ABSTRACT

ATTIOGBE, CYRIL EFOE. On Characterizing Nilpotent Lie algebras by their Multipliers. (Under the direction of Ernest L. Stitzinger.)

Authors have turned their attentions to special classes of nilpotent Lie algebras such as two-step nilpotent and filiform Lie algebras, in particular filiform Lie algebras are classified up to dimension eleven [8]. These techniques have not worked well in higher dimensions. For a nilpotent Lie algebra L, of dimension n, we consider central extensions $0 \to M \to C \to L \to 0$ with $M \subseteq c^2 \cap Z(C)$, where $c^2$ is the derived algebra of C and $Z(C)$ is the center of C. Let $M(L)$ be the M of largest dimension and call it the multiplier of L due to it’s analogy with the Schur multiplier. The maximum dimension that M can obtain is $\frac{1}{2} n(n-1)$ and this is met if and only if L is abelian.

Let $t(L) = \frac{1}{2} n(n-1) - \dim M(L)$. Then $t(L) = 1$ if and only if $L = H(1)$, where $H(n)$ is the Heisenberg algebra of dimension $2n + 1$.

A recent technique to classify nilpotent Lie algebra is to use the dimension of the multiplier of L. In particular, to find those algebras whose multipliers have dimension close to the maximum, we call this invariant $t(L)$. Algebras with $t(L) \leq 8$ have been classified [10]. It’s the purpose of this work to use this technique on filiform Lie algebras along with three main tools namely: Propositions 0, 3, and theorem 4. All algebras in this work will be taken over any field whereas in previous works, they have been taken over the field of real and complex numbers.
ON CHARACTERIZING NILPOTENT LIE ALGEBRAS
BY THEIR MULTIPLIERS

by

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A dissertation submitted to the Graduate Faculty of
North Carolina State University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

MATHEMATICS

Raleigh
2004

APPROVED BY:

________________________  ________________________
Chair of Advisory Committee  Co-chair of Advisory Committee
DEDICATION

To my beautiful and loving wife

Evonne Onike Cole-Attiogbe,

wonderful kids

Emilia Elizabeth Kokoe Attiogbe,

Cyril Efoe Attiogbe Jr.

and my late mother

Keziah Elizabeth Dohoo Baker

For helping me keep

the Ph.D. process

in perspective
BIOGRAPHY

Cyril Efoe Attiogbe was born in Freetown, Sierra-Leone, West Africa on September 17, 1959. He is married to a beautiful and loving wife by the name of Mrs. Evonne Onike Cole-ATTIOGBE in 1989, together they are blessed with two wonderful kids, Emilia Elizabeth Kokoe-ATTIOGBE and Cyril Efoe Attiogbe Jr. Emilia is a 9th grader at Southeast Raleigh High School and also wants to be a teacher like her dad, grandma and uncle. Cyril Jr., a 4th grader and honor roll student at Wilburn Elementary School wants to be an Aerospace Engineer. His favorite game is the Microsoft flight simulator 2004.

Cyril Efoe Attiogbe received his elementary education from the Murray Town Municipal School, Freetown, Sierra-Leone. He received his secondary education from the Sierra-Leone Grammar School, Freetown, Sierra-Leone, graduating in 1977. After completing high school he worked at the British Guardian Royal Exchange Assurance Company as a statistician for four years and was thus able to save up some money to pursue his dream of coming to America for further studies.

In 1982 he gained admission to St-Augustine’s College here in Raleigh, North Carolina, and being very intelligent and determined he graduated with honors with a GPA of 3.8(Suma Cum Laude) in 1986 and won a special talent award to graduate school at North Carolina Central University in Durham North Carolina. He graduated from Central in 1988 with masters in pure mathematics and a thesis in the area of Abstract Algebra. He then taught Algebra and Geometry to middle and high school students in the Wake County Public School System while also simultaneously serving as and instructor with the Upward Bound/Talent Search program in Rocky Mount, N.C.. He also taught
Statistics and Calculus for a couple of years at the Wake County Technical Community
College and after that was offered a position as an Assistant Professor of Mathematics in
1989 and has since been with St-Augustine’s College(15 years).

In 1998 Cyril Efoe Attiogbe was admitted to the Doctoral Program at North Carolina
State University. While enrolled at N.C. State he was able to take several courses in the
College of Engineering specifically in the department of Electrical and Computer
Engineering. He designed a “Pulse Amplitude Modulation Transmitter” (PAM) capable
of transmitting human voice to an AM radio at a frequency of 1450 KHz using a minimal
number of operational amplifiers. He also designed an equalizer for an electronics lab
project, he also did some signal processing research at the Engineering Graduate
Research Center at the Centennial Campus in the area of Global Positioning
System(GPS) and also conducted an investigation into the Detection of Radar
Signal in Noise and the extraction of Target information contained in the Signal in the
Spring of 1999. He also served as a Graduate Assistant Grader in the department of

Cyril Efoe Attiogbe earned a second masters in Mathematics from N.C. State University
in 2003 upon the successful completion of the PhD written exam in the areas of
Functional Analysis, Modern Algebra and Linear/Lie Algebras. He also passed the PhD
Language Requirement in French. His dissertation topic is “On Characterizing Nilpotent
Lie Algebras by Their multipliers”

Although Cyril Efoe Attiogbe works very hard, he is very thankful to God to have
blessed him with a family to spend time with. He also gives thanks to his parents who
believed in him and supported him. One of his main mentors was his late mother Mrs. Keziah Elizabeth Dohoo Baker who was also a teacher and disciplinarian. Another mentor was his dad Mr. Simon Attiogbe who taught him that no pain means no gain. Cyril Efoe Attiogbe has several honors including but not limited to “who’s who among students in American Universities and Colleges”, Special Talent Award by the graduate dean of North Carolina Central University, an Outstanding Professorship honoree, Delta Sigma Theta Sorority Inc., service award by Dr. Diane B. Suber, President of St-Augustine’s College. He is also a member of the Mathematical Society of America and the National Society of Black Engineers.

Cyril Efoe Attiogbe has also conducted several workshops on the General Knowledge and Specialty areas of the National Teachers Examination (NTE) for teacher Ed majors in the NTE lab at St-Aug. He also conducted workshops on Linear Regression and Modeling using the TI-85 Graphing calculator and the software “Derive” for the Upward Bound students on a model created by the school of Science and Math in Durham, N.C. He also created brochures on his native West African Country (Sierra-Leone) and gave a brief summary on its cultures to the Upward Bound students. In the spring of 2003 he did a Scholarly Presentation at St-Augustine’s college on Electronic Circuits and Real-life Application Problems in Linear Algebra.
ACKNOWLEDGMENTS

I would like to express my sincerest appreciation to the people who have inspired, advised and encouraged me; one does not get this chance too often so I’ll make good use of this opportunity. First and foremost, I would like to thank, God, for his grace and mercy, my wife Evonne Onike Cole-Attiogbe, for her love, support and relentless encouragement and sacrifices. I love you! Next I would like to thank my children Emilia Elizabeth Kokoe and Cyril Efoe Jr. for foregoing their wants and needs in allowing me to pursue my education and also for their endless support and love.

I would also like to sincerely thank my advisor and Committee Chair, Dr.E.L. Stitzinger for his wealth of knowledge, guidance, encouragement, understanding and continuous support throughout my tenure at N.C.State.

Thanks to the members of my graduate committee, Dr. K.C. Misra, Dr. L. Chung and last but by no means least Dr. E. Chukwu. Thanks also to the Graduate School Representative Dr. James B. Holland and other faculty members at N.C. State involved in my graduate education especially Dr. Robert Martin, Dr. Steven L. Campbell, Dr. A. Fauntleroy, Dr. K. Koh and Dr. H. Krim. Thanks to my fellow graduate students especially Frank Ingram, Dr. S. Deng, Stephen Zhou, Dr. Yun He and Zviad Kharebava for their friendship and knowledge.

I would like to thank my family, especially my parents, my late mother Mrs. Keziah Dohoo Baker and my father Mr. Simon Attiogbe for their love, prayers and for the many valuable lessons they have taught me. Thanks to my brother Archie for choosing St-Augustine’s College for my undergraduate education and as a result exposing me to the Raleigh area and N.C. State. Thanks to my other brother Chris for encouraging me at an
early age that I can do math. Thanks also to my late brother Prince for teaching me to play soccer and to ride a bike, two of my favorite hobbies. Thanks also to my sisters Simonette and Sylvia for encouraging and supporting me at the latter stages of the PhD process. Thanks also to my nephew Edgar who calls me a genius and thinks that I always plan ahead. Thanks also to my niece Ransoline for those comforting phone calls. Next I would like to thank my dearest friend Ronnie without whom my recent visit to the UK and France would have been less meaningful. Thanks to Dr. G. Payne my Division Chair at St-Aug. for her encouragement and support and also Dr. K. Jones my immediate boss and department head. Thanks to Dr. Y. Coston, Dr. James Nelson, Dr. W. Davis, Dr. A Lewis, Dr. R. Mathur, Dr. W.T. Fletcher, Dr. J. Shoaf, Dr. M. Moss and the late Dr. Mary Townes for their support while enrolled at St-Augustine’s College and North Carolina Central University. Thanks also to my late father-in-law Mr. O. O. Cole for always having confided in me. I would like to thank the Mathematics Department Staff at N.C. State. Especially Denise Seabrooks, Brenda Currin and Seyma Bennett for giving Harrelson hall a friendly atmosphere. Finally thanks to everyone else who has ever inspired me even those whom I have never met and the ones that I might have overlooked. If I did I am deeply sorry.
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1. Introduction

The classification of nilpotent Lie algebra has a long history, beginning with Umlauf, a student of Engel, in 1891. He found there were only finitely many non-isomorphic nilpotent Lie algebras of dimension less than or equal to 6. For dimensions greater than 6, there are infinitely many non-isomorphic Lie algebras [1], [9]. In recent years lists of Lie algebras of dimensions 7 and 8 have been studied [9]. Unfortunately, these lists contain errors as noted by [8], the list of Lie algebra of dimension 7 have been corrected more recently as noted in a complete list of nilpotent Lie Algebras of dimension 7 found in [9], dimensions of 8 and higher have been harder to classify.

Authors have turned their attentions to special classes of nilpotent Lie algebras such as two-step nilpotent [9], and filiform Lie algebras [8]. In particular filiform lie algebras are classified up to dimensions eleven [8]. These techniques have not worked well in higher dimensions. A recent technique to classify nilpotent lie algebras is to use the dimension of the multiplier of \( L \). (Note: We will define The Multiplier in the next section). In particular, to find those algebras whose multipliers have dimension close to the maximum, we call this invariant \( t(L) \). The philosophy of this work is to collect the algebras for values of \( t(L) \) and use those to find the next case. This is done for a given \( t(L) \) by considering \( H=L/Z(L) \) which has \( t(H) \leq t(L) \) and is filiform. From the previous cases found in table 22.1, we choose candidates for \( H \) and compute the central extensions to get candidates for \( L \). Algebras with \( t(L) \leq 8 \) have been classified in [10]. It’s the purpose of this work to use this technique on filiform lie algebras. Note: All algebras in this work will be taken over any field whereas in previous works, they have been taken over the field of real and complex numbers.
2. Preliminaries. Now we will define the Multiplier of L:

Consider all Lie algebras H such that \( H/Z(H) \cong L \). H is called a central extension of L. We want to consider the H of largest dimension. Without further restrictions H can be infinite, take \( L \oplus \) (infinite dimensional abelian Lie algebra) so we demand that \( Z(H) \subseteq H^2 \). An H of largest dimension is called a cover and \( Z(H) \) is called the multiplier of L, denote it by \( M(L) \). We see that \( M(L) \) is abelian since it is the center of some H.

Note: 1. All covers of L are isomorphic (Peggy Batten)

2. If L is abelian then \( \dim M(L) = \frac{1}{2}n(n-1) \) where \( n=\dim L \).

3. \( \dim M(L) \leq \frac{1}{2}n(n-1) \) for all L.

We let \( t = t(L) = \frac{1}{2}n(n-1) - \dim M(L) \).

**Definition.** Let L be a nilpotent Lie algebra with lower central series

\[ L^1 = L, \ L^2 = [L^1, L], \ldots, \ L^j = [L^{j-1}, L] \ldots \]

with \( L^s = 0 \), then L is called filiform if \( \dim L^j/L^{j+1} = 1 \)

For \( j = 2, \ldots, s-1 \), and \( \dim L/L^2 = 2 \).

Note that L is the maximal length a nilpotent Lie algebra can have.

Also note the upper central series

\[ 0 = Z_0 \subset Z_1 \subset \ldots \subset Z_{s-1} \subset Z_s = L \]

has dimension \( Z_{j+1}/Z_j = 1 \) when \( j=0, \ldots, s-2 \) and \( \dim L/Z_{s-1} = 2 \).

In a nilpotent Lie algebra \( Z_1 \cap B \neq O \) for any ideal \( B \neq 0 \), hence \( Z_1 \subseteq B \) since \( O \subsetneq Z_1 \cap B \subseteq Z_1 \) and \( \dim Z_1 = 1 \)
The following are examples of filiform lie algebra:

**Ex 1**

Let $L = (x,y,z)$ where $[x,y] = z$

$\quad \quad L^2 = (z), L^3 = [L^2,L] = 0, \quad Z(L) = \{z\} \Rightarrow L \supset L^2 \supset L^3 = 0$

$L$ is H(1) the three dimensional Heisenberg.

**Ex 2**

$L = (x,y,z,c)$ where $[x,y] = z, [x,z] = c$

$\quad \quad L^2 = (z,c), L^3 = [L^2,L] = c, \quad L^4 = [L^3, L] = 0 \Rightarrow Z(L) = \{c\}$

$\Rightarrow L \supset L^2 \supset L^3 \supset L^4 = 0$

**Ex 3**

$L = (x,y,z,c,r)$ where $[x,y] = z, [x,z] = c, [x,c] = r, [y,z] = r$

$\quad \quad L^2 = (z,c,r), L^3 = (c,r), L^4 = (r), L^5 = 0 \Rightarrow L^1 \supset L^2 \supset L^3 \supset L^4 \supset L^5 = 0$

**Ex 4**

The following is not filiform. $T_n = $ strictly upper triangular matrices of size $n \geq 4$. Since $\dim(T/T^2) = n-1$.

We will use the technique in [10] on filiform Lie algebra to classify Lie Algebras with small $t(L)$. Recall $t = t(L)$.

We will begin by introducing some inequalities necessary to our method.

**Proposition 0** Let $L$ be a nilpotent Lie algebra of dimension $n$. Then $t \leq \frac{1}{2} n(n-1)$

**Proof:** Since $\dim M(L) \leq \frac{1}{2} n(n-1)$, $t = \frac{1}{2} n(n-1) - \dim M(L)$ this implies $t \leq \frac{1}{2} n(n-1)$

**Proposition 1** Let $L$ be a nilpotent Lie algebra of dimension $n$.

Then $\dim L^2 + \dim M(L) \leq \frac{1}{2} n(n-1)$

**Proof:** see [10]
**Proposition 2**  Let $L$ be a nilpotent Lie algebra with $Z \subseteq Z(L) \cap L^2$ and $\dim Z = 1$
then $\dim M(L) + 1 \leq \dim M(L/Z) + \dim (L/L^2)$

*Proof:* see [10]

**Proposition 3**  Let $L$ be filiform Lie algebra of dimension $n$.
Let $K = Z(L)$ and $H = L/K$  then $t(H) + \dim L^2 \leq t(L)$

*Proof:* Since $L$ is filiform, $\dim K = 1$, we know that $t(L) = \frac{1}{2}n(n-1) - \dim M(L)$

$\Rightarrow \dim M(L) = \frac{1}{2}n(n-1) - t(L)$

now substitute for $\dim M(L)$ in Prop 2 [ie $\dim M(L) + 1 \leq \dim M(L/Z) + \dim (L/L^2)$]

$\Rightarrow \frac{1}{2}n(n-1) - t(L) + 1 \leq \dim M(L/Z) + \dim L - \dim L^2$

$\Rightarrow \frac{1}{2}n(n-1) - t(L) + 1 \leq \dim M(H) + n - \dim L^2$  (i)

note: $\dim M(H) = M(L) - 1$. Thus if $\dim M(L) = \frac{1}{2}n(n-1) - t(L)$
then $\dim M(H) = \frac{1}{2}(n-1)(n-2) - t(H)$  (ii)

now substitute (ii) in (i)

$\Rightarrow \frac{1}{2} n(n-1) - t(L) + 1 \leq \frac{1}{2}(n-1)(n-2) - t(H) + n - \dim L^2$

$\Rightarrow t(H) + \dim L^2 \leq \frac{1}{2}n(n-1) + t(L) - 1 + \frac{1}{2}(n-1)(n-2) + n$

$= \frac{1}{2}n^2 + \frac{1}{2}n + t(L) - 1 + \frac{1}{2}[n^2 - 3n + 2] + n$

$= \frac{1}{2}n^2 + \frac{1}{2}n + t(L) - 1 + \frac{1}{2}n^2 - 3/2n + 1 + n$

$= \frac{1}{2} n + t(L) - \frac{1}{2} n = t(L)$

thus $t(H) + \dim L^2 \leq t(L)$  ■

**Proposition 4**  Let $L$ be nilpotent. Then $\dim L^2 \leq t(L)$

*Proof:* This follows immediately from Proposition 3.
From Pete Hardy’s work and using propositions 1, 2 and 3, we arrive at the following theorems.

**Theorem 1** \( t(L) = 0 \) IFF \( L \) is abelian.

*Proof:* see [10]

Note: \( L \) is abelian and filiform implies \( \dim L = 2 \).

**Theorem 2** \( t(L) = 1 \) IFF \( L = H(1) \).

*Proof:* see [10]

Note: that \( H(1) \) is indeed filiform Lie algebra. See example 1

**Theorem 3** There are no filiform Lie algebras with \( t(L) = 2 \)

*Proof:* see [10]

We will take advantage of the following new results.

**Theorem 4** Let \( L \) be a Lie algebra, then \( c^2 + c \leq 2t \)

where \( c = \dim L^2 \)

*Proof:* In order to prove theorem 4 we need the following result.
Theorem 5 Let $L$ be a nilpotent Lie algebra with $H/Z(H) \cong L$. Then $\dim(H^2 \cap Z(H)) \leq \dim M(L/L^2) + \dim L^2 \{d(L/Z(L)-1\}$ where $d(X)=\dim(X/X^2)$ thus
\[d(L/Z(L))=\dim\{(L/Z(L))/(L/Z(L))^2\}\] 
Proof: In general if $H/Z(H) \cong L$ but $Z(H) \not\subseteq H^2$, $H= H^* \oplus S$ where $S$ is a central ideal and $Z(H) = Z(H^*) \oplus S$, $H^2 \cap Z(H) = H^* \cap Z(H^*)$ and $L \cong H/Z(H) = (H^* \oplus S)/(Z(H^*) \oplus S) \cong H^*/Z(H^*)$. Hence we assume that $H^2 \supseteq Z(H)$

We break the process into a collection of Lemmas. Let $H$ be a nilpotent Lie algebra with

$Z_1 = Z_1(H) = \{x \in H/ [x,y]=0 \ \forall \ y \in H\}$

$Z_2 = Z_2(H) = \{x \in H/ [x,y] \in Z_1(H) \ \forall \ y \in H\}$, $Z_2(H)/ Z(H) \cong Z(L)$

Since $H$ is nilpotent, if $H$ is not abelian then $0 \subset Z_1 \subset Z_2$ and $0 \neq Z_1 \neq Z_2$.

Lemma 1. Let $y \in Z_2$, $y \notin Z_1$, then $ad_y$ is a homomorphism from $H$ onto $N$ where $N \subseteq Z_1$.

Proof: Since $ad_y$ is linear we need to prove that $ad_y([x, z]) = [ad_y(x), ad_y(z)]$

Note $y \in Z_2$ implies $ad_y(h) = [y, h] \in Z_1$ hence $N \subseteq Z_1$

Thus $ad_y([x, z]) = [y, [x, z]] = -[x, [y, z]] - [z, [x, y]] = 0$

Also $ad_y(x) = [y, x] \in Z_1$ and $ad_y(z) = [y, z] \in Z_1$ which implies $[ad_y(x), ad_y(z)] = 0$ ■

Lemma 2. $[H^2, Z_2]=0$

Proof: Need to show that if $[x, y] \in H^2$ and $z \in Z_2$ then $[[x, y], z] = 0$

Note: $H^2 = \langle [x, y]\rangle$ where $x, y \in H$

now consider $[[x, y], z] +[[y, z], x] + [[z, x], y] = 0$
this implies \([x, y, z] = -[[y, z], x] - [[z, x], y] = 0\) since \([y, z] \in Z_1\) and \([z, x] \in Z_1\).

Lemma 3. Suppose \(y \in Z_2 \cap H^2, y \notin Z_1\), then \(Z_2 \subseteq \text{Ker ad}_y\)

Proof: Let \(z \in Z_2\) we need to show that \(z \in \text{Ker ad}_y\) which implies that \(\text{ad}_y(z) = [y, z] = 0\) \(\forall y \in H^2 \cap Z_2\).

Since \(z \in Z_2\) \(\text{ad}_y(z) = [y, z]\), \([H^2, Z_2] = 0\)

By Lemma 3 there is an induced homomorphism \(\sigma: H/Z_2(H) \rightarrow N\) given by

\[\sigma(t + Z_2(H)) = \text{ad}_y(t) = [y, t].\]

Now the theorem holds if \(k(L) \equiv \dim L^2 = 0\) for then

\(H^2 \subseteq Z(H)\) i.e., \(Z(H) \cap H^2 = H^2\). We proceed by induction: Suppose the result holds for all nilpotent Lie algebras \(K\) with \(\dim K^2 < \dim L^2\) and \(\dim(K/K^2) \leq \dim(L/L^2)\).

Lemma 4. \(\dim M(K/K^2) \leq \dim M(L/L^2)\)

Proof: Since \(\dim K^2 < \dim L^2\) and \(\dim(K/K^2) \leq \dim(L/L^2)\) letting \(\dim(K/K^2) = n\) and \(\dim(L/L^2) = m\) it follows that \(\dim M(K/K^2) = \frac{1}{2}n(n-1) \leq \dim M(L/L^2) = \frac{1}{2}m(m-1)\).

Now let \(y\) and \(N\) be as before (see Lemma 1). Since \(L/Z(L) \cong H/Z_2(H)\) and \(\sigma\) is a homomorphism from \(H/Z_2(H)\) onto \(N\), we consider \(\sigma\) also to be a homomorphism from \(L/Z(L)\) onto \(N\) therefore \(\sigma(L/Z(L))^2 = N^2 = 0\) therefore \((L/Z(L))^2 \subseteq \text{Ker } (\sigma)\).

Lemma 5. \(\dim N \leq \dim (L/Z(L)) - \dim (L/Z(L))^2 = \dim \{(L/Z(L))/(L/Z(L))^2\}\)

Proof: Since \(\sigma\) maps \(L/Z(L)\) onto \(N\) it follows that \(\dim (L/Z(L)) = \dim N + \dim \text{Ker } (\sigma) \geq \)
dim N + dim((L/Z(L))^2, since \{(L/Z(L))^2\leq Ker(\sigma)\}. ■

**Lemma 6.** Let E=H/N then, (i) y+ N \subseteq Z(E) and (ii) y + N \notin Z(H)/N.

(iii) \dim Z(H)/N \leq \dim(Z(H/N))

Proof: (i) This follows since \[ [x + N, y + N] = [x, y] + N = \text{ad}_y(x) + N \]

(Since \text{ad}_y(x) \in N i.e. N = \text{image of ad}_y)

(ii) Recall that y \notin Z(H) (=Z_1)

Suppose that y + N \in Z(H)/N. This implies that y + N = x + N, for some x \in Z(H) . Therefore y = x + n for some n \in N. Thus y \in Z(H), a contradiction.

(iii) Since Z(H)/N is always contained in Z(H/N), we get Z(H)/N \subseteq Z(H/N) by (i), (ii) of Lemma 6 ■

Let \( A \equiv E \, /Z(E) \) then \( L \cong H/Z(H) = (H/N)/(Z(H/N)) \Rightarrow (H/N)/(Z(H/N)) = E/Z(E)=A \)

Hence A is the homomorphic image of L and \dim A < \dim L by the last proof. Hence \( A/(A)^2 \) is a homomorphic image of \( L/L^2 \) and \( A/Z(A) \) is a homomorphic image of \( L/Z(L) \) and \( (A)^2 \) is a homomorphic image of \( L^2 \). In fact we use induction on A. Note this discussion yields:

**Lemma 7.** \( \dim(A/(A)^2) \leq \dim(L/L^2) \) and \( \dim(A)^2 \leq \dim L^2 \)

Proof : Recall y \in Z_2 \cap H^2 , y \notin Z_1 and y + N \subseteq Z(H/N) since N = \text{image ad}_y. So in the homomorphisms \( L \cong H/Z(H) \Rightarrow A \), the image of y in L is in L^2 but it is not 0 but it’s image in A is 0. Hence the homomorphism carries L^2 onto A^2 and it is not injective. Thus \dim L^2 > \dim A^2. By induction \( \dim(E^2 \cap Z(E)) \leq \dim M(A/A^2) + \dim A^2 \)

\( \dim[(A/Z(A))/(A/Z(A))^2 – 1] \).
By a previous “proof” $\dim M(A/A^2) \leq \dim M(L/L^2)$ and we have seen $\dim (A)^2 < \dim L^2$.

Since $A/Z(A)$ is the homomorphic image of $L$,

$$\dim [(A/Z(A))/(A/Z(A))^2] \leq \dim [(L/Z(L))/(L/Z(L))^2].$$

Substitution yields

$$\dim [(E)^2 \cap Z(E)] \leq \dim M(L/L^2) + [\dim (L^2) - 1] \{\dim [(L/Z(L))/(L/Z(L))^2 - 1]\}$$

Now $y + N \in Z(E) \cap (E)^2$ and $y + N \notin H^2 \cap Z(H)/N$ and since the latter is always contained in the former, $\dim H^2 \cap Z(H)/N < \dim [(E)^2 \cap Z(E)]$. Thus $\dim [(E)^2 \cap Z(E)] \leq \dim N + \dim ((E)^2 \cap Z(E)) - 1$

$$\leq \dim [(L/Z(L))/(L/Z(L))^2] + \dim M(L/L^2) + [\dim (L^2) - 1]$$

$$[\dim [(L/Z(L))/(L/Z(L))^2 - 1]\] - 1$$

$$= \dim M(L/L^2) - \dim L^2 + \dim L^2 \dim (L/Z(L))/(L/Z(L))^2 - 1]$$

$$= \dim M(L/L^2) + \dim L^2 \dim [(L/Z(L))/(L/Z(L))^2]$$

and the result is shown.

(Note that we have used the fact that $ad_y$ is a homomorphism from $H$ onto $N$ and $N \subseteq Z_1$

by lemma 1 as a consequence $N$ is abelian and by lemma 3 $Z_2 \subseteq \text{Ker } ad_y$. Consequently we have an induced homomorphism from $H/Z_2(H) \to N$ and an induced homomorphism from $(H/Z_2(H))/(H/Z_2(H))^2 \to N/N^2 \cong N$. Therefore $\dim N \leq \dim ((L/Z(L))/(L/Z(L))^2)$.)

When working with $M(L)$, $H^2 \supseteq Z(H)$ so the formula is $\dim M(L/L^2) = \dim H^2 \cap Z(H) \leq \dim M(L/L^2) + \dim L^2 [\dim (L/Z(L))/(L/Z(L))^2]$ or $\frac{1}{2}n(n-1) - t \leq \frac{1}{2}d(d-1) + c[\delta - 1]$

where $c = \dim L^2$, $d = \dim (L/L^2)$ ($c + d = n$) $\dim L = n$ and $\delta = \dim [(L/Z(L))/(L/Z(L))^2]$. We can now prove theorem 4.

Proof: Note: $\dim L^2 = c$, $\dim L = n$ where $L^2 \supseteq Z(L)$. For Lie algebra $n = c + d$

$$\delta = \dim (L/Z(L)) - \dim (L^2/Z(L)) = \dim (L/L^2)$$

we also know from the definition of $t = t(L)$ that $\dim M(L) = \frac{1}{2} n(n-1) - t$ and now substituting for $\dim M(L)$ in the inequality
\[ \text{dim} M(L) \leq \text{dim} M(L/L^2) + \text{dim} L^2[d(L/Z(L)) - 1] \]

\[ \Rightarrow \frac{1}{2} n(n-1) - t \leq \frac{1}{2} d(d - 1) + c[d - 1] \]

\[ \Rightarrow \frac{1}{2} (c + d)[c + d - 1] - t \leq \frac{1}{2} d(d - 1) + c[d - 1] \]

\[ \Rightarrow \frac{1}{2}[c^2 + cd - c + dc + d^2 - d] - t \leq \frac{1}{2} [d^2 - d] + (cd - c) \]

\[ \Rightarrow c^2 - c - 2t \leq -2c \]

\[ \Rightarrow c^2 + c \leq 2t \quad \blacksquare \]

Note theorem 5 has been proved in [17] by cohomological methods.

Henceforth, L will be a filiform Lie algebra, we define

\[ c = \text{dim} L^2, \quad t = t(L) \quad \text{and} \quad t^* = t(H), \quad H = L/Z(L) \]

and from proposition 3, \[ t^* + c \leq t. \]

We will now find the filiform Lie algebras for increasing values of t(L).
3. \( t(L) = 3 \)

This case has been computed by Hardy. We will do the work to check that our method is consistent with Hardy’s. By proposition 4 \((\dim L^2 \leq 3)\). If \(\dim L^2 = 3\) then \(\dim H^2 = 2\) \{since \(L^2 \cong (H/Z(H))^2 = (H^2 + Z(H))/Z(H) \cong H^2/Z(H)\)\} \{and \(t(H) = 0\) by proposition 3 \((t(H) + \dim L^2 \leq 3)\)\} which implies \(H\) is abelian, a contradiction. If \(\dim L^2 = 1\), then \(L^2 = Z(L)\) and \(L\) is Heisenberg. But the only Heisenberg that is filiform is the three dimensional one of example 1. It has \(t(L) = 1\) \((by\ Theorem\ 2)\), a contradiction. If \(\dim L^2 = 0\), then \(t(L) = 0\), another contradiction. Hence, consider \(\dim L^2 = 2\). Then \(\dim H^2 = 1\) and \(t(H) \leq 1\). If \(t(H) = 0\), then \(H\) is abelian which implies \(\dim H^2 = 0\), a contradiction \((since\ \dim H^2 = 1)\). If \(t(H) = 1\), then by Theorem 2, \(H = H(1)\) and \(L\) is a central extension of a one dimensional ideal \(K\) by \(H(1)\) and \(K \subseteq L^2\).

A basis for \(L\) is \(\{x, y, z, r\}\) with \([x, y] = z + \alpha_1 r\), \([x, z] = \alpha_2 r\), \([y, z] = \alpha_3 r\) and \(r \in Z(L)\).

Either \(\alpha_2\) or \(\alpha_3\) must be non-zero, for otherwise \(\dim L^2 = 1\) a contradiction. WLOG, let \(\alpha_2 \neq 0\). Then letting \(z' = z + \alpha_1 r\), \(y' = y - \alpha_3/\alpha_2 x\), \(r' = \alpha_2 r\) and relabelling yields \([x, y] = z\), \([x, z] = r\)

and \([y, z] = 0\). Now to compute the multiplier start with

\[
[x, y] = z + s_1 \quad [y, z] = s_3 \quad [y, r] = s_5 \\
[x, z] = r + s_2 \quad [x, r] = s_4 \quad [z, r] = s_6
\]

where \(s_1, s_2, \ldots, s_6\) generate \(M(L)\). Next we perform a change of variables i.e. \([x, y] = z + s_1 = z^*\) which implies \([x, y] = z^*\) similarly

\[
[x, z] = r + s_2 = r^* \quad and \quad [y, z] = s_3 \quad [x, r] = s_4 \quad [y, r] = s_5 \quad and \quad [z, r] = s_6
\]

thus \(s_1 = s_2 = 0\). Next we use Jacobi on all possible triples.
\[ [[x,y],z] + [[y,z],x] + [[[z,x],y] = 0 \]
\[ \Rightarrow \quad 0 + [s_3, x] + s_5 = 0 \]
\[ \Rightarrow \quad s_5 = 0. \]

Similarly, \( s_6 = 0 \). Therefore we get \( s_1 = s_2 = s_5 = s_6 = 0 \). Now going back to the original algebra which was \((s_1, s_2, s_3, s_4, s_5, s_6)\) now becomes \((s_3, s_4) = 2 = \text{dim} M(L)\). These are free variables and we can vary them and have different central extensions since \( t(L) = \frac{1}{2}n(n-1) - \text{dim} M(L) \), it follows that \( t(L) = 6 - 2 = 4 \). Note \( n = 4 \) since the basis for \( L = \{x, y, z, r\} \) we adopt the notation of Hardy and denote this algebra by \( L(3,4,1,4) \) where the first digit designates it came from the \( t(L) = 3 \) case, the second is the dimension of \( L \), the third is \( \text{dim} Z(L) \) and the last is \( t(L) \). Thus \( L(3,4,1,4) \) is the filiform Lie algebra with basis \( \{x, y, z, r\} \) and non-zero multiplication \([x, y] = z, [x, z] = r\).

We will use this algebra when \( t(L) = 4 \).

**Theorem 6** There are no filiform Lie algebras with \( t(L) = 3 \).
4. \( t(L) = 4 \)

By proposition 4 \( \dim L^2 \leq 4 \) and if \( \dim L^2 = 4 \), then \( \dim H^2 = 3 \) and \( t(H) = 0 \),
(by proposition 3) which implies \( H \) is abelian, a contradiction. If \( \dim L^2 = 3 \),
then \( \dim H^2 = 2 \) and \( t(H) \leq 1 \) (by proposition 3). There are no such filiform
algebras by previous work (i.e. \( t(H) = 0, t(H) = 1 \)). If \( \dim L^2 = 1 \), then \( L^2 = Z(L) \) and \( L \) is
Heisenberg since \( L \) is filiform, \( L \) is the three-dimensional Heisenberg. This contradicts
theorem 2. If \( \dim L^2 = 0 \), then \( L \) is abelian and \( t(L) = 0 \), a contradiction. The only
remaining possibility is \( \dim L^2 = 2 \), then \( \dim H^2 = 1 \) and \( t(H) \leq 2 \), by proposition 3.
By previous work, \( H = H(1) \). The case \( H = H(1) \) has been computed in the last section and
it was found that \( L = L(3,4,1,4) \), which satisfies \( t(L) = 4 \).

**Theorem 7** Let \( L \) be filiform with \( t(L) = 4 \). Then \( L = L(3,4,1,4) \).
5. t(L) = 5

By Theorem 4, we know that \( c^2 + c \leq 2t \) where \( c = \dim L^2 \) and \( t = t(L) \), thus if \( t = t(L) = 5 \) then \( c^2 + c \leq 10 \) which implies that \( c \leq 2 \) therefore \( c = 0, 1 \) or \( 2 \). If \( \dim L^2 = 0 = c \), then \( t(L) = 0 \) and \( L \) is abelian, a contradiction (see Theorem 1). If \( \dim L^2 = 1 = c \), then \( L^2 = Z(L) \) and \( L \) is Heisenberg (see Theorem 2) but the only Heisenberg Lie algebra that is filiform is the 3 dimensional one of example 1. It has \( t(L) = 1 \), a contradiction. If \( \dim L^2 = 2 \), then \( \dim H^2 = 1 \) and \( t(H) \leq 3 \). i.e \( t(H) = 0, 1, 2 \) or \( 3 \). If \( t(H) = 1 \), then \( H = H(1) \), \( L = L(3, 4, 1, 4) \) and \( t(L) = 4 \), a contradiction. If \( t(H) = 2 \), then there is no filiform Lie algebra (Theorem 3). Similarly if \( t(H) = 3 \) there is no filiform Lie algebra (Thm 6).

If \( t(H) = 0 \), then \( H \) is abelian, a contradiction (Theorem 1).

**Theorem 8** There are no filiform Lie algebras with \( t(L) = 5 \)
6. \( t(L) = 6 \)

By Theorem 4, we know that \( c^2 + c \leq 2t \) where \( c = \dim L^2 \) and \( t = t(L) \), thus if 
\( t = t(L) = 6 \) then \( c^2 + c \leq 12 \) which implies that \( c \leq 3 \) therefore \( c = 0, 1, 2 \) or 3. Also from Proposition 0 \( \{ i.e t \leq \frac{1}{2} (n)(n-1) \} \) and since \( n = c+2 \) for filiform we get 
\( t \leq \frac{1}{2} (c+2)(c+1) \). If \( c=0 \), then \( t=6 \leq \frac{1}{2} (2)(1)=1 \), a contradiction. If \( c=1 \), then 
\( t=6 \leq \frac{1}{2} (3)(2)=3 \), a contradiction. If \( c=2 \), then \( t=6 \leq \frac{1}{2} (4)(3)=6 \), and 
if \( c=3 \) \( t=6 \leq \frac{1}{2} (5)(4)=10 \). Thus \( c=2, 3 \) are the only cases that satisfy the inequality(Proposition 0), therefore \( c \geq 2 \).

If \( c = \dim L^2 = 2 \), then \( \dim H^2 = 1 \) and \( t(H) \leq 4 \) (Prop 3) i.e \( t(H) = 0, 1, 2, 3 \) or 4. If 
\( t(H)=0 \) then \( H \) is abelian, a contradiction(Theorem 1). If \( t(H) = 1 \), then \( H=H(1) \), 
\( H=L(3,4,1,4) \) and \( t(L) = 4 \), a contradiction. By previous work there are no 
filiform Lie algebras for \( t(H)=2 \) or 3(see theorems 3 and 4 respectively).

If \( t(H) = 4 \), then \( H=L(3,4,1,4) \) a contradiction(since the dimension of the derived 
algebra is 2 not 1). If \( c=3=\dim L^2 \) then \( \dim H^2 = 2 \) and \( t(H) \leq 3 \) (Prop 3) i.e \( t(H) = 0, 1, 2, 3 \).
If \( t(H) =0 \) the \( H \) is abelian, a contradiction(Theorem 1). If \( t(H) =1 \) then \( H=H(1) \), 
\( L=L(3,4,1,4) \) and \( t(L)=4 \), a contradiction. By previous work, if \( t(H)=2 \) or 3 there are 
no filiform Lie algebras(Theorems 3 ,6). If \( t(H)=4 \), then Proposition 3 is contradicted.

**Theorem 9** There are no filiform Lie algebras with \( t(L) = 6 \)
7. \( t(L) = 7 \)

Again by Theorem 4, \( c^2 + c \leq 2t \) where \( c = \text{dim } L^2 \) and \( t = t(L) \), thus if \( t = t(L) = 7 \), then \( c^2 + c \leq 14 \) which implies that \( c \leq 3 \) therefore \( c = 0, 1, 2 \) or 3. Again by Proposition 0 \{i.e. \( t \leq \frac{1}{2}(n)(n-1) \}\} and since \( n = c + 2 \) for filiform we get \( t \leq \frac{1}{2}(c+2)(c+1) \). Similarly if \( c = 0 \), then \( t = 7 \leq \frac{1}{2}(2)(1) = 1 \), a contradiction. If \( c = 1 \), then \( t = 7 \leq \frac{1}{2}(3)(2) = 3 \) another contradiction. If \( c = 2 \) then \( t = 7 \leq \frac{1}{2}(4)(3) = 6 \) another contradiction. Finally if \( c = 3 \), then \( t = 7 \leq \frac{1}{2}(5)(4) = 10 \), a true statement, therefore the inequality is true only for \( c = 3 \), thus if \( c = \text{dim } L^2 = 3 \), then \( \text{dim } H^2 = 2 \) and \( t(H) \leq 4 \) (proposition 3) i.e. \( t(H) = 0, 1, 2, 3 \) or 4. If \( t(H) = 0 \), then \( H \) is abelian, a contradiction. If \( t(H) = 1 \), then \( H = H(1) \), \( L = L(3,4,1,4) \) and \( t(L) = 4 \), a contradiction. Again by previous work, if \( t(H) = 2 \) or 3 there are no filiform Lie algebras (Theorems 3 and 4). If \( t(H) = 4 \), then \( H = L(3,4,1,4) \). (Henceforth \( t(H) = 4 \) and \( \text{dim } L^2 = 3 \)).

Now consider \( H = L(3,4,1,4) \). Then \( L \) can be described generally by the basis \{\( x, y, z, c, r \)\} and multiplication \( [x, y] = z \), \( [x, z] = c \), \( [y, z] = \alpha_3 r \), \( [x, c] = \alpha_4 r \), \( [y, c] = \alpha_5 r \) and \( [z, c] = \alpha_6 r \), \( r \in Z(L) \). The Jacobi identity shows that \( \alpha_3 = \alpha_5 = 0 \).

Now, assume \( c \notin Z(L) \), which implies \( \alpha_4 \neq 0 \). If \( \alpha_3 = 0 \), then replacing \( r \) by \( (1/\alpha_4)r \) and relabelling yields non-zero multiplication \( [x, y] = z \), \( [x, z] = c \), \( [x, c] = r \). To compute the multiplier, start with

\[
\begin{align*}
[x, y] &= z + s_1 & [y, c] &= s_5 & [z, r] &= s_9 \\
[x, z] &= c + s_2 & [z, c] &= s_6 & [s, r] &= s_{10} \\
[y, z] &= s_3 & [x, r] &= s_7 \\
[x, c] &= r + s_4 & [y, r] &= s_8
\end{align*}
\]

where \( s_1, \ldots, s_{10} \) generate the multiplier. By relabelling we get \( s_1 = s_2 = s_4 = 0 \).
Using Jacobi identity on all triples gives \( s_6 + s_8 = 0 \) and \( s_5 = s_9 = s_{10} = 0 \). Hence the multiplier has basis \( \{s_3, s_6, s_7\} \), \( \dim M(L) = 3 \) and \( t(L) = 7 \). This filiform algebra satisfies the requirements and we call it \( L(7,5,1,7) \). For the record, it has basis \( \{x, y, z, c, r\} \) and non-zero multiplication \([x, y] = z, [x, z] = c, [x, c] = r \). If \( \alpha_3 \neq 0 \), then replacing \( r \) by \((1/\alpha_3)r\) and relabelling yields non-zero multiplication

\[
[x, y] = z, [x, z] = c, [y, z] = r, [x, c] = \beta r, \quad \beta = (\alpha_4/\alpha_3) \neq 0.
\]

Then letting \( y^* = \beta y \), \( z^* = \beta z \), \( c^* = \beta c \) and \( r^* = \beta^2 r \) and relabelling yields non-zero multiplication

\[
[x, y] = z, [x, z] = c, [y, z] = r, [x, c] = r
\]

To compute the multiplier, start with

\[
\begin{align*}
[x, y] &= z + s_1 & [y, c] &= s_5 & [z, r] &= s_9 \\
[x, z] &= c + s_2 & [z, c] &= s_6 & [s, r] &= s_{10} \\
[y, z] &= r + s_3 & [x, r] &= s_7 \\
[x, c] &= r + s_4 & [y, r] &= s_8
\end{align*}
\]

where \( s_1, \ldots, s_{10} \) generate the multiplier.

By relabelling we get \( s_1 = s_2 = s_3 = 0 \). Using the Jacobi identity on all possible triple gives \( s_5 - s_7 = 0, s_6 + s_8 = 0 \) and \( s_9 = s_{10} = 0 \). Hence, the multiplier has basis \( \{s_4, s, s_6\} \), \( \dim M(L) = 3 \) and \( t(L) = 7 \). This filiform algebra satisfies the requirement and call it \( L'(7,5,1,7) \). For the record, it has basis \( \{x, y, z, c, r\} \) and non-zero multiplication

\[
[x, y] = z, [x, z] = c, [y, z] = r, [x, c] = r.
\]

**Theorem 10** Let \( L \) be filiform with \( t(L) = 7 \) then \( L = L(7,5,1,7) \) or \( L = L'(7,5,1,7) \).
8. \( t(L) = 8 \)

Again by Theorem 4, \( c^2 + c \leq 2t \) where \( c = \dim L^2 \) and \( t = t(L) \), thus if \( t = t(L) = 8 \) then 
\[ c^2 + c \leq 16 \]
which implies that \( c \leq 3 \) so \( c = 0, 1, 2 \) or 3 but again by proposition 0 we have \( t \leq \frac{1}{2}(c+2)(c+1) \) which implies that if \( c = 0 \), then \( t = 8 \leq \frac{1}{2}(2)(1) = 1 \), a contradiction. If \( c = 1 \), then \( t = 8 \leq \frac{1}{2}(3)(2) = 3 \), a contradiction. If \( c = 2 \), then 
\[ t = 8 \leq \frac{1}{2}(4)(3) = 6 \]
a contradiction. If \( c = 3 \), then \( t = 8 \leq \frac{1}{2}(5)(4) = 10 \), a true statement, 
therefore \( c = 3 \) is the only case that satisfies the inequality. Thus if \( c = \dim L^2 = 3 \), then 
\[ \dim H^2 = 2 \] and \( t(H) \leq 5 \) (Prop3). Therefore \( t(H) = 0, 1, 2, 3, 4 \) or 5. If \( t(H) = 0 \) then \( H \) is abelian, a contradiction (Theorem 1). If \( t(H) = 1 \), then \( H = H(1) \) \( L = L(3,4,1,4) \) and \( t(L) = 4 \), a contradiction. Again by previous work, if \( t(H) = 2 \) or 3 there are no filiform Lie algebras (Theorems 3 and 6). If \( t(H) = 4 \), then \( H = L(3,4,1,4) \), a contradiction since we have found that \( t(L) = 7 \). If \( t(H) = 5 \) there are no filiform Lie algebras (Theorem 6).

**Theorem 11**  There are no filiform Lie algebras with \( t(L) = 8 \)
9. t(L)=9

By Theorem 4, \( c^2 + c \leq 2t \) where \( c = \text{dim } L^2 \) and \( t = t(L) \), thus if \( t = t(L) = 9 \) then 
\[ c^2 + c \leq 18 \]
which implies that \( c \leq 3 \) therefore \( c = 0, 1, 2 \) or 3 but by proposition 0, we have 
\[ 9 \leq \frac{1}{2}(c+2)(c+1) \]
and by solving this inequality we know that \( c = 3 \) is the only case that satisfies the inequality. Thus if \( c = \text{dim } L^2 = 3 \), then \( \text{dim } H^2 = 2 \) and \( t(H) \leq 6 \) (prop 3).

By previous work there are no filiform Lie algebras if \( t(H) = 2, 3, 5 \) or 6 (theorems 3, 6, 8 and 9). If \( t(H) = 0 \), then \( H \) is abelian, a contradiction (theorem 1). If \( t(H) = 1 \), then 
\[ H = H(1), \ L = L(3,4,1,4) \] and \( t(L) = 4 \), a contradiction. If \( t(H) = 4 \), then 
\[ H = L(3,4,1,4) \], a contradiction, since from the \( t(L) = 7 \) case we know that \( L = L(7,5,1,7) \) or \( L = L'(7,5,1,7) \) which have \( t(L) = 7 \) but here \( t(L) = 9 \).

**Theorem 12** There are no filiform Lie algebras with \( t(L) = 9 \).
10. \( t(L) = 10 \)

Similarly, by Theorem 4, if \( t=t(L)=10 \) then \( c^2+c \leq 20 \) which implies that \( c \leq 4 \), therefore \( c=0,1,2,3 \) or \( 4 \) but by proposition 0, we have \( 10 \leq \frac{1}{2}(c+2)(c+1) \) and by solving this inequality we know that \( c=3 \) and \( c=4 \) are the two cases that satisfy the inequality. Thus if \( c=\text{dim}\,L^2=3 \), then \( \text{dim}\,H^2=2 \) and \( t(H) \leq 7 \) (prop 3). Again by previous work, there are no filiform Lie algebras if \( t(H)=2,3,5 \) or \( 6 \) (Theorems 3, 6, 8, 9 respectively). If \( t(H)=0 \), then \( H \) is abelian, a contradiction (Theorem 1). If \( t(H)=1 \), then \( H=H(1), L=L(3,4,1,4) \) and \( t(L)=4 \), a contradiction since from the \( t(L)=7 \) case we know that \( L=L(7,5,1,7) \) or \( L=L'(7,5,1,7) \) which have \( t(L)=7 \) but here \( t(L)=10 \). If \( t(H)=4 \), then \( H=H(3,4,1,4) \), a contradiction since from the \( t(L)=7 \) case we know that \( L=L(7,5,1,7) \) or \( L'(7,5,1,7) \) which have \( t(L)=7 \) but here \( t(L)=10 \). If \( t(H)=7 \), then \( H=H(7,5,1,7) \) or \( L'(7,5,1,7) \), a contradiction since the algebras \( H=L(7,5,1,7) \) or \( L'(7,5,1,7) \) were found to have \( t(L)=11 \) and here \( t(L)=t(L)=10 \).

Also if \( c=\text{dim}\,L^2=4 \), then \( \text{dim}\,H^2=3 \) and \( t(L) \leq 6 \) (prop 3).

Again by previous work, there are no filiform Lie algebras if \( t(H)=2,3,5 \) or \( 6 \). If \( t(H)=0 \), then \( H \) is abelian, a contradiction (Theorem 1). If \( t(H)=1 \), then \( H=H(1), L=L(3,4,1,4) \) and \( t(L)=4 \), a contradiction. If \( t(H)=4 \), then \( H=L(3,4,1,4) \), a contradiction by similar argument above.

**Theorem 13**  There are no filiform Lie algebras with \( t(L)=10 \).
11. $t(L)= 11$

By Theorem 4, if $t=11$ then $c^2 + c \leq 22$ which implies that $c \leq 4$ therefore $c=0,1,2,3$ or 4, but by proposition 0, we have $11 \leq \frac{1}{2}(c+2)(c+1)$ and by solving this inequality we know that $c=4$ is the only case that satisfies the inequality. Thus if $c=\dim L^2= 4$, then $\dim H^2=3$ and $t(H)\leq 7$ (prop 3). By looking at table 22.1, we found that the only time these parameters are satisfied was when $t(L)=7$. Again by table 22.1, there are no filiform Lie algebras if $t(H)=2,3,5$ or 6. If $t(H)=0$, then $H$ is abelian, a contradiction. If $t(H)=1$, then $H=H(1)$, $L=L(3,4,1,4)$ and $t(L)= 4$, a contradiction.

If $t(H)= 7$, then $L=L(7,5,1,7)$ or $L=L'(7,5,1,7)$.

We will consider these two cases. First let’s consider Case 1 where $H=L(7,5,1,7)$.

{We will later consider Case 2 where $H=L'(7,5,1,7)$, since it has different multiplication table}. This $L$ can be described generally by the basis $\{x, y, z, c, r, t\}$ and multiplication

\[
[x, y]=z, \ [x, z]=c, \ [x, c]=r, \ [x, r]=\alpha_4 t, \ [y, z]=\alpha_5 t, \ [y, c]= \alpha_6 t, \ [y, r]= \alpha_7 t, \ [z, c]= \alpha_8 t, \ [z, r]= \alpha_9 t, \ [c, r]=[\alpha_{10} t, \text{ and } t \in \mathbb{Z}(L). \text{ The Jacobi identity shows that } \alpha_6 =\alpha_9 =\alpha_{10}=0 \text{ and } \alpha_8 = -\alpha_7.}
\]

Relabelling yields the non zero multiplication $[x, y]=z, \ [x, z]=c, \ [x, c]=r, \ [x, r]= \alpha_4 t, \ [y, z]=\alpha_5 t, \ [y, c]= 0, \ [y, r]= \alpha_7 t, \ [z, c]= -\alpha_7 t, \ [z, r]=0, \text{ and } t \in \mathbb{Z}(L)$.

Now if $\alpha_7=0$ and $\alpha_4 \neq 0$. (Since if $\alpha_7 = \alpha_4=0$ then $r$ is also in the center $\mathbb{Z}(L)$, a contradiction because $L$ is filiform and only one element can be in $\mathbb{Z}(L)$ and we have already designated $t$ to be in $\mathbb{Z}(L)$ earlier.) Further relabelling yields

\[
[x, y]=z, \ [x, z]=c, \ [x, c]=r, \ [x, r]=\alpha_4 t, \ [y, z]=\alpha_5 t, \ [y, c]= 0, \ [y, r]= \alpha_7 t, \ [z, c]= -\alpha_7 t, \ [z, r]=0,
\]
Now by substituting $t' = \alpha_4 t$ which implies $t = (1/\alpha_4)t'$, we get

$[x, y] = z$, $[x, z] = c$, $[x, c] = r$, $[x, r] = t'$, $[y, z] = (\alpha_5/\alpha_4)t'$, $[y, c] = 0$, $[y, r] = 0$, $[z, c] = 0$, $[z, r] = 0$, $[c, r] = 0$, now let $yz = \alpha_5 t'$. This implies

$[x, y] = z$, $[x, z] = c$, $[x, c] = r$, $[x, r] = t'$, $[y, z] = \alpha_5 t'$, $[y, c] = 0$, $[y, r] = 0$, $[z, c] = 0$, $[z, r] = 0$, $[c, r] = 0$, thus $\alpha_4 = 1$ therefore we will now consider:

**Case 1 (i) where we suppose $\alpha_7 = 0$ and $1 = \alpha_4 \neq 0$.** (Since again if $\alpha_7 = \alpha_4 = 0$ then $r$ is also in the center $Z(L)$, a contradiction.)

Relabelling yields the non zero multiplication $[x, y] = z$, $[x, z] = c$, $[x, c] = r$, $[x, r] = t$, $[y, z] = \alpha s t$, $[y, c] = 0$, $[y, r] = 0$, $[z, c] = 0$, $[z, r] = 0$ and $t \in Z(L)$.

To compute the multiplier start with

- $[x, y] = z + s_1$
- $[y, z] = \alpha s t + s_5$
- $[z, r] = s_9$
- $[z, t] = s_{13}$
- $[x, z] = c + s_2$
- $[y, c] = s_6$
- $[c, r] = s_{10}$
- $[c, t] = s_{14}$
- $[x, c] = r + s_3$
- $[y, r] = s_7$
- $[x, t] = s_{11}$
- $[r, t] = s_{15}$
- $[x, r] = t + s_4$
- $[z, c] = s_8$
- $[y, t] = s_{12}$

where $s_1, \ldots, s_{15}$ generate the multiplier. Now we will use the Jacobi identity on all triples.

$\bullet$ $(xy)z + (yz)x + (zx)y = 0$

$\Rightarrow (\alpha s t)x - cy = 0$

$\Rightarrow - \alpha s s_{11} + s_6 = 0$

$\Rightarrow s_6 = \alpha s s_{11}$
• \((xy)c+(yc)x+(cx)y = 0 \Rightarrow zc-ry=0 \Rightarrow s_8- s_7 =0 \Rightarrow s_8=- s_7.\)

• \((xy)r+(yr)x+(rx)y = 0 \Rightarrow zr+s_7x-ty =0 \Rightarrow s_9+ s_{12}=0 \Rightarrow s_9=-s_{12}.\)

• \((xy)t+(yt)x+(tx)y = 0 \Rightarrow zt =0 \Rightarrow s_{13}=0.\)

• \((xz)r+(zr)x+(rx)z = 0 \Rightarrow cr- (\alpha_4 t)z =0 \Rightarrow s_{10}=0.\)

• \((xz)c+(zc)x+(cx)z = 0 \Rightarrow 0+s_8x-rz=0 \Rightarrow 0-rz=0 \Rightarrow s_9=0.\)

• \((xz)t+(zt)x+(tx)z = 0 \Rightarrow ct=0 \Rightarrow s_{14}=0 .\)

• \((xr)c+(rc)x+(cx)r = 0 \Rightarrow (\alpha_4 t)c- s_{10}x=0 \Rightarrow \alpha_4 s_{14}=0 \Rightarrow s_{14}=0.\)

• \((xr)t+(rt)x+(tx)r = 0 \Rightarrow (\alpha_4 t)t=0\)

• \((xc)t+(ct)x+(tx)c = 0 \Rightarrow rt=0 \Rightarrow s_{15}=0\)

• \((yz)c+(zc)y+(cy)z = 0 \Rightarrow (\alpha_5 t)c+s_8y-s_6z=0 \Rightarrow -\alpha_5 s_{14}=0 \Rightarrow s_{14}=0\)

• \((yz)r+(zr)y+(ry)z = 0 \Rightarrow (\alpha_5 t)r+(\alpha_7 t)z=0 \Rightarrow \alpha_5 0+\alpha_7 0=0\)

• \((yz)t+(zt)y+(ty)z = 0 \Rightarrow (\alpha_5 t)t=0 \Rightarrow 0=0\)

• \((yc)r+(cr)y+(ry)c = 0 \Rightarrow 0=0\)

• \((yc)t+(ct)y+(ty)c = 0 \Rightarrow 0=0\)

• \((rc)t+(ct)r+(tr)c = 0 \Rightarrow 0=0\)
From the preceding work we obtain the following equations.

(i)  \( s_6 = \alpha s_{11} \)  
(ii)  \( s_7 = -s_8 \)  
(iii)  \( s_9 = -s_{12} \)  
(iv)  \( s_{13} = 0 \)  
(v)  \( s_{10} = 0 \)

(vi)  \( s_9 = 0 \)  
(vii)  \( s_{14} = 0 \)  
(viii)  \( s_{15} = 0 \).  This implies

\[
\begin{align*}
  s_9 &= s_{10} = s_{12} = s_{13} = s_{14} = s_{15} = 0.
\end{align*}
\]

Further  \( s_6, s_8 \in (s_4, s_5, s_7, s_{11}) \) thus \( \text{dim}(L) = (s_4, s_5, s_7, s_{11}) = 4 \).

Therefore \( t(L) = \frac{1}{2}n(n-1) - \text{dim}(L) = \frac{1}{2}(6)(5) - 4 = 11 \), call this \( L = L(11, 6, 1, 11) \).

Now by substituting \( t' = \frac{1}{\alpha}t \) which implies \( t = (1/\alpha_4)t' \), we get

\[
\begin{align*}
  [x, y] &= z, \quad [x, z] = c, \quad [x, c] = r, \quad [x, r] = \alpha_4 t', \quad [y, z] = \alpha_5 t', \quad [y, c] = 0, \quad [y, r] = t', \quad [z, c] = t', \quad [z, r] = t', \\
  [z, r] &= 0, \quad [c, r] = 0, \quad \text{now let } yz = \alpha_5 t \text{ this implies} \\
  [x, y] &= z, \quad [x, z] = c, \quad [x, c] = r, \quad [x, r] = \alpha_4 t, \quad [y, z] = \alpha_5 t, \quad [y, c] = 0, \quad [y, r] = t, \quad [z, c] = -t, \quad [z, r] = 0, \\
  [c, r] &= 0, \quad \text{thus } \alpha_7 = 1 \text{ therefore we will now consider:}
\end{align*}
\]

**Case 1 (ii) where we suppose \( 1 = \alpha_7 \neq 0 \).**

Relabelling yields the non zero multiplication \( [x, y] = z, \quad [x, z] = c, \quad [x, c] = r, \quad [x, r] = \alpha_4 t, \quad [y, z] = \alpha_5 t, \quad [y, c] = 0, \quad [y, r] = t, \quad [z, c] = -t, \quad [z, r] = 0, \quad [c, r] = 0 \) and \( t \in Z(L) \).

To compute the multiplier start with

\[
\begin{align*}
  [x, y] &= z + s_1, \quad [y, z] = \alpha_5 t + s_5, \quad [z, r] = s_9, \quad [z, t] = s_{13} \\
  [x, z] &= c + s_2, \quad [y, c] = s_6, \quad [c, r] = s_{10}, \quad [c, t] = s_{14} \\
  [x, c] &= r + s_3, \quad [y, r] = t + s_7, \quad [x, t] = s_{11}, \quad [r, t] = s_{15} \\
  [x, r] &= \alpha_4 t + s_4, \quad [z, c] = -t + s_8, \quad [y, t] = s_{12}
\end{align*}
\]

where \( s_1, \ldots, s_{15} \) generate the multiplier. Now we will use the Jacobi on all triples.

- \( (xy)z + (yz)x + (zx)y = 0 \)

\[
\Rightarrow zz + (\alpha_5 t)x + s_5 x + (-c)y = 0
\]

\[
\Rightarrow -\alpha_5 s_{11} + s_6 = 0 \quad \Rightarrow s_6 = \alpha_5 s_{11}
\]
\( \cdot (xy)c+(yc)x+(cx)y = 0 \Rightarrow (z)c+ (s_6)x-(r)y=0 \Rightarrow -t-ry=0 \Rightarrow -t+t+ s_7=0 \Rightarrow s_7=0 \\

\( \cdot (xy)r+(yr)x+(rx)y = 0 \Rightarrow zr+tx- (\alpha_4t-s_4)y=0 \Rightarrow s_9- s_{11}+ \alpha_4 s_{12}=0 \Rightarrow s_{11}=s_9+ \alpha_4s_{12} \\

\( \cdot (xy)t+(yt)x+(tx)y = 0 \Rightarrow zt+ s_{12}x- s_{11}y=0 \Rightarrow s_{13}=0 \\

\( \cdot (xz)r+(zr)x+(rx)z = 0 \Rightarrow cr+s_9x- \alpha_4tz-s_4z=0 \Rightarrow s_{10}+ \alpha_4s_{13}=0 \Rightarrow s_{10}=- \alpha_4s_{13} \\

\( \cdot (xz)c+(zc)x+(cx)z = 0 \Rightarrow cc-tx-rz=0 \Rightarrow s_{11}=-s_9 \\

\( \cdot (xz)t+(zt)x+(tx)z = 0 \Rightarrow ct+s_{13}x-s_{11}z=0 \Rightarrow \alpha_4s_{14}=0 \\

\( \cdot (xr)c+(rc)x+(cx)r = 0 \Rightarrow (\alpha_4t)c-s_{10}x-rr=0 \Rightarrow s_{14}=0 \\

\( \cdot (xr)t+(rt)x+(tx)r = 0 \Rightarrow (\alpha_4t)t+s_{15}x-s_{11}r=0 \Rightarrow 0=0 \\

\( \cdot (xc)t+(ct)x+(tx)c = 0 \Rightarrow rt+s_{14}x-s_{11}c=0 \Rightarrow s_{15}=0 \\

\( \cdot (yz)c+(zc)y+(cy)z = 0 \Rightarrow (\alpha_5t)c -ty-s_6z=0 \Rightarrow -\alpha_5s_{14}+s_{12}=0 \Rightarrow s_{12}= \alpha_5s_{14} \\

\( \cdot (yz)r+(zr)y+(ry)z = 0 \Rightarrow s_5r+s_9y-tz=0 \Rightarrow s_{13}=0 \\

\( \cdot (yz)t+(zt)y+(ty)z = 0 \Rightarrow (\alpha_5t)t+s_{13}y-(-s_{12})z=0 \Rightarrow 0=0 \\

\( \cdot (yc)r+(cr)y+(ry)c = 0 \Rightarrow s_{6}r+s_{10}y+tc=0 \Rightarrow s_{14}=0 \\

\( \cdot (yc)t+(ct)y+(ty)c = 0 \Rightarrow s_6t+s_{14}y-s_{12}c=0 \Rightarrow 0=0 \\

\( \cdot (rc)t+(ct)r+(tr)c = 0 \Rightarrow -s_{10}t+s_{14}r-s_{15}c=0 \Rightarrow 0=0 \)
From the preceding work we obtain the following equations.

(i) \( s_6 = \alpha_5 s_{11} \)  
(ii) \( s_7 = 0 \)  
(iii) \( s_{11} = s_9 + \alpha_4 s_{12} \)  
(iv) \( s_{13} = 0 \)  
(v) \( s_{10} = -\alpha_4 s_{13} \)  
(vi) \( s_{11} = -s_9 \)  
(vii) \( s_{14} = 0 \)  
(viii) \( s_{15} = 0 \)  
(ix) \( s_{12} = \alpha_5 s_{14} \).

This implies \( s_7 = s_8 = s_9 = s_{10} = s_{11} = s_{12} = s_{13} = s_{14} = s_{15} = 0. \)

Further \((s_9, s_{11}, s_{12}) \in (s_4, s_5, s_6)\) thus \(\dim M(L) = (s_4, s_5, s_6) = 3.\)

Therefore \(t(L) = \frac{1}{2}n(n-1) - \dim M(L) = \frac{1}{2}(6)(5) - 3 = 12\), call this algebra \(L = L(11, 6, 1, 12).\)

Now we will consider case 2 where \(H = L'(7, 5, 1, 7)\).

This \(L\) can be described generally by the basis \(\{x, y, z, c, r, t\}\) and multiplication

\[
[x, y] = z, \ [x, z] = c, \ [x, c] = r, \ [x, r] = \alpha_4 t, \ [y, z] = r + \alpha_5 t, \ [y, c] = \alpha_6 t, \ [y, r] = \alpha_7 t, \ [z, c] = \alpha_8 t, \ [z, r] = \alpha_9 t, \ [c, r] = \alpha_{10} t, \ t \in \mathbb{Z}(L).
\]

The Jacobi identity shows that \(\alpha_9 = \alpha_{10} = 0\) and \(\alpha_6 = \alpha_4\) and \(\alpha_8 = -\alpha_7.\)

Relabelling yields the non zero multiplication \([x, y] = z, \ [x, z] = c, \ [x, c] = r, \ [x, r] = \alpha_4 t, \)

\([y, z] = r + \alpha_5 t, \ [y, c] = \alpha_4 t, \ [y, r] = \alpha_7 t, \ [z, c] = -\alpha_7 t, \ [z, r] = 0, \ [c, r] = 0 \) and \(t \in \mathbb{Z}(L).\)

Now by substituting \(t' = \alpha_7 t\) which implies \(t = (1/\alpha_7)t'\), we get

\([x, y] = z, \ [x, z] = c, \ [x, c] = r, \ [x, r] = (\alpha_4/\alpha_7)t', \ [y, z] = r + (\alpha_5/\alpha_7)t', \ [y, c] = (\alpha_4/\alpha_7)t', \ [y, r] = t', \)

\([z, c] = -t', \ [z, r] = 0, \ [c, r] = 0, \) now letting \(yc = \alpha_4 t'\) implies \(yc = \alpha_4 t \Rightarrow \alpha_7 = 1\)

also letting \(yr = \alpha_7 t'\) implies \(yr = \alpha_7 t \Rightarrow \alpha_7 = 1\)

similarly letting \(yr = r + \alpha_3 t'\) implies \(yr = r + \alpha_3 t \Rightarrow \alpha_7 = 1.\)

Thus \(\alpha_7 = 1\) therefore we will now consider:

**Case 2 (i) where we suppose \(1 = \alpha_7 \neq 0.\)**

Relabelling yields the non zero multiplication \([x, y] = z, \ [x, z] = c, \ [x, c] = r, \ [x, r] = \alpha_4 t, \)

\([y, z] = r + \alpha_5 t, \ [y, c] = \alpha_4 t, \ [y, r] = t, \ [z, c] = -t, \ [z, r] = 0, \ [c, r] = 0 \) and \(t \in \mathbb{Z}(L).\)
To compute the multiplier start with

\[
\begin{align*}
[x, y] &= z + s_1 & [y, z] &= r + \alpha s_5 + s_15 & [z, r] &= s_9 & [z, t] &= s_{13} \\
[x, z] &= c + s_2 & [y, c] &= \alpha s_4 + s_6 & [c, r] &= s_{10} & [c, t] &= s_{14} \\
[x, c] &= r + s_3 & [y, r] &= t + s_7 & [x, t] &= s_{11} & [r, t] &= s_{15} \\
[x, r] &= \alpha s_4 + s_4 & [z, c] &= -t + s_8 & [y, t] &= s_{12} \\
\end{align*}
\]

where \( s_1, \ldots, s_{15} \) generate the multiplier. Now we will use the Jacobi identity on all triples.

\[
\begin{align*}
\bullet \ (xy)z+(yz)x+(zx)y &= 0 \quad \Rightarrow \quad zz + (r + \alpha s_5)x - (-c)y = 0 \quad \Rightarrow \quad -\alpha s_5 = 0 \\
\bullet \ (xy)c+(yc)x+(cx)y &= 0 \quad \Rightarrow \quad (z)c + (\alpha s_4)x - (r)y = 0 \quad \Rightarrow \quad -\alpha s_4 = 0 \\
\bullet \ (xy)r+(yr)x+(rx)y &= 0 \quad \Rightarrow \quad zr + tx - (\alpha s_4)y = 0 \quad \Rightarrow \quad s_9 - s_{11} + \alpha s_{12} = 0 \\
\bullet \ (xy)t+(yt)x+(tx)y &= 0 \quad \Rightarrow \quad zt + s_{12}x - s_{11}y = 0 \quad \Rightarrow \quad s_{13} = 0 \\
\bullet \ (xz)r+(zr)x+(rx)z &= 0 \quad \Rightarrow \quad cr + s_9x - \alpha s_4z = 0 \quad \Rightarrow \quad s_{10} - \alpha s_{13} = 0 \\
\bullet \ (xz)c+(zc)x+(cx)z &= 0 \quad \Rightarrow \quad cc - tx - rz = 0 \quad \Rightarrow \quad s_{11} = -s_9 \\
\bullet \ (xz)t+(zt)x+(tx)z &= 0 \quad \Rightarrow \quad ct + s_{13}x - s_{11}z = 0 \quad \Rightarrow \quad s_{14} = 0 \\
\bullet \ (xr)c+(rc)x+(cx)r &= 0 \quad \Rightarrow \quad (\alpha s_4)c - s_{10}x - rr = 0 \quad \Rightarrow \quad \alpha s_{14} = 0 = \alpha s_{11} \quad \Rightarrow \quad s_{14} = s_{11} = 0 \quad \Rightarrow \quad s_9 = 0 \\
\bullet \ (xr)t+(rt)x+(tx)r &= 0 \quad \Rightarrow \quad (\alpha s_4)t + s_{15}x - s_{11}r = 0 \quad \Rightarrow \quad 0 = 0 \\
\end{align*}
\]
\( (xc)t+(ct)x+(tx)c = 0 \quad \Rightarrow \quad rt+s_{14}x-s_{11}c=0 \quad \Rightarrow \quad s_{15}=0 \)

\( (yz)c+(zc)y+(cy)z = 0 \quad \Rightarrow \quad (r+\alpha_5t)c -ty +(-\alpha_4t)z=0 \quad \Rightarrow \quad rc+\alpha_5tc-ty-\alpha_4tz=0 \)

\( \Rightarrow \quad a_5s_{14}+s_{12}+s_{13}=0 \quad \Rightarrow \quad s_{12}=0 \quad \Rightarrow \quad s_9=s_{11} \)

\( (yz)r+(zr)y+(ry)z = 0 \quad \Rightarrow \quad (r+\alpha_5t)r+s_{10}y-tz=0 \quad \Rightarrow \quad rr+\alpha_5tr+s_{13}=0 \)

\( \Rightarrow \quad a_5s_{15}+s_{13}=0 \quad \Rightarrow \quad s_{13}=0 \)

\( (yz)t+(zt)y+(ty)z = 0 \quad \Rightarrow \quad (r+\alpha_5t)t+s_{10}y-tz=0 \quad \Rightarrow \quad s_{15}=0 \)

\( (yc)r+(cr)y+(ry)c = 0 \quad \Rightarrow \quad \alpha_4tr+s_{10}y-tc=0 \quad \Rightarrow \quad -\alpha_4s_{15}+s_{14}=0 \quad \Rightarrow \quad 0=0 \)

\( (yc)t+(ct)y+(ty)c = 0 \quad \Rightarrow \quad (\alpha_4t)t+s_{14}y-s_{12}c=0 \quad \Rightarrow \quad 0=0 \)

\( (rc)t+(ct)r+(tr)c = 0 \quad \Rightarrow -s_{10}t+s_{14}r-s_{15}c=0 \quad \Rightarrow \quad 0=0 \)

From the preceding work we obtain the following equations.

(i) \( a_5s_{11}=0 \)  (ii) \( a_4s_{11}=0 \) (iii) \( s_{9}=s_{11} \) - \( a_4s_{12} \) (iv) \( s_{13}=0 \) (v) \( s_{10}=0 \)

(vi) \( s_{11}=-s_{9} \) (vii) \( s_{14}=0 \) (viii) \( a_4s_{14}=0 \) (ix) \( s_{11}=s_{14}=0 \) (x) \( s_{9}=0 \) (xi) \( s_{15}=0 \) (xii) \( s_{12}=0 \) (xiii) \( s_{13}=0 \)

This implies \( s_{7}=s_{8}=s_{9}=s_{10}=s_{11}=s_{12}=s_{13}=s_{14}=s_{15}=0 \), thus

\( \text{dimM}(L)=(s_4, s_5, s_6)=3. \)

Therefore \( t(L)=\frac{1}{2}n(n-1)-\text{dimM}(L)=\frac{1}{2}(6)(5)-3=12, \) call this algebra \( L=L'(11,6,1,12). \)

Now by substituting \( t'=a_4t \) which implies \( t=(1/\alpha_4)t' \), we get

\[ [x, y]=z, \ [x, z]=c, \ [x, c]=r, \ [x, r]=1/a_4t', \ [y, z]=r+(\alpha_5/a_4)t', \ [y, c]=(1/a_4)t', \ [y, r]=0, \]

\[ [z, c]=0, [z, r]=0, [c, r]=0, \text{ now letting } [x, r]=1/a_4t'=t'' \text{ implies } t'=a_4t'' \text{ thus relabelling} \]
implies \[x, y\]=z, [x, z]=c, [x, c]=r, [x, r]= 1/\alpha_4 t'=t', [y, z]= r+(\alpha_5/\alpha_4)(\alpha_4 t'')=r+\alpha_5 t, [y, c]= (1/ \alpha_4)(\alpha_4 t'')=t, [y, r]=0, [z, c]=0, [z, r]=0, [c, r]= 0, thus \alpha_4=1. Therefore we will now consider:

**Case 2 (ii) where we suppose \(\alpha_7 = 0\) and 1= \(\alpha_4 \neq 0\). (Again, \(\alpha_4\) and \(\alpha_7\) cannot both be zero since this implies \(r\) is also in the center, a contradiction)

Relabelling yields the non zero multiplication \([x, y]= z, [x, z]=c, [x, c]=r, [x, r]= t,\)

\([y, z]= r + \alpha_5 t, [y, c]= t, [y, r]= 0, [z, c]= 0, [z, r]= 0, [c, r]= 0\) and \(t \in \mathbb{Z}(L)\).

To compute the multiplier start with

\[
\begin{align*}
[x,y]=z+s_1 & \quad [y, z]=r + \alpha_5 t + s_5 & \quad [z,r]=s_9 & \quad [z, t]=s_{13} \\
[x, z]=c+s_2 & \quad [y, c]=t + s_6 & \quad [c, r]=s_{10} & \quad [c, t]=s_{14} \\
[x, c]=r+s_3 & \quad [y, r]= s_7 & \quad [x, t]=s_{11} & \quad [r, t]=s_{15} \\
[x, r]= t+s_4 & \quad [z, c]= s_8 & \quad [y, t]=s_{12}
\end{align*}
\]

where \(s_1, \ldots, s_{15}\) generate the multiplier. Now we will use the Jacobi identity on all triples.

\[
\begin{align*}
\text{• (xy)z+(yz)x+(zx)y = 0} & \iff zz+ (r+\alpha_5 t)x+(-c)y=0 & \iff -\alpha_5 s_{11}=0 \\
\text{• (xy)c+(yc)x+(cx)y = 0} & \iff (z)c+ (\alpha_4 t)x-(r)y=0 & \iff s_8-s_{11}+s_7 =0 & \iff s_8 = s_{11}-s_7 \\
\text{• (xy)r+(yr)x+(rx)y = 0} & \iff zr+ \alpha_7 t x-(\alpha_4 t)y=0 & \iff s_9 - \alpha_7 s_{11} + \alpha_4 s_{12}=0 \\
\implies & \iff s_9 = \alpha_7 s_{11} - \alpha_4 s_{12} =- s_{12}
\end{align*}
\]
\[ (xy)t+(yt)x+(tx)y = 0 \Rightarrow zt+s_{12}x-s_{11}y=0 \Rightarrow s_{13}=0 \]

\[ (xz)r+(zr)x+(rx)z = 0 \Rightarrow cr+s_{0}x-tz=0 \Rightarrow s_{10}+s_{13}=0 \Rightarrow s_{10}=-s_{13} \]

\[ (xz)c+(zc)x+(cx)z = 0 \Rightarrow cc-\alpha_{7}tx-rz=0 \Rightarrow -\alpha_{7}s_{11}+s_{9}=0 \Rightarrow s_{9}=0 \]

\[ (xz)t+(zt)x+(tx)z = 0 \Rightarrow ct+s_{13}x-s_{11}z=0 \Rightarrow s_{14}=0 \]

\[ (xr)c+(rc)x+(cx)r = 0 \Rightarrow (\alpha_{4}t)c-s_{10}x-rr=0 \Rightarrow \alpha_{4}s_{14}=0 \Rightarrow s_{14}=0 \]

\[ (xr)t+(rt)x+(tx)r = 0 \Rightarrow (\alpha_{4}t)t+s_{15}x-s_{11}r=0 \Rightarrow 0=0 \]

\[ (xc)t+(ct)x+(tx)c = 0 \Rightarrow rt+s_{14}x-s_{11}c=0 \Rightarrow s_{15}=0 \]

\[ (yz)c+(zc)y+(cy)z = 0 \Rightarrow (r+\alpha_{5}t)c-\alpha_{7}ty+(-\alpha_{4}t)z=0 \Rightarrow r\alpha_{5}tc-tz=0 \Rightarrow s_{10}=-\alpha_{5}s_{14} \]

\[ (yz)r+(zr)y+(ry)z = 0 \Rightarrow (r+\alpha_{5}t)r+s_{0}y-tz=0 \Rightarrow rr+\alpha_{5}tr+s_{13}=0 \]

\[ (yz)t+(zt)y+(ty)z = 0 \Rightarrow (r+\alpha_{5}t)t+s_{13}y+(-s_{12})z=0 \Rightarrow s_{15}=0 \]

\[ (yc)r+(cr)y+(ry)c = 0 \Rightarrow tr+s_{10}y-s_{7}c=0 \Rightarrow s_{15}=0 \]

\[ (yc)t+(ct)y+(ty)c = 0 \Rightarrow (t)t+s_{14}y-s_{12}c=0 \Rightarrow 0=0 \]

\[ (rc)t+(ct)r+(tr)c = 0 \Rightarrow -s_{10}t+s_{14}r-s_{15}c=0 \Rightarrow 0=0 \]
From the preceding work we obtain the following equations.

(i) \( \alpha_s 11 = 0 \)
(ii) \( s_8 = s_{11} - s_7 \)
(iii) \( s_9 = \alpha_7 s_{11} - \alpha_4 s_{12} = -s_{12} \)
(iv) \( s_{13} = 0 \)
(v) \( s_{10} = -s_{13} \)
(vi) \( s_9 = 0 \)
(vii) \( s_{14} = 0 \)
(viii) \( s_{15} = 0 \)
(ix) \( s_{10} = -\alpha_5 s_{14} \)
(x) \( \alpha_5 s_{15} = 0 \).

Now from (i) if \( \alpha_5 \) is not zero, this implies \( s_8 = s_9 = s_{10} = s_{12} = s_{13} = s_{14} = s_{15} = 0 \). Further \( s_{11} \in (s_5, s_6, s_7, s_8) \) thus

\[ \dim M(L) = (s_5, s_6, s_7, s_8) = 4. \]

Therefore \( t(L) = \frac{1}{2} n(n-1) - \dim M(L) = \frac{1}{2} (6)(5) - 4 = 11 \), we call this algebra \( L = L'(11,6,1,11) \).

From this work we have found the following algebras for \( t(L) = 11 \) and we summarize them in the following theorem.

**Theorem 14** Let \( L \) be filiform with \( t(L) = 11 \) then \( L = L(11,6,1,11) \) or \( L = L'(11,6,1,11) \).
12. $t(L)= 12$

By Theorem 4, we know that $c^2 + c \leq 2t$ where $c = \dim L^2$ and $t = t(L)$. If $t = 12$ then $c^2 + c \leq 24$ which implies that $c \leq 4$ therefore $c = 0, 1, 2, 3$ or $4$, but by proposition 0 (i.e. $t \leq \frac{1}{2}n(n-1)$) and since $n = c + 2$ for filiform we get $12 \leq \frac{1}{2}(c + 2)(c + 1)$ and by solving this inequality we know that $c = 4$ is the only case that satisfies the inequality. Thus if $c = \dim L^2 = 4$, then $\dim H^2 = 3$ and $t(H) \leq 8$ (proposition 3). In the $t(L) = 11$ case we have already computed these and we list them in the following theorem.

**Theorem 15** Let $L$ be filiform with $t(L) = 12$, then $L = L(11,6,1,12)$ or $L = L'(11,6,1,12)$. 
13. \( t(L) = 13 \)

By Theorem 4, we know that \( c^2 + c \leq 2t \) where \( c = \text{dim} L^2 \) and \( t = t(L) \). If \( t = 13 \) then
\[ c^2 + c \leq 26 \]
which implies that \( c \leq 4 \) therefore \( c = 0, 1, 2, 3 \) or \( 4 \), but by proposition 0 (i.e. \( t \leq \frac{1}{2} n(n-1) \)) and since \( n = c + 2 \) for filiform we get
\[ 13 \leq \frac{1}{2} (c + 2)(c + 1) \]
and by solving this inequality we know that \( c = 4 \) is the only case that satisfies the inequality. Thus if \( c = \text{dim} L^2 = 4 \), then \( \text{dim} H^2 = 3 \) and \( t(H) \leq 9 \) (proposition 3), but from Table 22.1, we know that the only time these conditions are satisfied is when \( t(L) = 7 \), but here \( t(L) = 13 \), hence a contradiction. We thus summarize our findings in the following Theorem.

**Theorem 16** There are no filiform Lie algebras with \( t(L) = 13 \).
14. $t(L)=14$

By Theorem 4, we know that $c^2 + c \leq 2t$ where $c = \dim L^2$ and $t = t(L)$. If $t = 14$ then
$c^2 + c \leq 28$ which implies that $c \leq 4$ therefore $c = 0, 1, 2, 3$ or $4$, but by proposition 0
(i.e. $t \leq \frac{1}{2}n(n-1)$) and since $n = c+2$ for filiform we get $14 \leq \frac{1}{2}(c + 2)(c + 1)$ and by solving
this inequality we know that $c = 4$ is the only case that satisfies the inequality. Thus if $c = \dim L^2 = 4$, then $\dim H^2 = 3$ and $t(H) \leq 10$ (proposition 3), but from Table 22.1, we
know that the only time these conditions are satisfied is when $t(L) = 7$, but here $t(L) = 14$, hence a contradiction. We thus summarize our findings in the following Theorem.

**Theorem 17** There are no filiform Lie algebras with $t(L) = 14$. 
15. $t(L)=15$

By Theorem 4, we know that $c^2 + c \leq 2t$ where $c = \dim L^2$ and $t = t(L)$. If $t = 15$ then $c^2 + c \leq 30$ which implies that $c \leq 5$ therefore $c = 0, 1, 2, 3, 4$ or $5$. But by proposition 0 (i.e. $t \leq \frac{1}{2}n(n-1)$) and since $n = c + 2$ for filiform we get $15 \leq \frac{1}{2}(c + 2)(c + 1)$ and by solving this inequality we know that $c = 4$ and $c = 5$ are the two cases that satisfy the inequality. Thus if $c = \dim L^2 = 4$, then $\dim L = 6$ which implies $\dim H = 5$, but from Table 22.1 this is a contradiction.

Also if $c = \dim L^2 = 5$, then $\dim H^2 = 4$ and $t(H) \leq 10$ (proposition 3), but again from Table 22.1, we know that there are no filiform Lie algebras satisfying these conditions, hence a contradiction. We thus summarize our findings in the following Theorem.

**Theorem 18** There are no filiform Lie algebras with $t(L) = 15$. 

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16. $t(L)=16$

By Theorem 4, we know that $c^2 + c \leq 2t$ where $c = \dim L_2$ and $t = t(L)$. If $t = 16$ then $c^2 + c \leq 32$ which implies that $c \leq 5$ therefore $c = 0, 1, 2, 3, 4$ or 5 but by proposition 0 (i.e. $t \leq \frac{1}{2}n(n-1)$) and since $n = c+2$ for filiform we get $16 \leq \frac{1}{2}(c + 2)(c + 1)$ and by solving this inequality we know that $c = 4$ or $c=5$. Thus (1) if $c = \dim L_2 = 4$, then $\dim H_2 = 3$ and $t(H) \leq 12$ (proposition 3). (2) If $c= \dim L_2 = 5$ then $\dim H_2 = 4$ and $t(H) \leq 11$ (proposition 3), but from previous work we know that the only time these conditions are satisfied is when $t(L)=11$ or $t(L)=12$. If $t(H)=11$, then $H=L(11,6,1,11)$ or $H=L´(11,6,1,11)$ and if $t(H)=12$, then $H=L(11,6,1,12)$ or $H=L´(11,6,1,12)$ but note that $c = \dim L_2 = 4$ is impossible since the dimension of $c$ must equal to 5 because $c$ has to be 2 less than the dimension of $L$ which has 7 elements as defined by the basis $\{x, y, z, c, r, t, u\}$ which is what we need here for $t(L)=16$. Therefore (1) above is totally eliminated leaving us with only (2). We will therefore now consider the two cases for $t(L)=11$. The first is when $H=L(11,6,1,11)$ and the second is when $H=L´(11,6,1,11)$.

**CASE 1** Let $H=L(11,6,1,11)$, this $L$ can be described generally by the basis $\{x, y, z, c, r, t, u\}$ and multiplication

$[x,y]=z$, $[x,z]=c$, $[x,r]=r$, $[x,t]=\alpha_4t+\beta_4u$, $[y,z]=\alpha_5t+\beta_5u$, $[y,c]=\beta_6u$, $[y,r]=t+\beta_7u$, $[y,t]=\beta_8u$, $[z,c]=-t+\beta_9u$, $[z,r]=\beta_{10}u$, $[z,t]=\beta_{11}u$, $[z,u]=\beta_{12}u$, $[c,t]=\beta_{13}u$, $[r,t]=\beta_{14}u$, and $u \in Z(L)$. Now we will use the Jacobi Identity on all triples.

$\cdot (xy)z+(yz)x+(zx)y = 0$

$\Rightarrow zz + (\alpha_5t+ \beta_5u)x +(-c)y=0$

$\Rightarrow \alpha_5tx+ \beta_5ux-cy=0$
\(-\alpha_5 \beta_1 u - (\beta_6 u) = 0 \quad \Rightarrow \beta_6 = \alpha_5 \beta_1\)

\[
\begin{align*}
\bullet \ (xy)c + (yc)x + (cx)y &= 0 \quad \Rightarrow zc + \beta_6 ux - (t - \beta_7 u) = 0 \quad \Rightarrow \beta_7 = -\beta_8 \\
\bullet \ (xy)r + (yr)x + (rx)y &= 0 \quad \Rightarrow zr + (t + \beta_7 u)x + (-\alpha_4 t - \beta_4 u)y = 0 \quad \Rightarrow zr + \beta_7 ux - \alpha_4 ty_4 u y = 0 \\
\Rightarrow \beta_9 u - \beta_{11} u - \alpha_4 (-\beta_{12} u) &\quad \Rightarrow \beta_9 = \beta_{11} - \alpha_4 \beta_{12} \\
\bullet \ (xy)t + (yt)x + (tx)y &= 0 \quad \Rightarrow zt + \beta_12 ux - \beta_{11} uy = 0 \quad \Rightarrow \beta_{13} u = 0 \\
\bullet \ (xy)u + (yu)x + (ux)y &= 0 \quad \Rightarrow 0 = 0 \\
\bullet \ (xz)c + (zc)x + (cx)z &= 0 \quad \Rightarrow cc + (-t + \beta_8 u)x - rz = 0 \quad \Rightarrow -tx rz = 0 \quad \Rightarrow \beta_{11} u + \beta_9 u = 0 \\
\Rightarrow \beta_9 &= -\beta_{11} \\
\bullet \ (xz)r + (zr)x + (rx)z &= 0 \quad \Rightarrow cr + \beta_9 ux - \alpha_4 tz - \beta_4 uz = 0 \quad \Rightarrow \beta_{10} u + \alpha_4 \beta_{13} u = 0 \\
\Rightarrow \beta_{10} &= -\alpha_4 \beta_{13} \\
\bullet \ (xz)t + (zt)x + (tx)z &= 0 \quad \Rightarrow ct + \beta_13 ux - \beta_11 uz = 0 \quad \Rightarrow \beta_{14} u = 0 \\
\bullet \ (xz)u + (zu)x + (ux)z &= 0 \quad \Rightarrow 0 = 0 \\
\bullet \ (xc)r + (cr)x + (rx)c &= 0 \quad \Rightarrow cr + \beta_{10} ux - \alpha_4 tc - \beta_4 uc = 0 \quad \Rightarrow -\alpha_4 tc = 0 \quad \Rightarrow \alpha_4 \beta_{14} u = 0 \\
\bullet \ (xc)t + (ct)x + (tx)c &= 0 \quad \Rightarrow rt + \beta_{14} ux + (-\beta_{11} u)c = 0 \quad \Rightarrow \beta_{15} u = 0 \\
\bullet \ (xc)u + (cu)x + (ux)c &= 0 \quad \Rightarrow 0 = 0
\end{align*}
\]
\begin{align*}
\bullet (xr)t+(rt)x+(tx)r = 0 & \Rightarrow (\alpha_4t + \beta_4u)t + \beta_{13}ux + (-\beta_{11}u)r = 0 \quad \Rightarrow 0=0 \\
\bullet (xr)u+(ru)x+(ux)r = 0 & \Rightarrow 0=0 \\
\bullet (xt)u+(tu)x+(ux)t = 0 & \Rightarrow 0=0 \\
\bullet (yz)c+(zc)y+(cy)z = 0 & \Rightarrow (\alpha_5t + \beta_5u)c + (-\beta_{11}u)y + (-\beta_{15}u)z = 0 \\
& \Rightarrow \alpha_5tc + \beta_5uc - \beta_{11}uy - \beta_{15}uz = 0 \quad \Rightarrow -\alpha_5 \beta_{14}u + \beta_{12}u = 0 \quad \Rightarrow \alpha_5 \beta_{14} = \beta_{12} \\
\bullet (yz)r+(zr)y+(ry)z = 0 & \Rightarrow (\alpha_5t + \beta_5u)r + \beta_{13}uy - \beta_{12}uz = 0 \\
& \Rightarrow -\alpha_5 \beta_{14}u + \beta_{13}u = 0 \quad \Rightarrow \beta_{13} = \alpha_5 \beta_{15} \\
\bullet (yz)t+(zt)y+(ty)z = 0 & \Rightarrow (\alpha_5t + \beta_5u)t + \beta_{13}uy - \beta_{12}uz = 0 \quad \Rightarrow 0=0 \\
\bullet (yz)u+(zu)y+(uy)z = 0 & \Rightarrow 0=0 \\
\bullet (yc)r+(cr)y+(ry)c = 0 & \Rightarrow \beta_6ur + \beta_{10}uy + (-\beta_{17}u)c = 0 \quad \Rightarrow -tc - \beta_{17}uc = 0 \quad \Rightarrow \beta_{14}u = 0 \\
\bullet (yc)t+(ct)y+(ty)c = 0 & \Rightarrow \beta_6ut + \beta_{14}uy + \beta_{17}uc = 0 \quad \Rightarrow 0=0 \\
\bullet (yc)u+(cu)y+(uy)c = 0 & \Rightarrow 0=0 \\
\bullet (yr)t+(rt)y+(ty)r = 0 & \Rightarrow (t + \beta_{17}u)t + \beta_{15}uy - \beta_{12}ur = 0 \quad \Rightarrow 0=0 \\
\bullet (yr)u+(ru)y+(uy)r = 0 & \Rightarrow 0=0 \\
\bullet (yt)u+(tu)y+(uy)t = 0 & \Rightarrow 0=0
\end{align*}
We will now summarize the equations found below:

(1) $\beta_6 = \alpha_5 \beta_{11}$  (2) $\beta_7 = -\beta_8$  (3) $\beta_9 = \beta_{11} - \alpha_4 \beta_{12}$  (4) $\beta_{13} u = 0$  (5) $\beta_9 = \beta_{11}$  (6) $\beta_{10} = -\alpha_4 \beta_{13}$  
(7) $\beta_{14} u = 0$  (8) $\alpha_4 \beta_{14} u = 0$  (9) $\beta_{15} u = 0$  (10) $\alpha_5 \beta_{14} = \beta_{12}$  (11) $\beta_{13} = \alpha_5 \beta_{15}$

From the preceding work we obtain the following equations.

(i) $\beta_6 = \beta_9 = \beta_{10} = \beta_{11} = \beta_{12} = \beta_{13} = \beta_{14} = \beta_{15} = 0$  (ii) $\beta_7 = -\beta_8$
(iii) $\beta_9 = -\beta_{11} = \beta_{13} = 0$  (iv) $\beta_{10} = -\alpha_4 \beta_{13} = 0$  (v) $\beta_{13} = \alpha_5 \beta_{15} = 0$.

Relabelling yields the non zero multiplication
\[ [x, y] = z, [x, z] = c, [x, c] = r, \quad [x, r] = \alpha_4 t + \beta_4 u, \quad [y, z] = \alpha_5 t + \beta_5 u, \quad [y, c] = 0, \quad [y, r] = t + \beta_5 u, \]
\[ [z, c] = -t + \beta_8 u, \quad [z, r] = 0, \quad [c, r] = 0, \quad [x, t] = 0, \quad [y, t] = 0, \quad [z, t] = 0, \quad [c, t] = 0, \quad [r, t] = 0, \quad u \in Z(L). \]

This implies that \( t \) is also in the center, a contradiction, thus we have not found any filiform lie algebra for this case.

**CASE 2.** We will now consider \( H = L \left( 11, 6, 1, 11 \right) \). This can be described generally by the basis \{x, y, z, c, r, t, u\} and multiplication

\[ [x, y] = z, \quad [x, z] = c, \quad [x, c] = r, \quad [x, r] = \alpha_4 t + \beta_4 u, \quad [y, z] = r + \alpha_5 t + \beta_5 u, \quad [y, c] = \alpha_4 t + \beta_6 u, \quad [y, r] = t + \beta_7 u, \]
\[ [z, c] = -t + \beta_8 u, \quad [z, r] = \beta_9 u, \quad [c, r] = \beta_{10} u, \quad [x, t] = \beta_1 u, \quad [y, t] = \beta_2 u, \quad [z, t] = \beta_3 u, \quad [c, t] = \beta_4 u, \quad [r, t] = \beta_5 u, \quad u \in Z(L). \]

Now we will use the Jacobi Identity on all triples.

\[ (xy)z + (yz)x + (zx)y = 0 \quad \Rightarrow \quad zz + (r + \alpha_5 t + \beta_5 u)x + (-c)y = 0 \quad \Rightarrow \quad rx + \alpha_5 tx + \beta_5 ux - cy = 0 \]
\[ \Rightarrow (-\alpha_4 t - \beta_4 u) + \alpha_5 (-\beta_1 u) + (-\alpha_4 t - \beta_5 u) = 0 \quad \Rightarrow \quad \beta_6 = \beta_4 u + \alpha_5 \beta_{11} \]

\[ (xy)c + (yc)x + (cx)y = 0 \quad \Rightarrow \quad zc + (\alpha_4 t + \beta_6 u)x - ry = 0 \quad \Rightarrow \quad -t + \beta_8 u + \alpha_4 tx + \beta_6 ux + t + \beta_7 u = 0 \]
\[ \Rightarrow \quad \beta_7 = \alpha_4 \beta_{11} - \beta_8 \]

\[ (xy)r + (yr)x + (rx)y = 0 \quad \Rightarrow \quad zr + (t + \beta_7 u)x + (-\alpha_4 t - \beta_4 u)y = 0 \quad \Rightarrow \quad \beta_9 u + tx + \beta_7 ux - \alpha_4 ty - \beta_4 uy = 0 \]
\[ \Rightarrow \quad \beta_9 u - \beta_{11} u - \alpha_4 (-\beta_{12} u) \quad \Rightarrow \quad \beta_9 = \beta_{11} - \alpha_4 \beta_{12} \]

\[ (xy)t + (yt)x + (tx)y = 0 \quad \Rightarrow \quad zt + \beta_{12} ux - \beta_{11} uy = 0 \quad \Rightarrow \beta_{13} u = 0 \]

\[ (xy)u + (yu)x + (ux)y = 0 \quad \Rightarrow 0 = 0 \]
\( (xz)c + (zc)x + (cx)z = 0 \implies cc + (-t + \beta_3 u)x - rz = 0 \implies -tx - rz = 0 \implies \beta_{11} u + \beta_9 u = 0 \)

\[ \Rightarrow \beta_9 = -\beta_{11} \]

\( (xz)r + (zr)x + (rx)z = 0 \implies cr + \beta_9 ux - \alpha_4 tz - \beta_4 uz = 0 \implies \beta_{10} u + \alpha_4 \beta_{13} u = 0 \implies \beta_{10} = -\alpha_4 \beta_{13} \)

\( (xz)t + (zt)x + (tx)z = 0 \implies ct + \beta_{13} ux - \beta_{11} uz = 0 \implies \beta_{14} u = 0 \)

\( (xz)u + (zu)x + (ux)z = 0 \implies 0 = 0 \)

\( (xc)r + (cr)x + (rx)c = 0 \implies cr + \beta_9 ux - \alpha_4 tc - \beta_4 uc = 0 \implies -\alpha_4 tc = 0 \implies -\alpha_4 \beta_{14} u = 0 \)

\( (xc)t + (ct)x + (tx)c = 0 \implies ct + \beta_{14} ux + (-\beta_{11} u)c = 0 \implies \beta_{15} u = 0 \)

\( (xc)u + (cu)x + (ux)c = 0 \implies 0 = 0 \)

\( (xr)t + (rt)x + (tx)r = 0 \implies (\alpha_4 t + \beta_4 u)t + \beta_{15} ux + (-\beta_{11} u)r = 0 \implies 0 = 0 \)

\( (xr)u + (ru)x + (ux)r = 0 \implies 0 = 0 \)

\( (xt)u + (tu)x + (ux)t = 0 \implies 0 = 0 \)

\( (yz)c + (zc)y + (cy)z = 0 \implies (a_5 t + \beta_5 u)c + (-t + \beta_8 u)y + (-\beta_6 u)z = 0 \)

\[ \Rightarrow (a_5 t + \beta_5 u)c + (-t + \beta_8 u)y + (-\beta_6 u)z = 0 \implies a_5 tc + \beta_5 uc - ty + \beta_8 uy - \beta_8 uz = 0 \]

\[ \Rightarrow -a_5 \beta_{14} u + \beta_{12} u = 0 \implies a_5 \beta_{14} = \beta_{12} \]

\( (yz)r + (zr)y + (ry)z = 0 \implies (a_5 t + \beta_5 u)r + \beta_9 uy + (-t - \beta_7 u)z = 0 \implies a_5 tr - tz - \beta_7 uz = 0 \)

\[ \Rightarrow -a_5 \beta_{15} u + \beta_{13} u = 0 \implies \beta_{13} = a_5 \beta_{15} \]
• $(yz)t+(zt)y+(ty)z = 0 \implies (\alpha t+ \beta u)t + \beta_1 uy - \beta_2 uz = 0 \implies 0=0$

• $(yz)u+(zu)y+(uy)z = 0 \implies 0=0$

• $(yc)r+(cr)y+(ry)c = 0 \implies (\alpha_4 t+ \beta_6 u)r+ \beta_1 uy+(-t- \beta_7 u)c=0 \implies \alpha_4 tr+ \beta_6 ur+ -tc- \beta_7 uc=0$

$\implies \beta_{14} = \alpha_4 \beta_{15}$

• $(yc)t+(ct)y+(ty)c = 0 \implies (\alpha_4+\beta_6 u)t+ \beta_1 uy+\beta_7 uc=0 \implies 0=0$

• $(yc)u+(cu)y+(uy)c = 0 \implies 0=0$

• $(yr)t+(rt)y+(ty)r = 0 \implies \beta_1 ut + \beta_1 uy - \beta_2 ur=0 \implies 0=0$

• $(yr)u+(ru)y+(uy)r = 0 \implies 0=0$

• $(yt)u+(tu)y+(uy)t = 0 \implies 0=0$

• $(zc)r+(cr)z+(rz)c = 0 \implies (-t+ \beta_8 u)r+ \beta_1 uz - \beta_9 uc=0 \implies -tr=0 \implies \beta_{15} u=0$

• $(zc)t+(ct)z+(tz)c = 0 \implies (-t+ \beta_8 u)t+ \beta_1 uz - \beta_13 uc=0 \implies 0=0$

• $(zc)u+(cu)z+(uz)c = 0 \implies 0=0$

• $(zr)t+(rt)z+(tz)r = 0 \implies \beta_8 ut+ \beta_1 uz - \beta_13 ur=0 \implies 0=0$

• $(zr)u+(ru)z+(uz)r = 0 \implies 0=0$
\[
(zt)u + (tu)z + (uz)t = 0 \quad \Rightarrow 0 = 0
\]

\[
(cr)t + (rt)c + (tc)r = 0 \quad \Rightarrow \beta_{10}u + \beta_{15}uc - \beta_{14}ur = 0 \Rightarrow 0 = 0
\]

\[
(cr)u + (ru)c + (uc)r = 0
\]

\[
(rt)u + (tu)r + (ur)t = 0 \quad \Rightarrow 0 = 0
\]

We will now summarize the equations found below:

\begin{enumerate}
\item \( \beta_6 = \beta_4u + \alpha_5 \beta_11 \) 
\item \( \beta_7 = \alpha_4 \beta_11 - \beta_8 \) 
\item \( \beta_9 = \beta_11 - \alpha_4 \beta_12 \) 
\item \( \beta_{13}u = 0 \) 
\item \( \beta_{13} = \alpha_4 \beta_15 \)
\end{enumerate}

From the preceding work we obtain the following equations.

\begin{enumerate}
\item \( \beta_9 = \beta_{10} = \beta_{11} = \beta_{12} = \beta_{13} = \beta_{14} = \beta_{15} = 0 \) 
\item \( \beta_7 = -\beta_8 \)
\item \( \beta_6 = \beta_4 \)
\end{enumerate}

Relabelling yields the non zero multiplication

\[
[x, y] = z, \ [x, z] = c, \ [x, c] = r, \ [x, r] = \alpha_4t + \beta_8u, \ [y, z] = r + \alpha_5t + \beta_3u, \ [y, c] = \alpha_4t, \ [y, r] = t - \beta_8u, \\
[z, c] = -t + \beta_8u, \ [z, r] = 0, \ [c, r] = 0, \ [x, t] = 0, \ [y, t] = 0, \ [z, t] = 0, \ [c, t] = 0, \\
[r, t] = 0, \text{ and } u \in \mathbb{Z}(L). \text{ Again this implies that } t \text{ is also in the center, a contradiction, thus we again have not found any filiform Lie algebras for this case and now summarize our findings in the following Theorem}
\]

**Theorem 19** There are no filiform Lie algebras for \( t(L) = 16 \)
17. $t(L)=17$

By Theorem 4, we know that $c^2 + c \leq 2t$ where $c = \dim L^2$ and $t = t(L)$. If $t = 17$ then $c^2 + c \leq 34$ which implies that $c \leq 5$ therefore $c = 0, 1, 2, 3, 4$ or 5 but by proposition 0 (i.e. $t \leq \frac{1}{2}n(n-1)$) and since $n = c+2$ for filiform we get $17 \leq \frac{1}{2}(c + 2)(c + 1)$ and by solving this inequality we know that $c = 5$ is the only case that satisfies the inequality. Thus if $c = \dim L^2 = 5$, then $\dim H^2 = 4$ and $t(H) \leq 12$ (proposition 3), but from Table 22.1, we know that the only time these conditions are satisfied is when $t(H)=12$. We will therefore consider the two cases for $t(H)=12$. The first case is when $H=L(11, 6, 1, 12)$ and the second is when $H=L^\prime(11, 6, 1, 12)$.

**CASE 1.** Let $H=L(11, 6, 1, 12)$, this $L$ can be described generally by the basis

$\{x, y, z, c, r, t, u\}$ and multiplication

$[x,y]=z$, $[x,z]=c$, $[x,c]=r$, $[x,r]=t$, $[y,z]=\alpha_5t+\beta_5u$, $[y,c]=\beta_6u$, $[y,r]=\beta_7u$,

$[z,c]=\beta_8u$, $[z,r]=\beta_9u$, $[c,r]=\beta_{10}u$, $[x,t]=\beta_{11}u$, $[y,t]=\beta_{12}u$, $[z,t]=\beta_{13}u$, $[c,t]=\beta_{14}u$,

$[r,t]=\beta_{15}u$, and $u \in Z(L)$. Now we will use the Jacobi Identity on all triples.

- $(xy)z+(yz)x+(zx)y = 0 \Rightarrow zz+ \alpha_5t+\beta_5u +(-c)y=0 \Rightarrow \alpha_5t-cy=0$

  $\Rightarrow -\alpha_5\beta_{11}u-(-\beta_6u)=0 \Rightarrow \beta_6=\alpha_5\beta_{11}$

- $(xy)c+(yc)x+(cx)y = 0 \Rightarrow zc+ \beta_6u-xr=0 \Rightarrow \beta_8u+ \beta_7u=0 \Rightarrow \beta_7=-\beta_8$

- $(xy)r+(yr)x+(rx)y = 0 \Rightarrow zr+ (\beta_7u)x + (-t)y=0 \Rightarrow \beta_9u-(-\beta_{12}u)=0 \Rightarrow \beta_9=-\beta_{12}$
\begin{itemize}
  \item \((xy)t+(yt)x+(tx)y\) = 0  \quad \Rightarrow \quad zt + \beta_{12}ux - \beta_{11}uy = 0  \quad \Rightarrow \quad \beta_{13}u = 0  \quad \Rightarrow \quad \beta_{13} = 0
  
  \item \((xy)u+(yu)x+(ux)y\) = 0  \quad \Rightarrow \quad 0 = 0
  
  \item \((xz)c+(zc)x+(cx)z\) = 0  \quad \Rightarrow \quad cc + \beta_{8}ux - rz = 0  \quad \Rightarrow \quad \beta_{9}u = 0  \quad \Rightarrow \quad \beta_{9} = 0
  
  \item \((xz)r+(zr)x+(rx)z\) = 0  \quad \Rightarrow \quad cr + \beta_{9}ux - tz = 0  \quad \Rightarrow \quad cr - (-\beta_{13}u) = 0  \quad \Rightarrow \quad \beta_{10}u + \alpha_{4}\beta_{13}u = 0

  \quad \Rightarrow \quad \beta_{10} = -\beta_{13}  \quad \Rightarrow \quad \beta_{10} = 0

  \item \((xz)t+(zt)x+(tx)z\) = 0  \quad \Rightarrow \quad ct + \beta_{13}ux - \beta_{11}uz = 0  \quad \Rightarrow \quad \beta_{14}u = 0  \quad \Rightarrow \quad \beta_{14} = 0

  \item \((xz)u+(zu)x+(ux)z\) = 0  \quad \Rightarrow \quad 0 = 0

  \item \((xc)r+(cr)x+(rx)c\) = 0  \quad \Rightarrow \quad rr + \beta_{10}ux - \alpha_{4}tc = 0  \quad \Rightarrow \quad -\alpha_{4}tc = 0  \quad \Rightarrow \quad \beta_{14} = 0

  \item \((xc)t+(ct)x+(tx)c\) = 0  \quad \Rightarrow \quad rt + \beta_{14}ux + (-\beta_{11}u)c = 0  \quad \Rightarrow \quad \beta_{15}u = 0  \quad \Rightarrow \quad \beta_{15} = 0

  \item \((xc)u+(cu)x+(ux)c\) = 0  \quad \Rightarrow \quad 0 = 0

  \item \((xr)t+(rt)x+(tx)r\) = 0  \quad \Rightarrow \quad (t + \beta_{4}u)t + \beta_{15}ux - \beta_{11}ur = 0  \quad \Rightarrow \quad tt + \beta_{4}ut + \beta_{15}ux - \beta_{11}ur = 0

  \quad \Rightarrow \quad 0 = 0

  \item \((xr)u+(ru)x+(ux)r\) = 0  \quad \Rightarrow \quad 0 = 0

  \item \((xt)u+(tu)x+(ux)t\) = 0  \quad \Rightarrow \quad 0 = 0

  \item \((yz)c+(zc)y+(cy)z\) = 0  \quad \Rightarrow \quad (\alpha_{5}t + \beta_{3}u)c + \beta_{8}uy + (-\beta_{6}u)z = 0  \quad \Rightarrow \quad \alpha_{5}tc = 0  \quad \Rightarrow \quad \alpha_{5}\beta_{14} = 0  \quad \Rightarrow \quad 0 = 0
\end{itemize}
• (yz)r+(zr)y+(ry)z = 0 \Rightarrow (\alpha_5t+\beta_5u)r+\beta_9uy-\beta_7uz=0 \quad \Rightarrow \alpha_5tr=0 \Rightarrow \alpha_5\beta_15u=0 \Rightarrow 0=0

• (yz)t+(zt)y+(ty)z = 0 \Rightarrow (\alpha_5t+\beta_5u)t+\beta_{13}uy-\beta_{12}uz=0 \Rightarrow 0=0

• (yz)u+(zu)y+(uy)z = 0 \Rightarrow 0=0

• (yc)r+(cr)y+(ry)c = 0 \Rightarrow \beta_6ur+\beta_{10}uy+(-\beta_7u)c=0 \Rightarrow 0=0

• (yc)t+(ct)y+(ty)c = 0 \Rightarrow \beta_6ut+\beta_{14}uy-\beta_{12}uc=0 \Rightarrow 0=0

• (yc)u+(cu)y+(uy)c = 0 \Rightarrow 0=0

• (yr)t+(rt)y+(ty)r = 0 \Rightarrow (\beta_7u)t+\beta_{13}uy-\beta_{12}ur=0 \Rightarrow 0=0

• (yr)u+(ru)y+(uy)r = 0 \Rightarrow 0=0

• (yt)u+(tu)y+(uy)t = 0 \Rightarrow 0=0

• (zc)r+(cr)z+(rz)c = 0 \Rightarrow \beta_8ur+\beta_{10}uz-\beta_9uc=0 \Rightarrow 0=0

• (zc)t+(ct)z+(tz)c = 0 \Rightarrow \beta_8ut-\beta_{14}uz-\beta_{13}uc=0 \Rightarrow 0=0

• (zc)u+(cu)z+(uz)c = 0 \Rightarrow 0=0

• (zr)t+(rt)z+(tz)r = 0 \Rightarrow \beta_9ut+\beta_{15}uz-\beta_{13}ur=0 \Rightarrow 0=0

• (zr)u+(ru)z+(uz)r = 0 \Rightarrow 0=0
We will now summarize the equations found below:

(1) $\beta_6 = \alpha_5 \beta_{11}$  
(2) $\beta_7 = -\beta_8$  
(3) $\beta_9 = -\beta_{12}$  
(4) $\beta_{13} = 0$  
(5) $\beta_9 = 0$  
(6) $\beta_{10} = 0$  
(7) $\beta_{14} = 0$  
(8) $\beta_{15} = 0$.

From the preceding work we obtain the following equations.

(i) $\beta_9 = \beta_{10} = \beta_{13} = \beta_{14} = \beta_{15} = 0$  
(ii) $\beta_7 = -\beta_8$  
(iii) $\beta_{12} = 0$  
(iv) $\beta_6 = \alpha_5$

Relabelling yields the non zero multiplication

$[x, y] = z$, $[x, z] = c$, $[x, c] = r$, $[x, r] = t$, $[y, z] = \alpha_5 t + \beta_5 u + s_5$, $[y, c] = \alpha_5 u + s_6$, $[y, r] = -\beta_8 u + s_7$, $[z, c] = \beta_8 u + s_8$, $[z, r] = s_9$, $[c, r] = s_{10}$, $[x, t] = u$, $[y, t] = 0$, $[z, t] = 0$, $[c, t] = 0$, $[r, t] = 0$, and $u \in Z(L)$. For $t$ to not be in $Z(L)$, either $\beta_{11}$ or $\beta_{12}$ is not zero, but $\beta_{12} = 0$ so $\beta_{11} \neq 0$ thus $[x, t] = u$ by the substitution $u' = \beta_{11} u$ which implies that $\beta_{11} = 1$.

To compute the multiplier start with

$[x, y] = z$, $[x, z] = c$, $[x, c] = r$, $[x, r] = t$, $[y, z] = \alpha_5 t + \beta_5 u + s_5$, $[y, c] = \alpha_5 u + s_6$, $[y, r] = -\beta_8 u + s_7$, $[z, c] = \beta_8 u + s_8$, $[z, r] = s_9$, $[c, r] = s_{10}$, $[x, t] = u$, $[y, t] = 0$, $[z, t] = 0$, $[c, t] = 0$, $[r, t] = 0$, $[c, u] = s_{19}$, $[r, u] = s_{20}$, $[t, u] = s_{21}$, where $s_1, \ldots, s_{21}$ generate the multiplier.
Now we will use the Jacobi identity on all triples.

• \((xy)z+(yz)x+(zx)y = 0\) \Rightarrow \(zz+ (\alpha_5t+ \beta_5u+s_5)x+(-c)y=0 \Rightarrow \alpha_5tx+ \beta_5ux+s_5x-cy=0\)

\(\Rightarrow \alpha_5(-u)-\beta_5s_16+ \alpha_5u+s_6=0 \Rightarrow s_6= \beta_5s_16\)

• \((xy)c+(yc)x+(cx)y = 0\) \Rightarrow \(zc+ (\alpha_5u+s_6)x-ry=0 \Rightarrow \beta_8u+s_8+ \alpha_5ux+s_6x-ry=0\)

\(\Rightarrow \beta_8u+s_8+ \alpha_5(-s_16)-(\beta_8u-s_7)=0 \Rightarrow s_8= \alpha_5s_16-s_7\)

• \((xy)r+(yr)x+(rx)y = 0\) \Rightarrow \(zr+(-\beta_8u+s_7)x+(-\alpha_4)y=0 \Rightarrow s_9-\beta_8ux+ s_7x-ty=0\)

\(\Rightarrow s_9-\beta_8(-s_16)-\alpha_4(-s_{12})=0 \Rightarrow s_9=\beta_8s_16+s_{12} \Rightarrow s_9=\beta_8s_16-s_{12}\)

• \((xy)t+(yt)x+(tx)y = 0\) \Rightarrow \(zt+(\alpha_4s_13)x+(-u)y=0 \Rightarrow s_{13}=(-s_{17})=0\)

\(\Rightarrow s_{13}=s_{17}\)

• \((xy)u+(yu)x+(ux)y = 0\) \Rightarrow \(zu+s_{17}x-s_{16}y=0 \Rightarrow s_{18}=0\)

• \((xz)c+(zc)x+(cx)z = 0\) \Rightarrow \(cc+( \beta_8u+s_8)x-rz=0 \Rightarrow \beta_8ux+s_8x+s_9=0\)

\(\Rightarrow \beta_8(-s_{16})+s_9=0 \Rightarrow s_9=\beta_8s_16\)

• \((xz)r+(zr)x+(rx)z = 0\) \Rightarrow \(cr+s_9x+(-t)z=0 \Rightarrow s_{10}=\alpha_4s_{13}\)

• \((xz)t+(zt)x+(tx)z = 0\) \Rightarrow \(ct+ s_{13}x+(-u)z=0 \Rightarrow s_{14}=u+u=0\)

\(\Rightarrow s_{14}=(-s_{18})=0 \Rightarrow s_{14}=s_{18} \Rightarrow s_{14}=0\)

• \((xz)u+(zu)x+(ux)z = 0\) \Rightarrow \(cu+s_{18}x-s_{16}z=0 \Rightarrow s_{19}=0\)
• \((xc)r+(cr)x+(rx)c = 0\)  \(\Rightarrow\) \(rr+s_{10}x+(-t)c=0\)  \(\Rightarrow\) \(-tc=0\)  \(\Rightarrow\) \(s_{14}=0\)

• \((xc)t+(ct)x+(tx)c = 0\)  \(\Rightarrow\) \(rt+(tx)c=0\)  \(\Rightarrow\) \(s_{15}=-s_{19}\)  \(\Rightarrow\) \(s_{15}=0\)

• \((xc)u+(cu)x+(ux)c = 0\)  \(\Rightarrow\) \(ru + s_{19}x- s_{16}c=0\)  \(\Rightarrow\) \(s_{20}=0\)

• \((xr)t+(rt)x+(tx)r = 0\)  \(\Rightarrow\) \(-ur=0\)  \(\Rightarrow\) \(s_{20}=0\)

• \((xr)u+(ru)x+(ux)r = 0\)  \(\Rightarrow\) \(tu+s_{20}x- s_{16}r=0\)  \(\Rightarrow\) \(s_{21}=0\)

• \((xt)u+(tu)x+(ux)t = 0\)  \(\Rightarrow\) \(s_{21}x- s_{16}t=0\)  \(\Rightarrow\) \(0=0\)

• \((yz)r+(zr)y+(ry)z = 0\)  \(\Rightarrow\) \((\alpha_5t+\beta_5u+s_5)r+s_{9}y+ (\beta_8u-s_7)z=0\)

\(\Rightarrow\) \((\alpha_5t+\beta_5u+s_5r)+\beta_8uz\)  \(\Rightarrow\) \(-\alpha_5s_{15}-\beta_5s_{20}- \beta_8 s_{18}=0\)  \(\Rightarrow\) \(\alpha_5s_{15}= \beta_8 s_{18}=0\)

• \((yz)c+(zc)y+(cy)z = 0\)  \(\Rightarrow\) \((\alpha_5t+\beta_5u)c+ \beta_8uy-\alpha_5uz=0\)

\(\Rightarrow\) \(-\alpha_5s_{14}-\beta_8s_{17}-\alpha_5s_{18}=0\)  \(\Rightarrow\) \(\beta_8 s_{17}=0\)

• \((yz)t+(zt)y+(ty)z = 0\)  \(\Rightarrow\) \((\alpha_5t+\beta_5u+s_5)t+s_{13}y+(-s_{12})z=0\)  \(\Rightarrow\) \(\beta_5ut =0\)

\(\Rightarrow\) \(-\beta_5s_{21}=0\)  \(\Rightarrow\) \(\beta_5s_{21}= 0\)

• \((yz)u+(zu)y+(uy)z = 0\)  \(\Rightarrow\) \((\alpha_5t+\beta_5u+s_5)u+s_{18}y-s_{17}z=0\)  \(\Rightarrow\) \(\alpha_5tu+ \beta_5uu=0\)  \(\Rightarrow\) \(\alpha_5s_{21}=0\)

• \((yc)r+(cr)y+(ry)c = 0\)  \(\Rightarrow\) \((\alpha_5u+s_6)r+s_{10}y+(\beta_8u-s_7)c=0\)  \(\Rightarrow\) \(\alpha_5ur+ s_{6}r+s_{10}y+\beta_8uc=0\)

\(\Rightarrow\) \(\alpha_5(-s_{20})+ \beta_8(s_{19})=0\)  \(\Rightarrow\) \(-\beta_8s_{19}=\alpha_5s_{20}\)
• \((yc)t+(ct)y+(ty)c = 0\) \(\Rightarrow (\alpha_5 u + s_6)t + s_{14}y + s_{12}c = 0\) \(\Rightarrow \alpha_5 u t = 0\) \(\Rightarrow \alpha_5 s_{21} = 0\)

• \((yc)u+(cu)y+(uy)c = 0\) \(\Rightarrow (\alpha_5 u + s_6)u + s_{19}y + s_{17}c = 0\) \(\Rightarrow \alpha_5 uu + s_6 u = 0\) \(\Rightarrow 0 = 0\)

• \((yr)t+(rt)y+(ty)r = 0\) \(\Rightarrow (-\beta_8 u + s_7)t + s_{15}y + (-s_{12})r = 0\) \(\Rightarrow -\beta_8 u t = 0\)

\(\Rightarrow \beta_8 s_{21} = 0\) \(\Rightarrow 0 = 0\)

• \((yr)u+(ru)y+(uy)r = 0\) \(\Rightarrow (-\beta_8 u + s_7)u + s_{20}y - s_{17}r = 0\) \(\Rightarrow -\beta_8 uu = 0\) \(\Rightarrow 0 = 0\)

• \((yt)u+(tu)y+(uy)t = 0\) \(\Rightarrow (s_{12})u + s_{12}y - s_{17}t = 0\) \(\Rightarrow 0 = 0\)

• \((zc)r+(cr)z+(rz)c = 0\) \(\Rightarrow (\beta_8 u + s_8)r + s_{10}z - s_9c = 0\) \(\Rightarrow \beta_8 ur = 0\) \(\Rightarrow 0 = 0\)

• \((zc)t+(ct)z+(tz)c = 0\) \(\Rightarrow (\beta_8 u + s_8)t + s_{14}z + (-s_{13})c = 0\) \(\Rightarrow \beta_8 ut = 0\) \(\Rightarrow 0 = 0\)

• \((zc)u+(cu)z+(uz)c = 0\) \(\Rightarrow (\beta_8 u + s_8)u + s_{19}z - s_{18}c = 0\) \(\Rightarrow 0 = 0\)

• \((zr)t+(rt)z+(tz)r = 0\) \(\Rightarrow s_9z + s_{15}z - s_{13}r = 0\) \(\Rightarrow 0 = 0\)

• \((zr)u+(ru)z+(uz)r = 0\) \(\Rightarrow s_9u + s_{20}z - s_{18}r = 0\) \(\Rightarrow 0 = 0\)

• \((zt)u+(tu)z+(uz)t = 0\) \(\Rightarrow s_{13}u + s_{21}z - s_{18}t = 0\) \(\Rightarrow 0 = 0\)

• \((cr)t+(rt)c+(tc)r = 0\) \(\Rightarrow s_{10}t + s_{15}c - s_{14}r = 0\) \(\Rightarrow 0 = 0\)

• \((cr)u+(ru)c+(uc)r = 0\) \(\Rightarrow s_{10}u + s_{29}c - s_{19}r = 0\) \(\Rightarrow 0 = 0\)

• \((rt)u+(tu)r+(ur)t = 0\) \(\Rightarrow s_{13}u + s_{21}r - s_{20}t = 0\) \(\Rightarrow 0 = 0\)
We now summarize the equations found below:

(1) \( s_6 = \beta_5 s_{16} \)
(2) \( s_8 = \alpha_5 s_{16} - s_7 \)
(3) \( s_9 = -\beta_8 s_{16} - s_{12} \)
(4) \( s_{13} = -s_{17} \)
(5) \( s_{18} = 0 \)
(6) \( s_9 = \beta_8 s_{16} \)
(7) \( s_{10} = -s_{13} \)
(8) \( s_{19} = 0 \)
(9) \( s_{15} = -s_{19} = 0 \)
(10) \( s_{20} = 0 \)
(12) \( s_{21} = 0 \)
(13) \( \beta_8 s_{17} = 0 \)

From the preceding work and equation (13) above we have two cases. \( \beta_8 s_{17} = 0 \) implies

Either Case (A) \( s_{17} = 0 \) or Case (B) \( s_{17} \neq 0 \) and \( \beta_8 = 0 \)

In Case (A), if \( s_{17} = 0 \) then \( s_{21} = s_{20} = s_{19} = s_{18} = s_{15} = s_{14} = s_{13} = s_{10} = 0 \) and \( s_6 = \beta_5 s_{16} \), \( s_8 = \alpha_5 s_{16} - s_7 \)
(12) \( s_{21} = 0 \)
(12) \( \beta_8 s_{17} = 0 \)

\( s_9 = \beta_8 s_{16} \), \( s_{12} = -\beta_8 s_{16} - s_9 \) which implies \( s_6, s_8, s_9, s_{12} \in (s_5, s_7, s_{16}) \) thus \( \text{dim}(L) = 3 \)

which implies \( t(L) = \frac{1}{2} n(n-1) - \text{dim}(L) = \frac{1}{2}(7)(6) - 3 = 18 \), we call this algebra

\( L = L(17, 7, 1, 18) \).

In Case (B), \( s_{17} \neq 0 \) and \( \beta_8 = 0 \) then \( s_{21} = s_{20} = s_{19} = s_{18} = s_{15} = s_{14} = 0 \) and \( s_6 = \beta_5 s_{16} \), \( s_8 = \alpha_5 s_{16} - s_7 \)
(12) \( s_{21} = 0 \)
(12) \( \beta_8 s_{17} = 0 \)

\( s_9 = \beta_8 s_{16} \), \( s_{12} = -s_9 \), \( s_{10} = s_{13} = s_{17} \) which implies \( s_6, s_8, s_9, s_{12}, s_{10}, s_{13} \in (s_5, s_7, s_{16}, s_{17}) \) thus

\( \text{dim}(L) = 4 \) which implies \( t(L) = \frac{1}{2} n(n-1) - \text{dim}(L) = \frac{1}{2}(7)(6) - 4 = 17 \), we call this algebra

\( L = L(17, 7, 1, 17) \).
**CASE 2.** Let $H=L'(11, 6, 1, 12)$, this $L$ can be described generally by the basis

\[ \{x, y, z, c, r, t, u\} \] and multiplication

\[ [x,y]=z, \ [x,z]=c, \ [x,r]=t, \ [y,z]=r+\alpha_5 t+\beta_5 u, \ [y,c]=t+\beta_6 u, \ [y,r]=\beta_7 u, \]

\[ [z,c]=\beta_8 u , \ [z,r]=\beta_9 u, \ [c,r ]=\beta_{10} u, \ [x, t]=\beta_{11} u, \ [y, t]=\beta_{12} u, \ [z, t]=\beta_{13} u, \ [c, t]=\beta_{14} u, \]

\[ [r, t]=\beta_{15} u, \text{ and } u\in Z(L). \] Now we will use the Jacobi Identity on all triples.

\[ \bullet \ (xy)z+(yz)x+(zx)y = 0 \quad \Rightarrow \quad zz+ (r+\alpha_5 t+\beta_5 u)x +(-c)y=0 \quad \Rightarrow r x+\alpha_5 t x+\beta_5 u x-(-t-\beta_6 u)=0 \]

\[ \Rightarrow -t-\alpha_5 \beta_{11} t+\beta_6 u=0 \quad \Rightarrow \beta_6 =\alpha_5 \beta_{11} \]

\[ \bullet \ (xy)c+(yc)x+(cx)y = 0 \quad \Rightarrow \quad zc+(t+\beta_6 u)x-ry=0 \quad \Rightarrow \beta_8 u t x+\beta_6 u x r y=0 \]

\[ \Rightarrow \beta_8 u-\beta_{11} u+\beta_7 u=0 \quad \Rightarrow \beta_7 =\beta_{11}-\beta_8 \]

\[ \bullet \ (xy)r+(yr)x+(rx)y = 0 \quad \Rightarrow \quad zr+\beta_7 u x+(-t) y =0 \quad \Rightarrow \beta_9 u t y =0 \quad \Rightarrow \beta_9 =\beta_{12} \]

\[ \bullet \ (xy)t+(yt)x+(tx)y = 0 \quad \Rightarrow \quad zt + \beta_{12} u x-\beta_{11} u y=0 \quad \Rightarrow \beta_{13} u=0 \quad \Rightarrow \beta_{13}=0 \]

\[ \bullet \ (xy)u+(yu)x+(ux)y = 0 \quad \Rightarrow \quad 0=0 \]

\[ \bullet \ (xz)c+(zc)x+(cx)z = 0 \quad \Rightarrow \quad cc+(\beta_8 u)x-r z=0 \quad \Rightarrow \beta_9 =0 \]

\[ \bullet \ (xz)r+(zr)x+(rx)z = 0 \quad \Rightarrow cr+ \beta_9 u x+(-t-\beta_4 u) z=0 \quad \Rightarrow \beta_{10} u t z =0 \quad \Rightarrow \beta_{10}=\beta_{13} \]

\[ \bullet \ (xz)t+(zt)x+(tx)z = 0 \quad \Rightarrow ct+ \beta_{13} u x- \beta_{11} u z=0 \quad \Rightarrow \beta_{14} u=0 \quad \Rightarrow \beta_{14}=0 \]

\[ \bullet \ (xz)u+(zu)x+(ux)z = 0 \quad \Rightarrow \quad 0=0 \]

\[ \bullet \ (xc)r+(cr)x+(rx)c = 0 \quad \Rightarrow \quad rr+ \beta_{10} u x+(-t) c=0 \quad \Rightarrow \beta_{14}=0 \]
\begin{align*}
\bullet (xc)t+(ct)x+(tx)c &= 0 \quad \Rightarrow rt + \beta_{14}ux + (-\beta_{11}u)c = 0 \quad \Rightarrow \beta_{15}u = 0 \quad \Rightarrow \beta_{15} = 0 \\
\bullet (xc)u+(cu)x+(ux)c &= 0 \quad \Rightarrow 0 = 0 \\
\bullet (xr)t+(rt)x+(tx)r &= 0 \quad \Rightarrow (t)t + \beta_{15}ux + (-\beta_{11}u)r = 0 \quad \Rightarrow 0 = 0 \\
\bullet (xr)u+(ru)x+(ux)r &= 0 \quad \Rightarrow 0 = 0 \\
\bullet (xt)u+(tu)x+(ux)t &= 0 \quad \Rightarrow 0 = 0 \\
\bullet (yz)c+(zc)y+(cy)z &= 0 \quad \Rightarrow (r+ \alpha_{5}t+ \beta_{3}u)c + (\beta_{9}u)y + (-\beta_{6}u)z = 0 \\
\quad \Rightarrow rc + \alpha_{5}tc + \beta_{3}uc + \beta_{9}uy - tz - \beta_{6}uz = 0 \quad \Rightarrow -\beta_{10}u - \alpha_{5}\beta_{14}u + \beta_{13}u = 0 \quad \Rightarrow \beta_{13} = \beta_{10} + \alpha_{5}\beta_{14} \\
\bullet (yz)r+(zr)y+(ry)z &= 0 \quad \Rightarrow (r+ \alpha_{5}t+ \beta_{3}u)r + \beta_{9}uy + (\beta_{7}u)z = 0 \quad \Rightarrow rr + \alpha_{5}tr + \beta_{3}ur = 0 \\
\quad \Rightarrow -\alpha_{5}\beta_{15}u = 0 \quad \Rightarrow \alpha_{5}\beta_{15} = 0 \\
\bullet (yz)t+(zt)y+(ty)z &= 0 \quad \Rightarrow (r+ \alpha_{5}t+ \beta_{3}u)t + \beta_{13}uy - \beta_{12}uz = 0 \quad \Rightarrow rt + \alpha_{5}tt + \beta_{3}ut = 0 \\
\quad \Rightarrow \beta_{15}u = 0 \quad \Rightarrow \beta_{15} = 0 \\
\bullet (yz)u+(zu)y+(uy)z &= 0 \quad \Rightarrow 0 = 0 \\
\bullet (yc)r+(cr)y+(ry)c &= 0 \quad \Rightarrow (t+\beta_{6}u)r + \beta_{10}uy + (-\beta_{7}u)c = 0 \quad \Rightarrow tr + \beta_{6}ur = 0 \\
\quad \Rightarrow \beta_{15} = 0 \\
\bullet (yc)t+(ct)y+(ty)c &= 0 \quad \Rightarrow (t+\beta_{6}u)t + \beta_{14}uy - \beta_{12}uc = 0 \quad \Rightarrow 0 = 0
\end{align*}
\[
\begin{align*}
&\text{• (yc)}u+(\text{cu})y+(\text{uy})c = 0 \quad \Rightarrow 0=0 \\
&\text{• (yr)}t+(\text{rt})y+(\text{ty})r = 0 \quad \Rightarrow \beta_{7}ut+\beta_{15}uy-\beta_{12}ur=0 \quad \Rightarrow 0=0 \\
&\text{• (yr)}u+(\text{ru})y+(\text{uy})r = 0 \quad \Rightarrow 0=0 \\
&\text{• (yt)}u+(\text{tu})y+(\text{uy})t = 0 \quad \Rightarrow 0=0 \\
&\text{• (zc)}r+(\text{cr})z+(\text{rz})c = 0 \quad \Rightarrow (\beta_{8}u)r+\beta_{10}uz-\beta_{9}uc=0 \quad \Rightarrow 0=0 \\
&\text{• (zc)}t+(\text{ct})z+(\text{tz})c = 0 \quad \Rightarrow (\beta_{8}u)t+\beta_{14}uz-\beta_{13}uc=0 \quad \Rightarrow 0=0 \\
&\text{• (zc)}u+(\text{cu})z+(\text{uz})c = 0 \quad \Rightarrow 0=0 \\
&\text{• (zr)}t+(\text{rt})z+(\text{tz})r = 0 \quad \Rightarrow \beta_{9}ut+\beta_{15}uz-\beta_{13}ur=0 \quad \Rightarrow 0=0 \\
&\text{• (zr)}u+(\text{ru})z+(\text{uz})r = 0 \quad \Rightarrow 0=0 \\
&\text{• (zt)}u+(\text{tu})z+(\text{uz})t = 0 \quad \Rightarrow 0=0 \\
&\text{• (cr)}t+(\text{rt})c+(\text{tc})r = 0 \quad \Rightarrow \beta_{10}ut+\beta_{15}uc-\beta_{14}ur=0 \quad \Rightarrow 0=0 \\
&\text{• (cr)}u+(\text{ru})c+(\text{uc})r = 0 \quad \Rightarrow 0=0 \\
&\text{• (rt)}u+(\text{tu})r+(\text{ur})t = 0 \quad \Rightarrow 0=0
\end{align*}
\]
We will now summarize the equations found below:

(1) $\beta_6 = \alpha_5 \beta_{11}$
(2) $\beta_7 = \beta_{11} - \beta_8$
(3) $\beta_9 = -\beta_{12}$
(4) $\beta_{13} = 0$
(5) $\beta_9 = 0$

(6) $\beta_{10} = -\beta_{13}$
(7) $\beta_{14} = 0$
(8) $\beta_{13} = \alpha_5 \beta_{14}$
(9) $\beta_{15} = 0$
(10) $\alpha_5 \beta_{15} = 0$

From the preceding work we obtain the following equations.

(i) $\beta_9 = \beta_{10} = \beta_{12} = \beta_{14} = \beta_{15} = 0$
(ii) $\beta_8 = 1 - \beta_7$
(iii) $\alpha_5 = \beta_6$

Relabelling yields the non zero multiplication

\[ [x,y] = z, \ [x,z] = c, \ [x,c] = r, \ [x,r] = t, \ [y,z] = r + \alpha_5 t + \beta_5 u, \ [y,c] = t + \beta_6 u, \ [y,r] = \beta_7 u, \]
\[ [z,c] = \beta_8 u, \ [z,r] = 0, \ [c,r] = 0, \ [x,t] = \beta_{11} u, \ [y,t] = 0, \ [z,t] = 0, \ [c,t] = 0, \]
\[ [r,t] = 0, \text{ and } u \in Z(L). \text{ Hence for } t \notin Z(L), \beta_{11} \neq 0 \text{ and we can use } xt = u \text{ for multiplier}. \]

To compute the multiplier start with

\[ [x,y] = z, \ [x,z] = c, \ [x,c] = r, \ [x,r] = t, \ [y,z] = r + \alpha_5 t + \beta_5 u + s_5, \ [y,c] = t + \beta_6 u + s_6, \]
\[ [y,r] = \beta_7 u + s_7, \ [z,c] = \beta_8 u + s_8, \ [z,r] = s_9, \ [c,r] = s_{10}, \ [x,t] = u, \ [y,t] = s_{12}, \]
\[ [z,t] = s_{13}, \ [c,t] = s_{14}, \ [r,t] = s_{15}, \ [x,u] = s_{16}, \ [y,u] = s_{17}, \ [z,u] = s_{18}, \ [c,u] = s_{19}, \ [r,u] = s_{20} \]
\[ [t,u] = s_{21}, \text{ where } s_1, \ldots, s_{21} \text{ generate the multiplier}. \]

Now we will use the Jacobi identity on all triples.

\[ (xy)z + (yz)x + (zx)y = 0 \quad \Rightarrow \quad zz + (r + \alpha_5 t + \beta_6 u + s_5)x + (-c)y = 0 \quad \Rightarrow \quad rtx + \beta_5 ux + s_5 x - cy = 0 \]
\[ \Rightarrow \quad rtx + \alpha_5 tx + \beta_5 ux - (t - \beta_6 u - s_6) = 0 \quad \Rightarrow \quad -t + \alpha_5 (-u) + t + \beta_6 u + s_6 = 0 \]
\[ \Rightarrow \quad s_6 = \beta_5 s_{16} \]
\( (xy)c + (yc)x + (cx)y = 0 \quad \Rightarrow \quad zc + (t + \beta_6 u + s_6)x - ry = 0 \)

\[ \Rightarrow \beta_8 u + s_8 - u - \beta_6 s_{16} + \beta_7 u + s_7 = 0 \quad \Rightarrow \quad s_8 + \beta_6 (-s_{16}) + s_7 = 0 \]

\[ \Rightarrow s_8 = \beta_6 s_{16} - s_7 \]

\( (xy)r + (yr)x + (rx)y = 0 \quad \Rightarrow \quad zr + (\beta_7 u + s_7)x - ty = 0 \)

\[ \Rightarrow s_9 + \beta_7 (-s_{16}) - s_{12} = 0 \quad \Rightarrow \quad s_{12} = s_9 - \beta_7 s_{16} \]

\( (xy)t + (yt)x + (tx)y = 0 \quad \Rightarrow \quad zt + s_{12}x + (-u)y = 0 \quad \Rightarrow \quad s_{13} + s_{17} = 0 \quad \Rightarrow \quad s_{13} = -s_{17} \)

\( (xy)u + (yu)x + (ux)y = 0 \quad \Rightarrow \quad zu + s_{17}x - s_{16}y = 0 \quad \Rightarrow s_{18} = 0 \)

\( (xz)c + (zc)x + (cx)z = 0 \quad \Rightarrow \quad cc + (\beta_8 u + s_8)x - rz = 0 \quad \Rightarrow \beta_8 ux + s_8 x + s_9 = 0 \)

\[ \Rightarrow \beta_8 (-s_{16}) + s_9 = 0 \quad \Rightarrow \quad s_9 = \beta_8 s_{16} \]

\( (xz)r + (zr)x + (rx)z = 0 \quad \Rightarrow \quad cr + s_9 x - tz = 0 \quad \Rightarrow s_{10} + s_{13} = 0 \quad \Rightarrow s_{10} = -s_{13} \)

\( (xz)t + (zt)x + (tx)z = 0 \quad \Rightarrow \quad ct + s_{13}x - uz = 0 \quad \Rightarrow s_{14} + s_{18} = 0 \quad \Rightarrow s_{14} = -s_{18} \)

\( (xz)u + (zu)x + (ux)z = 0 \quad \Rightarrow \quad cu + s_{18}x - s_{16}z = 0 \quad \Rightarrow s_{19} = 0 \)

\( (xc)r + (cr)x + (rx)c = 0 \quad \Rightarrow -tc = 0 \quad \Rightarrow s_{14} = 0 \)

\( (xc)t + (ct)x + (tx)c = 0 \quad \Rightarrow rt + s_{14}x - uc = 0 \quad \Rightarrow s_{15} + s_{19} = 0 \quad \Rightarrow s_{15} = 0 \)

\( (xc)u + (cu)x + (ux)c = 0 \quad \Rightarrow ru + s_{19}x - s_{16}c = 0 \quad \Rightarrow s_{20} = 0 \)
\[
\begin{align*}
(\text{xt})u+(\text{tu})x+(\text{ux})t &= 0 & \Rightarrow 0 = 0 \\
(\text{yz})r+(\text{zr})y+(\text{ry})z &= 0 & \Rightarrow (r+\alpha_5t+\beta_5u+s_5)r+s_9y+(-\beta_7u-s_7)z=0 \\
(\text{yz})c+(\text{zc})y+(\text{cy})z &= 0 & \Rightarrow (r+\alpha_5t+\beta_5u+s_5)c+(\beta_8u+s_8)y+(-t-\beta_6u-s_6)z=0 \\
(\text{yz})t+(\text{zt})y+(\text{ty})z &= 0 & \Rightarrow (r+\alpha_5t+\beta_5u+s_5)t+s_{13}y+(-s_{12})z=0 \\
(\text{yz})u+(\text{zu})y+(\text{uy})z &= 0 & \Rightarrow (r+\alpha_5t+\beta_5u+s_5)u+s_{18}y-s_{17}z=0 & \Rightarrow ru+\alpha_5tu+\beta_5uu=0 \\
(\text{yc})r+(\text{cr})y+(\text{ry})c &= 0 & \Rightarrow (t+\beta_6u+s_6)r+s_{10}y+(-\beta_7u-s_7)c=0 \\
\end{align*}
\]
$\Rightarrow -s_{15} + \beta_6(-s_{20}) + \beta_7(s_{19}) = 0 \Rightarrow s_{19} = 0$

$\bullet (yc)t + (ct)y + (ty)c = 0 \Rightarrow (t + \beta_6u + s_6)t + s_{14}y + (-s_{12})c = 0$

$\Rightarrow tt + \beta_6ut + s_6t - s_{12}c = 0 \Rightarrow \beta_6(-s_{21}) = 0 \Rightarrow \beta_6s_{21} = 0 \Rightarrow 0 = 0$

$\bullet (yc)u + (cu)y + (uy)c = 0 \Rightarrow (t + \beta_6u + s_6)u + s_{19}y - s_{17}c = 0 \Rightarrow tu = 0 \Rightarrow s_{21} = 0$

$\bullet (yr)t + (rt)y + (ty)r = 0 \Rightarrow (\beta_7u + s_7)t + s_{15}y + (-s_{12})r = 0 \Rightarrow 0 = 0$

$\Rightarrow \beta_7ut + s_7t - \beta_12ur = 0 \Rightarrow \beta_7(-s_{21}) - \beta_12(-s_{20}) = 0 \Rightarrow \beta_12s_{20} = \beta_7 s_{21}$

$\bullet (yr)u + (ru)y + (uy)r = 0 \Rightarrow (\beta_7u + s_7)u + s_{20}y - s_{17}r = 0 \Rightarrow \beta_7uu = 0 \Rightarrow 0 = 0$

$\bullet (yt)u + (tu)y + (uy)t = 0 \Rightarrow (s_{12})u + s_{21}y - s_{17}t = 0 \Rightarrow 0 = 0$

$\bullet (zc)r + (cr)z + (rz)c = 0 \Rightarrow (\beta_8u + s_8)r + s_{10}z - s_{9}c = 0 \Rightarrow \beta_8ur = 0 \Rightarrow 0 = 0$

$\bullet (zc)t + (ct)z + (tz)c = 0 \Rightarrow (\beta_8u + s_8)t + s_{14}z - s_{13}c = 0 \Rightarrow \beta_8ut = 0 \Rightarrow 0 = 0$

$\bullet (zc)u + (cu)z + (uz)c = 0 \Rightarrow (\beta_8u + s_8)u + s_{19}z - s_{18}c = 0 \Rightarrow 0 = 0$

$\bullet (sr)t + (rt)z + (tz)r = 0 \Rightarrow s_9z + s_{15}z - s_{13}r = 0 \Rightarrow 0 = 0$

$\bullet (sr)u + (ru)z + (uz)r = 0 \Rightarrow s_9u + s_{20}z - s_{18}r = 0 \Rightarrow 0 = 0$

$\bullet (zt)u + (tu)z + (uz)t = 0 \Rightarrow s_{13}u + s_{21}z - s_{18}t = 0 \Rightarrow 0 = 0$

$\bullet (cr)t + (rt)c + (tc)r = 0 \Rightarrow s_{10}t + s_{15}c - s_{14}r = 0 \Rightarrow 0 = 0$
\[ (cr)u+(ru)c+(uc)r = 0 \Rightarrow s_{10}u + s_{20}c - s_{19}r = 0 \Rightarrow 0 = 0 \]

\[ (rt)u+(tu)r+(ur)t = 0 \Rightarrow s_{15}u + s_{21}r - s_{20}t = 0 \Rightarrow 0 = 0 \]

We now summarize the equations found below:

1. \( s_6 = -\beta_5 s_{16} \)
2. \( s_8 = \beta_6 s_{16} - s_7 \)
3. \( s_{12} = s_9 - \beta_7 s_{16} \)
4. \( s_{13} = -s_{17} \)
5. \( s_9 = \beta_8 s_{16} \)
6. \( s_{13} = -s_{10} \)
7. \( s_{10} = -\beta_8 s_{17} + s_{13} \)
8. \( s_{19} = 0 \)
9. \( s_{21} = s_{20} = s_{19} = s_{18} = s_{15} = s_{14} = 0 \)

From the preceding work we have \( \text{dim}M(L) = (s_{17}, s_{16}, s_7, s_5) = 4 \).

Thus \( t(L) = t(L) = \frac{1}{2}n(n-1) - \text{dim}M(L) = \frac{1}{2}(7)(6) - 3 = 17 \), we call this algebra \( L = L^{(17, 7, 1, 17)} \).

And we summarize our result in the following theorem.

**Theorem 20** Let \( L \) be filiform with \( t(L) = 17 \) then \( L = L^{(17, 7, 1, 17)} \) or \( L = L^{\prime}(17, 7, 1, 17) \).
18. $t(L)=18$

By Theorem 4, we know that $c^2 + c \leq 2t$ where $c = \dim L^2$ and $t = t(L)$. If $t = 18$ then 
$c^2 + c \leq 36$ which implies that $c \leq 5$ therefore $c = 0, 1, 2, 3, 4$ or $5$ but by proposition 0
(i.e. $t \leq \frac{1}{2}n(n-1)$) and since $n = c+2$ for filiform we get $18 \leq \frac{1}{2}(c + 2)(c + 1)$ and by solving this inequality we know that $c = 5$ is the only case that satisfies the inequality. Thus if $c = \dim L^2 = 5$, then $\dim H^2 = 4$ and $t(H) \leq 13$ (proposition 3). In the $t(L)=17$ case we have already computed this algebra and we list it in the following theorem.

**Theorem 21** Let $L$ be filiform with $t(L)=18$, then $L=L(17, 7, 1, 18)$. 
19. $t(L)=19$

By Theorem 4, we know that $c^2 + c \leq 2t$ where $c = \text{dim}L^2$ and $t= t(L)$. If $t=19$ then $c^2 + c \leq 38$ which implies that $c \leq 5$ therefore $c = 0, 1, 2, 3, 4$ or $5$ but by proposition 0 (i.e. $t \leq \frac{1}{2}n(n-1)$) and since $n= c+2$ for filiform we get $19 \leq \frac{1}{2}(c + 2)(c + 1)$ and by solving this inequality we know that $c = 5$ is the only case that satisfies the inequality. Thus if $c = \text{dim}L^2 = 5$, then $\text{dim}H^2 = 4$ and $t(H) \leq 14$(proposition 3), but again from Table 22.1, we know that there are no filiform Lie algebras possible under these conditions, thus we have the following Theorem.

**Theorem 22.** There are no filiform Lie algebras with $t(L) = 19$. 
20. $t(L)=20$

By Theorem 4, we know that $c^2 + c \leq 2t$ where $c = \dim L^2$ and $t= t(L)$. If $t = 20$ then $c^2 + c \leq 40$ which implies that $c \leq 5$ therefore $c = 0, 1, 2, 3, 4$ or $5$ but by proposition 0 (i.e. $t \leq \frac{1}{2}n(n-1)$) and since $n= c+2$ for filiform we get $20 \leq \frac{1}{2}(c + 2)(c + 1)$ and by solving this inequality we know that $c = 5$ is the only case that satisfies the inequality. Thus if $c = \dim L^2 = 5$, then $\dim H^2 = 4$ and $t(H) \leq 15$ (proposition 3), but from Table 22.1, we know that there are no filiform Lie algebras possible under these conditions, thus we have the following Theorem.

**Theorem 23**  There are no filiform Lie algebras with $t(L) = 20$. 
21. \( t(L)=21 \)

By Theorem 4, we know that \( c^2 + c \leq 2t \) where \( c = \dim L^2 \) and \( t = t(L) \). If \( t = 21 \) then \( c^2 + c \leq 42 \) which implies that \( c \leq 6 \) therefore \( c = 0, 1, 2, 3, 4, 5 \) or \( 6 \) but by proposition 0 (i.e. \( t \leq \frac{1}{2}n(n-1) \)) and since \( n = c + 2 \) for filiform we get \( 21 \leq \frac{1}{2}(c + 2)(c + 1) \) and by solving this inequality we know that \( c = 5 \) and \( c = 6 \) are the two cases that satisfy the inequality. Thus if \( c = \dim L^2 = 5 \), then \( \dim H^2 = 4 \) and \( t(H) \leq 16 \) (proposition 3), but from Table 22.1, we know that there are no filiform Lie algebras possible under these conditions. Also if \( c = \dim L^2 = 6 \), then \( \dim H^2 = 5 \) and \( t(H) \leq 15 \) (proposition 3), but again from Table 22.1, we know that there are no filiform Lie algebras possible under these conditions, thus we have the following Theorem.

**Theorem 24** There are no filiform Lie algebras with \( t(L) = 21 \).
## 22. SUMMARY

The following table summarizes the findings of this paper:

### TABLE 22.1 The Filiform Lie Algebras With $t(L)=3$ Through $t(L)=21$.  

<table>
<thead>
<tr>
<th>$t(L)$</th>
<th>dimL</th>
<th>Basis</th>
<th>Non Zero Multiplication</th>
<th>Filiform Lie Algebra</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>4</td>
<td>${x,y,z,r}$</td>
<td>$[x,y]=z$, $[x,z]=r$</td>
<td>$L=L(3,4,1,4)$</td>
</tr>
<tr>
<td>7</td>
<td>5</td>
<td>${x,y,z,c,r}$</td>
<td>$[x,y]=z$, $[x,z]=c$, $[x,c]=r$ OR $[x,y]=z$, $[x,z]=c$, $[y,z]=r$</td>
<td>$L=L(7,5,1,7)$ OR $L=L'(7,5,1,7)$</td>
</tr>
<tr>
<td>11</td>
<td>6</td>
<td>${x,y,z,c,r,t}$</td>
<td>$[x,y]=z$, $[x,z]=c$, $[x,c]=r$, $[x,r]=\alpha_4 t$, $[y,z]=\alpha_5 t$, $[y,c]=0$</td>
<td>$L=L(11,6,1,11)$ OR $L=L'(11,6,1,11)$</td>
</tr>
<tr>
<td>12</td>
<td>6</td>
<td>${x,y,z,c,r,t}$</td>
<td>$[x,y]=z$, $[x,z]=c$, $[x,c]=r$, $[x,r]=t$, $[y,z]=\alpha_5 t$, $[y,c]=0$</td>
<td>$L=L(11,6,1,12)$ OR $L=L'(11,6,1,12)$</td>
</tr>
<tr>
<td>17</td>
<td>7</td>
<td>${x,y,z,c,r,t,u}$</td>
<td>$[x,y]=z$, $[x,z]=c$, $[x,c]=r$, $[x,r]=t+\beta_4 u$, $[y,z]=\alpha_5 t+\beta_4 u$, $[y,c]=\beta_4 u$</td>
<td>$L=L(17,7,1,17)$ $\beta_5=\alpha_5$ OR $L=L'(17,7,1,17)$ $\beta_5=\alpha_5$ and $\beta_7=1-\beta_7$</td>
</tr>
</tbody>
</table>
### TABLE 22.1 (continued)

<table>
<thead>
<tr>
<th>18</th>
<th>7</th>
<th>{x,y,z,c,r,t,u}</th>
<th>({x,y}=z, {x,z}=c, {x,c}=r,)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>({x,r}=t, {y,z}=\alpha t+\beta u, {y,c}=\beta u)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>({y,r}=\beta u, {z,c}=\beta u, {z,r}=0)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>({c,r}=0, {x,t}=0, {y,t}=0)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>({z,t}=0, {c,t}=0, {r,t}=0)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(L=L(17,7,1,18))</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(B_y=-\beta_8)</td>
<td></td>
</tr>
</tbody>
</table>
REFERENCES

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