

ABSTRACT

CAO, YINGFANG. Bayesian Based Structural Health Management and Reliability Analysis Techniques Utilizing Support Vector Machine. (Under the direction of Professor Mohammad N. Noori.)

Structural health and safety play a major role in all facets of human daily lives. Over the past few decades significant advancements have been made in structural damage detection and health management in a wide range of engineering disciplines and practices, including but not limited to aerospace, power generating plants, infrastructure systems, and manufacturing. Two main thrust areas of research in this field include the development of methodologies/algorithms for detection of damage and/or changes in the dynamic characteristics of the system, and the sensing/detection devices for capturing the required data/information. A third and evolving area is the integration of these two thrusts and the development of integrated systems that can “manage”, and “adapt” in an “intelligent” sense, the subsequent actions that need to take place in order to maintain the integrity of the system subject to the external environment and/or loading conditions. However, majority of the developed techniques fail to take into account the important effects of uncertainty presented in sensing, system modeling, and material behavior associated with dynamic systems. These uncertainties could greatly affect the structural performance and health management, which leads to challenging issues such as reliability and life prediction of the structure. In order to address these important factors, the application of the probabilistic and reliability analysis techniques to structural health management has emerged as an active research area in recent years.

Bayesian probabilistic analysis is such a technique in which the uncertainties could be related with a mathematical model -- probability distribution function- which interprets the measurement of confidence interval. The posterior probability distribution is known as an expression for the statistical knowledge of a system after a set of measurements is made. The Bayesian approach is a powerful way to continuously optimize the “posterior” probably density function (pdf) by adapting the predefined “priori” pdf based on “new” measurements. On the basis of Bayesian analysis, it is shown

to be possible to perform statistical based system identification, structural damage detection and reliability assessment, as part of structural health management.

In the first stage of this thesis work, a Bayesian based system identification approach was developed to identify system parameters provided so that inherent uncertainties and probabilities of system changes and/or environmental disturbances are taken into account. It is obvious that the nature of changes encountered the system models is critical to monitoring and managing the integrity of the structural systems. In this part of the work, system changes were modeled as random variables with certain statistical properties. The effects of priori definition and different data sampling techniques were studied. To explore the application of this Bayesian based system identification approach to structural health management, the probability density function (pdf) profiles of model parameters were studied to quantify the uncertainties associated with the estimated parameters. By analyzing the posterior pdf inference, the reliability parameters of interest could also be obtained from the available data.

Structural health monitoring, damage detection and structural reliability are usually considered as the sequential components in a structural health management chain. It is the ultimate goal of structural health management to achieve a significant improvement of the structural reliability. Therefore, the second stage of this thesis work was devoted to the study of system reliability. A reliability analysis package developed in MATLAB – PROBES was enhanced with its functionality in this work. The enhancements include the new capabilities of performing analysis to correlated, non-normally distributed random variables and the added functionality to obtain the statistical information for system performance function. However, it was noticed that due to the lack of a complete understanding and predicting the structural response under various environmental impacts, changes and variations occurring in a structure through its life time, and/or modifications and redesign of a structure's components during its service life, the inherent uncertainties also change continuously. Therefore, the ability to obtain the up to date reliability information is highly desirable. Probabilistic incorporated technique was found to be the effective and an efficient way to meet this goal. A

Bayesian based uncertainty analysis technique was introduced to give statistical information of failure probability. By utilizing the capability of Bayesian analysis to combine prior knowledge with the subjective information, the probability properties of system reliability could be optimized continuously. The probability test was performed to give statistical information of the current state reliability information.

In the last stage of this work, a Bayesian based approach combined with Support Vector Machine (SVM) was introduced to develop a novel technique of reliability analysis. Support Vector Machine is a learning algorithm for classification and regression. A unique characteristic of this method is its ability to provide good performance with small size of data sets. The failure probability obtained with this newly developed technique was compared to traditional reliability analysis methods including Monte Carlo simulation method, first order reliability method and second order reliability method. It was demonstrated that this SVM based approach leads to a more accurate and efficient reliability analysis algorithm. Effect of different kernel functions were investigated and compared for both linear and nonlinear cases. It is proved that utilization of SVM is a promising area for structural health management.

**BAYESIAN BASED STRUCTURAL HEALTH MANAGEMENT AND
RELIABILITY ANALYSIS TECHNIQUES UTILIZING SUPPORT VECTOR
MACHINE**

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BIOGRAPHY

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CHAPTER 1

1 Introduction

1.1 *Structural Health Management*

Nearly all in-service structures require some form of monitoring to maintain their integrity and health condition and to prolong their lifespan or to prevent catastrophic failure. The interest in the ability to monitor the physical condition of a structure and detect damage at the earliest possible stage is pervasive throughout the civil, mechanical and aerospace engineering communities. This is now commonly referred to as 'Structural Health Management and Damage Detection'.

Structural health management can be defined as the diagnostic and prognostics of the integrity or condition of a structure. The intent is to detect and locate damage or degradation in structural components and to provide this information quickly and in a form easily understood by the operators or occupants of the structure. The damage may result from fatigue, large earthquake, strong winds, explosion, vehicle impact or other external or internal loadings or changes. Early detection of damage or structural degradation prior to local failure can prevent "runaway" catastrophic failure of the system. In engineering applications, damage is understood intuitively as an imperfection or impairment of the function and working condition of a structure or machine. Damage can be described in many ways depending on the structure and its function. Hence damage detection has many definitions based on what type of damage is being measured [Staszewski, 1998 and Sone, 1995]. Since the health monitoring was defined as use of in-situ, nondestructive sensing and analysis of system characteristic, including structural response, for the purpose of detecting

changes, the term structural health monitoring and damage identification are usually interchangeable.

Research over the past several decades within the structural engineering and other related engineering fields have resulted in advancements and technologies that assist structural engineers in their attempts to ensure the safety and reliability of structures over their life spans. Two major reasons responsible for this growth of the health management of structural systems have been: a) the advances in sensors, data acquisition, data communications, and real-time data analysis, allowing for the implementation of highly reliable and accurate monitoring and diagnostics hardware with very advanced features [Housner et al., 1997], and (b) the growing interest in certain markets, such as military, aerospace, and civil infrastructures, which see the application of health monitoring systems as means of assuring the correct and safe performance of these engineered systems, as well as protecting their investments [Chang, 1999]. The structural health monitoring can be performed on both global and local basis. Global based health monitoring focuses on monitoring and verifying the performance of the entire system. By monitoring the output, such as vibration displacement, velocity or acceleration, of the system during its operation, the occurrence of the damage can be detected. On the other hand, local based health monitoring is interested in monitoring certain key elements of the total system. The applications of local based health monitoring usually rely on the use of localized non-destructive evaluation techniques, such as acoustic or ultrasonic methods, magnetic fields methods, radiography, eddy current methods, or thermal methods [Doherty, 1993]. All of these experimental techniques require that the vicinity of the damage is known a priori and that the portion of the structure being inspected is readily accessible [Doebeling, 1998]. Subject to these limitations, the need for additional

global damage detection methods has led to the development of the methods that examine changes in the vibration characteristics of the structure.

Damage detection, as determined by changes in the dynamic properties or responses of structures, is a subject that has received considerable attention in the literature. The basic idea is that modal parameters (frequencies, mode shapes, etc.) as functions of the physical properties will cause changes in the modal properties [Doebbling et al. 1996]. Damage effects on a structure can be classified [Doebbling et al. 1998] into linear or non-linear. A linear damage scenario is one in which the linearly elastic behavior of the structure is preserved after the occurrence of the damage. The changes of modal properties are resulted from the changes in geometry or material properties of the structure, but the behavior of the structure can still be modeled using linear equations of motions. On the other hand non-linear effects are the ones in which the system does no longer exhibit a linearly elastic behavior after the damage has been introduced. Loosening of fastening elements, where the separation of mating parts induce non-linear responses in the system, can be cited as these non-linear effects. Another example of nonlinear damage is the formation of a fatigue crack that subsequently opens and closes under the normal operating vibration environment.

As mentioned previously, the field of structural health management is very broad. Among different issues, an important one is to use signal processing techniques, which includes such methods as Fourier analysis, time-frequency analysis and wavelet analysis. There exist two major and complementary components that form the basis for structural health management: hardware or the experimental tools, and methodologies or analytical/computational tools. Previous analytical and computational studies on damage detection have been tailored toward identifying variations of modal parameters, such as natural frequencies, modal shapes, and

modal damping ratios obtained from the vibration signals [Doebling, et al 1996, 1998]. Some of these methods allow for assessing changes in these parameters, which then are related to some structural damage, but they do not provide information of the exact time of the occurrence of the damage neither detecting the location of a fault, which may be an important part in the total picture of what structural health monitoring implies. Methods such as the windowed Fourier Transforms and Wigner Distribution were among the first time-frequency methods to be implemented. In recent years, the application of the Wavelet Transform also provided a new tool for time frequency analysis, which has been effectively used in health management and damage detection of various structures. In this research work, different issues involved in the structural health management were investigated. Some novel ideas to adaptively update system parameters and to obtain system reliability were explored. In the following sections, backgrounds of different techniques involved will be introduced.

1.2 System Identification and its role in SHM

System identification is an approach for obtaining an empirical model of a dynamic system from measured inputs and outputs. Through this process one can find an optimal model which fits the measured data as well as possible. The model structure is built based on the prior knowledge and the personal experience of the physical system. According to the prior knowledge of the physical system, the model structure can be categorized into ‘white box model’, ‘gray box model’ and ‘black box model’. Among them, the ‘white box model’ has known properties from the prior knowledge and the physical principles and it not determined by the measured data. The ‘black box model’ is opposite to the ‘white box model’. It is determined by the measured data with little prior knowledge. The ‘gray box model’ is in

between of the 'white' and 'black' box model with which, the model structure is built based on the available prior knowledge with part of the parameters determined from the measured data. According to the model estimation method, system identification can be classified as parametric or nonparametric identification. Parametric identification methods are techniques to find the 'best' values of the parameters to make the simulated response close to those from the measurements. Some commonly used models include ARMAX, ARX and FIR model. Nonparametric identification methods estimate model parameters without assuming a parametric model set. Typical nonparametric identification methods include frequency analysis, correlation analysis and spectral analysis.

In general, changes in the real structures result in changes in the parameters of the structure models such as stiffness, natural frequency etc. Therefore, adaptively monitoring the changes of parameters can provide information on structure health condition. The great potential to apply system identification technique to structural health monitoring and damage detection has been shown extensively in the literatures. The research work by Saadat et al [2003] exclusively investigated the application of artificial neural network based system identification on the damage detection and structural health monitoring. The paper presented by Beck's group [2004] introduced a two-stage structural health monitoring approach in which the first stage was to identify the modal parameters for both the undamaged and damaged systems. In the second stage of their work, Bayesian system identification was used to update the structural model. The updated information was applied to decide if the system was damaged or not. Other noteworthy work can be found in Sohn and Law [2000], Liu and Rao [2000] to cite a few. To apply system identification technique to structural health monitoring, it is important to have a direct relationship between system parameters

and the real system model. Since the occurrence of damage will change the system response characteristics, the sudden changes in the system are desired to be detected as ‘spikes’. Some signal processing techniques such as time-frequency analysis and wavelet analysis have been widely applied to this field to detect structural damage. Literature contributions have been made by Hou and Noori [1999], Sone et al. [1999], Hou et al. [2000], Noori et al. [2001], Masuda et al. [2002].

1.3 Structural Reliability Analysis and its role in SHM

The fundamental goal of structural reliability analysis is to determine the safety condition of the structure. As one of the pioneers to develop the structural reliability, Freudenthal [1947] introduced the probabilistic theory to deal with structural safety issue. The researchers have studied this area over half century. Nowadays, the probability of failure has been one of the most important notations of the structural reliability. The traditional methodology of structural reliability is achieved by deterministically finding the design point according to some conservative assumptions. The theory of first reliability method (FORM) and second reliability method (SORM) was significantly improved by 1990 (Rackwitz, 2001). These two methods work by linearizing the limit state function at the design point which is the most possible failure point in the failure region. The simplicity and efficiency of these methods make them widely recognized in both academic and industrial fields. However, to use these methods, structural model is usually oversimplified to build up a limit state function. And more importantly, it is not applicable for some cases like non-asymptotic case. On the contrary, the Monte Carlo Simulation (MCS) method, another widely used structural reliability analysis tool, utilizes the numerical sampling to calculate the probability of failure. MCS is accurate but time consuming. Therefore, it is usually used for the cases with large

probability of failure. In 1983, Importance Sampling was introduced by Harbitz (1983) to perform reliability analysis. Importance sampling is a method that has some of the features of both FORM/SORM method and numerical Monte Carlo Simulation method, in which, sampling is performed close to the design point. It is more accurate and efficient for certain cases compared to FORM/SORM. The reliability analysis methods discussed herein have their own advantages. The detail application of these approaches will be introduced in the third chapter.

In practical world, structural safety and reliability is crucial for assuring sustainability of infrastructure systems and thus, economic benefits and saving lives. For certain structures, it is often required to calculate and predict the structural reliability and safety during their service life. In Wong & Yao (2001), the “value chain” concept was introduced in a holistic view of structural health management, with which the data interpretation, damage detection and structural reliability estimates were viewed as sequential components in a value chain. Among them, structural reliability and useful life expectancy are seen to be the crucial parts to realize the full value of the chain.

The ultimate goal of structural health management is to achieve a significant improvement in structural reliability. Even though techniques of structural health management have been extensively developed during the last decades majority of the literatures in this area is focused on how to collect information and detect the damage. More attention needs to be paid to integrate the current state of the structure health with reliability schemes to improve the service life of a structure. The ultimate goal of this research is to develop an adaptive technique that can be utilized to obtain the up to date system information. This technique can then be incorporated into an integrated strategy for improving the structural life span.

1.4 Bayesian Based Probabilistic Analysis

Bayesian theorem is a probabilistic theorem derived from the basic sum and product rules of probability theory. Traditionally, probability is interpreted by the frequency that an event happened in a large number of similar trial samples. The events happened are considered as the different realization of a random variables. In contrast to the frequency based probability theory, Bayes' theorem brought up another way to interpret probability. Firstly presented by Thomas Bayes in 1763, Bayes' theory interprets probability in terms of a certain degree of belief. A Bayesian probabilistic approach is a powerful way to formulate a model to describe the system of interest. It provides a rational basis to combine subjective priori information and objective system measurements. The posterior probability density function (pdf) of interest can be obtained by adapting the predefined priori pdf based on the observed information.

Classic inferential model do not permit the introduction of the prior knowledge into the calculation. However, some of the prior knowledge includes numerous useful information. It is often a desire to address a question with regarding to the past pertinent knowledge and newly collected data. Bayesian analysis provides such a logical, quantitative framework to combine prior and new data for inference. Inference means computing the probability distribution for a set of unknown variables from a set of known variables. The observing of this set of known variables form a distribution called the likelihood function. The characteristics of the variables are considered to have its own probability distribution which is the previously described priori distribution. Because of its process to update the pre-defined priori information based on the current state of system observable, the Bayesian

analysis is often referred to as an updating algorithm, with which new information is obtained using the existing information and some subjective assumption.

With the history of more than two century, a lot of attentions were addressed to Bayesian approach during 1950s and 1960s by researchers. Over the past decade both in the statistical fields and the methodology developments different areas of application of Bayesian theory have been actively explored. A variety of modern technologies are influenced broadly by Bayesian thinking which include the field of computer science, bioinformatics, economics, medicine, physics and so on (Fienberg, 2006; Poirier, 1988). The application of Bayesian approach to data analysis is of particular interest in this research work. It was first applied to system identification and structural damage detection. By utilizing the property of its functionality to model system parameters as random variables and its ability to incorporate subjective knowledge into system measurements, Bayesian analysis was used to model the system uncertainties and update the system information. The details of these works will be presented in the following chapters.

1.5 Support Vector Machine Learning Algorithm

In practice, structural systems under investigation always have certain complexity. Their behavior is either difficult or unlikely to be predicted under harsh environment. Similar to artificial neural network which has been used to model structural behavior, Support Vector Machine (SVM) possesses the well-known ability of being a universal approximator for any multivariate function to any desired degree of accuracy (Kecman, 2005). The SVM has become a promising algorithm and has gained popularity during the last decade.

Technically speaking, Support Vector Machine (SVM) is a statistical learning algorithm for classification and regression. The technical foundation of Support Vector Machine was

developed by Vapnik, Chervonenkis and their co-worker's back in 1960s (Vapnik and Chervonenkis, 1968). Unlike the traditional statistical inference, SVMs are so called 'nonparametric' models, which means the parameters involved in SVM are not predefined and their numbers depend on the training data used. A hyperplane is one that separates between different sets of objects having different class characteristics. A special property of Support Vector Machine classifier is to simultaneously minimize the empirical classification error and maximize the geometric margin. Therefore SVM is also known as maximum margin classifier.

During the past years, SVMs has been applied to various domains including handwriting recognition, speaker identification and text categorization and so on. In practice, structural systems being investigated always have certain complexity. Their behavior is either hard to predict using mathematical models or changing due to upgrade or redesign. A SVM based reliability analysis technique is introduced in this research work to classify safe condition and failure condition by utilizing a pre-trained hypothesis function. As it will be shown in the following chapters, the proposed SVM based reliability developed in this research is proved to be a time saving method compared with Monte Carlo Simulation and leads to more accurate results compared with other traditional reliability analysis techniques such as first order reliability method and second order reliability method. More over, in reality, due to the lack of complete knowledge about the structure or the changes in the characteristics of the structure through time, such as upgrades and redesigns, the information to define a reasonable limit state function could be limited and sometimes non-continuous. It is another advantage of SVM based reliability analysis that failure probability can be obtained without computation of the limit state function.

2 Bayesian Based System Identification

2.1 System Identification

System identification is about to build and evaluate a mathematical model of a system according to its input and output measurements. A system usually refers to a dynamic process to produce observable signals by the interaction of the different variables. With the development of the control theory, system identification was well developed and significantly applied to the control field. Over the past two decades, the different techniques of system identification have been well established and broadly applied to many different fields. Those different fields include financial analysis, speech system analysis and so on. The related literatures can be found in Lois (1997) and Nickel (2005).

Usually, there are different shapes of models available for a dynamic system according to different complexities. The key problem of system identification is to find the most suitable model for the physical system among the cluster of models. This model is considered as the optimal one with which it is simple enough to evaluate and accurate enough to capture the desired model behavior. A model reflects the relationship between system input and output. The system output is partially determined by the input because of the effects of environmental disturbance or other uncertainties. A general concept of a dynamic system model can be shown from Figure 2-1. A model is mathematically constructed according to the priori knowledge of the system dynamics and the assumptions according to the physical insights. “White Box”, “Gray Box” and “Black Box” are three color coded levels to describe the model structures in accordance with the priori information integrated to the model:

- “White Box Model” is built by the already known properties from the prior knowledge and physical principles. It is determined ideally from full knowledge of all components of the physical system instead of the measured data. The application of white box model is limited sometimes because of its complexity or limitation to obtain the fully required prior knowledge.
- “Gray Box Model” is built based on the available prior knowledge with part of the parameters determined from the system measurements. With this kind of model, a relatively simple structure can be obtained from the prior knowledge and system observation can be used to estimate the remaining unknown system parameters.
- “Black Box Model” is opposite to the ‘white box model’ and is fully determined by the measured data with little prior knowledge. The characteristic of analyzing the dynamic system without knowing the physical insight draws a lot research interests to black box system identification. There are varieties of literatures to cover topics from mathematical estimation theory to different algorithms including neural networks, fuzzy models and wavelets system identification and so on. The related literatures are Gupta and Sinha (1998), Haykin (1999), Nelles (2001), Judisky and Hjalmarsson et. al (1995) and Ashino, Mandai and Mrimoto (2004).

From the above introduction, it is easy to see that finding a model for a dynamic system according to the three color coded approaches is case dependent. Models built from different priori information usually come with varieties of formats. Different techniques to evaluate the model can be categorized into parametric and nonparametric system identification methods.

It is the intention of system identification to find the “most” suitable model structure for the dynamic system and to find the “best” value for the model structure. Parametric system identification method refers to techniques that estimate the parameter values for a given model. The application basis of parametric system identification is the determination of the system model. Once the model is defined, the process of finding the optimal value for system parameter vectors θ becomes the problem of parameter estimation. With the measured system input and output data, the optimal values for the system parameters are determined by the certain criteria, usually known as the prediction error:

$$\varepsilon(t, \theta) = y(t) - \hat{y}(t | \theta) \quad (2.1)$$

The process of finding the “best” model is the process to minimize the prediction error. A quadratic norm is typically chosen as a standard candidate of the error cost function due to its convenience for computation and analysis:

$$l(\varepsilon) = \frac{1}{2N} \sum_{t=1}^N \varepsilon^2(t, \theta) \quad (2.2)$$

There are varieties of ways to fit the model to the measured input-output data by minimizing the error cost function. Among them, Linear Regressions and the Least-square Method and the Maximum Likelihood Method are the most frequently used techniques (Ljung, 1999). The Least-square method has its unique feature to analytically find the optimal values. The Maximum Likelihood method is the one with statistical feature and will be studied and implemented in this research work.

Let random variables $y^N = y_1, y_2, \dots, y_N$ represent the N measured data from the system, θ denotes the M unknown parameters of the system. The likelihood function of the system generally refers to the probability distribution function of unknown model parameters given

the system measurements. Herein, the probability density function is used to make the expressions more explicit:

$$f_y(y^N | \theta) = f(y_1, y_2, \dots, y_N | \theta) \quad (2.3)$$

An estimator $\hat{\theta}(y^N)$ is often used to accomplish the parameter estimation task by utilizing the observable y^N . The estimation is performed by finding values which maximize the logarithm of the estimator:

$$\hat{\theta}_{ML}(y^N) = \log \max_{\theta} f_y(y^N | \theta) \quad (2.4)$$

Maximum likelihood method is a powerful method but it requires large sample size to obtain precise estimation. This method can be improved by integrating that with Bayesian approach as presented in section 3 of this chapter.

Nonparametric system identification techniques try to find an optimal system model without knowing the system parameters explicitly. Typical time domain methods include Impulse Response method, Step-Response analysis and Correlation analysis. The frequency domain methods include sine-wave testing method, Correlation analysis, Fourier analysis and Spectral analysis (Ljung, 1999). Those methods can be implemented both to the linear or non-linear systems depending on the different cases.

In practical world, system identification can be considered as an experimental process to estimate model parameters until its output matches with the observed physical system output as well as possible. The procedures of system identification include following basic steps:

- Data acquisition: By experiment design to collect good quality input-output data. The variables to be measured and the choice of input signals will affect the whole procedure.

- Selection and definition of model structure: As a mathematical representation of a structure, a suitable model is chosen using priori knowledge, application of identification and trial processes.
- Choice of a criteria of fit: Finding an optimal model of the system according to a given criteria, which reflects the quality of the model about how well it fits the measured data.
- Parameter estimation: Finding numerical values of the model parameters. This is an optimization problem.
- Model validation: Even though the model selected is considered as the best one to represent the system, the model still needs to be tested to reveal any deficiency.

Even if the model parameters obtained through this procedure are their optimal values, it may still not be possible to duplicate the behavior of the structures exactly. This is because of the so-called modeling error. Besides, the measured data is usually with a finite number and associated with environmental disturbance and measurement noise. This may induce the variation of the estimated value from this true value. This part of the variation is known as the variance error. Both modeling error and variance error introduce the uncertainties to the system identification process. In many applications such as aircraft and machinery health management, the consideration of these uncertainties is especially important.

Over the past century, techniques for system identification have been developed and implemented to various application fields. Major focuses of the techniques include the development of methodologies/algorithms which can adapt the model structures to different dynamic systems, and the data acquisition techniques for capturing the high quality input-output data/information. During last decades, an evolving area for improving both of these two techniques has drawn research interest to perform the system identification in an

“intelligent” way. The subsequent actions that need to take place in order to acquire the integrity of the system subject to the external environment and/or loading conditions. A critical and essential requirement for in-depth understanding of the nature of changes encountered in the system is the identification of the system’s model, and thereby the update of the knowledge about the changes in its dynamic characteristics. In order to address these important factors, the application of probabilistic techniques to identify and update the knowledge about a structural model has emerged as an active research area in recent years (Sohn and Law, 1997; Beck and Au, 2002). Due to its importance in data analysis, the Bayesian probabilistic system identification approach has received growing attention in recent years. Specific techniques such as the Markov chain Monte-Carlo method in Bayesian system identification have been explored by different researchers (Ninness and Brinsmead, 2001; Kerschen, Bolinval etc., 2003). Model updating for both linear and nonlinear systems have been investigated by Beck and Katafygiotis (1998), Yuen and Beck (2001). In this research work, the effects of different prior pdfs and amounts of available data were investigated. Following a brief introduction to the Bayesian system identification approach, numerical simulations will be presented in order to demonstrate these effects.

2.2 Fundamentals of Bayesian Analysis

Bayesian inference is statistical inference in which uncertainties are interpreted in terms of probabilities with degrees of belief. A Bayesian probabilistic approach is a powerful way to formulate a model to describe the system of interest. After observing some data, the “posterior” probably density function (pdf) can be obtained by adapting the predefined “priori” pdf based on those observed information.

Traditionally, probability is interpreted by the frequency that a certain event happened in a large number of trial samples. Bayes' theorem was first presented by Thomas Bayes in 1763. In contrast to the frequency based probability theory, Bayes' theorem brought up another way to interpret probability.

In a statistical view, let H stands for a hypothesis that contains the information before the observation. E stands for the new observation or event. According to the probability product rule, the joint probability is defined from the conditional probability:

$$P(E, H) = P(E | H) \times P(H) = P(H | E) \times P(E) \quad (2.5)$$

As a well-known theorem, Bayes' formula is actually a derivation of the above product rule and its expression is as follows:

$$P(H | E) = \frac{P(E | H) \times P(H)}{P(E)} \quad (2.6)$$

In formula (2.6), $P(H)$ is usually called the *prior probability* of H , $P(E|H)$ is the likelihood function and $P(H|E)$ is the *posterior probability* of H given E . The corresponding marginal probability can be obtained from the following equation:

$$P(E) = \int_{-\infty}^{+\infty} P(E, H) dH \quad (2.7)$$

The marginal probability is usually considered as a normalizing factor. To reduce the computation effort, for the cases that only the hypothesis part is of interest, the marginal of $P(E)$ can be omitted and Bayesian theorem can be expressed in its proportional form:

$$P(H | E) \propto P(E | H) \times P(H) \quad (2.8)$$

For the cases with more than two random variables, Bayes' theorem can be rewritten as:

$$P(H | E_1, E_2) \propto P(E_1 | H, E_2) \times P(E_2 | H) \times P(H) \quad (2.9)$$

With as many as of N random variables, the chain rule of probability is applied, and the Bayes' formula is written as the following form:

$$P(H | E_1, E_2, \dots, E_N) \propto P(E_1 | E_{N-1}, \dots, E_1, H) \times \dots \times P(E_2 | H) \times P(H) \quad (2.10)$$

In general, probability density function is the basic way to define a continuous random variable. Therefore, it is more convenient to perform Bayesian analysis using probability density function. A derivation from the probability distribution to probability density can be found in Papoulis (1984). The Bayesian theorem for probability density function is illustrated as in the following equation:

$$f_{H|E}(H | E) = \frac{f_{E|H}(E | H) \times f_H(H)}{f_E(E)} \quad (2.11)$$

The proportional form for the probability density function is as follows:

$$f_{H|E}(H | E) \propto f_{E|H}(E | H) \times f_H(H) \quad (2.12)$$

The generalized N random variables expression is:

$$f_{H|E}(H | E_1, E_2, \dots, E_N) \propto f_{E_1|H}(E_1 | E_{N-1}, \dots, E_1, H) \times \dots \times f_{E_2|H}(E_2 | H) \times f_H(H) \quad (2.13)$$

It was shown from above introduction that the general strategy of Bayesian analysis is to utilize the decomposition property of the conditional probability. The application of this theory is based on the assumption that the uncertainties modeled can be interpreted by certain probabilistic form. In the following section, a method of Bayesian based system identification is introduced. The statistics of system parameters are considered as the random variables with assumed pdfs. The posterior pdfs of those parameters are updated according to the system measurement and the likelihood function.

2.3 Bayesian Analysis Based System Identification

In Bayesian statistics, probability is interpreted as a rational measure of belief that is used to describe mathematically the uncertain relation between the statistician and the external world [Peterka, 1981]. Bayesian approach based system identification is thus different from the classical system identification conceptually. In Bayesian approach, the parameters to be evaluated are considered as random variables. The desired parameters can be inferred from the system measurements of other random variables and the likelihood function, which builds up the connection between the unknown and the observable parameters.

Suppose that θ is a finite set of parameter to be studied, Y_N is the measurement from the system, the equation (2.12) can be rewritten as:

$$f_{\theta|Y}(\theta|Y) \propto f_{Y|\theta}(Y|\theta) \times f_{\theta}(\theta) \quad (2.14)$$

Where, $f_{\theta}(\theta)$ is the predefined prior probability density function that represents the previous knowledge of the system, and $f_{Y|\theta}(Y|\theta)$ is the ‘likelihood function’, which reflects the ‘likelihood’ that the observed event should indeed take place [Ljung, 1999]. The prior information is modified according to the up to date measurements to obtain the current state knowledge of the system, which is the posterior probability density function $f_{\theta|Y}(\theta|Y)$. The generalization multi-variable form for equation (2.14) is as follows:

$$f_{\theta, Y_N}(\theta, Y_1, Y_2, \dots, Y_N) \propto f_{Y_N|\theta}(Y_N|Y_{N-1}, \dots, Y_1, \theta) \times \dots \times f_{Y_1|\theta}(Y_1|\theta) \times f_{\theta}(\theta) \quad (2.15)$$

Bayesian approach is applied to system identification because of its simplicity and stochastic feature. By utilizing this probabilistic theory, a degree of belief can be assessed for a system parameter in a straightforward way. The stochastic information obtained this way is on a

basis of subjective assumption and with the condition of the current available observations. The Figure 2-2 is a summarization for the principle of the Bayesian system identification. As discussed already, the priori pdf is generally assigned subjectively according to the exiting knowledge of the system parameters. There could be varieties of choices since it is based on the selected choice. The effects of different assignments will be checked in the following section by performing numerical simulations. It is shown that the critical part for Bayesian based system identification is the definition of the likelihood function. A proper form of the likelihood function is the one that not only builds up the connection between the parameters of interest and the system observables, but also provide an explicit mathematical formulation to reduce the computation cost. This term usually depends on the dynamic system and is also probabilistic based. In this Bayesian based approach, the external disturbance is incorporated into the system measurements. By performing basic conditional rules, it can be distinguished from the system model and evaluated separately if needed. Since the parameters of interest to be evaluated are obtained in probability distribution forms, the problems like prediction error could be irrelevant. The advantage of Bayesian approach to consider the unknown parameters as random variables is the feature that provides a rational basis for problem decision making (Peterka, 1981). Decision making is to choose an optimal point with which the system model can be best described. From the statistical view, the probability density associated with any variable is a measure of confidence that lies in the neighborhood of the mean value. The optimal estimation of the parameter is given by maximizing the posterior pdf as in equation (2.16).

$$\hat{\theta} = \max(f_{\theta|Y_N}(\theta | Y_N)) \quad (2.16)$$

In traditional system identification, the estimated parameter is one single value with the problem being processed in one shot. With Bayesian system identification, the adaptability of the method makes it possible to obtain the up to date system parameter information based on the current state observation. In general, changes in the real structures are reflected by the system output measurements, which induce the changes in the parameters of the structure models such as stiffness, natural frequency, etc. Therefore, adaptively monitoring the changes of system parameters can monitor the system change and provide information on the structural health conditions. This will be the potential application of the Bayesian based system identification to structural health management. In the following section, different issues including effects of priori selection, data sizes and sample conditions on Bayesian system identification will be addressed through the numerical simulation cases.

2.4 Numerical Simulation

As stated previously, system identification is to find a mathematical model for a physical system. This model should be good enough to describe the certain behavior of interest and simple enough to reduce the computation cost. In this research work, the ultimate goal is to study the system health condition and to improve the structural reliability, thus system models with specified parameters are the major concern. Therefore, it is a task of statisticians to solve a parametric system identification problem and to extract information according to their past experience and current observations. To start with a general example, the Bayesian system identification approach is applied first to a single degree of freedom mass-spring-damping system and further extended to a three-story-shear building.

A dynamic structural system is usually described by the equation of motion. The natural frequency and the damping ratio are crucial parameters in such case. Furthermore, the

change of the system natural frequency implies the change of the whole system. Hence it will be the main task to deal with those parameters in the following sections. To be consistent, let $\theta = \langle \zeta, \omega \rangle$. For a physical system, with a statistically simulated sample input, the output can usually be observed as Y_N . The choice of the priori pdf will be discussed in the following numerical cases. The choice of the likelihood function is critical though and it is often desired to build up the relationship between the system parameters and the system observables. This function is generally system dependent. For the systems to be studied herein, they are both linear systems with stationary Gaussian excitation. In such situation, a Gaussian distribution is commonly assigned to the probability density of Y_N given the system parameters. Gaussian is one of the most important distributions in probabilistic family. Many physical measurements can be well approximated by it even though the underlying mechanisms are different. The general form of its probability density function is:

$$f(x | \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}(x - \mu)^2 / 2\sigma^2\right] \quad (2.17)$$

The N-variant Gaussian density function with mean \bar{X} and covariance matrix Σ is expressed as:

$$f_X(X_1, \dots, X_N) = \frac{1}{(2\pi)^{N/2} |\Sigma|^{1/2}} \exp\left[-\frac{1}{2}(X - \bar{X})^T \Sigma^{-1} (X - \bar{X})\right] \quad (2.18)$$

For the current case, X is replaced by the output of systems Y_N . Since the likelihood function is desired to be a probability density function of Y_N given system parameters θ , the chain rule as in equation 2.15 needs to be applied to obtain the following equation:

$$f_{\theta|Y}(\theta | Y_N) \propto f_{\theta}(\theta) \times \prod_{k=1}^N f_{Y|\theta}(Y_k | Y_1, \dots, Y_{k-1}, \theta) \quad (2.19)$$

At this stage, the problem turns out to be a recursive parameter estimation problem. The least square function was used as the error cost function. A point estimator as discussed in equation 2.4 was used to find an optimal value of the parameters when needed. In the following part, the numerical results will be demonstrated and discussed.

2.4.1 Case study for a single degree of freedom system

The first simulation was done based on a SDOF oscillator characterized by the equation of motion: $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = -f(t)$, where ζ is the damping ratio and ω_n is the natural frequency. The configuration of the system is shown in Figure 2-3. The system is subject to the white noise excitation $f(t)$. The exact parameters used to generate simulation data Y_N are $\bar{\theta} = [\omega_n, \zeta] = [3, 0.02]$. To calculate the desired posterior pdf of the system parameters $f_{\theta|Y_N}(\theta | Y_N)$, the autocorrelation $R_{YY}(Y, \theta) = E[Y(t)Y(t + \tau) | \bar{\theta}]$ of the system response have to be evaluated. This part is the key factor to determine the likelihood function, thus it is of particular interest to analyze this first. From the definition of autocorrelation, it gives the information of dependency from one measurement $Y(t)$ to the next time series of the system $Y(t + \tau)$.

In Figure 2-4, the system excitation, the resulted system output and its correlation plots are plotted respectively. In the first step of the simulation, Gaussian pdf was assumed for both natural frequency and damping ratio. Posterior profile for single parameter was obtained given the other parameter was known. In Figure 2-5, the resulting posterior probability density function for natural frequency $f_{\theta|Y_N}(\omega_n | Y_N, \zeta = 0.02)$ and damping ratio $f_{\theta|Y_N}(\zeta | Y_N, \omega_n = 3)$ are illustrated respectively.

The optimal values obtained using maximum likelihood estimator were compared with the corresponding exact values. The standard deviations obtained from the statistical form are listed in Table 2-1. Error rates are also listed out in the table.

Table 2-1 Point estimation results for natural frequency and damping ratio

Parameters	Exact Value	Optimal Estimation	Error Rate	Standard Deviation
Natural Frequency ω_n	3 rad/s	3.12 rad/s	0.04	0.0036
Damping Ratio ζ	0.02	0.02005	0.0025	0.0004

From the data illustrated in the above table, it is shown that the system parameters estimated using Bayesian based system identification method are all in 5% of error range and accurate enough to be accepted.

Different Prior: To check results dependence on the priori distribution, simulations based on three different priori pdfs were performed to obtain the posterior pdfs. The results are plotted in Figure 2-6, Figure 2-7 and Figure 2-8.

In Figure 2-6, a uniform priori was defined for the natural frequency and the corresponding posterior pdf was obtained by performing the analysis. It is shown that the priori and posterior are in different form. Same phenomena happened to Rayleigh and even normal priori definition. In Figure 2-9, pdfs obtained using the previous 3 priori pdfs are plotted in one figure and obviously there are in close shape even though coming from totally different priori definition. At this stage, it is safe to state that the prior information defined subjectively has been corrected by the objective system observables. In the rest of this research work, the uniform priori was used because of its simplicity.

Different Data Length: It is a common rule that with more sampling data, more information can be captured and as a result, the mathematical model can be more accurate. However,

data acquisition is usually limited by physical memory, time to consume, data transmission and so on. Therefore it is always desired to find an optimal number of data lengths to balance the trade off between data information carried by data length and the accuracy of the system model. In Figure 2-10, results obtained from different data length $N=10$, $N=25$, $N=100$, $N=1000$ are illustrated. It is obvious that $N=100$ is a sufficient number to validate the result.

Different Data Sets: Similarly to the case stated in the above discussion, it is desired to have more objective information from the physical system to interpret the system characteristic more accurately. In practical situation, it is possible to have different sets of independent measurements. Hence it is worthwhile to check the effect of statistical sampling with 1 data set and more data sets. In Figure 2-11, 4 sets of independent samples were used to compare with the results using only one set of data. In the figure, the green line was obtained by taking the average of the pdf from 4 sets of data. It is shown that the optimal value using more data becomes closer to the actual value of ω_n . The width of the posterior pdf also becomes narrower with larger data. It indicates that the confidence has been increased in the estimation.

Cumulative Distribution Function: Probability distribution function is calculated from probability density function. It is more straightforward and contains information of mean and variance which might not be reflected in pdfs other than Gaussian distribution. It can act as a tool to evaluate the consistency of predicted pdf and the sample data. In the current case, results are plotted in Figure 2-12 and it shows the cumulative distribution function (CDF) prediction of the estimated data. The predicted CDF is found to be close to the sampled CDF.

Joint Probability Function: For systems with more than one parameter to be identified, the posterior pdf evaluated can be in the form of joint probability density function. To obtain the optimal values of the parameters, one way is to perform integration to obtain the marginal pdf and evaluate the parameter according to this resulting pdf. The other way is to extract the parameter information directly from the joint pdf. Figure 2-13 – Figure 2-18 illustrate the surface plots of the joint probability and the corresponding contour plots. These plots were based on data generated with different time step. It is shown that with finer time step, the information extracted can be more close to the actual value.

2.4.2 Three-story-shear building

To further explore the feasibility of this proposed approach on a higher degree of freedom system, a three-story-shear building, shown in the Figure 2-19, is employed. The equation of motion for this structure is:

$$M\ddot{y} + C\dot{y} + Ky = -F(t) \quad (2.20)$$

Where M , C and K are mass, damping and stiffness matrix respectively. $F(t)$ is the base excitation with Gaussian distribution. The original configuration of the structure is: $m_1 = m_2 = m_3 = 10\text{kg}$, $k_1 = 28 \text{ kN/m}$, $k_2 = k_3 = 24 \text{ kN/m}$ and the damping ratio is chosen to be 2%. By performing the general modal analysis techniques, the uncoupled modal equations can be obtained. Therefore, the Bayesian system identification procedure can be applied for the MDOF system in the same way as application to the SDOF system.

In Figure 2-20, the conditional posterior pdf $prob(K_1 | Y_N)$ is obtained by using the displacement measurement from the 1st floor. The actual value of the stiffness in the first floor is $K_1 = 28 \text{ kN/m}$, and the estimated value is $\hat{K}_1 = 27.85 \text{ kN/m}$. The error rate is about

0.007. The corresponding standard deviation is evaluated as: $\sigma_K = 0.0055$. The error rate falls to 5% range, hence it can be considered as an acceptable estimate.

In some cases, more than one parameters need to be identified. For example, stiffness of both the first and second floor is to be evaluated. The resulting posterior pdf is two-dimensional since it is a function of both K_1 and K_2 . Figure 2-21 is the contour plot of it with the contours corresponds to 10 percentage levels of the maximum probability.

2.5 Conclusions:

In this part of the research, a Bayesian system identification approach was introduced and demonstrated. The unknown parameters of dynamic systems were accurately estimated by optimizing the conditional posterior pdf. The associated covariance of the pdf was used to quantify the uncertainty. The choice of prior pdf has minor effects on the posterior pdf. By increasing data availability, the confidence of estimations can be increased. From the joint probability density function of different parameters, parameter information could be extracted from the contour plot. However, it was restricted by the data size. For more than two parameters, this will encounter the problem of visualization. In future studies, one may consider defining a different likelihood function with parameters independent of each other or finding marginal pdf for each parameter.

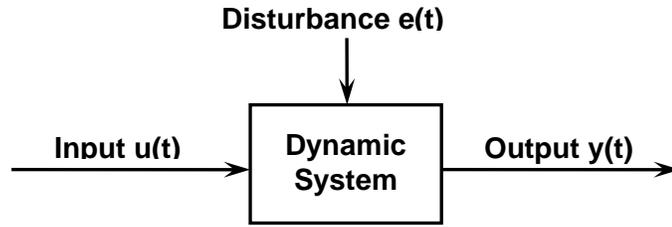


Figure 2-1 A General system model structure

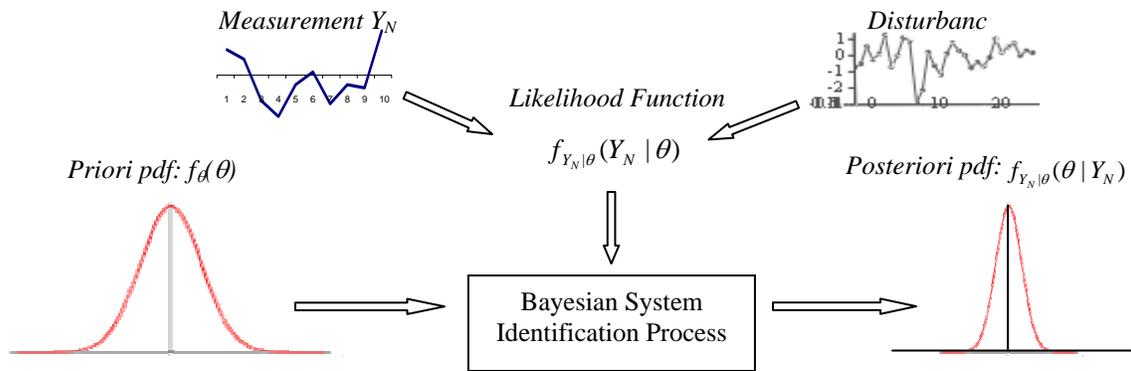


Figure 2-2 Principle for Bayesian system identification

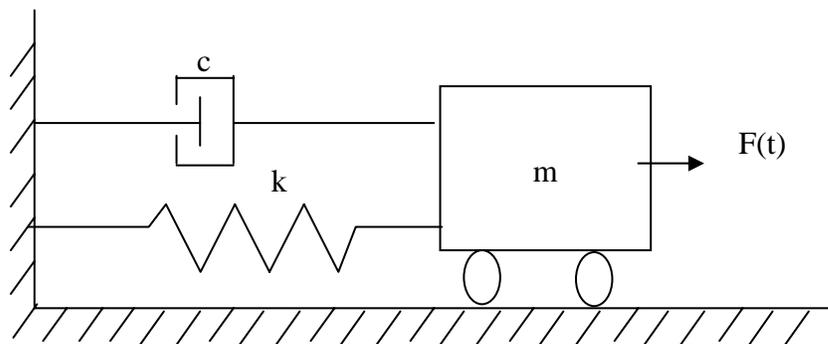


Figure 2-3 A single degree of freedom duffing oscillator

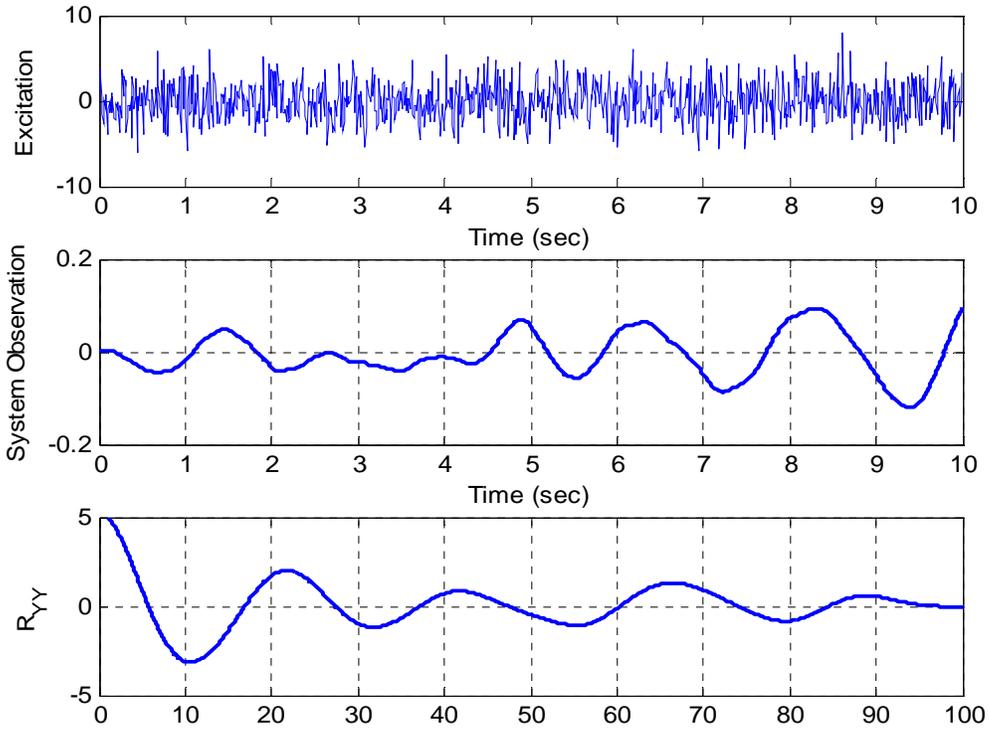


Figure 2-4 Excitation and Autocorrelation of the system

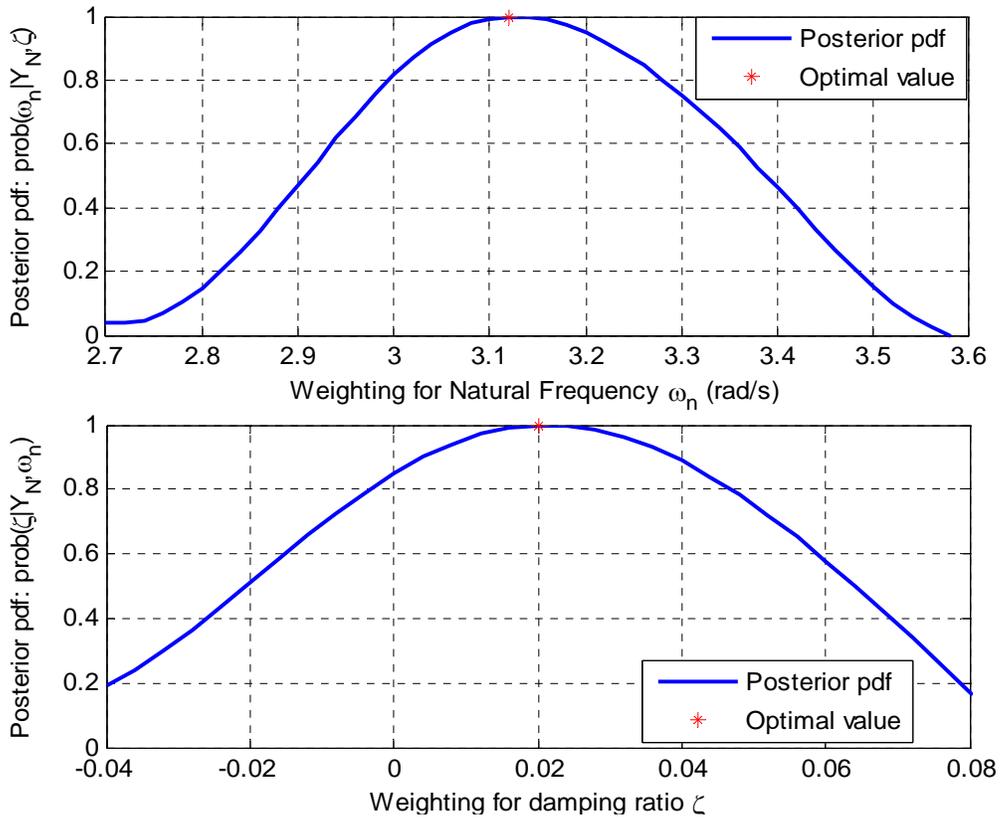


Figure 2-5 Posterior conditional pdf for the estimated parameter ω_n and ζ

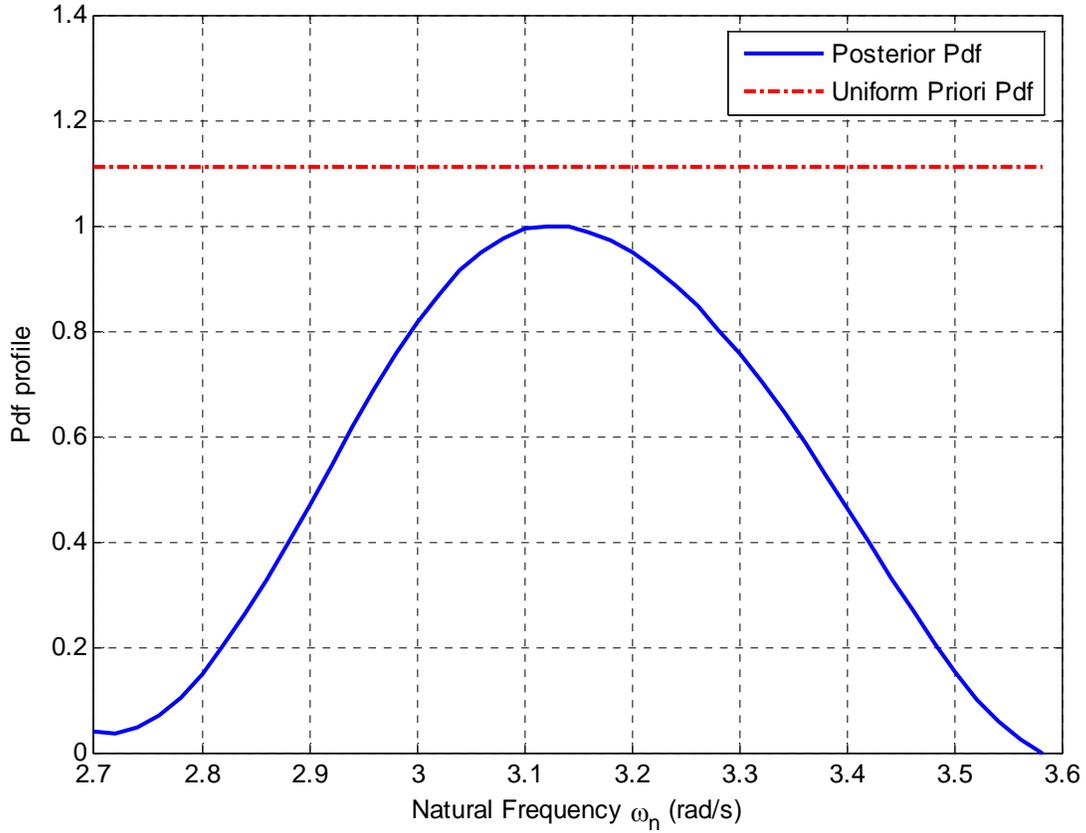


Figure 2-6 Uniform priori pdf and the resulted posterior pdf

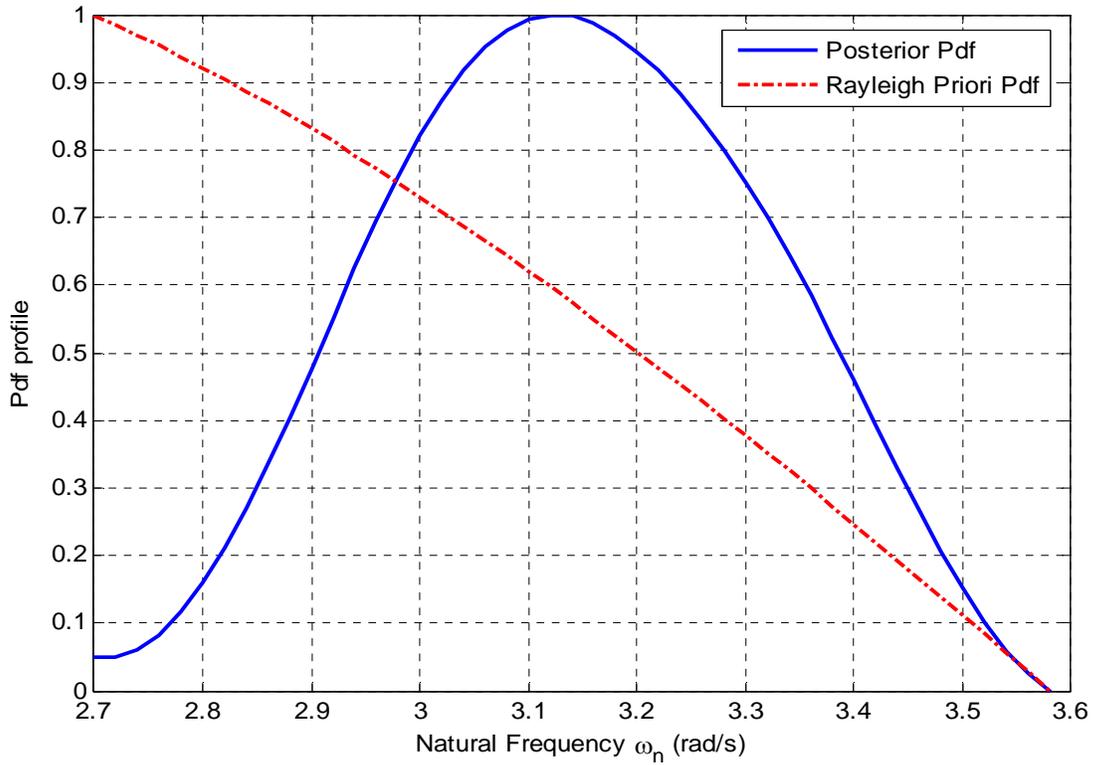


Figure 2-7 Rayleigh priori pdf and the resulted posterior pdf

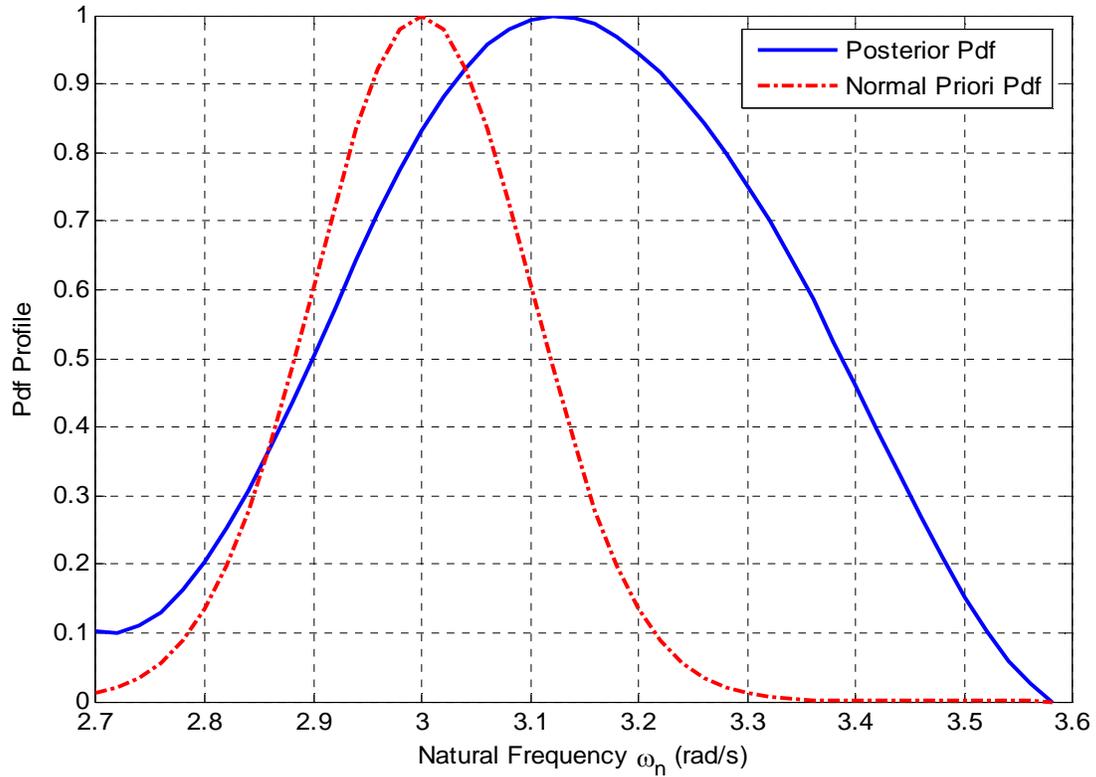


Figure 2-8 Normal priori pdf and the resulted posterior pdf

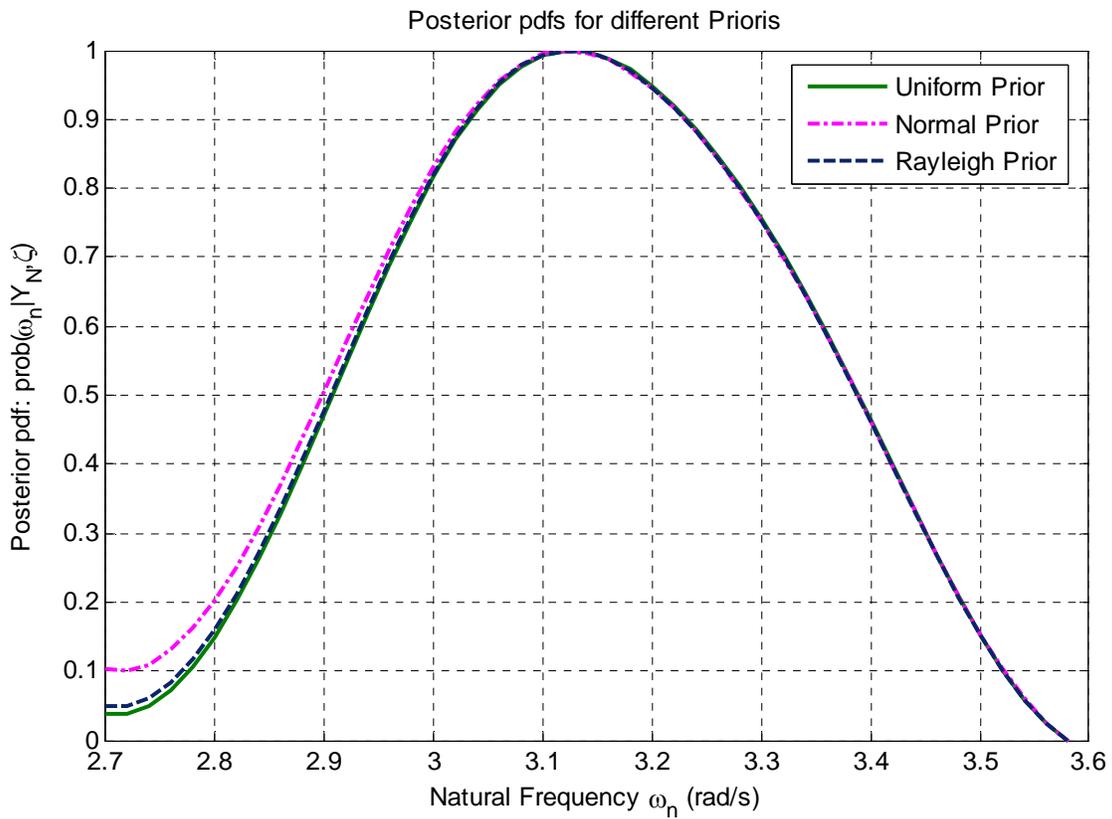


Figure 2-9 Posterior comparison

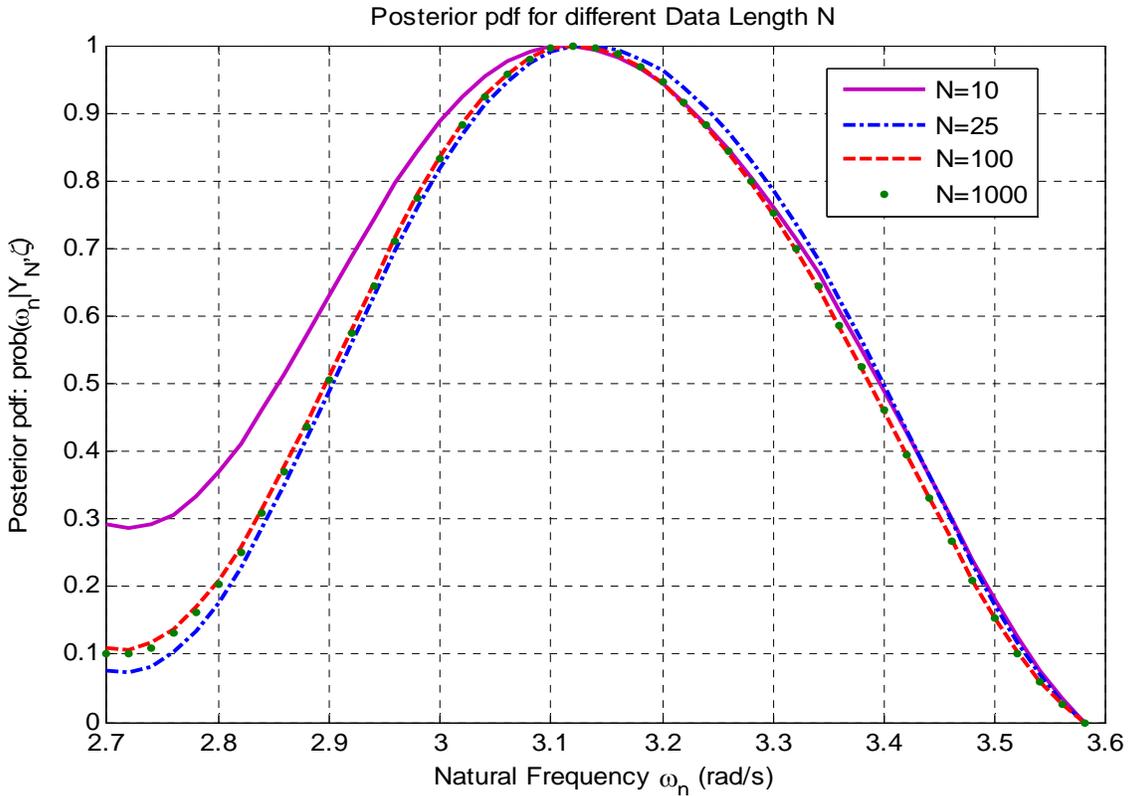


Figure 2-10 Posterior pdfs obtained from different length of data

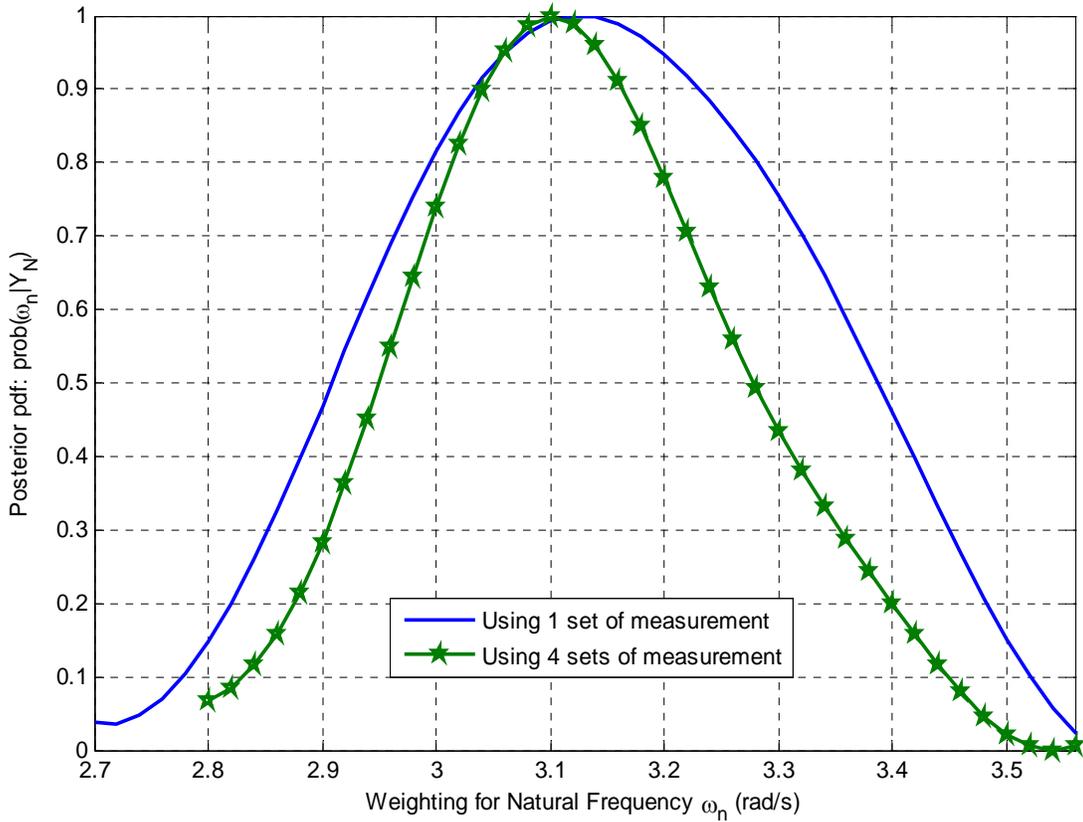


Figure 2-11 Pdfs comparison of using 1 set and 4 sets of measurement

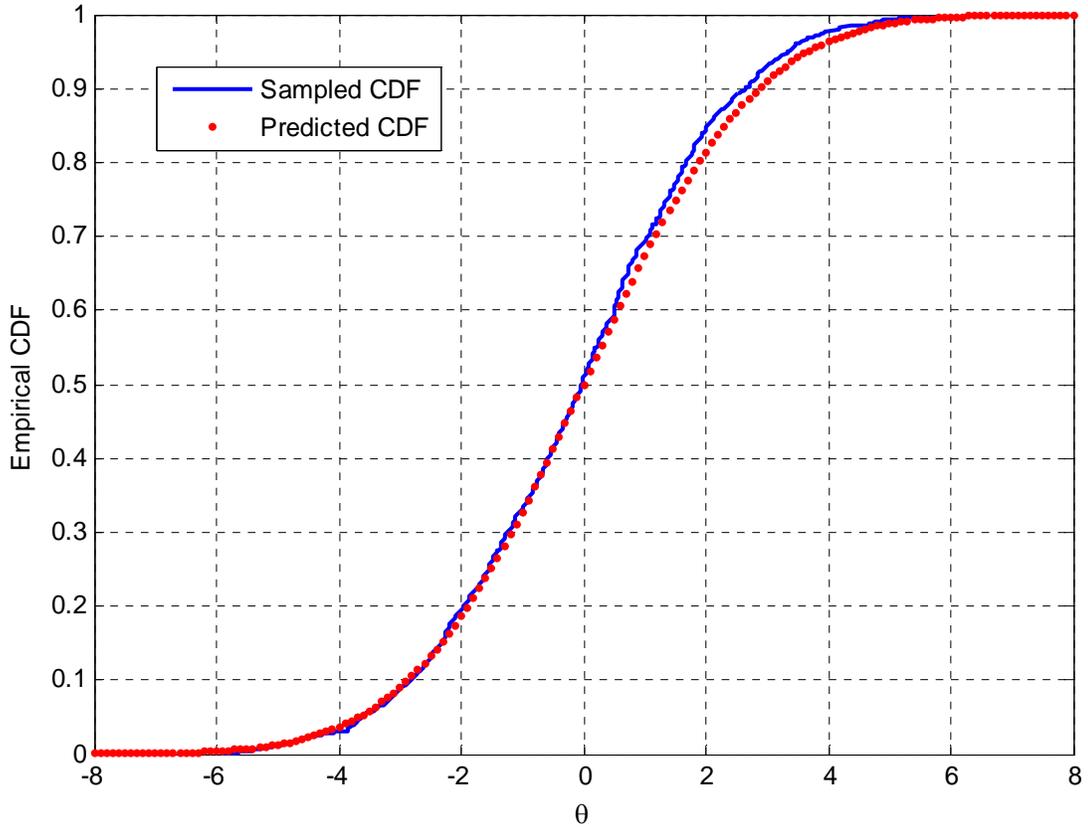


Figure 2-12 Cumulative distribution function comparison

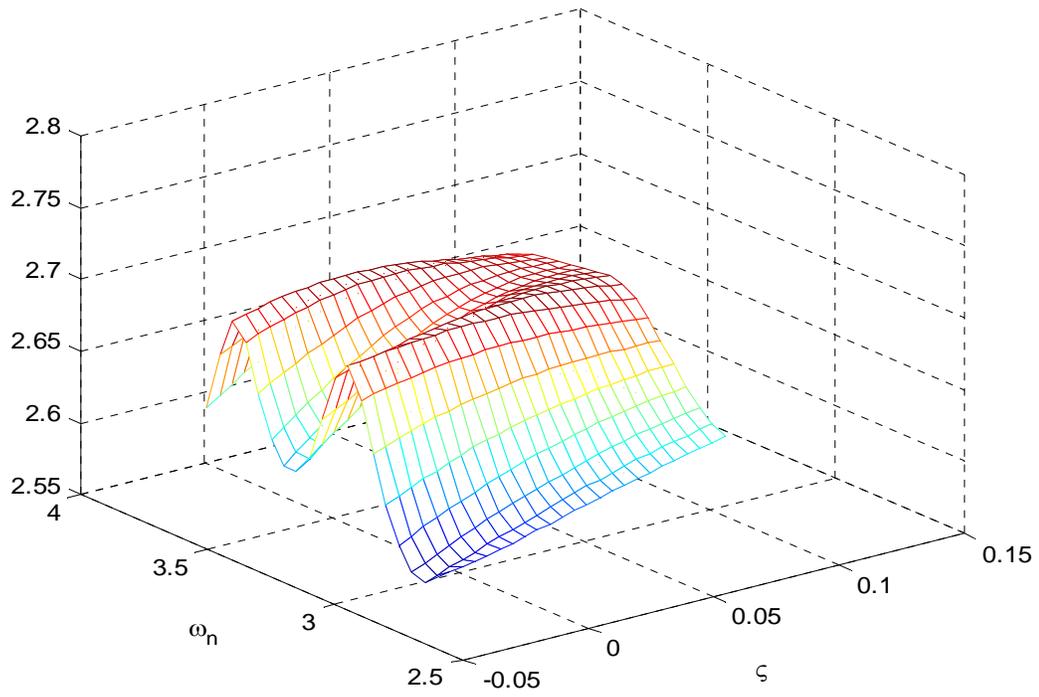


Figure 2-13 Joint probability density function for time step = 0.05

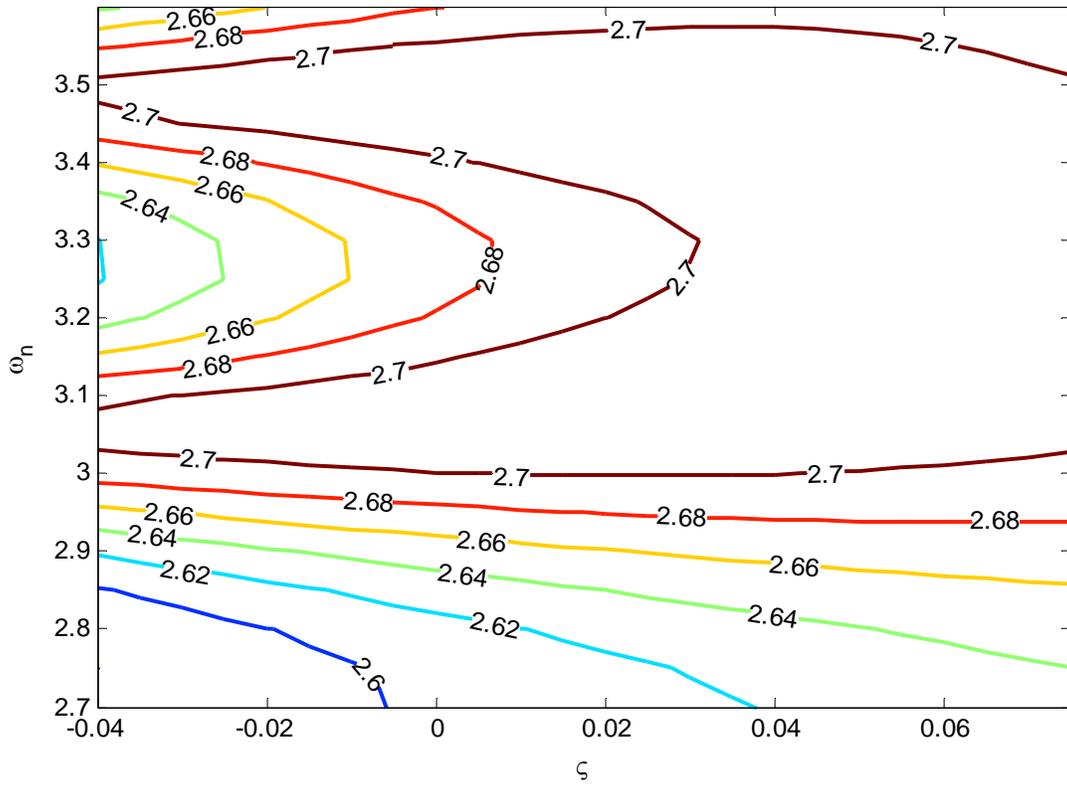


Figure 2-14 Contour plot for the joint pdf

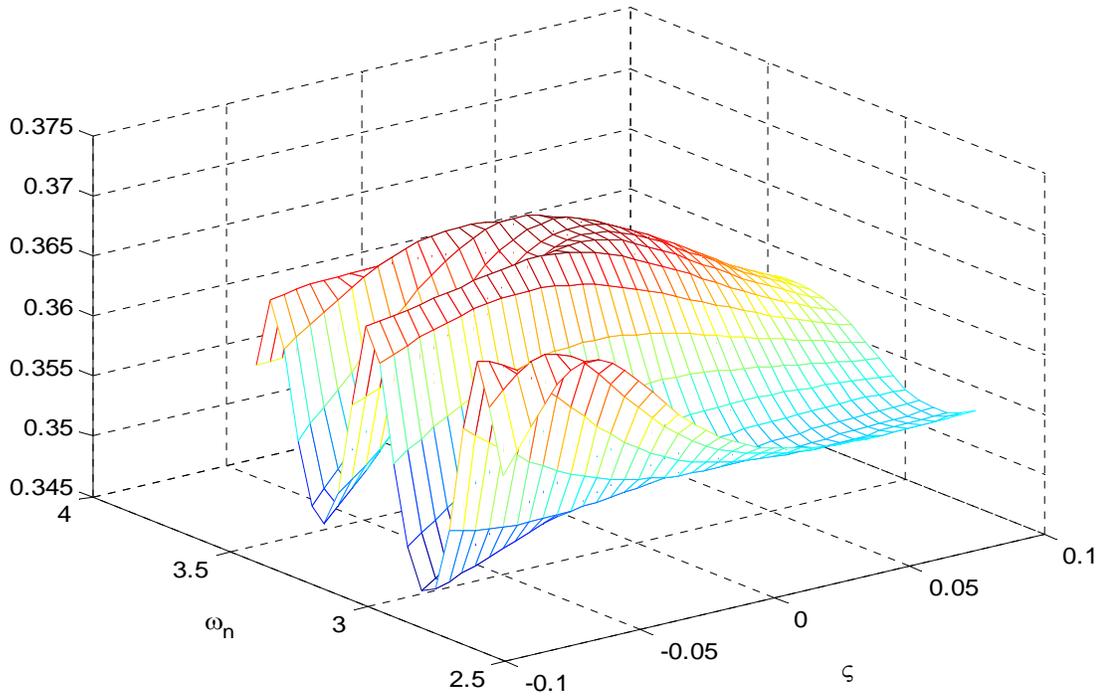


Figure 2-15 Joint probability density function for time step = 0.04

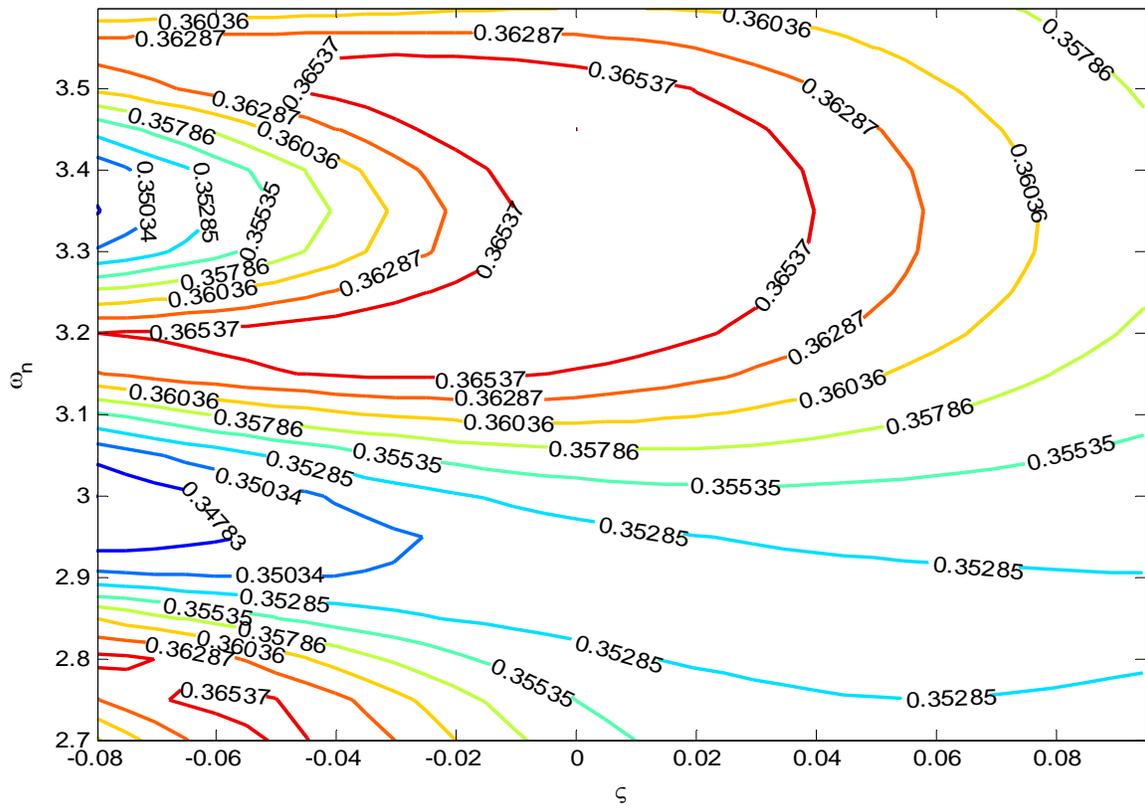


Figure 2-16 Contour plot for the joint pdf

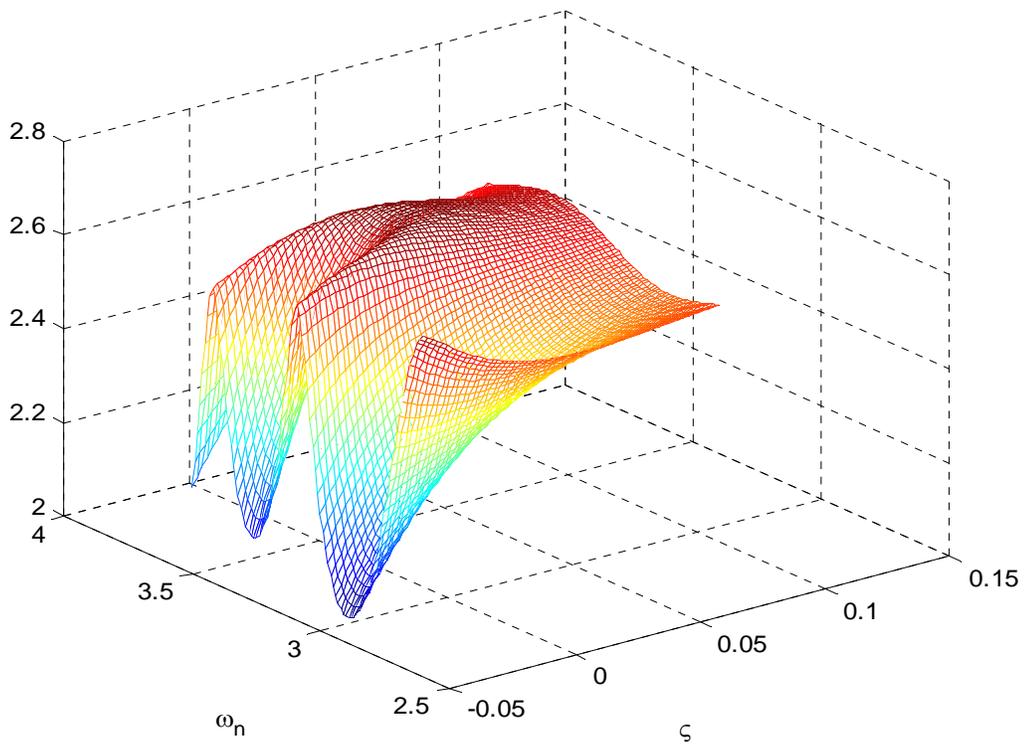


Figure 2-17 Joint probability density function for time step = 0.02

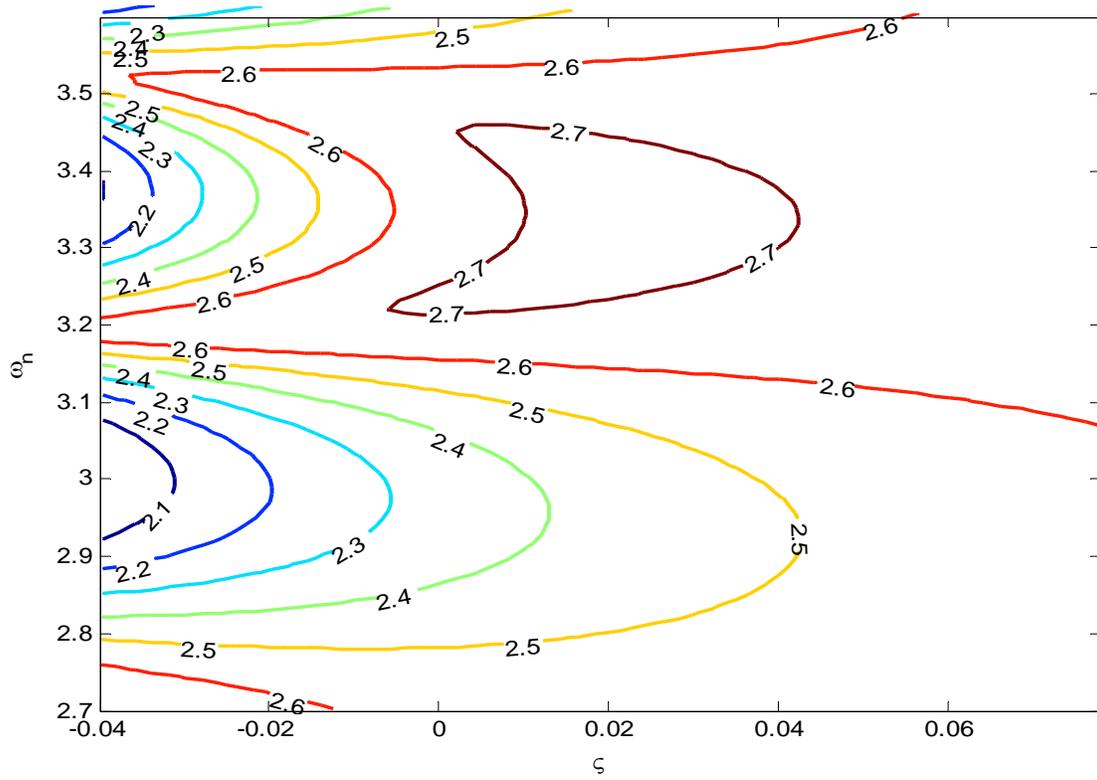


Figure 2-18 Contour plot for joint pdf

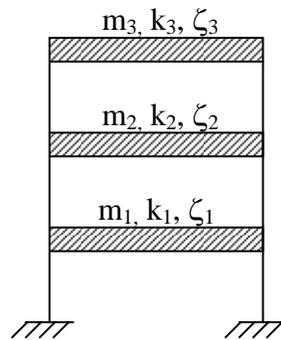


Figure 2-19 Configuration of a three-story-shear building

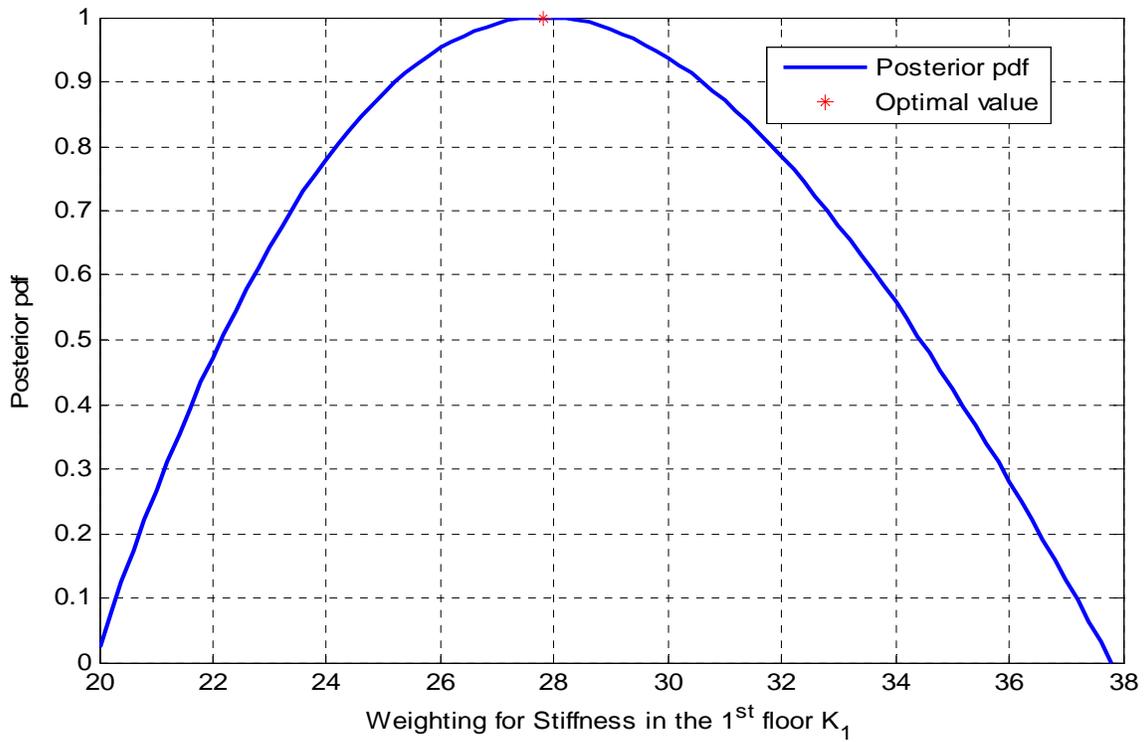


Figure 2-20 Posterior conditional pdf for the estimated stiffness K of 1st floor

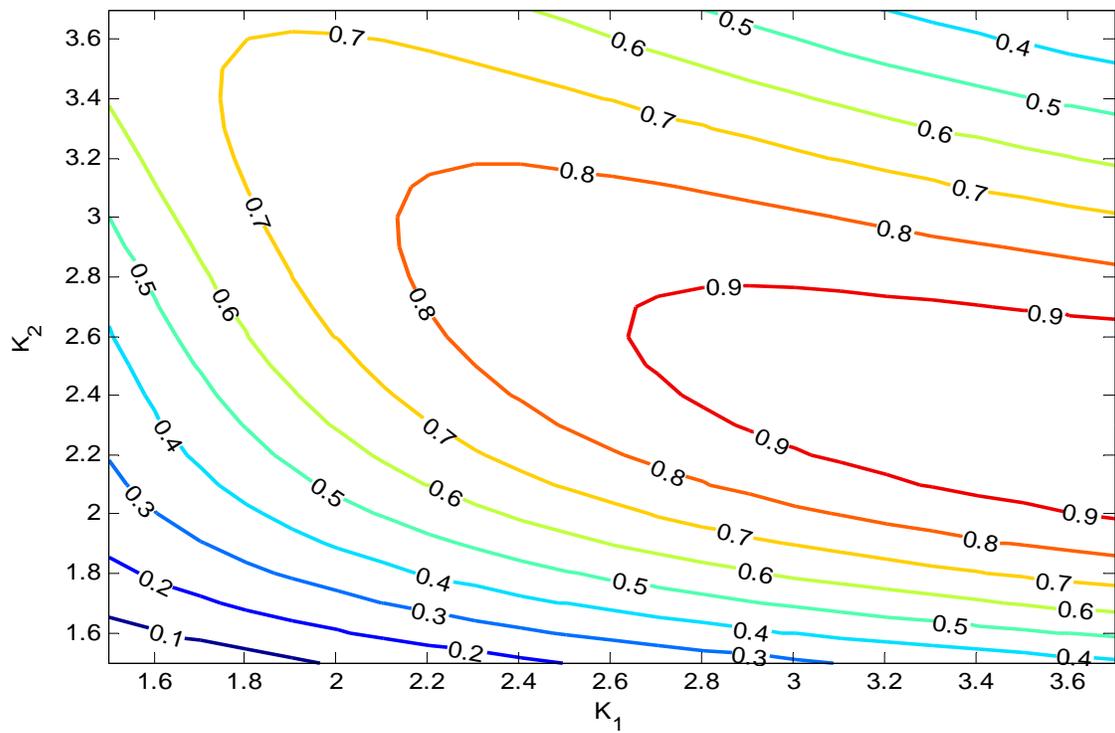


Figure 2-21 Contour plot of Joint pdf for the estimated stiffness K_1 and K_2

3 Statistical Reliability & Uncertainty Analysis

3.1 Structural Reliability and Its Role in SHM

Structural safety and reliability is crucial to support the human life and the well being of the society. For certain structures like aircrafts and bridges, it is often required to calculate and predict the structural reliability and safety during their service life. Structural reliability is usually considered as the final step of structural health management which has the ultimate goal of improving the reliability of the structures and increasing their service life. It is the fundamental goal of structural reliability analysis to determine the safety condition of a structure. One of the most important indicators to assess structural safety condition is the probability of failure. Let a n-dimensional vector of random variables $\mathbf{X}=[x_1, x_2, \dots, x_n]^T$ represent fundamental structural quantities describing the structure safety and behavior such as design loads, material resistance etc, the probability of failure is given as:

$$P_f = \int_{\Omega_f} f_x(\mathbf{X})d\mathbf{X} \quad (3.1)$$

Where $f_x(\mathbf{X})$ is the joint probability of the random variables, and Ω_f denotes the failure region defined by the limit state function of the system - $g(\mathbf{X})$. Given $g(\mathbf{X})=0$, the failure region is defined as $\Omega_f = \{\mathbf{X} \mid g(\mathbf{X}) < 0\}$. Herein, the limit state function defines the requirement under which the structure is either in the safe region or in the failure region. The boundary to divide these two regions is the so called “limit state”. There are different categories of limit state (Ditleven, O. and Madsen, H.O., 1996). In the current research work, the limit state is chosen to be a mathematical form to easily define the problem being investigated. The traditional methodologies for structural reliability analysis include but are

not limited to Monte Carlo Simulation (MCS) method, first order reliability method, second order reliability method, and importance sampling method. Those methods will be reviewed briefly in the following sections.

3.1.1 Monte Carlo Method

As described by Equation 3.1, in order to obtain the failure probability, one has to evaluate the integral of the joint pdf over the failure region. Because of its probabilistic aspect, Monte Carlo Simulation method is usually used to solve this integration. Monte Carlo simulation is an empirical method utilizing a large sample size to obtain the integral value through rejection scheme. With samples generated from the given random variable generators, it is an algorithm based on a recursive experiment to count the numbers of the samples falling into the interested region. The failure probability obtained from MCS is calculated as:

$$P_f = \frac{N_f}{N_T} \quad (3.2)$$

where N_T is the total sampling number and N_f is the number of samples falling into failure region.

In the literature, probability of failure evaluated using Monte Carlo method is usually referred to as the exact value. P_f is generally extremely small for most of the structural and mechanical structures. Thus, it requires a fairly huge number of samples to perform the simulation to achieve the satisfied accuracy. Subsequently, this increases the computation cost. Over the past two decades significant work has been done in order to develop a more efficient way for reliability analysis. These efforts have led to the development of other simulation methods such as FORM and SORM et al.

3.1.2 First Order Reliability Method (FORM)

First order reliability method (FORM) is an approach for obtaining the failure probability by approximating the limit state function at the most probable failure point. The most probable failure point, which is also called the design point, is defined in a standardized normal space that is also known as u-space. Generally speaking, the design point is located on the failure surface which is defined by the limit state function. In a u-space, a normally distributed pdf is rotationally symmetric and exponentially decaying. Therefore, the point on the limit state surface which has the shortest distance from the origin is the one with the highest probability of failure. This is the definition of the most probable failure point (MPP) and it can be found mathematically by iteration procedures. Once the design point \mathbf{u}^* is specified in a u-space, the probability of failure is analytically estimated as in following equation (Madsen, Krenk and Lind, 1986):

$$P_f = \Phi(-\beta) \quad (3.3)$$

In above formula, β is the ‘safety index’ which represents the shortest distance to define the MPP. A minimum distance represents a maximum possibility of failure. It is considered as a direct measure of the structural safety since the failure probability is simply evaluated as the standard normal cumulative distribution of that. The mathematical expression of β is shown in equation 3.4.

$$\beta = \|\mathbf{u}^*\| = \sqrt{u_1^{*2} + u_2^{*2} + \dots + u_n^{*2}} \quad (3.4)$$

The direct method to evaluate failure probability as in equation 3.3 is usually applied to the linear limit state function. In cases of nonlinear limit state function, Taylor expansion is generally applied at the design point. The first order based failure probability evaluation

algorithm is usually called First Order Reliability Method, also referred to as FORM. Its expression is written as in equation 3.5 (Wu, 1994):

$$g(\mathbf{u}) = a_0 + \sum_{i=1}^n a_i (u_i - u_i^*) \quad (3.5)$$

In order to evaluate the relative effects of each basic random variable on the failure probability results, the concept of sensitivity analysis has to be introduced. One of the efficient ways of calculating sensitivity factors is to utilize the definition of reliability index β . As defined in equation 3.4, β is the shortest distance from the origin to the failure surface with $g(\mathbf{u}) = 0$ in u-space. Therefore, the sensitivity of failure probability to each basic variable can be obtained from the following equation:

$$\alpha_i^* = \frac{\partial \beta}{\partial u_i^*} = \frac{\partial}{\partial u_i^*} \sqrt{\sum_{j=1}^n u_j^{*2}} \quad (3.6)$$

It can be easily shown that the sensitivity factors α_i^* are the unit projection of \mathbf{u}^* in u-space. Since $g(\mathbf{u})$ is the function of \mathbf{u} , sensitivity information for each random variable can also be derived from the following:

$$|\alpha_i^*| \propto \left(\frac{\partial g}{\partial u_i} \right)_{\mathbf{u}^*} \quad (3.7)$$

Since the sensitivity factor introduced herein is fully derived from the safety index β , it can be used as a sufficient importance measure of basic variables only if FORM can result a good approximation to the actual failure probability.

Due to its simplicity, first order reliability method is widely used to obtain the failure probability. Unfortunately, the accuracy of FORM is highly dependent on the nonlinearity of the limit state function. In cases with high nonlinearity, additional calculation needs to be performed to approximate the limit state function using a higher order parabolic. Varieties of

methods have been proposed to improve the accuracy of FORM. Among them, SORM is the most popular one.

3.1.3 Second Order Reliability Method (SORM)

As shown from the terminology, Second Order Reliability Method (SORM) is to linearize the nonlinear limit state function with a second order Taylor expansion. The failure surface of the system is obtained as a quadratic surface with the following form:

$$g(\mathbf{u}) = a_0 + \sum_{i=1}^n a_i (u_i - u_i^*) + \sum_{i=1}^n b_i (u_i - u_i^*)^2 + \sum_{i=1}^n \sum_{j=1}^{i-1} c_{ij} (u_i - u_i^*) (u_j - u_j^*) \quad (3.8)$$

From the above definition, it is shown that in order to solve reliability problem using SORM, one has to deal with second order derivative matrices. To avoid computing those matrices, the g-function is usually approximated in an incomplete form (Schuëller, 2003):

$$g(\mathbf{u}) \approx \tilde{g}(\mathbf{u}) = \beta - u_1 - \frac{1}{2} \sum \kappa_i u_i^2 \quad (3.9)$$

Because of its incompleteness, this form is not accurate enough to represent a highly non-linear limit state function. As a result, a well recognized asymptotic formula was therefore developed and presented by Breitung (1984) for large β :

$$p_f \approx \Phi(-\beta) \prod_{i=1}^{n-1} (1 - \beta \kappa_i)^{-1/2} \quad (3.10)$$

In equation 3.10, κ_i represents the main curvatures at the design point. Second order reliability method is generally working well for large curvatures and small numbers of basic variables. However, it still needs to be used carefully in cases that the nonlinear function cannot be well approximated using lower order polynomials. The complexity to calculate the main curvatures also makes it limited to compete with FORM. For cases that it is hard to

decide whether first or second order reliability approximation would be performed, it is a better to investigate the parameter range to make the appropriate decision (Zhao, 1999).

The first and second-order methods, as well as other MPP-based methods, attempt to approximate the original g-functions by simple functions. It has been reported that these MPP-based methods perform well for a wide range of applications. However, some nonlinear functions such as a sine wave function cannot be approximated well by low-order polynomials. In practice, when a performance function is implicitly defined and is highly nonlinear, it is desired but difficult to assess the error introduced by the polynomial approximation. Therefore, for a new type of g-function, the approximate solution needs to be checked using other more accurate methods. In cases of the MPP search fail, either Importance Sampling or Monte Carlo methods can be performed to find the failure probability.

3.1.4 Importance Sampling Method

Importance sampling method is a numerical simulation method where Monte Carlo simulation is applied for certain variables that area considered to be more important to the estimations. The fundamental concern of importance sampling lies in the choice of distributions and important regions to sample the input variables. According to the difference of the important region, most widely used techniques to approximate the optimal sampling density include design point based and kernel density estimator based importance sampling.

For design point based importance sampling method, input samplings are performed in the neighborhoods of the most probable point. Compared with the analytical methods such as FORM and SORM, the design point dose not have to be identified exactly as long as the

'important' region is determined properly. Design point based importance sampling is one of the most popular techniques to solve a reliability problem. However, in situations such as high dimensional or highly noisy limit state function, the design points are usually hard to be found or it consumes significant time to find them. Even worse, design points do not represent the most important region for some cases. Therefore, other adaptive schemes have been studied by researchers and kernel density estimator have been presented to obtain more robust importance sampling densities. Among the different techniques to obtain densities of importance sampling, Au and Beck (1999) presented an efficient approach to generate samples using Markov chain. The quality of this method is however affected by the mixing rate of the chain and the correlation of the samples. The sampling schemes for Monte Carlo simulation and importance sampling methods are illustrated in Figure 3-1 and Figure 3-2 respectively. It is shown that for importance sampling, the sample points are concentrated around the failure boundary instead of spreading around evenly in Monte Carlo simulation. It is noticed that the reliability methods of FORM, SORM, and ISM all require the most probable points to be found for further calculation. For first order reliability method, as long as the MPP is found, failure probability can be obtained directly by a point estimator. For second order reliability method, the failure probability is evaluated by calculating $n-1$ main curvatures around the most probable point. Although MPP region may be of interest for certain cases, the design point based importance sampling method is still working well and widely applied to general cases. Those different techniques have their own advantages and drawbacks. Thus, they are usually applied to different cases according to the limit state functions and practical applications. If these approaches fail to search the most probable point or where more than one MPP may exist, traditional Monte Carlo method is generally

applied to calculate the failure probability even though the computation cost is much higher than other methods.

3.2 Uncertainty assessment and updating of PROBES

As discussed in the previous section, limit state function for a structural system is defined by an n-dimensional vector with basic variables $X=[x_1, x_2, \dots, x_n]^T$ which defines fundamental structural behavior and safety. For the MPP based approaches, the design point is to be found in a standard normal u-space. However, the basic variables are usually not normal-distributed and always correlated. For non-normal variables, they can be transformed into standardized normal vectors. Therefore, it becomes a general desire to transform the basic variables into standard normal random variables. In this section of the research work, functionalities of random variable transformation and statistical assessment of failure probability were implemented to the software package 'PROBES'. Details are introduced in the following part.

3.2.1 Theoretical background to update 'PROBES'

The MATLAB software package 'PROBES' is the reliability analysis tool originally developed by Wu (2005). The objective of the software is to obtain failure probability given the limit state function and associated random variables. However, the original probabilistic based analysis package was designed only for random variables that are distributed normally and are not correlated. It was one of the tasks of this research work to update the software package to make it feasible for application in different practical cases other than independent normal variable cases. In another words, for random variables that are non-normal \correlated, it was the task to transform them into a set of uncorrelated, standard normal RVs. This part of research was part of a project subcontracted by Pratt and Whitney Corporation to

the National Institute of Aerospace in collaboration with researchers from Applied Research Associates in Raleigh, NC. The objective of this work was to realize the transformation shown in Figure 3-8. To perform the above transformation, the Nataf Model and orthogonal transformation were used and they are introduced next.

The Nataf Model:

The ‘Nataf Model’ was utilized in this work to transform a set of random variables into a new set of standard normal variables. Suppose that $\mathbf{X} = (X_1, X_2, \dots, X_n)$ is a set of correlated random variables with the covariance matrix $CovX(i, j) = \rho_{i,j}$. The Nataf model provides a way to transform this set of RVs into a set of standard normal variates $\mathbf{V} = (V_1, V_2, \dots, V_n)$ (Liu and Der-Kiureghian, 1986):

$$V_i = \Phi^{-1}\left[F_{X_i}(X_i)\right] \quad i = 1, \dots, n \quad (3.11)$$

Where $\Phi(\bullet)$ is the standard cumulative normal probability. $F_{X_i}(\bullet)$ is the cumulative distribution function (CDF) of the i^{th} random variable. \mathbf{V} is jointly normal with the covariance matrix $CovV(i, j) = \rho'_{i,j}$ obtained from equation (3.13):

$$\rho'_{i,j} = F\rho_{i,j} \quad (3.12)$$

In which, F is a function of $\rho_{i,j}$ and the marginal distributions of X_i and X_j . The formulae of F for selected distributions were developed in Liu and Der-Kiureghian’s work. In this transformation, a numerical integration was implemented to find the covariance matrix. The covariance expression for the non-normal random variables is:

$$\rho_{ij} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{x_i - \mu_i}{\sigma_i} \right) \left(\frac{x_j - \mu_j}{\sigma_j} \right) \phi_2(v_i, v_j, \rho'_{ij}) \frac{f_{x_i}(x_i) f_{x_j}(x_j)}{\phi(v_i) \phi(v_j)} dx_i dx_j \quad (3.13)$$

Where, $f_{x_i}(x_i)$ and $f_{x_j}(x_j)$ are the probability density functions of original input variates.

$\phi(\bullet)$ denotes the normal pdf with zero means, unit standard deviations. $\phi_2(v_i, v_j, \rho'_{ij})$ is the normal pdf as a function of ρ'_{ij} which is the covariance coefficient of the transformed normal random variables:

$$\phi_2(v_i, v_j, \rho'_{ij}) = \frac{1}{2\pi\sqrt{1-\rho'^2_{ij}}} \exp\left[-\frac{v_i^2 - 2\rho'_{ij}v_iv_j + v_j^2}{2(1-\rho'^2_{ij})}\right] \quad (3.14)$$

From equation 3.13 to 3.14, it is observed that an iterative procedure has to be performed to obtain the value of ρ'_{ij} . For given probability information of random variables x_i and x_j , the preceding equations can be solved to find ρ'_{ij} . In the transformation procedure, the marginal distribution is transformed first and the corresponding correlations can be estimated afterwards. The joint normal pdf is obtained using the rules of probability transformation as a function of n-dimensional normal pdf.

Orthogonal Transformation:

In order to transform correlated normal variables into a set of required uncorrelated variables, an orthogonal transformation process was applied. This process involves solving the eigenvectors of the covariance matrix of correlated variables and uses the eigenvector to generate the independent variables. From correlated normal variates $V = (V_1, V_2, \dots, V_n)$, the desired set of uncorrelated variates is obtained through the following transformation (Ang and Tang, 1984):

$$U = T^T \times V \quad (3.15)$$

In which, T is an orthogonal matrix that is composed of the eigenvectors corresponding to the eigenvalues of the covariance matrix $CovV(i, j) = \rho'_{i,j}$. Specifically, T is such that

$$T^T [Cov V] T = [\lambda] \quad (3.16)$$

Where $[\lambda]$ is eigenvalues and also the variances of the variates $U = (U_1, U_2, \dots, U_n)$.

Inversion:

In programs of 'PROBES' package, the simulation data were sampled in u-space, where the joint probability density function (PDF) is rotationally symmetric around the origin. Therefore, the inversion of the random variables becomes another major task to transform standard normal random variables from u-space back to their original space. The inversion procedure is illustrated as the diagram shown in Figure 3-4.

Mathematically, the corresponding algorithm is realized by inversion of equation (3.15) and equation (3.12), which yields the following two equations respectively:

$$V = T \times U \quad (3.17)$$

$$X_i = F_{X_i}^{-1}[\Phi(V_i)], \quad i = 1, \dots, n \quad (3.18)$$

3.2.2 Numerical case study:

To verify the updated functionality of software package PROBES to deal with correlated non normal random variables, the following numerical cases were designed. The analytical solution which is denoted as exact failure probability can be found from Ang and Tang's book.

Problem description-linear limit state function:

The objective of this numerical case was to calculate the safety of a structural element. Let X1 denote the resistance on the element, X2 denote the dead load effect and X3 denote the lifetime maximum live load effect. The limit state function is defined as:

$$g(X) = X1 - X2 - X3 \quad (3.19)$$

Suppose the element to be studied has basic variables with expectation relations of $E[X_2] = 100$, $E[X_1] = 2.831E[X_2]$ and $E[X_3] = 0.745E[X_2]$. The corresponding standard deviations are functions of dead load expectation: $V[X_1]=0.3114E[X_2]$, $V[X_2]= 0.1E[X_2]$, $V[X_3]= 0.1863E[X_2]$ respectively.

Case A: Inputs with Uncorrelated Normals:

Main purpose of this case was to verify the updated program that it is working well for linear limit state functions with uncorrelated normal variables. Simulation results are listed as in the following table, where the exact solution is the analytical result considered as the criteria to quantify the numerical results.

Table 3-1 Simulation results for uncorrelated normal inputs - linear case

Method	Exact solution	FORM	SORM	ISM
Failure probability	0.00196	0.00194	0.0019	0.00194
Error rate %	NA	1.02	3.06	1.02

Case B: Inputs with Uncorrelated Nonnormals:

This case was designed to verify that the updated program is working well for linear limit state function with uncorrelated non-normal variables. For this case, lognormal distribution was used for random variables X_1 , normal distribution was assigned to X_2 and type I asymptotic extreme distribution was used for X_3 .

Table 3-2 Simulation results for uncorrelated nonnormal inputs - linear case

Method	Exact solution	FORM	SORM	ISM
Failure probability	0.00214	2.3388e-003	0.0025	2.5519e-003
Error rate %	NA	9.3	16.8	19.2

Case C: Inputs with Correlated Normals:

This case was studied to verify the capability of updated 'PROBES' to handle correlated input random variables.

Table 3-3 Simulation results for correlated normal inputs - linear case

Method	Exact solution	FORM	SORM	ISM
Failure probability	3.63e-004	3.324e-004	3.459e-004	3.7240e-004
Error rate %	NA	8.4	4.7	2.6

From the simulation results illustrated in the above table, it was concluded that the modified program was sufficient for cases with correlated normal inputs given linear limit state function. With the cases studied, it was concluded that the updated program was feasible to systems that have linear limit state functions. Different error rates indicate that different methods should be chosen based on the practical needs.

Problem description-nonlinear limit state function:

To further study the applicability of the program to systems with nonlinear limit state function, another problem was designed. Let X_1 denote the yield strength of a steel beam, and X_2 denote the section modulus of the section. If the applied bending moment at the pertinent sections is X_3 , the limit state function is defined as:

$$g(\mathbf{X}) = X_1 \times X_2 - X_3 \quad (3.20)$$

Suppose a beam with which the expectations of characteristic variables are specified as $E[X_1] = 40\text{ksi}$, $E[X_2] = 50\text{in}^3$, is subject to bending moment of $E[X_3] = 1000\text{in-kip}$. The corresponding standard deviations for those variables are $V[X_1]=5.0\text{ksi}$, $V[X_2]=2.5\text{in}^3$, $V[X_3]=200\text{in-kip}$ respectively. It was desired to find the probability of failure of this beam under the described loading condition.

Case A: Inputs with Uncorrelated Normals:

This case was designed to verify that the updated program is working correctly for uncorrelated normal variables. Simulation results are listed as in the following table, where EXACT solution is the analytical result considered to be the criteria to quantify the numerical

results. From results listed in table 3-4, it is shown that the updated program works well for case with uncorrelated normal inputs.

Table 3-4 Simulation results for uncorrelated normal inputs - nonlinear case

Method	Exact solution	FORM	SORM	ISM
Failure probability	0.00114	0.0011	0.0012	1.1477e-3
Error rate %	NA	3.5	5.3	0.7

Case B: Inputs with Uncorrelated Nonnormals:

This case was designed to verify that the updated program was working well for uncorrelated non-normal variables. For this case, lognormal distribution was used for random variables X1 and X2. Type I asymptotic extreme distribution was used for X3. From the solution illustrated in table 3-2, it is concluded that the modified program was sufficient for cases with uncorrelated non-normal inputs.

Table 3-5 Simulation results for uncorrelated normal inputs – nonlinear case

Method	Exact solution	FORM	SORM	ISM
Failure probability	0.00302	0.0030	0.0031	3.0497e-3
Error rate %	NA	0.7	3.3	0.98

Case C: Inputs with Correlated Normals:

The following cases were designed to verify the capability of ‘PROBES’ to deal with cases with which the input random variables are correlated. Because of the nonlinearity of the limit state function, it was found that most probable point could not be found in certain time limit. For such cases, Monte Carlo simulation technique was used to check the feasibility of the transformation methods introduced in above section. Two ways to transform variables were compared and the resulting failure probability was checked based on the exact value of 0.00212.

As usual, let \mathbf{u} denote uncorrelated Standard normal variates, \mathbf{V} denote correlated standard normal variates, \mathbf{T} and $\boldsymbol{\lambda}$ stand for the transformation and eigenvalue matrix respectively.

Transformation utilizing both T and λ : $\mathbf{u} \sim N(0,1) \rightarrow \mathbf{V} (\mathbf{V} = \mathbf{T} * \mathbf{u} * \text{sqrt}(\lambda))$

According to the transformation theory, the relationship of \mathbf{u} and \mathbf{V} can be found as: $\mathbf{V} = \mathbf{T} * \mathbf{u} * \text{sqrt}(\lambda)$. Simulations were performed for different sample numbers and the resulted failure probabilities with corresponding error rates are listed in Table 3-6.

Table 3-6 Simulation results for correlated RVs with different sample numbers

Sample number	100,000	150,000	1,000,000	1,500,000
Failure probability	2.4738e-003	3.1641e-003	2.7516e-003	2.774e-003
Error rate %	12.466	8.9824	3.6955	3.0089

From the results listed herein, the error rates of failure probability were found to be in acceptable range. To further explore the tendency of the relationship of accuracy and sample number, the failure probability value versus sample number was plotted as in Figure 3-5.

From Figure 3-5, it is shown that the failure probability seem to be stable when the sample number is close to 1 million. Thus $n=1,000,000$ was used as the sample number for simulations performed afterwards. The statistics of \mathbf{V} obtained were checked for the cases of sample number equal to 1000,000 and 1,500,000. The simulation results are illustrated in Figure 3-6 and Figure 3-7 respectively.

Numerical results corresponding to sampling number 1,000,000 and 1,500,000 are shown in Table 3-7 and Table 3-8. The objective of this work was to transform nonnormal variates to standard normal variables. Thus it was desirable to obtain the variables with 0 mean and standard deviation value of 1. In this work, the statistics for those resulting variables and the error rates compared with the expected values are also listed in the tables.

Table 3-7 The statistics for variates V obtained from sampling number 1,000,000

Random variable	Mean μ	Error rate μ	Standard deviation σ	Error rate σ
V1	-0.00202285702314585	0.202%	1.18252789102296	18.253%
V2	-0.000202354200453275	0.020%	0.772916176332544	22.708%

V3	0.000172029172750838	0.017%	1.00063738425652	0.064%
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Table 3-8 The statistics for variates V obtained from sampling number 1,500,000

Random variable	Mean μ	Error rate μ	Standard deviation σ	Error rate σ
V1	-0.00185104373650243	0.185%	1.18279061817378	18.279%
V2	0.000401898540332085	0.040%	0.773488902291682	22.651%
V3	0.000453338588353564	0.045%	1.00062003589904	0.062%

Transformation utilizing T: $\mathbf{u} \sim N(0,1) \rightarrow \mathbf{V} (\mathbf{V}=\mathbf{T}*\mathbf{u})$.

From the listed results in Table 3-7 and Table 3-8, it is shown that resulting mean values of the random variables are much closer to those of standard normal variables. However, compared with the standard deviations, which is 1 for standard normal variables, not all the variances are in the desired range even for the simulation results with a large enough sampling number. Therefore, without implementing eigenvalue matrix λ , The simulation results are plotted in Figure 3-8 and the statistics are listed in Table 3-9.

Table 3-9 The statistics for variates V obtained from sampling number 1,000,000

Random variable	Mean μ	Error rate μ	Standard deviation σ	Error rate σ
V1	-0.00170962621981911	0.171%	0.999418478433203	0.058%
V2	-0.000261238149463504	0.026%	0.997830492983435	0.216%
V3	0.000172029172750838	0.017%	1.00063738425652	0.064%

It is obviously shown that statistics of the resulting random variables are much closer to that of the desired standard random variables. Therefore, the current transformation method is the one which matches the sampling method used in 'PROBES' and the implantation of this variable transformation has been contributed as one of the upgraded features of 'PROBES'.

3.2.3 Statistical analysis for the performance function

This part of the research work was performed as part of the work request from Pratt & Whitney who was one of the sponsors for the demonstrated research projects. The objective of the work was to obtain the percentile values of the performance function. In this work, different percentiles and statistics for performance function were evaluated. PDF and CDF profiles of the performance function were investigated. Since the failure probability at a given percentile level is the percentile itself, the information of failure structural reliability can be easily evaluated from its percentile distribution. According to the theory of reliability analysis, the limit state is defined as the difference between the system performance function $f(X, Y)$ and the design criteria Z_0 :

$$g(\mathbf{X}, \mathbf{Y}) = f(\mathbf{X}, \mathbf{Y}) - Z_0 \quad (3.21)$$

where \mathbf{X} is a vector of random design variables and \mathbf{Y} is a vector of other deterministic design parameters. The system fails when the value resulted from performance function is less than the certain design criteria Z_0 . The performance function is defined as:

$$Z = f(\mathbf{X}, \mathbf{Y}) \quad (3.22)$$

One of the performance functions that was of interest to Pratt and Whitney was as defined in the following equation:

$$Z = \sqrt{0.006 + (-0.07X)/(1 + (-0.6X)) + 0.03X^2} \quad (3.23)$$

The resulting statistics are listed as follows:

Summary of Percentiles:

- 1% value of $Z = 0$
- 5% value of $Z = 0$
- 10% value of $Z = 0$
- 25% value of $Z = 0.10659$
- 50% value of $Z = 0.25786$
- 75% value of $Z = 0.34935$

90% value of $Z = 0.51953$
95% value of $Z = 0.70942$
99% value of $Z = 1.4252$

Summary of Statistics:

$Z_{\text{mean}} = 0.28318$
 $Z_{\text{std}} = 0.36479$
 $Z_{\text{skew}} = 11.9573$
 $Z_{\text{kurt}} = 311.0482$

Where mean, standard deviation, skewness represent the 1st, 2nd and 3rd moment for a random series respectively. Kurtosis is the moment is the performance function distribution.

PDF and CDF Plots for $z = f(X,Y)$

The histogram plot is listed as in Figure 3-9 and the CDF plot is illustrated as in Figure 3-10. Those plots can be used to roughly evaluate failure probability given the performance function.

3.3 The effects of uncertainties in structural reliability analysis

Practically, the basic random variables such as temperature and pressure to evaluate failure probability are the measurements from sensors. Uncertainties introduced by the lack of knowledge and other human induced factors, along with the inherent modeling uncertainty affect the reliability assessment and the future decision making. There is a significant amount of literatures on different uncertainty categories and their modeling. One category of uncertainty is to separate uncertainties into two classes: aleatory and epistemic (Parry, 1996).

Aleatory Uncertainty

Aleatory Uncertainty is the uncertainty to model the events or phenomena occurring randomly. This uncertainty deals with randomness of events. A typical example includes randomness associated with natural or manufacturing induced variation with the material properties (Veley et. al 2001).

Epistemic Uncertainty

This class of uncertainty is induced due to the lack of system knowledge and inexact prediction of the simulation models. Epistemic uncertainty is also called ‘subjective uncertainty’. It reflects how well the assessment and estimation of the system model is. Updating the state of knowledge with real time data and a better parameter identification of the model can reduce this category of uncertainty.

As illustrated in Figure 3-11, an objective of this work was to address both of these uncertainties in the structural reliability analysis. In this framework, uncertainties for different sources are modeled as random variables. By integrating uncertainties into a particular program, the uncertainty can be propagated to the final output. An uncertainty model and how to update that using Bayesian approach for system input are discussed in the following section.

3.3.1 The Uncertainty Analysis Using Bayesian Approach

As discussed in the previous chapter, Bayesian analysis is a powerful approach to continuously optimize the posterior probability density function by adapting the predefined priori pdf based on new system measurements. Given that θ is a finite set of parameters and X is a measurement from the physical system, Bayesian expression is recalled as:

$$p(\theta | X) \propto p(X | \theta) \times p(\theta) \quad (3.24)$$

Same notation as before, $p(\theta)$ is the ‘prior probability’ and represents available knowledge of the system; $p(X | \theta)$ is the ‘likelihood function’, reflecting the ‘likelihood’ that the observed events could indeed take place. In a reliability analysis model, the inputs are modeled using random variables according to the available knowledge or sensor measurements. There is a recognized need to simulate the uncertainties presented in those

input parameters and to update them according to the new measurements. Bayesian analysis allows making inference from the model to incorporate both aleatory and epistemic uncertainties in the available information.

Let x , which could be a single or vector valued random variable, denote the inputs of a limit state function. Considering x as the measurements from sensors, it can be modeled using a Normal distribution: $x \sim N(\mu, \sigma^2)$. Herein, in this reliability analysis case, $[\mu, \sigma]$ is a set of mean and standard deviation corresponding to the finite set θ , which is defined in the general Bayesian analysis form. In general, the statistical average of sensor measurement x can be assigned to its mean value and the error information from the manufacturer can represent the associated uncertainty. It is manifested that this probabilistic model describes the randomness of the input data. To combine the epistemic uncertainty (in another word, model uncertainty) into the reliability analysis, parameters μ and σ are modeled as random variables with joint probability density function $p(\mu, \sigma)$. According to Bayesian analysis, by incorporating new sensor measurements into the Bayesian framework, the statistical information of μ and σ can be estimated and the pdf profile of x can be updated based on the new information of μ and σ . The procedure for updating the interested information is demonstrated in the following part.

Given the experimental measurements $\{x_k\}_{k=1:N}$, the joint posterior pdf of parameters μ and σ which are updated using the k^{th} measurements is expressed in the following equation:

$$p(\mu, \sigma | \{x_k\}) \propto p(\{x_k\} | \mu, \sigma) \times p(\mu, \sigma) \quad (3.25)$$

Where $p(\{x_k\} | \mu, \sigma)$ and $p(\mu, \sigma)$ are likelihood function and prior probabilistic distribution function respectively. For N sets of independent and identically distributed measurements, the likelihood function can be obtained as in equation (3.26) (Sivia, 1996):

$$p(\{x_k\} | \mu, \sigma) = (\sqrt{2\pi}\sigma)^{-N} \exp\left(-\frac{\sum_{k=1}^N (x_k - \mu)^2}{2\sigma^2}\right) \quad (3.26)$$

For a generic framework of Bayesian inference, limited information about prior pdf is available, therefore it is common to assume flat priors:

$$p(\mu, \sigma) = \begin{cases} \text{constant} & \sigma > 0 \\ 0 & \text{otherwise} \end{cases} \quad (3.27)$$

By multiplying the prior pdf, Equation (3.27), and the likelihood function as in Equation (3.25), the posterior pdf of μ and σ can be evaluated. By following the procedure of parameter identification as in the previous section, the updated estimation of μ and σ can be calculated and the updated uncertainty model can be obtained. In the Figure 3-12, a pdf profile of sensor measurements x (red line) and its updated pdf profile (blue line) are illustrated.

To quantify the uncertainty change, a box plot of the input uncertainty is plotted in Figure 3-13. In the figure, lines of boxes with 5%, median and 95% quartile are plotted respectively. From these two figures, it is shown that for this case, the uncertainty updated using additional data has been reduced.

3.4 Uncertainty results for failure probability

As stated in the previous section, one of the most desired results of reliability analysis is the failure probability. Different techniques to evaluate failure probability have been introduced. Among them, Monte Carlo simulation method is found to be more suitable for statistical reliability analysis. In this part of the work, MCS is utilized to propagate uncertainties and to obtain the failure probability. For structures in practice, due to aging problem and changes in

the environment, the current state of failure probability is needed. It is promising to develop an algorithm based on Bayesian analysis framework and it could be valuable to evaluate desired posterior pdf based on existing pdf of failure probability.

In current section, based on previously discussed input updating schemes, failure probability as the output of reliability analysis is assessed using Monte Carlo method. In general, the resulting failure probability is a scalar. However, an informative prior knowledge is usually desired for future applications of updating algorithm. To obtain the current state of information as the priori of next stage study, a statistical analysis of failure probability is introduced in the following with demonstration of numerical examples.

Linear Limit State Function: In this example, a linear limit state function of $g(\mathbf{X}) = x_1 - x_2$ is assumed. In which x_1 and x_2 are random variables defined by Normal distribution $N(10,0.1)$ and $N(9.8, 0.098)$ respectively. The updated statistics are listed as in Table 3-10:

Table 3-10 Updated probabilistic information of inputs

μ_{x1}	σ_{x1}	μ_{x2}	σ_{x2}
10.013	0.0804	9.8017	0.0937

Applying Monte Carlo simulation, Figure 3-14 is the resulting pdf profile of the failure probability with mean 0.0764. The associated uncertainty is also illustrated in the figure.

There are different ways to assign probability distribution to a random variable. From a statistical view, one way is to examine a plot of cumulative distribution function (cdf) of a certain sample of data. By overlaying a theoretical cdf on the corresponding cdf plot of sampled data, one can determine if this set of data follows the particular distribution or not.

Figure 3-15 is a plot of test case to compare cdf of empirical data with that of a Normal distribution. From the plot it is reasonable to make a hypothesis that the resulting failure probability follows a normal distribution. However, to make a more accurate decision, it is

wise to perform a hypothesis test. In table 3-11, a result of hypothesis test is listed. In which the test result of H=0 means the hypothesis is accepted with 95 percentage of confidence. The p value is the probability of observing the given sample result under the assumption that the null hypothesis is true. Cv is the cut-off value. The hypothesis of Normal distribution is acceptable for this case.

Table 3-11 Hypothesis test for linear limit state function case

Hypothesis	H	p value	Confidence	Cv
Normal	0	0.5354	95%	0.1608

Nonlinear Limit State Function: To check the feasibility of this statistical test program for different applications, a nonlinear example is demonstrated with limit state function expressed as follows:

$$g(X) = 0.00587 \times x_1 \times x_2^{0.543} \times 100^{x_3} \quad (3.28)$$

Where x1, x2 and x3 follow Lognormal, Normal and Chi-Square distribution respectively. The resulting cdf comparison of failure probability is shown in Figure 3-16. In this figure, the theoretical distribution used for comparison is lognormal distribution. From the comparison, a null hypothesis of P_f follows a lognormal distribution.

The hypothesis test results are listed in Table 3-12. For this case, the lognormal distribution is accepted as the theoretical distribution of failure probability.

Table 3-12 Hypothesis test for nonlinear case

Hypothesis	H	p value	Confidence	Cv
Lognormal	0	0.0591	95%	0.4889

From the numerical cases discussed above, it is shown that by having the statistical test, probabilistic information of the current state of failure probability can be defined. This

statistical information can be used for both decision-making and future time dependent updating of P_f .

3.5 Conclusions

With the random variable transformation algorithms implemented in the reliability analysis software package 'PROBES', the program was significantly expanded and an additional capability was incorporated so that PROBES can solve problems with the input random variables that are no longer limited to uncorrelated and normal distributed. Uncertainties of the input data can be modeled by normal distribution random variables. Where mean μ and variance σ can be modeled statistically and updated using Bayesian analysis based on the data measurements. With this method, both randomness and the modeling uncertainty are considered. With implementation of Monte Carlo simulation method, the probability of failure P_f can be obtained. Through statistical analysis, the probability distribution function (pdf) profile and the associated mean and variance can be calculated. According to the pdf profile, one can assign a distribution to the analyzed data. By performing the goodness of fit test, a confidence level of accepting this distribution can be specified to give a quantitative sense of making decision.

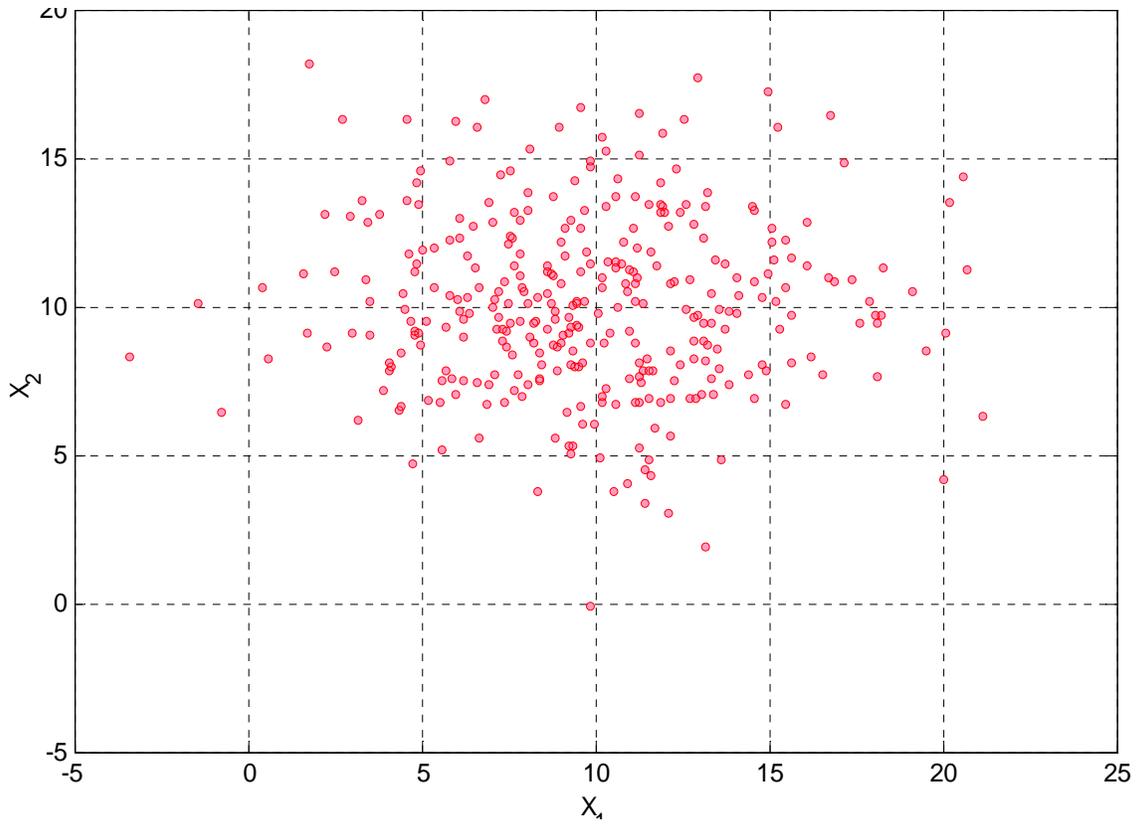


Figure 3-1 Monte Carlo sampling scheme

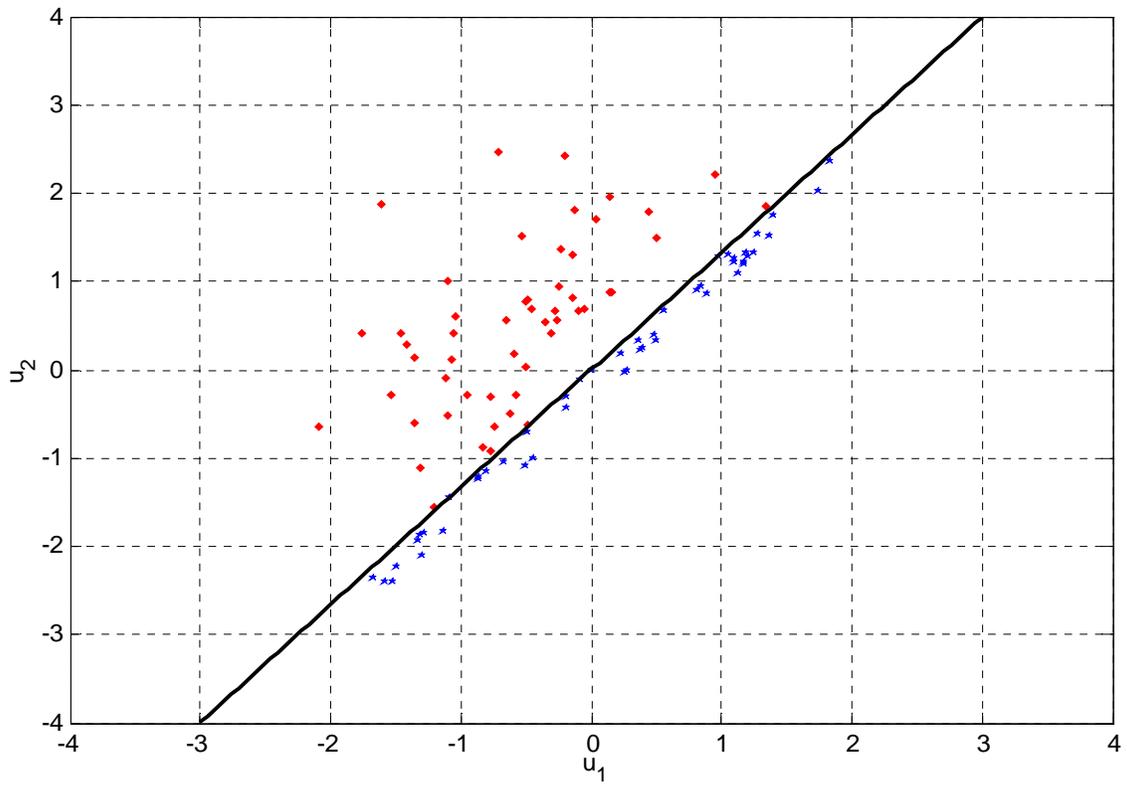


Figure 3-2 Importance sampling scheme

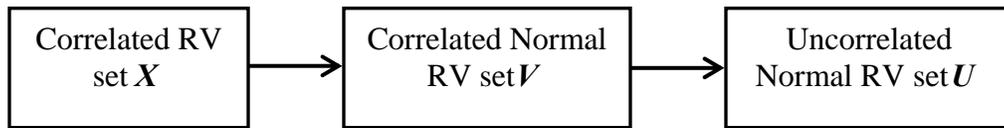


Figure 3-3 Random variable transformation diagram

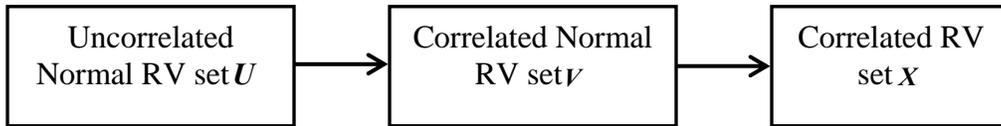


Figure 3-4 Random variable inverse transformation diagram

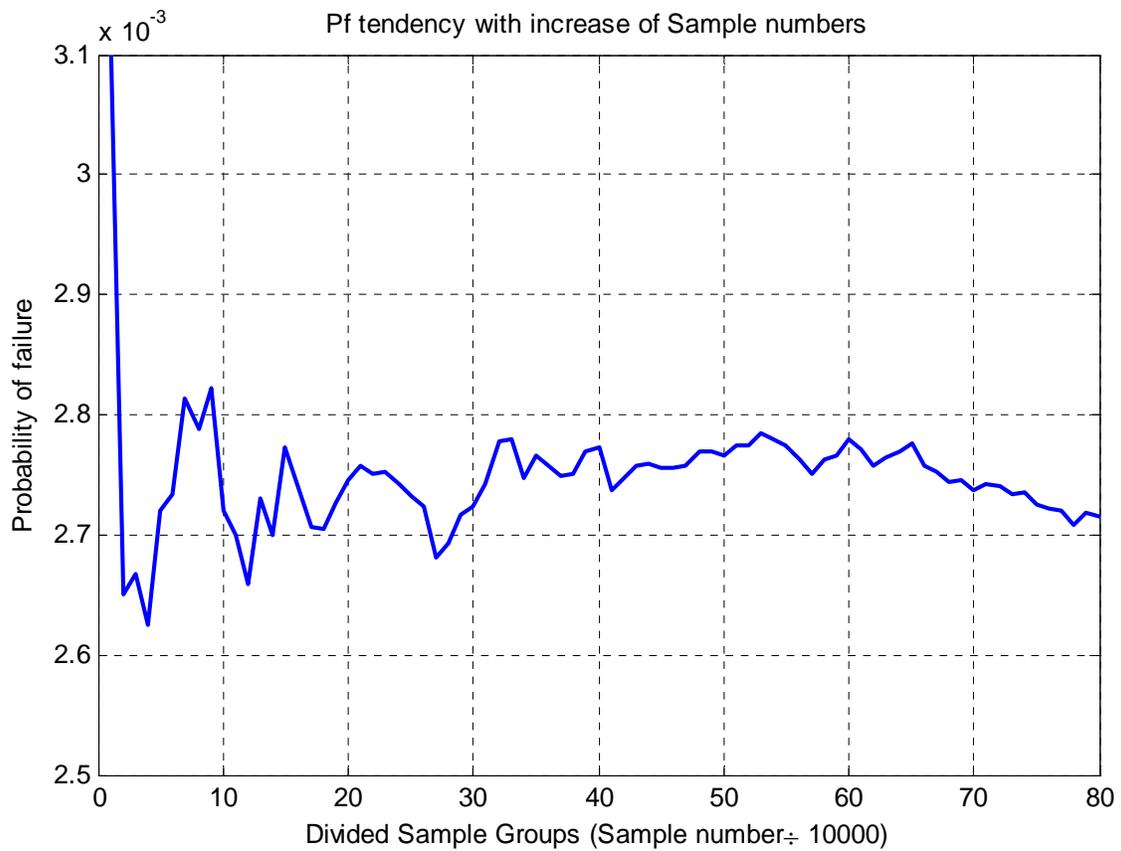


Figure 3-5 P_f tendency for sampling method $V=T*u*\sqrt{\lambda}$

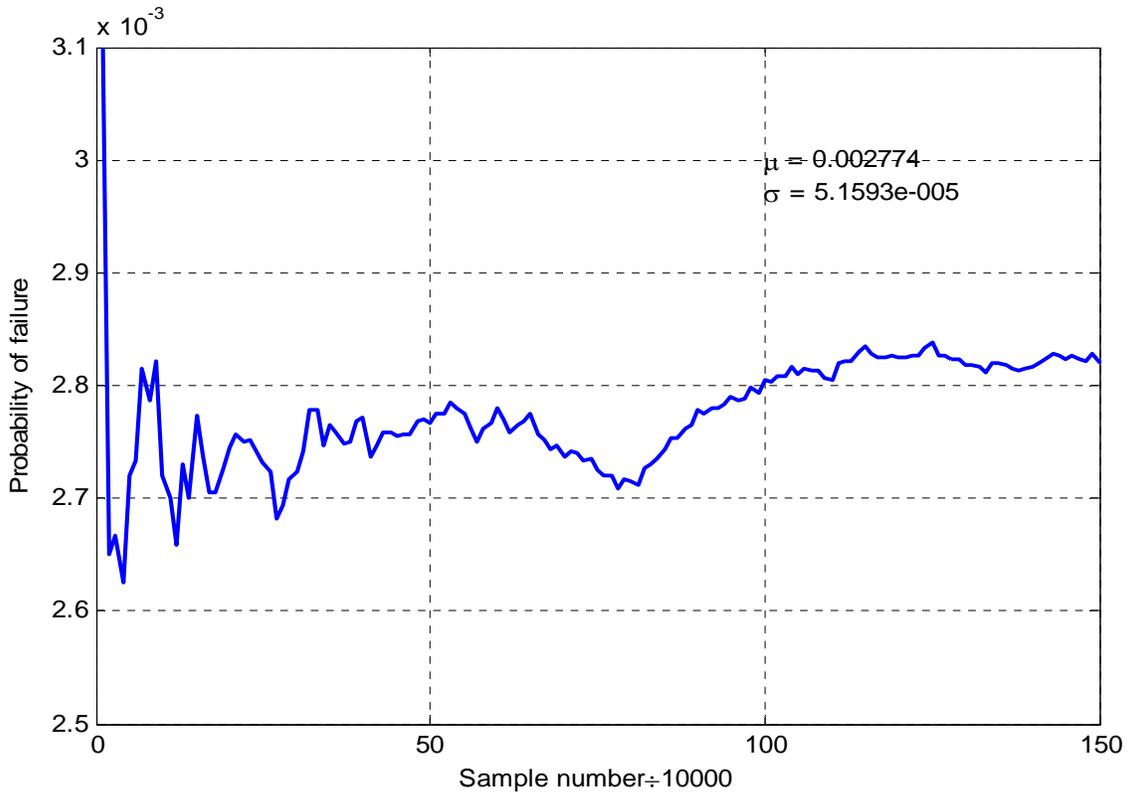


Figure 3-6 Sampling method: $V=T*u*\sqrt{\lambda}$ Sampling number 1,000,000

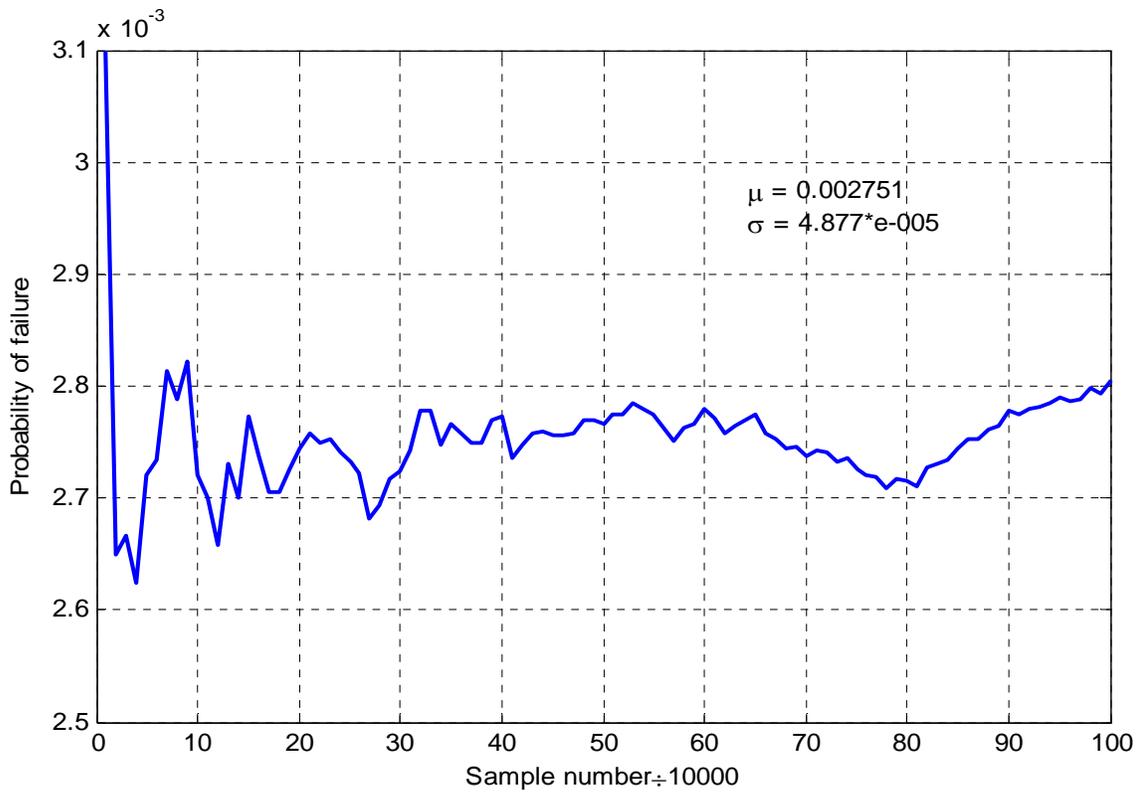


Figure 3-7 Sampling method: $V=T*u*\sqrt{\lambda}$ Sampling number 1,500,000

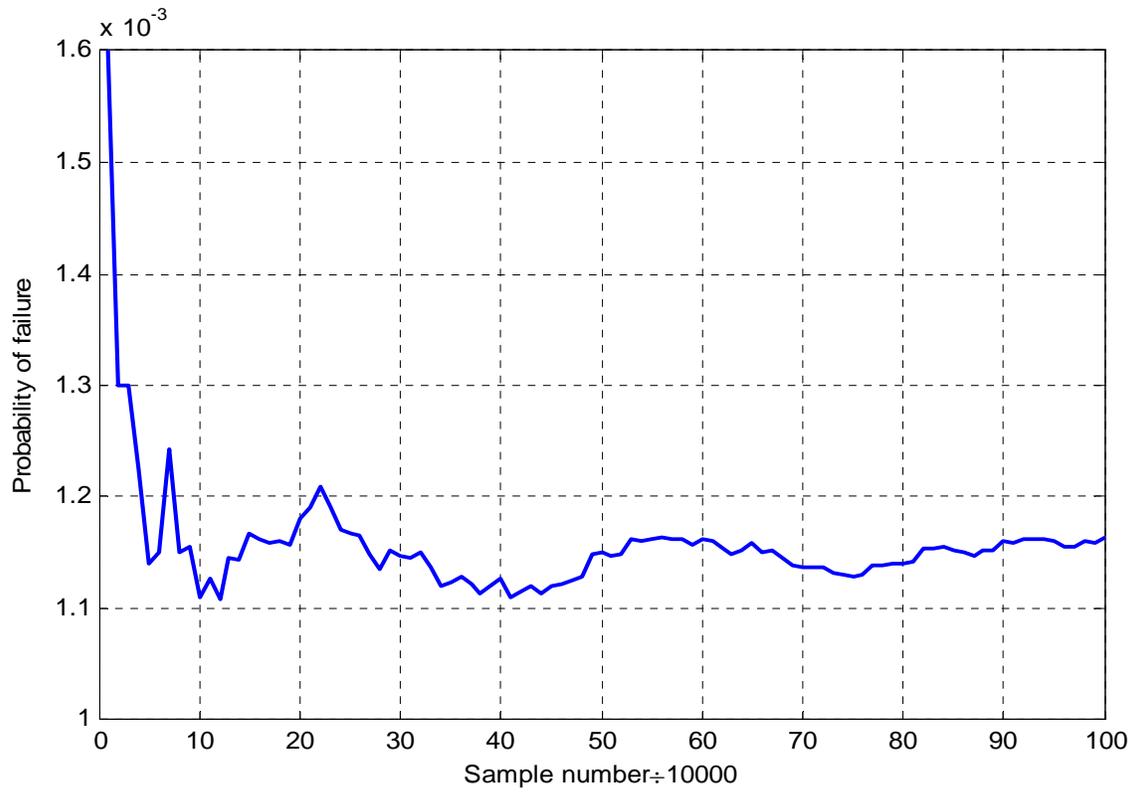


Figure 3-8 Sampling method: $V=T*u$; Sampling number 1,000,000

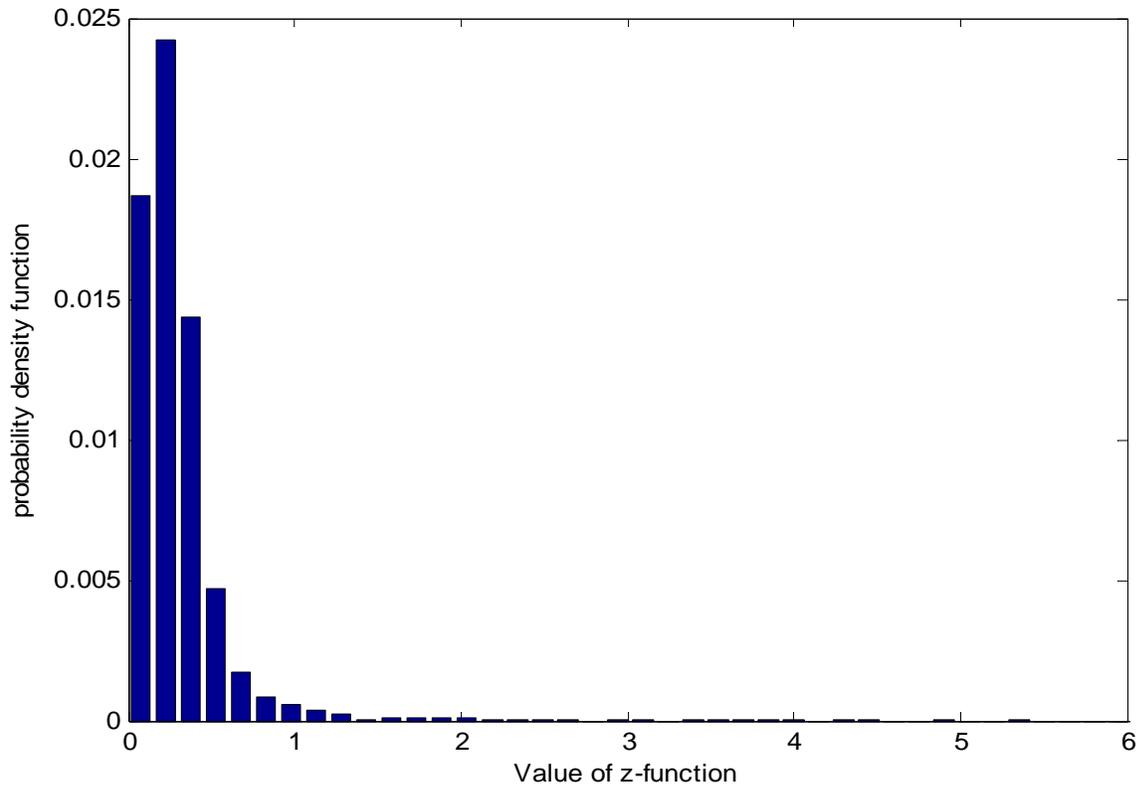


Figure 3-9 Pdf for performance z function

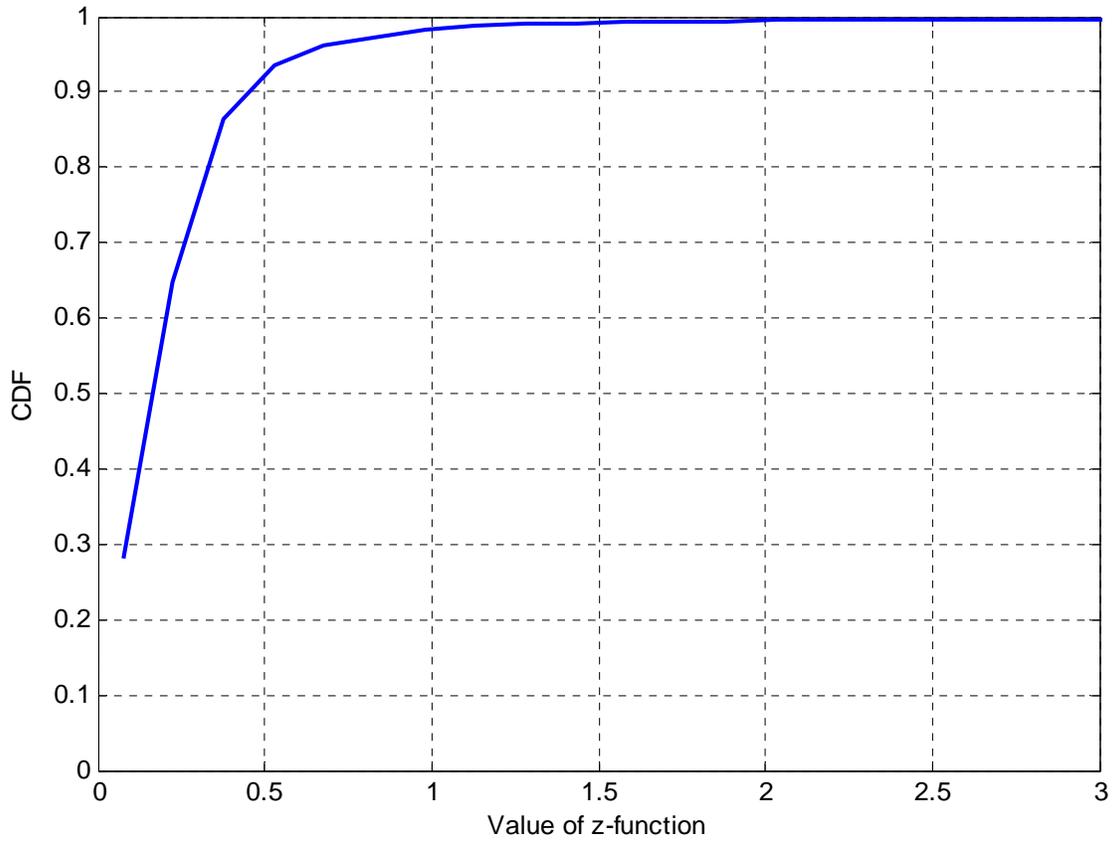


Figure 3-10 CDF for the performance z function

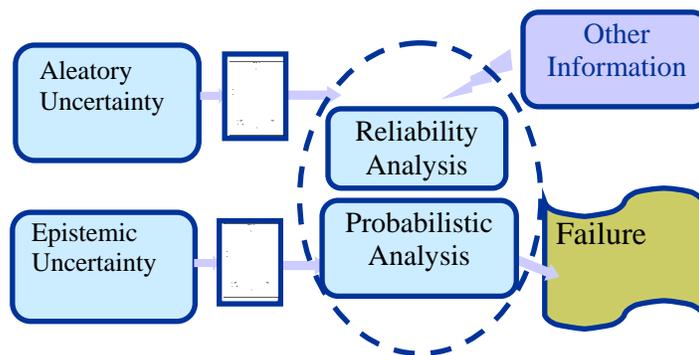


Figure 3-11 A framework of uncertainty model

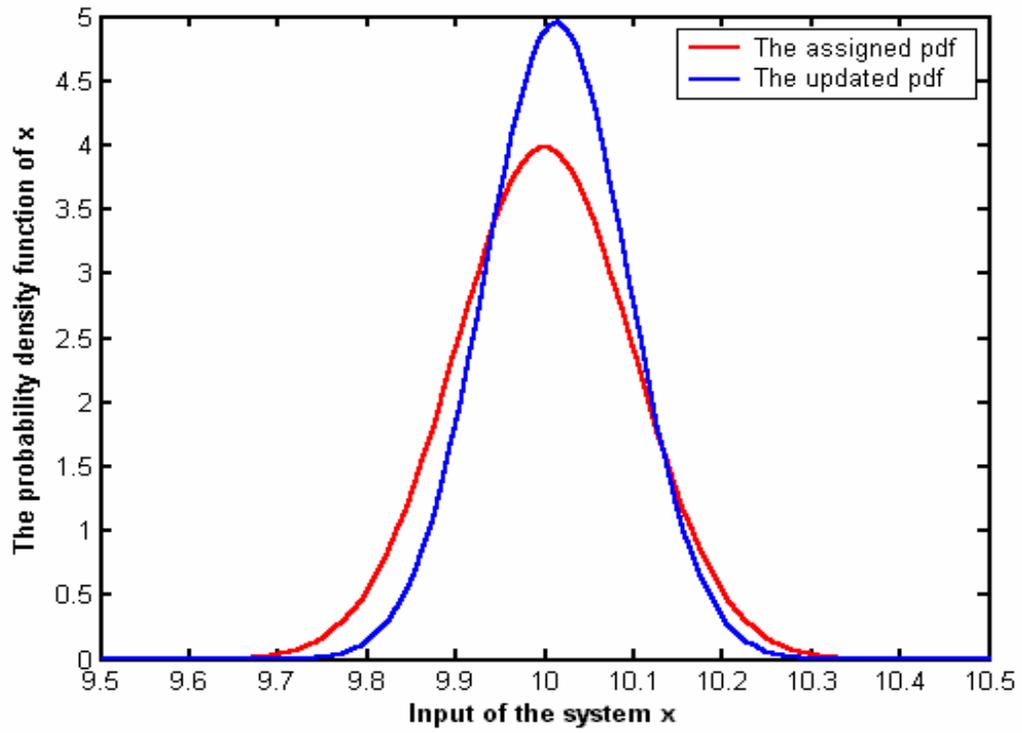


Figure 3-12 The input uncertainty updating

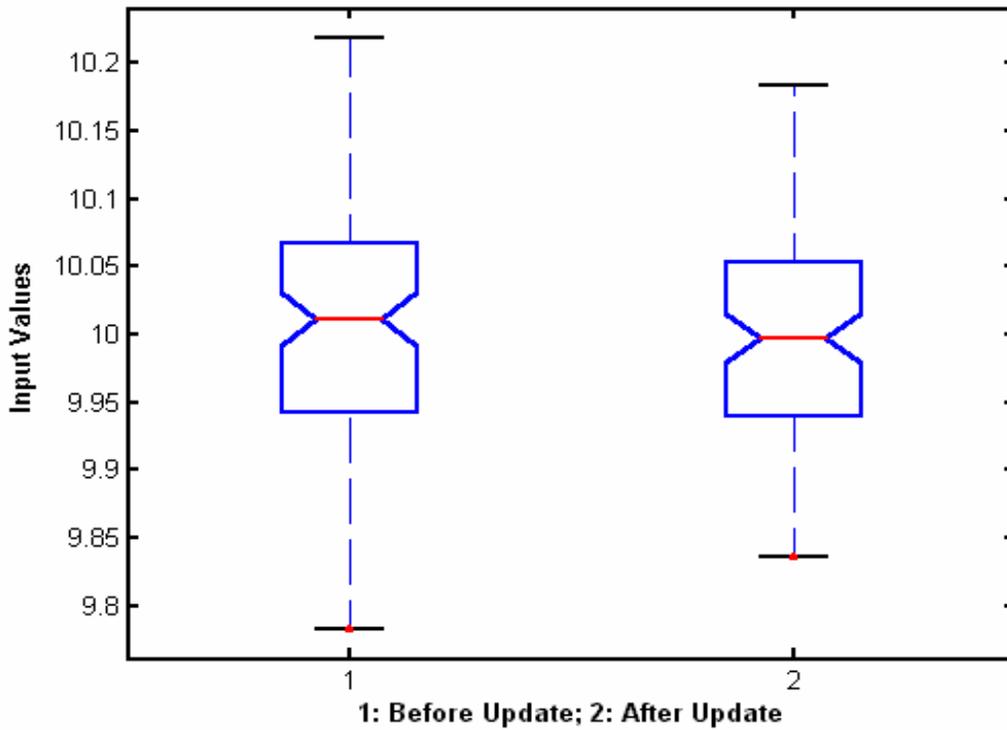


Figure 3-13 Box plot of the input uncertainty

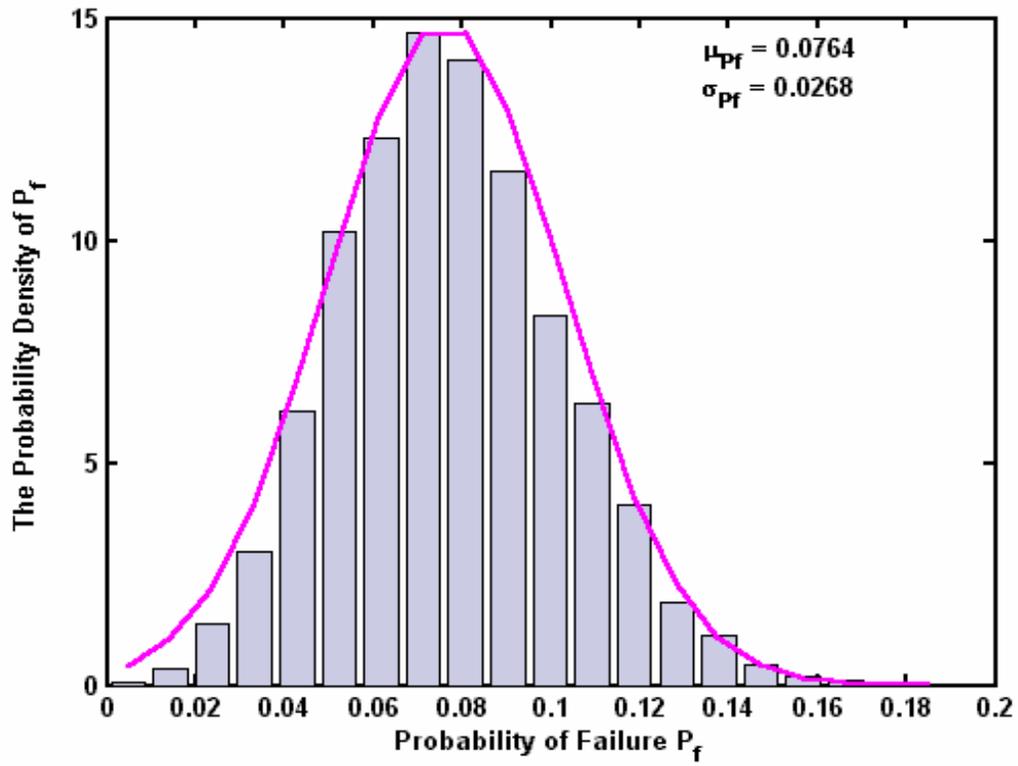


Figure 3-14 The pdf profile of failure probability

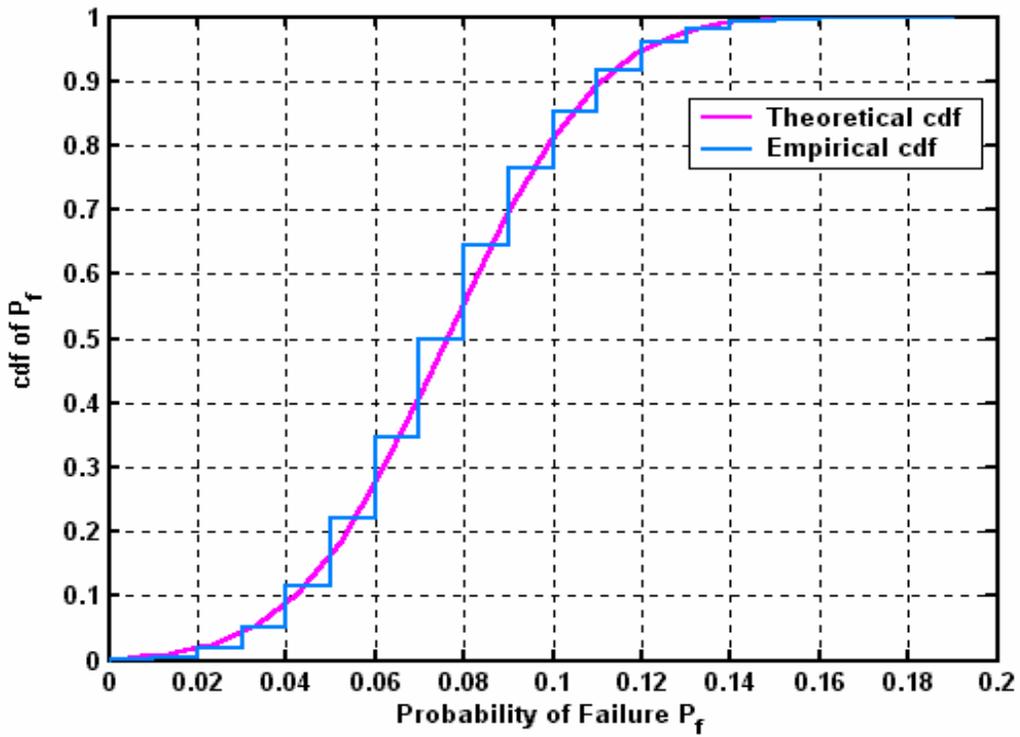


Figure 3-15 Comparison of the cumulative distribution

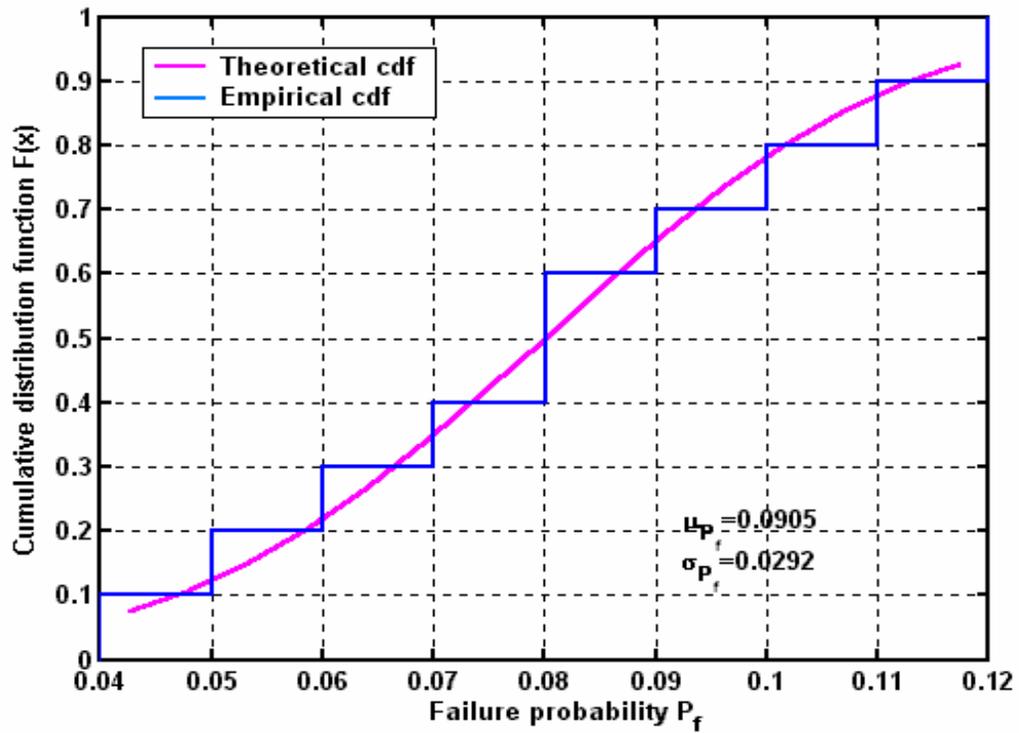


Figure 3-16 cdf for nonlinear limit state function case

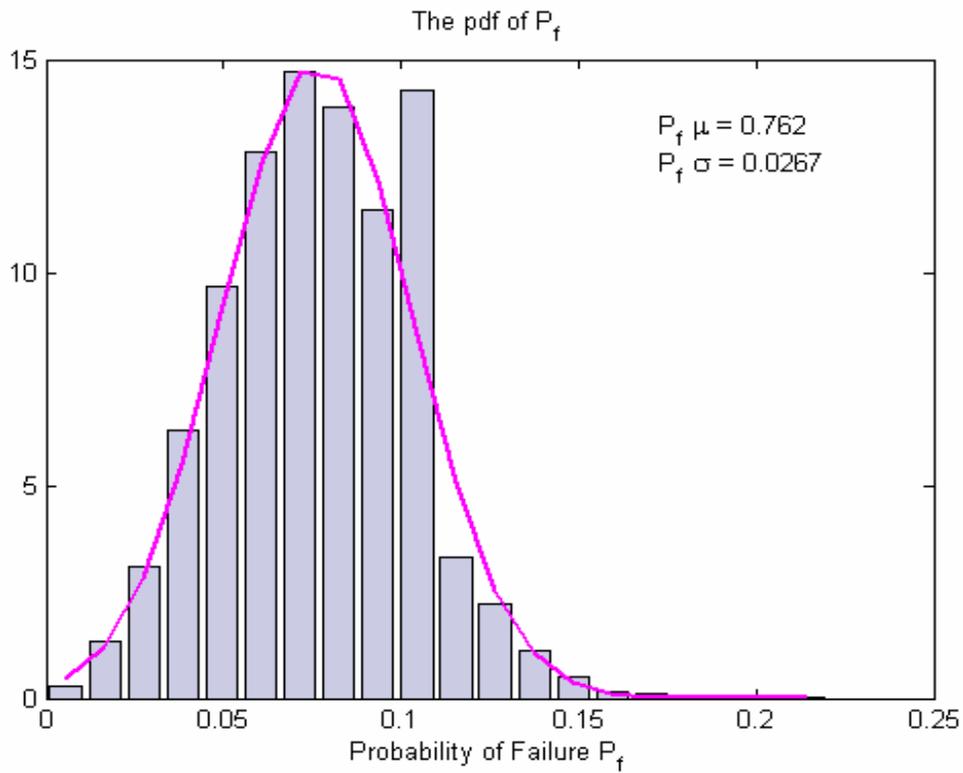


Figure 3-17 Pdf profile for failure probability

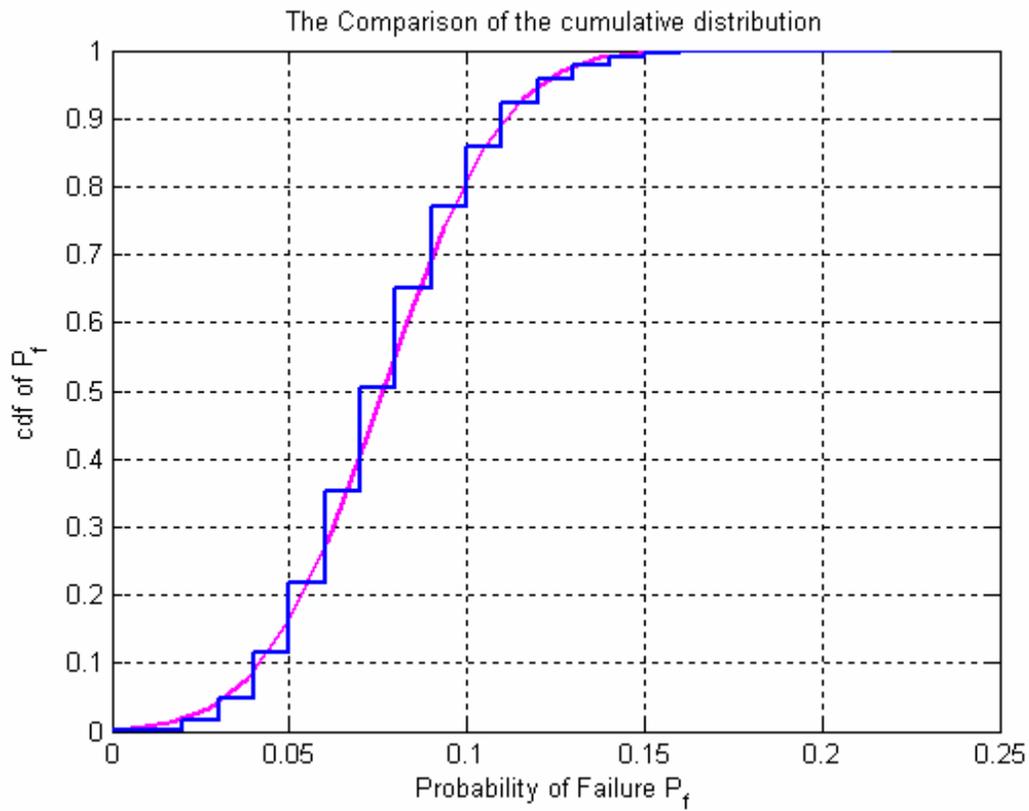


Figure 3-18 CDF comparison

4 Support Vector Machine Based Structural Reliability Analysis

Support Vector Machine (SVM) is a promising machine learning algorithm for data classification and regression. For the classification problem, the major feature of SVM is its capability of minimizing the training error while simultaneously maximizing the margin between two classes. This leads to the unique characteristic of its ability of generalization from the small data sets. In reliability analysis, on the basis of the simulated data sets, a hyperplane to classify the safety region and the failure region can be found by using SVM. Other testing data can be classified according to this classifier. The advantage of this approach is to classify the new data points without going through the calculation of the limit state function. In this section of the research work, the support vector machine algorithm was implemented to classify the safe region and the failure region. The failure probability for a set of test data was found based on the classifier. The reliability updating can be performed based on the learning machine function by updating the hyperplane with only a small set of new measurements.

4.1 Support Vector Machine

Support Vector Machine (SVM) is a learning algorithm for classification and regression. The technical foundation of Support Vector Machine was developed by Vapnik, Chervonenkis and their co-worker's back in 1960s. Over the last decade or so, SVM has become a promising algorithm and gained popularity. Unlike the traditional statistical inference, SVMs are so called 'nonparametric' models, which means the parameters involved in SVM are not predefined and their numbers depend on the training data used. It is a kind of

model designed to keep the value of training error fixed and minimize the confidence interval. To introduce the main concept of Support Vector Machine, D_0 is introduced to denote a set of training data with which \mathbf{x}_i stands for system input vector and y_{di} stands for the desired output. l is the training set size. D_0 is expressed as:

$$D_0 = \{\mathbf{x}_i, y_{di}\}_{i=1}^l \quad (4.1)$$

The training process of a learning machine is to find the relationship between input space x and output space y . In another words, a target function can be found for the possible mapping between input and output by applying Support Vector Machine: $f : (\mathbf{x}_i \rightarrow y_i)$. To illustrate the relationship more clearly, a brief mechanism of learning machine is summarized in Figure 4.1.

It is shown that during the training phase, a function $f(\mathbf{x}, \mathbf{w})$ to map the data could be found. This function is known as ‘hypothesis function’ to define the dependency between the input and output data. This dependency between two sets of data is defined by the weight vector \mathbf{w} . By adjusting the weight vector, the mapping relationship of the training data is found based on the certain error criteria. Herein in this work, the cost function used for the support vector machine is defined as:

$$R = \sum_{i=1}^l L_\varepsilon + \Omega(l, h) \quad (4.2)$$

For a given set of training data D_0 , the loss function L_ε is defined by the empirical measurement:

$$L_\varepsilon = R_{emp}(\mathbf{w}) = \frac{1}{2l} \sum_{i=1}^l |y_i - f(\mathbf{x}, \mathbf{w})| \quad (4.3)$$

The empirical risk R_{emp} is the average percentage of the misclassified fraction of the training data. Vapnik proved that for the classification learning problem, the bound in the following equation holds with a probability of at least $1-\eta$:

$$R(\mathbf{w}) \leq \Omega\left(\frac{h}{l}, \frac{\ln(\eta)}{l}\right) + R_{emp}(\mathbf{w}) \quad (4.4)$$

Where h is a non-negative integer called VC (Vapnik-Chervonenkis) dimension of a set of functions. The Ω term is the VC confidence which is defined by equation (4.5):

$$\Omega\left(\frac{h}{l}, \frac{\ln(\eta)}{l}\right) = \sqrt{\frac{h(\ln(2l/h)+1) - \ln(\eta/4)}{l}} \quad (4.5)$$

The VC dimension h is controlled by the number of parameters in the learning machine. With the increase of the VC dimension, the confidence interval increases (Kecman, 2005). The basic idea for Support Vector Machine is to choose an optimal model from an infinite number of functions. The free parameters of the model are given by a number of weights which can describe the set of training data.

4.2 Classification Using Support Vector Machine

For classification problems, SVM operates by finding a hyperplane in the space of possible inputs. The optimal hyperplane is chosen from a class of candidates. The nearest data points from both two classes have the largest distance to the optimal hyperplane. With this property, a new data points is desired to be classified accurately since the separation between the two classes is greater. The classification capability of Support Vector Machine is illustrated in Figure 4.2.

The hyperplane can be defined as:

$$f(\mathbf{x}, \mathbf{w}) = \mathbf{w}^T \mathbf{x} + b = \sum_{i=1}^n w_i x_i + b \quad (4.6)$$

$$\mathbf{w} \in R^n, b \in R$$

Where the weight vector \mathbf{w} defines the direction perpendicular to the hyperplane and scalar b defines the distance from origin to the hyperplane. A canonical hyperplane is generally considered with the following condition:

$$\min_{x_i \in X} |\mathbf{w}^T \mathbf{x}_i + b| = 1 \quad (4.7)$$

The separation function then satisfies the condition as in the equation (4.8):

$$y_i [\mathbf{w}^T \mathbf{x}_i + b] \geq 1 \quad (4.8)$$

$$i = 1, 2, \dots, l$$

$$d_i = \frac{\mathbf{w}^T \mathbf{x}_i + b}{\|\mathbf{w}\|} \quad (4.9)$$

The distance of a data point to hyperplane is expressed as in equation (4.9). Substitute this expression into equation (4.8), the formula turns out to be:

$$y_i d_i \geq \frac{1}{\|\mathbf{w}\|}, \quad i = 1, 2, \dots, l \quad (4.10)$$

Since the distance, which is also known as ‘margin’, between the closest points of two classes is with the form of $2/\|\mathbf{w}\|$, in order to obtain the largest distance, this distance has to be minimized subject to the condition of the equation (4.8). Lagrange multipliers are introduced to this minimization problem, and it is defined as:

$$L = \frac{1}{2} \mathbf{w}^T \mathbf{w} - \sum_{i=1}^l \alpha_i [y_i (\mathbf{w}^T \mathbf{x}_i + b) - 1] \quad (4.11)$$

Where α_i are non-negative Lagrange multipliers. The optimum results are obtained by the minimization of L with respect to w and b:

$$\frac{\partial L}{\partial b} = 0 \Rightarrow \sum_{i=1}^l \alpha_i y_i = 0 \quad (4.12)$$

$$\frac{\partial L}{\partial \mathbf{w}} = 0 \Rightarrow \mathbf{w} = \sum_{i=1}^l \alpha_i y_i \mathbf{x}_i \quad (4.13)$$

Substitute equation (4.12) and (4.13) into equation (4.11), the dual variable Lagrangian is obtained as:

$$L_D = \sum_{i=1}^l \alpha_i - \frac{1}{2} \sum_{i,j=1}^l y_i y_j \alpha_i \alpha_j \mathbf{x}_i^T \mathbf{x}_j \quad (4.14)$$

In order to find the optimal hyperplane, L_D has to be maximized with respect to Lagrange multipliers α_i and subjects to the following constraints:

$$\sum_{i=1}^l \alpha_i y_i = 0; \quad \alpha_i \geq 0 \quad (4.15)$$

Only those training data points which have non-zero α_i are called support vectors. Generally, support vectors of linear separation problem are located on the margin. Once the hyperplane is found, the given test points can be assigned to the proper class according to equation (4.8). However, in most cases, linear classifier is very limited. There is a need to transform input data into a high dimensional feature space. One general idea of dealing with nonlinear

separation problem is to choose a proper kernel function $K(\mathbf{X}_i, \mathbf{X}_j)$. By data transformation utilizing the kernel function, SVM can be operated in the higher dimensional feature space. Some widely used admissible kernel functions are listed as in following table:

Table 4-1 A list of Kernel functions

Kernel Type	Mathematical Form
Linear Function	$K(\mathbf{X}_i, \mathbf{X}_j) = \mathbf{X}_i^T \mathbf{X}_j$
Polynomial Function	$K(\mathbf{X}_i, \mathbf{X}_j) = [(\mathbf{X}_i^T \mathbf{X}_j) + 1]^d$
Cubic Function	$K(\mathbf{X}_i, \mathbf{X}_j) = \ \mathbf{X}_i - \mathbf{X}_j\ ^3$
Radial Basis Function (RBF)	$K(\mathbf{X}_i, \mathbf{X}_j) = e^{-\ \mathbf{X}_i - \mathbf{X}_j\ ^2 / 2\sigma^2}$
Distance Function	$K(\mathbf{X}_i, \mathbf{X}_j) = \ \mathbf{X}_i - \mathbf{X}_j\ $
Sigmoid Function	$K(\mathbf{X}_i, \mathbf{X}_j) = \tanh[(\mathbf{X}_i^T \mathbf{X}_j) + b]$
Thin Plate Spline	$K(\mathbf{X}_i, \mathbf{X}_j) = \ \mathbf{X}_i - \mathbf{X}_j\ ^2 \ln \ \mathbf{X}_i - \mathbf{X}_j\ $
Quadratic Function	$K(\mathbf{X}_i, \mathbf{X}_j) = \ \mathbf{X}_i - \mathbf{X}_j\ ^2$
Spline Function	$K(\mathbf{X}_i, \mathbf{X}_j) = 1 + \langle \mathbf{X}_i, \mathbf{X}_j \rangle + \frac{\langle \mathbf{X}_i, \mathbf{X}_j \rangle \min(\ \mathbf{X}_i, \mathbf{X}_j\)}{2} - \frac{(\min(\ \mathbf{X}_i, \mathbf{X}_j\))^3}{6}$

With the properly chosen kernel function, the optimization problem becomes:

$$L_D = \sum_{i=1}^l \alpha_i - \frac{1}{2} \sum_{i,j=1}^l y_i y_j \alpha_i \alpha_j K(\mathbf{X}_i, \mathbf{X}_j) \quad (4.16)$$

And the separating hyperplane can be given by:

$$f = \text{sgn}\left(\sum_{i=1}^l y_i \alpha_i K(\mathbf{X}_i, \mathbf{X}_j) + b\right) \quad (4.17)$$

Following the theory of Support Vector Machine, an advanced approach for reliability analysis based on SVM is proposed in the research work. The classification capability of SVM was utilized since the input data of limit state function can be considered as of two classes: one class of data is those in the safe region and another class of data is those in the failed region. The hyper parameters to define those two different input data classes can be found utilizing a small set of training data. As a result, other input data can be classified according to the trained hyper plane. Similar to Monte Carlo simulation, the failure probability is evaluated by the number count:

$$P_{f_SVM} = \frac{N_{\text{failure region}}}{N} \quad (4.18)$$

where N is the total number of testing data or collected data for practical cases and $N_{\text{failure region}}$ is the number of data which are classified into the failure region. The difference of the SVM based approach from other reliability analysis techniques introduced in the previous chapter is that the limit state function does not have to be available. Another advantage of the introduced approach is its capability of obtaining accurate result while only a small number of training data is available.

4.3 Simulation results

As stated in chapter 3, one of the desired results of reliability analysis is the failure probability. Different techniques to assess structure reliability include Monte Carlo Simulation (MCS), First Order Reliability Method (FORM), Second Order Reliability

Method (SORM) and so on. To verify the feasibility of the SVM based reliability analysis technique, different aspects for the approach were investigated and the resulting failure probability was compared with those obtained using traditional analytic methods including FORM and SORM.

A linear test case with limit state function of $g(X) = x_1 - x_2 + 0.5$ is initially assumed. In which x_1 and x_2 are random variables defined by Uniform distribution $U(0,1)$. According to the definition of failure probability, the simulation output of $g(X) < 0$ is considered as the failure case. Otherwise, it is considered as a safe case. In traditional Monte Carlo reliability assessment, the failure probability is calculated as the number of failed cases divided by the total number of samples. A large number of simulation samples is required for this method even though it is usually used to test results obtained using different techniques given the exact value is not available.

In SVM based reliability analysis approach, a small number of data set is used for training. Serial trials are performed to find out how small the data size could be. The classification error for the first trial with data size $N=10$ is 17.45%. When N is increased to be 50, the error is reduced to 6.05%. With the size increasing to 100, the error is dramatically reduced to 1.15%. The classification plot of data size equal to 10, 50 and 100 are shown in Figure 4-3, Figure 4-4 and Figure 4-5 respectively.

To further verify the effect of data size on the classification accuracy, $N=150, 200, 250, 350, 400$ and 500 are tested respectively. The classification results of these are plotted in Figure 4-6 to Figure 4-11 correspondingly. The classification error rates for different training data sizes are listed in Table 4-2. It is obviously shown that, as expected, with the increase of data size, the error rate tends to decrease. However, it is found that for cases of data size less than

a certain number, there oscillation in the result is encountered. Even when the classification error reaches to a relatively large number as happened for $N=100$, oscillation still happens.

Table 4-2 Data size vs. Error rate

N	100	150	200	250	350	400	500
Error(%)	1.15	0.65	0.65	0.8	0.6	0.55	0.55

To further examine the tendency of error rates and training data size, Figure 4-12 is plotted to illustrate the relationship of resulting failure probability vs. data size. It is shown that with increase of the training number, the value of failure probability tends to be stabilized after $N=100$. For different cases, it is worthwhile to have serial trials to determine an optimal training data size in order to maintain the accuracy of the solutions. In the current particular case, it is found that sample number of 150 is enough to obtain satisfactory results.

To compare the proposed approach for reliability analysis with other well-known analytical methods, the values of probability of failure and corresponding error rates are listed in Table 4-3. From this comparison, it can be stated that the SVM based reliability analysis is an effective technique through which one can obtain relatively accurate results while consume less time than MCS.

Table 4-3 Failure probability obtained using different reliability analysis methods

Reliability Method	Exact	P_{f_SVM}	P_{f_SORM}	P_{f_FORM}
Failure Probability	0.125	0.134	0.141	0.1701
Error rate (%)	N/A	6.71	12.8	36.08

The effects of different kernel functions were investigated in the following part. It was found that the linear kernel is too limited to be practically applied to obtain the failure probability. The classification plots to distinguish safe region and failure region are presented from Figure 4-14 to Figure 4-18. Figure 4-13 shows the training data to obtain the classifier.

From those figures, it is demonstrated that the classification results are dramatic and promising. The failure probabilities obtained by utilizing those different kernel functions are listed in Table 4-4 with the corresponding error rates. It is shown that the results obtained with using cubic function, distance function, thin plate spline and spline function as kernel function are within the acceptable error range. Due to the smoothness and simplicity, cubic function and distance function are considered as the best candidates among different kernel functions listed in Table 4-1.

Table 4-4 Failure probability obtained with different Kernel function – Linear Case

Kernel Type	Cubic	Distance	Thin Plate Spline	Quadratic	Spline
Failure Probability	0.132	0.1315	0.134	0.138	0.1325
Error Rate (%)	5.6	5.2	7.2	10.4	6

To check the feasibility of proposed approach to different cases other than linear one, a nonlinear case is illustrated herein. The original limit state function is defined as:

$$g(X) = 3 - 0.8x_1^2 - x_2 \quad (4.19)$$

The case was originally investigated with Monte Carlo based reliability method and the resulting failure probability is $Pf_{MC} = 6.9731e-002$. $N=5134$ simulation samples were used to obtain an accurate enough value. With SVM based reliability approach, 500 training data as shown in Figure 4-19 was applied to train the classifier.

Table 4-5 Failure probability from different Kernel function – Nonlinear Case

Kernel Type	Cubic	Radio Basis Function	Distance	Thin Plate Spline	Quadratic	Spline
Failure Probability	0.0693	0.0678	0.0668	0.0695	0.122	0.0660
Error Rate (%)	0.6181	2.7692	4.2033	0.3313	74.9581	5.3506

The resulting failure probability obtained with different kernel functions and corresponding error rates are listed in Table 4-5. It is found that expect for the quadratic kernel function, failure probabilities obtained from other kernel function based SVM approach are all in acceptable error range. The classification results are illustrated in Figure 4-20 to Figure 4-25. From the plotted figures, the SVM based reliability method is also proved to be a neat and efficient technique to obtain failure probability.

4.4 Conclusion

A Support Vector Machine (SVM) based approach is introduced to perform structural reliability analysis. A generalized reliability analysis model is introduced to verify the approach. A hyper plane to classify input data into safe region and failed region can be modeled from training data. As many as 2000 input data are tested. Different kernel functions are compared and applied to both linear and nonlinear cases. It is demonstrated that the proposed SVM based approach leads to an accurate and efficient procedure for reliability analysis. In mechanical and structural reliability, the probability of failure is usually very small, which requires very large samples to achieve the needed accuracy by using Monte Carlo simulation method. However, it is shown that much less training data are required for SVM based approach to obtain the accurate enough results. Moreover, in reality, due to the lack of complete knowledge about the structure or the changes in the characteristics of the structure through time, such as upgrades and redesigns, the information to define a reasonable limit state function could be limited and sometimes non-continuous. It is another advantage of SVM based reliability analysis that failure probability can be obtained without computation of the limit state function.

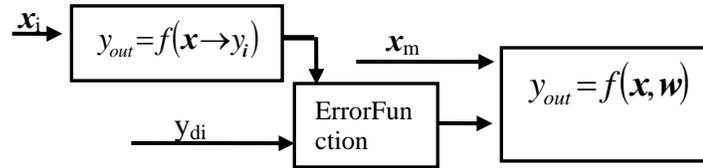


Figure 4-1 A mechanism of learning machine

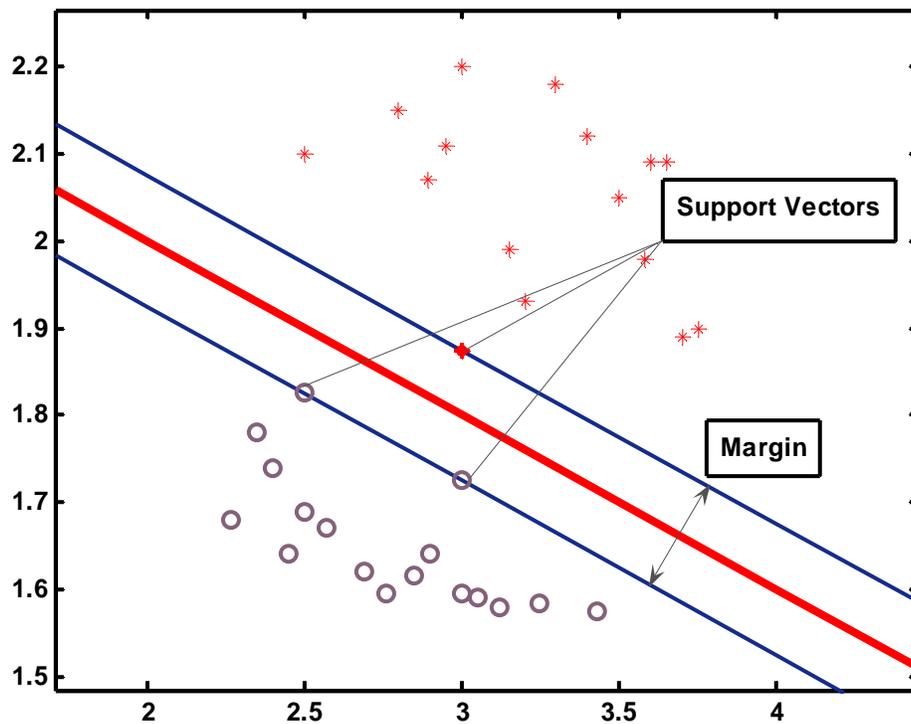


Figure 4-2 Supervised learning algorithm that provides a 2-class discriminant

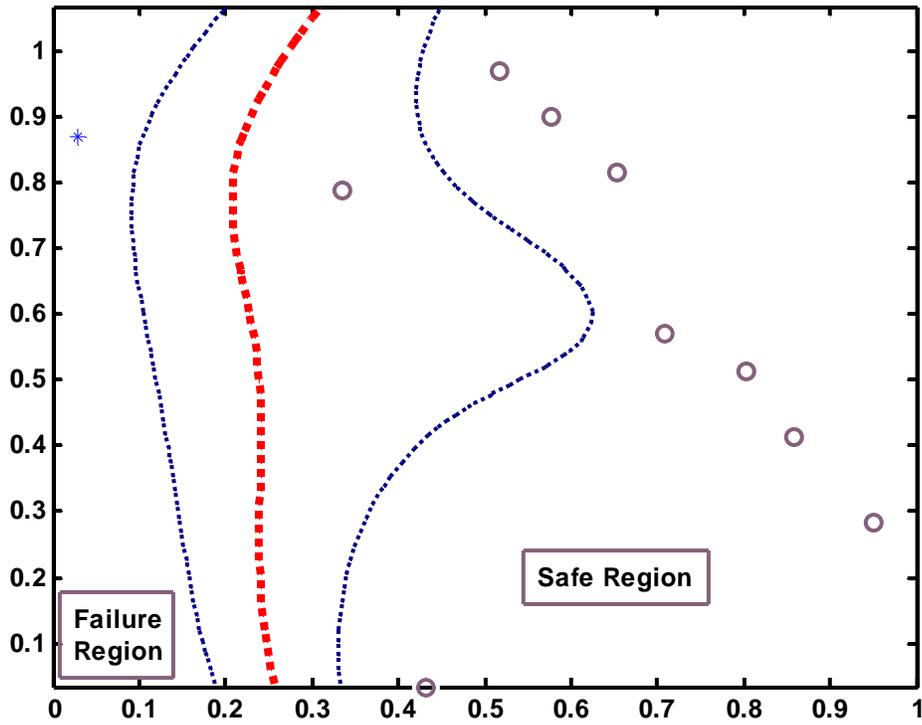


Figure 4-3 SVM reliability analysis training with data size N=10

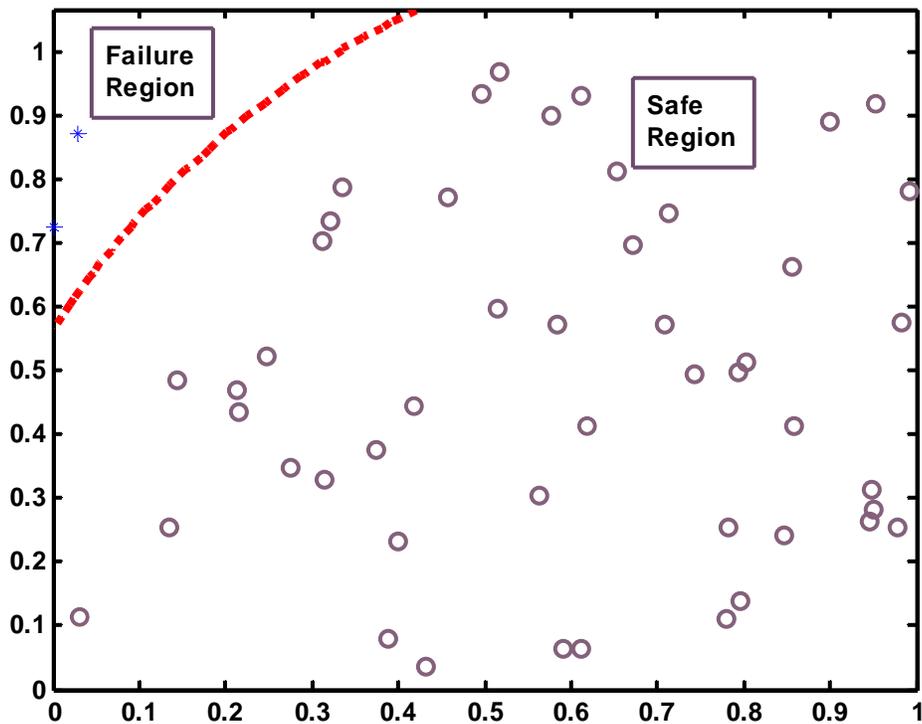


Figure 4-4 SVM reliability analysis training with data size N=50

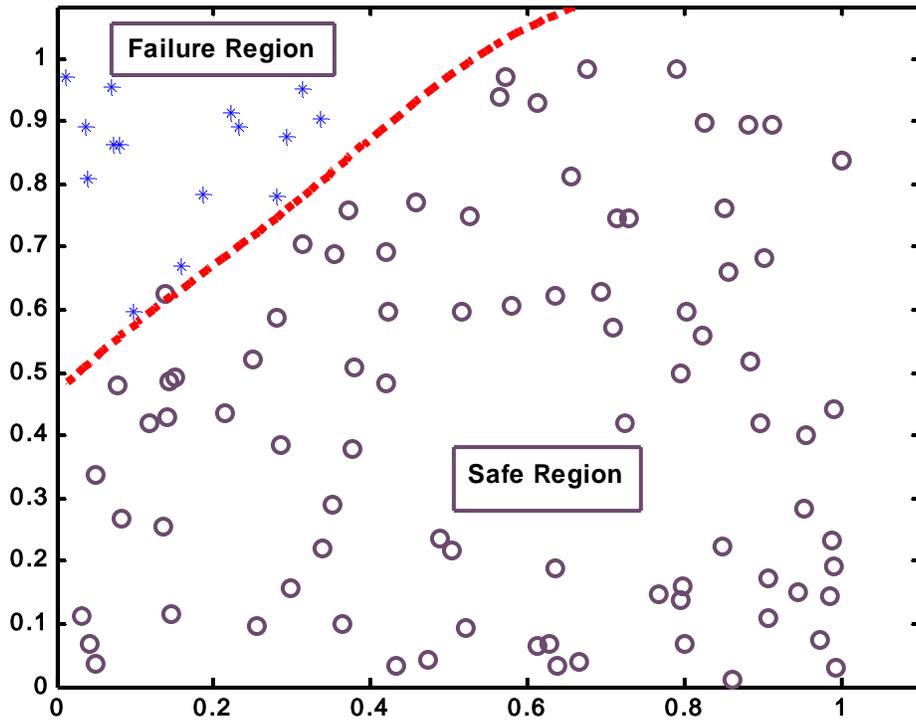


Figure 4-5 SVM reliability analysis training with data size N=100

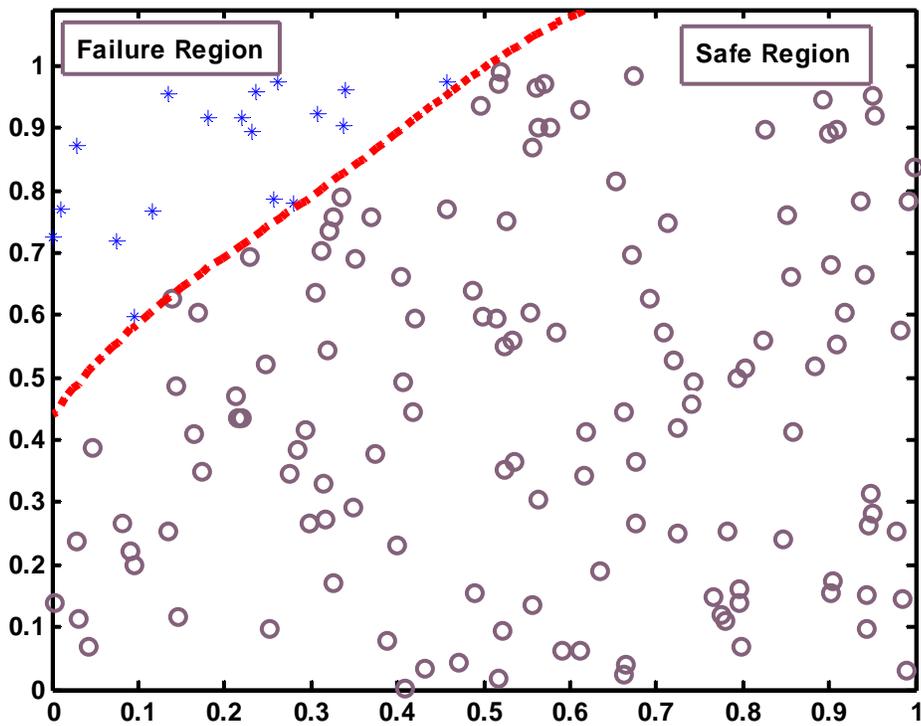


Figure 4-6 SVM reliability analysis training with data size N=150

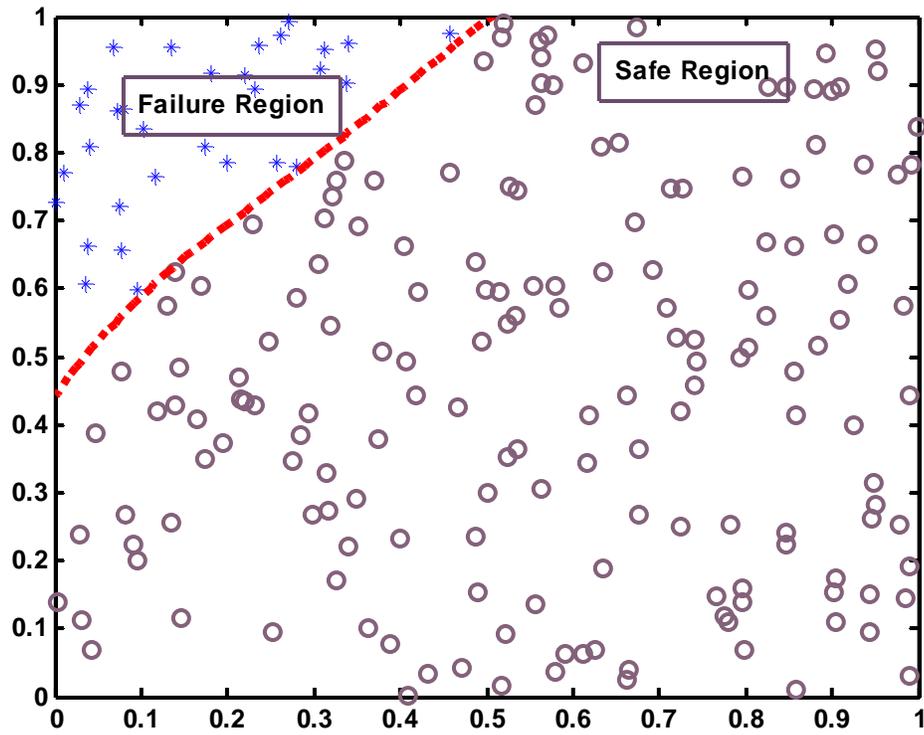


Figure 4-7 SVM reliability analysis training with data size N=200

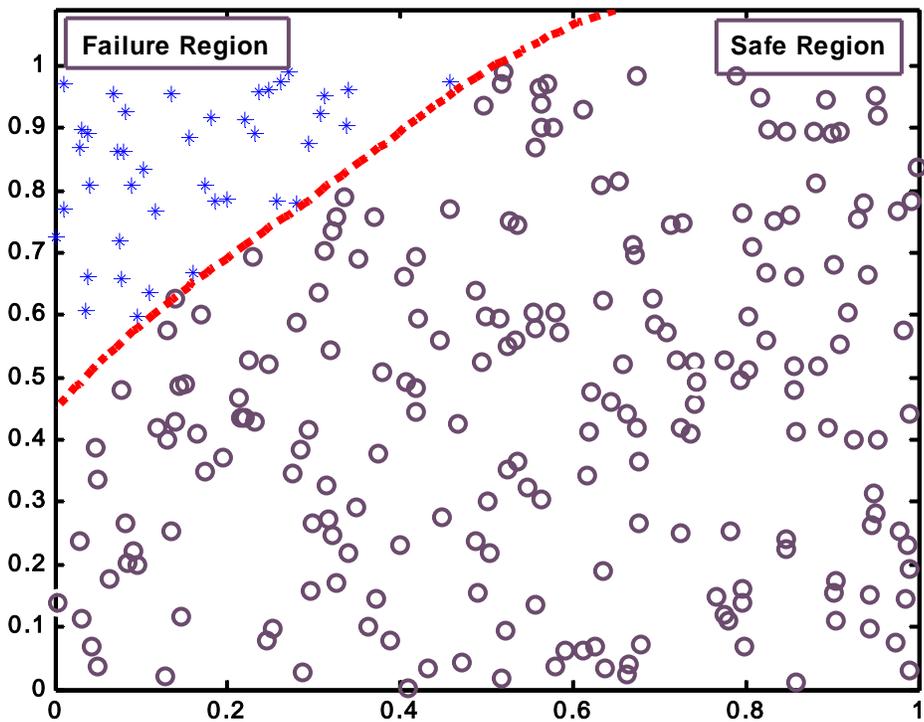


Figure 4-8 SVM reliability analysis training with data size N=250

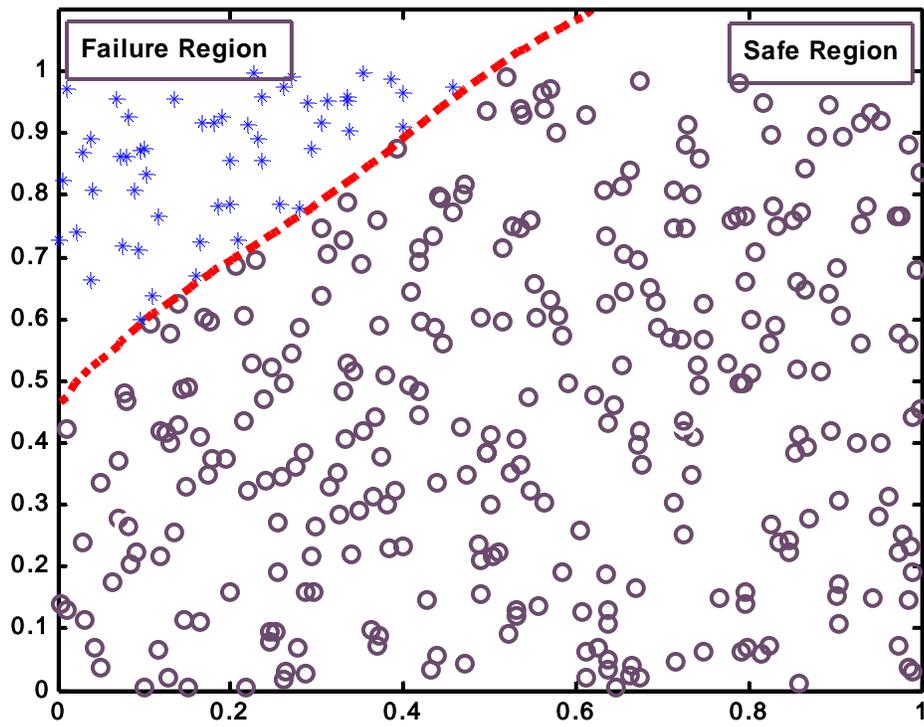


Figure 4-9 SVM reliability analysis training with data size N=350

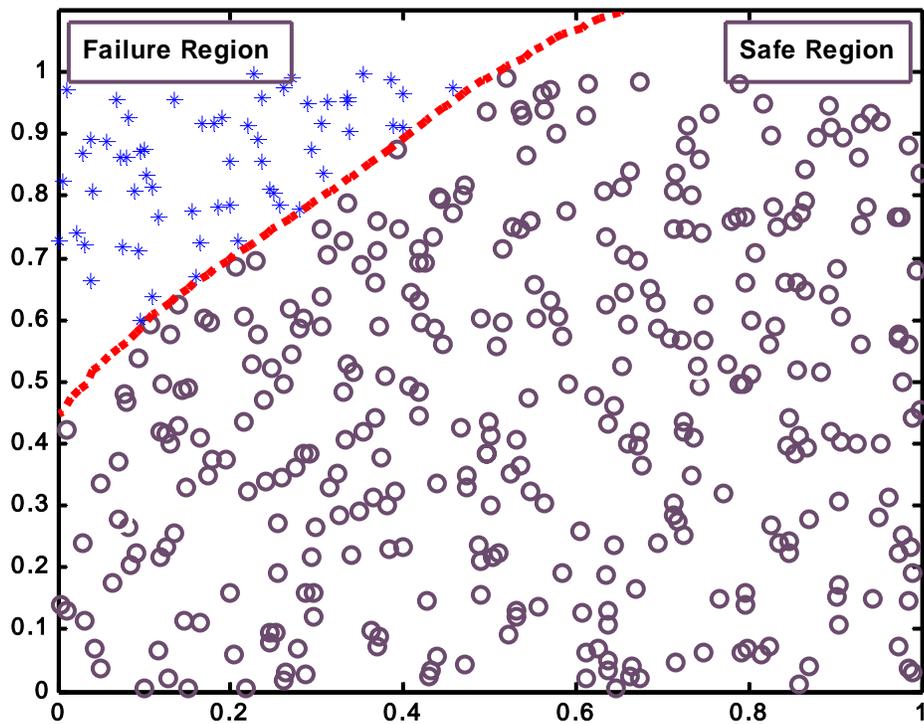


Figure 4-10 SVM reliability analysis training with data size N=400

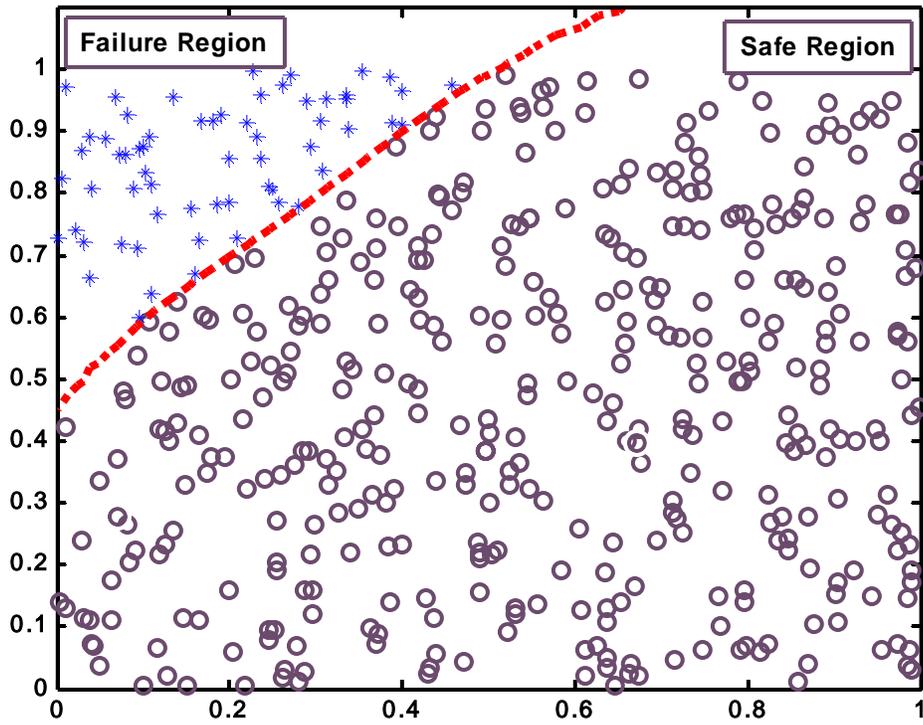


Figure 4-11 SVM reliability analysis training with data size N=500

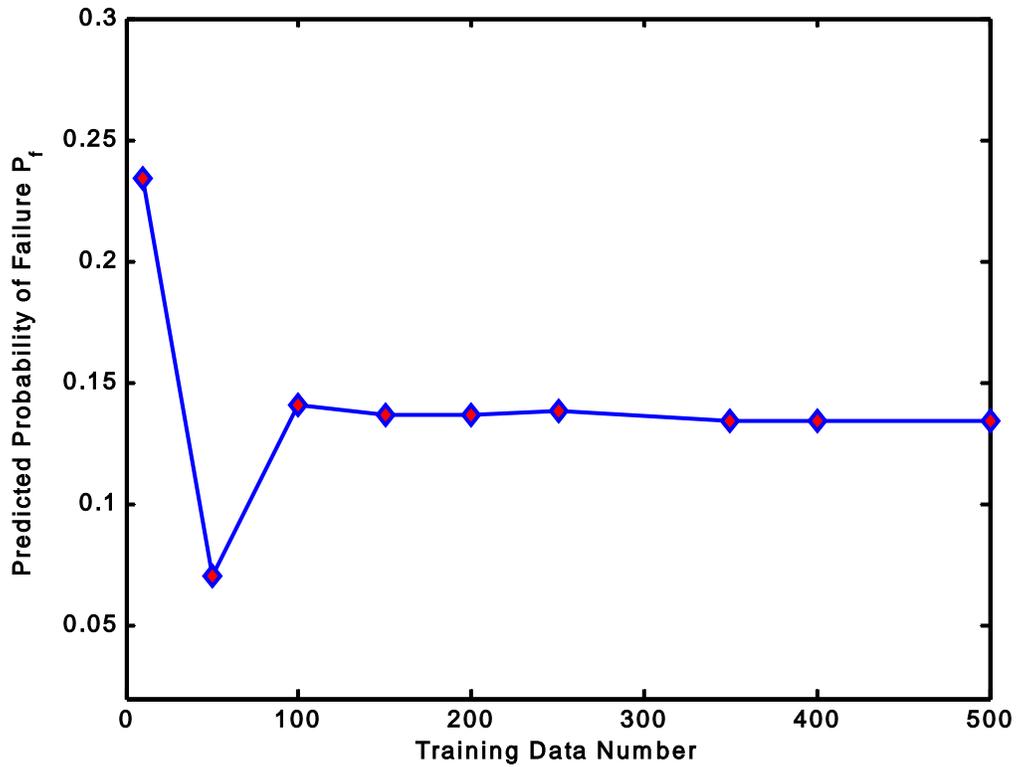


Figure 4-12 Probability of failure vs. Training data size N

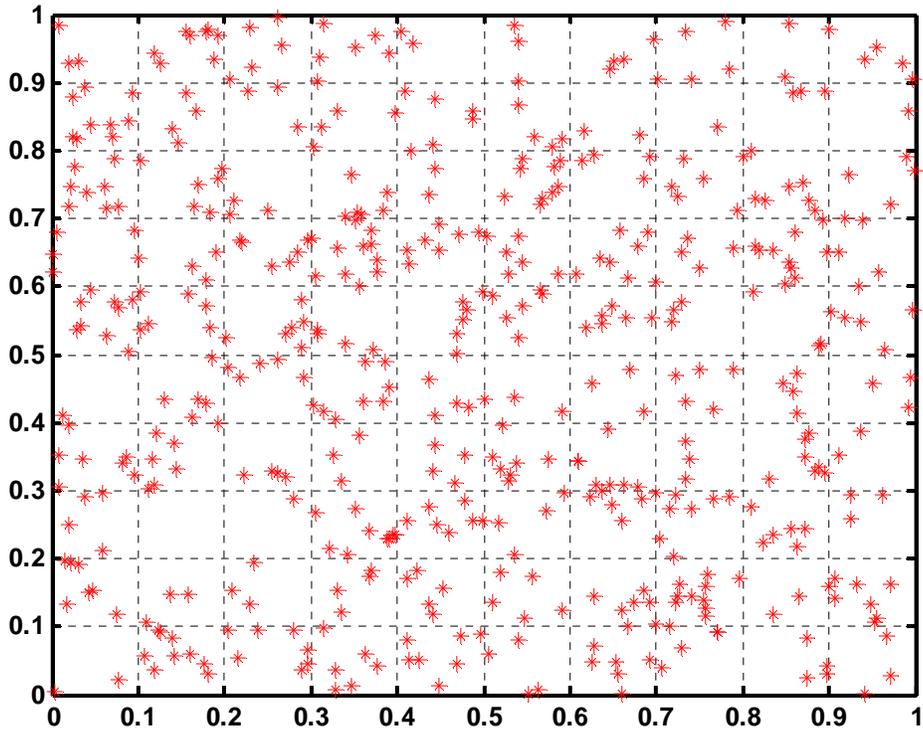


Figure 4-13 Training data for linear for case with linear limit state function

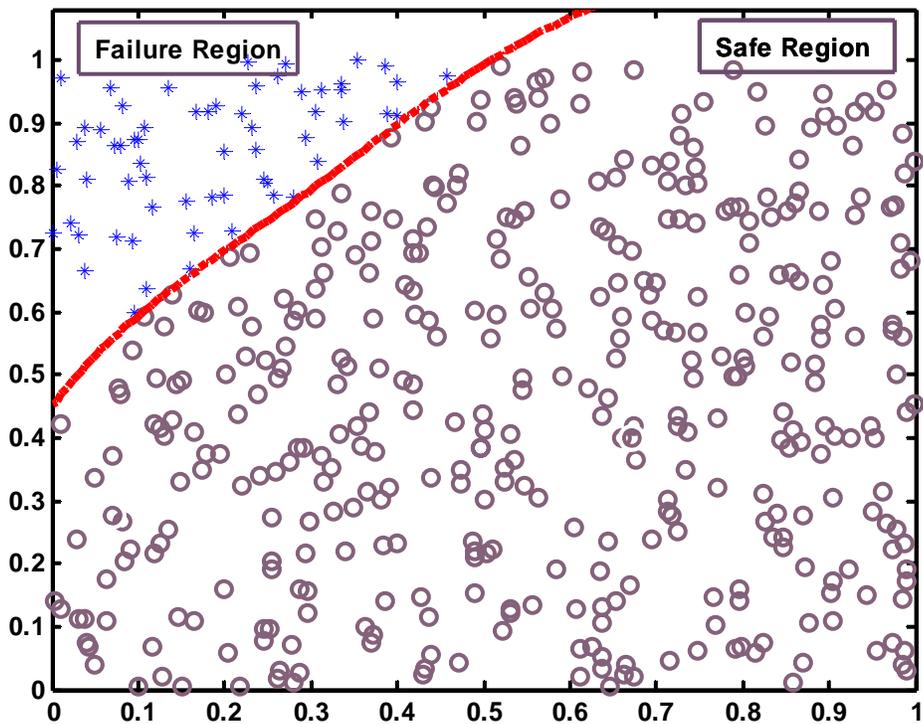


Figure 4-14 Kernel function of Thin Plate Spline for linear Case

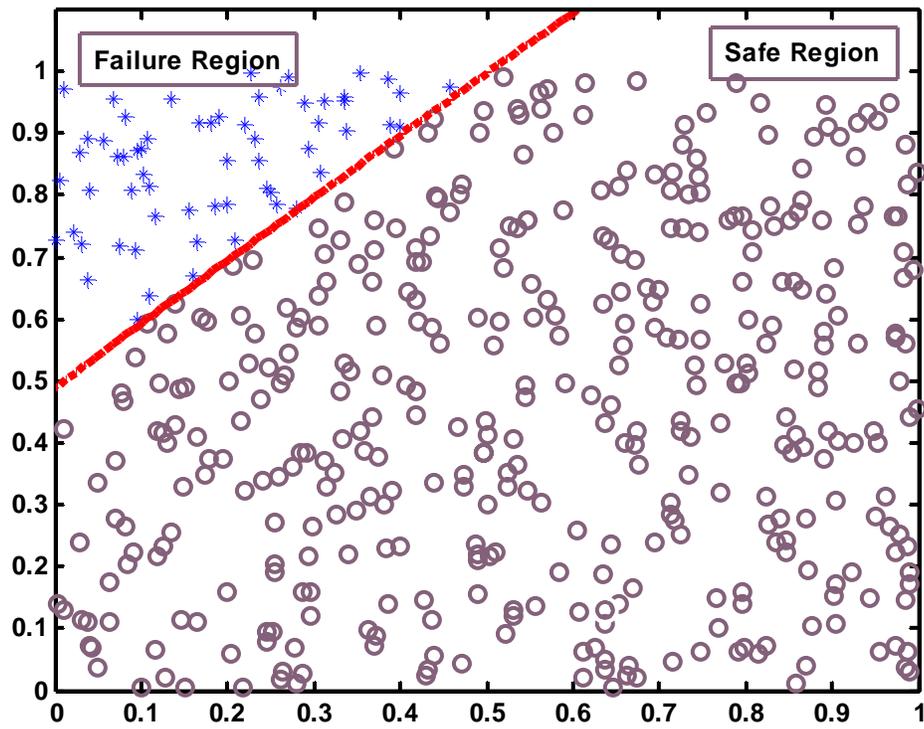


Figure 4-15 Spline Kernel function for Linear Case

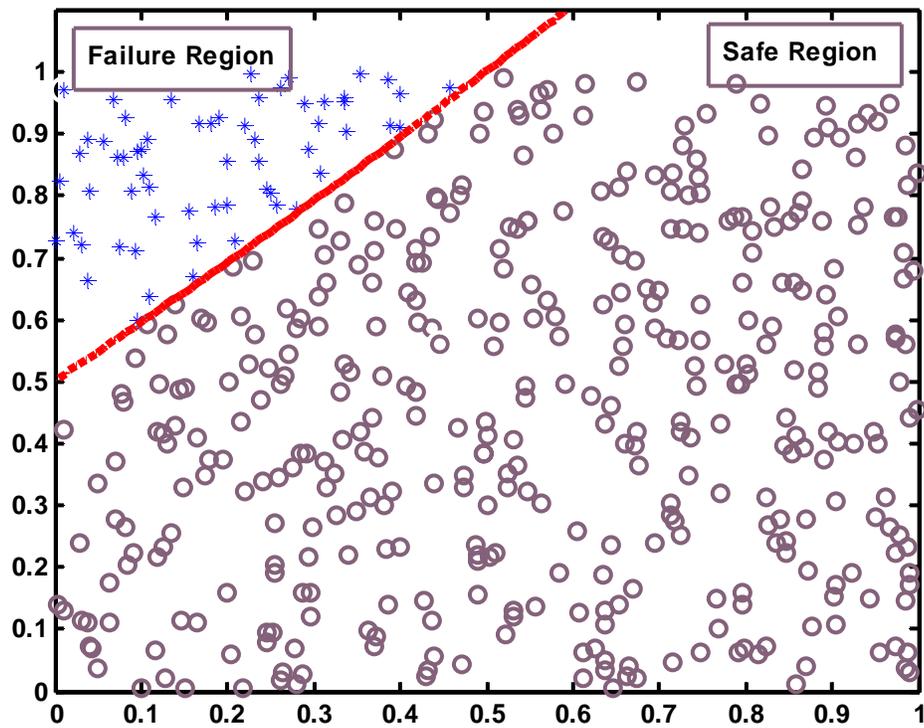


Figure 4-16 Cubic Kernel function for Linear Case

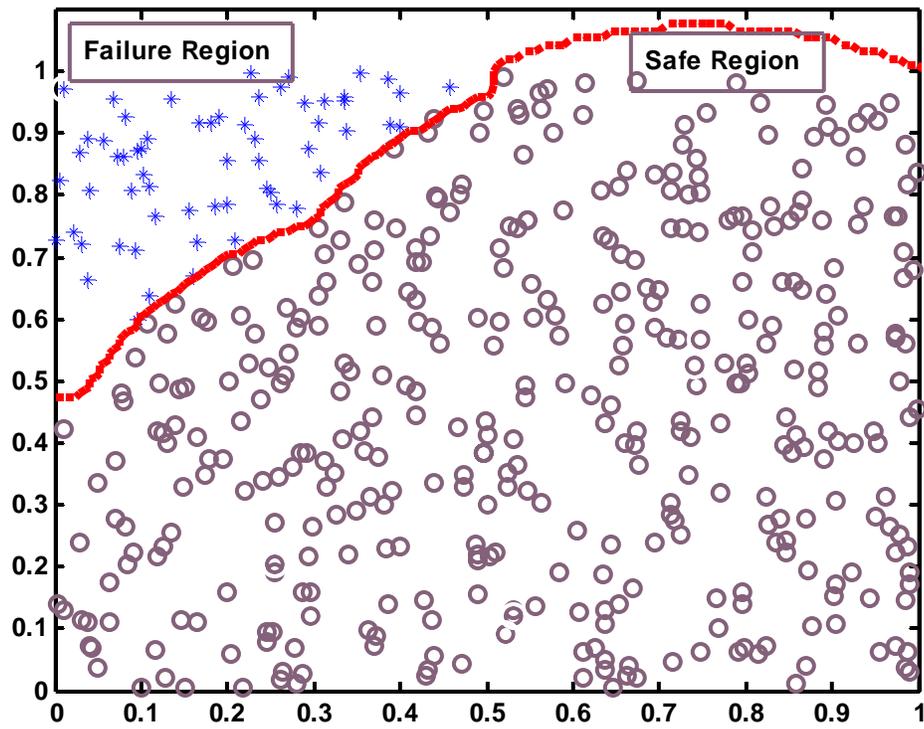


Figure 4-17 Quadratic Kernel function for Linear Case

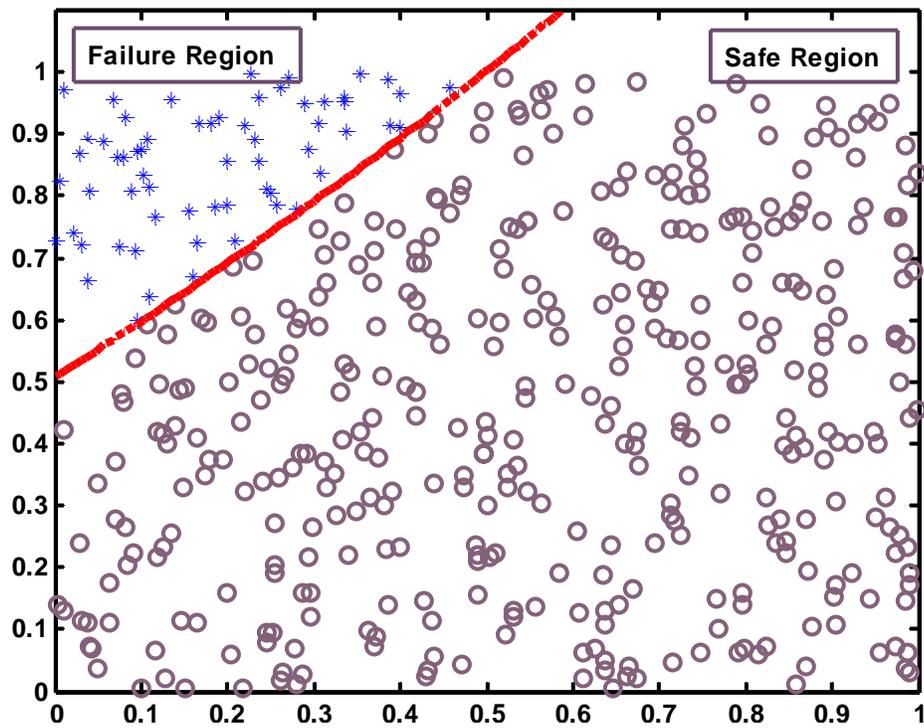


Figure 4-18 Distance Kernel function for Linear Case

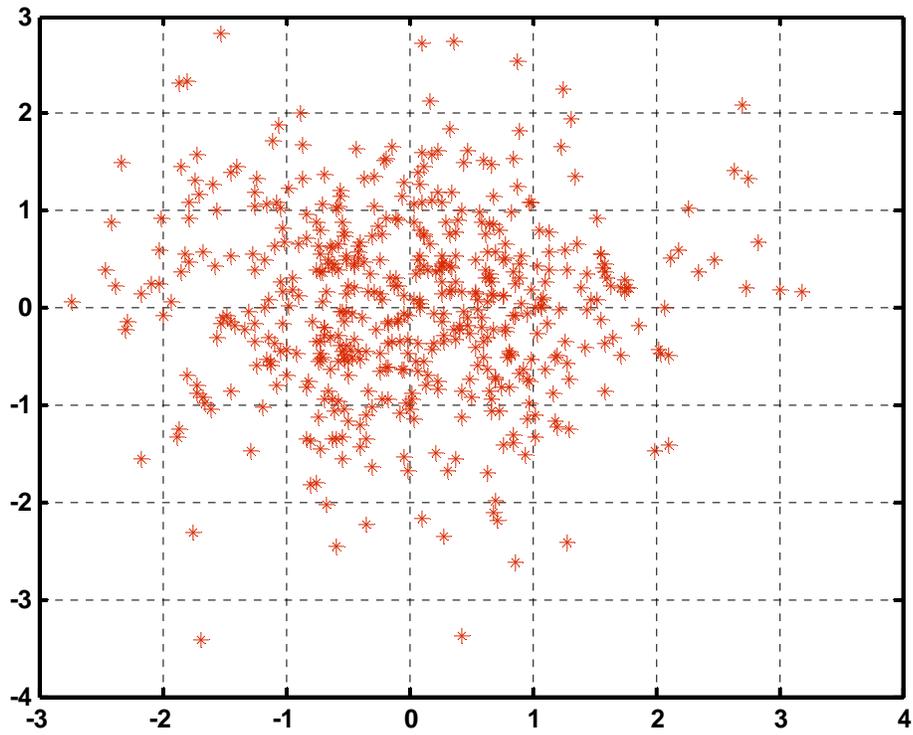


Figure 4-19 Training data for linear for case with linear limit state function

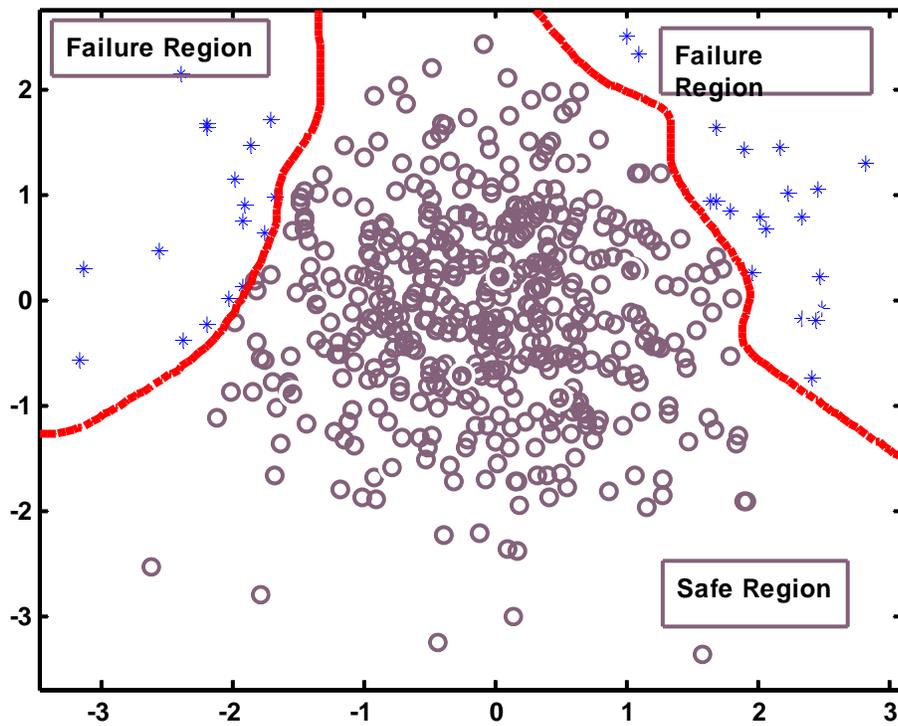


Figure 4-20 Radio Basis kernel function for nonlinear Case

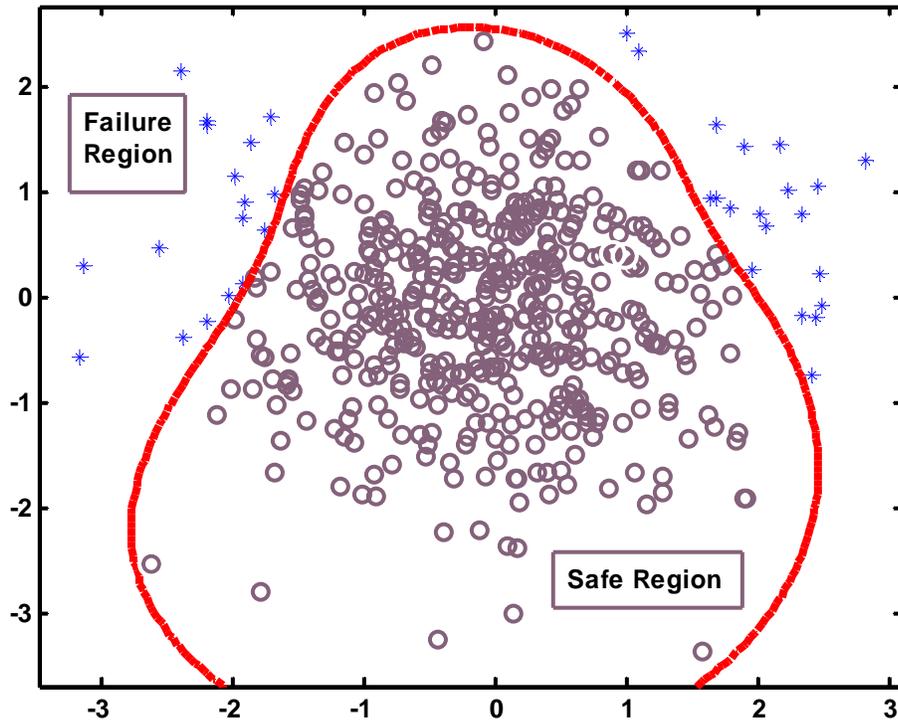


Figure 4-21 Thin plate Spline function for nonlinear Case

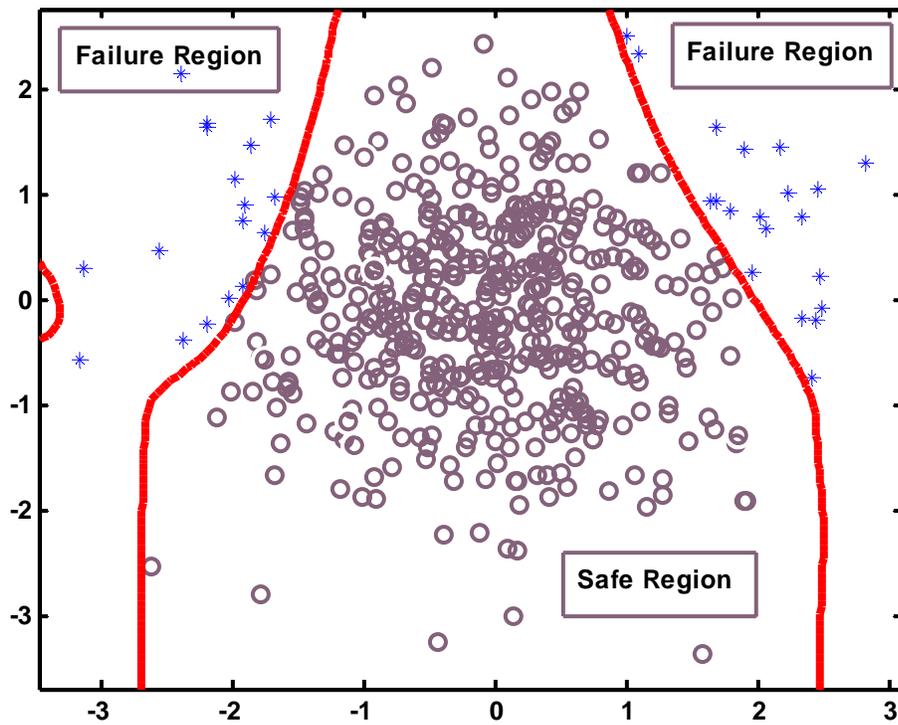


Figure 4-22 Spline function for nonlinear Case

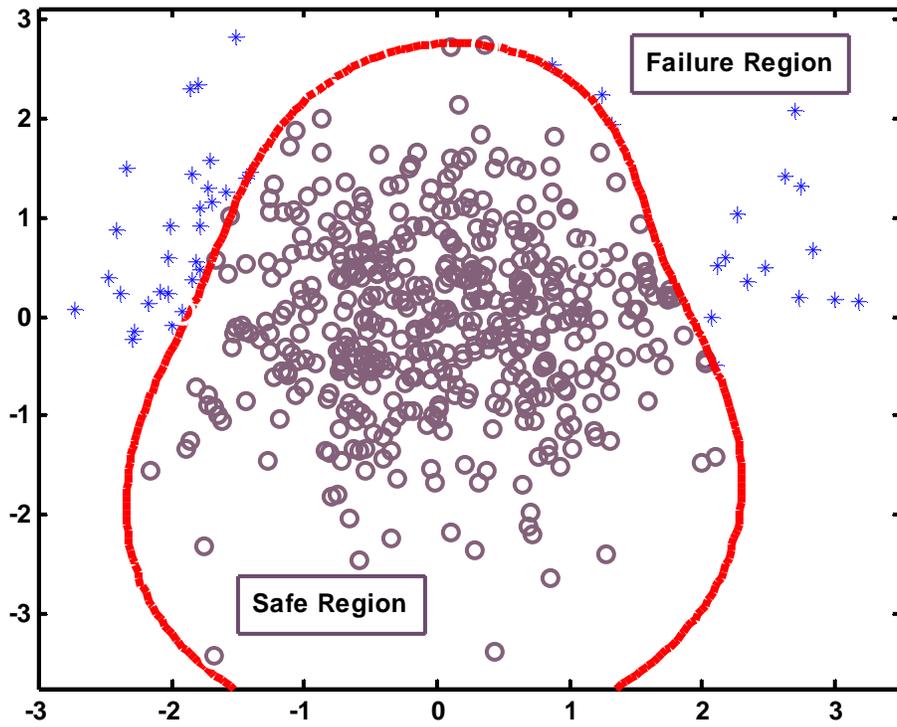


Figure 4-23 Cubic function for nonlinear Case

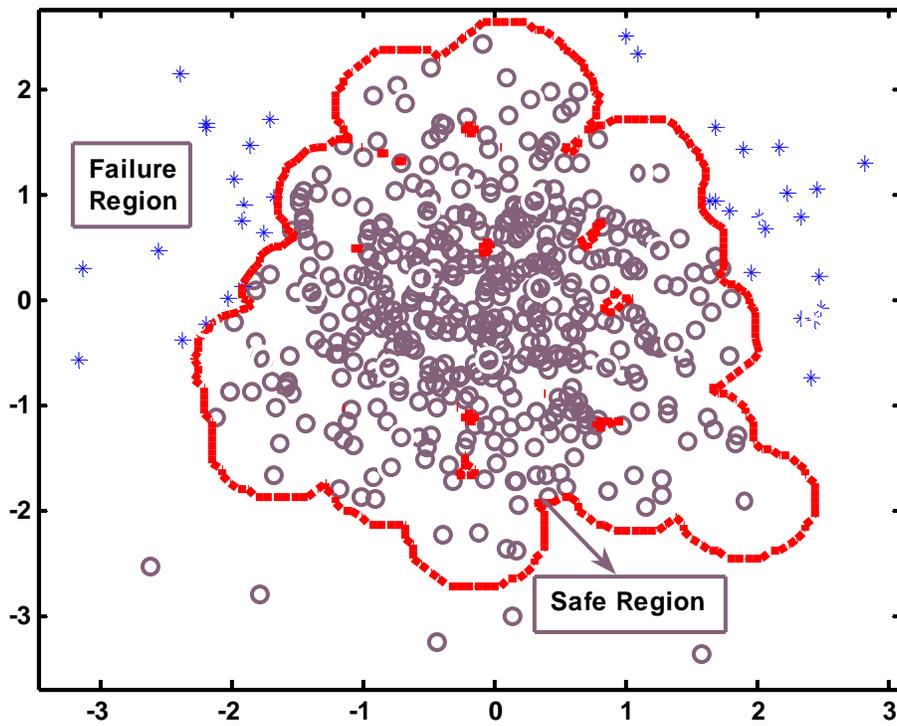


Figure 4-24 Quadratic function for nonlinear Case

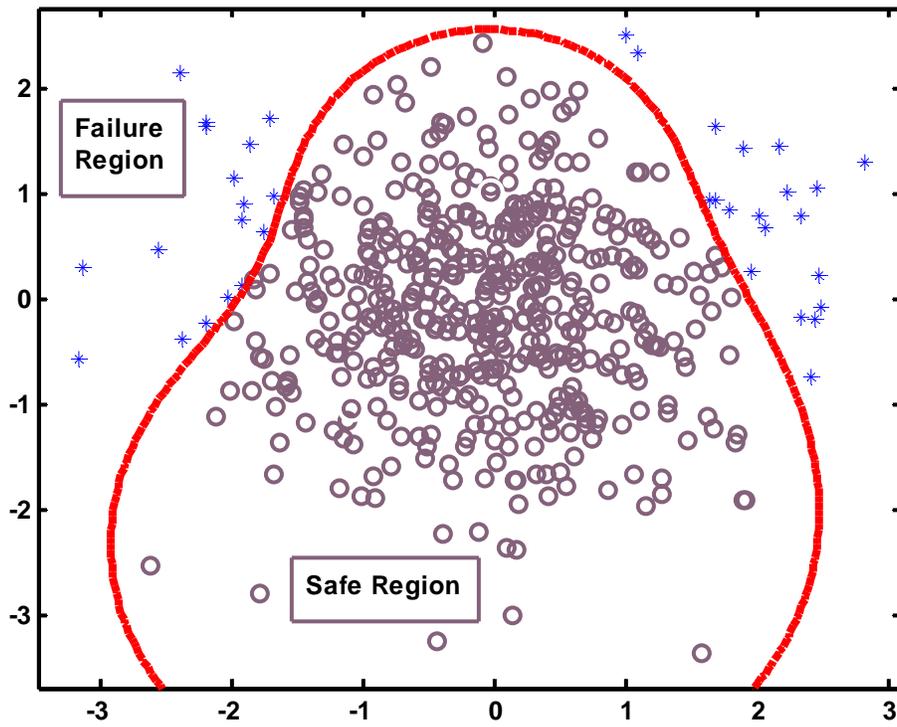


Figure 4-25 Distance Kernel function for nonlinear Case

CHAPTER 5

5 Conclusions

The general objective of this research work was to develop a generic framework of probabilistic uncertainty modeling and structural reliability analysis. In the first part of the research, a Bayesian based system identification approach was introduced and demonstrated. By utilizing the unique characteristic of Bayesian analysis to combine the subjective information and objective system measurements, the unknown parameters of dynamic systems can be accurately estimated with optimizing the conditional posterior probability density function. Since the choice of priori pdf is found to have minor effects on the posterior pdf, the subjective assumption can be corrected by the system observations. The associated covariance of the resulting pdfs for the system parameters can be used to quantify the uncertainty. By increasing data availability, the confidence of the estimations is found to be increased. In cases where is needed to identify more than one parameter, parameter information can be extracted from the contour plot of the joint probability density functions. For cases that the data size is crucial or better visualization results are desired, it is recommended to define a different likelihood function with parameters independent of each other or to evaluate marginal pdf for each individual parameter.

It is only in the early stage of structural health management that is needed to identify the desired system parameters. Other than techniques to evaluate system parameters or detect possible damage, structural reliability assessment is usually considered as the final step of structural health management with which, the ultimate goal is to improve the reliability of the structures and increase their service life. It is the fundamental goal of structural reliability analysis to determine the safety condition of the structure. One of the most important

notations to interpret structural safety condition is probability of failure. The MATLAB software package 'PROBES' is the reliability analysis tool originally developed by Wu (2005). The objective of the software is to obtain failure probability given the limit state function and associated random variables using different traditional reliability analysis methods including Monte Carlo simulation, first order reliability method, second order reliability method and importance sampling method. However, the original probabilistic based analysis package was designed only for random variables that are distributed normally and are not correlated. It was one of the tasks of this research work to update the software package to make it feasible to different practical cases other than independent normal variable cases. In another words, for random variables that are non-normal \ correlated, it was the task to transform them into a set of uncorrelated, standard normal RVs. With the random variable transformation algorithms implemented in the software, 'PROBES' was improved to be more powerful to solve problems where the input random variables are not any more limited to uncorrelated and normal distributed ones even though different simulation methods should be chosen with caution since different methods may be suitable to each specific case. With study of the first stage work to evaluate system parameters using Bayesian analysis, it was found that Bayesian approach is a powerful probabilistic inference method to obtain the current state system information from the previously available knowledge and system measurements. Following the updating of reliability analysis package 'PROBES', it was found that uncertainties introduced by the lack of knowledge and other human induced factors, along with the inherent modeling uncertainty affect the reliability assessment and the future decision making. In the third part of the research work, uncertainties of the input data were modeled with normal distribution random variables. The statistical parameters such as

mean μ and variance σ could be modeled statistically and updated using Bayesian analysis based on the current state data measurements. Both randomness and the modeling uncertainty were considered in this approach. The failure probability P_f was obtained with the implementation of Monte Carlo simulation method. Through statistical analysis, the probability distribution function (pdf) profile and the associated mean and variance can be calculated. According to the pdf profile, one can assign a distribution to the analyzed data. By performing the goodness of fit test, a confidence level of accepting this distribution can be specified to give a quantitative sense of making decision.

It was noticed that the traditional reliability methods of FORM, SORM, and ISM all require the most probable points to be found for further calculation. In some cases, it may not be possible to search the most probable point or there is more than one such point. Monte Carlo method is generally applied in such cases even though it costs a lot of computation time. In the final part of the work, a Support Vector Machine (SVM) based approach was introduced to perform structural reliability analysis. On the basis of the simulated data sets, a hyperplane to classify the safety region and the failure region could be found by using SVM. Other testing data could be classified according to this classifier. Failure probability is obtained by numbers falling into failure region divided by the total testing data. The advantage of this approach is to classify the new data points without going through the calculation of the limit state function. Generalized reliability analysis models were introduced to verify the approach. As many as 2000 input data were tested. Different kernel functions were compared and applied to both linear and nonlinear cases. It is demonstrated that the proposed SVM based approach leads to an accurate and efficient procedure for reliability analysis. In mechanical and structural reliability, the probability of failure is

usually very small, which requires very large samples to achieve needed accuracy by using Monte Carlo simulation method. However, it is shown that much less training data are required for SVM based approach to obtain the result accurate enough. Moreover, in reality, due to the lack of complete knowledge about the structure or the changes in the characteristics of the structure through time, such as upgrades and redesigns, the information to define a reasonable limit state function could be limited and sometimes discontinuous. It is proved to be another advantage of SVM based reliability analysis that failure probability can be obtained without computation of the limit state function.

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