KRISHNAMURTHY, SANDEEP H. Fundamental Limits and Joint Design of Wireless Systems with Vector Antennas. (Under the direction of Prof. Brian L. Hughes.)

Multiple-antenna systems have generated tremendous research interest in the recent past mainly because of their promise of significant gains in capacity and performance as compared to single-antenna systems. Most work on multiple antennas has focused on the design of coding and modulation schemes, channel estimation algorithms and decoding architectures. Information is sent by the transmitter as electromagnetic (EM) waves which subsequently undergo multipath fading before they reach the receiver. The EM properties of the antennas and the nature of the scattering environment jointly impact the performance of communication algorithms. However, there are relatively few works in the literature that consider this interrelation in the design of transmitter-receiver architectures. In this dissertation we study three such problems: the dependence of capacity on the EM properties of antennas and the scattering environment, the limits on performance of parameter estimation algorithms at the receiver and finally, the fundamental limits on the capacity that volume-limited multiple-antenna systems can achieve.

We first consider the joint design of multi-element antennas and capacity-optimal signalling for a multiple-input multiple-output (MIMO) wireless channel. We use EM theory and ray-tracing methods to derive a channel propagation model for antennas that can detect or excite more than one component of the electric field vector (known as vector antennas) in a discrete-multipath channel environment. This model provides insights into the inter-relation between the spatial multiplexing gain and the nature of the multipath environment for vector antennas. We then generalize this model to the case of antennas with more general electric-field patterns in a fading environment with clusters of scatterers. Capacity-optimal signalling and the impact of antenna electric field patterns on capacity are studied. We focus on joint antenna-signal design and derive optimality criterion for multi-element antenna systems for maximizing the ergodic capacity. We show that antennas that have orthogonal and equal norm electric-field patterns maximize the ergodic capacity. Vector antennas satisfy this
criteria, but a uniform linear array does not.

We next consider the problem of positioning and direction-of-arrival (DOA) estimation with ultrawideband (UWB) vector antennas. Due to the wideband nature of the antenna response and directional sensitivity of vector antennas, precise ranging and DOA estimation of a transmitting source can be jointly performed. We first derive a frequency-domain Cramér-Rao Bound formula in the asymptotic case of a large number of observation samples in stationary noise. We apply this formula to two UWB vector antennas and obtain closed-form lower-bound expressions for the ranging and DOA error covariances. A criterion based on the linearized confidence region is used to design signal pulses that give uniform resolving capability to the antennas for any DOA.

Finally, we consider the fundamental capacity limits that a multi-element antenna system that is restricted to occupy a finite volume can achieve. For simplicity, we consider the problem of a spherical volume current source radiating into space with a receiver in the far-field capable of detecting the electric field on a concentric spherical surface. The system is first described as a linear operator, and the exact singular values of the system are derived in closed form. The singular values and hence the capacity is shown to depend on the transmitter volume only through its radius. We calculate the capacity of such a system, and provide capacity formulas that are accurate at high signal-to-noise ratio.
Fundamental Limits and Joint Design of Wireless Systems with Vector Antennas

by

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To my parents
Biography

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Chapter 1

Introduction

The past 25 years have seen a phenomenal increase in the demand for wireless telephony. This is only expected to be matched by demand for wireless data access in the coming decade. There are two main reasons for this development. Firstly, the exponential growth in transistor density on silicon has enabled implementation of Very Large Scale Integrated (VLSI) circuits that run complex signal processing algorithms while consuming less power. This has had a direct impact on the cost of wireless equipment and the state-of-art in commercially available communication technology. Secondly, the success of second-generation cellular standards (GSM and CDMA) has provided newer perspectives on how to better utilize the wireless channel. Several ideas and tools from array signal processing and other areas have found application in the design of wireless systems.

Wireless communications is a challenging task mainly for three reasons. Firstly, the electromagnetic (EM) signal propagates over a time-varying channel. The receiver sees a highly-dispersed version of the transmitted signal due to multiple reflections (or multipaths) and scattering. This effect is known as fading and it severely degrades the quality of the wireless link. Secondly, multiple users share the wireless channel
which leads to *co-channel interference*. This has detrimental effect on the channel quality as well. Thirdly, the bandwidth of transmission is expensive and scarce. The communication system designer is faced with a challenging multi-dimensional optimization problem in overcoming the impairments and achieving the required quality within the resource limitations.

Diversity provides an important set of methods to counter the ill-effects of the wireless channel. The most important forms of diversity are time, frequency, space and multi-user diversity. The basic idea is that, when multiple versions of the same signal are available, the probability that all of them fade simultaneously is small compared to the probability of individual signal fading. One or more forms of diversity are used in all of the third-generation (3G) standards for cellular telephony and data and in emerging standards for home/office local area networks (LANs).

### 1.1 Wireless Communication Systems

In this section, we provide a brief overview of a wireless communication system first in the point-to-point context (i.e., single transmitter and a single receiver) and then in the context of multiple-access, where there are many transmitters and single receiver.

#### 1.1.1 Point-to-Point Wireless Links

A point-to-point wireless link consists of a single transmitter sending signals to a single receiver as shown in Fig. 1.1. In this figure, a base station or an access point communicates with a mobile station or a user terminal in a multipath fading environment. Both the ends are capable of both transmitting and receiving signals. Each of the incoming digital data bits \( \ldots, b_{-1}, b_0, b_1, \ldots \) (modelled as a random process) are
mapped onto complex constellation points $c_k$, $k = 1, 2, \ldots, |C|$ which belong to a set $C$ ($| \cdot |$ denotes the number of elements in set $C$). In this scheme, the data stream is divided into blocks of $\log |C|$ bits and are transmitted over a duration of $T_s$ seconds (known as the symbol duration). The symbol for the time period $[(n-1)T_s, nT_s]$ is chosen to be a point from the constellation $C$ and it is denoted by $x[n]$. This symbol is then multiplied with a real pulse $g(t)$ (known as the shaping function) with non-zero time support of $T_s$ seconds, therefore resulting in a signal with the following baseband representation

$$s(t) = \sum_{n=-\infty}^{\infty} x[n]g(t-nT_s).$$

(1.1)

This signal is then modulated from baseband by multiplying it with a sinusoid of frequency $f_c$ and transmitted over the antenna as an EM wave. As the signal propagates from the transmitter to the receiver, it often suffers multiple reflections and scattering due to obstacles in the propagation environment. Because these reflections typically experience different propagation delays and attenuation, multiple copies of the original signal arrive at the receiver. Most often, this phenomenon is approximated by a time-varying linear transformation. At the receiver, the signal is demodulated to baseband from center frequency $f_c$ and is passed through a matched-filter. The complex-baseband equivalent of the signal after demodulation and matched-filtering can be written as

$$y(t) = \int_{-\infty}^{\infty} h(t,\tau)s(t-\tau)d\tau + w(t)$$

(1.2)

where $h(t, \tau)$ is the causal impulse response function (i.e., $h(t, \tau) = 0$ for $\tau < t$) which incorporates the effects of the signal pulse shape, channel impulse response and the matched filter. The term $w(t)$ is the background noise that is independent
of the signal transmitted. The filtered signal is then sampled at a rate of $1/T_s$ and processed further in discrete-time. With a slight abuse of the notation, we denote the discrete-time sample at time $t = nT_s$ by $y[n]$. With similar notations for $h(t, \tau)$ and $w(t)$, we can approximate the input-output relationship using an $L$-tap channel model as

$$y[n] = \sum_{l=0}^{L-1} h[n, l] x[n - l] + w[n]. \quad (1.3)$$

The number of channel taps naturally is a function of the signalling rate and the time spread of $h(t, \tau)$ in its second argument. Due to the random nature of the channel fluctuations, the channel impulse response $h(t, \tau)$ is modelled as a random process that is independent of the signal $x(t)$ and the noise $w(t)$. The signal $x(t)$ is assumed to be independent of the noise $w(t)$ as well. The distribution of $h(t, \tau)$ depends on the channel dynamics. This is referred to as the fading channel. The time interval for which the channel impulse response remains approximately constant is called coherence time of the channel and is denoted by $T_c$. The length of the impulse response of the channel is called the delay spread. The coherence time indicates how fast the channel is changing; a smaller coherence time means a faster changing channel. The delay spread indicates the frequency diversity; the larger it is the better the ability of the channel to resolve multiple copies of the signal.

The objective of the detector is to infer the data transmitted (i.e., the sequence $\ldots, b_{-1}, b_0, b_1, \ldots$) from the received signal $y[n]$. Due to the random nature of the fading channel, the bits cannot be recovered with certainty and there is an associated bit-error rate (BER) which is defined as the expected probability a transmitted bit is detected incorrectly. For a fixed transmitter power, there is a trade-off between the data rate and the BER. The BER is a function of the signal-to-noise ratio (SNR)
which is usually defined as the ratio of the average signal power to the average noise power.

In order to combat the impairments of the channel and improve the BER, diversity techniques are often employed. We list the major diversity techniques that are employed in today’s systems.

1. **Time diversity** is achieved by averaging the random channel response over time through error-control coding or interleaving or both. Codes that add redundancy help to improve the BER, thereby leading to better signal recovery or lower power requirements. Errors due to signal fading typically occur in bursts. Interleaving helps to spread these errors over a longer duration of time and is often used in conjunction with coding. Convolutional codes, turbo-codes and low-density parity-check (LDPC) codes are some of the most common coding schemes. Repeated transmissions (like Automatic Repeat Request used in random access systems) are another form of time diversity.

2. **Frequency diversity** exploits the fact that signals separated in the frequency domain by more than a certain coherence bandwidth (defined as the inverse of the delay spread) fade differently. Thus multiple copies of the signal transmitted at different frequencies can be combined to improve performance. Frequency diversity can also be exploited by wideband systems that are capable of resolving and combining multiple delayed signal reflections at the receiver. Code-Division Multiple-Access (CDMA) systems such as the Direct-Sequence CDMA and Frequency-Hopping CDMA as well as the Orthogonal Frequency-Division Multiplexing (OFDM) systems are based on this approach.

3. **Spatial diversity** exploits the fact that when a system uses multiple antennas to transmit or receive signals in a multipath fading environment, the channel co-
Efficiencies associated with different antennas fade independently. Receive diversity is achieved through the use of multiple antennas at the receiver. Antennas separated by a certain distance or with certain decorrelation properties receive independently faded copies of the transmitted signal. Combining these copies in a suitable way leads to performance gains. The most notable techniques for combining are Maximum-Ratio Combining (MRC), Selection Diversity and Equal-Gain Combining (EGC). Transmit diversity is achieved through the use of multiple transmit antennas. The signals from sufficiently-separated antennas fade independently and the data stream can be coded across the antennas to obtain performance benefits [26, 77].

As mentioned before, there exists a trade-off between the data rate and the BER. In 1948, Shannon in his groundbreaking work [78] showed that there is a fundamental limit on the data rate at which one can transmit data reliably. This is known as the capacity of the channel and is defined in terms of certain information-theoretic quantities [12]. He also proved that for any data rate below capacity, there exists a coding scheme (a method for mapping the information bits into constellation points) for which the BER can be made arbitrarily small. We provide a more detailed discussion of capacity in Section 1.2, in the context of multiple-antenna systems.

1.1.2 Multiple-Access Systems

The point-to-point link (also known as single-user channel) was described in the previous section. In this section we discuss multiple-access systems. A system in which two or more transmitters share the same channel to transmit data to a single receiver is known as multiple-access system. Time-Division Multiple-Access (TDMA) is a channel access scheme in which time is divided into a number of slots and users are distinguished at the receiver by the times at which they transmit [67]. Frequency-
Division Multiple-Access is a scheme in which the available frequency band is subdi-
vided into a number of subbands and users share these to transmit their signals [67].
Code-Division Multiple-Access (CDMA) is a technique in which different pseudo-
random sequences are assigned to different users to modulate the data. These code
sequences spread the available energy over the entire bandwidth and all the users
share the both time and the bandwidth of signalling.

1.2 Multiple-Antenna Systems

Over the past decade, there has been an increasing interest in the possible use
of antenna arrays to improve the performance of wireless systems. It has long been
known that spatial diversity systems that employ multiple antennas at the receiver can improve performance over single antennas in multipath fading channels. It was shown in [26, 77] that transmit diversity achieved by using multiple antennas at the transmitter can also yield a better tradeoff between BER and data rate. In their seminal papers, Telatar [92] and Foschini and Gans [21] showed that the capacity achievable by using multiple antennas both at the transmitter and receiver scales with the SNR in decibels (dB) as the minimum of the number of transmit and receive antennas. (Other related work on capacity can be found in [61, 70, 71].) This result means that the spectral efficiency can be increased to tens of bits per second per Hertz of bandwidth. This was later demonstrated by the Bell Labs Space Time (BLAST) architecture [20], which spatially multiplexes several data streams and decodes each data stream by successively cancelling the interference caused by the other streams. Multiple antennas have been incorporated in many of the 3G standards. The cellular wireless access standards, cdma2000 and Wideband CDMA (HSDPA) both require multiple-antenna transmission at the base-station for high-data-rate transfer [34, 95]. Hiperlan and IEEE 802.11n require the use of multiple antennas with orthogonal frequency-division multiplexing to increase the range and data rates in a wireless local area network [18, 39]. For further details on MIMO systems, [62] serves as an excellent survey of the early work on this topic, and [52, 93] provide a good overview of current state of the area. For a sampling of recent work on MIMO systems, the reader is referred to the special issues [82, 83].

In this section we provide an overview of the results on the multiple-input multiple-output (MIMO) wireless channel. We then introduce space-time codes as a means of achieving robust performance at a low receiver complexity and present examples of approaches to systematic code design. We finally conclude this section with a brief sketch of the impact of MIMO technology on some 3G standards for indoor and
outdoor wireless access.

1.2.1 The MIMO Channel

MIMO wireless systems employ multiple antennas at both the transmitter and the receiver. The physical environment in between acts as conduit for electromagnetic signals that are emitted from the transmit antennas as in Fig. 2.1. The radiated signals undergo reflection, refraction, diffraction and thereby suffer amplitude and phase changes. Further, the environment is dynamic and this leads to unpredictability. Consider a MIMO system with $M$ transmit antennas and $N$ receive antennas. Suppose that at time instant $t$, the transmitter modulates a sinusoid with a signal with the baseband representation $x_i(t)$ from the $i$th antenna. This signal travels through a matrix channel with the impulse response at time $t$, $H(t, \tau)$ which incorporates the signal pulse shape, channel impulse response and the matched-filtering at the receiver. The signal induced in the $j$th receiver antenna denoted by $y_j(t)$ is modelled as linear function of the transmitted signal and is expressed as

$$y(t) = \int_{-\infty}^{\infty} H(t, \tau)x(t - \tau)d\tau + w(t),$$

(1.4)

where $x(t) = [x_1(t), \ldots, x_M(t)]^T$, $y(t) = [y_1(t), \ldots, y_N(t)]^T$ and $w(t)$ represents the interference and noise modelled as a random process. The quantities $x(t)$, $y(t)$ and $w(t)$ respectively are $M \times 1$, $N \times 1$ and $N \times 1$ complex vector-valued functions of time, and $H(t, \tau)$ is matrix-valued function of size $N \times M$. For any $t$, the channel impulse response $H(t, \tau)$ has a non-zero time support in the argument $\tau$ which depends on the delay spread of the channel. The amplitude, phase and delay spread of the channel are time-varying in general and this is reflected by the dependence of $H(t, \tau)$ on $t$. The signal is processed in discrete time after matched-filtering and sampling at the
rate of signalling (symbol rate or chip rate depending on the system), and the channel can be modelled as a $L$-tap impulse response. With a slight abuse of notation, we write the signal at the instant $n$ as

$$y[n] = \sum_{l=0}^{L-1} H[n, l] x[n - l] + w[n].$$  \hspace{1cm} (1.5)$$

The number of taps $L$ is a function of the sampling rate and the delay spread of the channel and bandwidth of the signal $x(t)$. Consider the case when the signal bandwidth is small enough such that there is not much appreciable delay spread. For this case the channel can be written as

$$y[n] = H[n] x[n] + w[n].$$  \hspace{1cm} (1.6)$$

The term $w[n]$ is complex white Gaussian noise process. To capture the randomness in the channel, $H[n]$ is modelled as a random process, where the nature of the channel dynamics and propagation environment determine the distribution. For example, if the channel changes from one signalling instant to another rapidly enough $H[n]$ can be modelled as an ergodic random process which is uncorrelated over time. If the channel changes relatively slowly, then it can be modelled as a block-fading channel, where $H[n]$ assumes a particular value and stays constant for $N_c$ symbol intervals and then takes a new realization. As before $N_c$ is called the coherence interval.

A common assumption is that $H[n]$ is an ergodic $N \times M$ matrix with i.i.d. circularly symmetric complex Gaussian entries with zero mean and unit variance, denoted as $\mathcal{CN}(0, 1)$. The noise vector $w$ is modelled as an $N \times 1$ vector with i.i.d. circularly-symmetric complex Gaussian entries with zero mean and variance $\sigma^2$, denoted by $\mathcal{CN}(0, \sigma^2)$. The transmitter has the power constraint $\mathbb{E} x^\dagger x \leq 1$, where $\dagger$ denotes the conjugate-transpose operation. The receive SNR $\rho$ is defined as the signal-to-
noise power ratio at each receive antenna (which is same for each antenna from our
definition of the channel distribution), and it can be shown that $\rho = 1/\sigma^2$.

In the previous section, we mentioned that capacity is the maximum data rate at
which we can transmit data reliably. However, for time-varying channels, there are
at least two different notions of capacity. When the channel changes quickly and the
data can be encoded many independent realizations of the channel, the appropriate
measure of capacity is the *ergodic capacity*. It is defined as the supremum of the
mutual information between $x$ and $y$. $C_e(\rho) = \sup_{p_x} E_H I(x; y|H)$, where the supre-
mum is over all input probability distributions $p_x$ subject to the power constraint
$E x^H x \leq 1$, $E_H$ is the expectation with respect to the distribution of $H$, and $I(x; y|H)$
is the conditional mutual information between the variables $x$ and $y$ conditioned on
the channel realizations $H$ [12]. The input distribution $p_x$ that achieves the supre-
mum is said to achieve ergodic capacity. When the channel changes slowly enough
that it can be regarded fixed over the duration of the codeword, a more appropriate
measure of capacity is the *outage capacity*. In this case, the transmitter cannot code
the information bits over different realizations of the the channel. When the trans-
mitter is attempting to send data at a rate $R$, the event when channel capacity falls
below $R$ is known as an *outage* event. The probability of the outage event is given by
$P_{out}(R) = \text{Prob}(I(x; y|H) < R)$, and the input distribution $p_x$ that minimizes this
probability is said to achieve the *outage capacity*.

Telatar [92] showed that the ergodic capacity is achieved with i.i.d. Gaussian
inputs at the transmitter and is given as a function of the receive SNR by

$$C(\rho) = E_H \log \det (I_N + (\rho/M)HH^H),$$

(1.7)

where $I_N$ denotes the $N \times N$ identity matrix, log is the base-2 logarithm, the ex-
pectation is with respect to the distribution of $H$, the $\det$ denotes the determinant
operation and \( H^\dagger \) denotes the conjugate-transpose of \( H \). It was further shown in [92] that at high SNRs, the capacity is given asymptotically by

\[
C(\rho) = \min(M, N) \log \rho + o(\log \rho),
\]

where \( o(\log \rho) \) denotes terms such that \( o(\log \rho)/\log \rho \to 0 \) vanishes as \( \rho \to \infty \). For a single-input single-output (SISO) channel \( (M = N = 1) \), the capacity scales as \( \log \rho + o(\log \rho) \). This means that the MIMO system is asymptotically equivalent to \( \min(M, N) \) parallel SISO channels or data pipes. This gain is due to the combined space-time diversity of the systems. For wideband signals, where \( L > 1 \), frequency diversity along with space-time diversity adds capacity benefits.

### 1.2.2 Spatial Multiplexing

In the BLAST architecture [20], the incoming data stream is demultiplexed into \( M \) substreams which are transmitted on the \( M \) antennas. The receiver uses successive nulling followed by symbol detection to decode all of the transmitted substreams. This modulation scheme employed here is known as spatial multiplexing. It is as if the matrix channel is equivalent to many parallel subchannels, and different data streams can be transmitted on each of those subchannels. Spatial multiplexing gain is defined as 

\[
r = \lim_{\rho \to \infty} C(\rho)/\log \rho
\]

and indicates the number of equivalent parallel subchannels the matrix channel supports at high SNRs [100]. The diversity gain is defined as

\[
d(r) = -\lim_{\rho \to \infty} \log \text{Prob}(C(\rho) < r \log \rho)/\log \rho.
\]

Recently it was shown the maximum diversity gain \( d(r) \) for any multiplexing gain \( r \) satisfies 

\[
d(r) \leq (M - r)(N - r), [100].
\]

Thus, there exists a trade-off between these
two types of gains.

1.2.3 Space-Time Codes

In order to exploit the promised benefits of MIMO systems, the data at the transmitter needs to be coded across both space and time. Several systematic code design approaches have been proposed in literature. We first examine the low-rate designs proposed in [90]. Suppose that data bits are transmitted for $T$ consecutive symbol intervals and the corresponding transmitted signals are stacked as an $M \times T$ matrix $X = [x[1], x[2], \ldots, x[T]]$. Further suppose that $X$ is selected from a set $C$ with finite number of elements, such that the transmission rate is given by $R = \log |C|/T$, where log is the base-2 logarithm. That is, for every $T$ seconds, $\log |C|$ bits of the incoming
bit stream is mapped on to one of the code matrices. The received data collected for
$T$ intervals are stacked similarly as $Y$ and can be written as

$$Y = HX + W,$$

where $W$ is the $N \times T$ noise matrix of i.i.d. $\mathcal{CN}(0,1/\rho)$ entries. Suppose that $X_0$ is
transmitted and maximum likelihood detection is employed at the receiver. It was
shown in [26, 90] that the expected pairwise-error probability of mistaking $X_0$ for
another valid code matrix $X_1$ is asymptotically given by

$$\text{Prob}(X_0 \rightarrow X_1) \approx \left( \det \left( I_N + \frac{\rho}{4} (X_0 - X_1)(X_0 - X_1)^\dagger \right) \right)^{-N},$$

at high SNR. The performance of codes at high SNRs is limited by the worst-case
pairwise error and designs that minimize this quantity are mostly considered. For
high SNR, the pairwise-error probability scales asymptotically as $\Lambda_{01}^{-N} \rho^{-rN}$, where $r$
is the rank of the difference matrix $X_0 - X_1$ and $\Lambda_{01} = (\lambda_1 \ldots \lambda_M)^{1/r}$ is the geometric
mean of the $r$ non-zero eigenvalues $\lambda_1, \ldots, \lambda_r$ of $(X_0 - X_1)(X_0 - X_1)^\dagger$. The minimum
rank $r$ of the pairwise differences and minimum $\Lambda_{ij}$ (for all pairs of matrices $X_i$ and
$X_j$ in the set $\mathcal{C}$ such that $i \neq j$) determine the average error probability at high SNR.

The minimum rank $r$ is called the diversity order and the corresponding $\Lambda_{ij}$ is known
as the coding gain (or the product distance). This is in contrast to a SISO system,
where the average pairwise-error probability at high SNR is proportional to worst-case $\Lambda_{ij}^{-1} \rho^{-1}$. In [2], Alamouti proposed a simple space-time code for $T = M = 2$
based on orthogonal designs. Codes based on orthogonal designs enable data symbols
to be decoded at the receiver in a very simple way; in fact for $N = 1$, decoding of
the symbols can be decoupled at the receiver. In [89], several other codes based on
orthogonal designs were been proposed.
Several approaches have been considered for the systematic design of high-rate codes. The most notable among them are the linear space-time spreading codes [30]. In this approach, the incoming bits linearly modulate code matrices known as dispersion matrices. The objective is to find dispersion matrices that maximize the ergodic capacity of the system subject to such modulation. Some other high-rate code designs based on the product distance criterion but with structured mappings from the bit sequence to code matrices have been considered. For example, [80] uses Cayley transform to map the data symbols onto unitary matrices, and codes that maximize the product distance are proposed.

1.3 Contributions of this Work

In this section, we give an overview of the main issues considered in this dissertation, as well as a brief outline of the approaches followed and the main results obtained.

Most MIMO systems currently use uniform linear arrays (LAs), which consist of uniformly spaced dipole antennas [5]. When LAs are used as receivers, each antenna measures the projection of the electric field (E-field) along its axis. Thus, using an array of parallel dipoles is equivalent to uniform spatial sampling of the E-field in the direction of the dipoles. It is natural to ask whether sampling the E-field in one direction is rich enough to capture all the useful information that is present in the electromagnetic (EM) field. In this context, several interesting questions arise.

Q1. Does sampling of the remaining two orthogonal components of the E-field provide additional information about the transmitted signal? What about sampling the magnetic field (H-field) components in addition to the E-field using loop antennas? Does it provide additional independent measurements of the
transmitted signal?

Q2. How does the EM field response of the antenna elements impact the spatial multiplexing gain and diversity gain? In the antenna design literature, several metrics such as power radiation pattern, radiation efficiency, mutual coupling and directivity are used. Do antennas that are optimal in the sense of one or more of these measures maximize capacity?

Q3. What properties of the antenna array most impact the capacity? Is it possible to define optimality criteria that multi-element antennas need to satisfy in order to maximize capacity given the *average* propagation conditions?

Multi-element antennas known as vector antenna (VAs) that are sensitive to more than one component of the electric and magnetic field vectors have been considered in the literature [13,31,44]. As mentioned before, a single dipole measures the E-field component in one direction. The other two components of the E-field can be measured by placing two more dipoles orthogonal to each other, and to the original dipole as shown in Fig. 1.3. A multi-antenna structure with three orthogonal dipoles is called a *tripole* [13]. Loop antennas [5] measure the component of the H-field parallel to their axis. Therefore, in principle, a 6-element vector antenna can be constructed by combining three electric dipoles to detect the electric field components, with three loops (also called magnetic dipoles) to detect the magnetic field components. Vector antennas that respond to all six EM field components have been investigated [31,44]. The use of such antennas to estimate the direction-of-arrival of electromagnetic signals in line-of-sight propagation has been extensively investigated in narrowband systems [37,46,56]. Recently, it was noted that these antennas can be used in a similar manner as spatially-separated antennas in a MIMO wireless link leading to capacity gains [3]. In [86], a two-antenna structure with orthogonal dipoles (also called “dual-polarized”)
was considered and it was demonstrated by measurements that these antennas lead to doubling of capacity at high SNR with respect to single antenna systems (or equivalently the dual-polarized antenna system has a spatial multiplexing gain of 2). In [47], a three-element antenna structure made up of two dipoles and a loop as shown in Fig. 1.4 was proposed. A four-element structure, shown Fig. 1.5, which adds a third orthogonal dipole to the three-element structure was also proposed. It was shown by measurements the three element and the four element structures respectively give spatial multiplexing gains of 3 and 4, respectively. These developments therefore provide a partial answer in the affirmative to question Q1 posed above. (It is yet to be demonstrated experimentally that a six-element “complete” structure with three dipoles and three loops give a spatial multiplexing gain of 6.)

In Chapter 2 of the dissertation, we consider questions Q2 and Q3 posed above. We first develop a signal propagation model for a wireless link with vector antennas in a discrete-multipath environment. We use the insights gained by this simple model to propose a propagation model for antenna systems with arbitrary EM field patterns in a channel environment with clusters of scatterers. This model jointly characterizes the antenna radiation properties and the scattering environment, thereby allowing us to study their impact on the capacity of the wireless link. In addition, mutual coupling that arises due to near-field EM interaction between the antenna elements can be
easily incorporated into the signal model. This framework is then used to derive an optimality criterion for multi-element antenna systems in order to achieve maximum capacity subject to power constraints. A high SNR capacity formula that allows us to compare the capacities of two different antenna systems is given. This gives us a metric to compare two different multi-element antenna systems. We show that, under rich scattering, multi-element antennas with elements having orthogonal electric field patterns and that are equal power radiators maximize the capacity. It is also shown that vector antennas are optimal in this sense.

As mentioned before, the problem of estimating the direction-of-arrival of electromagnetic signals in line-of-sight propagation has been extensively investigated in narrowband vector antenna systems [37, 46, 56]. These devices are relatively narrow-
band \(< 30\) MHz) and extremely large (many times the wavelength). Over the last few years, there has been a growing recognition that ultrawideband (UWB) technology \([91]\) offers several unique capabilities that enable a host of new radar-type positioning and communication applications. There is a rich literature on beamforming with spatial arrays of UWB antennas (e.g., \([29,38,91]\)) as applied to various radar and communications algorithms. Most work on ultrawideband systems has thus far focused on single-polarized electric dipole antennas. Dual-polarized antennas have been studied in certain radar applications, such as ground-penetrating and synthetic-aperture radars (e.g., \([8,10,17]\) and references therein). Single- and dual-polarized antennas can measure at most two components of the received signal. These antennas neglect additional independent components of data that might be available.
to improve the performance of the sensing, positioning, and communications algorithms. An “UWB vector antenna” that can independently detect or excite three or more EM field components enables the UWB system to access additional signaling dimensions, which can be used to enhance performance in the same way as antenna arrays. A three-element planar UWB-VA with two dipoles and one loop has recently been proposed in [68,69] as shown in Fig. 3.1.

An incoming EM wave at the receiver contains information about the source range (distance from the receiver) in its propagation delay, and direction information due to the spatial-sensitivity of the antenna elements. We therefore consider the problem of jointly estimating the range and the direction-of-arrival (DOA) of the incoming signal using vector antennas. Two new questions arise when using UWB antennas that are polarimetrically sensitive.

**Q4.** The antenna response function for an incoming plane wave is frequency dependent. In addition, there could be frequency-dependent mutual coupling between the antenna elements. How does this impact the design and performance of range and DOA estimators?

**Q5.** Unlike traditional UWB antennas, UWB-VAs are sensitive to different polarizations of the incoming signal. How can we best exploit the polarization at the receiver and what impact does it have on the resulting performance of the estimation algorithms?

In Chapter 3, we consider the questions Q4 and Q5 above. We consider the use of UWB-VAs to estimate the range and the DOA of a source located in the far-field of the receiver. We assume that the UWB signal is observed in the presence of an additive stationary noise process with a block-Toeplitz covariance matrix. The Cramèr-Rao Bound (CRB), which gives a lower bound on the error variance of any unbiased
estimator takes the form a simple frequency-domain integral in the asymptotic limit of a large number of sample points of the observed data. We study the impact of the properties of the UWB signal on the error covariances and on another performance metric called the *volume of linearized confidence region* which is a measure of the uncertainty associated with the location estimator in Cartesian coordinates. Finally, we propose signal designs that minimize the confidence volume.

The Rayleigh flat-fading channel model predicts a linear scaling of capacity with the minimum of the number of transmit and receive antennas in a MIMO wireless link [92]. However, this model assumes that the antennas can always be separated enough to achieve independent fading on each one. It is therefore natural to ask how these results are altered if the transmit and receive arrays are constrained to lie within some fixed volume. Finite volume restrictions can introduce antenna correlation, which can result in sublinear scaling of capacity [63]. This is because, as the antennas get closer the signal they receive become less independent. Therefore, there appears to be a dependence of capacity on the volume that the antennas are allowed to occupy.
This raises some additional questions:

**Q6.** What is the *best* capacity that one can attain when the antennas are restricted to occupy a fixed volume? How does this capacity depend on the volume?

**Q7** If we are interested constructing an $M$ element array limited to a fixed volume, then what kind of antennas achieve the maximum capacity?

In Chapter 4 of the dissertation, we consider the two questions above. For simplicity we consider a volume-limited transmitter and an all-powerful receiver capable of measuring all of the transmitted EM field in the far-field region. The transmitter is assumed to generate an arbitrary current distribution enclosed within a fixed volume. This current distribution serves as the generalization of any particular antenna geometry. The received electric field in the far-field region is expressed as a linear transformation of the current distribution using the dyadic Green's function formulation of [87]. The singular values of the transformation are given in closed form. With this approach, we can represent the original system as infinitely-many parallel Gaussian channels and well known techniques for finding the capacity can be employed. A capacity formula that is accurate at high SNR is given. It is argued that spatial basis function of the $M$ “strongest” eigenfunctions correspond to the $M$-element antenna system that maximizes capacity.

### 1.4 Organization

In the previous section we discussed the problems considered in this dissertation, their motivation and provided an outline of the approach followed. In this section, we briefly discuss the organization of the dissertation.

In Chapter 2, we consider the joint design of a multi-element antenna system and capacity-optimal signalling for a MIMO wireless channel. In Section 2.2, we derive
a channel propagation model for vector antennas in a discrete-multipath channel environment. In Section 2.3, this model is generalized to the case of antennas with more general electric field patterns in a fading environment with clusters of scatterers. Section 2.4 presents a discussion on capacity-optimal signalling. In Section 2.5, a capacity-optimizing criterion for multi-element antenna design is presented. Finally, we present simulation results and then summarise our conclusions in Section 2.6.

In Chapter 3, we consider the problem of estimating range and DOA of a target with ultrawideband vector antennas. In Section 3.2, we present some mathematical preliminaries. In Section 3.3, we derive frequency-domain Cramér-Rao Bound formula for the asymptotic case of a large number of sampled observations in stationary noise. In Section 3.4, we apply the Cramér-Rao Bound to two particular vector antenna designs, and derive closed-form error lower bounds for the estimation error covariance. In Section 3.5, we define different optimality criteria and derive conditions that the signals need to satisfy in order to satisfy the optimality criteria. Finally, we conclude the chapter with a summary of results.

In Chapter 4, we consider the fundamental capacity limits for any multi-element antenna system restricted to a fixed volume. In Section 4.2, we formulate a transmitter-receiver problem using EM theory and describe the channel as a linear transformation. In Section 4.3, we present the capacity expressions for the channel described. We present numerical results in Section 4.4, and finally conclude the chapter with a summary of results and some directions for future research.
Chapter 2

Optimal Array and Signal Design in Vector-Antenna Systems

This chapter considers the vector-antenna array system in a discrete-multipath channel and develops a signal model using the joint electromagnetic propagation of the signal. This model is extended to a more general setting, providing an explicit characterization of the channel model in terms the electromagnetic properties of the antenna and the nature of the scattering environment. This model is used to design capacity-optimal signalling and provide design criteria for multi-element antenna elements.

2.1 Introduction

Multiple-Input Multiple-Output (MIMO) systems have generated tremendous research interest in the past decade mainly because of two reasons: a promise of improvement in data rate and increased robustness of communication schemes to multipath fading and other physical environment impairments. Most work on multiple-antenna
systems has assumed a Rayleigh fading environment with i.i.d channel components for analytical tractability [92]. The capacity under the assumption that the channel is ergodic then scales as \( \min\{M, N\} \log \text{SNR} \), where \( M \) and \( N \) are the number of transmit and receive antennas, respectively, and \( \text{SNR} \) is the signal-to-noise power ratio at each receive antenna. To better model the physical channel, the joint impact of antenna array limitations and nature of the fading environment is taken into account by introducing correlation between the channel entries as a function of geometry of the antenna array (for example [79]).

The signal is transmitted as modulated bandpass electromagnetic (EM) waves which undergo scattering through multiple reflectors before reaching the receiver. A linear-time-varying (LTV) model of this phenomenon is widely used for system design and performance evaluation. Therefore, careful modelling of the channel is crucial in achieving all the benefits that spatial, polarization, frequency, and time diversity offer. Recent channel measurement campaigns (e.g. [9,66,84]) for the indoor channel have revealed that the signal propagates through a small number of clusters of scatterers. In general, the delay spread and angular spread are related to the geometry of the scatterers with respect to the transmit and receive antennas, and a larger delay spread often means a larger angular spread. This multipath diversity can be exploited to improve receive SNR and capacity.

Most of the literature on MIMO channels assumes a uniform linear arrays (LAs) of dipoles at both the transmitter and the receiver. The dipole antennas measure the projection of the electric field (E-field) vector along their axes and this results in a simple antenna pattern with a phase lag proportional to their relative locations. In the recent past several different multi-element antennas that do not belong to the class of LAs have been proposed. In particular, antenna systems that detect and excite more than one component of the EM field vector, called “vector antennas”,

have been considered [13, 31, 44].

The aim of this chapter is to develop simple models for communication channels that employ vector antennas (VAs) at the transmitter and receiver, and to use these models to gain insight into the impact of VAs on wireless system performance. We first develop a model for vector antennas in a discrete multipath channel. This model is then extended to the case of multi-element antennas with arbitrary radiation patterns in a fading environment with clusters of scatterers. This model takes into account the effect of both the antenna E-field patterns and the scattering geometry on the channel statistics. The advantages of developing a model for this dependence are three fold:

1. It allows us to quantify the impact of antenna E-field patterns on the capacity of multiple-antenna systems. Mutual coupling between different elements in an antenna array can lead to either improvements in capacity or impairments. This model includes the case when there is mutual coupling, and the impact of coupling effects on system performance is analyzed. In addition, the widely-used Kronecker model (separable transmit-receive correlation structure) is also a special case corresponding to a scattering environment with a single cluster.

2. It enables us to develop design criteria for multi-element antennas by optimizing suitable measures. In particular, we will derive a criterion that multi-element antennas need to satisfy in order to maximize the ergodic capacity.

3. The input signals can be jointly designed with the antenna system leading to better performance.

The remainder of this chapter is organized as follows. Section 2.2 presents the discrete multipath model for vector antennas. This model is extended to a more general setting in Section 2.3. Section 2.4 presents a discussion of the capacity for such a model. In Section 2.5, we derive an optimality criteria for multi-element antenna
design. Section 2.6 presents simulation results and we finally conclude the chapter with a summary of the results. The notation adopted in this chapter is as follows: If $A$ is a matrix, $A^\top$ denotes the transpose, $A^\dagger$ denotes the Hermitian transpose, $\overline{A}$ represents the complex conjugate matrix, $\det(A)$ is the determinant, $r(A)$ is the rank, $tr(A)$ denotes the trace. The term $A_{[m,n]}$ represents the $mn$-th entry of $A$; $A_{[m,:]}$ and $A_{[,n]}$ represent the $m$-th row and $n$-th column of $A$ in a similar fashion. The symbol $X \succeq 0$ indicates that the matrix $X$ is positive semidefinite. The symbol $\mathbf{E}$ denotes expectation with respect to the underlying distribution. Circularly symmetric jointly normal random variables with mean $m$ and variance $\sigma^2$ are denoted by $\mathcal{CN}(m, \sigma^2)$. The term $\log$ denotes the logarithm with respect to base 2, and $\ln$ denotes the natural logarithm. The function $\delta(\tau)$ denotes the Dirac-delta function.

### 2.2 Signal Model: Discrete-Multipath Channel

In this section, we develop a simplified model of multipath signal propagation which captures many key features of the vector antenna environment, but which is tractable enough to allow a simple analysis of different antennas, communication algorithms, and propagation conditions. For simplicity, we first consider vector antennas capable of detecting and exciting all six field components and a propagation environment which consists of discrete multipath scatterers. Most antennas of interest detect only a subset of the field components (or some function thereof), but this example will serve to illustrate our approach. In Section 2.3 we extend this model to antennas with arbitrary E-field radiation patterns and multipath channels with clusters of scatterers. Antennas with three and six electromagnetically-independent ports have been considered in [13, 31, 44], and narrowband vector sensors with 6 electromagnetically-independent ports are also commercially available [56]. Vector
antennas with 2, 3 and 4 elements have been proposed and the independence of the measured EM components has been demonstrated in [47,86].

Consider a channel in which the signal propagates from the transmitter to the receiver by $\Gamma$ dominant paths, each created by scattering and reflection from physical objects in the environment as shown in Fig. 2.1. We assume that each of the dominant scatterers is sufficiently far from both the transmit and receive arrays to justify a far-field approximation. Thus, each multipath component arrives as a polarized plane wave at the scatterer, and again as a plane wave at the receive array. This model is similar to the widely-used Raleigh model for multiple-input multiple-output wireless scalar antenna channels [70, 71]. Here the model is generalized to describe the joint behavior of the complete electric and magnetic fields.

Consider first a single vector antenna at the receiver. Each arriving multipath
component is a two-dimensional vector \( r(t) \), consisting of the horizontally and vertically polarized components of the electric field. For narrowband sinusoidal signals, it is convenient to write the signal in terms of its complex envelope \( z(t) \), where \( r(t) = \text{Re}\{\exp(j\omega_c t)z(t)\} \) and \( \omega_c \) is the carrier frequency.

If the multipath component is reflected or scattered by an object in the far-field, then the signal can be approximated as a plane wave at the receiver. Let \( E(t) \) and \( H(t) \) denote the 3-dimensional complex envelopes of the electric and magnetic field vectors at the receiver, and suppose that the multipath component arrives at the
sensor from the direction $-\mathbf{u}_r$, where

$$
\mathbf{u}_r(\theta, \phi) = \begin{bmatrix}
\cos \phi \sin \theta \\
\sin \phi \sin \theta \\
\cos \theta
\end{bmatrix}
$$

where $\phi$ and $\theta$ are the azimuth and elevation, respectively, of the incoming signal in receiver coordinates (cf. Fig. 2.2). With this spherical parametrization, we have $\phi \in [0, 2\pi]$ and $\theta \in [0, \pi)$. ($\theta = \pi/2$ corresponds to the horizontal or x-y plane and $\theta = 0$ corresponds to the zenith). For a narrowband plane wave propagating in a nonconductive, homogeneous, and isotropic medium, the received signal can be modelled by [56],

$$
y(t) = \begin{bmatrix} E(t) \\ \eta H(t) \end{bmatrix} = B(\theta, \phi)z(t) + n(t)
$$

where $n(t)$ represents zero-mean spatio-temporally white proper\textsuperscript{1} Gaussian noise process with a covariance $\mathbb{E}[n(t + \tau)n^\dagger(t)] = I_6 \delta(\tau)$, $\eta$ is the intrinsic impedance of the propagation medium, and

$$
B(\theta, \phi) = \begin{bmatrix}
-\sin \phi & \cos \phi \cos \theta \\
\cos \phi & \sin \phi \cos \theta \\
0 & -\sin \theta \\
\cos \phi \cos \theta & \sin \phi \\
\sin \phi \cos \theta & -\cos \phi \\
-\sin \theta & 0
\end{bmatrix}.
$$

The signal produced by a vector antenna at the transmitter can be modelled in

\textsuperscript{1}A stationary complex random process $n(t)$ is proper if the pseudo-covariance $\mathbb{E}[(n(t) - \mathbb{E}n(t))(n(t) - \mathbb{E}n(t))^\dagger]$ vanishes for all $t$ [55].
a similar way. Consider a vector antenna that acts as a point source generating the electric and magnetic fields

$$\mathbf{x}(t) = \begin{bmatrix} E'(t) \\ \eta H'(t) \end{bmatrix}.$$ 

Ignoring path losses and delays and invoking reciprocity, an obstacle located in the far field of the antenna “sees” a polarized plane wave with complex envelope proportional to

$$z'(t) = B^\dag(\theta', \phi')\mathbf{x}(t),$$

where $\phi'$ and $\theta'$ are the obstacle azimuth and elevation, respectively, in transmitter coordinates, and $\dag$ denotes the conjugate transpose. As the signal reflects or scatters off the obstacle, its amplitude and polarization state may be altered. Since any change in polarization can be represented as a $2 \times 2$ complex matrix [56], we can write $z(t) = \sigma D z'(t)$. The path loss is indicated by the positive number $\sigma$ and phase shift experienced and polarization transformation of the signal as it propagates to the receiver can further be lumped together into the matrix $D$ (specific assumptions reflecting different scenarios are introduced in Section 2.2.1).

Consider now a multipath environment in which the transmitted signal propagates to the receiver by multiple paths produced by $\Gamma$ different scatterers. Suppose that the $l$th path departs the transmit array with direction $\theta^T_l, \phi^T_l$, is reflected or scattered by an obstacle, and then arrives at the receiver from direction $\theta^R_l, \phi^R_l$. The combined signal at the receiver may then be described by

$$y(t) = \sqrt{\frac{\rho}{\Gamma}} \sum_{l=1}^{\Gamma} \sigma_l B(\theta^R_l, \phi^R_l) D_l B^\dag(\theta^T_l, \phi^T_l) \mathbf{x}(t - \tau_l) + n(t),$$

where $\tau_l$ is the propagation delay of the $l$th path, and $\sigma_l D_l$ is a $2 \times 2$ complex matrix.
that reflects the change in amplitude, phase, and polarization experienced by the $l$th multipath component as it propagates from the transmitter to the receiver. The factor $1/\sqrt{\Gamma}$ ensures that the received signal energy does not blow up as $\Gamma$ increases, and the term \( \rho \) is the “receive” SNR.

Thus far, we have considered only a single vector antenna at the transmitter and receiver. However, the model is easily generalized to arrays that consist of multiple spatially-separated, non-interacting vector antennas: Suppose that the receive array consists of $q$ spatially-separated vector antennas located at positions $x_1, \ldots, x_q$ in receiver coordinates, and the transmit array consists of $p$ such elements located at $x'_1, \ldots, x'_p$ in transmitter coordinates (with a total of $M = 6p$ elements at the transmit and $N = 6q$ elements at the receive). Let $y_i(t)$ be the signal detected at the $i$th receiver antenna, $x_j(t)$ be the signal transmitted from the $j$th transmit antenna, and

$$ y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_q(t) \end{bmatrix}, \quad x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_p(t) \end{bmatrix}. $$

A simple model for the resulting multiple-input, multiple-output (MIMO) channel is given by

$$ y(t) = \sqrt{\rho} \frac{\sum_{l=1}^\Gamma \sigma_l B_q(\theta_l^R, \phi_l^R) D_l B_p(\theta_l^T, \phi_l^T) x(t - \tau_l) + n(t)}, $$

where

$$ B_q(\theta, \phi) = d_q(\theta, \phi) \otimes B(\theta, \phi), $$

\(^2\)Consider the case when there is zero delay spread, $\tau_1 = \ldots = \tau_\Gamma$. If the channel and input are such that the expected signal power at each receive antenna is $\rho$, then $\rho$ is the SNR per receive antenna. We will discuss this in more detail later in the chapter.
⊗ is the Kronecker product, \( \mathbf{d}_q(\theta, \phi) \) is the classical narrowband array response

\[
\mathbf{d}_q(\theta, \phi) = \begin{bmatrix}
\exp(-j\omega_c d_1) \\
\vdots \\
\exp(-j\omega_c d_q)
\end{bmatrix},
\]

\( d_i = -\frac{\mathbf{u}_r \cdot \mathbf{x}_i}{c_0} \),

c_0 denotes the speed of light, and \( B_p(\theta, \phi) \) is given by an analogous expression at the transmitter. The noise \( n(t) \) is zero-mean and spatio-temporally white proper complex Gaussian process. Note that (2.2) requires the narrowband assumption at both the transmitter and receiver; i.e., the delays \( d_i \) are assumed to be sufficiently small relative to the inverse bandwidth of \( \mathbf{x}_j(t) \) to justify the approximation

\[
\mathbf{x}_j(t - d_i - \tau_l) \approx e^{-j\omega_c d_i} \mathbf{x}_j(t - \tau_l).
\]

Thus, the receive vector can be expressed as \( \mathbf{y}(t) = H(t) \star \mathbf{x}(t) + n(t) \), where

\[
H(t) = \frac{1}{\sqrt{\Gamma}} \sum_{l=1}^{r} \sigma_l B_q(\theta_l^R, \phi_l^R) D_l B_p^\dagger(\theta_l^T, \phi_l^T) \delta(t - \tau_l) \tag{2.3}
\]

where \( \star \) is the convolution operator.

### 2.2.1 Spatial Multiplexing Gain

We now use the model sketched above to explore the information-theoretic advantages of vector antennas over conventional scalar- and dual-polarized antennas in different propagation environments. In the previous section we considered the complete system with 6 element antennas. In this section, we assume that the transmitter consists of an array of \( p \) identical vector antennas with \( m(\leq 6) \) elements each, and a receiver that has an array of \( q \) identical vector antennas with \( n(\leq 6) \) elements.
each. Thus, we have $M = mp$ elements at the transmitter and $N = nq$ elements at the receiver. Let $B_R$ be the $m \times 2$ submatrix of (2.2) that corresponds to the $n$-component vector antennas in the receive array. Similarly, let $B_T$ denote the $m \times 2$ submatrix associated with the VAs in the transmit array. We call these the antenna response matrices of the antenna systems. Let $S_R^l \overset{\text{def}}{=} (\theta_R^l, \phi_R^l)$ be the direction of the $l$th scatterer with respect to the receive array (also called the “scattering angle”) and $S_T^l \overset{\text{def}}{=} (\theta_T^l, \phi_T^l)$ be the scattering angle of the $l$th scatterer with respect to the transmit array.

The polarization transformation $D_l$ at each multipath cluster controls the amount of mixing between the vertical and horizontal components of the electric field due to scattering. In general, $D_l$ is complex function of the properties of the scatterer as well as the transmitter-scatterer-receiver propagation geometry. Here we will consider two simple models for this mixing which attempt to capture the extreme cases:

1. $D_l$ has i.i.d $CN(0, 1)$ entries. This is termed “full mixing” (FM) because, the incident vertical and horizontal components on the scatterer are “mixed” completely. This captures the case when the scatterers re-scale and rotate the V and H components independently.

2. $D_l$ has i.i.d diagonal $CN(0, 1)$ entries. This is termed “no mixing” (NM) or “diagonal mixing” as vertical and horizontal components are scattered separately.

Under both the assumptions the $D_l$ are themselves independent for different $l$. We further assume that the path losses are equal, i.e. $\sigma_1 = \sigma_2 = \ldots = \sigma_\Gamma$.

We also assume that there is zero delay spread: $\tau_1 = \ldots = \tau_\Gamma$. If $x \in \mathbb{C}^M$ is the transmitted signal vector during a particular symbol interval, the received vector for the corresponding symbol interval $y \in \mathbb{C}^N$ is then given by the equation

$$y = \sqrt{\rho} H x + n$$  \hspace{1cm} (2.4)
where \( \mathbf{n} \) is the noise vector with i.i.d. \( \mathcal{CN}(0, 1) \) entries and

\[
H = \frac{\sigma_1}{\sqrt{\Gamma}} \sum_{l=1}^{\Gamma} B_R(S^R_l)D_lB_T(S^T_l)^\dagger.
\] (2.5)

The channel matrix \( H \) in general varies with time due to the changing scattering environment. If \( H \) is ergodic and the input data symbols are coded over a large time interval, the maximum data rate that can be transmitted through the channel reliably is given by the ergodic capacity. The ergodic capacity of (2.4) is obtained by maximizing the expected mutual information \( I(\mathbf{x}; \mathbf{y}) \) over all possible unit-power input distributions \( p_x \) such that \( \mathbb{E} \mathbf{x}^\dagger \mathbf{x} = 1 \). In general, the optimizing input will depend on the statistics\(^3\) of \( H \). A capacity lower bound (CLB) can be obtained by making the entries of the input vector i.i.d circularly symmetric Gaussian \( \mathcal{CN}(0, 1/M) \). The lower bound then can be written as [92],

\[
C_{lb} = \mathbb{E} \log \det(I_N + \rho M H H^\dagger).
\]

The channel matrix can be written in block matrix form as

\[
H = \frac{\sigma_1}{\sqrt{\Gamma}} \left( B_R(S^R_1), \ldots, B_R(S^R_{\Gamma}) \right) \begin{pmatrix} D_1 & 0 & \ldots & \ldots \\ 0 & D_2 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & 0 & D_{\Gamma} \end{pmatrix} \begin{pmatrix} B_T(S^T_1) \\ \vdots \\ B_T(S^T_{\Gamma}) \end{pmatrix}^\dagger.
\] (2.6)

The rank of the \( H \) matrix is known as spatial multiplexing gain which is also the slope of the capacity versus log-SNR curve for large SNR. It can be shown that if \( r_0 = r(H) \) with probability 1, then \( \lim_{\rho \to \infty} C_{lb} / \log \rho = r_0 \). The rank of \( H \) is trivially upper bounded by \( \min(M, N, 2\Gamma) \). Thus, \( 2\Gamma \geq \min(M, N) \) is a necessary condition

\(^3\)A more detailed discussion on capacity is presented in Section 2.4
in order for $H$ to have full rank. In the rest of the section, we consider different vector antenna systems and discuss the conditions under which the channel matrix achieves full rank. We have listed some properties of the $A_T$ and $A_R$ matrices of (2.6) for vector antennas in Appendix A. We next proceed with a few definitions and then illustrate the dependence of $r_0$ on the scattering environment through some examples.

**Definition 1:** Let the ranges for $\phi$ and $\theta$ be $[0, 2\pi)$ and $[0, \pi]$ respectively. $S_1 = (\theta_1, \phi_1)$ and $S_2 = (\theta_2, \phi_2)$ are distinct if $u_r(S_1) \neq u_r(S_2)$, i.e.,

1. $(\theta_1, \phi_1) \neq (\theta_2, \phi_2)$ for $\theta_1, \theta_2 \not\in \{0, \pi\}$
2. $\theta_1 \neq 0$ if $\theta_2 = 0$
3. $\theta_1 \neq \pi$ if $\theta_2 = \pi$

**Definition 2:** Two directions $S_1$ and $S_2$ are parallel if $(\theta_1, \phi_1) = (\theta_2, \phi_2)$ or $(\theta_1, \phi_1) = (\pi - \theta_2, \phi_2 \pm \pi)$.

**Example 1**

A “complete” vector antenna is a 6-element structure with three orthogonal dipoles and three orthogonal loops arranged such that their axes are aligned along the principal $x$, $y$ and $z$ coordinates. Consider a system with a complete vector antenna at the transmitter and a complete vector antenna at the receiver. From the trivial bound $\Gamma \geq 3$ scatterers are necessary for $H$ to have full rank. If we have three scatterers such that the scattering angles are distinct with respect to both the transmitter and receiver, then from property A1 in Appendix A, $A_R$ and $A_T$ are full rank. Thus with three such scatterers, we have $r(A_R) = r(A_T) = 6$. Under both full-mixing and no-mixing (or diagonal mixing) discussed at the beginning of this section, $r(D) = 1$ with probability 1. From Proposition 4 of Appendix I, $r(H) \geq r(A_R) + r(A_T) - 2\Gamma = 6$.
Figure 2.3: The planar 3-element vector antenna from [47] with three dipoles and a loop.

with probability 1. Thus \( r(H) = 6 \) with probability 1. Therefore, three scatterers with distinct scattering angles with respect to the transmitter and receiver are sufficient for a full rank \( H \). This result is in contrast with a 6–element uniform linear array where a minimum of 6 scatterers are necessary and the array and the scatterer geometry have to satisfy certain constraints in order that \( H \) is full rank [70].

The above result has some implications for any \( p \)-element vector antenna which is a subset of the complete six-element vector antenna system. If \( H \) has full rank, then all principal submatrices of \( H \) has full rank too. This would mean that three scatterers with distinct scattering angles and full rank mixing conditions would be sufficient to ensure that any \( p \times p \) subsystem, of the complete vector system is also full rank. The most notable subsets would be a \( 3 \times 3 \) system with tripoles at both the transmitter and receiver, and a \( 3 \times 3 \) system with a planar 2-dipole, single loop structure at the transmitter and receiver.

**Example 2**

Consider a tripole at transmitter aligned along the principal axis, and a planar 2-dipole, single-loop vector antenna at the receiver aligned along the horizontal plane as shown in Fig. 2.3. Assume that the scatterers lie in the horizontal plane. It is necessary to have at least \( \Gamma = 3 \) scatterers for \( \mathcal{A}_T \) to have full rank in the planar
case due to Property A4 in Appendix A. Further assume that no two of the three
directions corresponding to the three scatterers are parallel with respect to both the
transmitter and the receiver.

Consider three angles $S_1, S_2$ and $S_3$ such that no two of the unit vectors $u_r(S_1),
u_r(S_2)$ and $u_r(S_3)$ are parallel. Again from Property 4 of Appendix A, the first, third
and the fifth columns of $A_R$ are linearly independent and the remaining columns are
zeros with planar scattering. The $(2i - 1)$th and $2i$th rows of $DA_t^\dagger$ are
\begin{equation}
\begin{pmatrix}
-d_{11}^{(i)} \sin \phi_i & d_{11}^{(i)} \cos \phi_i^T & -d_{12}^{(i)} \\
-d_{21}^{(i)} \sin \phi_i^T & d_{21}^{(i)} \cos \phi_i^T & -d_{22}^{(i)}
\end{pmatrix}
\end{equation}
where the $d_{kl}^{(i)}$ are the elements of $i$th scattering matrix $D_i$, $i = 1, \ldots, \Gamma$ and $k, l = 1, 2$. The matrix formed by the first, third and fifth columns of $DA_t^\dagger$ is non-singular with
probability 1 if $d_{11}^{(i)}, d_{12}^{(i)}$ are $CN(0, 1)$ for $i = 1, 2, 3$. If we make $d_{21}^{(i)}, d_{22}^{(i)}$ identically
zero, $R(A_t^\dagger) = R(DA_t^\dagger)$ where $R(A)$ denotes the column-space of a matrix $A$. By
Proposition 4 in Appendix A, this is a sufficient condition for $H$ to achieve full rank
with probability 1.

**Example 3**

Consider the $6 \times 6$ MIMO system with complete 6-element vector antennas at
both transmitter and receiver, with the full-mixing assumption. The channel matrix
can be written as $H = (\sigma_1/\sqrt{\Gamma}) \sum_{l=1}^\Gamma H_l$, where $H_l = B(S_l^R)D_lB^\dagger(S_l^T)$. Assume that
the $S_l^R$ and $S_l^T$, $l = 1, 2, \ldots, \Gamma$ are independent and uniformly distributed on a sphere
(i.e., the probability density function of $(\theta, \phi)$ is $p(\theta, \phi) = (1/4\pi) \sin \theta$). Let $vec(A)$
denote the vector formed by stacking the columns of a matrix $A$ one below the other.
Then, $vec(H_l)$, $l = 1, 2, \ldots, \Gamma$ conditioned on $S_l^R$ and $S_l^T$ are i.i.d proper complex
Gaussian vectors because $vec(D_l)$ are proper [55, Lemma 3]. Conditioned on $S_l^R$, $S_l^T$,
the pseudo-covariance $E \left[ vec(H) vec(H)^T \right] S_i^2, S_j^2 = 0$. Now $\Re[vec((\sigma_1/\sqrt{\Gamma}) \sum_i H_i)]$ and $\Im[vec((\sigma_1/\sqrt{\Gamma}) \sum_i H_i)]$ are uncorrelated, and by the central limit theorem [4, Theorem 7.3.1], as $\Gamma \to \infty$, both terms converge in distribution to independent complex Gaussian random vectors with zero mean and identical covariance matrices. As a result of this, in the limit of large $\Gamma$, $vec(H)$ is a proper Gaussian random vector, uniquely specified by the covariance matrix. We proceed to compute the covariance matrix of the limiting $vec(H)$. The $(i, u)$th entry of $H_i$, $h_{iu}^\Gamma$ can be written as $h_{iu}^\Gamma = a_i(\theta_i^R, \phi_i^R) D_i b_u(\theta_i^T, \phi_i^T)$, where $a_i$ is the $i$th row of $B(S_i^R)$ and $b_u$ is the $u$th row of $B(S_i^T)$. $Eh_{iu}^\Gamma = 0$ and the covariance can be written as expectation over the random quantity $(\theta_i^R, \phi_i^R, \theta_i^T, \phi_i^T, D_i)$ as follows:

$$E[h_{iu}^\Gamma h_{jv}^\Gamma] = E_{\theta^R, \phi^R, \theta^T, \phi^T, D_i} \left( a_i(\theta_i^R, \phi_i^R) D_i b_u(\theta_i^T, \phi_i^T) b_v(\theta_j^T, \phi_j^T) D_j a_j(\theta_j^R, \phi_j^R) \right)$$

$$= E_{\theta^R, \phi^R, \theta^T, \phi^T} \left( a_i(\theta_i^R, \phi_i^R) a_j(\theta_j^R, \phi_j^R) b_u(\theta_i^T, \phi_i^T) b_v(\theta_j^T, \phi_j^T) \right)$$

$$= E_{\theta^R, \phi^R} \left( a_i(\theta_i^R, \phi_i^R) a_j(\theta_j^R, \phi_j^R) \right) E_{\theta^T, \phi^T} \left( b_u(\theta_i^T, \phi_i^T) b_v(\theta_j^T, \phi_j^T) \right)$$

We can observe that $Eh_{iu}^\Gamma h_{jv}^\Gamma$ is zero when $i \neq j$ or $u \neq v$. It can be shown that $E_{\theta^R, \phi^R} \left( a_j(\theta_j^R, \phi_j^R) a_j(\theta_j^R, \phi_j^R) \right) = 2/3$ for all $j = 1, \ldots, 6$. Thus, $Eh_{iu}^\Gamma h_{jv}^\Gamma = (4/9) \delta_{ij} \delta_{uv}$. If $h_{iu}$ is the $(i, u)$th entry of $H$, then $h_{iu} = (\sigma_1/\sqrt{\Gamma}) \sum_{l=1}^\Gamma h_{iu}^\Gamma$. Therefore, the limiting covariance matrix is given by $E(vec(H) vec(H)^T) = (4\sigma_1^2/9) I_{36}$. Thus, the 6-element vector antenna in the limit is equivalent to a $6 \times 6$ system with i.i.d $CN(0, 4\sigma_1^2/9)$ entries.

This has implications for any $m \times m$ subsystem of the complete vector antenna system. Since, the channel matrix of an $m \times m$ subsystem is a principal submatrix of the channel matrix of a complete vector antenna system, as $\Gamma \to \infty$ its channel matrix converges in distribution to an i.i.d complex Gaussian $m \times m$ channel matrix.
2.3 Signal Model: Clusters of Scatterers

In the previous section, we developed a joint EM signal propagation model for vector antennas in a discrete multipath channel. In this section, we extend this propagation model to include

1. multiple-element antenna systems with arbitrary EM radiation properties (as opposed to vector antennas which measure projections of the E-field and H-field along their principal axes), and

2. scattering environments consisting of \( L \) clusters with a large number of multi-path scatterers within each cluster.

Consider a transmit array with \( M \) antennas and a receive array with \( N \) antennas. At any time instant \( t \), the transmitter sends a narrowband pulse whose baseband equivalent vector is \( x (t) \in \mathbb{C}^M \). The narrowband signal propagates from the transmitter to the receiver through \( L \) scattering clusters. Each scattering cluster is a collection of several subpaths through which the signal propagates from the transmitter to the receiver. The received EM-wave induces a signal vector \( y \) in the antenna array \( (y \in \mathbb{C}^N) \). We assume that the \( l \)th scattering cluster has \( \Gamma_l \) subpaths, \( l = 1, 2, \ldots, L \). Let \( (\theta^T_{lj}, \phi^T_{lj}) \) and \( (\theta^R_{lj}, \phi^R_{lj}) \) be the direction of the \( j \)th subpath in the \( l \)th scattering cluster with respect to the transmit and receive antenna array respectively. The input-output equation is then given by

\[
y = \sqrt{\rho} H x + n \tag{2.7}
\]

where \( H \) is the channel matrix and using (2.5), can be written as

\[
H = \sum_{l=1}^{L} \frac{\sigma_l}{\sqrt{\Gamma_l}} \sum_{j=1}^{\Gamma_l} B_R(\theta^R_{lj}, \phi^R_{lj}) D_{lj} B^*_T(\theta^T_{lj}, \phi^T_{lj}) \tag{2.8}
\]
where $B_T (B_R)$ is the $N \times 2$ ($M \times 2$) antenna response matrix for the transmit (receive) array. Note that $(B_{R,[m,1]}(\theta, \phi)$ is the signal response in the $m$th receive antenna induced by the $u_\theta$ component of the E-field vector arriving from the direction $(\theta, \phi)$. Similarly, $B_{[m,2]}$ is the $u_\phi$ component response function. The electric-field pattern of the $i$th transmit is denoted by $f_T^i(\theta, \phi)$, where $1 \leq i \leq M$, and is defined as $f_T^i = B_{T,[i,1]}u_\theta + B_{T,[i,2]}u_\phi$ for all $(\theta, \phi) \in [0, \pi] \times [0, 2\pi)$. The electric-field pattern of the $j$th receive antenna denoted by $f_R^j(\theta, \phi)$ is defined in an analogous way. In the previous section we assumed that the antenna response matrices $B_R$ and $B_T$ have entries formed by the projection of the electric and magnetic field vectors onto the principal axis. In this section, we relax this assumption and assume that the antenna response matrices and therefore the E-field patterns are arbitrary, frequency non-selective functions.

We proceed with the following simplifying assumptions:

1. $D_{lj}$ is a $2 \times 2$ mixing matrix that describes the effect of scattering on polarizations in the $lj$th sub-path. The matrices are modelled as i.i.d., each with entries that are i.i.d proper complex Gaussian random variables with zero mean and unit variance.

2. We next assume that for any given $l$, the $(\theta_T^l, \phi_T^l)$ and $(\theta_R^l, \phi_R^l)$ are uniformly distributed over the set of angles corresponding to the $l$th cluster. We denote these sets by $S_{T,l}$ and $S_{R,l}$ respectively. We assume that the clusters are non-intersecting ($S_{T,l} \cap S_{T,l'}$ and $S_{R,l} \cap S_{R,l'}$ are both null-sets for $l \neq l'$).

Since, $vec(D_{lj})$ are proper complex Gaussian vectors, and the mapping $vec(B_R D_{lj} B_T^\dagger)$ conditioned on the $(\theta_T^l, \phi_T^l)$ and $(\theta_R^l, \phi_R^l)$ summed over $l$ is proper as well [55, Lemma 3]. Using reasoning similar to Example 4 of the previous section, and applying the
central limit theorem to the real and imaginary parts of

\[ vec \left( \frac{1}{\sqrt{T}} \sum_{l=1}^{\Gamma_t} B_R(\theta_{lj}^R, \phi_{lj}^R) D_{lj} B_T^\dagger(\theta_{lj}^T, \phi_{lj}^T) \right) \]

as \( \Gamma_t \to \infty \), the term above converges in distribution to a proper complex Gaussian vector. Since, \( vec(H) \) is a linear combination of the above terms for \( l = 1, 2, \ldots, L \), \( vec(H) \) converges to a proper complex Gaussian random vector with mean zero. The covariance matrix \( E[vec(H)vec(H)^\dagger] \) uniquely specifies the distribution of \( H \). Therefore, as \( \Gamma_t \to \infty \), the channel matrix \( H \) converges in distribution to a zero mean random matrix with jointly complex Gaussian entries. The covariance between \( ip \)th and the \( jq \)th entries of the limiting \( H \) can be written down (after making use of the fact that \( D_{lj} \) are uncorrelated over different subpaths and some straightforward steps) as

\[
E \overline{H_{[i,p]}}H_{[j,q]} = \sum_{l=1}^{L} \frac{\sigma_l^2}{|S_{T,l}|} \int_{S_{T,l}} \int_{S_{R,l}} (B_{R,[i,:]}(\theta^R, \phi^R)B_{R,[j,:]}^\dagger(\theta^R, \phi^R)) \cdot (B_{T,[p,:]}(\theta^T, \phi^T)B_{T,[q,:]}^\dagger(\theta^T, \phi^T)) dS_R dS_T
\]

(2.9)

where \( |S| \) is the Lebesgue measure of the set \( S \).

We now develop a signal model from the above equation. For each \( l \), consider the vector space spanned by the row vectors of \( B_T \), denoted by \( X_l = \text{span}\{B_{T,[i,:]}(\theta, \phi)\}_{i=1}^M \) for \( (\theta, \phi) \in S_{T,l} \). We assume that the antennas are finite-power radiators, and hence the elements of \( X_l \) are square-integrable over \( S_{T,l} \). Define an inner product as \( \langle x, y \rangle_{S_{T,l}} = (1/|S_{T,l}|) \int_{S_{T,l}} x(S_T)y^\dagger(S_T)dS_T \) for any two row vectors \( x \) and \( y \) in \( X_l \). Let \( m_l \) be the dimension of this space. Clearly \( m_l \leq M \), and therefore \( X_l \) is a Hilbert space. Let \( \Psi_{l,1}^T, \Psi_{l,2}^T, \ldots, \Psi_{l,m_l}^T \) denote any collection of \( m_l \) orthonormal basis functions for this space. Let \( [A_{T,l}]_{m,j} = \langle B_{T,[m,:]}(\theta_{lj}^R, \phi_{lj}^R) \rangle_{S_{T,l}} \) represent the projection
of $B_{T,[m,:]}$ onto the basis function $\Psi^T_{l,j}$ for all $1 \leq m \leq M$ and $1 \leq l \leq m_t$. Let $[A_{T,l}]_{m,j} = 0$ for all $1 \leq m \leq M$ and $m_t < l \leq M$. Denote by $A_{R,l}$, the $M \times M$ matrix formed by the entries $[A_{R,l}]_{m,j}$. It can be shown that $r(A_{R,l}) = m_t$ for $l = 1, 2, \ldots, L$. Similarly, starting with the row vectors of $B_R$, define $X'_l = \text{span}\{B_{R,[i,:]}\}_{i=1}^N$ and construct another sequence of $N \times N$ matrices $A_{R,l}$ with $r(A_{R,l}) = \dim(X'_l) = n_l \leq N$ for $l = 1, 2, \ldots, L$.

Next consider the random matrix $H' = \sum_{l=1}^L \sigma_l A_{R,l} W_l A_{T,l}^\dagger$, where $W_l$ is a $N \times M$ matrix with i.i.d complex $\mathcal{CN}(0,1)$ entries. Since, $\text{vec}(W_l)$ is a proper complex Gaussian random vector, $\text{vec}(H')$ is a zero-mean proper complex Gaussian vector. It can be shown that

$$E[H'_{[i,p]}H'^*_{[j,q]}] = \sum_{l=1}^L \sigma_l^2 ([A_{R,l}[i,:][A_{R,l}]^\dagger_{[j,:]})([A_{T,l}[p,:][A_{T,l}]^\dagger_{[q,:]}]$$

which is exactly the same as the expression (2.9) using the definition of the inner product. Therefore, $H$ is identically distributed as $H' = \sum_l \sigma_l A_{R,l} W_l A_{T,l}^\dagger$.

**Proposition 1.** The channel matrix $H$ is distributed identically as $\sum_{l=1}^L \sigma_l A_{R,l} W_l A_{T,l}^\dagger$, where $W_l$ are independent and are $N \times M$ matrices with i.i.d $\mathcal{CN}(0,1)$ entries.

Equivalently, we can write the channel matrix as

$$H = \begin{bmatrix} A_{R,1}, A_{R,2}, \ldots, A_{R,L} \end{bmatrix}_{A_R} H_{\text{net}} \begin{pmatrix} \sigma_1 W_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \sigma_L W_L \end{pmatrix}_{A_{T,L}^\dagger} \begin{bmatrix} A_{T,1}, A_{T,2}, \ldots, A_{T,L} \end{bmatrix}_{A_T}^\dagger \quad (2.10)$$
From the property $\text{vec}(ABC) = (C^T \otimes A)\text{vec}(B)$ applied to the product $A_{R,l}W_lA_{T,l}^\dagger$, we can show that

$$\text{vec}(H) = \left[\sigma_1A_{T,1} \otimes A_{R,1}, \ldots, \sigma_LA_{T,L} \otimes A_{R,L}\right]h,$$

where $h$ is a $LMN \times 1$ vector with i.i.d. $CN(0,1)$ entries. Therefore, the covariance of the channel matrix $H$ arranged as a vector is given by

$$\text{E} \text{vec}(H)\text{vec}(H)^\dagger = BB^\dagger = \sum_{i=1}^L \sigma_i^2 (A_{T,i}A_{T,i}^\dagger) \otimes (A_{R,i}A_{R,i}^\dagger).$$

This is an explicit parametrization of the channel covariance as a function of the antenna E-field patterns and the scattering environment. The $i$th row entries of $A_{R,i}$ are the coefficients of the expansion of the E-field pattern $f_i^R$ with respect to an orthonormal basis. Therefore, $A_{R,i}A_{R,i}^\dagger$ is the Gramian matrix of the E-field patterns restricted to $S_{R,i}$.

The relation of this model to several existing models is discussed next. Mutual coupling is seen as an impediment on performance by antenna designers. This model can naturally incorporate the presence of mutual coupling and we discuss how. Finally, we discuss the line-of-sight (LOS) and wideband extensions to the model.

2.3.1 Relation to Other Signal Models

In this section, we discuss how the channel model developed in Section 2.3 is related to other well known models in literature. Consider an $M$-antenna transmitter and an $N$-antenna receiver with one scattering cluster. This corresponds to our model with $L = 1$ cluster and $S_{T,1}$ and $S_{R,1}$ denoting the sets of transmit and receive
angles corresponding to the cluster. The channel matrix in (2.10) therefore becomes
\[
H = \sigma_1 A_{R,1} W_1 A_{T,1}^\dagger.
\]
Now, by the SVD, \( A_{R,1} = U_R D_R V_R^\dagger \), where \( U_R \) and \( V_R \) are unitary matrices of size \( N \times N \) and \( D_R \) is the diagonal matrix of singular values of \( A_{R,1} \). Similarly, by the SVD, let 
\[
A_{T,1} = U_T D_T V_T^\dagger
\]
where \( U_T \) and \( V_T \) are unitary matrices of size \( M \times M \) and \( D_T \) is a diagonal matrix. We have 
\[
A_{R,1} W_1 A_{T,1}^\dagger = U_R D_R V_R^\dagger W_1 V_T D_T U_T^\dagger.
\]
The matrix \( V_R^\dagger W_1 V_T \) is identically distributed as \( W_1 \), which in turn is identically distributed as \( U_R^\dagger W_1 U_T \). Therefore, \( H \) is identically distributed as 
\[
U_R^\dagger D_R V_R^\dagger W_1 V_T D_T U_T^\dagger.
\]
But, \( U_R^\dagger D_R U_R^\dagger \) and \( U_T^\dagger D_T U_T^\dagger \) are respectively the Hermitian square-roots of the matrices \( A_{R,1} A_{R,1}^\dagger \) and \( A_{T,1} A_{T,1}^\dagger \). Denote the Hermitian square-root of a matrix \( A \) by \( A^{1/2} \). The equivalent channel then can be written as 
\[
\sigma_1 (A_{R,1} A_{R,1}^\dagger)^{1/2} W_1 (A_{T,1} A_{T,1}^\dagger)^{1/2}.
\]
This is the familiar Kronecker covariance model (e.g., [11,79]).

As an example, next consider the transmitter and the receiver that employ linear arrays with dipole antennas aligned parallel to the z-axis with their centers on the on the x-axis. The antenna response matrix for a \( M \) element transmit array is given by 
\[
B_T(\theta, \phi) = \sin \theta \begin{pmatrix}
    e^{2\pi x_1' \cos \phi / \lambda} & 0 \\
    e^{2\pi x_2' \cos \phi / \lambda} & 0 \\
    \vdots & \vdots \\
    e^{2\pi x_M' \cos \phi / \lambda} & 0
\end{pmatrix},
\]
where \( x_i' \), \( i = 1, 2, \ldots, M \) denote the x-coordinates of the antennas. The receiver has a \( N \) element array, with a similar antenna response matrix. Assume that the scattering environment is planar with a large number of scatterers present only on the x-y plane. Therefore, \( S_{R,1} \) and \( S_{T,1} \) can be taken to be the unit circle on the x-y plane. As noted earlier, \( A_{R,1} A_{R,1}^\dagger \) is the Gramian matrix of antenna E-field vectors,
and the $l_k$th element of $A_{R,1}A_{R,1}^\dagger$ is given by

\[
[A_{R,1}A_{R,1}^\dagger]_{l_k} = \left\langle f_i^R, f_k^R \right\rangle_{S_{R,1}} = \frac{1}{2\pi} \int_0^{2\pi} \exp\left(j \frac{2\pi d_{lk} \cos \phi}{\lambda}\right) d\phi = J_0\left(\frac{2\pi d_{lk}}{\lambda}\right),
\]

(2.11)

where $d_{lk} = |x_i^l - x_k^l|$ is the inter-element spacing, and $J_0(x)$ is the cylindrical Bessel function of first kind and zeroth order [98]. A similar expression can be given for the transmit array. This model is also known as the Jakes’ two-ring model [79]. Thus, using our channel model in (2.10) we get the Kronecker model as a special case.

In [65] the transmitter is modelled as continuous current distribution which radiates EM waves. The EM signal vectors then undergo polarization transformation due to scattering. These in turn induce currents on the receiver. In our framework, if we consider the special case of a large number of densely-packed antennas with diagonal $A_{R,l}$ and $A_{T,l}$ (the singular values proportional to the amplitude of radiation of the corresponding signal vector) we essentially obtain the formulation in [65]. On the other hand, we can view our model as a special case of the approach presented in [65] by assuming that a finite number of antennas linearly modulate the spatial basis vectors. We note that both the approaches are therefore equivalent.

### 2.3.2 Mutual Coupling

In multiple-antenna systems, the antenna elements are often placed close to each other. When an element transmits a signal, the radiated E-field induces currents in the neighboring antennas, which in turn re-radiate the same signal. This alters the radiation pattern as compared to that when the antenna is in isolation. This is known as mutual coupling between the antenna elements. A linear approximation
to incorporate this effect is made through mutual impedance of the array [41, 48]. The model for mutual coupling in this section closely follows [41]. Let \( v_s \) be the \( M \)-dimensional voltage source excitation driving the \( M \)-transmit antennas, and let \( i \) be the currents in the antennas. Let \( Z_1 \) denote the mutual impedance of the transmit array, so that the voltage at the antennas can be expressed in terms of the antenna currents \( v = Z_1 i \), and let \( Z_s \) be the source impedance. The voltages at the \( M \) antennas are given by \( v = v_s - Z_s i \). Therefore, \( v = Z_1 (Z_1 + Z_s)^{-1} v_s \), assuming that \((Z_1 + Z_s)\) is invertible. The source impedance is often matched to the antenna impedance, i.e. \( Z_s = Z_1^\dagger \), to maximize the signal power transmitted. If \( x \) is the baseband equivalent of the signal excited by the source, the “effective” transmitted signal vector therefore becomes \( Z_T x \), where \( Z_T = c_T Z_1 (Z_1 + Z_s)^{-1} \), where the real scalar \( c_T \) is chosen so that \( tr(Z_T Z_T^\dagger) = M \). The constant \( c_T \) ensures that the mutual coupling model does not affect the power calculations. In general, computing \( Z_T \) in closed form for a given antenna system is difficult, and numerical techniques such as the method of moments are often used [5]. By reciprocity, the effect at the receiver can analogously be captured using \( Z_R = c_R Z_2 (Z_2 + Z_s')^{-1} \), \( Z_R \), \( Z_s' \) and \( c_R \) are defined similarly as \( Z_T \), \( Z_s \) and \( c_T \) respectively.

Mutual coupling can be incorporated into our model in a natural way. For example, the matrix \( Z_T \) can be viewed as a linear transformation of the signal prior to radiation, and the model in (2.10) can be changed by replacing \( A_{R,l} \) and \( A_{T,l} \) by \( Z_R A_{R,l} \) and \( Z_T A_{T,l} \) respectively.

### 2.3.3 Extension to LOS and Wideband Signals

When a line-of-sight (LOS) component is present, we can assume without loss of generality that \( L = 1 \) corresponds to the LOS component. The scattering matrix \( D_{11} \) then becomes a non-random rotation matrix which transforms transmitter coordinates
to the receiver coordinate system.

Next consider the case when the bandwidth of the transmitted signal is large enough such that there is non-negligible delay spread, but small enough so that the antenna E-field pattern and the scattering frequency response are flat. Assuming that the group delay of the rays within the $l$th cluster is $\tau_l$ (i.e., the delay of each individual subpath within the $l$th cluster is approximately $\tau_l$), we can write the channel in (2.8) as

$$H(\tau) = \sum_{l=1}^{L} \frac{\sigma_l \delta(\tau - \tau_l)}{\sqrt{T_l}} \sum_{j=1}^{\Gamma_l} B_R(\theta_{ij}^R, \phi_{ij}^R) D_{ij} B_T^\dagger(\theta_{ij}^T, \phi_{ij}^T)$$

(2.12)

### 2.4 Capacity-Optimal Signalling

In this section, we consider the design of signals that maximize the capacity when the channel is ergodic, the receiver has perfect knowledge of the channel and there is no feedback of the channel information to the transmitter. If the channel is ergodic and the receiver has perfect estimates of the channel realizations, the ergodic capacity is given by the supremum of the expected mutual information [92]

$$C_e = \sup_{S_x} I(x; y|H)$$

(2.13)

where

$$I(x; y|H) = E \log \det(I_N + \rho HQH^\dagger),$$

(2.14)

$Q = E xx^\dagger$, and $S_x$ is the set of positive semidefinite matrices to which $Q$ must belong. We consider the channel in (2.10) with the assumption that there is rich uniform scattering around both the transmitter and the receiver. This corresponds
to the case where there is just $L = 1$ scattering cluster with both $S_{R,1}$ and $S_{T,1}$
denoting the unit sphere. As discussed in Section 2.3.1, the channel model for this
case is equivalently given by the matrix $c_1(A_{R,1}A_{R,1}^\dagger)^{1/2}W_1(A_{T,1}A_{T,1}^\dagger)^{1/2}$.

In this chapter, we are mainly interested in calculating the capacity (2.13) subject
to the following two input constraints (ICs):

**IC 1** $S_x = \{Q \succeq 0 \mid \text{tr}(Q) \leq 1\}$, and

**IC 2** $S_x = \{Q \succeq 0 \mid \text{tr}(A_{T,1}^\dagger QA_{T,1}) \leq c\}$, where $c$ is positive real constant.

The first constraint IC 1 refers to the case where the total power *input* to the antennas
is limited. The second constraint IC 2 refers to the case where the total *radiated* power
from the transmitter antennas is restricted to be within a specified limit. Both cases
are of interest in different situations. For example, IC 1 is relevant when one is
concerned about reducing the battery power consumption in handheld units. IC 2
is more relevant when one is concerned about meeting regulatory power radiation
specifications. In this section, we consider the capacity optimization problem under
IC 1. In Section 2.5, we consider antenna design under IC 2.

We assume that the transmitter does not receive channel information feedback,
and knows both $A_{R,1}A_{R,1}^\dagger$ and $A_{T,1}A_{T,1}^\dagger$ (by initial calibration). With IC 1 as the
constraint, the capacity is obtained by solving

$$\sup_Q \mathbb{E} \log \det \left( I + \rho(A_{R,1}A_{R,1}^\dagger)^{1/2}W_1(A_{T,1}A_{T,1}^\dagger)^{1/2}Q(A_{T,1}A_{T,1}^\dagger)^{1/2}W_1A_{R,1}A_{R,1}^\dagger \right)^{1/2}$$
subject to $\text{tr}(Q) \leq 1$.

A unique solution to this problem exists as the objective function is concave and
the set of feasible points is bounded and convex. A complete analytical solution to
this problem is not known, but it has been shown in [40, 74] that the optimum $Q$
satisfies $Q = U_T\Lambda_QU_T^\dagger$, where $U_T$ is the unitary matrix obtained in the eigenvalue
decomposition of $A_{T,1}A_{T,1}^\dagger$, and $\Lambda_Q$ is a diagonal matrix satisfying the trace constraint $\text{tr}(\Lambda_Q) \leq 1$. The minimization problem above can be equivalently written as a minimization over $\Lambda_Q$ subject to a unit trace constraint, as a result the dimensionality of the set over which the numerical optimization needs to be carried over is reduced. As noted earlier, $A_{T,1}A_{T,1}^\dagger$ is the Gramian matrix of the antenna E-field vectors. Therefore, capacity-optimal signalling corresponds to transmitting independent data symbols along the eigenvectors of $A_{T,1}A_{T,1}^\dagger$, with powers specified by the diagonal elements of the optimal $\Lambda_Q$.

2.5 Antenna Design

The antenna electric-field patterns are fixed and cannot be adapted to changing scattering environments in indoor wireless or cellular contexts. So, instead of designing an array by optimizing the capacity specific for a particular realization of the scattering geometry, we seek an optimization for an average channel. Assuming that the scattering clusters move all around the antenna arrays due to relative motion, it would be reasonable to assume that scatterers are uniformly distributed on a sphere around both the transmitter and receiver array. We can use the model for clustered-multipath channel (2.10), with $L = 1$ cluster with $S_{T,1}$ and $S_{R,1}$ denoting the unit sphere. The channel model (2.10) for this scenario then becomes $H = \sigma_1 A_{R,1} W_1 A_{T,1}^\dagger$, where $W_1$ is a $N \times M$ matrix with i.i.d. $CN(0,1)$ entries and $A_{R,1}$ and $A_{T,1}$ have ranks $n_1$ and $m_1$ respectively. We then have $r(H) = \min(m_1, n_1)$ with probability 1, and hence $\min(m_1, n_1)$ is the spatial multiplexing gain. The antenna system achieves the maximum spatial multiplexing gain if $m_1$ and $n_1$ are both at least $\min(M, N)$.

**Proposition 2.** An antenna system achieves the maximum spatial multiplexing gain if $m_1 \geq \min(M, N)$ and $n_1 \geq \min(M, N)$. 
We are interested in maximizing the mutual information under the input constraint IC 2 given in (2.4), where the total radiated power by the transmitter is limited, i.e., $Q \in \{ X \succeq 0 \mid \text{tr}(A_{T,1}^\dagger X A_{T,1}) \leq c_3 \}$ for a positive constant $c_3$.

**Theorem 1.** Assume that $A_{R,1}$ and $A_{T,1}$ satisfy the constraints $\text{tr}(A_{R,1} A_{R,1}^\dagger) \leq c_2$ and $\text{tr}(A_{T,1}^\dagger Q A_{T,1}) \leq c_3$, for two positive reals $c_2$ and $c_3$. The mutual information defined in (2.14) for the channel $H = \sigma_1 A_{R,1} W_1 A_{T,1}^\dagger$ is upper bounded by

$$\mathbb{E} \log \det (I_N + \rho HQH^\dagger) \leq \mathbb{E} \log \det \left( I_N + \frac{\rho \sigma_1^2 c_2 c_3}{M} W_1 W_1^\dagger \right) \quad (2.15)$$

and the upper bound is achieved if and only if $A_{R,1} A_{R,1}^\dagger = (c_2/N) I_N$ and $A_{T,1}^\dagger Q A_{T,1} = (c_3/M) I_M$.

**Proof.** Let $D_R$ and $D_T$ be the diagonal matrices of eigenvalues of $A_{R,1} A_{R,1}^\dagger$ and $A_{T,1}^\dagger Q A_{T,1}$, respectively. Note that $\log \det (I + \rho HQH^\dagger)$ is identically distributed as $\log \det (I + (\rho \sigma_1^2/M) D_R W_1 D_T W_1)$. Define

$$I(D_R, D_T) = \mathbb{E} \log \det \left( I_N + (\rho \sigma_1^2/M) D_R W_1 D_T W_1^\dagger \right),$$

and therefore $I(D_R, D_T)$ is concave in the positive semi-definite argument $D_R$ for a fixed $D_T$. Consider the $N!$ row permutation matrices $P_i$ of size $N \times N$ indexed by $i$. We have by Jensen’s inequality

$$\frac{1}{N!} \sum_i \mathbb{E} I(P_i D_R P_i^\dagger, D_T) \leq \mathbb{E} \log \det \left( I + \frac{\rho \sigma_1^2}{M(N!)} \left( \sum_{i=1}^{N!} P_i D_R P_i^\dagger \right) W_1 D_T W_1^\dagger \right).$$

Note that for all $i$

$$I(P_i D_R P_i^\dagger, D_T) = \mathbb{E} \log \det (I + \rho \sigma_1^2 D_R P_i W_1 D_T (P_i W_1)^\dagger) = I(D_R, D_T).$$
because $P_i$ is real unitary. Hence the left side of the preceding inequality is $I(D_R, D_T)$. But, $(1/N!) \sum_i P_i D_R P_i^T = (tr(A_{R,1} A_{R,1}^\dagger)/N) I_N$, and the left side equals the right if and only if $A_{R,1} A_{R,1}^\dagger = (tr(A_{R,1} A_{R,1}^\dagger)/N) I_N$. Noting that $I(D_R, D_T)$ is concave in the positive semi-definite argument $D_T$, and proceeding in a similar fashion, we get the upper bound in the statement which is achieved if and only if $A_{T,1}^\dagger Q A_{T,1} = (tr(A_{T,1}^\dagger Q A_{T,1})/M) I_M$ in addition.

Observe that if we seek to design $A_{T,1}$, $A_{R,1}$ and $Q$ so as to maximize the mutual information subject to the constraints $tr(A_{R,1} A_{R,1}^\dagger) \leq c_2$ and $tr(A_{T,1}^\dagger Q A_{T,1}) \leq c_3$, then $A_{R,1} A_{R,1}^\dagger = (c_2/N) I_N$, $Q = (1/M) I_M$ and $A_{T,1} A_{T,1}^\dagger = c_3 I_M$ is one optimal solution. Since, $A_{R,1} A_{R,1}^\dagger$ and $A_{T,1} A_{T,1}^\dagger$ are the Gramians of the receiver and transmitter E-field patterns, one optimal choice of E-field patterns is $\langle f_{T,i}^T, f_{T,j}^T \rangle = c_3 \delta_{ij}$ and $\langle f_{R,i}^T, f_{R,j}^T \rangle = (c_2/N) \delta_{ij}$.

Most coding schemes are designed so as to make the inputs to the antennas i.i.d. so that $Q = (1/M) I_M$. For this case, the mutual information under the trace constraints $tr(A_{R,1} A_{R,1}^\dagger) \leq c_2$ and $tr(A_{T,1}^\dagger Q A_{T,1}) \leq M c_3$ is maximized if and only if $A_{R,1} A_{R,1}^\dagger = (c_2/N) I_N$ and $A_{T,1} A_{T,1}^\dagger = c_3 I_M$. Therefore, the antenna elements need to have orthogonal and equal norm E-field patterns in order to maximize the mutual information. We next try to quantify the mutual information loss for antenna elements that do not satisfy this requirement in the high SNR regime. To that end, we introduce some new notation. Let $p_{\det_n}(A)$ be the product of the $n$ largest eigenvalues of a $P \times P$ positive semi-definite matrix $A$. Therefore, $p_{\det_P}(A) = \det(A)$.

**Theorem 2.** For the channel matrix $H = \sigma_1 A_{R,1} W_1 A_{T,1}^\dagger$, let $r(A_{R,1}) = n_1 \leq N$, $r(A_{T,1}) = m_1 \leq M$ and define $p = \min(n_1, n_1)$. A lower bound on the ergodic
capacity is obtained by choosing $Q = (1/M)I_M$ and is given by

$$C_e \geq \log \det_p(A_{R,1}A_{R,1}^\dagger) + \log \det_p(A_{T,1}A_{T,1}^\dagger) + \frac{1}{\ln 2} \sum_{j=1}^p \sum_{i=1}^{p-j+1} \frac{1}{i}$$

$$-\frac{\gamma p}{\ln 2} + p \log \left(\frac{\rho \sigma^2}{M}\right) \overset{\text{def}}{=} C_{e;b}$$

(2.16)

where $\gamma = 0.5772\ldots$ is the Euler-Mascheroni constant [24].

**Proof.** We first note that $\det(I_N + \rho \sigma^2 A_{R,1}A_{R,1}^\dagger W_1 A_{T,1}A_{T,1}^\dagger Q A_{T,1} A_{T,1}^\dagger) = \det(I_N + (\rho \sigma^2 / M) D_R W_1 D_T W_1^\dagger)$, where $D_R$ and $D_T$ are respectively the diagonal matrices formed by the eigenvalues of $A_{R,1}A_{R,1}^\dagger$ and $A_{T,1}A_{T,1}^\dagger$ when $Q = (1/M)I_M$. Suppose $n_1 \leq m_1$ first. Partition $W_1$, $D_R$ and $D_T$ as

$$W_1 = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix},$$

$$D_R = \begin{bmatrix} D_{R1} & 0_{(N-n_1)\times n_1} \\ 0_{n_1\times(N-n_1)} & 0_{(N-n_1)\times(N-n_1)} \end{bmatrix},$$

$$D_T = \begin{bmatrix} D_{T1} & 0_{n_1\times(M-n_1)} \\ 0_{(M-n_1)\times n_1} & D_{T2} \end{bmatrix},$$

where $\alpha_{11}$, $\alpha_{12}$, $\alpha_{21}$ and $\alpha_{22}$ are submatrices of $W_1$ of sizes $n_1 \times n_1$, $n_1 \times (M-n_1)$, $(N-n_1)\times n_1$ and $(N-n_1)\times (N-n_1)$ respectively. Here $D_{R1}$ is a $n_1\times n_1$ diagonal matrix of non-negative real entries, and $D_{T1}$ and $D_{T2}$ are non-negative diagonal matrices of sizes $n_1 \times n_1$ and $(M-n_1) \times (M-n_1)$ respectively. The sizes of the zero matrices are indicated by their subscripts. Using the identity $\det(I + AB) = \det(I + BA)$, we have

$$\det\left(I_N + \frac{\rho \sigma^2}{M} D_R W_1 D_T W_1^\dagger\right) = \det\left(I_N + \frac{\rho \sigma^2}{M} D_R^{1/2} W_1 D_T W_1^\dagger D_R^{1/2}\right).$$
Making use of the partitioning, we have

\[
D_R^{1/2} W_1 D_T W_1^\dagger D_R^{1/2} = \begin{bmatrix}
D_R^{1/2} \alpha_{11} D_{T1} \alpha_{11}^\dagger D_R^{1/2} + D_R^{1/2} \alpha_{12} D_{T2} \alpha_{12}^\dagger D_R^{1/2} & 0_{n_1 \times (N-n_1)} \\
0_{(N-n_1) \times n_1} & 0_{(N-n_1) \times (N-n_1)}
\end{bmatrix}.
\]

We note that

\[
\log \det \left( I_N + (\rho \sigma_1^2 / M) D_R^{1/2} W_1 D_T W_1^\dagger D_R^{1/2} \right)
\]

\[
= \log \det \left( I_{n_1} + (\rho \sigma_1^2 / M) (D_{R1}^{1/2} \alpha_{11} D_{T1} \alpha_{11}^\dagger D_R^{1/2} + D_{R1}^{1/2} \alpha_{12} D_{T2} \alpha_{12}^\dagger D_R^{1/2}) \right).
\]

The term \( I_{n_1} + (\rho \sigma_1^2 / M) D_{R1}^{1/2} \alpha_{12} D_{T2} \alpha_{12}^\dagger D_R^{1/2} \) is a positive semi-definite matrix. We next use the property [35]: If \( A \) and \( B \) are two positive semi-definite matrices, then \( \det (A + B) \geq \det (B) \). Therefore, we get the inequality

\[
\log \det \left( I + (\rho \sigma_1^2 / M) D_R W_1 D_T W_1^\dagger \right)
\]

\[
\geq \log \det \left( (\rho \sigma_1^2 / M) D_{R1} \alpha_{11} D_{T1} \alpha_{11}^\dagger \right)
\]

\[
= n_1 \log (\rho \sigma_1^2 / M) + \log \det D_{R1} D_{T1} + \log \det \alpha_{11} \alpha_{11}^\dagger.
\]

(2.17)

We note that \( \alpha_{11} \alpha_{11}^\dagger \) has Wishart distribution \( \mathcal{W}(n_1, n_1) \). Therefore, using [27, Theorem 3.2.22],

\[
\mathbb{E} \ln \det \alpha_{11} \alpha_{11}^\dagger = \sum_{i=0}^{n_1-1} \psi(n_1 - i),
\]

where

\[
\psi(x) = -\gamma + \sum_{p=1}^{x-1} \frac{1}{p}
\]

for integer \( x \).

We have not made any assumption regarding the ordering of the diagonal entries in the matrices \( D_R \) and \( D_T \) yet. Therefore, we next assume that the diagonal entries of
$D_R$ and $D_T$ are arranged in descending order. This implies that $\det D_{R1} = \text{pdet}_{n_1}(D_R)$ and $\det D_{T1} = \text{pdet}_{n_1}(D_T)$. Taking expectation on both sides of (2.17), the we get the inequality in the theorem statement

$$
\mathbb{E} \log \det \left( I + (\rho \sigma_1^2 / M) D_R W_1 D_T W_1^\dagger \right)
\geq n_1 \log (\rho \sigma_1^2 / M) + \text{pdet}_{n_1}(D_R) + \text{pdet}_{n_1}(D_T) + \frac{1}{\ln 2} \sum_{j=1}^{n_1} \sum_{i=1}^{n_1-j-1} \frac{1}{i} - \frac{n_1 \gamma}{\ln 2}.
$$

For the case of $n_1 > m_1$, we can prove the result in an analogous manner by noting that $\det \left( I_N + (\rho \sigma_1^2 / M) D_R W_1 D_T W_1^\dagger \right) = \det \left( I_M + (\rho \sigma_1^2 / M) D_T W_1^\dagger D_R W_1 \right)$. The roles of $D_R$ and $D_T$ now get reversed and we can proceed as before.

The lower bound $C_{e,lb}$ defined in (2.16) is a lower bound on the ergodic capacity with IC 1. With the additional constraint $\text{tr}(A_{T,1} A_{T,1}^\dagger) \leq M c_3$, $C_{e,lb}$ is lower bound on the ergodic capacity with IC 2. Consider the special case of the transmit-receive system where $N = M = m_1 = n_1$. At high SNR, $C_{e,lb}$ can be asymptotically written as

$$
\log \det(A_{R,1} A_{R,1}^\dagger) + \log \det(A_{T,1} A_{T,1}^\dagger) + M \log \rho + o(\log \rho)
$$

where $o(\log \rho)$ denotes terms so that $\lim_{\rho \to \infty} o(\log \rho) / \log \rho \to \infty$. Therefore $C_{e,lb}$ is asymptotically tight for large SNR for this system if $A_{R,1}$ and $A_{T,1}$ have full rank. For any $P \times P$ positive semi-definite matrix $X$ such that $\text{tr}(X) \leq P$, $-\log \det(X)$ is positive and it can be shown that it is zero if and only if $X = I_P$. Therefore, we have the inequality

$$
\det(A_{R,1} A_{R,1}^\dagger) + \det(A_{T,1} A_{T,1}^\dagger) \leq M \log \left( \text{tr}(A_{R,1} A_{R,1}^\dagger) / M \right) + M \log \left( \text{tr}(A_{T,1} A_{T,1}^\dagger) / M \right),
$$

and equality is achieved if and only if the antenna elements have orthogonal and equal norm antenna patterns. Therefore, a mutual information gap expression due to
antenna elements not satisfying the orthogonality or equal norm conditions can be written as

\[- \log \det \left( \frac{A_{R,1} A_{R,1}^\dagger}{(\text{tr}(A_{R,1} A_{R,1}^\dagger) / M)} \right) - \log \det \left( \frac{A_{T,1} A_{T,1}^\dagger}{(\text{tr}(A_{T,1} A_{T,1}^\dagger) / M)} \right). \tag{2.18}\]

We can easily see that this gap is non-negative and is zero for antenna elements that satisfy the orthogonality and equal norm conditions. Therefore, even the lower bound in (2.16) is maximized when these conditions are satisfied.

The term \(\det(A_{R,1} A_{R,1}^\dagger)\) serves as a useful metric for comparing two multi-element antenna systems. For any \(N\) element antenna system, the \(i\)th row entries of \(A_{R,1}\) are the coefficients of the basis expansion of the E-field pattern vector for the \(i\)th antenna \(f_i^R\) in terms of a finite number (\(\leq N\)) of orthonormal basis functions. As discussed before, \(A_{R,1} A_{R,1}^\dagger\) is the Gramian matrix of the antenna patterns, with the \(ij\)th element given by the inner product \(\langle f_i^R, f_j^R \rangle = (1/4\pi) \int_0^{2\pi} \int_0^{\pi} f_i^R f_j^R \sin \theta d\theta d\phi\).

We next consider two multi-element antenna systems: the linear array system and the vector antenna system to see how they compare with respect to this measure.

Consider the linear array system discussed in Section 2.3.1. From (2.11), the \(ij\)th element of \(A_{R,1} A_{R,1}^\dagger\) is given by \(J_0(2\pi d_{ij}/\lambda)\), where \(d_{ij}\) is inter-element spacing between the pair \((i, j)\) and the antenna E-field patterns are normalized such that each element has unit norm. We plot \(\det A_{R,1} A_{R,1}^\dagger\) for the case when \(d_{ij} = |i - j| d\) for \(i \neq j\) (uniform linear array) for different values of \(d\) in Fig. 2.4. As \(d\) increases, \(\det A_{R,1} A_{R,1}^\dagger\) goes from zero to unity, and the the capacity gap defined in (2.18) drops from large values to zero. So, the uniform linear array satisfies the orthogonality and equal norm criteria in the asymptotic limit as \(d \to \infty\). The usual value for inter element spacing in antenna arrays is \(\lambda/2\) and this is markedly suboptimal.

Next consider any \(M\) element subsystem of the complete vector antenna system
discussed in Section 2.2. The normalized E-field pattern for this case is given a matrix formed by the $M$ relevant rows of (2.1). It can be easily seen that the $i_j$th element of $A_{R,1}A_{R,1}^\dagger$ is given by $\langle f_i^R, f_j^R \rangle = \delta_{i,j}$, which satisfies the orthogonality and equal norm criteria.

So far, we ignored mutual coupling. When mutual coupling is present, following the discussion in Section 2.3.2, the channel matrix becomes $H = \sigma_1 Z_R A_{R,1} W_1 A_{T,1}^\dagger Z_T^\dagger$. Therefore, the sum of the log det terms becomes

$$\log \det A_{R,1}A_{R,1}^\dagger + \log \det A_{T,1}A_{T,1}^\dagger + \log \det Z_RZ_R^\dagger + \log \det Z_TZ_T^\dagger.$$ 

Since $Z_T$ and $Z_R$ satisfy the trace constraints $tr(Z_TZ_T^\dagger) = M$ and $tr(Z_RZ_R^\dagger) = N$ respectively, we note the last two terms in the above expression are non-positive and are zero if only if $Z_T$ and $Z_R$ have orthonormal rows. We conclude that mutual coupling results in capacity loss as compared to uncoupled antennas.

### 2.6 Simulation Results

The main focus of this section is to demonstrate the following through simulations:

1. Vector antennas give similar capacity gains to spatially-separated antennas in a rich multipath environment, and perform even better than their spatially-separated counterparts in a scarce multipath environment.

2. Increasing the number of multipath clusters ($L$) increases the ergodic capacity and lowers the outage probability. Vector antennas achieve most of the capacity gains of an i.i.d. complex Gaussian channel for very few number of scattering clusters.
Figure 2.4: The function $\det A_{R,1}A_{R,1}^\dagger$ for a uniform linear array vs. the normalized inter-element spacing $d/\lambda$.

3. Capacity-optimal signalling gives significant gains as compared to the i.i.d. input distribution when the antenna correlation is large.

We define SNR ($= \rho$) as the average signal-to-noise power ratio per receive antenna when the input power is i.i.d. with $\mathcal{CN}(0, 1/M)$ entries. With this definition of the SNR, the channel matrix $H$ is re-normalized so as to make $\rho = \mathbb{E}(tr(HH^\dagger))/MN$ hold in all our simulations.

We first consider the 4x4 system with the vector antenna configuration from [47], depicted in Fig. 2.3. This antenna consists of three orthogonal dipoles and a loop co-located as shown. We consider the channel model in (2.8), with the mixing matrix $D_{lj}$ modelled as a matrix with i.i.d. $\mathcal{CN}(0, 1)$ entries. We consider $L = 1, 3, 5$ and 7 clusters, each with an angular spread of $\Delta$. We assume that each cluster has $\Gamma_j = 20, j = 1, \ldots, L$ subpaths with center angles $(\theta_l, \phi_l) = ((l - 1)\pi/L, 2(l - 1)\pi/L), l = 1, \ldots, L$ are uniformly spaced, and the subpath angles $(\theta_{lj}, \phi_{lj}), j = 1, \ldots, \Gamma_j$ are uniformly
distributed within $S_{T,l} = S_{R,l} = [\theta_l - \Delta/2, \theta_l + \Delta/2) \times [\phi_l - \Delta/2, \phi_l + \Delta/2)$. The time delay spread is assumed to be zero, i.e., $\tau_1 = \ldots = \tau_L$. We use Monte-Carlo simulations with $N_{\text{iter}} = 1000$ iterations and generate the capacity lower bound as the expectation of the mutual information $C_{lb}(\rho) = \mathbb{E} \log \det \left( I + \left( \rho/4 \right) HH^\dagger \right)$. We plot this mutual information with respect to SNR for different values of $L$ in Fig. 2.5 for $\Delta = 10$ degrees. For comparison we have plotted the capacity of a $4 \times 4$ i.i.d. $\mathcal{CN}(0,1)$ channel. The mutual information converges fairly fast to the i.i.d channel capacity and the number of clusters beyond $L = 5$ does not add much gain. Fig. 2.6 plots the capacity lower bound with $\Delta = 20$ degrees retaining rest of the parameters. We see that larger angular spread gives better capacity for each $L$ and makes the convergence faster. $L = 3$ is around 1 dB away from the i.i.d channel capacity, and increasing $L$ beyond 5 does not much gain.

Next, we plot the outage probability defined as $P_{\text{out}}(I) \overset{\text{def}}{=} \text{Prob}(\log \det(I_4 + (\rho/4)HH^\dagger) < I)$ for $\rho = 20$ dB for $L = 1, 3, 5$ and 7 with the channel generated as before. The number of Monte-Carlo samples is $N_{\text{iter}} = 2000$ and the rest of the parameters are retained from the previous simulation ($\Delta = 20$ degrees). Fig. 2.7 plots the outage probability for different values of mutual information. We see that the outage probability reduces rapidly with $L$. We conclude that multipath diversity leads to gains in outage capacity.

In Fig. 2.8, we plot $C_l$ for a $M$-element vector antennas at both the transmitter and receiver. We again assume the channel model in (2.8) with $L = 1$ and different numbers of subpaths $\Gamma_1$, assuming full mixing and uniform scattering ($S_{T,1}$ and $S_{R,1}$ denote unit spheres). We consider different $M \times M$ systems with $M = 1, 2, 3$ and 6. As before we assume zero delay spread. $M = 1$ corresponds to a single dipole along the $z$-axis, $M = 2$ corresponds to dual-polarized dipoles on the $x-y$ plane, $M = 3$ corresponds to the planar two dipole, single loop structure along the $x-y$
Figure 2.5: Mutual information with SNR for different number of clusters, $L = 1, 3, 5$ and 7 for angular spread $\Delta = 10$ degrees.

plane and finally $M = 6$ corresponds to the complete vector antenna structure with three dipoles and three loops. Also shown for comparison is the i.i.d. channel capacity with $CN(0, 1)$ entries. Note that $C_l$ increases rapidly with $\Gamma_1$ and approaches the i.i.d. channel capacity in the large-$\Gamma_1$ limit. Note that these gains do not require many multipaths: the lower bound of the 6-element vector antenna is everywhere within 4 dB of the i.i.d. channel capacity for $\Gamma_1 = 4$ multipath components, and within 1.5 dB of i.i.d. channel capacity for $\Gamma_1 = 8$. Although it is not evident from Fig. 2.8, it can be shown that the lower bound of a 6-element vector antenna increases more rapidly with $\Gamma_1$ than the lower bound of a spatial array of six scalar antennas, because the
vector antenna is capable of detecting two signaling dimensions with each multipath component, corresponding to the horizontally and vertically polarized signals. For different vector antenna structures, we can see that rate of convergence of lower bound to the i.i.d. channel capacity as $\Gamma_1$ increases, improves as we go from the 1–element to the 6–element antennas structures.

The model (2.8) was used to compare the mutual information plots calculated from channel measurements [47]. A $4 \times 4$ system with the 4-element antenna structure of Fig. 2.3 was employed. The channel model (2.8) was used to replicate the setup with $L = 1$ cluster, $\Gamma_1 = 17$ subpaths, zero delay spread, $S_{R,1} = S_{T,1} = \{(\theta, \phi) \in [0, 60^\circ) \times$
Figure 2.7: Outage probability with mutual information for SNR = 20 dB, \( L = 1, 3, 5 \) and 7 clusters and angular spread \( \Delta = 20 \) degrees.

\([0, 2\pi]\) was used. These parameters were chosen because 17 perfectly reflecting scatterers were placed around the antennas, and the antennas were within a distance of about 2 meters of each other and were mounted such that the cluster resembled a polar cap of half-angle 60°. The lower bounds for both the measured channel and the computer generated channel have been plotted in Fig. 2.9. The i.i.d. channel capacity has also been plotted for comparison. The predicted channel gives mutual information that is less than a dB away from that for the measured channel.

We finally consider the advantages due to use of optimal input signals as opposed to i.i.d. input signalling. We consider a 3 × 3 channel with uniform linear array
Figure 2.8: Capacity lower bound versus number of multipaths for 1-, 2-, 3- and 6-element vector antennas with full mixing and uniform scattering at both the transmitter and receiver with an inter element spacing of $\lambda/2$ between neighboring antennas. We use the channel model (2.8) for an environment with $L = 1$ and $\Gamma_1 = 20$ subpaths. The sets $S_{T,1}$ and $S_{R,1}$ are taken to be the unit spheres and the $\theta$ and $\phi$ for the different subpaths are generated from a uniform distribution. Monte Carlo simulations were performed with $N_{\text{iter}} = 1000$ channel realizations. The Gramian matrices $A_{R,1}A_{R,1}^H$ and $A_{T,1}A_{T,1}^H$ can be computed from the closed form expressions given in Section 2.2. We consider the capacity under input constraint IC 2 of Section 2.4 with the unit power constraint $c_3 = 1$. The optimal solution
Figure 2.9: Capacity lower bounds versus SNR for the measured channel and the model (2.8) (with $L = 1$ cluster, $\Gamma_1 = 17$ subpaths) for the 4-element planar vector antenna shown in Fig. 2.3.

is to design a transmit covariance such that $Q = (1/2)(A_{T,1}A_{T,1}^\dagger)^{-1}$. We plot the capacity with SNR and compare it with the mutual information obtained for i.i.d. input signalling $Q = (1/2)I_2$. The optimal signalling gives a high-SNR capacity gain corresponding to over 0.5 dB even when the scattering environment is rich.

2.7 Conclusions

In this chapter, we first developed a channel model for the joint EM signal propagation for vector antenna systems in a discrete multipath environment. We then
extended this signal model to a propagation scenario with clusters of scatterers and antennas with more general E-field patterns. The channel model and its statistical distribution was given as a function of the antenna E-field pattern and the scattering environment. We then outlined ergodic capacity results for such a channel under two different input constraints. The impact of the antenna E-field patterns on the ergodic capacity was studied and a criterion for maximizing the capacity at all SNRs was proposed. We showed that antennas with orthogonal and equal norm E-field patterns maximize the ergodic capacity. Vector antennas satisfy this criterion but linear arrays, for any finite inter-element separation, do not satisfy this criterion.
Our simulation results indicate that vector antennas are comparable to a uniform linear array under rich multipath scattering, and perform even better than uniform linear array when the scattering is scarce. They are also better equipped to exploit the multipath diversity as they give more gains in the presence of sparse scattering. The gains are both in ergodic and outage capacities. This model was then compared with the channel measurements from an experimental setup. For the specific configuration considered, the mutual information for the experimental channel was within a dB of the mutual information for the channel generated from the model. It was observed that even when the multipath scattering is very rich, optimal signalling gives gains in capacity as compared to i.i.d. inputs.
Chapter 3

Frequency-Domain Cramér-Rao Bound for Ultrawideband Vector Antennas

In this chapter, we consider the potential advantages of ultrawideband (UWB) vector antenna systems for range and direction-of-arrival estimation. UWB vector antennas are sensitive to the direction of incoming electromagnetic waves in addition to being able to resolve sharp pulses in time. We consider the problem of jointly estimating the range and direction-of-arrival of a source using UWB vector antennas. We derive asymptotic expressions for the Cramér-Rao Bound (CRB) which gives a lower bounds on the estimation error covariances. For two particular 3-element UWB vector antennas, we then use the CRB as a criterion to design signals.
3.1 Introduction

Over the last decade, there has been a growing recognition that ultra-wideband (UWB) radios offer several unique capabilities that can enable a host of new sensing, positioning, and communication applications. UWB has long been used in ground-penetrating radars, and is now being applied to new imaging devices (e.g. Time Domain’s RadarVision [81]) which enable law enforcement, fire and rescue personnel to see through walls and debris during emergencies. These devices can also improve safety in construction by locating steel bars, electrical wiring, and utility pipes hidden inside walls or underground. Recently, UWB medical imaging systems have been proposed which achieve unprecedented resolution in mammograms [6]. The precise ranging capability of UWB can provide accurate tracking for many applications, such as remote inventory, personnel and asset tracking, and collision avoidance radars and air bag proximity detectors for automobiles. In communications, UWB can transmit very high data rates over short distances without suffering the effects of multipath, and can relieve congested spectrum by effectively opening up new frequency bands in the noise floor (consistent with Part 15 rules, [19]). Several standardization efforts are now underway (e.g., IEEE 802.15.3 and 802.15.3 Sg3a) to develop UWB wireless personal area networks and local area networks that can overlay and co-exist with existing wireless systems.

Most work on ultra-wideband systems has thus far focused on single-polarized electric dipole antennas. Dual-polarized antennas have been studied in certain radar applications, such as ground-penetrating and synthetic-aperture radars (e.g., [8,10,17] and references therein). Single- and dual-polarized antennas can measure at most two components of the received signal. Since the signal detected at the receiver consists of six electromagnetic field components, however, these antennas neglect data that might be available to improve the performance of the sensing, positioning,
and communications algorithms. A “vector antenna” that can independently detect or excite more than one EM field component enables the UWB system to access additional signaling dimensions, which can be used to enhance performance in the same way as antenna arrays.

In this chapter, we develop some tools for the design of UWB systems for precise location estimation. We consider the problem of a receiver equipped with UWB vector antenna with the aim of jointly estimating the range (or distance) and direction-of-arrival (DOA) of a source located in the far-field. We formulate this as a vector parameter estimation problem with observations from a vector Gaussian stationary process. The Cramèr-Rao Bound (CRB) gives a lower bound on the error variance of any unbiased estimator. We prove a result on the asymptotic equivalence of block-Toeplitz matrices which enables us to derive closed-form expressions for the asymptotic CRB. This bound is asymptotic in the sense that it is the limit that is obtained for a large number of samples of the observation. The asymptotic CRB expressions is a frequency-domain expression, and it generalizes the frequency-domain result in [99] to the case vector observation processes. We observe that signals that minimize the ranging-error lower bound do not necessarily minimize the angular-error lower bound.

We use another performance measure that characterizes the uncertainty volume associated with the estimates, and express it in terms of the CRB. This quantity is then used to derive a criterion for designing signal pulse shapes.

The remainder of this chapter is organized as follows. In Section 3.2, we prove a result on the asymptotic properties of block-Toeplitz matrices. This result is then used to derive an asymptotic expression for the CRB in Section 3.3. In Section 3.4, the CRB formula is applied to two ultrawideband systems, and the impact of antenna radiation pattern and the signal shape on the bounds is discussed. In Section 3.5, we derive a criterion based on the CRB for designing signals. Finally, our conclusions
are summarized in Section 3.6. The notation used in this chapter is as follows. If $A$ is complex matrix, $A^\dagger$ is the conjugate-transpose of $A$, $A_{m,n}$ the $mn$-th element of $A$ and $\text{Tr}[A]$ is its trace. If $x$ is a vector, $x_i$ is the $i$-th element of $x$. The complex root unity is denoted by $j = \sqrt{-1}$. The term $\mathbb{E}$ denotes the expectation operator. A proper complex Gaussian random vector [55] with mean $m$ and covariance $\Sigma$ is denoted by $\mathcal{CN}(m, \Sigma)$.

3.2 Mathematical Preliminaries

In this section, we prove some results that will be useful in the subsequent development.

For any $N \times N$ complex matrix $A$, the strong norm is defined as the spectral norm

$$\|A\| = \sup_{x: x^\dagger x = 1} (x^\dagger A^\dagger Ax)^{1/2},$$

and the weak norm is defined to be the normalized Frobenius norm

$$|A| = (N^{-1} \text{Tr}[A^\dagger A])^{1/2}.$$

**Definition 1:** Two sequences of $Nr \times Ns$ matrices, $\{A_N\}$ and $\{B_N\}$, for positive integers $N, r$ and $s$, are said to be asymptotically equivalent if there exists a $c_1 < \infty$ such that $\|A_N\| \leq c_1$ and $\|B_N\| \leq c_1$ and

$$\lim_{N \to \infty} |A_N - B_N| = 0,$$

and this relation is denoted by $A_N \sim B_N$.

A useful result from [25] is provided below as a lemma.
Lemma 1. If \{A_N\}, \{B_N\}, \{C_N\} and \{D_N\} are sequences of $N \times N$ matrices and if $A_N \sim B_N$ and $C_N \sim D_N$, then

$$A_N C_N \sim B_N D_N,$$

and

$$\lim_{N \to \infty} (1/N) \text{Tr}[A_N] = \lim_{N \to \infty} (1/N) \text{Tr}[B_N].$$

If in addition, $A_N^{-1}$ and $B_N^{-1}$ exist and $||A_N^{-1}||, ||B_N^{-1}|| < c_1 < \infty$ for some positive $c_1$ for all $N$, then

$$A_N^{-1} \sim B_N^{-1}.$$ 

We next present the main result of this section on the asymptotic equivalence for block-Toeplitz matrices. Consider the block Toeplitz matrix

$$R_N = \begin{bmatrix}
T_0 & T_{-1} & \cdots & T_{-(N-1)} \\
T_1 & T_0 & \cdots & T_{-(N-2)} \\
\vdots & \vdots & \ddots & \vdots \\
T_{N-1} & T_{N-2} & \cdots & T_0
\end{bmatrix} \quad (3.1)$$

formed by a sequence \{T_k\} of $r \times s$ complex matrices for $k = -(N-1), \ldots, -1, 0, 1, \ldots, (N-1)$. Denote the $ij$th element of $T_k$ by $t_{ij}^k$.

Lemma 2. Consider the block-Toeplitz matrix $R_N$ defined in (3.1), formed with the sequence \{T_k\} of $r \times s$ complex matrices. If \{T_k\} is such that $\sum_{k=\infty}^{\infty} |t_{ij}^k| < \infty$ for all $i$ and $j$, then $R_N$ is asymptotically equivalent to the block-circulant matrix

$$C_N = (W_N \otimes I_r)^{\dagger} D_N(T)(W_N \otimes I_s), \quad (3.2)$$
where $W_N$ is the discrete-time Fourier Transform (DFT) matrix

$$[W_N]_{ij} = \frac{1}{\sqrt{N}} \exp \left( -j \frac{2\pi (i-1)(j-1)}{N} \right).$$

$D_N(T)$ is the block-diagonal matrix

$$D_N(T) = \begin{bmatrix}
T(\omega_1) & O & \cdots & O \\
O & T(\omega_2) & \cdots & O \\
\vdots & \vdots & \ddots & \vdots \\
O & O & \cdots & T(\omega_N)
\end{bmatrix} \quad (3.3)$$

where $\omega_i = 2\pi (i-1)/N$, and $T(\omega)$ is the discrete Fourier Transform of the matrix sequence $\{T_k\}$ defined as

$$T(\omega) = \sum_{k=-\infty}^{\infty} T_k e^{-jk\omega}. \quad (3.4)$$

Remarks: For $r = s = 1$, the result was proved by Gray [25]. For the particular case $r = s > 1$, an alternative asymptotic expression is stated without proof by Gazzah et al [22]. It can be shown that the expression in [22] is equivalent to (2); however, we shall omit this and instead give a direct proof of (2) since it is simpler, a bit more general, and makes the chapter more self-contained.

Proof. Let $kI_{N,r}$ denote an $N \times rN$ matrix such that $[kI_{N,r}]_{ij} = \delta_{j,k+i}$. It is straightforward to prove that the following properties hold for any $rN \times sN$ matrix $A$:

(A1) $kI_{N,r}A$ is the submatrix of $A$ comprising the $N$ rows $k, k+r, \ldots, k+(N-1)r$.

(A2) $kI_{N,r}^t kI_{N,r} A$ is equal to $A$ with all rows zeroed except $k, k+r, \ldots, k+(N-1)r$. 


\begin{align}
(A3) \quad A &= \sum_{i=1}^{r} \sum_{j=1}^{s} i I_{N,r}^T i I_{N,s} j I_{N,s} \\
(A4) \quad B \otimes I_r &= \sum_i i I_{N,r}^T B i I_{N,r} \text{ for any } N \times N \text{ matrix } B
\end{align}

To prove (2), observe that \(i I_{N,r} R_{N,j} I_{N,s}^T\) is an \(N \times N\) Toeplitz matrix formed from the scalar sequence \(t_k^{ij}, k = 0, \pm 1, \ldots\) for all \(i\) and \(j\). From [25], we therefore have \(i I_{N,r} R_{N,j} I_{N,s}^T \sim W_N^T D_N (T^{ij}) W_N\), where \(T^{ij}\) is the DFT of the sequence \(\{t_k^{ij}\}\). Observing that \(D_N (T^{ij}) = i I_{N,r} D_N (T) j I_{N,s}^T\), we conclude that

\[
R_N \overset{(A3)}{=} \sum_{ij} i I_{N,r}^T i I_{N,s} r_{n,j} I_{N,s}^T j I_{N,s} \\
\sim \sum_{ij} i I_{N,r}^T W_N^T D_N (T^{ij}) W_N j I_{N,s} \\
= \sum_{ij} i I_{N,r}^T W_N i I_{N,r} D_N (T) j I_{N,s}^T W_N j I_{N,s} \\
= \left( \sum_i i I_{N,r}^T W_N i I_{N,r} \right)^\dagger D_N (T) \left( \sum_j j I_{N,s}^T W_N j I_{N,s} \right) \\
\overset{(A4)}{=} (W_N \otimes I_r)^\dagger D_N (T) (W_N \otimes I_s),
\]

where the second step follows by observing \(A_N \sim C_N\) and \(B_N \sim D_N\) implies \(A_N + B_N \sim C_N + D_N\).

Now suppose that each matrix in the sequence \(\{T_k\}\) is function of a \(p \times 1\) real vector \(\vartheta\). The \(l\)-th element of \(\vartheta\) is denoted by \(\vartheta_l\). To indicate the dependence on \(\vartheta\), we denote the block-Toeplitz matrix in (3.1) as \(R_N (\vartheta)\). Similarly, the matrix discrete Fourier Transform given in (3.4) is denoted by \(T_{\vartheta}(\omega)\). If the discrete Fourier transform of the sequence \(\{\partial T_k/\partial \vartheta_l\}\) exists and is equal to \(\partial T_{\vartheta}(\vartheta)/\partial \vartheta_l\) for all \(l\) and \(\vartheta\), then

\[
\frac{\partial R_N (\vartheta)}{\partial \vartheta_l} \sim (W_N \otimes I_r)^\dagger D_N \left( \frac{\partial T}{\partial \vartheta_l} \right) (W_N \otimes I_r).
\]

If \(r = s\) and the smallest eigenvalue of \(T_{\vartheta}(\omega)\) is bounded away from zero, i.e.,
\( \sigma[T(\omega, \vartheta)] \geq c > 0 \) for all \( \omega \) and \( \vartheta \) and positive \( c \), then

\[
R_N^{-1}(\vartheta) \sim (W_N \otimes I_r)^\dagger D_N(T_{\vartheta}^{-1})(W_N \otimes I_r).
\]

(3.5)

### 3.3 The Cramér-Rao Bound

In this section, we introduce a general system model and apply the results of the previous section to derive expressions for the asymptotic Cramér-Rao Bound.

We consider a proper complex Gaussian vector random process \( y(n) \), in discrete-time \( n = 1, 2, \ldots, N \). We assume that this is obtained by uniform sampling of continuous-time observations in the time interval \([0, T] \). We next assume that the process \( y(n) \) has a \( p \times 1 \) parameter vector \( \vartheta \) embedded in it and we are interested in the problem of estimating \( \vartheta \) given the observation \( y(n) \). This problem arises in many situations where the receiver needs estimate various unknown quantities such as the signal timing or the direction-of-arrival (DOA) of an incoming signal, the channel gains associated with the transmitter-receiver link, etc. In Section 3.4, we consider the special case of this problem where \( \vartheta \) represents the range and DOA of an incoming plane wave.

Now suppose that \( y(n) \) is a stationary process so that it can be equivalently written as

\[
y(n) = m_\vartheta(n) + w_\vartheta(n), \quad n = 1, \ldots, N
\]

(3.6)

where \( m_\vartheta(n) \) is the \( r \times 1 \) mean vector for time \( n \), and \( w_\vartheta(n) \) is a proper zero-mean complex Gaussian process with its covariance dependent on \( \vartheta \). We assume that

\[\sum_{n=1}^{\infty} m_\vartheta(n)^\dagger m_\vartheta(n) \text{ is finite.} \]

It is convenient to stack the observations into a single \( Nr \times 1 \) observation \( y = (y(1)^T, \ldots, y(N)^T)^T \) (with a little notational abuse). By
stacking \( \mathbf{m}_\vartheta(n) \) and \( \mathbf{w}_\vartheta(n) \) in a similar way we get \( \mathbf{m}_\vartheta \) and \( \mathbf{w}_\vartheta \). The covariance matrix of \( \mathbf{w}_\vartheta \) has a block Toeplitz structure

\[
R_N(\vartheta) = \begin{bmatrix}
T_0 & T_{-1} & \cdots & T_{-(N-1)} \\
T_1 & T_0 & \cdots & T_{-(N-2)} \\
\vdots & \vdots & \ddots & \vdots \\
T_{N-1} & T_{N-2} & \cdots & T_0
\end{bmatrix}
\tag{3.7}
\]

where the \( r \times r \) matrix \( T_k \) is the autocorrelation function

\[
T_k = T_k(\vartheta) = \mathbb{E}[\mathbf{w}_\vartheta(n+k, \vartheta) \mathbf{w}_\vartheta^\dagger(n, \vartheta)] .
\]

Note that \( R_N(\vartheta) \) is Hermitian since \( T_k = T^\dagger_{-k} \).

To indicate its dependence on \( \vartheta \), the matrix DFT of the sequence \( \{T_k\} \) by \( \mathcal{T}_\vartheta \).

By Lemma 2, \( R_N \) is asymptotically equivalent to the block-circulant matrix

\[
C_N = (W_N \otimes I_r)^\dagger D_N(\mathcal{T})(W_N \otimes I_r) ,
\]

where \( D_N(\mathcal{T}) \) is the block-diagonal matrix

\[
D_N(\mathcal{T}) = \begin{bmatrix}
\mathcal{T}(\omega_1) & O & \cdots & O \\
O & \mathcal{T}(\omega_2) & \cdots & O \\
\vdots & \vdots & \ddots & \vdots \\
O & O & \cdots & \mathcal{T}(\omega_N)
\end{bmatrix}
\tag{3.8}
\]

and \( \omega_i = 2\pi(i-1)/N \), and

\[
\mathcal{T}(\omega) = \sum_{k=-\infty}^{\infty} T_k e^{-jk\omega} .
\tag{3.9}
\]
We make the following assumptions.

**A1.** The term \( w(n) \) represents a zero mean stationary Gaussian process such that \( R_N(\vartheta) \) is non-singular for all \( \vartheta \). Therefore, \( T_\vartheta(\omega) \) is non-singular for all \( \omega \) and \( \vartheta \). This assumption is relevant, for example, when there is stationary interference in addition to white noise in the observations.

**A2.** The discrete Fourier transform of the sequence \( \{ \frac{\partial T}{\partial \vartheta_l} \} \) exists and is equal to \( \frac{\partial T}{\partial \vartheta_l}(\omega, \vartheta) \) for all \( l \) and \( \vartheta \).

**A3.** The mean \( m_\vartheta \) is once differentiable in its \( \theta \) argument, i.e., \( \frac{\partial m_\vartheta}{\partial \vartheta_l} \) exists for all \( l \).

From assumption A2 we have
\[
\frac{\partial R_N(\vartheta)}{\partial \vartheta_l} \sim (W_N \otimes I_r) D_N \left( \frac{\partial T}{\partial \vartheta_l} \right) (W_N \otimes I_r). \tag{3.10}
\]

From assumption A1 we have
\[
R_N^{-1}(\vartheta) \sim (W_N \otimes I_r)^t D_N (T_\vartheta^{-1})(W_N \otimes I_r). \tag{3.11}
\]

Defining the \( r \times 1 \) DFT of the signal as
\[
M_\vartheta(\omega) = \sum_{n=-\infty}^{\infty} m_\vartheta(n) e^{-jn\omega},
\]
from assumption A3 (and the square-summability of \( m_\vartheta(n) \)), \( \frac{\partial M_\vartheta(\omega)}{\partial \vartheta_l} \) is finite for all \( \omega \) and \( \vartheta \).

Let \( \hat{\vartheta} = \hat{\vartheta}(y) \) be an estimator of \( \vartheta \) that provides estimates based on the observation \( y \). The performance of an estimator is measured by the mean-squared difference between the estimated and real values of \( \vartheta \). The estimator \( \hat{\vartheta}(y) \) is random vector
and is said to be unbiased if \( E(\hat{\vartheta}(y)) = \vartheta \). The Cramér-Rao Bound (CRB) provides a lower bound on the error covariance matrix of any unbiased estimator \([45]\). Defining the covariance matrix as

\[
C_{\hat{\vartheta}} = E\left[ (\hat{\vartheta} - \vartheta)(\hat{\vartheta} - \vartheta)^T \right],
\]

the CRB states that

\[
C_{\hat{\vartheta}} - \mathcal{I}_N^{-1}(\vartheta)
\]

is non-negative definite, where \( \mathcal{I}_N(\vartheta) \) is the Fisher-Information Matrix (FIM) which for complex Gaussian observation models is given by \([45, \text{pg. 525}]\)

\[
[I_N(\vartheta)]_{k,l} = 2\text{Re}\left[ \frac{\partial m^\dagger_{\vartheta}}{\partial \vartheta_k} R_N^{-1}(\vartheta) \frac{\partial m_{\vartheta}}{\partial \vartheta_l} \right] + \text{Tr}\left\{ R_N^{-1}(\vartheta) \frac{\partial R_N(\vartheta)}{\partial \vartheta_k} R_N^{-1}(\vartheta) \frac{\partial R_N(\vartheta)}{\partial \vartheta_l} \right\}.
\]

It follows from Lemma 2, (3.11) and (3.10) that

\[
\lim_{N \to \infty} \frac{1}{N} \text{Tr}\left\{ R_N^{-1}(\vartheta) \frac{\partial R_N(\vartheta)}{\partial \vartheta_k} R_N^{-1}(\vartheta) \frac{\partial R_N(\vartheta)}{\partial \vartheta_l} \right\} = \lim_{N \to \infty} \frac{1}{N} \text{Tr}\left\{ C_N^{-1}(\vartheta) \frac{\partial C_N(\vartheta)}{\partial \vartheta_k} C_N^{-1}(\vartheta) \frac{\partial C_N(\vartheta)}{\partial \vartheta_l} \right\} = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \text{Tr}\left\{ T_{\vartheta}^{-1}(\omega_i) \frac{\partial T_{\vartheta}(\omega_i)}{\partial \vartheta_k} T_{\vartheta}^{-1}(\omega_i) \frac{\partial T_{\vartheta}(\omega_i)}{\partial \vartheta_l} \right\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{Tr}\left\{ T_{\vartheta}^{-1}(\omega) \frac{\partial T_{\vartheta}(\omega)}{\partial \vartheta_k} T_{\vartheta}^{-1}(\omega) \frac{\partial T_{\vartheta}(\omega)}{\partial \vartheta_l} \right\} d\omega
\]
Similarly, we can show that
\[
\lim_{N \to \infty} \frac{1}{N} \text{Re} \left[ \frac{\partial \mathbf{m}_\theta}{\partial \vartheta_k} R_N^{-1}(\vartheta) \frac{\partial \mathbf{m}_\theta}{\partial \vartheta_l} \right] \\
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{Re} \left\{ \frac{\partial M_\theta^T(\omega)}{\partial \vartheta_k} T^{-1}(\omega) \frac{\partial M_\theta(\omega)}{\partial \vartheta_l} \right\} \, d\omega.
\]

We state the frequency-domain expression formally as a theorem.

**Theorem 3.** The asymptotic Fisher-Information Matrix denoted as
\[
\mathcal{I}_\infty(\vartheta) = \lim_{N \to \infty} (1/N) \mathcal{I}_N(\vartheta)
\]
can be written as the frequency-domain integral
\[
\lim_{N \to \infty} \frac{1}{N} [\mathcal{I}_N(\vartheta)]_{k,l} \\
= \frac{1}{2\pi} \int_{-\pi}^{\pi} 2\text{Re} \left\{ \frac{\partial M_\theta^T(\omega)}{\partial \vartheta_k} T^{-1}(\omega) \frac{\partial M_\theta(\omega)}{\partial \vartheta_l} \right\} \, d\omega + \text{Tr} \left\{ T^{-1}(\omega) \frac{\partial T_\theta(\omega)}{\partial \vartheta_k} T^{-1}(\omega) \frac{\partial T_\theta(\omega)}{\partial \vartheta_l} \right\} \, d\omega
\]

**3.3.1 Continuous-Time Signals**

Suppose next that the discrete time signal \( y(n), n = 1, \ldots, N \) is obtained from uniformly sampling a continuous time process \( y_c(t), t \in [0, T] \). Assume that \( \mathbf{m}_\theta(n) \) and \( \mathbf{w}_\theta(n) \) are obtained by sampling \( \mathbf{m}_\theta^c(t) \) and \( \mathbf{w}_\theta^c(t) \) respectively. The superscript \( c \) indicates that the signal is a continuous-time representation and the subscript \( \theta \) indicates the dependence of the signal on the parameter \( \theta \). The Fourier transform of a continuous-time signal \( x(t) \) is defined as
\[
X_c(f) = \int_0^T x(t) e^{j2\pi ft} \, dt.
\]
Let $M^c_\phi(f)$ represent the Fourier transform of $m^c_\phi(t)$. The Fourier transform of the auto-correlation function defined as $R(\tau) = \mathbb{E}w^c_\phi(t + \tau)w^c_\phi(t)$ is called the power spectral density function and is denoted by $S_\phi(f)$. From [58], when the sampling rate $N/T$ is higher than the Nyquist sampling rate, a simple relation between the continuous-time and discrete Fourier transforms exists as follows.

$$m_\phi(\omega) = \frac{1}{\sqrt{T}} m^c_\phi(\frac{N\omega}{2\pi T})$$

$$T(\omega) = \frac{1}{T} S_\phi(\frac{N\omega}{2\pi T}).$$

The asymptotic FIM given in Theorem 1 can be written using change of variables (in the limit as $N \to \infty$) as

$$I_\infty = \int_{-\infty}^{\infty} \left[2\text{Re} \left\{ \frac{\partial M^c_\phi(f)}{\partial \theta_k} S^{-1}_\phi(f) \frac{\partial M^c_\phi(f)}{\partial \theta_l} \right\} \right] df$$

$$+ \text{Tr} \left\{ S^{-1}_\phi(f) \frac{\partial S_\phi(f)}{\partial \theta_k} S^{-1}_\phi(f) \frac{\partial S_\phi(f)}{\partial \theta_l} \right\} df$$

### 3.4 Ultrawideband Vector Antennas

In this section, we apply the Cramér-Rao Bound expression to two wideband systems. The inter-relation between the antenna radiation patterns, signal design and performance are discussed.

We first develop a signal model for the UWB vector antenna receiver. In (2.12) we presented a signal model for the wideband channels under the assumption that antenna response function does not change over the frequency band of operation. The frequency band of operation is determined by the bandwidth of the transmitted signal and the channel response. However, for UWB signals with a bandwidth of the order of few Gigahertz, the antenna response can no longer be assumed to be flat.
over frequency. Therefore, the model in (2.12) does not apply directly. But, we can still use the development of Section 2.2 to obtain a signal model.

Consider a vector antenna structure located at the origin in the coordinate system indicated by Fig. 2.2. Let a scatterer be located at coordinates \((r, \theta, \phi)\) with respect to the receiver. We assume that the scatterer acts as a secondary source by reflecting the EM waves that impinge on it. (In the sequel we refer to this scatterer as the “source”). We assume a active radar scenario \([91]\), where there are antennas close to the receiver that transmit EM pulses, and the receiver processes the reflections of these pulses from these objects to estimate their locations. For simplicity we assume that there is just one scatterer or secondary source. In this case, if a pulse is sent starting at time \(t = 0\), a delayed and scattered version of this pulse is received at \(t = 2r/c\), where \(c\) is the speed of light. We assume that the reflected pulse arrives at the receiver as a plane-wave and is arbitrarily polarized (i.e., need not necessarily be linearly or circularly polarized). This signal can therefore be represented as a 2 × 1 vector

\[
s_r(t) = \begin{bmatrix} s_{r1}(t - 2r/c) \\ s_{r2}(t - 2r/c) \end{bmatrix},
\]

where \(s_{r1}(t - 2r/c)\) is the component along \(u_\theta\) and \(s_{r2}(t - 2r/c)\) is the component along \(u_\phi\).

An antenna response matrix similar in function to (2.1) which quantifies the response in the vector antenna to the signal input \(s_r(t)\) is defined for the case of UWB antennas. The signal vector induced in the vector antenna is written as the convolution

\[
\int_{-\infty}^{\infty} B(\tau, \theta, \phi)s_r(t - \tau)d\tau
\]
where $B(t, \theta, \phi)$ is the $N_R \times 2$ UWB antenna response function. This is a UWB generalization of the function defined in (2.1). We assume that the received signal is corrupted by a proper complex Gaussian noise process $w(t)$ with zero mean. For simplicity, we assume that covariance of this process has separable structure in space and time and is spatially white so that it is given by $E[w(t + \tau)w^\dagger(t)] = w(\tau)I_{N_R}$, where $w(\tau)$ is a scalar function of $\tau$. Therefore, the received signal in continuous time can be written as

$$r(t) = \int_{-\infty}^{\infty} B(\tau, \theta, \phi)s_r(t - \tau)d\tau + w(t).$$

Let the Fourier transform of the continuous-time signals $B(t, \theta, \phi)$, $s_r(t)$ and $w(t)$ be denoted by $B(f, \theta, \phi)$, $S_r(f)$ and $W(f)$ respectively. Let $S_r(f) = [S_{r1}(f), S_{r2}(f)]^T$ denote the Fourier transform in terms of the components. We assume that $w(t)$ is known, and the only unknown is the position vector $(r, \theta, \phi)$. Therefore, the parameter vector that needs to be estimated is $\theta = (r, \theta, \phi)^T$. From Section 3.3.1, the asymptotic FIM can be calculated directly noting that $M_{\theta}(f) = B(f, \theta, \phi)S_r(f)$ and $S(f) = W(f)I_3$. The FIM can now be written as

$$I_{\infty} = \int_{-\infty}^{\infty} \frac{2}{W(f)}\text{Re}\left[D_\theta^\dagger(f)D_\theta(f)\right]df,$$

where

$$D_\theta(f) = [-\frac{2}{c}B(\theta, \phi)S_r'(t), (\partial B(f, \theta, \phi)/\partial \theta)S_r(f), (\partial B(f, \theta, \phi)/\partial \phi)S_r(f)]$$

and $S_r'(f)$ is the Fourier transform of $(\partial s_{r1}(t)/\partial t, \partial s_{r2}(t)/\partial t)^T$.

We next make the following assumptions and present two UWB examples.

B1. We assume that there is no mutual coupling and further that the antenna re-
response function can be decoupled as \( \mathcal{B}(f, \theta, \phi) = J(f) \mathcal{B}^o(\theta, \phi) \) where \( \mathcal{B}^o(\theta, \phi) \) is a \( N_R \times 2 \) matrix function independent of \( f \) and \( J(f) \) is a scalar function of \( f \).

This is equivalent to assuming that all the antennas have identical frequency response and angular response of the antennas are frequency independent.

Before proceeding to the examples, we define a few quantities. Let \( \hat{\vartheta} = (\hat{r}, \hat{\theta}, \hat{\phi})^T \) be an unbiased estimator of \( \vartheta \). The diagonal entries of \( \mathcal{I}_\infty^{-1} \) are lower bounds on the variances of the estimates such that \( \text{var}(\hat{r}) \geq [\mathcal{I}_\infty^{-1}]_{1,1}, \text{var}(\hat{\theta}) \geq [\mathcal{I}_\infty^{-1}]_{2,2} \) and \( \text{var}(\hat{\phi}) \geq [\mathcal{I}_\infty^{-1}]_{3,3} \).

Following [56], Mean-Squared Range Error (MSRE) is defined as \( \text{var}(\hat{r}) \) and the Mean-Squared Angular Error (MSAE) is defined as \( \sin^2 \theta \cdot \text{var}(\hat{\phi}) + \text{var}(\hat{\theta}) \). Both the MSRE and MSAE can be lower bounded by replacing the \( \text{var} \) terms with the respective CRBs.

### 3.4.1 Example 1: An Ultrawideband Tripole

Consider a vector antenna consisting of \( N_R = 3 \) mutually orthogonal UWB dipoles with no mutual coupling and identical frequency responses \( J(f) \). This is the UWB counterpart of the tripole described in Section 2.2. In this case, we have

\[
\mathcal{B}(\theta, \phi) = J(f) \begin{bmatrix}
-\sin \phi & \cos \phi \cos \theta \\
\cos \phi & \sin \phi \cos \theta \\
0 & -\sin \theta
\end{bmatrix}
\]
After some calculations the CRB formulas can be written as

\[
\begin{align*}
\text{var}(\hat{r}) & \geq \frac{c^2}{8\beta^2 E s(1 - s \sin^2 \theta) - \mu^2 \sin^2 \theta - s \rho^2 \cos^2 \theta} \\
\text{var}(\hat{\phi}) & \geq \frac{1}{2E s(1 - s \sin^2 \theta) - \mu^2 \sin^2 \theta - \rho^2 \cos^2 \theta} \\
\text{var}(\hat{\theta}) & \geq \frac{1}{2E s(1 - s \sin^2 \theta) - \mu^2 \sin^2 \theta - \rho^2 \cos^2 \theta} \\
\text{MSAE} & \geq \frac{1}{2E s(1 - s \sin^2 \theta) - \mu^2 \sin^2 \theta - \rho^2 \cos^2 \theta}
\end{align*}
\]

where

\[
\begin{align*}
s &= \frac{||S_{r2}||^2}{||S_r||^2} = \frac{E_2}{E} \\
E &= ||S_r||^2 \\
\beta^2 &= \frac{||S'_r||^2}{||S_r||^2} \\
\rho &= \frac{2}{\beta E} \text{Re}\langle S_{r2}, S_{r1}' \rangle \\
\mu &= \frac{1}{E} \text{Re}\langle S_{r2}, S_{r1} \rangle
\end{align*}
\]

where we define the inner-product of two vectors by the frequency-domain integral

\[
\langle x(f), y(f) \rangle = \int_{-\infty}^{\infty} (j^2(f)/W(f)) y^*(f) x(f) df \quad \text{and} \quad ||x(f)||^2 = \langle x(f), x(f) \rangle.
\]

These definitions apply in a natural way to scalar functions. As defined before, \(S_r(f) = (S_{r1}(f), S_{r2}(f))^T\) and the primes denote the Fourier transform of partial derivatives with respect to \(t\) of the time-domain signals.

By Cauchy-Schwarz inequality, \(-1/2 \leq \mu \leq 1/2\). \(\mu = \pm 1/2\) if and only if linear polarization is used \((S_{r1}(f) = \text{constant} \cdot S_{r2}(f))\). The term \(\beta\) is called the root-mean-squared bandwidth.
3.4.2 Example 2: A Planar UWB Vector Antenna

Consider a three element vector antenna with two orthogonal UWB dipoles and a UWB loop colocated with the dipoles in the same plane as shown in Fig. 3.1. We again assume that there is no mutual coupling between the elements and that the frequency response of the antenna elements are identical. In this case, we have

$$\mathcal{B}(\theta, \phi) = \begin{bmatrix} -\sin \phi & \cos \phi \cos \theta \\ \cos \phi & \sin \phi \cos \theta \\ -\sin \theta & 0 \end{bmatrix}$$

Again, we omit the details of the calculation. The CRB formulas can be written as

$$\text{var}(\hat{r}) \geq \frac{c^2}{8\beta^2\mathcal{E}} \frac{(1 - s \sin^2 \theta)\{(1 - s) \cos^2 \theta + \sin^2 \theta\} - \mu^2 \sin^2 \theta}{\Delta}$$

$$\text{var}(\hat{\phi}) \geq \frac{1}{2\mathcal{E}} \frac{(1 + p^2)^{-1}[1 + \sin^2 \theta + p^2 \cos^2 \theta][\{(1 - s) \cos^2 \theta + s \sin^2 \theta\}]}{\Delta}$$

$$\text{var}(\hat{\theta}) \geq \frac{1}{2\mathcal{E}} \frac{(1 + p^2)^{-1}[1 + \sin^2 \theta + p^2 \cos^2 \theta][\{(1 - s) \cos^2 \theta + s \sin^2 \theta\}]}{\Delta} - \rho^2 \cos^2 \theta\{(1 - s) \cos^2 \theta + s \sin^2 \theta\}$$

where

$$\Delta = \left\{ \frac{1 + \sin^2 \theta + p^2 \cos^2 \theta}{1 + p^2} \left[ (1 - s \sin^2 \theta)\{(1 - s) \cos^2 \theta + \sin^2 \theta\} - \mu^2 \sin^2 \theta \right] - \rho^2 \cos^2 \theta\{(1 - s) \cos^2 \theta + s \sin^2 \theta\} \right\}$$

$$p = \frac{\|\mathbf{S}'_{r_2}\|}{\|\mathbf{S}'_{r_1}\|}$$

and $s, \mathcal{E}, \beta, \rho, \mu$ and the norm are defined as before.
3.5 Signal Design

In this section, we use the CRB and another measure derived from the CRB to design signal pulse shapes for the two UWB-VA described in the previous section.

Consider the two UWB-VAs described in the previous section. We assumed that there is no mutual coupling and identical frequency response $J(f)$ for antenna elements. The pulse shape of the signal input to the receiver is modified by the antenna response and the frequency-dependent noise. Let $p_1(t)$ and $p_2(t)$ be the whitened “effective” pulse shapes defined as follows. Let $P_1(f)$ and $P_2(f)$ denote the Fourier transforms of $p_1(t)$ and $p_2(t)$ respectively, then pulse shapes satisfy $[P_1(f), P_2(f)]^T = (J(f)/\sqrt{W(f)})S_r(f)$. We are interested in real-valued pulse shapes. We consider the impact of choosing different $p_1(t)$ and $p_2(t)$ on the MSRE and MSAE. We note that the
DOA is specified completely by $(\theta, \phi)$. With this definition the various inner-products that define $s$, $\mu$, $\rho$, etc. can be written as time-domain integrals. For example, by Parseval’s theorem [58] $\langle \mathbf{S}_r, \mathbf{S}'_{r2} \rangle = \int_{-\infty}^{\infty} p_1(t)p'_2(t)dt$. Therefore, $\rho = 0$ is equivalent to $\int_{-\infty}^{\infty} p_1(t)p'_2(t)dt = 0$ and $p = 1$ is equivalent to $\int_{-\infty}^{\infty} |p'_1(t)|^2dt = \int_{-\infty}^{\infty} |p'_2(t)|^2dt$.

Suppose we are interested in pulses limited in time to $[0, T_0]$. By inspection, the MSRE for the tripole becomes DOA independent if $\rho = 0$. The same for the planar antenna is achieved with $\rho = 0$ and $p = 1$. Equal energy differentiated Gaussian pulse shapes given by $p_1(t) = p_2(t) = A(t - T_0/2)e^{-(t-T_0/2)^2/2T_1^2}$, satisfy these two conditions. $A$ is chosen such that the signal energies add to $\mathcal{E}$ and the pulse duration is approximately $3T_1$, and it is is chosen according to the bandwidth specifications. (We note that the differentiated Gaussian pulses are strictly not limited to the interval $[0, T_0]$. Choosing $T_0 > 3T_1$ makes them approximately time limited.) These pulse shapes therefore render efficient estimators (estimators that achieve the CRB) yield DOA independent range-estimation performance. For the chosen pulses, we have $s = 1/2$, $\rho = \pm 1$, $\mu = 0$, $p = 1$, and the lower bound on MSRE then becomes

$$var(\hat{r}) \geq \frac{C^2}{8\beta^2\mathcal{E}} \quad (3.15)$$

Again by inspection, we note that MSAE for the tripole is DOA independent if $\rho = \pm 1$. The MSAE can be minimized subject to the constraints $0 \leq s \leq 1$ and $-1/2 \leq \mu \leq 1/2$ by choosing $\mu = 0$ and $s = 1/2$. The same $\mu$ and $s$ also minimize the MSAE for the planar 3-element under the additional constraints $-1 \leq \rho \leq 1$, and $p \geq 0$ for $\rho = \pm 1$ and $p = 1$. We note that the MSAE is identical for the two antennas for this choice of signal parameters.

For both the antennas, we can easily see that pulse shapes that minimize MSAE and render it DOA independent, lead to DOA-dependent MSRE and vice-versa. For example, when MSRE is made DOA independent by choosing $\rho = 0$, there exists a
singularity in MSRE at \( \sin \theta = \sqrt{s/(s^2 + \mu^2)} \). There appears to be trade-off in simultaneously achieving good performance in range and DOA estimation. This motivates the need for a “joint” criterion to design pulse shapes that “minimize” the estimation error. Towards that end we introduce the linearized confidence region in the next section.

### 3.5.1 Volume of Confidence Region

The linearized confidence region is used in [15,36,54] for source location estimation problems as a performance measure to design antenna arrays.

The confidence region is defined as the volume of the uncertainty region of Wald’s test [72, Chapter 6.e.3]. The maximum likelihood estimator of the parameter vector is asymptotically normal with the actual value of the parameter as its mean and the \( \mathcal{I}_\infty \) as its covariance matrix. Suppose \( \zeta \) is the vector parameter we are interested in estimating and \( \hat{\zeta} \) is its unbiased \( \zeta \). Testing the hypothesis \( H_0 : \hat{\zeta} = \zeta \) with \( H_0 \) as the null-hypothesis yields the confidence ellipsoid of the form \( (\hat{\zeta} - \zeta)^T \mathcal{I}_\infty (\hat{\zeta} - \zeta) \leq g \), where \( g \) is positive constant chosen to satisfy a given probability of confidence. The squared volume of this ellipsoid is proportional to \( \det (\mathcal{I}_\infty(\zeta))^{-1} \). If \( \zeta \) is a function of another parameter vector \( \vartheta \), then the squared volume of the confidence ellipsoid of the test \( H_0 : \zeta(\hat{\vartheta}) = \zeta(\vartheta) \) is proportional to \( \det (\mathcal{I}_\infty^{-1}(\vartheta)) = \det (\partial \zeta/\partial \vartheta)^T \mathcal{I}_\infty(\vartheta)^{-1} \partial \zeta/\partial \vartheta) \).

In [15,36,54] the source location defined in Cartesian coordinates as \( \zeta = [x, y, z]^T \). Let the spherical representation be \( \vartheta = [r, \theta, \phi]^T \). We have the relation

\[
    x = r \sin \theta \cos \phi \\
    y = r \sin \theta \sin \phi \\
    z = r \cos \theta.
\]
The volume of the linearized confidence region \([15,36,54]\), \(V_\zeta\) is shown to be proportional to

\[
V_\zeta^2 \sim \det (I_\infty (\zeta)^{-1})
= \det \left( \frac{\partial \zeta}{\partial \vartheta} I_\infty^{-1}(\vartheta) \frac{\partial \zeta^T}{\partial \vartheta} \right)
= \det \left( \frac{\partial \zeta}{\partial \vartheta} \right)^2 \det (I_\infty^{-1}(\vartheta)).
\]

We have \(\det \left( \frac{\partial \zeta}{\partial \vartheta} \right) = r^2 \sin \theta\), as a result

\[
V_\zeta \sim r^2 |\sin \theta| \cdot \det (I_\infty)^{-1/2}
\]

Therefore, the confidence volume is specified up to a proportionality constant. The confidence volume in Cartesian coordinates corresponds to actual physical volume, and the minimizing the confidence volume is equivalent to minimizing the uncertainty of estimates in the sense of the hypothesis test discussed above.

**Tripole Antenna**

We now evaluate the confidence volume for the tripole antenna. The determinant of the FIM evaluates to

\[
\det (I_\infty) = \frac{32\beta^2 \xi^3}{c^2} [s(1 - s \sin^2 \theta) - \mu^2 \sin^2 \theta - s \rho^2 \cos^2 \theta].
\]

In order that the confidence volume is DOA independent we need the following conditions to be satisfied:

\[
\rho = 1, \quad \tag{3.16}
\]
\[
s - s^2 - \mu^2 > 0. \quad \tag{3.17}
\]
Subject to these conditions and the constraints \(0 \leq s \leq 1\) and \(-1/2 \leq \mu \leq 1\), the confidence volume

\[
V_\zeta \sim \frac{cr^2}{\beta \sqrt{32E^3}} (s - s^2 - \mu^2)^{-1/2}
\]

is minimized for \(s = 1/2\) and \(\mu = 0\). These are the same conditions for making MSAE independent of DOA.

**Planar 3-element Antenna**

For the planar antenna we have

\[
\det(I_\infty) = \frac{32\beta^2 E^3}{c^2} \left\{ \frac{1 + \sin^2 \theta + p^2 \cos^2 \theta}{1 + p^2} \left[ (1 - s \sin^2 \theta) \{(1 - s) \cos^2 \theta + s \sin^2 \theta \} 
- \mu^2 \sin^2 \theta \right] - \rho^2 \cos^2 \theta \{(1 - s) \cos^2 \theta + s \sin^2 \theta \} \right\}
\]

With the choice of signals with \(p = 1\), the confidence volume becomes DOA independent if

\[
\rho = 1 \quad (3.18)
\]

\[
s = 1/2 \quad (3.19)
\]

\[
\mu^2 < 1/4 \quad (3.20)
\]

Subject to the same constraints on \(s\) and \(\mu\) as before, the confidence volume

\[
V_\zeta \sim \frac{cr^2}{\beta \sqrt{E^3}} \left( \frac{1}{4} - \mu^2 \right)^{-1/2}
\]

is minimized if \(\mu = 0\). Again, like in the case of tripole antenna, these conditions also make MSAE independent of DOA.
Figure 3.2: Square-roots of MSRE and MSAE lower bounds (dashed and solid lines respectively) plotted against the total SNR $\mathcal{E}$.

We note that signals pulses that minimize the DOA independent confidence volume for planar antenna are sufficient conditions to achieve the same result for the tripod. The MSAE and the confidence volumes are the same for both the antenna for this choice of signals. For DOA-independent range estimation, it is sufficient to have $s = 1/2$, $p = 0$, $\mu = 0$ and $p = 1$, and this results in

$$MSAE \geq \frac{2}{\mathcal{E}}$$

(3.21)

for both the dipole and planar antenna.

We next focus on designing signal pulses that satisfy the required conditions.
Consider the expression for $\rho$

$$
\rho = \frac{2 \text{Re}[\langle S_r, S'_r \rangle]}{\beta \mathcal{E}}
$$

$$
= \frac{2 \text{Re}[\langle S_r, S'_r \rangle]}{\sqrt{||S_{r1}||^2 + ||S_{r2}||^2} \cdot \sqrt{||S'_{r1}||^2 + ||S'_{r2}||^2}}
$$

$$
\leq \frac{2 ||S_r|| \cdot ||S'_r||}{\sqrt{||S_{r1}||^2 + ||S_{r2}||^2} \cdot \sqrt{||S'_{r1}||^2 + ||S'_{r2}||^2}}
$$

$$
= \frac{2 \sqrt{s} \cdot ||S'_{r1}||}{\sqrt{||S'_{r1}||^2 + ||S'_{r2}||^2}}
$$

With $s = 1/2$ and $||S'_{r1}|| = ||S'_{r2}||$ we note that the inequality above becomes equality if and only if $S_{r2} = C_1 \cdot S'_{r1}^\dagger$ where $C_1$ is a constant. To summarize the various conditions and the simplifications, the waveforms that give DOA independent angle estimates need to have the following properties:

- $||S_{r1}|| = ||S_{r2}||$
- $||S'_{r1}|| = ||S'_{r2}||$
- $S_{r2} = C_1 \cdot S'_{r1}^\dagger$ where $C_1$ is a constant
- $\langle S_{r1}, S_{r2} \rangle = 0$.

We are interested in real-valued pulse shapes $p_1(t)$ and $p_2(t)$. Choose

$$
p_1(t) = A \cos(\omega_s t) \cos(\omega_c t), \ t \in [0, 2\pi/\omega_s]
$$

and zero otherwise, and

$$
p_2(t) = A \sin(\omega_s t) \cos(\omega_c t), \ t \in [0, 2\pi/\omega_s]
$$

and zero otherwise. These pulses for large $\omega_c/\omega_s$ ratio approximately satisfy all the
Table 3.1: Square-root delay and angular errors for bandwidth of $\beta = 7.5$ GHz.

<table>
<thead>
<tr>
<th>SNR (dB)</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>delay (ps)</td>
<td>47.5</td>
<td>26.7</td>
<td>15.0</td>
<td>8.4</td>
<td>4.7</td>
</tr>
<tr>
<td>angular error (deg)</td>
<td>25.6</td>
<td>14.4</td>
<td>8.1</td>
<td>4.6</td>
<td>2.6</td>
</tr>
</tbody>
</table>

required conditions. By setting $\omega_c/\omega_s = n + 1/2$, for positive integer $n$, we can ensure that modulated waveform is zero at the boundary. The chosen waveforms $p_1(t)$ and $p_2(t)$ are known as Hanning-windowed pulses [58].

Numerical Example: The allowed spectral mask between $3.1 – 10.6$ GHz for UWB signalling [19] has a bandwidth of $7.5$ GHz. The lower bound on standard deviations of the estimates can be obtained by taking square-roots of MS Range Error (MSRE) and MSAE lower-bounds. For range estimation, equal energy differentiated Gaussian pulses are chosen such that the lower bound is DOA independent. For DOA independent angle estimation, we choose $p_1(t) = A\cos(\omega_s t)\cos(\omega_c t)$ and $p_2(t) = A\sin(\omega_s t)\cos(\omega_c t)$, with $\omega_c = 2\pi \cdot 6.875$ GHz, $\omega_s = 2\pi \cdot 1.25$ GHz. The pulses are plotted in Fig. 3.3. The $A$ is chosen such that the energy of the pulses add up to $E$.

The standard deviations of range and DOA error are defined as the square-roots of $MSRE$ and $MSAE$ respectively. The $MSRE$ and $MSAE$ are respectively given in (3.15) and (3.21). The standard deviations are plotted for the total SNR defined as $E$ in Fig. 3.2.

In table 3.1 we calculate the square-root of delay (delay defined as $cr/2$) and the angular error. We note that the minimum pulse width of the signal is 400 ps for the chosen bandwidth. This indicates that if the delay and DOA are being estimated for example in a UWB communication system, using a training bit-sequence, the squared-error scales as the inverse of SNR. Since, the total SNR is proportional to the number of training bits, we can get a lower bound on the number of training bits that need to be used. $\square$
Figure 3.3: The pulses $p_1(t)$ and $p_2(t)$ chosen to make the lower bound on MSAE DOA independent.

3.6 Conclusions

To demonstrate the advantages of using a polarimetric vector antenna instead of a single ultrawideband dipole or loop, we considered the problem of joint estimation of source range and DOA. We first proved a result on the asymptotic properties of block-Toeplitz matrices. We derived an asymptotic expression for the CRB as a frequency-domain integral. This result generalizes the expression obtained in [99] which was for a scalar Gaussian random process with separable covariance structure. Exact expressions for the asymptotic CRB for range and DOA estimation were derived for two UWB vector antennas under some simplifying assumptions. Signals that make the CRBs independent of the DOA were obtained for both range and DOA estimation. Another criterion called the confidence volume was derived and it was found that signals that are optimal in the sense of this criterion for the planar antenna are also optimal for the tripole. Both the antennas yield the same performance with
an efficient estimator for the chosen signal shapes.
Chapter 4

Capacity of Volume-Limited Current Distributions

This chapter considers the fundamental limits on the capacity of multiple-antenna systems limited by the volume that the antennas are allowed to occupy. A simplified model of a multiple-antenna transmitter with an unlimited number of antennas, but all of them restricted to a finite volume, is formulated as a vector wave problem in an electromagnetic theory setting. An infinite-dimensional input-output model is developed and the singular values of the associated linear operator are derived in closed form. Numerical calculations show that the capacity scaling is quadratic in the signal-to-noise power ratio (SNR). An explicit formula for capacity as a function of SNR and the transmitter size is given.

4.1 Introduction

Multiple-Input Multiple-Output (MIMO) systems have generated tremendous research interest in the past decade mainly because of two reasons: a promise of improve-
ment in data rate and improved robustness of communication schemes to multipath fading and other physical environment impairments. Most of the work on multiple-antenna systems assumes a Rayleigh flat-fading environment with i.i.d. channel components for analytical tractability. With this model, when the channel is ergodic and the receiver has perfect channel knowledge, the capacity scales as $\min\{M, N\} \log \text{SNR}$, where $M$ and $N$ are the number of transmit and receive antennas, respectively, and SNR is the signal-to-noise power ratio (SNR) at each receive antenna [92]. When the antennas are placed to close to each other, there is non-negligible correlation in the channel fading coefficients. A channel model that incorporates this by introducing correlation between the channel entries has been considered in literature (for example [79]). It has been shown in [11] that when the channel correlation has a certain symmetrical and separable structure (i.e., the correlation matrix of the random vector containing the channel matrix entries is decomposable into a Kronecker product of two matrices), the capacity still scales linearly with the number of antennas; however, there is a reduction in the slope of the capacity-SNR curve as compared to the uncorrelated case. When the transmit or receive antennas are restricted to finite volumes, as the number of antennas increases, the inter-antenna spacing becomes smaller and smaller. This leads to increased correlation which cannot be captured accurately by the Kronecker model [11] and a sub-linear scaling of capacity with the number of antennas results [63]. To maintain linear scaling, the size of the array needs to proportionally increase. Therefore, there appears to be a dependence of capacity on the volume that the antennas occupy. The main aim of this chapter is to develop tools to precisely characterize this dependence.

The Rayleigh fading model is not based on the underlying electromagnetic (EM) interactions between the antennas and the propagation environment. As a result, it is not adequate to provide insights into the spatial dependence of capacity. An
alternate approach that views antennas as current distributions has been adopted in [63] and [65]. A densely-packed antenna array in a finite volume was considered. It was observed that the capacity scaling is sub-linear in the number of antennas and saturates as the number of antennas becomes large. It was also observed that, the number of degrees of freedom (defined as the coefficient of the log\( \text{SNR} \) term in the capacity expression at high SNRs) is related the size of the antenna array. A different definition of degrees-of-freedom was given in [7] where it was defined as the minimum number of sampling functions required to approximate the electric field radiated from a source such that the error in the approximation asymptotically goes to zero in the large source-size limit. We will use the former definition of degrees-of-freedom in this chapter.

In [63], the antenna correlation for a uniform circular array of fixed radius \( a \) was expressed in terms of a Jakes-like correlation model. The limit of this channel when the number of elements becomes infinite for a finite \( a \) was considered. Based on an approximation to the Bessel functions, it was argued that the number of significant eigenvalues of the channel matrix tends to \( 2[\pi ea/\lambda] + 1 \), where \( \lambda \) is the operating wavelength and the remaining eigenvalues go to zero. In [65], a current distribution is assumed to consist of a large number of densely-packed infinitesimal one-dimensional currents (called uni-polarized currents). The Green’s function for the scalar potential problem (also known as the scalar-wave equation problem [5]) in its series form is considered for three geometries: (1) a linear array of length \( L \), (2) a circular array of radius \( a \), and (3) a spherical array of radius \( a \). An approximation similar to [63] is made and the the number of degrees-of-freedom for the three cases are found to be equal to (1) \( 2L/\lambda \), (2) \( 2\pi a/\lambda \), and (3) \( 4\pi^2 a^2/\lambda^2 \) for the three cases respectively. To extend the results to three-dimensional current sources (called tri-polarized currents), a reasoning based on the rank of the Green’s function is used. It was argued that, in
the case of tri-polarized currents, there is gain of 2 as compared to the uni-polarized currents.

The problem of transmitting information between a transmit volume and a receive volume through vector EM waves is formulated in an operation-theoretic framework in [53]. The number of dominant communication modes is calculated numerically using the Galerkin approximation. In [28], the same framework is used but with the scalar transfer function. An upper bound on the capacity of the channel between concentric spheres with the inner sphere transmitting and the outer sphere receiving is obtained.

In reality, the information from the transmitter travels as an EM field vector wave in the propagation environment. In this context, the implications of solving the scalar wave problem on the complete EM problem (also called the vector-wave problem [5]) are not easily apparent. Firstly, a solution to the complete vectorial EM problem would characterize the exact dependence of capacity on the geometry of the current distribution and the signal power. The analyses in [28, 63, 65] consider the scalar wave problem which gives the scalar potential due to a charge distribution and do not incorporate the vector EM propagation. Secondly, it is not clear whether linear arrays are the best antennas that one could use in a MIMO system in terms of capacity. The vectorial analysis would give insights on what kind of antennas maximize capacity and whether it is possible to build antennas that perform better than linear arrays. This chapter addresses the first of the two concerns. A multiple-antenna transmitter with no restriction on the number of antennas, but all of the antennas restricted to a finite volume and radiating into the surrounding space is considered. We assume that the volume is a spherical region of radius $a$. The transmitter can generate an arbitrary current distribution inside the volume subject to power constraints. Instead of considering a similar volume limited receiver and a scattering
environment in between, we simplify the problem considerably by assuming that we have an all-powerful receiver capable of measuring all the information transmitted in the far-field. The receiver measures the electric field (E-field) vector over its surface which is a sphere of radius $R \gg a$ such that it is concentric with the source and is in the far-field. We note that from Maxwell’s equations, specifying the E-field over a region along with the boundary conditions, completely specifies the magnetic field. The current distribution at the transmitter is taken to be the input to the channel and the E-field vector at the receiver is the output. The input-output relationship is expressed as a linear operator, and the exact singular values of this operator are derived. This description is then used to compute the information-theoretic capacity of the channel.

This chapter is organized as follows: In Section 4.2 we obtain the singular system of the linear operator which describes the signal propagation from the transmitter to the receiver. In Section 4.3, the capacity formulas are given. In Section 4.4, we present numerical results and summarize our conclusions. The notation adopted in the chapter is as follows: If $A$ is a matrix, $A^\top$ denotes the transpose, $A^\dagger$ denotes the Hermitian transpose, and $\text{tr}(A)$ denotes the trace. The symbol $\mathbb{E}$ denotes expectation with respect to the underlying distribution. The symbol $\sim$ indicates that the term on the left is proportional to the term on the right. All logarithms are with respect to base 2. The Kronecker delta function $\delta_n$ is equal to unity if $n = 0$, and equal to zero otherwise. Finally, $\delta_D(x)$ denotes the Dirac’s delta function.

### 4.2 Radiation into Free Space

In this section, we consider the problem of an arbitrary current distribution radiating into free space. Most of the notation and the development of EM theory
is adapted from [87]. Consider a three-dimensional current distribution enclosed in a volume $V'$, denoted by $J(R')$ for $R' \in V'$ as shown in Fig. 4.1. The vectors $E$ and $H$ are the complex baseband representations of a narrowband sinusoidal electric and magnetic field vectors respectively. If $\mathcal{E}(t)$ is the electric field vector, then it can be written as $\mathcal{E}(t) = \Re\{Ee^{j\omega t}\}$, and a similar expression can be given for the magnetic field $\mathcal{H}(t)$. The $E$ and the $H$ vectors at any point $R$ in space satisfy the Maxwell’s equations and they can be compactly represented as *inhomogeneous vector wave equations* ([87] and references therein) given by

\[
\nabla \times \nabla \times E - k^2 E = j\omega \mu_0 J \\
\nabla \times H - k^2 H = \nabla \times J
\]

(4.1)

where $k = \omega \sqrt{\mu_0 \varepsilon_0} = 2\pi / \lambda$ is called the *wave-number*, $\lambda$ denotes the free-space wavelength, $\omega$ is the angular frequency of the wave, $\varepsilon_0$ and $\mu_0$ are the electric permittivity and the magnetic permeability of free-space, respectively. Obtaining a solution via the Green’s function to the above vector differential equations for various boundary conditions is the focus of the body of literature on *dyadic* Green’s functions [87]. For a source radiating into free space in the absence of any other scattering object, and when the point of observation $R$ is outside the source, the solution to (4.1) takes a relatively simple form:

\[
E(R) = j\omega \mu_0 \int_{V'} G_{eo}(R, R') J(R') dR' \\
J(\lambda) = \nabla \times E(R)
\]

(4.2)  (4.3)

where $G_{eo}(R, R')$ is the matrix form of the Green’s function given in [87, Eqn. 4.144].

To solve for the E-field in [87], the Ohm-Rayleigh method is used to obtain a series representation for the Green’s function in terms of orthogonal functions in spherical
Figure 4.1: A volume $V'$ transmitting to an observation region $S$. In Section 4.2, $V'$ is sphere of radius $a$ and $S$ is a larger sphere of radius $R$ with a common center such that $R \gg a$.

coordinate system. In the spherical coordinate system, a point vector $\mathbf{R}$ can be represented by the triplet $(R, \theta, \phi)$ as in Fig. 2.2. The spherical coordinate system is specified with the help of an orthonormal triad with unit vectors $\mathbf{u}_r$, $\mathbf{u}_\theta$ and $\mathbf{u}_\phi$. In a similar way, let $\mathbf{R}'$ be parameterized by $(R', \theta', \phi')$. We use the development in [87] with a slight change in notation and write the expression for $G_{eo}$ as given in [87, Eqn. 10.24]

$$G_{eo}(\mathbf{R}, \mathbf{R}') = -\frac{1}{k^2} \mathbf{R} \mathbf{R}'^\dagger \delta_D(\mathbf{R} - \mathbf{R}') + \frac{jk}{4\pi} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} C_{m,n} \cdot$$

$$\begin{cases} 
M_{nm}^{(1)}(kR)M_{nm}^{+}(kR') + N_{nm}^{(1)}(kR)N_{nm}^{+}(kR'), & R > R' \\
M_{nm}(kR)M_{nm}^{(1)+}(kR') + N_{nm}(kR)N_{nm}^{(1)+}(kR'), & R < R'
\end{cases}$$
where \( \hat{R} \) is the unit vector in the direction of \( R \),

\[
C_{m,n} = (2 - \delta_m)(2n + 1)(n - m)! / n(n + 1)(n + m)!
\]

and

\[
M_{nm}(kR) = (-1)^{m+1} \frac{m}{\sin \theta} j_n(kR) P_{nm}(\cos \theta) f_m(\phi) \mathbf{u}_\theta - \frac{j_n(kR)}{\sin \theta} \frac{\partial P_{nm}(\cos \theta)}{\partial \theta} g_m(\phi) \mathbf{u}_\phi
\]

\[
N_{nm}(kR) = \frac{n(n + 1)}{kR} j_n(kR) P_{nm}(\cos \theta) g_m(\phi) \mathbf{u}_r + \frac{1}{kR} \frac{\partial}{\partial R} \left( R j_n(kR) \right) \left[ \frac{\partial P_{nm}(\cos \theta)}{\partial \theta} g_m(\phi) \mathbf{u}_\theta \right] + \left( -1 \right)^{m+1} \frac{m}{\sin \theta} P_{nm}(\cos \theta) f_m(\phi) \mathbf{u}_\phi,
\]

and expressions for \( M^{(1)} \) and \( N^{(1)} \) are obtained by replacing \( j_n(kR) \) with \( h_n^{(1)}(kR) \) in the expressions above for \( M \) and \( N \) respectively. Here \( j_n(x) \) is the spherical Bessel function of the first kind and \( n \)th order, \( h_n^{(1)}(x) \) is the spherical Hankel function of first kind, \( P_{nm}(x) \) is the \( m \)th associated Legendre function of \( n \)th order \([1, 98]\). The functions \( f \) and \( g \) are defined as

\[
f_m(\phi) = \begin{cases} 
\sin m\phi, & \text{if } m \text{ is even} \\
\cos m\phi, & \text{otherwise}
\end{cases}
\]

\[
g_m(\phi) = \begin{cases} 
\cos m\phi, & \text{if } m \text{ is even} \\
\sin m\phi, & \text{otherwise}
\end{cases}
\]
Let \( V_\infty = \mathbb{R}^3 \) be the entire space. From [87, Eqn. 10.18], we have

\[
\int_{V_\infty} M_{nm}(kR) \cdot N_{n'm'}(k'R) dV = 0
\]

\[
\int_{V_\infty} M_{nm}(kR) \cdot M_{n'm'}(k'R) dV = \int_{V_\infty} N_{nm}(kR) \cdot N_{n'm'}(k'R) dV = \frac{(1 + \delta_m)\pi^2 n(n+1)(n+m)!}{k^2(2n+1)(n-m)!} \delta_D(k - k') \cdot \delta_{n-n} \delta_{m-m'},
\]

(4.4)

where \( x \cdot y = y^\dagger x \), \( \delta_D(x) \) is the Dirac delta function, and \( \delta_n \) is the Kronecker delta function. The Green's function can therefore be written as sum of a sequence of orthogonal functions.

We now specialize this result to the case where the source is restricted to a spherical volume of finite radius \( a \) and the observation is a spherical surface of radius \( R \) in the far-field region concentric with the source. Let \( V_a \) represent a spherical volume region of radius \( a \), and \( S_R \) represent the spherical surface of radius \( R \) such that \( R \gg a \). By spherical symmetry of the integration volume, it can be shown that

\[
\int_{V_a} M_{nm}(kR) \cdot N_{n'm'}(k'R) dV = 0
\]

\[
\int_{V_a} M_{nm}(kR) \cdot M_{n'm'}(k'R) dV = \frac{2(1 + \delta_m)\pi n(n+1)(n+m)!}{(2n+1)(n-m)!} \delta_{m-m'} \cdot \int_0^a \left[ \frac{n+1}{2n+1} j_{n-1}(kR') j_{n-1}(k'R') + \frac{n}{2n+1} j_{n+1}(kR') j_{n+1}(k'R') \right] R'^2 dR'
\]

\[
\int_{V_a} N_{nm}(kR) \cdot N_{n'm'}(k'R) dV = \frac{2(1 + \delta_m)\pi n(n+1)(n+m)!}{(2n+1)(n-m)!} \delta_{m-m'} \cdot \int_0^a j_n(kR') j_n(k'R') R'^2 dR'.
\]
If \( kR \gg 1 \), we have the approximation \( h_n^{(1)}(kR) \approx (-j)^{n+1}e^{jkR}/k \) [87, Chapter 10]. Using this approximation in the expressions for \( M^{(1)} \) and \( N^{(1)} \), for a fixed \( R \), we can write the surface integrals over \( S_R \) as

\[
\int_{S_R} M^{(1)}_{nm}(kR) \cdot M^{(1)}_{nm}(k'R) dS = \int_{S_R} N^{(1)}_{nm}(kR) \cdot N^{(1)}_{nm}(k'R) dS = \frac{1}{R^2} \frac{2(1 + \delta_m)\pi n(n+1)(n+m)!}{(2n+1)(n-m)!}
\]

Now let \( ||x||_{V_a} = \sqrt{\int_{V_a} x(R) \cdot x(R) dR} \), where the volume integral is over all points \( R \in V_a \) and in a similar way \( ||x||_{S_R} = \sqrt{\int_{S_R} x(R) \cdot x(R) dS} \). Define

\[
\Psi_1(kR) = M(kR)/||M(kR)||_{V_a}, \quad \Psi_2(kR) = N(kR)/||N(kR)||_{V_a}, \\
\Psi_1^{(1)}(kR) = M^{(1)}(kR)/||M^{(1)}(kR)||_{S_R}, \text{ and} \quad \Psi_2^{(1)}(kR) = N^{(1)}(kR)/||N^{(1)}(kR)||_{S_R}.
\]

This is possible since, for any \( a > 0 \) and \( R > 0 \), each of the norms in the denominators of the definitions is positive. Therefore, we can write the Green’s function of (4.4) for the case \( R > R' \) as a sum of orthonormal functions as

\[
G_{eo}(R, R') = \frac{c_1}{R} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left[ \beta_n \Psi_1^{(1)}_{nm}(kR) \Psi_1^{\dagger}_{nm}(kR') \right] + \gamma_n \Psi_2^{(1)}_{nm}(kR) \Psi_2^{\dagger}_{nm}(kR') \quad (4.5)
\]
where \( c_1 = j\pi/2k \),

\[
\beta_n = \sqrt{\frac{n+1}{2n+1}}\alpha_{n-1} + \frac{n}{2n+1}\alpha_{n+1},
\]

\[
\gamma_n = \sqrt{\alpha_n}, \quad \text{and}
\]

\[
\alpha_n = \frac{k^2}{2\pi} \int_0^{ka} x^2 j_n^2(x) dx
= \frac{\pi (ka)^2}{4} j_{n+\frac{1}{2}}^2(ka),
\]

where the integral in \( \alpha_n \) is from [24, Eqn. 6.521].

We denote by \( T_G \) the linear integral transform that maps any square integrable current distribution in the volume \( V_a \) to an electric field at a point on the surface of \( S_R \). Let \( L_2^3(V_a) = \{ f(R) | \int_{V_a} f(R) \cdot f(R) \, dR < \infty \} \) denote the vector space of square-integrable vector functions with component-scalars defined over the volume \( V_a \). With the inner-product defined as \( \langle f, g \rangle_{V_a} = \int_{V_a} g^\dagger(R) f(R) dR \), and the induced norm defined as \( \|f\|_{V_a} = \sqrt{\langle f, f \rangle_{V_a}} \), \( L_2^3(V_a) \) is a Hilbert space. Similarly, let \( L_2^3(S_R) \) denote the Hilbert space of square integrable vector functions with component scalars defined over the surface \( S_R \), and the inner-product defined as \( \langle f, g \rangle_{S_R} = \int_{S_R} g^\dagger(R) f(R) dR \), and the induced norm defined as \( \|f\|_{S_R} = \sqrt{\langle f, f \rangle_{S_R}} \). For any two subsets \( X \subset L_2^3(V_a) \) and \( Y \subset L_2^3(S_R) \), let \( T_G : X \to Y \) denote a linear operator defined as

\[
y = T_G x \Leftrightarrow \quad y(R) = \int_{V_a} G_{\text{eo}}(R, R') x(R') dR', \quad x \in X, \ y \in Y. \tag{4.7}
\]

\( T_G \) is a bounded Hilbert-Schmidt-type operator [49] and we next prove that it is a compact operator.

**Theorem 4.** The linear operator \( T_G \) given in (4.6) is compact.

**Proof.** From [1, 10.1.50], \( \sum_{n=0}^{\infty} (2n+1) J_{n+0.5}^2(ka) = 2(ka)/\pi \). Thus, \( \sum_{n=1}^{\infty} (2n+1) \beta_n^2 < \)
2 \sum_{n=0}^{\infty} (2n + 1) \alpha_n < (ka)^3/2 and \sum_{n=1}^{\infty} (2n + 1) \gamma_n^2 < (ka)^3/2.

Define a norm \( ||G_{eo}(R, R')|| = \sqrt{\int_V \int_{S_R} tr(G_{eo}(R, R')G_{eo}^\dagger(R, R'))dRdR'} \), where \( tr(\cdot) \) indicates the matrix trace. From the boundedness of the sequences \( \beta_n \) and \( \gamma_n \)'s as shown above, we have \( ||G_{eo}(R, R')|| < c_1(ka)^3/R \). The functions \( \Psi_{1nm}, \Psi_{2nm}, \Psi_{1nm}^{(1)} \) and \( \Psi_{2nm}^{(1)} \) are dependent on the Bessel function of first kind, associated Legendre functions of cos \( \theta \) (and their first derivatives) and the exponential function \( \exp(j\phi) \). So, they are continuous on the set \( (R', \theta', \phi') \in [0, a] \times [0, \pi] \times [0, 2\pi] \). For any \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that \( |R' - R''| < \delta \Rightarrow |\Psi_1(kR') - \Psi_1(kR'')| < \epsilon \) and \( |\Psi_2(kR') - \Psi_2(kR'')| < \epsilon \). Thus, we have \( ||G_{eo}(R, R') - G_{eo}(R, R'')|| < c_1(ka)^3\epsilon/R \). Hence, \( G_{eo}(R, R') \) is continuous in the second argument. By a similar argument, it is continuous in the first argument as well. From [49, Theorem 2.16], a linear operator with continuous kernel is compact.

In the sequel, we ignore the presence of \( c_1/R \) in \( T_G \) as the only bearing this term has on our results is on the SNR. So, we set it to unity in the remainder of this chapter.

Suppose \( X \) and \( Y \) are Hilbert spaces and \( A : X \rightarrow Y \) is a compact linear operator. Let \( A^* : Y \rightarrow X \) be its adjoint. The non-negative square-root of the eigenvalues of the self-adjoint operator \( A^*A : X \rightarrow X \) are called the singular values of \( A \) [49, Theorem 15.15]. Denote the sequence of singular values by \( \{\mu_n\} \) such that \( \mu_1 \geq \mu_2 \geq \ldots \mu_n \geq \ldots \), including multiplicities for all positive integers \( n \in \mathbb{N} \). [49, Theorem 15.16] states that there exist orthonormal sequences \( \{f_n\} \) in \( X \) and \( \{g_n\} \) in \( Y \) such that

\[ Af_n = \mu_ng_n, \quad A^*g_n = \mu_nf_n \]

for all \( n \in \mathbb{N} \). The sequence \( (\mu_n, f_n, g_n) \) constitutes the singular system of \( A \). The singular values of the operator \( T_G \) defined in (4.6) are the sequences \( \{\beta_n\}_{n=1}^{\infty} \) and \( \{\gamma_n\}_{n=1}^{\infty} \).
with the \( n \)th singular value in each sequence having a multiplicity of \( 2n + 1 \). From (4.5) it is easily seen that the operator \( T_G \) has the pair of sequences \((\beta_n, \Psi_{1nm}, \Psi_{1nm}^{(1)})\) and \((\gamma_n, \Psi_{2nm}, \Psi_{2nm}^{(1)})\) as its unordered singular system.

### 4.3 Information-Theoretic Capacity

In this section, we study the capacity of the channel to which an arbitrary square-integrable current distribution in a spherical volume is the input and the E-field observed over a concentric spherical surface in the far-field is the output. We assume that there is an additive noise process corrupting the observation vectors at the receiver.

Consider a system with a volume-restricted current distribution \( x \in L^3_2(V_a) \) as the input and the electric field observed on the surface \( S_R \) as the output, related by \( y = T_G x \). The vector functions \( x \) and \( y \) are complex valued. Assume that the observations are corrupted by an additive white noise field \( z(R) \) defined on the surface \( S_R \). The noise field is assumed to be proper complex and that \( E[z(R')^\dagger z(R'')] = \sigma^2 \delta_D(R' - R'') \) and \( E[z(R')^T z(R'')] = 0 \) for any two coordinates \( R', R'' \in S_R \). By direct-sum decomposition, \( L^3_2(V_a) = X_1 \oplus X_1^\perp \), where \( X_1 = \text{span}\{\Psi_{1nm}, \Psi_{2nm}\} \) is the vector space spanned by the basis functions and \( X_1^\perp \) is the space of vectors orthogonal to all vectors in \( X_1 \). For any \( x \in L^3_2(V_a) \), we can uniquely decompose it into two components as \( x(R) = \sum_{nm}(x_{1nm}\Psi_{1nm}(R) + x_{2nm}\Psi_{2nm}(R)) + x'(R) \), where \( x' \in X_1^\perp \) for \( R \in V_a \). Since \( x' \) is in the null-space of the operator, it does not produce any output. As both \( x \) and \( y \) are infinite dimensional, the input-output equation can be
written as infinitely-many parallel Gaussian channels

\[ y_{1nm} = \beta_n x_{1nm} + z_{1nm} \]
\[ y_{2nm} = \gamma_n x_{2nm} + z_{2nm}, \quad m = -n, \ldots, n, n = 1, 2, \ldots \]  \hspace{1cm} (4.8)

Since the noise terms \( z_{inm}, i = 1, 2 \) are proper complex, the mutual information between the sequences \( y_{inm} \) and \( x_{inm} \) is maximized when \( x_{inm} \) are independent proper Gaussian random variables. The mutual information for a system of infinitely-many parallel AWGN channels is given by

\[
I(x; y) = \sum_{inm} I(x_{inm}; y_{inm}) \leq \sum_{nm} \left[ \log \left( 1 + \frac{\beta_n^2 \mathbb{E} |x_{1nm}|^2}{\sigma^2} \right) + \log \left( 1 + \frac{\gamma_n^2 \mathbb{E} |x_{2nm}|^2}{\sigma^2} \right) \right]
\]

and the upper bound is achieved if \( x_{1nm} \) and \( x_{2nm} \) are zero mean, jointly Gaussian and independent. The capacity of the system is the supremum of the above equality over all inputs \( x_{inm} \) that satisfy the power constraint \( \mathbb{E} \sum_{inm} |x_{inm}|^2 \leq P \). Denoting capacity by \( C(P) \) and setting \( p_{1nm} = \mathbb{E} |x_{1nm}|^2 \) and \( p_{2nm} = \mathbb{E} |x_{2nm}|^2 \), we have \( C(P) \) given by

\[
C(P) = \sup \sum_{nm} \left[ \log \left( 1 + \frac{p_{1nm} \beta_n^2}{\sigma^2} \right) + \log \left( 1 + \frac{p_{2nm} \gamma_n^2}{\sigma^2} \right) \right]
\]  \hspace{1cm} (4.9)

where the supremum is over all power distributions such that \( \sum_{nm} (p_{1nm} + p_{2nm}) \leq P \). Due to multiplicities in the singular values, the supremum in (4.9) is achieved for \( p_{inm} \) that do not change over \( m \). Therefore, (4.9) becomes

\[
C(P) = \sup \sum_n (2n + 1) \left[ \log \left( 1 + \frac{p_{1nm} \beta_n^2}{\sigma^2} \right) + \log \left( 1 + \frac{p_{2nm} \gamma_n^2}{\sigma^2} \right) \right]
\]  \hspace{1cm} (4.10)
where $p_{in} = p_{inm}$, for $i = 1, 2$, and $m = -n, \ldots, n$ and the supremum is over all power distributions such that $\sum_{n=1}^{\infty} (2n + 1)(p_{1n} + p_{2n}) \leq P$. The optimization is that of finding the supremum of a concave function over a compact convex set. As a result, a unique optimal solution exists and the optimum value of mutual information is known as the *water-pouring* capacity. A parametric solution for capacity can be given as follows [12].

\[
C(P) = \sum_n (2n + 1)(\log \mu \beta_n^2)^+ + (\log \mu \gamma_n^2)^+ \tag{4.11}
\]

\[
\sum_n (2n + 1) \left( \left( \mu - \frac{1}{\beta_n^2} \right)^+ + \left( \mu - \frac{1}{\gamma_n^2} \right)^+ \right) = \frac{P}{\sigma^2} \tag{4.12}
\]

where $\mu > 0$ is known as the *water-level* and $(x)^+ = \max(0, x)$.

We note the following:

- The $\beta_{nm}$ and $\gamma_{nm}$ depend only on the product $ka$, i.e. the *electrical radius* of the radiating sphere. The capacity is therefore a function only of the electrical radius $ka$ and the input SNR $P/\sigma^2$.

- In [28], the channel between the two concentric spheres is formulated as scalar wave problem, and the authors get only $\{\gamma_n\}$ as the sequence of singular values. The scalar wave solution gives the scalar potential at a point due to a charge distribution. On the other hand, in the case of the vector wave problem which models the vector EM propagation, we get a second sequence of singular values $\{\beta_n\}$.

One way to compute the water-pouring capacity in closed form is to make use of the empirical distribution function (e.d.f.) of the sequences $\{\beta_n^2\}$ and $\{\gamma_n^2\}$. The evaluation of the e.d.f. of the sequences appears intractable. So we first apply degrees-of-freedom type of analysis to the signal model with the help of some approximations.
to the Bessel function. We have \( J_n(z) \approx 0 \) for \( n > z \) and \( z \) large. By using this and from the dependence of the singular values on \( J_{n+1.5}(ka) \), have approximately \( 2([ka]-1)^2 \approx 2(ka)^2 \) significant terms in the capacity formula. This result means that at high SNR, the capacity is approximately equal to \( 2(ka)^2 \log \text{SNR} \). The same approximation was made use of in [65] for the scalar wave problem and an answer of \((ka)^2\) was deduced for the number of degrees-of-freedom. It was also conjectured in that paper that, the number of degrees-of-freedom for the vector wave problem is \( 2(ka)^2 \). This coincides with our result if we were to use the Bessel function approximation that we mentioned earlier.

In the next section, we provide a more accurate formula for the capacity as a function of \( ka \) and SNR through numerical calculations. We note that the notion of degrees-of-freedom does not give an accurate description of the spatial dependence of capacity. This is because, for any fixed \( z \), the Bessel function has the asymptotic formula \( j_n(z) \approx \sqrt{2/\pi}(z/2)^{n+0.5}/\Gamma(n+1.5) \), where \( \Gamma \) is the Gamma function [1]. For \( n \) large, the singular values decrease as \((n/e)^{-n}\). As the SNR increases, the water-pouring solution puts non-zero power in more and more orthonormal functions as a result of this. This means that the number of active parallel channels that contribute to capacity grows with SNR. We note that the notion of degrees-of-freedom can be accurate for a certain specific antenna geometry for large source size at large SNR. In [65], a line current distribution is considered such that its length is \( L \) and the current distribution is aligned with the \( z \)-axis. The observation is on a concentric spherical surface (and due to axial symmetry, the E-field varies only in the \( \theta \) coordinate). The solution to the scalar-wave problem for this current distribution has an integration kernel given by \( \exp(j \cos \theta z) \sin \theta \). By parameterizing \( t = \cos \theta \), the kernel becomes \( \exp(jtz) \) such that \(-L/2 \leq z \leq L/2\) and \(-1 < t \leq 1\). The singular system for this kernel has been studied in [51], where it has been shown that \([2L+1]\) singular values...
approach 1, and the remaining singular values approach 0, as $L \to \infty$. Therefore, in the asymptotic limit of aperture, the number of significant singular values is the number of degrees-of-freedom.

4.3.1 Antenna Design

The water-pouring solution (4.10) at any finite SNR allocates power to a finite number of parallel channels. The optimal signalling strategy is to transmit independent signals on the spatial basis functions with their respective powers. This indicates that the antennas that achieve capacity have current distributions given by the spatial basis functions $\Psi_{1mn}^{(1)}$ and $\Psi_{2mn}^{(1)}$. We can therefore easily see that if one is interested in implementing an $M$-element antenna system with a volume restriction, then the current distribution of the antennas corresponds to the parallel channels with the largest $M$ singular values of the sequence $\{\beta_n, \gamma_n\}_{n=1}^{\infty}$. We note that the analysis so far assumed that there is no restriction on the input current distribution other the square-integrability constraint. We ignored three important factors that affect realizability of a multi-element antenna: boundary conditions, resonance and mutual coupling. Boundary conditions arise mainly because, antennas are typically made of metallic conductor with a particular geometry on a non-conducting substrate. For example, the current distribution is forced to be zero at the edges. The geometry of the antennas determines their mutual impedance and this affects the radiation efficiency of the antenna. A desirable antenna is one which radiates most of its input energy in the frequency band of interest. This phenomenon is known as resonance. In general, it is not possible to get an antenna with an arbitrary geometry to resonate in an arbitrary frequency band. This imposes more restrictions on the geometries that are feasible. Finally, mutual coupling between elements arising out of near-field interactions also affects resonance and hence the feasibility of a certain geometry.
These constraints need to be suitably formulated in order to solve the constrained capacity optimization problem and this is an interesting extension to this work.

4.3.2 Extension to Transmit-Receive Systems

The analysis in the previous section can be extended to the case of a MIMO wireless link, where there is finite volume restriction on both the transmitter and the receiver. Consider a transmitter with an arbitrary current distribution within a spherical volume of radius $a$, and receiver with antennas enclosed within a spherical volume of radius $b$ as shown in Fig. 4.2. The infinitely-many parallel Gaussian channels model for a finite volume transmitter given in (4.8) can be written in terms of semi-infinite vectors and matrices as,

$$y = B_a x + z$$

where the semi-infinite input vector $x = (\ldots, x_{1nm}, x_{2nm}, \ldots)^T$ is a sequence of the scalar components, the outputs $y = (\ldots, y_{1nm}, y_{2nm}, \ldots)^T$, and the noise $z = (\ldots, z_{1nm}, z_{2nm}, \ldots)^T$ is an infinite-dimensional proper complex Gaussian random vector with zero mean and individual component variance $\sigma^2$. The matrix $B_a$ is a semi-infinite diagonal matrix with the sequence

$$\underbrace{\beta_1, \gamma_1, \ldots, \beta_n, \gamma_n, \ldots}_{3 \text{ times}} \underbrace{, \ldots, \beta_n, \gamma_n, \ldots}_{(2n+1) \text{ times}}$$

along its main diagonal. The subscript $a$ indicates the dependence of the transformation on the transmitter radius $a$. For channel coding theorems for infinite-dimensional matrix channels, the reader is referred to [75]. The E-field radiated as observed on a spherical surface $S_1$ concentric with the transmitter then has an equivalent repre-
sentation $B_a x$. Let $w$ be the vector corresponding to an E-field distribution over a spherical surface $S_2$ around the receiver, but in its far-field. The current distribution induced in the receiver can be written as $B_y^\dagger w$ due to reciprocity principle [5]. Now suppose, that $H$ is semi-infinite matrix that describes the transformation that maps an electric field vector on $S_1$ to another electric field vector on $S_2$. We assume that $H$ is bounded in the spectral norm (i.e., the largest singular value of $H$ is finite) and is nonsingular. This transformation captures the effect of fading and path-loss in the propagation environment. We assume that the E-field vector on $S_2$ is related to $B_a x$ by this transformation and it satisfies the equation $w = B_a x$. Let $y$ be the current distribution induced due to $w$. Then, we have the input-output relation $y = B_b^\dagger H B_a x$. This is under the assumption that there is no noise at the receiver. When an additive noise process corrupts the current distribution induced at the receiver, the signal becomes $y = B_b^\dagger H B_a x + n$, where $n$ is the equivalent vector representation of the noise process. Assuming that $H$ is ergodic, and is known to the receiver, the channel capacity depends on the eigenvalue distribution of $B_b^\dagger H B_a B_a^\dagger H^\dagger B_b$. An exact analysis of the dependence of capacity on $ka$ and SNR for this system is the natural extension to this work.

4.4 Numerical Results

In this section, we use numerical simulations to find out the dependence of the water-pouring capacity on $ka$ and the input power $P$. We normalize the channel $T_G$ such that for a current source with electrical radius unity ($ka = 1$), an input signal with unit mean power with uncorrelated and equal energy components results in unit signal output power (i.e., $E|x|^2 = 1 \Rightarrow E|B_a x|^2 = 1$). Define the signal-to-noise-power ratio (SNR), as $\text{SNR} = P/\sigma^2$. 
Figure 4.2: A MIMO link with transmitter current distribution in $V_a$, receiver with induced currents in a volume $V_b$, and a propagation channel described by $S_1, S_2$ and $H$.

The total receiver power depends on the input distribution as the gains on the different parallel channels are different in general. Therefore, in order to study the scaling of power received as a function of transmitter radius, we define the total received power for isotropic input (defined as an input with unit variance i.i.d components), as $P_R(ka) = \sum_{n=1}^{\infty} (2n+1)(\beta_n^2 + \gamma_n^2)$.

The main aim of this section is to demonstrate via numerical calculations the following.

- For a fixed size current distribution, the capacity of the channel scales as $(c_2 + c_1 \log \text{SNR}) \log \text{SNR}$ for large SNR, where $c_2 \sim (ka)^2$ and $c_1 \sim ka$.

- The total received power for isotropic input scales as $(ka)^3$.

Fig. 4.3 plots the water-pouring capacity with respect to SNR given by (4.10)
for different values of the electrical radius $ka = 1, 3, 5$ and $7$. The capacity at a 
picular SNR is denoted by $C(\text{SNR})$ with a little notational abuse. We also plot $C(\text{SNR})/\log \text{SNR}$ versus $\log \text{SNR}$ for different values of $ka$ in Fig. 4.4. From these 
plots, we can easily observe that the capacity scales non-linearly with $\log \text{SNR}$. The 
gap in the slopes for increasing values of $ka$ is indicative of the scaling with respect 
to $ka$. In order to get a more accurate picture, a polynomial approximation of degree 
2 was carried out to the capacity-$\log \text{SNR}$ curves. The coefficient of $\log \text{SNR}$ term in 
the polynomial approximation labelled as $c_2$ and the coefficient of the $\log \text{SNR}$ term 
labelled as $c_1$ were found for different values of $ka$. These scaling factors are plotted 
for different values of the electrical radius $ka$ in Fig. 4.5. The term $c_2$ is found to be 
approximately quadratic in $ka$ ($c_2 \approx 1.51(ka)^2$) and $c_1$ is found to be approximately 
linear in $ka$ ($c_1 \approx 1.1(ka)$).

In order to see how good the quadratic approximation is, the capacity-SNR 
curves and their second-degree polynomial fits are plotted in Fig. 4.6. The figure 
reveals that the approximation is less that 0.7 dB away from the actual curve for 
$ka = 8$ at SNRs up to 18 dB. The approximation correctly tracks the scaling of 
capacity with SNR. To see by how much the capacity predicted by the degrees-of-
freedom argument differs from the actual capacity, we plot the capacity approximation 
$C(\text{SNR}) \approx 2(ka)^2 \log \text{SNR}$ due to this approach. We can see that the degrees-of-
freedom approximation does not predict the scaling well, and it either under predicts 
or over predicts the capacity. As the SNR increases, the gap between the actual 
capacity and the degrees-of-freedom approximation grows with SNR.

Finally, the power gain ratio $P_R(ka)/P_R(1)$ is plotted for different values of the 
electrical radius $ka$ in Fig. 4.7. A polynomial fit (not shown) reveals that the power 
gain satisfies cubic law in $ka$. The reason for this is that, the channel gains (singu-
lar values) associated with the parallel Gaussian channel grow when the size of the
Figure 4.3: The capacity $C(\text{SNR})$ versus $\text{SNR}$ is plotted for different values of the electrical radius $ka = 1, 3, 5$ and 7.

4.5 Conclusions

The Rayleigh flat-fading model for MIMO wireless link predicts that the capacity scales linearly with the number of antennas. In reality, when the volumes that the antennas are allowed to occupy is finite, increasing the number of antennas beyond a point does not yield a linear scaling for capacity. The capacity is fundamentally limited by the size of the antenna. In this chapter we developed some tools that...
Figure 4.4: The ratio $C(\text{SNR})/\log \text{SNR}$ is plotted against SNR for different values of $ka$.

enable us to characterize this spatial dependence of capacity.

We first derived closed form expressions for the singular values of the linear operator channel that describes the problem of transmission of EM signals from a volume restricted current distribution onto a larger spherical receiver in the far-field. The singular values were given in terms of Bessel functions of the first kind. An analysis was developed to characterize the behavior and the dependence of water-pouring capacity on the electrical radius $ka$ and SNR. At large SNR, the capacity scaling law was shown to be $(c_2 + c_1 \log \text{SNR}) \log \text{SNR}$, where $c_2 \sim (ka)^2$ and $c_1 \sim ka$ via numerical calculations. It was also shown that the receive power obeys a cubic law in $ka$ when
Figure 4.5: The constants $c_1$ and $c_2$ computed from the capacity curves via polynomial fitting ("actual") and their approximations are plotted with respect to $ka$.

the input power is isotropic.

A more accurate estimate of the e.d.f of the singular values by precise calculations can result in better estimates of capacity, particularly at lower SNR. The tools developed here can be applied to the case of both transmitter and receiver restricted to finite volumes separated by a flat-fading channel. This case is more realistic as it is the natural generalization of the narrowband MIMO system with volume restrictions on both the transmitter and receiver.
Figure 4.6: Comparing the actual capacity (black), with polynomial approximation (blue) and the degrees-of-freedom approximation (green) for $ka = 3, 5$ and 7.
Figure 4.7: The power gain ratio $P_R(ka)/P_R(1)$ plotted for different values of the electrical radius $ka$. 
Chapter 5

Conclusions

In this dissertation, we mainly considered three problems relating to the fundamental limits and joint antenna-signal design in multiple-antenna wireless links. Our main focus was to develop system models that would incorporate the antenna EM field and the scattering environment properties.

We first considered the joint design of a multi-element antenna system and input signals that maximize capacity in a MIMO wireless channel. We used EM theory and ray-tracing approach to derive a channel propagation model for vector antennas in a discrete-multipath channel environment. This model provides insights into the inter-relation between the spatial multiplexing gain and the nature of the multipath environment for vector antennas. It was shown that vector antennas have similar performance gains as a spatially separated array of antennas in rich multipath, and perform even better that spatially separated antennas when the scattering is scarce. Motivated by this approach, we generalized the model to the case of antennas with more general electric field patterns in a fading environment with clusters of scatterers. Capacity optimal signalling and the impact of antenna electric field patterns on capacity were studied. We derived an optimality criterion for multi-element antennas...
to maximize the channel ergodic capacity. We showed that antennas that have orthogonal and equal norm electric field patterns maximize the ergodic capacity. Vector antennas satisfy this criteria, but a uniform linear array does not.

We next considered the problem of positioning and direction-of-arrival (DOA) estimation with ultrawideband (UWB) vector antennas. Due to their large bandwidth of operation and the use of sharp signal pulses, UWB systems have precise ranging capability. The directional sensitivity of vector antennas provide the ability to estimate the DOA jointly with the range. We first proved that a block-Toeplitz matrix is asymptotically equivalent to a block-circulant matrix. This result was then used to derive a frequency-domain Cramér-Rao Bound formula in the asymptotic case of a large number of observations samples under stationary noise. We applied this formula to two different UWB vector antennas and obtained closed form expressions for the lower bound on estimation error. A criterion that minimizes the volume of linearized confidence region was used to design signal pulses that give uniform resolving capability to the antennas for any DOA. It was shown that under suitable signal design, a planar 3-element UWB-VA with two dipoles and a loop performs as well as a 3-element tripole UWB-VA.

Finally, we considered the fundamental capacity limits of any multi-element antenna system that is restricted to occupy a finite volume. For simplicity, we considered the problem of a spherical volume-limited current source radiating into surrounding space. A channel with the spatial current distribution as the input and the radiated electric field as the output was formulated as a linear transformation, and the singular values of this transformation were derived in closed form. It was shown that this system is equivalent to infinitely-many parallel Gaussian channels. It was then shown that the singular values and hence the capacity depends on the size of transmitting volume only through its radius. We calculated the capacity the system, and pro-
vided capacity formulas that are accurate at high SNR. It was demonstrated that the
degrees-of-freedom approach, which approximates the channel by a finite-dimensional
transformation, is quite inaccurate for the problem at hand. Numerical simulations
were used to show that the capacity scales as $C \approx (c_1 \log \text{SNR} + c_2) \log \text{SNR}$, where
$\text{SNR}$ is the signal-to-noise power ratio and $c_1$ and $c_2$ scale linearly and quadratically
with the radius of the the transmitter respectively.
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Appendix A

Some Properties of Vector Antennas

In this appendix, we derive some properties of the channel matrix given in (2.6). These results are useful in the analysis of the impact of scattering on the rank of the channel matrix for specific choices of vector antennas. Towards this end, we introduce some notation. Let $A$ be a $m \times n$ complex matrix. Let $r(A)$ be the rank of $A$. Let the kernel of $A$ be defined as the set $K(A) = \{x \in \mathbb{C}^n : Ax = 0\}$, and the range of $A$ be defined as $R(A) = \{Ax : \forall x \in \mathbb{C}^n\}$, where $\mathbb{C}$ is the complex field.

From (2.6), the channel matrix is defined by the product, $H = A_R D A_T^\dagger$, where $A_R$, $D$ and $A_T$ have sizes $N \times 2\Gamma$, $2\Gamma \times 2\Gamma$ and $M \times 2\Gamma$ respectively. We assume that $D$ has full-rank in the sequel.

**Proposition 3.** For $H$, $A_R$, $A_T$ and $D$ defined in (2.6), if $r(D) = 2\Gamma$ we have,

$$r(H) \geq r(A_R) + r(A_T) - 2\Gamma$$

**Proof.** The proof follows straightforwardly from Frobenius’ inequality, [73, pages 134-
which states that for three complex matrices, $A$, $B$ and $C$ such that the products $AC$ and $CB$ are defined, we have $r(ACB) + r(C) \geq r(AC) + r(CB)$. Setting $A = \mathcal{A}_R^\dagger$, $C = \mathcal{D}$ and $B = \mathcal{A}_R^\dagger$ and observing that $\mathcal{D}$ is an invertible matrix and therefore $r(\mathcal{D}\mathcal{A}_R^\dagger) = r(\mathcal{A}_R^\dagger)$, the inequality in the proposition results.

**Proposition 4.** When $M = N$ in (2.6) (i.e., the number of transmit and receive antennas is the same), if $2\Gamma \geq r(\mathcal{A}_R) = r(\mathcal{A}_R^\dagger) = M = N$ and $R(\mathcal{A}_R^\dagger) = R(\mathcal{D}\mathcal{A}_R^\dagger)$, then $H$ is full-rank matrix.

**Proof.** Let $A$ and $B$ be two complex matrices of sizes $m \times n$ and $n \times m$ respectively. Now let $A$ and $B$ be such that $R(A^\dagger) = R(B)$ and let $r(A) = r(B) = m$ and $n > m$. Consider any $x \in \mathbb{C}^n$ be such that $x \in R(A^\dagger) \cap K(B^\dagger)$. Therefore, $B^\dagger x = 0$ and $x = By$ for some $y \in \mathbb{C}^m$. This implies $B^\dagger By = 0$. But, since $r(B) = m$, we have $y = 0$ and $x = 0$. Therefore, $R(A^\dagger) \cap K(B^\dagger) = \{0\}$. From [73, pages 133-134], we have the following result: $r(AB) = r(A)$ if and only if $R(A^\dagger) \cap K(B^\dagger) = \{0\}$. Therefore, we have $r(AB) = r(A)$. Setting $A = \mathcal{A}_R$ and $B = \mathcal{D}\mathcal{A}_R^\dagger$ and observing that $r(\mathcal{D}\mathcal{A}_R^\dagger) = r(\mathcal{A}_R^\dagger)$, we get the result in the proposition. 

We next consider single vector antennas (VAs) with their elements aligned along the x, y, and z axes of Fig. 2.2. We state and prove some rank conditions for specific assumptions on the scattering environment. The term $\mathcal{A}$ indicates both $\mathcal{A}_R$ and $\mathcal{A}_T$.

**A1:** Consider a 6-element VA with three orthogonal dipoles and three orthogonal loops collocated. If $S_1, S_2$ and $S_3$ represent three distinct directions, then $\text{rank}(B(S_1), \ldots, B(S_3)) = 6$.

**Proof:** Theorem 1 of [88].

**A2:** Consider a tripole antenna such that its elements are aligned along the principal axes. With $\Gamma = 2$ scatterers, and defining $\mathcal{A} = (B(S_1), B(S_2))$, $\mathcal{A}$ is rank deficient ($\text{rank}(\mathcal{A}) < 3$) if and only if one of the following conditions hold,
1. \( \theta_1, \theta_2 \in \{0, \pi\} \)

2. \((\theta_1, \phi_1) = (\theta_2, \phi_2), \text{ if } \phi_1, \phi_2 \not\in \{0, \pi\}\)

3. \((\theta_1, \phi_1) = (\pi - \theta_2, \phi_2 \pm \pi)\)

Proof:-

\[
\mathbf{A} = \begin{pmatrix}
-\sin \phi_1 & \cos \phi_1 \cos \theta_1 & -\sin \phi_2 & \cos \phi_2 \cos \theta_2 \\
\cos \phi_1 & \sin \phi_1 \cos \theta_1 & \cos \phi_2 & \sin \phi_2 \cos \theta_2 \\
0 & -\sin \theta_1 & 0 & -\sin \theta_2
\end{pmatrix}
\]

Let \( \Delta_1 \) be the determinant of the matrix formed deleting column 1 from \( \mathbf{A} \), etc. We have

\[
\Delta_1 = \sin \theta_1 \cos \theta_2 - \sin \theta_2 \cos \theta_1 \cos (\phi_1 - \phi_2)
\]

\[
\Delta_2 = \sin \theta_2 \sin (\phi_1 - \phi_2)
\]

\[
\Delta_3 = \sin \theta_2 \cos \theta_1 - \sin \theta_1 \cos \theta_2 \cos (\phi_2 - \phi_1)
\]

\[
\Delta_4 = -\sin \theta_1 \sin (\phi_1 - \phi_2)
\]

\( \Delta_1, \Delta_2 \geq 1 + \cos (\phi_1 - \phi_2) \geq 0. \) \( \mathbf{A} \) is rank deficient if and only if, \( \Delta_i = 0, i = 1, \ldots, 4. \)

Assume \( \theta_1, \theta_2 \not\in \{0, \pi\} \). Therefore, \( \phi_1 - \phi_2 = 0, \pm \pi. \) \( \theta_1 = \theta_2 \) or \( \theta_1 = \pi - \theta_2 \) accordingly.

Suppose, \( \theta_1 \in \{0, \pi\} \). This requires \( \theta_2 \in \{0, \pi\} \). So, we get the above conditions. \(\Box\)

A3: Consider a planar 3-element planar structure with 2-dipoles along the x- and y-axes, and a loop along the x-y plane, with its axis along the z-axis. For \( \Gamma = 2 \) scatterers and defining \( \mathbf{A} = (B(S_1), B(S_2)) \), \( \mathbf{A} \) is rank deficient \((\text{rank}(\mathbf{A}) < 3)\) if and only if one of the following conditions hold,

1. \( \theta_1, \theta_2 \in \{0, \pi\}, \)

2. \((\theta_1, \varphi_1) = (\theta_2, \varphi_2), \text{ if } \theta_1, \theta_2 \not\in \{0, \pi/2, \pi\}, \)
3. \((\theta_1, \varphi_1) = (\pi - \theta_2, \varphi_2 \pm \pi)\),

4. \(\theta_1 = \theta_2 = \pi/2\).

**Proof:** Let \(\Delta_1\) be the determinant of the matrix formed deleting column 1 from \(A\), etc. We have

\[
\begin{align*}
\Delta_1 &= \sin \theta_2 \cos \theta_1 \cos \theta_2 \sin (\phi_1 - \phi_2) \\
\Delta_2 &= \cos \theta_2 (\sin \theta_1 - \sin \theta_2 \cos (\phi_1 - \phi_2)) \\
\Delta_3 &= \sin \theta_1 \cos \theta_1 \cos \theta_2 \sin (\phi_1 - \phi_2) \\
\Delta_4 &= \cos \theta_1 (\sin \theta_2 - \sin \theta_1 \cos (\phi_1 - \phi_2))
\end{align*}
\]

\(A\) is rank-deficient if and only if \(\Delta_i = 0\) for \(i = 1, \ldots, 4\). Hence, we get the above conditions. \(\Box\)

**A4:** Consider the 3-element planar antenna with two dipoles along the x and y axes and a loop with its axis along the z direction as in property A3 above. For \(\Gamma = 3\) scatterers, \(A\) has full-rank if no two of the direction vectors \(u_r(S_1), u_r(S_2)\) and \(u_r(S_3)\) are parallel. Further, when the scatterers are restricted to the horizontal plane \((\varphi_l = 0)\), the first, third and the fifth columns are linearly independent and the remaining columns are zero.

**Proof:** No two of the three directions \(S_1, S_2\) and \(S_3\) are parallel. Therefore, if atleast one of the them is not on the plane containing the antenna, then by property 2, \(A\) is full-rank. Suppose, all the three directions are in the plane containing the antenna. The second, fourth and sixth columns of \(A\) are zeros. The determinant formed by the remaining columns is equal to \(\sin(\phi_2 - \phi_1) + \sin(\phi_1 - \phi_3) + \sin(\phi_3 - \phi_2)\). It can
be shown that this term is non-zero as no two of the directions are parallel. □