ABSTRACT

XU, QINGXIA. Modeling and Computing for Layered Pavements under Vehicle Loading. (Under the direction of Dr. M. Shamimur Rahman and Dr. Akhtarhusein A. Tayebali)

The objective of this research is to develop and implement some numerical models to analyze pavement responses under vehicle loading. Firstly, to study the pavement delamination problem, the pavement structure is modeled as an elastic finite layer system subjected to vertical and horizontal loadings over circular areas. By using the finite layer method, the maximum interface shear stress are determined; the maximum interface shear stress can be used to compare with the interface shear strength obtained through simple shear testing to determine reasonable pavement design parameters to prevent delamination failure. Secondly, the responses of a linear viscoelastic pavement system, with asphalt concrete layer of viscoelastic properties, subjected to vertical circular loadings, are analyzed by finite element method using three methods: (i) direct time integration; (ii) Fourier transform; (iii) Laplace transform. The inverse Fast Fourier Transform algorithm and the numerical inversion of Laplace transform method of Honig and Hirdes are used. The numerical results of the quasi-static responses by the three methods are presented and compared with respect to their accuracy and computational efficiency. To use the viscoelastic model in the pavement analysis, the parameters of the generalized Maxwell model based on the frequency sweep test results are determined by using the software IRIS, which is then assigned as the property of the asphalt concrete layer in a typical pavement structure subjected to a standard dual tire axle loading. Results for the distributions of stress and strain at various times are presented. In the last part of the research, a preliminary study is presented for permanent deformation of asphalt concrete. A simplified one-dimensional elasto-visco-plastic model is implemented and used to analyze the visco-plastic deformation of a cylindrical asphalt concrete sample under one-dimensional loading.
MODELING AND COMPUTING FOR LAYERED PAVEMENTS UNDER VEHICLE LOADING

by

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A dissertation submitted to the Graduate Faculty of North Carolina State University in partial fulfillment of the requirements for the Degree of Doctor of Philosophy

CIVIL ENGINEERING

Raleigh
2004

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ACKNOWLEDGEMENTS

First and foremost, I would like to give my sincere thanks to my advisor, Dr. M. Shamimur Rahman, for providing me such a wonderful opportunity to study at NCSU and giving me numerous valuable guidance and inspiration for my research.

I would like to thank Dr. Akhtarhusein A.Tayebali, my co-advisor, for his time, knowledge and providing me the testing data of his research project.

Many thanks go to Professor J. C. Small for providing me the detailed derivation of the finite layer theory, Professor H. H. Winter for offering me the software IRIS for the evaluation of viscoelastic model parameters, Professor Z. L. Feng for his help and support.

Thanks to my committee members, Dr. Roy Borden, Dr. Mohammed Gabr and Dr. N. Paul Khosla. I appreciate your time and effort.

Thanks to Dr. Moreshwar B. Kulkarni for his time and help.

Special thanks to my husband, Yuanxiong Huang, for his support, help and encouragement.
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Chapter 1 INTRODUCTION

A pavement’s function as a highway component is manifold. It is designed and constructed to provide safe, durable, smooth and economical highway surfaces, which would make possible the swift and convenient transportation. Generally, after a period of usage, the pavements would suffer some failures and distresses, which are caused by stress-strain-displacement fields caused in part by vehicle loadings. Good design and accurate pavement performance prediction are important to guarantee the performance during the design life of pavements. For all of these, evaluation of the response of pavements to vehicular loading is a very important consideration.

One common pavement failure is delamination as shown in Figure 1.1. In the design of pavement, the most commonly used method for rehabilitation of deteriorated pavement is to apply an asphalt concrete overlay onto them. Before paving the overlay, the top surface of the existing layer is cleaned and a tack coat is applied to bond the new surface being paved and the underlying layer. A strong bonding between layers is critical to dissipate shear stresses into the entire pavement structure. Lack of interface bonding may lead to slippage cracking, delamination under traffic loads, which in turn would activate distress mechanisms that will rapidly lead to total failure of the pavement.

Fatigue cracking as shown in Figure 1.2 is another kind of pavement distress. It is a series of interconnecting cracks caused by the fatigue failure of asphalt layer or stabilized base under repeated traffic loading. The cracking initiates at the bottom of the asphalt layer or stabilized base where the tensile strain is highest under a wheel load. The cracks propagate to the surface initially as one or more longitudinal parallel cracks. After repeated traffic loading, the cracks connect and form many-sided, sharp-angled pieces.

Another pavement failure mechanism, rutting, is a surface depression in the wheel paths, see Figure 1.3. It stems from the permanent deformation in any of the pavement layers or the subgrade, usually caused by the densification or lateral movement of the materials due to traffic loads. Rutting may also be caused by plastic movement of the
asphalt mix in hot weather. In properly compacted pavements, plastic flow in asphalt concrete layer is thought to be the primary rutting mechanism.

**Figure 1.1** Delamination failure in pavements.

**Figure 1.2** Fatigue cracking in pavements.
The objective of this study is to develop models and implement them for the computation of the pavement response needed for the evaluation of pavement delamination, fatigue life and permanent deformation. The structural analysis of a pavement system is an extremely complex problem: the system is layered; the loads are moving; the properties of asphalt concrete vary with composition, temperature, and the frequency and level of load; the domain is unbounded. In this study, some simplified models are developed and implemented. The outline of the presentation is described below.

Chapter 2 describes the finite layer model, its numerical implementation, and its application in the pavement delamination analysis. In the finite layer model, each layer of the pavement is treated as an elastic material; it is assumed that there is no slippage at the interface between two layers. The loadings applied to the pavement surface by the vehicle include vertical and horizontal pressure over a circular area. The maximum shear stress is evaluated by numerical analysis and compared with interface shear strength obtained by testing to determine whether delamination would occur for a certain pavement structure under a certain loading condition.
Chapter 3 introduces linear viscoelastic model and associated computational methods. Three techniques: direct time integration method, Fourier transformation method, and Laplace transformation method are used to solve the problem of viscoelastic layered system subjected to vertical circular loadings. The advantages and disadvantages of each technique are discussed. To apply the viscoelastic model into pavement fatigue analysis, the model parameters are firstly determined from frequency sweep testing data, then stress-strain field of a pavement structure with the AC layer having the testing derived viscoelastic model are analyzed. The fatigue life obtained by viscoelastic analysis is compared with that by elastic analysis.

Chapter 4 presents a review of the models used for pavement rutting analysis. For the viscoplastic component of the permanent deformation, a one-dimensional simplified model is presented and implemented followed by a case study.

In the closing chapter 5, a summary of the main aspects of this work is presented first, followed by the main conclusions. Also, recommendations are made for further studies.
Chapter 2 RESPONSE OF LAYERED PAVEMENTS:
LINEAR ELASTIC ANALYSIS

2.1 Introduction

Pavement engineers have been greatly interested in the behavior of layered elastic materials under certain loading conditions mainly due to the fact that pavements are composed of horizontal layers of materials of different types. When a vehicle is moving on the pavement, the contact region between the tire and the pavement is roughly circular in shape. The circular area is subjected to the vertical loading due to the self-weight of the vehicle and also horizontal loading due to braking, turning and acceleration.

Many analytical solutions have been developed for the response of elastic layered materials subjected to vertical and horizontal loads distributed over circular area. Burmister [1] presented the first solution for both two-layer and three-layer systems, which was also the first significant contribution in the application of the theories of continuum mechanics to the pavement structural design. He developed and presented the general equations and obtained the solution by assuming a stress function involving Bessel functions and exponentials. Jones [2] presented comprehensive results of stress analysis in tabular forms for a three-layer system under a uniform circular load. Ueshita and Myerhoff [3] obtained solutions for a three-layer system with infinitely deep underlying layer. Westmann [4] proposed the solutions for a uniform shear loading over a circular region being applied on a two-layered system with the second layer infinitely deep.

Some computer programs have been developed based on Burmister’s [1] theory, one of which is BISAR developed by Shell [5], which considers not only vertical loads but also horizontal loads, but this program can only analyze three-layer system. ELSYM5, another program, developed at the University of California, Berkley [6], can be used to analyze the elastic five-layer system under multiple vertical wheel loads, but cannot be used to analyze the horizontal loads.
When several layers of material are involved, analytic solutions become difficult, and presentation of results becomes complex because of the many different combinations of layer thicknesses and moduli. Finite element methods may be employed, but this is an inefficient method for solving problems involving horizontal layering, especially when horizontal loading over circular region is required.

In this study, the semi-analytical finite layer method developed by Small and Booker [7,8] is used to obtain solutions for the problem of normal and shear circular loadings on a layered system, which by itself is a 3-D problem. This provides a much more efficient way to solve such problems. By making use of Fourier or Hankel transforms, a three-dimensional problem is reduced into a one-dimensional problem resulting in a highly efficient solution.

2.2 General Approach and Formulation

The model for pavement system subjected to normal and shear circular loadings on the surface is presented in Figure 2.1. The solution of the problem is equivalent to the superposition of the solutions of the pavement system subjected to normal load and shear load respectively. The following assumptions are made:

(1) Each layer consists of homogeneous, isotropic or cross-anisotropic, elastic materials, which obey Hooke's law;

(2) The layers are of infinite extent in the horizontal direction, but of finite thicknesses in vertical direction.

(3) The base of the system is rough and rigid.

(4) There is no relative movement at layer interfaces.

Finite Layer theory is used to solve this problem. To illustrate the basic idea of finite layer method, let us start with the solution procedure of axial symmetric circular load on layered system.

2.2.1 Vertical Load over a Circular Area on Layered System

The vertical load applied to the surface is uniformly distributed over a circular area, which leads to an axially symmetric problem. The equations of the theory of
elasticity for the three-dimensional problem in cylindrical coordinates as used in this study are summarized in the following [7,8].

Equations of equilibrium:

\[
\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0
\]  
(2.1a)

\[
\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} = 0
\]  
(2.1b)

Stress-strain relationship:

\[
\sigma = D \varepsilon
\]  
(2.2)

where

\[\sigma = (\sigma_r, \sigma_\theta, \sigma_z, \tau_{rz})^T\] is the vector of stress components, \[\varepsilon = (\varepsilon_r, \varepsilon_\theta, \varepsilon_z, \gamma_{rz})^T\] is the vector of strain components and \[D\] is the matrix of elastic constants:

\[
D = \begin{bmatrix}
a & b & c & 0 \\
b & a & c & 0 \\
c & c & d & c \\
0 & 0 & 0 & f
\end{bmatrix}
\]

in which for isotropic materials \[a = d = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)}\], \[b = c = \frac{\nu E}{(1+\nu)(1-2\nu)}\] and \[f = \frac{E}{2(1+\nu)}\], in which \(E\) is the modulus of elasticity, and \(\nu\) is Poisson's ratio.

Strain-displacement relationship:

\[
\begin{bmatrix}
\varepsilon_r \\
\varepsilon_\theta \\
\varepsilon_z \\
\gamma_{rz}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial}{\partial r} & 0 & 0 \\
0 & \frac{1}{r} & 0 \\
0 & 0 & \frac{\partial}{\partial z} \\
\frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial r}
\end{bmatrix} \begin{bmatrix}
u_r \\
u_\theta \\
u_z
\end{bmatrix}
\]  
(2.3)
In order to solve the preceding equations (2.1), (2.2) and (2.3), the following Hankel transformations are used:

\[ U_z = \int_0^\infty r u_z J_0 (\alpha r) \, dr \]  
\[ (U_r, U_\theta) = \int_0^\infty r(u_r, u_\theta) J_1 (\alpha r) \, dr \] 

where \( J_0 \) and \( J_1 \) are zero order and first order Bessel functions of the first type respectively, and \( \alpha \) is the radial wave number.

The corresponding inverse transformations are

\[ u_z = \int_0^\infty \alpha U_z J_0 (\alpha r) \, d\alpha \]  
\[ (u_r, u_\theta) = \int_0^\infty \alpha U_r J_1 (\alpha r) \, d\alpha \] 

By applying Hankel transform to strain-displacement relationship equation (2.3), strain components can be expressed as the function of transformed displacement, then stresses in terms of the transformed displacements are available by using the stress-strain relationship. Substituting these values of stress into the equations of equilibrium results in governing equations in terms of only one spatial co-ordinate \( z \) and wave number \( \alpha \).

\[-\alpha \left[ \alpha a U_r + c \frac{\partial U_z}{\partial z} \right] - \frac{\partial}{\partial z} \left[ \alpha U_z - \frac{\partial U_r}{\partial z} \right] f = 0 \]  
\[-\alpha \left[ \alpha U_z - \frac{\partial U_r}{\partial z} \right] f + \frac{\partial}{\partial z} \left[ \alpha c U_r + d \frac{\partial U_z}{\partial z} \right] = 0 \]

To solve the equations (2.6a) and (2.6b) in transformed space, make the substitutions,

\[ H = \alpha a U_r + c \frac{\partial U_z}{\partial z} \]  
\[ T = \left[ -\alpha U_z + \frac{\partial U_r}{\partial z} \right] f \]
\[ N = \alpha c U_r + d \frac{\partial U_z}{\partial z} \quad (2.7c) \]

Substituting \( H, T, N \) into equations (2.6a) and (2.6b) results in

\[ M(\alpha, z) S = 0 \quad (2.8) \]

where

\[ S = (M, N, T)^T \]

\[ M(\alpha, z) = \begin{bmatrix} \alpha & 0 & -\partial / \partial z \\ 0 & -\partial / \partial z & -\alpha \end{bmatrix} \]

The equations (2.8) can be satisfied by introducing the Airy stress function \( \phi \) such that

\[ H = \partial^2 \phi / \partial z^2 \quad (2.9a) \]

\[ T = \alpha \partial \phi / \partial z \quad (2.9b) \]

\[ N = -\alpha^2 \phi \quad (2.9c) \]

Using these definitions of \( H, T, \) and \( N \) in addition to equations (2.7a)(2.7b)(2.7c)

leads to

\[ \alpha U_r = A \partial^2 \phi / \partial z^2 + B \alpha^2 \phi \quad (2.10a) \]

\[ \frac{\partial U_z}{\partial z} = -B \partial^2 \phi / \partial z^2 - C \alpha^2 \phi \quad (2.10b) \]

\[ \frac{\partial U_z}{\partial z} - \alpha U_z = F \alpha \partial \phi / \partial z \quad (2.10c) \]

where \( A = d/(ad - c^2), \ B = c/(ad - c^2), \ C = a/(ad - c^2), \) and \( F = 1/f. \)

Elimination of \( U_r, U_z \) from the equations (2.10a)(2.10b)(2.10c) leads to the following fourth-order ordinary differential equation in \( \phi \)

\[ A \frac{\partial^4 \phi}{\partial z^4} + \alpha^2 (2B - F) \frac{\partial^2 \phi}{\partial z^2} + \alpha^4 C \phi = 0 \quad (2.11) \]
Suppose that this differential equation has four eigenvalues $\lambda = \pm p$ and $\lambda = \pm q$, where

$$\left( \frac{p}{\alpha} \right)^2 = \left\{ \frac{1}{2} (2B - F) + \sqrt{(2B - F)^2 - 4AC} \right\}/2A \quad (2.12a)$$

$$\left( \frac{q}{\alpha} \right)^2 = \left\{ \frac{1}{2} (2B - F) - \sqrt{(2B - F)^2 - 4AC} \right\}/2A \quad (2.12b)$$

Then the general solution to equation (2.11) can be written as

$$\phi = L_a \cosh(pz) + M_a \cosh(qz) + L_b \sinh(pz) + M_b \sinh(qz) \quad (2.13)$$

Equation sets (2.9) (2.10) and equation (2.13) as well as the boundary conditions can be used to find the flexibility relationship for each layer as presented in the following.

For a given layer, there would be normal stress and shear stress at its top surface $N_a$, $T_a$ and bottom surface $N_b$, $T_b$ as in Figure 2.2. The subscript "a" and "b" denotes top surface and bottom surface of a layer respectively, and we use the notation $N_a = N(h)$, $N_b = N(-h)$ etc. Because $N_a$, $T_a$ and $N_b$, $T_b$ are related to the stress function $\phi$, the relationship between $L_a$, $M_a$, $L_b$, $M_b$ and $N_a$, $T_a$ and $N_b$, $T_b$ can be established. Therefore, $\phi$ can also be expressed as a function with respect to $N_a$, $T_a$ and $N_b$, $T_b$. Based on the relationship between the displacements $U_r$, $U_z$ and $\phi$ as expressed in equation set (2.10), we can finally get the flexibility relationship at the top and bottom surfaces for each layer, which can be written as

$$\delta^i = F^i \rho^i \quad (2.14)$$

where

$$\delta^i = (U_{za}, U_{ra}, -U_{zb}, -U_{rb})^T$$

$$P^i = (N_a, T_a, N_b, T_b)^T$$

$F^i$ is the flexibility matrix for the $i$th layer, which is a $4 \times 4$ symmetric matrix.
The layered system is subjected to a uniformly distributed circular pressure $\sigma_0$ over an area with radius $a$ on the top of surface layer, which can be treated mathematically in transformed domain as

$$N|_{z=0} = \int_0^\infty \sigma_0 r J_0(\alpha r) dr = \sigma_0 a J_1(\alpha a) \tag{2.15}$$

It is generally assumed that the layered system has a rough and rigid base; therefore, the displacements at the bottom of the last layer are all zeros, namely, $U_z = 0$, $U_r = 0$ there.

Since it is assumed that the two adjacent layers are bonded together and there is no slippage at the layer interface, stresses and displacements should be continuous just above and just below the interface. Continuity condition is shown in Figure 2.3, which can be written as

$$\begin{align*}
(N_b)_i &= (N_a)_{i+1} \\
(T_b)_i &= (T_a)_{i+1} \\
(U_{z_b})_i &= (U_{z_a})_{i+1} \\
(U_{x_b})_i &= (U_{x_a})_{i+1} \\
\end{align*} \tag{2.16}$$

In the analysis outlined above, it is shown that each layer can be treated as an element (compared to the finite element method), and each layer has a flexibility matrix associated with it. If every layer flexibility matrix is assembled into a global one, and so does the displacement vector, the following relationship can be obtained.

$$\delta = FP \tag{2.17}$$

Because of the displacement continuity at the interface as illustrated in equation (2.16), the global displacement vector $\delta$ consists largely of zeros. If no special displacement boundary conditions specified, $\delta$ would be all zeros. Solving the above set of equations, the normal stresses and shear stresses at the top and bottom of each layer can be obtained. To get the displacements, we only need to multiply the layer flexibility matrix by the stresses associated with that layer.
At this point, variables $N$, $T$, $U_r$, $U_z$ at the top and bottom of each layer in transformed domain are available. To get stresses, strains and displacements in space domain, inverse transformation is needed. For instance, $u_z, u_r$ can be solved by using equations (2.5a) (2.5b), $\sigma_z$ and $\tau_{rz}$ can be obtained by

$$\sigma_z = \int_0^\infty \alpha NJ_0(\alpha r) d\alpha$$  \hspace{1cm} (2.18)

$$\tau_{rz} = \int_0^\infty \alpha TJ_1(\alpha r) d\alpha$$  \hspace{1cm} (2.19)

To solve $\sigma_r$, firstly, use stress-strain relationship equation (2.2) to get $\sigma_r = a\varepsilon_r + b\varepsilon_\theta + c\varepsilon_z$. Then plug in $\varepsilon_r, \varepsilon_\theta$ and $\varepsilon_z$, which are expressed in terms of the already solved transformed variables. The solution can be written as follows:

$$\sigma_r = \int_0^\infty \alpha \left[ aU_r + c \frac{\partial U}{\partial z} \right] J_0(\alpha r) - (a - b)U_r \frac{J_1(\alpha r)}{r} d\alpha$$  \hspace{1cm} (2.20)

Similarly

$$\sigma_\theta = \int_0^\infty \alpha \left[ \alpha U_r \left( \frac{1}{A} - (a - b) \right) + \left( \frac{B}{A} \right) \right] J_0(\alpha r) + \left( a - b \right)U_r \frac{J_1(\alpha r)}{r} d\alpha$$  \hspace{1cm} (2.21)

In equations like (2.18), (2.19), (2.20), (2.21) etc., the integral upper limit is infinite. In numerical analysis, the infinite range of integration is truncated at such a point that the contribution from the omitted portion is negligible. Then the finite part of the range is equally divided into a number of sub-sections, as shown in Figure 2.4. The integrals are evaluated over each sub-section separately. For each sub-section, twenty $\alpha$ values are chosen according to Gauss quadrature rule. The final result is the sum of the solution for each $\alpha$, for example,

$$\sigma_r = \sum_{i=1}^K \alpha_i \left[ aU_r + c \frac{\partial U}{\partial z} \right] J_0(\alpha_ir) - (a - b)U_r \frac{J_1(\alpha_ir)}{r}$$  \hspace{1cm} (2.22)

In summary, the basic procedure is: (1) apply Hankel transform with respect to radial coordinate $r$ to reduce the problem to that of only one dimension in terms of
coordinate $z$; (2) solve the one dimensional problem for each wave number by the finite layer method; (3) apply inverse Hankel transform to get the solution in space domain. The flowchart is presented in Figure 2.5.

### 2.2.2 Horizontal Load over a Circular Area on Layered System

Uniformly distributed horizontal shear loading over a circular area on the surface of a layered system is a three dimensional problem. Theory of elasticity for the three-dimensional problem in Cartesian coordinates is employed.

Firstly, vectors $\sigma = (\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy}, \tau_{yz}, \tau_{zx})^T$, $\varepsilon = (\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \gamma_{xy}, \gamma_{yz}, \gamma_{zx})^T$, $u = (u_x, u_y, u_z)^T$ are used to denote the vector of stress components, vector of strain components and vector of displacement components respectively.

Equations of equilibrium:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0$$
$$\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = 0$$
$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0$$

Stress-strain law:

$$\sigma = D \varepsilon$$

where $D$ is the matrix of elastic constants:

$$D = \begin{bmatrix}
a & b & c & 0 & 0 & 0 \\
b & a & c & 0 & 0 & 0 \\
c & c & d & 0 & 0 & 0 \\
0 & 0 & 0 & f & 0 & 0 \\
0 & 0 & 0 & 0 & f & 0 \\
0 & 0 & 0 & 0 & 0 & f \\
\end{bmatrix}$$

Strain-displacement relationship:
\[ \varepsilon = -\partial u \] \hspace{1cm} (2.25)

where

\[ \partial = \begin{bmatrix}
\frac{\partial}{\partial x} & 0 & 0 \\
0 & \frac{\partial}{\partial y} & 0 \\
0 & 0 & \frac{\partial}{\partial z}
\end{bmatrix} \]

To solve the problem, a procedure similar to that of vertical load is used. The difference is that to reduce the problem to a one dimensional problem in transformed domain, a double Fourier transforms with respect to both \( x \) and \( y \) coordinates are employed.

Apply double Fourier transformations to displacements and stresses as follows:

\[ (U_x, U_y, U_z) = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (iu_x, iu_y, u_z)e^{i(\alpha x + \beta y)} \, dx \, dy \tag{2.26} \]

\[ (S_{xx}, S_{yy}, S_{zz}, T_{xy}, T_{yz}, T_{zx}) = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, i\tau_{xy}, i\tau_{yz}, i\tau_{zx})e^{i(\alpha x + \beta y)} \, dx \, dy \tag{2.27} \]

where \( \alpha \) denotes wave number in the \( x \) direction and \( \beta \) denotes wave number in the \( y \) direction.

The corresponding inverse Fourier transformations for equations (2.26) and (2.27) are

\[ (u_x, u_y, u_z) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (-iU_x, -iU_y, U_z)e^{-i(\alpha x + \beta y)} \, d\alpha d\beta \tag{2.28} \]

\[ (\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy}, \tau_{yz}, \tau_{zx}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left(S_{xx}, S_{yy}, S_{zz}, -iT_{xy}, -iT_{yz}, -iT_{zx}\right)e^{-i(\alpha x + \beta y)} \, d\alpha d\beta \tag{2.29} \]
It is observed that since the variables on the left hand sides of equations (2.28) and (2.29) satisfy the equations of elasticity, so do the variables on the right hand sides as follows.

\[
\begin{align*}
(u_x, u_y, u_z) &= (-iU_x(\alpha, \beta, z), -iU_y(\alpha, \beta, z), U_z(\alpha, \beta, z))e^{-i(\alpha x + \beta y)} \\
(\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy}, \tau_{xz}, \tau_{yz}) &= (S_{xx}(\alpha, \beta, z), S_{yy}(\alpha, \beta, z), S_{zz}(\alpha, \beta, z), -iT_{xy}(\alpha, \beta, z), -iT_{xz}(\alpha, \beta, z), -iT_{yz}(\alpha, \beta, z))e^{-i(\alpha x + \beta y)}
\end{align*}
\]  

(2.30)  

(2.31)

This means that equations (2.30) and (2.31) satisfy the equations of elasticity for every pair of wave number \(\alpha\) in the \(x\) direction and \(\beta\) in the \(y\) direction. It is further noticed that the following axis transformations

\[
\begin{align*}
\xi &= x \cos \varepsilon + y \sin \varepsilon \\
\eta &= -x \sin \varepsilon + y \cos \varepsilon \\
\alpha &= \rho \cos \varepsilon \\
\beta &= \rho \sin \varepsilon
\end{align*}
\]  

(2.32a)  

(2.32b)  

(2.32c)  

(2.32d)

lead to \(e^{-i(\alpha x + \beta y)} = e^{-i\rho \xi}\), which means that for a certain pair of wave numbers \(\alpha\) and \(\beta\), if an axis transformation is made with an angle of \(\varepsilon\) between the new coordinate system \((\xi, \eta)\) and the old coordinate system \((x, y)\) on the horizontal plane as indicated in Figure 2.6, all the displacements and stresses for a certain point \((x, y, z)\) would only be dependent on the coordinates \((\xi, z)\). Based on the advantage of axis transformation, it is possible to solve the problem in transformed domain with transformed coordinate system \(\xi, \eta, z\), then apply axis transformation back to \((x, y, z)\) still in transformed domain, and finally apply inverse Fourier transformation back to space domain. The following is about the solution procedure in transformed space with transformed axes.

Since the material is isotropic or cross anisotropic (transversely isotropic in the \(x-y\) plane), the same stress-strain relationship as equation (2.24) can also be written with respect to transformed axis \(\xi, \eta, z\) in transformed space as
\[ \dot{\sigma} = D \dot{\varepsilon} \]  \hspace{1cm} (2.33)

where

\[ \varepsilon = (\varepsilon_{\xi \xi}, \varepsilon_{\eta \eta}, \varepsilon_{\xi \eta}, \gamma_{\xi \eta}, \gamma_{\xi \eta}, \gamma_{\xi \xi})^T \]

\[ \sigma = (\sigma_{\xi \xi}, \sigma_{\eta \eta}, \sigma_{\xi \eta}, \tau_{\xi \eta}, \tau_{\xi \eta}, \tau_{\xi \xi})^T \]

Because all the stresses are independent of coordinate \( \eta \), the equations of equilibrium now become

\[ \frac{\partial \sigma_{\xi \xi}}{\partial \xi} + \frac{\partial \tau_{\xi \eta}}{\partial z} = 0 \]  \hspace{1cm} (2.34a)

\[ \frac{\partial \tau_{\xi \eta}}{\partial \xi} + \frac{\partial \sigma_{\xi \eta}}{\partial z} = 0 \]  \hspace{1cm} (2.34b)

\[ \frac{\partial \tau_{\eta \eta}}{\partial \xi} + \frac{\partial \tau_{\eta \eta}}{\partial z} = 0 \]  \hspace{1cm} (2.34c)

It should be noted that equations (2.34a)(2.34b) are uncoupled from equation (2.34c).

Since all the strains are independent of \( \eta \), then

\[ \varepsilon_{\eta \eta} = 0 \]  \hspace{1cm} (2.35)

Under the condition of equation (2.35), the stress-strain relationship equation (2.33) and the strain-displacement relationship, the following equations can be arrived at:

\[ \varepsilon_{\xi \xi} = A \sigma_{\xi \xi} - B \sigma_{\xi \eta} \]  \hspace{1cm} (2.36a)

\[ \varepsilon_{\xi \eta} = -B \sigma_{\xi \xi} + C \sigma_{\xi \eta} \]  \hspace{1cm} (2.36b)

\[ \gamma_{\xi \xi} = F \tau_{\xi \xi} \]  \hspace{1cm} (2.36c)
\[
\frac{\partial \tilde{u}_\eta}{\partial \xi} = F \tau_{\xi \eta} \quad (2.36d)
\]

\[
\frac{\partial \tilde{u}_\eta}{\partial z} = F \tau_{\eta z} \quad (2.36e)
\]

where \( A = d / (ad - c^2) \), \( B = c / (ad - c^2) \), \( C = a / (ad - c^2) \), \( F = 1 / f \)

In the same way as equations (2.30)(2.31), let the solutions be

\[
(\tilde{u}_\xi, \tilde{u}_\eta, \tilde{u}_z) = (-iU_\xi, -iU_\eta, U_z)e^{i\rho \xi} \quad (2.37)
\]

\[
(\tilde{\varepsilon}_{\xi \xi}, \tilde{\varepsilon}_{\xi \eta}, \gamma_{\xi \phi}) = (E_{\xi \xi}, E_{\xi \eta}, -iG_{\xi \phi})e^{i\rho \xi} \quad (2.38)
\]

\[
(\tilde{\sigma}_{\xi \xi}, \tilde{\sigma}_{\xi \eta}, \tilde{\tau}_{\xi \xi}, \tilde{\tau}_{\xi \eta}, \tilde{\tau}_{\eta \xi}) = (S_{\xi \xi}, S_{\xi \eta}, -iT_{\xi \xi}, S_{\xi \eta}, -iS_{\eta \xi})e^{i\rho \xi} \quad (2.39)
\]

It is observed that equations (2.36a)(2.36b)(2.36c) combined with equations (2.34a)(2.34b) can be solved together. Substituting the corresponding components in equations (2.38) (2.39) into stress-strain equations (2.36a)(2.36b)(2.36c) leads to

\[
E = \begin{bmatrix}
A & -B & 0 \\
-B & C & 0 \\
0 & 0 & F
\end{bmatrix}
\]

(2.40)

where \( E = (E_{\xi \xi}, E_{\xi \eta}, G_{\xi \phi})^T \) and \( S = (S_{\xi \xi}, S_{\xi \eta}, T_{\xi \xi})^T \).

Similarly, the equations of equilibrium (2.34a) and (2.34b) become

\[
M(\rho, z)S = 0 \quad (2.41)
\]

where

\[
M(\rho, z) = \begin{bmatrix}
\rho & 0 & -\frac{\partial}{\partial \xi} \\
0 & \frac{\partial}{\partial z} - \rho \\
-\frac{\partial}{\partial \xi} - \rho
\end{bmatrix}
\]

The strain-displacement relationship becomes

\[
E = -N(\rho, z)U \quad (2.42)
\]
where

\[
N(\rho, z) = \begin{bmatrix}
\rho & 0 & 0 \\
0 & \frac{\partial}{\partial z} & 0 \\
\frac{\partial}{\partial z} & -\rho
\end{bmatrix}
\]

\[U = (U_\xi, U_\zeta)^T\]

Introducing the Airy stress function \(\phi\) and let

\[
S_{\zeta\zeta} = \partial^2 \phi / \partial z^2
\]

\[
S_{\xi\phi} = \rho \partial \phi / \partial z
\]

\[
S_{\zeta\phi} = -\rho^2 \phi
\]

The stress-strain and strain-displacement relationships then lead to

\[
\rho U_\xi = A \partial^2 \phi / \partial z^2 + B \rho^2 \phi
\]

\[
\frac{\partial U_\zeta}{\partial z} = -B \partial^2 \phi / \partial z^2 - C \rho^2 \phi
\]

\[
\frac{\partial U_\xi}{\partial z} - \rho U_\zeta = F \rho \partial \phi / \partial z
\]

Equations (2.43)(2.44) are exactly in the same form as those of equations (2.9a-c), (2.10a-c). Following exactly the same procedure as that for vertical loading case, the element flexibility matrix for each layer and the global flexibility matrix for the whole system can be obtained.

For the uncoupled equilibrium equation (2.34c), plugging the corresponding components from equation (2.39) results in

\[
-\rho S_{\zeta\eta} + \frac{\partial S_{\phi\zeta}}{\partial z} = 0
\]

Plugging the corresponding components from equations (2.37)(2.39) into equations (2.36d)(2.36e) leads to
\[
\rho U_{\eta} = \frac{1}{f} S_{\xi \eta} \quad (2.46)
\]

\[
\frac{\partial U_{\eta}}{\partial z} = \frac{1}{f} S_{\varphi \eta} \quad (2.47)
\]

The above three equations (2.45)(2.46)(2.47) result in

\[
\frac{\partial^2 S_{\varphi \eta}}{\partial z^2} = \rho^2 S_{\varphi \eta} \quad (2.48)
\]

The general solution for the above ordinary differential equation is

\[
S_{\varphi \eta} = C_1 \cosh(\rho \varphi) + C_2 \sinh(\rho \varphi) \quad (2.49)
\]

where \(C_1\) and \(C_2\) are constants.

The equations (2.49) (2.45) and (2.46) lead to

\[
U_{\eta} = \frac{1}{f} \left[ C_2 \cosh(\rho \varphi) + C_1 \sinh(\rho \varphi) \right] \quad (2.50)
\]

which finally yields the flexibility relationship as

\[
\begin{bmatrix}
U_{\eta 
\end{bmatrix} = \frac{1}{f \rho} \begin{bmatrix}
cot \text{anh}(2 \rho h) & -\cos \text{ech}(2 \rho h) & S_{\varphi \alpha} \\
-\cos \text{ech}(2 \rho h) & \cot \text{anh}(2 \rho h) & S_{\varphi \beta}
\end{bmatrix} \begin{bmatrix}
S_{\varphi \alpha} \\
S_{\varphi \beta}
\end{bmatrix} \quad (2.51)
\]

Since the flexibility relationship for a single layer is obtained, it is easy to get the global flexibility relationship for multiple layers by the use of continuous conditions at the layer interface.

When uniformly distributed shear loading \(\tau\) is applied over a circular area, the boundary conditions in transformed domain in terms of coordinates \(x, y, z\) can be expressed as

\[
T_{xz} = \frac{1}{4 \pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} i \tau e^{i(\alpha x + \beta y)} dxdy \quad (2.52)
\]

Change the Cartesian coordinate to cylindrical coordinate as

\[
x = r \cos \theta \\
y = r \sin \theta
\]
\[ \alpha = \rho \cos \varepsilon \]  \hspace{1cm} (2.53) \\
\[ \beta = \rho \sin \varepsilon \]

Then equation (2.52) becomes

\[ T_{xz} = \frac{i}{4\pi^2} \int_{0}^{+\infty} \int_{0}^{2\pi} e^{i\rho r \cos (\theta - \varepsilon)} r dr d\theta \]  \hspace{1cm} (2.54)

Making substitution of \( \varphi = \theta - \varepsilon \) yields

\[ T_{xz} = \frac{i}{4\pi^2} \int_{0}^{+\infty} \int_{0}^{2\pi} e^{i\rho r \cos \varphi} r dr d\varphi \]  \hspace{1cm} (2.55)

Since it is known that Bessel functions of the first kind can be expressed in integral form [9] as

\[ J_n(z) = \frac{1}{i^n\pi} \int_{0}^{\pi} e^{iz\cos \theta} \cos(n\theta) d\theta \]  \hspace{1cm} (2.56)

Then

\[ T_{xz} = \frac{i \tau}{2\pi} \frac{a J_1(\rho a)}{\rho} \]  \hspace{1cm} (2.57)

where \( a \) is the radius of the circular loading area.

In \( \xi, \eta, z \) coordinate system, the boundary condition can be written as

\[ T_{\xi z} = T_{xz} \cos \varepsilon = \frac{i \tau}{2\pi} \frac{a J_1(\rho a)}{\rho} \cos \varepsilon \]  \hspace{1cm} (2.58)

\[ T_{\eta z} = T_{xz} \sin \varepsilon = \frac{i \tau}{2\pi} \frac{a J_1(\rho a)}{\rho} \sin \varepsilon \]  \hspace{1cm} (2.59)

If the boundary conditions like equations (2.58)(2.59) are used in equations (2.17)(2.51), \( U_z, U_\xi, S_{zz}, S_{\xi z} \) would be multiples of \( \cos \varepsilon \) and \( U_\eta, S_{\eta z} \) would be multiples of \( \sin \varepsilon \). If we let

\[ T_{\xi z} = T_{\xi z}^' \cos \varepsilon \]

\[ T_{\eta z} = T_{\eta z}^' \sin \varepsilon \]  \hspace{1cm} (2.60)
Then $T_{xz}$ can be written as
\[ T_{xz} = -T_{y} \sin \varepsilon + T_{y} \cos \varepsilon = -T_{y} \sin^2 \varepsilon + T_{y} \cos^2 \varepsilon \quad (2.61) \]

By inverse transformation
\[ \tau_{xz} = -i \int \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} T_{xz} e^{-i(\alpha x + \beta y)} d\alpha d\beta \quad (2.62) \]

Using cylindrical coordinate system as indicated in equation (2.53) and substituting equation (2.61) into equation (2.62) leads to
\[ \tau_{xz} = -i \int_{0}^{+\pi} \int_{0}^{+2\pi} \left( \frac{T_{y} + T_{y}'}{2} \right) \cos 2\varepsilon + \left( \frac{T_{y} - T_{y}'}{2} \right) e^{-i\sigma \cos(\theta - \varepsilon)} \rho d\rho d\varepsilon \quad (2.63) \]

Put $-\theta + \varepsilon = \phi$, then we get

1st term
\[ 1st \ term = -i \int_{0}^{+\pi} \int_{0}^{+2\pi} \left( \frac{T_{y} + T_{y}'}{2} \right) \cos 2\theta \cos 2\phi - \sin 2\theta \sin 2\phi e^{-i\sigma \cos \phi} d\phi d\rho \]

Since the ‘sin’ part of the above integral is zero and with the use of integral expression of Bessel function of the first order, the 1st term becomes
\[ 1st \ term = 2\pi i \int_{0}^{+\pi} \left( \frac{T_{y} + T_{y}'}{2} \right) \cos 2\theta J_{1}(\rho \sigma) \rho d\rho \]

Similarly, for 2nd term,
\[ 2nd \ term = -i \int_{0}^{+\pi} \int_{0}^{+2\pi} \left( \frac{T_{y} - T_{y}'}{2} \right) e^{-i\sigma \cos \phi} \rho d\rho d\phi \]
\[ = -2\pi i \int_{0}^{+\pi} \left( \frac{T_{y} - T_{y}'}{2} \right) J_{0}(\rho \sigma) \rho d\rho \]

Finally, $\tau_{xz}$ in $x, y, z$ coordinate system can be expressed in term of the solution in $\tilde{\xi}, \eta, z$ coordinate system in transformed domain as
\[ \tau_{xz} = 2\pi i \int_{0}^{+\pi} \left( \frac{T_{y} + T_{y}'}{2} \right) \cos 2\theta J_{1}(\rho \sigma) - \left( \frac{T_{y} - T_{y}'}{2} \right) J_{0}(\rho \sigma) \rho d\rho \quad (2.64) \]
In the same way, $\sigma_{zz}$ can be obtained as

$$
\sigma_{zz} = \int_{-\infty}^{+\infty} \int_{+\infty}^{+\infty} S_{zz} e^{-i(\alpha + \beta)} d\alpha d\beta
$$

$$
= \int_{0}^{+\infty} \int_{0}^{+\infty} S_{zz} \cos \omega e^{-j\rho \cos(\theta - \epsilon)} \rho d\rho d\epsilon
$$

$$
= 2\pi \cos \theta \int_{0}^{+\infty} S_{zz} J_1(\rho \theta) \rho d\rho
$$

(2.65)

Following the same procedure, the other variables can be obtained as

$$
\sigma_{xx} = 2\pi \int_{0}^{+\infty} \left\{ \left( \frac{3S_{zz} + S_{yy} + 2S_{\xi\eta}}{4} \right) \cos \theta J_1(\rho \theta) - \left( \frac{S_{\xi\xi} - S_{\eta\eta} - 2S_{\xi\eta}}{4} \right) \cos(3\theta) J_3(\rho \theta) \right\} \rho d\rho
$$

(2.66)

$$
\sigma_{yy} = 2\pi \int_{0}^{+\infty} \left\{ \left( \frac{S_{zz} + 3S_{yy} - 2S_{\xi\eta}}{4} \right) \cos \theta J_1(\rho \theta) + \left( \frac{S_{\xi\xi} - S_{\eta\eta} + 2S_{\xi\eta}}{4} \right) \cos(3\theta) J_3(\rho \theta) \right\} \rho d\rho
$$

(2.67)

$$
\tau_{xz} = 2\pi \int_{0}^{+\infty} \left\{ \left( \frac{T_{x^z} + T_{x^\xi}}{2} \right) \cos \theta J_2(\rho \theta) - \left( \frac{T_{x^\xi} + T_{x^\eta}}{2} \right) J_0(\rho \theta) \right\} \rho d\rho
$$

(2.68)

$$
\tau_{yz} = 2\pi \int_{0}^{+\infty} \left( \frac{T_{y^z} + T_{y^\xi}}{2} \right) \sin \theta J_2(\rho \theta) \rho d\rho
$$

(2.69)

$$
\tau_{xy} = 2\pi \int_{0}^{+\infty} \left\{ \frac{1}{2} \left( \frac{S_{\xi\xi} - S_{\eta\eta}}{2} - S_{\xi\eta} \right) \cos \theta J_1(\rho \theta) + \frac{1}{2} \left( \frac{S_{\xi\xi} - S_{\eta\eta} + 2S_{\xi\eta}}{4} \right) \cos(3\theta) J_3(\rho \theta) \right\} \rho d\rho
$$

(2.70)

$$
\bar{u}_x = 2\pi \int_{0}^{+\infty} \left\{ \left( \frac{U_{x^z} + U_{x^\xi}}{2} \right) \cos \theta J_2(\rho \theta) + \left( \frac{U_{x^\xi} - U_{x^\eta}}{2} \right) J_0(\rho \theta) \right\} \rho d\rho
$$

(2.71)

$$
\bar{u}_y = 2\pi \int_{0}^{+\infty} \left( \frac{U_{y^z} + U_{y^\xi}}{2} \right) \sin \theta J_2(\rho \theta) \rho d\rho
$$

(2.72)

$$
\bar{u}_z = 2\pi \int_{0}^{+\infty} \left[ U_z \cos \theta J_1(\rho \theta) \right] \rho d\rho
$$

(2.73)
2.3 Verification Problems

In order to validate the semi-analytical method illustrated in the preceding section 2.2, some test problems are analyzed. Stresses generated by the theory in this study were compared to those generated by conventional analytical methods for the same problem.

2.3.1 Stresses under a Uniform Vertical Load: Half Space

Vertical stresses beneath the center of a uniformly and vertically loaded circular area on the surface of a half space were found to be [10]

\[
\sigma_z = q \left[1 - \frac{z^3}{(a^2 + z^2)^{1.5}}\right]
\]  

(2.74)

where \(q\) is the uniform pressure and \(z\) denotes the depth and \(a\) is the radius of the area.

For the test problem, the uniform pressure and radius were selected as \(q = 100\)psi and \(a = 0.5\) inches respectively; the elastic Young’s modulus and Poisson’s ratio for the half space were chosen to be \(E = 100,000\)psi and \(v = 0.5\) respectively. The vertical stress distribution beneath the loading center can be obtained by analytical solution equation (2.74), while for the semi-analytical finite layer method, the half space is treated as a single layer with a very high value for layer thickness. The results by these two solutions are shown in Figure 2.7. As we can see, the results by the two methods are in good agreement.

2.3.2 Stresses under Uniform Vertical Load: 3-Layered System

The next validation test performed is for a uniform vertical circular surface load on a three-layered system. The vertical load applied at the surface was 100psi and the radius of loaded area was 1 inch. The first layer was 2.5 inches thick with an elastic modulus value of 200,000psi and a Poisson's ratio of 0.5. The second layer was also 2.5 inches thick with a Poisson's ratio of 0.5 and elastic modulus of 100,000psi. The third and final layer was infinitely deep with an elastic modulus value of 5,000psi and Poisson's ratio 0.5. The normal stress values found by semi-analytical method are on the axis of symmetry below the circular loaded area and at the interfaces between the layers, which are compared with those based on Jones' tables (2). Once again, the results by semi-
analytical finite layer method match almost perfectly with the conventional analytical method as presented in Figure 2.5.

2.3.3 Stresses under Uniform Horizontal Load: Half Space

This test problem is about a uniformly distributed horizontal pressure over a circular area on the surface of an elastic half space. The radius of the loading area is denoted by \( a \), normal stresses and shear stresses along certain coordinate lines (as shown in Figure 2.9) at the horizontal surface \( z = a \) are calculated by the semi-analytical finite layer method and compared with those presented by Barber [11]. During the analysis by the semi-analytical method, a single layer with very high value of thickness is used to represent the half space. It can be seen in Figure 2.9 that excellent agreement could be achieved for the two results.

2.3.4 Stresses under Uniform Horizontal Load: 2-Layered System

The last validation test was done on a two-layered system with a uniform horizontal circular surface load. The results used for validation in this case are from R. A. Westmann (6). The two-layered system he analyzed is shown in the Figure 2.10, the thickness of the first layer is represented by \( h \) and the second layer is a half space. Elastic Young’s modulus and Poisson’s ratio for both the upper layer and the half space are denoted by \( E_1, v_1 \) and \( E_2, v_2 \) respectively. A concentrated surface shear force \( Q \) is applied over a circular area with radius \( a \). During the analysis of this problem by the semi-analytical finite layer method, some measures are taken: firstly, the half space of infinite thickness is substituted with a layer of a very high value of thickness; secondly, the concentrated surface shear force \( Q \) is divided by the loading area to convert concentrated shear force to uniformly distributed shear force.

The results are parameterized by a load concentration factor \( h/a \) (ratio of first layer thickness to radius of loaded area) and Young’s modulus ratio \( E_1/E_2 \) (ratio of Young’s modulus of the first layer to that of the half space). Here, only the results for interface shearing stress coefficient \( I_{\tau_w} \) are compared for both methods under certain load concentration factor \( h/a \) and Young’s modulus ratio \( E_1/E_2 \). To do that, results for
interface shearing stress $\tau_z$ are first obtained by the semi-analytical method and then corresponding shear stress coefficients are solved according to the following equation

$$
\tau_z(r/a, \theta, h) = \frac{Q}{a^2} \cos(\theta) I_{\tau_z} 
$$

(2.75)

The results presented in Figure 2.10 are for the coordinate line $\theta = 0$ at the interface. As we can see, the results match well.

### 2.4 Pavement Delamination Analysis

In the design of pavement, the most commonly used method for rehabilitation of deteriorated pavements is to apply an AC overlay onto them. Before paving a rehabilitation asphalt layer, the top surface of the existing layer is cleaned and a tack coat is applied to bond the new surface being paved and the underlying layer. A strong bonding between layers is critical to dissipate shear stresses into the entire pavement structure. Lack of interface bonding may lead to slippage cracking, delamination and activate distress mechanisms that will rapidly lead to total failure of the pavement. Such failure has usually occurred in the wheel path and in the areas where the vehicles make sharp turns or apply sudden brakes. Typically a slippage crack is crescent shaped [12] as shown in Figure 2.11.

There might be some other reasons [12] for the crescent shaped crack to occur such as: tensile stress in the overlay behind the tire exceeded the tensile strength of the material, causing a crack behind the braking tire; compressive strength of the overlay was exceeded, causing shoving in front of the braking tire, etc. In this study, it is assumed that the delamination is caused only by inadequate bonding strength, i.e., shear stress produced by traffic load exceeds the shear strength of the layer interface.

Two measures can be taken to prevent delamination: (1) reducing the shear stress at the interface by increasing the overlay thickness; (2) increasing the interface shear strength. The first solution is less economical than the second one. For both of these conditions, it is necessary to determine the magnitude of shear stress at the interface for a certain pavement structure subjected to certain loading condition. The semi-analytical finite layer analysis can be a useful tool.
2.4.1 Shear Stress Developed at the Interface in Layered Pavement

The semi-analytical finite layer method as illustrated in preceding sections are used to examine the shear stress distribution at the interface in a multi-layered pavement system subjected to vertical and/or horizontal loadings at the surface.

The pavement system we consider consists of 5 layers and elastic properties for each layer are presented in Figure 2.12. The load applied to the surface is a dual tire load 4500 lb each with center to center distance 12". If the tire pressure is 100psi each, the contact radius of each tire is 3.785". Suppose that the two tires were put on the y-axis with the center of the first load stationed at point (0,0) and that of second load stationed at the point (0,12). The two tires moving in +x direction results in +x direction surface shear load on the pavement.

In the 1986 AASHTO Guide For Pavement Design [13], there is some information about the braking effect of a vehicle on the pavement including the coefficient of friction. It is shown in Table 2.1 [13] that the coefficient of friction varies with the speed of the vehicle. The maximum coefficient of friction is 0.68 at the speed of 30MPH. If the coefficient of friction at certain speed is available, the horizontal pressure applied to the pavement surface can be obtained by multiplying the coefficient of friction with the uniform pressure on one tire.

In this study, some factors affecting the interface shear stress, such as overlay thickness and loading combination are discussed and some results are presented as follows.

2.4.1.1 Effect of Constant Vertical Load and Various Magnitudes of Horizontal Load

On the surface of the pavement structure shown in Figure 2.12, let the 100psi vertical load held constant for each tire while six kinds of horizontal shear loads assigned to each tire: 0psi, 20psi, 30psi, 40psi, 50psi, 68psi. Each case corresponds to a certain vehicle speed. Distribution of shear stress $\tau_{xz}$ at the interface along y-axis ($z = 1.5''$, $x = 0$) is presented in Figure 2.13, where distance ratio denotes the ratio of distance from the origin to the radius of contact area.
If only 100psi vertical pressure with no horizontal pressure is assigned to each tire, then no shear stress $\tau_{xz}$ will occur at the points along y-axis. The greater the applied horizontal load, the greater the shear stress. The shear stress increases linearly with the applied horizontal load. It is shown that along y-axis the maximum shear stress occurs at the center of each tire.

The distribution of shear stress $\tau_{xz}$ along x-axis at the interface ($z = 1.5''$, $y = 0$) is shown in Figure 2.14. If only vertical load 100psi is assigned to each tire, maximum $\tau_{xz}$ occurs at the two edges of each tire. The two peak shear stresses are equal in value but in opposite direction. If $+x$ direction horizontal shear load is applied, the shear stress in the $+x$ part will increase while shear stress in the $-x$ part will decrease due to superposition. In the case of 68psi horizontal pressure plus 100psi vertical pressure, the maximum shear stress $\tau_{xz}$ is 58psi.

The shear stress resultants due to $\tau_{xz}$, $\tau_{yz}$ at the interface under loading combination of 68psi horizontal pressure plus 100psi vertical pressure are also calculated and their absolute values (without considering the direction) are presented in three-dimensional graph in Figure 2.15.

Comparing Figures 2.13 and 2.14, we can see that maximum interlayer shear stress $\tau_{xz}$ along x-axis is much higher than that along y-axis. The maximum shear stress resultants occur within the region close to the tire edges with coordinates (3.785, 0) and (3.785, 12.0). Therefore, the maximum shear stress resultant along x-axis should be very close to that of the whole interface. In later analysis, emphasis is focused on the shear stress distribution along x-axis at certain interface.

2.4.1.2 Effect of Overlay Thickness: Only Horizontal load Applied

Similar pavement structure in Figure 2.12 is used except with some changes about the thickness of the first layer and load combination. Only apply 68psi horizontal pressure to each tire and observe the variation of shear stress at the first interface under various overlay thickness 1.0'', 1.5'', 2.0'', 2.5'', 3.0'', 3.5''. The shear stress resultant $\tau$ versus distance ratio $d/a$ is shown in Figure 2.16, where $d$ is the distance from the origin to a point on x axis, $a$ is the radius of the load area. It is shown that when the layer is
thinner, say, $d=1.0''$ and $d=1.5''$, the maximum shear stress occurs at the point right below the center of the circular load. As the thickness increases the locus of the maximum $\tau$ will move along x-axis with distance to the origin increasing. It is obvious that the peak shear stress decreases as the thickness increases. For 1'' thickness, $\tau_{\text{max}}$ is about 42% of the applied pressure, for 3.5'' thickness, $\tau_{\text{max}}$ is less than 10% of the applied pressure. A conclusion can be drawn here that surface shear stress can affect only the upper shallow part of the pavement system.

### 2.4.1.3 Effect of Overlay Thickness: Both Vertical and Horizontal Loadings Applied

Set the thickness of the first layer of the pavement structure in Figure 2.12 with various values: 1.0", 1.5", 2.0", 2.5", 3.0" and 3.5". Each tire is applying normal stress 100psi and shear stress 68psi over the pavement surface. Variation of shear stress resultant $\tau$ along x-axis at the first interface is presented in Figure 2.17. $\tau_{\text{max}}$ is located exactly at the edge of the tire for thinner layer, say $d=1.0''$, 1.5''. $\tau_{\text{max}}$ decreases while the thickness of the first layer increases and the locus of $\tau_{\text{max}}$ moves a little outside, but not far from the tire edge, as indicated in Figure 2.17. Beyond certain depth, 3.5" for this case, shear stress is generated mainly by vertical load. Vertical load has a deeper influence zone than horizontal load.

From all the above analyses, we can see that higher loading leads to higher maximum interface shear stress and increasing overlay thickness is an effective way to reduce maximum interface shear stress. The maximum interface shear stress can be approximately found at the tire edges for a vehicle applying both normal and shear stresses to the pavement surface. After the maximum interface shear stress is available, it is used to compare with the bond strength in later delamination prevention testing.

### 2.4.2 Simple Shear Testing Results

The purpose of the simple shear testing is to find out the shear strength at the layer interface under certain confining pressures (normal pressures), temperatures and shearing rates. In this study, the samples were sheared at the rate of 0.625mm/min at 20°C.
The samples were from some pavement sections cored from the field. The pavement structure had five layers as shown in Table 2.2. The asphalt binder and aggregate properties for the top two layers SP12.5 and SP19, respectively, are presented in Table 2.3. Table 2.4 presents the aggregate gradation for the top two layers.

In the laboratory, those field pavement sections were cut and cored into two-layered cylinders with diameter of 6 inches and each layer thickness of 1.0 inch. Two types of samples are obtained: SP12.5 over SP19 and SP19 over SP37.5 as shown in the Figure 2.18. At the interface of the pavement sections, tack coat had been applied during the field pavement construction to bond the two adjacent layers together, no separation of layers were found in the cylindrical samples during the process of cutting and coring.

For the simple shear testing, the bottom of the cylindrical sample was stabilized in the mold, the shear force was applied laterally at constant strain rate onto the upper part of the sample to cause shear failure at the interface. During the testing, the failure generally occurred at the interface. Figure 2.19 - 2.21 present the typical shear failure at the interface. The typical shear stress versus displacement curve is shown in Figure 2.22. The shear stress $\tau$ was computed as follows:

$$\tau = \frac{P}{A}$$

where

$\tau =$ shear stress (psi)

$P =$ shearing loading (lb), and

$A =$ sample cross-sectional area (in$^2$)

The shear stress versus displacement curve was plotted for each specimen. For the same type of sample under same temperature and shearing rate, the failure envelope – peak shear stress versus normal stress curve was plotted. At the temperature 20ºC and shearing rate of 0.625mm/min, the results are listed in Table 2.5 for both types of samples. Figure 2.23 and Figure 2.24 present the relationships between normal stresses and shear strengths for both types of samples, respectively. For the SP12.5 over SP19 type of sample, some lower confining pressures correspond to higher shear strengths as shown in Figure 2.23, which is unreasonable theoretically, therefore, the failure envelope drawn
from the testing data has very low R value. This phenomenon can be explained this way: the samples were taken from the filed pavement sections, during its construction, the quantity of tack coat was not applied uniformly at the interface; researchers [14][15] have pointed out too much tack coat would cause the interface shear strength to decrease.

Following similar procedures, we can get the failure envelopes for the interface at other shearing rates and temperatures. Detailed testing results are available in Kulkarni’s [16] doctoral thesis, in which he used the analysis based on the finite layer theory in this chapter and the laboratory results to develop the guideline for the selection of tack coat and prime coat.

Because testing is not the focus of this study, the following part will discuss some general design guidelines for pavement delamination prevention.

2.4.3 Design Guideline for Pavement Delamination Prevention

The general procedure is as follows:

Step 1. Select a design pavement structure;

Step 2. Choose some representative temperatures T1, T2, …Ti…Tn.

Step 3. For a certain temperature Ti, determine the elastic properties of each layer;

Step 4. For the selected design pavement structure, choose a typical vehicle loading to calculate the maximum interface shear stress $\tau_{\text{max}}$ as well as the corresponding normal stress.

Step 5. Based on the failure envelope $\tau_f = \sigma \tan \phi + c$ obtained from the lab, plug into the normal stress to get the shear strength for each representative shearing rate;

Step 6. If $\tau_f > \tau_{\text{max}}$, it means delamination will not happen at this temperature, then go to step 2 to analyze next temperature; if $\tau_f < \tau_{\text{max}}$, consider increasing pavement thickness or using stronger interface binder and go to step 1 to repeat the whole procedure.
2.5 Summary

In the finite layer analysis, the horizontally layered pavements is taken as an elastic system with no slippage at the layer interface; vehicle loadings both vertically and horizontally are applied over a circular area on the pavement surface; which is a three dimensional problem. By using Hankel or Fourier transforms, this problem is reduced into a one-dimensional one and each layer can be taken as a single element. Compared with other numerical methods such as finite element method or finite difference method, the finite layer method is more accurate, efficient and easy to use, because its solution is semi-analytical; there is no need to generate complicated three-dimensional meshes; there are relatively lower requirements for computational time and memory.

Some parametric study is performed using the method for pavement delamination analysis. In this study, it is assumed that the delamination is caused only by inadequate bonding strength, i.e., shear stress produced by traffic load exceeds the shear strength of the layer interface. For a typical pavement structure subjected to a standard dual tire single axle load, it is found that higher loading leads to higher maximum interface shear stress; increasing overlay thickness is an effective way to reduce the maximum interface shear stress; the maximum interface shear stress can be approximately found at the tire edge in the moving direction; vertical load has a deeper influence zone than horizontal load as far as shear stress is concerned.

For the design guideline of pavement delamination prevention, the maximum interface shear stress should be compared with the interface shear strength determined by simple shear testing. Effort is required to duplicate the field conditions affecting the shear strength at the interface such as temperature, confining pressure and rate of shear.
References


Table 2.1  Braking Effect and Force Transferred to the Pavement due to Skid Resistance [13]

<table>
<thead>
<tr>
<th>Speed Of Vehicle (V) (MPH)</th>
<th>Coefficient of Friction (f)</th>
<th>Braking Distance ( D = \frac{V^2}{(30f)} ) (ft)</th>
<th>Braking Time ( t = \frac{1.3636D}{V} ) (Sec)</th>
<th>Deceleration ( a = -\frac{1.075V^2}{D} ) (ft/sec/sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Dry Pavement</td>
<td>Wet Pavement</td>
<td>Dry Pavement</td>
<td>Wet Pavement</td>
</tr>
<tr>
<td>20</td>
<td>0.66</td>
<td>0.4</td>
<td>20</td>
<td>33</td>
</tr>
<tr>
<td>25</td>
<td>0.675</td>
<td>0.38</td>
<td>31</td>
<td>55</td>
</tr>
<tr>
<td>30</td>
<td>0.68</td>
<td>0.35</td>
<td>44</td>
<td>86</td>
</tr>
<tr>
<td>35</td>
<td>0.675</td>
<td>0.34</td>
<td>60</td>
<td>127</td>
</tr>
<tr>
<td>40</td>
<td>0.66</td>
<td>0.32</td>
<td>81</td>
<td>167</td>
</tr>
<tr>
<td>45</td>
<td>0.64</td>
<td>0.31</td>
<td>105</td>
<td>218</td>
</tr>
<tr>
<td>50</td>
<td>0.62</td>
<td>0.30</td>
<td>134</td>
<td>278</td>
</tr>
<tr>
<td>55</td>
<td>0.60</td>
<td>0.30</td>
<td>168</td>
<td>336</td>
</tr>
<tr>
<td>60</td>
<td>0.58</td>
<td>0.29</td>
<td>207</td>
<td>414</td>
</tr>
<tr>
<td>65</td>
<td>0.56</td>
<td>0.29</td>
<td>251</td>
<td>486</td>
</tr>
<tr>
<td>70</td>
<td>0.54</td>
<td>0.28</td>
<td>302</td>
<td>583</td>
</tr>
</tbody>
</table>

Note: Horizontal Force Transferred from Wheel to the Pavement on Applying Brakes

\[ H = f \times \text{Weight On One Wheel} \]
## Table 2.2  SPS-9A Project 370900, Highway US-1 Northbound, Sanford, NC, Pavement Structure – New Construction

<table>
<thead>
<tr>
<th>Layer No.</th>
<th>Layer Thickness Ins. (mm)</th>
<th>Material Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>N/A</td>
<td>Subgrade Soils – Clay/Loam</td>
</tr>
<tr>
<td>2</td>
<td>7 (178)</td>
<td>Lime Stabilized Subgrade</td>
</tr>
<tr>
<td>3</td>
<td>3-1/2 (89)</td>
<td>Asphalt Concrete Base Course, Type HB (SP37.5)</td>
</tr>
<tr>
<td>4</td>
<td>3 (76)</td>
<td>Asphalt Concrete Binder Course, Type HDB (SP19.0)</td>
</tr>
<tr>
<td>5</td>
<td>2-1/2 (64)</td>
<td>Asphalt Concrete Surface Course, Type HDS (SP12.5)</td>
</tr>
</tbody>
</table>

## Table 2.3  Asphalt binder and aggregates used in the pavement structure

<table>
<thead>
<tr>
<th>Supplier</th>
<th>Location/Source</th>
<th>Material Type</th>
<th>Blend (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Martin Marietta</td>
<td>Lemon Springs Quarry</td>
<td>#67</td>
<td>50 15</td>
</tr>
<tr>
<td>Martin Marietta</td>
<td>Lemon Springs Quarry</td>
<td>#78</td>
<td>22 55</td>
</tr>
<tr>
<td>Martin Marietta</td>
<td>Lemon Springs Quarry</td>
<td>REG. SCRGS.</td>
<td>17 19</td>
</tr>
<tr>
<td>Lee Paving</td>
<td>Rambeaut Pit</td>
<td>N. SAND</td>
<td>10 10</td>
</tr>
<tr>
<td>Lee Paving</td>
<td>Sanford</td>
<td>Bagfines</td>
<td>1 1</td>
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<td>Lee Paving</td>
<td>Sanford</td>
<td>PG64-22</td>
<td></td>
</tr>
</tbody>
</table>
### Table 2.4  Aggregate gradations

| Sieve size (U.S Sieve size SP12.5 | SP19 | ASTM Spec. (
Designation) | mm | % passing | % passing (D 3515) |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>3/4&quot;</td>
<td>19</td>
<td>100</td>
</tr>
<tr>
<td>1/2&quot;</td>
<td>12.5</td>
<td>95</td>
</tr>
<tr>
<td>3/8&quot;</td>
<td>9.5</td>
<td>83</td>
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<td>No.4</td>
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<tr>
<td>No.8</td>
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<td>30</td>
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<tr>
<td>No.16</td>
<td>1.18</td>
<td>23</td>
</tr>
<tr>
<td>No.30</td>
<td>0.6</td>
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<tr>
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<td>0.3</td>
<td>11</td>
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<tr>
<td>No.100</td>
<td>0.15</td>
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</tr>
<tr>
<td>No.200</td>
<td>0.075</td>
<td>4.9</td>
</tr>
</tbody>
</table>

### Table 2.5  Shear strengths under various normal stresses with shearing rate 0.625mm/min at 20ºC

<table>
<thead>
<tr>
<th>Normal Stress (psi)</th>
<th>SP12.5 over SP19 Shear Stress (psi)</th>
<th>SP19 over SP37.5 Shear Stress (psi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>67.7</td>
<td>43.6</td>
</tr>
<tr>
<td>2.7</td>
<td>74.1</td>
<td>60.0</td>
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<td>21.3</td>
<td>60.6</td>
<td>63.7</td>
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<tr>
<td>42.9</td>
<td>71.1</td>
<td>65.3</td>
</tr>
<tr>
<td>43.4</td>
<td>79.6</td>
<td>74.5</td>
</tr>
</tbody>
</table>
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Figure 2.24  Relationship between normal stress and shear strength with shearing rate 0.625mm/min at 20°C for the SP19 over SP37.5 samples
Chapter 3 RESPONSE OF LAYERED PAVEMENTS:
LINEAR VISCOELASTIC ANALYSIS

3.1 Introduction

It is well known that most of the paving materials for flexible pavement are not purely elastic. For example, the mechanical response of asphalt concrete is time-dependent, depending on the loading rate and entire loading history. Generally, the behavior of this kind of material can be described by the theory of linear viscoelasticity. In the field of flexible pavement design, in which vehicle loads are applied on the surface of a layered pavement – subgrade system, it is desirable to analyze the viscoelastic response of the structure under loading condition. For instance, with tensile strains determined from these analyses, the fatigue life of the pavement may be assessed using the fatigue criteria developed by some researchers [1-3]. It is of interest therefore to accurately predict the viscoelastic response of pavement structure.

Correspondence principle is often used to analyze the behavior of viscoelastic material [4-6]. It is based on the fact that the governing equations of viscoelasticity can be converted to the equations of elasticity by integral transforms such as Fourier transform or Laplace transform. Bland [7] and Lee [8] have proposed analytical solutions to problems involving creep displacements by applying the correspondence principle.

For complex geometries and loading conditions, it is often impossible to get the closed form solution. Numerical solution techniques, such as the finite element method (FEM), the finite difference method (FDM) or the semi-analytical method should be adopted instead. There are three approaches to deal with the time dependent response of a viscoelastic material.

The first approach is the direct time domain analysis. White [9] is the one who employed this algorithm. He used the hereditary integral constitutive relationship for the viscoelastic material and discretized the time domain by the finite difference method to perform a finite element analysis in a quasi-static problem. Booker and Small [10] analyzed the response of layered viscoelastic materials under three-dimensional loading.
conditions by a semi-analytical finite layer method using finite difference approach in the time domain. The other two approaches employ Fourier transform and Laplace transform, respectively. They are implemented numerically in similar three steps as follows:

1. Apply Laplace/Fourier transform with respect to time and reduce the problem to a time-independent one.
2. Formulate and solve the problem in the Laplace/Fourier domain by numerical methods, such as FEM, FDM or semi-analytical method.
3. Numerically invert the transformed-domain solution to obtain the time-domain solution.

The application of FEM in Fourier domain to solve viscoelastic problems has been proposed by a number of authors [11-13]. In this procedure, a complex, frequency dependent modulus is usually used in representing the properties of the viscoelastic material, and the input load is transformed to the frequency domain via the Fast Fourier Transform (FFT) algorithm. After the problem in frequency domain is solved, the inverse FFT is applied to the solution to get the response in the time domain.

Some authors have presented solutions to viscoelastic problems by using the numerical method in conjunction with Laplace transform. By combining the FEM with Laplace transform, Adey and Brebbia [14] solved viscoelastic problems; Holzlöhner [15] analyzed dynamic viscoelastic and discrete structural dynamic problems. Chen [16] used the hybrid Laplace transform/FEM to perform the quasi-static and dynamic analysis of viscoelastic Timoshenko beams. In his analysis, the Laplace transform with respect to time was applied to the coupled equations and the finite element model was developed with respect to the spatial coordinate. Aköz [17] used the hybrid Laplace-Carson and FEM to analyze the quasi-static and dynamic response of the viscoelastic Timoshenko beams. Utilization of the FDM in conjunction with Laplace transform enabled Cost [18] to solved problems of viscoelasticity. Using the semi-analytical method in conjunction with Laplace transform, Booker and Small [19] solved the problem of viscoelastic layered system under three-dimensional loading conditions.
With these techniques available to carry out the analyses of the viscoelastic material as mentioned above, it is possible to analyze pavement response in which asphalt concrete may be treated as a viscoelastic material. Some researchers have performed such kind of study. Taking into account the effect of a moving load, Perloff and Moavenzadeh [20] determined the surface deflection of a viscoelastic half-space, Chou and Larew [21] solved the stresses and displacements in a viscoelastic two-layered system, Elliott and Moavenzadeh [22] analyzed a three-layered system. However, these studies could not handle multi-layered system. Most recently, Hopman [23] developed a viscoelastic multi-layer computer program VEROAD by using the correspondence principle in Fourier domain. In his research, Burger’s model was used to describe the viscoelastic behavior. However, the parameters for the Burger’s model are not easy to determine through laboratory testing, thus the program may not be as useful.

In this study, the author used three different approaches to deal with time dependent response of viscoelastic layer. The linear viscoelastic layered pavement system subjected to axisymmetric vertical circular loadings on the surface was analyzed for quasi-static responses. The generalized Maxwell model or Kelvin model were used to describe the viscoelastic behavior. Hankel transform was used to simplify problem involving three-dimensional loading to that involving only a single spatial dimension. Then the resulting transformed space-time problem was solved by the finite element method with three techniques: (a) direct time analysis in which the time interval was discretized by the finite difference method; (b) Fourier transform applied to the time terms; (c) Laplace transform applied to the time terms. The solution flow chart is presented in Figure 3.1. For the latter two conditions, numerical inversion of Fourier transform or Laplace transform with respect to time are required. In this study, inverse FFT and the numerical inversion of Laplace transform method developed by Honig and Hirdes [24] were used. In the literature [19], Talbot method for the inversion of Laplace transform was used, which is less reliable as compared to the one developed by Honig and Hirdes because its accuracy depends on four free parameters. Finally, the numerical results for the quasi-static responses by three techniques were presented and compared.
Computational efficiency and accuracy for each technique were discussed. Advantages of one method over the other were also evaluated.

The author also applied the viscoelastic model in the pavement analysis. Based on the Frequency Sweep Testing data about asphalt concrete samples, the corresponding generalized Maxwell models were established with model parameters determined by the software IRIS [26]. Representative pavement structures were selected with the lab-based viscoelastic properties designated for the asphalt concrete layer and then finite element viscoelastic analyses in the Fourier domain were performed. The pavement fatigue life based on maximum tensile strain theory by viscoelastic analysis was compared with that calculated by elastic analysis.

As compared to chapter 2, horizontal shear load is not taken into account in this chapter. This is justifiable for the reasons stated below. For the viscoelastic effects to be significant, the loading time introduced should be larger. In case of slow moving vehicle (< 20 MPH), where this effect may be important, the horizontal shear force generated due to braking and turning is small. In case of fast moving vehicle, where the shear force developed during braking and turning may be large, the viscoelastic effects will be insignificant due to relatively short loading time.

3.2 Mechanical Models for Viscoelastic Material Behavior

The behavior of a viscoelastic material is a combination of elastic deformation and viscous flow. Take a one-dimensional viscoelastic bar as an example, at the time \( t=0 \), a constant stress is applied to it and kept unchanged until \( t=t' \). The response curve is shown in Figure 3.3, at \( t=0 \), a more or less instantaneous strain occurs, after which the material creeps until \( t' \), and then, when the stress is removed, it creeps back. If, alternatively, an instantaneous strain is applied at \( t=0 \) and maintained constant, there results an instantaneous stress which is finite, after which, the stress decays with time as illustrated in Figure 3.4.

A material is considered to be linearly viscoelastic if the stress \( \sigma \) is proportional to the strain \( \varepsilon \) at a given time \( t \) and the linear superposition principle is satisfied.
The details of the theory of viscoelasticity are available in the books written by Findley [4], Christensen [5], Flugge [6].

3.2.1 **Basic Elements: Spring and Dashpot**

Linear springs and linear viscous dashpots are the basic components for linear viscoelastic models. The linear spring, as indicated in Figure 3.5 obeys Hooke’s law when it is subjected to a force:

$$\sigma = E\varepsilon$$  \hspace{1cm} (3.1)

in which $E$ is the Young’s modulus.

As indicated in Figure 3.6, the dashpot is an ideal viscous element whose stress is proportional to the strain rate. By Newton’s law of viscosity, we have

$$\sigma = \eta \frac{d\varepsilon}{dt} = \eta \dot{\varepsilon}$$  \hspace{1cm} (3.2)

where the constant $\eta$ is called the coefficient of viscosity, and the dot here and elsewhere denotes the derivative with respect to time.

The behavior of viscoelastic material is a kind of combination of that of the springs and dashpots. Viscoelastic models can be built up by putting the springs and dashpots in series, parallel, or various combinations of these.

3.2.2 **The Maxwell Model**

The Maxwell model is shown in Figure 3.7, which is composed of a spring and a dashpot connected in series. Stress-strain relationships are represented by the spring as

$$\sigma_1 = E\varepsilon_1$$  \hspace{1cm} (3.3)

and by the dashpot as

$$\sigma_2 = \eta \dot{\varepsilon}_2$$  \hspace{1cm} (3.4)

Since both the spring and the dashpot are connected in series, they are subjected to the same amount of stress, the total stress

$$\sigma = \sigma_1 = \sigma_2$$  \hspace{1cm} (3.5)
The total strain is

\[ \varepsilon = \varepsilon_1 + \varepsilon_2 \]  

(3.6)

Differentiating (3.6) with respect to time \( t \), we have

\[ \dot{\varepsilon} = \dot{\varepsilon}_1 + \dot{\varepsilon}_2 \]  

(3.7)

Differentiating (3.3) with respect to time to get the expression for \( \dot{\varepsilon}_1 \), and substituting \( \dot{\varepsilon}_1 \) and \( \dot{\varepsilon}_2 \) into (3.7) yields

\[ \frac{\sigma}{E} + \frac{\dot{\sigma}}{\eta} = \dot{\varepsilon}_1 + \dot{\varepsilon}_2 = \dot{\varepsilon} \]  

(3.8)

For the case of creep under a constant stress \( \sigma_0 \) applied at \( t = 0 \), the solution of this equation is

\[ \varepsilon = \sigma_0 \left( \frac{1}{E} + \frac{1}{\eta} t \right) = \sigma_0 J(t) \]  

(3.9)

in which \( J(t) = \frac{1}{E} + \frac{1}{\eta} t \) is called the creep compliance.

For the case of relaxation under a constant \( \varepsilon_0 \) applied at \( t = 0 \), the solution of (3.8) is

\[ \sigma = E \varepsilon_0 \frac{E \varepsilon_0}{\eta} = E(t) \varepsilon_0 \]  

(3.10)

The stress will gradually relax and, after a long period of time, will become zero. \( E(t) = E \varepsilon_0 \frac{E \varepsilon_0}{\eta} \) is called the relaxation modulus.

### 3.2.3 The Kelvin Model

The Kelvin model can be constructed by connecting a spring and a dashpot in parallel, as illustrated in Figure 3.8. This model is also called the Voigt model sometimes. Since both elements are connected in parallel, we have

\[ \varepsilon = \varepsilon_1 = \varepsilon_2 \]  

(3.11)
and the total stress is

\[ \sigma = \sigma_1 + \sigma_2 \]  \hspace{1cm} (3.12)

Substituting equations (3.3) and (3.4) into (3.12) and using (3.11) results in

\[ \dot{\sigma} = \dot{E} \varepsilon + \eta \varepsilon \]  \hspace{1cm} (3.13)

If the model is subjected to a constant \( \sigma_0 \) applied at \( t = 0 \), the solution to (3.13) is

\[ \varepsilon = \sigma_0 \left(1 - e^{-\frac{Et}{\eta}}\right) = \sigma_0 J(t) \]  \hspace{1cm} (3.14)

It is observed that the strain increases with a reducing rate and approaches the value of \( \frac{\sigma_0}{E} \) asymptotically when \( t \) tends to infinity. The initial strain rate at \( t = 0^+ \) is finite \( \left. \frac{d\varepsilon}{dt} \right|_{t=0^+} = \frac{\sigma_0}{\eta} \), and the strain rate approaches the value \( \varepsilon \big|_{t=\infty} = 0 \) asymptotically when \( t \) tends to infinity.

Neither the Maxwell nor Kelvin model described above accurately represents the behavior of most viscoelastic materials. For instance, the Maxwell model does not show the decreasing strain rate under constant stress, which is a characteristic of primary creep. The Kelvin model does not exhibit instantaneous strain on loading or unloading.

### 3.2.4 Generalized Maxwell Model

The generalized Maxwell model is composed of springs and dashpots connected in parallel as shown in Figure 3.9. For the single spring component, the stress-strain relationship can be written as

\[ \sigma_0 = E_0 \varepsilon \]  \hspace{1cm} (3.15)

For each Maxwell component, we have

\[ \frac{\dot{\sigma}_i}{E_i} + \frac{\sigma_i}{\eta_i} = \varepsilon \]  \hspace{1cm} (3.16)
Because the single spring and every Maxwell component are connected in parallel, the total stress is

$$\sigma = \sigma_0 + \sum_{i=1}^{n} \sigma_i$$  \hspace{1cm} (3.17)

For the case of relaxation under a constant $\varepsilon_0$ applied at $t=0$, we have

$$\sigma = \left( E_0 + \sum_{i=1}^{n} E_i e^{\frac{E_i}{\eta_i}} \right) \varepsilon_0 = E(t) \varepsilon_0$$  \hspace{1cm} (3.18)

in which $E(t) = \left( E_0 + \sum_{i=1}^{n} E_i e^{\frac{E_i}{\eta_i}} \right)$ is the relaxation modulus for the generalized Maxwell model.

The creep compliance $J(t)$ for this model is not as easy to solve as the relaxation modulus $E(t)$. There is a detailed illustration about its derivation in the literature [6]. In this study, only the relaxation modulus is used to describe the behavior of the generalized Maxwell model.

### 3.2.5 Generalized Kelvin Model

The generalized Kelvin model is presented in Figure 3.10, in which a single spring and an arbitrary number of Kelvin components are arranged in series. For the single spring, the stress-strain relationship can be expressed by (3.15), and for every single Kelvin model we have

$$\sigma = E_i \varepsilon_i + \eta_i \dot{\varepsilon}_i$$  \hspace{1cm} (3.19)

Since the stress is equal for the single spring and every Kelvin elements we have

$$\sigma = \sigma_0 = \sigma_1 = \ldots = \sigma_n$$  \hspace{1cm} (3.20)

For the case of creep under a constant stress $\sigma_0$ applied at $t = 0$, the total strain is the sum of the strain of each Kelvin element and the single spring, which can be written as
\[ \varepsilon = \left( \frac{1}{E_0} + \sum_{i=1}^{n} \frac{1}{E_i} \left( 1 - e^{-\frac{E_i}{\eta_i}} \right) \right) \sigma_0 = J(t) \sigma_0 \]  

(3.21)

in which \( J(t) = \frac{1}{E_0} + \sum_{i=1}^{n} \frac{1}{E_i} \left( 1 - e^{-\frac{E_i}{\eta_i}} \right) \) is the creep compliance for the generalized Kelvin model.

For the relaxation modulus of this model, we are going to skip that part due to the complexity of derivation process. Detailed derivation can be found in the literature [6]. In this study, only the creep compliance is used to describe the behavior of the generalized Kelvin model.

The generalized Maxwell model and generalized Kelvin model can more realistically represent the behavior of the viscoelastic material as compared to the Maxwell model or Kelvin model. Actually, these two models can be described by the same type of constitutive equation if they are composed with the same number of springs and dashpots [4]. Thus they are mechanically equivalent and quantitatively the same with proper choice of spring and dashing parameters.

### 3.2.6 Integral Form of Constitutive Relations

Assume that a stress \( \sigma_0 \) is applied suddenly at \( t = 0 \), and then the stress function in terms of time \( \sigma(t) \) varies arbitrarily as time increases, as shown in Figure 3.11. To get the viscoelastic response under this kind of stress, the stress can be broken up into a sequence of infinitesimal step function as \( \sigma_0 \Delta(t), d\sigma \cdot \Delta(t - t') \), where \( \Delta \) is the sign for a unit step function and \( d\sigma = (d\sigma / dt) (t = t') dt' \). The corresponding strain at time \( t \) is then the sum of the strain caused by the entire step loadings at time \( t' < t \), which can be written as

\[ \varepsilon(t) = \sigma_0 J(t) + \int_{0}^{t} J(t - t') \frac{d\sigma}{dt'} dt' \]  

(3.22)

This equation shows that the strain at any time \( t \) depends on the complete past stress history.
Through integration by parts, (3.22) can be changed into another form as

$$\varepsilon(t) = \sigma(t)J(0) + \int_0^t \sigma(t') \frac{dJ(t-t')}{dt'} dt'$$  \hspace{1cm} (3.23)

Both (3.22) and (3.23) are called a *hereditary integral*. We can also write the stress in terms of the strain and relaxation modulus in the convolution integral form as follows.

$$\sigma(t) = \varepsilon_0 E(t) + \int_0^t E(t-t') \frac{d\varepsilon}{dt'} dt'$$  \hspace{1cm} (3.24)

$$\sigma(t) = \varepsilon(t)E(0) + \int_0^t \varepsilon(t') \frac{dE(t-t')}{dt'} dt'$$  \hspace{1cm} (3.25)

### 3.2.7 Differential Operator Form of Constitutive Relations

The creep and relaxation integral equations of the stress strain constitutive relations are not the only forms. The governing equation of any model of the Kelvin or Maxwell type can also be expressed as

$$\sigma + p_1 \frac{d\sigma}{dt} + p_2 \frac{d^2\sigma}{dt^2} + \ldots + p_n \frac{d^n\sigma}{dt^n} = q_0 \varepsilon + q_1 \frac{d\varepsilon}{dt} + q_2 \frac{d^2\varepsilon}{dt^2} + \ldots + q_n \frac{d^n\varepsilon}{dt^n}$$  \hspace{1cm} (3.26)

A comparison of equation (3.26) with equations (3.8) & (3.13) shows that by choosing $p_0 = 1, \quad p_1 = \frac{\eta}{E}, \quad q_1 = \eta$ and the other coefficients equal to zero, equation (3.26) represents a constitutive relation described by the Maxwell model. By choosing $p_0 = 1, \quad q_0 = E, \quad q_1 = \eta$ and the other coefficients equal to zero, equation (3.26) represents a constitutive relation described by the Kelvin model.

The equation (3.26) can be written in a compact way as

$$\sum_{k=0}^n p_k \frac{d^k\sigma}{dt^k} = \sum_{k=0}^n q_k \frac{d^k\varepsilon}{dt^k}$$  \hspace{1cm} (3.27)

Equation (3.27) may also be written as

$$P\sigma = Q\varepsilon$$  \hspace{1cm} (3.28)
where $P$ and $Q$ are differential operators:

$$P = \sum_{k=0}^{n} p_k \frac{d^k}{dt^k}$$  \hfill (3.29a)

$$Q = \sum_{k=0}^{n} q_k \frac{d^k}{dt^k}$$  \hfill (3.29b)

### 3.2.8 Constitutive Relations in Fourier Transformed Domain

The Fourier transform pair is given by

$$f(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt$$  \hfill (3.30)

and

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\omega) e^{i\omega t} d\omega$$  \hfill (3.31)

The bar here and thereafter denotes a Fourier transform with respect to time.

Applying the transform (3.30) to (3.27) yields

$$\tilde{\sigma}(\omega) = \tilde{\varepsilon}(\omega) \sum_{k=0}^{n} p_k (i\omega)^k = \varepsilon(\omega) E^*(i\omega)$$  \hfill (3.32)

where $\tilde{\sigma}$ and $\tilde{\varepsilon}$ are the transformed stress and strain respectively, and

$$E^*(i\omega) = \frac{\sum_{k=0}^{n} q_k (i\omega)^k}{\sum_{k=0}^{n} p_k (i\omega)^k}$$  \hfill (3.33)

$E^*(i\omega)$ is sometimes referred to as the dynamic modulus. However, this terminology can be misleading since $E^*(i\omega)$ has no connection with the inertia terms in the governing equation.

For the case of general Maxwell model, the dynamic modulus can be written as
\[
E^*(i\omega) = E_0 + \sum_{k=1}^{n} \frac{i \cdot \omega}{E_k + i\omega\eta_k}
\]  \hfill (3.34)

For the case of general Kelvin model, the dynamic compliance can be expressed as

\[
J^*(i\omega) = \frac{1}{E_0} + \sum_{k=1}^{n} \frac{1}{E_k + i\omega\eta_k}
\]  \hfill (3.35)

### 3.2.9 Viscoelastic Behavior in Three Dimensions

#### 3.2.9.1 Time Domain Stress Strain Relations

According to Boltzman’s superposition principle, the stress-strain relationship can be expressed as follows using a tensorial notation [5]:

\[
\sigma_{ij}(t) = \int_{-\infty}^{t} \varepsilon_{kl}(t-\tau) d\tau \frac{dM_{ijkl}(\tau)}{d\tau}
\]  \hfill (3.36)

where \(\sigma_{ij}\) is the stress tensor and \(\varepsilon_{kl}\) is the strain tensor, \(t\) and \(\tau\) are time variables. It is assumed that \(\varepsilon_{ij} = 0\) for \(t < 0\). \(M_{ijkl}(t)\) can be expressed in the following form:

\[
M_{ijkl}(t) = \frac{1}{3} [3K(t) - 2G(t)] \delta_{ij} \delta_{kl} + G(t)(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})
\]  \hfill (3.37)

in which \(K(t)\) and \(G(t)\) are independent relaxation bulk modulus and shear modulus functions, respectively; \(\delta_{ij}\) is the Kronecker delta. Introducing the deviatoric components of stress \(\sigma_{ij}\) and strain \(\varepsilon_{ij}\) leads to

\[
\sigma_{ij} = 2 \int_{-\infty}^{t} \varepsilon_{ij}(t-\tau) d\tau \frac{dG(\tau)}{d\tau}
\]  \hfill (3.38)

whereas the volumetric stress \(\sigma_{kk}\) and strain \(\varepsilon_{kk}\) are related as

\[
\sigma_{kk} = 3 \int_{-\infty}^{t} \varepsilon_{kk}(t-\tau) d\tau \frac{dK(\tau)}{d\tau}
\]  \hfill (3.39)

where
\[ s_{ij} = \sigma_{ij} - \frac{1}{3} \delta_{ij} \sigma_{kk} \quad (3.40) \]

\[ e_{ij} = e_{ij} - \frac{1}{3} \delta_{ij} e_{kk} \quad (3.41) \]

From equations (3.38) (3.39) (3.40) (3.41), the following equation is derived:

\[ \sigma_{ij} = 2 \int_{-\infty}^{t} e_{ij}(t-\tau) \frac{dG(\tau)}{d\tau} d\tau + \int_{-\infty}^{t} \delta_{ij} e_{kk}(t-\tau) \frac{dK(\tau)}{d\tau} d\tau \quad (3.42) \]

\( K(t) \) and \( G(t) \) are related to the characteristics of dilation and shear, respectively, and can be drawn from the viscoelastic models composed of springs and dash-pots as shown schematically in Figure 3.9 and Figure 3.10.

### 3.2.9.2 Fourier Transformed Stress Strain Relations

Using equation (3.30), the Fourier transform of the deviatoric stress strain relation expressed by equation (3.38) is given by

\[ \tilde{s}_{ij}(\omega) = 2 \int_{-\infty}^{t} \int_{-\infty}^{t} \tilde{e}_{ij}(t-\tau) \frac{dG(\tau)}{d\tau} d\tau \int_{-\infty}^{t} \tilde{e}_{ij}(t-\tau) \frac{dK(\tau)}{d\tau} d\tau e^{-i\omega t} d\tau \quad (3.43) \]

which can be further reduced to the simple form as

\[ \tilde{s}_{ij}(\omega) = 2 \tilde{G}(\omega) \tilde{e}_{ij}(\omega) \quad (3.44) \]

Similarly, applying the Fourier transform to the dilatational part of the stress strain relations gives

\[ \tilde{\sigma}_{kk}(\omega) = 3 \tilde{K}(\omega) \tilde{e}_{kk}(\omega) \quad (3.45) \]

\( \tilde{G}(\omega) \) and \( \tilde{K}(\omega) \) are often referred as complex shear modulus and complex bulk modulus, respectively. The three-dimensional constitutive relation written in tensor form is

\[ \tilde{\sigma}_{ij} = 2 \tilde{G} \tilde{e}_{ij} + \delta_{ij} \tilde{K} \tilde{e}_{kk} \quad (3.46) \]

### 3.2.9.3 Laplace Transformed Stress Strain Relations

The Laplace transform is given as
The Laplace transform of equations (3.38) (3.39) gives, for \( \varepsilon_{ij} = 0 \) for \( t < 0 \),

\[
\sigma_{kk} = 3sK\varepsilon_{kk} \tag{3.49}
\]

Written in tensor form, the stress-strain relationship can be expressed as

\[
\sigma_{ij} = 2G\varepsilon_{ij} + \delta_{ij}K\varepsilon_{kk} \tag{3.50}
\]

### 3.3 Response of Layered Viscoelastic System to Vertical Circular Loading

For linear viscoelastic materials, the three-dimensional stress-strain constitutive relations can be written in integral form as shown in equations (3.42), which are in the time domain. Equations (3.46) and (3.50) illustrate the stress-strain relations in the Fourier transformed domain and Laplace transformed domain, respectively. Therefore, for a viscoelastic material under certain loading and/or boundary conditions, the responses can be obtained through direct time domain analysis, Fourier transformed domain analysis or Laplace transformed domain analysis. In this study, the three aforementioned analyses are carried out for the problem of the layered viscoelastic materials under vertical circular loading conditions, and results are compared to check the advantages and disadvantages of each approach.

#### 3.3.1 Method 1: Direct Time Integration

The equilibrium equations for the layered system subjected to axisymmetric loading in cylindrical coordinates can be expressed as

\[
\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{3.51}
\]

\[
\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} = 0 \tag{3.52}
\]
which are similar to those in chapter 2 except that each stress component here is time-dependent.

Stress-strain relationship for viscoelastic material in time domain is

\[ \sigma(t) = D(t \rightarrow 0) \varepsilon(t) - \int_{0}^{\infty} \frac{d}{d\tau} \varepsilon(\tau) d\tau \]  

(3.53)

where \( \sigma(t) \) and \( \varepsilon(t) \) are the vectors of time-dependent stress components and strain components respectively, and \( D_A(t) \) is the stress-strain relationship matrix for axial-symmetric problems.

\[
D_A(t) = \begin{bmatrix}
K(t) + \frac{1}{4}G(t) & K(t) - \frac{1}{4}G(t) & K(t) - \frac{1}{4}G(t) & 0 \\
K(t) - \frac{1}{4}G(t) & K(t) + \frac{1}{4}G(t) & K(t) - \frac{1}{4}G(t) & 0 \\
K(t) - \frac{1}{4}G(t) & K(t) - \frac{1}{4}G(t) & K(t) + \frac{1}{4}G(t) & 0 \\
0 & 0 & 0 & G(t)
\end{bmatrix}
\]

(3.54)

Strain-displacement relationship is as follows:

\[
\begin{bmatrix}
\varepsilon_r \\
\varepsilon_\theta \\
\varepsilon_z \\
\gamma_{rz}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial}{\partial r} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \frac{\partial}{\partial z} \\
0 & \frac{\partial}{\partial z} & 0
\end{bmatrix} \begin{bmatrix}
u_r \\
u_\theta \\
u_z
\end{bmatrix}
\]

(3.55)

Apply the Hankel transformations with respect to spatial coordinate \( r \) to the displacement components as follows:

\[ U_z = \int_{0}^{\infty} ru_z J_0(ar) dr \]

(3.56)

\[ (U_r, U_\theta) = \int_{0}^{\infty} r(u_r, u_\theta) J_1(ar) dr \]

(3.57)

where \( J_0 \) and \( J_1 \) are zero order and first order Bessel functions of the first type respectively. The corresponding inverse transforms are
\[ u_z = \int_0^\infty a U_z J_0(\alpha r) d\alpha \] (3.58)

\[ (u_r, u_\theta) = \int_0^\infty a U_r J_1(\alpha r) d\alpha \] (3.59)

Substituting (3.58) (3.59) into the strain-displacement relationship (3.55) results in

\[ \varepsilon_{rr} = \int_0^\infty \alpha (a J_0(\alpha r) - \frac{1}{r} J_1(\alpha r)) U_r d\alpha \] (3.60)

\[ \varepsilon_{\theta\theta} = \int_0^\infty \alpha \frac{1}{r} J_1(\alpha r) U_r d\alpha \] (3.61)

\[ \varepsilon_{zz} = \int_0^\infty \alpha J_0(\alpha r) \frac{\partial U_z}{\partial z} d\alpha \] (3.62)

\[ \gamma_{rz} = \int_0^\infty \alpha \left( \frac{\partial U_r}{\partial z} - a U_z \right) J_1(\alpha r) d\alpha \] (3.63)

Substituting the above strain components into the stress-strain law in equation (3.53), the stress components can be expressed in terms of transformed displacement components. If these values of stress are used in the equilibrium equation (3.51)(3.52), we get

\[- \alpha M - \frac{\partial}{\partial z} T = 0 \] (3.64)

\[- \alpha T + \frac{\partial}{\partial z} N = 0 \] (3.65)

where \( M \), \( T \) and \( N \) are substitutions for the following expressions:

\[ M = \alpha \left( a(0) U_z(t) - \int_0^t \frac{da(t-\tau)}{d\tau} U_z(\tau) d\tau \right) + c(0) \frac{\partial U_z}{\partial z}(t) - \int_0^t \frac{dc(t-\tau)}{d\tau} \frac{\partial U_z}{\partial z}(\tau) d\tau \] (3.66)

\[ T = \alpha \left( f(0) U_z(t) - \int_0^t \frac{df(t-\tau)}{d\tau} U_z(\tau) d\tau \right) - \left( f(0) \frac{\partial U_z}{\partial z}(t) - \int_0^t \frac{df(t-\tau)}{d\tau} \frac{\partial U_z}{\partial z}(\tau) d\tau \right) \] (3.67)
\begin{equation}
N = \alpha \left( c(0)U_r(t) - \int_0^t \frac{dc(t-\tau)}{d\tau} U_r(\tau) d\tau \right) + a(0) \frac{\partial U_z}{\partial z}(t) - \int_0^t \frac{da(t-\tau)}{d\tau} \frac{\partial U_z}{\partial z}(\tau) d\tau \tag{3.68}
\end{equation}

where \( a(t), c(t) \) and \( f(t) \) are substitutions for the following expressions:

\begin{align*}
a(t) &= K(t) + \frac{\alpha}{2} G(t) \tag{3.69} \\
c(t) &= K(t) - \frac{\alpha}{2} G(t) \tag{3.70} \\
f(t) &= G(t) \tag{3.71}
\end{align*}

It is observed that equations (3.64)-(3.65) can be written in matrix form as

\begin{equation}
RD(0)VV(t) - R \int_0^t \frac{dD(t-\tau)}{d\tau} VV(\tau) d\tau = 0 \tag{3.72}
\end{equation}

where

\begin{align*}
R &= \begin{bmatrix} \alpha & 0 & -\frac{\partial}{\partial z} \\ 0 & -\frac{\partial}{\partial z} & -\alpha \end{bmatrix} \\
D(t) &= \begin{bmatrix} a(t) & c(t) & 0 \\ c(t) & a(t) & 0 \\ 0 & 0 & f(t) \end{bmatrix} \\
V &= \begin{bmatrix} \alpha & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial z} & -\alpha \end{bmatrix}^T \\
W &= (U_r, U_z)^T
\end{align*}

Consistent with the associated boundary and continuity equations, the solution of equation (3.72) can be obtained by the finite element approximation. From the principle of virtual displacement, we can get the following relationship
\[
\int_\Omega \delta W^T(t) V^T D(0) V W(t) dz - \int_0^t \int_\Omega \delta W^T(t)V^T \frac{dD(t-\tau)}{d\tau} V W(\tau) dz d\tau \\
- \int_\Omega \delta W^T(t) p dz - \int_\Omega \delta W^T(t) q dz = 0
\] (3.73)

where \( \Omega \) is the region along \( z \)-axis because the problem is reduced to that of one dimension after the spatial transform; \( p \) is the body force; \( q \) is the surface traction.

Dividing the layered system along vertical direction into finite elements, the displacements within each element can be expressed as

\[
U_r = \sum_{i=1}^n N_i U_{ri}
\] (3.74)

\[
U_z = \sum_{i=1}^n N_i U_{zi}
\] (3.75)

where \( U_{ri} \) and \( U_{zi} \) are the corresponding radial and vertical displacements at node \( i \) and \( N_i \) is the shape function associated with node \( i \). The displacement vector \( W^e \) for each element can then be expressed in matrix form as

\[
W^e = N_i \{ W^e \}
\] (3.76)

By substituting equation (3.76) into equation (3.73), the following expression is arrived

\[
\sum_{e=1}^N \delta \{ W^e^T(t) \} \left[ \int \int_\Omega N^T V^T D(0) V N \{ W^e(t) \} dz - \int_0^t \int_\Omega N^T V^T \frac{dD(t-\tau)}{d\tau} V N \{ W^e(\tau) \} dz d\tau - \int_\Omega N^T p dz - \int_\Omega N^T q dz \right] = 0
\] (3.77)

For arbitrary \( \delta \{ W^e^T(t) \} \) in equation (3.77), the following elemental equation is obtained:

\[
K^e(0) \{ W^e(t) \} - \int_0^t \frac{dK^e(t-\tau)}{d\tau} \{ W^e(\tau) \} d\tau = F^e
\] (3.78)

where
\[
K^e(t) = \int_{\Omega^e} N^T V^T D(t) V N dz
\]
(3.79)

\[
F^e = \int_{\Omega^e} N^T p dz - \int_{\Omega^e} N^T q dz
\]
(3.80)

During the assembly of the elemental equations, it is assumed that the stresses and displacements are continuous at the layer interface, which yields the following set of equations:

\[
K(0)\{W(t)\} - \int_0^t \frac{dK(t-\tau)}{d\tau} \{W(\tau)\} d\tau = F
\]
(3.81)

For the problem in consideration, the body force is ignored and no surface traction applied. Therefore, most components of the global force vector \( F \) are zero except for the node on the layer surface where the vertical nodal force is nonzero. The global force vector \( F \) can be written as

\[
F = (0, F_{zz}, 0, 0, \ldots, 0)^T
\]
(3.82)

If a uniform pressure \( f_{zz} \) is applied on the layer surface over a circular region, then

\[
F_{zz} = \int_0^\infty r f_{zz} J_0(\alpha r) dr
\]
(3.83)

A fully discretization is achieved in equation (3.81) by discretizing the convolution integral by the finite difference method as follows

\[
\int_t^{t+\Delta t} \frac{dK(t-\tau)}{d\tau} \{W(\tau)\} d\tau = (1-\theta)\Delta t \frac{dK(t-\tau)}{d\tau} \left| \left. \frac{\{W(t)\} - \theta\Delta t \frac{dK(t-\tau)}{d\tau} \{W(\tau)\}}{\tau=t+\Delta t} \right| \right. \{W(\tau)\} d\tau
\]
(3.84)

The above equation leads to

\[
\left. \left( K(0) - \theta\Delta t \frac{dK(t-\tau)}{d\tau} \right) \right|_{\tau=t+\Delta t} \{W(\tau)\} = F(\tau+\Delta t) + (1-\theta)\Delta t \frac{dK(t-\tau)}{d\tau} \left| \left. \frac{\{W(t)\} - \theta\Delta t \frac{dK(t-\tau)}{d\tau} \{W(\tau)\}}{\tau=t+\Delta t} \right| \right. \{W(\tau)\}
\]
(3.85)

Therefore, the solutions at any time can be solved by a forward marching process. However, it is observed that there is an accumulation of history terms on the right-hand-side of equation (3.85) as time increases. Thus the demand on computer storage and
computer time will increase as time increases. This problem can be overcome if each component of \( D(t) \) can be expressed as a Prony series and therefore bypasses the need to store the entire history of displacements, strains and stresses.

In this study, the spatial domain is discretized by finite element method using linear shape functions, and the time domain is discretized by finite difference method using the trapezoidal rule.

For an element lying between coordinates \( z_i \) and \( z_{i+1} \), using linear interpolation for displacements \( U_r \) and \( U_z \) results in

\[
U_r = U_{r(i)} \frac{z_{i+1} - z}{z_{i+1} - z_i} + U_{r(i+1)} \frac{z - z_i}{z_{i+1} - z_i} \quad (3.86)
\]

\[
U_z = U_{z(i)} \frac{z_{i+1} - z}{z_{i+1} - z_i} + U_{z(i+1)} \frac{z - z_i}{z_{i+1} - z_i} \quad (3.87)
\]

Then, the vector of displacement can be written as

\[
W^e = \frac{1}{z_{i+1} - z_i} \begin{bmatrix} (z_{i+1} - z) & 0 & (z - z_i) & 0 \\ 0 & (z_{i+1} - z) & 0 & (z - z_i) \end{bmatrix} \{W^e\} \quad (3.88)
\]

where

\[
\{W^e\} = \begin{bmatrix} U_{r(i)} & U_{z(i)} & U_{r(i+1)} & U_{z(i+1)} \end{bmatrix}^T \quad (3.89)
\]

Equations (3.86)-(3.89) leads to the following expression:

\[
B = \frac{1}{h} \begin{bmatrix} \alpha(z_{i+1} - z) & 0 & \alpha(z - z_i) & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \quad (3.90)
\]

\[
K^e(t) = \int_{z_i}^{z_{i+1}} B^T D(t) B \, dz = \begin{bmatrix} \alpha^2 a(t) + f(t) & -\alpha c(t) + f(t) \alpha & \alpha^2 a(t) + f(t) & \alpha c(t) + f(t) \alpha \\ -\frac{3}{2} \alpha c(t) + f(t) \alpha & \frac{6}{h} - \frac{3}{2} \alpha c(t) + f(t) \alpha & \frac{3}{2} \alpha c(t) + f(t) \alpha & \frac{6}{h} - \frac{3}{2} \alpha c(t) + f(t) \alpha \\ \frac{3}{2} \alpha c(t) + f(t) & -\frac{3}{2} \alpha c(t) + f(t) \alpha & \frac{3}{2} \alpha c(t) + f(t) & -\frac{3}{2} \alpha c(t) + f(t) \alpha \\ \frac{3}{2} \alpha c(t) + f(t) & -\frac{3}{2} \alpha c(t) + f(t) \alpha & \frac{3}{2} \alpha c(t) + f(t) & -\frac{3}{2} \alpha c(t) + f(t) \alpha \end{bmatrix}
\]

where
\[ h = z_{i+1} - z_i \]

After the elemental stiffness matrix and elemental load vector are assembled into global ones, the time domain discretization is performed. Assume that both the bulk relaxation modulus \( K(t) \) and shear relaxation modulus \( G(t) \) comply with the law of generalized Maxwell model or generalized Kelvin model or are kept constant, then each component of stress-strain relationship matrix \( D(t) \) can be expressed as a Prony series.

For illustration purpose, the detailed time forward marching algorithm is as follows for a material with \( K(t) \) and \( G(t) \) drawn from the same viscoelastic models but with different parameters. For that case, \( D(t) \) can be written as

\[
D(t) = \begin{bmatrix} a & c & 0 \\ c & d & 0 \\ 0 & 0 & f \end{bmatrix} \phi(t) = D_0 \phi(t)
\]

(3.91)

where \( \phi(t) \) is given as a Prony series:

\[
\phi(t) = \sum_{i=1}^{n} \phi_i e^{-\alpha_i t}
\]

(3.92)

Plugging (3.91) into (3.81) results in

\[
K(0) \begin{bmatrix} \{W(t)\} - \frac{1}{\tau_0} \int_0^t \phi(t - \tau) \{W(\tau)\} d\tau \end{bmatrix} = F(t)
\]

(3.93)

where

\[
K(0) = \sum_{i=1}^{n} \phi_i \int B^T D_0 B dz
\]

(3.94)

In order to obtain the general fully discrete formulation at each time step \( t = t_i \), we begin considering a fully discrete formulation at \( t = t_i > 0 \):
Let

\[
K(0) \cdot \left[ \frac{1}{t_1} \int_0^{t_1} \phi(t_1 - \tau) \left\{ W(\tau) \right\} d\tau \right] = F(t_1)
\]  \hspace{1cm} (3.95)

Since

\[
\dot{\phi}(t_1 - \tau) = \sum_{i=1}^{\infty} a_i \phi_i \cdot e^{-a_i(t_1 - \tau)}
\]  \hspace{1cm} (3.97)

then

\[
r(t_1) = \int_0^{t_1} \sum_{j=1}^{\infty} a_j \phi_j \cdot e^{-a_j(t_1 - \tau)} \left\{ W(\tau) \right\} d\tau
\]  \hspace{1cm} (3.98)

\[
= \sum_{j=1}^{\infty} r_j (t_1)
\]  \hspace{1cm} (3.99)

where

\[
r_j (t_1) = \int_0^{t_1} \alpha_j \phi_j \cdot e^{-\alpha_j(t_1 - \tau)} \left\{ W(\tau) \right\} d\tau, \quad j = 1, 2, \cdots, n
\]  \hspace{1cm} (3.100)

The integral equation (3.98) is approximated by the trapezoidal rule. Setting \( \Delta t = t_1 \), we obtain

\[
r(t_1) = \sum_{j=1}^{\infty} r_j (t_1) = \frac{\Delta t}{2} \left[ \sum_{j=1}^{\infty} a_j \phi_j \cdot \left[ e^{-a_j t_1} \cdot \left\{ W(0) \right\} + \left\{ W(t_1) \right\} \right] \right]
\]  \hspace{1cm} (3.101)

Substituting equation (3.101) into (3.95) and rearranging leads to

\[
K(0) \cdot \left[ 1 - \frac{\Delta t}{2 \cdot \left( \sum_{i=1}^{\infty} \phi_i \right)} \cdot \left\{ W(t_1) \right\} \right] = F(t_1) + \frac{\Delta t}{2} \left( \sum_{j=1}^{\infty} \alpha_j \cdot \phi_j \cdot e^{-a_j t_1} \right) \cdot K(0) \cdot \left\{ W(0) \right\}
\]  \hspace{1cm} (3.102)

\( \left\{ W(t_1) \right\} \) can be obtained by solving the above set of linear equations, and then

\( r_j (t_1), j = 1, 2, \cdots, n \) can be calculated. These values will be used in the next time step.
For \( t = t_2 \), we have

\[
K(0) \cdot \left[ W(t_2) - \frac{1}{t_2} \int_0^{t_2} \phi(t_2 - \tau) \left\{ W(\tau) \right\} d\tau \right] = F(t_2) \tag{3.103}
\]

Define

\[
r(t_2) = \int_0^{t_2} \phi(t_2 - \tau) \left\{ W(\tau) \right\} d\tau = \int_0^{t_2} \phi(t_2 - \tau) \left\{ W(\tau) \right\} d\tau + \int_{t_1}^{t_2} \phi(t_2 - \tau) \left\{ W(\tau) \right\} d\tau \tag{3.104}
\]

By making use of \( r(t_j) \) and approximating the second integral by the trapezoidal rule, \( r(t_2) \) can be written as

\[
r(t_2) = \int_0^t \sum_{j=1}^n \alpha_j \phi_j \cdot e^{-\alpha_j (t_2 - \tau)} \left\{ W(\tau) \right\} d\tau + \int_{t_1}^{t_2} \sum_{j=1}^n \alpha_j \phi_j \cdot e^{-\alpha_j (t_2 - \tau)} \left\{ W(\tau) \right\} d\tau
\]

\[
= \int_0^t \sum_{j=1}^n \alpha_j \phi_j \cdot e^{-\alpha_j (t_1 - \tau)} \cdot e^{-\alpha_j (t_2 - \tau)} \left\{ W(\tau) \right\} d\tau + \frac{t_2 - t_1}{2} \cdot \sum_{j=1}^n \alpha_j \phi_j \cdot \left\{ e^{-\alpha_j (t_2 - t_1)} \cdot \left\{ W(t_1) \right\} + \left\{ W(t_2) \right\} \right\}
\]

\[
= \sum_{j=1}^n e^{-\alpha_j \Delta t} \cdot r(t_1) + \frac{\Delta t}{2} \sum_{j=1}^n \alpha_j \phi_j \cdot \left\{ e^{-\alpha_j (t_2 - t_1)} \cdot \left\{ W(t_1) \right\} + \left\{ W(t_2) \right\} \right\} \tag{3.105}
\]

Substituting (3.105) into (3.103), we obtain

\[
K(0) \cdot \left[ 1 - \frac{\Delta t}{2 \cdot (\sum_{i=1}^n \phi_i) \sum_{j=1}^n \alpha_j \phi_j} \right] \cdot \left\{ W(t_2) \right\} = F(t_2) + K(0) \cdot \left\{ \sum_{j=1}^n e^{-\alpha_j \Delta t} \cdot r(t_1) \right\} + \frac{\Delta t}{2 \cdot (\sum_{i=1}^n \phi_i)} \left\{ \sum_{j=1}^n \alpha_j \phi_j \cdot e^{-\alpha_j \Delta t} \right\} \left\{ W(t_1) \right\}
\]

\[
\tag{3.106}
\]

where \( r(t_j), j = 1, 2, \ldots, n \) are determined by equation (3.101). Again, \( W(t_j) \) is determined by solving equation (3.106), and then \( r(t_j) \) in equation (3.105) is calculated. These values will be stored for the next time step.

Similarly, at \( t = t_1 \), we have
\[
\begin{align*}
K(0) - \frac{\Delta t}{2} \left( \sum_{j=1}^{n} \alpha_j \phi_j \right) \left[ \mathbf{W}(t) \right] = F(t) + K(0) \sum_{j=1}^{n} \left( e^{-\alpha_j \Delta t} \cdot \left[ r_j(t) + \frac{\Delta t}{2} \alpha_j \phi_j \cdot \left[ \mathbf{W}(t) \right] \right] \right)
\end{align*}
\] (3.107)

After solving equation (3.107) for \( \mathbf{W}(t) \), \( r_j(t) \) is obtained from the equation as follows.

\[
\begin{align*}
r_j(t) = e^{-\alpha_j \Delta t} \cdot r_j(t_{i-1}) + \frac{\Delta t}{2} \alpha_j \phi_j \cdot \left[ e^{-\alpha_j \Delta t} \cdot \left[ \mathbf{W}(t) \right] \right]
\end{align*}
\] (3.108)

Therefore, the fully discrete formulation in the case of a linear viscoelastic material can be obtained from equation (3.102) for \( t = t_i \) and equation (3.107) for \( t > t_i \). The advantage of this implementing way is that only the displacement \( \mathbf{W} \) and state variable \( r_j, j = 1,2, \cdots, n \) at the previous time step are involved in the computation of the displacement at the current time step. After getting the solution of displacement, stress can be calculated.

### 3.3.2 Method 2: Fourier Domain Analysis

To remove the time variable from the governing equations and boundary conditions, the method of Fourier transform is utilized. After the Fourier transform being applied to all the filed variables, the equations of equilibrium become

\[
\begin{align*}
\frac{\partial \tilde{\sigma}_r}{\partial r} + \frac{\partial \tilde{\tau}_{rz}}{\partial z} + \frac{\tilde{\sigma}_r - \tilde{\sigma}_\theta}{r} &= 0 \\
\frac{\partial \tilde{\tau}_{rz}}{\partial r} + \frac{\partial \tilde{\sigma}_z}{\partial z} + \frac{\tilde{\tau}_{rz}}{r} &= 0
\end{align*}
\] (3.109, 3.110)

The stress-strain relationship

\[
\tilde{\sigma} = D \tilde{\varepsilon}
\] (3.111)

where \( \tilde{\sigma} = (\tilde{\sigma}_r, \tilde{\sigma}_\theta, \tilde{\sigma}_z, \tilde{\tau}_{rz})^T \); \( \tilde{\varepsilon} = (\tilde{\varepsilon}_r, \tilde{\varepsilon}_\theta, \tilde{\varepsilon}_z, \tilde{\gamma}_{rz})^T \); and
in which $\kappa(\omega)$ and $G(\omega)$ are the complex bulk modulus and complex shear modulus for the viscoelastic material respectively.

Strain-displacement relationship:

$$
\begin{bmatrix}
\varepsilon_r \\
\varepsilon_\theta \\
\varepsilon_z \\
\gamma_{rz}
\end{bmatrix} = -\begin{bmatrix}
\frac{\partial}{\partial r} & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & \frac{\partial}{\partial z} \\
0 & \frac{\partial}{\partial z} & 0
\end{bmatrix} \begin{bmatrix}
\ddot{u}_r \\
\ddot{u}_\theta \\
\ddot{u}_z
\end{bmatrix}
$$

(3.113)

where $\ddot{u}_r, \ddot{u}_\theta, \ddot{u}_z$ are Fourier transformed displacements.

By applying Hankel transformations to the transformed displacement variables as follows

$$
\ddot{U}_r = \int_{0}^{\infty} r \ddot{u}_r J_0(\alpha r) dr
$$

(3.114)

$$
(\ddot{U}_r, \ddot{U}_\theta) = \int_{0}^{\infty} r (\ddot{u}_r, \ddot{u}_\theta) J_1(\alpha r) dr
$$

(3.115)

and then by following similar procedure as that of the time domain analysis, we can get

$$
-\alpha \left[ \alpha a \ddot{U}_r + c \frac{\partial \ddot{U}_z}{\partial z} \right] - \frac{\partial}{\partial z} \left[ \alpha \ddot{U}_z - \frac{\partial \ddot{U}_r}{\partial z} \right] \ddot{f} = 0
$$

(3.116)

$$
-\alpha \left[ \alpha \ddot{U}_r - \frac{\partial \ddot{U}_z}{\partial z} \right] \ddot{f} + \frac{\partial}{\partial z} \left[ \alpha c \ddot{U}_r + a \frac{\partial \ddot{U}_z}{\partial z} \right] = 0
$$

(3.117)

Equations (3.116)(3.117) can be written in matrix form as
\[ R \tilde{D}(\omega) V \tilde{W}(\omega) = 0 \]  

where

\[
\tilde{D}(\omega) = \begin{bmatrix}
\tilde{a}(\omega) & \tilde{c}(\omega) & 0 \\
\tilde{c}(\omega) & \tilde{a}(\omega) & 0 \\
0 & 0 & \tilde{f}(\omega)
\end{bmatrix}
\]

From the principle of virtual displacement, we arrive at

\[
\int_{\Omega} \delta W(\omega) V^T \tilde{D}(\omega) V W(\omega) \, dz - \int_{\Omega} \delta W(\omega) p \, dz - \int_{\Omega} \delta W(\omega) q \, dz = 0
\]  

where the definition of \( \Omega \), \( p \) and \( q \) are the same as those defined in the time domain analysis. Adopting the same finite element discretization algorithm as what has been done in the time domain analysis, we can get the following relationship:

\[
K^e \begin{bmatrix} \tilde{W}(\omega) \end{bmatrix} = \begin{bmatrix} F \end{bmatrix}
\]  

where

\[
K^e(\omega) = \int_{\Omega} N^T V^T \tilde{D}(\omega) V N \, dz
\]

Assembling the elemental equation into a global one and applying boundary conditions leads to

\[
\tilde{K}(\omega) \begin{bmatrix} \tilde{W}(\omega) \end{bmatrix} = \tilde{F}
\]  

where

\[
\tilde{F} = (0, F_z, 0, 0, \ldots, 0)^T
\]

If a time varying uniform pressure \( f_z(t) \) is applied on the layer surface over a circular region, then
\[
F_{zz} = \int_{-\infty}^{\infty} r f_{zz} J_0 (\alpha r) e^{-i\omega t} dr dt \tag{3.123}
\]

For the Fourier transform, the algorithm of DFT is used. The boundary load \( F_{zz} \) is expanded to a number of terms with each term corresponding to a frequency. The set of equations (3.122) is solved for each frequency and then the inverse DFT is applied to the resulted solutions for all the frequencies to obtain the solution in the transformed space-time domain. Finally, the inverse Hankel transform is applied to get the real solutions.

### 3.3.3 Method 3: Laplace Domain Analysis

The principle for this analysis is similar to that of the transformed space-Fourier domain analysis: by applying Laplace transform with respect to the time variable, the problem is converted to an associated elastic problem; finite element method is used to discretize the space domain and then the yielding equations are solved for a number of \( s \); inverse Laplace transform is applied to get the time domain solution. In this study, only the applied load history whose Laplace transform is available in analytical form is considered. The inverse Laplace transform developed by Honig and Hirdes [24] is utilized.

### 3.3.4 Numerical Experiments

In order to examine the preceding algorithms, two test problems are given. These two test problems were once analyzed by Booker and Small [10], where they used a hybrid Laplace transform/finite layer method. One example problem is also analyzed.

#### 3.3.4.1 Test Problem 1: Single Layer System under Constant Load

The problem is about a uniform vertical load \( q \) applied over a circular region with radius \( a \) on the surface of a single layer of viscoelastic material. The load is applied at time \( t=0 \) and then maintained constant. The bottom of the layer is rough and rigid, which means zero displacements there, and the thickness of the layer \( h=2a \), as indicated in Figure 3.12. The viscoelastic material is of a constant Poisson’s ratio of \( \nu = 0.3 \) and the
relaxation modulus is derived from the generalized Maxwell model as shown in Figure 3.12 and can be expressed as

$$E(t) = E_0 + E_1 e^{\eta_1 t} + E_2 e^{\eta_2 t} + E_3 e^{\eta_3 t}$$ (3.124)

The corresponding relaxation modulus at $t=0$ is noted as $E_{\text{initial}}$. By applying the Fourier and Laplace transform to the constitutive relations of the generalized Maxwell model, the corresponding moduli are as follows

$$\tilde{E}(\omega) = E_0 + \frac{i \cdot \omega}{i \omega / E_1 + 1/\eta_1} + \frac{i \cdot \omega}{i \omega / E_2 + 1/\eta_2} + \frac{i \cdot \omega}{i \omega / E_3 + 1/\eta_3}$$ (3.125)

$$\tilde{E}(s) = E_0 + \frac{s}{s / E_1 + 1/\eta_1} + \frac{s}{s / E_2 + 1/\eta_2} + \frac{s}{s / E_3 + 1/\eta_3}$$ (3.126)

With the modulus and Poisson’s ratio available, it is easy to get the stress-strain relationship matrix and the analyses can be carried out according to the aforementioned three algorithms.

It should be pointed out that the Fourier domain analysis is performed over a time duration $T$. The loading history over the time duration $T$ is sampled at $N$ equally spaced time instants. As $T$ and $N$ increase, the required computer storage and computational time increase too. Therefore, both $T$ and $N$ must be within a range. For the other two algorithms, this problem does not exist because the time domain analysis is a marching forward process, solution at next step can be obtained based on the solution at the current step; the Laplace domain analysis permits one to get the solution at a desired time without prior knowledge of its values at all previous time.

In Figure 3.12, the normalized surface deflection at the loading center $u_{z,t}$ vs. time $\lambda_{t}$ is presented based on the results by the three approaches. It is observed that the solutions by the time-domain analysis and the Laplace-domain analysis are in very good agreement. However, there are some minor discrepancies between these two and the Fourier-domain analysis when the time factor is very small, as the time factor increases, results by the three approaches match very well.
Stresses and strains beneath the centerline \( r = 0 \) for \( q = 1.0 \) and \( E_{\text{initial}} = 1.0 \) are plotted in Figure 3.13 (a) and (b), respectively. The stress distributions along the vertical depth show no variation with time while the strain distributions vary with the time.

### 3.3.4.2 Test Problem 2: Two-Layered System under Constant Load

This problem deals with a two-layered system with the first layer of viscoelastic properties subjected to a vertical circular load. Loading condition and the profile of the layered system are shown in Figure 3.14. For the viscoelastic properties of the first layer, constant bulk modulus \( \kappa \) is used to describe the volumetric behavior, and the deviatoric behavior is characterized by the viscoelastic model as illustrated in Figure 3.14. The variable \( G_{\text{initial}} \) is defined as the relaxation shear modulus at \( t = 0 \). The constant bulk modulus \( \kappa \) is expressed as a multiple of \( G_{\text{initial}} \), which is \( \kappa = \frac{8}{7} G_{\text{initial}} \). The material properties for the elastic layer were chosen so that they were the same as those of the viscoelastic layer at time \( t = 0 \), which means that the bulk modulus is the same as that of the viscoelastic layer and the elastic shear modulus is \( G_{\text{initial}} \). In the analysis, the parameter \( G_0 \) is chosen to be 2.1, and the load is selected to be a unit one.

The comparisons of the results for the surface deflection, stress-depth distribution and strain-depth distribution are demonstrated in Figure 3.14 and Figure 3.15 (a) and (b). It is observed that solutions by three analyses are in perfect match.

### 3.3.4.3 Example Problem: Multi-Layer System under Repeated Load

In this example, a multi-layer system subjected to a time varying haversine loading of Figure 3.16 is analyzed. The multi-layer system consists of 5 layers with the uppermost layer with viscoelastic characteristics and the other four underlying layers treated as elastic. The bottom of the last layer is rough and rigid. The viscoelastic layer is of constant Poisson’s ratio \( \nu = 0.4 \) and thickness \( h_1 = 5.0 \) inch, and its relaxation modulus is drawn from the seven parameter generalized Kelvin model shown in Figure 3.17. For this kind of model, as the number of components in series increases, it is easy to get the creep compliance but not the relaxation modulus in the time domain. The time domain
relaxation modulus, though can be written in the form of Prony series, is in very complicated form and increases the difficulty for numerical analysis. For this problem, only Fourier domain and Laplace domain analyses are carried out.

The viscoelastic parameters are chosen to be the following values: \( E_0 = 678300 \text{psi} \), 
\( E_1 = E_2 = E_3 = E_0/7 = 96900 \text{psi} \), \( \eta_1 = 969000 \text{psiu} \), \( \eta_2 = 96900 \text{psiu} \), \( \eta_3 = 9690 \text{psiu} \). The thickness and elastic properties of the other four layers are listed below:

Layer 2: \( h_2 = 3.5 \text{ in.}, E = 624000 \text{ psi}, \nu = 0.4 \)
Layer 3: \( h_3 = 8.0 \text{ in.}, E = 35000 \text{ psi}, \nu = 0.3 \)
Layer 4: \( h_4 = 7.0 \text{ in.}, E = 100000 \text{ psi}, \nu = 0.2 \)
Layer 5: \( h_5 = 20.0 \text{ in.}, E = 5000 \text{ psi}, \nu = 0.4 \)

The shear modulus in Fourier domain and Laplace domain are expressed respectively as follows:

\[
\tilde{G}(\omega) = \frac{1}{E_0 + \frac{1}{E_1 + i\omega\eta_1} + \frac{1}{E_2 + i\omega\eta_2} + \frac{1}{E_3 + i\omega\eta_3}}
\]  \hspace{1cm} (3.127)

\[
\hat{G}(s) = \frac{1}{E_0 + \frac{1}{E_1 + s\eta_1} + \frac{1}{E_2 + s\eta_2} + \frac{1}{E_3 + s\eta_3}}
\]  \hspace{1cm} (3.128)

The Laplace transform of the haversine loading \( \sin kt \) shown in Figure 3.16 is as

\[
\hat{q}(s) = \frac{k}{(s^2 + k^2)(1 - e^{-m/k})}
\]  \hspace{1cm} (3.129)

In this study, the magnitude of the haversine loading is chosen to be 100 psi; the radius of the circular area is 3.785 in; \( k = 2\pi \). The time variation of surface deflection at the loading center is of interest, which is presented in Figure 3.18 by the Fourier domain analysis and the Laplace domain analysis. It is shown that the phase of the deflection is lagging behind that of the load, which is due to the viscoelasticity of the top layer. The values of deflection obtained by the Laplace domain analysis are somehow higher than those by the Fourier domain analysis. It is observed that there is some non-causal effect for the Fourier domain analysis where the material responses before the application of the
force. The non-causality was also observed by Lunden [25] and Barkanov [11] during their Fourier domain analysis. It can be explained only by the accuracy of the discrete Fourier transform, which is connected with the number of samples and sampling interval.

### 3.4 Three Methods: A Comparison

The quasi-static responses of a linear viscoelastic layered system subjected to vertical circular loading are successfully solved by three schemes: direct time integration method, Fourier transform method and Laplace transform method.

In comparison with the direct time integration method, use of the transformation for the time-dependent linear problem offers a simple, straightforward method of solution by reducing the problem to associated elastic one and at the same time easily handling time-dependent boundary conditions. For some viscoelastic models such as the generalized Kelvin model, it might not be possible to determine the relaxation modulus with respect to time as the number of components increases, therefore, solving the problem by applying the direct time integration will be of great difficulty. However, the transform methods can easily circumvent that difficulty.

For the Fourier domain analyses, it must be performed over a time duration, as the time duration increases to a certain extent, the computational storage requirement for all the frequencies will be hard to meet; for the other two methods, this problem will not occur: the time integration method is a step-by-step method, the Laplace transform method allows one to invert a particular response variable at a time without prior knowledge of its values at all previous times.

The Laplace transform technique seems the best one among three algorithms, however, for the inverse Laplace transform method developed by Honig and Hirdes [24] and utilized in this study, it allows the inversion of only one response variable for each analysis. Therefore, it would require the most computer time to determine all the response variables.
3.5 Application of Viscoelastic Model in Pavement Analysis

3.5.1 Derivation of Viscoelastic Model Parameters Through Experimental Data

In order to use a linear model of viscoelasticity to calculate the response of the asphalt concrete pavement, it is usually necessary to have available an explicit equation for the linear relaxation modulus in the time domain or dynamic modulus in the frequency domain. The linear relaxation modulus can be measured through the step strain experiment; however, the testing is usually time-consuming and less accurate as compared to the dynamic mechanical experiment. The dynamic mechanical experiment is often referred as Frequency Sweep Test in the pavement field, in which the sample is deformed sinusoidally at small amplitudes and the stress response is measured to obtain the dynamic moduli, i.e. the storage modulus and the lose modulus.

In this study, the generalized Maxwell model is used to fit the Frequency Sweep Test moduli of an asphalt concrete sample. The mathematical expression for dynamic modulus is as equation (3.34), and the storage modulus $E'$ and lose modulus $E''$ are the real part and image part of equation (3.34), respectively, and can be written as

\[
E'(\omega) = E_0 + \sum_{i=1}^{N} E_i \frac{(\omega T_i)^2}{1 + (\omega T_i)^2}
\]

\[
E''(\omega) = \sum_{i=1}^{N} E_i \frac{\omega T_i}{1 + (\omega T_i)^2}
\]

where $T_i = \frac{\eta_i}{E_i}$ is also called relaxation time.

The common procedure to obtain the viscoelastic model is to determine the parameters, such as $E_0$, $E_i$, $T_i$, and $N$, of a generalized Maxwell model based on a set of experimental storage and loss modulus data from Frequency Sweep test. However, this is an ill-posed problem and is not at all a straightforward curve-fitting operation. This means that without consideration of the physical meaning and some restrictions, infinitely many parameter sets can be found that are equally satisfactory. In the general Maxwell model, the parameters for all the springs and dashpots must be positive to have physical
meaning. To best fit the testing data, negative moduli $E_i$ might be produced as the number of relaxation time $N$ increases, which are generally thought to be physically unrealistic [27, 28].

In the literature various techniques have been proposed to determine the parameters of the dynamic modulus. Laun [29] presented the linear regression method, Honerkamp [30] proposed the linear regression with regularization method, and Baumgaertel [31] used a nonlinear regression method. The linear regression method may lead to negative moduli (ill-posed problem) when $N$ is high. Linear regression with regularization can give reasonable fitting curve but usually with a large $N$ value. Nonlinear regression is found to give a good fit of the data with a minimum number of parameters. For the simplicity of numerical calculation, we want a constitutive model with a good fit with experimental data and having the few possible parameters. Starting from this point, nonlinear regression method is the best one.

In this study, parameters for the generalized Maxwell model is determined based on experimental dynamic modulus data by using the software IRIS (nonlinear regression method) developed by Winter [26].

Based on the time-temperature superposition principle [32], the raw data used to draw the master curve (dynamic moduli $E^*$ versus various frequencies) at the temperature of $20\,^\circ C$ for an asphalt concrete beam subjected to the Axial Frequency Sweep testing are listed in the Table 3.1. The corresponding storage moduli $E'$, loss moduli $E''$ as well as phase angle are also listed in Table 3.1. Those data were provided by Huang [33]. By inputting those data into the program IRIS, a 9-parameter general Maxwell model is derived and the model parameters are shown in Table 3.2.

The comparison between the testing data and the fitting curves are shown in Figure 3.19-3.21. It should be noted here that because of the limitations of testing equipment, material properties at either relatively high or relatively low frequencies are not available, extrapolations beyond the accessible frequencies are required. For the determination of equilibrium modulus $E_0$, which corresponds to the modulus at zero frequency, or in other words, corresponds to the modulus at very high temperature
according to the time-temperature superposition principle, it is assumed that at zero frequency or very high temperature, the asphalt loses its function to bind the mixes together, and the strength of asphalt concrete at this condition is purely attributed to that of the mixes. The stiffness range for loose sand and gravel is 50MPa ~ 140MPa (or approximately 7250psi ~ 20300psi) [34]. For dense sand and gravel, the range is 100Mpa ~ 200Mpa (or approximately 14500psi ~ 29000 psi) [34]. For an asphalt concrete with the asphalt almost losing its binding capability, we can treat the asphalt concrete as a loose sand and gravel. Therefore, the extrapolation of equilibrium modulus $E_0$ 12500psi is acceptable.

### 3.5.2 Layered Viscoelastic Asphalt Concrete Pavement Response

In this section, the above viscoelastic model obtained from experimental data is used to define the AC properties in a typical three layer system as shown in Figure 3.22. Standard dual tire single axle load 18kip is applied on the surface of the pavement. The tire pressure is assumed to be 100psi. Loading time, which is defined as the time required for a vehicle driving through the distance of the diameter of the contact area, is used to account for the effect of the vehicle speed (load frequency). The stresses and strains along the vertical section below the tire center (position (0, 0)) and the tire edge (position (0, 3.785)) at various times are shown in Figures 3.23-3.30 and Figures 3.31-3.38, respectively. It is seen that loading time does have some effects on the stress and strain field. As the loading time increases, the tensile strain at the bottom of the AC layer increases too.

At present, the design criteria for the fatigue life of asphalt concrete is based on the regression model developed by the Asphalt Institute and can be expressed as [32]

$$N_f = 0.00432C\varepsilon_i^{-3.291} |E^*|^{-0.854}$$  \hspace{1cm} (3.132)

where $N_f$ denotes the fatigue life; $C$ is the constant; $\varepsilon_i$ = the maximum principal tensile strain at the bottom of the asphalt concrete layer in a pavement structure subjected to certain vehicle loading as shown in Figure 3.39; $|E^*|$ is the dynamic modulus determined
by a dynamic test under a sinusoidal loading with a frequency $f$, which is equivalent to the elastic modulus of the asphalt layer in a layer system.

For elastic layer system, the general procedure to do fatigue life analysis is as follows: get the corresponding loading time for a certain vehicle speed; convert loading time to frequency; obtain the dynamic modulus of the asphalt concrete corresponding to that frequency from frequency sweep testing data or curve, for instance, Table 3.1 or Figure 3.21; input the layer properties and loading condition into the elastic pavement analysis program (as illustrated in Chapter 2) to get the maximum principal tensile strain at the bottom of the asphalt concrete layer; plug the maximum tensile strain and the elastic modulus of the asphalt concrete into equation (3.132) to get the fatigue life.

If asphalt concrete is taken as a viscoelastic material, the procedure described below is followed for the fatigue life analysis: determine the viscoelastic model for the asphalt concrete; use the algorithm described in this chapter to do viscoelastic analysis for a given pavement structure to get the maximum tensile strain at the bottom of asphalt concrete layer for a series of loading time; based on the derived viscoelastic model, calculate the relaxation modulus for a certain loading time; plug into equation (3.132) the obtained maximum tensile strain and relaxation modulus to get the fatigue life for a certain loading time.

In this study, using the pavement structure in Figure 3.22, pavement fatigue life based on viscoelastic analysis is compared with that based on elastic analysis. The results for various loading time (vehicle speed) are shown in Table 3.3. For viscoelastic analysis, the value of $|E^*|$ at certain loading time is obtained from relaxation modulus since we have the general Maxwell model available as shown in Table 3.2; for the elastic analysis, loading time is converted to frequency by the relationship $t = \frac{1}{f}$ and then the frequency is converted to angular frequency by $\omega = 2\pi f$, $|E^*|$ can then be obtained by interpolation on the lab curve as in Figure 3.21.

It is shown in Table 3.3 that the fatigue life calculated by elastic analysis is $3 \sim 4$ times of that calculated by viscoelastic analysis. The author has not found any similar study or references so far and cannot make final conclusions. Because during the process
of deriving the viscoelastic model from the master curve of dynamic modulus (Figure 3.21), some extrapolations for low frequencies and high frequencies were made, which would definitely affect the results. To get accurate results, the master curve should cover as many low frequencies and high frequencies as possible.
References


Figure 3.1 Solution flow chart for visco-elastic material analysis
Figure 3.2  Viscoelastic bar

Figure 3.3  Creep

Figure 3.4  Stress relaxation
Figure 3.5  A linear spring

Figure 3.6  A Newtonian dashpot

Figure 3.7  The Maxwell model

Figure 3.8  The Kelvin model
Figure 3.9 The generalized Maxwell model

Figure 3.10 The generalized Kelvin model

Figure 3.11 Convolution integral
Figure 3.12 Time–deflection relationship for constant circular loading on single viscoelastic layer
Figure 3.13  (a) Stresses and (b) strains on centerline beneath the circular loading for single layer
Figure 3.14  Time–deflection relationship for constant circular loading on two-layered viscoelastic-elastic system.
Figure 3.15  (a) Stresses and (b) strains on centerline beneath the circular loading for a two-layered system
Figure 3.16  Haversine loading of $\sin kt$

Figure 3.17  Seven parameter generalized Kelvin model
Figure 3.18  Surface vertical deflection – time relationship and load-time history
Table 3.1  Raw Data for an Asphalt Concrete Beam Subjected to Axial Frequency Test at Temperature 20ºC

<table>
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<tr>
<th>Frequency (Hz)</th>
<th>Angular Frequency ω (rad/s)</th>
<th>Phase Angle ψ (º)</th>
<th>Dynamic Modulus E* (psi)</th>
<th>Storage Modulus E' (psi)</th>
<th>Loss Modulus E'' (psi)</th>
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Figure 3.19  Storage modulus $E'$ by testing and fitting curve

Figure 3.20  Loss Modulus $E''$ by testing and fitting curve
Figure 3.21  Master curve of dynamic modulus $|E^*|$ by testing and curve-fitting

Figure 3.22  Three-layer pavement system
Figure 3.23  Stress-zz distribution below the tire center

Figure 3.24  Stress-yy distribution below the tire center
Figure 3.25  Stress-xx distribution below the tire center

Figure 3.26  Stress-xz distribution below the tire center
Figure 3.27  Stress-\(yz\) distribution below the tire center

Figure 3.28  Strain-\(zz\) distribution below the tire center
Figure 3.29  Strain-yy distribution below the tire center

Figure 3.30  Strain-xx distribution below the tire center
Figure 3.31  Stress-\(zz\) distribution below the tire edge point (0, 3.785)

Figure 3.32  Stress-\(yy\) distribution below the tire edge point (0, 3.785)
Figure 3.33  Stress-xx distribution below the tire edge point (0, 3.785)

Figure 3.34  Stress-xz distribution below the tire edge point (0, 3.785)
Figure 3.35  Stress-yz distribution below the tire edge point (0, 3.785)

Figure 3.36  Strain-zz distribution below the tire edge point (0, 3.785)
Figure 3.37  Strain-yy distribution below the tire edge point (0, 3.785)

Figure 3.38  Strain-xx distribution below the tire edge point (0, 3.785)
Figure 3.39  Design criteria for pavement fatigue life

Table 3.2  Parameters of the generalized Maxwell model for an AC beam sample

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<th>$T_i$ or $\eta_i / E_i$ (s)</th>
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<td>-</td>
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Table 3.3  Comparison of fatigue life obtained by viscoelastic analysis and elastic analysis

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<th>Viscoelastic Analysis</th>
<th>Elastic Analysis</th>
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Chapter 4 MODELING OF PERMANENT DEFORMATION OF ASPHALT CONCRETE PAVEMENTS

4.1 Introduction

Rutting is an important factor in flexible pavement design. It has adverse effect on pavements by influencing the drainage properties leading to reduced frictional properties and hydroplaning. The hydroplaning phenomenon consists of the building up of a thin layer of water between the pavement and the tire and results in the tire losing contact with the surface, with the consequent loss of steering control [1]. A thorough understanding of rutting phenomenon is desirable in order to improve pavement design and performance.

Rutting stems from the permanent deformation in any of the pavement layers or subgrade. In flexible pavements, both the top asphalt concrete layer and the subgrade contribute to rutting. In the AC layer, the rutting is caused by a combination of densification and shear flow. The initial rutting of the pavements is caused by densification of the granular material in both the AC layer and the subgrade. The subsequent rutting is a result of shear flow of the mix in the AC layer. In properly compacted pavements, shear flow in AC layer is thought to be the primary rutting mechanism, which is the focus of this study.

The evaluation of AC mixes for their tendency to cause rutting has been an active area of research for several years. Some empirical rutting prediction methods such as VESYS method [2] have been widely used. However, all of the relationships proposed contain several parameters that are arbitrary and cannot be generated from fundamental concepts. The corresponding experimental tests such as incremental static test, dynamic test and creep test, etc, have been proposed to determine the parameters in those empirical models. At present, no rational method of predicting the permanent deformation or rutting of AC pavements exists. The cornerstone of rational methods of pavement design is the ability to predict the behavior of the paving materials by analytical methods. Up until now, very limited research has been done on analytical
methods to predict pavement rutting. It is mainly because it is extremely difficult to handle the wide variety of material characteristics that asphalt concrete exhibit under different real conditions, under millions of load repetitions.

To transcend the empirism of pavement rutting prediction, the following are required. Firstly, a constitutive model, which is able to capture the primary mechanism causing rutting, be available; secondly, numerical algorithm such as finite element method, finite difference method or other methods can implement those constitutive models, and therefore analyze the material response under certain loading conditions. If these two requirements were met, it would eliminate the need to have multi million dollar testing facilities to evaluate pavements.

In this study, the research about AC constitutive model for pavement rutting prediction and related numerical algorithm are reviewed. Based on other researchers' study and available numerical technique, the author did some finite element analysis for laboratory AC sample rutting prediction under 1-D loading conditions.

4.2 Literature Review

4.2.1 Study by Uzan

To derive a rutting prediction constitutive model, Uzan [3] first analyzed the components of the deformation of asphalt concrete based on repeated creep and recovery tests. With repeated square wave loads with 10-sec loading and 10-sec unloading. He divided the deformation into 4 components: plastic and/or viscoplastic components exist under loading only while elastic and viscoelastic components appear under both loading and unloading conditions as shown in Figure 4.1. The separation of the deformation into components cannot be done directly from the test results. He suggested to use a model to compute the different components as stated below. The elastic and plastic components are time independent, they develop with the application and removal of the load, without any delay. The elastic one can be obtained directly from the unloading sequence; then the plastic one can be obtained from the loading sequence by subtracting the elastic part from the instantaneous deformation. The viscoelastic and viscoplastic components are time dependent and both develop during the loading. But only the viscoelastic part recovers
during the unloading sequence. Since the recovery represents only a portion of the
viscoelastic component, the viscoplastic components cannot be separated based on the
test results only. Uzan used a viscoelastic model to fit the recovery curve and then
subtract the plastic, elastic and viscoelastic component from the loading sequences to get
the viscoplastic component.

Uzan purposed a constitutive equation to calculate the deformation under a step
loading function as follows

$$\frac{\varepsilon(t)}{\sigma} = D_e + D_p + D_{ve} \cdot \frac{t^m}{1 + a \cdot t^m} + D_{vp} \cdot t^n$$  \hspace{1cm} (4.1)

where

- $D_e = \text{elastic compliance (time independent)}$
- $D_p = \text{plastic compliance (time independent)}$
- $D_{ve}, a, m = \text{viscoelastic parameters and}$
- $D_{vp}, n = \text{viscoplastic parameters}$

In this formula, we can see any component can be modeled separately from others.
The viscoelastic component was computed by using the power law [4][5]. In the
computation of plastic and viscoplastic deformation, he assumed that both components
increase with number of cycles, $N$, which are empirical assumption without considering
the basic principles of plastic theory such as plastic hardening rule and associate or non-
associate flow rule.

4.2.2 Study by UC-Berkeley

UC Berkeley proposed a constitutive relationship to predict the development of
permanent deformation as shown schematically in one-dimensional form in Figure 4.2.
The deformation is divided into three components: elastic, viscoelastic, and plastic.

The literature [6] pointed out that the elastic response characteristics (related to
modulus $E_0$) could be determined by the following three testing methods: (1) simple
shear constant height test – determine the initial shear stiffness by measurements of stress
and deformation with fast application of a shear stress while maintaining the specimen height constant; (2) uniaxial strain test – rapid application of an axial stress while maintaining the specimen perimeter constant to provide additional information on the elastic response; (3) volumetric test – determine the bulk modulus from measurements of the hydrostatic stress and radial strain by rapid application of a hydrostatic stress.

The values of $E_i, \eta_i \ (i = 1, \text{to} \ n)$ could be drawn from complex modulus which is generally obtained by sinusoidally applying shear stresses to result in small strains over a range in temperatures from $4^\circ C$ to $60^\circ C$ and at frequencies from 0.02 to 10 Hz.

For the determination of plastic deformation response characteristics (represented by the slider in Figure 4.2), the literature proposed to employ simple shear constant height test since the test is performed at three stress levels and the recovery of deformation is observed after each stress application for a period of time sufficient to permit equilibrium to be obtained.

Though the constitutive relationship and corresponding testing methods to determine the model parameters were presented in the literature [6], no testing data available in it. Even for the testing methods, the author of this study has some doubts about them. Firstly, the literature did not mention how to determine the parameters of $E_i, \eta_i \ (i = 1, \text{to} \ n)$ from complex modulus, namely, what kind of algorithm should be used to do the curve fitting? Secondly, for the determination of plastic parameters such as yield stress, flow rate as well as hardening rule, is the simple shear constant height test at three levels adequate to capture all the plastic characteristics? Finally, because various testing methods were required to determine the model parameters and each testing required specific confining conditions, would the testing results be consistent for all the testing?

4.2.3 Study by Ramsamooj

Ramsamooj [7] presented a mechanical-empirical elastoplastic model for predicting the plastic deformation of asphalt concrete under cyclic loading of both triaxial compression and extension tests. The model utilizes multiyield surfaces and isotropic hardening. Rowe’s stress dilatancy theory [8] is used to obtain the relationship between
the permanent volumetric and permanent vertical strains. For triaxial compression test, the relationship turned out to be

\[
d\varepsilon_v^p = \frac{d\varepsilon_v^p}{d\varepsilon_v^p} = 1 - \frac{\sigma_3}{K\sigma_3}
\]

(4.2)
in which \(\sigma_1\) = vertical stress; \(\sigma_3\) = horizontal stress; \(\varepsilon_v^p\) = plastic volumetric strain; \(\varepsilon_1^p\) = plastic vertical strain; and \(K = \tan^2(45 + \phi/2)\) and \(\phi\) is defined as the equivalent angle of friction between particles.

For triaxial extension test, the relationship can be expressed as

\[
d\varepsilon_v^p = \frac{d\varepsilon_v^p}{d\varepsilon_3^p} = 1 - \frac{K\sigma_3}{\sigma_1}
\]

(4.3)
in which \(\sigma_1, \sigma_3\) and \(\varepsilon_v^p\) are defined above, and \(\varepsilon_5^p\) = plastic vertical strain.

Rowe’s stress dilantancy theory is also used to derive the hardening law for the changes in the sizes of the plastic moduli caused by cyclic loading. For triaxial compression tests, the plastic modulus \(H\) in any cycle is given by

\[
H = H_i(1 + \lambda\varepsilon_v^p)
\]

(4.4)
in which \(H_i\) = plastic modulus at a specified deviator stress level in the first cycle; \(\lambda\) = hardening parameters, slope of \(\ln p\) versus \(\varepsilon_v^p\) in consolidation tests.

After the hardening law is determined, Ramsamooj [7] presented the formulas to calculate the plastic volumetric strain increment and the plastic vertical strain increment in any cycle after the first one. He gave a complete description of the plastic deformation of asphalt concrete under cyclic loading, including the elastic, viscoelastic and plastic components. A computer program RUT was developed to determine the plastic deformation under cyclic loading. By using RUT, the following results become available:

- The total permanent vertical, volumetric, and lateral strains as a function of the number of cycles of loading;
• The volumetric and vertical permanent strains and its components, viz., viscous, plastic, and microcracking of the specimen as a function of the number of cycles of load.

However, RUT requires many input data obtained from testing as listed below:

• Average plastic vertical strain over the first 100 cycles;
• Value of friction angle at the critical state;
• Complex modulus;
• Fracture toughness;
• Tensile strength;
• Initial creep rate;
• Viscoelastic parameters;
• Hardening parameters $\lambda$.

Because several types of testings are required to characterize the rutting model stated above, its application is limited.

4.2.4 Study by Schwartz

Schwartz [9] developed a comprehensive constitutive model for asphalt concrete based on an extended form of the Schapery continuum damage formulation [10][11]. That model considers the viscoelastic, damage, and viscoplastic components of asphalt concrete behavior over the full range conditions of interest for the mechanistic prediction of flexible pavement distresses. Special study is done on the viscoplastic response component by performing unconfined uniaxial creep and recovery test at intermediate and high temperatures.

In his study, the axial viscoplastic strain for uniaxial constant-stress loading is assumed to follow a strain-hardening model of the form:

$$\dot{E}_{vp} = \frac{g(\sigma)}{Ae_{vp}^{\nu}} \quad (4.5)$$
in which $\dot{\varepsilon}_{vp}$ is the viscoplastic strain rate, $\varepsilon_{vp}$ is the total viscoplastic strain, $g(\sigma)$ is the uniaxial stress loading function, and $A, p$ are material constants. By rearranging and integrating equation (4.5), the following relationship can be obtained

$$\varepsilon_{vp} = \left(\frac{p+1}{A}\right)^{\frac{1}{p+1}} \left(\int_0^t g(\sigma) dt\right)^{\frac{1}{p+1}}$$

(4.6)

For constant stress creep conditions, $g(\sigma)$ is independent of time and equation (4.6) becomes:

$$\varepsilon_{vp} = \left(\frac{p+1}{A}\right)^{\frac{1}{p+1}} g(\sigma)^{\frac{1}{p+1}} t^{\frac{1}{p+1}}$$

(4.7)

Schwartz assumed a power law of the form $g(\sigma) = B\sigma^q$ in which $B$ and $q$ are material properties to reduce equation (4.7) to:

$$\varepsilon_{vp} = \left(\frac{p+1}{D}\right)^{\frac{1}{p+1}} (\sigma^q)^{\frac{1}{p+1}} t^{\frac{1}{p+1}}$$

(4.8)

in which $D=A/B$.

In Schwartz’s study, the creep and recovery tests were conducted to evaluate the viscoplastic material behavior, in which one type of tests was denoted as uniform time tests where the loading time 10 second and recovery time 200 second were kept constant but the loading stress level varied. The tests were performed in uniaxial compression and at the temperature of 35ºC. The loading history was designed to produce failure after approximately 8 or 9 cycles of loading. Strain vs. time responses are shown in Figure 4.3. Three replicate specimens were tested as indicated by the lighter gray lines in the figure. The average of the replicates represented by the darker black line. Viscoplastic strains for each loading cycle were determined from the recovery after each load cycle. Data analysis was based on a nonlinear least-squares optimization of equation (4.8) to find simultaneously the best-fit viscoplastic values $p, q,$ and $D$ that minimized the total squared errors. The material parameters for the 12.5mm MSHA asphalt mixture tested were found to be $p = 1.8169$, $q = 2.3361$, and $D = 9.203E+10$. The revised best-fit models
as determined from the nonlinear least-squares optimization in log-log space was summarized in Figure 4.4.

Schwartz considered the viscoplastic strain as the main and only component in the permanent strain as we can see from above literature review, which may not be true according to Uzan’s [3] study. Schwartz’s viscoplastic model is mechanic-empirical kind of model.

4.2.5 Numerical Techniques for Analysis of Permanent Deformation

If the plastic deformation were the only component of the permanent deformation as the elastic-plastic model shown in Figure 4.5, the mathematical description and computation would be greatly simplified. Detailed finite element algorithm can be found in the literature [13].

If the permanent deformation includes only the viscoplastic component as described by the model shown in Figure 4.6, where viscous element is interlocked by parallel-connected plastic element and the viscous effect manifest themselves only in combination with plastic properties after exceeding of yield limit, current numerical technique such as FEM can be readily used.

For the model shown in Figure 4.7, where both plastic component and viscoplastic component contribute to the permanent deformation, although it permits to describe a wide class of phenomena, but leads to the considerable complication of mathematical description and computation.

4.2.6 Discussion

According to all the literature review mentioned above, we can see most of the permanent deformation models are still at the stage of empirical level. Uzan’s [3] model is more complete with respect to the components of asphalt concrete permanent deformation and requires fewer types of testing to determine the model parameters. But his model is only of one dimension and empirical, it is difficult to extend to multi-dimension. Ramsamooj’s [7] model has more theoretical background, but too many model parameters to be determined. Schwartz’s [9] model is also based on empirism.
Ideally, the model we need should have physical meanings, easily determined model parameters, and be easy to implement numerically. None of the above models meets those demands.

### 4.3 One Dimensional Constitutive Model for Rutting Analysis

Based on Uzan’s research, the author of this study presents a one-dimensional constitutive model as shown in Figure 4.8 for asphalt concrete rutting analysis. The deformation under a step loading function can be expressed as

\[
\frac{\varepsilon(t)}{\sigma} = D_w + D_{ep} + D_{exp}
\]  

(4.9)

where 

- \(D_w\) = viscoelastic compliance
- \(D_{ep}\) = elastic-plastic compliance
- \(D_{exp}\) = elasto-visco-plastic compliance

As an integrated element, the model in Figure 4.8 is highly nonlinear and numerically unsolvable at present. But if any component can be modeled separately from others, then for the viscoelastic component, numerical techniques stated in Chapter 3 can be used to solve the viscoelastic deformation, which is recoverable. For the elastic-plastic and elasto-visco-plastic components, by using the algorithm stated in literature (12), the deformations are also solvable as long as the model parameters can be determined. Because of the lack of experimental data, the author just did some algorithm implementation and case study about visco-plastic deformation.

#### 4.3.1 Algorithm Implementation about Analysis of Visco-Plastic Deformation

For the model shown in Figure 4.6, the total strain is the summation of elastic strain and the visco-plastic strain as

\[
\varepsilon = \varepsilon_e + \varepsilon_{vp}
\]  

(4.10)

The stress in the spring can be determined by
where $E$ is the elastic modulus of the spring.

Stress in the slider depends on the value of the yield stress $Y$. The first time the stress in the slider is greater than the initial yield stress $\sigma_y$, the visco-plastic deformation starts to develop, but its rate of increment depends on the material’s strain hardening properties. In this study, only the simplest linear isotropic hardening model as presented in Figure 4.9 is considered. After the yielding happens, the yielding stress is given by

$$Y = \sigma_y + H' \varepsilon_{vp}$$

(4.12)

in which $H'$ is the strain hardening parameter, $\varepsilon_{vp}$ is the current visco-plastic strain.

Therefore, the stress in the slider can be expressed as

$$\sigma_p = \sigma \quad \text{if } \sigma < Y$$

$$\sigma_p = Y \quad \text{if } \sigma \geq Y$$

(4.13)

In the dashpot, the stress-strain relationship can be written as

$$\sigma_d = \eta \frac{d \varepsilon_{vp}}{dt}$$

(4.14)

in which $\eta$ is the coefficient of viscosity, $t$ denotes the time.

Before the initial yielding, $\varepsilon_{vp} = 0$, therefore the total stress can be expressed as

$$\sigma = E \varepsilon$$

(4.15)

After the initial yielding, the total stress can be written as

$$\sigma = \sigma_d + \sigma_p$$

(4.16)

Substituting equations (4.13) (4.14) into (4.16), we can derive

$$\sigma_y + H' \varepsilon_{vp} + \eta \frac{d \varepsilon_{vp}}{dt} = 0$$

(4.17)

Substituting equation (4.10) into the above equation and using equation (4.11) yields
If we introduce a parameter $\gamma$ as

$$
\gamma = \frac{1}{\eta}
$$

(4.19)

and plug it into equation (4.18) and rearrange the equation, we get

$$
\dot{\varepsilon} = \frac{\sigma}{E} + \gamma \left[ \sigma - \left( \sigma_y + H \varepsilon_{vp} \right) \right]
$$

(4.20)

in which the dot means the derivative with respect to the time. The equation (4.20) can be rewritten as

$$
\dot{\varepsilon} = \dot{\varepsilon}_e + \dot{\varepsilon}_{vp}
$$

(4.21)

where

$$
\dot{\varepsilon}_e = \frac{\sigma}{E}
$$

(4.22)

$$
\dot{\varepsilon}_{vp} = \gamma \left[ \sigma - \left( \sigma_y + H \dot{\varepsilon}_{vp} \right) \right]
$$

(4.23)

One dimensional finite element algorithm for the analysis of the above elasto-visco-plastic model is stated below. Starting from the initial time $t=0$, we can solve for the initial total displacement $\phi^0$, initial total stress $\sigma^0$, initial total strain $\varepsilon^0$. At the initial time, the total viscoplastic strain $\varepsilon_{vp}^0 = 0$. The general procedure to solve the solutions at time step $t_{n+1}$ with the solutions at time step $t_n$ is as follows:

Step 1. At the time step $t = t_n$, the values such as $\sigma^n$, $\varepsilon^n$, $\varepsilon_{vp}^n$, and $f^n$ for each element have been solved, then calculate the viscoplastic strain rate for each element according to equation (4.23) as

$$
\dot{\varepsilon}_{vp} = \gamma \left[ \sigma^n - \left( \sigma_y + H \dot{\varepsilon}_{vp}^n \right) \right]
$$

Step 2. Calculate the displacement increment $\Delta \phi^n$ as
\[ \Delta \phi^n = [K]^{-1} \Delta V^n \]

in which \( \Delta V^n = AE \dot{\varepsilon}_{vp} \Delta t_n \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \Delta f^n \) and also calculate the stress increment \( \Delta \sigma^n \) and the viscoplastic strain increment \( \Delta \varepsilon_{vp}^n \) for each element as

\[ \Delta \sigma^n = E \left( \frac{\Delta \phi_1^n - \Delta \phi_2^n}{L} - \dot{\varepsilon}_{vp} \Delta t_n \right) \]

\[ \Delta \varepsilon_{vp}^n = \dot{\varepsilon}_{vp} \Delta t_n \]

Step 3 Determine the total displacement, total stress, and total viscoplastic strain as

\[ \phi^{n+1} = \phi^n + \Delta \phi^n \]

\[ \sigma^{n+1} = \sigma^n + \Delta \sigma^n \]

\[ \varepsilon_{vp}^{n+1} = \varepsilon_{vp}^n + \Delta \varepsilon_{vp}^n \]

Step 4 Calculate the visco-plastic strain rate for each element for next time step \( t = t_{n+1} \)

\[ \varepsilon_{vp}^{n+1} = \gamma \left[ \sigma_{vp}^{n+1} - (\sigma_v^p + H \varepsilon_{vp}^{n+1}) \right] \]

Step 5 Calculate the residue force for each element

\[ \psi^{n+1} = A \sigma_{vp}^{n+1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + f^{n+1} \]

and add it into the pseudo force vector for next time step as

\[ \Delta V^{n+1} = AE \dot{\varepsilon}_{vp} \Delta t_{n+1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \Delta f^{n+1} + \psi^{n+1} \]

Step 6. Check each element the viscoplastic strain rate \( \dot{\varepsilon}_{vp} \) to see whether it is less than the allowable tolerance. If yes, it means steady solutions achieved and we can either end the solution procedure or keep on analyzing the next external force increment \( \Delta f \). If no, go to step 1 and repeat.
4.3.2 Numerical Example about Visco-Plastic Deformation

Assume there is an asphalt concrete specimen as presented in Figure 4.10, 4 inches diameter by 6 inches tall cylinder, stabilized at one end and subjected to uniform compressive pressure 1000psi at the other end. The parameters in the model shown in Figure 4.6 are as follows: elastic modulus $E = 4.73 \times 10^5$psi, initial yield stress $\sigma_y = 500$psi, the coefficient of viscosity $\gamma = \frac{1}{\eta} = 0.000001$ s/psi, linear isotropic hardening parameter $H = 1.0 \times 10^5$psi. Under single loading increment, the displacement versus time and the plastic strain versus time are shown in Figure 4.11 and Figure 4.12, respectively.

For multiple loading increments, each one is applied only after no more deformation is developing under previous loadings. In this study, 4 compressive loading increments each 1000psi are considered. The visco-plastic strain versus time for multiple loading increments is shown in Figure 4.13.

Figure 4.14 presents the variation of the displacement during a loading-unloading-reloading process. The process is as follows: 1000psi is applied until no further deformation develops; then the 1000psi is removed; and then 2000psi is applied. We can see that the elastic deformation immediately recovers after the loading being removed, while the unloading process has no effect on the plastic deformation.

From the analyses about the example problem, we can see the limitations of the model in Figure 4.6: under the same loading increment, the permanent deformation increment is the same for each loading increment except the first one, which is not the case for lab results. The main reason for this phenomenon is we assumed the linear isotropic hardening for simplification of programming.

4.4 Summary

At present, most researches about permanent deformation of asphalt concrete stay at the mechanic-empirical stage. The author presented a nonlinear model to evaluate the permanent deformation. However, it is difficult to determine the model parameters by testing methods and implement it by numerical techniques. The one-dimensional elasto-
visco-plastic model was analyzed in this study, but the author found it was not adequate to capture the characterization of the permanent deformation of the asphalt concrete. Additional effort in both material characterization and analysis is needed for a more reliable pavement performance prediction.
References


5. Lai, J. S. “Predicting permanent deformation of asphalt concrete from creep tests”, *Transportation Research Board*, Transportation Research Record 616.


Figure 4.1  Strain decomposition from creep and recovery test (by Uzan [3])

Figure 4.2  Schematic representation of non-linear viscoelastic model with slider[6]
Figure 4.3  Strain vs. time response for the creep and recovery test at 35°C (by Schwartz [9])

Figure 4.4  Best fit model in log-log space. (by Schwartz [9])
Figure 4.5  Elasto-plastic model

Figure 4.6  Elasto-visco-plastic model
Figure 4.7  Schematic rutting analysis model

Figure 4.8  Schematic model for pavement rutting analysis
Figure 4.9   Elastic linear strain hardening model

\[ E = 4.73 \times 10^5 \text{psi} \]

\[ \sigma_p = 500 \text{psi} \]

\[ \gamma = \frac{1}{\eta} = 0.00001 \text{s/psi} \]

\[ H' = 1.0 \times 10^5 \text{psi} \]

Figure 4.10   Elasto-Visco-plastic asphalt concrete sample
Figure 4.11  Displacement versus time curve for single load increment

Figure 4.12  Visco-plastic strain versus time curve for single load increment
Figure 4.13  Visco-plastic strain versus time curve for multiple loading increments

Figure 4.14  Displacement versus time for loading-unloading-reloading process
Chapter 5 CLOSURE

This thesis is on the numerical modeling and computation for the layered asphalt concrete pavement system under vehicle loadings. The objective of this research has been to use the pavement responses obtained through the numerical model analysis to study some pavement distresses: delamination, fatigue and rutting. The main contribution is the implementation and experimentation of three schemes for numerically solving viscoelastic pavement system subjected to vertical circular loading.

For the delamination analysis, the horizontally layered pavement is treated as an elastic system with no slippage at the layer interface. The finite layer algorithm developed by Small and Booker is followed to solve the response of the pavement subjected to circular vertical and horizontal loadings. Through parameter study, it is found that higher loading leads to higher maximum interface shear stress and increasing overlay thickness is an effective way to reduce the maximum interface shear stress; the maximum interface shear stress can be approximately found at the tire edge in the moving direction for a vehicle with standard dual tire single axle load. Based on the numerical analysis and direct shear testing, design guidelines to prevent pavement delamination, which take into account the temperature and shear rate, can be developed.

For the problem of the layered viscoelastic system subjected to circular vertical loading, finite element method is used to do spatial discretization and then three methods: direct time integration, Fourier transform and Laplace transform are used to handle the time-dependent property of the viscoelastic material. After numerical implementation and experimentation, it is found that each method has both advantages and disadvantages.

In comparison with the direct time integration method, use of the transformation for the time-dependent linear problem offers a simple, straightforward method of solution by reducing the problem to associated elastic one and at the same time easily handling time-dependent boundary conditions.

For the Fourier domain analysis, the results have some non-causal effect. It must be performed over a time duration, as the time duration increases to a certain extent, the
computational storage requirement for all the frequencies will be hard to meet; for the other two methods, this problem will not happen: the time integration method is a step-by-step method, the Laplace transform method allows one to determine a particular response variable at a desired time without prior knowledge of its values at all previous times.

For the Laplace transform method, the inverse transform utilized in this study allows the inversion of only one response variable, therefore, it would require the most computer time to determine all the responses.

To do viscoelastic analysis for layered asphalt concrete pavement, we have to have the viscoelastic model parameters. The software IRIS is a useful tool to perform this task if the master curves for storage modulus and lose modulus are provided. To get accurate model, the master curve should extend as far as possible in both low frequency and high frequency directions.

According to a case study about the fatigue life obtained by viscoelastic analysis and elastic analysis, it is found the former one gets lower fatigue life than the latter one.

For the pavement rutting analysis, there is no well-developed model available at the present time. For the visco-plastic model in section 4.3.1, it is not adequate to describe the asphalt concrete rutting phenomenon. One reason is due to the linear isotropic hardening model, which makes numerical implementation easier but does not reflect real material properties.

In the future, a few more direct shear testing would be conducted to check the effect of temperature and shearing rate on the interface shear strength. For the pavement rutting analysis, it is expected to use laboratory-obtained hardening rule instead of linear hardening rule in the numerical implementation; optimization and back-analysis methods may be used to determine the model parameters.