Abstract

LUKE, NICHOLAS STEPHEN. Modeling Shear Wave Propagation in Biotissue: An Internal Variable Approach to Dissipation. (Under the direction of Professor H. T. Banks).

The ability to reliably detect coronary artery disease based on the acoustic noises produced by a stenosis can provide a simple, non-invasive technique for diagnosis. Current research exploits the shear wave fields in body tissue to detect and analyze coronary stenoses. A mathematical model of this system, utilizing an internal strain variable approximation to the quasi-linear viscoelastic constitutive equation proposed by Fung in [12], was presented in [5]. The methods and ideas outlined in that paper are expanded upon in this work.

As an initial investigation, a homogeneous two-dimensional viscoelastic geometry is considered. Being uniform in $\theta$, this geometry behaves as a one dimensional model, and the results generated from it are compared to the one dimensional results from [5]. Several variations of the model are considered, to allow for different assumptions about the elastic response. A statistical significance test is employed to determine if the extra parameters needed for certain variations of the model are necessary in modeling efforts.

After validating the model with the comparison to previous findings, more complicated geometries are developed. Simulations involving a heterogeneous geometry with a uniform ring running through the original medium, a $\theta$ dependent model which considers a rigid occlusion formed along the inner radius of the geometry, and a model which combines the ring and occlusion are presented. In an attempt to move towards the ultimate goal of detecting the location of a stenosis from the data gathered at the chestwall, an inverse problem methodology is introduced and results from the inverse problem are shown.
Modeling Shear Wave Propagation in Biotissue: An Internal Variable Approach to Dissipation

by

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This work is dedicated to my family and friends, whom have always been there for me.
Biography

Nicholas Luke was born on February 11, 1979, in Watertown, New York, to parents Robert and Patricia Luke. He has two siblings, older brother Jon, and younger brother Russell. As an “Air Force Brat”, he had the opportunity to experience the climate and culture of different regions, both foreign and domestic.

In 1993, his father retired from the Air Force, and the family relocated to Greensboro, NC, where Nicholas graduated from Ben L. Smith High School as Salutatorian of the class of 1997. Having spent much of his life adjusting to new and different environments, the author was in no hurry to leave his surroundings in Greensboro, and decided to attend a local University. He received the North Carolina Teaching Fellows Scholarship, and graduated summa cum laude from North Carolina Agricultural and Technical State University with bachelor’s degrees in Applied Mathematics (with a minor in physics), and Mathematics Education.

He received a fellowship from The David and Lucile Packard Foundation, which would allow him to pursue a Ph.D at his choice of institutions of higher learning. He began his graduate studies in applied mathematics at North Carolina State University in the fall of 2001, and received a Master’s of Science Degree in Applied Mathematics in 2004. With the completion of this dissertation, the author will have earned his Doctor of Philosophy degree from N. C. State.

Beginning in September, 2006, he will be employed as a post-doctoral fellow at the United States Environmental Protection Agency.
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Contents

List of Figures viii
List of Tables xii

1 Introduction 1

2 Model Formulation 4
  2.1 Physical Geometry of the Model 4
  2.2 The Equations of Motion 5
    2.2.1 The Continuity Equation 5
    2.2.2 The Momentum Equation 7
    2.2.3 An Alternative Formulation of the Momentum Equations 13
    2.2.4 The Equations of Motion 15
  2.3 Formulation of the Constitutive Equations 16
    2.3.1 Elastic Response 17
    2.3.2 The Reduced Relaxation Function 20
    2.3.3 The Internal Variable Model Approximation 20
  2.4 A Generic Theoretical Model 21
  2.5 Computational Models 22
    2.5.1 ER1-II SV 23
    2.5.2 ER2-II SV 24
    2.5.3 ER3-II SV 25
    2.5.4 ER1-2ISV 26

3 Calibration of the Model with One Dimensional Results 29
  3.1 Computational Methods 29
    3.1.1 A Sample Grid Refinement Study 32
  3.2 Computational Results 34
  3.3 Statistical Model Comparison 37

4 Simulations Involving More Complex Geometries 42
  4.1 Investigating a Heterogeneous Medium 42
4.1.1 Examining the Effect of Density Within the Model .................. 43
4.1.2 The Importance of Viscoelasticity in Modeling Efforts .............. 48
4.1.3 Improvement of the Heterogeneous Model .......................... 59
4.2 Incorporating a $\theta$ Dependency Into the Model ..................... 70
4.3 Combining the Ringed Geometry Model and the $\theta$ Dependency Model ... 84

5 Inverse Problem Methodology and Results 101
5.1 Inverse Problem Methodology ........................................ 101
5.2 Inverse Problem Results ............................................. 103
  5.2.1 An Initial Investigation of the Inverse Problem ................... 103
  5.2.2 Refining the Mesh for the Inverse Problem ....................... 104
  5.2.3 Increasing the Number of Data Nodes in the Inverse Problem .... 105
  5.2.4 Revisiting the Inverse Problem on a Coarser Mesh ............. 106
5.3 Investigating the Effects of the Material Parameters on the Inverse Problem 107

6 Conclusion 110

Bibliography 112
## List of Figures

1.1 Turbulent blood flow generated by a stenosis ......................................... 2
2.1 Illustration of the physical geometry considered in our modeling efforts. .... 5
2.2 An arbitrary element in a polar coordinate system ................................. 6
2.3 Illustration of the various unit vectors used in our derivation. ............... 8
2.4 Stresses exerted on an arbitrary polar element .................................... 11
3.1 Normalized Acceleration data presented in [5]. .................................... 30
3.2 Positive and negative characteristics interacting with domain boundaries .. 31
3.3 $w_1$ impulse data used as boundary data in [5] .................................. 32
3.4 Comparison of ER1-IISV results to data. ........................................... 35
3.5 Comparison of ER2-IISV results to data. ........................................... 35
3.6 Comparison of ER3-IISV results to data. ........................................... 36
3.7 Comparison of ER1-2ISV results to data. .......................................... 36
3.8 Comparison of 1D acceleration data to 2D approximation of acceleration computed using model ER3-2ISV. ................................................. 40
4.1 Model Geometry with a uniform ring of a different material .................. 43
4.2 Modified input function with zero steady state .................................... 44
4.3 Velocity results for the various heterogeneous geometries at $t=0.00195$ .... 45
4.4 Velocity results for the various heterogeneous geometries at $t=0.00399$ .... 45
4.5 Velocity results for the various heterogeneous geometries at $t=0.00602$ .... 46
4.6 Velocity results for the various heterogeneous geometries at $t=0.00806$ .... 46
4.7 Velocity results for the various heterogeneous geometries at $t=0.01213$ .... 47
4.8 Velocity results for the various heterogeneous geometries at $t=0.02433$ .... 47
4.9 Results for $w_1$ for varying values of the relaxation parameter, $\nu$ at $t=0.0015$ 49
4.10 Results for $w_1$ for varying values of the relaxation parameter, $\nu$ at $t=0.0039$ 50
4.11 Results for $w_1$ for varying values of the relaxation parameter, $\nu$ at $t=0.0121$ 50
4.12 Results for $w_1$ for varying values of the relaxation parameter, $\nu$ at $t=0.0182$ 51
4.13 Results for $w_1$ for varying values of the relaxation parameter, $\nu$ at $t=0.0243$ 51
4.14 Results for $w_1$ for varying values of the relaxation parameter, $\nu$ at $t=0.0304$ 52
4.15 Results for $w_1$ for varying values of the relaxation parameter, $\nu$ at $t=0.0365$ 52
4.16 Results for \( w_1 \) for varying values of the relaxation parameter, \( \nu \) at \( t=0.0487 \)

4.17 Results for \( w_1 \) for varying values of the relaxation parameter, \( \nu \) at \( t=0.0609 \)

4.18 Results for \( v_1 \) for varying values of the relaxation parameter, \( \nu \) at \( t=0.0009 \)

4.19 Results for \( v_1 \) for varying values of the relaxation parameter, \( \nu \) at \( t=0.0032 \)

4.20 Results for \( v_1 \) for varying values of the relaxation parameter, \( \nu \) at \( t=0.0097 \)

4.21 Results for \( v_1 \) for varying values of the relaxation parameter, \( \nu \) at \( t=0.0146 \)

4.22 Results for \( v_1 \) for varying values of the relaxation parameter, \( \nu \) at \( t=0.0194 \)

4.23 Results for \( v_1 \) for varying values of the relaxation parameter, \( \nu \) at \( t=0.0243 \)

4.24 Results for \( v_1 \) for varying values of the relaxation parameter, \( \nu \) at \( t=0.0325 \)

4.25 Results for \( v_1 \) for varying values of the relaxation parameter, \( \nu \) at \( t=0.0504 \)

4.26 Results for \( v_1 \) for varying values of the relaxation parameter, \( \nu \) at \( t=0.0634 \)

4.27 Results for \( v_1 \) for the geometry comprised of different materials, \( t=0.00195 \)

4.28 Results for \( v_1 \) for the geometry comprised of different materials, \( t=0.0039 \)

4.29 Results for \( v_1 \) for the geometry comprised of different materials, \( t=0.00602 \)

4.30 Results for \( v_1 \) for the geometry comprised of different materials, \( t=0.00806 \)

4.31 Results for \( v_1 \) for the geometry comprised of different materials, \( t=0.0121 \)

4.32 Results for \( v_1 \) for the geometry comprised of different materials, \( t=0.0243 \)

4.33 Results for \( v_1 \) for the geometry comprised of different materials, \( t=0.0304 \)

4.34 Results for \( v_1 \) for the geometry comprised of different materials, \( t=0.0365 \)

4.35 Results for \( v_1 \) for the geometry comprised of different materials, \( t=0.0487 \)

4.36 Improved results for \( v_1 \) for the ringed geometry, \( t=0.0039 \)

4.37 Improved results for \( v_1 \) for the ringed geometry, \( t=0.0081 \)

4.38 Improved results for \( v_1 \) for the ringed geometry, \( t=0.0121 \)

4.39 Improved results for \( v_1 \) for the ringed geometry, \( t=0.0182 \)

4.40 Improved results for \( v_1 \) for the ringed geometry, \( t=0.0223 \)

4.41 Improved results for \( v_1 \) for the ringed geometry, \( t=0.0365 \)

4.42 Geometry with occlusion along inner radius

4.43 Results for \( v_1 \) for the geometry with an occluded inner radius, \( t=0.0012 \)

4.44 Results for \( v_1 \) for the geometry with an occluded inner radius, \( t=0.0039 \)

4.45 Results for \( v_1 \) for the geometry with an occluded inner radius, \( t=0.0121 \)

4.46 Results for \( v_1 \) for the geometry with an occluded inner radius, \( t=0.0162 \)

4.47 Results for \( v_1 \) for the geometry with an occluded inner radius, \( t=0.0223 \)

4.48 Results for \( v_1 \) for the geometry with an occluded inner radius, \( t=0.0407 \)

4.49 Results for \( v_2 \) for the geometry with an occluded inner radius, \( t=0.0012 \)

4.50 Results for \( v_2 \) for the geometry with an occluded inner radius, \( t=0.0039 \)

4.51 Results for \( v_2 \) for the geometry with an occluded inner radius, \( t=0.0121 \)

4.52 Results for \( v_2 \) for the geometry with an occluded inner radius, \( t=0.0162 \)

4.53 Results for \( v_2 \) for the geometry with an occluded inner radius, \( t=0.0223 \)

4.54 Results for \( v_2 \) for the geometry with an occluded inner radius, \( t=0.0407 \)

4.55 Smaller scale results for \( v_1 \) for the geometry with an occluded inner radius, \( t=0.0012 \)

4.56 Smaller scale results for \( v_1 \) for the geometry with an occluded inner radius, \( t=0.0039 \)
4.57 Smaller scale results for $v_1$ for the geometry with an occluded inner radius, $t=0.0121$ ................................................................. 79
4.58 Smaller scale results for $v_1$ for the geometry with an occluded inner radius, $t=0.0162$ ................................................................. 79
4.59 Smaller scale results for $v_1$ for the geometry with an occluded inner radius, $t=0.0223$ ................................................................. 80
4.60 Smaller scale results for $v_1$ for the geometry with an occluded inner radius, $t=0.0407$ ................................................................. 80
4.61 Smaller scale results for $v_2$ for the geometry with an occluded inner radius, $t=0.0012$ ................................................................. 81
4.62 Smaller scale results for $v_2$ for the geometry with an occluded inner radius, $t=0.0039$ ................................................................. 81
4.63 Smaller scale results for $v_2$ for the geometry with an occluded inner radius, $t=0.0121$ ................................................................. 82
4.64 Smaller scale results for $v_2$ for the geometry with an occluded inner radius, $t=0.0162$ ................................................................. 82
4.65 Smaller scale results for $v_2$ for the geometry with an occluded inner radius, $t=0.0223$ ................................................................. 83
4.66 Smaller scale results for $v_2$ for the geometry with an occluded inner radius, $t=0.0407$ ................................................................. 83
4.67 Propagation of radial velocity $v_1$ wave through a geometry with a highly dense, purely elastic material and an occlusion on the inner radius, $t=0.0039$ 85
4.68 Propagation of radial velocity $v_1$ wave through a geometry with a highly dense, purely elastic material and an occlusion on the inner radius, $t=0.0081$ 85
4.69 Propagation of radial velocity $v_1$ wave through a geometry with a highly dense, purely elastic material and an occlusion on the inner radius, $t=0.0121$ 86
4.70 Propagation of radial velocity $v_1$ wave through a geometry with a highly dense, purely elastic material and an occlusion on the inner radius, $t=0.0223$ 86
4.71 Propagation of radial velocity $v_1$ wave through a geometry with a highly dense, purely elastic material and an occlusion on the inner radius, $t=0.0284$ 87
4.72 Propagation of radial velocity $v_1$ wave through a geometry with a highly dense, purely elastic material and an occlusion on the inner radius, $t=0.0406$ 87
4.73 Propagation of tangential velocity $v_2$ wave through a geometry with a highly dense, purely elastic material and an occlusion on the inner radius, $t=0.0039$ 88
4.74 Propagation of tangential velocity $v_2$ wave through a geometry with a highly dense, purely elastic material and an occlusion on the inner radius, $t=0.0081$ 88
4.75 Propagation of tangential velocity $v_2$ wave through a geometry with a highly dense, purely elastic material and an occlusion on the inner radius, $t=0.0121$ 89
4.76 Propagation of tangential velocity $v_2$ wave through a geometry with a highly dense, purely elastic material and an occlusion on the inner radius, $t=0.0223$ 89
4.77 Propagation of tangential velocity $v_2$ wave through a geometry with a highly dense, purely elastic material and an occlusion on the inner radius, $t=0.0284$ 90
4.78 Propagation of tangential velocity $v_2$ wave through a geometry with a highly dense, purely elastic material and an occlusion on the inner radius, $t=0.0406$ 90
4.79 Radial velocity ($v_1$) results for a geometry with a slightly more dense, less viscoelastic ring and an occlusion, $t=0.0081$ .......................... 92
4.80 Radial velocity ($v_1$) results for a geometry with a slightly more dense, less viscoelastic ring and an occlusion, $t=0.0121$ .......................... 92
4.81 Radial velocity ($v_1$) results for a geometry with a slightly more dense, less viscoelastic ring and an occlusion, $t=0.0162$ .......................... 93
4.82 Radial velocity ($v_1$) results for a geometry with a slightly more dense, less viscoelastic ring and an occlusion, $t=0.0223$ .......................... 93
4.83 Tangential velocity ($v_2$) results for a geometry with a slightly more dense, less viscoelastic ring and an occlusion, $t=0.0081$ .......................... 94
4.84 Tangential velocity ($v_2$) results for a geometry with a slightly more dense, less viscoelastic ring and an occlusion, $t=0.0121$ .......................... 94
4.85 Tangential velocity ($v_2$) results for a geometry with a slightly more dense, less viscoelastic ring and an occlusion, $t=0.0162$ .......................... 95
4.86 Tangential velocity ($v_2$) results for a geometry with a slightly more dense, less viscoelastic ring and an occlusion, $t=0.0223$ .......................... 95
4.87 Radial velocity ($v_1$) results for a geometry with a less dense, more viscoelastic ring and an occlusion, $t=0.0081$ .......................... 96
4.88 Radial velocity ($v_1$) results for a geometry with a less dense, more viscoelastic ring and an occlusion, $t=0.0121$ .......................... 97
4.89 Radial velocity ($v_1$) results for a geometry with a less dense, more viscoelastic ring and an occlusion, $t=0.0162$ .......................... 97
4.90 Radial velocity ($v_1$) results for a geometry with a less dense, more viscoelastic ring and an occlusion, $t=0.0223$ .......................... 98
4.91 Tangential velocity ($v_2$) results for a geometry with a less dense, more viscoelastic ring and an occlusion, $t=0.0081$ .......................... 98
4.92 Tangential velocity ($v_2$) results for a geometry with a less dense, more viscoelastic ring and an occlusion, $t=0.0121$ .......................... 99
4.93 Tangential velocity ($v_2$) results for a geometry with a less dense, more viscoelastic ring and an occlusion, $t=0.0162$ .......................... 99
4.94 Tangential velocity ($v_2$) results for a geometry with a less dense, more viscoelastic ring and an occlusion, $t=0.0223$ .......................... 100
List of Tables

3.1 Grid Refinement for $u_1$ and $u_2$ using ER1-IISV .......................... 33
3.2 Grid Refinement for $v_1$ and $v_2$ using ER1-IISV .......................... 33
3.3 Grid Refinement for $w_1$ using ER1-IISV .......................... 34
3.4 Optimal parameters for model ER1-IISV .......................... 35
3.5 Optimal parameters for model ER2-IISV .......................... 35
3.6 Optimal parameters for model ER3-IISV .......................... 36
3.7 Optimal parameters for model ER1-2ISV .......................... 36
3.8 Data Comparison 1D IISV vs. 1D 2ISV .......................... 38
3.9 $\chi^2$ values .................................................. 38
3.10 Data Comparison 2D ER1-IISV vs. 2D ER1-2ISV .......................... 39
3.11 Data Comparison 2D ER1-IISV vs. 2D ER2-IISV .......................... 39
3.12 Data Comparison 2D ER1-IISV vs. 2D ER3-IISV .......................... 40
3.13 Data Comparison ER1-2ISV vs. ER3-2ISV .......................... 41
3.14 Data Comparison ER3-IISV vs. ER3-2ISV .......................... 41

4.1 Parameter values used in improved heterogeneous model .................. 60

5.1 Optimization results for the model with 64 tangential nodes and 12 sensor points .................................................. 103
5.2 Optimization results for the model with 128 tangential nodes and 12 sensor points .................................................. 104
5.3 Optimization results for the model with 128 tangential nodes and 64 sensor points .................................................. 106
5.4 Optimization results for the model with 64 tangential nodes and 32 sensor points .................................................. 107
5.5 Material parameter sets used in inverse problem .................................................. 108
5.6 Optimization results for the varied material parameter sets .............. 108
Chapter 1

Introduction

Coronary artery disease (CAD), also known as coronary heart disease (CHD), is the most common form of heart disease. The National Institute of Health estimates that approximately seven million Americans suffer from the disease, and roughly 500,000 deaths per year can be attributed to it [2]. It is caused by atherosclerosis, the gradual buildup of plaque (cholesterol, calcium, and platelets) within the artery. This accumulation of plaque, known as a stenosis, restricts the flow of blood, leading to a decrease in the oxygen supply to the heart muscle. The end result of an arterial stenosis is permanent damage to the heart muscle, possibly leading to death. Because CAD affects so many people in the United States, and worldwide, its detection and treatment is a matter of high priority.

Current detection techniques include the angiogram which is a reliable, yet expensive, invasive technique and prone to interobserver variability (see e.g., [13],[33]). Ultra-fast CT techniques are also employed; this is a non-invasive imaging technique effective in detecting and scoring the severity of calcium deposits in the coronary arteries. Unfortunately, CT testing equipment is very expensive, and it only detects calcium deposits and not the soft plaques that make up many of the most dangerous lesions.

A plausible alternative to the angiogram and CT scan for detection of stenoses utilizes the detection of acoustic waves propagating from the stenosis. It is well known that arterial stenoses produce sounds due to turbulent blood flow in partially occluded arteries. In principle, turbulent normal wall forces exist at and downstream from an arterial stenosis. These wall forces, which are extremely small, exert a pressure on the wall of the artery
which then causes a small displacement in the surrounding body tissue (see Figure 1.1). The vibrations of the surrounding body tissues, which occur in two forms, a compressional wave and a shear wave, produce sounds [39]. In larger arteries such as the carotid arteries, these acoustic sounds can be detected by physicians using a stethoscope. However, detecting acoustic signals in smaller arteries deep inside the body has proved difficult for two reasons: these acoustic noises attenuate significantly as they travel through the intervening tissues, and many complex sounds within the body can overwhelm conventional acoustic detection systems.

During the late 1990’s, MedAcoustics Inc., a company financed by venture capital, developed and patented a polyvinylidifluoride (PVDF) array of piezoelectric thin film accelerometer sensor elements. These sensors are band-limited, so as to filter out irrelevant signals (e.g., heartbeat) and so, when placed on a person’s acoustic window (located on the chest surface), the array has the ability to detect acoustic shear waves due to arterial stenoses. Utilizing this technology, doctors would have an inexpensive means of deciding whether or not an angiography is needed. As the hardware was designed to collect acceleration data from the wave propagation at the chest wall, it did not give any insight into how the waves traveled through the chest cavity. A mathematical framework was needed to model the wave propagation from the stenosis to the chestwall, and allow for the eventual detection of stenoses. Initial development of the mathematical model were presented in [5], of which this dissertation is an extension.

The idea of utilizing acoustic waves in biological and medical applications is not new. In fact, there has been much research conducted in the field of transient elastography, a method which utilizes acoustic waves and sensors to map an image of selected tissue (see
e.g., [11],[16],[38]). In [9] the authors discuss the ability to detect tumors in breast tissue. Similarly [42] considers tumor detectability in liver tissue. An inverse problem methodology involving data from transient elastography is presented in [20], [21], and [29].

A difficulty that arises when considering biological tissue as a medium, is the formulation of constitutive equations for such tissue. It is well accepted that body soft tissue medium behaves like a viscoelastic medium [12],[26],[34],[44]. However, there are several techniques available to incorporate viscoelasticity into a model. In [4] and [24], the authors propose that applications of fractional calculus be used in viscoelastic modeling. A mathematical framework that examines several methods for dealing with viscoelasticity (in lung tissue) is presented in [41]. Finally, Fung introduced a quasi-linear viscoelastic constitutive law in [12]. While there are many ways to represent the relaxation function in Fung’s law, the work presented in this document is based on the internal variable approach discussed in [5].

The remainder of this dissertation is organized as follows. In Chapter 2, we present a derivation of the model. We begin with the equations of motion, and present the formulation (including different variations) of the constitutive equations.

We examine the validity of our model in Chapter 3 by comparing its generated results to data generated for a one dimensional geometry. We present a comparison of results generated by each variation of our model with the one dimensional data presented in [5]. A statistical significance test is employed to examine the significance of extra parameters involved in the more complicated models.

In Chapter 4, we present several simulations produced using the model. We consider heterogeneities by introducing a uniform ring running through the middle of the geometry. Dependence on $\theta$ is introduced by considering a rigid occlusion located along a portion of the inner radius. Finally, we examine a geometry which combines the heterogeneity and inner occlusion.

With the ultimate goal of this project being detection of stenoses, we obviously must examine inverse problems utilizing the model. Chapter 5 contains a presentation of our initial efforts involving inverse problem methodology. We use an inverse problem in an attempt to find the location of a simulated occlusion, and we also examine the role of the material parameters on the successful termination of the inverse problem.

Finally, in Chapter 6, we present our concluding remarks and future directions of this project.
Chapter 2

Model Formulation

2.1 Physical Geometry of the Model

Although the motivation of this problem involves the detection of stenoses within the human body, several complexities exist with the consideration of the chest cavity. The medium within the chest cavity is heterogeneous and not uniform. The shape of the chest cavity is generally elliptical, but varies between subjects. For these reasons, we consider, for our initial efforts, a more simplistic geometry.

The simplified physical geometry is a cylindrical gel mold. The synthetic gel of which the geometry is comprised, has material properties similar to those of soft tissues. A surgical tube which mimics an arterial vein with stenosis, passes axisymmetrically through the center of the mold. A source disturbance (representative of the disturbance caused by stenosis) is generated within the tube. Shear waves propagate through the gel, and the shear acceleration is measured (by sensors) at the outer surface of the gel.

Let $R_1$ and $R_2$ be the inner and outer radius of the gel, respectively (see Figure 2.1). We assume that the source disturbance is time dependent, has only a radial component, and the disturbance is a pure shear force. The outer surface of the gel is a free surface and the gel is initially at rest.

Since we are interested in modeling the wave propagation from the inner radius to the outer radius, we can neglect the axial direction, and concentrate on a cross section
of the cylindrical mold. This cross sectional area takes the shape of a circle, thus it will be beneficial to derive our model using polar coordinates.

2.2 The Equations of Motion

2.2.1 The Continuity Equation

We begin the formulation of our model by deriving the continuity equation in polar coordinates. The continuity equation is a mathematical expression of the physical property of mass conservation. It states that the rate of change of mass in any particular element is equal to the difference between the flow of mass convected in and the flow of mass convected out. We consider an arbitrary element in a polar coordinate system, such as the one depicted in Figure 2.2. Recall ([14]), that the area of the sector of a circle is given by the formula $A = \frac{1}{2}r^2\theta$, and thus, the area of an arbitrarily small element will have the form

$$dA = \frac{1}{2}(r + dr)^2d\theta - \frac{1}{2}r^2d\theta = r(dr)(d\theta) + \frac{1}{2}(dr)^2(d\theta)$$

Also, recall that the equation for mass flow ([17]) takes the form $Q = \rho vdA$, where $\rho$ is the density of the element, $v$ is velocity, and $dA$ is the area of the face of the element. We determine the mass rate change by multiplying the derivative of the density with respect to time by the area of the element. When we examine the flow of mass convected into our element, we must examine the flow in both the $r$ and $\theta$ directions, thus, the flow of mass
convected into the element will take the form

$$\rho v_1 \bigg|_r r(d\theta) + \rho v_2 \bigg|_{\theta} dr.$$  

Similarly, the flow convected out can be expressed as

$$\rho v_1 \bigg|_{r+dr} (r + dr)(d\theta) + \rho v_2 \bigg|_{\theta+d\theta} dr.$$  

So, the expression for the conservation of mass will be

$$\frac{\partial \rho}{\partial t} r(dr)(d\theta) + \frac{1}{2} \frac{\partial \rho}{\partial t} (dr)^2(d\theta) = (\rho v_1 \bigg|_r r(d\theta) + \rho v_2 \bigg|_{\theta} dr) - (\rho v_1 \bigg|_{r+dr} (r + dr)(d\theta)) + \rho v_2 \bigg|_{\theta+d\theta} dr.$$  

Dividing through by the quantity $r(dr)(d\theta)$, we have

$$\frac{\partial \rho}{\partial t} + \frac{1}{2} \frac{\partial \rho}{\partial t} \frac{dr}{r} = \frac{\rho v_1 \bigg|_r - \rho v_1 \bigg|_{r+dr}}{dr} - \frac{\rho v_1 \bigg|_{r+dr}}{r} + \frac{(\rho v_2 \bigg|_{\theta} - \rho v_2 \bigg|_{\theta+d\theta})}{r(dr)(d\theta)}.$$  

Since our element, $dA$, is arbitrarily small, we allow $dr, d\theta \to 0$, yielding

$$\frac{\partial \rho}{\partial t} = -\frac{\partial \rho v_1}{\partial r} - \frac{1}{r} \frac{\partial \rho v_2}{\partial \theta} - \frac{\rho v_1}{r}.$$  

So, the continuity equation takes the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_1}{\partial r} + \frac{1}{r} \frac{\partial \rho v_2}{\partial \theta} + \frac{\rho v_1}{r} = 0. \quad (2.1)$$

Figure 2.2: An arbitrary element in a polar coordinate system
2.2.2 The Momentum Equation

We now wish to derive the momentum equations in polar coordinates by considering the conservation of momentum, which can be expressed by

\[
\text{(rate of momentum accumulation)} = \text{(momentum flow convected in)} -(\text{momentum flow convected out}) + \text{(sum of forces acting on system)}.
\]

Before further examining the conservation of momentum, we must address a small complication in the geometry of our arbitrary element. The derivation of the momentum equations relies on velocities in both the tangential and radial directions. Unfortunately, the tangential and radial directions at a point on either side of our element do not have the same orientation as those at a point in the center of our element. To rectify this problem, we introduce the unit vectors \( e_r^{(-)} \), \( e_r^{(+)} \), \( e_\theta^{(-)} \), and \( e_\theta^{(+)} \), where the subscripts \( r \) and \( \theta \) refer to the radial and tangential directions respectively, and the superscripts \((-)\) and \((+)\) denote that the vectors originate from a point on the right and left hand side of the arbitrary element (as illustrated in Figure 2.3). These unit vectors have been defined such that the following relations are true:

\[
\begin{align*}
e_r^{(-)} \cdot \vec{e}_r &= \cos \frac{d\theta}{2} \approx 1 & e_\theta^{(-)} \cdot \vec{e}_r &= \sin \frac{d\theta}{2} \approx \frac{d\theta}{2} \\
e_r^{(-)} \cdot \vec{e}_\theta &= -\sin \frac{d\theta}{2} \approx -\frac{d\theta}{2} & e_\theta^{(-)} \cdot \vec{e}_\theta &= \cos \frac{d\theta}{2} \approx 1 \\
e_r^{(+)} \cdot \vec{e}_r &= \cos \frac{d\theta}{2} \approx 1 & e_\theta^{(+)} \cdot \vec{e}_r &= -\sin \frac{d\theta}{2} \approx -\frac{d\theta}{2} \\
e_r^{(+)} \cdot \vec{e}_\theta &= \sin \frac{d\theta}{2} \approx \frac{d\theta}{2} & e_\theta^{(+)} \cdot \vec{e}_\theta &= \cos \frac{d\theta}{2} \approx 1.
\end{align*}
\]

These relations play an important role in the following pages as we derive the equations of momentum.

(a) Rate of Momentum Accumulation

We begin the derivation of the momentum equations by examining the left hand side of (2.2). The rate of momentum accumulation is simply the rate of change of momentum within the element with respect to time, \( \frac{\partial (\rho \vec{v})}{\partial t} \). From the definition of density, we have that \( m = \rho V \) (where \( m \) is mass, and \( V \) is volume), and using this in our expression, we have

\[
\frac{\partial}{\partial t} (\rho V \vec{v}).
\]
The volume for an arbitrarily small element takes the form \( dV = (r(dr)(d\theta)) + \frac{1}{2}(dr)^2(d\theta)dz \). However, since we are only working in two dimensions, we may neglect the \( dz \) term, and our expression becomes
\[
\frac{\partial(\rho \vec{v})}{\partial t} - (r(dr)(d\theta) + \frac{1}{2}(dr)^2(d\theta)).
\]

We divide through by \( r(dr)(d\theta) \), yielding
\[
\frac{\partial(\rho \vec{v})}{\partial t} + \frac{1}{2} \frac{\partial(\rho \vec{v})}{\partial t} \frac{dr}{r}.
\]

Allowing \( dr, d\theta \to 0 \), we obtain the expression for the rate of momentum accumulation, which takes the form
\[
\frac{\partial(\rho \vec{v})}{\partial t}.
\]

(2.3)

(b) Momentum Flow Convected In - Momentum Flow Convected Out

We shall now examine the change in momentum flow. Momentum flow is computed by multiplying the mass flow at a given point by the velocity at the same point. Mass flow was discussed when we derived the continuity equation, and the velocity at any given point will take the form \( (v_1 \hat{e}_r + v_2 \hat{e}_{\theta}) \). Since we are working in two dimensions,
we must account for momentum flow in the radial direction, as well as momentum flow in the tangential direction. To compute the momentum flow in a specific direction, we take the dot product of the momentum flow and the unit vector for the corresponding direction. We will now derive the momentum flow in the radial direction.

\[
MF_r = (\rho v_1 r(d\theta))(v_1 \vec{e}_r + v_2 \vec{e}_\theta) \cdot \vec{e}_r - (\rho v_1 (r + dr)d\theta)(v_1 \vec{e}_r + v_2 \vec{e}_\theta) \cdot \vec{e}_r \bigg|_{r+dr} + (\rho v_2 (dr))(v_1 \vec{e}_r^{(-)} + v_2 \vec{e}_\theta^{(-)}) \cdot \vec{e}_r \bigg|_r - (\rho v_2 (dr))(v_1 \vec{e}_r^{(+)} + v_2 \vec{e}_\theta^{(+)}) \cdot \vec{e}_r \bigg|_{\theta+d\theta} \\
= (\rho v_1^2(r(d\theta))\vec{e}_r \cdot \vec{e}_r + (\rho v_1 v_2 r(d\theta))\vec{e}_\theta \cdot \vec{e}_r \bigg|_r - (\rho v_2^2(r + dr)(d\theta))\vec{e}_r \cdot \vec{e}_r \\
+ (\rho v_2 v_1 (dr)(d\theta))\vec{e}_\theta \cdot \vec{e}_r \bigg|_{r+dr} + (\rho v_2^2(dr)\vec{e}_\theta^{(-)} \cdot \vec{e}_r - (\rho v_2^2(dr)\vec{e}_\theta^{(+)} \cdot \vec{e}_r) \bigg|_{\theta+d\theta} \\
- (\rho v_2 v_1 (dr)\vec{e}_r^{(+)} \cdot \vec{e}_r + \rho v_2^2(dr)\vec{e}_\theta^{(+)} \cdot \vec{e}_r) \bigg|_{\theta+d\theta}.
\]

Using the previously stated unit vector relations, our expression simplifies to

\[
MF_r = \rho v_1^2 \bigg|_r r(d\theta) - \rho v_1^2 \bigg|_{r+dr} (r + dr)d\theta + \rho v_2 v_1 (dr) \bigg|_{\theta} + \rho v_2^2 \bigg|_{\theta+d\theta} \frac{d\theta}{2} (dr) \\
- \rho v_2 v_1 (dr) \bigg|_{\theta+d\theta} - \rho v_2^2 \bigg|_{\theta+d\theta} \frac{d\theta}{2} (dr) \\
= \rho v_1^2 \bigg|_r - \rho v_1^2 \bigg|_{r+dr} r(d\theta) - \rho v_1^2 \bigg|_{r+dr} (d\theta)(d\theta) + \rho v_2 v_1 \bigg|_{\theta} - \rho v_2 v_1 \bigg|_{\theta+d\theta} dr + \\
\rho v_2^2 \frac{d\theta}{2} (dr) \bigg|_{\theta+d\theta} + \rho v_2^2 \frac{d\theta}{2} (dr) \bigg|_{\theta+d\theta}.
\]

We divide through by \(r(d\theta)(d\theta)\), yielding

\[
MF_r = \frac{\rho v_1^2}{dr} - \rho v_1^2 \bigg|_{r+dr} - \rho v_1^2 \bigg|_{r+dr} + \rho v_2 v_1 \bigg|_{\theta} - \rho v_2 v_1 \bigg|_{\theta+d\theta} + \frac{\rho v_2^2}{2r} \frac{d\theta}{2} + \frac{\rho v_2^2}{2r} \frac{d\theta}{2}.
\]

Let \(dr, d\theta \to 0\), and we have

\[
MF_r = -\frac{\partial \rho v_1^2}{\partial r} - \frac{\rho v_1^2}{r} - \frac{\partial \rho v_2 v_1}{\partial \theta} + \frac{\rho v_2^2}{r}.
\]

We derive the momentum flow in the tangential direction in a similar manner, beginning with the dot product of the momentum flow with the unit vector \(\vec{e}_\theta\).

\[
MF_\theta = (\rho v_1 r(d\theta))(v_1 \vec{e}_r + v_2 \vec{e}_\theta) \cdot \vec{e}_\theta - (\rho v_1 (r + dr)d\theta)(v_1 \vec{e}_r + v_2 \vec{e}_\theta) \cdot \vec{e}_\theta \bigg|_{r+dr} + (\rho v_2 (dr))(v_1 \vec{e}_r^{(-)} + v_2 \vec{e}_\theta^{(-)}) \cdot \vec{e}_\theta \bigg|_r - (\rho v_2 (dr))(v_1 \vec{e}_r^{(+)} + v_2 \vec{e}_\theta^{(+)}) \cdot \vec{e}_\theta \bigg|_{\theta+d\theta} \\
= (\rho v_1^2 r(d\theta))\vec{e}_r \cdot \vec{e}_\theta + (\rho v_1 v_2 r(d\theta))\vec{e}_\theta \cdot \vec{e}_\theta \bigg|_r - (\rho v_2^2 (r + dr)(d\theta))\vec{e}_r \cdot \vec{e}_\theta \\
+ (\rho v_2 v_1 (dr)(d\theta))\vec{e}_\theta \cdot \vec{e}_\theta \bigg|_{r+dr} + (\rho v_2^2 (dr)\vec{e}_\theta^{(-)} \cdot \vec{e}_\theta - (\rho v_2^2 (dr)\vec{e}_\theta^{(+)} \cdot \vec{e}_\theta) \bigg|_{\theta+d\theta} \\
- (\rho v_2 v_1 (dr)\vec{e}_r^{(+)} \cdot \vec{e}_\theta + \rho v_2^2 (dr)\vec{e}_\theta^{(+)} \cdot \vec{e}_\theta) \bigg|_{\theta+d\theta}.
\]
Employing the unit vector relationships again, leads us to

\[
MF_\theta = \rho v_1 v_2 \left| \frac{r}{r} \right| \begin{aligned}
&= \rho v_1 v_2 \left| \frac{r}{r+dr} \right| (r+dr)(d\theta) + \rho v_2 v_1 \left| \frac{\theta}{\theta+d\theta} \right| \frac{d\theta}{2} + \rho v_2^2 \left| \frac{\theta}{\theta+d\theta} \right| \frac{d\theta}{2} \\
&- \rho v_2 v_1 \left| \frac{\theta}{\theta+d\theta} \right| \frac{d\theta}{2} - \rho v_2^2 \left| \frac{\theta}{\theta+d\theta} \right| \frac{d\theta}{2} \\
&= \left( \rho v_1 v_2 \left| \frac{r}{r+dr} \right| r \right)(d\theta) - \left( \rho v_1 v_2 \left| \frac{r}{r+dr} \right| (d\theta) + \left( \rho v_2^2 \left| \frac{\theta}{\theta+d\theta} \right| \frac{d\theta}{2} \right) - \left( \rho v_2 v_1 \left| \frac{\theta}{\theta+d\theta} \right| \frac{d\theta}{2} \right)
\end{aligned}
\]

We divide by \( r(dr)(d\theta) \), leading to

\[
MF_\theta = \frac{\rho v_1 v_2}{dr} - \frac{\rho v_1 v_2}{r+dr} + \frac{\rho v_2 v_1}{r} \frac{d\theta}{2} - \frac{\rho v_2 v_1}{\theta+d\theta} \frac{d\theta}{2}.
\]

Allowing \( dr, d\theta \to 0 \), we find the expression for momentum flow in the tangential direction will be

\[
MF_\theta = -\frac{\partial \rho v_1 v_2}{\partial r} - \frac{\rho v_1 v_2}{r} \frac{\partial \rho v_2 v_1}{\partial \theta} - \frac{\rho v_2 v_1}{r}.
\]

So, the momentum flow portion of the momentum equations takes the form

\[
MF_r = -\frac{\partial \rho v_1 v_2}{\partial r} - \frac{1}{r} \frac{\partial \rho v_2 v_1}{\partial \theta} - \frac{1}{r} (\rho v_1^2 - \rho v_2^2) \quad (2.4)
\]

\[
MF_\theta = -\frac{\partial \rho v_1 v_2}{\partial r} - \frac{1}{r} \frac{\partial \rho v_2 v_1}{\partial \theta} - \frac{1}{r} (\rho v_1 v_2 + \rho v_2 v_1). \quad (2.5)
\]

(c) Sum of Forces Acting on System

Now, we turn our efforts to the summation of the forces acting upon the system. We examine an arbitrary element, such as the one depicted in Figure 2.4. Each face of the element undergoes a stress \( t_i \) composed of an axial stress and a shear stress. We define these stress elements as follows:

\[
t_A = \sigma_{11} \left| \frac{\varepsilon_r}{r} + \sigma_{12} \left| \frac{\varepsilon_\theta}{r} \right.ight.
\]

\[
t_B = \sigma_{11} \left| \frac{\varepsilon_r}{r+dr} + \sigma_{12} \left| \frac{\varepsilon_\theta}{r+dr} \right. \right.
\]

\[
t_C = \sigma_{21} \left| \frac{\varepsilon_r}{\theta} \left. + \sigma_{22} \right| \frac{\varepsilon_\theta}{\theta} \right. \right.
\]

\[
t_D = \sigma_{21} \left| \frac{\varepsilon_r}{\theta+d\theta} + \sigma_{22} \left| \frac{\varepsilon_\theta}{\theta+d\theta} \right. \right. \right.
\]
We know from basic physics that stress = \frac{\text{force}}{\text{area}}, thus, we need to multiply the stresses by the areas upon which they act in order to get our forces. Similarly to the momentum flow derivations, we must take the dot product of our forces with the unit vectors \( \vec{e}_r \) and \( \vec{e}_\theta \) to derive the sum of the forces in the radial and tangential directions, respectively. We begin by examining the summation of the forces in the radial direction.

\[
\sum F_r = t_B \cdot \vec{e}_r (r + dr) d\theta - t_A \cdot \vec{e}_r r(d\theta) + t_D \cdot \vec{e}_r dr - t_C \cdot \vec{e}_r dr
\]

\[
+ \left( \sigma_{11} \left|_{r+dr} \right. \frac{r}{2} + \sigma_{12} \left|_{r+dr} \right. \left( \frac{dr}{2} \right) \right)
\]

Simplifying the dot products, we now have

\[
\sum F_r = \sigma_{11} \left|_{r+dr} \right. r(d\theta) + \sigma_{11} \left|_{r+dr} \right. (dr) (d\theta) - \sigma_{11} \left|_r \right. r(d\theta) + \sigma_{21} \left|_{\theta+\theta} \right. dr
\]

\[
+ \sigma_{22} \left|_{\theta+\theta} \right. dr \left( -\frac{d\theta}{2} \right) - \sigma_{21} \left|_\theta \right. dr - \sigma_{22} \left|_{\theta+\theta} \right. dr \left( \frac{d\theta}{2} \right)
\]

\[
= \sigma_{11} \left|_{r+dr} \right. r(d\theta) + \sigma_{11} \left|_{r+dr} \right. (dr)(d\theta) + (\sigma_{21} \left|_{\theta+\theta} \right. - \sigma_{21} \left|_\theta \right. ) dr
\]

\[
- \sigma_{22} \left|_{\theta+\theta} \right. dr \left( -\frac{d\theta}{2} \right) - \sigma_{22} \left|_\theta \right. dr \left( \frac{d\theta}{2} \right).
\]
We divide by \( r(dr)(d\theta) \) and we obtain

\[
\sum F_r = \frac{\sigma_{11}|_{r+dr} - \sigma_{11}|_r}{dr} + \frac{\sigma_{11}|_{r+dr} - \sigma_{11}|_r}{r} + \frac{\sigma_{21}|_{\theta+d\theta} - \sigma_{21}|_{\theta}}{r - d\theta} + \frac{\sigma_{22}|_{\theta+d\theta} - \sigma_{22}|_{\theta}}{2r}.
\]

We allow \( dr \) and \( d\theta \) to tend to zero, yielding

\[
\sum F_r = \frac{\partial \sigma_{11}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{21}}{\partial \theta} + (\sigma_{11} - \sigma_{22}).
\]

Similarly, the summation of forces in the tangential direction can be derived by examining the following.

\[
\sum F_\theta = t_B \cdot \tilde{e}_\theta (r + dr)d\theta - t_A \cdot \tilde{e}_\theta (dr) - \tau_B \cdot \tilde{e}_\theta dr - t_C \cdot \tilde{e}_\theta dr
\]

\[
= (\sigma_{11}|_{r+dr} \tilde{e}_r + \sigma_{12}|_{r+dr} \tilde{e}_\theta) \cdot \tilde{e}_\theta (r + dr)d\theta - (\sigma_{11}|_r \tilde{e}_r + \sigma_{12}|_r \tilde{e}_\theta) \cdot \tilde{e}_\theta (dr)
\]

\[
+ (\sigma_{21}|_{\theta+d\theta} \tilde{e}_{\theta}(+) + \sigma_{22}|_{\theta+d\theta} \tilde{e}_{\theta}(-)) \cdot \tilde{e}_\theta dr - (\sigma_{21}|_{\theta} \tilde{e}_{\theta}(-) + \sigma_{22}|_{\theta} \tilde{e}_{\theta}) \cdot \tilde{e}_\theta dr
\]

Utilizing our unit vector relationships, we find

\[
\sum F_\theta = \sigma_{12}|_{r+dr} (dr)(\theta) + \sigma_{12}|_{r+dr} r(d\theta) - \sigma_{12}|_r r(d\theta) + \sigma_{21}|_{\theta+d\theta} (dr) \left( \frac{d\theta}{2} \right)
\]

\[
+ \sigma_{22}|_{\theta+d\theta} (dr) \left( \frac{d\theta}{2} \right) - \sigma_{22}|_\theta (dr)
\]

\[
= (\sigma_{12}|_{r+dr} - \sigma_{12}|_r) r(d\theta) + \sigma_{12}|_{r+dr} (dr) (d\theta) + \left( \sigma_{22}|_{\theta+d\theta} - \sigma_{22}|_{\theta} \right) (d\theta)
\]

\[
+ \sigma_{21}|_{\theta+d\theta} (dr) \left( \frac{d\theta}{2} \right) + \sigma_{21}|_{\theta} (dr) \left( \frac{d\theta}{2} \right).
\]

We divide by \( r(dr)(d\theta) \)

\[
\sum F_\theta = \frac{\sigma_{12}|_{r+dr} - \sigma_{12}|_r}{dr} + \frac{\sigma_{12}|_{r+dr} - \sigma_{12}|_r}{r} + \frac{\sigma_{22}|_{\theta+d\theta} - \sigma_{22}|_{\theta}}{r - d\theta} + \frac{\sigma_{21}|_{\theta+d\theta} - \sigma_{22}|_{\theta}}{2r} + \frac{\sigma_{21}|_{\theta}}{2r}.
\]

As \( dr, d\theta \to 0 \), this simplifies to

\[
\sum F_\theta = \frac{\partial \sigma_{12}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{22}}{\partial \theta} + \frac{1}{r} (\sigma_{12} + \sigma_{21}).
\]

Thus, the equations representing the sum of the forces in both directions are

\[
\sum F_r = \frac{\partial \sigma_{11}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{21}}{\partial \theta} + \frac{1}{r} (\sigma_{11} - \sigma_{22})
\]

\[
\sum F_\theta = \frac{\partial \sigma_{12}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{22}}{\partial \theta} + \frac{1}{r} (\sigma_{12} + \sigma_{21})
\]
Finally, to obtain the momentum equations, we return to the conservation of momentum concept illustrated in (2.2). We use (2.3) for the rate of momentum accumulation, we replace momentum flow convected in - momentum flow convected out with (2.4) and (2.5) and sum of forces acting on the system is represented by (2.6) and (2.7). Putting everything together, we arrive at the following momentum equations

\[
\frac{\partial \rho v_1}{\partial t} = -\frac{\partial \rho v_1^2}{\partial r} - \frac{1}{r} \frac{\partial \rho v_1 v_2}{\partial \theta} - \frac{1}{r} (\rho v_1^2 - \rho v_2^2) + \frac{\partial \sigma_{11}}{\partial r} + \frac{\partial \sigma_{21}}{\partial \theta} + \frac{1}{r} (\sigma_{11} - \sigma_{22}) \tag{2.8}
\]

\[
\frac{\partial \rho v_2}{\partial t} = -\frac{\partial \rho v_1^2}{\partial r} - \frac{1}{r} \frac{\partial \rho v_1 v_2}{\partial \theta} - \frac{1}{r} (\rho v_1 v_2 + \rho v_2 v_1) + \frac{\partial \sigma_{12}}{\partial r} + \frac{\partial \sigma_{22}}{\partial \theta} + \frac{1}{r} (\sigma_{12} + \sigma_{21}) \tag{2.9}
\]

### 2.2.3 An Alternative Formulation of the Momentum Equations

The following generic version of the momentum equations in vector form is presented in [8],

\[
\frac{\partial}{\partial t} \rho \ddot{\mathbf{v}} = -\nabla \cdot \rho \dddot{\mathbf{v}} \otimes \dddot{\mathbf{v}} + \nabla \cdot \dddot{\mathbf{r}} - \nabla \dddot{p} + \rho \dddot{\mathbf{g}} \tag{2.10}
\]

where $\rho \dddot{v} \otimes \dddot{v}$ is the dyadic product of $\rho \ddot{v}$ and $\dddot{v}, \dddot{r}$ is the stress tensor, $\nabla \dddot{p}$ is the pressure force per unit volume, and $\rho \dddot{g}$ is the gravitational force on the fluid element per unit volume. We will use (2.10) to derive the momentum equation in polar coordinates. However, before we begin the derivation, we must examine a couple of concepts that will be important to our derivation.

First, since we are working in polar coordinates, we must recall the following relationships, involving the derivative of the unit vectors $\ddot{e}_r$ and $\ddot{e}_\theta$,

\[
\frac{\partial}{\partial r} \ddot{e}_r = 0 \quad \frac{\partial}{\partial \theta} \ddot{e}_r = \ddot{e}_\theta
\]

\[
\frac{\partial}{\partial r} \ddot{e}_\theta = 0 \quad \frac{\partial}{\partial \theta} \ddot{e}_\theta = -\ddot{e}_r.
\]

Also, we consider the quantity $\nabla \cdot \dddot{T}$ for a generic tensor $\dddot{T}$ in polar coordinates. Recall that in polar coordinates, the gradient operator can be written in vector notation as $[\frac{1}{r} \frac{\partial}{\partial r} (r \cdot), \frac{1}{r} \frac{\partial}{\partial \theta} (\cdot)]$. We define our tensor as

\[
\dddot{T} = \begin{bmatrix} \dddot{T}_1 \\ \dddot{T}_2 \end{bmatrix},
\]

where,

\[
\dddot{T}_1 = \dddot{T}_{11} \ddot{e}_r + \dddot{T}_{12} \ddot{e}_\theta
\]

\[
\dddot{T}_2 = \dddot{T}_{21} \ddot{e}_r + \dddot{T}_{22} \ddot{e}_\theta.
\]
So, we have

\[
\vec{\mathbf{z}} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} \vec{e}_r \\ \vec{e}_\theta \end{bmatrix}.
\]

Now that we have defined the gradient operator \( \vec{\nabla} \), and a tensor \( \vec{\mathbf{z}} \), we are ready to examine their “dot” product. Notice,

\[
\vec{\nabla} \cdot \vec{\mathbf{z}} = \begin{bmatrix} \frac{1}{r} \frac{\partial}{\partial r} (r r) & \frac{1}{r} \frac{\partial}{\partial \theta} (\cdot) \end{bmatrix} \begin{bmatrix} \vec{T}_1 \\ \vec{T}_2 \end{bmatrix}
\]

\[
= \frac{1}{r} \frac{\partial}{\partial r} (\vec{T}_1) + \frac{\partial}{\partial \theta} (\vec{T}_2)
\]

\[
= \frac{1}{r} \frac{\partial}{\partial r} (T_{11} \vec{e}_r + T_{12} \vec{e}_\theta) + \frac{\partial}{\partial \theta} (T_{11} \vec{e}_r + T_{12} \vec{e}_\theta) + \frac{1}{r} \frac{\partial}{\partial \theta} (T_{21} \vec{e}_r + T_{22} \vec{e}_\theta)
\]

\[
= \frac{1}{r} T_{11} \vec{e}_r + \frac{1}{r} T_{21} \vec{e}_r + \frac{\partial T_{11}}{\partial r} \vec{e}_r + T_{11} \frac{\partial \vec{e}_r}{\partial r} + \frac{1}{r} \frac{\partial T_{21}}{\partial \theta} \vec{e}_\theta + T_{21} \frac{\partial \vec{e}_\theta}{\partial \theta}
\]

Utilizing the derivatives of unit vectors presented earlier in this section, this becomes

\[
\vec{\nabla} \cdot \vec{\mathbf{z}} = \frac{1}{r} T_{11} \vec{e}_r + \frac{1}{r} T_{12} \vec{e}_\theta + \frac{\partial T_{11}}{\partial r} \vec{e}_r + \frac{1}{r} \frac{\partial T_{21}}{\partial \theta} \vec{e}_\theta
\]

\[
+ \frac{1}{r} \frac{\partial T_{21}}{\partial \theta} \vec{e}_r + \frac{1}{r} T_{21} \vec{e}_\theta + \frac{1}{r} \frac{\partial T_{22}}{\partial \theta} \vec{e}_\theta + \frac{1}{r} T_{22} (\vec{e}_r)
\]

\[
= \left( \frac{1}{r} T_{11} + \frac{1}{r} \frac{\partial T_{11}}{\partial r} \vec{e}_r + \frac{1}{r} \frac{\partial T_{21}}{\partial \theta} \vec{e}_\theta - \frac{1}{r} T_{22} \vec{e}_r ight)
\]

\[
+ \left( \frac{1}{r} T_{12} + \frac{1}{r} \frac{\partial T_{12}}{\partial r} \vec{e}_r + \frac{1}{r} T_{21} + \frac{1}{r} \frac{\partial T_{22}}{\partial \theta} \vec{e}_\theta \right)
\]

\[
= \begin{bmatrix} \frac{\partial T_{11}}{\partial r} + \frac{1}{r} \frac{\partial T_{11}}{\partial \theta} + \frac{1}{r} (T_{11} - T_{22}) \\ \frac{\partial T_{12}}{\partial r} + \frac{1}{r} \frac{\partial T_{12}}{\partial \theta} + \frac{1}{r} (T_{12} + T_{21}) \end{bmatrix}^T
\]

We are now ready to reexamine (2.10). In our formulation, we neglect the pressure term and the gravity term, so we have

\[
\frac{\partial}{\partial t} (\rho \vec{v})^T = -\vec{\nabla} \cdot \rho \vec{v} \vec{v} + \vec{\nabla} \cdot \vec{\mathbf{z}}
\]

\[
\begin{bmatrix} \frac{\partial \rho v_1}{\partial \mathbf{r}} \\ \frac{\partial \rho v_2}{\partial \mathbf{r}} \end{bmatrix}^T = -\begin{bmatrix} \frac{1}{r} \frac{\partial}{\partial r} (r \rho v_1) & \frac{1}{\rho} \frac{\partial}{\partial \theta} (\rho v_1) \\ \frac{1}{r} \frac{\partial}{\partial r} (r \rho v_2) & \frac{1}{\rho} \frac{\partial}{\partial \theta} (\rho v_2) \end{bmatrix} \begin{bmatrix} \rho v_1 \rho v_2 \\ \rho v_2 \rho v_1 \end{bmatrix}
\]

\[
+ \begin{bmatrix} \frac{1}{r} \frac{\partial}{\partial r} (r \rho v_1) & \frac{1}{\rho} \frac{\partial}{\partial \theta} (\rho v_1) \\ \frac{1}{r} \frac{\partial}{\partial r} (r \rho v_2) & \frac{1}{\rho} \frac{\partial}{\partial \theta} (\rho v_2) \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}
\]

\[
\begin{bmatrix} \frac{\partial \rho v_1}{\partial \mathbf{r}} \\ \frac{\partial \rho v_2}{\partial \mathbf{r}} \end{bmatrix}^T = \begin{bmatrix} -\frac{1}{r} \frac{\partial \rho v_1}{\partial \mathbf{r}} - \frac{1}{r} \frac{\partial \rho v_1}{\partial \mathbf{r}} - \frac{1}{r^2} (\rho v_1^2 - \rho v_2^2) \\ -\frac{1}{r} \frac{\partial \rho v_1}{\partial \mathbf{r}} - \frac{1}{r} \frac{\partial \rho v_2}{\partial \mathbf{r}} - \frac{1}{r^2} (\rho v_1 v_2 + \rho v_2 v_1) \end{bmatrix}^T
\]

\[
+ \begin{bmatrix} \frac{\partial \sigma_{11}}{\partial \mathbf{r}} + \frac{1}{r^2} \frac{\partial \sigma_{11}}{\partial \mathbf{r}} + \frac{1}{r^2} (\sigma_{11} - \sigma_{22}) \\ \frac{\partial \sigma_{12}}{\partial \mathbf{r}} + \frac{1}{r^2} \frac{\partial \sigma_{12}}{\partial \mathbf{r}} + \frac{1}{r^2} (\sigma_{12} + \sigma_{21}) \end{bmatrix}^T.
\]
So, the momentum equations take the form
\[
\frac{\partial \rho v_1}{\partial t} = -\frac{\partial \rho v_1^2}{\partial r} - \frac{1}{r} \frac{\partial \rho v_2 v_1}{\partial \theta} - \frac{1}{r} (\rho v_1^2 - \rho v_2^2) + \frac{\partial \sigma_{11}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{21}}{\partial \theta} + \frac{1}{r} (\sigma_{11} - \sigma_{22})
\]
\[
\frac{\partial \rho v_2}{\partial t} = -\frac{\partial \rho v_1 v_2}{\partial r} - \frac{1}{r} \frac{\partial \rho v_2^2}{\partial \theta} - \frac{1}{r} (\rho v_1 v_2 + \rho v_2 v_1) + \frac{\partial \sigma_{12}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{22}}{\partial \theta} + \frac{1}{r} (\sigma_{12} + \sigma_{21}).
\]

It should be noted, that we do not assume that the density, \( \rho \), is constant in the derivation of our model. The derivations presented in the preceding section (as well as those which follow) will hold for constant density, as well as spatially dependent densities.

### 2.2.4 The Equations of Motion

Now that we have derived the continuity equation and the momentum equations, we are ready to derive the equations of motion. Once again, we will have a separate equation for each direction of motion. We will first derive the equation of motion in the radial direction by beginning with (2.8) written in the following form,
\[
\frac{\partial \rho v_1}{\partial t} + \frac{\partial \rho v_1^2}{\partial r} + \frac{1}{r} \frac{\partial \rho v_2 v_1}{\partial \theta} + \frac{1}{r} (\rho v_1^2 - \rho v_2^2) = \frac{\partial \sigma_{11}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{21}}{\partial \theta} + \frac{1}{r} (\sigma_{11} - \sigma_{22}).
\]

We examine the left hand side of this equation. Applying the product rule, we obtain
\[
\frac{\partial \rho v_1}{\partial t} + v_1 \frac{\partial \rho}{\partial t} + v_1 \frac{\partial \rho v_1}{\partial r} + \frac{v_1}{r} \frac{\partial \rho v_1}{\partial \theta} + \rho v_2 \frac{\partial v_1}{\partial r} + \frac{v_1}{r} \frac{\partial \rho v_1}{\partial \theta} + \frac{\rho v_2^2}{r} - \frac{\rho v_2^2}{r}.
\]

Simple factoring yields
\[
\rho \left( \frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial r} + \frac{v_2}{r} \frac{\partial v_1}{\partial \theta} - \frac{v_2^2}{r} \right) + v_1 \left( \frac{\partial \rho}{\partial t} + \frac{\partial \rho v_1}{\partial r} + \frac{\partial \rho v_2}{\partial r} + \frac{\rho v_2}{r} \right).
\]

Using the continuity equation (2.1), we find that the second term becomes zero, leaving us with
\[
\rho \left( \frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial r} + \frac{v_2}{r} \frac{\partial v_1}{\partial \theta} - \frac{v_2^2}{r} \right) = \frac{\partial \sigma_{11}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{21}}{\partial \theta} + \frac{1}{r} (\sigma_{11} - \sigma_{22}).
\]

Similarly, the derivation of the equation of motion in the tangential direction begins with (2.9), written as
\[
\frac{\partial \rho v_2}{\partial t} + \frac{\partial \rho v_1 v_2}{\partial r} + \frac{1}{r} \frac{\partial \rho v_2^2}{\partial \theta} + \frac{1}{r} (\rho v_1 v_2 + \rho v_2 v_1) = \frac{\partial \sigma_{12}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{22}}{\partial \theta} + \frac{1}{r} (\sigma_{12} + \sigma_{21}).
\]

Application of the product rule to the left hand side of the equation produces
\[
\rho \left( \frac{\partial v_2}{\partial t} + v_2 \frac{\partial \rho}{\partial t} + \rho v_1 \frac{\partial v_2}{\partial r} + \frac{v_2}{r} \frac{\partial \rho v_2}{\partial \theta} + \frac{\rho v_2^2}{r} \right) + \frac{\rho v_1 v_2}{r} - \frac{\rho v_2 v_1}{r}.
\]
which we factor into
\[
\rho \left( \frac{\partial v_2}{\partial t} + v_1 \frac{\partial v_2}{\partial r} + \frac{v_2}{r} \frac{\partial v_2}{\partial \theta} - \frac{v_1 v_2}{r} \right) + v_2 \left( \frac{\partial \rho}{\partial t} + \frac{\partial \rho v_1}{\partial r} + \frac{1}{r} \frac{\partial \rho v_2}{\partial \theta} + \frac{\rho v_1}{r} \right).
\]

Once again, the continuity equation will cause the second term to be zero, thus we have
\[
\rho \left( \frac{\partial v_2}{\partial t} + v_1 \frac{\partial v_2}{\partial r} + \frac{v_2}{r} \frac{\partial v_2}{\partial \theta} - \frac{v_1 v_2}{r} \right) = \frac{\partial \sigma_{12}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{22}}{\partial \theta} + \frac{1}{r} (\sigma_{12} + \sigma_{21}).
\]

Hence, the equations of motion will be
\[
\rho \left( \frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial r} + \frac{v_1}{r} \frac{\partial v_1}{\partial \theta} - \frac{v_1 v_1}{r} \right) = \frac{\partial \sigma_{11}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{21}}{\partial \theta} + \frac{1}{r} (\sigma_{11} - \sigma_{22})
\]
\[
\rho \left( \frac{\partial v_2}{\partial t} + v_1 \frac{\partial v_2}{\partial r} + \frac{v_2}{r} \frac{\partial v_2}{\partial \theta} - \frac{v_1 v_2}{r} \right) = \frac{\partial \sigma_{12}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{22}}{\partial \theta} + \frac{1}{r} (\sigma_{12} + \sigma_{21}).
\]

Since we are dealing with only small displacements, the higher order terms will become negligible, and thus we ignore them to obtain,
\[
\rho \frac{\partial^2 u_1}{\partial t^2} = \frac{\partial \sigma_{11}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{21}}{\partial \theta} + \frac{1}{r} (\sigma_{11} - \sigma_{22}) \quad (2.11)
\]
\[
\rho \frac{\partial^2 u_2}{\partial t^2} = \frac{\partial \sigma_{12}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{22}}{\partial \theta} + \frac{1}{r} (\sigma_{12} + \sigma_{21}). \quad (2.12)
\]

These equations will be coupled with a set of constitutive equations to form our model.

2.3 Formulation of the Constitutive Equations

Having derived the equations of motion, we now need to formulate the constitutive equations to complete our model. Constitutive equations typically quantify the relationship between the stresses to which a material is subjected and the strain response. To formulate a set of constitutive equations for our model, we begin with the generalized quasi-linear viscoelastic model proposed by Fung [12],
\[
S_{ij}(t) = \int_{-\infty}^{t} G_{ijkl}(t - \tau) \frac{\partial S_{kl}^{(e)}}{\partial \tau} [\tilde{E}^{(e)}(\tau)] d\tau,
\]
where \( S_{ij} \) is the Kirchhoff stress tensor, \( \tilde{E} \) is the Green’s strain tensor, \( G_{ijkl} \) is the reduced relaxation function, and \( S_{kl}^{(e)} \) is the “elastic” stress tensor.

Since its introduction, this quasi-linear viscoelastic (QLV) theory has been applied to several tissues including, among others, tendons [18], ligaments [46], articular cartilage
under tension [45], heart valves [37], papillary muscle [35], and smooth muscle [36]. A
benefit to using the QLV theory is that, unlike simpler models for viscoelasticity, it allows
for the consideration of a continuous spectrum of relaxation times and frequencies. The
QLV theory has been shown capable of modeling relaxation, creep, and hysteresis within
viscoelastic materials. Fung has shown (as have the modeling efforts listed above) that the
QLV theory works well for fitting stress-strain relationships numerically to experimental
data.

While Fung’s theory is mainly used for stress-strain fits, we are interested in using
it in a full dynamical model. Unfortunately, the QLV, as presented by Fung, is difficult to
compute within full dynamical partial differential equations, so we will utilize a different
approach. Our alternative to Fung’s kernel is a parameter dependent kernel with a contin-
uous distribution of parameters and internal variables. In the case of a finite combination
of δ measures, we employ a finite dimensional summation of exponential functions as our
approximation kernel. This method is based on the internal variable approach from [5],
in which the authors have shown that a summation of exponentials approximates Fung’s
kernel fairly well. Further details of our internal variable approach are presented in section
2.3.3

2.3.1 Elastic Response

$S_{ij}^{(c)}$ is the elastic stress tensor corresponding to the strain tensor $E$; it is a function
of the strain components $E_{ij}$, which, in polar coordinates are expressed as [10]

$$E_{11} = \frac{\partial u_1}{\partial r}$$
$$E_{22} = \frac{1}{r} \frac{\partial u_2}{\partial \theta} + \frac{u_1}{r}$$
$$E_{21} = E_{12} = \frac{1}{r} \frac{\partial u_1}{\partial \theta} + \frac{\partial u_2}{\partial r} - \frac{u_2}{r}.$$

$S_{ij}^{(c)}$ is defined to be the stress that is reached instantaneously when the strains are suddenly
increased from 0 to $E_{ij}$.

Following the one-dimensional arguments presented in [12], we assume that the
“elastic” responses can be approximated by pseudo-elastic stresses, where the pseudo-elastic
stresses take the form of the derivative of a pseudo-strain energy function $(\rho_p W)$ with regards
to components of the strain tensor, $E_{ij}$, i.e.,

$$S^{(e)}_{ij} \sim S_{ij} = \frac{\partial (\rho_0 W)}{\partial E_{ij}}.$$  

In two dimensional experiments involving skin tissue, Fung chose the following form for the pseudo-strain-energy function [12]

$$\rho_0 W^{(2)} = \frac{1}{2} (\alpha_1 E_{11}^2 + \alpha_2 E_{22}^2 + \alpha_3 E_{12}^2 + \alpha_3 E_{21}^2 + 2\alpha_4 E_{11} E_{22})$$

$$+ \frac{c}{2} \exp(a_1 E_{11}^2 + a_2 E_{22}^2 + a_3 E_{12}^2 + a_3 E_{21}^2 + 2a_4 E_{11} E_{22})$$

$$+ \gamma_1 E_{11}^3 + \gamma_2 E_{22}^3 + \gamma_4 E_{11}^2 E_{22} + \gamma_5 E_{11} E_{22}^2),$$

where the $\alpha$'s, $a$'s, $\gamma$'s, and $c$ are constants. According to Fung, the pseudo-strain energy function for arterial walls and lung tissue also takes the form of (2.13). In Fung’s experiments, investigators fixed $E_{12} = E_{21} = 0$, and they also found that the higher order $\gamma$ terms can be set to zero without a significant loss of accuracy. In this case, (2.13) simplifies to

$$\rho_0 W^{(2)} = f(\alpha, E) + c(\exp[F(a, E)]),$$

where

$$f(\alpha, E) = \alpha_1 E_{11}^2 + \alpha_2 E_{22}^2 + \alpha_3 E_{12}^2 + \alpha_3 E_{21}^2 + 2\alpha_4 E_{11} E_{22}$$

$$F(a, E) = a_1 E_{11}^2 + a_2 E_{22}^2 + a_3 E_{12}^2 + a_3 E_{21}^2 + 2a_4 E_{11} E_{22}.$$  

Note that $f(\alpha, E)$ and $F(a, E)$ are of the same form, and only differ by the values of the coefficients $\alpha$ and $a$.

It is noted that the first term was introduced because the data seemed biphasic. The second term expresses the behavior of the material at high stresses and the first term remedies the situation at a lower stress level. If we are concerned mainly with higher stresses and strains in the physiological range, and do not care for great accuracy at very small stress levels, then we might disregard the first term in (2.14), yielding

$$\rho_0 W^{(2)} = c(\exp[F(a, E)]).$$
The elastic responses for the full pseudo-elastic strain energy function will take the form

\[ S_{11}^{(e)} = \frac{\partial \rho_0 W^{(2)}}{\partial E_{11}} = 2(\alpha_1 E_{11} + \alpha_4 E_{22}) + 2c(a_1 E_{11} + a_4 E_{22}) \exp(a_1 E_{11}^2 + a_4 E_{22}^2) + a_3 E_{12}^2 + a_3 E_{21}^2 + 2a_4 E_{11} E_{22}) \]

\[ S_{12}^{(e)} = \frac{\partial \rho_0 W^{(2)}}{\partial E_{12}} = 2(\alpha_1 E_{12} + 2\alpha_3 E_{12}) \exp(a_1 E_{11}^2 + a_4 E_{22}^2 + a_3 E_{12}^2 + a_3 E_{21}^2 + 2a_4 E_{11} E_{22}) \]

\[ S_{21}^{(e)} = \frac{\partial \rho_0 W^{(2)}}{\partial E_{21}} = 2(\alpha_1 E_{21} + 2\alpha_3 E_{21}) \exp(a_1 E_{11}^2 + a_4 E_{22}^2 + a_3 E_{12}^2 + a_3 E_{21}^2 + 2a_4 E_{11} E_{22}) \]

\[ S_{22}^{(e)} = \frac{\partial \rho_0 W^{(2)}}{\partial E_{22}} = 2(\alpha_1 E_{11} + \alpha_4 E_{22}) + 2c(a_1 E_{11} + a_4 E_{22}) \exp(a_1 E_{11}^2 + a_4 E_{22}^2) + a_3 E_{12}^2 + a_3 E_{21}^2 + 2a_4 E_{11} E_{22}) \]

This full version of the elastic response involves 9 unknown parameters, \(c, \alpha_1, \alpha_2, \alpha_3, \alpha_4, a_1, a_2, a_3\), and \(a_4\). Depending on the attributes which we want to consider in our model, we can change the pseudo-strain energy function and resulting elastic responses accordingly. We now present different elastic response model simplifications that may be considered throughout our presentation here. These various models will be labeled models ER1, ER2, and ER3, respectively, and are defined as follows.

**ER1** In this model, we neglect the small stress term \(f(\alpha, E)\), use the pseudo strain energy function (2.14), and fix \(E_{12} = E_{21} = 0\) as did Fung, to obtain:

\[ S_{11}^{(e)} = 2c(a_1 E_{11} + a_4 E_{22}) \exp(a_1 E_{11}^2 + a_4 E_{22}^2 + 2a_4 E_{11} E_{22}) \]

\[ S_{12}^{(e)} = S_{21}^{(e)} = 0 \]

\[ S_{22}^{(e)} = 2c(a_4 E_{11} + a_2 E_{22}) \exp(a_1 E_{11}^2 + a_4 E_{22}^2 + 2a_4 E_{11} E_{22}) \]

This model involves 4 unknown parameters: \(c, a_1, a_2, \text{ and } a_4\).

**ER2** In this case, we neglect the small stress term \(f(\alpha, E)\), and use the pseudo strain-energy function (2.14), with \(E_{12} = E_{21} \) nonzero, yielding:

\[ S_{11}^{(e)} = 2c(a_1 E_{11} + a_4 E_{22}) \exp(a_1 E_{11}^2 + a_4 E_{22}^2 + 2a_3 E_{12}^2 + 2a_4 E_{11} E_{22}) \]

\[ S_{12}^{(e)} = S_{21}^{(e)} = 2c(a_3 E_{12}) \exp(a_1 E_{11}^2 + a_4 E_{22}^2 + 2a_3 E_{12}^2 + 2a_4 E_{11} E_{22}) \]

\[ S_{22}^{(e)} = 2c(a_4 E_{11} + a_2 E_{22}) \exp(a_1 E_{11}^2 + a_4 E_{22}^2 + 2a_3 E_{12}^2 + 2a_4 E_{11} E_{22}) \]

This model involves 5 unknown parameters: \(c, a_1, a_2, a_3, \text{ and } a_4\).
For the final model, we keep the small stress term \( f(\alpha, E) \), use the pseudo strain energy function (2.13), and fix \( E_{12} = E_{21} = 0 \), to result in:

\[
S_{11}^{(c)} = 2(\alpha_1 E_{11} + \alpha_4 E_{22}) + 2c(\alpha_1 E_{11} + \alpha_4 E_{22}) \exp(a_1 E_{11}^2 + a_2 E_{22}^2 + 2a_4 E_{11} E_{22})
\]
\[
S_{12}^{(c)} = S_{21}^{(c)} = 0
\]
\[
S_{22}^{(c)} = 2(\alpha_4 E_{11} + \alpha_2 E_{22}) + 2c(\alpha_4 E_{11} + \alpha_2 E_{22}) \exp(a_1 E_{11}^2 + a_2 E_{22}^2 + 2a_4 E_{11} E_{22}).
\]

This case involves 7 unknown parameters: \( c, \alpha_1, \alpha_2, \alpha_4, a_1, a_2, \) and \( a_4 \).

2.3.2 The Reduced Relaxation Function

\( G_{ijkl} \) is the reduced relaxation function tensor of rank 4, where the word “reduced” refers to the condition \( G_{ijkl} = 1 \) when \( t = 0 \). If we make the assumption that the material is isotropic \([3]\), then the reduced relaxation function will only have two independent components, and we can rewrite it as

\[
G_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}).
\] (2.16)

Since the assumption of isotropy implies that \( S_{kl}^{(c)} = S_{lk}^{(c)} \), we have

\[
G_{ijkl}(t - \tau) \frac{\partial}{\partial \tau} S_{kl}^{(c)}(\tau) = \lambda (t - \tau) \frac{\partial}{\partial \tau} (S_{11}^{(c)}(\tau) + S_{22}^{(c)}(\tau)) \delta_{ij} + 2\mu (t - \tau) \frac{\partial}{\partial \tau} S_{ij}^{(c)}(\tau).
\]

Thus, we can express Fung’s generalized viscoelastic relation for isotropic materials as

\[
S_{ij}(t) = \int_{-\infty}^{t} \left[ \lambda (t - \tau) \frac{\partial}{\partial \tau} (S_{11}^{(c)}(\tau) + S_{22}^{(c)}(\tau)) \delta_{ij} + 2\mu (t - \tau) \frac{\partial}{\partial \tau} S_{ij}^{(c)}(\tau) \right] d\tau.
\]

2.3.3 The Internal Variable Model Approximation

In \([5]\), the authors begin with Fung’s one dimensional quasi-linear viscoelastic model

\[
\sigma(t) = \int_{-\infty}^{t} G(t - \tau) \frac{\partial}{\partial \tau} \sigma^{(c)}(\lambda(\tau)) d\tau,
\]

but then adopt an internal variable approach and show that the one and two linear internal variable models were equivalent to approximating \( G \) by one or two exponential terms, respectively.

We will utilize the same methodology in the two dimensional case. We begin by approximating the parameters \( \lambda \) and \( \mu \) in (2.16) with exponentials, such that

\[
\lambda(t) = C_\lambda e^{-\nu_\lambda t}; \quad \mu(t) = C_\mu e^{-\nu_\mu t}.
\]
If we define
\[
\varepsilon_{\lambda} = \int_{-\infty}^{t} \lambda(t - \tau) \frac{\partial}{\partial \tau} (S_{11}^{(e)}(\tau) + S_{22}^{(e)}(\tau)) d\tau
\]
\[
\varepsilon_{ij}^{\mu} = \int_{-\infty}^{t} 2\mu(t - \tau) \frac{\partial}{\partial \tau} S_{ij}^{(e)}(\tau) d\tau,
\]
then \( S_{ij} = \varepsilon_{\lambda} \delta_{ij} + \varepsilon_{ij}^{\mu} \), where the dynamics of \( \varepsilon_{\lambda} \) and \( \varepsilon_{ij}^{\mu} \) are given by the following linear differential equations
\[
\frac{d\varepsilon_{\lambda}}{dt} = -\nu \varepsilon_{\lambda} + C_{\lambda} \frac{d}{dt} (S_{11}^{(e)} + S_{22}^{(e)})
\]
\[
\frac{d\varepsilon_{ij}^{\mu}}{dt} = -\nu \varepsilon_{ij}^{\mu} + 2C_{\mu} \frac{d}{dt} (S_{ij}^{(e)}).
\]
The terms \( \varepsilon_{\lambda} \) and \( \varepsilon_{ij}^{\mu} \) are therefore internal variables similar to those defined in the one dimensional case. The generalization of this idea for multiple internal variables is as follows.

Define \( S_{ij} \) using a sum of internal strain variables, as follows
\[
S_{ij} = \delta_{ij}(\varepsilon_{\lambda 1} + \varepsilon_{\lambda 2} + \cdots) + (\varepsilon_{ij}^{\mu 1} + \varepsilon_{ij}^{\mu 2} + \cdots).
\]
The dynamics of each internal variable may be given as
\[
\frac{d\varepsilon_{\lambda k}}{dt} = f_{\lambda k}(\varepsilon_{\lambda k}) + C_{\lambda k} \frac{d}{dt} (S_{11}^{(e)} + S_{22}^{(e)}), \quad \varepsilon_{\lambda k}(0) = 0
\]
\[
\frac{d\varepsilon_{ij}^{\mu k}}{dt} = f_{\mu k}(\varepsilon_{\mu k})^{ij} + 2C_{\mu k} \frac{d}{dt} (S_{ij}^{(e)}), \quad \varepsilon_{\mu k}(0) = 0.
\]
where \( f_{\lambda k} \) and \( f_{\mu k} \) are linear or nonlinear functions of \( \varepsilon_{\lambda k} \) and \( \varepsilon_{\mu k} \), respectively. The constants \( C_{\lambda k}, C_{\mu k} \), and any constants associated with \( f_{\lambda k} \) and \( f_{\mu k} \) are unknown parameters to be determined by matching with data.

2.4 A Generic Theoretical Model

Putting the equations of motion and the constitutive equations derived in the previous sections together, we are now prepared to examine the theoretical model. The
following system of equations represents a general framework for our theoretical model:

\[
\rho \frac{\partial^2 u_1}{\partial t^2} = \frac{\partial}{\partial r} \left( \epsilon_\lambda + \epsilon_\mu^{11} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left( \epsilon_\mu^{21} \right) + \frac{1}{r^2} \left( \epsilon_\mu^{12} - \epsilon_\mu^{22} \right)
\]

\[
\rho \frac{\partial^2 u_2}{\partial t^2} = \frac{\partial}{\partial r} \left( \epsilon_\lambda + \epsilon_\mu^{12} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left( \epsilon_\lambda + \epsilon_\mu^{22} \right) + \frac{1}{r^2} \left( \epsilon_\mu^{12} + \epsilon_\mu^{21} \right)
\]

\[
\frac{\partial \epsilon_{\lambda_k}}{\partial t} = -\nu_\lambda \epsilon_{\lambda_k} + C_\lambda \frac{\partial}{\partial t} \left( S_{11}^{(c)} + S_{22}^{(c)} \right)
\]

\[
\frac{\partial \epsilon_{\mu_k}}{\partial t} = -\nu_\mu \epsilon_{\mu_k} + C_\mu \frac{\partial}{\partial t} \left( S_{11}^{(c)} \right)
\]

\[
\frac{\partial \epsilon_{\mu_k}}{\partial t} = -\nu_\mu \epsilon_{\mu_k} + C_\mu \frac{\partial}{\partial t} \left( S_{12}^{(c)} \right)
\]

\[
\frac{\partial \epsilon_{\mu_k}}{\partial t} = -\nu_\mu \epsilon_{\mu_k} + C_\mu \frac{\partial}{\partial t} \left( S_{22}^{(c)} \right)
\]

Initial Conditions

We assume that the medium is initially at rest, thus \( u_1(0 : r, \theta) = 0, u_2(0 : r, \theta) = 0, v_1(0 : r, \theta) = 0, v_2(0 : r, \theta) = 0 \), and \( S_{ij}(0, r, \theta) = 0 \). For convenience, we also assume that the internal strain variables are zero at time \( t = 0 \).

Boundary Conditions

*Inner Radius:* The stress (or force) along the inner radius of our model will be governed by the impulse function which represents the disturbance due to the stenosis. Thus \( \sigma_{11}(t, R_1, \theta) = f_{11}(t, R_1, \theta) \), and \( \sigma_{22}(t, R_1, \theta) = f_{22}(t, R_1, \theta) \).

*Outer Radius:* The outer radius of the model represents a free surface, so \( \sigma_{ij}(t, R_2, \theta) = S_{ij}(t, R_2, \theta) = 0 \).

This generic theoretical model is easily configured to account for the different assumptions presented in section 2.3.1 by substituting the appropriate expressions for \( S_{ij}^{(c)} \).

### 2.5 Computational Models

To numerically integrate the model, we choose to convert it into a system of first order PDEs. This is accomplished through a series of substitutions. The variables \( v_1, v_2, w_1, w_2, w_3, \) and \( w_4 \), which are defined below, will be used to formulate all computational models. The variables which we use are defined by:

\[
v_1 = \frac{\partial u_1}{\partial t}, \quad v_2 = \frac{\partial u_2}{\partial t}, \quad w_1 = \frac{\partial u_1}{\partial r}, \quad w_2 = \frac{\partial u_1}{\partial \theta}, \quad w_3 = \frac{\partial u_2}{\partial r}, \quad w_4 = \frac{\partial u_2}{\partial \theta},
\]

\[
F(\bar{u}, \bar{\omega}, \bar{a}, r) = a_1 w_1^2 + a_2 \left( \frac{w_4 + u_1}{r} \right)^2 + 2a_3 \left( \frac{w_2 - u_2 + w_3}{r} \right)^2 + 2a_4 w_1 \left( \frac{w_4 + u_1}{r} \right)
\]
where \( \bar{a} = [a_1, a_2, a_3, a_4] \), \( \bar{w} = [w_1, w_2, w_3, w_4] \), and \( \bar{u} = [u_1, u_1] \). The state variables \( z_i \) vary, depending on the assumptions made in the model formulation, and will be defined within the presentation of each specific computational model.

### 2.5.1 ER1-1ISV

We begin by considering the simplest of the four models which we shall examine. This model employs the first elastic response, making the assumptions that \( E_{12} = E_{21} = 0 \) and that the small stress term is negligible \( (\alpha_i = 0) \). One internal strain variable is utilized in this scenario. We define the \( z_i \) variables as follows:

\[
\begin{align*}
  z_1 &= \epsilon_\lambda - 2cC_\lambda \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] \left[ (a_1 + a_4) w_1 + (a_2 + a_4) \frac{1}{r} (w_4 + u_1) \right] \\
  z_2 &= \epsilon_{\mu}^{11} - 4cC_\mu \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] \left[ a_1 w_1 + \frac{1}{r} (w_4 + u_1) \right] \\
  z_3 &= \epsilon_{\mu}^{22} - 4cC_\mu \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] \left[ a_4 w_1 + \frac{1}{r} (w_4 + u_1) \right].
\end{align*}
\]

Using the parameters defined above, we are able to rewrite our system as a first order system as follows:

\[
\begin{align*}
  \frac{\partial u_1}{\partial t} &= v_1 \\
  \frac{\partial u_2}{\partial t} &= v_2 \\
  \frac{\partial z_1}{\partial t} &= -\nu\lambda \left( z_1 + 2cC_\lambda \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] \left[ (a_1 + a_4) w_1 + (a_2 + a_4) \frac{1}{r} (w_4 + u_1) \right] \right) \\
  \frac{\partial z_2}{\partial t} &= -\nu\mu \left( z_2 + 4cC_\mu \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] \left[ a_1 w_1 + \frac{1}{r} (w_4 + u_1) \right] \right) \\
  \frac{\partial z_3}{\partial t} &= -\nu\mu \left( z_3 + 4cC_\mu \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] \left[ a_4 w_1 + \frac{1}{r} (w_4 + u_1) \right] \right) \\
  \frac{\partial v_1}{\partial t} &= \frac{1}{\rho} \left[ \frac{\partial}{\partial r} \left( z_1 + z_2 + 2cC_\lambda \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] \left[ (a_1 + a_4) w_1 + (a_2 + a_4) \frac{1}{r} (w_4 + u_1) \right] \right) \right. \\
&\quad + \frac{1}{r} \left( z_2 - z_3 + 4cC_\mu \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] \left[ a_1 w_1 + \frac{1}{r} (w_4 + u_1) \right] \right) \\
&\quad \left. + 4cC_\mu \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] \left[ a_4 w_1 + \frac{1}{r} (w_4 + u_1) \right] \right) \\
  \frac{\partial v_2}{\partial t} &= \frac{1}{\rho r \partial \theta} \left[ z_1 + z_2 + 2cC_\lambda \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] \left[ (a_1 + a_4) w_1 + (a_2 + a_4) \frac{1}{r} (w_4 + u_1) \right] \right. \\
&\quad + \frac{1}{r} \left( z_2 - z_3 + 4cC_\mu \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] \left[ a_1 w_1 + \frac{1}{r} (w_4 + u_1) \right] \right) \\
&\quad \left. + 4cC_\mu \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] \left[ a_4 w_1 + \frac{1}{r} (w_4 + u_1) \right] \right) \\
&\quad + 4cC_\mu \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] \left[ a_4 w_1 + \frac{1}{r} (w_4 + u_1) \right].
\end{align*}
\]
\[
\frac{\partial w_1}{\partial t} = \frac{\partial v_1}{\partial r} \\
\frac{\partial w_4}{\partial t} = \frac{\partial v_2}{\partial q}.
\]

2.5.2 ER2-1ISV

The next model which we present assumes that the small stress term is negligible and utilizes one internal strain variable. We define the \( z_i \) variables to be

\[
\begin{align*}
    z_1 &= \epsilon_\lambda - 2cC_\lambda \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] \left[ (a_1 + a_4) w_1 + (a_2 + a_4) \frac{1}{r}(w_4 + u_1) \right] \\
    z_2 &= \epsilon_{11} - 4cC_\mu \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] \left[ a_1 w_1 + \frac{a_4}{r}(w_4 + u_1) \right] \\
    z_3 &= \epsilon_{12} \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] \left[ a_3 \left( \frac{w_2 - u_2}{r} + w_3 \right) \right] \\
    z_4 &= \epsilon_{22} - 4cC_\mu \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] \left[ a_4 w_1 + \frac{a_2}{r}(w_4 + u_1) \right].
\end{align*}
\]

Substitution of these defined variables (as well as those defined at the beginning of this section) into the theoretical model yields the following computational model.

\[
\begin{align*}
    \frac{\partial u_1}{\partial t} &= v_1 \\
    \frac{\partial u_2}{\partial t} &= v_2 \\
    \frac{\partial z_1}{\partial t} &= -\nu_\lambda \left( z_1 + 2cC_\lambda \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] \left[ (a_1 + a_4) w_1 + (a_2 + a_4) \frac{1}{r}(w_4 + u_1) \right] \right) \\
    \frac{\partial z_2}{\partial t} &= -\nu_\mu \left( z_2 + 4cC_\mu \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] \left[ a_1 w_1 + \frac{a_4}{r}(w_4 + u_1) \right] \right) \\
    \frac{\partial z_3}{\partial t} &= -\nu_\mu \left( z_3 + 4cC_\mu \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] \left[ a_3 \left( \frac{w_2 - u_2}{r} + w_3 \right) \right] \right) \\
    \frac{\partial z_4}{\partial t} &= -\nu_\mu \left( z_4 + 4cC_\mu \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] \left[ a_4 w_1 + \frac{a_2}{r}(w_4 + u_1) \right] \right) \\
    \frac{\partial u_1}{\partial t} &= \frac{1}{\rho} \left[ \frac{\partial}{\partial r} \left( z_1 + z_2 + 2cC_\lambda \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] \left[ (a_1 + a_4) w_1 + (a_2 + a_4) \frac{1}{r}(w_4 + u_1) \right] \right) \\
    &\quad + 4cC_\mu \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] \left[ a_1 w_1 + \frac{a_4}{r}(w_4 + u_1) \right] \right) \\
    &\quad + \frac{1}{r} \left( z_3 + 4cC_\mu \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] \left[ a_3 \left( \frac{w_2 - u_2}{r} + w_3 \right) \right] \right) \\
    &\quad + z_2 - z_3 + 4cC_\mu \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] \left[ a_4 w_1 + \frac{a_2}{r}(w_4 + u_1) \right] \\
    &\quad - 4cC_\mu \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] \left[ a_4 w_1 + \frac{a_2}{r}(w_4 + u_1) \right].
\end{align*}
\]
\[
\frac{\partial v_2}{\partial t} = \frac{1}{\rho} \left[ \frac{\partial}{\partial r} \left( z_3 + 4\alpha \exp[F(\bar{u}, \bar{\varphi}, a, r)] \left[ a_3 \left( \frac{w_2 - u_2}{r} + w_3 \right) \right] \right) + \frac{1}{r} \left( \frac{\partial}{\partial \theta} (z_1 + z_4 + 2\alpha \exp[F(\bar{u}, \bar{\varphi}, a, r)] \left[ (a_1 + a_4) w_1 + (a_2 + a_4) \frac{1}{r} (w_4 + u_1) \right] \right) \right)
\]

\[
\frac{\partial w_1}{\partial t} = \frac{\partial v_1}{\partial r},
\frac{\partial w_2}{\partial t} = \frac{\partial v_1}{\partial \theta},
\frac{\partial w_3}{\partial t} = \frac{\partial v_2}{\partial r},
\frac{\partial w_4}{\partial t} = \frac{\partial v_2}{\partial \theta}.
\]

### 2.5.3 ER3-1ISV

We also wish to examine the model which employs the third elastic response, with one internal strain variable. In this model, we keep the small stress terms, and assume that \(E_{12} = E_{21} = 0\). We define the \(z_i\) variables for this model in the following manner:

\[
z_1 = \epsilon_\lambda - 2\alpha \left[ (a_1 + a_4) w_1 + (a_4 + a_2) \frac{w_4 + u_1}{r} \right]
\]

\[
-2\alpha \exp[F(\bar{u}, \bar{\varphi}, a, r)] \left[ (a_1 + a_4) w_1 + (a_2 + a_4) \frac{1}{r} (w_4 + u_1) \right]
\]

\[
z_2 = \epsilon_{\mu}^{11} - 4\alpha \left[ a_1 w_1 + a_4 \frac{w_4 + u_1}{r} \right] - 4\alpha \exp[F(\bar{u}, \bar{\varphi}, a, r)] \left[a_1 w_1 + a_4 \frac{w_4 + u_1}{r} \right]
\]

\[
z_3 = \epsilon_{\mu}^{22} - 4\alpha \left[ a_2 w_1 + a_4 \frac{w_4 + u_1}{r} \right] - 4\alpha \exp[F(\bar{u}, \bar{\varphi}, a, r)] \left[a_2 w_1 + a_4 \frac{w_4 + u_1}{r} \right]
\]

Utilizing these substitution parameters, we present the the following computational model for ER3-1ISV:

\[
\frac{\partial u_1}{\partial t} = v_1,
\frac{\partial u_2}{\partial t} = v_2,
\frac{\partial z_1}{\partial t} = -\nu \left[ z_1 + 2\alpha \left[ (a_1 + a_4) w_1 + (a_4 + a_2) \frac{w_4 + u_1}{r} \right] + 2\alpha \exp[F(\bar{u}, \bar{\varphi}, a, r)] \left[a_1 w_1 + a_4 \frac{1}{r} (w_4 + u_1) \right] \right]
\]
\[
\frac{\partial z_2}{\partial t} = -\nu_\mu \left( z_2 + 4C_\mu \left[ \alpha_1 w_1 + \frac{\alpha_4 w_4 + u_1}{r} \right] + 4cC_\mu \exp \left[ F(\bar{a}, \bar{\omega}, \bar{n}, r) \right] [a_1 w_1 + \frac{a_4}{r}(w_4 + u_1)] \right)
\]
\[
\frac{\partial z_3}{\partial t} = -\nu_\mu \left( z_3 + 4C_\mu \left[ \alpha_4 w_1 + \frac{\alpha_2 w_4 + u_1}{r} \right] + 4cC_\mu \exp \left[ F(\bar{a}, \bar{\omega}, \bar{n}, r) \right] [a_4 w_1 + \frac{a_2}{r}(w_4 + u_1)] \right)
\]
\[
\frac{\partial v_1}{\partial t} = \frac{1}{\rho} \left[ \frac{\partial}{\partial r} \left( z_1 + z_2 + 2cC_\lambda \left[ (\alpha_1 + \alpha_4) w_1 + (\alpha_2 + \alpha_4) \frac{w_4 + u_2}{r} \right] \right) + 2cC_\lambda \exp \left[ F(\bar{a}, \bar{\omega}, \bar{n}, r) \right] \left[ (a_1 + a_4) w_1 + (a_2 + a_4) \frac{1}{r}(w_4 + u_1) \right] + 4C_\mu \left[ \alpha_1 w_1 + \alpha_4 \frac{w_4 + u_1}{r} \right] + 4cC_\mu \exp \left[ F(\bar{a}, \bar{\omega}, \bar{n}, r) \right] \left[ a_1 w_1 + \frac{a_4}{r}(w_4 + u_1) \right] \right)
\]
\[
\frac{\partial v_2}{\partial t} = \frac{1}{\rho r \partial \theta} \left[ z_1 + z_3 + 2C_\lambda \left[ (\alpha_1 + \alpha_4) w_1 + (\alpha_4 + \alpha_2) \frac{w_4 + u_1}{r} \right] \right] + 2cC_\lambda \exp \left[ F(\bar{a}, \bar{\omega}, \bar{n}, r) \right] \left[ (a_1 + a_4) w_1 + (a_2 + a_4) \frac{1}{r}(w_4 + u_1) \right] + 4C_\mu \left[ \alpha_4 w_1 + \alpha_2 \frac{w_4 + u_1}{r} \right] + 4cC_\mu \exp \left[ F(\bar{a}, \bar{\omega}, \bar{n}, r) \right] \left[ a_4 w_1 + \frac{a_2}{r}(w_4 + u_1) \right]
\]
\[
\frac{\partial w_1}{\partial t} = \frac{\partial v_1}{\partial r}
\]
\[
\frac{\partial w_4}{\partial t} = \frac{\partial v_2}{\partial \theta}
\]

2.5.4 ER1-2ISV

Finally, we consider the model which assumes the first elastic response, with two internal strain variables. The \( z_i \) variables for this model are given by:

\[
z_1 = \epsilon_{\lambda_1} - 2cC_{\lambda_1} \exp \left[ F(\bar{a}, \bar{\omega}, \bar{n}, r) \right] \left[ (a_1 + a_4) w_1 + (a_2 + a_4) \frac{w_4 + u_1}{r} \right]
\]
\[
z_2 = \epsilon_{\lambda_2} - 2cC_{\lambda_2} \exp \left[ F(\bar{a}, \bar{\omega}, \bar{n}, r) \right] \left[ (a_1 + a_4) w_1 + (a_2 + a_4) \frac{w_4 + u_1}{r} \right]
\]
\[
\begin{align*}
  z_3 & = \epsilon_{\mu_1}^{11} - 4cC_{\mu_1} \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] \left[ a_1 w_1 + a_4 \frac{w_4 + u_1}{r} \right] \\
  z_4 & = \epsilon_{\mu_2}^{12} - 4cC_{\mu_2} \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] \left[ a_4 w_1 + a_2 \frac{w_4 + u_1}{r} \right] \\
  z_5 & = \epsilon_{\mu_1}^{22} - 4cC_{\mu_1} \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] \left[ a_1 w_1 + a_4 \frac{w_4 + u_1}{r} \right] \\
  z_6 & = \epsilon_{\mu_2}^{22} - 4cC_{\mu_2} \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] \left[ a_4 w_1 + a_2 \frac{w_4 + u_1}{r} \right].
\end{align*}
\]

Making the appropriate substitutions and manipulations to the theoretical model presented in section 2.4, we find the system of first order equations that serve as the computational model for ER1-2ISV to be:

\[
\begin{align*}
  \frac{\partial u_1}{\partial t} & = v_1 \\
  \frac{\partial u_2}{\partial t} & = v_2 \\
  \frac{\partial z_1}{\partial t} & = -\nu_{\lambda_1} \left( z_1 + 2cC_{\lambda_1} \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] \left[ (a_1 + a_4) w_1 + (a_2 + a_4) \frac{1}{r} (w_4 + u_1) \right]\right) \\
  \frac{\partial z_2}{\partial t} & = -\nu_{\lambda_2} \left( z_2 + 2cC_{\lambda_2} \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] \left[ (a_1 + a_4) w_1 + (a_2 + a_4) \frac{1}{r} (w_4 + u_1) \right]\right) \\
  \frac{\partial z_3}{\partial t} & = -\nu_{\mu_1} \left( z_3 + 4cC_{\mu_1} \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] \left[ a_1 w_1 + \frac{a_4}{r} (w_4 + u_1) \right]\right) \\
  \frac{\partial z_4}{\partial t} & = -\nu_{\mu_2} \left( z_4 + 4cC_{\mu_2} \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] \left[ a_1 w_1 + \frac{a_4}{r} (w_4 + u_1) \right]\right) \\
  \frac{\partial z_5}{\partial t} & = -\nu_{\mu_1} \left( z_5 + 4cC_{\mu_1} \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] \left[ a_4 w_1 + \frac{a_2}{r} (w_4 + u_1) \right]\right) \\
  \frac{\partial z_6}{\partial t} & = -\nu_{\mu_2} \left( z_6 + 4cC_{\mu_2} \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] \left[ a_4 w_1 + \frac{a_2}{r} (w_4 + u_1) \right]\right) \\
  \frac{\partial v_1}{\partial t} & = \frac{1}{\rho} \left[ \frac{\partial}{\partial r} (z_1 + z_2 + z_3 + z_4 + 2c(C_{\lambda_1} + C_{\lambda_2}) \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] [(a_1 + a_4) w_1 \\
  + (a_2 + a_4) \frac{1}{r} (w_4 + u_1)] + 4c(C_{\mu_1} + C_{\mu_2}) \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] [a_1 w_1 \\
  + \frac{a_4}{r} (w_4 + u_1)] \right] + \frac{1}{r} (z_3 + z_4 - z_5 - z_6 + 4c(C_{\mu_1} + C_{\mu_2}) \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] [(a_1 + a_4) w_1 \\
  + (a_2 + a_4) \frac{1}{r} (w_4 + u_1)] - 4c(C_{\mu_1} + C_{\mu_2}) \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] [a_4 w_1 \\
  + \frac{a_2}{r} (w_4 + u_1)]) \right] \\
  \frac{\partial v_2}{\partial t} & = \frac{11}{\rho r} \left[ \frac{\partial}{\partial \theta} (z_1 + z_2 + z_5 + z_6 + 2c(C_{\lambda_1} + C_{\lambda_2}) \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] [(a_1 + a_4) w_1 \\
  + (a_2 + a_4) \frac{1}{r} (w_4 + u_1)] + 4c(C_{\mu_1} + C_{\mu_2}) \exp[F(\bar{a}, \bar{w}, \bar{a}, r)] [a_4 w_1 \\
  + \frac{a_2}{r} (w_4 + u_1)] \right].
\end{align*}
\]
\[
\frac{\partial w_1}{\partial t} = \frac{\partial v_1}{\partial r} \\
\frac{\partial w_A}{\partial t} = \frac{\partial v_2}{\partial \theta}
\]
Chapter 3

Calibration of the Model with One Dimensional Results

Now that the models have been formulated, we wish to test their utility, differences, and appropriateness. By considering a homogeneous material geometry which is independent of $\theta$, we reduce the two dimensional problem to a one dimensional problem. With this in mind, if our model has been derived correctly, we should be able to reproduce the acceleration results presented in [5], depicted in Figure 3.1, by considering a homogeneous geometry with no $\theta$ dependencies.

3.1 Computational Methods

We utilize the MacCormack finite difference scheme (see [23, 40]) to numerically integrate each system presented in section 2.5. The MacCormack scheme is a two-step integration scheme. The first step utilizes forward differences and is followed by a step of backward differences. The scheme is second order in space and time, which is verified in section 3.1.1.

To update boundary terms, we use direction cosines, also known as the method of boundary characteristics [19]. If we define the components of vector $U$ to be our model
parameters, then we can rewrite the first order system in matrix-vector notation as follows:

\[ U_t = AU_r + BU_q + Q, \]

where \( A \) is the matrix of coefficients corresponding to \( U_r \), \( B \) is the matrix of coefficients corresponding to \( U_q \), and \( Q \) is a vector of terms that do not involve any derivatives. For example, if we consider the simplest model, ER1-IIIV, we have the following matrix for \( A \),

\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\frac{b_1}{p} & 0 & \frac{1}{p} & \frac{1}{p} & 0 & 0 & 0 & \frac{b_2}{p} & \frac{b_3}{p}
\end{bmatrix}
\]

\[
\lambda_A = 0 \pm \sqrt{\frac{b_2}{p}}
\]

where \( b_1, b_2, \) and \( b_3 \) are expressions of \( w_1, w_4, u_1, r \), and the undetermined constants.

To apply the method of direction cosines, we examine the eigenvalues of the matrices \( A \), and \( B \), depending on the boundary being considered. Let us begin by considering the boundary along the inner and outer radii. To employ direction cosines on these boundaries,
Figure 3.2: Positive and negative characteristics interacting with domain boundaries

we must consider the matrix $A$. The eigenvalues of the matrix correspond to the characteristics of the problem. If the eigenvalue is positive, the characteristic speed is positive, giving the characteristic a positive slope. If the eigenvalue is negative, then the characteristic has a negative slope, and negative speed. Figure 3.2 represents our geometry, and depicts how characteristics with positive and negative speeds ($C_+$ and $C_-$, respectively) interact with the inner and outer radius boundaries.

Along the inner radius, represented by the left boundary in Figure 3.2, the characteristic with positive slope points into the domain. This means that information propagates from the boundary, and so the interior points will depend on boundary information. We must prescribe a physical boundary condition for any characteristics that point into the domain. The characteristic with negative slope points outward from the domain. This implies that the boundary data is dependent on the interior data. For characteristics that point out of the domain, the boundary condition must be updated numerically, based on the interior data.

Revisiting the ER1-IISPV model, we have determined that the eigenvalues of the $A$ matrix are $\lambda_A = 0, \pm \sqrt{\frac{b_2}{\rho}}$. That is, there is one positive eigenvalue, one negative eigenvalue, and seven eigenvalues which are zero. As explained above, on the inner radius, the positive eigenvalue corresponds to a characteristic which will point into the domain. Thus, we shall have one prescribed physical boundary condition on the inner radius. In order to reproduce the data presented in [5], our physical boundary condition on the inner radius is $w_1(R_1, t) = f(t)$, where $f(t)$ is the same impulse function used in [5], depicted in Figure 3.3. Boundary values for the other eight variables are determined from interior data points.
at the previous time step using the second order discretization

\[ U^{n+1} = U^n + \Delta t [AU_r + BU_\theta + Q]^n + \frac{1}{2}\Delta t^2 [AU_r + BU_\theta + Q]^n. \]  

(3.1)

Examining the outer boundary, we use a similar argument. Since there is only one negative eigenvalue, there can only be one characteristic pointing into the domain, and thus, we can only prescribe one physical boundary condition. We define the physical boundary condition to be \( w_1(R_2, t) = 0 \). Once again, the other boundary terms are updated using (3.1).

Finally, we can use the periodicity of the geometry to deal with \( U \) at \( \theta = 0 \) and \( \theta = 2\pi \). Since our geometry is circular, we know that the parameter values at \( \theta = 0 \) and \( \theta = 2\pi \) must be the same. This constitutes the final set of boundary conditions for our model.

### 3.1.1 A Sample Grid Refinement Study

To verify the accuracy of our code, we performed a grid refinement study. Since our goal is to use this 2 dimensional model to reproduce the results presented in [5], we choose exact solutions similar to those utilized in the grid refinement presented in that
paper. We defined our exact solutions to be

\[ u_1(t, r, \theta) = -2(1 - \cos(300t)) \left( \frac{r - R_1}{R_2 - R_1} \right)^2 \]

\[ u_2(t, r, \theta) = -2(1 - \cos(300t)) \left( \frac{r - R_1}{R_2 - R_1} \right)^2 \]

\[ \epsilon_\lambda(t, r, \theta) = \epsilon_\mu(t, r, \theta) = \epsilon_\nu(t, r, \theta) = .1 \sin(300t) \left( \frac{R_2 - r}{R_2 - R_1} \right)^2 \]

From our defined exact functions, we are able to calculate \( v_1, v_2, w_1, w_2, z_1, z_2, z_3 \), and the corresponding forcing functions appropriately. The model is integrated from \( t = 0 \) to \( t = 1 \).

To perform the refinement, we fix the ratios \( \Delta t / \Delta r \) and \( \Delta t / \Delta \theta \), run the forward problem, compute the \( L^\infty \) error for each model variable, halve the spatial and temporal step sizes, and repeat. The order of accuracy is determined by examining the ratio

\[ \frac{\ln \left( \frac{||E_n(u_i)||}{||E_{n+1}(u_i)||} \right)}{\ln \left( \frac{h_n}{h_{n+1}} \right)} \]

where \( U_i \) represents the model variable being examined, and \( h_{n,r} \) and \( h_{n,\theta} \) are the grid mesh sizes. Tables 3.1 - 3.3 display the grid refinement results for \( u_1, u_2, v_1, v_2, \) and \( w_1 \), as computed using the ER1-IISV version of the model.

**Table 3.1: Grid Refinement for \( u_1 \) and \( u_2 \) using ER1-IISV**

| \( n \) | \( ||E_n(u_1)|| / ||E_{n+1}(u_1)|| \) | order | \( ||E_n(u_2)|| / ||E_{n+1}(u_2)|| \) | order |
|---|---|---|---|---|
| 10 | 2.0576e-2 | 4.506 | 2.1718 | 2.066e-2 | 4.5199 | 2.1762 |
| 20 | 4.5667e-3 | 4.171 | 2.0603 | 4.5717e-3 | 4.1599 | 2.0565 |
| 40 | 1.0949e-3 | 4.106 | 2.0377 | 1.0989e-3 | 4.0658 | 2.0208 |
| 80 | 2.6665e-4 | 4.216 | 2.0759 | 2.7080e-4 | 4.0239 | 2.0086 |
| 160 | 6.3243e-5 | 4.856 | 2.2798 | 6.7298e-5 | 4.0111 | 2.0040 |
| 320 | 1.3021e-5 | 1.6778e-5 | | | |

**Table 3.2: Grid Refinement for \( v_1 \) and \( v_2 \) using ER1-IISV**

| \( n \) | \( ||E_n(v_1)|| / ||E_{n+1}(v_1)|| \) | order | \( ||E_n(v_2)|| / ||E_{n+1}(v_2)|| \) | order |
|---|---|---|---|---|
| 10 | 5.6112 | 4.2625 | 2.0916 | 5.6114 | 4.2626 | 2.0917 |
| 20 | 1.3164 | 4.1013 | 2.0361 | 1.3164 | 4.1013 | 2.0361 |
| 40 | 0.3209 | 4.0442 | 2.0159 | 0.3209 | 4.0439 | 2.0157 |
| 80 | 7.9373e-2 | 4.0216 | 2.0078 | 7.9373e-2 | 4.0203 | 2.0073 |
| 160 | 1.9735e-2 | 4.0148 | 2.0039 | 1.9735e-2 | 4.0097 | 2.0035 |
| 320 | 4.9155e-3 | 4.9237e-3 | | | |

The grid refinement results displayed in Tables 3.1 - 3.3 verify that our implementation of the MacCormack scheme, with the previously described boundary conditions,
is second order accurate, as expected. Grid refinement studies for the other models were conducted, with results analogous to those presented.

3.2 Computational Results

Having successfully completed a grid refinement study, we now wish to see if we can use the model to reproduce the one dimensional acceleration results. We integrate our model as described in section 2.5, and use it to generate an acceleration solution for the outer boundary of our geometry. This solution is what we will compare to the results from [5]. Since our parameters are unknown, we must seek a set of optimal parameters for each variation of the model, and use the optimal parameter set to generate the solution mentioned above. We use Matlab’s \textit{fminsearch} command (an implementation of the Nelder-Mead simplex algorithm) to minimize the difference between the model output and the original acceleration data, and to identify the optimal parameters which generate the model output.

The optimal parameters for each variation of the model are displayed in Tables 3.4-3.7. Also, in Figures 3.4-3.7 we compare the acceleration data from [5] and the optimized solution from the models presented in section 2.5. Each figure contains two plots. The top plot compares the normalized acceleration data to the normalized solution generated by our model, and the bottom plot compares the normalized fast Fourier transform (FFT) of the output data to the normalized fast Fourier transform of our solution. This plot indicates which frequencies are excited by each model. In all plots, the original data is marked with a solid line; the model solution is indicated with a dashed line.

Table 3.4 and Figure 3.4 exhibit data from model ERI-1ISV, Table 3.5 and Figure
Table 3.4: Optimal parameters for model ER1-1ISV

<table>
<thead>
<tr>
<th>$C_\lambda$</th>
<th>$\nu_\lambda$</th>
<th>$C_\mu$</th>
<th>$\nu_\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6366.919</td>
<td>768.266</td>
<td>1.542</td>
<td>1811.847</td>
</tr>
</tbody>
</table>

Figure 3.4: Comparison of ER1-1ISV results to data.

Table 3.5: Optimal parameters for model ER2-1ISV

<table>
<thead>
<tr>
<th>$C_\lambda$</th>
<th>$\nu_\lambda$</th>
<th>$C_\mu$</th>
<th>$\nu_\mu$</th>
<th>$a_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6683.567</td>
<td>795.935</td>
<td>2.788</td>
<td>3678.888</td>
<td>-1.888</td>
</tr>
</tbody>
</table>

Figure 3.5: Comparison of ER2-1ISV results to data.
Table 3.6: Optimal parameters for model ER3-IISV

<table>
<thead>
<tr>
<th>$C_\lambda$</th>
<th>$\nu_\lambda$</th>
<th>$C_\mu$</th>
<th>$\nu_\mu$</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$\alpha_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1853.634</td>
<td>250.178</td>
<td>2462.783</td>
<td>2061.129</td>
<td>1.273</td>
<td>1.882</td>
<td>0.661</td>
</tr>
</tbody>
</table>

Figure 3.6: Comparison of ER3-IISV results to data.

Table 3.7: Optimal parameters for model ER1-2ISV

<table>
<thead>
<tr>
<th>$C_{\lambda_1}$</th>
<th>$C_{\lambda_2}$</th>
<th>$\nu_{\lambda_1}$</th>
<th>$\nu_{\lambda_2}$</th>
<th>$C_{\mu_1}$</th>
<th>$C_{\mu_2}$</th>
<th>$\nu_{\mu_1}$</th>
<th>$\nu_{\mu_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1187.432</td>
<td>1432.646</td>
<td>215.841</td>
<td>215.842</td>
<td>386.639</td>
<td>3456.749</td>
<td>38.514</td>
<td>2093.164</td>
</tr>
</tbody>
</table>

Figure 3.7: Comparison of ER1-2ISV results to data.
3.5 present data from model ER2-IISV, Table 3.6 and Figure 3.6 display data from model ER3-IISV, and data from model ER1-2ISV is shown in Table 3.7 and Figure 3.7. Clearly, models ER1-2ISV and ER3-IISV produce a better approximation to the data than the other models do.

3.3 Statistical Model Comparison

With the successful implementation of the four different versions of our model, it is natural to ask which model gives the best fit. In other words, are the extra parameters introduced in the more complicated models really necessary? Examination of the graphs of each model’s output in comparison to the data, by itself, is not a sufficient method of comparing the significance of additional parameters to our modeling efforts. For this reason, we will use a statistical test presented in [6]. This method involves the examination of two models with parameter sets \( \tilde{Q} \) and \( Q_0 \), where \( Q_0 \subset \tilde{Q} \). Using a least squares minimization routine, we find \( \tilde{q} \in Q_0 \) and \( \bar{q} \in \tilde{Q} \), such that \( \tilde{q} \) and \( \bar{q} \) minimize the following cost functional over their respective parameter sets.

\[
J(q) = \sum_{i=1}^{N} |u(t_i; q) - z_i|^2,
\]

where \( z_i \) is the measured data, \( u(t_i; q) \) is the modeled approximation, and \( N \) is the number of data points considered. For each of the statistical significance tests which we present in this section, there were 2048 data points, so \( N = 2048 \).

Next, we use the values \( J(\tilde{q}) \) and \( J(\bar{q}) \) to create a test statistic. The test statistic \( U_N \) is defined as

\[
U_N = N \frac{J(\tilde{q}) - J(\bar{q})}{J(\bar{q})}.
\]

We are now ready to test the significance of the extra parameters used in the model formulated with parameter set \( \tilde{Q} \). We formulate a null hypothesis, that the correct parameter set is \( Q_0 \) (i.e. all of the extra parameters in \( \tilde{Q} \) are equal to zero). From [6], we know that the test statistic \( U_N \) approaches a \( \chi^2 \) distribution as \( N \) tends to \( \infty \). Thus, we compare the test statistic to a \( \chi^2 \) table with the degrees of freedom equal to the difference in the number of parameters in the two sets. If our test statistic \( U \) is greater than the \( \tau \) value on the \( \chi^2 \) table for a given \( \alpha \), then the test suggests that we reject the null hypothesis with a
(1 − α) × 100% confidence level. The statistical model comparison results are now presented for various models, including the model presented in [5].

1 Dimensional Model - 1 ISV vs. 2 ISV from [5]

Here, we compare the one dimensional model employing one internal strain variable to the one dimensional model employing two internal strain variables. Using the cost functionals in Table 3.8, we compute the test statistic $U = 235,653.29$ and compare it to the $\chi^2$ values in Table 3.9. Since the difference in the number of parameters between the two models is two, we need to look at the values of the $\chi^2$ table with two degrees of freedom. Comparison of our test statistic to $\chi^2(2)$ shows that $U > \tau$ for all confidence levels. Thus,

**Table 3.8: Data Comparison 1D 1ISV vs. 1D 2ISV**

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Cost $J$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_0$ $(C_1, 0, \nu_1, 0, \alpha)$</td>
<td>$J(\tilde{q}) = 23.2324$</td>
</tr>
<tr>
<td>$Q$ $(C_1, C_2, \nu_1, \nu_2, \alpha)$</td>
<td>$J(\tilde{q}) = 0.200167$</td>
</tr>
</tbody>
</table>

we reject our null hypothesis, leading us to the conclusion that the extra unknown parameters are statistically significant to the model. This confirms what the graphs presented in the one dimensional project led us to believe; that the model employing two internal strain variables has a better fit to the data.

**2 Dimensional Model - ER1-1ISV vs. ER1-2ISV**

We expect that the results from the 2 dimensional model should be similar to the results presented for the 1 dimensional model. Examination of Figure 3.7 suggests that a model employing 2 internal strain variables gives a good fit to our data. We now present a statistical test to verify this assumption. Table 3.10 displays the parameter sets for the two models, and their corresponding cost values.

Utilizing the cost values, we can compute the test statistic $U = 85412.52$. Since the model employing two internal strain variables has 4 parameters more than the original
model, we compare this test statistic with the $\chi^2$ values for 4 degrees of freedom. Clearly, the test statistic is larger than the $\tau$ values for each $\alpha$ level, thus, we reject the null hypothesis at all confidence levels. This strongly suggests that the model employing 2 internal strain variables provides a better fit to data than the original model does, and so the extra parameters are statistically significant.

2 Dimensional Model - ER1-1ISV vs. ER2-1ISV

In the original model, we assumed that $E_{12} = E_{21} = 0$. The model that incorporates the second elastic response assumed that $E_{12} = E_{21} \neq 0$, adding an extra parameter to the system. We wish to test this extra parameter for significance in the model. The parameter sets and optimal costs for the models employing ER1 and ER2 are presented in Table 3.11. With the optimal costs for each model, we can compute a test statistic $U = 1.5298$.

<table>
<thead>
<tr>
<th>parameters</th>
<th>cost $J$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_0$</td>
<td>$(C_{\lambda}, \nu_{\lambda}, C_{\mu}, \nu_{\mu}, 0, 0, 0, 0)$</td>
</tr>
<tr>
<td>$Q$</td>
<td>$(C_{\lambda}, \nu_{\lambda}, C_{\mu}, \nu_{\mu}, \nu_{\lambda 2}, C_{\mu 2}, \nu_{\mu 2})$</td>
</tr>
</tbody>
</table>

Table 3.11: Data Comparison 2D ER1-1ISV vs. 2D ER2-1ISV

With only one additional parameter, we compare the test statistic to the $\tau$ values for $\chi^2(1)$ in Table 3.9. The test statistic is less than $\tau$ for all $\alpha$ values in the table, thus we accept the null hypothesis, that the extra parameter, $a_3$, should be taken equal to zero.

2 Dimensional Model - ER1-1ISV vs. ER3-1ISV

Finally, we examine the assumption that we can ignore the small stress term in the elastic response. ER1 was formed with the assumption that it could be ignored, while ER3 was formed assuming that it was necessary to the model. The inclusion of the small stress term adds three additional parameters to the model. The parameters and optimal costs for each model formulation can be seen in Table 3.12. From the optimal costs, we compute a test statistic $U = 82.920.87$, and we are ready to conduct the statistical test. Since there are three additional parameters, we'll compare the test statistic with the $\tau$ values of $\chi^2(3)$. The test statistic is significantly larger than the $\tau$ values at all levels of $\alpha$,
Table 3.12: Data Comparison 2D ER1-1ISV vs. 2D ER3-1ISV

<table>
<thead>
<tr>
<th>parameters</th>
<th>cost $J$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_0$ $(C_{\lambda}, \nu_{\lambda}, C_{\mu}, \nu_{\mu}, 0, 0, 0)$</td>
<td>$J(\tilde{q}) = 18.47514$</td>
</tr>
<tr>
<td>$Q$ $(C_{\lambda}, \nu_{\lambda}, C_{\mu}, \nu_{\mu}, \alpha_1, \alpha_2, \alpha_4)$</td>
<td>$J(\tilde{q}) = 0.4446687$</td>
</tr>
</tbody>
</table>

Figure 3.8: Comparison of 1D acceleration data to 2D approximation of acceleration computed using model ER3-2ISV.

thus we can reject the null hypothesis at a 99.9% confidence level. This suggests that the model is better formulated with the inclusion of the small stress term.

**Combining Significant Models**

The preceding results lead us to believe that we get a better fit from the model employing the third elastic response, and also that incorporating 2 internal strain variables allows for an improved model. These results prompted us to consider combining the two models, that is, we now examine a model that utilizes two internal strain variables, along with the third elastic response. Figure 3.8 displays a comparison of the acceleration data with the new model’s approximation. Examination of the graph shows a good fit to the data, but it is hard to determine whether it is a “better” fit than the previous models.

Tables 3.13 and 3.14 display the functional costs needed to compare the combined model with the individual models. Using the cost functionals in Table 3.13, we compute the test statistic to compare the model using two internal strain variables with the first elastic response to the model using two ISV’s with the third elastic response. The test statistic will have value $U = 8.747$. Comparison of this test statistic with the $\chi^2(3)$ table, we see
that the extra parameters are statistically significant with a 90% confidence.

Table 3.13: Data Comparison ER1-2ISV vs. ER3-2ISV

<table>
<thead>
<tr>
<th></th>
<th>parameters</th>
<th>cost $J$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_0$</td>
<td>$(C_{\lambda_1}, \nu_{\lambda_1}, C_{\mu_1}, \nu_{\mu_1}, C_{\lambda_2}, \nu_{\lambda_2}, C_{\mu_2}, \nu_{\mu_2}, 0, 0, 0)$</td>
<td>$J(\hat{\theta}) = 0.43261$</td>
</tr>
<tr>
<td>$Q$</td>
<td>$(C_{\lambda_1}, \nu_{\lambda_1}, C_{\mu_1}, \nu_{\mu_1}, C_{\lambda_2}, \nu_{\lambda_2}, C_{\mu_2}, \nu_{\mu_2}, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$</td>
<td>$J(\bar{\theta}) = 0.430770$</td>
</tr>
</tbody>
</table>

Table 3.14: Data Comparison ER3-1ISV vs. ER3-2ISV

<table>
<thead>
<tr>
<th></th>
<th>parameters</th>
<th>cost $J$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_0$</td>
<td>$(C_{\lambda_1}, \nu_{\lambda_1}, C_{\mu_1}, \nu_{\mu_1}, 0, 0, 0, \alpha_1, \alpha_2, \alpha_3)$</td>
<td>$J(\hat{\theta}) = 0.444668$</td>
</tr>
<tr>
<td>$Q$</td>
<td>$(C_{\lambda_1}, \nu_{\lambda_1}, C_{\mu_1}, \nu_{\mu_1}, C_{\lambda_2}, \nu_{\lambda_2}, C_{\mu_2}, \nu_{\mu_2}, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$</td>
<td>$J(\bar{\theta}) = 0.430770$</td>
</tr>
</tbody>
</table>

Similarly, examination of Table 3.14 leads us to the test statistic $U = 66.075$ for the comparison of ER3-1ISV vs. ER3-2ISV. We use the $\chi^2(4)$ values to conduct the hypothesis testing, and we see that the extra parameters are significantly significant for all confidence levels. Since the transition from model ER1-2ISV to model ER3-2ISV only had a 90% confidence level in the statistical significance testing, and because ER3-2ISV is more computationally intensive than ER1-2ISV, we shall employ model ER1-2ISV for all simulations presented in future sections.
Chapter 4

Simulations Involving More Complex Geometries

In the previous chapters, we have developed a model to simulate wave propagation in biotissue, utilizing an internal variable approach, and validated the model by comparing its output to previously established one dimensional results. We also conducted a statistical significance test to determine which version of the model was most accurate. Now we wish to extend our model to true two dimensional scenarios. In this chapter, we present various modifications of the model to deal with different situations which are of interest when modeling wave propagation in biotissue. Since we previously established that the model employing the first elastic response, with two internal strain variables was one of the more useful versions of our model, all results reported within this chapter have been generated using that model, ER1-2ISV.

4.1 Investigating a Heterogeneous Medium

Thus far, all of our modeling efforts have been conducted on a homogeneous geometry. This, of course, is the simplest type of geometry to deal with, but can only be used as a building block in our modeling efforts. Obviously, when examining waves in biotissue,
we cannot assume that the medium is homogeneous. For our application (the detection of CAD), we are interested in waves traveling through the chest cavity. Common knowledge tells us that the chest cavity is not a homogeneous medium; it is comprised of general bio-tissue, lung tissue, bone, cartilage, and other materials. To incorporate heterogeneity into our geometry, we begin with the simplest possible situation and introduce a ring comprised of one of the different materials in the middle of our original geometry. This is illustrated in Figure 4.1.

\[
\begin{align*}
\rho(r) &= \begin{cases} 
\rho_1 : & r_1 \leq r \leq r_2 \\
1100 : & \text{elsewhere}
\end{cases}
\end{align*}
\]

where \( \rho_1 \) is equal to the density of the ring’s material. In our simulations, we examined rings made of a low density material, a medium density material, and a highly dense material. The densities for these materials (in \( \text{kg/m}^3 \)) are 300, 1900, and 7800, respectively. It is important to realize that simply changing the density within the ring is not a realistic method of representing a material different from the biotissue that comprises our geometry (i.e., one
must also allow for different material properties in the constitutive laws). It does, however, allow us to examine the role of density in our modeling efforts.

Simulations were once again conducted using the MacCormack finite difference scheme. Boundary values were updated using the direction cosines method outlined in section 3.1. We did, however, utilize a different input function to produce the following results. Figure 4.2 depicts the modified version of the original input used in these calculations. The input function originally employed, and depicted in Figure 3.3, consists of an impulse that returned to a steady state of approximately 0.2. This implies that the geometry continued to be driven (at a constant rate) after the impulse passed through it. The modified version we employ here simply shifts the original input function so that the function returns to a steady state of zero after the impulse occurs.

![Figure 4.2: Modified input function with zero steady state](image)

Results for the velocity, \( v_1 \), computed using the ringed geometry are depicted in Figures 4.3-4.8

Figures 4.3 and 4.4 depict the activity of the wave before it encounters the heterogeneous ring (represented by the vertical lines in these figures). Logically, since the medium in which the wave is traveling is the same for all four models (a general biotissue with \( \rho = 1100 \)), we see no discrepancies between the wave propagation in this area. Examination of Figures 4.5 and 4.6, which illustrate the wave propagation within the ring of various materials, suggests that the wave travels differently through the different materials. The wave traveling through the ring of low density material appears to propagate much faster than
Figure 4.3: Velocity results for the various heterogeneous geometries at $t=0.00195$

Figure 4.4: Velocity results for the various heterogeneous geometries at $t=0.00399$
Figure 4.5: Velocity results for the various heterogeneous geometries at $t=0.00602$

Figure 4.6: Velocity results for the various heterogeneous geometries at $t=0.00806$
Figure 4.7: Velocity results for the various heterogeneous geometries at $t=0.01213$

Figure 4.8: Velocity results for the various heterogeneous geometries at $t=0.02433$
the waves traveling through the other materials. Following the low density material is the wave moving in general biotissue (the homogeneous model), trailed by the wave traveling through the mid density material. It appears that the wave progressing through the highly dense material propagates with the slowest speed. This demonstrates the fact that wave speed is inversely proportional to a material’s density. That is, a wave’s speed is greater in a low density material than it is in a higher density material. We see the behavior of the waves after passing through the ring in Figure 4.7. The amplitude of each wave has decreased dramatically (due to the viscoelastic, i.e., dissipative, nature of the material), and each wave is tending to a steady-state of zero, illustrating that the wave has completely passed through the geometry (as depicted in Figure 4.8). The results computed for $u_1$ and $w_1$ were analogous to the results presented here for $v_1$. It should be noted that the results shown in Figures 4.3-4.8 represent only a small fraction of the total time interval used in our simulation. The simulation was conducted for $t = 0 \ldots 0.16658$, however, there were not many changes occurring in the results at higher time values.

4.1.2 The Importance of Viscoelasticity in Modeling Efforts

Although we have derived our model under the assumption that biotissue has viscoelastic properties, there are other groups conducting research in this field that utilize constitutive equations related to elastic media for their modeling effort [20], [21], [29]. This raises the question as to whether the more complicated viscoelastic models, as proposed by Fung and others, really are needed. We therefore wish to examine the difference between a model employing an elastic assumption and one which assumes viscoelasticity. The Maxwell, Voigt, and Kelvin models for viscoelasticity are rather easily reduced to an elastic model by setting the viscous portion to be equal to zero. Similarly, we should be able to reduce our model to a purely elastic model with little difficulty.

When we derived our model using the internal strain variables, we defined

$$\lambda(t) = C_\lambda e^{-\nu_\lambda t}; \quad \mu(t) = C_\mu e^{-\nu_\mu t}.$$ 

We used these parameters, $\lambda$, and $\mu$ to help define the reduced relaxation function. In other articles, authors use $Ce^{-k\tau}$ to approximate the reduced relaxation function. In their definition, the parameter $\tau$ represents the relaxation time for the material. So, it is clear that our relaxation parameter $\nu_k$ is essentially $\frac{1}{\tau}$, i.e., the inverse of the relaxation time, $\tau$. 
Now, if our material is purely elastic, it does not exhibit relaxation. Thus, for an elastic material, the relaxation time approaches $\infty$. We can reduce our model from a viscoelastic model to an elastic model by defining $\nu_k = \lim_{\tau \to \infty} \frac{1}{\tau} = 0$. To examine the effect of $\nu$ on our system, we run our simulations with values of $10\nu^*_k, \nu^*_k, \nu^{*}_{10}, \nu^*_k$, and $\nu_k = 0$, where $\nu^*_k$ represents the optimized parameter from Table 3.7.

All simulations and boundary conditions were treated in the same manner as before, and we used the modified input function from Figure 4.2 to define the prescribed boundary condition, $w_1$. Figures 4.9-4.17 display time snapshot results for $w_1$ for each value of $\nu$ at various time steps, and Figures 4.18-4.26 display the results for the radial velocity ($v_1$).

![Figure 4.9: Results for $w_1$ for varying values of the relaxation parameter, $\nu$ at $t=0.0015$](image)

Examination of Figures 4.9-4.26 help us examine the effect that the relaxation parameter has on our model. In both sets of figures, we notice that as time goes on, the results produced using larger values for the relaxation parameter show more dissipation, and approach a steady state sooner. When $\nu = 10 \times \nu^*_k$, we notice that the wave has almost completely dissipated in a short period of time (shown in Figure 4.11). It takes longer for the $\nu^*_k/10$ result to dissipate completely, and although the result from the simulation using $\nu^*_k/50$ doesn’t completely dissipate in the time frame shown in our figures, the amplitude of
Figure 4.10: Results for $w_1$ for varying values of the relaxation parameter, $\nu$ at $t=0.0039$

Figure 4.11: Results for $w_1$ for varying values of the relaxation parameter, $\nu$ at $t=0.0121$
Figure 4.12: Results for $w_1$ for varying values of the relaxation parameter, $\nu$ at $t=0.0182$

Figure 4.13: Results for $w_1$ for varying values of the relaxation parameter, $\nu$ at $t=0.0243$
Figure 4.14: Results for $w_1$ for varying values of the relaxation parameter, $\nu$ at $t=0.0304$

Figure 4.15: Results for $w_1$ for varying values of the relaxation parameter, $\nu$ at $t=0.0365$
Figure 4.16: Results for $w_1$ for varying values of the relaxation parameter, $\nu$ at $t=0.0487$

Figure 4.17: Results for $w_1$ for varying values of the relaxation parameter, $\nu$ at $t=0.0609$
Figure 4.18: Results for $v_1$ for varying values of the relaxation parameter, $\nu$ at $t=0.0009$

Figure 4.19: Results for $v_1$ for varying values of the relaxation parameter, $\nu$ at $t=0.0032$
Figure 4.20: Results for $v_1$ for varying values of the relaxation parameter, $\nu$ at $t=0.0097$

Figure 4.21: Results for $v_1$ for varying values of the relaxation parameter, $\nu$ at $t=0.0146$
Figure 4.22: Results for $v_1$ for varying values of the relaxation parameter, $\nu$ at $t=0.0194$

Figure 4.23: Results for $v_1$ for varying values of the relaxation parameter, $\nu$ at $t=0.0243$
Figure 4.24: Results for $v_1$ for varying values of the relaxation parameter, $\nu$ at $t=0.0325$

Figure 4.25: Results for $v_1$ for varying values of the relaxation parameter, $\nu$ at $t=0.0504$
Figure 4.26: Results for $v_1$ for varying values of the relaxation parameter, $\nu$ at $t=0.0634$

its wave has obviously been reduced in comparison to the $\nu = 0$ simulation. When we set $\nu = 0$, we see no dissipation in the amplitude of the wave. The wave simply reflects from one boundary to the other, traveling back and forth like a pendulum. This is what we would expect when dealing with an elastic medium. Thus, our results validate our assumption that setting $\nu = 0$ effectively reduces our model from viscoelastic in nature to elastic.

It should be noted that the difference in the boundary interactions between the two figures arises from the different boundary conditions for each quantity. The variable $w_1$ is defined on the inner and outer boundary, thus our simulations must show a fixed point on each boundary. Since each boundary is a fixed point, the reflection of the wave changes its polarity (amplitude becomes negative). The boundary interaction corresponding to $v_1$ (best illustrated in Figure 4.23) is different. Since $v_2$ was not prescribed a boundary condition at either boundary, both boundaries act as free boundaries. When a wave encounters a free boundary, the amplitude appears to double, and the reflected wave has the same polarity as the incident wave.

While our results demonstrate that our model can be easily reduced to an elastic model, they also indicate that the results produced by an elastic model vary significantly from the results generated by a model incorporating viscoelasticity. Since we know dissi-
pation occurs in biomaterials, the viscoelastic model clearly is a more realistic model. The variance between the elastic and viscoelastic simulations motivate our efforts in using a viscoelastic approach to modeling wave propagation in biotissue.

4.1.3 Improvement of the Heterogeneous Model

In our previous efforts to model wave propagation through a heterogeneous medium, presented in section 4.1.1, we considered the introduction of a ring of material with properties different than the general biotissue considered in our homogeneous modeling efforts. As an initial attempt to include heterogeneities in our model, we examined the role that the density of the material plays in the simulation. We defined the ring running throughout the geometry to have the same viscoelastic properties as the general biotissue, but a different density. We found that the density affected the speed at which the wave propagated through the medium. The wave traveled more quickly through the less dense material, and more slowly through the material having higher density.

While it is encouraging to see that our simulations display the desired behavior when the density is varied, changing the density alone is not a very realistic method for representing the heterogeneity in our model. It is true that the ring comprised of a material different than the general biotissue (such as lung tissue, bone, etc.) will have a different density, but it should also have different relaxation and elasticity parameters. In order to improve our heterogeneous model, we must consider a change in all parameters representative of material properties.

We wish to consider rings comprised of materials with different attributes; a highly dense, purely elastic material; a medium density, slightly viscoelastic material; and a low density, highly viscoelastic material. Finding suitable parameter values for the material properties in the available literature proves to be a daunting task. While parameter values are readily available for various types of materials, all parameters reported are representative of the material as a whole and not of the internal variables, on which we have based our model. Thus, the parameter values reported throughout the literature are of little use in our modeling efforts. In order to find parameter values suitable to our needs, we begin with values presented in literature and manipulate the parameter values until they produce results which are commensurate with our expectations.

To gain a better understanding of how each parameter affects the simulation re-
results, we examine the results produced by changing one parameter at a time (while keeping all others set to their original value). We examine the results generated by increasing each parameter, as well as decreasing each parameter. We find as expected that the parameters $\nu_{\lambda_1}$, $\nu_{\lambda_2}$, and $\nu_{\mu_1}$ directly impact the rate of dissipation in the model. When these parameters are increased, the amplitude of the resulting wave dissipates more quickly, and dissipation occurs more slowly when these parameters are decreased. The parameter $\nu_{\mu_2}$ does not appear to affect dissipation in the same manner as the other $\nu$ parameters. It does, however, seem to have an inverse relationship to the initial amplitude and wave propagation speed. Investigation of the parameters $C_{\lambda_1}$, $C_{\lambda_2}$, and $C_{\mu_1}$ shows that an increase in the parameter value results in an increase in both the initial amplitude and propagation speed. The parameter $C_{\mu_2}$ also shows an increase in amplitude and propagation speed when the parameter is increased, but it also seems to directly affect the rate of dissipation.

Using this knowledge of how each parameter affects the resulting simulation of our model, we are able to manipulate the parameter values so that the resulting simulations exhibit a behavior which meets our expectations. Table 4.1 displays parameter values used to generate such results.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>homogeneous</th>
<th>mid dens-low visc</th>
<th>low dens-high visc</th>
<th>high dens-purely elas</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{\lambda_1}$</td>
<td>1187.432</td>
<td>726.943</td>
<td>1421.623</td>
<td>713.623</td>
</tr>
<tr>
<td>$\nu_{\lambda_1}$</td>
<td>215.841</td>
<td>1.961</td>
<td>2857.14</td>
<td>0</td>
</tr>
<tr>
<td>$C_{\lambda_2}$</td>
<td>1432.646</td>
<td>978.193</td>
<td>1873.444</td>
<td>824.172</td>
</tr>
<tr>
<td>$\nu_{\lambda_2}$</td>
<td>215.841</td>
<td>1.852</td>
<td>2860.7</td>
<td>0</td>
</tr>
<tr>
<td>$C_{\mu_1}$</td>
<td>386.639</td>
<td>213.72</td>
<td>405.2</td>
<td>24.991</td>
</tr>
<tr>
<td>$\nu_{\mu_1}$</td>
<td>38.514</td>
<td>16.667</td>
<td>100.023</td>
<td>0</td>
</tr>
<tr>
<td>$C_{\mu_2}$</td>
<td>3456.749</td>
<td>2980.75</td>
<td>3560.827</td>
<td>1872.634</td>
</tr>
<tr>
<td>$\nu_{\mu_2}$</td>
<td>2093.164</td>
<td>158.73</td>
<td>7299.27</td>
<td>0</td>
</tr>
</tbody>
</table>

In determining the parameters shown above in Table 4.1, we examine the behavior of the wave when the entire geometry was made of the desired material. We then compare the new simulation with our original simulation to determine whether the expected changes occur. Figures 4.27-4.35 depict a comparison of the results generated by the four different sets of parameters at different times.

When $t = 0.0019s$ (Figure 4.27), we see that the initial amplitude of the wave varies between the different material. The initial amplitude is greatest in the low density
Figure 4.27: Results for $v_1$ for the geometry comprised of different materials, $t=0.00195$
Figure 4.29: Results for $v_1$ for the geometry comprised of different materials, $t=0.00602$

Figure 4.30: Results for $v_1$ for the geometry comprised of different materials, $t=0.00806$
Figure 4.31: Results for $v_1$ for the geometry comprised of different materials, t=0.0121

Figure 4.32: Results for $v_1$ for the geometry comprised of different materials, t=0.0243
Figure 4.33: Results for $v_1$ for the geometry comprised of different materials, t=0.0304

Figure 4.34: Results for $v_1$ for the geometry comprised of different materials, t=0.0365
Figure 4.35: Results for $v_1$ for the geometry comprised of different materials, t=0.0487

highly viscoelastic material, followed by the general biotissue, then the medium density, slightly viscoelastic material, and finally, the model representing the highly dense, purely elastic material has the smallest initial amplitude. This is consistent with our expectation that the same amount of force will cause a greater disturbance in a softer material than it would in a harder material. Examination of Figures 4.28-4.30 allows us to see that the wave traveling in the low density tissue is propagating through the medium the fastest, and its amplitude is dissipating at a larger rate than the waves in the other materials. We also see considerable dissipation within the wave propagating in the general biotissue (labeled homogeneous). Also, we notice that the wave traveling in the highly dense, purely elastic geometry is propagating much more slowly than the other waves.

By the time $t = 0.0304$, we see that the waves in the least dense material and general biotissue have almost completely dissipated. We also notice that the wave moving through the geometry comprised of the medium density, slightly viscoelastic material has reflected off of the outer boundary, and has dissipated to an amplitude less than that of the wave moving in the highly dense, purely elastic material. As time passes on, we notice that there has been no dissipation in the wave propagating within the purely elastic medium, and the wave will simply reflect back and forth between the boundaries until the end of the
simulations. We should note that all of these observations coincide with what we expect to see. Having determined parameter values which produce the desired results for each type of material, we are now ready to reevaluate the model with a ringed geometry. As explained in our previous efforts, we prescribe a uniform ring throughout the middle of our original geometry. This means that as the wave travels through the geometry, it will begin by propagating in the general biotissue. It will encounter the ring of a different material, and after traveling through the ring, it will be reintroduced into the general biotissue medium.

While traveling in the various media, we expect our earlier observations to hold true (i.e., faster propagation in less dense material). We also expect the properties of the wave when it enters and exits the ring to exhibit the behavior of transmissions and reflections across a general interface as described in [1]. When the wave travels from a material of higher density into one of lower density, the wave will be reflected as well as transmitted. The transmitted portion of the wave has an amplitude which is roughly half of what the incident wave was. The reflected portion has an amplitude comparable to the transmitted portion, but it has the opposite polarity. A wave moving from a material of lower density into one of higher density also will have a transmitted and reflected portion. The transmitted wave, in this scenario, is larger than the incident wave was, while the reflected wave is roughly half of the size of the incident wave. The reflected and transmitted waves have the same polarity.

Figures 4.36-4.41 display snapshots of the propagation of waves in a heterogeneous geometry. Examining Figure 4.36, we see that all waves are behaving in the same manner. In Figure 4.37, which depicts the waves after they have been introduced into their respective rings, we see that the wave traveling in the low density material propagates with the fastest speed, and the wave introduced to the highly dense material has the slowest speed. Furthermore, we see that the wave traveling into the low density, highly viscoelastic material, a material with lower density than the general biotissue, has a transmitted portion with larger amplitude, and a reflected portion with the same polarity. Also, the waves which encounter the more dense materials (mid density and highly dense) have transmitted waves with smaller amplitudes, and reflected waves with opposite polarity. These observations are exactly what we'd expect to see. As time progresses (Figures 4.38-4.41, we notice that the wave traveling through the ring of low density dissipates (almost completely), the wave traveling through the ring with medium density reflects off of the outer edge of the ring and dissipates as it travels. The wave traveling through the ring of highly dense, purely elastic
Figure 4.36: Improved results for $v_1$ for the ringed geometry, $t=0.0039$.

Figure 4.37: Improved results for $v_1$ for the ringed geometry, $t=0.0081$. 
Figure 4.38: Improved results for $v_1$ for the ringed geometry, $t=0.0121$.

Figure 4.39: Improved results for $v_1$ for the ringed geometry, $t=0.0182$. 
Figure 4.40: Improved results for $v_1$ for the ringed geometry, $t=0.0223$.

Figure 4.41: Improved results for $v_1$ for the ringed geometry, $t=0.0365$. 
material shows no dissipation, but every time it encounters the edge of the ring, part of its amplitude is transmitted, and the rest is reflected back into the ring. As this occurs each time the wave interacts with the ring/biotissue interface, the amplitude of the wave inside of the ring is slowly decreasing, and will eventually disappear.

By changing all of the parameters that have physical representations, we are able to vastly improve our modeling technique for the ringed geometry.

4.2 Incorporating a $\theta$ Dependency Into the Model

In each situation that we have examined to this point, our models have been independent of the variable $\theta$. Essentially, this means that everything that we have considered can be reduced to a one dimensional model. This also is not practical, as the chest cavity is not uniform. We need to introduce $\theta$ dependencies to our model. As an initial attempt to introduce the $\theta$ dependency, we begin with our original geometry. We assume that a buildup of plaque (cholesterol, calcium, and platelets) has formed along the wall of the inner radius of our geometry (see Figure 4.42). We assume that this buildup is completely rigid and impermeable, thus any impulse along the inner radius will have no effect on the region occluded by the buildup.

![Figure 4.42: Geometry with occlusion along inner radius](image)

In order to numerically solve our system under these conditions, we shall define the impulse along the inner radius as follows: $w_1(t; R_1, \theta) = f_1(t) * s(\theta)$, where $f_1(t)$ is the
original impulse function, (depicted in Figure 3.3) and \( s(\theta) \) is defined as

\[
s(\theta) = \begin{cases} 
0 & : \theta_1 \leq \theta \leq \theta_2 \\
1 & : \text{elsewhere}
\end{cases}
\]

For our computational efforts, we’ll assume that the occlusion affects the area between \( \theta_1 = \frac{2\pi}{3} \) and \( \theta_2 = \frac{5\pi}{6} \). Computations are still carried out by the MacCormack scheme, and boundary values are once again updated using direction cosines. However, our input function, \( w_1(t; R, \theta) \), as defined above, has discontinuities at the edges of the occlusion (\( \theta_1 \) and \( \theta_2 \)). These discontinuities will cause problems with the integration scheme, and must be treated with certain precautions [19],[27]. Although the method of direction cosines only allows us to prescribe one boundary condition (corresponding to the one eigendirection that points into the domain) on the inner radius, the presence of discontinuities in the input allows us to prescribe additional conditions. To ensure that the problem is well defined, we must define the velocity across the discontinuity. In [19], Hirsch notes that in this predicament, we should define the tangential velocity to be zero across the discontinuity. This means that we may prescribe \( v_2(t; 0, \theta_1 \pm \epsilon) = 0 \) and \( v_2(t; 0, \theta_2 \pm \epsilon) = 0 \). We run our simulation using the discontinuous impulse function, as well as the new boundary condition, and our results are displayed in Figures 4.43-4.54.

Figures 4.43-4.48 depict the radial velocity throughout the geometry at various times. Notice that the velocity propagates outward uniformly, with the exception of the area of the occlusion. Examining Figures 4.49-4.54, we see that velocity in the tangential direction only occurs on the border between the occlusion, and the normal wall. This is what we would expect, since this is the only place where a dependency on \( \theta \) exists. The results shown in Figures 4.43-4.48 would suggest that there is nothing happening in the sector of the geometry affected by the occlusion. This is not the case. The velocity in the rest of the geometry diffuses into the portion of the geometry affected by the occlusion, however, the values of the velocity in the affected region are a couple magnitudes smaller than that in the rest of the medium, thus it does not show on this graph. Figures 4.55-4.60 shows the same results as Figures 4.43-4.48, except on a smaller scale. Similarly, Figures 4.61-4.66 illustrates the tangential velocity results on a smaller scale. When we examine the results on a smaller scale, we can easily see that the radial and tangential velocities are both being transported throughout the geometry.
Figure 4.43: Results for $v_1$ for the geometry with an occluded inner radius, $t=0.0012$ 

Figure 4.44: Results for $v_1$ for the geometry with an occluded inner radius, $t=0.0039$
Figure 4.45: Results for $v_1$ for the geometry with an occluded inner radius, $t=0.0121$

Figure 4.46: Results for $v_1$ for the geometry with an occluded inner radius, $t=0.0162$
Figure 4.47: Results for $v_1$ for the geometry with an occluded inner radius, $t=0.0223$

Figure 4.48: Results for $v_1$ for the geometry with an occluded inner radius, $t=0.0407$
Figure 4.49: Results for $v_2$ for the geometry with an occluded inner radius, $t=0.0012$

Figure 4.50: Results for $v_2$ for the geometry with an occluded inner radius, $t=0.0039$
Figure 4.51: Results for $v_2$ for the geometry with an occluded inner radius, $t=0.0121$

Figure 4.52: Results for $v_2$ for the geometry with an occluded inner radius, $t=0.0162$
Figure 4.53: Results for $v_2$ for the geometry with an occluded inner radius, $t=0.0223$

Figure 4.54: Results for $v_2$ for the geometry with an occluded inner radius, $t=0.0407$
Figure 4.55: Smaller scale results for $v_1$ for the geometry with an occluded inner radius, $t=0.0012$

Figure 4.56: Smaller scale results for $v_1$ for the geometry with an occluded inner radius, $t=0.0039$
Figure 4.57: Smaller scale results for $v_1$ for the geometry with an occluded inner radius, $t=0.0121$

Figure 4.58: Smaller scale results for $v_1$ for the geometry with an occluded inner radius, $t=0.0162$
Figure 4.59: Smaller scale results for $v_1$ for the geometry with an occluded inner radius, $t=0.0223$

Figure 4.60: Smaller scale results for $v_1$ for the geometry with an occluded inner radius, $t=0.0407$
Figure 4.61: Smaller scale results for $v_2$ for the geometry with an occluded inner radius, $t=0.0012$

Figure 4.62: Smaller scale results for $v_2$ for the geometry with an occluded inner radius, $t=0.0039$
Figure 4.63: Smaller scale results for $v_2$ for the geometry with an occluded inner radius, $t=0.0121$

Figure 4.64: Smaller scale results for $v_2$ for the geometry with an occluded inner radius, $t=0.0162$
Figure 4.65: Smaller scale results for $v_2$ for the geometry with an occluded inner radius, $t=0.0223$.

Figure 4.66: Smaller scale results for $v_2$ for the geometry with an occluded inner radius, $t=0.0407$. 
It is interesting to see in Figure 4.66, that the polarity of the wave propagating along the location of the occlusion seems to change almost simultaneously at the inner and outer radius. We believe that this occurs because the movement is introduced from within the center of the geometry, rather than at either the inner or outer boundary. Since the disturbance is introduced within the interior, it is feasible that the wave reflects from the inner and outer boundary simultaneously, causing a change in the polarity.

4.3 Combining the Ringed Geometry Model and the \( \theta \) Dependency Model

Now that we have demonstrated an ability to independently incorporate heterogeneities and theta dependencies into our model, we'd like to combine these components into one model. We now consider a geometry with a heterogeneous ring running through the middle, and a hard occlusion blocking a portion of the inner radius. We once again define the ring by modifying the material parameters of our model as described in section 4.1. Similarly, the dynamics of the occlusion will be handled in the same manner as before. We will use the input function which attains a steady state of zero (depicted in Figure 4.2), which will allow us to observe reflection and transmission behavior at the ring/biotissue interface. If our model is implemented correctly, our expected observations should be a combination of the two models. We expect to see little activity in the portion of the geometry affected by the occlusion. Reflected and transmitted waves should be observable, and they should behave as previously described. We first examine the model including a ring of highly dense, purely elastic material. The results for the radial velocity, \( v_1(t; r, \theta) \), are presented in Figures 4.67-4.72, and Figures 4.73-4.78 display the results for the tangential velocity, \( v_2(t; r, \theta) \).

In examining Figure 4.67, we see that the behavior of the wave propagation before encountering the ring is the same as it was for the model which only incorporated the occlusion. From Figures 4.68 and 4.69, we see that upon encountering the interface with the ring, the wave is both transmitted and reflected. As expected, the reflected portion of the wave changes polarity and the portion which is transmitted displays a slower propagation speed and a considerably smaller wavelength. The behavior of transmitted and reflected
Figure 4.67: Propagation of radial velocity $v_1$ wave through a geometry with a highly dense, purely elastic material and an occlusion on the inner radius, $t=0.0039$

Figure 4.68: Propagation of radial velocity $v_1$ wave through a geometry with a highly dense, purely elastic material and an occlusion on the inner radius, $t=0.0081$
Figure 4.69: Propagation of radial velocity $v_1$ wave through a geometry with a highly dense, purely elastic material and an occlusion on the inner radius, $t=0.0121$

Figure 4.70: Propagation of radial velocity $v_1$ wave through a geometry with a highly dense, purely elastic material and an occlusion on the inner radius, $t=0.0223$
Figure 4.71: Propagation of radial velocity $v_1$ wave through a geometry with a highly dense, purely elastic material and an occlusion on the inner radius, $t=0.0284$

Figure 4.72: Propagation of radial velocity $v_1$ wave through a geometry with a highly dense, purely elastic material and an occlusion on the inner radius, $t=0.0406$
Figure 4.73: Propagation of tangential velocity $v_2$ wave through a geometry with a highly dense, purely elastic material and an occlusion on the inner radius, $t=0.0039$

Figure 4.74: Propagation of tangential velocity $v_2$ wave through a geometry with a highly dense, purely elastic material and an occlusion on the inner radius, $t=0.0081$
Figure 4.75: Propagation of tangential velocity $v_2$ wave through a geometry with a highly dense, purely elastic material and an occlusion on the inner radius, $t=0.0121$

Figure 4.76: Propagation of tangential velocity $v_2$ wave through a geometry with a highly dense, purely elastic material and an occlusion on the inner radius, $t=0.0223$
Figure 4.77: Propagation of tangential velocity $v_2$ wave through a geometry with a highly dense, purely elastic material and an occlusion on the inner radius, $t=0.0284$

Figure 4.78: Propagation of tangential velocity $v_2$ wave through a geometry with a highly dense, purely elastic material and an occlusion on the inner radius, $t=0.0406$
portions of our wave is again displayed in Figures 4.70 and 4.71. While dissipation does not occur within the ring, we do notice that the activity occurring within the ring is reducing as time progresses. Each time that the portion of the wave traveling within the ring encounters an interface with the general biotissue, it is reflected and transmitted, causing a reduction in its amplitude. Successive interactions with the ring/general biotissue boundary causes the energy of the wave to decrease and eventually vanish.

Investigating Figures 4.73-4.78, we see that the tangential velocity results are similar to those presented earlier. We notice that the only places where the tangential velocity is nonzero where the occlusion along the inner radius begins and ends ($\theta = \frac{3\pi}{8}$ and $\theta = \frac{5\pi}{8}$). It is interesting to note that, unlike the original occluded model, there doesn’t seem to be much tangential activity occurring within the highly dense, purely elastic ring. Also, in our original results, the tangential velocity seemed to reflect (almost simultaneously) from the inner and outer radius, displaying a change in polarity. This same phenomenon seems to occur between the inner radii of our geometry and the ring, as well as between the outer radii.

Next, we examine an occluded geometry with a ring comprised of a material which is slightly more dense (1900 $\frac{kg}{m^3}$) and less viscoelastic than the general biotissue. The velocity results for this model can be viewed in Figures 4.79-4.86. Since we established that the wave propagation in the general biotissue, before encountering the ring, will be the same regardless of ring material, our first depicted result (Figure 4.79) is of the initial interaction between the wave and the ring. Similarly to the results presented in Figure 4.68, we see a reflection of opposite polarity emanating from the biotissue/ring interface. Upon reaching the outer boundary of the ring, (Figure 4.81), the wave is once again reflected and transmitted. The reflection in this instance is much weaker than what was observed in our previously presented results. This can be attributed to the viscoelasticity of the material of which the ring is comprised. Since the entire geometry has some degree of viscoelasticity, dissipation occurs throughout the geometry, and the observed activity seems to cease at an earlier time than the previously considered model. The tangential velocity, shown in Figures 4.83, 4.84, 4.85, and 4.86, appears to behave in a similar manner to the previous model. Activity occurs along the boundary of the occlusion, and there appears to be a simultaneous reflection between the inner radius of the geometry and the inner radius of the ring.

Finally, we examine an occluded geometry with a ring formed of a material which
Figure 4.79: Radial velocity ($v_r$) results for a geometry with a slightly more dense, less viscoelastic ring and an occlusion, $t=0.0081$

Figure 4.80: Radial velocity ($v_r$) results for a geometry with a slightly more dense, less viscoelastic ring and an occlusion, $t=0.0121$
Figure 4.81: Radial velocity ($v_1$) results for a geometry with a slightly more dense, less viscoelastic ring and an occlusion, $t=0.0162$.

Figure 4.82: Radial velocity ($v_1$) results for a geometry with a slightly more dense, less viscoelastic ring and an occlusion, $t=0.0223$. 
Figure 4.83: Tangential velocity ($v_2$) results for a geometry with a slightly more dense, less viscoelastic ring and an occlusion, $t=0.0081$

Figure 4.84: Tangential velocity ($v_2$) results for a geometry with a slightly more dense, less viscoelastic ring and an occlusion, $t=0.0121$
Figure 4.85: Tangential velocity \( (v_2) \) results for a geometry with a slightly more dense, less viscoelastic ring and an occlusion, \( t=0.0162 \)

Figure 4.86: Tangential velocity \( (v_2) \) results for a geometry with a slightly more dense, less viscoelastic ring and an occlusion, \( t=0.0223 \)
is less dense, and more viscoelastic than the general biotissue. The resulting velocities from this model are presented in Figures 4.87-4.94. Unlike the two previously considered scenarios, the reflection from the initial interaction with the material of the ring has the same polarity as the transmitted portion of the wave. This is what we would expect of a wave traveling from a material into one with a lower density. It should be pointed out that since this geometry is comprised of materials which are more viscoelastic than the previous models, the amplitude of our wave dissipates more quickly. Once again, the tangential velocity exhibits the same behavior as in the previous models, with the exception that it appears to dissipate more quickly (It is not as prevalent at t=0.0223 s, as it was in the previous models).

Figure 4.87: Radial velocity ($v_1$) results for a geometry with a less dense, more viscoelastic ring and an occlusion, t=0.0081
Figure 4.88: Radial velocity \( (v_1) \) results for a geometry with a less dense, more viscoelastic ring and an occlusion, \( t=0.0121 \)

Figure 4.89: Radial velocity \( (v_1) \) results for a geometry with a less dense, more viscoelastic ring and an occlusion, \( t=0.0162 \)
Figure 4.90: Radial velocity \((v_1)\) results for a geometry with a less dense, more viscoelastic ring and an occlusion, \(t=0.0223\)

Figure 4.91: Tangential velocity \((v_2)\) results for a geometry with a less dense, more viscoelastic ring and an occlusion, \(t=0.0081\)
Figure 4.92: Tangential velocity ($v_2$) results for a geometry with a less dense, more viscoelastic ring and an occlusion, $t=0.0121$

Figure 4.93: Tangential velocity ($v_2$) results for a geometry with a less dense, more viscoelastic ring and an occlusion, $t=0.0162$
Figure 4.94: Tangential velocity \(v_2\) results for a geometry with a less dense, more viscoelastic ring and an occlusion, \(t=0.0223\)
Chapter 5

Inverse Problem Methodology and Results

Having demonstrated the ability to simulate wave propagation in biotissue under various assumptions, we now turn our investigation towards an inverse problem methodology. Inverse problems are often used to identify feasible parameter values within a model when data from the process being modeled is available. We wish to utilize an inverse problem in conjunction with the occluded model presented in section 4.2 in an effort to determine the location of the occlusion.

5.1 Inverse Problem Methodology

Unfortunately, experimental acceleration data, generated under the occluded inner radius assumption, is not currently available to us. In lieu of experimental data, we solve the forward problem with $\theta_1 = \frac{\pi}{4}$ and $\theta_2 = \frac{3\pi}{4}$, as described in section 4.2, and use the results on the outer boundary as simulated data. We select $M$ nodes along the outer radius to represent points where sensors were placed, and we use the acceleration data at those nodes, as if it were measured experimentally by the sensors. In true experimental data, there is some degree of noise incorporated in the data. As this is the case, we consider different
levels of noise with our simulated data. The noisy data is generated in the following manner.

\[ z = v \ast (1 + \alpha R) \]

where \( v \) is the pure simulated data, \( \alpha \) is the noise level, and \( R \) is a normally distributed vector with norm zero and standard deviation 1. Our results, which are presented in the next section, consider noise levels of 0%, 5%, 10%, and 25%.

Using the newly simulated data, we want to see if we can approximate the location \((\theta_1, \theta_2)\) of the occlusion along the inner radius. We wish to determine optimal parameters, \(\theta_1^*\) and \(\theta_2^*\), which minimize the following cost functional.

\[ J = \sum_{j=1}^{M} \sum_{i=1}^{N} |(u(t_i; \theta_1^*, \theta_2^*))_j - (z_i)_j|^2 \]

where \( M \) is the number of nodes at which data is known, \( N \) is the number of temporal nodes used, \( u(t_i; \theta_1^*, \theta_2^*) \) is the modeled data, and \((z_i)_j\) is the simulated data which we are trying to replicate. The cost functional can be minimized using one of many different optimization routines. For our efforts, we utilize an implementation of the Nelder-Mead algorithm.

The Nelder-Mead simplex algorithm maintains a simplex \( S \) of approximations to an optimal point. In this algorithm the vertices \( \{x_j\}_{j=1}^{N+1} \) are sorted according to the objective function values

\[ f(x_1) \leq f(x_2) \leq \cdots \leq f(x_{N+1}) \]

\( x_1 \) is called the best vertex and \( x_{N+1} \) the worst. If several vertices have the same objective value as \( x_1 \), the best vertex is not uniquely defined.

The algorithm attempts to replace the worst vertex \( x_{N+1} \) with a new point. The new point is identified by reflection, expansion, contraction, or shrinking of the simplex. The objective function is reevaluated for the new simplex, the vertices are reordered, and the step is repeated. The algorithm terminates if either \( f(x_{N+1}) - f(x_1) \) is sufficiently small or a user-specified number of function evaluations has been expended. For further information on the Nelder-Mead simplex algorithm, see [22] or [32]. Upon successful termination of the algorithm, the optimal parameters and their corresponding function value are returned.
5.2 Inverse Problem Results

5.2.1 An Initial Investigation of the Inverse Problem

As an initial attempt, we consider the model with 64 spatial nodes in the radial and tangential directions (i.e., a 64 x 64 grid), and 2048 temporal nodes. We select 12 tangential nodes to serve as sensor locations. The forward problem is executed, generating the acceleration data at these 12 locations. Appropriate noise levels are incorporated into the simulated data, and the inverse problem is conducted. A variety of initial parameter sets are examined with the non noisy data, including the exact parameters which we seek ($\frac{\pi}{4}$ and $\frac{3\pi}{4}$). The results for our initial consideration of the inverse problem are shown in Table 5.1.

Table 5.1: Optimization results for the model with 64 tangential nodes and 12 sensor points.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\frac{\pi}{4}$</th>
<th>$\frac{3\pi}{4}$</th>
<th>$\frac{\pi}{4}$</th>
<th>$\frac{3\pi}{4}$</th>
<th>$\frac{\pi}{4}$</th>
<th>$\frac{3\pi}{4}$</th>
<th>$\frac{\pi}{4}$</th>
<th>$\frac{3\pi}{4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$</td>
<td>0.785398</td>
<td>2.356194</td>
<td>0.790306</td>
<td>2.422462</td>
<td>0.6%</td>
<td>2.81%</td>
<td>0.6%</td>
<td>2.81%</td>
</tr>
<tr>
<td>$\phi$</td>
<td>1.047197</td>
<td>2.094395</td>
<td>0.929387</td>
<td>2.336599</td>
<td>18.3%</td>
<td>0.833%</td>
<td>18.3%</td>
<td>0.833%</td>
</tr>
<tr>
<td>$\phi$</td>
<td>0.523598</td>
<td>2.617993</td>
<td>0.523598</td>
<td>2.617993</td>
<td>33.3%</td>
<td>11.1%</td>
<td>33.3%</td>
<td>11.1%</td>
</tr>
<tr>
<td>$\phi$</td>
<td>0.004266</td>
<td>1.570796</td>
<td>0.711527</td>
<td>2.36885</td>
<td>9.4%</td>
<td>0.53%</td>
<td>9.4%</td>
<td>0.53%</td>
</tr>
<tr>
<td>$\phi$</td>
<td>1.570796</td>
<td>1.570796</td>
<td>0.711527</td>
<td>2.36885</td>
<td>9.4%</td>
<td>0.53%</td>
<td>9.4%</td>
<td>0.53%</td>
</tr>
<tr>
<td>$\phi$</td>
<td>0.05</td>
<td>1.570796</td>
<td>5.2285</td>
<td>0.773126</td>
<td>1.5%</td>
<td>1.5%</td>
<td>1.5%</td>
<td>1.5%</td>
</tr>
<tr>
<td>$\phi$</td>
<td>1.570796</td>
<td>1.570796</td>
<td>0.773126</td>
<td>2.40282</td>
<td>1.5%</td>
<td>1.5%</td>
<td>1.5%</td>
<td>1.5%</td>
</tr>
<tr>
<td>$\phi$</td>
<td>0.1</td>
<td>1.570796</td>
<td>20.3825</td>
<td>0.795944</td>
<td>1.3%</td>
<td>0.9%</td>
<td>1.3%</td>
<td>0.9%</td>
</tr>
<tr>
<td>$\phi$</td>
<td>1.570796</td>
<td>1.570796</td>
<td>0.795944</td>
<td>2.3343</td>
<td>0.9%</td>
<td>0.9%</td>
<td>0.9%</td>
<td>0.9%</td>
</tr>
<tr>
<td>$\phi$</td>
<td>0.25</td>
<td>1.570796</td>
<td>444.205</td>
<td>0.746128</td>
<td>5%</td>
<td>1.25%</td>
<td>5%</td>
<td>1.25%</td>
</tr>
<tr>
<td>$\phi$</td>
<td>1.570796</td>
<td>1.570796</td>
<td>0.746128</td>
<td>2.326742</td>
<td>1.25%</td>
<td>1.25%</td>
<td>1.25%</td>
<td>1.25%</td>
</tr>
</tbody>
</table>

In examining Table 5.1, we see that the optimization routine returns optimal parameters that are reasonably close to our expectations. With the exception of the results generated using a starting point of $[\pi/6, 5\pi/6]$ (which returned the initial parameters), all of the optimal parameters seem to be within a reasonable neighborhood of our expected results. It is a little disheartening that when given the exact parameters as initial parameters (displayed in the first row), the minimization routine returns different parameters. We believe that this is an artifact of our mesh size. Since our simulations are conducted on a discrete mesh, then $\theta$ parameters falling within a certain distance of each mesh point will
generate the same value of the cost functional. It is our belief that these individual regions create problems with the uniqueness of the simplex method. As such, the minimization routine does a fair job of finding the region, but cannot identify the individual parameter.

5.2.2 Refining the Mesh for the Inverse Problem

In an attempt to improve the results of our inverse problem, we refine the mesh on which the problem is solved. We once again consider the model with 64 spatial nodes in the radial direction, and 2048 temporal nodes, but we halve the step size in the tangential direction, creating 128 spatial nodes. The forward problem is once again executed, and the acceleration data is generated for the same 12 nodes previously used. Table 5.2 displays the results generated on this refined mesh.

Table 5.2: Optimization results for the model with 128 tangential nodes and 12 sensor points.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>initial parameters</th>
<th>cost $J$</th>
<th>Optimal parameters</th>
<th>Relative Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.785398, 2.356194</td>
<td>1.8189e-7</td>
<td>0.784171, 2.334105</td>
<td>0.15%</td>
</tr>
<tr>
<td>0</td>
<td>1.047197, 2.094395</td>
<td>823.261</td>
<td>0.960476, 2.362739</td>
<td>22.0%</td>
</tr>
<tr>
<td>0</td>
<td>0.523598, 2.617993</td>
<td>1834.886</td>
<td>0.566959, 2.331651</td>
<td>27.8%</td>
</tr>
<tr>
<td>0</td>
<td>1.570796, 1.570796</td>
<td>3767.799</td>
<td>1.551161, 1.619883</td>
<td>97.5%</td>
</tr>
<tr>
<td>0.05</td>
<td>1.570796, 1.570796</td>
<td>3785</td>
<td>1.551161, 1.619883</td>
<td>97.5%</td>
</tr>
<tr>
<td>0.1</td>
<td>1.570796, 1.570796</td>
<td>3845.76</td>
<td>1.551161, 1.619883</td>
<td>97.5%</td>
</tr>
<tr>
<td>0.25</td>
<td>1.570796, 1.570796</td>
<td>4177.312</td>
<td>1.551161, 1.619883</td>
<td>97.5%</td>
</tr>
</tbody>
</table>

An examination of the results presented in Table 5.2 leads us to the conclusion that a grid refinement alone isn’t enough to improve our optimization efforts. It is encouraging that the results generated when using the exact parameters as the initial parameters are considerably closer to the exact parameters. This seems to support our assumption that a grid refinement will allow the optimization routine to produce results which are more commensurate with our expectations. Of course, this improvement occurs because the individual regions in which the optimization routine was getting stuck are reduced with the
mesh size. We see slight improvements in the results generated with initial parameter sets $[\pi/3, 2\pi/3]$ and $[\pi/6, 5\pi/6]$. The results generated when $[\pi/2, \pi/2]$ is the initial parameter are disconcerting. While this parameter set produced some of the best results in our previous examination, the results produced on this refined mesh with data produced from 12 nodes, are unacceptable. We feel that this decline in accuracy can be attributed to the refinement of the mesh, coupled with the number of nodes on which the data was collected. For the previously considered, more coarse mesh, there were at most 6 nodes in between the data collection points. For this refined mesh, there are at most 12 nodes between data collection points. This increased number of nodes between the points at which the data is generated might make it more difficult to distinguish between the optimal parameters. Perhaps our efforts would be improved by increasing the number of nodes acting as sensors.

5.2.3 Increasing the Number of Data Nodes in the Inverse Problem

In our next set of optimization results, we again consider a mesh which is refined in the tangential direction. There are 64 radial nodes, 128 tangential nodes, and 2048 temporal nodes. Now, data is collected and compared at every tangential node between $\theta = 0$ and $\theta = \pi$ (the upper portion of our geometry). This results in 64 data collecting nodes, thus $M = 64$. The forward problem is executed, and the acceleration data is computed at these 64 nodes. We conduct the inverse problem, using the Nelder-Mead algorithm, and the results are listed in Table 5.3

The results presented in Table 5.3 are much improved over those presented in Table 5.1. Using the exact parameters as our initial parameters returns the exact parameters as optimal. This is what we want to see happen. Results generated by each set of initial parameters are improved over previous findings, although the initial parameter set $[\pi/6, 5\pi/6]$ still seems to have problems identifying the optimal value for $\theta_1$. The results from the noisy data sets are also improved, which demonstrates that we can find the location of the occlusion when the data has lower levels of noise incorporated within it.

We should point out that the minimized values of the cost functional are generally higher for these results than they were previously. This can be attributed to the increase in data points. Previously reported results were generated using 2048 time points at 12 data collecting nodes, for a total of 24576 points. Optimizations conducted under the current assumptions used 2048 time points at 64 data collecting nodes, for a total of 131072 points.
Table 5.3: Optimization results for the model with 128 tangential nodes and 64 sensor points.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>initial parameters</th>
<th>cost J</th>
<th>Optimal parameters</th>
<th>Relative Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.785398 2.356194</td>
<td>3.8364e-10 2.356194</td>
<td>0.785395 2.356194</td>
<td>0% 0%</td>
</tr>
<tr>
<td>0</td>
<td>1.047197 2.094395</td>
<td>1678.35</td>
<td>0.769036 2.449461</td>
<td>2.09% 3.95%</td>
</tr>
<tr>
<td>0</td>
<td>0.523598 2.617993</td>
<td>3686.05</td>
<td>0.559187 2.360285</td>
<td>28.8% 0.25%</td>
</tr>
<tr>
<td>0</td>
<td>1.570796 1.570796</td>
<td>3.8364e-10 2.339852</td>
<td>0.789832 2.339852</td>
<td>0.5% 0.69%</td>
</tr>
<tr>
<td>0.05</td>
<td>1.570796 1.570796</td>
<td>944.714</td>
<td>0.779875 2.280263</td>
<td>0.7% 3.2%</td>
</tr>
<tr>
<td>0.1</td>
<td>1.570796 1.570796</td>
<td>1205.91</td>
<td>0.779875 2.280263</td>
<td>0.7% 3.2%</td>
</tr>
<tr>
<td>0.25</td>
<td>1.570796 1.570796</td>
<td>2113.954</td>
<td>0.776577 2.347815</td>
<td>1.12% 0.36%</td>
</tr>
</tbody>
</table>

Since we do not normalize our cost functional (by dividing the cost by the total number of data points), we would expect for the costs generated using more points to be considerably higher.

5.2.4 Revisiting the Inverse Problem on a Coarser Mesh

We have established that the inverse problem seems to work very well on a refined mesh, when considering a higher number of data collecting nodes. Perhaps the finer mesh is not as important, and the increase in the amount of data is the factor that dramatically improves our efforts in regard to the inverse problem. To investigate this notion, we once again consider the scenario described in section 5.2.1. We have a 64 x 64 node spatial grid, and 2048 temporal nodes. We wish to examine this coarser grid with more data collecting points. We once again allow all nodes between $\theta = 0$ and $\theta = \pi$ to act as sensors. We run the forward problem, generating data on the 32 nodes which make up the upper part of our geometry. The inverse problem is carried out, and the results are presented in Table 5.4.

These results are improvements over those presented in Table 5.1, suggesting that an increased number of data collection nodes is essential to the success of the inverse problem. Comparison of Table 5.3 to Table 5.4 shows that the results generated on the refined mesh are generally better than those which were generated here. This reiterates our previous
Table 5.4: Optimization results for the model with 64 tangential nodes and 32 sensor points.

<table>
<thead>
<tr>
<th>α</th>
<th>initial parameters</th>
<th>cost J</th>
<th>Optimal parameters</th>
<th>Relative Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.785398</td>
<td>7.39132e-9</td>
<td>0.805033</td>
<td>2.4%</td>
</tr>
<tr>
<td></td>
<td>2.356194</td>
<td>2.3856469</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1.047197</td>
<td>968.6399</td>
<td>0.785398</td>
<td>0%</td>
</tr>
<tr>
<td></td>
<td>2.094395</td>
<td>2.487094</td>
<td></td>
<td>5.5%</td>
</tr>
<tr>
<td>0</td>
<td>0.523598</td>
<td>1394.25</td>
<td>0.536688</td>
<td>31.66%</td>
</tr>
<tr>
<td></td>
<td>2.617993</td>
<td>2.356194</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1.570796</td>
<td>7.3913183e-9</td>
<td>0.7258030</td>
<td>7.5%</td>
</tr>
<tr>
<td></td>
<td>1.570796</td>
<td>2.341794</td>
<td></td>
<td>0.61%</td>
</tr>
<tr>
<td>0.05</td>
<td>1.570796</td>
<td>41.2131</td>
<td>0.765034</td>
<td>2.5%</td>
</tr>
<tr>
<td></td>
<td>1.570796</td>
<td>2.360902</td>
<td></td>
<td>0.19%</td>
</tr>
<tr>
<td>0.1</td>
<td>1.570796</td>
<td>175.137</td>
<td>0.765034</td>
<td>2.5%</td>
</tr>
<tr>
<td></td>
<td>1.570796</td>
<td>2.360902</td>
<td></td>
<td>0.19%</td>
</tr>
<tr>
<td>0.25</td>
<td>1.570796</td>
<td>16531.27</td>
<td>1.610066</td>
<td>104%</td>
</tr>
<tr>
<td></td>
<td>1.570796</td>
<td>1.649336</td>
<td></td>
<td>30%</td>
</tr>
</tbody>
</table>

conclusion that results are improved by refining the mesh.

5.3 Investigating the Effects of the Material Parameters on the Inverse Problem

We now wish to examine the effects of the material property parameters on the inverse problem. For each formulation of the inverse problem presented in the previous section, results were generated using the exact same material properties that were employed in creating the simulated data. Because we are considering internal strains, it may be difficult to find the exact parameters which model the properties of any given material. Thus, we would like to examine the effectiveness of our inverse problem method if the material properties in the inverse problem are not the same as those used to make the simulated data.

Simulated data is created by running the forward problem on the refined mesh with 64 data collecting nodes. The material parameters employed in the creation of the simulated data are the set used for a general, homogeneous soft tissue (see Table 4.1). For the inverse problem, we consider material parameter sets which are 5%, 10%, 25%, and 50% variations of the homogeneous soft tissue material parameters (See Table 5.5). We
also use the material parameters for the low density, medium density, and highly dense materials, as reported in Table 4.1. Each effort uses $[\pi/2, \pi/2]$ as the initial parameter for the Nelder-Mead scheme. Optimization results using these parameter sets are presented in Table 5.6.

Table 5.5: Material parameter sets used in inverse problem

<table>
<thead>
<tr>
<th>Parameter</th>
<th>5%</th>
<th>10%</th>
<th>25%</th>
<th>50%</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{\lambda_1}$</td>
<td>1246.803</td>
<td>1306.175</td>
<td>1484.29</td>
<td>1781.115</td>
</tr>
<tr>
<td>$\nu_{\lambda_1}$</td>
<td>226.633</td>
<td>237.425</td>
<td>269.801</td>
<td>323.762</td>
</tr>
<tr>
<td>$C_{\lambda_2}$</td>
<td>1504.278</td>
<td>1575.911</td>
<td>1790.807</td>
<td>2148.969</td>
</tr>
<tr>
<td>$\nu_{\lambda_2}$</td>
<td>226.633</td>
<td>237.425</td>
<td>269.801</td>
<td>323.762</td>
</tr>
<tr>
<td>$C_{\mu_1}$</td>
<td>405.971</td>
<td>425.303</td>
<td>483.299</td>
<td>579.958</td>
</tr>
<tr>
<td>$\nu_{\mu_1}$</td>
<td>40.43</td>
<td>423.654</td>
<td>48.143</td>
<td>57.771</td>
</tr>
<tr>
<td>$C_{\mu_2}$</td>
<td>3629.58</td>
<td>3802.424</td>
<td>4320.936</td>
<td>5185.124</td>
</tr>
<tr>
<td>$\nu_{\mu_2}$</td>
<td>2197.822</td>
<td>2302.48</td>
<td>2616.455</td>
<td>3139.746</td>
</tr>
</tbody>
</table>

Table 5.6: Optimization results for the varied material parameter sets

<table>
<thead>
<tr>
<th>Parameter Set</th>
<th>Cost $J$</th>
<th>Optimal $\theta$</th>
<th>Relative Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>5%</td>
<td>919.551</td>
<td>0.779875</td>
<td>0.7%</td>
</tr>
<tr>
<td>10%</td>
<td>1251.319</td>
<td>0.779875</td>
<td>0.7%</td>
</tr>
<tr>
<td>25%</td>
<td>3278.713</td>
<td>0.779875</td>
<td>0.7%</td>
</tr>
<tr>
<td>50%</td>
<td>7745.295</td>
<td>0.792454</td>
<td>0.8%</td>
</tr>
<tr>
<td>low density</td>
<td>Error</td>
<td>1.556137</td>
<td>98.2%</td>
</tr>
<tr>
<td>mid density</td>
<td>714712.822</td>
<td>0.058904</td>
<td>92.5%</td>
</tr>
<tr>
<td>high density</td>
<td>268180.61</td>
<td>-0.025731</td>
<td>103.3%</td>
</tr>
</tbody>
</table>

We see from the results shown in Table 5.6, that altering the material parameters by 5, 10, 25, or 50% of their original values does not affect the resulting optimal parameters very much. The optimal $\theta$ results for each of these data sets are good approximations of the exact parameters.

When using the parameter sets for the low density, mid density, and highly dense
materials, the results are not as promising. When conducting the inverse problem using the low density parameters, the minimization routine could not work. The code used for the inverse problem encountered an overflow error, thus making it difficult to adjust the simplex. The minimization routine did not finish, and so we have no results to compare. When employing the parameter sets representing the medium density, and highly dense materials, the minimization routine did terminate successfully, but the results are not good. The highly dense parameters returned a negative parameter, which is not possible in our geometry. The optimal $\theta$ values returned from both the highly dense and medium density parameter sets differ greatly from what we expect.

These results lead us to believe that when attempting to locate the occlusion, it is not necessary to know the exact values for the material parameters, since values which are close to exact seem to produce similar results. It is unclear, however, in what regard individual changes affect the results from the inverse problem. In the results presented here, each parameter was altered by a certain percentage of its original value, thus, the ratio between the parameters remains the same. It is possible that the inverse problem worked for these altered parameter sets simply because they all had the same ratios between the parameters. Further investigation is necessary to determine the validity of these results.
Chapter 6

Conclusion

We have developed a mathematical framework to model the propagation of shear waves through biological tissue. Using previously published one dimensional results as a basis of comparison, we were able to examine the utility and appropriateness of our model, and we determined which variation of the model was best suited to the data by employing the statistical significance testing methodology presented in [6].

After examining the appropriateness of our model, we presented a variety of simulations. We demonstrated the importance of incorporating viscoelasticity in modeling efforts. Our simulation involving a ringed geometry met our expectations, displaying the behavior of a wave that encounters an interface with a material with different properties. The simulation of the wave propagating from an inner radius with a rigid occlusion also performed as expected. The simulations presented within this document should easily be extended to consider more complicated geometries (i.e., nonuniform ring in geometry, multiple rings, multiple occlusions, etc.), which may be of interest in future investigations on this subject.

Having displayed the ability to simulate wave propagations in more complicated geometries, we turned our efforts to formulating an inverse problem. We discovered that the inverse problem is more accurate on a more refined mesh, when using several positions on the outer radius to collect data. It is possible that in our efforts, we utilized more data points than necessary. Future investigation may be needed to determine exactly how many sensors would be necessary to accurately conduct the inverse problem.
We have also determined that the inverse problem is reliable for data with up to a 10% noise level. We conducted an investigation into the effect of the material parameters on the inverse problem. We found that the inverse problem works when each parameter is varied by up to 50% of its original value. Further investigation is needed to determine the effect of the individual parameters, and variation levels higher than 50%.

While the methods and results reported within this dissertation are a considerable extension of the project originally presented in [5], there are still several things to be done to reach the ultimate goal of stenosis detection. We would like to reconsider the inverse problem methodology presented in Chapter 5, utilizing experimental data, rather than simulated data. We would like to implement a probabilistic multi-scale approach to our model, similar to that presented in [7]. An existence, uniqueness, and continuous dependence framework should be investigated. Finally, we need to extend the model and methodology to accommodate a three dimensional geometry.
Bibliography


