra of all functionals $A = A(\phi^B, \phi^*_B)$ of the fields, anti-fields, and their derivatives. If A_1, A_2 are in $\mathcal{R}$ then the anti-bracket (A_1, A_2) defined above is then again an element of $\mathcal{R}$. The bracket defined in this manner is odd, carries ghost number $+1$, and obeys the (graded) Jacobi identity. More explicitly, if ϵ_F denotes the parity of an element F of $\mathcal{R}$ we have for $A, B, C \in \mathcal{R}$:

$$(A, B) = -(-1)^{(\epsilon_A+1)(\epsilon_B+1)}(B, A)$$

$$(-1)^{(\epsilon_A+1)(\epsilon_C+1)}(A, (B, C)) + (-1)^{(\epsilon_B+1)(\epsilon_A+1)}(B, (C, A)) + (-1)^{(\epsilon_C+1)(\epsilon_B+1)}(C, (A, B)) = 0.$$

For our exposition below we find it useful to extend the definition of the anti-bracket to the space of formal power series $\mathcal{R}[[t]] = \{\sum_{i=0}^{\infty} a_i t^i | \quad a_i \in \mathcal{R}\}$. We define an anti-bracket on $\mathcal{R}[[t]]$ by:

$$(\sum_{i=0}^{\infty} a_i t^i, \sum_{j=0}^{\infty} b_j t^j) = \sum_{i,j=0}^{\infty} (a_i, b_j) t^{i+j}.$$

Clearly, $\mathcal{R}[[t]]$ is both a vector space over the field k of complex numbers and a module over the algebra $k[[t]]$.

It will be fairly obvious from the exposition below that in our formalism $\mathcal{R}$ could be any vector space over an arbitrary field k of characterisic zero provided one has an anti-bracket defined on $\mathcal{R}$ subject to the conditions above.

We now proceed to analyze the master equation using chain extensions. Let $S_0 \in \mathcal{R}$ be a solution of the master equation

$$(S_0, S_0) = 0 \tag{3.37}$$

and let $s_0 = (S_0, \cdot)$ denote the corresponding BRST differential. The problem of deforming S_0 is the problem of finding a formal power series $S(t) = \sum_0^\infty S_i t^i$ in $\mathcal{R}[[t]]$ such that $(S(t), S(t)) = 0$ and such that $\epsilon_{S_i} = 0$ for all i. Expansion of the master equation in the deformation parameter gives:

$$\begin{aligned} (S_0, S_0) &= 0 \\ (S_0, S_1) &= 0 \\ (S_1, S_1) + (S_0, S_2) + (S_2, S_0) &= 0 \\ &\cdots \end{aligned}$$

In general, $\sum_{i+j=n}(S_i, S_j) = 0$. In order to solve for the S_n inductively, we notice that:

$$0 = \sum_{i+j=n, i,j\geq 0} (S_i, S_j) = \sum_{i+j=n, i,j\geq 1} (S_i, S_j) + (S_0, S_n) + (S_n, S_0) \tag{3.38}$$

Since the anti-bracket is graded commutative and $\mathrm{gh}(S) = 0$, we have $(S_0, S_n) = (S_n, S_0)$,thus:

$$\sum_{i+j=n, i,j\geq 1} (S_i, S_j) = -2(S_0, S_n) \tag{3.39}$$

Set $R_n = \sum_{i+j=n,i,j\geq 1}(S_i, S_j)$, obviously R_n is determined by the first $n-1$ terms: $S_1, S_2 \cdots S_{n-1}$ of the sequence S and R_n is a s_0-coboundary, which we call the n-th obstruction to the deformation of S_0. By definition the n-th order deformation of S_0 is a sequence $S_1, S_2 \cdots, S_n$ in $\mathcal{R}$ such that

$$\left(\sum_0^n S_i t^i, \sum_0^n S_j t^j\right) \equiv 0 \quad (mod \quad t^{n+1}). \tag{3.40}$$

Given an n-th order deformation of S_0 clearly the obstruction to obtaining an $(n+1)$-th order deformation of S_0 is $R_{n+1} = \sum_{i+j=n+1, i,j\geq 1}(S_i, S_j)$.

Define an submodule I in $\mathcal{R}[[t]]$ by $I = \mathcal{R}[[t]]t^{n+1} = \{\sum_{i\geq n+1} a_i t^i | \quad a_i \in \mathcal{R}\}$; thus we have a short exact sequence:

$$0 \longrightarrow I \longrightarrow \mathcal{R}[[t]] \longrightarrow \mathcal{R}[[t]]/(t^{n+1}) \longrightarrow 0. \tag{3.41}$$

Define a k-linear mapping D on $\mathcal{R}$ with values in $\mathcal{R}[[t]]$ by $D(x) = (\sum_0^n S_i t^i, x)$ for each $x \in \mathcal{R}$. The scalar multiplication of the field k on $\mathcal{R}[[t]]$ can clearly be extended to a module multiplication of the power series ring $k[[t]]$ on $\mathcal{R}$. Moreover the operator D can be extended to a $k[[t]]$ module homomorphism of $\mathcal{R}[[t]]$ with values in $\mathcal{R}[[t]]$. Notice that $D(I) \subseteq I$ and as a consequence there is an induced operator on $\bar{\mathcal{R}}[[t]] = \mathcal{R}[[t]]/I$ which we denote by$\bar{D}$. We will show below that $D^2(I) \subset I$ and consequently that $\bar{D}^2 = 0$. Thus we obtain a triple $(I, \bar{\mathcal{R}}[[t]], \bar{D})$; we intend to construct a chain extension for this triple.

To do this we first introduce a "superpartner set of $\mathcal{R}$" which we denote by $\mathcal{R}[1]$ and which is defined by $\mathcal{R}[1] = \{a^* | \quad a \in \mathcal{R}\}$ where $a^* \leftrightarrow a$ is a one to one correspondence and $\epsilon(a^*) = \epsilon(a) + 1$.

Let $X_0 = \mathcal{R}[[t]]$ and $X_1 = \mathcal{R}[1][[t]]t^{n+1} = \{\sum_{i\geq n+1} a_i^* t^i | \quad a_i^* \in \mathcal{R}\}$, so that one has a homological resolution of $\mathcal{R}[[t]]/I$:

$$0 \longrightarrow X_1 \xrightarrow{l_1} X_0 \longrightarrow 0. \tag{3.42}$$

Here $l_1 : X_1 \longrightarrow X_0$ is defined by $l_1(x) = \sum_{i\geq n+1} a_i t^i$ for each x such that $x = \sum_{i\geq n+1} a_i^* t^i$. We can now construct a contracting homotopy similar to earlier definitions. In order to avoid confusion, we denote the contracting homotopy in that which follows by h. Since the sequence (3.42) is split, there exists a natural direct sum decomposition of $X_0 = I \oplus X_0/I$. Define $\mathcal{F} = X_0/I$, $\eta : X_0 \longrightarrow \mathcal{F}$ by $\eta = proj|_{\mathcal{F}}$ and λ by $\lambda = i_{\mathcal{F}\to X_0}$ (the inclusion mapping). Finally, since $X_0 = I \oplus \mathcal{F}$, define the contracting homotopy h by $h(x) = -\sum_{i\geq n+1} a_i^* t^i$ for each $x \in I, x = \sum_{i\geq n+1} a_i t^i$, and define h to be zero on $\mathcal{F}$. It's easy to show that $\lambda \circ \eta - 1_{X_*} = l_1 \circ h + h \circ l_1$, thus we may define a chain extension for the triple $(I, \mathcal{F}, \bar{D})$ via the proposition below.

Let $S_D = \sum_{i=0}^n S_i t^i \in X_0 = \mathcal{R}[[t]]$ so that $D : X_0 \to X_0$ is given by $D(x) = (S_D, x)$ for all $x \in X_0$. It is easy to show that

$$(S_D, (S_D, x)) = \frac{1}{2}((S_D, S_D), x)$$

for all $x \in X_0$ and that $D^2(I) \subseteq I$. It follows that one has an induced operator $\bar{D}$ on $\mathcal{R}[[t]]/I$. Using this fact we obtain the following proposition.

Proposition 3.4 *Assume that $\mathcal{R}$ is a graded linear space (graded with respect to ghost number above) and that it has an anti-bracket structure which is graded commutative and which satisfies the graded Jacobi identity as above. Let $\mathcal{R}[[t]]$ denote the corresponding space of formal power series and let I denote the $k[[t]]$ submodule $\mathcal{R}[[t]]t^{n+1}$. If $S_0, S_1, \cdots, S_n$ are even elements of $\mathcal{R}$ and D is the operator on $\mathcal{R}[[t]]$ defined by $D(x) = (\sum_{i=0}^{n} S_i t^i, x)$, $x \in \mathcal{R}[[t]]$, then $D(I) \subseteq I$ and the induced operator $\bar{D}$ on $\mathcal{R}[[t]]/I$ is nilpotent. Moreover the triple $(I, \mathcal{R}[[t]]/I, \bar{D})$ admits a chain extension with chain maps l_1, l_2, l_3 such that the map $S = l_1 + l_2 + l_3$ satisfies $S^2 = 0$. The explicit homological resolution X_* is defined in the proof below.*

Proof Recall that $X_0 = \mathcal{R}[[t]], X_1 = \mathcal{R}[1][[t]]t^{n+1}$, and define a mapping $l_2 : X_0 \to X_0$ by $l_2(x) = D(x) = (\sum_{i=0}^{n} S_i t^i, x), x \in X_0$. Since we intend to apply Theorem 3, recall that in that context, $\mathcal{B} = l_1(X_1)$ and consequently, $\mathcal{B} = I$ in the present case. Thus $l_2(\mathcal{B}) l_2(\mathcal{R}[[t]]t^{n+1}) = D(\mathcal{R}[[t]]t^{n+1}) \subseteq \mathcal{R}[[t]]t^{n+1} = I = \mathcal{B}$. This is one of the conditions required to apply Theorem 3. Also note that for $x \in X_0$, $l_2^2(x) = D^2(x) = \frac{1}{2}((S_D, S_D), x)$ and by (3.40) $((S_D, S_D) = 0 \quad (mod(t^{n+1})$. Thus $l_2^2(X_0) \subseteq I = \mathcal{B}$.

Thus the hypothesis of Theorem 3 holds and consequently the theorem asserts the existence of maps l_1, l_2, l_3 which define a chain extension, in particular their sum has square zero.

From the proposition we know that the triple $(I, \mathcal{R}[[t]]/I, \bar{D})$ admits a chain extension with chain maps l_1, l_2, l_3 defined by Theorem 3. In applications, it is useful to have explicit formulas for these maps. Each mapping l_i is a mapping from X_* to X_* which is uniquely determined by its values on $\mathcal{R}$ and $\mathcal{R}[1]$. Our next theorem provides explicit formulas which show how to obtain the maps $l_i, i = 1, 2, 3$.

Theorem 3.5 *The chain maps l_1, l_2, l_3, which define the chain extension $S = l_1 + l_2 + l_3$,* *guaranteed by the proposition above, are uniquely determined by their values on the linear spaces $\mathcal{R}$ and $\mathcal{R}[1]$ and are given in a concise form as follows. The mapping l_1 is determined by its values on $\mathcal{R}[1]$ with $l_1(a^*) = a$. The map l_2 is given by its values on $\mathcal{R}$ and $\mathcal{R}[1]$, respectively by*

$$l_2(a) = \sum_{j=0}^{n}(S_i, a)t^i, \quad l_2(a^*) = -\sum_{j=0}^{n}(S_i, a)^* t^i.$$

Finally, l_3 is uniquely determined by its values on $\mathcal{R}$,

$$l_3(a) = -\frac{1}{2}\sum_{n+1\leq i+j\leq 2n}((S_i, S_j), a)^* t^{i+j}.$$

Proof By the proposition above and Theorem 3 we have,

$$l_2(a^*) = hl_2l_1(a^*) = hl_2(a) = h(\sum_0^n S_it^i, a) = h(\sum_0^n(S_i, a)t^i) = -\sum_0^n(S_i, a)^*t^i.$$

Before evaluating l_3 ,we replace l_2 by D, which appears in the canonical transformation and in the corollary above. Then by Theorem 3, $l_3(a) = hl_2^2(a) = h(D, (D, a))$. Since

$$D(x) = (\sum_{i=0}^{n} S_it^i, x),$$

we have:

$$\begin{aligned} l_3(a) &= \frac{1}{2}h((\sum_{i=0}^{n} S_it^i, \sum_{j=0}^{n} S_jt^j), a) \\ &= \frac{1}{2}h(\sum_{k=0}^{2n}\sum_{i+j=k}((S_i, S_j), a)t^{i+j}) \\ &= -\frac{1}{2}\sum_{k=n+1}^{2n}\sum_{i+j=k}((S_i, S_j), a)^*t^{i+j}) \\ &= -\frac{1}{2}\sum_{n+1\leq i+j\leq 2n}((S_i, S_j), a)^*t^{i+j} \\ &= -\frac{1}{2}R_{n+1} - \frac{1}{2}\sum_{n+2\leq i+j\leq 2n}((S_i, S_j), a)^*t^{i+j}. \end{aligned} \tag{3.43}$$

This finishes the proof of the theorem.

Remark. Observe that the last calculation above identifies the obstruction R_{n+1} as a summand of the value of l_3 on $\mathcal{R}$. In particular, if l_3 is zero then the obstruction to further deformation vanishes.

If one wishes to consider infinitesimal deformations, simply take $n = 1$, then by the theorem, we have:

$$S(a) = S_0(a) + S_1(a)t - ((S_1, S_1), a)t^2$$

$$S(a^*) = a - (S_0, a)^* - (S_1, a)^* t.$$

Extend these as derivations and we obtain a consistent deformation of the space of fields $\mathcal{R}$ by enlarging $\mathcal{R}$ to the space $\mathcal{R}[1]$,or physically speaking, the deformation is achieved by adjoining the superpartners to the space of original fields.

Chapter 4

Deformation theory, sh-Lie algebras

In the last two decades, deformations of various types of structures have assumed an ever increasing role in mathematics and physics. For each such deformation problem a goal is to determine if all related deformation obstructions vanish and many beautiful techniques been developed to determine when this is so. Sometimes genuine deformation obstructions arise and occasionally that closes mathematical development in such cases, but in physics such problems are dealt with by introducing new auxiliary fields to kill such obstructions. This idea suggests that one might deal with deformation problems by enlarging the relevant category to a new category obtained by appending additional algebraic structures to the old category.

In the present chapter, we consider deformations of Lie algebras. In order to be complete we review basic facts on Lie algebra deformations; more detail may be found in the book edited by M.Hazawinkel and M.Gerstenhaber [8].

Let A be a k-algebra and α be its multiplication, i.e., α is a k-bilinear map $A \times A \longrightarrow A$ defined by $\alpha(a, b) = ab$. A deformation of A may be defined to be a formal power series $\alpha_t = \alpha + t\alpha_1 + t^2\alpha_2 + \cdots$ where each $\alpha_i : A \times A \longrightarrow A$ is a k-bilinear map and the "multiplication" α_t is formally of the same "kind" as α, e.g., it is associative or Lie or whatever is required. One technique used to set up a deformation problem is to extend a k-bilinear mapping $\alpha_t : A \times A \longrightarrow A[[t]]$ to a $k[[t]]$-bilinear mapping $\alpha_t : A[[t]] \times A[[t]] \longrightarrow A[[t]]$. A mapping $\alpha_t : A[[t]] \times A[[t]] \longrightarrow A[[t]]$ obtained in this manner is necessarily uniquely determined by it's values on $A \times A$. In fact we would not regard the mapping $\alpha_t : A[[t]] \times A[[t]] \longrightarrow A[[t]]$ to be a deformation of A unless it is determined by it's values on $A \times A$.

From this point on, we assume that (A, α) is a Lie algebra,i.e., we assume that $\alpha(\alpha(a, b), c) + \alpha(\alpha(b, c), a) + \alpha(\alpha(c, a), b) = 0$. Thus the problem of deforming a Lie algebra A is equivalent to the problem of finding a mapping $\alpha_t : A \times A \longrightarrow A[[t]]$ such that $\alpha_t(\alpha_t(a, b), c) + \alpha_t(\alpha_t(b, c), a) + \alpha_t(\alpha_t(c, a), b) = 0$. If we set $\alpha_0 = \alpha$ and expand this Jacobi identity by making the substitution $\alpha_t = \alpha + t\alpha_1 + t^2\alpha_2 + \cdots$, we get the equation

$$\sum_{i,j=0}^{\infty} [\alpha_j(\alpha_i(a, b), c) + \alpha_j(\alpha_i(b, c), a) + \alpha_t(\alpha_i(c, a), b)]t^{i+j} = 0 \tag{4.1}$$

and consequently a sequence of deformation equations;

$$\sum_{i,j \geq 0, i+j=n} [\alpha_j(\alpha_i(a, b), c) + \alpha_j(\alpha_i(b, c), a) + \alpha_t(\alpha_i(c, a), b)] = 0. \tag{4.2}$$

The first two equations are:

$$\alpha_0(\alpha_0(a, b), c) + \alpha_0(\alpha_0(b, c), a) + \alpha_0(\alpha_0(c, a), b) = 0 \tag{4.3}$$

$$\begin{aligned} \alpha_0(\alpha_1(a, b), c) + \alpha_0(\alpha_1(b, c), a) + \alpha_0(\alpha_1(c, a), b) + \alpha_1(\alpha_0(a, b), c) \\ + \alpha_1(\alpha_0(b, c), a) + \alpha_1(\alpha_0(c, a), b) = 0 \end{aligned} \tag{4.4}$$

We can reformulate the discussion above in a slightly more compact form. Given a sequence $\alpha_n : A \times A \longrightarrow A$ of bilinear maps, we define "compositions" of various of the α_n as follows:

$$\alpha_i\alpha_j : A \times A \times A \longrightarrow A \tag{4.5}$$

is defined by

$$(\alpha_i\alpha_j)(x_1, x_2, x_3) = \sum_{\sigma \in unsh(2,1)} (-1)^{\sigma} \alpha_i(\alpha_j(x_{\sigma(1)}, x_{\sigma(2)}), x_{\sigma(3)}) \tag{4.6}$$

for arbitrary $x_1, x_2, x_3 \in A$.

Thus the deformation equations are equivalent to following equations:

$$\alpha_0^2 = 0 \tag{4.7}$$

$$\alpha_0\alpha_1 + \alpha_1\alpha_0 = 0 \tag{4.8}$$

$$\alpha_1^2 + \alpha_0\alpha_2 + \alpha_2\alpha_0 = 0 \tag{4.9}$$

$$\cdots$$

$$\Sigma_{i+j=n}\alpha_i\alpha_j = 0 \tag{4.10}$$

$$\cdots . \tag{4.11}$$

Define a bracket on the sequence $\{\alpha_n\}$ of mappings by $[\alpha_i, \alpha_j] = \alpha_i\alpha_j + \alpha_j\alpha_i$ and a "differential" d by $d = ad_{\alpha_0} = [\alpha_0, \cdot]$, the "adjoint representation" relative to α_0. Notice that the second equation in the list above is equivalent to the statement that α_1 defines a cocycle $\alpha_1 \in Z^2(A, A)$ in the Lie algebra cohomology of A. Moreover it is known that the second cohomology group $H^2(A, A)$ classifies the equivalence class of infinitesimal deformations of A [8]. This being the case we refer to the triple (A, α_0, α_1) as being initial conditions for deforming the Lie algebra A. Notice that the third equation in the above list can be rewritten as

$$[\alpha_1, \alpha_1] = -[\alpha_0, \alpha_2] = -d\alpha_2 \tag{4.12}$$

When this equation holds one has then that $[\alpha_1, \alpha_1]$ is a coboundary and so defines the trivial element of $H^3(A, A)$ for any given deformation α_t. Thus if $[\alpha_1, \alpha_1]$ is not a coboundary, then we may regard $[\alpha_1, \alpha_1]$ as the first obstruction to deformation and in this case we can not deform A at second order. In general, to say that there exists a deformation of (A, α_0, α_1) up to order $n-1$, means that there exists a sequence of maps $\alpha_0, \cdots, \alpha_{n-1}$ such that $\sum_{\sigma\in unsh(2,1)}(-1)^\sigma\alpha_t(\alpha_t(x_{\sigma(1)}, x_{\sigma(2)}), x_{\sigma(3)}) = 0 \pmod{t^n}$. If this is the case and if there is an obstruction to deformation at nth order, then it follows that $\rho_n = -(\Sigma_{i+j=n,i,j>0}\alpha_i\alpha_j)$ is in some sense the obstruction and $[\rho_n]$ is a nontrivial element of $H^3(A, A)$. If $[\rho_n] \neq 0$,then the process of obtaining a deformation will terminate at order $n-1$ due to the existence of the obstruction ρ_n. In principal, it is possible that one could return to the beginning and select different terms for the α_i but when this fails what can one say? This is the issue in the remainder of this section.

Indeed the central point of this section is to show that when there is an obstruction to the deformation of a Lie algebra, one can use the obstruction itself to define one of the structure mappings of an sh-Lie algebra. Without loss of generality, we consider a deformation problem which has a first order obstruction.

The required sh-Lie structure lives on a graded vector space X_* which we define below. This space in degree zero is given by $X_0 = A[[t]] = (\{\Sigma a_i t^i| \quad a_i \in A\}$. The spaces $\mathcal{B} = < t^2 >= A[[t]] \cdot t^2 = \{\Sigma_{i \geq 2} a_i t^i | a_i \in A\}$ and $\mathcal{F} = X_0/\mathcal{B} =$ are also relevant to our construction. Notice that $\mathcal{F}$ is isomorphic to $\{a_0 + a_1 t| \quad a_0, a_1 \in A\}$ as a linear space and that $X_0, \mathcal{B}$ are both $k[[t]]$-modules while $\mathcal{F}$ is a $k[[t]]/ < t^2 >$ module (recall that k is underlying field of A). To summarize, we have following short exact sequence:

$$0 \longrightarrow \mathcal{B} \longrightarrow X_0 \longrightarrow \mathcal{F} \longrightarrow 0.$$

Suppose that the initial Lie structure of A is given by $\alpha_0 : A \times A \longrightarrow A$ and denote a fixed infinitesimal deformation by $[\alpha_1] \in H^2(A, A)$. One of the structure mappings of our sh-Lie structure will be determined by the mapping $\tilde{l}_2 : X_0 \times X_0 \longrightarrow X_0$ defined as follows: for any $a, b \in A$, let

$$\tilde{l}_2(a, b) = \alpha_0(a, b) + \alpha_1(a, b)t \tag{4.13}$$

and extend it to X_0 by requiring that it be $k[[t]]$-bilinear. Obviously, $\tilde{l}_2$ induces a Lie bracket $[,]$ on $\mathcal{F}$, but if the obstruction $[\alpha_1, \alpha_1]$ is not zero, then $\tilde{l}_2^2 \neq 0$ and consequently $\tilde{l}_2$ can not be a Lie bracket on X_0 (since it doesn't satisfy the Jacobi identity).

To deal with this obstruction we will show that we can use α_0, α_1 to construct an sh-Lie structure with at most three nontrivial structure maps l_1, l_2, l_3 such that the value of l_3 on $A \times A \times A$ is the same as that of $[\alpha_1, \alpha_1]$. In particular, l_3 will vanish if and only if the obstruction $[\alpha_1, \alpha_1]$ vanishes. Thus the sh-Lie algebra encodes the obstruction to deformation of the Lie algebra (A, α_0).

The required sh-Lie algebra lives on a certain homological resolution (X_*, l_1) of $\mathcal{F}$, so our first task is to construct this resolution space for $\mathcal{F}$. To do this let's introduce a "superpartner set of A," denoted by $A[1]$, as follows: for each $a \in A$, introduce a^* such that $a^* \leftrightarrow a$ is a one to one correspondence and define $\epsilon(a^*) = \epsilon(a) + 1$. Let $X_1 = A[1][[t]]t^2$ and define a map $l_1 : X_1 \longrightarrow X_0$ by

$$l_1(x) = \Sigma_{i \geq 2} a_i t^i \in X_0, \qquad x = \Sigma_{i \geq 2} a_i^* t^i \in X_1.$$

Notice that this is just the $k[[t]]$ extension of the $a^* \leftrightarrow a$ map. Since l_1 is injective, we obtain a homological resolution $X_* = X_0 \oplus X_1$ due to the fact that the complex defined by:

$$0 \longrightarrow X_1 \xrightarrow{l_1} X_0 \longrightarrow 0 \tag{4.14}$$

has the obvious property that $H(X_*) = H_0(X_*) \simeq \mathcal{F}$.

The sh-Lie algebra being constructed will have the property that $l_n = 0, n \geq 4$. Generally sh-Lie algebras can have any number of nontrivial structure maps. The fact that all the structure mappings of our sh-Lie algebra are zero with the exception of l_1, l_2, l_3 is an immediate consequece of the fact that we are able to produce a resolution of the space $\mathcal{F}$ such that $X_k = 0$ for $k \geq 2$. In general such resolutions do not exist and so one does not have $l_n = 0$ for $n \geq 4$.

In order to finish the preliminaries, we now construct a contracting homotopy s such that following commutative diagram holds:

$$\begin{array}{ccccccc} 0 \longrightarrow & X_1 & \overset{s}{\underset{l_1}{\overset{\longleftarrow}{\longrightarrow}}} & X_0 & \longrightarrow 0 \\ & \lambda \Big\uparrow \Big\downarrow \eta & & \lambda \Big\uparrow \Big\downarrow \eta & \\ 0 \longrightarrow & 0 & \longrightarrow & \mathcal{F} & \longrightarrow 0 \end{array}$$

Clearly the linear space X_0 is the direct sum of $\mathcal{B}$ and a complementary subspace which is isomorphic to $\mathcal{F}$; consequently we have $X_0 \simeq \mathcal{B} \oplus \mathcal{F}$. Define $\eta = proj \mid_{\mathcal{F}}, \lambda = i_{\mathcal{F} \rightarrow X_0}$ and a contracting homotopy $s : X_0 \longrightarrow X_1$ as follows: write $X_0 = \mathcal{B} \oplus \mathcal{F}$, set $s|_{\mathcal{F}} = 0$, and let $s(x) = -x^*$ for all $x \in \mathcal{B}$. It is easy to show that $\lambda \circ \eta - 1_{X_*} = l_1 \circ s + s \circ l_1$. In order to obtain the sh-Lie algebra referred to above, we apply a theorem of [5]. The hypothesis of this theorem requires the existence of a bilinear mapping $\tilde{l}_2$ from $X_0 \times X_0$ to X_0 with the properties that for $c, c_1, c_2, c_3 \in X_0$ and $b \in \mathcal{B}$ (i) $\tilde{l}_2(c, b) \in \mathcal{B}$ and (ii) $\tilde{l}_2^2(c_1, c_2, c_3) \in \mathcal{B}$. To see that (i) holds notice that if $p(t), q(t) \in X_0 = A[[t]]$, then $\tilde{l}_2(p(t), q(t)t^2) = r(t)t^2$ for some $r(t) \in A[[t]] = X_0$. Also note that the fact that $\tilde{l}_2$ induces a Lie bracket on $\mathcal{F} = X_0/\mathcal{B}$ implies that $\tilde{l}_2^2$ is zero modulo $\mathcal{B}$ and (ii) follows. Thus X_* supports an sh-Lie structure with only three nonzero structure maps l_1, l_2, l_3 (see the remark at the end of [5]).

Theorem 4.1 *Given a Lie algebra A with Lie bracket α_0 and an infinitesimal obstruction $[\alpha_1] \in H^2(A, A)$ to deforming (A, α_0), there is an sh-Lie algebra on the graded space (X_*, l_1)*

with structure maps $\{l_i\}$ such that $l_n = 0$ for $n \geq 4$. The graded space X_* has at most two nonzero terms $X_0 = A[[t]], X_1 = A[1][[t]]t^2$. Finally, the maps l_1, l_2, l_3 may be given explicitly in terms of the maps α_0, α_1.

Remark : The mapping l_1 is simply the differential of the graded space (X_*, l_1). The mapping l_2 restricted to $X_0 \times X_0$ is the mapping $\tilde{l}_2$ defined directly in terms of α_0, α_1 above. On $X_1 \times X_0$, l_2 is determined by $l_2(a^*t^2, b) = t^2(\alpha_0(a,b)^* + \alpha_1(a,b)^*t)$ for $a^* \in A[1], b \in A$. Finally, l_3 is uniquely determined by its values on $A \times A \times A \subset X_0 \times X_0 \times X_0$ and is explicitly a multiple of the obstruction to the deformation of (A, α_0), in particular, $l_3(a_1, a_2, a_3) = -\frac{1}{2}t^2[\alpha_1, \alpha_1](a_1, a_2, a_3), a_i \in A$.

Proof The sh-Lie structure maps are given by Theorem 7 of [5]. The fact that $l_n = 0, n \geq 4$ is an observation of Markl which was proved by Barnich [3] (see the remark at the end of [5]). A generalization of Markl's remark is available in a paper by Al-Ashhab [1] and in that paper more explicit formulas are given for l_1, l_2, l_3. Examination of these formulas provide the details needed for the calculations below.

First of all, we examine the mapping $l_2 : X_* \times X_* \longrightarrow X_*$. Now $l_2 : A \times A \longrightarrow X_*$ is determined by $\tilde{l}_2 : X_0 \times X_0 \longrightarrow X_0$, consequently we need only consider the restricted mapping:

$$l_2 : X_1 \times X_0 \longrightarrow X_1. \tag{4.15}$$

Moreover, since X_0 is a module over $k[[t]]$, X_1 is a module over $k[[t]]t^2$, and $\tilde{l}_2$ respects these structures we need only consider its values on pairs (a^*t^2, b) with $a^*t^2 \in X_1, b \in X_0$. By Theorem 2.2 of [1], we have

$$\begin{aligned} l_2(a^*t^2, b) &= -sl_2l_1[(a^*t^2) \otimes b] \\ &= -sl_2[l_1(a^*t^2) \otimes b + (-1)^{\epsilon(a^*)}(a^*t^2) \otimes l_1(b)] \\ &= -sl_2[(at^2 \otimes b)] = -s[t^2 l_2(a \otimes b)] \\ &= -s[t^2(\alpha_0(a,b) + \alpha_1(a,b)t)] \\ &= -s[\alpha_0(a,b)t^2 + \alpha_1(a,b)t^3] \\ &= \alpha_0(a,b)^*t^2 + \alpha_1(a,b)^*t^3 \\ &= t^2(\alpha_0(a,b)^* + \alpha_1(a,b)^*t). \end{aligned} \tag{4.16}$$

From this deduction, we that the mapping l_2 can essentially be replaced by the modified map:

$$\bar{l}_2 : A[1] \times A \longrightarrow A[1][[t]], \qquad \bar{l}_2(a^*, b) = \alpha_0(a,b)^* + \alpha_1(a,b)^* t. \tag{4.17}$$

We clarify this remark below by showing that a new sh-Lie structure can be obtained with $\bar{l}_2$ playing the role of l_2.

The next mapping we examine is the mapping

$$l_3 : X_0 \times X_0 \times X_0 \longrightarrow X_1 \tag{4.18}$$

Since l_3 is $k[[t]]$-linear,we need only consider mappings of the type:
$l_3 : A \times A \times A \longrightarrow X_1$ where for $x_1, x_2, x_3 \in A$,

$$\begin{aligned}
l_3(x_1,x_2,x_3) &= sl_2^2(x_1,x_2,x_3) \\
&= \sum_{\sigma \in unsh(2,1)} (-1)^\sigma sl_2(l_2(x_{\sigma(1)}, x_{\sigma(2)}), x_{\sigma(3)}) \\
&= \sum_{\sigma \in unsh(2,1)} (-1)^\sigma sl_2(\alpha_0(x_{\sigma(1)}, x_{\sigma(2)}) + \alpha_1(x_{\sigma(1)}, x_{\sigma(2)})t, x_{\sigma(3)}) \\
&= \sum_{\sigma \in unsh(2,1)} (-1)^\sigma s[\alpha_0(\alpha_0(x_{\sigma(1)}, x_{\sigma(2)}), x_{\sigma(3)}) \\
&+ t\alpha_1(\alpha_0(x_{\sigma(1)}, x_{\sigma(2)}), x_{\sigma(3)}) + t\alpha_0(\alpha_1(x_{\sigma(1)}, x_{\sigma(2)}), x_{\sigma(3)}) \\
&+ t^2\alpha_1(\alpha_1(x_{\sigma(1)}, x_{\sigma(2)}), x_{\sigma(3)}), x_{\sigma(3)})] \\
&= s(\sum_{\sigma \in unsh(2,1)} (-1)^\sigma \alpha_0(\alpha_0(x_{\sigma(1)}, x_{\sigma(2)}), x_{\sigma(3)}) \\
&+ t(\sum_{\sigma \in unsh(2,1)} (-1)^\sigma \alpha_1(\alpha_0(x_{\sigma(1)}, x_{\sigma(2)}), x_{\sigma(3)}) \\
&+ \sum_{\sigma \in unsh(2,1)} (-1)^\sigma \alpha_0(\alpha_1(x_{\sigma(1)}, x_{\sigma(2)}), x_{\sigma(3)})) \\
&+ t^2(\sum_{\sigma \in unsh(2,1)} (-1)^\sigma \alpha_1(\alpha_1(x_{\sigma(1)}, x_{\sigma(2)}), x_{\sigma(3)}))] \\
&= s((\alpha_0^2 + t(\alpha_0\alpha_1 + \alpha_1\alpha_0) + t^2\alpha_1^2)(x_1,x_2,x_3)) \\
&= s(t^2\alpha_1^2(x_1,x_2,x_3)) \\
&= t^2(\alpha_1^2(x_1,x_2,x_3))^*
\end{aligned} \tag{4.19}$$

Or $l_3(x_1,x_2,x_3) = -\frac{1}{2}t^2([\alpha_1,\alpha_1](x_1,x_2,x_3))^*$ which is precisely the "first deformation obstruction class".

Recall that we know from Theorem 7 of [5] that we have an sh-Lie structure. The point of these calculations is that it enables us to obtain the modified sh-Lie structure of Corollary 10 below and it is this structure which is relevant to Lie algebra deformation. Thus we already know that the mappings l_1, l_2, l_3 satisfy the relations:

$$l_1 l_2 - l_1 l_2 = 0 \tag{4.20}$$

$${l_2}^2 + l_1 l_3 + l_3 l_1 = 0 \tag{4.21}$$

$${l_3}^2 = 0 \tag{4.22}$$

$$l_2 l_3 + l_3 l_2 = 0. \tag{4.23}$$

Observe that if we let $\tilde{X}_* = \tilde{X}_1 \oplus \tilde{X}_0 = A[1][[t]] \oplus A[[t]]$, then the formulas defining l_1, l_2, l_3 defined on X_* make sense on the new complex $\tilde{X}_*$. Indeed the calculations above show that l_1, l_3 are uniquely determined by their values on "constants" in the sense that they could be first defined on elements of $A[1] \oplus A \subseteq A[1][[t]] \oplus A[[t]]$ and then extended to $A[1][[t]] \oplus A[[t]]$ using the fact that l_1, l_3 are required to be $k[[t]]$ linear. l_2 is not obviously $k[[t]]$ linear. The whole point of corollary 10 below is that the sh-Lie structure defined by Theorem 10 can be redefined to obtain sh-Lie maps on the graded space $\tilde{X}_*$ which are obviously $k[[t]]$ linear and consequently this "new" structure is intimately related to deformation theory. Thus, as we say above, the modified map $\bar{l}_2$ can be extended to the new complex $\tilde{X}_*$ and is uniquely determined by its values on "constants". If we denote the extensions of l_1, l_3 to $\tilde{X}_*$ by $\bar{l}_1, \bar{l}_3$, then clearly these mappings satisfy the same relations (63)-(66) as the maps l_1, l_2, l_3 and consequently if we define $\bar{l}_n = 0, n \geq 4$ it follows that $(\tilde{X}_*, \tilde{l}_1, \tilde{l}_2, \tilde{l}_3, 0, 0 \cdots)$ is an sh-Lie algebra. This proves the following corollary.

Corollary 4.2 *There is an sh-Lie structure on $A[1][[t]] \oplus A[[t]]$ whose structure mappings $\{\bar{l}_1, \bar{l}_2, \bar{l}_3, 0, \cdots\}$are precisely the mappings $\{l_1, l_2, l_3, 0, \cdots\}$ when restricted to $A[1][[t]]t^2 \oplus A[[t]]$. Moreover, the structure mappings of $A[1][[t]] \oplus A[[t]]$ have the property that they are uniquely determined by their values on $A[1] \oplus A$ and $k[[t]]$ linearity.*

From the discussion above the set of mappings $\{\bar{l}_1, \bar{l}_2, \bar{l}_3\}$ is essentially a deformation of an sh-Lie algebra. In addition, the construction of the mapping $\bar{l}_2$ is equivalent to defining an initial condition for a Lie algebra deformation.

This means that a Lie algebra which can't be deformed in the category of Lie algebra may admit an sh-Lie algebra deformation by first imbedding it into an appropriate sh-Lie algebra.

Conclusion

Here, at the end of the dissertation, we comment briefly regarding the developing role chain extensions could play in mathematics and physics. We have seen that chain extensions may be regarded as a generalization of ordinary algebraic extensions of Lie algebras. This being the case an interesting question is that of classifying such extensions perhaps using methods similar to the use of Chevalley-Eilenberg cohomology to classify Lie algebra extensions. More generally, it would be of interest to develop a theory of chain extensions to parallel that of ordinary Lie algebra extensions and to consider applications of this theory beyond those initiated in this paper.

Another question of interest is that of the relationship between chain extensions and sh-Lie algebras. Preliminary calculations suggest a link between chain extensions and sh-Lie algebras and consequently that there may be a way of dealing with deformations of Lie algebras via chain extensions, but more work is required to be conclusive. More generally, the notion of a chain extension has provided us with a new technique for the investigation of deformation problems, but it's significance is not yet fully understood. Its role in physics is yet to be fully understood, but we believe that our reformulation of consistent deformations is an indication of its value.

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