ABSTRACT

DONG, KE. Advanced Design Techniques in Linear Parameter Varying Control. (Under the direction of Associate Professor Fen Wu).

To improve the analysis and control synthesis approach of linear fractional transformation (LFT) parameter-dependent systems, two types of non-quadratic Lyapunov function and switching control scheme are introduced and studied in this thesis. A gain-scheduled controller with parameter variation rate, a nonlinear gain-scheduled controller and an online switching linear parameter varying (LPV) controller are derived, and the advantages of proposed LPV control techniques are demonstrated through numerical and physical examples.

In the first part of this thesis, we introduce a quadratic LFT parameter-dependent Lyapunov function, which includes affine parameter-dependent functions as special cases. Using full-block S-procedure, new LPV synthesis conditions have been derived in terms of finite number of linear matrix inequalities (LMIs). The constructed controller depends on parameters and their variation rate in general form compared with traditional LFT form. It is shown that the proposed approach can achieve better performance in a ship steering example by exploiting parameter variation rates.

In the same spirit of exploiting more general type of Lyapunov function to achieve better controller, an analysis and synthesis algorithm for LPV systems using convex hull Lyapunov function (CHLF) and maximum Lyapunov function is presented. Using duality of LPV systems and conjugate properties of CHLF, sufficient LPV analysis and synthesis conditions have been derived in terms of LMIs with linear search over scalar variables. Because of the special structure of CHLF and maximum Lyapunov function, the output feedback controller turns out to be a nonlinear gain-scheduled controller. A second-order plant is used to demonstrate advantages and benefits of the new approach.

The other main contribution in this thesis is the application of switching control to LPV systems with online optimization method. Arbitrary switching among subsystems is achieved, as well as performance improvement using multiple Lyapunov functions. A gain-scheduled controller working for the next switching interval is designed at each switching instant. A bumpless transfer compensator is also designed to minimize the output jump caused by switching. The synthesis conditions for both switching controller and bumpless transfer compensator are formulated as LMIs. The new LPV switching control scheme is
applied to an uninhabited combat aerial vehicle (UCAV) problem.

All our proposed approaches are efficient in computation, where the conditions are all formulated as LMIs or LMI-like ones. With slightly increased computational complexity, the proposed new approaches for analysis and synthesis of LFT parameter-dependent systems can achieve significant performance improvement comparing to existing approaches.
Advanced Design Techniques in Linear Parameter Varying Control

by

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A dissertation submitted to the Graduate Faculty of
North Carolina State University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Mechanical Engineering

Raleigh
2006

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To my wife Yuan Zhang
with all my love.
Biography

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Acknowledgements

My greatest gratitude goes to my advisor, Dr. Fen Wu. His financial support and consistent encouragement made my NCSU experience fruitful and unique. Without his patience, insight and foresighted suggestion, it is impossible to fulfill this thesis. I will be forever thankful for this opportunity to work with him.

I would like to extend my appreciation to the other members of my thesis committee: Dr. Paul I. Ro, Dr. Stephen L. Campbell, and Dr. Gregory D. Buckner. Thanks for providing constructive comments and advices on my research and presentation.

I acknowledge the National Science Foundation for their financial support under the grant CMS-0324397 through Dr. Fen Wu.

I would also like to express my sincere thanks to my fellows and friends. The last three years graduate study in NCSU is exciting and memorable with their help and friendship.

Finally, I would like to address my special thanks to my family for their unconditional love and selfless support. They were always with me all these years, and without them I would not have strength to pursue this research. In particular, I would like to appreciate my wife for her unending patience, support and motivation. I owe her love much more than I would ever be able to express.
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<table>
<thead>
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<th>Notation</th>
<th>Description</th>
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<tbody>
<tr>
<td>$0_{n \times m}$</td>
<td>the zero element of $R_{n \times m}$</td>
</tr>
<tr>
<td>$\Delta \star M$</td>
<td>star product</td>
</tr>
<tr>
<td>$C^{m \times n}$</td>
<td>set of complex $m \times n$ matrices</td>
</tr>
<tr>
<td>$C^0(U, V)$</td>
<td>set of continuous functions from $U$ to $V$</td>
</tr>
<tr>
<td>$C^1(U, V)$</td>
<td>set of continuously differentiable functions from $U$ to $V$</td>
</tr>
<tr>
<td>$\text{diag}(a_1, a_2, \cdots, a_n)$</td>
<td>the $n$ by $n$ diagonal matrix with diagonal elements $a_1, a_2, \cdots, a_n$</td>
</tr>
<tr>
<td>$\mathcal{F}_l(\cdot, \cdot)$</td>
<td>lower linear fractional transformation</td>
</tr>
<tr>
<td>$\mathcal{F}_u(\cdot, \cdot)$</td>
<td>upper linear fractional transformation</td>
</tr>
<tr>
<td>$\partial f(x)$</td>
<td>the subdifferential of $f(x)$, being the set of all subgradients</td>
</tr>
<tr>
<td>$\text{Im}(M)$</td>
<td>the image space of a linear mapping $y = Mx$</td>
</tr>
<tr>
<td>$I_n$</td>
<td>the $n$-dimensional identity</td>
</tr>
<tr>
<td>$I[k_1, k_2]$</td>
<td>$= {k_1, k_1 + 1, \cdots, k_2}$ for two integer $k_1, k_2, k_1 &lt; k_2$</td>
</tr>
<tr>
<td>$\ker(M)$</td>
<td>the orthogonal complement of the matrix $M$</td>
</tr>
<tr>
<td>$L_2$</td>
<td>space of square integrable functions</td>
</tr>
<tr>
<td>$M^T$</td>
<td>the transpose of the matrix $M$</td>
</tr>
<tr>
<td>$M^{-1}$</td>
<td>the inverse of the invertible matrix $M$</td>
</tr>
<tr>
<td>$M &gt; 0 (M \geq 0)$</td>
<td>the matrix $M$ is positive definite (positive semi-definite)</td>
</tr>
<tr>
<td>$M &lt; 0 (M \leq 0)$</td>
<td>the matrix $M$ is negative definite (negative semi-definite)</td>
</tr>
<tr>
<td>$R$</td>
<td>set of real numbers</td>
</tr>
<tr>
<td>$R_+$</td>
<td>set of non-negative real numbers</td>
</tr>
<tr>
<td>$R^n$</td>
<td>set of $n$-dimensional real vectors</td>
</tr>
<tr>
<td>$R^{m \times n}$</td>
<td>set of real $m \times n$ matrices</td>
</tr>
<tr>
<td>$S^{n \times n}$</td>
<td>set of symmetric matrices in $R_n$</td>
</tr>
</tbody>
</table>
\( \mathbf{S}^{n \times n}_+ \) is the set of positive definite matrices in \( \mathbf{R}^{n \times n} \)

\[ \|u\|_2 = \left[ \int_{0}^{\infty} u^T(t)u(t)dt \right]^\frac{1}{2} \text{ for } u \in \mathcal{L}_2 \]

\[ \|x\| = (x^T x)^\frac{1}{2} \text{ for vector } x \]
List of Abbreviations

LFT  linear fractional transformation
LPV  linear parameter varying
LMI  linear matrix inequality
LTI  linear time invariant
LTV  linear time varying
PDLF parameter dependent Lyapunov function
CHLF convex hull Lyapunov function
BT   bumpless transfer
UCAV uninhabited combat aerial vehicle
VSTOL vertical/short take-off and landing
Chapter 1

Introduction

Most mechanical systems have nonlinear characteristics, which cannot be ignored in many control designs. One control design approach with great potential for nonlinear system is linear parameter varying (LPV) control. The study of LPV control provides a systematic control design framework for nonlinear systems, which is motivated from the gain scheduling control methodology [80, 75]. The classical gain-scheduling control approach involves the design of several linear time invariant (LTI) controllers for a family of linearized systems with fixed parameter values and the interpolation of the controller gains. This heuristic design procedure does not take the parameter variations into account. In particular, it cannot guarantee stability and performance except for slow varying parameters.

The notation of LPV systems was first introduced by [80]. LPV control theory is useful to simplify the interpolation and realization problems associated with conventional gain-scheduling. The use of a single or parameter dependent quadratic Lyapunov function in the analysis and control design for LPV systems has been developed in a robust control framework [6, 5, 97, 99]. The parameter variation was also incorporated into the analysis conditions [97, 99]. However, the solution to the general LPV control synthesis problem is formulated as infinite number of linear matrix inequalities (LMIs). A gridding based method is typically used to solve this problem.

On the other hand, a linear fractional transformation (LFT) control design technique was developed by [61] and [1] using the scaled small-gain theorem. Comparing to the gridding based approach, it can provide stability and performance guarantee over entire
parameter set. The control synthesis conditions are fully characterized by a finite set of LMIs. The limitation of this approach is the special structure of the LPV plant.

1.1 Motivation and Objectives

LPV control theory is advantageous because it provides stability and performance guarantee over wide range of changing parameters, and has found applications in various industrial problems. A thorough review of gain-scheduling and LPV control research can be found in [76].

The LPV system is allowed to have general parameter dependence with continuity. The analysis test in [6] introduced potential conservatism by measuring performance under arbitrarily fast variations of scheduling parameters. After that, parameter variation information was incorporated into the analysis conditions in [97, 99]. The synthesis conditions with parameter dependent Lyapunov function lead to the controllers scheduled by both parameters and their variation rate. All these conditions for general LPV control synthesis problem is formulated as parameter-dependent LMIs, which usually involve infinite number of inequalities. A brutal force gridding method can be used to grid the parameter set and convert the infinite optimization problem to be finite. Affine parameter-dependent LMIs in [23] provided an alternative solution.

In broad terms, the gridding based design approach of a LPV controller for a nonlinear plant can be described as a three-step procedure.

1. Approximating the nonlinear plant by a LPV plant. The most common approach is based on Jacobian linearization of the nonlinear plant about a family of operating points (equilibrium points). In this way, a parameterized family of linearized plants can be derived. Another approach is called quasi-LPV scheduling, in which the nonlinear plant can be rewritten as a LPV plant by treating part of states of the original nonlinear plant as scheduling parameters. Quasi-LPV does not need Jacobian linearization.

2. Using linear design tools to design linear controller for the LPV model which is derived in the first step. A family of linear controllers are derived in this process. Traditionally, the designs are based on gridding that linear time invariant (LTI) controller is designed
for each fixed value of parameter. Therefore, there may be an interpolation process to get a LPV controller from a set of LTI controller designed at isolated parameter values.

3. Implementation of the controller and performance assessment. The actual gain-scheduling involves adjustment of the controller’s gain according to the current value of the parameters. Because the LPV controller is designed based on LPV plant, the close loop performance of nonlinear plant and LPV controller is not necessarily guaranteed in the design procedure. Typically, the nonlocal stability and performance is evaluated through simulation studies.

From the above design procedure, some limitations do exist in the general gridding based LPV control design method. General LPV control design often involves several ad hoc steps, from problem formulation to controller interpolation method, which may be increasingly troublesome as complicated controllers are designed. Generally speaking, the solution to general LPV control analysis and synthesis problem is undetermined and has high computational complexity.

Parallel to the general LPV approach, a systematic gain-scheduling control design technique was developed by [61] and [1]. When the plant depends on parameters in LFT form, a gain-scheduling controller with same structure as plant can be derived by finite number of LMIs. This promising approach is applicable whenever the parameters are measurable in real-time. Furthermore, the control synthesis problem is a convex one which can be optimized by efficient interior-point algorithms. Further ramifications have been proposed to count for real gain-scheduling parameters [79]. Recently, a general form of multipliers was introduced in [78, 94] to reduce the conservatism associated with block diagonal scaling. Nevertheless, it often requires the use of general scheduling functions. Alternative LPV analysis and synthesis approaches can be found in [32, 15, 39, 90].

From the above overview of existing LPV control approaches, gain-scheduling control of LFT system has the advantage that control synthesis conditions can be formulated as finite number of LMIs, which provide stability and performance guarantee over entire parameter set comparing to the gridding based LPV approach. The research of this dissertation focuses on the LPV systems with LFT structure. The objectives of this research are to develop new approaches to design gain-scheduling controller with better performance for LFT systems compared to existing approaches. We also expect the conditions to be efficient
in computation.

First, we hope to use different types of Lyapunov functions other than single quadratic Lyapunov functions to solve synthesis problem of gain-scheduling controllers. Although quadratic Lyapunov function is sufficient for optimizing the performance of LTI systems, non-quadratic Lyapunov functions can be used to synthesize controllers with better performance for linear time varying (LTV) systems. In LPV control synthesis problem, parameter-dependent Lyapunov function is widely used, which is in the form of \( V(x, \Theta) = x^T P(\Theta) x \). Some other types of Lyapunov function are being taken into LPV research. However, most synthesis problems with non-quadratic Lyapunov function have not been solved or have significant computational complexity. In this research, we try to get benefit from non-quadratic Lyapunov function with solvable conditions.

The second goal is to improve LPV controller performance by incorporating other control design schemes, such as switching control. In the past, common LPV controller is designed over entire parameter set. Nevertheless, a sub-controller designed over a parameter subset is expected to perform better in this subset comparing to common controller. Partitioning the parameter set to several subsets and designing sub-controllers over each subset will lead to a better performance. In this case, switching among sub-systems is becoming a critical issue. In this research, we try to improve LPV controller performance by incorporating switching control schemes and address the stability and performance optimization problems in switching LPV control.

In order to design controller efficiently, the controller synthesis condition should be convex one, which can be solved by efficient optimization algorithms. Other than improving stability and performance from the existing approaches, we also hope the new conditions to be convex or convex-like one. We will try to formulate the new conditions to be LMIs or BLMIs, which can be solved by iteration or linear search with reasonable computational effort. Proposed new approaches are expected to have wide applications in mechanical or aerospace engineering fields. We will apply improved approaches to some mechanical or aerospace problems to demonstrate the advantages of the new methods.
1.2 Background in LPV Control with LFT Structure

As stated in previous sections, there are two parallel directions in LPV control techniques, one is general LPV control, the other is control of LFT parameter dependent systems.

1.2.1 General LPV systems

Background of general LPV control and LPV control with LFT structure is introduced in this section. A set of all possible parameter trajectories is defined first.

Definition 1 (Parameter Variation Set) Given a compact set $\mathcal{P} \subset \mathbb{R}^s$, the parameter variation set denotes the set of all piecewise continuous functions mapping $\mathbb{R}^+$ (time) into $\mathcal{P}$ with a finite number of discontinuities in any interval.

Definition 2 (Linear Parameter Varying (LPV) Systems) A LPV system with $n$-th order dynamics is defined as

\[
\begin{bmatrix}
\dot{x}(t) \\
e(t) \\
y(t)
\end{bmatrix} =
\begin{bmatrix}
A(\Theta(t)) & B_1(\Theta(t)) & B_2(\Theta(t)) \\
C_1(\Theta(t)) & D_{11}(\Theta(t)) & D_{12}(\Theta(t)) \\
C_2(\Theta(t)) & D_{21}(\Theta(t)) & D_{22}(\Theta(t))
\end{bmatrix}
\begin{bmatrix}
x(t) \\
d(t) \\
u(t)
\end{bmatrix},
\]

where $\Theta(t) \in \mathcal{P} \subset \mathbb{R}^s$ and functions $A \in \mathcal{C}^0(\mathbb{R}^s,\mathbb{R}^{n \times n}), B_1 \in \mathcal{C}^0(\mathbb{R}^s,\mathbb{R}^{n \times n_d}), B_2 \in \mathcal{C}^0(\mathbb{R}^s,\mathbb{R}^{n \times n_u}), C_1 \in \mathcal{C}^0(\mathbb{R}^s,\mathbb{R}^{n_e \times n}), C_2 \in \mathcal{C}^0(\mathbb{R}^s,\mathbb{R}^{n_y \times n}), D_{11} \in \mathcal{C}^0(\mathbb{R}^s,\mathbb{R}^{n_e \times n_d}), D_{12} \in \mathcal{C}^0(\mathbb{R}^s,\mathbb{R}^{n_e \times n_u}), D_{21} \in \mathcal{C}^0(\mathbb{R}^s,\mathbb{R}^{n_y \times n_d})$ and $x(t), \dot{x}(t) \in \mathbb{R}^n$, $d(t) \in \mathbb{R}^{n_d}$ is the disturbance, $e(t) \in \mathbb{R}^{n_e}$ is the controlled output, $u(t) \in \mathbb{R}^{n_u}$ is the control input and $y(t) \in \mathbb{R}^{n_y}$ is the measurement for control.

The representation (1.1) is the general form of LPV systems. In some applications, a simplified representation may be used, for example, if all states are measurable without
sensor noise, the measurement channel $y(t)$ can be removed. This kind of representation will be used in state feedback control synthesis problems.

The following assumption is made through entire thesis.

(A1) The triple $(A(\Theta), B_2(\Theta), C_2(\Theta))$ is parameter-dependent stabilizable and detectable for all $\Theta \in \Theta$,

(A2) $[C_2(\Theta) \hspace{1mm} D_{21}(\Theta)]$ and $[B_2^T(\Theta) \hspace{1mm} D_{12}^T(\Theta)]$ have full row rank for all $\Theta$,

(A3) $D_{11}(\Theta) = 0$ and $D_{22}(\Theta) = 0$.

The assumptions (A1)-(A2) guarantee the existence of a stabilizing output feedback LPV controller. Assumption (A3) simplifies the controller formula and our presentation.

There are several approaches to design controllers for general LPV systems. In references [97, 99], by using a parameter-dependent Lyapunov function in the form of $V = x^T P(\Theta) x$, $H_\infty$ synthesis condition of an LPV controller in the form of

$$
\begin{bmatrix}
\dot{x}_k(t) \\
u(t)
\end{bmatrix} =
\begin{bmatrix}
A_k(\Theta(t), \dot{\Theta}(t)) & B_k(\Theta(t)) \\
C_k(\Theta(t)) & D_k(\Theta(t))
\end{bmatrix}
\begin{bmatrix}
x_k(t) \\
y(t)
\end{bmatrix},
$$

where $x_k \in \mathbb{R}^{n_k}$, for the LPV system (1.1) is given by

$$
\begin{bmatrix}
N^T_R(\Theta) \\
N^T_S(\Theta)
\end{bmatrix}
\begin{bmatrix}
0 & R(\Theta) & 0 & 0 \end{bmatrix}^T
\begin{bmatrix}
0 & \gamma^{-1}I & 0 & 0 \\
R(\Theta) & -\dot{R}(\Theta) & 0 & 0 \\
0 & 0 & -\gamma I & 0 \\
0 & 0 & 0 & -\gamma I
\end{bmatrix}
\begin{bmatrix}
A^T(\Theta) & C^T(\Theta) \\
I & 0 \\
B_1^T(\Theta) & D_{11}^T(\Theta) \\
0 & I
\end{bmatrix}
\begin{bmatrix}
N_R(\Theta) < 0, \hspace{1mm} (1.2) \\
N_S(\Theta) < 0,
\end{bmatrix}
$$

where $R(\Theta), S(\Theta)$ are continuously differentiable functions, and

$$
\begin{bmatrix}
R(\Theta) & I \\
I & S(\Theta)
\end{bmatrix} \geq 0, \hspace{1mm} (1.4)
$$

The induced $L_2$-norm of closed loop LPV system is represented by $\gamma$, which is defined as follows.
Definition 3 (Induced $\mathcal{L}_2$-norm of stable LPV systems) Given a stable closed loop LPV system

\[
\begin{bmatrix}
\dot{x}_{cl}(t) \\
e(t)
\end{bmatrix} =
\begin{bmatrix}
A_{cl}(\Theta(t), \dot{\Theta}(t)) & B_{cl}(\Theta(t), \dot{\Theta}(t)) \\
C_{cl}(\Theta(t), \dot{\Theta}(t)) & D_{cl}(\Theta(t), \dot{\Theta}(t))
\end{bmatrix}
\begin{bmatrix}
x_{cl}(t) \\
d(t)
\end{bmatrix},
\]

for zero initial condition $x_{cl} = 0$, define the induced $\mathcal{L}_2$ norm as

\[
\max_{\Theta \in \Theta, \|d\|_2 \neq 0} \frac{\|e\|_2}{\|d\|_2}.
\]

The above conditions (1.2)-(1.4) all depend on parameters which are characterized by infinite number of inequalities. To solve the conditions, a gridding based method is needed. First, parameterize function space using finite number of basis functions, then grid the parameter space.

1.2.2 LFT Parameter-dependent systems

Definition 4 (Linear Fractional Transformation/Star Product) Let

\[
M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \in \mathbb{C}^{p_1+p_2 \times q_1+q_2} \text{ and } \Delta_l \in \mathbb{C}^{q_2 \times p_2} \text{ and } \Delta_u \in \mathbb{C}^{q_1 \times p_1}. \text{ Assume that } (I - M_{22} \Delta_l)^{-1} \text{ exists, a lower LFT is defined as }
\]

\[
\mathcal{F}_l(M, \Delta_l) = M_{11} + M_{12} \Delta_l (I - M_{22} \Delta_l)^{-1} M_{21}
\]

Assume that $(I - M_{11} \Delta_u)^{-1}$ exists, an upper LFT is defined as

\[
\mathcal{F}_u(M, \Delta_u) = M_{22} + M_{21} \Delta_u (I - M_{11} \Delta_u)^{-1} M_{12}
\]

LFT can be represent by star product with $\Delta_u \ast M = \mathcal{F}_u(M, \Delta_u)$ and $M \ast \Delta_l = \mathcal{F}_l(M, \Delta_l)$.

A Linear fractional transformation (LFT) parameter-dependent system is a special case of LPV system, which depends on the varying parameters in linear fractional transformation.
Definition 5 (Linear Fractional Transformation (LFT) Systems) A LFT system with n-th order dynamics is defined as

\[
\begin{bmatrix}
\dot{x}(t) \\
q(t) \\
e(t) \\
y(t)
\end{bmatrix} = \begin{bmatrix}
A & B_0 & B_1 & B_2 \\
C_0 & D_{00} & D_{01} & D_{02} \\
C_1 & D_{10} & 0 & D_{12} \\
C_2 & D_{20} & D_{21} & 0
\end{bmatrix}
\begin{bmatrix}
x(t) \\
p(t) \\
d(t) \\
u(t)
\end{bmatrix}, \quad (1.5)
\]

where the time-varying parameter \(\Theta(t)\) has the following structure

\[
\Theta = \{ \text{diag} \{ \theta_1 I_{r_1}, \theta_2 I_{r_2}, \ldots, \theta_s I_{r_s} \} : \theta_i \in C^1(\mathbb{R}_+, \mathbb{R}), |\theta_i| \leq 1, i = 1, 2, \ldots, s \} \quad (1.7)
\]

with \(\sum_{i=1}^s r_i = n_p\), and functions \(A \subset \mathbb{R}^{n \times n}, B_0 \subset \mathbb{R}^{n \times n_p}, B_1 \subset \mathbb{R}^{n \times n_d}, B_2 \subset \mathbb{R}^{n \times n_u}, C_0 \subset \mathbb{R}^{n_p \times n}, C_1 \subset \mathbb{R}^{n_e \times n}, C_2 \subset \mathbb{R}^{n_y \times n}, D_{00} \subset \mathbb{R}^{n_p \times n_p}, D_{01} \subset \mathbb{R}^{n_p \times n_d}, D_{02} \subset \mathbb{R}^{n_p \times n_u}, D_{10} \subset \mathbb{R}^{n_e \times n_p}, D_{11} \subset \mathbb{R}^{n_e \times n_d}, D_{12} \subset \mathbb{R}^{n_e \times n_u}, D_{20} \subset \mathbb{R}^{n_y \times n_p}, D_{21} \subset \mathbb{R}^{n_y \times n_d}, D_{22} \subset \mathbb{R}^{n_y \times n_u}\) and \(x, \dot{x} \in \mathbb{R}^n, d \in \mathbb{R}^{n_d}\) is the disturbance, \(e \in \mathbb{R}^{n_e}\) is the controlled output, \(p, q \in \mathbb{R}^{n_p}\) are the pseudo-input and output, \(u \in \mathbb{R}^{n_u}\) is the control input and \(y \in \mathbb{R}^{n_y}\) is the measurement for control.

We assume that

(A4) The LFT representation (1.5)-(1.6) is well-posed, i.e., \((I - D_{00}\Theta)\) is invertible for any allowable parameter values.

Define a set of positive definite similarity scalings associated with the structure \(\Theta\),

\[
L_\Theta = \{ L > 0 : L\Theta = \Theta L, \forall \Theta \in \Theta \subset \mathbb{R}^{n_y \times n_y} \} \quad (1.8)
\]
By absorbing parameter $\Theta$, the state-space equation of LFT systems can also be written as (1.1), where

$$
\begin{bmatrix}
A(\Theta) & B_1(\Theta) & B_2(\Theta) \\
C_1(\Theta) & D_{11}(\Theta) & D_{12}(\Theta) \\
C_2(\Theta) & D_{21}(\Theta) & D_{22}(\Theta)
\end{bmatrix} = 
\begin{bmatrix}
A & B_1 & B_2 \\
C_1 & 0 & D_{12} \\
C_2 & D_{21} & 0
\end{bmatrix} + 
\begin{bmatrix}
B_0 \\
D_{10} \\
D_{20}
\end{bmatrix} \Theta(I - D_{00}\Theta)^{-1}
\times 
\begin{bmatrix}
C_0 & D_{01} & D_{02}
\end{bmatrix}.
$$

(1.9)

In references [1, 61], by using common quadratic Lyapunov function, $H_\infty$ synthesis condition of a LFT controller in the form of

$$
\begin{bmatrix}
\dot{x}_k(t) \\
u(t) \\
q_k(t)
\end{bmatrix} = 
\begin{bmatrix}
A_k & B_{k1} & B_{k0} \\
C_{k1} & D_{k11} & D_{k10} \\
C_{k0} & D_{k01} & D_{k00}
\end{bmatrix} 
\begin{bmatrix}
x_k(t) \\
y(t) \\
p_k(t)
\end{bmatrix},
$$

(1.10)

$$
p_k(t) = \Theta(t)q_k(t),
$$

(1.11)

where $x_k \in \mathbb{R}^{n_k}$, for the LPV system (1.5)-(1.6) is given by

$$
\begin{bmatrix}
RA^T + AR & RC_{0}^{T} & RC_{1}^{T} & B_0 & B_1 \\
C_0R & -J & 0 & D_{00} & D_{01} \\
C_1R & 0 & -J & D_{10} & D_{11} \\
B_0^T & D_{00}^T & D_{01}^T & -L & 0 \\
B_1^T & D_{01}^T & D_{11}^T & 0 & -J
\end{bmatrix} 
\begin{bmatrix}
N_R \\
I
\end{bmatrix} < 0
$$

(1.12)

$$
\begin{bmatrix}
A^T S + SA & SB_0 & SB_1 & C_0^T & C_1^T \\
B_0^T & -L & 0 & D_{00}^T & D_{01}^T \\
B_1^T & 0 & -J & D_{01}^T & D_{11}^T \\
C_0 & D_{00} & D_{01} & -J & 0 \\
C_1 & D_{10} & D_{11} & 0 & -J
\end{bmatrix} 
\begin{bmatrix}
N_S \\
I
\end{bmatrix} < 0
$$

(1.13)

$$
\begin{bmatrix}
R & I \\
I & S
\end{bmatrix} \geq 0
$$

(1.14)

$$
\begin{bmatrix}
L & I \\
I & J
\end{bmatrix} \geq 0
$$

(1.15)

where $R, S \in \mathbb{R}^{n \times n}, L, J \in \mathbb{L}_\Theta, N_R = \ker \begin{bmatrix} B_2^T & D_{02}^T & D_{12}^T \end{bmatrix}, N_S = \ker \begin{bmatrix} C_2 & D_{20} & D_{21} \end{bmatrix}$. A less conservative result called full-block S-procedure is presented in [78], which use a full-
block multiplier instead of scaling matrix $L$ and $J$ in (1.12) -(1.15). However, the controller structure designed by full-block S-procedure is same as scaling matrix method, (1.10)-(1.11).

### 1.3 Thesis Outline

The detailed outline of this thesis is as follows:

Chapter 2 proposes a new control design approach for LFT systems using parameter-dependent Lyapunov functions (PDLF). Instead of designing a controller with LFT parameter dependency, general parameter-dependent controllers are considered to achieve better closed-loop performance by exploiting parameter variation rates information. Using full-block multipliers, new LPV synthesis conditions have been derived in terms of finite number of LMIs. The new approach is applied to a ship steering problem, which demonstrate advantages and benefits of the proposed approach.

Chapter 3 studies stability and performance properties of LFT systems using duality theory and tools from convex analysis. A pair of conjugate functions, the convex hull Lyapunov functions (CHLF) and the maximum of a family of quadratic Lyapunov functions, are used for analysis and synthesis of LFT systems. A sufficient synthesis condition for gain-scheduled output feedback control problem is formulated as a set of LMIs with linear search over scalar variables. However, the resulting gain-scheduled control law is of nonlinear type. Finally, an example is used to demonstrate the advantages of the proposed approach.

Chapter 4 studies the switching control of LFT parameter dependent systems using multiple Lyapunov functions with online optimization techniques to improve performance and enhance control design flexibility. A gain-scheduled controller working for the next switching interval is designed at each switching instant. An approach to design bumpless transfer compensator is presented to minimize the jump of output caused by switching. The control synthesis conditions for both online controller design and bumpless transfer are formulated as LMIs. The online switching control scheme is applied to uninhabited combat aerospace vehicle (UCAV) problem.

Finally, Chapter 5 contains a summary of the main results, as well as provides a remark on the future work.
Chapter 2

Control of LFT systems using PDLP

In this chapter, a new control synthesis approach will be developed for LPV systems with LFT parameter dependency. Different from previous methods, our proposed approach is to design a general parameter-dependent controller to achieve better controlled performance. Parallel to the LFT analysis work in [32, 39], quadratic LFT parameter-dependent Lyapunov functions and full-block multipliers will be used to convert the infinite dimensional LPV control synthesis conditions into a finite set of LMIs. The advantages of the proposed approach will be demonstrated using a ship steering example.

Section 2.2 provides some preliminaries for the development of new control synthesis approach. A quadratic LFT parameter-dependent Lyapunov function is introduced in Section 2.3. In Section 2.4, we present the LPV control synthesis condition for LFT systems as a finite set of LMIs and follow by a detailed proof. Section 2.5 uses a ship control example to demonstrate the advantages of this new LPV control design approach. Finally, the chapter is summarized in Section 2.6.
2.1 Introduction

As introduced in Chapter 1, LFT control in [61] and Apkarian and Gahinet [1] can provide systematic approach, which guarantee the stability and performance over entire parameter set. The LFT gain-scheduled controller synthesis condition is fully characterized by finite number of LMIs. Therefore, the control synthesis problem is a convex problem which can be optimized by efficient interior-point techniques [1]. The resulting controller gain is time-varying and smoothly scheduled by the parameter measurement. Further ramifications have been proposed to count for real gain-scheduling parameters [79]. Recently, a method based on full block multipliers was introduced in [78, 94] to reduce the conservatism associated with block diagonal scaling. However, full block multipliers requires the use of general scheduling functions.

On the other hand, A single quadratic Lyapunov function was first used in the analysis and synthesis for parameter-dependent plants [61, 1], which is motivated from linear time invariant control. Although it may be sufficient to use a single quadratic Lyapunov function to analyze and synthesize LTI systems, parameter-dependent or non-quadratic Lyapunov functions have been used to achieve either larger stability region, i.e. domain of attraction, or better $L_2$ performance for general LPV systems [6, 97, 99]. Whereas the analysis test in [6] introduced potential conservatism by measuring performance against arbitrarily fast variations in scheduling parameters, known bounds on the rate of parameter variation were incorporated into the analysis conditions in [97, 99]. In general, the solution to the this type of LPV control analysis and synthesis problems is formulated as a parameter-dependent LMIs, which is a special type of convex optimization problem with high computational complexity.

By incorporating the parameter variation rate, the controller synthesis condition becomes less conservative than conventional gain-scheduling controller in general LPV control [97, 99]. This advantage comes from the usage of PDLF. By applying the same idea to LFT systems, not only the performance improvement but also the parameter-independent synthesis condition are expected. The key issues are what type of PDLF should be chosen, how to derive computationally efficient parameter-independent synthesis conditions and how to solve the conditions efficiently.
2.2 Preliminaries

We need some preliminary results to derive new LFT control synthesis conditions. The first lemma states a fundamental fact that the multiplication of any two LFTs is also an LFT, and its proof can be found in [103]. From [103], the summation of any two LFTs or the inversion of LFT is also an LFT.

Lemma 1 Given two LFTs

\[
\begin{bmatrix}
\Delta_1 \\
M_21&M_22
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
\Delta_2 \\
N_21&N_22
\end{bmatrix}
\]

with compatible dimensions, then

\[
(\Delta_1 \ast M)(\Delta_2 \ast N) = \begin{bmatrix}
\Delta_1 \\
\Delta_2
\end{bmatrix} \ast W,
\]

where

\[
W =
\begin{bmatrix}
M_{11} & M_{12}N_{21} & M_{12}N_{22} \\
0 & N_{11} & N_{12} \\
M_{21} & M_{22}N_{21} & M_{22}N_{22}
\end{bmatrix}.
\]

The second lemma provides a basis for the null space of an LFT. It has been shown that the null space is also represented by an LFT [91].

Lemma 2 Given an LFT as

\[
L = \Delta \ast \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}.
\]

Assume \( M_{22} \in \mathbb{R}^{m \times r} \) and has rank \( m \), the singular value decomposition of \( M_{22} = U \begin{bmatrix} \Sigma & 0 \end{bmatrix} V^* \), where \( V^* \) is the complex-conjugate transpose of matrix \( V \). Partition \( V \) into the first \( m \) and last \( (r - m) \) columns as \( V = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \), then the null space of \( L \) is

\[
\ker(L) = \text{Im} \left( \Delta \ast \begin{bmatrix} M_{11} - M_{12}V_1\Sigma^{-1}U^*M_{21} & M_{12}V_2 \\ V_1\Sigma^{-1}U^*M_{21} & V_2 \end{bmatrix} \right).
\]
**Proof:** Define a transformed version $\bar{L}$ of $L$ by applying $V$ to the input and multiplying the output by $U^*$.

$$
\bar{L} = U^*L \begin{bmatrix} V_1 & V_2 \end{bmatrix}
= \Delta \ast \begin{bmatrix} M_{11} & M_{12}V_1 & M_{12}V_2 \\ U^*M_{21} & \Sigma & 0 \end{bmatrix} = \begin{bmatrix} \bar{L}_1(\Delta) & \bar{L}_2(\Delta) \end{bmatrix},
$$

where

$$
\bar{L}_1(\Delta) = \Delta \ast \begin{bmatrix} M_{11} & M_{12}V_1 \\ U^*M_{21} & \Sigma \end{bmatrix}, \\
\bar{L}_2(\Delta) = \Delta \ast \begin{bmatrix} M_{11} & M_{12}V_2 \\ U^*M_{21} & 0 \end{bmatrix}.
$$

Partition the input of $\bar{L}$ into the first $m$ inputs $w_1$, and other $r - m$ inputs $w_2$, then the output $z$ of $\bar{L}$ is

$$
z = \begin{bmatrix} \bar{L}_1(\Delta) & \bar{L}_2(\Delta) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.
$$

Since $\Sigma$ is non-singular, then $\bar{L}_1$ is invertible. Therefore

$$
\bar{L}_1^{-1} = \Delta \ast \begin{bmatrix} M_{11} - M_{12}V_1\Sigma^{-1}U^*M_{21} & M_{12}V_1\Sigma^{-1} \\ -\Sigma^{-1}U^*M_{21} & \Sigma^{-1} \end{bmatrix}.
$$

Applying Theorem 3.6 in [91], the null space of the linear map $\bar{L}(w_1, w_2) = \bar{L}_1w_1 + \bar{L}_2w_2$ is

$$
\ker(\bar{L}) = \text{Im}\left( \begin{bmatrix} -\bar{L}_1^{-1}\bar{L}_2 \\ I \end{bmatrix} \right).
$$

Note that

$$
-\bar{L}_1^{-1}\bar{L}_2 = \begin{bmatrix} \Delta & \Delta \\ \Delta & \Delta \end{bmatrix} \ast \begin{bmatrix} M_{11} - M_{12}V_1\Sigma^{-1}U^*M_{21} & M_{12}V_1\Sigma^{-1}U^*M_{21} & 0 \\ 0 & M_{11} & M_{12}V_2 \\ -\Sigma^{-1}U^*M_{21} & \Sigma^{-1}U^*M_{21} & 0 \end{bmatrix}
= \Delta \ast \begin{bmatrix} M_{11} - M_{12}V_1\Sigma^{-1}U^*M_{21} & M_{12}V_2 \\ \Sigma^{-1}U^*M_{21} & 0 \end{bmatrix},
$$

then we get the desired result by applying Theorem 3.7 in [91]

$$
\ker(L) = \text{Im}\left( V \begin{bmatrix} -\bar{L}_1^{-1}\bar{L}_2 \\ I \end{bmatrix} \right) = \text{Im}\left( \Delta \ast \begin{bmatrix} M_{11} - M_{12}V_1\Sigma^{-1}U^*M_{21} & M_{12}V_2 \\ V_1\Sigma^{-1}U^*M_{21} & V_2 \end{bmatrix} \right).
$$
The third lemma converts an uncertain matrix inequality to a finite set of inequalities using full-block multipliers [78]. For completeness, its proof is also provided here.

**Lemma 3** Given a quadratic matrix inequality

\[ G^T(\Theta)MG(\Theta) < 0 \]  \hspace{1cm} (2.1)

with \( G(\Theta) = \Theta \ast \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \) and \( \Theta \in \Theta \). The condition (2.1) holds if and only if there exists a full-block multiplier \( \Pi \) such that

\[
\begin{bmatrix}
* \\
* 
\end{bmatrix}^T \text{diag} \{\Pi, M\} \begin{bmatrix}
G_{11} & G_{12} \\
I & 0 \\
G_{21} & G_{22}
\end{bmatrix} < 0,
\]  \hspace{1cm} (2.2)

and for any \( \Theta \in \Theta \)

\[
\begin{bmatrix}
* 
\end{bmatrix}^T \Pi \begin{bmatrix}
I \\
\Theta
\end{bmatrix} \geq 0.
\]  \hspace{1cm} (2.3)

**Proof:** To apply the full block S-procedure, we introduce

\[
N = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & M
\end{bmatrix}, \quad S = \text{Im} \begin{bmatrix}
G_{11} & G_{12} \\
I & 0 \\
G_{21} & G_{22}
\end{bmatrix}, \quad S_0 = \text{Im} \begin{bmatrix}
G_{12} \\
0 \\
G_{22}
\end{bmatrix},
\]

\[
U = \begin{bmatrix}
-\Theta & I
\end{bmatrix}, \quad T = \begin{bmatrix}
I & 0 \\
0 & I
\end{bmatrix}.
\]

Hence \( S_U = \text{ker}(UT) \cap S \) is

\[
\left\{ \begin{bmatrix}
G_{11} & G_{12} \\
I & 0 \\
G_{21} & G_{22}
\end{bmatrix} \begin{bmatrix}
w_1 \\
w_2
\end{bmatrix} : (I - \Theta G_{11})w_1 - \Theta G_{12}w_2 = 0 \right\}.
\]
Note that $S_U \cap S_0 = \{0\}$ if and only if $(I - \Theta G_{11})$ is nonsingular. Then $S_U$ can be represented as

$\text{Im} \begin{bmatrix} G_{11} (I - \Theta G_{11})^{-1} \Theta G_{12} + G_{12} \\ (I - \Theta G_{11})^{-1} \Theta G_{12} \\ G_{21} (I - \Theta G_{11})^{-1} \Theta G_{12} + G_{22} \end{bmatrix}.

From the full block S-procedure theorem [78], we have the following two equivalent conditions:

1. $N < 0$ on $S_U$,
2. $N + T^T \Pi T < 0$ on $\mathcal{S}$ and $\Pi > 0$ on $\ker(U)$.

Condition 1 is exactly $G^T(\Theta) M G(\Theta) < 0$, and condition 2 leads to inequalities (2.2) and (2.3).

This lemma breaks the original inequality (2.1) into two inequalities, (2.2) and (2.3). One of them depends on parameter $\Theta$, the other does not depend on $\Theta$, which provides us an alternative way to deal with parameter-dependent conditions. Actually, parameter-dependent inequality (2.3) is much simpler than (2.1) and easier to convexify.

### 2.3 Quadratic LFT Lyapunov Function

In this section, we introduce a new type of Lyapunov function to be used in this chapter, which is called quadratic LFT Lyapunov Function. We consider a parameter-dependent Lyapunov function in the form of

$$V_{\text{cl}} = x_{\text{cl}}^T X_{\text{cl}}(\Theta) x_{\text{cl}}$$

where

$$X_{\text{cl}}(\Theta) = \begin{bmatrix} S(\Theta) & N(\Theta) \\ N^T(\Theta) & ? \end{bmatrix}, \quad X_{\text{cl}}^{-1}(\Theta) = \begin{bmatrix} R(\Theta) & M(\Theta) \\ M^T(\Theta) & ? \end{bmatrix},$$

The subscript $\text{cl}$ means close loop system. We have $M(\Theta) N^T(\Theta) = I - R(\Theta) S(\Theta)$. ? denote the elements of matrices we do not care about.

Define

$$R(\Theta) = T_R^T(\Theta) P T_R(\Theta), \quad S(\Theta) = T_S^T(\Theta) Q T_S(\Theta),$$
$R(\Theta)$ and $S(\Theta)$ are specified as quadratic LFT form of scheduling parameter $\Theta$, i.e.,

$$R(\Theta) = T^T_R(\Theta)PT_R(\Theta), \quad S(\Theta) = T^T_S(\Theta)QT_S(\Theta)$$

(2.4)

where $T_R(\Theta), T_S(\Theta)$ are pre-specified LFT functions of parameter $\Theta$,

$$T_R(\Theta) = \Theta \ast \begin{bmatrix} T_{R11} & T_{R12} \\ T_{R21} & T_{R22} \end{bmatrix} = T_{R22} + T_{R21}\Theta(I - T_{R11}\Theta)^{-1}T_{R12},$$

$$T_S(\Theta) = \Theta \ast \begin{bmatrix} T_{S11} & T_{S12} \\ T_{S21} & T_{S22} \end{bmatrix} = T_{S22} + T_{S21}\Theta(I - T_{S11}\Theta)^{-1}T_{S12},$$

and $T_{R22} \in \mathbb{R}^{n_r \times n}$, $T_{S22} \in \mathbb{R}^{n_s \times n}$ with $n_r, n_s \geq n$. Note that $T_{R11}$ and $T_{S11}$ must be chosen such that the LFT forms are well-defined for all $\Theta \in \Theta$. By Lemma 1, it is easy to verify that $X_{cl}$ is also in quadratic LFT parameter-dependent form.

The quadratic LFT type functions are quite general, and include affine parameter-dependent functions as a special case. For a two parameters example, if we choose $T_R(\Theta) = \begin{bmatrix} \theta_1I & \theta_2I & I \end{bmatrix}^T$ and restrict matrix $P$ as

$$P = \begin{bmatrix} 0 & P_1 \\ P_2^T & P_0 \end{bmatrix}, \quad P_0 = P_0^T > 0,$$

then $R(\Theta) = P_0 + (P_1 + P_1^T)\theta_1 + (P_2 + P_2^T)\theta_2$ is in affine function form. The quadratic LFT parameterization is also advantageous because the resulting LFT control synthesis condition can be formulated as finite number of LMIs. The quadratic LFT PDLF also includes some polynomial functions by choosing off-diagonal elements of $T_{R11}$ and $T_{S11}$ non-zeros, for example,

$$T_R(\Theta) = \begin{bmatrix} \theta \\ \theta \end{bmatrix} \ast \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & p \\ 1 & 0 & 0 \end{bmatrix} = p\theta^2$$

2.4 Gain-scheduled Output Feedback Systems

Consider an LFT parameter-dependent plant (1.5)-(1.6), with assumption (A1)-(A4).
It is assumed that the vector-valued parameter $\theta$ and its derivative are measurable in real-time. In addition, its time derivative is bounded and satisfies the constraint $-\bar{\nu}_i \leq \dot{\theta}_i \leq \bar{\nu}_i$, $i = 1, 2, \cdots, s$. For notational purposes, denote

$$\mathcal{V} = \{v: -\bar{\nu}_i \leq v \leq \bar{\nu}_i, \ i = 1, 2, \cdots, s\}$$

, i.e., $\mathcal{V}$ is a given convex polytope in $\mathbb{R}^s$ that contains the origin.

By absorbing parameter $\Theta$, the state-space equation of LFT systems can also be written as

$$\begin{bmatrix}
\dot{x}(t) \\
e(t) \\
y(t)
\end{bmatrix} =
\begin{bmatrix}
A(\Theta(t)) & B_1(\Theta(t)) & B_2(\Theta(t)) \\
C_1(\Theta(t)) & D_{11}(\Theta(t)) & D_{12}(\Theta(t)) \\
C_2(\Theta(t)) & D_{21}(\Theta(t)) & D_{22}(\Theta(t))
\end{bmatrix}
\begin{bmatrix}
x(t) \\
d(t) \\
u(t)
\end{bmatrix},$$

which is same as (1.1), where two state-space representations have the relationship of (1.9).

The class of LPV controllers we are interested is in the form of

$$\begin{bmatrix}
\dot{x}_k(t) \\
u(t)
\end{bmatrix} =
\begin{bmatrix}
A_k(\Theta(t), \dot{\Theta}(t)) & B_k(\Theta(t)) \\
C_k(\Theta(t)) & D_k(\Theta(t))
\end{bmatrix}
\begin{bmatrix}
x_k(t) \\
y(t)
\end{bmatrix}, \quad (2.5)$$

where $x_k \in \mathbb{R}^{n_k}$. The dimension of controller state $n_k$ is yet to be determined. Unlike the conventional LFT control techniques, we do not assume LFT parameter dependency in the controller formula.

In references [97, 99], $H_\infty$ synthesis condition of an LPV controller (2.5) for the LPV system (1.1) is given by (1.2)-(1.4). However, the set of parameter-dependent LMIs (1.2)-(1.4) usually involves infinite number of constraints and is difficult to solve.

### 2.4.1 Controller Synthesis Conditions

We derive the control synthesis condition using quadratic LFT Lyapunov function. The idea to derive the condition is replacing $R(\Theta)$ and $S(\Theta)$ in (1.2)-(1.4) by its quadratic LFT form (2.4).

**Theorem 1** Consider an LFT system (1.5)-(1.6) with its parameter and derivative set $\Theta$ and $\mathcal{V}$, the conditions (1.2)-(1.4) are solvable using quadratic LFT functions $R(\Theta), S(\Theta)$ if
and only if there exist positive-definite matrices \( P \in S^{(n_r+n) \times (n_r+n)} \) and \( Q \in S^{(n_s+n) \times (n_s+n)} \), and full-block multipliers \( \Pi_P, \Pi_Q \in S^{5np \times 5np} \) and \( \Pi \in S^{2np \times 2np} \), such that

\[
\begin{bmatrix}
* \\
* \\
\end{bmatrix}
^T \begin{pmatrix}
0 & P & 0 & 0 \\
P & 0 & 0 & 0 \\
0 & 0 & \gamma^{-1}I & 0 \\
0 & 0 & 0 & -\gamma I
\end{pmatrix}
\begin{pmatrix}
\hat{G}_{P11} & \hat{G}_{P12} \\
I & 0 \\
\hat{G}_{P21} & \hat{G}_{P22}
\end{pmatrix} < 0, \quad (2.6)
\]

\[
\begin{bmatrix}
* \\
* \\
\end{bmatrix}
^T \begin{pmatrix}
0 & Q & 0 & 0 \\
Q & 0 & 0 & 0 \\
0 & 0 & \gamma^{-1}I & 0 \\
0 & 0 & 0 & -\gamma I
\end{pmatrix}
\begin{pmatrix}
\hat{G}_{Q11} & \hat{G}_{Q12} \\
I & 0 \\
\hat{G}_{Q12}^T & \hat{G}_{Q22}
\end{pmatrix} < 0, \quad (2.7)
\]

\[
\begin{bmatrix}
* \\
* \\
\end{bmatrix}
^T \begin{pmatrix}
P & 0 & 0 & 0 \\
0 & Q & 0 & 0 \\
0 & 0 & 0 & I \\
0 & 0 & I & 0
\end{pmatrix}
\begin{pmatrix}
G_{11} & G_{12} \\
I & 0 \\
G_{21} & G_{22}
\end{pmatrix} \geq 0, \quad (2.8)
\]

and for any \((\Theta, \dot{\Theta}) \in \Theta \times V\),

\[
\begin{bmatrix}
* \\
\end{bmatrix}
^T \Pi_P \begin{pmatrix}
I \\
\dot{\Theta}
\end{pmatrix} \geq 0, \quad (2.9)
\]

\[
\begin{bmatrix}
* \\
\end{bmatrix}
^T \Pi_Q \begin{pmatrix}
I \\
\dot{\Theta}
\end{pmatrix} \geq 0, \quad (2.10)
\]
\[
\begin{bmatrix}
* \\
\end{bmatrix}^T \Pi \begin{bmatrix}
I & 0 \\
0 & I \\
\Theta & 0 \\
0 & \Theta \\
\end{bmatrix} > 0,
\] (2.11)

where \( \Theta = \text{diag}\{\Theta, \Theta, \Theta, \Theta, \Theta\} \) and

\[
\begin{bmatrix}
\hat{G}_{P11} & \hat{G}_{P12} \\
\hat{G}_{P21} & \hat{G}_{P22} \\
\end{bmatrix} = 
\begin{bmatrix}
\hat{R}_{11} & \hat{R}_{12}M_{R21} & \hat{R}_{12}M_{R22}N_{R21} & \hat{R}_{12}M_{R22}N_{R22} \\
0 & M_{R11} & M_{R12}N_{R21} & M_{R12}N_{R22} \\
0 & 0 & N_{R11} & N_{R12} \\
\end{bmatrix}
\]

with

\[
\begin{bmatrix}
\hat{R}_{11} & \hat{R}_{12} \\
\hat{R}_{21} & \hat{R}_{22} \\
\end{bmatrix} = 
\begin{bmatrix}
0 & 0 & T_{R11} & 0 & T_{R12} & 0 & 0 \\
T_{R11} & T_{R11} & 0 & T_{R12} & 0 & 0 & 0 \\
0 & 0 & T_{R11} & 0 & T_{R12} & 0 & 0 \\
-T_{R21} & T_{R21} & 0 & T_{R22} & 0 & 0 & 0 \\
0 & 0 & T_{R21} & 0 & T_{R22} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
M_{R11} & M_{R12} \\
M_{R21} & M_{R22} \\
\end{bmatrix} = 
\begin{bmatrix}
D_{00}^T & B_0^T & D_{10}^T \\
C_0^T & A^T & C_1^T \\
0 & I & 0 \\
D_{01}^T & B_1^T & D_{11}^T \\
0 & 0 & I \\
\end{bmatrix}
\]

\[
\Theta \ast \begin{bmatrix}
N_{R11} & N_{R12} \\
N_{R21} & N_{R22} \\
\end{bmatrix} = N_{R}(\Theta),
\]
\[
\begin{bmatrix}
\hat{G}_{Q11} & \hat{G}_{Q12} \\
\hat{G}_{Q21} & \hat{G}_{Q22}
\end{bmatrix} = \begin{bmatrix}
\hat{S}_{11} & \hat{S}_{12}M_{S21} & \hat{S}_{12}M_{S22}N_{S21} & \hat{S}_{12}M_{S22}N_{S22} \\
0 & M_{S11} & M_{S12}N_{S21} & M_{S12}N_{S22} \\
0 & 0 & N_{S11} & N_{S12} \\
\hat{S}_{21} & \hat{S}_{22}M_{S21} & \hat{S}_{22}M_{S22}N_{S21} & \hat{S}_{22}M_{S22}N_{S22}
\end{bmatrix}
\]

with

\[
\begin{bmatrix}
\hat{S}_{11} & \hat{S}_{12} \\
\hat{S}_{21} & \hat{S}_{22}
\end{bmatrix} = \begin{bmatrix}
0 & 0 & T_{S11} & 0 & T_{S12} & 0 & 0 \\
T_{S11} & T_{S11} & 0 & T_{S12} & 0 & 0 & 0 \\
0 & 0 & T_{S11} & 0 & T_{S12} & 0 & 0 \\
T_{S21} & T_{S21} & 0 & T_{S22} & 0 & 0 & 0 \\
0 & 0 & T_{S21} & 0 & T_{S22} & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & 0 & I & 0
\end{bmatrix},
\]

\[
\begin{bmatrix}
M_{S11} & M_{S12} \\
M_{S21} & M_{S22}
\end{bmatrix} = \begin{bmatrix}
D_{00} & C_0 & D_{01} \\
B_0 & A & B_1 \\
0 & I & 0 \\
D_{10} & C_1 & D_{11} \\
0 & 0 & I
\end{bmatrix}, \quad \Theta \ast \begin{bmatrix}
N_{S11} & N_{S12} \\
N_{S21} & N_{S22}
\end{bmatrix} = N_S(\Theta),
\]
and

$$
\begin{bmatrix}
G_{11} & G_{12} \\
G_{21} & G_{22}
\end{bmatrix}
= \begin{bmatrix}
T_{R11} & 0 & T_{R12} & 0 \\
0 & T_{S11} & 0 & T_{S12} \\
T_{R21} & 0 & T_{R22} & 0 \\
0 & T_{S21} & 0 & T_{S22} \\
0 & 0 & I & 0 \\
0 & 0 & 0 & I
\end{bmatrix}.
$$

**Proof:** For quadratic LFT functions $R(\Theta)$ and $S(\Theta)$, the time-derivative of $T_S$ can be expressed as an LFT

$$
\dot{T}_S = T_{S21}(I - \Theta T_S^T T_S)\dot{\Theta}(I - T_{S11}\Theta)^{-1}T_{S12},
$$

and

$$
\begin{bmatrix}
0 & S \\
S & \dot{S}
\end{bmatrix}
= \begin{bmatrix}
T_S & \dot{T}_S \\
0 & T_S
\end{bmatrix}^T \begin{bmatrix}
0 & Q \\
Q & 0
\end{bmatrix} \begin{bmatrix}
T_S & \dot{T}_S \\
0 & T_S
\end{bmatrix}.
$$

(2.12)

Note that

$$
\dot{S}(\Theta, \dot{\Theta}) = \begin{bmatrix}
T_S & \dot{T}_S & 0 & 0 \\
0 & T_S & 0 & 0 \\
0 & 0 & I & 0 \\
0 & 0 & 0 & I
\end{bmatrix} = \begin{bmatrix}
\Theta & \dot{\Theta}
\end{bmatrix}^T \begin{bmatrix}
\Theta & \Theta
\end{bmatrix} * \begin{bmatrix}
\dot{\Theta}
\Theta
\end{bmatrix} = \begin{bmatrix}
\dot{S}_{11} & \dot{S}_{12} \\
\dot{S}_{21} & \dot{S}_{22}
\end{bmatrix}.
$$
One can also write the outer factor matrices in (1.3) as LFTs

\[
M_S(\Theta) = \begin{bmatrix}
A(\Theta) & B_1(\Theta) \\
I & 0 \\
C_1(\Theta) & D_{11}(\Theta) \\
0 & I
\end{bmatrix} = \Theta \ast \begin{bmatrix}
D_{00} & C_0 & D_{01} \\
B_0 & A & B_1 \\
0 & I & 0 \\
D_{10} & C_1 & D_{11} \\
0 & 0 & I
\end{bmatrix} = \Theta \ast \begin{bmatrix}
M_{S11} & M_{S12} \\
M_{S21} & M_{S22}
\end{bmatrix},
\]

\[
N_S(\Theta) = \Theta \ast \begin{bmatrix}
N_{S11} & N_{S12} \\
N_{S21} & N_{S22}
\end{bmatrix} = \ker \left( \Theta \ast \begin{bmatrix}
D_{00} & C_0 & D_{01} \\
D_{20} & C_2 & D_{21}
\end{bmatrix} \right),
\] (2.13)

where the null space in eqn. (2.13) is determined using Lemma 2. Therefore,

\[
\hat{G}_Q(\Theta, \dot{\Theta}) = \hat{S}(\Theta, \dot{\Theta}) M_S(\Theta) N_S(\Theta)
\]

\[
= \begin{bmatrix}
\Theta \\
\Theta \\
\Theta \\
\Theta
\end{bmatrix} \ast \begin{bmatrix}
\hat{G}_{Q11} & \hat{G}_{Q12} \\
\hat{G}_{Q21} & \hat{G}_{Q22}
\end{bmatrix}.
\]

Then the condition (1.3) can be rewritten as

\[
\hat{G}_Q^T(\dot{\Theta}) \begin{bmatrix}
0 & Q & 0 & 0 \\
Q & 0 & 0 & 0 \\
0 & 0 & \gamma^{-1}I & 0 \\
0 & 0 & 0 & -\gamma I
\end{bmatrix} \hat{G}_Q(\dot{\Theta}) < 0,
\]

which is equivalent to (2.7) and (2.10) by Lemma 3.

Similar to eqn. (2.12), we have

\[
\begin{bmatrix}
0 & R \\
R & -\dot{R}
\end{bmatrix} = \begin{bmatrix}
T_R & -\dot{T}_R \\
0 & T_R
\end{bmatrix}^T \begin{bmatrix}
0 & P \\
P & 0
\end{bmatrix} \begin{bmatrix}
T_R & -\dot{T}_R \\
0 & T_R
\end{bmatrix},
\]

then the same approach can be applied to prove conditions (2.6) and (2.9), and coupling conditions (2.8) and (2.11). \( Q.E.D. \)
Theorem 1 provides a necessary and sufficient condition for controller synthesis. Condition (2.6)-(2.11) are equivalent to (1.2)-(1.4) by choosing a quadratic LFT Lyapunov function. Note that (2.6)-(2.8) does not depend on parameters. However, same as (1.2)-(1.4), (2.9)-(2.11) are still parameter-dependent, which involves infinite number of LMIs.

2.4.2 Computational Consideration

Note that the conditions (2.9)-(2.11) are specified for any parameter and its derivative within $\Theta \times \mathcal{V}$, and consist of infinite number of constraints. However, by introducing additional constraints on the matrices $\Pi_P, \Pi_Q$ and $\Pi$, these infinite constraint sets can be made finite. For example, if we partition the full-block multiplier $\Pi_Q$ as

$$
\Pi_Q = \begin{bmatrix} \Pi_{Q11} & \Pi_{Q12} \\ \Pi_{Q12}^T & \Pi_{Q22} \end{bmatrix},
$$

and assume $\Pi_{Q22} < 0$, then condition (2.10) will be convex with respect to parameter $\Theta$ and its derivative $\dot{\Theta}$. It is therefore adequate to check the matrix inequality (2.10) at vertices $\hat{\Theta}_i$ of the set $\Theta \times \mathcal{V}$, that is

$$
[\begin{bmatrix} \Pi_{Q11} & \Pi_{Q12} \\ \Pi_{Q12}^T & \Pi_{Q22} \end{bmatrix} \begin{bmatrix} I \\ \hat{\Theta}_i \end{bmatrix}] \geq 0, \quad i = 1, 2, \ldots, 2^s, \quad \Pi_{Q22} < 0. \quad (2.14)
$$

Nevertheless, the matrix inequalities (2.7) and (2.14) now become sufficient for

$$
\hat{G}_Q^T(\hat{\Theta}) \begin{bmatrix} 0 & Q & 0 & 0 \\ Q & 0 & 0 & 0 \\ 0 & 0 & \gamma^{-1}I & 0 \\ 0 & 0 & 0 & \gamma I \end{bmatrix} \hat{G}_Q(\hat{\Theta}) < 0
$$

to hold. On the other hand, the condition (2.10) can be completely removed given $\Theta$ as structured uncertainty. Specifically, one can choose $\Pi_{Q11} = -\Pi_{Q22} > 0$, $\Pi_{Q11} \dot{\Theta} = \hat{\Theta} \Pi_{Q11}$, $\Pi_{Q12}$ skew-symmetric and commutable with $\hat{\Theta}$. With some performance degradation, it will significantly reduce the number of LMIs and optimization variables in the LPV control synthesis condition. Similar arguments are applicable to multipliers $\Pi_P$ and $\Pi$.

The user-defined LFTs $T_R(\Theta)$ and $T_S(\Theta)$ are essentially the basis functions to parameterize Lyapunov functions. Note that the dimensions of $T_R, T_S$ matrices are not
necessarily the same as the plant dimension $n$. Intuitively, this would lead to a search of feasible matrices $P, Q$ in higher dimensional space with increased number of optimization variables. Using $T_R(\Theta), T_S(\Theta)$ as tuning nob, one can improve the performance of controlled LPV systems by incorporating parameter variation information. However, it is not clear at this stage how to select right $T_R, T_S$ matrices.

2.4.3 Controller Formulation

After solving quadratic LFT matrix functions $R(\Theta)$ and $S(\Theta)$, the LPV controller can be constructed through the following scheme [94]:

1. Let $M(\Theta)N^T(\Theta) = I - R(\Theta)S(\Theta)$ and define
   $\begin{align*}
   F(\Theta) &= - \left[ D_{12}^T(\Theta)D_{12}(\Theta) \right]^{-1} \left[ \gamma B_2^T(\Theta)R^{-1}(\Theta) + D_{12}^T(\Theta)C_1(\Theta) \right],
   \\
   L(\Theta) &= - \left[ \gamma S^{-1}(\Theta)C_2^T(\Theta) + B_1(\Theta)D_{21}(\Theta) \right] \left[ D_{21}(\Theta)D_{21}^T(\Theta) \right]^{-1},
   \end{align*}$

2. Construct the state-space matrices of one $n$th-order, strictly proper controller as
   $\begin{align*}
   A_K(\Theta, \dot{\Theta}) &= -N^{-1}(\Theta) \left\{ -S(\Theta) \frac{dR}{dt} - N(\Theta) \frac{dM^T}{dt} + A^T(\Theta) \\
   &+ S(\Theta) \left[ A(\Theta) + B_2(\Theta)F(\Theta) + L(\Theta)C_2(\Theta) \right] R(\Theta) \\
   &+ \gamma^{-1}S(\Theta) \left[ B_1(\Theta) + L(\Theta)D_{21}(\Theta) \right] B_1^T(\Theta) \\
   &+ \gamma^{-1}C_1^T(\Theta) \left[ C_1(\Theta) + D_{12}(\Theta)F(\Theta) \right] R(\Theta) \right\} M^{-T}(\Theta),
   \\
   B_K(\Theta) &= N^{-1}(\Theta)S(\Theta)L(\Theta),
   \\
   C_K(\Theta) &= F(\Theta)R(\Theta)M^{-T}(\Theta),
   \\
   D_K(\Theta) &= 0.
   \end{align*}$

The controller depends on not only the parameter $\Theta$, but also the variation rate $\dot{\Theta}$. Therefore, a better performance is expected. This controller formula is based on the assumption that $D_{11}(\Theta) = 0$. For non-zero case, the details of deriving the controller can be found in [93]. The idea is to solve $D_K(\Theta)$ first. Then the original plant is manipulated to be $D_{11}(\Theta) = 0$. Finally, all controller state space can be solved by the scheme of $D_{11}(\Theta) = 0$ case.

If the user-defined function $T_R(\Theta)$ and $T_S(\Theta)$ are restricted as constant matrices, the controller state space will only depend on parameters. No parameter variation rates are
involved in this special case. The result by choosing $R(\Theta)$ and $S(\Theta)$ as constant matrices will cover the existing LFT control design approach by full-block multiplier [78, 94].

2.5 Application to Ship Steering

Autopilot for ship steering is based on feedback from a heading measurement, using a gyrocompass, to a steering device, which drives the rudder. Although a PID regulator may work reasonably well, its performance is often poor in some severe weather, such as heavy wind or big waves, and when the ship’s speed is changing. Because the ship’s dynamics change significantly with the speed, the autopilots can be improved by modeling the ship’s dynamics as parameter varying system.

In this section, we will design various LPV controllers for a ship control problem. This example will be used to demonstrate the design procedure of the proposed parameter-dependent controller and its advantages over conventional LFT control designs.

2.5.1 Ship Dynamics Modeling

The ship steering problem has been analyzed in [3, 32], and its dynamics can be obtained by applying Newton’s equations to the motion of the ship. For simplicity, only lateral dynamics is considered in ship steering problem (see Fig. 2.1).

Figure 2.1: Dynamics of ship
Let $V$ be the velocity, $v$ and $x$ are the components of $x$ and $y$ directions correspondingly, and $w$ is angular velocity of the ship. The following dynamics equations are derived,

\[
\frac{dx}{dt} = \frac{v}{l}a_{11}x + va_{12}w + \frac{v^2}{l}b_1u
\]  
\[
\frac{dw}{dt} = \frac{v}{l^2}a_{21}x + \frac{v}{l}a_{22}w + d + \frac{v^2}{l^2}b_2u
\]  
\[
\frac{d\psi}{dt} = w
\]

(2.15)  
(2.16)  
(2.17)

where $\psi$ denotes the heading of the ship and $w$ is the angular velocity. $u$ denotes the rudder angle as control input, $d$ is the disturbance moment caused by wind or wave, $l$ is the length of the ship. It is assumed that $v(t) \geq 0$.

The parameters in the equations are constant for different ships and different operating conditions [3],

<table>
<thead>
<tr>
<th>Ship</th>
<th>Minesweeper</th>
<th>Cargo</th>
<th>Tanker(Full)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length(m)</td>
<td>55</td>
<td>161</td>
<td>350</td>
</tr>
<tr>
<td>$a_{11}$</td>
<td>-0.86</td>
<td>-0.77</td>
<td>-0.45</td>
</tr>
<tr>
<td>$a_{12}$</td>
<td>-0.48</td>
<td>-0.34</td>
<td>-0.43</td>
</tr>
<tr>
<td>$a_{21}$</td>
<td>-5.2</td>
<td>-3.39</td>
<td>-4.1</td>
</tr>
<tr>
<td>$a_{22}$</td>
<td>-2.4</td>
<td>-1.63</td>
<td>-0.81</td>
</tr>
<tr>
<td>$b_1$</td>
<td>0.18</td>
<td>0.17</td>
<td>0.10</td>
</tr>
<tr>
<td>$b_2$</td>
<td>-1.4</td>
<td>-1.63</td>
<td>-0.81</td>
</tr>
</tbody>
</table>

By adding disturbance and measurement, the ship dynamics (2.15)-(2.17) can be
converted to an LFT parameter-dependent system

\[
\begin{bmatrix}
\dot{x} \\
\dot{w} \\
\dot{\psi} \\
\dot{q} \\
y
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0.001 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{11}/l & a_{12} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & b_1/l & 0 \\
a_{21}/l^2 & a_{22}/l & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & b_2/l^2 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
x \\
w \\
\psi \\
p \\
d \\
u
\end{bmatrix},
\]  
(2.18)

\[p = v(t)I_4q,\]  
(2.19)

where \(p, q \in \mathbb{R}^4, y, d, u, e \in \mathbb{R}.\)

Define

\[\delta(t) = \frac{v(t) - v_{\text{nom}}}{h(v(t) + v_{\text{nom}})} \in [-1, 1].\]  
(2.20)

Then the ship dynamics can be rewritten as a LFT parameter-dependent system with varying parameter \(\delta.\) We will design controllers for the minesweeper with parameters in the first column of Table 2.1 and \(h = 0.7, v_{\text{nom}} = 3(m/s),\) which specifies the forward speed variation range from 0.529 (m/s) to 17 (m/s). The parameter \(\delta\) is obviously measurable in real-time, and can be used by LPV controllers. Note that the plant model has its \(D_{00}\) term non-zero, therefore its control synthesis problem can not be solved using previous LPV control approaches like [23] and [89].

![Figure 2.2: System structure of ship steering](image-url)
The control design objectives include tracking the heading reference of the ship under disturbances with reasonable control force. They are quantified by rational weighting functions, which apply to the error output and control input channels.

\[
W_e(s) = \frac{0.15(s + 1)}{s + 0.001}, \quad W_u(s) = \frac{250s + 5}{s + 12500}.
\]

The close loop system structure is shown in Fig. 2.2.

### 2.5.2 $\mathcal{L}_2$ gain comparison

For the ship steering problem, five different gain-scheduling control techniques will be considered. The first three cases are based on the proposed control approach in this chapter.

1. LFT parameter-dependent Lyapunov function with parameter variation rate $\nu = 0.1, 10, 100$. For this case, we select $T_R(\delta), T_S(\delta)$ as

\[
T_R(\delta) = T_S(\delta) = \delta I_4 \star \begin{bmatrix} 0 & I_4 \\ I_4 & 0 \\ 0 & I_4 \end{bmatrix}.
\]

In fact, this corresponds to quadratic parameter-dependent Lyapunov functions $R(\delta) = R_0 + \delta R_1 + \delta^2 R_2$ and $S(\delta) = S_0 + \delta S_1 + \delta^2 S_2$.

2. LFT parameter-dependent Lyapunov function with parameter variation rate $\nu = 0.1, 10, 100$. $T_R(\delta)$ and $T_S(\delta)$ are chosen as

\[
T_R(\delta) = T_S(\delta) = \delta I_4 \star \begin{bmatrix} 0.9I_4 & 1.8I_4 \\ I_4 & I_4 \\ 0 & I_4 \end{bmatrix},
\]

3. Constant Lyapunov function with $T_R = T_S = I_4$,

4. LFT control with full-block multipliers [78, 94],

5. LFT control with commutable scaling matrices [1].
Table 2.2: Optimal performance of continuous-time LPV control.

<table>
<thead>
<tr>
<th>method</th>
<th>parameter variation rate $\nu$</th>
<th>$\mathcal{L}_2$ gain</th>
<th>CPU time (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>QLFT LF (case 1)</td>
<td>0.01</td>
<td>1.557</td>
<td>920.16</td>
</tr>
<tr>
<td></td>
<td>0.1</td>
<td>1.706</td>
<td>885.27</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2.171</td>
<td>750.98</td>
</tr>
<tr>
<td>QLFT LF (case 2)</td>
<td>0.01</td>
<td>1.868</td>
<td>905.78</td>
</tr>
<tr>
<td></td>
<td>0.1</td>
<td>2.300</td>
<td>1315.9</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2.742</td>
<td>1271.9</td>
</tr>
<tr>
<td>SQLF</td>
<td>$\infty$</td>
<td>3.642</td>
<td>480.30</td>
</tr>
<tr>
<td>LFT (multiplier)</td>
<td>$\infty$</td>
<td>3.591</td>
<td>3.906</td>
</tr>
<tr>
<td>LFT (scaling)</td>
<td>$\infty$</td>
<td>602.9</td>
<td>3.469</td>
</tr>
</tbody>
</table>

The comparison of required computational time of different gain-scheduling control methods was performed on a Pentium III 900 MHz PC. As can be seen, the LFT control with scaling matrices involves 4 LMIs and 51 optimization variables, and demands the least CPU time; while the proposed approach in cases 1 and 2 has 8 LMIs and 909 variables, thus is computationally more expensive. From Table 2.2, it is observed that the new control design approach gets much better performance than the previous LFT control methods, although the new approach consumes more computational effort. As parameter variation rates decrease, the performance bounds are further improved. However, the results clearly depend on the selection of $T_R(\delta)$ and $T_S(\delta)$. Case 3 and Case 4 are essentially the same. The small difference between their calculated performance is due to numerical accuracy. Finally, the performance achieved by conventional LFT control approach [1] is found very conservative.

2.5.3 Simulation Results

To demonstrate the time-domain performance of these controllers, we choose a particular speed trajectory as

$$v_1(t) = 8.77 - 8.23 \cos(0.02t).$$

which varies between 0.54 (m/s) and 17 (m/s) in sinusoid form. With the relation of $v(t)$ and $\delta(t)$ in (2.20), the trajectories of parameter $\delta$ and its derivative are shown in Fig. 2.3.
For reference tracking, an step input is chosen while the measurement noise and disturbance is set to zero. Figure 2.4 shows that the tracking output using the proposed parameter-dependent controller is much smaller than the others although the control force is relatively large.

From the simulation results, it is very hard to track reference when the ship’s speed is low. Even under a small step input, we still need a relative large rudder angle change. From Fig. 2.3(a), the ship’s speed is very low at the beginning. We can see that large rudder angle input is needed even if we want to tack 0.01 rad step reference. However, under same constraints, i.e. weighting functions, our proposed controller can work much better than other approaches.

Next, we choose another speed trajectory as
\[ v_2(t) = 8.77 + 8.23 \sin(0.02t). \]
which is 8.77 m/s at the beginning. It is much larger than previous case.

From Fig. 2.6, we can see a smaller control input is needed to achieve 0.1 rad step input. Always, our proposed controller performs much better than existing LFT control approaches.

Now we take disturbance into account and investigate the performance of different LPV controllers. A uniform random noise with magnitude 1 is chosen for disturbance \( d \) and a uniform random noise with magnitude 0.001 is chosen for sensor noise \( d_m \). We choose \( v_1(t) \) as speed trajectory. For reference tracking, an step with magnitude 0.01 is chosen as input. Fig. 2.7 (a) shows the tracking output comparison. Our proposed controller still performs better under disturbance. Fig. 2.7 (b)-(d) shows the control inputs from different controllers. We can see that our proposed controller is less sensitive to the disturbance. The noise magnitude of our controller output is much smaller than other controllers under the same measurement noise.

2.6 Summary

In this chapter, we have developed a new approach for gain-scheduling control of LFT systems with bounded parameter variation rates. With the help of quadratic LFT parameter-dependent Lyapunov functions and full-block multipliers, the control synthesis
condition is formulated as finite number of LMI s with stringent controlled performance. The extension of proposed LPV control technique to discrete-time LFT systems also exploits parameter variation information for less conservative gain-scheduling control designs [95]. However, it remains a question how to find the most suitable matrices $T_R(\Theta), T_S(\Theta)$ for the best performance.

The proposed LPV control method was applied on a ship control example. It has been shown that the new approach is very powerful, and always provides better performance than existing LFT control approaches.
(a) Trajectory of $\delta$

(b) Trajectory of $\dot{\delta}$

Figure 2.3: Trajectories of parameter $\delta$ and its derivative (Case of $v_1(t)$)

(a) step response  
(b) control force

Figure 2.4: Step response and control input of different LPV controllers (Case of $v_1(t)$)
Figure 2.5: Trajectories of parameter $\delta$ and its derivative (Case of $v_2(t)$)

Figure 2.6: Step response and control input of different LPV controllers (Case of $v_2(t)$)
(a) step response with disturbance

(b) control force of case 1

(c) control force of case 3

(d) control force of case 5

Figure 2.7: Step response and control input of different LPV controllers with disturbance (Case of $v_1(t)$)
Chapter 3

Control of LFT systems using CHLF

In this chapter, we will study stability and performance properties of LFT systems from duality perspective. When the stability condition seems difficult to solve for original system, resorting to its dual formulation might be advantageous. Once a Lyapunov function was found in the dual space of an LFT system, then its conjugate function can be used for analysis purpose of the original LFT system. This provides us alternative approach to search for a suitable Lyapunov function for stability and performance enhancement.

First, the equivalency between linear differential inclusions (LDIs) and a class of LFT systems is investigated. The control development of LFT systems can be translated to the development on LDIs. Secondly, a new control synthesis approach will be developed for robust and gain-scheduling control of LFT systems. Using a convex hull Lyapunov function and a maximum function of a family of quadratic functions, the control synthesis conditions can be formulated as a set of LMIs plus scalar variables and solved by linear search and LMI optimization techniques. Different from previous LFT control techniques, our proposed approach results in nonlinear parameter-dependent controllers for better controlled performance. The performance improvement of the new control design technique will be demonstrated using a simple example.
The chapter is organized as follows: Section 3.2 provides some preliminary results for this research. The analysis of LFT systems through duality is studied in Section 3.3. In Sections 3.4 and 3.5, robust state feedback and gain-scheduling output feedback control of LFT systems are addressed separately. The control synthesis conditions are formulated as LMIs with a set of scalar variables using convex hull Lyapunov function and maximum functions. Section 3.6 uses a simple second-order example to demonstrate the proposed control design approach for LFT systems. Finally, the chapter concludes in Section 3.7.

3.1 Introduction

Approximating a LFT system in terms of linear differential inclusions (LDIs) is an alternative way to solve the LFT control problem. However, the effectiveness of the LDI approach depends on how accurate the approximation is. For a class of LFT systems, the LDI representation can be equivalent to the original LFT systems [8]. This exciting equivalence may lead to an alternative way to analyze and synthesize the LFT systems, which may lead to a better result.

Another important issue in the LDI approach is what kind of Lyapunov function will be used for analyzing the LDI systems. Like early development of LFT control, a single quadratic Lyapunov function can be used to solve the problem, which was completed with the LMI optimization techniques [8, 58]. Some literatures has indicated that search of Lyapunov functions should be widened beyond single quadratic forms [12, 40, 101]. In recent years, more research has been taken into account of searching non-single quadratic Lyapunov functions, such as piecewise quadratic Lyapunov functions [98, 41], homogeneous polynomial Lyapunov functions [12, 40, 101] and convex hull Lyapunov functions [29, 36].

A convex hull Lyapunov function based on a set of quadratic functions is first introduced in [36], which is motivated by problems of estimating the domain of attraction and constructing controllers to enlarge the domain of attraction. A nice feature of this type of functions is that it is continuously differentiable, which makes it possible to construct continuous feedback laws based on the gradient of the function or a given set of linear feedback laws. This kind of Lyapunov function has been applied to constrained control systems [36, 37, 38]. The application to stability analysis of LDI systems is also presented in [29].
Duality is a well established concept in linear time invariant (LTI) systems. For example, the dual system of $\dot{x} = Ax$ is $\dot{\xi} = A^T \xi$; the pair $(A, B)$ is stabilizable if and only if $(A^T, B^T)$ is detectable; $\|D + C(sI - A)^{-1}B\|_\infty = \|D^T + B^T(sI - A^T)^{-1}C^T\|_\infty$; and so on. The system $\dot{x} = Ax$ is stable if and only if there is a positive definite matrix $P$ such that $A^T P + PA < 0$. Moreover, $A^T P + PA < 0$ is equivalent to $AP^{-1} + P^{-1}A^T < 0$, which is a stability condition for the dual system $\dot{\xi} = A^T \xi$. From convex analysis, it is well-known that $\frac{1}{2} \xi^T P^{-1} \xi$ is a conjugate function of $\frac{1}{2} x^T Px$. These nice dual properties of LTI systems motivate us to study the duality properties of LFT systems and use the convex hull Lyapunov function for LFT analysis and synthesis problems. Our major contribution is to apply convex hull Lyapunov function to LFT system control and extend the approach to solve output feedback synthesis problem.

3.2 Preliminaries

In this section, we will provide some preliminary results relevant to our subsequent development, including some useful lemmas, duality of LFT systems and CHLF. The first result states that a set of LFT matrices is solely characterized by its vertices, and its proof can be found in [8].

**Lemma 4**  Given $\Theta = \{\text{diag} \{\theta_1, \theta_2, \cdots, \theta_s\} : \theta_i \in C^1(R_+, R)\}$, define two sets as

\[ \Omega = \{ A + B\Theta(I - D\Theta)^{-1}C : \Theta \in \Theta, \ |\Theta| \leq 1 \}, \]

\[ \Sigma = \{ A + B\Theta(I - D\Theta)^{-1}C : \Theta \in \Theta, \ |\theta_i| = 1, \ i \in I[1, s] \}. \]

Then $\Omega = \text{Co} \Sigma$, where Co stands for convex hull.

The second lemma can be shown through Schur complement and simple argument.

**Lemma 5**  Given a positive definite function $h(x) = \max_{i \in I[1, n]} h_i(x)$, and positive matrices $X, Y > 0$. Then the inequality

\[ 2\partial h_i(x)Ax + x^T C^T X C x + \partial h_i(x)BYB^T \partial^T h_i(x) < 0 \]
whenever \( h_i(x) \geq h_j(x), \forall j \in I[1,n], \) is necessary and sufficient condition for

\[
2\partial h(x)Ax + x^T C^T X Cx + \partial h(x)BY B^T \partial^T h(x) < 0
\]

to hold for all \( x \in \mathbb{R}^n \).

where \( \partial \) means subdifferential.

### 3.2.1 Duality of LFT Systems and Conjugate Functions

Consider an LFT system given by

\[
\begin{bmatrix}
\dot{x} \\
q \\
e
\end{bmatrix} = \begin{bmatrix}
A & B_0 & B_1 \\
C_0 & D_{00} & D_{01} \\
C_1 & D_{10} & D_{11}
\end{bmatrix} \begin{bmatrix}
x \\
p \\
d
\end{bmatrix}, \tag{3.1}
\]

\[
p = \Theta q, \tag{3.2}
\]

where \( x, \dot{x} \in \mathbb{R}^n, d \in \mathbb{R}^n \) is the disturbance, \( e \in \mathbb{R}^n \) is the controlled output, \( p, q \in \mathbb{R}^n \) are the pseudo-input and output. The LFT representation is well-posed, i.e., \((I - D_{00}\Theta)\) is invertible for any allowable parameter values. The time-varying parameter \( \Theta \) obeys the following structure

\[
\Theta = \{ \text{diag} \{ \theta_1, \theta_2, \ldots, \theta_s \} : \theta_i \in C^1(\mathbb{R}_+, \mathbb{R}), |\theta_i| \leq 1, i \in I[1,s] \}. \tag{3.3}
\]

Different from other chapters, we assume \( \Theta \) is in scalar diagonal form. For this class of LFT systems, we can convert them to LDI systems by using Lemma 4.

By absorbing \( \Theta \) into the state space, the LFT system (3.1)-(3.2) can be rewritten as

\[
\begin{bmatrix}
\dot{x} \\
e
\end{bmatrix} = \begin{bmatrix}
A(\Theta) & B_1(\Theta) \\
C_1(\Theta) & D_{11}(\Theta)
\end{bmatrix} \begin{bmatrix}
x \\
d
\end{bmatrix}. \tag{3.4}
\]

From Lemma 4, the LPV system (3.4) can be represented by

\[
\begin{bmatrix}
\dot{x} \\
e
\end{bmatrix} \in \text{Co} \left\{ \begin{bmatrix} A_p & B_{1,p} \\ C_{1,p} & D_{11,p} \end{bmatrix} \right\} \begin{bmatrix}
x \\
d
\end{bmatrix},
\]
which is an LDI at the vertices of $\Theta$, where $p$ is the index of parameter set’s vertices. Therefore, 
\[
\begin{bmatrix}
A_p & B_{1,p} \\
C_{1,p} & D_{11,p}
\end{bmatrix}
\]
are the vertical values of
\[
\begin{bmatrix}
A(\Theta) & B_1(\Theta) \\
C_1(\Theta) & D_{11}(\Theta)
\end{bmatrix}.
\]
As shown in [55], the dual system of original LFT system (3.1)-(3.2) is
\[
\begin{bmatrix}
\dot{\xi} \\
\dot{\tilde{q}} \\
z
\end{bmatrix} =
\begin{bmatrix}
A^T & C^T_0 & C^T_1 \\
B^T_0 & D^T_{00} & D^T_{01} \\
B^T_1 & D^T_{10} & D^T_{11}
\end{bmatrix}
\begin{bmatrix}
\xi \\
\tilde{p} \\
w
\end{bmatrix},
\]
(3.5)
where $\tilde{\Theta}$ has the same structure as $\Theta$. Similarly, one can express (3.5)-(3.6) as an LDI
\[
\begin{bmatrix}
\dot{\xi} \\
z
\end{bmatrix} \in \text{Co} \left\{ \begin{bmatrix}
A^T_p & C^T_{1,p} \\
B^T_{1,p} & D^T_{11,p}
\end{bmatrix} \right\}
\begin{bmatrix}
\xi \\
w
\end{bmatrix}
\]
using the convex hull of $\tilde{\Theta}$ vertices.

Next, we examine a control system with feedback structure in Fig. 3.1(a) where both the plant $G$ and controller $K$ are assumed to have the same LFT parameter dependency on a block structured $\Theta$. The dual system of above system is shown in Fig. 3.1(b) where both plant $G^T$ and controller $K^T$ are the dual systems of the original $G$ and $K$. Then $K$ stabilizes $G$ with respect to the block structure $\Theta_{cl} = \text{diag}\{\Theta, \Theta\}$ if and only if $K^T$ stabilizes $G^T$ with respect to the same block structure $\tilde{\Theta}_{cl} = \text{diag}\{\tilde{\Theta}, \tilde{\Theta}\}$.

![Figure 3.1](image_url)

Figure 3.1: Duality of closed-loop LFT system.
be found in [73, 9]. Given any function \( f : \mathbb{R}^n \to \mathbb{R} \), its conjugate function is defined as

\[
f^*(\xi) = \sup_{x \in \mathbb{R}^n} \{ \xi^T x - f(x) \}
\]

for \( \xi \in \mathbb{R}^n \). In this research, we are mainly interested in the functions \( f \) that are convex, positive definite and positively homogeneous of degree \( p > 1 \). For this class of functions, we have following properties about its conjugate functions:

I. \( f^*(\xi) \) is finite for every \( \xi \in \mathbb{R}^n \),

II. \( f^* \) is a convex, positive definite, and positively homogeneous of degree \( q > 1 \) where \( 1/p + 1/q = 1 \),

III. \( f^*(\alpha \xi) = \alpha^q f^*(\xi) \),

IV. For \( \alpha > 0 \), the conjugate of \( g(x) = \alpha f(x) \) is \( g^*(\xi) = \alpha f^*(\xi/\alpha) \),

V. \( \xi^T \in \partial f(x) \) if and only if \( x^T \in \partial f^*(\xi) \).

The usefulness of conjugate Lyapunov function has been shown in studying stabilization problem for saturated control systems in [29]. Consider a linear differential inclusion (LDI)

\[
\begin{bmatrix} \dot{x} \\ e \end{bmatrix} \in \text{Co} \left\{ \begin{bmatrix} A_p & B_p \\ C_p & D_p \end{bmatrix} \right\} \begin{bmatrix} x \\ d \end{bmatrix},
\]

where the subscript \( p \) stands for the vertices of the LDI and \( p \in I[1, P] \). Its dual system is also an LDI system in the form of

\[
\begin{bmatrix} \dot{\xi} \\ z \end{bmatrix} \in \text{Co} \left\{ \begin{bmatrix} A_p^T & C_p^T \\ B_p^T & D_p^T \end{bmatrix} \right\} \begin{bmatrix} \xi \\ w \end{bmatrix},
\]

and \( \text{dim}(x) = \text{dim}(\xi), \text{dim}(e) = \text{dim}(w), \text{dim}(d) = \text{dim}(z) \). We then have the following theorem that establishes the equivalent performance analysis conditions in the original space and its dual space.

**Theorem 2** Given \( \gamma > 0 \). Let \( V : \mathbb{R}^n \to \mathbb{R} \) be a closed proper convex, positive definite, positively homogeneous function of degree \( p > 1 \), and \( V^* \) is its conjugate function. Then,
the condition

\[
\begin{bmatrix}
2\partial V(x)A_p x & \partial V(x)B_p & x^T C_p^T \\
B_p^T \partial V(x) & -\gamma I & D_p^T \\
C_p x & D_p & -\gamma I
\end{bmatrix} < 0, \quad p \in I[1, P] \tag{3.7}
\]

is equivalent to

\[
\begin{bmatrix}
2\partial V^*(\xi)A_p^T \xi & \partial V^*(\xi)C_p^T \xi^T B_p \\
C_p \partial V^*(\xi) & -\gamma I & D_p \\
B_p^T \xi & D_p^T & -\gamma I
\end{bmatrix} < 0, \quad p \in I[1, P]. \tag{3.8}
\]

Moreover, if one of the above conditions satisfied and \( V(x) \) is differentiable, then \( \|e\|_2 < \gamma \|d\|_2 \).

**Proof:** \( V \) is a closed proper convex function, range \( \partial V^* = \text{dom} \ \partial V = \text{range} \ \partial V = \text{dom} \ \partial V^* = \mathbb{R}^n \), which means any \( x \in \mathbb{R}^n \) can be represented by \( \partial V^* \) for all \( x^T \in \partial V^*(\xi) \).

\((\Rightarrow)\) Pick any \( x \neq 0 \) and \( \xi^T \in \partial V(x) \). Then \( \xi \neq 0 \), since \( 0 \in \partial V(x) \) would imply \( x \) minimizes \( V(x) \). Thus

\[
\begin{bmatrix}
2\xi^T A_p x & \xi^T B_p & x^T C_p^T \\
B_p^T \xi & -\gamma I & D_p^T \\
C_p x & D_p & -\gamma I
\end{bmatrix} < 0. \tag{3.9}
\]

From the property (V) of conjugate functions, \( \xi^T \in \partial V(x) \) implies \( x^T \in \partial V^*(\xi) \). Then (3.8) can be derived by replacing \( x \) in (3.9) with \( \partial V^*(\xi) \), since condition (3.9) is true for all \( x^T \in \partial V^*(\xi) \).

\((\Leftarrow)\) vice versa.

For the proof of the last claim, we define \( \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \text{Co} \left\{ \begin{bmatrix} A_p & B_p \\ C_p & D_p \end{bmatrix} \right\} \). Note that condition (3.7) implies

\[
\begin{bmatrix}
2\partial V(x)A x & \partial V(x)B & x^T C^T \\
B^T \partial V(x) & -\gamma I & D^T \\
C x & D & -\gamma I
\end{bmatrix} < 0 \tag{3.10}
\]
Inequality (3.10) implies that
\[
\begin{bmatrix}
-\gamma I & D^T \\
D & -\gamma I
\end{bmatrix} < 0
\]
therefore \((\gamma I - \gamma^{-1}D^TD)\) is positive definite, so we can always take its inverse and matrix square root. Define \(\tilde{d}(t) := \gamma^{1/2}d(t),\) and \(\tilde{e}(t) := \gamma^{-1/2}e(t),\) then we can write the state-space representation of the scaled LDI system as,
\[
\begin{bmatrix}
\dot{x} \\
\dot{\tilde{e}}
\end{bmatrix} = \begin{bmatrix} A & \tilde{B} \\
\tilde{C} & \tilde{D}
\end{bmatrix} \begin{bmatrix} x \\
\tilde{d}
\end{bmatrix} \tag{3.11}
\]
where \(\tilde{B} := \gamma^{-1/2}B, \tilde{C} := \gamma^{-1/2}C\) and \(\tilde{D} := \gamma^{-1}D.\) Define the unitary transformation
\[
\begin{bmatrix}
d \\
\hat{e}
\end{bmatrix} = \begin{bmatrix} \bar{D}^T & [I - \bar{D}\bar{D}^T]^{1/2} \\
[I - \bar{D}\bar{D}^T]^{1/2} & -\bar{D}
\end{bmatrix} \begin{bmatrix} \tilde{e} \\
\tilde{d}
\end{bmatrix}.
\]
Note that \(\hat{e}^T\hat{e} + \hat{d}^T\hat{d} = \tilde{e}^T\tilde{e} + \tilde{d}^T\tilde{d}.\) Applying the above transformation to the LDI system (3.11), we get
\[
\begin{bmatrix}
\dot{x} \\
\dot{\hat{e}}
\end{bmatrix} = \begin{bmatrix} \hat{A} & \hat{B} \\
\hat{C} & 0
\end{bmatrix} \begin{bmatrix} x \\
\tilde{d}
\end{bmatrix}
\]
where
\[
\hat{A} := A + \gamma^{-2}B(I - \gamma^{-2}D^TD)^{-1}D^T C,
\]
\[
\hat{B} := \gamma^{-1/2}B(I - \gamma^{-2}D^TD)^{-1/2},
\]
\[
\hat{C} := \gamma^{-1/2}[I + \gamma^{-2}D(I - \gamma^{-2}D^TD)^{-1}D^T]^{1/2}C.
\]
Therefore, the inequality (3.10) is equivalent to
\[
\begin{bmatrix}
2\partial V(x)\hat{A}x & \partial V(x)\hat{B} & x^T\hat{C}^T \\
\hat{B}^T\partial^T V(x) & -I & 0 \\
\hat{C}x & 0 & -I
\end{bmatrix} < 0 \tag{3.12}
\]
By Schur complement, the above inequality is equivalent to
\[
2\partial V(x)\hat{A}x + \partial V(x)\hat{B}\hat{B}^T\partial^T V(x) + x^T\hat{C}^T\hat{C}x < 0,
\]
which implies
\[
2\partial V(x)\hat{A}x + \partial V(x)\hat{B}\hat{B}^T\partial^T V(x) + x^T\hat{C}^T\hat{C}x - \|\hat{B}^T\partial^T V(x) - \hat{d}\|^2 \\
= 2\partial V(x)(\hat{A}x + \hat{B}\hat{d}) + \hat{e}^T\hat{e} - \hat{d}^T\hat{d} < 0 \\
= 2\partial V(x)(\hat{A}x + \hat{B}\hat{d}) + \gamma^{-1}e^Te - \gamma d^Td < 0.
\]
If function $V(x)$ is differentiable, the subdifferential $\partial V(x)$ can be replaced by $\frac{\partial V(x)}{\partial x}$, which makes inequality (3.7)-(3.8) same as well known bounded real lemma. From the proposition of [51], the LDI system is uniformly asymptotically stable with $\|e\|_2 < \gamma \|d\|_2$. Q.E.D.

### 3.2.2 Convex Hull Lyapunov Function

For a set of positive definite matrices $Q_i \in \mathbb{R}^{n \times n}$, $i \in [1, n]$, we will define a convex hull Lyapunov function as

$$V(x) = \min_{\lambda \in \Lambda} \frac{1}{2} x^T Q^{-1}(\lambda) x,$$

where

$$Q(\lambda) = \sum_{i=1}^{n} \lambda_i Q_i, \quad \Lambda = \left\{ \lambda \in \mathbb{R}^n : \sum_{i=1}^{n} \lambda_i = 1, \lambda_i \geq 0, i \in [1, n] \right\}.$$

The convex hull Lyapunov function (CHLF) (3.13) has its level set as the convex hull of the level sets of $x^T P_i x$ [35]. Fig. 3.2 (a) shows a level set of a CHLF with two basis functions (two basis ellipsoids).

![Level set of convex hull Lyapunov function](a) Level set of convex hull Lyapunov function

![Level set of maximum Lyapunov function](b) Level set of maximum Lyapunov function

**Figure 3.2:** Level set of two conjugate Lyapunov functions

The optimal value of $\lambda$, where the function $V(x)$ achieves its minimum, is represented as $\lambda^* = \arg \min_{\lambda \in \Lambda} x^T Q^{-1}(\lambda) x$. As a continuous function of state $x$, $\lambda^*$ can be
calculated by the following optimization algorithm:

\[
\min_{\lambda} \quad \alpha \\
\text{s.t.} \quad \begin{bmatrix} \alpha & x^T \\ x & \sum_{i=1}^{n} \lambda_i R_i \end{bmatrix} \geq 0,
\]

\[
\sum_{i=1}^{n} \lambda_i = 1, \quad \lambda_i \geq 0.
\] (3.14)

Also, we have the following technical lemma for this convex hull Lyapunov function.

**Lemma 6** The function \( g(x) = Q^{-1}(\lambda^*)x \), where \( \lambda^* = \arg \min_{\lambda \in \Lambda} x^T Q^{-1}(\lambda)x \), is continuously differentiable and

\[
\frac{\partial g}{\partial x} = Q^{-1}(\lambda^*(x)).
\] (3.15)

Moreover, we have

\[
\frac{\partial V}{\partial x} = x^T Q^{-1}(\lambda^*(x)).
\]

**Proof:** \( g(x) \) is continuous and differentiable because the function (3.13) is continuous and differentiable [36]. Next, we need to show \( \frac{\partial g}{\partial x} = Q^{-1}(\lambda^*) \).

Suppose that \( g(x) \) is differentiable at \( x_0 \) with partial derivative \( \frac{\partial g}{\partial x} |_{x=x_0} \) and let \( k \) be given. Since \( g(kx) = kg(x) \) for all \( x \in \mathbb{R}^n \), then we have

\[
g(kx_0 + \Delta x) - g(kx_0) = k \left( g \left( x_0 + \frac{\Delta x}{k} \right) - g(x_0) \right)
\]

\[
= k \left( \frac{\partial g}{\partial x} |_{x=x_0} \right) \frac{\Delta x}{k} + o \left( \frac{\|\Delta x\|}{k} \right)
\]

\[
= \left( \frac{\partial g}{\partial x} |_{x=x_0} \right) \Delta x + o(\|\Delta x\|).
\]

It follows that \( \frac{\partial g}{\partial x} |_{x=kx_0} = \frac{\partial g}{\partial x} |_{x=x_0} \). So we only need to consider those \( x \) on the boundary of \( L_V(1) \), \( \partial L_V(1) \).

Pick up \( x_0, x \in \partial L_V(1) \) with \( x = x_0 + \Delta x \). To simplify the notation, denote \( P_0 \) as the optimal \( Q^{-1}(\lambda) \) at \( x_0 \) and \( P \) as the optimal \( Q^{-1}(\lambda) \) at \( x \). Assuming

\[
Px - P_0x = \hat{P}_0(x_0) \Delta x + o(\|\Delta x\|)
\]
In this proof, we only consider two quadratic functions, that is, \( Q(\lambda) = \lambda Q_1 + (1 - \lambda)Q_2 \).

For the case of \( n > 2 \), it can be proved in a similar way.

If \( \lambda^* = 0 \) or \( 1 \), we have \( \hat{P}_0 = 0 \) because only one quadratic function is involved in this area. When the optimizer \( \lambda^* \) of \( Q^{-1}(x_0) \) is located in the area of \([0, 1 - \delta]\), we perturb \( Q_1 \) by a small \( \delta > 0 \) to \( \hat{Q}_1 = (1 - \delta)Q_1 + \delta Q_2 \) and \( \hat{Q}_2 = Q_2 \) and obtain a perturbed function with \( \hat{\lambda} \in [0, 1] \)

\[
\hat{Q}(\hat{\lambda}) = \lambda \hat{Q}_1 + (1 - \lambda)\hat{Q}_2
\]
\[
= \hat{\lambda}(1 - \delta)Q_1 + (1 - \hat{\lambda}(1 - \delta))Q_2
\]
\[
= \lambda Q_1 + (1 - \lambda)Q_2, \quad \forall \lambda = \hat{\lambda}(1 - \delta) \in [0, 1 - \delta].
\]

This implies the following two sets \( \hat{Q} \) and \( Q \) are identical,

\[
\begin{align*}
\hat{Q} &= \{ Q(\hat{\lambda}) | \hat{Q}(\hat{\lambda}) = \hat{\lambda} \hat{Q}_1 + (1 - \hat{\lambda})\hat{Q}_2, \hat{\lambda} \in [0, 1] \}, \\
Q &= \{ Q(\lambda) | Q(\lambda) = \lambda Q_1 + (1 - \lambda)Q_2, \lambda \in [0, 1 - \delta] \},
\end{align*}
\]

Thus the optimal points from \( \hat{Q} \) and \( Q \) will be the same, \( \hat{Q}(\lambda^*(x_0)) = Q(\lambda^*(x_0)) \), which means the function value of \( Q(\lambda^*) \) remain unchanged after perturbation for \( \lambda^* \in [0, 1 - \delta] \).

For perturbed function, we have \( \hat{P}_0 = 0 \) at \( \hat{\lambda}^* = 1 \) by continuity. Since the function value remain unchanged in \( \lambda^* \in [0, 1 - \delta] \), we have \( \hat{P}_0 = 0 \) for unperturbed function at \( \lambda^* = 1 - \delta \) for arbitrarily small \( \delta \). So \( \hat{P}_0 = 0 \) for all \( \lambda^* \in [0, 1] \). Finally

\[
P x - P_0 x_0 = (P x - P_0 x) + (P_0 x - P_0 x_0) = \hat{P}_0(x_0)\Delta x + o(|\Delta x|) + P_0\Delta x
\]
\[
= P_0\Delta x + o(|\Delta x|),
\]

Therefore the partial derivative of \( g(x) \) at \( x_0 \) is \( P_0 \).

Finally, we get from eqn. (3.15)

\[
\frac{\partial V}{\partial x} = \frac{1}{2} \left\{ x^T \frac{\partial Q^{-1}(\lambda^*(x))x}{\partial x} + x^T Q^{-1}(\lambda^*(x)) \right\} = x^T Q^{-1}(\lambda^*(x)).
\]

Q.E.D.

Using the definition of conjugate function, it can be shown that the conjugate of convex hull Lyapunov function (3.13) is a maximum function in the form of

\[
V^*(\xi) = \max_{\lambda \in \Lambda} \frac{1}{2} \xi^T Q(\lambda) \xi = \max_{i \in [1, n]} \frac{1}{2} \xi^T Q_i \xi.
\]

(3.16)
The maximum Lyapunov function has its level set as the intersection of the level sets of $x^TP_xx$ [35]. Fig. 3.2 (b) shows a level set of a maximum Lyapunov function with two basis functions (two basis ellipsoids).

### 3.3 LFT Systems Analysis using CHLF

For analysis and control synthesis problems of LFT systems, one can often resort to its dual formulation with the help of conjugate Lyapunov functions. Generally speaking, the dual approach provides alternative solutions when the analysis and synthesis problems are hard to solve for original systems. The following theorem provides a dual analysis condition for the original systems (3.1)-(3.2).

**Theorem 3** Given $\gamma > 0$, suppose that there exist positive definite matrices $Q_i \in \mathbb{S}_{+}^{n_x \times n_x}$ and scalars $\epsilon_{ij,p} \geq 0$ for $i, j \in [1, n]$ and $p \in [1, P]$ such that

$$
\begin{bmatrix}
Q_i A_p^T + A_p Q_i + \sum_{j=1}^{n} \epsilon_{ij,p} (Q_i - Q_j) & Q_i C_{1,p}^T & B_{1,p} \\
C_{1,p} Q_i & -\gamma I & D_{11,p} \\
B_{1,p}^T & D_{11,p}^T & -\gamma I
\end{bmatrix} < 0, \quad \forall i, j \in [1, n] \tag{3.17}
$$

and for $p \in [1, P]$. Then the stability of LFT system (3.1)-(3.2) is established for any $\Theta \in \Theta$ by using the convex hull Lyapunov function (3.13), and $\|e\|_2 < \gamma \|d\|_2$.

**Proof:** Given the convex hull Lyapunov function $V(x)$ and its conjugate function $V^*(\xi)$, the LFT system (3.1)-(3.2) is stable and has its $L_2$ gain from $d$ to $e$ less than $\gamma$ if

$$
\begin{bmatrix}
2\partial V^*(\xi) A_p^T \xi & \partial V^*(\xi) C_{1,p}^T & \xi^T B_{1,p} \\
C_{1,p} \partial^T V^*(\xi) & -\gamma I & D_{11,p} \\
B_{1,p}^T \xi & D_{11,p}^T & -\gamma I
\end{bmatrix} < 0
$$
using Theorem 2. Note that $V^*(\xi) = \max_{i \in [1,n]} \frac{1}{2} \xi^T Q_i \xi$. Then by Lemma 5, the above inequality can be rewritten as

$$
[\xi]^T \begin{bmatrix}
Q_i A_p^T + A_p Q_i & Q_i C_{1,p}^T & B_{1,p} \\
C_{1,p} Q_i & -\gamma I & D_{11,p} \\
B_{1,p}^T & D_{11,p}^T & -\gamma I
\end{bmatrix} \begin{bmatrix}
\xi \\
I \\
I
\end{bmatrix} < 0 \tag{3.18}
$$

with constraints $\xi^T Q_i \xi - \xi^T Q_j \xi \geq 0$, $\forall i, j \in I[1,n]$. Finally, the sufficiency of (3.17) to guarantee eqn. (3.18) can be verified using S-procedure. \(Q.E.D.\)

Note that condition (3.17) has variables $Q_i$ and $\epsilon_{ij,p}$ involved, which are not jointly convex. A feasible approach to solve this matrix inequality is performing linear search over scalar variables $\epsilon_{ij,p}$. The resulting condition then becomes LMIs and can be solved efficiently.

### 3.4 Robust State Feedback Control

Given an open-loop linear parameter-dependent system (1.5)-(1.6), where $\Theta$ is defined as (3.3). The LFT system can also be transformed to its equivalent form (1.1), where two state-space representations have the relationship of (1.9). We assume that (A1)-(A4) hold. For simplicity, $D_{11}$ is assumed to be zero in the following derivation. The difference between $D_{11} = 0$ and $D_{11} \neq 0$ is mainly in the controller construction phase.

For robust state feedback control problem, it is assumed that $\Theta$ is an uncertainty, and all state information is available ($y = x$) for feedback control use.

#### 3.4.1 Robust State Feedback Control Law

We would like to design a nonlinear state feedback controller in the form of

$$
u = F(\lambda^*(x))x \tag{3.19}
$$

such that $L_2$ gain of the closed-loop system is minimized.

The state feedback law only depends on current optimal value of $\lambda$, which can be calculated through algorithm (3.14) online. Therefore, the feedback law (3.19) actually depends on current state $x$, which means it is a nonlinear state feedback controller.
With the state feedback control law (3.19), the state-space data of the closed-loop system is

\[
\begin{bmatrix}
\dot{x} \\
e
\end{bmatrix} = 
\begin{bmatrix}
A(\Theta) + B_2(\Theta)F(\lambda^*(x)) & B_1(\Theta) \\
C_1(\Theta) + D_{12}(\Theta)F(\lambda^*(x)) & D_{11}(\Theta)
\end{bmatrix}
\begin{bmatrix}
x \\
d
\end{bmatrix}
\in \text{Co}
\begin{bmatrix}
A_\Theta + B_{2,\Theta}F(\lambda^*(x)) & B_{1,\Theta} \\
C_{1,\Theta} + D_{12,\Theta}F(\lambda^*(x)) & D_{11,\Theta}
\end{bmatrix}
\begin{bmatrix}
x \\
d
\end{bmatrix},
\] (3.20)

### 3.4.2 Controller Synthesis

We have the following condition for synthesizing robust state feedback control law through the conjugate Lyapunov functions.

**Theorem 4** Suppose that there exist positive definite matrices \(Q_i \in \mathbb{S}_+^{n_x \times n_x}\), rectangular matrices \(Y_i \in \mathbb{R}^{n_u \times n_x}\), and scalars \(\epsilon_{ij,p} \geq 0\) for \(i, j \in I[1, n]\) and \(p \in I[1, P]\) such that

\[
\begin{bmatrix}
Q_iA_p^T + Y_i^TB_{2,p}^T + A_pQ_i + B_{2,p}Y_i \\
\quad + \sum_{j=1}^{n} \epsilon_{ij,p}(Q_i - Q_j) \\
C_{1,p}Q_i + D_{12,p}Y_i \\
B_{1,p}^T
\end{bmatrix} < 0, \quad i, j \in I[1, n]
\]

for all \(p \in I[1, P]\). Define

\[
Y(\lambda^*) = \sum_{i=1}^{n} \lambda_i^*Y_i, \quad Q(\lambda^*) = \sum_{j=1}^{n} \lambda_j^*Q_i
\]
where $\lambda^*$ is determined by

\[
\min_{\lambda} \quad \alpha \\
\text{s.t.} \quad \begin{bmatrix} \alpha & x^T \\ x & \sum_{i=1}^{n} \lambda_i Q_i \end{bmatrix} \geq 0, \\
\sum_{i=1}^{n} \lambda_i = 1, \quad \lambda_i \geq 0.
\]

Then the closed loop system (3.20) is stable $\forall \Theta \in \Theta$ and its $L_2$ gain is less than $\gamma$ under the state feedback law

\[
u = F(\lambda^*(x))x = Y(\lambda^*)Q^{-1}(\lambda^*)x.
\]

Moreover, if the vector function $\lambda^*(\cdot)$ is continuous, then $u = F(\lambda^*(x))x$ is a continuous feedback law.

**Proof:** From the optimization problem (3.22), it is clear that the optimizer $\lambda^*$ is the minimizer of function $V(x)$, i.e.

\[
\lambda^* = \arg \min_{\lambda \in \Lambda} x^T Q^{-1}(\lambda)x = \arg \min_{\lambda \in \Lambda} V(x).
\]

Define the 1-level set of $V$ as

\[
L_V := \{ x \in \mathbb{R}^n : V(x) \leq 1 \},
\]

for simplicity and without loss of generality, assume that $\lambda_i^* > 0$ for $i \in I[1, n_0]$ and $\lambda_i^* = 0$ for $i \in I[n_0, n]$, and denote the 1-level set of the ellipsoid $x^T Px$ as

\[
\mathcal{E}(P) := \{ x \in \mathbb{R}^n : x^T Px \leq 1 \},
\]

Then define a extreme point set in dom $V$

\[
X_i := \partial L_V \cap \partial \mathcal{E}(Q_i^{-1}) = \{ x \in \mathbb{R}^n : V(x) = x^T Q_i^{-1}x = 1 \}
\]

where $\cup_{i=1}^{n} X_i$ contains all the extreme points of $L_V$. We also define a set in dual space dom $V^*$ as

\[
\Xi_i := \{ \xi \in \mathbb{R}^n : \xi^T Q_i \xi = \max_{i \in I[1,n]} \frac{1}{2} \xi^T Q_i \xi = 1 \}
\]
Then $\bigcup_{i=1}^{n} \Xi_i = L_{V^*}$. For example, in two-dimensional case with two basis ellipsoids, as shown in Fig. 3.2, segment $AB$ is a extreme point set $X_1$ and segment $A'B'$ is a set in this dual space $\Xi_1$. By definition, the conjugate of $V(x_i)$ with $x_i \in X_i$ will be mapped into $V^*(\xi_i), \xi_i \in \Xi_i$.

$V(x_i)$ is a closed proper convex function, then range $\partial V(x_i) = \text{dom} \ V^*(\xi_i) = \Xi_i$. By property $V$, $x_i \in \partial V^*(\xi_i) \iff \xi_i \in \partial V(x_i)$. From [37], $x = \sum_{i=1}^{N} \lambda^* x_i$ and $\nabla V(x) = Q_i^{-1} x_i = Q(\lambda^*)^{-1} x$ for $i \in I[1, n_0]$. We also have the following observation,

$$F(\lambda^*) x = Y(\lambda^*)Q^{-1}(\lambda^*)x = \sum_{i=1}^{n} \lambda^*_i Y_i Q_i^{-1} x_i = \sum_{i=1}^{n} \lambda^*_i F_i x_i$$

For all $\xi_i \in \partial V(x_i)$, we have $Y_i \xi_i \in F_i x_i$.

Then by Lemma 5, inequality (3.21) can be written as

$$\begin{bmatrix} \xi_i \\ I \end{bmatrix}^T \begin{bmatrix} Q_i A_i^T + Y_i B_{2,p}^T \\ + A_i p Q_i + B_{2,p} Y_i \\ C_{1,p} Q_i + D_{12,p} Y_i \\ B_{1,p}^T \end{bmatrix} \begin{bmatrix} * \\ * \\ -\gamma I \\ * \end{bmatrix} \begin{bmatrix} \xi_i \\ I \end{bmatrix} < 0, \forall i \in I[1, N], p \in I[1, P]$$

(3.23)

Because (3.23) holds $\forall i \in I[1, N]$, $\xi_i$ can represent any $\xi \in \mathbb{R}^n$ with some index $i$. Then (3.23) holds $\forall \xi \in \partial V(x)$. Following the same procedure of Theorem 2, (3.23) is equivalent to

$$\begin{bmatrix} x_i^T A_i^T Q_i^{-1} x_i + x_i^T F_i^T B_{2,p} Q_i^{-1} x_i \\ + x_i^T Q_i^{-1} A_i p x_i + x_i^T Q_i^{-1} B_{2,p} F_i x_i \\ x_i^T Q_i^{-1} C_{1,p} x_i + x_i^T Q_i^{-1} D_{12,p} F_i x_i \\ B_{1,p}^T \\ D_{11,p}^T -\gamma I \end{bmatrix} < 0,$$

(3.24)

where (3.24) holds $\forall i \in I[1, N], p \in I[1, P]$. Replace $Q_i^{-1} x_i$ by $Q^{-1}(\lambda^*) x$ and summarize inequalities with coefficient $\lambda^*$, we get

$$\begin{bmatrix} x \\ I \end{bmatrix}^T \begin{bmatrix} (A_p + B_{2,p} F(\lambda^*))^T Q^{-1}(\lambda^*) \\ + Q^{-1}(\lambda^*)(A_p + B_{2,p} F(\lambda^*)) \\ Q^{-1}(\lambda^*)(C_{1,p} + D_{12,p} F(\lambda^*)) -\gamma I \\ B_{1,p}^T \end{bmatrix} \begin{bmatrix} * \\ * \\ -\gamma I \\ 0 \end{bmatrix} \begin{bmatrix} x \\ I \end{bmatrix} < 0.$$

(3.25)
This leads to
\[
2\partial V(x) \dot{x} + \gamma^{-1} e^T e - \gamma d^T d \\
= \dot{x}^T Q^{-1} x + x^T Q^{-1} \dot{x} + \gamma^{-1} e^T e - \gamma d^T d \\
\leq x^T [(A + B_2 F)^T Q^{-1} + Q^{-1} (A + B_2 F) + \gamma^{-1} (C_1 + D_{12} F)^T (C_1 + D_{12} F)] \\
+ \gamma^{-1} Q^{-1} B_1 B_1^T Q^{-1} x \\
< 0
\]
by Lemma 6 and Schur complement. Since the convex hull Lyapunov function is bounded from below and above for any \( \lambda \in \Lambda \), the closed loop system is indeed stable and has its \( L_2 \) gain less than \( \gamma \). \( Q.E.D. \)

The controller gain constructed in (3.19) depends on \( \lambda^* \), which can be solved from the optimization problem (3.22) either online or offline. Online solving LMIs is computationally expensive and have better accuracy. On the other hand, if solved offline, one can grid the entire state space and solve (3.22) at each gridding point. Then a look-up table can be built up for functional relation between the state \( x \) and \( \lambda^* \). When implementing the controller, \( \lambda^* \) will be chosen from the look-up table through interpolation. Offline method requires much less online computational effort but is less accurate.

### 3.5 Gain-Scheduling Output Feedback Control

When parameter \( \Theta \) is time-varying and measurable in real time, one can design a nonlinear output-feedback controller in the form of
\[
\begin{bmatrix}
\dot{x}_k \\
u
\end{bmatrix} =
\begin{bmatrix}
A_k(\Theta, \lambda^*) & B_k(\Theta, \lambda^*) \\
C_k(\Theta, \lambda^*) & D_k(\Theta, \lambda^*)
\end{bmatrix}
\begin{bmatrix}
x_k \\
y
\end{bmatrix}
\] (3.26)
for the open-loop LFT system (1.5)-(1.6). Note that the nonlinear controller gain depends on the parameter \( \Theta \), and the state information through function \( \lambda^* \) indirectly. Then the closed-loop system becomes
\[
\begin{bmatrix}
\dot{x}_c \\
e
\end{bmatrix} =
\begin{bmatrix}
A_d(\Theta, \lambda^*) & B_d(\Theta, \lambda^*) \\
C_d(\Theta, \lambda^*) & D_d(\Theta, \lambda^*)
\end{bmatrix}
\begin{bmatrix}
x_c \\
d
\end{bmatrix}
\] (3.27)
with the state-space data defined as
\[
\begin{bmatrix}
A_{cl}(\Theta, \lambda^*) & B_{cl}(\Theta, \lambda^*) \\
C_{cl}(\Theta, \lambda^*) & D_{cl}(\Theta, \lambda^*)
\end{bmatrix} = 
\begin{bmatrix}
A(\Theta) & 0 \\
0 & 0 \\
C_1(\Theta) & 0 \\
0 & D_{11}(\Theta)
\end{bmatrix} + 
\begin{bmatrix}
0 & B_2(\Theta) \\
I & 0 \\
0 & D_{12}(\Theta)
\end{bmatrix}
\begin{bmatrix}
A_k(\Theta, \lambda^*) & B_k(\Theta, \lambda^*) \\
C_k(\Theta, \lambda^*) & D_k(\Theta, \lambda^*)
\end{bmatrix} 
\begin{bmatrix}
0 & I & 0
\end{bmatrix}.
\]

### 3.5.1 Convex Hull and Maximum Lyapunov Function

For gain-scheduling control synthesis problem, we will use a convex hull function and a maximum function as follows.

\[ 
V(x) = \frac{1}{2} x^T R^{-1}(\lambda^*_V(x_k)) x, \quad \lambda^*_V = \arg\min_{\lambda^*_V \in \Lambda} x_k^T R^{-1}(\lambda^*_V) x_k, 
\]

\[ 
U(x_k) = \frac{1}{2} x_k^T S(\lambda^*_U(x_k)) x_k, \quad \lambda^*_U = \arg\max_{\lambda^*_U \in \Lambda} x_k^T S(\lambda^*_U) x_k = \arg\max_{\ell \in I[1,m]} x_k^T S_\ell x_k, 
\]

where \( x, x_k \) are the plant and the controller states, and

\[
R(\lambda_V) = \sum_{i=1}^{n} \lambda_{V_i} R_i, \quad S(\lambda_U) = \sum_{\ell=1}^{m} \lambda_{U_\ell} S_\ell.
\]

Note that functions \( R(\lambda^*_V) \) and \( S(\lambda^*_U) \) only depend on controller state \( x_k \). As a continuous function of state \( x_k \), \( \lambda^*_V \) can be calculated by the following optimization algorithm:

\[
\min_{\lambda_V} \alpha \\
\text{s.t.} \begin{bmatrix}
\alpha \\
x_k \sum_{i=1}^{n} \lambda_{V_i} R_i
\end{bmatrix} \geq 0, \\
\sum_{i=1}^{n} \lambda_{V_i} = 1, \quad \lambda_{V_i} \geq 0.
\]

On the other hand, \( \lambda^*_U \) has its \( \ell \)th element equal to 1 and others to be zero when \( j \)th quadratic function \( x_k^T S_\ell x_k \) is activated. Therefore the function \( \lambda^*_U(x_k) \) is piecewise constant with discontinuity.

The convex hull and maximum Lyapunov function we consider in this section is in the form of \( V_{cl}(\tilde{x}_{cl}) = \frac{1}{2} \tilde{x}_{cl}^T X_{cl}(x_k) \tilde{x}_{cl} \), where

\[
\tilde{x}_{cl} = \begin{bmatrix} x^T (x_k - x)^T \end{bmatrix}^T, \quad X_{cl}(x_k) = \begin{bmatrix} R^{-1}(\lambda^*_V) & \left( S(\lambda^*_U) - R^{-1}(\lambda^*_V) \right) \end{bmatrix} > 0,
\]
3.5.2 Controller Synthesis

The following theorem provides a sufficient condition for the existence of the non-linear control law (3.26) in a switched quasi-LPV form.

**Theorem 5** Suppose that there exist positive definite matrices $R_i \in \mathbb{S}^{n_x \times n_x}$, $i \in [1,n]$ and $S_\ell \in \mathbb{S}^{n_\ell \times n_\ell}$, $\ell \in [1,m]$, rectangular matrices $Y_i \in \mathbb{R}^{n_x \times n_\ell}$ and $Z_i \in \mathbb{R}^{n_\ell \times n_y}$ and scalars $\bar{\epsilon}_{ij,p} \geq 0$ for $i, j \in [1,n]$, $\hat{\epsilon}_{\ell k,p} \geq 0$ for $\ell, k \in [1,m]$ and $p \in [1,P]$ such that

\[
\begin{pmatrix}
R_i A_p^T + Y_i^T B_{2,p} + A_p R_i + B_{2,p} Y_i \\
\sum_{j=1}^{n} \epsilon_{ij,p} (R_i - R_j) \\
C_1,p R_i + D_{12,p} Y_i \\
B_{1,p}^T 
\end{pmatrix}
\begin{pmatrix}
R_i C_{1,p}^T + Y_i^T D_{12,p}^T B_{1,p} \\
-\gamma I \\
D_{11,p} \\
-\gamma I 
\end{pmatrix} < 0, \quad \forall i, j \in [1,n]
\]

(3.29)

\[
\begin{pmatrix}
A_p^T S_\ell + C_{2,p}^T Z_\ell + S_\ell A_p + Z_\ell C_2,p \\
\sum_{k=1}^{m} \hat{\epsilon}_{\ell k,p} (S_\ell - S_k) \\
B_{1,p}^T S_\ell + D_{12,p}^T Z_\ell \\
C_1,p 
\end{pmatrix}
\begin{pmatrix}
S_\ell B_{1,p} + Z_\ell D_{21,p} + C_{1,p}^T \\
-\gamma I \\
D_{11,p}^T \\
-\gamma I 
\end{pmatrix} < 0, \quad \forall \ell, k \in [1,m],
\]

(3.30)

\[
\begin{pmatrix}
R_i & I \\
I & S_\ell 
\end{pmatrix} > 0, \quad \forall i \in [1,n], \quad \ell \in [1,m],
\]

(3.31)

for all $p \in [1,P]$. Define

\[
Y(\lambda_V^*) = \sum_{i=1}^{n} \lambda_{V_i}^* Y_i, \quad Z(\lambda_U^*) = \sum_{\ell=1}^{m} \lambda_{U_\ell}^* Z_\ell,
\]

Then the closed loop system (3.27) is stable for all $\Theta \in \Theta$ and its $L_2$ gain is less than $\gamma$.
with an $n_x$th-order nonlinear output feedback control gain

\[ A_k(x_k, \Theta) = (S - R^{-1})^{-1} \{ A^T R^{-1} + S (A + B_2 F + LC_2) \]

\[ + \gamma^{-1} S (B_1 + LD_{21}) B_1^T R^{-1} + \gamma^{-1} C_1^T (C_1 + D_{12} F) \} , \quad (3.32) \]

\[ B_k(x_k, \Theta) = -(S - R^{-1})^{-1} SL , \quad (3.33) \]

\[ C_k(x_k, \Theta) = F , \quad (3.34) \]

\[ D_k(x_k, \Theta) = 0 , \quad (3.35) \]

where

\[ F(x_k, \Theta) = Y(\lambda^*_V) R^{-1}(\lambda^*_V) , \quad L(x_k, \Theta) = S^{-1}(\lambda^*_U) Z(\lambda^*_U) \]

**Proof:** Given the controller gain (3.32)-(3.35), we apply a similar transformation

\[ x_{cl} = T \tilde{x}_{cl} = \begin{bmatrix} I & 0 \\ I & I \end{bmatrix} \tilde{x}_{cl} \]

to the closed-loop states, where $x_{cl} = \begin{bmatrix} x^T \\ x_k^T \end{bmatrix}^T$. Define $W = (S - R^{-1})^{-1}$, then

\[
\begin{pmatrix}
T^{-1} A_{cl} T & T^{-1} B_{cl} \\
C_{cl} T & D_{cl}
\end{pmatrix}
= \begin{pmatrix}
T^{-1} & 0 \\
I & I
\end{pmatrix}
\begin{pmatrix}
A & B_2 F \\
-W S L C_2 & A + B_2 F + W S L C_2 + M
\end{pmatrix}
\begin{pmatrix}
B_1 \\
C_1
\end{pmatrix}
= : \begin{pmatrix}
\hat{A}_{cl} & \hat{B}_{cl} \\
\hat{C}_{cl} & 0
\end{pmatrix},
\]

where

\[
\hat{A}_{cl} = \begin{bmatrix}
A + B_2 F & B_2 F \\
M & A + W S L C_2 + M
\end{bmatrix}, \quad \hat{B}_{cl} = \begin{bmatrix}
B_1 \\
-(B_1 + W S L D_{21})
\end{bmatrix},
\]

\[
\hat{C}_{cl} = \begin{bmatrix}
(C_1 + D_{12} F) & D_{12} F
\end{bmatrix},
\]

\[
M = (W S - I)(A + B_2 F) + W A^T R^{-1} + \gamma^{-1} W S (B_1 + LD_{21}) B_1^T R^{-1} + \gamma^{-1} W C_1^T (C_1 + D_{12} F).\]
To prove the sufficiency, it is enough to show that $\tilde{A}_d$ is stable for any $\Theta \in \Theta$, and $\|e\|_2 < \gamma \|d\|_2$ when conditions (3.29)-(3.31) hold. To this end, we consider a Lyapunov function $V_d(\tilde{x}_d) = \frac{1}{2} \tilde{x}_d^T X_d(x_k) \tilde{x}_d$ in which

$$\tilde{x}_d = \begin{bmatrix} x^T (x_k - x)^T \end{bmatrix}^T, \quad X_d(x_k) = \begin{bmatrix} R^{-1}(\Lambda^*_V) & S(\Lambda^*_V) - R^{-1}(\Lambda^*_V) \end{bmatrix} > 0,$$

where $R(\Lambda^*_V), S(\Lambda^*_V)$ as defined in (3.28). Clearly, $X_d$ is positive definite and bounded from below and above because of the coupling condition (3.31). By Lemma 6 and completing square, we get

$$2\partial V_d(\tilde{x}_d) \dot{\tilde{x}}_d + \gamma^{-1} e^T e - \gamma d^T d$$

$$= 2(\dot{x}_k - \dot{\tilde{x}})^T S(\Lambda^*_V)(x_k - x) + 2\dot{x}_k^T R^{-1}(\Lambda^*_V)x_k$$

$$+ 2 \left( \dot{x}_k^T R^{-1}(\Lambda^*_V)x_k + x^T \frac{\partial R^{-1}(\Lambda^*_V)x_k}{\partial x_k} \dot{x}_k \right) + \gamma^{-1} e^T e - \gamma d^T d$$

$$= \dot{x}_d^T X_d \tilde{x}_d + \dot{x}_d^T X_d \dot{x}_d + \gamma^{-1} e^T e - \gamma d^T d$$

$$= \dot{x}_d^T X_d \tilde{x}_d + \dot{x}_d^T X_d \dot{x}_d + \gamma^{-1} X_d \dot{\tilde{x}}_d \dot{\tilde{x}}_d + 2d^T \dot{B}_d X_d \dot{x}_d - \gamma d^T d$$

$$= \dot{x}_d^T X_d \tilde{x}_d + \dot{x}_d^T X_d \dot{x}_d + \gamma^{-1} X_d \dot{B}_d \dot{B}_d^T X_d + \gamma^{-1} \dot{C}_d \dot{\tilde{C}}_d \dot{x}_d - \|\gamma^{-1} \dot{B}_d^T X_d \dot{x}_d - \gamma \frac{1}{2} \|^2.$$  

Note that

$$\dot{x}_d^T (x_d \dot{A}_d + \dot{A}_d X_d + \gamma^{-1} X_d \dot{B}_d \dot{B}_d^T X_d + \gamma^{-1} \dot{C}_d \dot{\tilde{C}}_d) \dot{x}_d$$

$$= \dot{x}_d^T \begin{bmatrix} I & I \\ 0 & I \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} \begin{bmatrix} I & I \\ 0 & I \end{bmatrix} \dot{x}_d$$

$$= x_d^T \Delta_1 x_k + (x_k - x)^T \Delta_2 (x_k - x),$$  \hspace{1cm} (3.36)

where

$$\Delta_1 = R^{-1}(A + B_2 F) + (A + B_2 F)^T R^{-1} + \gamma^{-1} R^{-1} B_1 B_1^T R^{-1} + \gamma^{-1} (C_1 + D_{12} F)^T (C_1 + D_{12} F),$$

$$\Delta_2 = S(A + L C_2) + (A + L C_2)^T S + \gamma^{-1} S(B_1 + LD_{21})(B_1 + LD_{21})^T S + \gamma^{-1} C_1^T C_1.$$

By Schur complement, $x_k^T \Delta_1 x_k < 0$ is equivalent to

$$\begin{bmatrix} x_k \\ I \end{bmatrix}^T \begin{bmatrix} \left( A_p + B_{2,p} F \right)^T R^{-1} & (C_{1,p} + D_{12,p} F)^T R^{-1} B_{1,p} \\ + R^{-1}(A_p + B_{2,p} F) & \left( C_{1,p} + D_{12,p} F \right)^T R^{-1} B_{1,p} \end{bmatrix} \begin{bmatrix} x_k \\ I \end{bmatrix} < 0.$$  \hspace{1cm} (3.37)
Following the procedure of Theorem 4 proof, the sufficient condition to guarantee $x_k^T \Delta_1 x_k < 0$ is (3.29).

Similarly, eqn. (3.30) implies $(x_k - x)^T \Delta_2 (x_k - x) < 0$ using Lemma 5 and $S$-procedure. Then we confirmed that (3.36) is negative-definite. Therefore the closed loop system is stable and $\|e\|_2 < \gamma \|d\|_2$. \quad Q.E.D.

Note that the number $n$ of quadratic functions in $V(x)$, and $m$ of $U(x)$ is not necessary to be same. Although the synthesis condition in Theorem 5 is also sufficient, it is generally less conservative than the single quadratic Lyapunov function case. When $n = m = 1$, Theorem 5 recovers the single quadratic Lyapunov function result in [6] as a special case.

Since the output feedback law depends on both the current controller state and scheduling parameter, it can be thought as a nonlinear gain-scheduling controller. Moreover, the parameter $\lambda_V^*$ is a continuous function of state $x_k$ and the parameter value of $\lambda_U^*$ is discontinuous, this nonlinear controller can be thought as a switched LPV controller with $\lambda_U^*$ serving as the switching signal. Unlike parameter-dependent Lyapunov function cases [97, 95], the controller gain in our proposed approach does not require parameter rates information, which is desirable from control implementation point of view. The performance improvement mainly benefits from nonlinear nature of the controller. Same as robust state feedback case, $\lambda_V^*$ and $\lambda_U^*$ can be computed either online or offline.

### 3.6 Example

In this section, a second order LFT plant will be used to demonstrate the proposed approach. Both robust state feedback and gain-scheduling output feedback control will be designed for the LFT system. For robust state feedback case, a disturbance rejection problem will be considered. For gain-scheduled output feedback case, a tracking problem
will be considered. The plant is in the form of

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
p
\end{bmatrix}
= \begin{bmatrix}
-1 & -0.8 & 1 & 0 & 0 \\
1 & -1.6 & 0 & -3 & -2 \\
0 & 1 & 0.5 & 0 & 1 \\
1 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
p \\
d \\
u
\end{bmatrix}
\]

(3.38)

\[p = \Theta q.\] (3.39)

The plant has LFT dependency of a scalar time-varying parameter \( \Theta \in [-1, 1] \). The parameter trajectory is chosen as Fig. 3.3.

![Parameter trajectory](image.png)

Figure 3.3: Parameter trajectory

### 3.6.1 Robust State Feedback Control

In this section, robust state feedback control problem is considered, where \( \Theta \) is treated as an uncertainty. We assume all the states are measurable with no noise, therefore the measurement \( y \) will be ignored. The error output is the first state. The state feedback
control law only depends on the states of plant states and weighting functions.

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
q \\
e
\end{bmatrix}
= 
\begin{bmatrix}
-1 & -0.8 & 1 & 0 & 0 \\
1 & -1.6 & 0 & -3 & -2 \\
0 & 1 & 0.5 & 0 & 1 \\
1 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
p \\
d \\
u
\end{bmatrix}
\] (3.40)

\[p = \Theta q.\] (3.41)

The plant has LFT dependency of a scalar time-varying parameter \( \Theta \in [-1,1] \) and the trajectory is shown in Fig. 3.3. The control design objectives include minimizing the error output under disturbances with reasonable control force. They are quantified by rational weighting functions on the error output and control input channels.

The weighting functions are chosen to be

\[W_e(s) = \frac{0.1s + 4}{s + 0.4}, \quad W_u(s) = \frac{100s + 100}{s + 125}.\]

Three different robust controllers will be examined:

1. Quadratic Lyapunov function (QLF) with scaling matrix \([1]\);
2. Single quadratic Lyapunov function using condition (3.21);
3. Convex hull Lyapunov function with two quadratic functions

\[V(x) = \min_{\lambda \in \Lambda} \frac{1}{2} x^T [\lambda Q_1 + (1 - \lambda) Q_2]^{-1} x,\]

using condition (3.21).

The \(L_2\) gains calculated from these approaches are listed in the following table. Case 3 is solved by optimizing the synthesis condition (3.21) with \(\epsilon_{12,1} = 1.2, \epsilon_{21,2} = 5.4\) and all other \(\epsilon_{ij,p} = 0\). The linear search range of \(\epsilon\) is \([0,10]\) with accuracy of 0.1. Although the linear search method derives a suboptimal result, it is still much better than existing approaches.

As shown in table 3.1, the robust control performance can be significantly improved using our proposed approach. The CPU time of case 3 is based on a known suboptimal \(\epsilon\), which excludes the computational time spending on linear search. The linear search may consume much more computational effort, which depends on the search range.
Table 3.1: Optimal performance of robust state feedback control.

<table>
<thead>
<tr>
<th>Method</th>
<th>$\mathcal{L}_2$ gain</th>
<th>CPU time (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>QLF with scaling matrix</td>
<td>3.922</td>
<td>1.453</td>
</tr>
<tr>
<td>Single QLF</td>
<td>3.921</td>
<td>0.266</td>
</tr>
<tr>
<td>Convex hull Lyapunov function</td>
<td>3.343</td>
<td>0.375</td>
</tr>
</tbody>
</table>

For simulation purpose, we choose the disturbance input as an impulse. The time domain simulation of robust controlled response is shown in Figure 3.4. The dashed line corresponds to the controller using quadratic Lyapunov function with scaling matrix [1], and the solid line is for our approach. Case 2 has same simulation result as case 1. Note that online implementation of the proposed controller is used to get better comparison with other approaches. Although the response of control input has a little bit larger magnitude, the error output using proposed robust controller is much smaller.

Fig. 3.5 shows the trajectory of optimal $\lambda_1$, which is varying between 0 and 1. Optimal $\lambda$ can be either calculated online or looked up from a pre-calculated table. Online calculation may need large computational effort when implementing controller, especially in large systems.

(a) tracking output          
(b) control force

Figure 3.4: Impulse response and control input of different robust controllers.
3.6.2 Gain-scheduled Output Feedback Control

Next, we will design a gain-scheduled output feedback controller for the LFT system. For this study, $\Theta$ is a scheduling parameter, which is measurable in real time and can be scheduled in the controller gains. We want the measured output $y$ to track a reference signal $\text{ref}$. The system structure is shown in Fig. 3.6.

![System diagram](image_url)

**Figure 3.6: System structure of gain-scheduled output feedback**

The weighting functions have been modified to

$$W_e(s) = \frac{0.003s + 1.2}{s + 0.004}, \quad W_u(s) = \frac{s + 1}{s + 12500}. $$

Four gain-scheduling control approaches will be studied:
1. Quadratic Lyapunov function with scaling matrix [1];

2. Quadratic Lyapunov function with full-block multiplier [78];

3. Single quadratic Lyapunov function using conditions (3.29)-(3.31);

4. Convex hull Lyapunov function $V(x)$ with two quadratic functions and single quadratic function $V(x)$

\[
V(x) = \frac{1}{2} x^T [\lambda^*_V R_1 + (1 - \lambda^*_V) R_2]^{-1} x, \\
\lambda^*_V = \arg \min_{\lambda_V \in \Lambda} \frac{1}{2} x_k^T [\lambda_V R_1 + (1 - \lambda_V) R_2]^{-1} x_k, \\
U(x_k) = \frac{1}{2} x_k^T S x_k,
\]

using conditions (3.29)-(3.31).

The $L_2$ gains of different gain-scheduling control approaches are compared in the following table.

<table>
<thead>
<tr>
<th>Method</th>
<th>$L_2$ gain</th>
<th>CPU time (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>QLF with scaling matrix</td>
<td>3.523</td>
<td>1.391</td>
</tr>
<tr>
<td>QLF with full-block multiplier</td>
<td>3.522</td>
<td>0.906</td>
</tr>
<tr>
<td>Single QLF</td>
<td>3.521</td>
<td>0.328</td>
</tr>
<tr>
<td>Convex hull Lyapunov function</td>
<td>3.000</td>
<td>0.406</td>
</tr>
</tbody>
</table>

In case 4, the gain-scheduling control is designed by solving conditions (3.29)-(3.31) with $\bar{\epsilon}_{12,1} = 1.0, \bar{\epsilon}_{21,2} = 7.0$ and all other $\bar{\epsilon}_{ij,p} = 0$. The sub-optimal value of $\bar{\epsilon}$ is calculated by linear search in the range of $[0, 10]$ with accuracy of 0.1. Since $U(x)$ is a single quadratic function, scalars $\hat{\epsilon}_{ij,p}$ are not necessary for the solution.

From table 3.2, it can be seen that the $L_2$ gain of gain-scheduling control performance can also be significantly improved by our proposed approach. The CPU time of case 4 is based on a known $\bar{\epsilon}$, which excludes the computational time spending on linear search. The linear search may consume much more computational effort, which depends on the search range.

In time-domain simulation, we use step input as reference and the same parameter trajectory in 3.3 as before. The time domain response of different controllers are given in
Fig. 3.7 with dashed line for the gain-scheduled control using quadratic Lyapunov function with scaling matrix [1], and solid line for our proposed approach. Case 2 and case 3 have same simulation results as case 1. Again, the proposed gain-scheduling controller provides better tracking capability with reasonable larger control force. The proposed controller achieves not only faster transition, but also less steady state error. All the controllers are synthesized based on same weighting functions.

Figure 3.7: Step response and control input of different gain-scheduled controllers.

For a fair comparison, it should be pointed out that the computational time of proposed robust and gain-scheduling control approach is close to other approaches for the scaler parameter $\Theta$ case as shown tables 3.1 and 3.2. However, if $\Theta$ contains more parameters $\theta_i$, the vertex points of the set $\Theta$ will increase exponentially. This will result in large number of LMIs and require heavier computational cost. In addition, increasing the number of quadratic functions in convex hull Lyapunov function and the maximum function also leads to increased number of LMIs.

Fig. 3.8 provides the comparison of two different Lyapunov function level set. Because the close loop system is eighth order, it is difficult to plot a 8-dimensional level set. Fig. 3.8 is the intersection of the close loop level set 1 with the plant state plane. The Lyapunov function level set of proposed method is shown much larger than the other methods, which is the reason why our new controller can perform better.

Next we consider the step response with disturbance. A uniform random noise with
magnitude 0.5 is chosen for disturbance $d$ and a uniform random noise with magnitude 0.05 is chosen for sensor noise $d_n$. As shown in Fig. 3.9 (a), our proposed controller still performs better under disturbance. Fig. 3.9 (b)-(c) shows the control inputs from different controllers. Our proposed controller is less sensitive to the disturbance. The noise magnitude of our controller output is much smaller than other controllers under the same measurement noise.

### 3.7 Summary

In this chapter, we first studied parameter-dependent LFT systems using duality theory. It has been shown that conjugate Lyapunov functions provide attractive alternative for LFT stability and performance analysis purpose. Then we employed a convex hull Lyapunov function and a maximum function of a family of quadratic functions for control synthesis problems. Both robust state feedback and gain-scheduling output feedback control of LFT systems have been considered. The control synthesis conditions were formulated and solved using LMI optimizations combined with linear search over scalar variables. The resulting nonlinear controllers depend on plant/controller states and the scheduling parameters, but not on the parameter rate information. As demonstrated on a simple example, the newly proposed control design approach provides better controlled performance than existing LPV controllers and is relatively easy to implement.
Figure 3.9: Step response of different controllers with disturbance.
Chapter 4

Control of LFT systems through online switching control

The performance of LPV systems can be improved by partitioning the parameter set into several subset, and designing sub-controller based on each subset. The critical issue in this design method is the switching between subcontrollers and how to assign different Lyapunov functions to each subset. Using multiple Lyapunov functions, we will study in this chapter the switching control design of LPV systems with LFT parameter dependence. Instead of off-line designs, we will use LMI optimization approach to design switching control laws online. The resulting synthesis condition decouples the interconnection among subsystems, and can be solved using efficient interior-point algorithms. In this approach, the stability of the switched control system is guaranteed and the closed-loop performance will be bounded. Moreover, a bumpless transfer compensator will be designed to minimize the output jump of switched system at switching instants. Different from traditional anti-windup BT designs, the BT compensator is also generated through online optimization techniques.

This chapter is organized as follows: Section 4.2 provides some preliminary results for this research and develops analysis conditions for stability and performance of switched LFT system. The online control synthesis method of switched LFT systems is studied in
Section 4.3. Gain-scheduled state feedback and output feedback control of switched LFT systems are addressed separately. The control synthesis conditions are formulated as LMIs. An online anti-windup BT compensator design method is proposed in Section 4.4 to reduce the output jump generated by switching. Section 4.5 uses an uninhabited combat aerospace vehicle (UCAV) example to demonstrate the proposed control design approach for switched LFT systems. Finally, a summary of this chapter is provided in Section 4.6.

4.1 Introduction

Using the scaled small-gain theorem, LFT control design technique was developed in [61, 1] using single Lyapunov functions. However, for a LFT system with a large parameter variation range, a single stabilizing Lyapunov function may not exist. If it does exist, the performance is often sacrificed in some parameter subregions to obtain a single gain-scheduling controller over the entire parameter region. One possible approach to improve the performance over entire region is to design several gain-scheduling controllers, each working in a specific parameter subregion, and switch among them to achieve better performance. However, by partitioning the parameter region, the LFT system become a new class of systems, namely, switched LFT systems.

Switched system is closely related to parameter-dependent systems, and can be described by an interaction between continuous time systems and discrete switching events, which usually depend on states or time [48]. It is conceivable that the dynamic behavior of switched system is more complicated than either continuous or discrete systems. It is possible to get instability by switching between stable systems, and switching among several unstable systems may produce a stable trajectory [13]. Therefore, stability analysis in switched systems is an important and challenging problem, and has received considerable attention in the recent literature [13, 48].

Various switching logics have been proposed for the stabilization of diverse switched system. They can be classified into two categories: state-dependent and time-dependent. Our research focuses on the time-dependent switched systems. One practical way to stabilize time-dependent switched system is switching with dwell time, which requires sufficiently slow switching such that the transient effects are dissipated after each switching instant [48, 50, 34], as shown in Fig. 4.1 (b). Obviously, arbitrary switching can not be achieved
in switching control with dwell time.

For a switched system with a family of stable LTI subsystems, the existence of a single Lyapunov function provides sufficient conditions for stability under arbitrary switching sequences [48]. However, using single Lyapunov function is too conservative and may not achieve good performance in some subsystems. Parallel to recent development of parameter-dependent systems, nontraditional stability conditions have been taken into account using either piecewise continuous Lyapunov functions [64, 92, 70, 41] or discontinuous Lyapunov functions [10].

Multiple Lyapunov functions have been shown to be very useful tools for stability analysis of switching systems [21, 13, 10]. The results of switched LTI systems have been generalized to the analysis and control of switched LPV systems [51]. Multiple parameter-dependent Lyapunov functions are used to improve performance of switched LPV system over entire parameter region in [96, 51]. However, because switched LFT systems are interacted by several subsystems, the stability conditions in previous literatures are mostly coupled between subsystems, which leads to possible non-convexity or complicated computation of synthesis conditions.

On the other hand, several controllers, each suitable for a specific subsystem, will be generated by using multiple Lyapunov functions. The controller gain switching will cause transient variations of control input and error output. One reasonable approach to reduce switching bump is to design an anti-windup BT compensator [31, 45, 11]. Anti-windup control is motivated by minimizing the difference between the actual plant input and the controller output limited by actuator. Many approaches have been proposed in anti-windup control for constrained systems [102, 35, 44]. A general framework that unifies a large class of existing anti-windup control schemes was developed in [45, 11]. Bumpless transfer(BT) design is close to anti-windup control design for saturated systems. BT compensator is trying to minimize the difference between the current controller’s output and the next controller’s output[11]. Detail background of BT compensator design can be found in Section 4.4.
4.2 Online Analysis of LFT Switching Systems

Consider a standard LFT system with varying parameter $\Theta$. By partitioning the parameter set into $N$ subsets, the original system can be described using multiple LFT systems

$$
\begin{bmatrix}
\dot{x}(t) \\
q(t) \\
e(t)
\end{bmatrix}
= 
\begin{bmatrix}
A_{\alpha} & B_{\alpha,0} & B_{\alpha,1} \\
C_{\alpha,0} & D_{\alpha,00} & D_{\alpha,01} \\
C_{\alpha,1} & D_{\alpha,10} & D_{\alpha,11}
\end{bmatrix}
\begin{bmatrix}
x(t) \\
p(t) \\
d(t)
\end{bmatrix},
$$

(4.1)

$$
p(t) = \Theta(t)q(t),
$$

(4.2)

where $x, \dot{x} \in \mathbb{R}^n$, $d \in \mathbb{R}^{n_d}$ is the disturbance, $e \in \mathbb{R}^{n_e}$ is the controlled output, $p, q \in \mathbb{R}^{n_\theta}$ are the pseudo-input and output. Each subsystem activates in the specific parameter subregion.

Define the indicator function

$$
\alpha(t) = [\alpha_1(t), \cdots, \alpha_N(t)]^T
$$

with

$$
\alpha_i(t) = \begin{cases} 
1 & \text{when the switched system is described by the } i\text{th model } G_i \\
0 & \text{otherwise}
\end{cases}
$$

Then the switching system (4.1) can also be written as

$$
G_{\alpha} = \sum_{i=1}^{N} \alpha_i(t) G_i
$$

(4.3)

A simple example will be used to illustrate the above modeling strategy. Consider a single LFT plant $G$:

$$
\begin{bmatrix}
\dot{x} \\
q \\
e
\end{bmatrix} = 
\begin{bmatrix}
-1 & 1 & 2 \\
2 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
p \\
d
\end{bmatrix}
\quad
p = \theta q
$$

with $\theta \in [-1, 1]$. We want to partition the parameter set to two subset $[-1, 0.5]$ and $[0.5, 1]$. Then the corresponding switching LFT system can be constructed by the following steps.
1. Partition the parameter set and separate the original system. With partitioning the parameter set to two subset \([-1, 0.5]\) and \([0.5, 1]\), the original LFT system can be separated into two subsystems.

\[
\begin{bmatrix}
\dot{x} \\
q \\
e
\end{bmatrix} = \begin{bmatrix}
-1 & 1 & 2 \\
2 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
p \\
d
\end{bmatrix}
\]

\[p = \tilde{\theta}_1 q\]

with \(\tilde{\theta}_1 \in [-1, 0.5]\) and

\[
\begin{bmatrix}
\dot{x} \\
q \\
e
\end{bmatrix} = \begin{bmatrix}
-1 & 1 & 2 \\
2 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
p \\
d
\end{bmatrix}
\]

\[p = \tilde{\theta}_2 q\]

with \(\tilde{\theta}_2 \in [0.5, 1]\).

2. Set the parameter set of subsystems to [-1,1] by scaling the state space matrices of subsystems. Then we get \(G_1\):

\[
\begin{bmatrix}
\dot{x} \\
q \\
e
\end{bmatrix} = \begin{bmatrix}
-1 & 1 & 2 \\
0.8 & 0.6 & 0 \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
p \\
d
\end{bmatrix}
\]

\[p = \theta_1 q\]

with \(\theta_1 = \frac{4}{3}(\tilde{\theta}_1 + 1) - 1 \in [-1, 1]\) and \(G_2\):

\[
\begin{bmatrix}
\dot{x} \\
q \\
e
\end{bmatrix} = \begin{bmatrix}
-1 & 1 & 2 \\
8 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
p \\
d
\end{bmatrix}
\]

\[p = \theta_2 q\]

with \(\theta_2 = 4(\tilde{\theta}_2 - 1) + 1 \in [-1, 1]\).

3. The switching LFT system can be written as

\[G_\alpha = \alpha_1 G_1 + \alpha_2 G_2,\]
which is equivalent to the original single LFT system $G$. We have $\alpha_1 = 1, \alpha_2 = 0$ as $\theta \in [-1,0.5]$ and $\alpha_1 = 0, \alpha_2 = 1$ as $\theta \in [0.5,1]$.

The LFT representation is well-posed, i.e., $(I - D_{\alpha,00}\Theta)$ is invertible for any allowable parameter values. The time-varying parameter of subsystem, $\Theta$, has the structure of (1.7), where $\sum_{i=1}^a s_m = n_\theta$. If $\|\Theta\| \geq 1$, one can always scale down $\Theta$ by modifying plant state space matrices correspondingly. It is assumed that the vector-valued parameter $\theta$ are measurable in real-time. Note that $\Theta$ in subsystems has same structure as original parameter $\Theta$. We also assume that (A1)-(A4) hold. For simplicity, $\begin{bmatrix} D_{\alpha,00} & D_{\alpha,01} \\ D_{\alpha,10} & D_{\alpha,11} \end{bmatrix}$ is assumed to be zero in the following derivation. The difference between $\begin{bmatrix} D_{\alpha,00} & D_{\alpha,01} \\ D_{\alpha,10} & D_{\alpha,11} \end{bmatrix}$ and $\begin{bmatrix} D_{\alpha,00} & D_{\alpha,01} \\ D_{\alpha,10} & D_{\alpha,11} \end{bmatrix} \neq 0$ is mainly in the controller construction phase.

Motivated by the previous result using multiple Lyapunov functions [21, 13, 10], the following lemma is a sufficient condition to guarantee system stability.

**Lemma 7** Given the switched LFT system of (4.1)-(4.2), suppose there is a Lyapunov-like function $V_m : \mathbb{R}^n \to \mathbb{R}$ associated with each switching interval $[t_m, t_{m+1}]$ such that $V_m$ is positive definite and $\dot{V}_m < 0, \forall x \neq 0, \Theta \in \Theta$, and in addition

$$V_m(x(t_m^+)) \leq V_{m-1}(x(t_m^-)), \tag{4.4}$$

where $t_m$ denotes the $m$th switching instant when the system switches from $G_j$ to $G_i$. Then, the system (4.1)-(4.2) is asymptotically stable when $d(t) \equiv 0$.

Lemma 7 allows the discontinuity of plant states at switching instant. Given the states are continuous at switching, condition (4.4) can be further relaxed to

$$V_m(x(t_m)) \leq V_{m-1}(x(t_m)).$$

Note that the Lyapunov-like function $V_m$ is monotonically non-increasing on each switching time interval $[t_m, t_{m+1}]$, when the $i$th subsystem is activated. As a distinction from previous
multiple Lyapunov function results, it is possible to use different Lyapunov functions for the same subsystem if it is activated at different time intervals. The bottom line is the value of the new Lyapunov function $V_m$ at the switching time $t_m$ must be less than or equal to the value of old Lyapunov function $V_{m-1}$, which is used for the previous time interval. The difference between switching with Lemma 7 and switching with dwell time is shown in Fig. 4.1.

![Diagram](image)

(a) Switching with Lemma 7  
(b) Switching with dwell time

Figure 4.1: Lyapunov function value of different switching

The following theorem introduces a performance analysis condition based on scaled small-gain method of LFT control [1, 61]. To this end, we define a set of positive definite similarity scalings associated with the structure $\Theta$,

$$L_\Theta = \{ L > 0 : L\Theta = \Theta \alpha L, \forall \Theta \alpha \in \Theta \subset \mathbb{R}^{n_\theta \times n_\theta} \}$$

Before we present the next theorem, a new performance index is defined, namely nominal $L_2$ gain. The traditional $L_2$ gain evaluates the energy amplification from disturbance $d$ to error output $e$. If a system has its $L_2$ gain less than $\gamma$, it means

$$\int_0^\infty (e^T e - \gamma d^T d) < 0$$

But in this chapter, we consider a switched system with unexpected switching time, which means we do not know how long current subsystem will be active and which subsystem will be active at next switching time. So we treat $L_2$ gain of current active subsystem as a
nominal value, which may become a real $L_2$ gain only if $G_i$ is active from $t_m$ until $t = \infty$. Then the nominal performance level $\gamma_m$ of subsystem $G_i$ active in time interval from $t_m$ to $t_{m+1}$ is defined as

$$\int_{t_m}^{\infty} (e^T e - \gamma_m d^T d) < 0$$

with assumption that $G_i$ remains active from $t_m$ to $\infty$.

**Theorem 6** For a given switching sequence $\{t_m\}_{m=1}^\infty$, the switching LFT system of (4.1)-(4.2) is asymptotically stable $\forall \Theta \in \Theta$ and has its $L_2$ gain from $d$ to $e$ less than $\gamma = \max_m \gamma_m$ if there exist a sequence of positive definite matrices $X_m \in \mathbb{R}^{n \times n}$ and $L_m \in L_\Theta$ such that

$$\begin{bmatrix}
A_i^T X_m + X_m A_i & X_m B_{0,i} & X_m B_{1,i} & C_{0,i}^T & C_{1,i}^T \\
B_{0,i}^T X_m & -L_m & 0 & D_{00,i}^T & D_{10,i}^T \\
B_{1,i}^T X_m & 0 & -\gamma_m I & D_{01,i}^T & D_{11,i}^T \\
C_{0,i} & D_{00,i} & D_{01,i} & -L_m^{-1} & 0 \\
C_{1,i} & D_{10,i} & D_{11,i} & 0 & -\gamma_m I
\end{bmatrix} < 0 \quad (4.5)$$

where $t_m$ denotes the $m$th switching instant when system switches from $G_j$ to $G_i$.

**Proof:** By Schur complement and result from [1], condition (4.5) is equivalent to

$$A_i^T X_m + X_m A_i + X_m \begin{bmatrix} B_{0,i} & B_{1,i} \end{bmatrix} \begin{bmatrix} L_m & 0 \\ 0 & \gamma_m^{-1} I \end{bmatrix}^{-1} \begin{bmatrix} B_{0,i}^T \\ B_{1,i}^T \end{bmatrix} X_m + \begin{bmatrix} C_{0,i}^T & C_{1,i}^T \end{bmatrix} \begin{bmatrix} L_m & 0 \\ 0 & \gamma_m^{-1} I \end{bmatrix} \begin{bmatrix} C_{0,i} \\ C_{1,i} \end{bmatrix} < 0$$

which implies

$$\frac{dV_m}{dt} < 0$$

$$\| \begin{bmatrix} L_m & 0 \\ 0 & \gamma_m^{-1} I \end{bmatrix} \|_{1}^{\frac{1}{2}} G_i(s) \begin{bmatrix} L_m & 0 \\ 0 & \gamma_m I \end{bmatrix}^{-\frac{1}{2}} \|_{\infty} < 1$$
with \( V_m = x^T X_m x > 0, \forall x \neq 0 \). Then
\[
\frac{dV_m}{dt} + \frac{1}{\gamma_m} e^T e - \gamma_m d^T d < 0
\]
Integrating both sides of the above inequality from \( t_m^- \) to \( t_{m+1}^- \), we have the following inequality satisfied in each time interval.
\[
V_m(t_{m+1}^-) - V_m(t_m^+) + \int_{t_m^-}^{t_{m+1}^-} \left( \frac{1}{\gamma} e^T e - \gamma d^T d \right) < 0
\] (4.7)
Also condition (4.6) guarantees
\[
V_m(x(t_m^+)) \leq V_m(x(t_m^-)).
\]
Adding (4.7) together from \( t = 0 \) to \( \infty \),
\[
V_\infty(\infty) - V_1(0) + \int_0^\infty \left( \frac{1}{\gamma} e^T e - \gamma d^T d \right) < 0.
\]
Then we conclude that the system (4.1)-(4.2) is asymptotically stable \( \forall \Theta \in \Theta \) and
\[
\int_0^\infty (\gamma^{-1} e^T e - \gamma d^T d) < 0.
\]
Q.E.D.

Theorem 6 provides stability and performance guarantees for switched LFT systems. The idea behind Theorem 6 is finding a suitable Lyapunov function at the beginning \( t_m^- \) of every switching interval. For this purpose, it is necessary to know which subsystem is activated at \( t_m^- \) and the value of previous Lyapunov function at \( t_m^- \). These information can be obtained from the information of switching signal \( \alpha(t) \) and measurement of plant states at \( t_m^- \).

4.3 Gain-Scheduled Controller Design using Online Optimization

The online gain-scheduled controller design method will be proposed in this section. Both of state feedback and output feedback are considered.
4.3.1 State Feedback

Consider a class of switched LFT systems given by

\[
\begin{bmatrix}
\dot{x}_p(t) \\
q_p(t) \\
e(t)
\end{bmatrix} =
\begin{bmatrix}
A_{p,\alpha} & B_{p0,\alpha} & B_{p1,\alpha} & B_{p2,\alpha} \\
C_{p0,\alpha} & D_{p00,\alpha} & D_{p01,\alpha} & D_{p02,\alpha} \\
C_{p1,\alpha} & D_{p10,\alpha} & D_{p11,\alpha} & D_{p12,\alpha}
\end{bmatrix}
\begin{bmatrix}
x_p(t) \\
p_p(t) \\
d(t) \\
u(t)
\end{bmatrix},
\] (4.8)

\[p_p(t) = \Theta_\alpha(t)q_p(t),\] (4.9)

where \(u \in \mathbb{R}^n\) is the control input and we assume states \(x_p\) is measurable.

When parameter \(\Theta\) is time-varying and measurable in real time, one can design a switching state-feedback controller in the form of

\[
\begin{bmatrix}
u(t) \\
q_k(t)
\end{bmatrix} =
\begin{bmatrix}
F_{1,\alpha,m} & F_{0,\alpha,m} \\
H_{1,\alpha,m} & H_{0,\alpha,m}
\end{bmatrix}
\begin{bmatrix}
x_p(t) \\
p_k(t)
\end{bmatrix},
\] (4.10)

\[p_k(t) = \Theta_\alpha q_k(t).\] (4.11)

The closed loop system then is given by

\[
\begin{bmatrix}
\dot{x}_p \\
q \\
e
\end{bmatrix} =
\begin{bmatrix}
A_{\alpha} & B_{0,\alpha} & B_{1,\alpha} \\
C_{0,\alpha} & D_{00,\alpha} & D_{01,\alpha} \\
C_{1,\alpha} & D_{10,\alpha} & D_{11,\alpha}
\end{bmatrix}
\begin{bmatrix}
x_p \\
p \\
d
\end{bmatrix},
\] (4.12)

\[p = \begin{bmatrix}
\Theta_\alpha \\
\Theta_\alpha
\end{bmatrix} q,
\] (4.13)

where \(p = \begin{bmatrix} p_p \\ p_k \end{bmatrix}\), \(q = \begin{bmatrix} q_p \\ q_k \end{bmatrix}\) and the closed-loop data are

\[
\begin{bmatrix}
A_{\alpha} & B_{0,\alpha} & B_{1,\alpha} \\
C_{0,\alpha} & D_{00,\alpha} & D_{01,\alpha} \\
C_{1,\alpha} & D_{10,\alpha} & D_{11,\alpha}
\end{bmatrix} =
\begin{bmatrix}
A_{p,\alpha} & B_{p0,\alpha} & 0 & B_{p1,\alpha} \\
C_{p0,\alpha} & D_{p00,\alpha} & 0 & D_{p01,\alpha} \\
0 & 0 & 0 & 0
\end{bmatrix}
+ \begin{bmatrix}
B_{p2,\alpha} & 0 \\
D_{p2,\alpha} & 0 \\
0 & I
\end{bmatrix}
\times
\begin{bmatrix}
F_{1,\alpha} & F_{0,\alpha} \\
H_{1,\alpha} & H_{0,\alpha}
\end{bmatrix}
\begin{bmatrix}
I & 0 & 0 \\
0 & 0 & I
\end{bmatrix}.
\] (4.14)
For simplicity of notation, define
\[
\begin{bmatrix}
B_{p_1,\alpha} & B_{p_2,\alpha}
\end{bmatrix}
= \begin{bmatrix}
B_{p_0,\alpha} & 0 & B_{p_1,\alpha} & 0
\end{bmatrix}
\]
\[
C_{p_1,\alpha} = \begin{bmatrix}
C_{p_0,\alpha} \\
0 \\
C_{p_1,\alpha}
\end{bmatrix}, \quad \bar{D}_{p_{12,\alpha}} = \begin{bmatrix}
D_{p_{02,\alpha}} & 0 \\
0 & I \\
D_{p_{12,\alpha}} & 0
\end{bmatrix}
\]

**Theorem 7** Given a switching sequence \( \{t_m\}_{m=1}^{\infty} \), the switched LFT system of (4.8)-(4.9) is stable by switching state feedback control (4.10)-(4.11) \( \forall \Theta \in \Theta \) and the closed-loop \( L_2 \) gain from \( d \) to \( e \) less than \( \gamma = \max_m \gamma_m \) if there exists a sequence of positive definite matrices \( R_m \in \mathbb{R}^{n_p \times n_p} \) and \( L_m, J_m \in L_\Theta \) such that

\[
\begin{bmatrix}
R_m A_{p,i}^T + A_{p,i} R_m & R_m C_{p_0,i}^T & R_m C_{p_1,i}^T & B_{p_0,i} & B_{p_1,i} \\
C_{p_0,i} R_m & -J_m & 0 & D_{p_{00,i}} & D_{p_{01,i}} \\
C_{p_1,i} R_m & 0 & -\gamma_m I & D_{p_{10,i}} & D_{p_{11,i}} \\
B_{p_0,i}^T & D_{p_{00,i}}^T & D_{p_{10,i}}^T & -L_m & 0 \\
B_{p_1,i}^T & D_{p_{01,i}}^T & D_{p_{11,i}}^T & 0 & -\gamma_m I
\end{bmatrix}
\begin{bmatrix}
N_{R,i} \\
I
\end{bmatrix} < 0 \quad (4.15)
\]

\[
\begin{bmatrix}
L_m & I \\
I & J_m
\end{bmatrix} \geq 0 \quad (4.16)
\]

\[
\begin{bmatrix}
V_m-1(t_m^-) \\
x_p(t_m^+)^T
\end{bmatrix} \begin{bmatrix}
x_p(t_m^-) \\
R_m
\end{bmatrix} \geq 0 \quad (4.17)
\]

with \( V_m-1(t_m^-) = x_p^T(t_m^-) R_m^{-1} x_p(t_m^-) \), where \( N_{R,i} = \ker \begin{bmatrix}
B_{p_2,\alpha}^T & D_{p_{02,\alpha}}^T & D_{p_{12,\alpha}}^T
\end{bmatrix} \) and \( t_m \) denotes the \( m \)th switching instant when system switches from \( G_j \) to \( G_i \).

**Proof:** Substituting closed loop state space (4.14) into condition (4.5), (4.15)-(4.16) can be derived by applying elimination approach from [1] and [25]. (4.15)-(4.16) are equivalent to (4.5) with closed loop state space. As we known, the closed loop Lyapunov function can
be constructed as

$$V_m = x_p^T R_m^{-1} x_p$$

Then (4.6) is equivalent to

$$V_{m-1}(t_m^-) - x_p(t_m^+)^T R_m^{-1} x_p(t_m^+) \geq 0$$

Apply Schur complement to the above inequality, condition (4.17) can be derived. \(Q.E.D.\)

Let

$$L_{cl,m} = \begin{bmatrix} L_m & L_{12,m} \\ L_{12,m}^T & ? \end{bmatrix}, \quad J_{cl,m} = \begin{bmatrix} J_m & J_{12,m} \\ J_{12,m}^T & ? \end{bmatrix},$$

where $L_{12,m}, J_{12,m}$ are commutable with $\Theta$ and $L_{12,m} J_{12,m}^T = I - L_m J_m, L_{cl,m} J_{cl,m} = I$, ? denotes the matrix elements we do not care about. The gain-scheduled state feedback controller works for the interval between $t_m$ and $t_{m+1}$ can be constructed as

$$\begin{bmatrix} F_{1,\alpha,m} \\ H_{1,\alpha,m} \end{bmatrix} = - \left( \bar{D}_{p1,\alpha}^T \begin{bmatrix} L_{cl,m} & \gamma^{-1} I \end{bmatrix} \bar{D}_{p1,\alpha} \right)^{-1}$$

$$\times \left\{ \bar{B}_{p2,\alpha}^T R_m^{-1} + \bar{D}_{p1,\alpha}^T \begin{bmatrix} L_{cl,m} & \gamma^{-1} I \end{bmatrix} \bar{C}_{p1,\alpha} \right\}$$

$$\begin{bmatrix} F_{0,\alpha,m} \\ H_{0,\alpha,m} \end{bmatrix} = 0$$

(4.18)

(4.19)

The state feedback subcontroller working in the interval $t_m$ to $t_{m+1}$ is synthesized at the beginning of interval, $t_m^+$. It not only depends on next active subsystem $G_i$, but also depends on previous Lyapunov function $V_{m-1}$. Same subsystem activated in different time interval may require different controllers. The details about derivation of controller formula can be found in [94, 95].
4.3.2 Output Feedback

Consider a class of switched LFT systems given by

\[
\begin{bmatrix}
\dot{x}_p(t) \\
p(t) \\
e(t) \\
y(t)
\end{bmatrix} = \begin{bmatrix}
A_{p,\alpha} & B_{p0,\alpha} & B_{p1,\alpha} & B_{p2,\alpha} \\
C_{p0,\alpha} & D_{p00,\alpha} & D_{p01,\alpha} & D_{p02,\alpha} \\
C_{p1,\alpha} & D_{p10,\alpha} & D_{p11,\alpha} & D_{p12,\alpha} \\
C_{p2,\alpha} & D_{p20,\alpha} & D_{p21,\alpha} & 0
\end{bmatrix} \begin{bmatrix}
x_p(t) \\
p_p(t) \\
d(t) \\
u(t)
\end{bmatrix},
\]

(4.20)

\[p(t) = \Theta_{\alpha}(t)q(t),\]

(4.21)

where \(y \in \mathbb{R}^{n_y}\) is the measurement, \(u \in \mathbb{R}^{n_u}\) is the control input.

When parameter \(\Theta\) is time-varying and measurable in real time, one can design a switching output-feedback controller in the form of

\[
\begin{bmatrix}
\dot{x}_k(t) \\
u(t) \\
p_k(t)
\end{bmatrix} = \begin{bmatrix}
A_{k,\alpha,m} & B_{k1,\alpha,m} & B_{k0,\alpha,m} \\
C_{k1,\alpha,m} & D_{k11,\alpha,m} & D_{k10,\alpha,m} \\
C_{k0,\alpha,m} & D_{k01,\alpha,m} & D_{k00,\alpha,m}
\end{bmatrix} \begin{bmatrix}
x(t) \\
y(t) \\
p_k(t)
\end{bmatrix},
\]

(4.22)

\[p_k(t) = \Theta_{\alpha}(t)q_k(t),\]

(4.23)

The closed loop system then is given by

\[
\begin{bmatrix}
\dot{x} \\
q \\
e
\end{bmatrix} = \begin{bmatrix}
A_{\alpha,m} & B_{0,\alpha,m} & B_{1,\alpha,m} \\
C_{0,\alpha,m} & D_{00,\alpha,m} & D_{01,\alpha,m} \\
C_{1,\alpha,m} & D_{10,\alpha,m} & D_{11,\alpha,m}
\end{bmatrix} \begin{bmatrix}
x \\
p \\
d
\end{bmatrix},
\]

(4.24)

\[p = \begin{bmatrix}
\Theta_{\alpha} \\
\Theta_{\alpha}
\end{bmatrix} q,\]

(4.25)

where the closed-loop data are

\[
\begin{bmatrix}
A_{\alpha,m} & B_{0,\alpha,m} & B_{1,\alpha,m} \\
C_{0,\alpha,m} & D_{00,\alpha,m} & D_{01,\alpha,m} \\
C_{1,\alpha,m} & D_{10,\alpha,m} & D_{11,\alpha,m}
\end{bmatrix} = \begin{bmatrix}
A_{p,\alpha} & 0 & B_{p0,\alpha} & 0 & B_{p1,\alpha} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
C_{p0,\alpha} & 0 & D_{p00,\alpha} & 0 & D_{p01,\alpha} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I \\
C_{p1,\alpha} & 0 & D_{p10,\alpha} & 0 & D_{p11,\alpha}
\end{bmatrix} + \begin{bmatrix}
0 & B_{p2,\alpha} & 0 \\
I & 0 & 0 \\
0 & D_{p02,\alpha} & 0 \\
0 & 0 & I \\
0 & D_{p12,\alpha} & 0
\end{bmatrix},
\]

(4.26)

\[
\begin{bmatrix}
A_{k,\alpha,m} & B_{k1,\alpha,m} & B_{k0,\alpha,m} \\
C_{k1,\alpha,m} & D_{k11,\alpha,m} & D_{k10,\alpha,m} \\
C_{k0,\alpha,m} & D_{k01,\alpha,m} & D_{k00,\alpha,m}
\end{bmatrix} \begin{bmatrix}
0 & I & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & I & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]
Define

\[
\begin{bmatrix}
B_{p1,\alpha} & B_{p2,\alpha}
\end{bmatrix} = \begin{bmatrix}
B_{p0,\alpha} & 0 & B_{p1,\alpha} & B_{p2,\alpha} & 0
\end{bmatrix}, \quad 
\begin{bmatrix}
D_{p0,\alpha} & 0 & D_{p2,\alpha}
\end{bmatrix} = 
\begin{bmatrix}
0 & 0 & I & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
\bar{C}_{p1,\alpha} \\
\bar{C}_{p2,\alpha}
\end{bmatrix}
= 
\begin{bmatrix}
C_{p0,\alpha} \\
0 \\
C_{p1,\alpha} \\
C_{p2,\alpha} \\
0
\end{bmatrix}, \quad 
\begin{bmatrix}
\bar{D}_{p12,\alpha}
\end{bmatrix}
= 
\begin{bmatrix}
D_{p02,\alpha} & 0 \\
0 & I \\
D_{p12,\alpha}
\end{bmatrix}
\]

**Theorem 8** Given a switching sequence \(\{t_m\}_{m=1}^{\infty}\), the switched LFT system of (4.20)-(4.21) is stable by switching output feedback control (4.22)-(4.23) \(\forall \Theta \in \Theta\) and the closed-loop \(L_2\) gain from \(d\) to \(e\) less than \(\gamma = \max_m \gamma_m\) if there exist positive definite matrices \(R_m, S_m \in \mathbb{R}^{n_p \times n_p}\) and \(L_m, J_m \in L_\Theta\) such that

\[
\begin{bmatrix}
R_mA_{p,i}^T + A_{p,i}R_m & R_mC_{p0,i}^T & R_mC_{p1,i}^T & B_{p0,i} & B_{p1,i}
\end{bmatrix}^T 
\begin{bmatrix}
C_{p0,i}R_m & 0 & D_{p00,i} & D_{p01,i} \\
0 & 0 & -\gamma_m I & D_{p10,i} & D_{p11,i} \\
B_{p0,i} & D_{p00,i}^T & D_{p01,i}^T & -L_m & 0 \\
B_{p1,i} & D_{p10,i}^T & D_{p11,i}^T & 0 & -\gamma_m I
\end{bmatrix} 
\begin{bmatrix}
N_{R,i} \\
I
\end{bmatrix} < 0 \quad (4.27)
\]

\[
\begin{bmatrix}
A_{p,i}^T S_m + S_m A_{p,i} & S_mB_{p0,i} & S_mB_{p1,i} & C_{p0,i}^T & C_{p1,i}^T
\end{bmatrix}^T 
\begin{bmatrix}
B_{p0,1}^T S_m & -L_m & 0 & D_{p00,i}^T & D_{p01,i}^T \\
B_{p1,1}^T S_m & 0 & -\gamma_m I & D_{p10,i}^T & D_{p11,i}^T \\
C_{p0,i} & D_{p00,i} & D_{p01,i} & -L_m & 0 \\
C_{p1,i} & D_{p10,i} & D_{p11,i} & 0 & -\gamma_m I
\end{bmatrix} 
\begin{bmatrix}
N_{S,i} \\
I
\end{bmatrix} < 0 \quad (4.28)
\]
\[
\begin{bmatrix}
R_m & I \\
I & S_m
\end{bmatrix} \geq 0 
\quad (4.29)
\]

\[
\begin{bmatrix}
L_m & I \\
I & J_m
\end{bmatrix} \geq 0 
\quad (4.30)
\]

\[
\begin{bmatrix}
V_{m-1}(t_m^-) & [x_p(t_m^+) - x_k(t_m^+)]^T S_m & x_k^T(t_m^+) & 0 \\
S[x_p(t_m^+) - x_k(t_m^+)] & S_m & 0 & 0 \\
x_k(t_m^+) & 0 & R_m & I \\
0 & 0 & I & S_m
\end{bmatrix} \geq 0 
\quad (4.31)
\]

with \( V_{m-1}(t_m^-) = \begin{bmatrix} x_p^T(t_m^-) & x_k^T(t_m^-) \end{bmatrix} \begin{bmatrix} S_{m-1} & -S_{m-1} \\
-S_{m-1} & S_{m-1} + (R_m - S_m^{-1})^{-1} \end{bmatrix} \begin{bmatrix} x_p(t_m^-) \\
x_k(t_m^-) \end{bmatrix} \), where

\[
N_{R,i} = \ker \begin{bmatrix} B_{p2,\alpha}^T & D_{p02,\alpha}^T & D_{p12,\alpha}^T \end{bmatrix}, \quad N_{S,i} = \ker \begin{bmatrix} C_{p2,\alpha} & D_{p20,\alpha} & D_{p21,\alpha} \end{bmatrix}
\]

and \( t_m \) denotes the \( m \)th switching instant when system switches from \( G_j \) to \( G_i \).

**Proof:** Substituting closed loop state space (4.26) into condition (4.5), (4.27)-(4.30) can be derived by applying elimination approach from [1] and [25]. (4.27)-(4.30) are equivalent to (4.5) with closed loop state space. As we known, \( R_m, S_m \) solved from (4.27)-(4.30) is \((1,1)\) block of closed loop Lyapunov function, which can be constructed as

\[
X_{cl,m} = \begin{bmatrix} S_m & N_m \\
N_m^T & ? \end{bmatrix}, \quad X_{cl,m}^{-1} = \begin{bmatrix} R_m & M_m \\
M_m^T & ? \end{bmatrix}
\]

where \( M_m \) and \( N_m \) can be arbitrarily chosen with constraint \( M_m N_m^T = I - R_m S_m \). However, arbitrary \( M_m \) and \( N_m \) may make (4.6) of closed loop system to be non-convex and difficult to solve. By choosing \( N_m = -S_m \) and \( M_m = R_m - S_m^{-1} \), the condition (4.6) turns out to be a convex condition as (4.31). A Lyapunov-like function \( V_m \) with special \( N_m \) and \( M_m \), which is operating in the interval between \( t_m \) and \( t_{m+1} \), can be constructed as

\[
V_m = x_{cl}^T X_{cl,m} x_{cl} = x_{cl}^T \begin{bmatrix} S_m & -S_m \\
-S_m & S_m + (R_m - S_m^{-1})^{-1} \end{bmatrix} x_{cl}
\]
where \( x_{cl} = \begin{bmatrix} x_p \\ x_k \end{bmatrix} \). Then (4.6) is equivalent to

\[
\begin{bmatrix} x_p(t_m^+)^T & x_k(t_m^+)^T \end{bmatrix} \begin{bmatrix} S_m & -S_m \\ -S_m & S_m + (R_m - S_m^{-1})^{-1} \end{bmatrix} \begin{bmatrix} x_p(t_m^+) \\ x_k(t_m^+) \end{bmatrix} \leq V_{m-1}(t_m^-)
\]

\[
V_{m-1}(t_m^-) - (x_p(t_m^+) - x_k(t_m^+))^T S_m (x_p(t_m^+) - x_k(t_m^+)) - x_k(t_m^+)^T (R_m - S_m^{-1})^{-1} x_k(t_m^+) \geq 0
\]

Apply Schur complement to the above inequality, condition (4.31) can be derived. \( Q.E.D. \)

After solving \( R_m, S_m \) and \( L_m, J_m \), one can construct the LFT controller gain for the \( m \)th time interval. Let

\[
L_{cl,m} = \begin{bmatrix} L_m & L_{12,m} \\ L_{12,m}^T & ? \end{bmatrix}, \quad J_{cl,m} = \begin{bmatrix} J_m & J_{12,m} \\ J_{12,m}^T & ? \end{bmatrix},
\]

where \( L_{12,m}, J_{12,m} \) are commutable with \( \Theta \) and \( L_{12,m} J_{12,m}^T = I - L_m J_m, L_{cl,m} J_{cl,m} = I \). The gain-scheduled controller works for the interval between \( t_m \) and \( t_{m+1} \) can be constructed as

\[
A_{k,\alpha,m} = -N_m^T \left( A_{p,\alpha}^T + S_m \left( A_{p,\alpha} + B_{p2,\alpha} F_{\alpha,m} + L_{\alpha,m} C_{p2,\alpha} \right) R_m \right)
\]

\[
+ S_m \left( \bar{B}_{p1,\alpha} + L_{\alpha,m} \bar{D}_{p21,\alpha} \right) \begin{bmatrix} J_{cl,m} \\ \gamma_m^{-1} I \end{bmatrix} \bar{B}_{p1,\alpha}^T
\]

\[
+ C_{p1,\alpha}^T \begin{bmatrix} L_{cl,m} \\ \gamma_m^{-1} I \end{bmatrix} \left( C_{p1,\alpha} + \bar{D}_{p12,\alpha} F_{\alpha,m} \right) \end{bmatrix} M_m^{-T} \tag{4.32}
\]

\[
\begin{bmatrix} B_{k1,\alpha,m} & B_{k0,\alpha,m} \\ C_{k1,\alpha,m} & C_{k0,\alpha,m} \end{bmatrix} = N_m^{-1} S_m L_{\alpha,m} \tag{4.33}
\]

\[
\begin{bmatrix} D_{k11,\alpha,m} & D_{k10,\alpha,m} \\ D_{k01,\alpha,m} & D_{k00,\alpha,m} \end{bmatrix} = 0 \tag{4.35}
\]

\[
\begin{bmatrix} C_{k1,\alpha,m} \\ C_{k0,\alpha,m} \end{bmatrix} = F_{\alpha,m} R_m M_m^{-T} \tag{4.34}
\]
with

\[ F_{α,m} = - \left( \bar{D}_{p12,α}^T \left[ L_{cl,m} \gamma_{m}^{-1} I \right] \bar{D}_{p12,α} \right)^{-1} \]

\[ \times \left\{ \bar{B}_{p2,α} R_{m}^{-1} + \bar{D}_{p12,α} \left[ L_{cl,m} \gamma_{m}^{-1} I \right] \bar{C}_{p1,α} \right\} \]

\[ L_{α,m} = - \left\{ S_{m}^{-1} \bar{C}_{p2,α} + \bar{B}_{p1,α} \left[ J_{cl,m} \gamma_{m}^{-1} I \right] \bar{D}_{p21,α} \right\} \left( \bar{D}_{p21,α} \left[ J_{cl,m} \gamma_{m}^{-1} I \right] \bar{D}_{p21,α}^T \right)^{-1} \]

The boundary condition (4.31) in Theorem 8 is in a convex form due to the selection of a special pair of \( N_m \) and \( M_m \). As a result, the online synthesis condition for switching control is formulated as an LMI optimization problem, and can be solved efficiently. Note that the condition (4.31) requires plant state information at \( t_m \). In order to synthesize the controller for the next interval starting at \( t_{m}^+ \), we must know which subsystem is activated at \( t_{m}^+ \), the value of previous Lyapunov function at \( t_{m}^- \), and plant and controller state information at \( t_m \).

An alternative way of convexifying the boundary condition is called controller state reset [54]. Usually, the controller states can be set arbitrarily in state space. By setting the controller state to zero at time \( t_m \), the boundary condition (4.31) can be simplified to

\[ x_p^T(t_{m}^+) S_m x_p^T(t_{m}^+) \leq V_{m-1}(t_{m}) \]

The controller state reset approach is not proposed in this research.

### 4.4 Online Switching with Bumpless Transfer

As is well known, switching system is a kind of interaction between continuous time system and discrete switching events. The discrete switching signal \( α(t) \) may cause sudden changes on the system or controller’s states, which will cause the system output have a big bump at switching instant. The bump of output will exist no matter the state is continuous or not because it is mainly from the discontinuity of system state space. The discontinuity of output is undesirable and needs to be minimized for smooth operation. Fortunately, many work has been done on the bumpless transfer controller design, as well as anti-windup design [31, 45]. By designing an additional compensator for the closed-loop
of plant and nominal controller, the bumpless transfer will reduce the difference between commanded control input and actual control input. Therefore it minimizes the transient output variations at switching instants. In traditional bumpless transfer designs, an anti-windup BT compensator is used for each closed loop subsystem and all anti-windup BT compensators are running parallel regardless which subsystem is active. This is not feasible for our situation because the closed loop subsystem with plant and switching controller is not known a priori but at the beginning of the switching interval. Following switching control design approach, the bumpless transfer compensator for switching control will be also synthesized on-line after the switching controller is constructed at time $t_m$.

![Bumpless transfer controller structure](image)

Figure 4.2: Bumpless transfer controller structure

The structure of proposed bumpless transfer compensator is shown in Fig. 4.2 (a), where $\tilde{u}$ is an artificial input. When system is switching from $G_j$ to $G_i$ at $t_m$, $\tilde{u}$ will be set as the previous control input at $t_m$, denoted by $u(t_m)$, and keep this value until certain
condition is satisfied. Then by resetting \( \tilde{u} \) to \( u \), the bumpless transfer compensator will be disabled. The structure can also be seen as Fig. 4.2 (b). From Fig. 4.2 (b), the bumpless transfer problem actually can be treated that new control force is trying to track the old one. The error we try to minimize includes \( e \) and \( u - \tilde{u} \). The weighting of them is user defined.

The condition for disabling the compensator is problem dependent and one of which is to check if \( \| u(t) - u(t_m) \| < \epsilon \). It is also possible to run the bumpless transfer compensator for a fixed amount of time after \( t_m \). The online anti-windup BT compensator can only be synthesized after nominal switching controller is generated.

### 4.4.1 Problem Formulation

To simplify the presentation, we will drop the subscript \( \alpha \) and \( m \) in the sequel, but keep in mind that BT compensator synthesis is based on current closed loop system with \( G_i \) and \( K_{i,m} \). To this end, the switching controller \( K_{nom} \) from Section 4.3.2 is modified to

\[
\begin{bmatrix}
\dot{x}_k(t) \\
u(t) \\
q_k(t)
\end{bmatrix} =
\begin{bmatrix}
A_k & B_{k1} & B_{k0} \\
C_{k1} & D_{k11} & D_{k10} \\
C_{k0} & D_{k01} & D_{k00}
\end{bmatrix}
\begin{bmatrix}
x(t) \\
y(t) \\
p_k(t)
\end{bmatrix} +
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix},
\]

(4.36)

\[
p_k(t) = \Theta(t)q_k(t),
\]

(4.37)

where the variables \( v_1, v_2, v_3 \) are the auxiliary inputs that will be provided by a bumpless transfer compensator.

The bumpless transfer compensator is in the form of

\[
\begin{bmatrix}
\dot{x}_{bt} \\
v_1 \\
v_2 \\
v_3 \\
p_{bt}
\end{bmatrix} =
\begin{bmatrix}
A_{bt} & B_{bt1} & B_{bt0} \\
C_{bt1} & D_{bt11} & D_{bt10} \\
C_{bt0} & D_{bt01} & D_{bt00}
\end{bmatrix}
\begin{bmatrix}
x_{bt} \\
w
\end{bmatrix}
\]

(4.38)

\[
p_{bt} = \Theta(t)q_{bt},
\]

(4.39)

where the compensator state \( x_{bt} \in \mathbb{R}^{n_{bt}} \).

Let the system \( P \) be the interconnection of the open-loop system \( G \) and the nominal controller \( K_{nom} \), but excluding bumpless transfer compensator. Then its state-space model
is given by

\[
\begin{bmatrix}
    \dot{x} \\
    u \\
    e \\
    q
\end{bmatrix} =
\begin{bmatrix}
    A & B_0 & B_1 & B_2 \\
    C_0 & D_{00} & D_{01} & D_{02} \\
    C_1 & D_{10} & D_{11} & D_{12} \\
    C_2 & D_{20} & D_{21} & 0
\end{bmatrix}
\begin{bmatrix}
    x \\
    q \\
    d \\
    \tilde{u} \\
    v_1 \\
    v_2 \\
    v_3
\end{bmatrix}
\]

\begin{equation}
q = \begin{bmatrix}
    \Theta \\
    \Theta
\end{bmatrix}
p
\end{equation}

(4.40)

where \( x \in \mathbb{R}^n \) with \( n = n_p + n_k \), \( \Theta \in \Theta \) and its state-space matrices are

\[
A = \begin{bmatrix}
    A_p + B_{p2}D_kC_{p2} & B_{p2}C_k \\
    B_kC_{p2} & A_k
\end{bmatrix}
\]

\[
B_0 = \begin{bmatrix}
    B_{p0} + B_{p2}D_{k11}D_{p20} & B_{p2}D_{k10} \\
    B_{k1}D_{p20} & B_{k0}
\end{bmatrix},
\quad
B_1 = \begin{bmatrix}
    B_{p1} + B_{p2}D_{k11}D_{p21} & 0 \\
    B_{k1}D_{p21} & 0
\end{bmatrix},
\]

\[
C_0 = \begin{bmatrix}
    C_{p0} + D_{p02}D_{k11}D_{p20} & D_{p02}C_{k1} \\
    D_{k01}C_{p2} & C_{k0}
\end{bmatrix},
\quad
C_1 = \begin{bmatrix}
    C_{p1} + D_{p12}D_{k11}C_{p2} & D_{p12}C_{k1}
\end{bmatrix}
\]

\[
D_{00} = \begin{bmatrix}
    D_{p00} + D_{p02}D_{k11}D_{p20} & D_{p02}D_{k10} \\
    D_{k01}D_{p20} & D_{k00}
\end{bmatrix},
\quad
D_{01} = \begin{bmatrix}
    D_{p01} + D_{p02}D_{k11}D_{p21} & 0 \\
    D_{p01}D_{k21} & 0
\end{bmatrix},
\]

\[
D_{10} = \begin{bmatrix}
    D_{p10} + D_{p12}D_{k11}D_{p20} & D_{p12}D_{k10} \\
    D_{k11}D_{p20} & D_{k10}
\end{bmatrix},
\quad
D_{11} = \begin{bmatrix}
    D_{p11} + D_{p12}D_{k11}D_{p21} & 0 \\
    D_{k11}D_{p21} & -I
\end{bmatrix},
\]

\[
C_2 = \begin{bmatrix}
    D_{k11}C_{p2}C_{k1}
\end{bmatrix},
\quad
D_{20} = \begin{bmatrix}
    D_{k11}D_{p20} & D_{k10}
\end{bmatrix},
\quad
D_{21} = \begin{bmatrix}
    D_{k11}D_{p21}
\end{bmatrix}
\]

\[
B_2 = \begin{bmatrix}
    0 & B_{p2} & 0 \\
    I_{n_k} & 0 & 0
\end{bmatrix},
\quad
D_{02} = \begin{bmatrix}
    0 & D_{p02} & 0 \\
    0 & 0 & I_{n_k}
\end{bmatrix},
\quad
D_{12} = \begin{bmatrix}
    0 & D_{p12} & 0
\end{bmatrix}
\]

Denote \( x_{cl}^T = [x^T \ x_{bd}^T] \). Then the final closed-loop system \( T = \mathcal{F}_I(P, K_{bd}) \) is described by

\[
\begin{bmatrix}
    \dot{x}_{cl} \\
    p_{cl} \\
    e
\end{bmatrix} =
\begin{bmatrix}
    A_{cl} & B_{0,cl} & B_{1,cl} \\
    C_{0,cl} & D_{00,cl} & D_{01,cl} \\
    C_{1,cl} & D_{10,cl} & D_{11,cl}
\end{bmatrix}
\begin{bmatrix}
    x_{cl} \\
    q_{cl} \\
    d
\end{bmatrix}
\]

\begin{equation}
q_{cl} = \text{diag}\{\Theta, \Theta, \Theta\}_{p_{cl}}
\end{equation}

(4.43)
and has its state-space data related to the interconnected system $P$ and the bumpless transfer compensator $K_{bt}$ as follows

$$
\begin{bmatrix}
A_{cl} & B_{0,cl} & B_{1,cl} \\
C_{0,cl} & D_{00,cl} & D_{01,cl} \\
C_{1,cl} & D_{10,cl} & D_{11,cl}
\end{bmatrix}
= \begin{bmatrix}
A & B_0 & B_1 \\
C_0 & D_{00} & D_{01} \\
C_1 & D_{10} & D_{11}
\end{bmatrix} + \begin{bmatrix}
P_{1}^T \\
P_{2}^T \\
P_{3}^T
\end{bmatrix}
\begin{bmatrix}
A_{bt} & B_{bt1} & B_{bt0} \\
C_{bt1} & D_{bt11} & D_{bt10} \\
C_{bt0} & D_{bt01} & D_{bt00}
\end{bmatrix}
\begin{bmatrix}
Q_1 & Q_2 & Q_3
\end{bmatrix}
$$

As expected, the closed-loop state-space data depends on the bumpless transfer compensator gain in affine form. Define a set

$$
D_{\Theta} = \{ L > 0 : L \begin{bmatrix}
\Theta \\
\Theta
\end{bmatrix} = \begin{bmatrix}
\Theta \\
\Theta
\end{bmatrix} L, \forall \Theta \in \Theta \subset \mathbb{R}^{n_\Theta \times n_\Theta}\}
$$

### 4.4.2 Synthesis Condition

The following theorem provides a synthesis condition for the bumpless transfer compensator in terms of LMI optimization.

**Theorem 9** Given the LFT open-loop system $G$ with a stabilizing switching controller $K_{nom}$. If there exist positive-definite matrices $R_{11} \in \mathbb{S}_+^{n_p \times n_p}$ and $S \in \mathbb{S}_+^{n \times n}$, $J_{11} \in L_{\Theta}, L \in$
\( D_\Theta \) satisfying

\[
\begin{bmatrix}
N_{R11} & 0 \\ 0 & I
\end{bmatrix}^T
\begin{bmatrix}
R_{11}A_p^T + A_pR_{11} & R_{11}C_p^T & R_{11}C_p^T & B_{p0}J_{11} & B_{p1} \\
C_p0R_{11} & -J_{11} & 0 & D_{p00}J_{11} & D_{p01} \\
C_p1R_{11} & 0 & -\gamma I_{n_e} & D_{p10}J_{11} & D_{p11} \\
J_{11}B_{p0}^T & J_{11}D_{p01}^T & J_{11}D_{p10}^T & J_{11} & 0 \\
B_{p1}^T & D_{p01}^T & D_{p11}^T & 0 & -\gamma I_{n_d}
\end{bmatrix}
\begin{bmatrix}
N_{R11} \\ 0 \\ I
\end{bmatrix} < 0
\]

\[(4.45)\]

\[
\begin{bmatrix}
N_S & 0 \\ 0 & I
\end{bmatrix}^T
\begin{bmatrix}
A^TS + SA & SB_0 & SB_1 & C_0^TL & C_1^T \\
B_0^TS & -L & 0 & D_{00}^TL & D_{10}^T \\
B_1^TS & 0 & -\gamma I_{n_d} & D_{01}^TL & D_{11}^T \\
LC_0 & LD_{00} & LD_{01} & -L & 0 \\
C_1 & D_{10} & D_{11} & 0 & -\gamma I_{n_e}
\end{bmatrix}
\begin{bmatrix}
N_S \\ 0 \\ I
\end{bmatrix} < 0
\]

\[(4.46)\]

\[
\begin{bmatrix}
R_{11} & I_{np} & 0 \\
I_{np} & S & 0
\end{bmatrix} \geq 0
\]

\[(4.47)\]

\[
\begin{bmatrix}
J_{11} & I_{n\theta} & 0 \\
I_{n\theta} & L & 0
\end{bmatrix} \geq 0
\]

\[(4.48)\]
where \( \mathcal{N}_{R11} = \ker \begin{bmatrix} B_{p2}^T & D_{p01}^T & D_{p11}^T \end{bmatrix} \) and \( \mathcal{N}_S = \ker \begin{bmatrix} C_2 & D_{20} & D_{21} \end{bmatrix} \), then there exists an \( n_p \)th-order bumpless transfer compensator \( K_{bt} \) to stabilize the closed-loop system and have the induced \( L_2 \) norm from \( d \) to \( e \) less than \( \gamma \).

**Proof:** Denote

\[
\Theta = \begin{bmatrix} A_{bt} & B_{bt} & B_{bt0} \\ C_{bt1} & D_{bt11} & D_{bt10} \\ C_{bt0} & D_{bt01} & D_{bt00} \end{bmatrix}
\]

We apply condition (4.5) to the closed-loop system \( T \), and have the following inequality

\[
\Phi + \mathcal{P}^T \Theta \mathcal{Q} + \mathcal{Q}^T \Theta^T \mathcal{P} < 0 \quad (4.49)
\]

with

\[
\Phi = \begin{bmatrix} \begin{bmatrix} A^T X_d + X_d A & X_d B_0 & X_d B_1 & C_0^T L_{cl} & C_1^T \\ B_0^T X_d & -L_{cl} & 0 & D_{00}^T L_{cl} & D_{10}^T \\ B_1^T X_d & 0 & -\gamma I & D_{01}^T L_{cl} & D_{11}^T \\ L_{cl} C_0 & L_{cl} D_{00} & L_{cl} D_{01} & -L_{cl} & 0 \\ C_1 & D_{10} & D_{11} & 0 & -\gamma I \end{bmatrix} \\ \end{bmatrix}
\]

\[
\mathcal{P} = \begin{bmatrix} P_1 X_d & 0 & 0 & P_2 L_{cl} & P_3 \end{bmatrix},
\]

\[
\mathcal{Q} = \begin{bmatrix} Q_1 & Q_2 & Q_3 & 0 & 0 \end{bmatrix}
\]

Partition the matrix \( X_d \) compatibly to the states of interconnected system \( G \) and anti-windup compensator \( K_{bt} \) as \( n = n_p + n_k \) and \( n_{bt} \), and let

\[
X_{cl} = \begin{bmatrix} S & N \\ NT & ? \end{bmatrix}, \quad X_{cl}^{-1} = \begin{bmatrix} R & M \\ M^T & ? \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & M \\ R_{12}^T & R_{22} & ? \end{bmatrix}
\]

where \( MN^T = I - RS \). Partition the matrix \( L_{cl} \) compatibly to the parameters of interconnected system \( G \) and bumpless transfer compensator \( K_{bt} \) as \( n_\theta = n_{\theta,p} + n_{\theta,k} \) and \( n_{\theta, bt} \), and let

\[
L_{cl} = \begin{bmatrix} L & L_2 \\ L_2^T & ? \end{bmatrix}, \quad L_{cl}^{-1} = \begin{bmatrix} J & J_2 \\ J_2^T & ? \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} & J_2 \\ J_{12}^T & J_{22} & ? \end{bmatrix}
\]
where $J_2L_2^T = I - JL$. According to Elimination Lemma [8, 61], the inequality (4.49) is equivalent to

$$N_P^T \Phi N_P < 0 \quad \text{and} \quad N_Q^T \Phi N_Q < 0$$

where $N_P$ and $N_Q$ are the null spaces of matrices $P$ and $Q$, which are

$$N_P = \text{diag} \{ X_{cl}^{-1}, I, I, J_{cl}, I \}$$

$$N_Q = \text{diag} \{ X_{cl}^{-1}, I, I, J_{cl}, I \}$$

Through lengthy algebraic manipulations, it can be shown that

$$N_P^T \Phi N_P = \begin{bmatrix}
N_{R11,1} & 0 & 0 & N_{R11,2} & 0 & N_{R11,3} & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & I
0 & 0 & 0 & 0 & 0 & 0 & I
\end{bmatrix}$$

$$\times
\begin{bmatrix}
\begin{bmatrix}
RAT & RA_T + AR & RC_T & RC_T & B_0 & J_B & B_1
\begin{bmatrix}
C_0R & -J & 0 & D_{00} & J & 0
C_1R & 0 & -\gamma J & D_{10} & J & 0
JB_0^T & JD_{00} & JD_{10} & J & 0
B_1^T & D_{01} & D_{11} & 0 & -\gamma I
\end{bmatrix}
\end{bmatrix}
\end{bmatrix}$$

$$\times
\begin{bmatrix}
N_{R11,1} & 0 & 0 & N_{R11,2} & 0 & N_{R11,3} & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & I
0 & 0 & 0 & 0 & 0 & 0 & I
\end{bmatrix}$$

$$< 0$$
\[ N_{\Phi}^T \Phi N_{Q} = \begin{bmatrix} N_S & 0 & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} A^T S + SA & SB_0 & SB_1 \\ B_0^T S & -L & 0 \\ B_1^T S & 0 & -\gamma I_{n_d} \end{bmatrix} \begin{bmatrix} C_0^T L & C_1^T \\ D_{00}^T L & D_{10}^T \\ D_{01}^T L & D_{11}^T \end{bmatrix} \begin{bmatrix} N_S & 0 \\ 0 & I \end{bmatrix} < 0 \]

which are the same as the conditions (4.45) and (4.46), respectively.

Given the definition for matrices \( X_{cl} \), \( X_{cl}^{-1} \) and \( L_{cl} \), \( J = L_{cl}^{-1} \), the coupling condition between \( R \) and \( S \) would be
\[
\begin{bmatrix} R & I \\ I & S \end{bmatrix} \geq 0 \quad \text{and} \quad \text{rank}(R - S^{-1}) \leq n_{bt}
\]

Since only the \((1,1)\) element of \( R \) matrix is constrained in the LMIs (4.45)-(4.48), it is always possible to augment matrix \( R_{11} \) to \( R \) while remain satisfying the above coupling condition. For example, one may choose
\[
R = \begin{bmatrix} R_{11} \\ 0 \end{bmatrix} S^{-1} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} S^{-1} \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ I \end{bmatrix} \begin{bmatrix} 0 \\ I \end{bmatrix}
\]

The resulting \( R \) matrix is positive-definite because of the condition (4.47). Also \( R - S^{-1} \geq 0 \) is satisfied for the selected \( R \) matrix. The rank condition is trivially satisfied if one chooses \( n_{bt} = n_p \). The situation between \( L \) and \( J_{11} \) is similar to the relationship between \( S \) and \( R_{11} \). So we obtain the desired synthesis condition for the bumpless transfer compensator. Q.E.D.

### 4.4.3 Bumpless Transfer Compensator Formula

After solving the synthesis condition (4.45)-(4.48), the bumpless transfer compensator can be determined by solving the feasibility problem from the closed-loop LMI (4.49), which is the approach taken in [31]. Alternatively, the bumpless transfer compensator can be constructed explicitly by the following theorem. This construction approach is advantageous because it avoids possible numerical ill-condition when solving the above feasibility LMI problem.
**Theorem 10** Given $R, J$ constructed from the solutions $R_{11}, J_{11}$ of the LMIs (4.45)-(4.48) and solution $S, L, \gamma$. Let $MNT = I_n - RS$ with $M, N \in S^{n \times np}$ and $H^T = \begin{bmatrix} I_{np} & 0 \end{bmatrix}$, $G^T = \begin{bmatrix} I_{n_0} & 0 \end{bmatrix}$, then one $n_p$th-order bumpless transfer compensator with $n_0$ parameter dimension can be constructed through the following scheme:

1. Compute a feasible $\hat{D}_{bt} \in R^{n_u \times n_u}$ such that

   $$\Pi = -\begin{bmatrix} -L & G \\ GT & GTJG \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \gamma I > 0,$$

   $$\begin{bmatrix} \hat{D}_{00} & \hat{D}_{10} \\ \hat{D}_{01} & \hat{D}_{11} \end{bmatrix} + \hat{D}_{12}\hat{D}_{bt}\hat{D}_{21}$$

2. Compute the least-square solutions of the following linear equations for $\hat{B}_{bt} \in R^{n \times n_u}, \hat{C}_{bt} \in R^{n \times np}$

   $$\begin{bmatrix} 0 & \hat{D}_{21} & 0 \end{bmatrix} \begin{bmatrix} \hat{B}_{bt}^T \\ \hat{D}_{12}^T \\ 0 \end{bmatrix} = -\begin{bmatrix} \hat{C}_2 \\ B_0 \ 0 \ B_1 \end{bmatrix}^T \begin{bmatrix} \hat{D}_{bt} \hat{C}_2 \\ C_1 \end{bmatrix}$$

   $$-L \quad G$$

   $$G^T \quad G^TJG$$

   $$\hat{D}_{12}^T \quad \hat{D}_{bt}$$

   $$0 \quad \hat{D}_{21}$$
\[
\begin{bmatrix}
0 & 0 & \hat{D}_{t2}^T \\
0 & -\Pi & \\
\hat{D}_{t2} & \\
\end{bmatrix}
\begin{bmatrix}
\hat{C}_{bt} \\
\end{bmatrix}
= 
\begin{bmatrix}
\hat{B}_2^T \\
\hat{B}_1^T H \\
0 C_0^T G & C_1^T \\
\end{bmatrix}^T \begin{bmatrix}
\bar{B}_2^T \\
G^T J B_0^T H \\
B_1^T H \\
0 C_0^T G & C_1^T \\
\end{bmatrix} H
\]

and the matrix \( \hat{A}_{bt} \in \mathbb{R}^{n \times n_p} \) as

\[
\hat{A}_{bt} = -A^T H - \hat{C}_2^T \hat{D}_{bt} \hat{B}_2^T - L_1 \Pi^{-1} L_2^T
\]

3. Convert the transformed bumpless transfer compensator gain to its original state-space data by

\[
\begin{bmatrix}
A_{bt} & B_{bt} \\
C_{bt} & D_{bt} \\
\end{bmatrix}
= \begin{bmatrix}
N & SB_2 & 0 \\
0 & [0 \ I \ 0] & 0 \\
0 & LD_{02} & L_2 \\
\end{bmatrix}^{-1}
\begin{bmatrix}
\hat{A}_{bt} & \hat{B}_{bt} \\
\hat{C}_{bt} & \hat{D}_{bt} \\
\end{bmatrix}
- \begin{bmatrix}
S A R H & 0 & SB_0 J G \\
0 & 0 & 0 \\
L C_0 R H & 0 & L D_{00} J G \\
\end{bmatrix}
\times \begin{bmatrix}
M^T H & 0 & 0 \\
C_2 R H & I & D_{20} J G \\
0 & 0 & J_2^T G \\
\end{bmatrix}^{-1}
\]

**Proof:** The derivation of the bumpless transfer controller formula follows closely the procedure outlined in [24].

\[
Z_1 = \begin{bmatrix}
I_n & RH \\
0 & M^T H \\
\end{bmatrix}, \quad Z_2 = \begin{bmatrix}
S & H \\
N^T & 0 \\
\end{bmatrix}, \quad W_1 = \begin{bmatrix}
I_n & J G \\
0 & J_2^T G \\
\end{bmatrix}, \quad W_2 = \begin{bmatrix}
L & G \\
L_2^T & 0 \\
\end{bmatrix}
\]
Then it can be shown that \( X_{cd}Z_1 = Z_2 \) and \( L_{cd}W_1 = W_2 \). Also we have the following congruent transformation

\[
Z^T_1 X_{cd}Z_1 = \begin{bmatrix} S & H \\ H^T & H^T RH \end{bmatrix}, \quad W^T_1 L_{cd}W_1 = \begin{bmatrix} L & G \\ G^T & G^T JG \end{bmatrix},
\]

\[
\begin{bmatrix}
Z^T_1 X_{cd}A_{cd}Z_1 & Z^T_1 X_{cd}B_{0,cd}W_1 & Z^T_1 X_{cd}B_{1,cd} \\
W^T_1 L_{cd}C_{0,cd}Z_1 & W^T_1 L_{cd}D_{00,cd}W_1 & W^T_1 L_{cd}D_{01,cd} \\
C_{1,cd}Z_1 & D_{10,cd}W_1 & D_{11,cd}
\end{bmatrix}
\]

\[
= \begin{bmatrix} \hat{A} & \hat{B}_0 & \hat{B}_1 \\ \hat{C}_0 & \hat{D}_{00} & \hat{D}_{01} \\ \hat{C}_1 & \hat{D}_{10} & \hat{D}_{11} \end{bmatrix} + \begin{bmatrix} I_n & 0 & 0 \\ 0 & \hat{B}_2 & 0 \\ 0 & 0 & \hat{D}_{12} \end{bmatrix} \begin{bmatrix} \hat{A}_{bt} & \hat{B}_{bt} & 0 & I & 0 \\ \hat{C}_{bt} & \hat{D}_{bt} & \hat{C}_2 & 0 & \hat{D}_{20} \\ 0 & 0 & \hat{C}_1 & \hat{D}_{10} \end{bmatrix}
\]

\[
= \begin{bmatrix} SA & 0 & SB_0 & 0 & SB_1 \\ H^T A & H^T ARH & H^T B_0 & H^T B_0 JG & H^T B_1 \\ LC_0 & 0 & LD_{00} & 0 & LD_{01} \\ G^T C_0 & G^T C_0 RH & G^T D_{00} & G^T D_{00} JG & G^T D_{01} \\ C_1 & C_1 RH & D_{10} & D_{10} JG & D_{11} \end{bmatrix}
\]

\[
+ \begin{bmatrix} I_n & 0 & 0 \\ 0 & B_{p2} & 0 \\ 0 & 0 & I_n \\ 0 & D_{p02} & 0 \\ 0 & D_{p12} & 0 \end{bmatrix} \begin{bmatrix} \hat{A}_{bt} & \hat{B}_{bt1} & \hat{B}_{bt0} \\ \hat{C}_{bt1} & \hat{D}_{bt11} & \hat{D}_{bt10} \\ \hat{C}_{bt0} & \hat{D}_{bt01} & \hat{D}_{bt00} \end{bmatrix} \begin{bmatrix} 0 & I & 0 & 0 & 0 \\ C_2 & 0 & D_{20} & 0 & D_{21} \\ 0 & 0 & 0 & I & 0 \end{bmatrix}
\]

where

\[
\begin{bmatrix} \hat{A}_{bt} & \hat{B}_{bt1} & \hat{B}_{bt0} \\ \hat{C}_{bt1} & \hat{D}_{bt11} & \hat{D}_{bt10} \\ \hat{C}_{bt0} & \hat{D}_{bt01} & \hat{D}_{bt00} \end{bmatrix} = \begin{bmatrix} SARH & 0 & SB_{0,JG} \\ 0 & 0 & 0 \\ LC_0 RH & 0 & LD_{00} JG \end{bmatrix}
\]

\[
+ \begin{bmatrix} N & SB_2 & 0 \\ 0 & [I & 0] & 0 \\ 0 & LD_{02} & L_2 \end{bmatrix} \begin{bmatrix} A_{bt} & B_{bt1} & B_{bt0} \\ C_{bt1} & D_{bt11} & D_{bt10} \\ C_{bt0} & D_{bt01} & D_{bt00} \end{bmatrix} \begin{bmatrix} M^T H & 0 & 0 \\ C_2 RH & I & D_{20,JG} \\ C_2 RH & I & D_{20,JG} \end{bmatrix}
\]
Multiply diag \( \{ Z_i^T, I, I, I \} \) from the left side, and its conjugate transpose from the right side of eq. (4.49), we get

\[
\begin{bmatrix}
H_{11} + \mathcal{L}_1 \Pi^{-1} \mathcal{L}_1^T & * \\
H_{21} + \mathcal{L}_2 \Pi^{-1} \mathcal{L}_2^T & H_{22} + L_2 \Pi^{-1} L_2^T
\end{bmatrix} < 0 
\tag{4.51}
\]

The intermediate variables in above equation are defined as

\[
\Pi = -\begin{bmatrix}
-W_1^T L_{cl} W_1 & 0 & 0 & -\gamma I \\
0 & -\gamma I & 0 & * \\
* & 0 & -\gamma I & 0
\end{bmatrix}
\begin{bmatrix}
\hat{D}_{00}^T & \hat{D}_{10}^T & \hat{D}_{12} \hat{D}_{21} \\
\hat{D}_{01}^T & \hat{D}_{11}^T & & \\
-W_1^T L_{cl} W_1 & 0 & & \\
0 & & & 
\end{bmatrix}
\]

\[
\mathcal{H}_{11} = SA + A^T S + \hat{B}_{bt} \hat{C}_2 + \hat{C}_2^T \hat{B}_{bt}^T
\]

\[
\mathcal{H}_{22} = H^T A R H + H^T R A^T H + \hat{B}_{2} \hat{C}_{bt} + \hat{C}_{bt}^T \hat{D}_{12}
\]

\[
\mathcal{H}_{21} = H^T A + \hat{A}_{bt} + \hat{B}_{2} \hat{D}_{bt} \hat{C}_2
\]

\[
\mathcal{L}_1 = \begin{bmatrix}
S & B_0 & 0 & B_1 \\
C_0^T & C_0^T G & C_1^T & + C_2^T \hat{D}_{bt} \hat{D}_{12}^T
\end{bmatrix}
\]

\[
\mathcal{L}_2 = \begin{bmatrix}
H^T B_0 & H^T B_0 J G & H^T B_1 \\
H^T R & 0 & C_0^T G & C_1^T & + \hat{C}_{bt} \hat{D}_{12}^T
\end{bmatrix}
\]

Clearly, the lower \((3 \times 3)\) matrix of the inequality (4.51) must be negative definite, this determines feasible \(\hat{D}_{bt}\). Let the \((2, 1)\) element equal to zeros, one can solve for \(\hat{A}_{bt}\). This also leads to decoupled LMIs from the inequality (4.51). Then \(\hat{B}_{bt}, \hat{C}_{bt}\) terms can be solved from the \((1, 1)\) and \((2, 2)\) elements of the decoupled inequality (4.51). Note that both inequalities have regular solutions [24].

The \((1, 1)\) element of the above matrix inequality corresponds to LMI (4.46) after elimination of the variables \(\hat{B}_{bt}\) and \(\hat{D}_{bt}\). It can also be shown that the \((2, 2)\) element is equivalent to LMI (4.45) by eliminating \(\hat{C}_{bt}\) and \(\hat{D}_{bt}\). \(Q.E.D.\)

### 4.5 Application to Uninhabited Combat Aerial Vehicle

This section demonstrates the advantage of online switching control using the UCAV6 linear models, which have been developed at Texas A&M University as part of a
UCAV basic research project with the Office of Naval Research [100]. The UCAV6 is representative of a subsonic, vertical/short take-off and landing (VSTOL) uninhabited combat aerospace vehicle (UCAV) with medium altitude cruise and weapons delivery capabilities.

4.5.1 Dynamics Modeling of UCAV

In this section, we build up an UCAV longitudinal LFT model, which is constructed based on a set of linear models [100]. Some details about aircraft dynamics and control can be found in [83, 84, 85]. The longitudinal dynamics of UCAV is shown in Fig. 4.3

![Longitudinal Dynamics of UCAV](image)

Figure 4.3: Longitudinal Dynamics of UCAV.

There are twelve sets of longitudinal linear models corresponding to the flight conditions listed in Table 4.3. The states and inputs of the original model are listed in Table 4.1 and 4.2.

<table>
<thead>
<tr>
<th>State</th>
<th>Units</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u$</td>
<td>ft/s</td>
<td>velocity along x-axis of aircraft</td>
</tr>
<tr>
<td>$w$</td>
<td>ft/s</td>
<td>velocity along z-axis of aircraft</td>
</tr>
<tr>
<td>$q$</td>
<td>deg/s</td>
<td>pitch rate</td>
</tr>
<tr>
<td>$\theta$</td>
<td>deg</td>
<td>Euler angle rotation of aircraft reference frame about inertial y-axis</td>
</tr>
</tbody>
</table>
Table 4.2: Longitudinal aircraft inputs

<table>
<thead>
<tr>
<th>Input</th>
<th>Units</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta_e$</td>
<td>%</td>
<td>elevator stick input</td>
</tr>
<tr>
<td>$\delta_t$</td>
<td>%</td>
<td>throttle input</td>
</tr>
<tr>
<td>$\delta_n$</td>
<td>deg</td>
<td>nozzle angle input</td>
</tr>
</tbody>
</table>

Table 4.3: Twelve LTI models at different flight conditions

<table>
<thead>
<tr>
<th>Flight Condition</th>
<th>$V$ (ft/s)</th>
<th>Mach</th>
<th>Altitude</th>
<th>$\alpha$ (deg)</th>
<th>$\delta_e$(%)</th>
<th>$\delta_t$(%)</th>
<th>$\delta_n$(deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_1$</td>
<td>10</td>
<td>0.009</td>
<td>100</td>
<td>0.701</td>
<td>39.446</td>
<td>89.567</td>
<td>88.0</td>
</tr>
<tr>
<td>$G_2$</td>
<td>25</td>
<td>0.022</td>
<td>100</td>
<td>0.955</td>
<td>39.492</td>
<td>90.149</td>
<td>86.0</td>
</tr>
<tr>
<td>$G_3$</td>
<td>50</td>
<td>0.045</td>
<td>100</td>
<td>2.565</td>
<td>37.598</td>
<td>90.977</td>
<td>82.0</td>
</tr>
<tr>
<td>$G_4$</td>
<td>75</td>
<td>0.067</td>
<td>100</td>
<td>5.786</td>
<td>36.099</td>
<td>90.931</td>
<td>80.0</td>
</tr>
<tr>
<td>$G_5$</td>
<td>100</td>
<td>0.090</td>
<td>100</td>
<td>6.649</td>
<td>33.886</td>
<td>89.970</td>
<td>80.0</td>
</tr>
<tr>
<td>$G_6$</td>
<td>150</td>
<td>0.134</td>
<td>100</td>
<td>11.692</td>
<td>28.661</td>
<td>82.867</td>
<td>70.0</td>
</tr>
<tr>
<td>$G_7$</td>
<td>200</td>
<td>0.179</td>
<td>100</td>
<td>18.515</td>
<td>65.724</td>
<td>56.260</td>
<td>0.0</td>
</tr>
<tr>
<td>$G_8$</td>
<td>250</td>
<td>0.224</td>
<td>100</td>
<td>12.026</td>
<td>57.600</td>
<td>45.753</td>
<td>0.0</td>
</tr>
<tr>
<td>$G_9$</td>
<td>300</td>
<td>0.273</td>
<td>5000</td>
<td>9.728</td>
<td>53.575</td>
<td>49.225</td>
<td>0.0</td>
</tr>
<tr>
<td>$G_{10}$</td>
<td>500</td>
<td>0.446</td>
<td>5000</td>
<td>4.044</td>
<td>47.941</td>
<td>62.851</td>
<td>0.0</td>
</tr>
<tr>
<td>$G_{11}$</td>
<td>700</td>
<td>0.638</td>
<td>5000</td>
<td>1.607</td>
<td>39.182</td>
<td>77.315</td>
<td>0.0</td>
</tr>
<tr>
<td>$G_{12}$</td>
<td>900</td>
<td>0.820</td>
<td>5000</td>
<td>0.525</td>
<td>35.250</td>
<td>88.366</td>
<td>0.0</td>
</tr>
</tbody>
</table>

The original models are linear time invariant, which are linearized at different operation points. The plant model used in this example is interpolated from original LTI models in the following way.

1. Pick up first three LTI models with flight conditions $V = 10, 25, 50 (ft/s)$.

2. Construct LFT models by polynomial curve fitting of three neighboring LTI models and treat velocity of aircraft $V$ (ft/s) as scheduling parameter.

$$
\hat{G} = \hat{G}_1 + \left(-\frac{11}{3} \hat{G}_1 + \frac{64}{15} \hat{G}_2 - \frac{3}{5} \hat{G}_3\right) \hat{\theta} + \left(\frac{8}{3} \hat{G}_1 - \frac{64}{15} \hat{G}_2 + \frac{8}{5} \hat{G}_3\right) \hat{\theta}^2
$$

with $\hat{\theta} \in [0, 1]$. Define

$$
\hat{G}_1 = \hat{G}(\hat{\theta}), \quad \hat{\theta} \in [0, 3/8] \\
\hat{G}_2 = \hat{G}(\hat{\theta}), \quad \hat{\theta} \in [3/8, 1]
$$
Then system $G_1$ is derived by re-scaling $\tilde{G}_1$ to $\theta_1 \in [-1, 1]$, and $\tilde{G}_2$ is derived by the same way, where $\theta_1 = (V - 17.5)/7.5 \in [-1, 1]$ and $\theta_2 = (V - 37.5)/12.5 \in [-1, 1]$.

3. Combine $G_1$ and $G_2$ to be our switching plant with switching surface at $V = 25$.

$$G_\alpha = \alpha_1(t)G_1 + \alpha_2(t)G_2$$  \hspace{1cm} (4.52)

where $\alpha = [1; 0]$ when $10 \leq V \leq 25$ and $\alpha = [0; 1]$ when $25 \leq V \leq 50$.

Our control objective is to let $w$ to track a reference input, and minimize the effect of disturbances and measurement noise on the tracking error. Borrowed from the time-invariant ideas, these objectives are quantified by rational weighting functions and the weighted interconnection is given in Figure 4.4, where

$$W_e(s) = \frac{0.1s + 1}{s + 0.1}, \quad W_u(s) = \frac{50(s + 1000)}{s + 10000}I_3$$

Figure 4.4: Weighted structure of the UCAV tracking problem.

4.5.2 Performance Level Comparison of Different Methods

The trajectory of aircraft speed is chosen as fig. 4.5. The switching between two subsystems will occur when velocity of aircraft cross the value of 25ft/s. From fig. 4.5 (a), we can see system switches twice at 1.88s and 13.12s respectively. At the first switching instant, the system switches from $G_1$ to $G_2$. At the second switching, the system switches back to $G_1$.

For the UCAV problem, three different switching control techniques will be examined. They include:
Figure 4.5: Aircraft velocity trajectory.

Table 4.4: Performance comparison

<table>
<thead>
<tr>
<th>Method</th>
<th>Nominal Performance Level</th>
<th>CPU Time (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1-1.88 (s)</td>
<td>1.88-13.12(s)</td>
</tr>
<tr>
<td>Common Lyapunov function</td>
<td>12.775</td>
<td>12.998</td>
</tr>
<tr>
<td>Online synthesis</td>
<td>10.001</td>
<td>11.519</td>
</tr>
</tbody>
</table>

1. Switching gain-scheduled output feedback using common Lyapunov function

2. Switching gain-scheduled output feedback using multiple Lyapunov functions by online optimization

3. Switching gain-scheduled output feedback using multiple Lyapunov functions by online optimization with bumpless transfer (BT) compensator

The nominal performance levels are compared in Table 4.4. For the first case using common Lyapunov function, the sub-controller $K_1$ and $K_2$ are synthesized offline separately based on subsystem $G_1$ and $G_2$. The $L_2$ gain for closed loop system $\mathcal{F}_l(G_1, K_1)$ and $\mathcal{F}_l(G_2, K_2)$ are 12.775 and 12.998, respectively. Therefore, the overall $L_2$ performance should be less than 12.998. The CPU time in the first case is offline synthesis time. Less computational effort is needed when implementing the controller.

For the second case using multiple Lyapunov function, the sub-controller $K_{1,m}$ and
$K_{2,m}$ are synthesized online separately based on subsystem $G_1, G_2$ and switching instant $t_m$. The nominal $L_2$ gain for closed loop system $F_l(G_1, K_{1,0})$, $F_l(G_2, K_{2,1})$ and $F_l(G_1, K_{1,2})$ are 10.001, 11.519 and 10.004, respectively. We can not guarantee performance level in this case because future switchings are unknown. However, if no switching occurs after 20s, the overall $L_2$ performance should be less than 11.519. The CPU time in this case is online synthesis time, which is the time needed to synthesize the controller working for next switching interval.

The online bumpless transfer compensator is used in third case. The performance level has been shown better than the second case because of BT compensator, although the computational time is larger.

4.5.3 Simulation Results

In the simulation study, the condition to disable the anti-windup BT compensator is chosen as $\|u(t) - u(t_m)\| < \epsilon$, where $\epsilon = 0.001$.

Comparing three different control approaches, we observe online control design method using multiple Lyapunov function can achieve better performance than single Lyapunov function. The bumpless transfer compensator reduced the bump caused by switching successfully. As expected, the bumpless transfer only acts as it is needed. In bumpless transfer design of this example, we only minimize the bump of error output. The bump of controller output is enlarged because we did not quantify the control force as a part of evaluated error. If minimizing control force is needed in anti-windup BT design, weighted control force can be added into the error output channel of (4.42)-(4.43). But there is a trade-off as $H_\infty$ control, we may not minimize error output and control force at same time.

Next we consider the step response with disturbance. A uniform random noise with magnitude 0.001 is chosen for disturbance $d$ and a uniform random noise with magnitude 0.05 is chosen for sensor noise $d_n$. As shown in Fig. 4.7 (a), our proposed controller still performs better under disturbance. Fig. 4.7 (b)-(g) shows the control inputs from different controllers. Our proposed controller is less sensitive to the disturbance. The noise magnitude of our controller output is much smaller than other controllers under the same measurement noise, especially for elevator stick and nozzle angle input.
4.6 Summary

In this chapter, we first developed analysis condition of stability and performance of switched LPV systems with LFT parameter dependence. Then we employed online optimization method to develop control synthesis conditions using multiple Lyapunov functions. Both gain-scheduling state feedback and output feedback control of switched LFT systems have been considered. The controller can be synthesized at switching time, which works till next switching time. The control synthesis conditions were formulated and solved using LMI optimizations. It has been shown that online switching control design method provide great performance improvement comparing with single Lyapunov function and has less computational complexity comparing with other multiple Lyapunov function method. The new approach can guarantee stability and optimize system performance under arbitral switching between subsystems. An online method to design a bumpless transfer compensator is also presented. It can be used to reduce the jump of system output caused by switching between subcontrollers. As demonstrated on an UCAV example, the newly proposed control design approach provides better controlled performance than existing controller synthesized from single Lyapunov function and is easy to implement.
Figure 4.6: Tracking response and control input of different gain-scheduled controllers.
(a) tracking output

(b) elevator stick input of case 1

(c) elevator stick input of case 2

(d) throttle input of case 1

(e) throttle input of case 2
(f) nozzle angle input of case 1

(g) nozzle angle input of case 2

Figure 4.7: Tracking response of different controllers with disturbance.
Chapter 5

Conclusion

Advanced LPV methodologies were studied in this thesis by introducing particular Lyapunov functions and switching control scheme. The main goal of this research was to advance LPV control techniques, and apply new techniques to practical problems, such as ship steering and UCAV. To conclude, we briefly summarize our main contributions, and outline some directions of future research.

5.1 Thesis Contributions

This thesis makes several contributions in the areas of stability analysis and controller synthesis for LPV systems with LFT structure. These contributions are discussed below.

1. LPV Controller Synthesis with Parameter Variation Rate

Parameter-dependent Lyapunov functions has been used in the analysis and control design for parameter-dependent plants, which shows great advantages comparing to single quadratic Lyapunov functions. However, the solution to general LPV control analysis and control synthesis problems is formulated as infinite-dimensional parameter-dependent LMIs, which have high computational complexity. Gridding of parameter space is needed to solve the conditions, which may lead to the problem-dependent controller interpolation.
Motivated from the existing systematic approach for LFT system control and LPV control using PDLF, the study of applying PDLF to LFT systems is proposed in this research. Different from the existing research, we restrict the PDLF in the form of quadratic LFT. [95] This type of Lyapunov functions include polynomial parameter dependency as a special case. Part of quadratic LFT PDLF is pre-defined by user, which can be used to tune performance of the closed-loop system. The remaining problem is how to find a optimal quadratic LFT PDLF for the given LFT system.

The biggest advantage of applying quadratic LFT PDLF to the analysis and control design for LFT systems is that the problem can be converted systematically to convex conditions with finite number of LMIs. The derivation of the exciting result mainly benefits from the nice properties of LFT and full block S-procedure. The derived convex analysis and synthesis condition can be solved globally by LMI optimization. The stability and performance can be guaranteed over entire parameter region and no controller interpolation is needed when implementation.

Same as general PDLF, a quadratic LFT PDLF can also provide less conservatism than common quadratic Lyapunov functions. [95] Quadratic LFT PDLF can lead to LPV controller scheduled by both parameter and its variation rate, which requires both of them are measurable in real time. Known bounds on the rate of parameter variation may significantly reduce the conservatism of the optimized results, which leads to a controller with better performance. Lastly, if the system is well-posed, observable and controllable, then the algorithms will always be feasible.

The LFT control scheme with PDLF was applied to the ship steering model. The model is varying in a wide range of ship’s velocity. The corresponding LPV model has quadratic dependence in parameter, which is equivalent to a LFT model with four dimensional parameter block. A LPV controller scheduled by ship’s velocity and acceleration is designed using the proposed approach. The results showed that the new LPV controller could improve the steering performance significantly comparing to the existing LFT approaches using common quadratic Lyapunov functions.

2. Nonlinear LPV Controller Synthesis

Parallel to parameter-dependent Lyapunov function approaches, the use of homogeneous or higher-order Lyapunov function in the analysis and control design for LPV
systems has been taken into account. Although better performance is expected, it is difficult to formulate the condition to be efficient in computation, i.e. LMIs. CHLF is a special type of quadratic-like Lyapunov function, whose level set is the convex hull of a set of ellipsoids. This nice property provides the possibility to extend the level set of single quadratic functions. The conjugate function of CHLF is in quadratic form, which is easy to deal with. It is possible to formulate the analysis and synthesis condition to LMI like condition using duality tools.

Our contribution is to apply CHLF approach to the analysis and control synthesis of LFT system and solve the gain-scheduled output feedback synthesis problem. The synthesis condition was formulated and solved using LMI optimization combined with linear search over scalar variables, which is efficient in computation. Although it is difficult to get global optimal solution by linear search, the suboptimal solution still perform better than single quadratic Lyapunov function approach. On the other hand, the synthesis condition by CHLF recovers the existing condition using single quadratic Lyapunov function by choosing single basis function of CHLF. Therefore, the solution of CHLF approach will never be worse than single quadratic Lyapunov function approach. [16]

The controller synthesized by CHLF is a nonlinear LPV controller. Benefit from the continuously differentiable property of CHLF, the nonlinear controller is continuous. Different from Chapter 1, the nonlinear LPV controller has a better close loop $H_\infty$ performance than normal gain-scheduling controller without other additional information, i.e. parameter variation rate. An second order plant is used to demonstrate the advantage of our proposed approach. Both of robust state feedback and gain-scheduled output feedback are examined on this plant.

3. Online Switching LPV Control

As is well known, switching control can be used to improve LPV control by partitioning the parameter set into several subsets. Sub-controllers are designed based on each partitioned subsystem. A single Lyapunov function can provide stability under arbitrary switching. However, using single Lyapunov function is too conservative and may not achieve good performance in some subsystems. Multiple Lyapunov function has been shown to be very useful for both stability analysis and performance optimization. When multiple Lyapunov functions are used, the critical issue in applying
switching control is the stability problem and coupling between subsystems. Switching with dwell time is a practical way to stabilize the time-dependent switched system using multiple Lyapunov functions. However, arbitrary switching cannot be achieved in switching with dwell time.

Proposed online switching method can provide stability guarantee over arbitrary switching as well as performance improvement by multiple Lyapunov functions. \[17\] Our proposed method can assign different Lyapunov function to different subsystem, which renders the subcontroller to be optimized in parameter subset. Neither knowledge nor constraint on switching instant is required because controllers are synthesized after switching event occurs.

Motivated from anti-windup compensator design for constraint control systems, a online BT compensator design method was proposed in this research. Because of the online optimization characteristic of proposed switching LPV control design method, the BT compensator could only be synthesized after synthesis of the switching controller. Similar to traditional anti-windup compensator design, the online BT compensator has the same order and same dimension of parameter block as plant.

The proposed online switching method with BT compensator was applied to a UCAV take-off and landing control problem. The LFT model of UCAV was built upon a set of LTI models by curve fitting. The result showed that online switching controller can achieve better performance than single Lyapunov function method with arbitrary switching. BT compensator reduced the output bump successfully without sacrificing the performance of online switching controller.

5.2 Future Research

Besides the results presented in this thesis, there are some possible improvement on our research. In LFT control using quadratic LFT PDLF, the pre-defined functions $T_R(\Theta)$ and $T_S(\Theta)$ may not be the optimal solution. How to optimize the functions needs further investigation. By using CHLF, the performance may be improved for a class of LFT systems. However, in some cases, the proposed approach may get identical result as single quadratic Lyapunov function. The optimization algorithm may push the different basis ellipsoids to a single ellipsoid, which makes the level set of CHLF to be a single ellipsoid. Specifying
the class of LFT system which could benefit from CHLF needs to be investigated. In our future work of switching LPV control, we will try to drop the information of plant state at switching instant in the gain-scheduled output feedback design approach. Estimated plant state may be used to construct switching controller and the overall stability of the closed loop system can still be guaranteed although Lyapunov function value may increase at some switching instant.

Another future direction of research would be to apply sum of square (SOS) tools and higher order Lyapunov functions to LPV control problem [40, 63, 71, 70], which is expected to achieve better performance and generate nonlinear LPV controllers. The promising feature of SOS decomposition is the polynomial-time computational complexity. Adaptive control schemes can also be taken into account of extending LPV control approaches to robust control problem [57, 3]. By estimating unknown parameters of the plant, LPV controller, which depends on estimated parameters, may robustly stabilize the plant with uncertainty.
Bibliography


