ABSTRACT

SLATEN, KELLI MARLENE. Effective Teaching and Uses of Instructional Representations in Secondary Geometry: A Comparison of a Novice and an Experienced Mathematics Teacher. (Under the direction of Sarah B. Berenson and Karen F. Hollebrands.)

The purpose of this qualitative case study is to investigate the uses of instructional representations of a novice secondary mathematics teacher and an experienced mathematics teacher in the content area of secondary geometry. Instructional representations are the tools teachers use to communicate their mathematical knowledge to students (Berenson & Nason, 2003). Classroom observations and semi-structured interviews were conducted and analyzed in order to find emergent patterns among the participants’ uses of instructional representations. Patterns are reported from each participant’s uses of instructional representations and from a cross-case analysis of both participants’ uses of instructional representations.

The Pirie-Kieren theory (Pirie & Kieren, 1994b) for students’ growth of mathematical understanding serves as the framework for the study. The Pirie-Kieren theory describes eight potential levels of student understanding based on the cognitive changes that occur during the processes of learning mathematics. These eight levels were adapted for the purposes of this study and used to describe how the participants’ uses of instructional representations allowed and fostered opportunities for students to engage within those levels of mathematical understanding.

In order to facilitate students’ growth of mathematical understanding, effective teachers have a well-developed knowledge base for teaching, including knowledge of multiple instructional representations and the connections between them (Lesh, Post, & Behr,
1987; Moseley & Brenner, 1997; NCTM, 2000; Rider, 2004; Wilson, Shulman, & Richert, 1987). The results of this study reveal the importance of examining *how* they use those representations in order to better understand how teacher knowledge contributes to effective teaching and student learning.
EFFECTIVE TEACHING AND USES OF INSTRUCTIONAL REPRESENTATIONS IN SECONDARY GEOMETRY: A COMPARISON OF A NOVICE AND AN EXPERIENCED MATHEMATICS TEACHER

by

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DEDICATION

To my father.
BIOGRAPHY

Kelli Marlene Slaten was born in New Kensington, PA, on July 10, 1969, the daughter of Robert Patrick Slaten and Ruby Marlene Broome. She resided in Charlotte, NC from the age of 4 and relocated to Nashville, TN in 1990. After receiving her Bachelor of Science degree in Recording Industry Management from Middle Tennessee State University in 1995, she worked as a sound engineer and electronics technician in Seattle, WA and later in Portland, OR. While living in Portland, she began working in education as a lab technician and assistant for the Electronics Department at Clark College in Vancouver, WA. It was then she decided to formally pursue her studies in mathematics and become a teacher.

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INTRODUCTION

“Teachers must know their subject well enough to reveal its mysteries to students and should
be able to assess where each student is in his or her knowledge of the subject and teach in
such a way that each can learn.”

-Nancy, a novice secondary mathematics teacher

This chapter presents an introduction to the research study by providing several
perspectives of teacher knowledge. Components of a knowledge base for teaching are
described in relation to the practice of teaching, and more specifically, to teaching with
instructional representations. The purpose of this study and the research focus are then
addressed in terms of teacher knowledge and the need for research concerning teachers’ uses
of instructional representations, particularly within the content area of secondary geometry.

Background

Historically, teacher knowledge and how it contributes to student learning has been
conceptualized and evaluated based on quantifiable and observable aspects of teaching and
learning to teach. These aspects include teaching behaviors used during instructional practice
and educational experiences such as the number of subject courses taken during college,
grades, and degrees earned (Ball, Lubienski, & Mewborn, 2001; Begle, 1972; Carter, 1990;
Eisenberg, 1977; Shulman, 1986, 1987). Traditionally, teacher effectiveness has been
evaluated based on the procedures teachers use during instruction. Their knowledge of the
subjects they teach was not considered as an important factor of teacher effectiveness
(Grossman, 1990; Shulman, 1986). These methods of evaluating teacher knowledge ignore the complexities of the practice of teaching to be an effective teacher (Ball et al., 2001; Even, 1993; Shulman, 1987). Moreover, research in the domain of teacher knowledge has not had much effect on the improvement of classroom teaching (Hiebert, Gallimore, & Stigler, 2002). Specific connections between teacher knowledge and student learning have not yet been established in mathematics education research.

The ways teachers’ knowledge affects their practice and their students’ learning are a result of many individual factors, more than just prescribed behaviors and undergraduate educational experiences. A well-defined conceptualization of teacher knowledge is needed in order to efficiently evaluate what constitutes effective teaching (Ball, 2000; Grossman, Wilson, & Shulman, 1989; Hiebert et al., 2002; Shulman, 1987; Wilson et al., 1987). The aspects of teacher knowledge used in practice that do contribute to effective student learning need to be examined and researched. More recently, researchers in the domain of teacher knowledge have shifted their focus to the various inter-related components of teacher knowledge and what constitutes a knowledge base for teaching (Borko & Putnam, 1996; Carter, 1990; Grossman, 1990; Hiebert et al., 2002; Shulman, 1986, 1987).

Knowledge Base for Teaching

Conceptually, the idea of a knowledge base for teaching can be defined through the components that structure teacher knowledge. Different models of teacher knowledge are suggested by researchers based on different interpretations of those components (Grossman, 1990). Shulman (1987) considers the following seven components as a minimal list for a knowledge base for teaching: content knowledge, general pedagogical knowledge,
pedagogical content knowledge, curriculum knowledge, knowledge of learners, knowledge of educational contexts, and knowledge of the purposes of education. Fennema and Franke (1992) describe four inter-related components that make up teacher knowledge as it occurs within the classroom setting: knowledge of mathematics, pedagogical knowledge, content specific knowledge, and knowledge of learners’ cognitions in mathematics. Another related model of teacher knowledge proposed by Grossman (1990) consists of four general areas of teacher knowledge: subject matter knowledge, general pedagogical knowledge, pedagogical content knowledge, and knowledge of context. Simon (1995) proposes a model of teaching based on a pedagogical perspective where teachers’ pedagogical decisions are guided by their goals for student learning. These decisions are informed by their knowledge of mathematics, their knowledge of how students learn particular mathematical concepts, and knowledge of mathematical representations, tasks, and materials. This model, called a Mathematics Teaching Cycle, incorporates a knowledge base of subject matter knowledge with knowledge of pedagogical decision making based on the goals for student learning and the relationship of this knowledge base to instructional design.

Each of these models has several characteristics in common. Each model includes aspects of teacher knowledge such as content knowledge, pedagogical knowledge, and knowledge of how students learn. Content knowledge, or subject matter knowledge, is the knowledge teachers have about the subject they teach such as concepts, procedures, content domains and the organization of those domains. Pedagogical knowledge is the knowledge teachers have about effective teaching procedures such as planning, classroom management, how students learn, and classroom routines. The combination of these aspects yields another component of teacher knowledge: pedagogical content knowledge. This is the knowledge
teachers have about how to teach content-specific material such as the concepts that are easy or difficult for students to learn and the uses of representations for teaching different concepts (Grossman, 1990; Shulman, 1986, 1987; Wilson et al., 1987). The components consistently included in models of teacher knowledge that have received considerable attention from researchers are subject matter knowledge (SMK), pedagogical content knowledge (PCK) and knowledge of how students learn. Indeed, Shulman (1987) claims:

The key to distinguishing the knowledge base of teaching lies at the intersection of content and pedagogy, in the capacity of a teacher to transform the content knowledge he or she possesses into forms that are pedagogically powerful and yet adaptive to the variations in ability and background presented by the students. (p. 15)

The uniqueness of a knowledge base for teaching lies in how teachers use and combine these components of teacher knowledge.

Components of Teacher Knowledge

Subject Matter Knowledge

The role of teachers’ subject matter knowledge (SMK) is a major focus of research in mathematics education (Ball, 1990; Ball et al., 2001; Ball & McDiarmid, 1990; Grossman et al., 1989; Wilson et al., 1987). Research finds that teachers who have more knowledge of their subjects tend to focus on the conceptual and problem-solving aspects of their subjects, whereas those teachers with less SMK focus on rules, facts, and procedures (Borko & Putnam, 1996; Ma, 1999). Current reform efforts require teachers to have deep and flexible knowledge of the subjects they teach. This deep and flexible content knowledge implies
effective teaching, which in turn, encourages student learning, as will be elaborated in the next chapter.

Part of having deep and flexible SMK for teaching requires knowledge of the structures of that subject. These include content knowledge, substantive knowledge and syntactic knowledge (Grossman, 1990; Grossman et al., 1989; Shulman, 1986). Content knowledge refers to knowledge of the major ideas and concepts of a subject as well as the relationships that exist between them. Knowledge of the major paradigms and frameworks of a subject that guide its organization and focus of inquiry is defined as substantive knowledge. Syntactic knowledge is knowledge of the rules that establish truth and validity within that subject. This implies teachers know more than just the rules and procedures of a subject; they must also be able to explain why the rules exist and how the procedures work (Shulman, 1986). That is, effective teachers have both substantive and syntactic knowledge of the mathematics they teach.

*Pedagogical Content Knowledge*

Pedagogical content knowledge (PCK) is knowledge of how to teach specific subject matter and how to make the subject matter accessible for students. Teachers use their SMK to guide their pedagogical practices and decisions. PCK, as defined by Grossman (1990), consists of four components: conceptions of purposes for teaching a particular subject, knowledge of student’s understanding, knowledge of instructional strategies, and knowledge of the curriculum. Teachers’ knowledge and beliefs about the purposes for teaching a subject are reflected in their instructional goals. Effective teachers know what makes a subject easy or difficult for students and they use that knowledge to generate appropriate explanations and representations. They know what curriculum materials are available for their subject and how
to incorporate those materials into effective instructional approaches. Effective teaching implies teachers have well-developed PCK, yet this knowledge develops over time (Grossman, 1990).

**Instructional Representations**

The concept of using *representations* as a means of communicating abstract mathematical ideas is a prominent focus of research in teacher knowledge and student learning (Borko & Putnam, 1996). Representations are considered the signs, characters, icons, actions or objects that stand for something else (Goldin, 2003). The National Council of Teachers of Mathematics (NCTM) (2000) considers representation as both a process and a product. It is the act of representing and the form of the representation itself. These representations can be categorized as either internal or external. Internal representations are those that are created and held mentally by an individual and external representations are the signs, objects, or actions that symbolize, depict, or encode something other than itself (Goldin & Shteingold, 2001). For the purposes of this study, the topic of representations focuses only on the external category, specifically those external representations used by teachers during classroom instruction.

External representations refer to the observable embodiments of abstract ideas. Zhang (1997) defines external representations as:

…the knowledge and structure in the environment, as physical symbols, objects, or dimensions (e.g., written symbols, beads of abacuses, dimensions of a graph, etc.), and as external rules, constraints, or relations embedded in physical configurations.
(e.g., spatial relations of written digits, visual and spatial layouts of diagrams, physical constraints in abacuses, etc.). (p. 180)

Other general examples of external representations include verbal representations such as written words, symbolic representations such as equations and formulas, graphical representations, manipulatives, and visual representations such as pictures and diagrams (Lesh, Post, & Behr, 1987). Several examples of external representations are given in Figure 1. The overall objective in utilizing external representations is to communicate about mathematical ideas and concepts. The uses of external representations in teaching mathematics extend the need for communication to include the need for teachers to communicate mathematical ideas in a manner that students can understand.

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<td>$y = 5x + 3$</td>
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*Figure 1. Examples of external representations*

External representations are important in the teaching and learning of mathematics for a variety reasons. Dufour-Janvier, Bednarz, and Belanger (1987) address four reasons for
teaching and learning with external representations: they are an inherent part of mathematics, they can provide multiple perspectives for the same concept, they can be used as tools in problem solving, and they can make learning mathematics more interesting.

External representations also play an important role in the conceptualization of teacher knowledge (Orton, 1988). Since teachers use external representations to communicate mathematical concepts and ideas to students, it is important to include them as a part of teacher knowledge. An important aspect of teacher knowledge and teaching mathematics with external representations is teachers’ uses of *instructional representations*. Instructional representations can be described as the words, pictures, graphs, objects, numbers, symbols, and contexts (including examples, metaphors, and analogies) that teachers use during instruction to communicate abstract mathematical ideas to students (Berenson & Nason, 2003). These are the external representations created and/or used by a teacher during instruction. Teachers’ uses of instructional representations not only provide a view of their SMK and PCK as it is used in practice, but the activity of representing is considered a fundamental and core activity of teaching of mathematics (Ball, 1993, 2000; Ball et al., 2001). Instructional representations are situated within classroom settings and other educational situations. For example, a sheet of paper is not an instructional representation until it is used for instructional purposes by a teacher. A high school geometry teacher who instructs students to physically make a cylinder with the sheet of paper in order to illustrate how to find surface area is using that sheet of paper as an instructional representation. Instructional representations can include any physical objects if they are used within a classroom by the teacher for instructional purposes.
Instructional Representations and Pedagogical Content Knowledge

Teachers’ PCK is considered a central component of their knowledge base for teaching (Grossman, 1990; Shulman, 1986). By conceptualizing the PCK of teachers, the gap left by focusing on teachers’ credentials and behaviors is addressed and we gain a deeper understanding of the knowledge needed for teaching (Ball et al., 2001). Moreover, evaluation of the relationship between effective teaching and PCK becomes more challenging, but necessary. “Although less easily quantified than other indices such as the type and number of courses taken, it is the pedagogically functional mathematical knowledge that seems to be central to effective teaching” (Ball et al., 2001). Effective teaching emanates from teachers’ PCK and knowledge of how to communicate their mathematical knowledge to their students.

The concept of PCK leads to new ways of probing teachers’ knowledge. One potential avenue of research in teachers’ PCK is their uses of instructional representations. Teaching mathematics in a way that encourages student understanding requires the development of teachers’ PCK. Many researchers include teachers’ knowledge of appropriate uses of instructional representations as a part of PCK (Borko & Putnam, 1996). Shulman (1986) describes the uses of representations within the category of PCK:

Within the category of pedagogical content knowledge I include, for the most regularly taught topics in one’s subject area, the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations – in a word, the ways of representing and formulating the subject that make it comprehensible to others (p. 9).
Purpose of Study

The aim of this study is to enrich our understanding of the ways teachers use instructional representations during mathematics instruction, specifically secondary geometry. Instructional representations are the basis for communication between teachers and students about subject matter, but they are presented by teachers to their students. Thus, the implications for student learning from teachers’ uses of instructional representations are also considered. The patterns of similarities and differences in the uses of instructional representations between a novice teacher and an experienced teacher will contribute to a conceptualization of teacher knowledge, including the components of SMK, PCK, and knowledge of how students learn. Fennema and Franke (1992) address the need for including instructional representations in research on teacher knowledge: “Do teachers know the representations of the content they ordinarily teach? Does knowing these representations make any difference in how teachers teach or what students learn? Where does knowledge of such representations fit into the knowledge structure of teachers? Such questions remain unanswered.” (p. 154). Teachers’ uses of instructional representations provide insight into not only how teachers understand the mathematics they teach, but also how they think about teaching the mathematical concepts, what they think their students should understand about those concepts, and how they think students learn. In other words, the ways teachers utilize instructional representations provide a conceptual link between their SMK and PCK.

Teachers’ uses of instructional representations are considered by many researchers to be an important part of both their SMK and PCK (Ball et al., 2001; Berenson & Nason, 2003; Orton, 1988; Wilson et al., 1987). Teachers’ pedagogical decisions regarding their uses of instructional representations are guided by the depth of their SMK. Furthermore, there are
important mathematical aspects embedded within particular instructional representations and the ability to use the mathematical and the pedagogical aspects of an instructional representation requires deep and flexible mathematical knowledge (Ball et al., 2001). Therefore, a knowledge base for teaching should not be built solely on the knowledge obtained from research and teacher evaluation. It should also be built on practitioner knowledge, the knowledge teachers gain from active participation in the practice of teaching (Hiebert et al., 2002), which includes the use of instructional representations. This knowledge base for teaching can be connected to student learning in order to deepen our understanding of what constitutes effective teaching.

Research Focus

Prior research studies have expounded on students’ learning through the use of instructional representations (Dufour-Janvier et al., 1987; Gagatsis, Chrisou, & Elia, 2004), yet there is little research about how teachers use instructional representations in practice. Furthermore, researchers tend to focus on students’ lack of understanding of a particular mathematical concept rather than on the instructional representations utilized during instruction (Dufour-Janvier et al., 1987). However, because instructional representations are how teachers communicate their SMK to students, the relationship between teachers’ uses of instructional representations and student learning needs to be addressed in the research. This issue to how teachers’ use instructional representations leads to the following research question for this study:

*How do a novice teacher’s uses of instructional representations differ from that of an experienced teacher, particularly in the teaching of secondary geometry?*
A fundamental difficulty in learning to teach is how teachers’ learn to use their SMK (Ball, 2000). This includes the idea that teachers learn how to generate multiple instructional representations that take into account students’ different learning styles, abilities, and prior knowledge. An assumption is held by many researchers that teachers enter the profession with a good foundation of SMK, but they don’t yet know how to make it accessible to all students. Instructional representations provide tools for teachers to use their SMK and PCK in flexible ways in order to make mathematical ideas comprehensible to all students. This study explores how two secondary mathematics teachers used instructional representations in their attempts to make their subject matter knowledge of geometry accessible and understandable to their students.

The subject of geometry is included throughout pre-K - 12 school mathematics. The NCTM (2000) addresses the need and the importance of both geometry and representation in school mathematics. Despite the apparent centrality of geometry in school mathematics curricula, students from the United States exhibit low levels of achievement in geometry compared with students in other industrialized countries (Clements, 2003). The results from the Third International Mathematics and Science Study (TIMSS) show that students from the United States scored below the international average in geometry and that within the United States, students scored lower in geometry than in any other content area tested (Mullis, Martin, Gonzalez, Gregory, Garden, O'Connor, Chrostowski, & Smith, 2000). A previous president of the NCTM, Lee Stiff, expresses his concern that algebra gets more attention than geometry in school mathematics curricula, thereby contributing to the lower achievement in geometry on the TIMSS (Helfand, 2000). Stiff explains that geometry has not received as much attention as algebra because algebra is considered a “gateway” subject. The TIMSS
results, as well as similar outcomes from national assessments such as the National Assessment of Educational Progress (NAEP), prompted another past president of the NCTM to declare geometry the “forgotten strand” in school mathematics (Lappan, 1999). This suggests a need for more attention from mathematics education researchers in the content area of geometry concerning both teacher knowledge and students’ growth of mathematical understanding.

Much of teachers’ SMK is obtained before college and is shaped by previous experiences (Ball & McDiarmid, 1990). Mathematics teachers at all grade levels may be weak in their SMK of geometry. However, prior studies in teacher knowledge and geometry focus on elementary and middle grades teachers, not secondary mathematics teachers (eg. Mayberry, 1983; Swafford, Jones, & Thornton, 1997). Teachers’ knowledge has a direct impact on how they structure their instructional practices and how effective those practices are for student learning (Fennema & Franke, 1992). Yet researchers have not focused on this aspect of why students in the United States perform poorly in geometry. Instead, prior research studies focus on two problems in the teaching and learning of geometry: student performance and the curriculum (Clements, 2003; Clements & Battista, 1992; Usiskin, 1987). This suggests future research on teacher knowledge be considered within the context of teaching in specific content areas. Research on secondary teacher knowledge centers more frequently on content areas of algebra, such as function (Even, 1993; Wilson, 1993). Overall, the research that focuses on teacher knowledge in geometry and other content areas tends to focus more exclusively on elementary and middle grades teachers (Brown, Cooney, & Jones, 1990). A gap exists in the body of literature about secondary teacher knowledge within the content area of geometry. This study contributes to the body of literature about teacher
knowledge in geometry and to a conceptualization of teacher knowledge through teachers’
uses of instructional representations. This study investigates teachers’ uses of instructional
representations in terms of the components of teacher knowledge and from the theoretical
perspective set forth by Pirie and Kieren (1994b) (elaborated in Chapter 3) about how
students grow in their mathematical understanding. For the purposes of this study, the term
effective teaching refers to those teaching practices that positively contribute to students’
growth in mathematical understanding.
LITERATURE REVIEW

The goal of this literature review is to examine previous research relating to this study and research question. The roles of the components of teacher knowledge and the importance of instructional representations are addressed, as well as the research comparing experienced and novice teachers. Teaching and learning to teach rely on teachers’ abilities to make their mathematical knowledge understandable for students. Instructional representations are discussed as the tools teachers use to communicate their knowledge of mathematics to students. The implications of the literature in relation to this study are then discussed.

Role of Subject Matter Knowledge in Teaching

The role of SMK in teaching is a key theme in research on teacher knowledge. Historically, teacher knowledge and its effectiveness have been evaluated based on observable teaching procedures and behaviors. This approach does not account for teachers’ SMK, which is an important component of teacher knowledge. The SMK teachers use in various teaching procedures was taken for granted or often ignored. The omission of attention from researchers on teachers’ SMK is considered by Shulman (1986) to be the “missing paradigm” in educational research: “what we miss are questions about the content of the lessons taught, the questions asked, and the explanations offered” (p. 8). Even (1993) agrees that traditional measures, such as counting the number of mathematics courses taken and standardized test scores, do not reveal the depth of preservice teachers’ SMK. Although preservice mathematics teachers take a substantial number of mathematics courses, this does not guarantee depth of SMK. Further illustrating this gap in research, Wilson, Shulman, and
Richert (1987) address the lack of a clear understanding of what SMK teachers do possess and how they represent their knowledge to students. A significant amount of research has since been published on the role of SMK in teaching, yet there is still no well-defined conception of teacher knowledge. Most researchers do, however, agree that SMK must be included.

Although the issue of teachers’ SMK in the area of mathematics receives extensive attention from researchers, our understanding of the nature and role of teachers’ SMK in mathematics continues to be an unsolved problem (Ball et al., 2001). This unsolved problem prevents the improvement of the teaching and learning of mathematics. Past reform movements in the United States have failed to make any significant difference in the ways mathematics is taught in American classrooms. Several factors have been identified as contributors to this failure: “…the misrepresentation of mathematics; culturally embedded views of knowledge, learning, and teaching; curriculum materials and assessments; and teacher education and professional development” (Ball et al., 2001). Underlying all of these factors is the lack of understanding of what teachers need to know about mathematics in order to teach it effectively (see Figure 2).
In a study of novice teachers and the role of SMK in teaching, Wilson, Shulman and Richert (1987) found “…teachers need more than a personal understanding of the subject matter they are expected to teach. They must also possess a specialized understanding of the subject matter, one that permits them to foster understanding in most of their students” (p. 104). What teachers know about their subject is not, by itself, sufficient for effective teaching; they must also find ways to communicate their knowledge to their students. This can be quite difficult for novice teachers who need to “unpack” their mathematical knowledge to teach the content they learned as pre-college students. Content from college courses may be more readily accessible for preservice and novice teachers since they have been more recently exposed to that subject matter (Grossman, 1990). In studies involving novice mathematics teachers and their SMK, the novice teachers are found to rely on procedural and rule-based approaches to teaching mathematics and lack conceptual understanding of the mathematics they will teach (Ball, 1990, 2000; Ma, 1999). That is, novice teachers tend to lack conceptual knowledge of school mathematics, but they require
that kind of knowledge in order to foster growth in their students’ mathematical understanding.

Effective mathematics teachers have SMK that is comprised of both procedural and conceptual understandings of the subject (Berenson & Nason, 2003). Procedural understanding of mathematics comes from computational skills and knowledge of procedures, such as algorithms. Conceptual understanding comes from knowing the relationships and connections among mathematical concepts that give meaning to mathematical procedures (Eisenhart, Borko, Underhill, Brown, Jones, & Agard, 1993). The distinction lies between mathematical skills and understanding: procedural knowledge is based on mathematical skills and conceptual knowledge is based on understanding. Although historically, the two types of knowledge have been treated as separate entities by researchers, the current approach in educational research is the exploration of the relationships between them (Hiebert & Lefevre, 1986). Research shows that mathematics teachers need both procedural and conceptual knowledge if they are to help their students develop these kinds of mathematical knowledge.

Role of Pedagogical Content Knowledge in Teaching

The type of knowledge that separates teachers from experts in particular subject areas, such as mathematics, is teachers’ PCK. Teachers differ from experts “…not necessarily in the quality or quantity of their subject matter knowledge, but in how that knowledge is organized and used” (Fuller, 1996, p. 4). Teachers must be able to combine their pedagogical knowledge and their SMK in order to address instructional goals and student needs within a particular subject (Grossman, 1990; Shulman, 1986). Since beginning
teachers have not yet developed their pedagogical knowledge, their PCK may be weak. Many novice teachers do not have a variety of strategies for representing the content to students, nor do they yet have adequate knowledge of how students learn; these aspects of teacher knowledge are usually learned from practice and experience (Feiman-Nemser & Parker, 1990). Researchers acknowledge that teachers’ PCK grows and develops from the practice of teaching and from the experience of transforming their SMK for teaching. Teachers’ PCK develops over time as they gain more knowledge about teaching from practice (Grossman, 1990; NCTM, 2000; Wilson et al., 1987).

Although PCK combines components of a knowledge base for teaching, the relationships between these components remain unknown. Wilson et al., (1987) describe the lack of knowledge about these relationships: “Influenced by both subject matter and pedagogical knowledge, pedagogical content knowledge emerges and grows as teachers transform their content knowledge for the purpose of teaching. How these kinds of knowledge relate to one another remains a mystery to us” (p. 118). The inclusion of teachers’ PCK in a conception of a knowledge base for teaching contributes to solving this mystery by filling in gaps left by focusing on teachers’ credentials and improving our understanding of teacher knowledge (Ball et al., 2001).

Just as teachers’ SMK consists of procedural and conceptual knowledge, so too, does their PCK. These types of knowledge are essential dimensions of PCK in that, teachers who rely on procedural knowledge lack the ability to represent mathematical concepts in ways that students can connect their prior and current knowledge to these new concepts (Ball & Wilson, 1990). For beginning teachers, learning to teach for students’ procedural and conceptual understanding can be a difficult process despite their desire to teach for both
procedural and conceptual understanding (Eisenhart et al., 1993). Many factors influence their reliance on procedural approaches to teaching such as the expectations of administrators, cooperating teachers, curriculum materials, and the curriculum itself.

Role of Instructional Representations

Representations play an important and necessary role in the teaching and learning of mathematics. “The ways in which mathematical ideas are represented is fundamental to how people can understand and use those ideas” (NCTM, 2000). Representations are tools that support student understanding and tools for applying mathematics. Representations are used to communicate abstract mathematical concepts. These concepts can be modeled and explicated through teachers’ uses of instructional representations. Effective teachers know how to use instructional representations in ways that support student understanding of mathematical concepts. Teachers’ knowledge about how to use instructional representations is embedded in their knowledge base for teaching. Instructional representations connect what mathematics teachers know about mathematics and what they know about teaching mathematics (Berenson & Nason, 2003). The main purpose of using external representations in the teaching of mathematics is to help students learn how to use them in the doing of mathematics (Dufour-Janvier et al., 1987; Goldin & Shteingold, 2001). Furthermore, teachers need to know a wide variety of representations to use during instruction. Wilson et al., (1987) term this variety of instructional representations a representational repertoire “…that consists of the metaphors, analogies, illustrations, activities, assignments, and examples that teachers use to transform the content for instruction” (p. 120). In other words, instructional
representations are fundamental to teaching and connect several components of teacher knowledge: SMK, PCK, and knowledge of how students learn (see Figure 3).

![Instructional representations and components of teacher knowledge](image)

**Figure 3.** Instructional representations and components of teacher knowledge

Since instructional representations play such an important role in the development of student understanding, it is important to consider the ways instructional representations potentially help or hinder understanding. Students’ mathematical understanding can be encouraged through careful construction and sequencing of instruction with representations (Gagatsis et al., 2004; Post & Cramer, 1989). Thus, effective teachers consider their own
uses of instructional representations and how these uses affect their students’ growth in understanding. Students need time to work with representations in order to develop their understanding of mathematical concepts. For example, regarding the teaching of mathematics with conventional representations, such as symbols, the NCTM (2000) explains “research indicates…that students at all levels need to work at developing their understandings of the complex ideas captured in conventional representations. A representation as seemingly clear as the variable $x$ can be difficult for students to comprehend” (p. 67). This implies that teachers have at their disposal a wide variety of instructional representations and know the appropriate time to use them based on their students’ current levels of understanding.

Additionally, effective teachers know the strengths and weaknesses of instructional representations and the relationships that among them (Ball, 1993; NCTM, 2000; Wilson et al., 1987). Teachers who are effective in helping their students’ growth in understanding have the ability to use different instructional representations of the same concept. They can also illustrate how these instructional representations connect different ideas and aspects of a concept.

The uses of multiple instructional representations and the translations between them are important pedagogical acts that teachers use to illustrate different perspectives of a particular mathematical concept. Different representations highlight different aspects of a mathematical concept or relationship, thus a variety of instructional representations helps support student understanding (NCTM, 2000). However, some representations have related aspects that support each other and some representations have related aspects that interfere with one another (Dufour-Janvier et al., 1987; Goldin & Shteingold, 2001; Kaput, 1987). Therefore, it is important for teachers to be aware of and understand which instructional
representations support each other and support student learning. Moreover, effective teachers are able to illustrate the connections between instructional representations of the same concept and can emphasize what aspects of a representation are relevant in the context of what concept is being taught (Leinhardt, 1989). In a study about multiple representations of functions, college students who were exposed to a multi-representational approach to learning function were found to have increased student achievement and were more able to use representations in algebraic problem solving contexts (Rider, 2004). A similar study found that middle grades students show higher levels of algebraic reasoning about variables after experiencing instruction that focused on multiple representations (Moseley & Brenner, 1997). Thus, the use of multiple instructional representations can encourage students’ growth in procedural and conceptual knowledge. Just as the Pirie-Kieren theory (1994b), which is further explicated in the next chapter, addresses the growth of students’ mathematical understanding, the appropriate uses of instructional representations are considered crucial in students’ conceptual development of mathematical understanding (Post & Cramer, 1989). In light of the research results, Ball (1993) suggests that instructional representations highlight both the conceptual and procedural features of a concept. That is, effective teachers have both conceptual and procedural knowledge of mathematical concepts and knowledge of the different ways to represent those concepts.

Many researchers in mathematics education study the SMK of teachers in relation to their pedagogical knowledge. The assumption is held that teachers’ SMK influences their pedagogical practices and these pedagogical practices, in turn, affect student learning outcomes (e.g., Even, 1993; Knuth & Elliott, 1997; Quinn, 1997; Swafford et al., 1997; Van Dooren, Verschaffel, & Onghena, 2002; Wilson, 1993). The ways in which teachers utilize
instructional representations are a pedagogical practice that reveal important aspects of their SMK and their PCK (Ball et al., 2001; Berenson & Nason, 2003; Wilson et al., 1987). In a study about elementary preservice teachers’ understanding of what and how to teach the concept of ratio, Berenson and Nason (2003) examined the instructional representations the preservice teachers included in their lesson plans. The researchers suggest that instructional representations “…provide valuable insights into the depth and accuracy of the knowledge prospective teachers bring to instructional settings” (p. 89). For example, the teachers’ uses of accurate multiple instructional representations suggest deep SMK, whereas shallow SMK is indicated by teachers “owning” the instructional representations, using too many open-ended questions without providing instructional representations, or overuse of algorithmic or procedural approaches. In a related study, Ward, Anhalt, and Vinson (2003) analyzed the mathematical representations of elementary preservice teachers included in their lesson plans as a way of developing their PCK. They advocate using instructional representations in preservice teacher education and assert that this approach plays an important role in the conceptualization of a knowledge base for teaching.

Ball (1993) notes that many teachers do not use certain instructional representations such as manipulatives and diagrams due to their lack of deep SMK. Hashweh (1985) further contends the deep SMK is a prerequisite for knowing and using instructional representations. These findings point to the significance of considering instructional representations as part of a conceptualization of teacher knowledge. Instructional representations of subject matter are an important component of teacher knowledge because they are essential for mathematical learning, the cognitive actions of students, and the discipline of mathematics itself (Orton, 1988). Other researchers consider teachers’ abilities to represent subject matter as an
essential component of their PCK. In other words, the conceptualization of a knowledge base for teaching includes the connection between teachers’ SMK, PCK, and their knowledge of how students learn made from instructional representations.

Teaching and Learning Geometry

The teaching and learning of geometry involves more than geometric shapes and their related properties; it also involves visual and spatial reasoning, modeling, and relationships to other content areas in mathematics. Geometry is more than a theoretical branch of mathematics in which students learn the particulars of the content area; it is an area of mathematics education that can be used to foster different ways of thinking. These ways of thinking, such as spatial reasoning and visualization, must be included as a pedagogical goal in the teaching of geometry (Duval, 1998). The National Council of Teachers of Mathematics advocates the inclusion of geometry throughout the pre-K – 12 grades and emphasizes the importance of learning mathematical reasoning skills through the study of geometry (NCTM, 2000). The NCTM further emphasizes the need for students to develop spatial reasoning and visualization skills in the learning of geometry. However, results from the Third International Mathematics and Science Study (TIMSS) reveal that students from the United States often perform poorly in geometry, especially with spatial reasoning problems (Battista, 1999). Due to the spatial and visual nature of geometry, the ways teachers use instructional representations in the teaching of geometry cannot be overlooked in a conceptualization of teacher knowledge. This suggests that effective teachers are aware of the aspects of instructional representations that make them easy or difficult for students to understand.
Geometry is a content area rich with representations, particularly visual representations. Geometry is also the study of space and that space, as an abstract concept, must be represented in such a way that students can understand. This can create potential difficulties in the teaching and learning of geometry: “Representation of geometrical objects is a central problem in geometry. But this poses with it a new problem, that of the links between geometrical objects and their representation” (Mesquita, 1998, p. 184). Many of the visual external representations used in the teaching of geometry take on a double status: that of a specific, concretized example or that of a general example with no empirical constraints. In other words, a figure of a triangle can be considered as an abstract object that represents the concept of a triangle or as a particular and specific example. This can create difficulties in the teaching and learning of geometry concerning the uses of instructional representations. To further complicate the issue, geometry concepts and representations are used in other content areas of mathematics and in real-world situations. For instance, the representations used in the learning of geometry “…can help students make sense of area and fractions, histograms and scatterplots can give insights about data, and coordinate graphs can serve to connect geometry and algebra” (NCTM, 2000, p. 41). Therefore, effective teachers of geometry must be able to use and explain instructional representations that are connected to other areas of mathematics and real-world phenomena.

Although instructional representations are essential to the teaching and learning of geometry, prior studies about the teaching and learning of geometry focus on student performance, curriculum, and the knowledge of elementary and middle grades teachers (Brown et al., 1990; Clements, 2003; Clements & Battista, 1992; Usiskin, 1987). Another area of research concerning students’ understanding of geometry and related representations
is the use of technology in the teaching and learning of geometry. The importance of visual representations in the teaching and learning of geometry has led researchers to consider these representations within technological contexts (Laborde, Kynigos, Hollebrands, & Strasser, 2006). The dynamic capabilities of computer software are an example of a technological context used for much of this research concerning student learning of geometry concepts. For instance, Hollebrands (2002, 2003) analyzed how a dynamic software environment influenced students’ understandings of geometric transformations. The results of this research suggest that technology can be used to enhance student learning. Clements and Battista (1992) claim that technology can encourage students’ higher-level reasoning skills in geometry due to the interactive nature of working within a computer environment. Although technology has tremendous potential as a tool for effective teaching practices and students’ growth in understanding, its use is not specifically addressed in this study.

Experienced and Novice Teachers

The research comparing experienced teachers and novice teachers tends to focus on the differences in instructional practices and pedagogical decision making. Within these studies are dimensions of SMK and PCK relating to specific mathematics subject areas. Many of these studies are based on an information-processing approach that take in account “…differences in thinking between expert teachers and candidates in initial teacher-preparation programs” (Carter, 1990). One of the purposes for studying experienced teachers is to obtain a basis for training novice teachers; however, most of this research in mathematics education concentrates on elementary and middle grades mathematics teachers (Brown et al., 1990; Leinhardt, 1989). Overarching findings from previous studies show:
expert teachers have more domain-specific knowledge than novice teachers, they are organized and use classroom time more efficiently than novice teachers, and they gained much of their expert knowledge from their teaching experience (Brown et al., 1990; Carter, 1990; Koehler & Grouws, 1992). Thus, SMK and PCK are developed from the experience of teaching.

Prior studies comparing novice and experienced elementary and secondary mathematics teachers’ instructional practices and pedagogical reasoning used in planning and teaching found that novice teachers have less developed cognitive knowledge structures for both content and pedagogy (Borko & Livingston, 1989; Livingston & Borko, 1990). Pedagogical reasoning involves both SMK and PCK. It is how teachers transform curricular materials for teaching based on their knowledge of the subject matter and how students learn and it is the creation and use of appropriate multiple instructional representations (Fuller, 1996). Novice teachers do not yet have a well-developed knowledge base for teaching, and thus do not have well-developed pedagogical reasoning skills. The SMK and PCK needed for effective and efficient teaching come with the experience of teaching and develop over time.

Moreover, this growth in SMK and PCK can be difficult for beginning teachers:

When novice teachers become frustrated with the difficulties inherent in teaching mathematics meaningfully, the time constraints involved in covering the content of the curriculum, and the need to cope with individual differences, they often resort to teaching the way they were taught despite their desire to do otherwise. (Fuller, 1996 p.6-7)

Other studies focus on the differences in SMK and/or PCK between novice and experienced teachers. Concerned over the absence of subject matter knowledge in expert-
novice research, Feiman-Nemser and Parker (1990) studied conversations about subject matter between beginning and experienced elementary and secondary teachers. In their analysis of these conversations, the researchers address four aspects of learning to teach subject matter: the development of personal SMK, learning to think about subject matter from students’ perspectives, learning to represent subject matter, and learning to manage students in order to help them learn the subject matter. Experienced teachers can be an excellent source for the development of novice teachers’ SMK and PCK.

Implications

The goals for teaching and learning mathematics must include making connections between informal and formal mathematics (Ball, 1993). Yet research finds that many teachers lack a deep understanding of the mathematics they teach that allow for these connections. Without these connections, students may develop misconceptions or an incomplete understanding of a concept (Martin, 2000). By analyzing teachers’ uses of instructional representations, this study contributes to an understanding of how teachers make their SMK known to students and if these uses illustrate the connections between informal and formal mathematics. The study further highlights differences between novice and experienced teachers in their uses of instructional representations. The lack of clear understanding of teachers’ SMK and how they represent their knowledge to students (Ball et al., 2001; Wilson et al., 1987) is addressed in this study by examining the ways teachers use instructional representations and how those uses connect to student learning. Furthermore, the lack of research concerning secondary mathematics teachers’ understanding of geometry needs to be addressed. Although the subject of geometry should be emphasized throughout
the school mathematics curriculum (NCTM, 2000), most studies consider only elementary
and middle grades teachers. This study contributes to filling this gap by studying the uses of
instructional representations used by two mathematics teachers in secondary geometry.
THEORETICAL FRAMEWORK

This chapter begins with a discussion of the Pirie-Kieren (1994b) theoretical framework concerning the cognitive changes that occur during the processes of growth of mathematical understanding. A discussion of the ways the theory has been applied in mathematics education research on student understanding follows and includes a review of literature that supports and extends the theory. Connections between the theoretical framework and the research study are elaborated to explain how the framework can be applied to the study of secondary mathematics teachers and their uses of instructional representations when teaching concepts from secondary honors geometry.

The Pirie-Kieren Dynamical Theory for the Growth of Mathematical Understanding

Explaining mathematical understanding and the means in which it can be developed is a complex, theoretical problem in mathematic education research. Researchers utilize diverse approaches in the study of mathematical understanding; for example, many consider understanding as an acquisition or set of levels or stages that can be achieved through either hierarchical or cyclic processes (e.g., Hiebert & Lefevre, 1986; Serpinska, 1990; Sfard, 1991; Skemp, 1976). The Pirie-Kieren theory adopts the approach that the processes of learning and understanding are whole, dynamic, recursive, and non-linear. Building their theory from constructivist and enactivist views of learning, Pirie and Kieren (1994b) elaborate the constructivist idea of an individual’s understanding as the continual process of organizing and reorganizing knowledge structures (von Glasersfeld, 1987). Acknowledging the interdependence of all participants in a particular environment, the Pirie-Kieren theory
elaborates an enactivist view of learning as an interactive process where understanding is a continual process, not an achieved state (Martin, 2001; Towers, 2001; Towers, Martin, & Pirie, 2000; Varela, Thompson, & Rosch, 1991).

The Pirie-Kieren theory is first and foremost a theory about how a student’s mathematical understanding grows and develops concerning a specific mathematical concept or topic. The theory is illustrated through a model (see Figure 4) of embedded layers that consist of eight potential levels in the growth of understanding for a specific person and a given topic or concept:

- **Primitive Knowing** refers to the knowledge that an individual brings to a setting. The process of growth of understanding begins at this level and it contains what a teacher assumes a student can do initially, at the beginning of instruction.

- **Image Making** refers to the level where a student can make distinctions in his or her Primitive Knowing and can use that knowledge in new ways that involve actions and activities with that knowledge.

- **Image Having** occurs when the student can use this new knowledge without the actions needed in the previous level. That is, a student can use a mental construct of a concept without the need for the activities that contributed his or her Image Making.

- **Property Noticing** is the level where a student can connect images by manipulating or combining aspects of those images and can construct contextually relevant properties that are related to his or her Image Having.

- **Formalizing** occurs when a student can abstract methods or common qualities from prior images that where characterized by Property Noticing. At this level, a concept
takes on a formal mathematical characteristic for the student, who is now capable of enunciating a formal definition or algorithm.

- *Observing* refers to the level where a student can reflect on and coordinate his or her formal understanding as theorems and can make predictions without the need for actions.

- *Structuring* refers to the level where a student is able to think about formal observations as theory and is aware of the inter-connections between theorems. The student can now justify mathematical arguments through logic and proof without the need for physical or algorithmic actions.

- *Inventising* is the outermost level, where a student has a fully structured understanding and no longer requires the use of the ideas and images that contributed to the understanding. The student is now able to create new mathematical questions that can potentially lead to a new concept.
Understanding grows and develops through continuously unfolding and dynamic processes that can be mapped through observation using the Pirie-Kieren model (Martin, Towers, & Pirie, 2006; Pirie & Kieren, 1994b). The levels in this model, referred to as activity models (Pirie & Kieren, 1994b), can be applied to any level of mathematics and the inner, lower levels do not necessarily imply lower levels of mathematics. In other words, growth of mathematical understanding is iterative between the layers of the model and the activity levels do not imply the level of mathematics in which a student is working. Instead, the layers are nested and the understandings held in the lower levels are embedded in understandings of the outer levels. The first three inner levels, Primitive Knowing, Image
Making, and Image Having are informal, local and context-dependent. The next three levels, Property Noticing, Formalizing, and Observing are potential formal levels that are less context dependent and thus, more general and abstract in the nature of understanding (Kieren, Reid, & Pirie, 1995). Furthermore, there are features of the theory that extend and deepen our understanding of the relationships between the layers. These features further illustrate how a student’s mathematical understanding of a concept moves among and between the activity levels.

The first feature is illustrated by the bold rings in the Pirie-Kieren model (see Figure 4) which denote ‘don’t need’ boundaries between the layers. In this theory, previous levels of understanding are embedded in a student’s current level of understanding and readily available if needed. However, a student’s growth in understanding across a ‘don’t need’ boundary implies the student no longer needs to mentally or physically refer to specific actions or images about a concept. The first ‘don’t need’ boundary occurs between Image Making and Image Having when a student no longer needs the actions that brought about the growth in understanding from making images. The second boundary occurs between Property Noticing and Formalizing. Once a student understands a concept formally, there is no longer a need for the images the student associated with that concept. The third boundary occurs between Observing and Structuring. When a student has gained a structural understanding, he or she no longer needs the meanings associated with the inner levels of understanding. These boundaries illustrate how the levels of understanding are embedded, but this does not imply that a student will not ever again need to access their inner levels of understanding. When a student is presented a problem that is not immediately solvable, the student will need to access inner levels of understanding or “fold back”.

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The second feature of the Pirie-Kieren theory is *folding back*. Folding back is considered a key feature of the Pirie-Kieren theory and an activity that is vital to growth in understanding (Martin & Pirie, 1998; Martin, 2000; Pirie & Kieren, 1994b). When a student folds back to an inner level of understanding, not only is the current level of understanding extended, but the inner level has also changed because it has been informed by the outer, albeit incomplete, level of understanding. This process “thickens” the inner level of understanding and thus, a student gains a broader and deeper understanding of a concept. The process of folding back can also allow for the reconstruction of inner levels of understanding (Pirie & Kieren, 1994b).

The third feature is based on two complementary processes called *acting* and *expressing*. Acting is based on mental or physical activity and provides continuity within the inner levels of understanding. Expressing is the ability to make the nature of the acting activities overt to one’s self and others. These two aspects of growth in understanding have a symbiotic relationship and are contained within every level beyond Primitive Knowing and are necessary for growth in understanding (Pirie & Kieren, 1994b). More importantly, although students may be able to act within a level, they must be able to express in order to move into the next level of understanding.

**Related Literature**

The Pirie-Kieren theory is used and extended in numerous research studies concerning students’ growth of mathematical understanding. Several studies focus on features of the theory, such as folding back and Formalizing (Martin & Pirie, 1998; Martin, 2000; Pirie & Kieren, 1994a). Others explore the effects of teacher interventions on student
understanding (Kieren et al., 1995; Martin, 2001; Towers, 2001; Towers et al., 2000). The theory is also extended to such domains as preservice teacher knowledge and students’ collective mathematical understanding (Berenson, Cavey, Clark, & Staley, 2001; Cavey, 2002; Cavey, Berenson, Clark, & Staley, 2001; Clark, 2001; Martin, 2001; Martin et al., 2006). These diverse studies illustrate the different ways the Pirie-Kieren theory can be used to think about teaching, learning, and understanding. The following review of the literature that uses the Pirie-Kieren theory considers the feature of folding back, mathematics preservice teacher knowledge, and teacher interventions.

Folding Back

As a key feature of the Pirie-Kieren theory, the activity of folding back is the focus of several recent research studies. Folding back occurs when a student is faced with a problem that is not immediately solvable within the current level of understanding. Martin and Pirie (1998) describe how the activity of folding back can be classified as either effective or ineffective. Effective folding back occurs when a student who is working with a problem that is not immediately solvable folds back to an inner level of understanding and is able to use the resulting growth in understanding to solve the original problem. Folding back is ineffective if the student folds back but is unable to use previous understandings to solve the original problem. The authors claim that effective folding back is more likely when a student is given the opportunity and time to work with their inner levels of understanding in order to enrich it, and then return to the original problem. That is, a student who needs to fold back must re-evaluate his or her current knowledge instead of just recalling memorized ideas (Martin, 2001). Furthermore, effective folding back can be further facilitated by subtle questions or prompts from a teacher (Martin & Pirie, 1998). Thus, effective teachers have a
deep understanding of the mathematics they teach and know when to use appropriate instructional representations that can encourage effective folding back.

Folding back is also a key feature in the growth of understanding of mathematics preservice teachers’ knowledge of what and how to teach school mathematics (Berenson et al., 2001; Cavey, 2002; Cavey et al., 2001). Assessing the SMK of a preservice teacher who did not use many instructional representations in her lesson plan for ratio and proportion, Berenson and Nason (2003) ascertain that she could grow in her mathematical understanding by folding back and collecting more images of the concept. She would then be able to notice properties and formalize her understanding and deepen her SMK. Teachers who are weak in their SMK may have the need to fold back and make images during teaching (Berenson & Nason, 2003). On the other hand, growth in PCK can be observed through the activity of folding back when learning to teach. Utilizing teaching tasks designed to develop preservice teachers’ content and pedagogical knowledge, “…prospective teachers rely on their teaching mathematics understanding and are thereby forced to fold back to their understandings of mathematics, teaching strategies, and mathematical learning…” (Cavey, 2002 p. 1086). The growth in understanding of both SMK and PCK in preservice teachers who are engaged in activities related to teaching can be observed using the Pirie-Kieren theory through their processes of folding back.

*Teacher Preparation*

The first five levels within the model of the Pirie-Kieren theory are extended to the domain of secondary teacher preparation through the use of teaching tasks (Berenson et al., 2001). The authors assert that preservice teachers’ education is most effective when the tasks of teaching, such as lesson planning and student teaching, are included as part of their
experiences. Thus, the inner five levels are modified to illustrate growth in understanding of what and how to teach secondary mathematics. In their adaptation, primitive knowledge is the knowledge preservice teachers have of the subject matter they will teach and of the teaching strategies that can be used to teach those topics. Preservice teachers make images of what and how to teach during teaching task activities, and they have images when they can mentally manipulate ideas about what and how to teach. Property noticing occurs when preservice teachers can combine aspects of their images about what and how to teach, especially in the context of lesson planning. Formalizing occurs when preservice teachers understand the relationships among their images of what and how to teach. Through their observations of secondary preservice teachers engaged in teaching tasks, Berenson et al., (2001) conjecture that the activity of folding back is critical to their growth in understanding of what and how to teach. However, preservice teachers with deep SMK may have difficulty folding back to reconstruct their images of how to teach a concept. Yet their lack of understanding of teaching strategies makes it easier for them to fold back and revise their images of how to teach.

Teacher Interventions

Research investigating teaching interventions and their affect on students’ growth in mathematical understanding is another extension of the Pirie-Kieren theory. In the attempt to coordinate teachers’ actions and students’ learning, researchers find that teacher interactions which encourage effective folding back in students encourage growth in their understanding and ability to solve problems (Martin & Pirie, 1998; Martin, 2000; Towers, 2001; Towers et al., 2000). For example, instead of telling and directing a student who is having difficulty folding back, a teacher should offer carefully worded, open-ended questions and comments
in order to guide the student’s folding back (Martin, 2000; Towers et al., 2000). In general, students’ growth in understanding depends on, but is not determined by, teachers’ actions. Appropriate teacher interventions do not guarantee growth in student understanding, but they can help students effectively fold back. Teachers who are effective recognize students’ need to fold back and collect images from their inner levels of understanding and they encourage their students to do this instead of memorizing the material (Martin, 2000). Teacher interventions that encourage effective folding back are less directive and encourage exploration by the student.

Relation to Study

How do teachers’ uses of instructional representations relate to the Pirie-Kieren theory for growth in mathematical understanding? Research addresses the importance of teacher knowledge concerning appropriate and multiple uses of instructional representations. Thus, effective teachers have a personal knowledge base for teaching that includes a variety of instructional representations that can encourage growth in students’ mathematical understanding. Positive growth of students’ mathematical understanding is often considered a result of effective teaching. The NCTM (2000) states:

Effective mathematics teaching requires a serious commitment to the development of students' understanding of mathematics. Because students learn by connecting new ideas to prior knowledge, teachers must understand what their students already know. Effective teachers know how to ask questions and plan lessons that reveal students' prior knowledge; they can then design experiences and lessons that respond to, and build on, this knowledge. (p. 17)
The implication is effective teachers use instructional representations in ways that connect to students’ Primitive Knowing. Furthermore, using instructional representations that force students to focus only on the properties and formal definitions and algorithms of a concept can be detrimental to students’ growth in mathematical understanding. The premature introduction of certain instructional representations can cause students to develop misconceptions and can give the impression that mathematics is nothing more than a formal language (Dufour-Janvier et al., 1987). Presenting students with only formalized ideas can hinder students’ learning of new ideas, resulting in disjoint understanding (Pirie & Kieren, 1994a). The types of instructional representations used by mathematics teachers influence the level of the Pirie-Kieren model in which students work with a particular concept. In turn, teachers’ uses of instructional representations affect the directions and patterns of growth in their students’ mathematical understanding. When students use available representations, their mathematical ideas and conceptions can become more concrete and thus, useable for reflection (NCTM, 2000). The instructional representations provided by teachers then have an influence on students’ abilities to make images and fold back. This implies that the ways teachers use instructional representations can be categorized based on the activity levels of the Pirie-Kieren model.

When considering the ways teachers use instructional representations, the feature of folding back is also important to take into account. Folding back is a key feature of the Pirie-Kieren theory and research concerning this feature addresses teacher interventions such as questioning and prompting in terms of encouraging students to fold back. However, there is little or no research about how teachers’ uses of instructional representations may or may not encourage their students’ growth of mathematical understanding. Folding back is considered
a key activity for growth of understanding and effective teachers can use instructional representations that encourage students to fold back to prior knowledge. For example, in the introductory chapter of this study, an example of an instructional representation was given where a teacher has his students use pieces of paper to make cylinders in order to connect the surface area of that cylinder to the area of a rectangle, in this case, the piece of paper. This teacher is encouraging his students to fold back to their Primitive Knowing of the area of a rectangle and then use this knowledge to make images of the surface area of a cylinder through the action of forming a rectangular object into a cylindrical object.

In order to help students’ growth of mathematical understanding, effective teachers should have a well-developed knowledge base for teaching. Instructional representations play an important role as the tools teachers use to convey their SMK to students in a way that makes sense to them and highlights the important mathematical aspects of a concept. Effective teaching requires teachers know their students’ current levels of Primitive Knowing in order to choose the most appropriate instructional representations for a given concept (Hashweh, 1985). Instructional representations not only reveal aspects of teachers’ SMK and PCK, but the ways teachers’ use instructional representations influences the direction and completeness of their students’ growth of mathematical understanding. This study considers the trends and patterns in the uses of instructional representations by a novice teacher and an experienced teacher as viewed through the Pirie-Kieren theory. This study uses the Pirie-Kieren theory to examine the instructional practice of two secondary mathematics teachers, not students’ growth in mathematical understanding. However, Pirie and Kieren (1994b) acknowledge the diversity of applications of their theory:
We have used it in a variety of learning environments as a tool to observe the mathematical behavior of students as they work on a single mathematical task and as they build and organize mathematical knowledge structures over periods of time. The theory has enabled us to comment closely on the levels at which different students are making sense of their mathematical activities and thoughts. Such insight into students’ understandings has been used to provide a frame for planning and engaging in mathematics lessons and, in addition, to make observations about curriculum development. (p. 181-182)

Furthermore, Towers et al., (2000) assert the “…power of the Pirie-Kieren theory is that it is a theory for, not a theory of, the growth of mathematical understanding, and as such it is validated by its usefulness to someone seeking to make sense of the growing mathematical understanding of learners” (p. 229). This study further extends the existing body of literature using the Pirie-Kieren theory by applying the theory to analyze teachers’ uses of instructional representations. By examining this phenomenon through a framework developed for observing students’ growth of understanding, a connection can potentially be established between teachers’ everyday instructional practices and the tenets put forth by the Pirie-Kieren theory to explain how students grow in their mathematical understanding.
METHODOLOGY

This chapter describes the methodology for data collection and analysis that was utilized in this study. The study is based on qualitative research methodology, in particular, case study research. In order to present a detailed view of a research study, the topic under investigation needs to be explored in its natural setting (Creswell, 1998). In this study, the topic of teachers’ uses of instructional representations was examined within the classroom setting. The data are collected from semi-structured interviews and classroom observations of two secondary mathematics teachers, one a novice teacher and the other an experienced teacher. The methodological approach of case study research is described as well as the data collection process, the analysis, and issues of validity.

Case Study

Case study research is characterized by the in-depth examination of a “case” that is bounded by time and place, is informed from multiple sources of data, and is situated within a particular context or setting (Creswell, 1998). Stake (1995) identifies three kinds of case studies: intrinsic, instrumental, and collective. Intrinsic case studies focus on the case itself, whereas instrumental case studies focus on the issues involved with the case. Collective case studies are those that involve more than one case. Instrumental case study is appropriate for this study because the issues of teacher knowledge and student learning are central to the examination and analysis of teachers’ uses of instructional representations, which is considered the case in this study.
The case of using instructional representations is situated within a classroom setting and is bounded by the classroom setting and the time allotted for the class. To provide insight into the uses of instructional representations of a novice teacher and an experienced teacher, the Pirie-Kieren theory serves as the framework for analysis and discussion. Given that the Pirie-Kieren theory takes the perspective that the growth of understanding is situated in a particular person, in a particular setting, and on a specific concept, it is necessary to consider teachers’ uses of instructional representation as situated in practice within the classroom setting. By using an existing theoretical approach to explore a case, further theoretical development is encouraged instead of constrained, which is a danger in social science research (Vaughan, 1992). This study extends the Pirie-Kieren theory, thereby facilitating theoretical development.

Participants

Two secondary geometry teachers were chosen to participate in this study. Nancy (pseudonym) was in her final year of a four-year teacher education program at a large southeastern university at the time of data collection. During the study, she was student teaching in a local area high school as part of her requirements for obtaining a secondary teacher license in mathematics. David (pseudonym) was teaching in a different local area high school than Nancy, but he was taking graduate courses in preparation for doctoral study at the same university. Each of the participants was teaching an Honors Geometry course during the time of the study. Both participants were chosen based on faculty recommendations and on this researcher’s personal knowledge. Two different university faculty members in the mathematics education department recommended each of the
participants because of a strong content knowledge of mathematics and devoted sense of purpose in teaching. This researcher co-taught a methods course in which Nancy was enrolled in the beginning of her teacher education program and also took mathematics education courses with both Nancy and David. Both participants signed informed consent forms agreeing to take part in the study and allowing their data to be used for publication purposes.

Data Collection and Sources

The data sources used in this study are semi-structured interviews with the participants and classroom observations of their teaching. Semi-structured interviews are conducted with a written protocol, but allow the interviewer to ask further questions based on the interviewees’ responses. This type of interviewing technique is considered suitable for case studies that involve small numbers of participants (Drever, 1995). Each participant was interviewed twice concerning their ideas about and knowledge of using instructional representations, once before the classroom observations began and again after the final classroom observation. The interviews were audio-taped and then transcribed for analysis. The interview protocols are found in the Appendices.

Each participant was observed on several different occasions during classroom instruction in a secondary Honors Geometry course. Nancy was observed five times during the last 6 weeks of the 2005 Fall semester and David was observed four times during the last 6 weeks of the 2006 Spring semester. During these observations, detailed field notes were taken with a focus on how these two participants used instructional representations during their instruction.
Method of Analysis

The aim of this study is to deepen our understanding of the ways in which teachers use instructional representations during instruction. In alignment with case study research, data were analyzed using categorical aggregation (Stake, 1995). In this form of data analysis, instances from the data are sought that relate to the issues of the case. Specifically, the Pirie-Kieren theory was used to code the data from each participant, which were then sorted categorically. This initial coding is called a within-case analysis (Creswell, 1998) and was used to find emergent patterns and themes within each participant’s uses of instructional representations. A cross-case analysis (Creswell, 1998) was then done to compare and contrast these emergent themes. Patterns of participants’ uses of instructional representations were sought within the analysis that further elaborates the research on effective teaching and the components of teacher knowledge: SMK, PCK, and knowledge of how students learn.

Classroom observations were coded according to the types of instructional representation used, the activity level in which the instructional representations were used, and the amount of instructional time spent using the coded instructional representations. For the purposes of this study, instructional representations were coded according to the following types: diagrams, spoken or written words, non-mathematical objects, teacher actions, numbers, and symbols. Once coded for type, the instructional representations were then coded according to their uses within particular activity levels. These activity levels were adapted from the eight potential activity levels in the Pirie-Kieren model for growth of mathematical understanding. The activity levels in the Pirie-Kieren model are determined by student actions when working with a particular concept. In this study, the levels are adapted based on how a teacher’s uses of instructional representations foster opportunities for
students to work within those levels. In other words, the teacher is not working within a given activity level; instead, the teacher is facilitating potential avenues for students’ growth of mathematical understanding through the use of instructional representations.

In the Pirie-Kieren model, the innermost level, Primitive Knowing (PK), is the knowledge that a student brings to a setting and is where growth of understanding begins. In this study, an instructional representation is used within the activity level PK if it is used to encourage students to review or recall concepts that have been previously taught by the teacher to that particular group of students. The next two levels of the Pirie-Kieren model are Image Making (IM) and Image Having (IH). In the level IM, a learner uses activities to form mental images and then can use those images without the need for those activities in the level IH. In this study, these two levels are combined into one activity level (IMH) to describe uses of instructional representations that are used as context-dependent images, models, words, or activities. A distinction between the levels in terms of how a teacher uses instructional representations is not appropriate for this study because the adaptation is based on how the teacher uses instructional representations, not on how students work with those representations. Since these inner levels are local and context-dependent in the Pirie-Kieren model, then any use of an instructional representation that is not formalized mathematically or is not used to recall previously taught concepts is used within the combined activity level IMH.

The next two levels in the Pirie-Kieren model are potential formal levels that are more general and abstract than the previous inner levels. Property Noticing (PN) is the level where a learner can focus on and connect the properties of the images formed in the previous two levels. In the adaptation of this activity level used for this study, instructional
representations are used within PN if they are used to prompt students to notice particular properties of a concept and if they are used to convey to mathematical meaning to students. Formalizing (F), in the Pirie-Kieren model, is the level where a concept takes on a formal mathematical characteristic for the learner, who can now enunciate a formal definition or algorithm. Thus, in this study, using a formal instructional representation such as an applied algorithm or symbolic formula is considered to be used within the activity level F.

The outer three layers of the Pirie-Kieren model are abstract, mathematically formal levels. Observing (O) refers to the level where a learner can express formalized understandings as theorems, Structuring (S) is the level in which a learner can connect theorems through the use of logic and proof, and Inventising (I), where a learner can extend understandings to new create new ideas. In this study, fostering opportunities for students to connect a formalized instructional representation to a theorem is considered used within the activity level O, while connecting a formalized instructional representation to a proof is within the activity level S, and encouraging students to focus beyond a concept to a new area of mathematics with an instructional representation is used within the activity level I.

Validity

Researchers are concerned about the validity and credibility of their research. Different research methods have differing ways establishing the validity of a study. In case study research, particular procedures, such as triangulation, are used to reduce the likelihood of misinterpretation (Stake, 1995). The process of triangulation uses multiple perceptions in order to clarify meanings. In this study the multiple perceptions are from the multiple data sources, the theories used in the research, separate researcher reviews of the data, and from
an external check called peer review where an uninterested party reviews the research (Creswell, 1998). Separate researcher reviews of the data were used for check-coding (Miles & Huberman, 1994) where several of this researcher’s peers coded the same section of data separately as a reliability check of the coding. Member checks were used by allowing the participants of the study to read and comment on the research report in order to lend credibility to the interpretations and conclusions (Creswell, 1998).
PRESENTATION OF FINDINGS

The purpose of this chapter is to report the findings related to the uses of instructional representations by a novice mathematics teacher and an experienced mathematics teacher. The interview and observational data were used to analyze each participant’s uses of instructional representations in order to find emergent patterns and themes concerning types of instructional representations, activity levels in which instructional representations were used, and use of instructional time. Furthermore, using the observational data, a mapping was generated of each classroom observation for both participants. Each mapping illustrates the associations between the types of instructional representations and activity levels used by the participants during a particular observation, but do not account for instructional time. Also, the mappings do not include the activity levels *Observing* (O) and *Inventising* (I) because neither participant used instructional representations within those activity levels in any classroom observation. These mappings were used during analysis in conjunction with the collected data. Also used in the analysis were each participant’s purposes of instruction, whether introducing a new concept or revisiting previously taught concepts. Both participants informed their students of their purpose of instruction in each observation.

This chapter is divided into three sections: the case of David’s uses of instructional representations, the case of Nancy’s uses of instructional representations, and a cross-case comparison of their uses of instructional representations. Each section first describes the types of instructional representations used by each participant and the activity levels in which each participant used those instructional representations. The instructional time each
participant used within particular activity levels is then discussed including the amount of
time each participant used for purposes of folding back to previously taught concepts.

The nature of Nancy’s and David’s uses of instructional representations was considered
in terms of the activity levels of the Pirie-Kieren theory for growth of students’ mathematical
understanding. The activity levels used for analysis are adapted from the eight potential
levels used to describe growth in student understanding (see Table 1) and include the feature
of folding back. These adaptations are described in more detail in the previous chapter. For
the purposes of this study, types of instructional representations were categorized according
to the following types:

- Diagrams
- Spoken or written words
- Non-mathematical objects
- Teacher actions
- Numbers
- Symbolic formulas

The descriptions from data of the types of instructional representations the two participants
used within these activity levels are focused on the specific concepts taught by each
participant during the data collection phase.
Table 1
Activity levels using instructional representations (IR) adapted from the Pirie-Kieren model for growth of mathematical understanding

<table>
<thead>
<tr>
<th>Primitive Knowing (PK)</th>
<th>Formalizing (F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Review</td>
<td>• Describe IR (mathematical language)</td>
</tr>
<tr>
<td>• Recall</td>
<td>• Generate patterns</td>
</tr>
<tr>
<td></td>
<td>• Apply algorithms and formulas</td>
</tr>
<tr>
<td>Image Making/Image Having (IMH)</td>
<td><strong>Observing</strong> (O)</td>
</tr>
<tr>
<td>• Connect IR to contexts</td>
<td>• Connect formal IR to a theorem</td>
</tr>
<tr>
<td>• Apply IR as metaphor/analogy</td>
<td><strong>Structuring</strong> (S)</td>
</tr>
<tr>
<td>• Describe IR (non-mathematical language)</td>
<td>• Connect formal IR to a proof</td>
</tr>
<tr>
<td><strong>Property Noticing</strong> (PN)</td>
<td><strong>Inventising</strong> (I)</td>
</tr>
<tr>
<td>• Focus attention on IR properties</td>
<td>• Focus IR beyond concept</td>
</tr>
<tr>
<td>• Compare/contrast</td>
<td></td>
</tr>
<tr>
<td>• Connect IR to mathematical meaning</td>
<td></td>
</tr>
</tbody>
</table>

The Case of David’s Uses of Instructional Representations

David has over 20 years experience teaching secondary mathematics. He received his first undergraduate degree in history education. After teaching secondary history for a year he decided to teach secondary mathematics. He went back to college and received a second undergraduate degree in mathematics. Since then, David has earned a Master’s degree and during the time of this study, was working towards his National Board Certification. David is an atypical mathematics teacher in that he does not prepare traditional written lesson plans before instruction. Instead, he researches the concept itself, particularly the historical development of that concept. During the first interview, David was questioned about his lesson planning practices:
I: Do you spend much time planning?

D: It depends on what you mean by planning. I do not, any longer write down a lesson plan...What I do is, if I’m going to teach a topic, I research it...I go research it in print, I research it on the web, I try and find different ways of thinking about it...but that helps me understand it...I am constantly looking for something that will make it easier for my students to understand a difficult topic. So I am always planning, but I very rarely write something down.

This practice of researching mathematics was further addressed in the second interview:

I: Last time we talked about how you prepare and how you go and research ideas.

D: I always research...it has become so much easier to do it now, if nothing else, I can find just a few distilled paragraphs that I can look at, get an example for, you know, whatever it is I need, from several different sources and be able to present that viewpoint, that method, that version of history, how it happened and why it was done and what it’s used for...I tell stories about the mathematicians and how things came about, but I also extend it to be more content-oriented.

This section reports the findings relating to David’s uses of instructional representations in a secondary honors geometry class. The data are described in terms of the patterns that emerged from David’s uses of instructional representations concerning type, activity level, and instructional time. These findings are dependent upon the specific concept that was taught during a particular observation and on the purpose of instruction for that particular observation (see Table 2). The first and third classroom observation each had one
instructional purpose: an introduction of a new concept and review of a previously taught concept, respectively. Instruction during the second and fourth classroom observations both began with a review of previously taught concepts followed by the introduction of a new concept.

Table 2
*David’s classroom observations*

<table>
<thead>
<tr>
<th>Observation</th>
<th>Purpose</th>
<th>Concept</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Introduction</td>
<td>3-Dimensional Perspectives</td>
</tr>
<tr>
<td>2</td>
<td>Review</td>
<td>Surface Areas of Circles and Polygons</td>
</tr>
<tr>
<td></td>
<td>Introduction</td>
<td>Lateral and Total Surface Areas</td>
</tr>
<tr>
<td>3</td>
<td>Review</td>
<td>Lateral and Total Surface Areas</td>
</tr>
<tr>
<td>4</td>
<td>Review</td>
<td>Similarity of 2-Dimensional Shapes</td>
</tr>
<tr>
<td></td>
<td>Introduction</td>
<td>Similar Solids</td>
</tr>
</tbody>
</table>

The results from the analysis of David’s uses of instructional representations are described by first describing the emergent patterns in the types of instructional representations he used for reviewing and introducing concepts. The activity levels of David’s uses of instructional representations are then discussed followed by a discussion of how he used his instructional time during the classroom observations.
Types of Instructional Representations

The types of instructional representations David used during his teaching of specific concepts in a secondary honors geometry course were analyzed from the collected data and the generated mappings. The types of instructional representations he used for the purpose of introducing a new concept are discussed first followed by a discussion of the types of instructional representations he used for the purpose of reviewing a previously taught concept.

In the first classroom observation, David introduced the concept of 3-dimensional perspective. During this observation, the instructional focus was on isometric and orthogonal illustrations of 3-dimensional figures. Field notes taken during the observation show David introduced a 3-dimensional figure by drawing it on the board which he then used to illustrate its associated orthogonal drawings (see Figure 5). First, he drew a 2-dimensional representation of the 3-dimensional figure on the chalkboard (A) and then verbally described the figure from the top view (B). Afterwards, he drew the top, front, and right orthogonal perspectives of the 3-dimensional figure (C) he had drawn in (A).
<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>David draws the following image on the board and labels the front “F”. He tells the students the image is like “stairs”:</td>
</tr>
<tr>
<td></td>
<td><img src="image1.png" alt="Image" /></td>
</tr>
<tr>
<td>B</td>
<td>D: “From the top, I’d get square-square” and draws the following image on the board:</td>
</tr>
<tr>
<td></td>
<td><img src="image2.png" alt="Image" /></td>
</tr>
<tr>
<td>C</td>
<td>He continues drawing the front “F”, right “R”, and top “T”:</td>
</tr>
<tr>
<td></td>
<td><img src="image3.png" alt="Image" /></td>
</tr>
</tbody>
</table>

*Figure 5. David’s IR for introducing orthogonal perspective*

In this instructional sequence David used diagrams as well as spoken words for the majority of his instructional representations. This trend of using diagrams and spoken words continued throughout this first observation.

In the second classroom observation, after presenting a brief review of the surface area of circles and polygons, David began an introduction of the concept of lateral and total surface areas. The observational data reveal that he relied on using diagrams and words for many of his instructional representations also in this lesson (see Figure 6).
A He writes the word *prism* on the board. He tells students they have seen prisms before, the ones that break up light.

He writes the word *cylinder* on the board. He tells students to show him a cylinder using a piece of paper.

D: “It’s a circle all the way down. It’s like a slinky, like stacked circles.”

He writes the word *tube* next to the word *cylinder*.

<table>
<thead>
<tr>
<th>B</th>
<th>He draws an image of a prism on the board:</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>He draws several more prisms on the board:</td>
</tr>
<tr>
<td>D</td>
<td>D: “What do we notice? What about the top?”</td>
</tr>
<tr>
<td></td>
<td>A student comments that all of the sides are rectangles.</td>
</tr>
<tr>
<td>E</td>
<td>David adds arrows that point to the sides of the prisms:</td>
</tr>
</tbody>
</table>

*Figure 6. David’s IR to introduce lateral and total surface areas*

He wrote the words *prism* and *cylinder* on the board, but further described each written word using non-mathematical language and non-mathematical objects (A). After providing these informal descriptions, David drew a figure of a triangular prism (B) followed by the figures
of three different prisms (C). He then used spoken words (D) to focus on the properties of those prisms and drew in a set of three arrows over the set of prisms illustrating the property that all of the sides of these prisms are rectangles (E). Both of the above sections of observational data show David’s reliance on the use of diagrams and written or spoken words when introducing different concepts.

He spoke during an interview about his propensity for introducing concepts using a more context-dependent approach and less reliance on giving formulas:

D: I try and give them a context first without the particulars, just a general overview of things. If there’re any formulas, if there’re any shortcuts, if there’s anything like that, they have to derive them; I do not give them formulas…I’m rabid about not telling them the formulas. They have to find it. If they find, they can use it, if they can find another one, they can use that, whatever it is. I like to teach them the shape of the thing that they’re talking about. I like to give them the outline, I like to give them the feel for it and let them play with it…I think it’s important to put it in context, otherwise it’s just numbers, it’s just methods.

When David introduced new concepts, he used many diagrams and words to help students get an idea of the concepts before giving them any formal mathematical algorithms.

The introduction of a concept was not the only purpose of instruction where David relied on the use of diagrams and words. He used these types of instructional representations during review of previously taught concepts. For instance, before introducing the concept of similar solids, he reviewed the concept of similarity of 2-dimensional figures. During that
review, he addressed similarity of circles and ellipses (see Figure 7) immediately after reviewing similarity of squares and rectangles.

<table>
<thead>
<tr>
<th>A</th>
<th>He draws the images of a circle and an ellipse on the board:</th>
</tr>
</thead>
</table>
| B | D: “Are they similar?”
He tells students that circles are like squares and ellipses are like rectangles.
He writes the word *Oval* on the board. A short discussion ensues about the relationship of “ovals” to “egg-shapes”.
D: “Let’s look at what makes circles similar”.
<table>
<thead>
<tr>
<th>C</th>
<th>He writes and draws the following IR on the board:</th>
</tr>
</thead>
</table>
| D | D: “What is proportional in these two circles?”
He writes on the board:
*Proportional*
- *radius*
- *diameter*
- *circumference*
- *Area*

*Figure 7. David’s IR for reviewing similarity of 2-dimensional figures*
He first drew figures of a circle and an ellipse (A) and then used spoken and written words to focus on properties of these figures (B) in relation to the concepts previously discussed. He continued to examine the properties of circles using written words along with a diagram (C) and spoken and written words (D) that addressed the property of proportionality of similar circles. During this instructional review, David used diagrams and words to compare circles and ellipses and to draw attention to the properties of similar circles.

The mappings that were produced from each observation show associations between activity levels and the types of instructional representations (IR) David used during each observation (see Figures 8, 9, 10, 11). The phenomenon of his uses of diagrams and words can readily be seen in each mapping. The mappings also show that David used diagrams and words within a variety of activity levels, whether he was introducing a new concept or reviewing previously taught concepts.
Figure 8. Activity levels vs. type of IR from David’s first observation

Figure 9. Activity levels vs. type of IR from David’s second observation
Figure 10. Activity levels vs. type of IR from David’s third observation

Figure 11. Activity levels vs. type of IR from David’s fourth observation
The overall trend that emerges from the data of the types of instructional representations David used is a reliance on diagrams and spoken words over any other type of instructional representation. According to David in the interview data, he wants his students to get a general idea of a concept instead of just giving them algorithms and formulas. Furthermore, the data show he used his preferred types of instructional representations within particular activity levels.

**Activity Levels of David’s Instructional Representations**

The types of instructional representations David used during the classroom observations and the way he used them relate to the activity levels adapted from the Pirie-Kieren model for student growth of mathematical understanding. Observational data and the mappings shown above reveal that, in addition to his preference for using specific types of instructional representations, he used them within particular activity levels. For instance, in the mapping of David’s first observation (see Figure 8), he uses of diagrams and words were used with the activity levels PK, IMH, and PN. Figure 5 shows a sequence of David’s uses of diagrams and words during that particular observation that were used within the activity levels IMH and PN. The figure he drew on the board along with his use of spoken words describing the figure as “stairs” (A) were used within IMH because he described the figure using non-mathematical language in order to facilitate students’ understanding of a 2-dimensional drawing of a 3-dimensional object. Next, his drawn figure of the top perspective of the previous figure (B) was used within IMH because it is a basic, non-labeled diagram. However, his spoken words (B) were used within PN because his description relied upon the mathematical features of the diagram. The labeled figures of the orthogonal perspectives (C) of the original figure were also used within PN since they illustrate the different perspectives
of orthogonal drawings and encourage students’ understanding of the connections between a 3-dimensional object and its associated orthogonal drawings.

Similarly, in the second classroom observation, David’s instructional representations used to introduce the concepts of lateral and total surface areas were primarily diagrams and words (see Figure 6) used within the activity levels IMH and PN. The written and spoken words David used (A) in the beginning of the introduction were within the activity level IMH because the words *prism* and *cylinder* were described using non-mathematical words and objects. The figure of the prism (B) was also used within IMH because it is a basic diagram of a prism utilized to show students what a prism looks like when drawn in 2-dimensional space. The next set of three prisms (C), his spoken words (D), and the final addition of arrows to the diagrams (E) are all used within PN because these instructional representations foster opportunities for students to compare and contrast properties of the different prisms.

When introducing new concepts, David used diagrams and spoken and written words most often within the activity levels of IMH and PN. However, he also used these types of instructional representations within IMH and PN during review of concepts he had previously taught. Before introducing the concept of similar solids, he reviewed the concept of similarity of 2-dimensional figures (see Figure 7). His basic diagram of a circle and an ellipse (A) were used within IMH, but his spoken words (B) were used within PN because the focus shifted and students’ attention was encouraged to move from the figures themselves to the properties of those figures and why they are similar. The next figure (C) was used within IMH because it is a basic diagram and the written words (B and C) are within PN because the focus was shifted from the figure to the properties of similar circles, particularly proportionality.
Another instance where David used instructional representations within the activity levels IMH and PN during review is found in the data from the second classroom observation. Before introducing the concept of lateral and total surface areas, David spent time reviewing how to determine the area of regular polygons (see Figure 12).

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
</table>
| A | David asks “What’s the formula? Do you remember?”  
He writes on the board:  
\[ A_p = bh \]  
\[ = \frac{1}{2}ap \] |
| B | He asks “Which one is the base? Height?”  
He draws the following polygon on the board: |
| C | He says “It sounds like an animal”.  
No response from students. He says “Apothem”. |
| D | David adds the following lines and variables to the polygon drawn on the board: |
| E | He says that the bases of the triangles make the perimeter. |

*Figure 12. David’s IR for reviewing area of regular polygons*
During this review, he used formalized instructional representations, but connected them to words and diagrams that were used within IMH and PN. He began by writing a symbolic formula on the board (A) which was used facilitate students’ understanding within the activity level PK because he asked students to recall the formula. That is, David had used this particular symbolic formula in previous instruction. He then used spoken words (B) within PN because he asked questions to focus attention on the formula and a diagram (B) within IMH where he drew a basic diagram of a regular polygon. He used spoken words (C) within IMH by using a non-mathematical word to describe the vocabulary word *apothem*. His diagram and spoken words (D and E) were used within PN because these instructional representations relate properties of the polygon to its area.

In the third classroom observation, David reviewed the concept of lateral and total surface areas in preparation for a test on that concept (see Figure 13). David began by reminding students of the previously taught concept (A) and then drew basic diagrams of two different prisms (B) which were used within IMH. He proceeded with spoken and written words (C and D) where he focused on the properties of the prisms and the information that needed to be known in order to find the surface area; therefore these written and spoken words were used in ways to promote students’ understanding within the activity level PN.
David begins class by telling students what they have already covered: surface area of prisms.

He draws two prisms on the board:

D: “What numbers do I need to know?”

D: “We need to know the number of sides.”

He writes # of sides on the board.

Figure 13. David’s IR for reviewing surface area

The mappings of David’s activity levels versus the type of instructional representations used during each observation (see Figures 8, 9, 10, 11) show that not only did he rely heavily on diagrams and spoken and written words for his instructional representations, but he also tended to use those diagrams and words to foster opportunities for students’ understanding specifically within the activity levels PK, IMH, and PN. Regardless of David’s purpose of instruction, whether introduction of a new concept or review of a previously taught concept, he used his instructional representations within the less formal activity levels. He used many diagrams and spoken and written words in ways that would encourage his students to connect those instructional representations to contexts and non-mathematical meanings and also to compare and contrast those instructional representations to notice properties relating to the concept being addressed.
David’s Use of Instructional Time

Time spent within the activity levels was included in the analysis of the classroom observations. Time intervals in minutes were recorded in the field notes during each of the four observations. This information was subsequently used in the analysis to track how David spent his instructional time when using instructional representations. The recorded times include only when David was teaching with instructional representations and not time used for student activities, individual seatwork, or other classroom activities. His uses of instructional time were divided into two categories according to David’s purpose of instruction: review of a previously taught concept or introduction of a new concept and then further divided among the time spent within the activity levels of his instructional representations (See Tables 3, 4, 5, 6).

Table 3
Instructional time (in minutes) of David’s first observation

<table>
<thead>
<tr>
<th>Purpose</th>
<th>PK</th>
<th>IMH</th>
<th>PN</th>
<th>F</th>
<th>S</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Review</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>Introduction</td>
<td>13</td>
<td>13</td>
<td></td>
<td></td>
<td></td>
<td>26</td>
</tr>
<tr>
<td>Total</td>
<td>2</td>
<td>13</td>
<td>13</td>
<td></td>
<td></td>
<td>28</td>
</tr>
</tbody>
</table>
Table 4
*Instructional time (in minutes) of David's second observation*

<table>
<thead>
<tr>
<th>Purpose</th>
<th>PK</th>
<th>IMH</th>
<th>PN</th>
<th>F</th>
<th>S</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Review</td>
<td>4</td>
<td>3</td>
<td>9.5</td>
<td></td>
<td></td>
<td>16.5</td>
</tr>
<tr>
<td>Introduction</td>
<td>11</td>
<td>6.5</td>
<td>4</td>
<td></td>
<td></td>
<td>22.5</td>
</tr>
<tr>
<td>Total</td>
<td>4</td>
<td>14</td>
<td>16</td>
<td>4</td>
<td></td>
<td>38</td>
</tr>
</tbody>
</table>

Table 5
*Instructional time (in minutes) of David's third observation*

<table>
<thead>
<tr>
<th>Purpose</th>
<th>PK</th>
<th>IMH</th>
<th>PN</th>
<th>F</th>
<th>S</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Review</td>
<td>2</td>
<td>21.5</td>
<td>8.5</td>
<td></td>
<td></td>
<td>32</td>
</tr>
<tr>
<td>Introduction</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>2</td>
<td>21.5</td>
<td>8.5</td>
<td></td>
<td></td>
<td>32</td>
</tr>
</tbody>
</table>

Table 6
*Instructional time (in minutes) of David's fourth observation*

<table>
<thead>
<tr>
<th>Purpose</th>
<th>PK</th>
<th>IMH</th>
<th>PN</th>
<th>F</th>
<th>S</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Review</td>
<td>3.5</td>
<td>7.5</td>
<td>16.5</td>
<td>2</td>
<td>2</td>
<td>31.5</td>
</tr>
<tr>
<td>Introduction</td>
<td>1.5</td>
<td>5.5</td>
<td></td>
<td></td>
<td></td>
<td>7</td>
</tr>
<tr>
<td>Total</td>
<td>3.5</td>
<td>9</td>
<td>22</td>
<td>2</td>
<td></td>
<td>38.5</td>
</tr>
</tbody>
</table>
The time data show that David’s use of instructional time depended upon his purpose of instruction. Furthermore, the data show how much time he spent using instructional representations within particular activity levels. When David introduced a new concept, the time spent using instructional representations within IMH and PN was evenly divided: 25.5 minutes in IMH and 25 minutes in PN. For each observation that included instructional time for review, the majority of David’s instructional time was spent using instructional representations within PN: 12.5 minutes in IMH and 47.5 minutes in PN. Overall, David spent significantly more instructional time using instructional representations to facilitate students’ understanding within PN than within IMH, regardless of the purpose of instruction: 38 minutes in IMH and 72.5 minutes in PN (See Table 7).

Table 7
David’s total instructional time according to purpose of instruction

<table>
<thead>
<tr>
<th>Purpose</th>
<th>PK</th>
<th>IMH</th>
<th>PN</th>
<th>F</th>
<th>S</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Introduction</td>
<td>25</td>
<td>25</td>
<td>4</td>
<td></td>
<td></td>
<td>54.5</td>
</tr>
<tr>
<td>Review</td>
<td>9.5</td>
<td>12.5</td>
<td>47.5</td>
<td>10.5</td>
<td>2</td>
<td>82</td>
</tr>
<tr>
<td>Total</td>
<td>9.5</td>
<td>38</td>
<td>72.5</td>
<td>14.5</td>
<td>2</td>
<td>136.5</td>
</tr>
</tbody>
</table>

In contrast to the amount of instructional time David spent using instructional representations within the activity levels IMH and PN, he spent relatively little time within any of the other activity levels. He did not use any instructional representations within the activity level O and once he used an instructional representation within S in the fourth
observation when he symbolically proved that all circles are similar. The time David spent using instructional representations within PK was used only during review of previously taught concepts (9.5 minutes altogether). This is due to the nature of the activity level PK, whereby instructional representations are used for the purpose of recall or review. The time David spent using instructional representations within F was used both during review of previously taught concepts and the introduction of a new concept. However, he spent more time using instructional representations within F for the purpose of review (10.5 minutes) than for introducing a new concept (4 minutes).

Further analysis of David’s use of instructional time shows that he spent more time using instructional representations for the purpose of review than for the purpose of introducing a new concept. Over all four observations, David spent 80.5 minutes of his instructional time reviewing previously taught concepts and 56 minutes introducing new concepts. This time spent reviewing previously taught concepts is related to the feature of folding back in the Pirie-Kieren theory. As stated in a previous chapter, for the purposes of this study, folding back occurs when instructional representations are used by a teacher to connect an instructional representation to a previously taught concept. David spent most of his instructional time reviewing previously taught concepts; therefore he spent most of his instructional time folding back to review. In the second interview, David spoke about his reasons for reviewing a concept before beginning a new one:

D: I don’t believe in letting them do it for a week before I chime in…I just don’t believe that letting someone twist in the wind for even a day is really fruitful. If they’re going to twist in the wind, I want them going in a direction…I at least want to remind them of what’s gone before. Nail it down to something.
From the observational data, David used many instructional representations to fold back for the instructional purpose of review. Furthermore, his uses of instructional representations tended to be used within the activity level PN. As shown above in Table 7, David spent most of his instructional time used for the purpose of review within PN. For example, in the second classroom observation, David reviewed the concept of surface areas of circles and polygons before introducing the concept of lateral and total surface areas. At the beginning of the observation, David focused on comparing the area of a circle to its circumference (see Figure 14). He began by asking for the formula for the area of a circle and compared it to the formula for the circumference of a circle (A). Here he used words and formulas to focus students’ attention on properties of circles. He then drew a figure of a number line to illustrate the length of the circumference of a circle (B). He continued by connecting this idea to the area of a circle by using a square in place of the number line (C). This comparison further illustrated the difference between the area of a circle and its circumference by using words and diagrams within PN.
A: D: “Let’s start with a circle. What is the area of a circle?”

He writes on the board:

\[ A = \pi r^2 \]

D: “How do you know it’s not \(2\pi r\)?”

B

He draws the following diagram on the board:

D: “It’s like a number line stretched out.”

He explains that the above line will get about three times as long if he multiplies by \(\pi\).

C

He draws the following diagram on the board:

D: “The area is the one with a square in it; it’s a radius by radius sort of thing. We make the area bigger by \(\pi\) instead of the line”.

Figure 14. David’s IR for reviewing area of circles

David began the fourth classroom observation reviewing the concept of similarity of 2-dimensional figures before he introduced the concept of similar solids (see Figure 15). He used spoken and written words (A, B, C, D) to promote students’ understanding within PN because the words focus on the properties of similar 2-dimensional figures.
<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>D: “What makes shapes similar?”</td>
</tr>
<tr>
<td>B</td>
<td>He writes on the board:</td>
</tr>
</tbody>
</table>
|   | *Angles are congruent*  
|   | *Shapes must be proportional*  
|   | *Corresponding sides of the shapes* |
| C | He erases the word *shapes* from the last line and draws an arrow to the previous line: |
|   | *Angles are congruent*  
|   | *Shapes must be proportional*  
|   | *Corresponding sides of the* |
| D | He writes the words *same shape* on the board. |

*Figure 15. David’s IR for reviewing similarity of 2-dimensional figures*

David spent the majority of his instructional time reviewing previously taught concepts. This implies he used instructional representations during review for the purpose of folding back and he tended to use them to promote students’ understanding within the activity level PN.

*Summary of David’s Uses of Instructional Representations*

David’s uses of particular types of instructional representations were categorized within specific activity levels and were further dependent upon his purposes for instruction of a specific concept. The patterns that emerge from the data concern the types of instructional representations David used most often, the activity levels in which he used them, and his uses of instructional time. Overall David used diagrams and spoken and written words for the majority of his instructional representations and they were most often used to facilitate students’ understanding within the activity levels IMH and PN. This trend occurred regardless of David’s purpose of instruction, whether a review of a previously taught concept
or the introduction of a new concept. However, David spent more time using instructional representations within the activity level PN, especially when he fostered opportunities for students’ folding back to review. David did not spend much instructional time using instructional representations within the activity level F, but when he did, he used them primarily during review of previously taught concepts.

The Case of Nancy’s Uses of Instructional Representations

At the time of this study, Nancy was student teaching and a senior in her teacher education program. Her decision to become a mathematics teacher was influenced by both her family (her mother is a teacher) and her previous teachers. Nancy realized the complexity of learning to teach from her student teaching experiences. The second interview with Nancy was conducted towards the end of her student teaching internship and after the five observations had been completed. She was questioned about her student teaching experiences:

I: So you just got finished with student teaching. How did that work out for you?

D: It was definitely a learning experience…you have to deal with all this behind-the-scenes stuff and nobody ever sees the grading papers, the discipline, the contacting parents, the administrative meetings and all kinds of stuff. Just keeping up with papers and it’s all just, virtually impossible. I think that I was well-prepared for the content, even though I had to relearn a lot of geometry for myself…I did well with the content and writing lesson plans wasn’t too much…but then you get to the nitty-gritty and things kind of fall apart…I’ve already started getting things ready for next year…I’m trying to psyche
myself up for next year and make it so that I don’t have so much to do. I don’t know, if student teaching is intended for you to learn how you want to teach, then I definitely think that it’s been a good experience for me.

The findings related to Nancy’s uses of instructional representations in a secondary geometry class are reported in this section. The patterns that emerged from Nancy’s uses of instructional representations concerning type, activity level, purpose, and instructional time are described based on the collected data and generated mappings. The findings are dependent upon the specific concept being taught or the purpose of instruction for a particular observation (see Table 8). In the first and fifth classroom observations Nancy introduced to new concepts. In the second, third, and fourth classroom observations Nancy extended previously taught concepts by extending them through applications or connections to other concepts.
Table 8
Nancy’s classroom observations

<table>
<thead>
<tr>
<th>Observation</th>
<th>Purpose</th>
<th>Concept</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Introduction</td>
<td>Area Formulas for 2-Dimensional Figures</td>
</tr>
<tr>
<td>2</td>
<td>Extend</td>
<td>Area of Circle Sections and Irregular Polygons</td>
</tr>
<tr>
<td>3</td>
<td>Extend</td>
<td>Circles and the Pythagorean Theorem</td>
</tr>
<tr>
<td>4</td>
<td>Extend</td>
<td>Volume Applications</td>
</tr>
<tr>
<td>5</td>
<td>Introduction</td>
<td>Volume and Surface Area of Spheres</td>
</tr>
</tbody>
</table>

The results from the data analysis of Nancy’s uses of instructional representations are first described in terms of the types of instructional representations she used for introducing or extending concepts. The activity levels of Nancy’s uses of instructional representations are then described followed by a discussion of her use of instructional time.

Types of Instructional Representations

The types of instructional representations Nancy used for specific concepts in a secondary honors geometry course were analyzed from the collected data and the generated mappings. The types of instructional representations she used for the purpose of introducing a new concept are discussed first followed by a discussion of the types of instructional representations she used for the purpose of extending a previously taught concept.
Nancy introduced the concept of the area of 2-dimensional figures in the first classroom observation. After a brief activity where students used index cards to find the relationship between the area of a rectangle and the area of a parallelogram, Nancy began her Power Point presentation (see Figure 16). In her first slide (A), Nancy used written words, diagrams, and formulas to address the area of rectangles and parallelograms. She proceeded to address the area of triangles in the next slide (B). Again, she used written words, diagrams, and formulas for her instructional representations. With the written words, she connected the area of a triangle to the area of a rectangle and then provided an example (C) using numbers obtained from the student activity.
Nancy projects the next Power Point slide:

Rectangles and Parallelograms

**Area of a Rectangle**

$A = bh$

**Area of a Parallelogram**: $A = bh$

Note: height is $\perp$ to the base

Nancy projects another slide:

**Area of a Triangle**

- What’s the ratio of the area of one of the two triangles you formed when you cut the index card in half to the entire rectangle?
  - 1:2
- So the area of one of these triangles is 7.5 in$^2$
- The general formula for the area of

$$A = \frac{1}{2} bh$$

Remember! The height has to be perpendicular to the corresponding base.

On the board, she writes:

A of $\triangle$: A of Rect.

1:2

$A = \frac{1}{2} bh$

$A = \frac{1}{2} \times 5 \times 8$

$A = 4 \times 5 = 20$

*Figure 16. Nancy’s IR for introducing area of rectangles and parallelograms*

Figure 17 shows Nancy continued her uses of words, diagrams, formulas, and numbers later during the same observation.
| A | On the board, she draws the following rhombus and writes the word rhombus: 

[rhombus]

She explains the congruency by reminding students of the SSS postulate. |
|---|---|
| B | She writes on the board:

\[
A = \frac{1}{2} \cdot (1)(2) \\
A = 1 \text{ un}^2 \\
4 \cdot A = 4 \text{ un}^2 \\
A = \frac{1}{2} \cdot d_1 \cdot d_2
\] |
| C | Nancy proceeds to show students how this formula is derived and writes the following on the board:

(one of the \( \Delta \))

\[
\Delta A = \frac{1}{2} \left( \frac{1}{2} d_1 \right) \left( \frac{1}{2} d_2 \right)
\]

\[
\square A = 4 \left( \frac{1}{2} \left( \frac{1}{2} d_1 \right) \left( \frac{1}{2} d_2 \right) \right)
\]

\[
A = 4 \left( \frac{1}{4} \right) d_1 \cdot \frac{1}{2} d_2
\]

\[
A = \frac{1}{2} d_1 \cdot d_2
\] |
| D | Nancy leads the students through an example on the board:

**Rhombus**

\( d_1 = 5 \text{ in.} \)

\( d_2 = 7 \text{ in.} \)

**Find area.**

\[
A = \frac{1}{2} \cdot d_1 \cdot d_2 = \frac{1}{2} \cdot (5)(7)
\]

\[
= 17.5 \text{ in.}^2
\]

*Figure 17.* Nancy’s IR for introducing area of rhombuses
She wrote the word *rhombus* on the board and drew a diagram of a rhombus with its associated congruency marks (A). She then used numbers to find the area of one of the four triangles in the rhombus (B) which led to her draw another diagram and show the symbolic derivation for the formula for the area of a rhombus (C). She proceeded to show another specific example using numbers (D).

During Nancy’s introduction to the concept of area of 2-dimensional figures, she used words, diagrams, numbers, and formulas for the majority of her instructional representations. The fifth observation is another observation where Nancy introduced a new concept: the volume and surface area of spheres. During this observation, Nancy relied on the use of formulas and numbers for her instructional representations (see Figure 18).
Nancy introduced this concept by providing the symbolic formulas for finding the volume and surface area of a sphere. She proceeded to work through an example of this concept using a cube and sphere. First, she wrote the formula for the surface area of a sphere (A) and

\[
S.A. = 4\pi r^2
\]

She redrews the projected images of the cube and the sphere and labels the lengths of the side of the cube and the radius of the sphere:

\[
\begin{align*}
12 \text{ cm} & \\
6 \text{ cm}
\end{align*}
\]

She returns to the surface area equation and writes underneath:

\[
\begin{align*}
S.A. &= 4\pi (6)^2 \\
S.A. &= 4\pi \times 36 \\
S.A. &= 144\pi = 452.4cm^2
\end{align*}
\]

\[
V_{\text{cube}} = lwh
= 12 \cdot 12 \cdot 12 = 12^3
= 1728cm^3
\]

\[
V_{\text{sphere}} = \frac{4\pi r^3}{3}
= \frac{4\pi (6)^3}{3}
= \frac{(4\pi (216))}{3}
= 904.78cm^3
\]

\[
V_{\text{cube}} - V_{\text{sphere}} = V_{\text{leftovers}}
= 1728 - 904.78 = 823.22cm^2
\]

**Figure 18.** Nancy’s IR for volume and surface area of spheres
then she drew a diagram of a cube and a sphere and labeled them with the information provided from the example (B). She continued with the example using the symbolic formulas and numbers (C). Nancy maintained her uses of symbolic formulas and numbers with specific examples for the duration of the classroom observation.

The other three classroom observations involved Nancy using previously taught concepts to either show connections between those concepts or extend them through new applications. During the third observation, Nancy connected the previously taught concepts of the Pythagorean Theorem and the tangent lines and inscribed angles of circles. After introducing the first example, Nancy reviewed special right triangles, the Pythagorean Theorem, and the area of a sector of a circle (see Figure 19).
A Nancy first does some review and writes on the board:

\[
\begin{align*}
45-45-90 & \quad 30-60-90 \\
\text{leg} &= a \\
\text{hyp} &= 2a \\
\text{hyp} &= \sqrt{2}a \\
\text{longest leg} &= \sqrt{3}a \\
\text{shortest leg} &= a \\
\end{align*}
\]

\[a^2 + b^2 = c^2\]

\[A_{\text{sector}} = \frac{a \pi r^2}{360} = \text{central } \angle\]

B Over the slide image, Nancy fills in the length of TA and circles the information on the board about the 30-60-90 triangle.

She writes on the board:

\[
\frac{12\sqrt{3}}{\sqrt{3}} = \sqrt{3}a
\]

\[
12 = a
\]

Nancy then writes 12 for the length of AN over the projected slide image.

Under \(A_{\text{sector}}\) on the board, she writes:

\[
\frac{60\pi 12^2}{360} = 75.4 \text{ units}^2
\]

C Nancy writes on the board:

\[A = \pi r^2 = \pi 12^2 = 452.4 \text{ units}^2\]

She asks “What’s our last step?”

On the board she writes:

\[
\begin{align*}
452.4 \\
- 75.4 \\
377 \text{ units}^2
\end{align*}
\]

---

\textit{Figure 19. Nancy’s IR for circle sections and the Pythagorean Theorem}

In the second classroom observation, she extended the previously taught concepts of the areas of circles and regular polygons to the concept of the area of parts of circles and irregular polygons. After briefly describing sectors, segments, and annuluses of circles to her students, she proceeded to solve examples of each one using words, diagrams, symbolic
formulas, and numbers. As a case in point, Nancy’s uses of words, diagrams, symbolic formulas, and numbers during this observation can be seen in Figure 20. She introduced the annulus of a circle using words, a symbolic formula, and a diagram (A). She proceeded to solve the example using the numbers in an applied algorithm (B) using the previously given formula (A).

<table>
<thead>
<tr>
<th>A</th>
<th>Nancy projects the next slide:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td><strong>Area of an annulus</strong></td>
</tr>
<tr>
<td></td>
<td>• What do you think?</td>
</tr>
<tr>
<td></td>
<td>– Area of big circle – Area of little circle</td>
</tr>
<tr>
<td></td>
<td>– $\pi R^2 - \pi r^2$</td>
</tr>
</tbody>
</table>

![Diagram of an annulus with radii labeled 4 cm and 20 cm.]

<table>
<thead>
<tr>
<th>B</th>
<th>Nancy writes on the board:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\pi R^2 - \pi r^2$</td>
</tr>
<tr>
<td></td>
<td>$\pi(20^2) - \pi(4^2)$</td>
</tr>
<tr>
<td></td>
<td>$\pi(400) - \pi(16)$</td>
</tr>
<tr>
<td></td>
<td>$400 \pi - 16 \pi = 384 \pi = 1206.37 \text{cm}^2$</td>
</tr>
</tbody>
</table>

*Figure 20. Nancy’s IR for the annulus of a circle*
From the observational data, Nancy used many words, diagrams, symbolic formulas, and numbers, especially with specific examples. In the first interview with Nancy, she spoke about providing students with specific examples along with the relevant diagrams:

N: …the examples take a long time to draw…this week I’m going to try giving them the examples…so they can write how to work them out beside that, so the drawings they’ll have on their paper because I’m just printing off Power Point slides. It’s not like it’s a lot of extra work. So I’m just going to do that and they can write the theorems and then we’ll go through some more examples hopefully…we haven’t really been able to get to the class work; I’ve been doing the examples as guided practice except that it’s on the board.

The mappings of activity levels versus the type of instructional representations Nancy used during each observation (see Figures 21, 22, 23, 24, 25) show the variety in her uses of instructional representations. Diagrams, words, numbers, and formulas are the types of instructional representations she used in every observation, regardless of the purpose of instruction.
Figure 21. Activity levels vs. type of IR from Nancy’s first observation

Figure 22. Activity levels vs. type of IR from Nancy’s second observation
Figure 23. Activity levels vs. type of IR from Nancy’s third observation

Figure 24. Activity levels vs. type of IR from Nancy’s fourth observation
In summary, Nancy used a wide variety of instructional representations but showed a penchant for using words, diagrams, numbers, and formulas, especially with specific examples. Nancy worked through numerous examples in each observation, regardless of her purpose of instruction. However, the activity levels in which she used her instructional representations differed based whether she was introducing a new concept or extending previously taught concepts.

*Activity Levels of Nancy’s Instructional Representations*

The various types of instructional representations Nancy used during the classroom observations and the way she used them relate to the activity levels adapted from the Pirie-Kieren model for student growth of mathematical understanding. Observational data and the mappings shown above reveal that, although she used a variety of types of instructional
representations, she used them within particular activity levels which were dependent upon her purpose of instruction. For instance, in the first classroom observation, Nancy introduced the area of rectangles and parallelograms. Figure 16 shows Nancy’s uses of written words, diagrams, formulas, and numbers. Her written words and diagrams (A and B) and numbers (C) were used to focus students’ attention on the concept of area and its related properties, thus these instructional representations are within the activity level PN. Her use of symbolic formulas (A and B) is within F because she provided her students with the mathematical formula for finding the area of rectangles and parallelograms. Nancy informed students, via Power Point slide (see Figure 26), at the beginning of this observation that the learning objectives were to find and use area formulas.

8.1 Areas of Rectangles and Parallelograms
8.2 Areas of Triangles, Trapezoids, and Kites

- To find formulas for areas of rectangles and parallelograms
- To find formulas for areas of triangles, trapezoids, and kites
- Use these formulas to find the area of geometric shapes

Figure 26. Nancy’s learning objectives from the first observation
In other words, Nancy intended for students to learn the symbolic formulas for the areas of 2-dimensional geometric shapes in this introductory lesson. Figure 17 shows observational data from the same lesson where Nancy introduced finding the area of a rhombus. Her diagram of a rhombus (A) was used within the activity level PN because the congruency marks focus students’ attention on the properties associated with the relationships of the sides and diagonals of a rhombus. She used spoken words (A) to encourage students to connect these properties to the previously taught concept of congruent triangles and therefore, these spoken words were used within PK. The diagram (C) was used to promote students’ understanding within PN because the relationship between the lengths of the diagonals was illustrated. The numbers and symbolic formulas she used for the area of a rhombus (B, C, and D) were all used within F due the degree of mathematical formalization of these instructional representations.

She continued this trend of using instructional representations within the activity levels PN and F in the fifth classroom observation. In this observation, Nancy introduced the volume and surface area of spheres. Again, using Power Point slides, she stated her learning objectives for students to learn the symbolic formulas for this concept, which she subsequently provided (see Figure 27).
Figure 27. Nancy’s learning objectives and symbolic formulas from the fifth observation

She continued in this observation by working an example (see Figure 18). Her instructional representation of the symbolic formula (A) was used to facilitate students’ understanding within the activity level F. Her figures (B) were used within IMH because the numbers she used to label the figures connect them to the context of the example.
Conversely, her uses of numbers and formulas (C) were used within F because she applied an algorithm in the solving of the example.

When introducing concepts, Nancy used her instructional representations in ways that facilitate students’ understanding within the activity levels PN and F. However, when Nancy’s purpose of instruction was to extend previously taught concepts, she used instructional representations within a wider variety of activity levels. For example, in the fourth classroom observation, Nancy extended the previously taught concept of volume to application problems involving volume. After a brief review of the volume of cones, Nancy proceeded to her first example (see Figure 28). In this example, she used a piece of paper to make a box and find the volume of that box.
| A | She tells students that they are going to do a physical example.  
She draws the following rectangle on the board: |
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
|   | 11"
|   | 8 1/2"
|   |  
|   |  
|   | 6 1/2"
|   | 9"
| B | She asks the students “What shape is this? What is the formula for the volume?”  
On the board she writes:  
\[ V = lwh \]  
\[ V = 9(6.5)1 \]  
\[ V = 58.5in^3 \]  
| C | She adds the following labels to the rectangle already drawn on the board:  
She asks “What’s the height going to be?” and holds up the paper model.  
| D | Under the formula for volume that is already written on the board, she writes:  
\[ V = 9(6.5)1 \]  
\[ V = 58.5in^3 \]  

*Figure 28. Nancy’s IR for volume applications*
She drew a rectangle on the board and labeled the lengths of the sides and then proceeded to make a box using a piece of white paper (A). The diagram and the non-mathematical object were both used within the activity level IMH because Nancy used them to connect to the context of the problem. Her spoken words and written symbolic formula (B) were used within PK because students had to recall the previously taught formula. Nancy then wrote more information on the diagram and showed the paper box to the students again (C). These instructional representations were used within IMH because she used them to connect to the context of the problem, but her spoken words (C) were used within PN because they were used to focus students’ attention on the properties of the diagram and box. She concluded by applying the volume formula (D), therefore it was used within F.

In the second classroom observation, Nancy extended the previously taught concepts of areas of circles and regular polygons by addressing areas of circle segments and irregular polygons. She began by providing written descriptions of circle segments (see Figure 29).
Nancy shows a slide providing written definitions for sector, segment, and annulus and students copy the information in their notes.

Any Way You Slice It

- Sector of a circle – the region between two radii of a circle and the included arc (slice of pizza!)
- Segment of a circle – the region between a chord of a circle and the included arc (there’s got to be a food sliced like this!)
- Annulus – the region between two concentric circles (doughnut!)

Figure 29. Nancy’s written definitions for circle segments

Nancy used written words to provide the mathematical definitions of circle segments and then connected the definitions of sector and annulus to food items. These written words were used within both IMH and F because she described each circle segment mathematically and with non-mathematical objects. Nancy’s uses of instructional representations when extending previously taught concepts incorporated more uses within IMH along with the reliance on PN and F as when she introduced a concept.

The mappings generated from the data show that Nancy used a wide variety of types of instructional representations regardless of the purpose of instruction for a particular observation. The mappings also show her uses of instructional representations were used to promote students’ understanding within activity levels PK, IMH, PN, and F in every
observation. She addressed using a variety of instructional representations in the first interview:

I: What do think about the term “instructional representation”? If you were describing it to a friend, how would you explain it?

N: Instructional representations are just different ways to represent ideas so that different learning styles, that’s my understanding, can understand exactly what you are meaning. So you have just giving the definition, you have symbols, you have drawing a picture on the board or getting up and doing a kinesthetic activity. So the way I see it, representing different things to different learning styles, so that everybody can understand.

I: What about an example?

N: For example, parallelograms, the definition is a shape with two pairs of, with opposite sides that are parallel. That’s the definition. A drawing would be another representation and you draw it on the board, you draw two sets of parallel lines and there’s your parallelogram. Or you could cut out, or we’re going to do this when we’re doing area, and you have an index card and you cut it along its diagonal and then you rearrange the shapes and you have a parallelogram with the same area as the card…or you could say “where is a parallelogram in this classroom?”, so they see that geometry does actually apply to real life.

For Nancy, using a variety of types of instructional representations contributes to student understanding by addressing the different learning styles of individual students.
Nancy’s Use of Instructional Time

Time intervals were recorded in the field notes for each of the five classroom observations with Nancy. This information was subsequently used in the analysis to track how she spent her instructional time when using instructional representations. The recorded time interval (in minutes) include only when Nancy was teaching with instructional representations and not the time used for student activities or individual seatwork. Nancy had one instructional purpose in each observation; she either introduced a concept or extended a previously taught concept. The time she spent using instructional representations is summarized in Table 9, which shows the total instructional time she used within the different activity levels for each classroom observation. In the first and fifth classroom observations Nancy introduced to new concepts. In the second, third, and fourth classroom observations Nancy extended previously taught concepts. The activity levels O, S, and I are not included because Nancy did not use any instructional representations within those activity levels in any of the five observations.
Table 9
Instructional time (in minutes) of Nancy’s observations

<table>
<thead>
<tr>
<th>Observation</th>
<th>PK</th>
<th>IMH</th>
<th>PN</th>
<th>F</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.5</td>
<td>4</td>
<td>11</td>
<td>10</td>
<td>26.5</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>12.5</td>
<td>13</td>
<td>16</td>
<td>42.5</td>
</tr>
<tr>
<td>3</td>
<td>5.5</td>
<td>2</td>
<td>1.5</td>
<td>7</td>
<td>16</td>
</tr>
<tr>
<td>4</td>
<td>6.5</td>
<td>8</td>
<td>5.5</td>
<td>9.5</td>
<td>29.5</td>
</tr>
<tr>
<td>5</td>
<td>1.5</td>
<td>5.5</td>
<td>8.5</td>
<td>21</td>
<td>36.5</td>
</tr>
<tr>
<td>Total</td>
<td>16</td>
<td>32</td>
<td>39.5</td>
<td>61.5</td>
<td>151</td>
</tr>
</tbody>
</table>

The time data show that Nancy’s uses of instructional representations not only varied by type, but also by the activity level in which they were used. The total time spent using instructional representations within each level illustrates that much of her instructional time was spent within level F. Specifically, she used instructional representations within activity level F for 61.5 minutes out of the 151 minutes total instructional time. When Nancy introduced new concepts in observations 1 and 5, she spent more time using instructional representations within PN and F. However, when Nancy extended previously taught concepts she spent more time using instructional representations within PK and IMH than when she introduced a concept. Table 10 shows a comparison of total instructional time between when she introduced a concept and when she extended previously taught concepts.
This comparison shows that when Nancy’s purpose of instruction was to extend previously taught concepts, she spent more time using instructional representations within PK and IMH than when she introduced concepts. However, regardless of her purpose of instruction, Nancy’s instructional time using instructional representations was almost evenly divided between the levels PN and F in all five observations.

Nancy spent more time using instructional representations within PK and IMH when her purpose of instruction was to extend previously taught concepts. This time spent addressing previously taught concepts is related to the feature of folding back in the Pirie-Kieren theory. That is, Nancy used her instructional representations as a means to fold back to extend by using them to address previously taught concepts within the activity levels PK and IMH. Although Nancy did use instructional representations to fold back to extend, she recognized that she needed to learn how to do so more effectively. In the second interview, Nancy acknowledged her need to learn how to revisit previously taught concepts during her instruction:

N: I do need to learn how to back up and say things in a different way instead of just saying the same thing over again. And I think that just comes with time,
that it comes with having to face these questions from students and not knowing what to say and then going back and re-explaining yourself, after doing research.

Summary of Nancy’s Uses of Instructional Representations

Nancy’s uses of a variety of types of instructional representations were categorized within specific activity levels and these uses were dependent upon her purpose of instruction for a specific concept. The patterns that emerge from the data concern the types of instructional representations used by Nancy, the activity levels in which she used them, and her use of instructional time. Overall, Nancy used a wide variety of types of instructional representations, but specifically used words, diagrams, numbers, and formulas in all five classroom observations. The variety in her uses of instructional representations was seen through her uses of specific examples during instruction. When introducing concepts, Nancy used her instructional representations to promote students’ understanding within the activity levels PN and F. However, when Nancy’s purpose of instruction was to extend previously taught concepts, she used instructional representations in a wider variety of activity levels, including PK, IMH, PN, and F. The wider variety of uses of instructional representations occurred when Nancy used them to fold back to extend previously taught concepts. That is, she used instructional representations within the activity levels PK and IMH most often when she was folding back to extend during instruction. However, she spent much of her instructional time using instructional representations to promote students’ understanding within the activity level F.
Comparison of David’s and Nancy’s Uses of Instructional Representations

The findings from the comparative analysis of David’s and Nancy’s uses of instructional representations are presented in this section. These findings are described based on the similarities and differences in their uses of instructional representations, including purpose of instruction. David and Nancy each had two purposes of instruction. David either introduced a new concept or reviewed previously taught concepts. Nancy also introduced new concepts, but she extended previously taught concepts by connection or application. The data are first described in terms of the emergent patterns from comparing David’s and Nancy’s uses of particular types of instructional representations. The comparison of activity levels of their uses of instructional representations is then discussed followed by the comparison of their uses of instructional time including the feature of folding back based on the adaptation from the Pirie-Kieren theory for this study.

Comparison of David’s and Nancy’s Types of Instructional Representations

The data used to analyze David’s uses of particular types of instructional representations show his preference for using diagrams and words over any other types of instructional representations. David’s purpose of instruction, whether introducing a new concept or reviewing previously taught concepts, did not affect his use of diagrams and words as his instructional representations. On the other hand, the data used to analyze Nancy’s uses of particular types of representations show that she used a wider variety of types of instructional representations than David. However, she relied on the use of words, diagrams, numbers, and formulas more frequently than any other type of instructional representation. This commonality in the types of her instructional representations was consistent whether her purpose of instruction was to introduce a new concept or to extend
previously taught concepts. Although David’s and Nancy’s uses of instructional representations differed in the types each participant used most frequently during instruction, both of them were consistent in their uses of particular types regardless of their purpose of instruction.

Comparison of Activity Levels of David’s and Nancy’s Instructional Representations

The types of instructional representations used by these two participants were analyzed for their uses within the activity levels adapted from the Pirie-Kieren theory (see Table 1). The words and diagrams used by David as his preferred types of instructional representations were also used within specific activity levels. Data analysis shows that regardless of David’s purpose of instruction, he used his instructional representations to facilitate students’ understanding within the less formal activity levels PK, IMH, and PN. Overall, David used his instructional representations in a manner as to encourage students to connect those instructional representations to contexts and non-mathematical meanings and also to compare and contrast the mathematical properties of the concept he was teaching. Unlike David’s uses of instructional representations within particular activity levels, which did not change according to his purpose of instruction, Nancy’s uses of instructional representations within particular activity levels did differ according to her purpose of instruction. When Nancy’s purpose of instruction was to introduce a concept, she used her instructional representations most frequently within the activity levels PN and F. When her purpose of instruction was to extend previously taught concepts, her instructional representations were used within a wider variety of activity levels, but more frequently within the less formal levels PK and IMH.
Comparing these two participants’ uses of instructional representations within specific activity levels reveals differences their uses within those activity levels. David consistently used his instructional representations within the less formal activity levels PK, IMH, and PN, regardless of his purpose of instruction. Nancy, on the other hand, used her instructional representations within PN and F more frequently during an introduction to a concept. Her introductory lessons consisted of instructional representations that focused students’ attention on the mathematical properties, definitions, and formulas of the concept she introduced. Yet when Nancy’s purpose of instruction was to extend previously taught concepts, her instructional representations were used over a larger range of activity levels with more instructional representations used within PK and IMH.

*Comparison of David’s and Nancy’s Use of Instructional Time*

The comparison of David’s and Nancy’s uses of instructional time reveals a greater diversity between their uses of instructional representations than the types they used or the activity levels in which they used them. The analysis of David’s use of instructional time shows that the time he spent using instructional representations within specific activity levels depended upon his purpose of instruction. When introducing a concept, David’s instructional time was evenly divided between using instructional representations within the activity levels IMH and PN. When reviewing a previously taught concept, David spent more time using instructional representations within PN. Although David did not spend much time using instructional representations within F, when he did, more time was spent using instructional representations within F for the purpose of reviewing a previously taught concept than for introducing a new concept. However, if his purpose of instruction is not taken into consideration, David spent more instructional time overall using instructional representations

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within PN. The analysis of Nancy’s use of instructional time shows that the time she spent within specific activity levels depended upon her purpose of instruction just as with David’s use of instructional time. Yet she spent her time using instructional representations in different activity levels based on her purpose of instruction as compared to David. When introducing a concept, Nancy spent more time using instructional representations promoting students’ understanding within activity levels PN and F. When her purpose of instruction was to extend previously taught concepts, she spent more time within PK and IMH. If her purpose of instruction is not taken into consideration, Nancy spent a large amount of her overall instructional time using instructional representations within F.

The instructional time David and Nancy used to address previously taught concepts relates to the feature of *folding back* in the Pirie-Kieren theory. David used instructional representations to connect to previously taught concepts when his purpose of instruction was review. Before introducing new topics, David spent most of his instructional time reviewing related concepts that were previously taught. Furthermore, when reviewing a previously taught concept, David spent more time using instructional representations within the activity level PN. On the other hand, Nancy used instructional representations to connect to previously taught concepts when her purpose of instruction was to extend those concepts. When Nancy extended previously taught concepts by either extending them through applications or by combining concepts, she spent more time using instructional representations within the activity levels PK and IMH.

*Summary of Comparisons*

The patterns of similarities and differences between David’s and Nancy’s uses of instructional representations reveal their different approaches to teaching particular concepts.
from their respective secondary honors geometry curricula. A common finding between David’s and Nancy’s uses of instructional representations is their consistency in using particular types of instructional representations regardless of their purpose of instruction. David consistently used words and diagrams and although Nancy used a wider variety of instructional representations, she consistently used words, diagrams, numbers, and formulas. Another similarity concerns David’s and Nancy’s use of instructional time. Each participant’s uses of instructional representations within particular activity levels were dependent upon the purpose of instruction. However, the activity levels in which they used their instructional representations differed depending upon their purpose of instruction.

When the purpose of instruction was to introduce a new concept, David’s instructional time using instructional representations was divided evenly between the activity levels IMH and PN, but Nancy’s instructional time was spent more within the activity levels PN and F. When the purpose of instruction was to review a previously taught concept, David spent more instructional time using instructional representations within PN. When Nancy’s purpose of instruction was to extend previously taught concepts, her instructional time using instructional representations was spent in PK and IMH. Their use of instructional time also differs if the purpose of instruction is not taken into consideration. In their overall uses of instructional time, David spent more time using instructional representations promoting students’ understanding within the activity level PN and Nancy spent much of her time using instructional representations promoting students’ understanding within F.

David and Nancy also diverge in their uses of instructional representations within different activity levels. David frequently used instructional representations within the activity levels PK, IMH, and PN, regardless of his purpose of instruction. Nancy, however,
frequently used instructional representations within PN and F when she introduced a new concept. If her purpose of instruction was to extend previously taught concepts, then her instructional representations were used most often within PK and IMH. When folding back to connect instructional representations to previously taught concepts, David used his instructional representations within PN and Nancy used instructional representations within PK and IMH. Essentially, many of the similarities and differences between David’s and Nancy’s uses of instructional representations are determined by their purpose of instruction.
SUMMARY AND DISCUSSION

The purpose of this chapter is to provide a summary and a discussion of the findings related to the comparison of the uses of instructional representations by a novice mathematics teacher and an experienced mathematics teacher. The chapter is divided into two sections. First, a summary of the results of how a novice teacher’s uses of instructional representations compared to those of an experienced teacher are addressed based on the results of the qualitative analysis. Second, a discussion of the results and contribution towards a conceptualization of a knowledge base for teaching is offered.

The theoretical framework used in the analysis of the data contributes to an explanation of the similarities and differences of uses of instructional representations of two secondary mathematics teachers. The framework concerns the cognitive changes that occur during the processes of growth of students’ mathematical understanding (Pirie & Kieren, 1994b). These cognitive changes are defined using eight potential levels to describe the growth of understanding for a specific person and a given topic or concept. In this study, these eight levels were adapted to describe how two secondary geometry teachers used instructional representations. In particular, the levels were adapted to describe the how the teachers’ uses of instructional representations allow and foster opportunities for students to engage in those levels of mathematical understanding.

Summary of the Study

Researchers consider the practice of teaching to be effective when teachers’ knowledge contributes to positive student learning outcomes (Ball et al., 2001; Borko &
Putnam, 1996; Fennema & Franke, 1992; Shulman, 1986; Wilson et al., 1987). Yet the relationships among the components of knowledge teachers use during instructional practice that contribute to effective teaching, and thus to student learning, are not well understood (Ball, 2000; Grossman et al., 1989; Hiebert et al., 2002; Shulman, 1987; Wilson et al., 1987). The use of instructional representations is considered by many researchers to be an important part of teacher knowledge (Ball et al., 2001; Berenson & Nason, 2003; Orton, 1988; Wilson et al., 1987). This study utilized case study methodology in order to address the following research question concerning teachers’ uses of instructional representation:

How do a novice teacher’s uses of instructional representations differ from that of an experienced teacher, particularly in the teaching of secondary geometry?

The analysis of the similarities and differences between the two participants’ uses of instructional representations revealed two relevant themes: purpose of instruction and use of instructional time. The first theme concerns whether the participant’s purpose of instruction was to introduce a new mathematical concept or revisit a previously taught concept. The participants’ uses of instructional representations differed based on the purpose of instruction for a particular concept. The second theme concerns the relationship between uses of instructional representations and instructional time.

Teacher Knowledge and Student Learning

Concerning teacher knowledge, in particular SMK, researchers find that teachers who have more knowledge of their subjects tend to focus on the conceptual and problem-solving aspects of their subjects, whereas those teachers with less SMK focus on rules, facts, and procedures (Borko & Putnam, 1996; Ma, 1999). Thus, teachers’ pedagogical decisions are guided by the depth of their SMK. David, an experienced mathematics teacher, said his
pedagogical decisions are guided by teaching students within a context instead of giving them formulas. Nancy, a novice mathematics teacher, provided her students formulas at the beginning of lessons and used those formulas in teacher-guided procedural examples. In studies involving novice mathematics teachers and their SMK, novice teachers tend to rely on procedural and rule-based approaches to teaching mathematics and lack conceptual understanding of the mathematics they will teach (Ball, 1990, 2000; Ma, 1999).

Although both participants in this study were recommended based on their strong mathematical content knowledge, how each one used their SMK in practice is revealed in how each one used instructional representations. A fundamental difficulty in learning to teach is how to use SMK and transform it in ways that students can comprehend (Ball, 2000; Grossman, 1990; Wilson et al., 1987). When Nancy’s purpose of instruction was to introduce a new concept, she used instructional representations and her instructional time to encourage students’ understanding within the activity levels PN and F. In other words, she fostered opportunities for students to notice properties and work with symbolic formulas when introducing a concept. However, when Nancy extended previously taught concepts, her instructional representations and her instructional time were used within IMH, thereby facilitating image making and image having in her students’ growth of mathematical understanding. David, on the other hand, approached introducing a new concept and reviewing previously taught concepts with instructional representations that encouraged image making, image having, and property noticing with his students. He spent his instructional time using instructional representations within IMH and PN when introducing a topic and within PN when reviewing previously taught concepts.
The different approaches to teaching taken by the participants illustrate deductive and inductive approaches to teaching. Deductive teaching methods entail proceeding from a general, theoretical perspective to a specific and application-oriented perspective and inductive teaching methods begin from a particular perspective through the use of specific problems and proceed to a more general perspective (Moore, 1929; Prince & Felder, 2006). Nancy’s approach to instruction using instructional representations was more deductive than inductive. She introduced concepts using general definitions and formulas and extended previously taught concepts by addressing specific aspects of those concepts. On the other hand, David’s approach to teaching using instructional representations was a more inductive approach since he introduced concepts using specifics and then progressed towards the general by building on the properties of those concepts. Although the deductive approach to teaching mathematics is the most common, the nature of mathematics as a discipline requires an integration of inductive teaching methods (Christiansen, 1969). Furthermore, Prince and Felder (2006) argue that “inductive methods are consistently found to be at least equal to, or in general more effective than, traditional deductive methods for achieving a broad range of learning outcomes.” (p. 123)

The goals for teaching and learning mathematics must include making connections between informal and formal mathematics (Ball, 1993). Considering his uses of instructional representations, David made these connections by starting with a more informal, and thus inductive, approach when introducing a concept and encouraged students to move towards a more formal level of understanding when reviewing previously taught concepts. Nancy introduced concepts using instructional representations within more formal levels and then encouraged students to move towards a more informal level of understanding when
extending previously taught concepts, thereby using a deductive approach. The manner in which the connections between informal and formal mathematics are made during instruction has implications for students’ growth of mathematical understanding. The premature introduction of certain instructional representations can cause students to develop misconceptions and can give the impression that mathematics is nothing more than a formal language (Dufour-Janvier et al., 1987). Moreover, presenting students with only formalized ideas can hinder students’ learning of new ideas, resulting in disjoint understanding (Pirie & Kieren, 1994a). Using instructional representations that force students to focus only on the properties and formal definitions and algorithms of a concept can be detrimental to students’ growth in mathematical understanding. However, using instructional representations in ways that encourage students’ folding back to their current levels of understanding of previously taught concepts is recognized by researchers as a characteristic of effective teachers (Hashweh, 1985; Martin & Pirie, 1998; Martin, 2000; NCTM, 2000). Both participants in this study facilitated students’ folding back in their uses of instructional representations. David encouraged folding back by reviewing previously taught concepts and Nancy encouraged folding back by extending and connecting previously taught concepts.

Effective Teaching and Student Learning

Teacher knowledge not only affects pedagogical decisions and instructional practices, but it also contributes to how effective those decisions and practices are for student learning (Fennema & Franke, 1992). Effective teachers know what makes a subject easy or difficult for students in order to generate appropriate explanations and representations (Borko & Putnam, 1996; Grossman, 1990; Shulman, 1986). This type of knowledge, PCK, is the knowledge of how to communicate SMK to students and is considered central to effective
teaching (Ball et al., 2001). Since instructional representations are tools teachers use to communicate about abstract mathematical concepts, it is important to consider how teachers’ uses of instructional representations contribute to students’ growth of mathematical understanding. Students’ growth of mathematical understanding can be encouraged through careful construction and sequencing of instruction with representations (Gagatsis et al., 2004; Post & Cramer, 1989).

One method of constructing and sequencing instruction with representations is through the use of multiple instructional representations. The use of multiple instructional representations is considered by researchers as a necessary component of teacher knowledge that contributes to students’ growth of mathematical understanding (Moseley & Brenner, 1997; NCTM, 2000; Rider, 2004; Wilson et al., 1987). Although each of the participants in this study used multiple instructional representations, both tended to use the same types of instructional representations. David consistently used diagrams and words and Nancy consistently used diagrams, words, numbers, and formulas. Both David and Nancy used these types of instructional representations regardless of whether their purpose of instruction was to introduce a concept or to review or extend previously taught concepts.

Another use of instructional representations considered by researchers to contribute to student learning is the use of connections between instructional representations of the same concept and an emphasis on relevant aspects of a particular instructional representation (Leinhardt, 1989). In terms of his use of instructional time, David spent more time using instructional representations within the activity level PN, thereby, fostering opportunities for his students to attend to the properties of an instructional representation in relation to the concept he was teaching. If his purpose of instruction is considered, he spent his instructional
time using instructional representations within IMH and PN when he introduced a concept and within PN when reviewing a previously taught concept. Nancy spent a large portion of her instructional time using instructional representations within the activity level F, thus encouraging her students to attend to the formalized, mathematical generalizations of the concept she was teaching. If her purpose of instruction is taken into consideration, when she introduced a new concept her time was spent using instructional representations within PN and F, but when she extended a previously taught concept, her time was spent using instructional representations within IMH. Although both participants encouraged their students to notice properties of instructional representations, they did so differently according to their purpose of instruction. A major difference between their uses of instructional representations lies in how they each used their instructional representations according to their instructional goals, particularly concerning instruction involving previously taught concepts.

The use of instructional representations to review or extend previously taught concepts is related to the feature of folding back as adapted from the Pirie-Kieren theory (Pirie & Kieren, 1994b) for the purposes of this study. Effective teaching requires teachers know their students’ current level of PK in order to choose the most appropriate instructional representations for a given concept (Hashweh, 1985). When the participants of the study reviewed or extended previously taught concepts, they were facilitating students’ folding back to their PK of those concepts. David introduced and folded back to review concepts using instructional representations within the activity levels IMH and PN. Nancy, on the other hand, introduced concepts using instructional representations within PN and F. When she used them to fold back and extend previously taught concepts, she used her instructional
representations within IMH. These results imply that David approached using instructional representations in ways that build upon students’ current levels of understanding of a concept and then fosters their understanding towards more general and abstract aspects of that concept. Nancy introduced concepts using instructional representations in ways that encourage students to learn mathematical definitions and formulas and then fold back to extend those concepts by focusing on more local, context-dependent aspects of those concepts.

Conclusion

The aim of this study was to deepen our understanding of the ways secondary mathematics teachers use instructional representations when teaching honors geometry. Teaching mathematics in a way that encourages student understanding requires the development of teachers’ SMK, PCK, and knowledge of how students learn. Effective teaching that results in student learning requires teachers have well-developed components of teacher knowledge, yet this knowledge develops over time with the practice of teaching (Feiman-Nemser & Parker, 1990; Grossman, 1990; NCTM, 2000; Wilson et al., 1987). The results of this study reveal differences in the uses of instructional representations between an experienced mathematics teacher and a novice mathematics teacher. These differences are due in part to the differences in their teacher knowledge and experience. David has many years experience teaching secondary mathematics and continues to research the mathematics of the concepts he teaches. His reliance on lower activity levels in which he used instructional representations allowed his students to spend time building their own images and to notice relevant properties of the concept he was teaching or reviewing. Nancy was
student teaching during the time of this study and therefore had limited experiences with teaching. Her reliance on more formal and abstract activity levels in which she used her instructional representations allowed her students to spend time learning the formal mathematical aspects of the concepts she was introducing. Her uses of instructional representations within lower activity levels during her extension of previously taught concepts allowed her students to spend time folding back to make images of those concepts. Both participants’ uses of instructional representations within these activity levels show how they applied their teacher knowledge in practice.

When discussing how students learn from the use of representations, researchers assert that effective teachers know multiple types of representations and can make connections between them (Lesh et al., 1987; Moseley & Brenner, 1997; NCTM, 2000; Rider, 2004; Wilson et al., 1987). Other researchers address the need for teachers to know the strengths and weaknesses of different representations and the appropriate time to use particular representations based on that knowledge (Ball, 1993; Hashweh, 1985; NCTM, 2000; Wilson et al., 1987). While these studies point to the importance of what instructional representations teachers know and use, the results of this study reveal the importance of examining how they use those representations. When examined from a theoretical perspective of student learning, the differences between how teachers use instructional representations reveal deeper insight into teacher knowledge and how that knowledge contributes to effective teaching and growth of students’ mathematical understanding. Furthermore, by examining teachers’ uses of instructional representations based on their purposes of instruction, relationships are revealed concerning their goals for instruction and their projected learning trajectories for their students. Fennema and Franke (1992) question
our knowledge of teachers’ uses of representations: “Do teachers know the representations of the content they ordinarily teach? Does knowing these representations make any difference in how teachers teach or what students learn? Where does knowledge of such representations fit into the knowledge structure of teachers? Such questions remain unanswered.” (p. 154). The results of this study provide avenues of research that can further answer these questions.


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APPENDICES
APPENDIX A

Novice Teacher Informed Consent Form

Dear Teacher:

The Center for Research in Mathematics and Science Education (919-515-1634) investigates issues related to the preparation of teachers at North Carolina State University. A major goal of these investigations is to better understand the needs of undergraduates as they prepare to enter the teaching profession. You have been selected to participate in this study because of your successful performance as a teacher and student. We are therefore requesting permission from you to participate in this study.

The research will be conducted in several ways. This fall/spring I will want to video/audio tape three interviews with you that will take about an hour of your time. When you begin student teaching or teaching your regularly scheduled classes, I will want to observe you in class and collect samples of your work relating to your use of instructional representations. If you are student teaching in the Fall semester, I may want to observe your methods classes and collect samples of your work. All information collected by videotape/audiotape, written work, field notes, and interviews during this study will be used only for this research study and will in no way impact your grades. No identifying names will be used in any future reports of this research. The audiotapes will be erased upon completion of data analysis.

The information gained from this research can help faculty and researchers here and at other universities understand how we can do a better job preparing teachers. These results will be used to inform changes to courses and the program.

Your decision to participate will in no way affect your standing or grades in class. At any time during the study you may withdraw your permission to participate in the research project. At the conclusion of the study, a summary of the results will be made available to you. Should you have any questions or desire further information, please do not hesitate to call. Thank you in advance for your cooperation and support.

Please indicate whether or not you are willing to participate by signing your name.

___Yes, I agree to participate in the research study of mathematics instruction.

___No, I do not agree to participate in the research study of mathematics instruction.

________________________________________________       ______________________
Signature                   Date
APPENDIX B

Experienced Teacher Informed Consent Form

Dear Teacher:

The Center for Research in Mathematics and Science Education (919-515-1634) investigates issues related to the preparation of teachers at North Carolina State University. A major goal of these investigations is to better understand the needs of undergraduates as they prepare to enter the teaching profession. You have been selected to participate in this study because of your successful performance as a teacher and student. We are therefore requesting permission from you to participate in this study.

The research will be conducted in several ways. This fall/spring I will want to video/audio tape three interviews with you that will take about an hour of your time. I will also want to observe you in class and collect samples of your work relating to your use of instructional representations. All information collected by videotape/audiotape, written work, field notes, and interviews during this study will be used only for this research study and will in no way impact your grades. No identifying names will be used in any future reports of this research. The audiotapes will be erased upon completion of data analysis.

The information gained from this research can help faculty and researchers here and at other universities understand how we can do a better job preparing teachers. These results will be used to inform changes to courses and the program.

Your decision to participate will in no way affect your standing or grades in class. At any time during the study you may withdraw your permission to participate in the research project. At the conclusion of the study, a summary of the results will be made available to you. Should you have any questions or desire further information, please do not hesitate to call. Thank you in advance for your cooperation and support.

Please indicate whether or not you are willing to participate by signing your name.

___ Yes, I agree to participate in the research study of mathematics instruction.

___ No, I do not agree to participate in the research study of mathematics instruction.

________________________________________________       _________________
Signature                   Date
APPENDIX C

Novice Teacher Interview Protocol 1

1. How is your student teaching going so far?

2. Last Fall you took EMS 203. What, from that course, was most helpful to you in your own learning about teaching?

3. In EMS 203, there were eight assignments where you developed instructional representations. What did you think of those assignments?

4. How would you describe “instructional representations” to a friend?
   a) Can you explain further?
   b) Can you give me an example?
   c) Tell me about some of the instructional representations you used today in your teaching.

5. What guided your choice of instructional representations in your lesson planning this week?

6. If you were planning a lesson on the concept of congruency for a geometry class, what specific representations would you use? (paper and markers will be provided)
   a) Any others you can think of?
   b) Why did you select this one? And this one?

7. What role do your instructional representations play in student learning?
   a) Have you noticed other approaches that help students learn?
   b) Can you give me an example?

8. Where do you find the representations you use during instruction?
   a) Of the representations you have used so far in your teaching, which have worked the best?
   b) How do you know? Why do you think they worked well?
   c) Which representations have not worked well for you? Any ideas why?
   d) Have you ever made one up on the spot?

Do you have any questions for me or anything you would like to add? Thanks very much for your time.
APPENDIX D

Novice Teacher Interview Protocol 2

1. How do you feel about your student teaching experience?

2. Did you try to use similar representations for each geometry class or were they different?
   a. Why or why not?

3. Did the use of two different textbooks affect which representations you used?
   a. What about how you used your representations?

4. What guides your choice of representations in your PowerPoint slides?
   a. Where do you get the images you use in those slides?

5. Do you think there are situations where students should create their own representations? Explain.

6. In the EMS 203 course, you built a collection of representations for weekly assignments. Do you go use any of those when planning a lesson?
APPENDIX E

Experienced Teacher Interview Protocol 1

1. How would you describe “instructional representations” to a friend?
   a) Can you explain further?
   b) Can you give me an example?
   c) Tell me about some of the instructional representations you used today in your teaching.

2. What guided your choice of instructional representations in your lesson planning this week?

3. If you were planning a lesson on the concept of congruency for a geometry class, what specific representations would you use? (paper and markers will be provided)
   a) Any others you can think of?
   b) Why did you select this one? And this one?

4. What role do your instructional representations play in student learning?
   a) Have you noticed other approaches that help students learn?
   b) Can you give me an example?

5. Where do you find the representations you use during instruction?
   a) Of the representations you have used so far in your teaching, which have worked the best?
   b) How do you know? Why do you think they worked well?
   c) Which representations have not worked well for you? Any ideas why?
   d) Have you ever made one up on the spot?

6. Do you think there are situations where students should create their own representations? Explain.

Do you have any questions for me or anything you would like to add? Thanks very much for your time.
APPENDIX F

Experienced Teacher Interview Protocol 2

1. What topic are you teaching today?

2. How have you prepared? Do you plan what representations you will use?

3. How would you have prepared 10 years ago? Did you plan what representations you would use then?

4. How have your teaching strategies changed over time in terms of how you use representations?

5. What areas of concern do you have when planning lessons?

6. Have you ever reevaluated your ideas about a concept based on planning or researching that idea for teaching?

7. How far do you go into a mathematical idea when you research it for teaching? Can you give me an example?