ABSTRACT

ADU-GYAMFI, KWAKU. Connections among Representations: The Nature of Students’ Coordinations on a Linear Function task. (Under the direction of Karen S. Norwood, Ed.D.)

This study focused on students’ translations between and coordinations among representations of linear function after having being explicitly taught functions via multiple representational approaches. The purpose of the study was to investigate the nature of connections performed among representations of linear function by students whose instructional experiences have been based on multiple representational approaches. In class examinations and task based interviews were conducted with a purposefully chosen sample of students from the Intermediate Algebra through Multiple Representations course in order to elucidate patterns in students’ connections among representations of linear function.

Two types of translations and coordinations were elucidated from students work with the given representations of linear function, namely; procedural and foundational. In the case of translations, this study showed that students’ who performed procedural translations knew of procedures for working within both given source and specified target representations and for translating from the given source to the specified target representations but lacked knowledge of the preservation of meaning requirement of the source representation under a translation. In the case of coordination, the study showed that students’ who performed procedural coordinations were restricted in their ability to relate specific representations of linear function and or to utilize specific constructs in relating the correct pairs of representations of linear function together.
CONNECTIONS AMONG REPRESENTATIONS: THE NATURE OF STUDENTS’ COORDINATIONS ON A LINEAR FUNCTION TASK

by

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To My Grandmother

Nana Grace
BIOGRAPHY

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# TABLE OF CONTENTS

LIST OF TABLES........................................................................................................ viii

LIST OF FIGURES..................................................................................................... ix

PART I: INTRODUCTION ........................................................................................... 1

Definition of Key Terms ......................................................................................... 3
Description of Dynagraph....................................................................................... 5
Problem Statement................................................................................................ 6
Purpose of the Study and Research Question(s).................................................. 8

PART II: REVIEW OF LITERATURE........................................................................ 10

The Function Concept-Introduction...................................................................... 11
Students’ Difficulty with Functions........................................................................ 12
Connections among Function Representations.................................................... 14
Multiple Representations....................................................................................... 16
Multiple Representations: Definition and History............................................... 17
Multiple Representations in Mathematics............................................................ 18
Issues Associated with Multiple Representations................................................. 20
Connection among Representations...................................................................... 21
Research on Multiple Representations................................................................. 23
Framework-Theoretical Foundation...................................................................... 28
Framework.............................................................................................................. 30

PART III: METHODOLOGY....................................................................................... 34

Rationale.................................................................................................................. 34
Pilot Study: Usefulness of Instruments................................................................. 35
The Researcher’s Role............................................................................................ 37
Main Study............................................................................................................... 38
The Context: An Intermediate Algebra Course.................................................... 38
The Participants: Intermediate Algebra Students............................................... 39
Data....................................................................................................................... 41
Data Analysis........................................................................................................... 43
Ethical Issues.......................................................................................................... 44
Limitations of the study......................................................................................... 45
PART IV: CONNECTIONS AMONG REPRESENTATIONS: THE NATURE OF STUDENTS’ COORDINATIONS ON A LINEAR FUNCTION

INTRODUCTION ........................................................................................................... 47

MULTIPLE REPRESENTATIONS ............................................................................... 49

FUNCTION .................................................................................................................. 50

METHODOLOGY ......................................................................................................... 52

THE CONTEXT: AN INTERMEDIATE ALGEBRA COURSE ....................................... 52

THE PARTICIPANTS: INTERMEDIATE ALGEBRA STUDENTS .................................. 53

DATA SOURCES .......................................................................................................... 54

DATA ANALYSIS .......................................................................................................... 55

RESULTS ....................................................................................................................... 56

STUDENT BY STUDENT ANALYSIS ................................................................. 56

TASK BY TASK ANALYSIS ................................................................................. 65

DISCUSSION AND CONCLUSION ............................................................................ 70

PART V: CONNECTIONS AMONG REPRESENTATIONS: THE NATURE OF
STUDENTS’ CONNECTIONS BETWEEN STANDARD
REPRESENTATIONS OF LINEAR FUNCTION .......................................................... 76

INTRODUCTION ........................................................................................................... 77

FUNCTION .................................................................................................................. 79

MULTIPLE REPRESENTATIONS ............................................................................... 80

METHODOLOGY ......................................................................................................... 81

THE CONTEXT: AN INTERMEDIATE ALGEBRA COURSE ....................................... 82

THE PARTICIPANTS: INTERMEDIATE ALGEBRA STUDENTS .................................. 82

DATA SOURCES .......................................................................................................... 83

DATA ANALYSIS .......................................................................................................... 84

RESULTS ....................................................................................................................... 85

ITEM BY ITEM ANALYSIS .................................................................................... 85

ANALYSIS OF RESULTS BY STUDENTS ............................................................ 92

DISCUSSION AND CONCLUSION ............................................................................ 98

LIST OF REFERENCES ................................................................................................ 104

APPENDICES ................................................................................................................. 112

APPENDIX A: INTERVIEW PROTOCOL .................................................................... 113

APPENDIX B: FUNCTION MATCH TASK ............................................................... 113

APPENDIX C: STUDENT WORK SHEET ................................................................. 115

APPENDIX D: COORDINATION CHART ................................................................. 118

APPENDIX E: RUBRIC FOR ANALYZING STUDENTS’ COORDINATION CHART ....... 119

APPENDIX F: TEST INSTRUMENT ............................................................................ 120

APPENDIX G: TRANSLATION CHART ................................................................. 122

APPENDIX H: RUBRIC FOR ANALYZING STUDENTS’ TRANSLATION CHART ....... 123
LIST OF TABLES

PART I
TABLE 1 Components of a coordination ................................................................. 4

PART III
TABLE 1 Data sorted into categories based on research question ....................... 43
TABLE 2 Data analysis methods ...................................................................... 44

PART IV
TABLE 1 Components of a coordination ................................................................. 48
TABLE 2 Summary description of participants .................................................. 53
TABLE 3 Summary results for Task1 ................................................................. 67
TABLE 4 Summary results for Task2 .................................................................. 70

PART V
TABLE 1 Summary description of participants .................................................. 83
TABLE 2 A Summary chart of students work on the instrument ....................... 85
LIST OF FIGURES

PART I

FIGURE 1 Multiple representations of a given linear function ...................................... 3
FIGURE 2 Example of a dynagraph ............................................................................. 6
FIGURE 3 Example of a dynagraph with a slope of 2 ................................................. 6

PART II

FIGURE 1 Example of a symbolic representation of a linear function ......................... 22
FIGURE 2 Example of a graphical representation of a linear function ......................... 22
FIGURE 3 Example of a tabular representation of a linear function ............................ 23
FIGURE 4 Graphical representation of the SOLO Taxonomy ...................................... 32

PART IV

FIGURE 1 Summary of John’s matches on the task ................................................... 57
FIGURE 2 John’s coordination chart ........................................................................... 58
FIGURE 3 Summary of Rita’s matches on the task .................................................... 59
FIGURE 4 Rita’s coordination chart ............................................................................ 60
FIGURE 5 Summary of Jen’s matches on the task ..................................................... 60
FIGURE 6 Jen’s coordination chart ............................................................................ 61
FIGURE 7 Summary of Ama’s matches on the task .................................................. 61
FIGURE 8 Ama’s coordination chart .......................................................................... 62
FIGURE 9 Summary of Shama’s matches on the task .............................................. 63
FIGURE 10 Shama’s coordination chart ..................................................................... 64
PART V

FIGURE 1 A sample student worksheet on item 1.................................................. 86
FIGURE 2 Ama’s worksheet on item 1.................................................................. 86
FIGURE 3 Jen’s worksheet on item 1................................................................. 87
FIGURE 4 A sample student worksheet on item 2........................................... 87
FIGURE 5 A sample student worksheet on item 3........................................... 88
FIGURE 6 A sample student worksheet on item 4........................................... 89
FIGURE 7 A sample student worksheet on item 5........................................... 90
FIGURE 8 A sample student worksheet on item 6 ........................................... 91
FIGURE 9 Ama’s worksheet on item 6.......................................................... 92
FIGURE 10 Ama’s translation chart............................................................. 93
FIGURE 11 John’s translation chart ............................................................. 94
FIGURE 12 Jen’s translation chart................................................................. 95
FIGURE 13 Rita’s translation chart............................................................... 96
FIGURE 14 Shama’s translation chart......................................................... 97
INTRODUCTION

Multiple representational approaches entail the explication of notions and ideas associated with mathematical concepts (such as functions) through more than one of its representations (table, graph, symbolic). For example, such an approach will explicate slope (m) as a single quantity signified in three different ways; a constant difference among y-values in the table, invariant angle of line in relation to the horizontal axis for the graph, and the coefficient of the x variable in the symbolic. Furthermore procedures such as figuring out rise over run (in the graph), finding the consistent difference among the “y” \( f(x) \) for every unit increase in the “x” (table) or identifying the coefficient of x (in the symbolic) are emphasized for establishing the abstract equivalency of this quantity in all three representations of the function.

Research that has permeated reform-oriented curricula over the past two decades suggests that utilizing multiple representational approaches to highlight connections among representations such as graph, table and the symbolic may help students develop strong conceptions of notions and ideas associated with mathematics concepts (Even, 1998; Porzio, 1998; Knuth, 2000a, 2000b; NCTM, 1989, 2000; Moseley & Brenner, 1997). Such a viewpoint seem to equate, or at least closely approximate, conceptual understanding with flexibility in making connections among representations (Hilbert & Carpenter, 1992; Janvier, 1987; Kaput, 1989; NCTM, 1989, 2000). While these connections have being posited as a center piece of mathematical understanding (Brenner, et al, 1997; Knuth, 2000; Piez & Voxman, 1997; Porzio, 1999) it is less clear if emphasizing them during instruction necessarily ensures that students develop such an understanding for an algebraic concept such as linear function more so connect among
standard representations of the concept.

The concept of linear function is a major concept in first year algebra and represents the first formal example of function student’s most likely encounter. Knuth (2000a) contended that the study of linear functions sets the stage for more advanced work in school mathematics. Constructs of linear functions that seem of essence are “slope”, “y-intercept” (Moschkovich, 1996, 1998, 1999) and relation between dependent and independent variables. These ideas stem in part from the coordinated images of two independently varying linear quantities, “x” and “y” (Carlson, 1998, 2002), (where x and y co-vary) and a focus on the relations among these quantities. For example, the slope (m) is the invariant ratio between “y” and “x” increments; slope can thus be conceived as an invariant rate of change across co-variation. The “y-intercept” (c), could be described as the value of “y” when the coordinated value of ‘x’ is zero.

These ideas are signified in the standard representations commonly used to represent the concept. A foundational conception of linear function has been linked to an ability to anticipate connections among these representations in given situations of the concept (Carlson, 1998). Consequently, this dissertation focuses on the nature of students’ connections among standard representations of linear function after having been explicitly taught functions via a multiple representational approach. This is done via through two different studies; the first focuses on students’ coordination among representations of linear function, the second focuses on students translations between representations of linear function.

**Definition of Key Terms**

To clarify the nature of the research problem, I define key terms and phrases used
in the previous and subsequent sections. This section serves the purpose of explicating those terms and phrases.

The term “multiple representations” as used in the study refers to external mathematical embodiments of ideas and concepts to provide the same information in more than one form (Ozgun-Koca, 1998). For example, in figure 1 (the case of the function, \( f(x) = 2x + 6 \)), the term would refer to a combination of at least 2 standard representations (such as, the table, the graph, the symbolic) to provide the same information (e.g. \( y \)-intercept = 6) in more than one form (e.g. where it crosses the vertical axis; the output \( f(x) \) corresponding to an input value \( x \) of 0; value of \( f(x) \) when \( x = 0 \))

![Figure 1. Multiple representations of a given linear function](image)

The phrase “making connection among multiple representations” will be used in reference to the explicit act of anticipating and abstracting the same information in structurally equivalent but apparently different representations (i.e. informationally equivalent representations). For example, anticipating slope as an invariant quantity in the graph (rise over run), table (invariant ratio of increment) and symbolic representation
(coefficient of x) of the function. This would entail not only being able to flexibly move from one representation to another (translation) but also to anticipate the invariance of constructs across the different representations (coordination).

A “coordination” would be used in reference to the process involved in matching two equivalent representations together. In this study, a coordination would be specified by two components; the two representations being related or matched and the invariant property or construct used (Table 1).

Table 1. Components of a coordination

<table>
<thead>
<tr>
<th>Representations related</th>
<th>Invariant property used</th>
<th>Type of Coordination</th>
</tr>
</thead>
<tbody>
<tr>
<td>Graph &amp; Symbolic (GS)</td>
<td>Slope (sL)</td>
<td>GS-sL</td>
</tr>
<tr>
<td>Symbolic &amp; table (ST)</td>
<td>y-Intercept (yInt)</td>
<td>ST-yInt</td>
</tr>
<tr>
<td>Table &amp; dynagraph (TD)</td>
<td>Input/output points (pT)</td>
<td>TD-Pt</td>
</tr>
</tbody>
</table>

For example, in a “GS-sL coordination”, the representations matched/related are graph & symbolic (GS) and the invariant property or construct used is slope. Similarly in an “ST-yInt coordination” the representations related are symbolic and table and the invariant property used is y-intercept. Being able to successfully perform a specified coordination will thus entail being able to correctly relate (match) the associated representations together via the specified invariant property or construct.

The term “translation” would be used in reference to the process involved in going from one form of representation to another (Janvier, 1987). A translation will thus always involve two modes of representations; a given source representation and a specified target representation. Performing a translation (e.g., $G \rightarrow T$) in this regard will
entail interpreting the meaning of the given source representation (e.g. Graph) and producing an equivalent target representation (e.g. Symbolic) with the same meaning. Being able to successfully perform a specified translation will thus entail success in mapping constructs of the specified source representation onto constructs of the specified target representation. In this light, performing a reversible translation (complementary translation) would entail translating from a given source representation to a specified target representation and vice versa; when position of source and target are switched.

The term “standard representations” will be used in reference to the three representations of functions commonly employed in a MATH101 course (i.e., symbolic, table, graph). These standard representations are employed because they are associated with linear functions in most mathematics texts. The term “non-standard representations” on the other hand will be used in reference to representations of functions that are not likely to be encountered by students in a MATH101 course. A typical example of such a representation is a dynagraph (Goldenberg, et al., 1992).

**Description of Dynagraph**

A dynagraph (Figure 2) is a computer screen based on abstraction of functions as co-variation that can be dynamically manipulated (Figure 2). Unlike the graphs on a Cartesian coordinate system, a dynagraph consists of two horizontal parallel number lines, one directly on top of the other. One of the number lines represents domain variables and the other represents the image (co-variables). In a dynagraph, the domain variable (point B) can be dynamically manipulated by the mouse as the user moves a cursor on one number line. This manipulation is coordinated with the image variable (f(B)), according to a rule that can be configured in advance.
Figure 2: Example of a dynagraph

Configuring the magnitude and direction of movement of these two variables (domain and image) is used to determine the type of dynagraph represented. For example, if every movement in the domain is associated with twice as much corresponding movement in the image variable (same direction as the domain), a linear function with slope 2 is represented (Figure 3).

Figure 3: Example of a dynagraph with a slope of 2

Problem Statement

Students encounter the concept of function in one form or another from grade one through college. Yet function remains one of the most elusive and difficult of concepts in school mathematics. Research studies over the years have sought to investigate ways in which student’s difficulty with the concept could be mitigated (e.g. Dufour-Janvier, Bednarz & Belanger, 1987; Hitt, 1998). Reported findings from these and other studies
suggest that being able to connect among multiple representations is essential to a foundational conception of function (e.g. Brenner et al 1999; Hollar & Norwood, 1999; Keller & Hirsch, 1998; Knuth, 2000a; Knuth, 2000b; Rider, 2004).

In light of this, reform curricula have advocated for instructional approaches that help students make these connections. Even though multiple representational approaches have emerged as a leading candidate for helping students to make these connections, it is less clear if it is sufficient. Whether students in a course that employs these approaches to teach function concepts would necessarily be able to make these connections for standard representations of linear function and whether these connections once developed is a sufficient indicator of a foundational conception of linear function remains a contentious issue. More importantly it is less clear the nature of students’ connections after experiencing such an approach to function instruction. This current state is problematic, because it does not clearly distinguish between a students’ connections as a case of conceptual understanding or of a desired procedural skill resulting from students work with these representations as ends in themselves (Dufour-Janvier et al. 1987).

For example, students may be able to determine zeros of given linear function and may correctly develop procedures for abstracting zeros for the table and even connect zeros to the x-intercept in a graph but may not really have the understanding to explain why two or more given linear functions are similar or different. Also students may develop invariant strategies for moving from one representation to another but may not understand the underlying theme behind a translation-an essential attribute in problem solving success and mathematical understanding. Thus focusing instruction on connecting among multiple representations and not knowing its’ effect on students’
conceptions may not have the desired impact on students; this is because they may lack the foundational conception required for meaningfully operating with the associated constructs of the concept.

This apparent gap in the literature in an era where these connections have emerged as a center piece of mathematics instruction and function understanding is rather appalling. The argument is that unless teachers have a way of distinguishing between the essences of students’ connection when multiple representational approaches are employed in instruction, they are going to be less likely to foster the type of foundational conception needed for students to successfully work with notions and ideas under-girding the function concept. Thus there is a need to further study the nature of students’ connections after going through a course that emphasize multiple representational approaches as a pathway to their understanding of associated mathematical concepts. To delineate students’ connections, the study necessitates engaging students in both standard and non-standard contexts of linear function. A research of this nature would add to the literature on both learning and teaching via multiple representations because it will inform the mathematics education community of the reasoning processes and connections students make when instruction focuses on multiple representational approaches.

**Purpose of the Study and Research Question(s)**

In this study, the researcher explores students’ work, actions, and articulations on test and task situations of linear function. The purpose of the study was to investigate the nature of students’ connections among standard representations of linear functions after having completed a course that emphasizes multiple representational approaches as a pathway to their understanding of associated function concepts.
The study was guided by the questions below;

- What is the nature of students’ coordination among representations of linear function?

- What is the nature of students’ translation among standard representations of linear function?
REVIEW OF LITERATURE

The function concept has been distinguished as an essential component in mathematics (Selden & Selden, 1992). Although the concept remains one of the most prominent in a students’ mathematics education, studies indicate that many students emerge from freshman-college and developmental college courses with documented difficulties with the concept (Cooney & Wilson, 1993; Carlson, 1998, 2000; Thompson, 1994a). These difficulties have been blamed in part to a lack of foundational function conception—one that allows for meaningful interpretation and use of functions in different representations and settings (Hitt, 1998; Knuth, 2000).

In order to address this current state, numerous instructional approaches have been suggested by the mathematics education community. One such approach that has gained in prominence over the years is the use of multiple representations to emphasize connections among the table, the graph and the symbolic (NCTM, 1989, 2000). It has been argued that these connections are at the center of deeper and flexible understanding (Keller & Hirsch, 1998; Porzio, 1997, 1999; Even, 1998; Cunningham, 2005). This chapter examines ideas and research pertaining to functions and multiple representations. The examination is done in two main sections; the first explores issues pertaining to the concept of function; the second elaborates on ideas, issues and research pertaining to multiple representations. The chapter also explicates the theoretical foundation on which the study is based and the framework utilized in analyzing data for the study.
The Function Concept

Introduction

The function concept has been described as central to mathematics and to students’ success in mathematics (Selden & Selden, 1992). The importance of functions was brought to the fore front of the mathematics education and research community from as early as the 1920’s. At which time The National Committee on Mathematical Requirements of the Mathematical Association of America recommended that functions be given a central focus in school mathematics (Cooney & Wilson, 1993). In the light of this, an understanding of the concept was deemed as necessary for any student hoping to comprehend mathematics at the college level (e.g. calculus and differential equations) (Carlson, 1998). Thus one of the goals of undergraduate mathematics was to develop in students a sense for function (Eisenberg, 1992).

Understanding function entails among other things, understanding ways possible to represent and connect functions (Rider, 2002). Sierpinska (1992) argued that such an understanding of functions emanates from being able to anticipate and coherently articulate in any given representation of the concept, the quantities that were co-varying and the invariant or regularity governing their variation. For example in the case of the function, “y = 2x”, this may entail being able to; anticipate what changes (x, y) and by how much (e.g. x is changing by 1 while y is changing by 2); coordinate the changing quantities (e.g. for every unit change in x there is an associated corresponding 2 unit change in y); identify regularities or invariant relation between the changing quantities (e.g. the ratio of the changes of y to x is invariant and equal to 2 in the case of f(x)=2x ). This ability has been described as co-variational reasoning (Saldhana & Thompson, 1998;
Thompson, 1994; Carlson, 1998; 2002). This entailed being able to co-ordinate the images of two varying quantities and attend to the ways they change in relation to each other.

This study focuses on students’ connections among representations of linear function because this family represents the first formal example of function students’ encounter. Moreover linear function consists of the various aspects of functions identified by Selden & Selden (1992); definitions (ordered pairs, correspondence, dependency etc), representations (graph, table, symbolic, etc), and conceptions types (action, process, object). It also exhibits all the multifaceted structure identified by Dreyfus & Eisenberg (1992) as under-girding function; that is sub-concepts (e.g. domain, range, pre-image, image, inputs, outputs, etc), and representations (e.g. tabular, graphical, symbolic, diagrams, mappings, etc). Linear function can therefore be considered as prototypical of the function concept and an understanding of linear function viewed as foundational to an understanding of the function concept (Moschkovich, 1998).

**Students’ Difficulty with Functions**

Numerous studies have been conducted on students’ understanding of the function concept (e.g. Briendenbach, Dubinsky, Hawks & Nichols, 1992; Carlson, 1998; Cuoco, 1994; Eisenberg & Dreyfus, 1994). Research over the years has documented students’ interpretations of functions (linear functions in particular) in their varied representations (Hitt, 1998, 1994; Knuth, 2000b). Studies indicate that while students often learn how to manipulate and use function ideas in tasks and problem situations, most students possessed limited knowledge on functions (Moschkovich, 1998, 1993; Vinner & Dreyfus, 1989). Many students, even those who were high performing and those who had taken a
fair number of mathematics courses experienced difficulties working with notions and ideas of function in the three standard representations used to signify the concept (Eisenberg & Dreyfus, 1994; Hitt, 1998; Nemirovsky, 1996).

For example, a study by Hitt (1998) focused on students conceptions of functions. Participants (n = 30) were students beginning a postgraduate course in mathematics education. The researcher reported that for most of these students their knowledge of functions was restricted to functions that were continuous and definable by a single symbolic expression. Data from the study indicated that these students were not able to anticipate sub-concepts of functions (e.g. domain, range, etc) in representations other than the symbolic and regarded symbolic expressions as the most significant part of functions.

Similar student conceptions have been reported in other studies on functions (e.g. Carlson, 1998; Even, 1990, 1993; Vinner & Dreyfus, 1989). Some of the more prominent ones include the belief that functions needed to be established by a symbolic rule; the view that different symbolic representations of functions that yielded the same values (e.g. \( f(x) = 2x \) and \( g(0) = 0, g(x+1) = g(x) + 2x + 1 \) were different functions (Sfard, 1992); the conception that different representations (e.g. graph, symbolic) of the same function signified different examples of functions; the belief that the only admissible representations of function were the graph and the symbolic; the conception that graphs of functions needed to be smooth and continuous (Dubinsky et al., 1989; Kieran, 1999; Sfard, 1992). These limited conceptions have been attributed to a compartmentalized view on functions, the result of an overwhelming reliance on the symbolic (Carlson, 1998, Vinner & Dreyfus, 1989).
Such limited conceptions didn’t allow for coherent articulation of the different representations used to signify the concept (Hitt, 1998, 1994). Research indicates that many students fail to comprehend notions and ideas associated with function in their varied representations and are unable to link these representations in given problem situations (Cunningham, 2005; Dreyfus & Eisenberg, 1987, 1991; Keller & Hirsch, 1997; Vinner & Dreyfus, 1989). Students’ difficulty in making connections among representations of function has been documented in several studies (e.g. Aspinwall, Shaw & Presmeg, 1997; Eisenberg & Dreyfus, 1991; Porzio, 1999; Sfard, 1992). Making such connections have however become associated with a foundational understanding of function (Moskovich, Schoenfeld, & Arcavi, 1993; Eisenberg & Dreyfus, 1986; Even, 1998).

Connections among Function Representations

There is increasing recognition among the mathematics education community that central to function understanding is an ability to make connections among representations used to signify the concept (Cunningham, 2005). Even (1998) reported that knowledge of the connections among representations of functions was intertwined with knowledge about the different approaches to functions, and knowledge of the underlying notions and constructs of the function concept. Similarly, Eisenberg (1992) asserted that being able to connect among the graph and symbolic representations was a main component of a robust understanding of the function concept. It is contended that deeper and flexible understanding of function entailed coherent articulation of the different representations used to signify the concept in specific tasks or problem situations (Hitt, 1998). At issue
then is the nature of this articulation for students who have been taught through multiple representational approaches.

A study by Galbraith and Haines (2000) revealed that students’ connections among standard representations of function were poor and that students performed poorly when having to move from the graph to the symbolic. Similarly Knuth (2000b) reported that student’ experienced difficulties translating between the symbolic and the graph and as a result failed to anticipate the link or connection between these representations in given problem situation. Research indicates that being able to translate between representations of associated concept in given problem situation was essential for problem solving success (Gagatsis & Shiakalli, 2004). Six different types of translation have been identified by research for the three standard representations of linear functions. These are in terms of their symmetric or complementary pairs (Janvier, 1987); graph to symbolic ($G\rightarrow S$) and symbolic to graph ($S\rightarrow G$); symbolic to table ($S\rightarrow T$) and table to symbolic ($T\rightarrow S$), graph to table ($G\rightarrow T$) and table to graph ($T\rightarrow G$) (Cunningham, 2005; Knuth, 2000a; Kaput 1989; Porzio, 1995).

Researchers contend that facility in performing complementary paired translations (i.e., reversible translations) was foundational to an understanding of function (Gagatsis & Shiakalli, 2004; Janvier, 1987). This facility however appears to be difficult for students to develop (Aisnworth, 1999). For example Schoenfeld et al (1993) reported even though a student developed facility in translating in the graph to symbolic direction, symbolic to the table direction and from the table to the graph direction during problem solving, the student was not able to perform translations that demonstrated reversibility (i.e., complementary pairs). At issue then is whether students in a course that employs
multiple representational approaches in instruction would necessarily be able to perform these complementary paired translations or reversible translations for standard representations of linear function.

**Multiple Representations**

Before the term “multiple representations” can be explained, a description of what representation(s) is (are) must be given. Representations are artifacts, objects or devices whether external or internal for maintaining a relation with an object or event in its absence (Duval, 1999; Kaput, 1989; Olson & Campbell, 1993). Goldin (1998) noted that a representation is any configuration that can denote, symbolize or represent something other than itself. Pratt & Garton (1993) argued that a central attribute of representations is that they are not just objects in themselves. Rather in their representational capacity they evoke something else hence they allow our thinking to be about a unique object or event and no other (Olson & Campbell, 1993). For example, graphs, tables and the symbolic may be considered as representations of linear function because they could be used to evoke constructs (such as slope, y-intercepts, input/output) pertaining to the linear function concept.

In any treatise of representations, Pratt & Garton (1993) contended that three important distinctions needed to be made, they were; distinction between external and internal representation; distinction between representation as a process and as an object; distinction between the extents to which the representation relates to the object it represents. In this study, the term representation is used solely in reference to an external object whose relationship with the objects they signify are established through shared
conventions (mathematical conventions). Examples of such representations include the
standard representations used to signify the function concept. Zhang (1997) argued that
these types of representations enable us to talk about mathematical relations, can be
easily exhibited or communicated to other people and are involved in most mathematical
tasks.

Multiple Representations; Definition and History

Ainsworth (1999) coined the term multiple representations for two or more of these
representations used simultaneously to signify the same object. Multiple representations
refer to “external mathematical embodiments of concepts to provide the same
information in more than one form” (Ozgun-Koca, 1998, p1). For example, the term may
be used in the case of a concept like rate of change to include difference quotients in the
tables, slopes of graphs in the Cartesian coordinate plane, and formal algebraic
derivatives in the symbolic (Porzio, 1999). These embodiments may be utilized to
provide the same information (e.g. y-intercept) in more than one form; the point at which
the line crosses the vertical axis in a coordinate plane; the output (f(x)) corresponding to
an input value (x) of 0 in the table; value of f(x) when x = 0 in the symbolic.

The idea of using multiple representations has been a recurring theme in the
mathematics education community for some time (Goldin & Shteingold, 2001). As far
back as the early 1920’s, the National Committee on Mathematical Requirements of the
Mathematics Association of America in their reorganization of mathematics report of
1923 recommended that students develop the ability to work with multiple
representations of algebraic and geometric concepts (Bidwell & Clason, 1970, pp. 403-
407). During the early 1960’s, Dienes’ (1960) suggested that mathematical concepts and
notions be presented in multiple forms in order for students to obtain the mathematical essence of an abstraction. Thompson (1994a) contended that though the importance of multiple representations in students’ understanding had been common knowledge for at least a century, it gained in prominence as a teaching tool only in the early 1980’s as a result of technological advances.

It was during this time that The National Council of Teachers of Mathematics (NCTM) published “Curriculum and Evaluation Standards for School Mathematics” (NCTM, 1989). The NCTM identified multiple representations as one of the key components in the curriculum that needed to be emphasized during mathematics instruction (NCTM, 1989). In particular, “all students should be able to represent and analyze relationships using tables, verbal rules, equations and graphs; translate among tabular, symbolic and graphical representations of functions” (p.154). In the NCTM’s more recent publication, “Principles and Standards for School Mathematics” there is a call for all students to be able to; create and use representations to organize record and communicate mathematical ideas; select, apply and translate among mathematical representations to solve problems; use representations to model and interpret physical, social, and mathematical phenomena (NCTM, 2000, p.67). NCTM advocated that all instructional programs (K-12) should help students develop the ability to make connections among multiple representations of mathematical concepts (NCTM, 2000).

Multiple Representations in Mathematics

Researchers have over the years argued about the centrality of multiple representations in mathematics (Brenner et al., 1999; Dreyfus & Eisenberg, 1982; Knuth, 2000; Piez & Voxman, 1997; Porzio, 1999). Greeno & Hall (1997) contended that
mathematics by its very nature was one of the academic subjects in which multiple representations was needed for communicating and understanding mathematical notions. Because unlike other fields of knowledge (e.g. physics, botany), there was no way of gaining access to the objects of mathematics except through these representations (Duval, 1999). Researchers argued that most mathematical activities involved multiple forms of these representations used in concert and in combination with one another (De Jong & van Joolingen, 1998).

The whole of mathematics is little else than exploring mathematical representations and their interrelations (Mckendree, Small, Stenning & Conlon, 2002). Dufour-Janvier et al. (1987) contended that mathematical teaching and learning, together with all its’ elements including curricula advertently or inadvertently exposes students of all ages and school levels to multiple representations of concepts. For example, graphs, tables and symbolic representations used in concert with concepts like linear functions were not linear functions per se but were conventional representations via which notion and ideas associated with linear function could be abstracted. In order for students to develop facility in abstracting the mathematical concept under-girded in each representation, Cunningham (2005) argued that instruction should formally allow students to work and reason with multiple representations of these concepts.

It is argued that allowing students to incorporate many different types of representations into their own sense making process would help them develop insights into understandings of the essence as well as the many facets of a concept and make them more capable mathematician (Brenner, et al, 1997). Even (1998) contended that such an access might help students identify and represent the same concept in different
representations, flexibly move from one representation to another, and this ultimately may allow them to develop deeper and more foundational understanding of associated concepts. Similarly, Keller & Hirsch (1998) admitted that the different concretizations possible with multiple representations were potentially beneficial in that, they selectively emphasized and de-emphasized different aspects and facets of complex concepts and facilitated cognitive linking of representations. It is believed by Piez & Voxman (1997) that students gained a more thorough understanding of function when the concept is explored via multiple representational approaches.

**Issues Associated with Multiple Representations**

Even though there is general agreement among math educators today on the need for students to work with and reason via multiple representations, how to effectively utilize these representations in mathematics instruction remains contentious (Even, 1998). Thompson (1994a) believed that the idea of multiple representations as currently construed was not carefully thought out. It has been asserted that the problem pertinent to multiple representations was not so much its importance in mathematics teaching and learning (Keller & Hirsch, 1998). Ainsworth & van Labeke (2004) noted that the issue is not whether multiple representations were effective but the circumstances that influence the effectiveness of its use. This issue as noted by Duval (1999) revolves around the question of how best to help students distinguish between the representations and the mathematical objects they purport to signify.

Researchers like Thompson (1994a) have suggested that the key to effectively utilizing multiple representations was to find situations that were sufficiently favorable for multiple representational activities and to orient students to draw connections among
their representational activities in regard to the situation that engendered them. Ainsworth (1999) argued for approaches that ensured that learners understood the semantics of each representation, which aspects of the concept were represented, how to relate the representations to each other, and how to translate between the representations. Duval (1999) similarly recommended methods that allowed students to compare multiple representations of the same concept, to connect multiple representations of the same concept and to discriminate the specific ways of working within each of the different representation.

**Connection among Multiple Representations**

Multiple representational approaches highlight and place emphasis on the connections among standard representations of concepts and on procedures for abstracting information from these representations (Porzio, 1997; Keller & Hirsch, 1998). These connections have arguably being said to develop deeper and more flexible understanding of mathematical concepts (NCTM, 1989, 2000). Even (1998) argued that multiple representations connectedness develop insights into understandings of the essence as well as the many facets of a concept. Some researchers contend that these connections allowed for abstraction of invariant attributes and aspects of concepts and as a result led to deeper and more flexible understanding of associated concepts (Ainsworth, 1999; Eisenberg & Dreyfus, 1992; Kaput, 1992).

For example, a concept like linear function, may be abstracted through the; symbolic as a dependency relationship (Figure 1); the graph as a coordination or mapping between input-output values (Figure 2); the table as a correspondence between a continuum (sets) of input and output values (Figure 3). Thus through connecting multiple
representations, it is arguably said that deeper and more flexible understanding of linear function would likely emerge (Even, 1998). Kozma (2003) noted that while experts were made to connect among multiple representations, novices on the other hand were most of the time constrained by the surface features of each individual representation. Ainsworth (1999) contended that this may arise due to the fact that when these approaches are emphasized, learners are faced with the complex task of not only understanding the form of the representations (how each representation encodes and present information) but also of understanding how these representations relate to the concept they represent and to each other.

$$F(x) = 2x + 6$$

Figure 1 Example of a symbolic representation of a linear function

Figure 2 Example of a graphical representation of a linear function
At issue then is whether emphasizing multiple representational approaches during instruction necessarily ensures that students’ develop an understanding of the format of each of the representations and of the invariance of associated constructs across representations. Thus a focus of this research is on the nature of students’ coordinations among standard representations of linear function after going through a course that emphasizes multiple representational approaches.

**Research on Multiple Representations**

A significant body of research has been published on the use of multiple representations in the teaching and learning of mathematics (e.g., Hollar & Norwood, 1999; Keller & Hirsch, 1998; Knuth, 2000a; Knuth, 2000b; McGowan & Tall, 1999). Some of these studies have focused on the impact of technology as a multiple representations generating tool on students’ understanding of function (e.g., Hollar & Norwood, 1999; Patterson & Norwood, 2004). Others have focused on the impact of curricula on students’ understandings of representations associated with function (Aviles,
2400; Rider, 2004; Brenner et. al., 1997; Rich, 1995; Porzio, 1997). Still other studies have concentrated on students’ preferences for representations (e.g., Keller & Hirsch, 1998; Piez & Voxman, 1997).

A central theme emerging from these and other studies is the issue of students’ connections among representations. Reported findings from these studies suggest that students with compartmentalized view of functions were limited in their ability to make connections among multiple representations of associated concepts (e.g. Carlson, 1998; Hitt, 1998; Knuth, 2000). Other findings suggested that successful problem solvers were necessarily successful at making connections among different representations in given problem situations (e.g., Brenner, Herman, Ho & Zimmerman, 1999; Tchoshanov, 1997). Studies in which these connections were specifically highlighted and emphasized (multiple representational approaches) however reveal mixed results (e.g., Aviles, 2004; Moseley & Brenner, 1997; Piez & Voxman, 1997; Porzio, 1999; Rider, 2004).

For example, a study by Tchoshanov (1997) explored the effect of different approaches to mathematics instruction on students understanding and connections among representations of trigonometric functions. Participants for the study were students from a pre calculus class in a high school in Russia. Students were put into three comparison groups; the first (“symbolic group”) was instructed using symbolic representations; the second (“visual group) was taught using only visual representations; the third and final group (“representational group) was taught via multiple representational approaches. The researcher reported that on a given test on trigonometric functions and proof, the representational group out scored students in the other groups; they scored 26% higher than the visual groups and 43% higher than the symbolic groups.
In a similar study, Porzio (1997) examined and compared the effects of three different instructional approaches on students understanding and connections among representations of function. Participants for the study were undergraduate students (n = 100) chosen from a large Midwestern University; one course (MA151), with 40 students utilized symbolic representations for instruction; the second (MA151G) with 24 students utilized all three standard representations of function; the third (MA151C) with 36 students also utilized these standard representations of function but placed special emphasis on students connections among these representations. On a test designed to assess knowledge and connections among standard representations, students’ from MA151C were better than students from the other groups at anticipating and connecting among graph, table and symbolic representations.

A study by Rich (1995) explored the effect of multiple representational approaches on the learning and retention of mathematical concepts. Participants were students (n = 59) from three regularly scheduled high school calculus classes; a traditional class with 20 students (control group); a class with 21 students (treatment group1) where different representations are used but the connections among these representations not highlighted nor emphasized; and a class with 18 students (treatment group2) where connections among these representations were highlighted and emphasized. On the post-test, the researcher observed no significant differences between students in the three groups. On the surprise retention-test, however the researcher reported that students in the treatment groups demonstrated significantly higher retention ability than students in the traditional class. Students in the treatment groups tended to show larger effects at the one-week retention posttest than at the immediate posttest.
A study on the impact of multiple representational curricula on students understanding of functional relationships was conducted by Brenner, Mayer, Moseley, Brar, Duran, Reed & Webb (1997). Students (n=128) from three junior high schools in Southern California participated in the study. These students were placed into treatment and control groups; students in the treatment group (n=72) were taught using a unit of reform curriculum that emphasized connections among multiple representations; students in the control group (n=56) were taught using a traditional curriculum. On a test instrument made up of routine problems, non-routine problems and linking representations problems, the researchers reported that students in the control group produced significantly greater pre to post-test gains than did the treatment group (p < 0.01) on routine problems. In the case of the “non-routine problems” and the linking representation problems however, students from the treatment groups produced greater pre to posttest gains than did the control group (p < 0.01).

Similar results were reported in a study conducted by Aviles (2001) on students understanding of linear function. The researcher compared the mathematical performance of students in two classes of a college algebra course at a University in Puerto Rico. The focus of the study was on the impact of multiple representational approaches (via technology) on students understanding of linear function. Participants for the study were college students (n=52) assigned to either a control group (n=23) or a treatment group (n=29). The control group was taught via a traditional curriculum while the treatment group was taught via multiple representational approaches. The researcher reported that on a test given to both groups before and after instruction, the treatment group performed
significantly better than the control group on items assessing their ability to connect among the graph, table and symbolic representations of linear function.

While the above mentioned studies seem to suggest an increase in the levels of student connections when multiple representational approaches are emphasized (e.g. Porzio, 1999; Aviles, 2004; Moseley & Brenner, 1997), other studies seem to indicate otherwise. For example, in one such study, Piez & Voxman (1997) reported that of the 20 students in a class where multiple representational approaches were emphasized, only one was able to correctly work with both graphs and symbolic representations in given problem situations. A similar study conducted by Ozgun-Koca (1998) revealed that, even though students acknowledged that mathematics problems could be solved in a variety of ways via different representations, most of them found it easier to work with one representation than to deal with multiple representations. Similar finding was reported in a study by Poppe (1993). The study explored the effects of multiple representational approaches to calculus instructions on students understanding and connections among representations during problem solving. The researcher reported that although students realized that tables, graphs and other representations were equally important when solving given problem situations, students failed to use them in unfamiliar problem situations unless suggested to do so.

Thus a question that remains in light of this is the nature of students’ connections among representations after having experienced instruction that emphasizes multiple representational approaches. What could not be ascertained form these studies is the nature of connections students were making among these representations when instruction emphasizes multiple representational approaches. Thus a focus of this
research is on the nature of students’ translations and coordinations among representations of linear function after experiencing instruction that emphasizes multiple representational approaches.

**Conceptual Framework**

This section discusses both the theoretical perspective (basis) under-girding the study and the conceptual framework via which the results of the study would be interpreted.

**Theoretical Foundation**

The research is based on constructivism as a theory of knowing. This view on knowledge and how we come to know is shared by a majority of researchers (Janvier, 1987). The study is grounded in a radical orientation to constructivism (von Glasersfeld, 1984). This orientation is under-girded by the perspectives that; knowledge is actively constructed by the cognizing subject rather than passively transmitted by way of communication or through the senses; the function of cognition is adaptive tending towards fit or viability; cognition serves the subject’s organization of the experiential world not the discovery of an objective reality (von Glasersfeld, 1995, p. 50).

These perspectives represent the theoretical foundation upon which this study is built. The first perspective embraces the idea of knowledge as the result of our own construction. von Glasersfeld (1974, 1984) contended that knowledge was the invariant that arose as the cognizing subject mutually or cyclically balanced changes in his (her) perceptual view. For example, “knowledge of function” could be described as an invariant a learner abstracts, from distinguishing between and coordinating among records of experience of representations and examples of function. The second and third
perspectives embrace the idea of knowledge as an instrument of adaptation that helps the learner to avoid external perturbations and internal contradiction (von Glasersfeld, 1980, 1999; Goldin, 1999; Noddings, 1999).

Knowledge was thus a repertoire of actions and thoughts, an invariant, a cognizing subject constructs as he (she) sorts through records of experience, and which in the past have been viable for making sense of and in dealing with experiential situations (von Glasersfeld, 1987; 1999). Sfard (1992) uses the term conception for a learners’ knowledge with respect to a concept. A conception therefore was not merely a store of facts or procedures to be obtained, but rather an anticipated regularity in the learner’s experiential world (Simon & Tzur, 2004). For example, a conception of linear function may be the anticipated regularity a learner forms from coordinating the images of two independently varying linear quantities.

The regularity in this case may be that; any increment in one of the variables considered as independent is associated with a corresponding consistent increment in the other variable considered as dependent on the other variable (co-variation); the consistent ratio between dependent and independent increments is invariant and is quantifiable by a fixed number (slope); dependent and independent variable values are related by a rule or regularity. A mathematical conception in this light was thus the anticipated invariant or repertoire of actions and thoughts a learner constructs for making sense of representations and constructs pertaining to a concept.

The study focuses on students’ conceptions on linear functions, the specific focus is on students connections among representations of linear functions (Duval, 1999) and their anticipations on constructs pertaining to linear function (Moschkovich, 2000).
Hiebert & Carpenter (1992) contended that insights into such conceptions may be gained via a focus on students’ work with external representations in task or problem situations. Thus by analyzing students work and articulations on task and problem situations involving representations of linear function it is conceivable that the nature of their connections and anticipations for the linear function may be inferred; the SOLO Taxonomy (Biggs & Collis, 1982) was employed as the framework for this analysis.

Framework

In order to better understand the nature of students’ connections among standard representations of linear function, research literature necessitated an analysis of students’ translations between and coordinations among these representations of linear function (Cunningham, 2005; Hitt, 1998; Duval, 1999). The framework employed for the analysis thus served as a lens via which students’ actions, articulations, anticipations and work with standard representations could be elucidated and the nature of their connections characterized. The SOLO Taxonomy (Biggs & Collis, 1982) was employed as the framework for characterizing students’ connections among representations of linear function. The acronym SOLO, which stands for Structure of Observed Learning Outcome, identifies characteristics of increasing quantity and quality of thought and categorizes mental activity by the observable product of student work.

Biggs & Collis (1991) contended that as the depth of a student’s knowledge increased, the quality of work the student produced as outcomes of their learning also displayed similar levels of increasing complexity. These levels of complexity are represented by five qualitative categories in the taxonomy; prestructural, unistructural, multistructural, relational and extended abstract. Figure 4 provides a graphical description
of the different levels in order of increasing complexity (adapted from Biggs, 1999).

![SOLO Taxonomy Diagram](image)

Figure 4 Graphical representation of the SOLO Taxonomy

A student at the prestructural level in relation to an assessed component or a questioned skill lacks knowledge of the element being assessed. A student at this level maybe easily misled or distracted by irrelevant aspects or detail of the assessed component and as a result may not be able to correctly complete any aspect of the component. A student at the unistructural level in relation to an assessed component or a questioned skill would only be able to correctly complete just a single aspect of the component or questioned skill. Such a student is able to focus on only one aspect of the element being assessed and is not easily distracted by irrelevant aspects. A student at the multistructural level in relation to an assessed component or a questioned skill would be able to correctly complete more than one aspect of the component but may not be able to relate these aspects together. Such a student is able to focus on several separate aspects of the element being assessed but is unable to anticipate the relationship between these aspects.
A Student at the relational level in relation to an assessed component or a
questioned skill would be able to not only correctly complete more than one aspect of the
task but also relate the different aspects of the task together. Such a student has an
understanding of more than one aspect of the element being assessed and of their
significance to the whole and relationship to other aspects. A student at the extended
abstract level in relation to an assessed component would be able to not only correctly
work with and relate the different aspects of the component together but can also
generalize to new and novel situations. Students at this level may be described as being
able to see the element being assessed from an overall point of view.

In this research, the elements or components assessed were translations between
and coordinations among representations of linear function. When applying the SOLO
Taxonomy to the assessed components, students at either the unistructural or
multistructural level would be described as having developed procedural conceptions of
the element being assessed. These students may arguably be said to have developed
compartmentalized notions and ideas (conceptions) of the elements being assessed. Such
conceptions allowed them to successfully work with the elements only in specific
instances or situations. Students at either the relational or extended abstract level on the
other would be described as having developed foundational conceptions of the elements
being assessed. These students may arguably be said to have abstracted regularities (from
their experiences) that allowed them to successfully anticipate the element being assessed
irrespective of the situation or instance.

The SOLO Taxonomy has been adapted for different research studies. For
example; to determine students’ cognitive development in algebra (Crowley, 2001), to
evaluate students understanding of chance in statistics education (Watson & Moritz, 1998); to evaluate learning outcomes in counseling education (Burnett, 1999); to evaluate the impact of curricula on students’ connections among function representations (Rider, 2004). In this study, the SOLO Taxonomy was utilized as a lens for viewing students’ work on the given test (Appendix F) and task situation (Appendix B).
METHODOLOGY

This chapter provides information on the specific research methods employed in the study. The design of the study as well as aspects such as the rationale for the research method, pilot study, participants, contexts, data collection and analysis procedures are discussed.

Rationale

In order to make knowledge claims about students’ mental processes or conceptions, it was necessary to observe and investigate the mathematical activities and communications of students (Creswell, 2003, pp. 20). This was needed for understanding and inferring students’ meanings, knowledge structures, and cognitive processes (Goldin, 1998) with respect to linear function. Ernest (1998) argued that the general research tradition that allowed for such investigations was qualitative research. Qualitative research is an approach to research in which the researcher as a primary instrument of data collection seeks to establish the meanings of phenomenon based primarily on the views, experiences and perspectives of participants (Punch, 2000; Eisner & Peshkin, 1990; Strauss, 1990). It’s strategies of inquiry include narratives, phenomenologies, ethnographies, grounded theories and or case studies (Creswell, 2003; Stake, 1994, 1995). The specific strategy of inquiry employed in this study meets Creswell (1998) standard for a case study.

In general case studies are used to explore the interpretative and subjective dimensions of a phenomenon (Strauss, 1990; Stake, 1994). As students’ conceptions are observable, subjective and multifaceted, this strategy of inquiry offers a viable means through which to make inferences about another person’s conceptions. It is argued that
such cases could contribute to the extension of a theory through supporting existing principles or challenging those principles. Case studies involve an in-depth exploration of a program, an event, an activity, a process or one or more individuals using a variety of data collection procedures such as open-ended interviews, direct observation and documents (Creswell, 2003, Stake, 1995).

This study employed a set of case studies of students work and reasoning processes on given tests and task situations involving linear functions. This method enabled the investigation of students’ translations between and their co-ordinations among representations of linear functions. Through students’ articulations and written work on given task and test items, the study addresses the nature of students’ connections among representations of linear functions. These interviews and tasks were open ended and were used as a means through which students’ mental processes could be elicited and their capability to successfully complete the given tasks determined; an interview protocol (Appendix B) was used to guide the interview process. Goldin (1998) argued that task-based interviews allowed for more in-depth questioning of subjects and helped alleviate some of the difficulties students may have with the questions posed. This data generating tool was tested in a pilot study during the summer session of 2005.

Pilot Study: Usefulness of instruments

The pilot was designed to determine the effectiveness of these instruments in generating the data needed to answer the research questions. Students (n=10) between the ages of seventeen and nineteen voluntarily elected to take part in the pilot study. These students were enrolled in a section of an Intermediate Algebra through Multiple Representations course in a large southeastern university at the time of the study. The
pilot proceeded through a series of one-on-one interviews with the students as they completed the given task.

Students were prompted to articulate their thinking processes as they worked on the problems, which involved both standard and non-standard representations of linear function. The task featured had two main parts. First students were asked to match linear function examples based on their standard representations then on their dynagraphs (Goldenberg, Lewis, & O’Keefe, 1992). Second, students were asked to match these representations of linear function (both standard and dynagraphs) using any preferred order for matching. Data from the pilot were analyzed first in terms of students’ work with standard representations and then the dynagraph.

Findings from the pilot indicated that students who participated in the study were more successful in matches involving the graph, table and symbolic representations than in matches involving the dynagraph. A notable finding was that students who successfully matched all three standard representations were more successful at working with the dynagraphs. It was observed that these students were using their knowledge of these standard representations to constrain their interpretation of the dynagraph. Though the available instrument provided data to account for whether or not students could match the given representations of linear function, it could not be used to adequately account for the nature of students’ connections among these representations.

In the pilot, the instrument failed to provide conclusive evidence on students’ translations and coordinations among the given representations of linear function.

In order to address this, a test instrument (Appendix F) addressing the different translation tasks identified in the research literature was incorporated in the study design.
The instrument (Appendix F) was employed to ascertain students’ strategies in performing reversible translations (complementary pairs) between standard representations. Also the task and interview protocol used to guide the interview process were adjusted to allow a focus on not only students’ matches but also their coordination among the given representations of linear function. This addition and adjustment were done for the explicit purpose of affording students ample opportunities at concept articulation and allowing the researcher opportunities at gaining insight into the nature of students’ anticipations. This study built on the pilot by examining students’ matches as well as their coordinations, and translations between representations of linear function.

The Researcher’s Role

My perceptions on multiple representational approaches have being shaped by my experiences. From the period of spring of 2001 to spring 2004, I served as an instructor in sections of an intermediate algebra through multiple representations course; this course is the exact same course reported in this study. As an instructor, I utilized multiple representations approaches to emphasize connections among representations of functions and to help students develop anticipations for these connections for functions. I believe my experiences and interactions in the course enhanced my awareness of and sensitivity to students’ understanding of these connections especially when the mode of instruction placed continual emphasis on connecting among such representations.

Due to these experiences, I bring certain biases to this study. Though every effort for ensuring objectivity will be done, I am of the opinion that these biases may in a way shape the way I view, understand and interpret the data I collect. I come to this study with a perspective that connections are essential for conceptual understanding and that
multiple representational approaches may represent an effective means for enhancing the connections that students make among these representations. Also because the theoretical foundation for the study is premised on the perspective that no individual has direct access to anyone else’s knowledge only to knowledge or experience the individual himself construct of those experiences (radical constructivism). I am of the opinion that any findings I end up with on the nature of students coordinations or translations however withstanding are personal models that I myself construct based on my experience with these students and as a result may only closely approximate the nature of conceptions developed by these students on linear function.

**Main Study**

**The Context: An Intermediate Algebra Course**

Students in sections of an Intermediate Algebra through Multiple Representations course at a large southeastern university in the United States were selected for the study. While the course does not satisfy a mathematics requirement or credit towards a degree, a pass in the course is required for enrollment in further mathematics courses and for graduation. That is, knowledge and skills acquired in the course are considered essential for work in successive courses. Enrollment in the course was based on students’ scores on a mathematics placement test given by the university. Students who scored below the university’s stipulated cutoff point were automatically enrolled in the course. A basic requirement for all students enrolled in the course was that they must have completed Algebra I and II in their respective high schools and should have taken the pre-course assessment given on the first day of class. This was used to gather baseline information
on their level of understanding and to ensure that their placement in the course was warranted.

The course was based on a functional approach to algebra instruction, in which the emphasis is on representations of functions (e.g. table, graph, symbolic) and the associated connections among these representations. A typical instructional unit in the course embodies the use of graphical, symbolic and tabular representations to explicate the notions and ideas associated with different function types. For instance, instead of solely utilizing the symbolic approach to develop function concepts, students were presented with tables and graphs at the same time as the symbolic, with instructional focus on their understanding of how these representations are linked. This approach is based on the assumption that by exposing students’ to different ways of signifying and connecting function ideas and representations they may be able to discern and abstract invariant relations and procedures across these representations and use that to develop foundational conceptions (Thompson, 1994a).

The Participants: Intermediate Algebra Students

The participants in the study were selected from a pool of students in several sections of the course. Guided by the researcher, instructors in the various sections acted as recruiting agents. Students in each section were informed of the study by their instructors during class time and ten volunteer students were selected from the available pool. The instructors were not to include in the pool students with poor attendance. Also nontraditional students were excluded from the study to ensure that students age was not variable in the study. Out of the total pool of volunteer students selected, only eight took part in all aspects of the study.
The results of the study are based on the work of five (Shama, Ama, Jen, Rita, John) of the total pool of eighth students. These students were selected because of their differing ability levels (based on their class scores and teacher recommendation) and because they provided elaborated explanations of their work and actions and coherently articulated their reasoning process. John was a first year college student enrolled in the MATH 101 course at the time of the study. The course instructor reported that John was low achieving and a likely candidate to repeat the course.

On the test given after the linear function unit, John score of 57% was well below the course average of 75%. John’s course grade at the time of the study was well below the C minimum required to pass the course. Rita was a first year college student enrolled in the MATH 101 course at the time of the study. The course instructor described Rita as an average student overall. On the test given after the unit on linear function, Rita scored 81%. Rita’s grade for the course at the time of the study was borderline C. Jen was a first year college student enrolled in the MATH 101 course. Jen was viewed by the course instructor as an above average student. On the test given after the linear function unit, Jen had a score of 90. Jen’s grade for the course at the time of the study was a B-.

Ama was a second year college student enrolled in the MATH 101 course at the time of the study. This was the second time Ama was taking the course, the course instructor described Ama as an above average student. On the test given after the unit on linear function, Ama had a score of 87%. Ama’s grade for the course at the time of the study was a B. Shama was a first year college student enrolled in the MATH 101 course at the time of the study. The course instructor reported that Shama was high achieving and self driven and one who was likely to get an A+ grade in the course. On the test given
after the linear function unit, Shama had a perfect score of a 100. Shama’s grade for the course at the time of the study was an A+.

Data

Three main sources provided data: Task instrument (Appendix B), test instruments (Appendix F), student worksheeets (Appendix C). The task instrument (Appendix B) was used to generate data on students’ coordinations among representations of linear functions (Keller & Hirsch, 1998; Even, 1998; Knuth, 2000). The two tasks on the instrument were open ended and were used as a means via which students’ mental processes could be elicited and their capability to successfully coordinate given representations of linear function determined. The tasks on the instrument were made up of problem situations involving both standard and non-standard representations of linear function. These problems were created with the Geometer’s sketchpad (Jackiw, 1991) and were posed to students via a computer screen.

Both task involved coordinating dynagraphs, graphs, tables and symbolic representations of linear function. The difference between the two tasks was that one task had examples of linear function of the same slopes but different y-intercepts while the other had examples of linear function with different y-intercepts but the same slopes. The two tasks were posed to students in the form of a task based interview. Students were required on these interviews to articulate their thinking processes and to answer questions posed by the researcher as they completed the task. In order to ensure consistency among participants, an interview protocol (Appendix A) was used.

These interviews were conducted one week to the end of course exam and were held in a location chosen by the researcher. This location was far removed from the classroom settings and was selected to ensure that students had access to the study
instruments. Students for the study were asked to select days and times that fitted their schedules. These time slots were coordinated so that the researcher could interview one student at a time; each interview lasted between 20 and 30 minutes. Each student was observed and interviewed as they worked on the given tasks. These observations were done for the purpose of gathering field notes (Bogden & Biklen, 2003) on students’ interactions with the task medium (computer screen) and for asking probing questions on specific actions employed by students. The interviews were used to elicit individual students’ conceptions and ways of working with representations of linear function. Students’ work on the task and interviews were videotaped for further examination.

Students’ worksheets (Appendix, C) were used to generate data on students’ answers to the given tasks. The worksheets were given to students in the course of the interviews and were used to follow their generation of solutions and to determine their success on the given tasks. These worksheets were structured in such a way that each of the tasks featured had a space allotted to it. An extra space was provided for any extra work by students. Each student was required to complete the worksheet by recording their matches on each task.

The test instrument (Appendix F) was utilized to generate data on students’ translations between standard representations of linear function. This instrument served as a means via which students strategies when moving from one mode of standard representation (given) to another (specified) could be explored and their ability to successfully translate between representations of linear function determined. The instrument was administered to students three weeks to their end of course exam. Students were required to show all work and answer all the questions posed on the
Data Analysis

Qualitative data analysis is viewed by Bogden & Biklen (2003) as a process of systematic searching and arrangement of interview transcripts and other data forms accumulated in the course of the study in order to come up with meaningful descriptions of participants’ thinking. To this end, data from the study were searched and arranged in a manner that allowed for meaningful interpretation of the nature of students connections among representations of linear function. The analysis was conducted in three phases; sorting, coding, rating. In the first phase of the analysis, student interviews were transcribed and coordinated with other forms of data collected (videotapes, audiotapes, worksheets, etc.). Each student data was then stripped off any form of formal identification and pseudonyms were employed. The various forms of student data were then organized into categories based on the research questions they addressed (Table 1).

<table>
<thead>
<tr>
<th>Categories</th>
<th>Data Sources</th>
<th>Questions Addressed</th>
</tr>
</thead>
</table>
| Coordination | 1. Work sheets  
            2. Videotapes  
            3. Field notes | 1. *What is the nature of students coordination among representations of linear function?* |
| Translation | 1. Work sheets  
            2. Test Instrument | 2. *What is the nature of students translations between the three standard representations of linear function?* |

The data was then analyzed by categories; a data analysis plan was devised for this purpose (Table 2). In the second phase, data within each category was analyzed. Each student’s articulations, strategies and actions within a category was analyzed and coded. The coded data on each student was then represented by category in charts; student data for the coordination category was represented in a coordination chart (Appendix D),
while student data for the translation category was represented in a translation chart (Appendix G). These charts facilitated the comparison of students’ data within each category.

Table 2: Data analysis methods

<table>
<thead>
<tr>
<th>Categories</th>
<th>Data Sources</th>
<th>Method of Analysis</th>
</tr>
</thead>
</table>
| Coordination| 1. Student work sheets  
2. Student videotapes  
3. Student transcripts | 1. Annotated notes of transcripts  
2. Coordination Chart  
3. SOLO rubric  
4. Summary Sheet |
| Translation | 1. Test instrument  
2. Student Work sheets | 1. Translation Chart  
2. SOLO rubric  
3. Summary Sheet |

In the third phase, student charts within each category were analyzed via the SOLO Taxonomy (Biggs & Collis, 1982). Two rubrics (Appendix E, Appendix H) based on an adaptation of the SOLO Taxonomy (Table 1) to the instruments in each category were used for this purpose. The specific rubric for each category applied the levels of the taxonomy to items on the associated instrument and gave exemplars of responses at the different levels on the SOLO corresponding to the assessed component.

Ethical Issues

In order to safeguard the rights of students who participated in the study, the following steps were taken. First and foremost, research permission was obtained form the institutional review board (IRB). Before the study, the research objectives were clearly explained to participants verbally and in writing in order to emphasize their role in the research. Students were allowed to participate in the study only after they had signed a consent form with the objectives and their rights clearly spelled out to them. Students were informed of all data collection instruments and activities. Data collected in the study
were stored in a secure and safe location by the researcher. Pseuodonyms were employed on all data reports to ensure anonymity of all students. Student rights, interests and wishes determined any decisions made in regards to reporting data from the study.

Limitations of the Study

I believe that one of the limitations of the study was the effectiveness of the study instruments in generating the relevant data need for the study questions. To minimize this factor, before the study was done, a pilot study was conducted with a comparable sample of students to check the instruments for their effectiveness. Modifications were done to these instruments as a result of the findings from the pilot study. The pilot gave me a better sense on ways to effectively utilize these instruments to gather much needed data.

Another limitation has to do with the way and manner students were sampled for the study. Even though students who volunteered for the study were screened and a purposeful sample selected, it was not possible to ensure that the students who volunteered in the first place were necessarily a purposeful sample of students in the course. As a result, research findings on these students may or may not necessarily reflect those of most students from the course.
CONNECTIONS AMONG REPRESENTATIONS: THE NATURE OF STUDENTS’ COORDINATIONS ON A LINEAR FUNCTION TASK

KWAKU ADU-GYAMFI
INTRODUCTION

Reform in the field of mathematics education with its focus on conceptual understanding is deeply rooted in finding ways of empowering students to learn to do mathematics (Thomasenia, 2000). A view on mathematical learning that has permeated research and reform oriented curricula over the past two decades seemingly suggests that central to mathematics understanding is facility in coordinating among multiple representations of associated mathematical concepts (Kaput, 1992; Ainsworth, 1999). This facility has been described as the cognitive architecture via which students can recognize the same mathematical concept in different representations (Duval, 1999; Even, 1998; Porzio, 1998; Knuth, 2000a; NCTM, 1989, 2000).

Facility in coordinating among multiple representations entailed among other things understanding the connection between the ways in which different representations exhibit the same properties of the mathematical concept. Ainsworth (1999) contended that this included being able to notice both regularities and discrepancies between representations of associated concepts and being able to relate these representations together via these regularities. While multiple representational approaches have been posited as a means for helping students develop facility in coordinating among representations of associated concepts (Brenner, et al, 1997; Knuth, 2000a; Piez & Voxman, 1997; Porzio, 1999) it is less clear if this is sufficient in the case of an algebraic concept such as linear function.

The concept of linear function is a major concept in first year algebra and sets the stage for more advanced work in school mathematics (Knuth, 2000b). Constructs related to linear functions that seem of essence are “slope” (m) and “y-intercept” (c)
(Moschkovich, 2000) and relation between dependent and independent variables. These constructs are signified in the three standard representations used to signify the concept. A foundational conception of linear function has been linked to an understanding that these constructs are not unique to these representations but to the function concept itself (Carlson, 1998). At issue then is the nature of students’ coordinations among these representations of linear function by students.

Consequently this study focuses on the coordinations of students’ in an Intermediate Algebra through Multiple Representations course among the dynagraph, graph, table, and symbolic representations of linear function. In this study the researcher explored the work and reasoning processes of students’ as they matched given representations of linear function. The purpose of the study was to investigate the nature of coordinations performed by these students’ among given representations of linear function. The question used to guide the research was;

1.) What is the nature of coordination performed among representations of linear function by students’ in an Intermediate Algebra through Multiple Representations course?

Multiple Representations

Multiple representations refer to “external mathematical embodiments of concepts to provide the same information in more than one form” (Ozgun-Koca, 1998, p1). For example, the term may be used in the case of a concept like rate of change to include difference quotients in the tables, slopes of graphs in the Cartesian coordinate plane, and formal algebraic derivatives in the symbolic (Porzio, 1999). A central reason for using these embodiments is to provide the same information (e.g. y-intercept) in more than one
form; the point at which a curve crosses the vertical axis in a coordinate plane; the output corresponding to an input zero in the table; the value of the independent variable when the dependent variable is zero in the symbolic.

The National Council of Teachers of Mathematics (NCTM) identified multiple representations as one of the key components in the curriculum that needed to be emphasized during mathematics instruction (NCTM, 1989). In particular, NCTM noted that “all students should be able to represent and analyze relationships using tables, verbal rules, equations and graphs; translate among tabular, symbolic and graphical representations of functions” (p.154). In the NCTM’s more recent publication, “Principles and Standards for School Mathematics” there is a call on all instructional programs (K-12) to help students develop the ability to make connections among multiple representations of mathematical concepts (NCTM, 2000). Though emphasizing connections among multiple representations have been recommended as a viable means for helping students to develop facility in coordinating representations of associated mathematical concepts (Porzio, 1997) it is less clear if this is sufficient.

For example, Kozma (2003) reported that while experts were able to coordinate and abstract invariant features from the various representations when these connections were emphasized, novices on the other hand, were most of the time constrained by the surface features of each individual representation. Ainsworth (1999) believed that when these connections were emphasized, learners were faced with the complex task of not only understanding the format of each representation but also of how these representations related to the concept they represented and also to each other. As a result they may fail to notice the regularities and invariance within and across these
representations and as a result experience difficulties coordinating among these representations. At issue then is the nature of students’ coordinations among multiple representations of linear function after completing a course where connections among multiple representations are emphasized.

**Function**

This study focuses on coordinations among representations of linear function because this “family” represents the first formal example of function students’ encounter. Moreover linear function consists of the various aspects of functions identified by Selden & Selden (1992); definitions (ordered pairs, correspondence, dependency, etc.), representations (graphs, tables, symbolic, etc.) and conception types (action, process, and object). It also exhibits the multifaceted structure identified by Dreyfus & Eisenberg (1992) as under girding functions, that is sub-concepts (e.g. domain, range, pre-image, image, inputs, outputs, etc.) and representations (table, graph, symbolic, diagrams, mappings, etc.). Linear function can therefore be considered as prototypical of the function concept and an understanding of linear function viewed as foundational to an understanding of the function concept (Moschkovich, 1998).

Numerous studies have been conducted on students’ understanding of the function concept (e.g. Briendenbach, Dubinsky, Hawks & Nichols, 1992; Carlson, 1998; Cuoco, 1994; Eisenberg & Dreyfus, 1994). Research has documented students’ interpretations of functions (linear functions in particular) in their varied representations (Carlson, 1998; Hitt, 1998, 1994; Knuth, 2000; Moschkovich, 1998, 1993; Vinner & Dreyfus, 1989). Studies indicate that while students often learn how to manipulate and use function ideas in tasks and problem situations, most students possessed limited knowledge on functions
(Briendenbach, Dubinsky, Hawks & Nichols, 1992; Cuoco, 1994). For example, students may be able to perform procedures such as determining zeros of given linear function from the table, graph and symbolic representations but may not really have the understanding to explain why two or more given linear functions are similar or different; an essential attribute in a coordination.

Many students, even those who are high performing and those who had taken a fair number of mathematics courses, lacked the foundational understanding needed for working with notions and ideas associated with functions in the different representations used to signify the concept (Carlson, 1998; Eisenberg & Dreyfus, 1994). There is increasing recognition among the mathematics education community that understanding the link between the different representations associated with the concept was foundational to an understanding of the function concept (Cunningham, 2005; Eisenberg, 1992). Even (1998) reported that knowledge of the link between these representations was intertwined with knowledge of the different approaches to functions, and knowledge of the constructs and notions invariant to the function concept.

**METHODOLOGY**

In order to make knowledge claims about the nature of students’ coordinations, it was necessary to observe and investigate the mathematical activities and communications of students (Creswell, 2003, pp. 20). The specific strategy of inquiry employed in this study meets Creswell (1998) standard for a case study. This study employed a set of case studies of students work and/or reasoning processes on given tests and task situations involving linear functions. This method enabled the investigation of students’ translations coordinations among representations of linear functions. Through students’ articulations
and or written work on a given task situation, the study addresses the nature of students’ coordination among standard representations of linear functions.

The Context: An Intermediate Algebra Course

Students in sections of an Intermediate Algebra through Multiple Representations course at a large southeastern university in the United States were selected for the study. While the course does not satisfy a mathematics requirement or credit towards a degree, a pass in the course is required for enrollment in further mathematics courses and for graduation. That is, knowledge and skills acquired in the course are considered essential for work in successive courses. A basic requirement for all students enrolled in the course was that they must have completed Algebra I and II in their respective high schools and should have taken (and failed) the pre test given on the first day of the course. This was used to gather baseline information on their level of understanding and to ensure that their placement in the course was warranted.

The Participants: Intermediate Algebra Students

The participants in the study were selected from a pool of students in several sections of the course. Guided by the researcher, instructors in the various sections acted as recruiting agents. Students in each section were informed of the study by their instructors during class time and volunteer students were selected from the available pool. These were mostly first year college students, who had taken both the pre-test administered at the beginning and the post test administered at the end of instruction. The instructors were not to include in the pool students with poor attendance and nontraditional students.
Out of the total pool of 14 volunteer students, only eight took part in all aspects of the study. The analysis presented in this study focuses on the work of five students (Shama, Ama, Jen, Rita, John). These students were selected because of their differing ability levels and because they provided elaborated explanations of their work and actions and coherently articulated their reasoning process during the study. Table 1 provides a brief description of these students; their scores on the exam given during the linear function unit, their grades at the time of the study, and their ratings compared to other students in the course.

Table 1: Summary description of participants

<table>
<thead>
<tr>
<th></th>
<th>Linear unit score</th>
<th>Course grade</th>
<th>Ratings</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shama</td>
<td>100%</td>
<td>A+</td>
<td>High Achieving</td>
</tr>
<tr>
<td>Ama</td>
<td>87%</td>
<td>B</td>
<td>Above Average</td>
</tr>
<tr>
<td>Jen</td>
<td>90%</td>
<td>B-</td>
<td>Above average</td>
</tr>
<tr>
<td>Rita</td>
<td>81%</td>
<td>C</td>
<td>Average</td>
</tr>
<tr>
<td>John</td>
<td>57%</td>
<td>F</td>
<td>Below average</td>
</tr>
</tbody>
</table>

Data Sources

Two main sources provided data for the study: Task based instrument (Appendix B), Student worksheets (Appendix C). The task based instrument was used to generate data on students’ coordinations among representations of linear functions (Keller & Hirsch, 1998; Even, 1998; Knuth, 2000). The two tasks on the instrument were open ended and were used as a means via which students’ mental processes could be elicited and their capability to successfully perform coordinations among the given representations of linear function determined. The tasks on the instrument were made up
of problem situations involving both standard and non-standard representations of linear function. Both task involved dynagraphs and standard representations of linear function. While one task involved representations of linear function with the same slopes but different y-intercepts, the other involved representations of linear function with different y-intercepts but the same slope.

These tasks were created with the Geometer’s sketchpad (Jackiw, 1991) and were posed to students via a computer screen during a task based interview. Students were required on these interviews to articulate their thinking processes and to answer questions posed by the researcher as they completed the given tasks. In order to ensure consistency among participants, an interview protocol (Appendix A) was used. These interviews were conducted one week to the end of course exam and were held in a location chosen by the researcher. This location was far removed from the classroom settings and was selected to ensure that students had access to the study instruments. Participants were asked to select days and times that fitted their schedules. These time slots were coordinated so that the researcher could interview one student at a time; each interview lasted between 20 and 30 minutes.

Each student was observed and interviewed as they worked on the given tasks. These observations were done for the purpose of gathering field notes (Bogden & Biklen, 2003) on students’ interactions with the task medium (computer screen) and for asking probing questions on specific actions employed by students. The interviews were used to elicit individual students’ conceptions and ways of working with representations of linear function. Students’ work on the task and interviews were videotaped for further analysis.
Students’ worksheets (Appendix, C) were used to generate data on students’ matches on the given tasks. The worksheets were given to students in the course of the task based interviews and were used to follow their generation of solutions and to determine their success on the given tasks. These worksheets were structured in such a way that each of the tasks featured had a space allotted to it. An extra space was provided for any extra work by students. Students’ were required to record their matches on the spaces provided on the work sheets.

Data Analysis

Data for the study were searched and arranged in a manner that allowed for meaningful analysis of students reasoning process. The analysis was conducted in three phases; sorting, coding, rating. In the first phase of the analysis, student interviews were transcribed and coordinated with other forms of data collected (videotapes, audiotapes, worksheets, etc.). Each student data was then stripped off any form of formal identification and pseudonyms were employed.

In the second phase, students’ transcripts, worksheets and videotapes were analyzed; student’s articulations, strategies, actions and matches on the task were analyzed; transcripts of the interviews were then analyzed and coded while viewing student videotapes. Matrices of students’ matches on each task were then constructed; these matrices detailed the different matches students came up with on the task. The matrices were then paired up with student work sheets and other forms of student data and a coordination chart (Appendix D) was then created on each student. These charts facilitated the comparison of students’ data by type of representations matched (e.g., graph & symbolic (GS), symbolic & table (ST), etc), the invariant property or construct
employed (e.g., slope (sL), y-intercepts (yInt) and pointwise (pT)), types of coordination performed (i.e., GT-sL, ST-yInt, etc.), and the correctness or accuracy of the coordination.

In the third phase, student charts were analyzed and rated. The analysis was done with a focus on the research question, “what is the nature of students’ coordination among given representations of linear function?” A rubric (Appendix E) based on the SOLO Taxonomy (Biggs & Collis, 1983) and adapted for the purpose of the study was used. The rubric applied the levels of the SOLO taxonomy to the task and gave exemplars of students’ coordinations at the different levels on the SOLO Taxonomy. The levels of the SOLO were viewed as cognitive expressions of students’ coordinations among the given representations of linear function. Hence by achieving a particular level on the SOLO, the nature of coordination performed among the given representations could be characterized.

RESULTS

In this section, I present the results of the analysis of student’s data. The results are presented in two main sections, the first describes these results on a student by student basis and the second describes the results on a task by task basis.

Student by Student Analysis

The task entailed matching given examples of linear function (same slopes and y-intercepts). Students were expected to coordinate among the given representations in order to come up with correct matches. In the analysis presented below, coordinations involving the dynagraph were considered only when a student was able to successfully relate all three standard representations.
John, Task1. The matrix detailing John’s matches on task1 is shown (Figure 1).

The matrix revealed that John had a match for all three possible paired combinations of standard representations of linear function (i.e., GS, GT and ST); in the case of the dynagraph however, he had only one of the three possible matches.

<table>
<thead>
<tr>
<th></th>
<th>Graph (G)</th>
<th>Symbolic (S)</th>
<th>Table (T)</th>
<th>Dynagaph (D)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Graph (G)</td>
<td></td>
<td>GS</td>
<td>GT</td>
<td>GD</td>
</tr>
<tr>
<td>Symbolic (S)</td>
<td>GS</td>
<td></td>
<td>ST</td>
<td></td>
</tr>
<tr>
<td>Table (T)</td>
<td>GT</td>
<td>ST</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dynagaph (D)</td>
<td>GD</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 1: Summary of John's matches on the task

In coming up with these matches John attempted four distinct coordination. Three of these were coordinations involving standard representations (i.e., GS-sL, ST-sL, GT-yInt) and one the dynagraph (GD-pT). In the GS match, John performed the “GS-sL” coordination. In the said coordination, John identified slope from the symbolic and then related the symbolic to a graph with similar slope.

John: This equation $y = x + 3$ would match graph1 because the slope here is one over one..........here (graph1) is up one over one, up one over one.

In the ST match, an “ST-sL coordination” was attempted by John. In the said coordination, the slope of the symbolic was first determined and then related to a table with similar slope. In both GS-sL and the ST-sL coordination, though John correctly interpreted slope within each of the representations, he was unable to relate the correct pair of representations together. This was because his choice of slope as an invariant quantity was not favorable in the context of the given problem situation was not favorable.
In the GT match however, John not only correctly interpreted y-intercepts within each of the representations but was also able to relate the correct pair of graph and table together using the said construct.

**John:** I believe that graph _ and table _ are the same due to the fact that zero x is right here and negative four is right here and that is where the y-intercept is and by that I mean table _ and graph _ would be the same by the fact that zero and y four are on both.

On the said match, John attempted a “GT-yInt coordination”. Here the y-intercept of the graph was determined and then compared with a table with similar y-intercept. John’s coordination chart on task 1 is shown (Figure 2). Based on available data (i.e., transcripts, coordination chart, summary matrix, videotapes, etc.) and the SOLO Rubric (Appendix E), the nature of coordination performed by John on the task was characterized as unistructural. This was because though he correctly identified slope and or y-intercept within the associated representations, he was only able to utilize these constructs in relating just one pair of standard representations together.

<table>
<thead>
<tr>
<th>Match</th>
<th>Construct used</th>
<th>Coordination</th>
<th>Match correct</th>
</tr>
</thead>
<tbody>
<tr>
<td>GS</td>
<td>sL  pT  yInt</td>
<td>GS-sL</td>
<td>NO</td>
</tr>
<tr>
<td>GT</td>
<td>sL  pT  yInt</td>
<td>GT-yInt</td>
<td>YES</td>
</tr>
<tr>
<td>ST</td>
<td>sL  pT  yInt</td>
<td>ST-sL</td>
<td>NO</td>
</tr>
<tr>
<td>GD</td>
<td>sL  pT  yInt</td>
<td>GD-pT</td>
<td>Yes</td>
</tr>
</tbody>
</table>

Figure 2: John’s coordination chart

*Rita, task1.* The matrix detailing Rita’s matches on task1 is shown (Figure 3).
The matrix revealed that Rita had matches for only two of the possible three paired combinations of standard representations on the task; notably missing was the symbolic and table match (i.e., ST). In coming up with her matches on task1, Rita attempted three distinct coordination; two of these involved standard representations (i.e., GS-yInt and GT-pT), and one involved the dynagraph (GD-pT).

<table>
<thead>
<tr>
<th></th>
<th>Graph (G)</th>
<th>Symbolic (S)</th>
<th>Table (T)</th>
<th>Dynagraph (D)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Graph (G)</td>
<td></td>
<td>GS</td>
<td>GT</td>
<td>GD</td>
</tr>
<tr>
<td>Symbolic (S)</td>
<td>GS</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Table (T)</td>
<td>GT</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dynagraph (D)</td>
<td>GD</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Figure 3: Summary of Rita's matches on the task**

In the graph and symbolic match (GS), Rita attempted a “GS-yInt coordination”. While in the case of the graph and table match a “GT-pT coordination” was attempted.

**Rita:** This graph 1 has a positive slope….this is in the negative quadrant (y-intercepts) so this y = x-4 ..so graph 1 would match with equation y=x-4

**Rita:** Okay so table ___ goes with graph _ because when x is…..when x hits the x axis at 4 y is zero, x is zero, sorry…okay when it is y, x is at 4…when y is zero, x is 4 so that is table _

In both the GS-yInt and GT-pT coordination, Rita not only successfully interpreted the associated construct from within the representations but was also able to utilize these constructs in relating the correct pairs of standard representations together. Rita’s coordination chart on task1 is shown (Figure 4).
Based on available data (i.e., transcripts, coordination chart, summary matrix, videotapes, etc.) and the SOLO Taxonomy (Biggs & Collis, 1982), the nature of coordination performed among standard representations by Rita on task 1 was characterized as multistructural. This was because though she correctly related more than one pair of standard representations together she was not able to relate all three standard representations together.

Jen, task 1. The matrix detailing Jen’s matches on task 1 is shown (Figure 5). The matrix revealed that Jen had matches for only two of the possible three standard representations; notably missing was the symbolic and table match (i.e., ST). In coming up with her matches, Jen attempted three distinct coordination; two of these involved standard representations (i.e., GS-pT and GT-pT), and one involved the dynagraph (TD-pT).
In matching graph and symbolic (GS), Jen attempted a “GS-pT coordination”.

While in the case of the graph and table match, a “GT-pT coordination” was attempted.

**Jen:** This equation, $y = x - 4$ goes to graph .... when the x value is four, the y value is zero, if you put in the four it goes over four and up zero on graph....

**Jen:** Graph____ matches with table____ because the (4, 0) are on both

In both the GS-pT and GT-pT coordination, Jen not only correctly identified x-intercepts points from within the different representations but was able to successfully utilize these points in relating the correct pairs of representations.

Based on available data (i.e., transcripts, coordination chart, summary matrix, videotapes, etc.) and the SOLO Taxonomy (Biggs & Collis, 1982), the nature of Jen’s coordination among standard representations on task1 was characterized as multistructural. This was because though she correctly related more than one pair of standard representations together using the same invariant property she was not able to coordinate among all three standard representations. Jen’s coordination chart on task1 is shown (Figure 6).

<table>
<thead>
<tr>
<th>Task: 1</th>
<th><strong>Match</strong></th>
<th><strong>Constructs used</strong></th>
<th><strong>Coordination</strong></th>
<th><strong>Answer correct</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>GS</td>
<td>$sL$</td>
<td>$pT$</td>
<td>$yInt$</td>
<td>$GS-pT$</td>
</tr>
<tr>
<td>GT</td>
<td>$sL$</td>
<td>$pT$</td>
<td>$yInt$</td>
<td>$GT-pT$</td>
</tr>
<tr>
<td>TD</td>
<td>$sL$</td>
<td>$pT$</td>
<td>$yInt$</td>
<td>$TD-pT$</td>
</tr>
</tbody>
</table>

Figure 6: Jen’s coordination chart
Ama, Task 1. The matrix detailing Ama’s work on task 1 (Figure 7) revealed the different matches she performed. The matrix further showed that Ama came up with matches for all the different paired combinations of standard representations.

<table>
<thead>
<tr>
<th></th>
<th>Graph (G)</th>
<th>Symbolic (S)</th>
<th>Table (T)</th>
<th>Dynagraph (D)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Graph (G)</td>
<td></td>
<td>GS</td>
<td>GT</td>
<td></td>
</tr>
<tr>
<td>Symbolic (S)</td>
<td>GS</td>
<td></td>
<td>ST</td>
<td></td>
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<tr>
<td>Table (T)</td>
<td>GT</td>
<td>ST</td>
<td></td>
<td>TD</td>
</tr>
<tr>
<td>Dynagraph (D)</td>
<td></td>
<td></td>
<td></td>
<td>TD</td>
</tr>
</tbody>
</table>

Figure 7: Summary of Ama's matches on the task

In coming up with her matches on task1, Ama attempted four distinct coordination; three of these involved standard representations (i.e., GS-pT, GT-yInt, ST-yInt) and one the dynagraph (TD-pT). In the GT match, Ama first attempted the “GT-sL coordination” but had to abandon it midway into her match because of her difficulty in determining slopes from the graph and resorted to a GT-yInt coordination instead.

Ama: For the two tables I am going to start with the point slope form… I pick two points….. and slope is one for table1… Now I know there is a way that you can determine slope depending on what quadrant the graph is in.. I don’t really know

Ama: So for graph1, y-intercept is …. alright based of the y-intercept I can match up table …. with graph ….

In the GS match, Ama attempted the “GS-pT coordination”, while in the ST match she attempted the “ST-yInt coordination”. In all three coordination involving standard representations, Ama not only correctly identified the associated invariant property from
within the representation but was able to successfully utilize the property to relate correct pairs of representations together. Ama’s coordination chart on task 1 is shown (Figure 8).

**Figure 8: Ama's coordination chart**

Based on the available data and the SOLO taxonomy (Biggs & Collis, 1982), the nature of Ama’s coordination among representations on task 1 was characterized as multistructural. This was because though she correctly related more than one pair of standard representations together using the same construct, she was not able to coordinate among all three representations via the same construct.

**Shama, Task 2.** The matrix detailing Shama’s work on task 2 (figure 9) showed the different matches Shama performed on the task.

**Figure 9: Summary of Shama's matches on the task**

The matrix further showed that she had a match for all three possible paired combinations of standard representations of linear function; on the dynagraph however she had only
one of the possible three matches. In coming up with her matches on task 1, Shama attempted four distinct coordination. Three of these involved standard representations of linear function (GS-yInt, GT-yInt, ST-yInt) and one involved the dynagraph (i.e., TD-P).

In all her coordination involving standard representations, Shama employed the same invariant property or construct. This was the case in the “GS-yInt coordination”, the “GT-yInt coordination”, the “ST-yInt coordination” and the “TD-yInt coordination”. For example, in the GS-yInt coordination, the representations related were graph and symbolic and the invariant property or construct employed was y-intercept. In the said coordination, the y-intercept of the symbolic was determined and a graph with similar y-intercepts found and related.

**Shama:** The $y = x+3$ matches to graph ___ because of the y-intercept… If $x$ is zero, then you get a positive three on both the equation and the graph…

In all three coordinations attempted, Shama not only correctly discerned y-intercepts from within the associated representations but was also able to relate all three representations together via this construct. In the case of the dynagraph however, even though she correctly interpreted y-intercepts using the table she was not able to discern how y-intercepts were represented in the dynagraph.

**Shama:** I am thinking that if I am going of the same point the (0, 3) of the table then it has to be on the other side….so I am thinking if $x$ is zero like on the table then actually you should get a positive three and dynagraph 1 doesn’t look right and so the second dynagraph is right.

Shama’s articulation seemed to indicate that the y-intercept points from the table were used in a pointwise manner to coordinate table and dynagraph. Shama was not able to perform any coordination involving the dynagraph and the graph or the dynagraph and
the symbolic. Based on the available data (i.e., transcripts, coordination chart, summary matrix, videotapes, etc.) and the SOLO Taxonomy (Biggs & Collis, 1982), the nature of coordination performed by Shama on the task was categorized as relational.

This was because she not only correctly related all three standard representations together but also discerned and successfully utilized the same invariant property or construct in all her coordination. Even though Shama, successfully performed one coordination involving the dynagraph, she could not be classified at the extended abstract level in that though she was only able to perform just one of the possible three coordination involving the dynagraph. Shama’s coordination chart on task 1 is shown (Figure 10).

<table>
<thead>
<tr>
<th>Task: 1</th>
<th>Construct used</th>
<th>Coordination</th>
<th>Answer correct</th>
</tr>
</thead>
<tbody>
<tr>
<td>GS</td>
<td>sL pT yInt</td>
<td>GS-yInt</td>
<td>Yes</td>
</tr>
<tr>
<td>GT</td>
<td>sL pT yInt</td>
<td>GT-yInt</td>
<td>Yes</td>
</tr>
<tr>
<td>ST</td>
<td>sL pT yInt</td>
<td>ST-yInt</td>
<td>Yes</td>
</tr>
<tr>
<td>TD</td>
<td>sL pT yInt</td>
<td>TD-pT</td>
<td>Yes</td>
</tr>
</tbody>
</table>

Figure 10: Shama's coordination chart

**Task by Task analysis**

Task 1. The task involved given examples of linear function with the same slopes but different y-intercepts. Students were expected to coordinate among representations (standard and non standard) associated with these linear functions examples in order to come up with a correct match for each of the linear function given. A total of eighteen
coordinations were attempted on task 1; of this thirteen involved standard representations of linear function. The pair of representations most consistently coordinated among by students was “graph & table” (GT). All five students attempted and successfully performed a coordination involving this pair of standard representations. The pair least coordinated among by students was “symbolic & table”. Only two students attempted and successfully coordinated among this pair of representations.

On task 1, all three constructs of linear function were identified in coordinations attempted by students (i.e., y-intercept (yInt), slope (sL) and input/output (pT)). In particular, y-intercept was utilized in seven of the thirteen coordinations attempted among standard representations. The input/output construct was utilized in four of the thirteen coordinations while slope was utilized in just two of the thirteen coordinations attempted. In general, all attempted coordinations involving the y-intercepts were successfully performed by students from the study. In the case of the input/output construct, only three of the coordination attempted was successfully performed (75%). None of the attempted coordination involving the slope construct was successfully performed. Students reasons for not utilizing a construct in a coordination varied based on either the type of representations being coordinated or the kind of examples of linear function given. For example, Ama chose not to use slope in coordinations involving the graph in view of her difficulties working with the construct from the graph, while Shama on the other hand chose not to use slope because the examples of linear function given all had the same slopes.

In general, the nature of coordinations performed among standard representations by students in the study varied according to the hierachal levels identified in the SOLO
Taxonomy (Biggs & Collis, 1983); with a majority of these students classified at the multistructural level. In particular, one student, John was classified at the unistructural level, three others, Ama, Rita and Jen were classified at the multistructural level and one more, Shama was classified at the relational level. The classification was based first off the number of correct coordination performed among standard representations by the student then on the number of invariant property or construct utilized by the student in his/her coordinations among standard representations.

A summary of the results of the analysis on task 1 is shown (Table 2).

### Table 2: Summary results for Task 1

<table>
<thead>
<tr>
<th>Student</th>
<th>Matches made</th>
<th>Invariant property used</th>
<th>Number of correct coordination Performed</th>
<th>Nature of coordination</th>
</tr>
</thead>
<tbody>
<tr>
<td>John</td>
<td>GS, GT, TS,</td>
<td>Slope (sL) y-intercept (yInt)</td>
<td>1</td>
<td>Unistructural</td>
</tr>
<tr>
<td>Ama</td>
<td>GS, GT, TS,</td>
<td>Input/output, (pT) y-intercepts (yInt)</td>
<td>3</td>
<td>Multistructural</td>
</tr>
<tr>
<td>Jen</td>
<td>GS, GT,</td>
<td>Input/output (pT)</td>
<td>2</td>
<td>Multistructural</td>
</tr>
<tr>
<td>Rita</td>
<td>GS, GT</td>
<td>y-intercept (yInt) input/output (pT)</td>
<td>2</td>
<td>Multistructural</td>
</tr>
<tr>
<td>Shama</td>
<td>GS, GT, TS</td>
<td>y-intercepts</td>
<td>3</td>
<td>Relational</td>
</tr>
</tbody>
</table>

In general, students classified at the unistructural level successfully coordinated among just one pair of standard representations. Students at the multistructural level successfully coordinated among more than one pair of standard representations but were not able to
relate all three representations together via the same invariant property. Whereas relational level students successfully coordinated among all three pairs of standard representations and were able to relate all three representations together via the same construct or invariant property.

**Task 2.** The task involved examples of linear function with the same y-intercepts but different slopes. Students were expected to coordinate among representations (standard and non standard) associated with these linear functions examples in order to come up with correct matches for each of the linear function given. A total of eighteen coordinations were attempted by students on task 2, of this, fourteen were coordinations involving standard representations of linear function. The two pairs of standard representations most consistently coordinated among by students from the study were graph & table (GT) and graph & symbolic (GS); three of the students attempted and successfully coordinated among these two pairs of representations. The pair of standard representations least coordinated among by students was the symbolic & table (ST); only two of the students attempted a coordination involving this pair of representations.

On task 2, two different types of invariant properties or constructs were identified in coordinations attempted among standard representations by students, these were slope (sL), and input/output (pT); notably missing were coordinations involving the y-intercept construct. In particular, slope was identified as an invariant property in eight of the coordinations attempted, while input/output was identified in six of the coordinations attempted. Of the eight attempted coordinations involving slope, only three were successfully performed (43%). In the case of the input/output construct, five of the six attempted coordinations were successfully performed (83%). In general, student’s reason
for not using a construct in a coordination was based off the nature of examples given on the task. For example all five students chose not to use the y-intercept construct in coordinations performed on the task because all the given examples were of the same y-intercepts.

In general, the nature of coordinations performed among given representations of linear function by students from the study varied according to the hierarchal levels identified in the SOLO Taxonomy (Biggs & Collis, 1982); from prestructural through to relational. In particular, one student, Rita was classified at the prestructural level, one more, Ama was classified at the unistructural level, two more, John and Jen were classified at the multistructural level and one more, Shama at the relational level. The classification was based first off the number of correct coordination performed among standard representations by the student then on the number of invariant property utilized by the student in his/her coordinations.

Prestructural students were unable to successfully coordinate among any of the standard representations given. Unistructural students successfully coordinated among just one pair of standard representations. Multistructural students successfully coordinated among more than one pair of standard representations but were unable to relate all three representations together. Relational students successfully coordinated among all three pairs of standard representations via the same invariant property or construct. A summary of the results of the analysis on task 1 is shown (Table 3).
Table 3: Summary results for Task 2

<table>
<thead>
<tr>
<th>Student</th>
<th>Matches made</th>
<th>Invariant property used</th>
<th>Number of correct coordination Performed</th>
<th>Nature of coordination</th>
</tr>
</thead>
<tbody>
<tr>
<td>John</td>
<td>GS, GT,</td>
<td>Slope (sL)</td>
<td>2</td>
<td>Multistructural</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Input/output (pT)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ama</td>
<td>GS, GT, TS,</td>
<td>Slope (sL)</td>
<td>1</td>
<td>Unistructural</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Input/output (pT)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Jen</td>
<td>GS, GT,</td>
<td>Slope (sL)</td>
<td>2</td>
<td>Multistructural</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Input/output (pT)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rita</td>
<td>GS, GT</td>
<td>Slope (sL)</td>
<td>0</td>
<td>Prestructural</td>
</tr>
<tr>
<td>Shama</td>
<td>GS, GT, TS</td>
<td>Input/output (pT)</td>
<td>3</td>
<td>Relational</td>
</tr>
</tbody>
</table>

**DISCUSSION AND CONCLUSION**

The goal for this study was to investigate the nature of coordination performed among representations of linear function by students selected from an Intermediate Algebra through Multiple Representations course.

In general the students who participated in this study were more successful in working with the examples of linear function given on task 1 than they were with examples on task 2. In particular 87% of the coordinations attempted among standard representations were successfully performed by these students on task 1. In the case of task 2, only 53% of these coordinations were successfully performed by the students.

These results may have arisen due to the fact that each task emphasized or deemphasized a particular construct of linear function and as a result success on these tasks demanded a level of expertise akin to facility in working with these constructs (of linear function) in the representations given on the task.
For example task 1 had examples of linear function with the same slope but different y-intercepts; thus success on task 1 entailed discerning and being able to utilize a construct other than slope to relate the given representations together. In the case of task 2, the given examples of linear function were of the same y-intercepts but different slopes; thus success on the task entailed discerning and being able to utilize a construct other than y-intercepts to relate the given representations together. Data from students work on these two tasks revealed that for a majority of the students the regularity or invariant they abstracted from their experiences of connecting among representations of linear function (in the course) allowed them to work much more successfully with the y-intercept construct than with the slope construct.

For example, the study showed that while all five students readily discerned that the y-intercept construct was not suitable for relating given representations on task 2, only two of the students noted that slope was not a suitable construct to use in relating the representations of linear function given on task 1. Also while all coordinations involving the y-intercept construct was successfully performed by students from the study, only 43% of the coordination involving the slope construct was successfully performed by these students. It is not clear from the data obtained whether these findings were as a result of instruction favoring a particular construct of linear function over another or a case of students compartmentalizing connections for specific constructs of linear function.

In general students who participated in this study were successful at coordinating among the graph & table (GT) pair and or the graph & symbolic (GS) pair. The study further showed that in matches involving these two pairs of representations (GT and GS), a majority of the students’ successfully utilized appropriate constructs or invariant
properties to relate the correct pairs of representations together. These findings though in sharp contrast with findings from other studies on functions (e.g., Knuth, 2000b; Schoenfeld, Smith & Arcavi, 1993; Yerushalmy, 1991) was however consistent with the view that focusing instruction on connecting among multiple representations of functions may allow students to recognize invariants (constructs associated with function concepts) while connecting among representations of associated concepts (Schwarz & Dreyfus, 1995).

The study however showed that in matches involving the symbolic & table pair (ST), a majority of the students experienced difficulties coordinating among these two standard representations of linear function. In particular only two of the students successfully related these two representations together via an appropriate construct or invariant property. The other three either failed to come up with a construct for relating these representations or failed in their attempt at utilizing a construct to relate the correct pair of representations together. Cunningham (2005) suggested that findings of this nature may be as a result of instruction favoring connection among particular representations at the expense of others. It was not possible to infer based on the data available if these findings were a result of instruction favoring connections among particular pairs of representations (e.g., graph & table and or the graph & symbolic) at the expense of others (e.g., symbolic & table). Thus in these respect, it was not possible to discern why these students were readily able to relate the graph & table pairs and or the graph & symbolic pairs together via an invariant property but failed in their attempt at coordinating among the table & symbolic.
In general, the nature of coordination performed among the given representations of linear function by students from the study varied according to the hierarchal levels of the SOLO Taxonomy. In particular, students’ coordinations on the task were classified as prestructural, unistructural, multistructural, or relational depending on the number of representations correctly matched by the student and the number of invariant property or constructs employed by the student in their matches. This study showed that students classified as relational were able to correctly match all the given standard representations on the task via an appropriate construct. Students at this level not only knew of specific ways of working with linear function constructs within a given representation but were also able to notice regularities and discrepancies between these representations and exploit them in their match of equivalent representations.

The nature of coordinations performed by students classified at this level was said to be foundational. In that their articulations and work on the given task situation indicated that they had developed foundational conceptions of the notions and ideas associated with the linear function concept. Thus for these students, the regularity or invariant constructed from records of experiences of connecting function representations allowed them to distinguish between the constructs associated with linear function and the representations used to signify these constructs.

For students classified as either multistructural or unistructural on the SOLO (Biggs & Collis, 1982), this study showed that though they knew of specific ways of working with constructs of linear function within associated representations, their knowledge of the regularity or discrepancy between these representations was compartmentalized to specific representations or constructs. Data for the study showed
that for a majority of student at this level, they either failed to match specific representations of linear function or were unable to employ the same invariant property in matching the representations on the task.

The nature of coordination performed by students classified as either multistructural or unistructural was considered to be procedural. In that their articulations and work on the given task situation indicated compartmentalized conceptions of the connections among these representations of linear functions and or the notions and ideas associated with the linear function concept. Thus for students at this level, the regularity they abstracted from their experiences of connecting among function representation allowed them to relate specific representations via specific constructs. A notable finding was that majority of the students from the study were identified as performing procedural coordinations among these representations of linear function.

The results of this study showed that though multiple representational approaches had the potential of helping students develop facility in coordinating among representations of linear function this potential is not necessarily achieved for all students. Thus though emphasizing connections among multiple representations maybe necessary for students to develop facility in coordinating among representations of linear functions, findings from this study indicated that this may not be entirely sufficient. This study identified two types of coordinations in the articulations and work of students from the intermediate algebra through multiple representations course; namely procedural coordinations and foundational coordinations. A notable finding was that majority of the students from this study performed procedural coordinations among the given representation of linear function on the task.
Since the sample of students selected for the study was not entirely representative of the population of students from the course it was not possible to infer if these findings were consistent across the population of students from the course or unique to just students who participated in the study. In order to clearly distinguish between these two types of coordinations, further research is needed. Such a research should be based on a representative sample of the population of students from the course and should have a classroom observation component where the types of connections emphasized and or deemphasized among these representations during instruction are noted.
CONNECTIONS AMONG REPRESENTATIONS: THE NATURE OF STUDENTS’ CONNECTIONS BETWEEN STANDARD REPRESENTATIONS OF LINEAR FUNCTION

Kwaku Adu-Gyamfi
INTRODUCTION

Reform initiatives in the field of mathematics education underscore the importance of multiple representational approaches to mathematics teaching and learning (NCTM, 2000). Such approaches entail the explication of notions and ideas associated with mathematical concepts such as functions via multiple representations (i.e., graph, table, symbolic). Research that has permeated reform-oriented curricula over the past two decades suggests that utilizing multiple representational approaches to highlight connections among representations such as graphs, tables and symbolic may help students develop strong conceptions of notions and ideas associated with mathematical concepts (Even, 1998; Knuth, 2000b; Moseley & Brenner, 1997; NCTM, 1989, 2000; Porzio, 1998). Such a viewpoint seems to closely approximate conceptual understanding with flexibility in connecting representations of concepts (Hilbert & Carpenter, 1992; Kaput, 1989; NCTM, 1989, 2000).

While these approaches have been posited as a viable means for helping students’ develop foundational conceptions of associated mathematical concepts it is less clear if this is sufficient in the case of an algebraic concept such as linear function. The concept of linear function is a major concept in first year algebra and represents the first formal example of function student’s most likely encounter. Knuth (2000a) contended that the study of linear functions sets the stage for more advanced work in school mathematics. A foundational understanding (conceptual understanding) of linear function has been linked to an ability to flexibly translate between the three standard representations used to signify the concept. At issue then is the nature of students’ translations between standard representations of linear function after going through instruction that emphasizes multiple
Consequently, this study focuses on the nature of students’ translation between the graph, the table and the symbolic representation of linear function after having being explicitly taught functions via a multiple representational approach. The term *translation* refers to the process involved in going from one mode of representation to another (Janvier, 1987). A translation as used will thus always involve two modes of representations; a given source representation and a specified target representation. Performing a translation (e.g., G → T) in this regard will entail interpreting the meaning of the given source representation (e.g. Graph) and producing a target representation (e.g. Symbolic) with the same meaning. Being able to successfully perform a specified translation (flexibility in translating) will thus entail being able to map constructs of the source representation onto constructs of the target representation while preserving the meaning of the source representation.

In this study, the researcher analyzed the strategies and work of students in an intermediate algebra through multiple representations course on given problem situations of linear function. The study was limited to these students because unlike their counterparts in other courses the main method of instruction is via multiple representational approaches. The SOLO Taxonomy (Biggs & Collis, 1991) was employed as the guiding framework for the analysis. The taxonomy (Appendix H) was used to categorize students’ translations between standard representations of linear function. The purpose of the study was to investigate the nature of students’ translations after having experienced multiple representational approaches to function instruction.
The research question below guided the study:

- What is the nature of students’ translations between representations of linear function?

**Function**

The function concept has been described as central to mathematics and to students’ success in mathematics (Selden & Selden, 1992). The importance of functions was brought to the forefront of the mathematics education and research community from as early as the 1920’s. At which time the National Committee on Mathematical Requirements of the Mathematical Association of America recommended that functions be given a central focus in school mathematics (Cooney & Wilson, 1993). In light of this, an understanding of the concept was deemed as necessary for any student hoping to comprehend mathematics at the college level (e.g. calculus and differential equations) (Carlson, 1998). Thus one of the goals of undergraduate mathematics was to develop in students a sense for function (Eisenberg, 1992).

Understanding the function concept entailed among other things, understanding ways possible to represent and connect function representations (Hitt, 1998; Janvier, 1987; Rider, 2002). There is increasing recognition among the mathematics education community that flexibility in translating between the three standard representations of function is a central component in function understanding (Cunningham, 2005; Dreyfus & Eisenberg, 1996; Eisenberg, 1992; Hitt, 1998). Even (1998) posited that knowledge of the link between the different representations was intertwined with knowledge of the different approaches to functions, and knowledge of the constructs and notions underlying the function concept. Six different types of translations have been identified
from research as possible between the three standard representations, these are; graph to symbolic (G → S), symbolic to graph (S → G), symbolic to table (S → T), table to symbolic (T → S), graph to table (G → T), and table to graph (T → G) (Cunningham, 2005; Porzio, 1995; Knuth, 2000; Kaput 1989; Gagatsis & Shiakalli, 2004; Janvier, 1987).

Researchers contend that being able to successfully perform these six translations in complementary pairs or reversibly (i.e., G → S & S → G) was essential to a foundational understanding of function (Eisenberg, 1992; Hitt, 1998). At issue then is the nature of students’ reversible translations between the graph, the table and the symbolic after going through a course where multiple representational approaches are emphasized.

**Multiple Representations**

A significant body of research has been published on the use of multiple representations in the teaching and learning of mathematics (e.g., Hollar & Norwood, 1999; Keller & Hirsch, 1998; Knuth, 2000a; Knuth, 2000b; McGowan & Tall, 1999). A central theme emerging from these and other studies is the issue of students’ connections among representations of associated concepts. Reported findings from these studies suggest that students with compartmentalized view of functions were limited in their ability to translate between the table, graph and symbolic representation of the concept in given problem situation (e.g. Knuth, 2000; Hitt, 1998; Carlson, 1998). Others report that successful problem solvers were necessarily successful at translating between different representations in their solution process (e.g., Brenner, Herman, Ho & Zimmerman, 2000; Tchoshanov, 1997). Studies in which multiple representational approaches were specifically emphasized during instruction (via multiple representational approaches) however show mixed results (e.g., Moseley & Brenner, 1997; Piez & Voxman, 1997).
For example, whereas studies comparing students in multiple representational approaches classes (treatment group) to students in traditional approaches classes (control group) report of significant increases in students skill in moving between representations in favor of the treatment group (e.g. Porzio, 1999; Rider, 2004; Aviles, 2004; Moseley & Brenner, 1997), studies done with students from just classes where multiple representational approaches were emphasized on the other hand seemed to indicate otherwise (e.g., Ozgun-Koca, 1998; Schoenfeld, Smith & Arcavi, 1993; Yerushalmy, 1991). Thus though these approaches have been identified and recommended as a viable means for helping students to develop flexibility in translating between representations of associated function concepts (Porzio, 1997) it is less clear if this is necessarily sufficient. At issue then is the nature of students’ translations between standard representations of linear function after completing a course where multiple representational approaches are emphasized as a pathway to their understanding of function concepts.

**METHODOLOGY**

In order to make knowledge claims about the nature of students’ translations, it was necessary to investigate the mathematical activities of students (Creswell, 2003, pp. 20). The specific strategy of inquiry employed in this study meets Creswell (1998) standard for a case study. Case studies involve an in-depth exploration of a program, an event, an activity, a process or one or more individuals using a variety of data collection procedures such as open-ended interviews, direct observation and work (Creswell, 2003, Stake, 1995). This study employed a set of case studies of students work processes on given tests situation of linear function. This method enabled the investigation of students’ translations between representations of linear function. Through students’ strategies and
work on the given test situation, the study addresses the nature of students’ translations between standard representations of linear function.

**The Context: An Intermediate Algebra Course**

Students in sections of an Intermediate Algebra through Multiple Representations course at a large southeastern university in the United States were selected for the study. A basic requirement for all students enrolled in the course was that they must have completed Algebra I and II in their respective high schools and should have taken (and failed) the pre test given on the first day of the course. This was used to gather baseline information on their level of understanding and to ensure that their placement in the course was warranted.

**The Participants: Intermediate Algebra Students**

The participants in the study were selected from a pool of students in several sections of the course. Guided by the researcher, instructors in the various sections acted as recruiting agents. Students in each section were informed of the study by their instructors during class time and volunteer students were selected from the available pool. These were mostly first year college students, who had taken both the pre-test administered at the beginning and the post test administered at the tail end of instruction. The instructors were not to include in the pool students with poor attendance. Also nontraditional students were excluded from the study to ensure that students age was not variable in the study.

Out of the total pool of 14 volunteer students, only eight took part in all aspects of the study. The analysis presented in this study focuses on the work of five (Shama, Ama, Jen, Rita, John) of the eight students who took part in all aspects of the study.
These students were selected because of their differing ability levels and because they provided elaborated explanations of their work and actions and coherently articulated their processes during the study. Table 1 provides a brief description of the students; their scores on the exam given during the linear function unit, their grades at the time of the study, and their ratings compared to other students from the course.

Table 1: Summary description of participants

<table>
<thead>
<tr>
<th></th>
<th>Linear unit score</th>
<th>Course grade</th>
<th>Student Rating</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shama</td>
<td>100%</td>
<td>A+</td>
<td>Excellent</td>
</tr>
<tr>
<td>Ama</td>
<td>87%</td>
<td>B</td>
<td>Above Average</td>
</tr>
<tr>
<td>Jen</td>
<td>90%</td>
<td>B-</td>
<td>Above average</td>
</tr>
<tr>
<td>Rita</td>
<td>81%</td>
<td>C</td>
<td>Average</td>
</tr>
<tr>
<td>John</td>
<td>57%</td>
<td>F</td>
<td>Below average</td>
</tr>
</tbody>
</table>

**Data Sources**

A test instrument (Appendix F) based on recommendations and findings from research literature on functions was utilized to generate data on students’ translations. The test instrument was made out of problems commonly encountered in the intermediate algebra through multiple representations course. The items featured on the instrument addressed each of the six translations identified in the research literature on functions (Cunningham, 2005). This instrument served as a means via which students strategies when moving from one mode of representation (given source) to another (specified target) could be investigated and their ability to successfully preserve the meaning of the source representation under a translation could be elucidated. The test instrument was
administered to students in the form of an in class test three weeks to their end of course exam. Students were required to show all their work on each of the six items on the instrument.

**Data Analysis**

Qualitative data analysis is viewed by Bogden & Biklen (2003) as a process of systematic searching and arrangement of data forms accumulated in the course of the study in order to come up with meaningful descriptions of participants’ processes. To this end, student’s work samples werestriped off any form of formal identification and then analyzed. The analysis first focused on the strategies employed by students on each item and then on students answers to the problems on each item. On the item by item analysis each student’s strategy on each item was analyzed and a preliminary list of possible coding categories of students’ strategies was then developed. Student work sample was then re-examined and the codes applied on an item by item basis to the data. Common themes in strategy across students work on these items were identified. These themes and the strategy associated with it were then represented by student in a translation chart (Appendix G). The chart detailed each student’s strategies and the accuracy of their solutions on each item of the instrument.

The chart made it possible to categorize students based on the number of reversible or complementary translations successfully performed. The SOLO taxonomy (Biggs & Collis, 1982) was used for this purpose. A rubric designed based off the different levels of the taxonomy (Appendix G) was employed to analyze and categorize the translation chart of each student; it was not possible to give an extended abstract rating to students work on the test instrument since the study design did not allow for that.
RESULTS

In this part I present the result of the analyses of students’ data. The results are presented in two main sections with their corresponding subsections. The first, presents the results on an item by item basis. The second presents these results on a student by student basis.

Item by Item Analysis

This part presents results pertaining to the item by item analysis of students work on each item. On each of the six items on the instrument, students were provided with a given source representation (e.g., table) and were required to come up with a specified target representation (e.g., symbolic) of the source. Table 2 provides a summary of students’ strategies and work on each item of the instrument. The subsequent paragraphs describe the result of the analyses on each item.

Table 2. A summary chart of students work on the test instrument

<table>
<thead>
<tr>
<th>Item No</th>
<th>Translation</th>
<th>Prevalent Strategy</th>
<th>No of students using prevalent strategy (N=5)</th>
<th>No of successful students with strategy</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>T→S</td>
<td>Fitting</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>S→T</td>
<td>Computing</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>G→T</td>
<td>Reading off</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>T→G</td>
<td>Plotting</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>G→S</td>
<td>Curve Fitting</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>S→G</td>
<td>Sketching</td>
<td>5</td>
<td>3</td>
</tr>
</tbody>
</table>

*Item1.* The item was used to investigate students’ translations from the table to the symbolic (i.e., T→S translation). The strategy prevalent in students’ work samples in the
case of the T→S translation was the identification of constructs or invariant properties such as slope, y-intercepts from the table and the re-encoding or mapping of these constructs onto the symbolic; a sample student worksheet is shown (Figure 1). This strategy was coded as “Fitting”.

![Table and Equation]

*Figure 1: A sample student worksheet on item 1*

Four of the five students employed the fitting strategy to perform the T→S translation. Of these four only two of the students, John and Shama managed to preserve the meaning of the source representation under the translation. The other two students, Rita and Ama came up with non-equivalent target representation of the source. Ama’s work sheet on item 1 is shown (Figure 2). Though the fifth student, Jen correctly performed the T→S translation, she did this via a different strategy. In performing the said translation, Jen introduced an intermediary target representation (the graph) in order to get to the specified target representation. Jen’s work sheet illustrating her strategy is shown (Figure 3).
It was not possible to provide a code for Jen’s strategy as. Instead, this strategy was interpreted in terms of two other strategies, plotting and curve fitting; these strategies are described in depth in subsequent paragraphs.

_Item2_. The item was employed to investigate students’ translations from the symbolic to the table (i.e., S→T translation). The prevalent strategy employed by students on the T→S translation was mapping construct in the symbolic onto construct in the table via the generation of corresponding values from the symbolic. A sample student worksheet is shown (Figure 4). This strategy was coded as “computing or tabulating”.

1. Find the equation of the function represented by the table below.

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>11</td>
</tr>
<tr>
<td>-1</td>
<td>9</td>
</tr>
<tr>
<td>0</td>
<td>7</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

\[ m = \frac{-1}{3-y} = \frac{-2}{-1} = 2 \]
\[ y - (1) = 2(x - 4) \]
\[ y + 1 = 2x - 8 \]
\[ y = 2x - 9 \]

Figure 2: Ama’s worksheet on item 1

Figure 3: Jens’ worksheet on item 1
All five students utilized the computing strategy to perform the T$\rightarrow$S translation. Of these five, only four, Jen, John, Shama, and Rita, successfully preserved the meaning of the given symbolic representation (source) under the said translation; Ama failed in her attempt.

Item 3. The item was employed to investigate student’s translations from the graph to the table (i.e., G$\rightarrow$T translation). The prevalent strategy employed by students in the G$\rightarrow$T translation entailed locating specified points on the graph and then reading of their corresponding values for the table; a sample student worksheet is shown (Figure 5). This strategy was coded as “reading off”. All five students, Shama, Ama, Rita, John, and Jen successfully employed the “reading off” strategy to perform the G$\rightarrow$T translation. These students successfully located the given points on the graph and correctly read off their corresponding values for the table.
Item 4. The item was used to explore students’ translations from table to graph (i.e., G→T translation). The prevalent strategy employed by students in the G→T translation entailed converting corresponding values in the table into points that defined (on) a line in the coordinate plane. A sample student worksheet is shown (Figure 6). This strategy was coded as “plotting”.

Figure 5: A sample student worksheet on item 3

Figure 6: A sample student worksheet on item 4
All five students from the study successfully utilized the plotting strategy to perform the T→G translation. They successfully converted corresponding values in the table into equivalent points on a line in the coordinate plane.

An interesting result from students work samples on this item was that while Rita, Jen, Ama and John plotted all seven points on the graph to come up with their target representation, Shama on the other hand plotted only two of the points; the x-intercept and the y-intercept. Since this part of the study did not have an interview component to it, it was not possible to determine if Shama’s choice of those point was because of her intuitive understanding of the invariant aspect of the Cartesian coordinate system or just an automated skill on her part; however the fact that she focused on just two instead of all seven points in coming up with her graph was worthy of mention.

**Item 5.** The item was employed as a means to investigate students’ translations from graph to symbolic (i.e., G→S translation). The prevalent strategy employed by students to perform the G→S translation entailed the identification of constructs or invariant properties of linear function from the graph and the re-encoding or fitting of these constructs onto the symbolic; a sample of student’s work is shown (Figure 7).

In particular all five students, Ama, Jen, Shama, John and Rita successfully utilized the curve fitting strategy to perform the G→S translation. These students correctly identified slope and y-intercept construct using the given graph and came up with equivalent symbolic representations.
Figure 7: A sample student worksheet on item 5 illustrating the curve fitting strategy

Item 6. The item was employed as a means to explore students’ translations from the symbolic to the graph (i.e., S→G translation). The prevalent strategy employed by students to perform the S→G translation entailed the mapping of constructs of linear identified in the symbolic onto the graph via a line defined by points generated from the symbolic. A sample student work sheet is shown (Figure 8). This strategy was coded as “curve sketching”. All five students employed the said strategy in performing the S→G translation.

Student work sample on the said item revealed three variations of the sketching strategy. In the first variation, student generated point covered all three quadrants of the coordinate grid (Figure 8); this was the case for two of the students, John and Rita. In the second variation, the generated points covered only the first quadrant (Figure 9); this was the case for two of the students, Ama and Jen. In the third variation, only two points were generated, the x and y intercepts; this was the case for Shama. Only three of the five students, Shama, Rita and Jen were able to preserve the meaning of the source representation under the said translation (S→T).
Figure 8: A sample student worksheet on item 6

The other two students, Ama and John came up with non-equivalent graphs of the symbolic.

Figure 9: Ama’s work sheet on item 6

Analysis of Results by Students

This section describes the results of the analyses on each student.
*Ama Translation.* Ama’s translation chart (Figure 10) revealed that she had a strategy or rule for mapping constructs of the source onto constructs of the specified target representation on all six translations.

<table>
<thead>
<tr>
<th>Translation</th>
<th>Strategy</th>
<th>Meaning of source preserved</th>
</tr>
</thead>
<tbody>
<tr>
<td>S → T</td>
<td>Tabulating/Computing</td>
<td>No</td>
</tr>
<tr>
<td></td>
<td>computing x/y pairs</td>
<td></td>
</tr>
<tr>
<td></td>
<td>tabulating x/y pairs</td>
<td></td>
</tr>
<tr>
<td>T → S</td>
<td>Fitting — y = mx + b</td>
<td>No</td>
</tr>
<tr>
<td></td>
<td>Slope — slope formula</td>
<td></td>
</tr>
<tr>
<td></td>
<td>point slope formula</td>
<td></td>
</tr>
<tr>
<td>T → G</td>
<td>Plotting</td>
<td>Yes</td>
</tr>
<tr>
<td></td>
<td>Points</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Line</td>
<td></td>
</tr>
<tr>
<td>G → T</td>
<td>Reading off</td>
<td>Yes</td>
</tr>
<tr>
<td></td>
<td>locating points</td>
<td></td>
</tr>
<tr>
<td></td>
<td>tabulating coordinate points</td>
<td></td>
</tr>
<tr>
<td>G → S</td>
<td>Curve fitting</td>
<td>Yes</td>
</tr>
<tr>
<td></td>
<td>slope — slope formula</td>
<td></td>
</tr>
<tr>
<td></td>
<td>y-intercepts</td>
<td></td>
</tr>
<tr>
<td>S → G</td>
<td>Curve sketching — pointwise</td>
<td>No</td>
</tr>
<tr>
<td></td>
<td>Solve for y</td>
<td></td>
</tr>
<tr>
<td></td>
<td>coordinate points — 1st quadrant</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Line</td>
<td></td>
</tr>
</tbody>
</table>

*Figure 10: Ama's translation chart*

The chart further showed that she preserved the meaning of the source representation on only three of the six translations she performed. Based on her chart, the nature of translation performed by Ama on the given instrument was categorized as unistructural. This was because she successfully performed only one (i.e., G → T & T → G) of the possible three reversible or complementary translations given. In the case of the other two, (i.e., S → T & T → S; S → G & G → S), though Ama had a strategy or rule for mapping constructs of the source onto constructs of the target, she failed to preserve the meaning.
of the source representation in each case and as a result ended up with non-equivalent target representations of the given source.

*John Translation.* John’s translation chart (Figure 11) revealed that he had a strategy or rule for mapping constructs of the source onto constructs of the specified target representation on all six translations.

<table>
<thead>
<tr>
<th>Translation</th>
<th>Strategy</th>
<th>Meaning of source preserved</th>
</tr>
</thead>
</table>
| S→T | Computing/tabulating  
• x/y pairs | Yes |
| T→S | Fitting—\(y = mx + b\)  
• Slope-slope formula  
• y-intercept =x zero | Yes |
| T→G | Plotting  
• x/y points  
• Line | Yes |
| G→T | Reading off  
• Coordinate points.  
• Corresponding values | Yes |
| S→G | Curve sketching  
• Solve for \(y\)  
• x and y points (1-3 quadrant)  
• line | No |
| G→S | Curve fitting  
• slope  
• y-intercepts | Yes |

Figure 11: John's translation chart

The chart further showed that he preserved the meaning of the source representation on just five of the six translations he performed. Based on his translation chart, the nature of John’s translation between standard representations on the instrument was categorized as multistructural. This was because he successfully performed two (i.e., S→T & T→S; T→G & G→T) of the possible three reversible or complementary translations. In the case of the third pair (i.e., S→G & G→S), John failed to preserve the meaning of the source representation in the S→G translation.
Jen Translation. Jen’s translation chart (Figure 12) revealed that she had a strategy or rule for mapping constructs of the source onto constructs of the specified target representation on all six translations.

<table>
<thead>
<tr>
<th>Translation</th>
<th>Strategy</th>
<th>Meaning of source preserved</th>
</tr>
</thead>
<tbody>
<tr>
<td>S → T</td>
<td>Tabulating/Computing</td>
<td>Yes</td>
</tr>
<tr>
<td></td>
<td>• computing x/y pairs</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• tabulating x/y pairs</td>
<td></td>
</tr>
<tr>
<td>T → S</td>
<td>Fitting</td>
<td>Yes</td>
</tr>
<tr>
<td>T → G → S</td>
<td>• coordinate points</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• slope—rise over run</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• y-intercept—slope intercept</td>
<td></td>
</tr>
<tr>
<td></td>
<td>formula (y = mx + b)</td>
<td></td>
</tr>
<tr>
<td>T → G</td>
<td>Plotting</td>
<td>Yes</td>
</tr>
<tr>
<td></td>
<td>• x/y points</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• line</td>
<td></td>
</tr>
<tr>
<td>G → T</td>
<td>Reading off</td>
<td>Yes</td>
</tr>
<tr>
<td></td>
<td>• coordinate points</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• tabulating</td>
<td></td>
</tr>
<tr>
<td>S → G</td>
<td>Curve sketching—(pointwise)</td>
<td>Yes</td>
</tr>
<tr>
<td></td>
<td>• Solve for y</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• plot x and y points—(x-intercepts)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• line through points</td>
<td></td>
</tr>
<tr>
<td>G → S</td>
<td>Curve fitting (y = mx + b)</td>
<td>Yes</td>
</tr>
<tr>
<td></td>
<td>• slope—rise over run</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• slope intercept formula</td>
<td></td>
</tr>
</tbody>
</table>

Figure 12: Jen's translation chart

The chart further showed that she preserved the meaning of the source representation on all six translations attempted. Based on her translation chart, the nature of Jen’s translation between standard representations on the instrument was classified as relational. This was because she successfully performed all three possible reversible or complementary pairs of translations given (i.e., S → T & T → S; T → G & G → T, S → G & G → S).
Rita Translation. Rita’s translation chart (Figure 13) revealed that she had a strategy or rule for mapping constructs of the source onto constructs of the specified target representation on all six translations.

<table>
<thead>
<tr>
<th>Translation</th>
<th>Strategy</th>
<th>Meaning of source preserved</th>
</tr>
</thead>
<tbody>
<tr>
<td>S$\rightarrow$T</td>
<td><strong>Computing</strong>&lt;br&gt;• input/output pairs&lt;br&gt;• tabulating pairs</td>
<td>Yes</td>
</tr>
<tr>
<td>T$\rightarrow$S</td>
<td><strong>Fitting—slope intercept form</strong>&lt;br&gt;• slope—rate of change&lt;br&gt;• y-intercept—point slope formula</td>
<td>No</td>
</tr>
<tr>
<td>T$\rightarrow$G</td>
<td><strong>Plotting</strong>&lt;br&gt;• coordinate points&lt;br&gt;• line</td>
<td>Yes</td>
</tr>
<tr>
<td>G$\rightarrow$T</td>
<td><strong>Reading off</strong>&lt;br&gt;• line&lt;br&gt;• corresponding points</td>
<td>Yes</td>
</tr>
<tr>
<td>S$\rightarrow$G</td>
<td><strong>Curve sketching (point wise)</strong>&lt;br&gt;• solve for $y$&lt;br&gt;• coordinate points&lt;br&gt;• line</td>
<td>Yes</td>
</tr>
<tr>
<td>G$\rightarrow$S</td>
<td><strong>Curve fitting (slope, y-intercepts)</strong>&lt;br&gt;• slope—rate of change&lt;br&gt;• y-intercepts—point slope formula</td>
<td>Yes</td>
</tr>
</tbody>
</table>

Figure 13: Rita’s translation chart

The chart further showed that Rita preserved the meaning of the source representation on five of the six translations she attempted. Based on her translation chart, the nature of Rita’s translation between standard representations on the instrument was classified as multistructural. This was because she successfully performed just two (i.e., S$\rightarrow$G & G$\rightarrow$S; T$\rightarrow$G & G$\rightarrow$T) of the possible three reversible or complementary translations given. In the case of the third pair (i.e., S$\rightarrow$T & T$\rightarrow$S), Rita failed in her attempt to come up with an equivalent symbolic representation of the given table (i.e., T$\rightarrow$S).
Shama Translation. Shama’s translation chart (Figure 14) revealed that she had a strategy or rule for mapping constructs of the source onto constructs of the specified target representation on all the translations.

<table>
<thead>
<tr>
<th>Translation</th>
<th>Strategy</th>
<th>Meaning of source preserved</th>
</tr>
</thead>
</table>
| S→T         | **Tabulating/Computing**  
  - computing x/y pairs  
  - tabulating x/y pairs | Yes                        |
| T→S         | **Fitting—y=mx+b**      
  - Slope  
  - y-intercept | Yes                        |
| T→G         | **Plotting**            
  - plotting points  
  - line | Yes                        |
| G→T         | **Reading off**         
  - locating points  
  - tabulating values | Yes                        |
| S→G         | **Curve sketching-(x/y-intercepts)**  
  - solve for y  
  - x and y intercepts  
  - Line | Yes                        |
| G→S         | **Curve fitting**       
  - slope  
  - y-intercepts | Yes                        |

Figure 14: Shama's translation chart

The chart further showed that she preserved the meaning of the source representation on all six translations. Based on her translation chart, the nature of Shama’s translation between standard representations on the instrument was classified as relational. This was because she successfully performed all three reversible or complementary translations given (i.e., S→T & T→S; T→G & G→T, S→G & G→S).
Discussion and Conclusion

A number of studies have focused on students’ translations between modes of representations of function (Dubinsky & Harrel, 1992; Hitt, 1998; Sfard, 1992; Yerushalmy, 1997). The goal for this study was to investigate the nature of intermediate Algebra through Multiple representations students’ translations between representations of linear function.

Students who participated in this study attempted all six translation items given on the instrument (Appendix E). In particular, this study showed that these students’ utilized the values given in the table to map out lines in the coordinate plane (i.e., T\rightarrow G) and to define relationships between corresponding values (T\rightarrow S). They also utilized qualitative properties of given graphs to generate corresponding values for the table (i.e., G\rightarrow T) and to define relationships between points on lines in the coordinate plane (i.e., G\rightarrow S). The students also utilized the given symbolic as a means for generating coordinate points for graphs in the coordinate plane (i.e., S\rightarrow G) and for computing corresponding values for the table (S\rightarrow T) (Shwarz & Dreyfus, 1995).

These findings though in sharp contrast with findings from some of the available studies on function representations (e.g., Dubinsky & Harrel, 1992; Hitt, 1998; Knuth, 2000a; Sfard, 1992; Sierpinska, 1992; Vinner & Dreyfus, 1989; Yerushalmy, 1997) was however consistent with the view in the mathematics education community to the effect that students’ would be better able to move between function representations when multiple representational approaches are emphasized (Porzio, 1998; Moseley & Brenner, 1997; NCTM, 2000). A probable explanation may be that when multiple representational approaches are utilized, students may be directly or indirectly exposed to the different
modes of representations and the connections among them and may as a result develop invariant or anticipatory strategies for moving between these modes in given problem situations.

In general, students who participated in this study used similar strategies in moving between modes of representation of linear function. These strategies were consistent in the work of all five students and varied based on the nature of the given source and or specified target. For example, while point generation/reading was the prevalent strategy used by students in the \( G \rightarrow T \) translation, plotting/sketching was the strategy employed by these students in the \( T \rightarrow G \) translation. In the case of the \( T \rightarrow S \) translation interpreting/fitting was the strategy consistently used by students while for the \( S \rightarrow T \) translation, computing/tabulating was the prevalent strategy used.

One interesting observation made was that students’ strategies on each of the six translations were aligned to the translation process model proposed by Janvier (1987a). Because the study had no interview or observation component in it’s design it was not possible to generate data that would completely account for whether these students were explicitly taught these processes (in the manner outlined by Janvier) or whether these strategies were invariants that arose from their experience of connecting among the different representations in given situations (via multiple representational approaches). The instructors from the course claimed that they specifically taught students how to connect the different representations, but had no idea as to what the translation process model proposed by Janvier entailed.

This not withstanding, the finding that students from the study employed these processes or strategies to move between the different modes of representations of linear
function seem to lend credence to the view that each translation (e.g., \(T \rightarrow G\)) has associated with it specific processes (plotting) or rules for performing it (Janvier, 1987a). Further research is needed to clarify or ascertain if these processes or strategies are consistent in the work of students who are flexible in their translations between standard representations of functions. This may provide insight into ways translation ability can be fostered in students and may provide insights into how multiple representational approaches may be effectively utilized to help students develop flexibility in translating between the table, the graph and the symbolic; one of the recommendations of the National Council for Teachers of Mathematics (NCTM, 2000).

While it is not entirely clear if multiple representational approaches explicitly teaches students’ these processes in the manner outlined by Janvier (1987), what is clear from research is that students from classes in which these approaches are utilized were more likely to develop strategies approximate to these processes than students in traditional courses (Porzio, 1999; Rider, 2004). This may account for the reason why comparative studies involving multiple representational groups and traditional groups report of significant increase in student’s translations in favor of multiple representational groups (e.g., Aviles, 2004; Moseley & Brenner, 1997; Rider, 2004).

A surprising result from the study was that though all five students’ utilized appropriate strategies on each of the items on the instrument, only two successfully performed all six specified translations. This study showed that students who failed in their attempt at performing a specified translation did so because they failed to preserve the meaning of the source representation under the said translation. These results may have arisen because these students may have developed anticipatory strategies for
moving from one representation of linear function to another but may have lacked knowledge of the equivalency requirement of the source and target under a translation. Janvier (1987) contended that flexibility in translating between representations entailed much more than having a procedure for use in different modes, it entailed being able to look at the source representation from a target point of view (and vice versa) and being able to derive the results.

Thus though multiple representational approaches had the potential of helping students develop flexibility in translating between representations of functions, results from this study indicate that not all students achieve this potential. This may account for the reason why some of the research studies, report of student’s difficulty in performing specified translations even after experiencing instruction that emphasizes connections among multiple representations (e.g., Ozgun-Koca, 1998; Piez & Voxman, 1997; Yerushalmy , 1991; Schoenfeld, Smith & Arcavi, 1997). Janvier (1987) contended that in order for students to develop flexibility in translating between representations in given problem situations, instruction needed to explicitly highlight the different components of a translation and the rules or conventions under-girding these components. Further research is needed to clarify if students from a multiple representational approach course explicitly highlight these components.

In general, the nature of students’ translations varied according to the hierarchal levels of the SOLO Taxonomy (Biggs & Collis, 1983). None of the students from the study performed translations that could be categorized at either the prestructural or unistructural level. In particular, the translations performed by three of the students’, Ama, John, and Rita was categorized as multistructural, that of the remaining two
students, Jen and Shama were categorized as relational. The main distinction between students at these two levels was in the number of reversible translations performed or completed on the test instrument. For example while Shama and Jen successfully completed all three reversible or complementary pairs of translations on the instrument Ama, John and Rita on the other hand were successful only on specific instances of the source and target representations.

The nature of translations performed between representations of linear function by students’ categorized at the multistructural level was considered to be procedural. In that though they had appropriate strategies for mapping constructs of given source onto specified target representations they were not able to consistently preserve the meaning of these sources under these translations. Thus for these students, the regularity they abstracted from their experiences of connecting among representations may have allowed them to work in the different modes of representations (source and target) and to move between these modes but not to consistently monitor their progress in the course of a translation to ensure equivalency of both the source and the target representation.

The nature of translations performed by students’ categorized at the relational level on the hand was considered to be foundational. In that, they not only had appropriate strategies or processes to use for mapping constructs of given source onto specified target representations but were also able to consistently preserve the meaning of the source representation under each of these reversible translations. It was not entirely possible to clarify if students considered to be performing foundational translations had knowledge of the equivalency requirement of the source and target representations. Thus in the absence of that it was impossible to infer if these students had developed flexibility
in translating between these representations of linear function; an indicator of a foundational conception of linear function.

In order to clearly distinguish between the two identified types of translations, further research is needed. This may help furnish teachers with benchmarks or lenses for viewing students’ translations when multiple representational approaches or connections among multiple representations are emphasized during instruction. And may also help them plan appropriate pedagogical strategies to foster students’ flexibility in translating between representations of associated mathematical concepts.

This study revealed a variety of informative insights on students’ translations when multiple representations are emphasized. The study showed that while multiple representational approaches had the potential of helping students develop flexibility in translating reversibly between representations of linear function, this potential was not achieved for all of the students. Thus though this approaches were necessary for fostering students flexibility in translating between representations, it was not entirely sufficient. Two types of translations were identified in the work of students from this study, namely procedural translation and foundational translation. Further research is needed to clarify the distinction between these two.
LIST OF REFERENCES


Appendix A
INTERVIEW PROTOCOL

Participant:

Date:

To prepare: Have camera set up; pencil and scratch paper; PC with the Geometer’s sketchpad-----

• file 1
• file 2.

Purpose: To gain insight into the nature of students’ coordination among representations of linear functions.

Introduction:
Hello, I’m __________. I really appreciate you taking the time to talk with me today for about 30 minutes. This is not a graded assignment, so do not feel pressured to only share what you think is the “right” answer. Please keep in mind that my primary interest is what you are thinking as you attempt to solve each problem.

Roles:
Since I am interested in how you think about math, it would really help me if you would talk as much as you can. While you are working and talking, I probably will ask questions to be sure I am following you. Please keep in mind that I am asking because I am really interested. Chances are I probably just missed something you said or did, or you did something that would require more clarification. Also, if I ask a question you do not understand or there is a part of the activity you are unclear about please ask me. I will be happy to clear up any misunderstandings.

Recordings:
I might take a few notes to help me remember what you have shared, but I would like to video tape our conversation so I will not be distracted from your work. The video camera will be focused on the computer screen and not on you or me.

Awareness of computer and software:
I am not sure if you have ever used a computer in your math class, but today we will be using it to access The Geometer’s Sketchpad software. Our activity uses Geometer’s Sketchpad to work with symbolic, graphs, tables and dynagraphs, which are ways of representing functions. If you have never used sketchpad before or are having trouble remembering how to use it, I will be happy to help you at any point during the task. Feel free to also use your paper and pencil whenever you please.

Familiarity with dynagraph:
This is a dynagraph, which is another way of representing functions. You will be using the arrow to move point x (show student how this is done). Notice that when you move point x, point f(x) also moves. Keep in mind that each line of the dynagraph is a number line. Each position on the number line represents magnitudes of respective variable (recall that the point itself can only represent ONE value at a time). Positions along the line traversed by point x represents the magnitude of the independent variables while position along the line of point f(x) represents the magnitude of the dependent variable.
(Display FILE 1—question #1)
Say: Please complete the given task using the representations provided.
Ask:
• Why did you match graph__ or table__ with equation__ or dynagraph__?
• What features of graph__ caused you to match it with table__?
• How do you know table__ matches equation__ and dynagraph__?
• What features of dynagraph__ caused you to match it with equation__?
• What makes equation__ different from equation__?
• What makes graph__ different from graph__?
• What makes table__ different from table__?
• What makes dynagraph__ different from dynagraph____?

(Display FILE 2—question #2)
Say: Now you are given 3 new equations with their corresponding graphs, tables and dynagraph. Follow the directions in question #2 to complete this task.
Ask:
• Why did you match graph__ or table__ with equation__ or dynamap__?
• What features of graph__ caused you to match it with table__?
• How do you know table__ matches equation__ and dynamap__?
• What features of dynamap__ caused you to match it with equation__?
• Can you identify the slope in all 4 representations? Justify your answer?

• END OF INTERVIEW
Say: Thank you. I really appreciate your willingness to take your time to work through this activity today. You did an excellent job sharing how you were thinking. I hope you have a great day.
(Once student has left the room, turn off the recording equipment. Be sure to save anything that was written on in a folder labeled with the interview session date and time.)
APPENDIX B - Function Match Task

1. Match each graph to an equation, a dynagraph, and a table.

### Graph 1
- $y = x + 3$
- $y = x - 4$

### Graph 2

### Table 1
<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>0</td>
<td>7</td>
</tr>
<tr>
<td>-1</td>
<td>8</td>
</tr>
<tr>
<td>-2</td>
<td>9</td>
</tr>
<tr>
<td>-3</td>
<td>10</td>
</tr>
</tbody>
</table>

### Table 2
<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>-2</td>
<td>1</td>
</tr>
<tr>
<td>-3</td>
<td>0</td>
</tr>
</tbody>
</table>

### Dynagraph 1
- $A$

### Dynagraph 2
- $B$
2. Match each dynagraph to a graph, a table, and an equation.

**Dynagraph 1**
- \( f(x) \)
- \( g(x) \)

**Dynagraph 2**
- \( f(x) \)
- \( g(x) \)

**Dynagraph 3**
- \( f(x) \)
- \( g(x) \)

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Table 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( y )</td>
</tr>
<tr>
<td>-5</td>
<td>9</td>
</tr>
<tr>
<td>-4</td>
<td>8</td>
</tr>
<tr>
<td>-3</td>
<td>7</td>
</tr>
<tr>
<td>-2</td>
<td>6</td>
</tr>
<tr>
<td>-1</td>
<td>5</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
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<td>4</td>
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</tr>
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<td>5</td>
<td>-1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>0</td>
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<tr>
<td>1</td>
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<tr>
<td>2</td>
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<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>5</td>
</tr>
</tbody>
</table>
### Appendix C
#### Student Work Sheet

Name___________________________

1. Fill in the table with your matches from Task # 1

<table>
<thead>
<tr>
<th>Dynagrapgh</th>
<th>Graph</th>
<th>Table</th>
<th>Symbolic</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Fill in the table with your matches from Task # 2

<table>
<thead>
<tr>
<th>Dynagrapgh</th>
<th>Graph</th>
<th>Table</th>
<th>Symbolic</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
APPENDIX D

Coordination Chart (coordination category)

Student: _______________

<table>
<thead>
<tr>
<th>Coordination</th>
<th>Match</th>
<th>Invariant property</th>
<th>Match correct</th>
</tr>
</thead>
<tbody>
<tr>
<td>GS</td>
<td>SL</td>
<td>P</td>
<td>yInt</td>
</tr>
<tr>
<td>GT</td>
<td>SL</td>
<td>P</td>
<td>yInt</td>
</tr>
<tr>
<td>ST</td>
<td>SL</td>
<td>P</td>
<td>yInt</td>
</tr>
<tr>
<td>GD</td>
<td>SL</td>
<td>P</td>
<td>yInt</td>
</tr>
<tr>
<td>TD</td>
<td>SL</td>
<td>P</td>
<td>yInt</td>
</tr>
<tr>
<td>SD</td>
<td>SL</td>
<td>P</td>
<td>yInt</td>
</tr>
</tbody>
</table>

Comments:
### Appendix E- Rubric for Analyzing Students’ Coordination Charts

<table>
<thead>
<tr>
<th>Phase of concept development</th>
<th>Examples of how it emerges when evaluating the nature of Students’ Coordinations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre-structural (lacks knowledge of the assessed component) – may have pieces of unconnected knowledge but make no sense to the student</td>
<td>No clue May identify Slopes and or y-intercepts but within one standard representation (G, T, S) Task1: Recognize the same slopes in more than one standard representations but still not coordinating correct functions Task 2: Recognize the same y-intercepts in more than one standard representation but still not correctly coordinating functions</td>
</tr>
<tr>
<td>Uni-structural (focuses on a single coordination) – has an understanding of one coordination but not of its significance to the whole or relationship to other coordination</td>
<td>Correctly matches ONE pair of standard representations (e.g., GT-sL) by a construct • Slopes • Y-intercepts • Input/output points (points in table to points on graph) • Both Might show some understanding of the third standard representation but incorrectly matches it with the other pair</td>
</tr>
<tr>
<td>Multi-structural (focuses on several separate aspects) – has an understanding of more than one aspect of the task but not their significance to the whole or relationship to other aspects. Student has gained flexibility in solving problems with more than one representation, but little structural knowledge of how those representations relate to each other.</td>
<td>Correctly matches more than ONE pair of standard representations but do not relate the three standard representations together • May correctly match two representations by slopes and two different representations by y-intercepts, for example may be able to match T-G pointwise, and may be able to match G-S with constructs but can’t recognize the constructs from the tables • May plug values into symbolic and match pointwise to numerical values in table but not correspond those points to coordinates on a graph or even consider constructs</td>
</tr>
<tr>
<td>Relational (Relates different aspects together) – Has an understanding of more than one aspect and of their significance to the whole and relationship to other aspects</td>
<td>Correctly matches all of the standard representations together using the same invariant property (input/output, slopes, y-intercepts, etc)</td>
</tr>
<tr>
<td>Extended Abstract (Seeing the concept from an overall viewpoint) – student makes connections within the task and can go beyond it to generalize the underlying principles</td>
<td>Can use the invariant features in the dynagraph to match to the standard representations and can connect all the representations (dynagraph and standard) via that same invariant feature in all of them</td>
</tr>
</tbody>
</table>
Appendix F – Test Instrument

Answer all questions *(Show all work)*

Name___________________

1. Determine the rule for the function represented by the table below.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>11</td>
</tr>
<tr>
<td>-1</td>
<td>9</td>
</tr>
<tr>
<td>0</td>
<td>7</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>-1</td>
</tr>
</tbody>
</table>

2. Complete the table below using the provided function rule.

\[ 2x + 3 = y \]

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>15</td>
</tr>
</tbody>
</table>

3. Complete the table below using the provided graph
4. Draw the graph of the function represented by the table below

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>-2</td>
</tr>
<tr>
<td>-3</td>
<td>0</td>
</tr>
<tr>
<td>-2</td>
<td>2</td>
</tr>
<tr>
<td>-1</td>
<td>4</td>
</tr>
<tr>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
</tr>
</tbody>
</table>

5. Determine the rule for the function represented by the graph below

6. Given that $3y - 6x = 9$ represents the rule of a function, graph the function using the grid below.
# APPENDIX G

## Translation Chart (translation category)

Student: _______________

<table>
<thead>
<tr>
<th>Translation</th>
<th>Strategy</th>
<th>Meaning preserved</th>
</tr>
</thead>
<tbody>
<tr>
<td>S→T</td>
<td></td>
<td>Y N</td>
</tr>
<tr>
<td>T→S</td>
<td></td>
<td>Y N</td>
</tr>
<tr>
<td>S→G</td>
<td></td>
<td>Y N</td>
</tr>
<tr>
<td>G→S</td>
<td></td>
<td>Y N</td>
</tr>
<tr>
<td>T→G</td>
<td></td>
<td>Y N</td>
</tr>
<tr>
<td>G→T</td>
<td></td>
<td>Y N</td>
</tr>
</tbody>
</table>

Comments:
## Appendix H- Rubric for Analyzing Students’ Translation Charts

<table>
<thead>
<tr>
<th>Phase of Concept Development</th>
<th>Examples of how it emerges when evaluating the nature of students’ translations</th>
</tr>
</thead>
</table>
| Pre-structural (lacks knowledge of the assessed component) – may have pieces of unconnected knowledge but make no sense to the student | Student may be able to calculate slope and/or y-intercept but cannot correctly determine the rule for the function  
  • May substitute values in for the wrong variable (y for x, x for y), may substitute all values in for x  
  • Transposing variables or axes (x for y and y for x). Choosing random values in the graph without linking them to the values in the table  
  • Transpose x and y, draw a graph of a random function  
  • Student may identify y-intercept or slope but cannot write the symbolic formula.  
  • Student may solve for x. Student may solve for y and correctly transpose the symbolic formula but still not be able to graph |
| Uni-structural (focuses on a single aspect of a translation) – has an understanding of more than one aspect of the assessed component but not of their significance to the whole or relationship to other aspects—Student understands translations in terms of strategies to use in giving direction but does not understand translations in terms of the different modes of representations and complementary pairs within each mode. | Student can correctly translates from a given source representation to a specified target representation and may solve MORE THAN ONE of the problems in Appendix E but those problems do not display reversibility (i.e., they can do #2 (SÆT) and/or #4 (TÆG), and/or #6 (SÆG) or any combination of those but cannot do any of the reverse procedures in #1, #3 or #5 |
| Multi-structural (focuses on several separate aspects of a translation) – Student has gained flexibility in moving from one mode of representation to another, but little structural knowledge of the underlying notion behind a translation --has an understanding of the idea of complementary translations and of strategies for performing them but not of the equivalency of representations under a translation. | Student correctly performs at least one and at most two translations that displays reversibility (complementary translations) i.e., can do #1 & 2 AND/OR #3 & 4, OR/AND #5&6 |
| Relational (Relates different translations together) – has an understanding of the different modes, strategies and the underlying ideas behind a translation | Students correctly performs all three complementary or reversible Translations i.e., can do #1&2 AND #3 &4 AND #5&6 |