We study elastoplastic transitions in solutions of the antiplane shear model of granular flow, and describe a time-periodic solution that arises when the antiplane shear model is discretized in space. The antiplane shear model is a simplification of the continuum equations representing the flow of granular materials. The modeling of granular flow has many applications, from agricultural silos to geomechanics: improved accuracy in modeling will lead to safer and more economical designs for silos and industrial hoppers, and make oil drilling a more efficient process.

We construct approximate solutions to the antiplane shear model with piecewise linear initial data, which feature transitions between elastic and plastic states. These transitions travel with fixed speed. Numerical simulations demonstrate that the same elastoplastic transitions are the prominent features of the numerical solution.

The periodic solution of discretized antiplane shear appears at a critical value of the elasticity parameter for antiplane shear. The bifurcation to a periodic solution appears to be a Hopf bifurcation. The periodic solution contains elastoplastic transitions, as well as a shear band that appears over part of the period. Away from the shear band, the periodic solution has four distinct regions, three elastic and one plastic. Refinement of the spatial discretization further resolves these states.
Granular Flow Models: Analysis and Numerical Simulations

by

Robert E. Wieman

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Approved By:

__________________________  __________________________
Dr. Hien T. Tran            Dr. Stephen Schecter

__________________________  __________________________
Dr. Michael Shearer         Dr. Pierre A. Gremaud
Chair of Advisory Committee
This work is dedicated to Dr. Henry Matlack Wieman, who thought I might make a good mathematician.
Biography

Robert Edward Wieman was born in Detroit, Michigan and raised in Owego, New York. He attended the University of California in Berkeley where he received a Bachelor of Arts degree in mathematics and physics. He went on to pursue a graduate degree in applied mathematics at North Carolina State University.
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Chapter 1

Introduction

The study of granular materials is at an exciting point, where mathematical analysis can contribute to the understanding of a myriad of interesting phenomena. Recent experiments challenge traditional engineering theory, and there is still debate over physical theories of granular flow [1], [3]. Mathematical analysis as well as numerical results may advance the process of winnowing out less applicable ideas and synthesizing those that remain.

The continuum models of steady state granular flow considered in this thesis are hyperbolic conservation laws (for many models, hyperbolicity can be lost, but we do not run into that issue here). Therefore, it is appropriate to consider the admissibility of shocks as weak solutions of the partial differential equations arising from these models.

Attempts to analyze the partial differential equations arising from the dynamics of granular flow models are complicated by the fact that these equations are dynamically ill-posed. In particular, they are unstable with respect to perturbations in certain directions in Fourier space [19]. Although in the linear theory, perturbations in these unstable directions would lead to non-existence of solutions, nonlinear effects control the amplification of perturbations [20].

For much of this thesis, we study a model of antiplane shear, representing a simplification of the full dynamic granular flow equations that retains the characteristic ill-posedness of the full equations. The antiplane shear model is a system of three unknowns (one velocity and two stress components) in two space dimensions and time; by contrast, the general granular flow equations in two space dimensions would have five functions (two velocity and three stress components). But like the equations of general granular flow, the
antiplane shear model is unstable to perturbations in a wedge of directions in Fourier space. The model is thus a more analyzable system with the same key features as a full-blown granular flow model.

The numerical simulations in this thesis are results of spatial discretization of the equations of the antiplane shear model. If the partial differential equations governing antiplane shear are discretized in space, the ill-posedness is removed. Spatial discretization removes the effect of high frequency perturbations, arguably analogous to the effect that finite grain size has in the physical flow. Corresponding to the ill-posedness of the underlying continuum model, the discretized antiplane shear model can exhibit a sharp jump in the variables over one mesh width, representing a physical shear band.

Models of granular flow often assume plastic deformation; indeed, because the deformations in granular materials are primarily irreversible, they are excellent examples of ‘plastic’ materials [23]. Our model includes elasticity; we find it exhibits very complicated behavior, including a periodic solution for certain values of the shear modulus. This is particularly exciting because oscillatory behavior may contribute to hopper failure, and help explain the transition from mass flow to core flow. Thus, the elasticity of granular materials may play a greater role in their behavior than previously thought.

1.1 Phenomena in the Flow of Granular Materials

A granular material is any material composed of a large number of individual solid particles. Granular materials are ubiquitous in industry and agriculture. It is estimated that approximately one half of the products, and three quarters of the raw materials, of the chemical industry are in granular form [16]. When one considers the catastrophic power of avalanches, the economic significance of petroleum extraction, or the importance of grain storage from the dawn of civilization to the present, it becomes evident that predicting and controlling the behavior of granular materials is of critical importance. Despite this, our understanding of the behavior of granular materials is much less complete than that of fluids, for example. Our limited knowledge has a high price: in North America each year, over a thousand bins, hoppers, and silos fail, and the efficiency of industrial solids processing plants is poor [10].

Granular materials in motion resemble liquids, and at rest resemble solids, but also exhibit behavior distinct from either, and granular materials are sometimes viewed
as another form of matter entirely. For example, because the weight of grain in a silo is distributed to the walls, the pressure approaches a constant as the depth increases, rather than increasing linearly with depth like a liquid [9]. A similar effect is observed in granular material flowing from a hopper: the material can form arches, supporting the grains above it and arresting the outflow.

Granular material typically flows from a hopper in one of two modes: mass flow, which occurs for narrow hoppers and in which all the material is moving, and core flow, which occurs when the angle of the hopper is wider and in which stagnant regions of material form on the walls while a core of granular material near the center of the hopper continues to flow [16]. Stagnant regions are usually undesirable in practical applications, and demonstrate that granular flow is qualitatively different from fluid flow. In addition to these steady state flows, pulsating flow is often observed [19]. Hopper failure is thought to be the result of such oscillations, but this connection has not been studied in theory, analysis, or simulation.

One of the phenomena of granular flow demanding attention is the occurrence of localization of flow. Often, the displacement of a granular material is localized in a narrow band of material, called a shear band [23]. Because shear bands and associated phenomena (such as core flow) are so prevalent, it is critical that a granular flow model include solutions corresponding to shear bands.

1.2 Continuum Models of the Flow of Granular Materials

There are, in general, two approaches taken in modeling granular materials. One is to attempt to model the behavior of individual grains and their interactions with each other: these are referred to as molecular dynamic, micromechanical, or discrete element models. There is a broad literature on this topic, including attempts to simulate real deformations and flows [6], [14], [2], [11], [15].

The other is to treat the granular material as a continuum, using constitutive laws which reflect the observed macroscopic behavior [7], [17], [8]. The two approaches are complementary: molecular dynamics simulations can provide ensemble averages to compare to the results of a continuum model (for example, [18]). This thesis is solely concerned with continuum modeling, but spatial discretization introduces a length scale that may be thought of as a grain size; the resulting ordinary differential equations provide an alternative
to the micromechanical modeling of individual grains.

1.2.1 Modeling Principles

Force Balance

In a continuum with constant density \( \rho \), conservation of momentum is represented by the formula

\[
\rho \frac{Dv}{Dt} = \rho f + \nabla \cdot T,
\]

where \( f \) is a body force, such as gravity, \( Dv/Dt \) is the material derivative of velocity, and \( T \) is the stress tensor. Constitutive laws constraining the components of the stress tensor, and relating the stress components to velocity, are used to close the system. In the absence of body forces, steady state force balance is given by

\[
\nabla \cdot T = 0.
\]

The stress tensor is symmetric as a result of conservation of angular momentum. (A more complicated continuum model, taking into account the possibility of grains spinning relative to the grains surrounding them, is the Cosserat continuum model, which has an asymmetric stress tensor: see [23].) In two space dimensions, we denote the stress tensor by

\[
T = \begin{pmatrix}
\sigma_x & \tau_{xy} \\
\tau_{xy} & \sigma_y
\end{pmatrix},
\]

where \( \tau \) is the shear stress; \( \sigma_x \) and \( \sigma_y \) are normal stresses in the \( x \) and \( y \) directions, respectively.

In three space dimensions, the form of the stress tensor is

\[
T = \begin{pmatrix}
\sigma_x & \tau_{xy} & \tau_{xz} \\
\tau_{xy} & \sigma_y & \tau_{yz} \\
\tau_{xz} & \tau_{yz} & \sigma_z
\end{pmatrix}.
\]

Constitutive Laws

Granular materials typically exhibit plastic behavior, in that if the shear stress along any plane in a granular material reaches a certain threshold, the material will begin to slip along that plane, and the shear stress will not increase past the threshold. This
threshold is analogous to the standard physics problem of a mass on an inclined plane (see Figure 1.1). If the coefficient of friction is $\mu$, then the block moves if

$$F = \mu N,$$

(1.2.5)

where $F$ is the tangential force down the slope, and $N$ is the normal force. The angle $\delta$ at which this occurs is defined by the relation $\mu = \tan \delta$.

In two dimensions, the analogy can be pursued further. For an idealized granular material, called a Coulomb material, the shear stress $\tau$ takes the role of $F$, and the normal stress $\sigma$ that of $N$. The angle $\delta$ defined by $\mu = \tan \delta$ is called the angle of friction. Along every plane in the granular material, the following inequality holds:

$$\tau \leq \mu \sigma = \sigma \tan \delta.$$

(1.2.6)

The granular material will flow plastically only if there is a plane along which equality is attained; this is the first constitutive law, the Coulomb yield condition. When the inequality is strict, the material behaves elastically, but this case is often ignored: in hopper flow, for example, it is generally assumed that all the material is flowing, although experimentally this is not always the case. There are competing generalizations of the Coulomb yield condition to three dimensions; the Tresca and von Mises conditions are commonly used [23], but we shall not consider them here.

The angle of friction $\delta$ is also called the angle of repose, since it is the maximum angle that a free-standing pile of the material can make. It is empirically observed that a
free-standing granular material forms this characteristic angle of steepness. Viewing Figure 1.1 as a slope of granular material, the link between the angle of repose and the Coulomb yield condition is clear; one can imagine the block as a grain, which will fall down the slope if the angle of the slope is $\delta$ or greater.

To derive the Coulomb yield condition, let us assume that the granular material is in a state of incipient flow everywhere; that is, at every point within the granular material, the inequality 1.2.6 achieves equality for some plane. By maximizing $\tau/\sigma$ over all planes through a point, and setting that ratio to $\tan \delta$, we obtain an algebraic constraint on the components $\sigma_x$, $\sigma_y$, and $\tau_{xy}$:

$$
\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2 = \left(\frac{\sigma_x + \sigma_y}{2}\right)^2 \sin^2 \delta.
$$

This constraint is the Coulomb yield condition: it describes a cone in $(\sigma_x, \sigma_y, \tau_{xy})$ space, as depicted in Figure 1.2.

For steady state flow (1.2.2) in two dimensions, the yield condition (1.2.7) closes the system; the equations for velocity decouple from the stress equations.
Regardless of which yield condition is used in three dimensions, the steady state equations do not decouple, and another constitutive law is required to relate the velocity to the stress: such a relation is known as a flow rule. (A notable exception is 3-d axisymmetric granular flow, in which the circumferential shear stresses vanish, and an assumption about the circumferential normal stress can decouple the stress from the velocity [16].) A flow rule is also necessary to close the dynamic system (1.2.1). The flow rule relates stress and strain rate through the strain rate tensor, \( V \):

\[
V = \frac{1}{2} (\nabla v + \nabla v^T).
\]  

(1.2.8)

A typical flow rule is the associative or coaxial flow rule, which asserts that the eigenvectors of the stress tensor \( T \) and the strain rate tensor \( V \) are parallel. However, experimental results [23] indicate that a nonassociative flow rule, for which the eigenvectors of \( T \) and \( V \) do not align, is more realistic.

**Rankine states**

The cone of Figure 1.2 indicates that in general, a value of \( \sigma_x \) and \( \tau \) determines not one but two possible values of \( \sigma_y \). For each point along the axis of the cone, half of the circular cross-section of the cone has \( \sigma_x \geq \sigma_y \) while for the other half, the opposite inequality holds. These two halves of the cone are referred to as the passive and active Rankine states, respectively. If \( y \) signifies the vertical direction, the passive state reflects the behavior of a granular material that is being compressed horizontally more than it is being compressed vertically, such as might happen in a funnel or hopper. The active state is one for which the vertical stress is dominant, as one might expect in a free pile of granular material, or in a standpipe. Transitions between active and passive are expected in practical applications, such as a conical hopper atop a cylindrical standpipe; modeling such a transition is an open problem [5].

**1.2.2 Ill-Posedness in Continuum Models**

The equation (1.2.1) with an appropriate yield condition and the coaxial flow rule is dynamically ill-posed. The instability is analogous to the equation

\[
u_t = u_{xx} - u_{yy},
\]

(1.2.9)
where perturbations in some directions of Fourier space behave as the backwards heat equation. Thus, infinitesimal plane wave perturbations in a particular direction will be amplified, but the solution does not grow uncontrollably [19]. The instantaneous but finite amplification of infinitesimal perturbations in unstable directions in Fourier space is comparable to shear band formation, where a large (but not unbounded) displacement is concentrated in a narrow region.

1.3 Summary of Results

1.3.1 Shocks in Hopper Flow

We prove that in two-dimensional and three-dimensional axisymmetric steady state hopper flow, the Hugoniot locus, which describes possible stresses on either side of a discontinuity, is a closed curve, topologically equivalent to a figure of eight, wrapping around the conical yield surface of Figure 1.2. Shocks between active and passive Rankine states, thought to be part of a solution for a funnel-standpipe combination, do not satisfy the Lax entropy condition.

1.3.2 Elastoplastic Antiplane Shear

The spatially discretized elastoplastic shear model admits shear band solutions; for a given set of model parameters, the shear band solutions are the intersection points of a hyperbola and a straight line (so there are zero, one, or two possible shear band solutions; there is at most one stable shear band solution.) The most significant issue in numerical implementation of the discretized elastoplastic antiplane shear model is ensuring that the yield condition is not violated. This can be done a number of ways, and in fact, even if not explicitly enforced, the equations will tend to correct an overstep. However, explicit enforcement is more stable, allowing larger time steps. The long-term solutions are the same regardless of what enforcement scheme is used. The speed of elastic waves in the elastic part of the model is well-known; we also compute the speed of plastic waves, which depends on the particular value of the stress.
1.3.3 Elastoplastic Transitions in Antiplane Shear

There is a surprising parallel with [21], where a continuous piecewise linear solution was constructed for piecewise linear initial data for a model of longitudinal displacement in an elastoplastic rod. The same approach can be applied to determine the local behavior of the elastoplastic antiplane shear problem with piecewise linear initial data. A piecewise linear initial condition is seen to generate elastic waves, elastoplastic transitions, and/or plastic waves, each propagating with a determinable speed. These waves resemble parts of the periodic solution that arises for low values of the shear modulus in elastoplastic antiplane shear.

1.3.4 Periodic Solution in Antiplane Shear

We use numerical simulations to systematically explore the effect of variation of the parameters of the antiplane shear problem on the solution. The parameters varied are the shear modulus $E$, which determines the elasticity of the granular material; the angle $\alpha$, which determines the degree of nonassociativity of the system, and the mesh size of the spatial discretization, $\Delta x$.

A periodic solution develops in discretized elastoplastic antiplane shear for values of $E$ below a critical value $E_{\text{crit}}$. This solution consists of a shear band which appears and disappears, and four distinct regions outside the shear band, separated by sharp jumps in the velocity and stress. The size of the jump in velocity is relatively insensitive to changes in $E$, while the sizes of the jumps in the components of stress scale with $E$. The critical shear modulus $E_{\text{crit}}$ shrinks with refinement of the mesh, as expected. The dependence of the solution on the parameter $\alpha$ is more complicated. Below a certain value of $\alpha$, the nature of the solution drastically changes to one that is roughly sinusoidal, rather than the four-stage solution for larger values.
Chapter 2

Background

2.1 Antiplane Shear Model

2.1.1 Motivation and development of model

The antiplane shear model is simpler than a general continuum model of a granular material, while retaining the ill-posedness typical of these models. Thus, the antiplane shear model is unstable with respect to perturbations in a wedge of directions in Fourier space, but consists of only one velocity component and two stress components.

It presumes that all movement of the granular material is in one direction, which we take to be parallel to the $z$ axis:

$$v_x = v_y = 0. \quad (2.1.1)$$

In addition, the antiplane shear model assumes that there is no dependence of velocity or stress on $z$. Consequently,

$$v \cdot \nabla v = \begin{pmatrix} 0 \\ 0 \\ v_z \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial v_x}{\partial x} \\ \frac{\partial v_x}{\partial y} \\ 0 \end{pmatrix} = 0, \quad (2.1.2)$$

so the material time derivative $Dv/ Dt = \partial v/\partial t$. These assumptions also eliminate most of the stress components from the momentum balance equation, leaving only the shear stresses $\tau_{xz}$ and $\tau_{yz}$. Finally, the material is assumed to be incompressible ($\rho$ is constant), so that (1.2.1) becomes

$$\frac{\partial v_z}{\partial t} = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y}. \quad (2.1.3)$$
For convenience of notation, we will rename \( v_z \) as simply \( v \) (as the velocity is a scalar in this model), and the vector \((\tau_{xx}, \tau_{yz})^T\) of shear stresses as \( \tau \), so that we can use subscripts to denote derivatives and (2.1.3) can be rewritten

\[
v_t = \nabla \cdot \tau.
\]  

(2.1.4)

We need constitutive laws to close the system (which is now just one partial differential equation with three unknowns.)

The yield condition constrains the magnitude of the shear stresses:

\[
\|\tau\| \leq 1.
\]  

(2.1.5)

The flow rule changes form depending on whether the material is deforming elastically or plastically. With the reduction in variables, the flow rule is no longer a relationship between tensors, but between the vectors \( \tau \) and \( rv = (v_x, v_y)^T \). In elastic deformation, the time derivative of stress is proportional to the spatial derivative of velocity:

\[
\tau_t = E \nabla v.
\]  

(2.1.6)

In plastic deformation, the time derivative of stress must be such that the yield condition (2.1.5) is not violated. The flow rule

\[
\tau_t = E \left( \nabla v - \frac{\tau \cdot \nabla v}{\cos \alpha} R_\alpha^T \tau \right),
\]  

(2.1.7)

where \( R_\alpha \) is the rotation matrix \( \left( \begin{array}{cc}
\cos \alpha & -\sin \alpha \\
\sin \alpha & \cos \alpha
\end{array} \right) \), \( 0 \leq \alpha < \pi/2 \), satisfies the yield condition:

**Lemma 1.** If \( \|\tau(x,0)\| \leq 1 \) for all \( x \), and \( \tau_t \) is given by (2.1.7), then \( \|\tau(x,t)\| \leq 1 \) for all \( x \) at any \( t \geq 0 \).

**Proof.** We first observe that the time derivative of \( \|\tau\| \) is

\[
\|\tau\|_t = \sqrt{\tau \cdot \tau_t} = (\tau \cdot \tau_t)/\|\tau\|,
\]  

(2.1.8)

so \( \tau \cdot \tau_t \) has the same sign as \( \|\tau\|_t \). (We can safely ignore the case \( \|\tau\|_t = 0 \).) We assume without loss of generality that \( \|\tau\| = 1 \) at some point \((x_0, t_0)\). Then

\[
\tau \cdot \tau_t = \tau \cdot E \left( \nabla v - \frac{\tau \cdot \nabla v}{\cos \alpha} R_\alpha^T \tau \right)
= E(\tau \cdot \nabla v)(1 - \|\tau\|^2) = 0.
\]  

(2.1.9)

Thus, for all \( \tau \) such that \( \|\tau\| = 1 \), \( \|\tau\|_t = 0 \), and if \( \|\tau\| > 1 \), \( \|\tau\|_t < 0 \). From this we conclude that \( \|\tau\| \) is nonincreasing if \( \|\tau\| = 1 \). Therefore, \( \|\tau\| \) does not rise above 1. \( \square \)
The elastic and plastic flow rules can be expressed together as
\[ \tau_t = E \left( \nabla v - \frac{\chi(||\tau||)(\tau \cdot \nabla v)}{\cos \alpha} R_0^T \right), \] (2.1.10)
where the transition from elastic to plastic is expressed by the indicator function \( \chi \):
\[ \chi(||\tau||) = \begin{cases} 0 & \text{if } ||\tau|| < 1 \\ 1 & \text{if } ||\tau|| \geq 1. \end{cases} \] (2.1.11)
The + subscript in (2.1.10) indicates the positive part of the function: \( (\tau \cdot \nabla v)_+ = \tau \cdot \nabla v \) if \( \tau \cdot \nabla v > 0 \), and 0 otherwise. This expresses loading; if the material is at yield but is unloading \( (\tau \cdot \nabla v \leq 0) \), the elastic flow rule applies.

A transition from the plastic to the elastic form of the equations can only occur through unloading:

**Corollary 1.** If \( \tau \) evolves according to (2.1.10), and \( ||\tau(x, t)|| \geq 1 \) and \( \tau(x, t) \cdot \nabla v(x, t) > 0 \) for some \( (x, t) \), then (2.1.10) will reduce to the elastic evolution equation (2.1.6) at \( (x, t^*) \) for some \( t^* > t \) only if \( \tau(x, t^*) \cdot \nabla v(x, t^*) \leq 0 \).

**Proof.** As seen in (2.1.9), \( \tau \cdot \tau_t = 0 \) if \( ||\tau|| = 1 \) and \( \tau \cdot \nabla v > 0 \), and therefore \( ||\tau|| \) will remain at 1 until \( \tau \cdot \nabla v \leq 0 \). \( \square \)

Thus, plastic to elastic transitions are characterized by \( \tau \cdot \nabla v \) becoming negative. On the other hand, for the elastic or unloading states \( (||\tau|| < 1 \text{ or } \tau \cdot \nabla v \leq 0) \), \( \tau \cdot \tau_t = E(\tau \cdot \nabla v) \), so unloading states have decreasing \( ||\tau|| \), while for loading elastic states, \( ||\tau|| \) increases until \( ||\tau|| = 1 \). Thus, a transition from elastic to plastic occurs when \( ||\tau|| \) becomes 1 (the elastic state must be loading for \( ||\tau|| \) to increase, so \( \tau \cdot \nabla v > 0 \) already).

### 2.1.2 One-Dimensional Model

A linear function \( v = ax + by \) coupled with a constant \( \tau = R_0 \nabla v/||\nabla v|| \) is a steady state solution of the equations (2.1.4) and (2.1.10):
\[ v_t = \nabla \cdot \tau = \nabla \cdot \left( R_0 \begin{pmatrix} a \\ b \end{pmatrix} / (a^2 + b^2) \right) = 0 \] (2.1.12)
\[ \tau_t = E \left( \nabla v - \frac{\tau \cdot \nabla v}{\cos \alpha} \nabla v/||\nabla v|| \right) = E \left( \nabla v - \frac{||\nabla v|| \nabla v}{||\nabla v||} \right) = 0. \] (2.1.13)
The material is plastic for all \( x \), and loading because of the constraint \( \alpha < \pi/4 \). This solution represents a uniform shear, and might represent a solution for the system (2.1.4), (2.1.10) on the domain \( \{ x \in [0, 1]; y \in \mathbb{R}; t > 0 \} \) with boundary conditions \( v(0, y, t) = by \), \( v(1, y, t) = a + by \), for example, as in [25].

We introduce a perturbation \( w(x,t) \) in time and only one space dimension of this uniform shear: \( v = ax + by + w(x,t) \). Without loss of generality, we also normalize \( a^2 + b^2 = 1 \). Relabeling the components of \( \tau = \begin{pmatrix} p(x,t) \\ q(x,t) \end{pmatrix} \), the equations of motion become

\[
\begin{align*}
    w_t &= \nabla \cdot \tau = p_x, \\
    \tau_t &= E \begin{pmatrix}
        a + w_x \\
        b
    \end{pmatrix} - \frac{\chi(||\tau||) \left( \begin{pmatrix} a + w_x \\ b \end{pmatrix} \cdot \frac{\tau}{\cos \alpha} \right)}{\cos \alpha} \cos \alpha + R^T_{\alpha} \tau,
\end{align*}
\]

with the same yield condition, expressed by the indicator function \( \chi(||\tau||) \) defined in (2.1.11).

We observe that the equations are unchanged if a constant is added to \( w \).

**Spatial Discretization**

If the \( x \)-derivatives in (2.1.14)–(2.1.15) are replaced by finite differences, the partial differential equation becomes a system of ordinary differential equations. For \( N \in \mathbb{N} \), we let \( \Delta x = 1/N \). We use a staggered grid, so the function \( w(x,t) \) is approximated by values \( w_n(t) \) at the points \( x_n = n\Delta x \), \( n = 0, 1, 2, \ldots, N \), while the stress \( \tau(x,t) \) is approximated by values \( \tau_n(t) \) at the points \( x_{n+1/2} = n\Delta x + \frac{1}{2}\Delta x \). This grid is illustrated in Figure 2.1.
If we define a difference operator by

\[ D_n(f(x)) = (f_n - f_{n-1})/\Delta x, \]  

the discretized form of (2.1.14)–(2.1.15) is

\[ \frac{dw_n}{dt} = D_n(p) \]  
\[ \frac{d\tau_n}{dt} = E \left( \frac{a + D_{n+1}(w)}{b} - \frac{\chi(\|\tau_n\|)(p_n(a + D_{n+1}(w_n)) + q_n b)_{+} R_{\alpha}^T \tau_n}{\cos \alpha} \right) \]

for \( n = 0, \ldots, N - 1 \).

Applying periodic boundary conditions introduces the constraints \( w_N = w_0, \tau_N = \tau_0 \). Since adding a constant to \( w \) does not change (2.1.17)–(2.1.18), periodic boundary conditions are equivalent to Dirichlet conditions on \( w \).

### 2.1.3 Shear Band Solution

The equations (2.1.17)–(2.1.18) admit steady state solutions which are evocative of shear bands. By this we mean that \( D_n(w) \) is constant except for one value of \( n \), at which the

Figure 2.2: The shear band solution of (2.1.17)–(2.1.18)
perturbation $w$ makes a large jump, as shown in Figure 2.2. The velocity $v = ax + by + w(x, t)$ (and therefore displacement) exhibits a sudden jump along a particular value of $x$, much like a physical shear band; see Figure 2.3.

Because of the periodic boundary conditions, the location of this shear band is undetermined. (In practice, the location of the shear band will be determined by the initial conditions of the dynamic problem.) These shear band solutions are entirely plastic (that is, $\|\tau_n\| = 1$ and $\tau_n \cdot (a + D_n(w), b)^T \geq 0$ for all $n$).

As shown in equation (2.1.13), the expression $\tau = R_\alpha(\nabla v)/\|\nabla v\|$ satisfies the steady state form of (2.1.15). In the same manner, the expression

$$\tau_n = \frac{R_\alpha \left( a + D_{n+1}(w) \right)}{b}$$

(2.1.19)

satisfies (2.1.18), the discretized form of (2.1.15). The steady state form of (2.1.17), $D_n(p) = 0$, simply asserts that $p$ must be constant in an equilibrium solution.

For a plastic solution, the stresses $(p, q)$ must lie on the unit circle; coupled with the condition $D_n(p) = 0$, this implies that an equilibrium solution of the system (2.1.17)–(2.1.18) that is entirely plastic can have at most two states of stress: for each such solution,
Figure 2.4: Location of $\tau$ (in (a)) with respect to $\nabla v$ (in (b)).

there is an angle $\theta$ such that $p_n = \cos \theta$, and $q_n = \pm \sin \theta$ for all $n$. Each of these states corresponds to a single value of $\nabla v_n = (a + D_n(w), b)^T$, since the vector $\tau_n$ is a fixed rotation of the unit vector $\nabla v/\|\nabla v\|$; see Figure 2.4. We restrict $\theta$ to the range

$$0 \leq \theta \leq \pi/2,$$  \hfill (2.1.20)

which is the case depicted in Figure 2.4, appropriate if the underlying uniform shear satisfies $b < 0$. (There is a symmetric case for uniform shears with $b > 0$, for which $\pi/2 \leq \theta \leq \pi$. Hereafter we assume the case $b < 0$.) The values of variables at the shear band are subscripted “sb”, while those for all other $n$ are subscripted “nsb”. As can be seen in Figure 2.4, solutions of this type must satisfy

$$\theta - \alpha < 0,$$  \hfill (2.1.21)

in order to meet the condition $[\nabla v_{sb}]_2 = b < 0$.

Whether a given angle $\theta < \alpha$ generates an equilibrium solution depends on whether it satisfies the boundary conditions. In order to satisfy the boundary condition $w_N = w_0$, the sum over $n$ of the $D_n(w)$ must be zero. This is an algebraic relation between $D_n(w)_{sb}$ and $D_n(w)_{nsb}$:

$$D_n(w)_{sb} + (N - 1)D_n(w)_{nsb} = 0.$$  \hfill (2.1.22)
In Figure 2.4, the coordinates of $\nabla v_{\text{nsb}}$ and $\nabla v_{\text{sb}}$ are given by

$$\nabla v_{\text{nsb}} = \left( a + D_n(w)_{\text{nsb}} \right) = \left( b \cot(\theta - \alpha) \right),$$  \hspace{1cm} (2.1.23)

$$\nabla v_{\text{nsb}} = \left( a + D_n(w)_{\text{nsb}} \right) = \left( -b \cot(\theta + \alpha) \right).$$ \hspace{1cm} (2.1.24)

Therefore, the boundary condition (2.1.22) can be written in terms of $\mu$ as

$$(b \cot(\theta - \alpha) - a) + (N - 1)(-b \cot(\theta + \alpha) - a) = 0.$$ \hspace{1cm} (2.1.25)

Thus,

$$\Delta x \cot(\theta - \alpha) - (1 - \Delta x) \cot(\theta + \alpha) = a/b.$$ \hspace{1cm} (2.1.26)

We seek a solution $\theta$ to (2.1.26). As $\Delta x \to 0$, $D_n(w)_{\text{sb}} \to \infty$, so (2.1.23) indicates that $\theta \to \alpha$. In [22], a formula for $\theta$ is obtained to first order in $\Delta x$, rewritten here as

$$\theta = \alpha + \frac{1}{\cot \phi + \cot 2\alpha} \Delta x,$$ \hspace{1cm} (2.1.27)

in which the angle $\phi$ is defined by

$$(a, b) = (\cos \phi, \sin \phi).$$ \hspace{1cm} (2.1.28)

Uniqueness of the solution to (2.1.26) will be analyzed further in Chapter 4, but for typical values of $\alpha$ and $\phi$, (2.1.26) has a unique solution for $\theta$.

### 2.1.4 Periodic Solution

The shear band solution is stable for suitably high values of the shear modulus $E > E_{\text{crit}}$ [22]. In terms of $\theta$,

$$E_{\text{crit}} \approx \frac{8 \sin^2(\alpha - \theta) \sin(\theta) \cos(\alpha) \sin(\theta + \alpha)}{b^2 \Delta x}.$$ \hspace{1cm} (2.1.29)

Thus, as $\Delta x \to \infty$, using (2.1.27) and (2.1.28) we express $E_{\text{crit}}$ as

$$E_{\text{crit}} = \frac{4 \sin^2 2\alpha}{\cos^2 \phi + \sin 2\phi \cot 2\alpha + \cot^2 2\alpha} \Delta x + O(\Delta x^2).$$ \hspace{1cm} (2.1.30)

For example, for $\alpha = \pi/4$, (2.1.30) reduces to

$$E_{\text{crit}} = \frac{4\Delta x}{\cos^2 \phi}.$$ \hspace{1cm} (2.1.31)
For $E < E_{\text{crit}}$, the shear band solution is not stable. Instead, for $E$ near but less than $E_{\text{crit}}$, a periodic solution develops, in which the shear band changes its strength over time, and for some values of $\alpha$, disappears entirely for part of the period [22]. This periodic solution is described and analyzed extensively in Chapter 6.

2.2 Plastic Antiplane Shear Model

2.2.1 Description of the Plastic Model

The plastic antiplane shear model is the limit of the elastoplastic system as the shear modulus $E$ goes to infinity. In order for $\tau_t$ to remain bounded in (2.1.10), the factor

$$\nabla v - \frac{\chi(\|\tau\|)(\tau \cdot \nabla v)}{\cos \alpha}R_\alpha^T \tau \to 0,$$

which, as seen in (2.1.13), occurs if

$$\tau = R_\alpha \frac{\nabla v}{\|\nabla v\|}.$$

With an explicit representation for $\tau$, the system reduces to a partial differential equation in the one variable $v$:

$$v_t = \nu \cdot \tau = \nu \cdot R_\alpha \frac{\nabla v}{\|\nabla v\|}.$$

2.2.2 One-Dimensional Plastic Model

As in the elastoplastic case, we observe that a linear function $v = ax + by$ is a steady state solution of the equation (2.2.3):

$$v_t = \nabla \cdot \frac{R_\alpha \begin{pmatrix} a \\ b \end{pmatrix}}{\sqrt{a^2 + b^2}} = \nabla \cdot \frac{\begin{pmatrix} a \cos \alpha - b \sin \alpha \\ a \sin \alpha + b \cos \alpha \end{pmatrix}}{\sqrt{a^2 + b^2}} = 0.$$

We again introduce a perturbation $w(x,t)$ to this uniform shear, in which case (2.2.3) reduces to

$$w_t = \nabla \cdot \frac{R_\alpha \begin{pmatrix} a + w_x \\ b \end{pmatrix}}{\sqrt{(a + w_x)^2 + b^2}} = \partial_x \left( \frac{(a + w_x) \cos \alpha - b \sin \alpha}{\sqrt{(a + w_x)^2 + b^2}} \right).$$

The behavior of this partial differential equation depends on the values of $a$ and $b$ relative to $\alpha$; for certain ranges of these parameters, the trivial solution as well as any solution
satisfying Dirichlet boundary conditions is ill-posed to perturbations, demonstrating that the model still retains the ill-posedness typical of granular flow models [25]. (For other ranges of parameters, the trivial solution is stable, and initial data satisfying a bound on the upper value of \( w_x \) converges to the trivial solution as \( t \to \infty \).)

2.2.3 Shear Bands

Using the same difference operator defined by (2.1.16), the discretization of (2.2.5) is

\[
\frac{dw_n}{dt} = \frac{1}{\Delta x} \left( \frac{(a + D_{n+1}(w)) \cos \alpha - b \sin \alpha}{\sqrt{(a + D_{n+1}(w))^2 + b^2}} - \frac{(a + D_n(w)) \cos \alpha - b \sin \alpha}{\sqrt{(a + D_n(w))^2 + b^2}} \right). 
\]

(2.2.6)

The shear band solutions to the discretized elastoplastic antiplane shear model, found in Subsection 2.1.3, are entirely plastic solutions. It is perhaps not surprising then that these shear band solutions are also solutions of the plastic model.

2.3 Solutions to Riemann-type Problems in Elastoplastic Model of Longitudinal Displacement in a Rod

The periodic solution described in 2.1.4 consists of nearly linear sections separated by sharp transitions, which propagate through the domain; see Figure 6.1. The form of the periodic solution prompts the investigation of propagating transitions in the equations of elastoplastic antiplane shear.

The paper [21] deals with equations describing the longitudinal motion of a rod. However, many parallels can be drawn from that model to elastoplastic antiplane shear. Results that reflect those in that paper will be pointed out where they occur, but in order to motivate the approach taken, let me summarize the problem analyzed in [21]. The system in [21] also has three variables, a velocity \( v \), a stress \( \sigma \), and a history variable \( \gamma \), rather than our \( v, p, \) and \( q \).
The evolution equations are

\begin{align*}
v_t &= \sigma_x, \\
\sigma_t + k\gamma_t &= v_x, \\
\gamma_t &= \begin{cases} 
0 & \text{if } \sigma < \gamma \text{ (elastic)} \\
(\sigma)_+ & \text{if } \sigma = \gamma \text{ (plastic)}.
\end{cases}
\end{align*}

(2.3.1)

Consider continuous solutions with discontinuous \(x\)-derivatives of the “SI problem”, specifically (2.3.1) with the initial conditions

\begin{align*}
v(x, 0) &= \begin{cases} 
a_Lx \text{ for } x \leq 0 \\
a_Rx \text{ for } x \geq 0
\end{cases} \\
\sigma(x, 0) &= \begin{cases} 
b_Lx \text{ for } x \leq 0 \\
b_Rx \text{ for } x \geq 0
\end{cases} \\
\gamma(x, 0) &= \begin{cases} 
c_Lx \text{ for } x \leq 0 \\
c_Rx \text{ for } x \geq 0
\end{cases}.
\end{align*}

(2.3.2)

This problem is scale invariant, meaning that the equations and initial conditions are unchanged under the scaling

\begin{align*}
\tilde{v}(x, t) &= \eta^{-1}v(\eta x, \eta t), \\
\tilde{\sigma}(x, t) &= \eta^{-1}\sigma(\eta x, \eta t), \\
\tilde{\gamma}(x, t) &= \eta^{-1}\gamma(\eta x, \eta t).
\end{align*}

(2.3.3)

Scale invariant solutions \(U = (v(x, t), \sigma(x, t), \gamma(x, t))^T\) take the form \(U(x, t) = tF(x/t)\).

From this it follows that, between characteristics, the solution in a wedge of either plastic or elastic deformation is in fact linear: \(U(x, t) = xU_0 + tU_1\). In addition, the form of \(U\) determines that any transition from elastic to plastic (or vice versa) occurs along a radial line.

Solutions to (2.3.1)–(2.3.2) are found by first dividing the domain into the two regions \(x < 0\) and \(x > 0\). Then all the transitions, including elastic and plastic waves as well as changes in state, that could result from the initial conditions for each region are characterized by a locus of values of \(v_x\) and \(\sigma_x\) that could occur along the \(x = 0\) line as a result of these transitions. The loci for the two regions intersect at one point: thus there is one specified set of transitions resulting in a solution with continuous \(v_x\) and \(\sigma_x\) at \(x = 0\). If the line \(x = 0\) is in an elastic state, then \(x = 0\) is characteristic for the third variable
\( \gamma \), so \( \gamma_x \) may jump along \( x = 0 \). On the other hand, if the line \( x = 0 \) is in a plastic state, then \( \gamma_x = \sigma_x \); continuity of \( \sigma_x \) implies the continuity of \( \gamma_x \). Therefore, a continuous solution to (2.3.1)–(2.3.2) can be constructed by choosing the transitions corresponding to the intersection point of the aforementioned loci of \((v_x, \sigma_x)\) along \( x = 0 \).

The problem (2.1.14)–(2.1.15) has some critical differences: for example, the plastic deformation does not admit a solution of the form \( U(x, t) = tF(x/t) \), because of the dependence on \( \tau \) in the evolution equations. Even in the elastic state, which does admit such solutions, the conditions for transition to the plastic state are not invariant along radial lines. Despite these differences, much of the analysis in [21] can be applied, albeit locally, to elastoplastic antiplane shear. In particular, we will find that the same method to construct a continuous solution can be applied to find a local description of the elastoplastic antiplane shear model at a jump in the derivatives \((v_x, p_x, q_x)\). See [4] for more general analysis of perturbations to (2.3.1)–(2.3.2).
Chapter 3

Shocks in Steady Hopper Flow

3.1 Hugoniot locus

3.1.1 2-d Hopper Flow

We restate the Coulomb yield condition (1.2.7) for convenience:

$$\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2 = \left(\frac{\sigma_x + \sigma_y}{2}\right)^2 \sin^2 \delta. \quad (3.1.1)$$

As shown in Figure 1.2, the yield condition (3.1.1) describes a cone.

The steady state momentum balance equation (1.2.2) in two dimensions is

$$\partial_x \left( \frac{\sigma_x}{\tau_{xy}} \right) + \partial_y \left( \frac{\tau_{xy}}{\sigma_y} \right) = \begin{pmatrix} 0 \\ -g \end{pmatrix}, \quad (3.1.2)$$

where $g$ is acceleration due to gravity and $x$ is the downward vertical direction. This system can be viewed as a system of conservation laws, with $x$ playing the role of time. The Rankine-Hugoniot condition for such a system, using square brackets to denote jumps across a discontinuity interface $x = sy$ would be

$$[\tau_{xy}] = s[\sigma_x] \quad (3.1.3)$$

$$[\sigma_y] = s[\tau_{xy}]. \quad (3.1.4)$$

Where the brackets denote the jump across the discontinuity:

$$[f(x, y)] = \lim_{x \to sy^+} f(x, y) - \lim_{x \to sy^-} f(x, y). \quad (3.1.5)$$
Eliminating $s$ from (3.1.3)–(3.1.4), we have the Hugoniot locus

$$[\sigma_x][\sigma_y] = [\tau_{xy}]^2. \quad (3.1.6)$$

Equation (3.1.6) relates six unknowns, the values of $(\sigma_x, \sigma_y, \tau_{xy})$ on each side of the discontinuity. If the stress components are specified on one side of the discontinuity, the equation of the Hugoniot locus (3.1.6) also describes a cone; the intersection of the cones (3.1.1) and (3.1.6) is the locus of possible states on the other side of the discontinuity satisfying the yield condition.

**Theorem.** Suppose the stress components $(\sigma_x^L, \sigma_y^L, \tau_{xy}^L)$ are fixed on one side of the discontinuity $x = sy$, satisfying the yield condition (3.1.1) with $\delta < \pi/2$. Then the Hugoniot locus (3.1.6) is a closed curve, topologically equivalent to a figure-of-eight.

**Proof.** The cone (3.1.1) has axis $\sigma_x = \sigma_y$, $\tau_{xy} = 0$, with opening angle $\beta$, where $\beta$ satisfies the equation $\tan \beta = \sin \delta$. For $\delta < \pi/2$, $\tan \beta < 1$ and therefore $\beta < \pi/4$.

On the other hand, the cone (3.1.6) can be written

$$\left(\left[\frac{\sigma_x}{2}\right] - \left[\frac{\sigma_y}{2}\right]\right)^2 + [\tau_{xy}]^2 = \left(\left[\frac{\sigma_x}{2}\right] + \left[\frac{\sigma_y}{2}\right]\right)^2. \quad (3.1.7)$$

Therefore, this cone has a parallel axis to (3.1.1), although its vertex has been shifted to $(\sigma_x^L, \sigma_y^L, \tau_{xy}^L)$. The opening angle is $\pi/4$.

The intersection between two cones of different opening angle is nontrivial. However, we can demonstrate that this intersection is contained within a parabolic cylinder.

First we simplify by denoting

$$\xi = \frac{\sigma_x - \sigma_y}{2} \quad (3.1.8)$$

$$\nu = \frac{\sigma_x + \sigma_y}{2} \quad (3.1.9)$$

$$\xi^L = \frac{\sigma_x^L - \sigma_y^L}{2} \quad (3.1.10)$$

$$\nu^L = \frac{\sigma_x^L + \sigma_y^L}{2}. \quad (3.1.11)$$

Then, the cones (3.1.1) and (3.1.6) are

$$\xi^2 + \tau_{xy}^2 = \nu^2 \sin^2 \delta, \quad (3.1.12)$$

$$(\xi - \xi^L)^2 + (\tau_{xy} - \tau_{xy}^L)^2 = (\nu - \nu^L)^2. \quad (3.1.13)$$
Figure 3.1: Hugoniot Locus of Hopper Flow in \((\sigma_x, \tau)\) space, from initial values \(\sigma_x \approx 3.40, \quad \tau \approx 0.19\)

Expanding (3.1.13) and applying (3.1.12) gives

\[-\xi \xi^L - \tau_{xy} \tau_{xy}^L + \nu L = (\nu^2 + (\nu^L)^2) \cos^2 \delta / 2.\]  

(3.1.14)

This is a parabolic cylinder, rewritable as

\[\xi \xi^L + \tau_{xy} \tau_{xy}^L = -\frac{\cos^2 \delta}{2} \left( \nu - \frac{\nu L}{\cos^2 \delta} \right)^2 + \frac{(\nu L)^2}{2} \left( \frac{1}{\cos^2 \delta} - \cos^2 \delta \right).\]  

(3.1.15)

The axis of the parabolic cylinder (3.1.15) is the plane \(\nu = \nu L / \cos^2 \delta\), which is perpendicular to the axes of both cones (which are the \(\nu\)-axis and the line parallel to it \((\xi = \xi^L, \tau_{xy} = \tau_{xy}^L, \nu \in \mathbb{R})\)). Therefore the parabolic cylinder cuts through the cones. The projection of the yield cone (3.1.12) onto the \((\xi \xi^L + \tau_{xy} \tau_{xy}^L, \nu)\) plane is

\[\xi \xi^L + \tau_{xy} \tau_{xy}^L = \nu^2 \sin^2 \delta.\]  

(3.1.16)

The projection of the Hugoniot locus into \((\sigma_x, \tau_{xy})\) space is shown in Figure 3.1.

The yield condition constrains \(\sigma_y\) to a cone, which results in the locus looping back on itself. Most important is determining if there are Lax shocks which permit a transition from passive to active state (that is, from the top of the cone to the bottom.)

### 3.2 Shock speeds

Once the Hugoniot locus is determined from (3.1.6), the shock speed associated with any point on the locus can easily be determined by (3.1.3). To determine the admissibility of a shock, we compare the speed of the shock with the speed of the characteristics
on each side of the shock [12]. For hopper flow, the speeds of the characteristics are given by

\[ \lambda_{\pm} = \tan \left( \psi \pm \frac{\pi}{4} \right), \tag{3.2.1} \]

where \( \psi \) is half the angle around the yield cone (\( \psi = 0 \) is the extreme of the passive case, the highest point on the cone in Figure 1.2.) The results are shown in Figure 3.2. The asymptotes for the characteristic speeds lie at \( \psi = \pm \pi/4 \), the boundary between passive and active states. As a result, the Lax shocks (those for which \( \lambda_+^L > s_+ > \lambda_+^R \), and similarly for the minus branch) from a left state in the passive region have a right state in the passive region as well: there are no Lax shocks between passive and active states.

Thus, if there are discontinuities between passive and active states in steady flow, they violate the Lax entropy condition. It is not clear how to identify mathematically the physically occurring active to passive discontinuities. One possibility is a kinetic relation of the type discussed by various authors in the context of dynamic phase transitions and other applications (see the book by LeFloch [13] for an extensive bibliography.)
Chapter 4

Antiplane Shear

4.1 Instability of Piecewise Linear Solutions in Plastic Antiplane Shear

Consider the plastic antiplane shear model (2.2.3):

\[ v_t = \nabla \cdot \tau, \quad \tau = R_0 \frac{\nabla v}{||\nabla v||}, \]  

(4.1.1)

In Subsection 2.2.2, it was shown that a uniform shear \( v = ax + by \) satisfies (4.1.1). We went on to introduce a one-dimensional perturbation to the uniform shear, discretized in space, and found shear band solutions to the discretized problem.

Another approach is to consider solutions of (4.1.1) with continuous velocity but a discontinuity in the stress \( \tau \) along a line \( y = sx \). We take \( v_s \) to be piecewise linear:

\[ v_s = \begin{cases} 
   a_Lx + b_Ly, & y > sx \\
   a_Rx + b_Ry, & y < sx
\end{cases} \]  

(4.1.2)

We treat \( x \) as timelike, and \( q \) as a function of \( p \), so that \( p_x + q_y = 0 \) has the form of a conservation law. The Rankine-Hugoniot condition is

\[ s = \frac{[q]}{[p]}, \]  

(4.1.3)

where the bracket notation \([f(x, y)]\) indicates the jump in \( f \) across the line \( y = sx \). In this section, we shall show that any such solution to (4.1.1) is unstable with respect to perturbations perpendicular to the front \( y = sx \).
**Theorem.** Suppose $v_s(x, y)$, as defined in (4.1.2), is a solution of (4.1.1) satisfying (4.1.3). Then this solution is unstable with respect to a perturbation in time and one space dimension $w(y - sx, t)$.

**Proof.** Let $v = v_s + w(y - sx, t)$. Then the dynamic equation (4.1.1) is

$$
\partial_t v = \nabla \cdot \left( R_\alpha \frac{\nabla v}{\|\nabla v\|} \right).
$$

(4.1.4)

The perturbation $w(y - sx, t)$ is one dimensional in space, perpendicular to the front $y = sx$. We define the variable $\xi$ to represent the distance from the discontinuity:

$$
\xi = y - sx,
$$

(4.1.5)

and the unit vector perpendicular to the discontinuity

$$
\hat{\xi} = \nabla \xi / \|\nabla \xi\| = \frac{(-s, 1)}{\sqrt{1 + s^2}}.
$$

(4.1.6)

Then (4.1.4) can be written as

$$
\partial_t w = \partial_\xi \left( R_\alpha \frac{\nabla v_s + (\partial_\xi w)\hat{\xi}}{\|\nabla v_s + (\partial_\xi w)\hat{\xi}\|} \cdot \hat{\xi} \right).
$$

(4.1.7)

This can be abbreviated as

$$
\partial_t w = \partial_\xi (F(\partial_\xi w)),
$$

(4.1.8)

where

$$
F(u) = \left( R_\alpha \frac{\nabla v_s + u\hat{\xi}}{\|\nabla v_s + u\hat{\xi}\|} \cdot \hat{\xi} \right).
$$

(4.1.9)

Let $u = \partial_\xi w$, and apply $\partial_\xi$ to both sides of (4.1.8):

$$
\partial_t u = \partial_\xi (F'(u)\partial_\xi u).
$$

(4.1.10)

Equation (4.1.10) is a diffusion equation; if $F'(u) > 0$, it behaves like the heat equation, but if $F'(u) < 0$, then (4.1.10) is unstable, like the backwards heat equation.

In Figure 4.1(a), we show that if the stresses associated with $v_s$ on each side of the discontinuity and the vector $\hat{\xi}$ are plotted on the same axes, $\hat{\xi}$ will bisect the two stresses, since the slope of the line connecting the stresses is $s$ (by (4.1.3)) and $\hat{\xi}$ is constructed to be perpendicular to lines with slope $s$. (Which stress is on which side of the discontinuity is irrelevant, so the stresses are labeled $\tau_A$ and $\tau_B$ in the figure.)
The maximum of $F(u)$ can be seen by inspection of (4.1.9) to occur at $u_{\text{max}}$, where
\[ \frac{\nabla v_s + u_{\text{max}} \xi}{\| \nabla v_s + u_{\text{max}} \xi \|} = R^T_{\alpha} \xi. \] (4.1.11)

In Figure 4.1(b), we see that since $\xi$ bisects the two stresses, $R^T_{\alpha} \xi$ bisects the two velocity gradients. Therefore, the value of $u_{\text{max}}$ on each side of the discontinuity has a different sign.

The structure of $F$ (given by (4.1.9)) is such that $F$ is increasing for $u < u_{\text{max}}$ and decreasing for $u > u_{\text{max}}$; see Figure 4.2. Since $u_{\text{max}}$ switches sign across the discontinuity, $F$ is increasing at $u = 0$ on one side of the shock, but is decreasing on the other side of the front. Therefore, the solution is unstable on one side of the front with respect to arbitrarily small perturbations $w$.

4.2 Shear Band Solution of Spatially Discretized Antiplane Shear

We now quantify the shear band solution described in 2.1.3. Recall the discretized form of the antiplane shear problem, (2.2.6), rewritten here as
\[ \frac{dw_n}{dt} = \frac{1}{\Delta x} \left[ R_{\alpha} \frac{\nabla_{n+1} v}{\| \nabla_{n+1} v \|} - R_{\alpha} \frac{\nabla_n v}{\| \nabla_n v \|} \right], \] (4.2.1)
where $\nabla_n v = (a + D_n(w), b)^T$. Since $a^2 + b^2 = 1$, we define the angle $\phi$ by (2.1.28), restated here:

$$a = \cos \phi, \quad b = \sin \phi.$$  

Consider a shear band solution to (4.2.1) under a periodic boundary condition:

$$w_0 = w_N.$$  

Such a solution is identified by two values of $D_n(w) = (w_n - w_{n-1})/\Delta x$: the difference operator $D_n(w)$ takes the value $D_{n\text{sb}}$ for all values of $n$ except one, for which $D_n(w) = D_{sb}$. Note that the position of the shear band (the value of $n$ for which $D_n(w) = D_{sb}(w)$) is undetermined.

We proceed to determine conditions on the parameters $(\alpha, \phi, N)$ so that a shear band solution exists, and determine how many such shear band solutions there are for a given set of parameters.

As in 2.1.3, only solutions with a single jump in velocity are considered. Recall that $\alpha$ is chosen in the range

$$0 \leq \alpha \leq \pi/2.$$  

We assume the angle $\phi$ of the underlying uniform shear is constrained so that $b = \sin \phi < 0$ (cf. the discussion in Subsection 2.1.3; the case $b > 0$ generates a symmetric result). We further constrain $\phi$ to the range

$$-\pi/2 \leq \phi < 0,$$  

because the case $-\pi \leq \phi < -\pi/2$ gives rise to the same shear bands, with shifted values for $(D_{n\text{sb}}, D_{sb})$. In Figure 4.3, we show an example of velocity gradients $(\nabla v_{n\text{sb}}, \nabla v_{sb})$ and the
Figure 4.3: Example of Velocity and Stress in a Shear Band Equilibrium Solution. Given the underlying shear \(v_0 = ax + by\) and the two values \((D_{nsb}, D_{sb})\) of the difference operator \(D_n(w)\), the velocity gradients are given by \(\nabla v_{nsb} = (a + D_{nsb}, b), \nabla v_{sb} = (a + D_{sb}, b)\). If the associated stresses \((\tau_{nsb} = R_\alpha \frac{\nabla v_{nsb}}{||\nabla v_{nsb}||}, \tau_{sb} = R_\alpha \frac{\nabla v_{sb}}{||\nabla v_{sb}||})\) have the same first coordinate, then these values of \((D_{nsb}, D_{sb})\) give a steady state solution of (4.2.1).

The corresponding stress vectors \((\tau_{nsb}, \tau_{sb})\) for a shear band solution with parameters satisfying (4.2.4) and (4.2.5). Note that the point \((a', b')\) represents a underlying uniform shear in the case \(-\pi \leq \phi < -\pi/2\); that choice of \(\phi\) would result in the same shear band solution, with the length of the chord between \((a', b')\) and \((a, b)\) added to both \(D_{nsb}\) and \(D_{sb}\).

**Theorem.** Let \((\alpha, \phi, N)\) be a parameter set satisfying (4.2.4), (4.2.5), and \(N > 2\). Then:

1. if \(-\alpha < \phi\), then there are two pairs \((D_{nsb}, D_{sb})\) corresponding to steady state shear band solutions of (4.2.1) and (4.2.3); these pairs are in quadrants II and IV.

2. if \(\phi < -2\alpha\), then there are no steady state shear band solutions of (4.2.1) and (4.2.3).

3. if \(-2\alpha < \phi < -\alpha\), the number of solutions depends on \(N\); for each \((\alpha, \phi)\), there exists a value \(N_{crit}\) such that for \(N < N_{crit}\), there are no steady state shear band solutions; for \(N = N_{crit}\), there is exactly one shear band solution, and for \(N > N_{crit}\), there are two shear band solutions. For all these solutions, the pair \((D_{nsb}, D_{sb})\) is in quadrant IV.

**Proof.** A periodic boundary condition gives

\[
w_N - w_0 = \sum_{n=0}^{N} D_n(w) \Delta x = 0, \quad (4.2.6)
\]
Therefore

\[ D_{sb} = -(N - 1)D_{nsb}. \quad (4.2.7) \]

Thus, \( D_{sb} \) is given in terms of \( D_{nsb} \).

On the other hand, the steady state condition \( \frac{dw_n}{dt} = 0 \) implies that the first coordinate of \( R_\alpha(\nabla_n v/\|\nabla_n v\|) \) is constant for all \( n \). Given a value for \( D_{nsb} \), we can compute \( R_\alpha(\nabla_{nsb} v/\|\nabla_{nsb} v\|) \), determine the other unit vector with the same first coordinate, and scale this to solve for \( D_{sb} \). This process is immensely simplified by separating \( \nabla_{nsb} v \) into the components

\[
\nabla_{nsb} v = \begin{pmatrix} a \\ b \end{pmatrix} + D_{nsb} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix} + D_{nsb} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (4.2.8)
\]

Then the stress vector \( \tau_{nsb} \) is

\[
\tau_{nsb} = R_\alpha(\nabla_{nsb} v)/\|\nabla_{nsb} v\| = \frac{1}{\|\nabla_{nsb} v\|} \left( \begin{pmatrix} \cos(\phi + \alpha) \\ \sin(\phi + \alpha) \end{pmatrix} + D_{nsb} \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \right). \quad (4.2.9)
\]

As illustrated in Figure 4.3, the stress vector \( \tau_{sb} \) is the reflection of \( \tau_{nsb} \) around the \( x \)-axis. This is simply

\[
\tau_{sb} = \frac{1}{\|\nabla_{sb} v\|} \left( \begin{pmatrix} \cos(-\phi + \alpha) \\ \sin(-\phi + \alpha) \end{pmatrix} + D_{nsb} \begin{pmatrix} \cos(-\alpha) \\ \sin(-\alpha) \end{pmatrix} \right). \quad (4.2.10)
\]

To recover \( \nabla_{sb} v/\|\nabla_{sb} v\| \), \( \tau_{sb} \) is rotated counterclockwise by \( \alpha \):

\[
\nabla_{sb} v/\|\nabla_{sb} v\| = \frac{1}{\|\nabla_{sb} v\|} \left( \begin{pmatrix} \cos(-\phi + 2\alpha) \\ \sin(-\phi + 2\alpha) \end{pmatrix} + D_{nsb} \begin{pmatrix} \cos(-2\alpha) \\ \sin(-2\alpha) \end{pmatrix} \right) \quad (4.2.11)
\]

\[
\nabla_{sb} v = \frac{\|\nabla_{sb} v\|}{\|\nabla_{nsb} v\|} \left( \begin{pmatrix} \cos(-\phi + 2\alpha) \\ \sin(-\phi + 2\alpha) \end{pmatrix} + D_{nsb} \begin{pmatrix} \cos(-2\alpha) \\ \sin(-2\alpha) \end{pmatrix} \right). \quad (4.2.12)
\]

The \( y \)-component of \( \nabla_{sb} v \) is \( b = \sin \phi \) (as seen in Figure 4.3). Therefore, the unknown ratio \( \|\nabla_{sb} v\|/\|\nabla_{nsb} v\| \) in (4.2.12) can be solved for in terms of \( \phi \) and \( \alpha \):

\[
\frac{\|\nabla_{sb} v\|}{\|\nabla_{nsb} v\|} = \frac{\sin \phi}{\sin(-\phi + 2\alpha) + D_{nsb} \sin(-2\alpha)}. \quad (4.2.13)
\]

The \( x \)-component of \( \nabla_{sb} v \) is \( \cos \phi + D_{sb} \). Thus, (4.2.12) and (4.2.13) give that

\[
D_{sb} = -\cos \phi - \sin \phi \frac{\cos(\phi + 2\alpha) + D_{nsb} \cos(2\alpha)}{\sin(\phi + 2\alpha) + D_{nsb} \sin(2\alpha)}, \quad \text{or} \quad (4.2.14)
\]

\[
\left( D_{sb} + \frac{\sin(\phi + 2\alpha)}{\sin(2\alpha)} \right) \left( D_{nsb} + \frac{\sin(\phi + 2\alpha)}{\sin(2\alpha)} \right) = \frac{\sin^2 \phi}{\sin^2(2\alpha)}. \quad (4.2.15)
\]
The equation \(4.2.15\) describes a hyperbola of slope pairs \((D_{nsb}, D_{sb})\) satisfying the condition \(\frac{dw_n}{dt} = 0\). If this pair also satisfies \(4.2.7\), then the boundary condition \(w_0 = w_N\) is also satisfied. Thus, given the parameters \((\alpha, \phi, N)\), the possible shear band solutions are characterized by the intersection points of the line \((4.2.7)\) with the hyperbola \((4.2.15)\), as seen in Figure 4.4.

The vertex of the hyperbola is
\[
D_{nsb} = D_{sb} = -\frac{\sin \phi + \sin(\phi + 2\alpha)}{\sin(2\alpha)}.
\] (4.2.16)

The linear constraint \((4.2.7)\) passes through the second and fourth quadrants, so all admissible pairs will be in one of those quadrants. If the vertex \((4.2.16)\) is in the third quadrant, then there are necessarily exactly two intersection points of the line and hyperbola. The vertex \((4.2.16)\) is 0 when \(\phi = -\alpha\). For the range
\[
-\alpha < \phi < 0,
\] (4.2.17)
the vertex \((4.2.16)\) is negative, so there are two intersection points, one with \(D_{nsb} < 0\), \(D_{sb} > 0\), and the other with the signs reversed. This is case 1.

The asymptotes of the hyperbola \((4.2.15)\) are
\[
D_{nsb} = -\frac{\sin(\phi + 2\alpha)}{\sin(2\alpha)} \quad \text{and} \quad D_{sb} = -\frac{\sin(\phi + 2\alpha)}{\sin(2\alpha)}.
\] (4.2.18)
If
\[ \phi < -2\alpha, \]  
then the right-hand side of the asymptotes is positive, and the hyperbola is contained in the first quadrant, so it does not intersect the line (4.2.7), and there are no shear band solutions. This is case 2.

Finally, in the intermediate case
\[ -2\alpha < \phi < -\alpha, \]  
the vertex of the hyperbola is in the first quadrant, but the asymptotes are negative. For suitably large \( N \), the line (4.2.7) will intersect the hyperbola (4.2.15) twice, while for \( N \) suitably near 2, the line will not intersect the hyperbola at all. (The slope of the line tangent to the hyperbola at the vertex is -1, which is the slope of the line (4.2.7) if \( N = 2 \). If the vertex is in the first quadrant, it is clear that the line (4.2.7) with \( N = 2 \) will not intersect it.) Between the two extremes is a value of \( N \) so that the line (4.2.7) is tangent to the hyperbola (4.2.15) at some point in the second quadrant; this is \( N_{\text{crit}} \), and completes the third case.

Although there are cases in Theorem 4.2 where two shear band solutions satisfy the boundary condition, only one such solution is seen in numerical simulations, because only the solution in the second quadrant (\( D_{\text{nsb}} < 0, D_{\text{sb}} > 0 \)) with the largest value of \( D_{\text{sb}} \) is stable [25]. It is also interesting to consider the limit of the two pairs \( (D_{\text{nsb}}, D_{\text{sb}}) \) as \( N \to \infty \) (for cases 1 and 3). The stable solution approaches a discontinuity, with
\[ D_{\text{nsb}} \to -\frac{\sin(\phi + 2\alpha)}{\sin(2\alpha)}, \quad D_{\text{sb}} \to \infty, \]  
while the unstable solution approaches a constant \( w \), a trivial perturbation:
\[ D_{\text{nsb}} \to 0, \quad D_{\text{sb}} \to \frac{\sin^2(\phi) - \sin^2(\phi + 2\alpha)}{\sin(2\alpha)\sin(\phi + 2\alpha)}. \]

### 4.3 Numerical Implementation of Elastoplastic Antiplane Shear

#### 4.3.1 Enforcing the Yield Condition \( \|\tau\| \leq 1. \)

One approach in implementing a numerical scheme to solve (2.1.14)–(2.1.15) is to determine which form of the equations apply at each point in time and space, and solve
explicitly forward in time. As a result of the proof of Lemma 1 in Subsection 2.1.1, if a step results in $\|\tau\| > 1$, ensuing steps will not increase $\|\tau\|$ any further:

**Corollary 2.** $\|\tau\|_t > 0$ if and only if $\|\tau\| < 1$ and $\tau \cdot \nabla v > 0$.

*Proof.* Equation (2.1.8) demonstrates that the sign of $\|\tau\|_t$ is the same as that of $\tau \cdot \tau_t$. If $\tau \cdot \nabla v > 0$ (loading) and $\|\tau\| \geq 1$ (plastic), (2.1.9) expresses $\tau \cdot \tau_t$ as

$$\tau \cdot \tau_t = E(\tau \cdot \nabla v)(1 - \|\tau\|^2).$$

(4.3.1)

Thus, if $\|\tau\| > 1$ and $\tau \cdot \nabla v > 0$, $\tau \cdot \tau_t < 0$, and therefore $\|\tau\|_t < 0$ also. On the other hand, if $\tau \cdot \nabla v < 0$, then (2.1.6) gives that

$$\tau \cdot \tau_t = E(\tau \cdot \nabla v) < 0.$$  

(4.3.2)

Thus, only when $\tau \cdot \nabla v > 0$ and $\|\tau\| < 1$, that is, the case of elastic loading, is $\|\tau\|_t > 0$. □

This approach is not as stable as desired; very small timesteps (on the order of three or more orders of magnitude less than the space discretization size) are required to obtain results for which noise does not overwhelm the system [24].

Another approach is to enforce the yield locus, so that if any step results in $\|\tau\| > 1$, $\tau$ is renormalized so that $\|\tau\| = 1$. Such a case should result in the plastic state, since the yield locus has been reached. Unfortunately, machine arithmetic may compute the norm of the renormalized $\tau$ as less than 1, and the elastic form of the equations would be incorrectly applied. This miscalculation can be avoided by setting a flag variable, so that the plastic form of the equations is always used immediately after a renormalization. This approach results in greater stability than depending on Corollary 2. Another approach to avoiding the renormalization error in machine arithmetic would be to renormalize $\tau$ so that $\|\tau\| = 1 + \epsilon$, with $\epsilon$ one order of magnitude larger than machine error.

A final approach is to exploit the simple form of the yield condition by using polar coordinates. In that case, $p$ and $q$ are replaced with variables $r$ and $\phi$. The advantage of this method is that, in the plastic case, the evolution for $r$ is simply $r_t = 0$, and the transition to elasticity is a condition solely on $\phi$ (that is, that $\tau \cdot \nabla v \leq 0$), so any computed value of $r > 1$ can be ignored (simply solving the plastic equations with $r$ set to 1), without introducing renormalization and the associated roundoff problems.
4.3.2 Comparison with Other Differential Algebraic Equations (DAE) Problems

The elastoplastic antiplane shear problem is a partial differential equation whose properties change according to algebraic constraints.

Some DAE techniques use the algebraic constraints to limit the admissibility of results of an unconstrained method: projected Newton, for example, forces Newton steps to remain within the boundaries of the problem. We adopt a method similar to this, although when the boundary is reached, we continue solving the problem forward in time, using the plastic regime equations.

Other DAE techniques, applied when a problem’s solution is expected to lie along an algebraic constraint, reconfigure the problem in such a way to look for solutions along the constraint. The completely plastic antiplane shear problem is addressed this way in [20] and [25]. These techniques are inappropriate for the elastoplastic problem, in which the interaction between elastic and plastic states is significant.

4.4 Elastic and Plastic Characteristic Wave Speeds

Determining and comparing the elastic and plastic wave speeds in (2.1.14)–(2.1.15) may help us characterize the periodic solution described in 2.1.4, as well as give insight into the dependence of the periodic solution on the parameters of the problem.

4.4.1 Elastic Wavespeed

In the elastic regime, the equations of motion are simply

\[ w_t = p_x, \quad \tau_t = E \left( \begin{array}{c} a + w_x \\ b \end{array} \right), \]

Consider the mixed partials of \( \tau \):

\[ \tau_{xt} = \begin{pmatrix} p_x \\ q_x \end{pmatrix}_t = \begin{pmatrix} w_t \\ q_t \end{pmatrix}_t = \begin{pmatrix} w_{tt} \\ q_{xt} \end{pmatrix}, \]

\[ \tau_{tx} = \begin{pmatrix} p_t \\ q_t \end{pmatrix}_x = E \begin{pmatrix} a + w_x \\ b \end{pmatrix}_x = E \begin{pmatrix} w_{xx} \\ 0 \end{pmatrix}. \]
So equality of mixed partials implies $w_{tt} = E w_{xx}$, (and $q_{xt} = 0$.) A solution of the form $w(x, t) = \tilde{w}(x \pm ct)$ satisfies this equation if $w_{tt} = c^2 \tilde{w}'' = E w_{xx} = E \tilde{w}''$, or $c^2 = E$. So the elastic wavespeed is $c_e = \pm \sqrt{E}$. (Since $p_t = E(a + w_x)$ and $p_x = w_t = \pm c_e \tilde{w}''$, $p_{tt} = E w_{xt} = \pm c_e^2 \tilde{w}'' = c_e^2 p_{xx}$ also.)

### 4.4.2 Plastic Wavespeed

In the plastic regime,

$$w_t = p_x$$

$$\tau_t = E \left( \begin{pmatrix} a + w_x \\ b \end{pmatrix} - \frac{1}{\cos \alpha} \begin{pmatrix} a + w_x \\ b \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \right) R^T_\alpha \begin{pmatrix} p \\ q \end{pmatrix}. \tag{4.4.6}$$

Of course, the situation is more complicated, but we can proceed as before:

$$p_{tx} = E w_{xx} \left( 1 - \frac{p}{\cos \alpha} \left[ R^T_\alpha \begin{pmatrix} p \\ q \end{pmatrix} \right]_1 \right) + \text{first order terms} \tag{4.4.7}$$

$$p_{xt} = w_{tt} \text{, and so} \tag{4.4.8}$$

$$w_{tt} = E w_{xx} \left( 1 - \frac{p}{\cos \alpha} \left( p \cos \alpha + q \sin \alpha \right) \right) + \text{first order terms}. \tag{4.4.9}$$

To calculate wave speeds, we extract the principal symbol of (4.4.9) by neglecting the first order terms. The plastic wavespeed can only be defined when

$$1 - \frac{p}{\cos \alpha} \left( p \cos \alpha + q \sin \alpha \right) = q^2 - pq \tan \alpha > 0; \tag{4.4.10}$$

otherwise (4.4.9) is not hyperbolic.

Consider wave perturbations to a constant stress $\tau_{\text{nsb}}$, where

$$\tau_{\text{nsb}} = \begin{pmatrix} \cos \theta \\ -\sin \theta \end{pmatrix}. \tag{4.4.11}$$

(We think of this stress as the stress outside a shear band, as shown in Figures 2.4 and 4.3.) In that case, $\theta$ is constrained by (2.1.20) and (2.1.21), which we combine here for convenience:

$$0 \leq \theta < \alpha \leq \pi/2. \tag{4.4.12}$$

**Theorem.** Let $0 \leq \alpha < \pi/2$ and $E > 0$ be given, let $c_e = \sqrt{E}$, and let $\tau_{\text{nsb}}$, as given by (4.4.11), be the stress outside the shear band in a steady state shear band solution to (4.4.5)-(4.4.6). (As a result, $\tau_{\text{nsb}}$ is subject to the constraint (4.4.12).) Then:
1. the criterion (4.4.10) for definition of the plastic wave speed $c_p$ of wave perturbations to $\tau_{nsb}$ is satisfied.

2. the plastic wave speed $c_p$ is given by

$$c_p = c_e \sqrt{\frac{\sin \theta}{\cos \alpha}} \sin(\theta + \alpha). \quad (4.4.13)$$

3. The plastic wave speed $c_p$ is equal to the elastic wave speed $c_e$ if and only if

$$\theta = \pi/2 - \alpha. \quad (4.4.14)$$

If $\theta < \pi/2 - \alpha$, then $c_p < c_e$; if $\theta > \pi/2 - \alpha$, then $c_p > c_e$.

Proof. The criterion (4.4.10), for $p = \cos \theta$, $q = -\sin \theta$, is

$$\frac{\sin \theta}{\cos \alpha} \sin(\theta + \alpha) > 0. \quad (4.4.15)$$

Given the constraint (4.4.12), the condition (4.4.15) holds if $0 < \theta < \pi - \alpha$. But this is less strict than (4.4.12), because $\alpha \leq \pi/2$. Thus, if (4.4.12) holds, (4.4.10) holds as well.

From (4.4.9), the wave speed is

$$c_p = \sqrt{E \left(1 - \frac{p (p \cos \alpha + q \sin \alpha)}{\cos \alpha}\right)} = c_e \sqrt{1 - \frac{\cos \theta}{\cos \alpha} (\cos \theta \cos \alpha - \sin \theta \sin \alpha)}, \quad (4.4.16)$$

which can be rewritten using standard trigonometric formulae as

$$c_p^2 = c_e^2 \left(\frac{\sin \theta}{\cos \alpha} \sin(\theta + \alpha)\right). \quad (4.4.17)$$

Equation (4.4.17) allows us to find a general criterion for equal plastic and elastic wavespeeds: the wavespeeds are equal when

$$\frac{\sin \theta}{\cos \alpha} \sin(\theta + \alpha) = 1, \quad (4.4.18)$$

$$\sin \theta \sin(\theta + \alpha) = \cos \alpha. \quad (4.4.19)$$

Using the trigonometric formulae

$$\cos(A - B) - \cos(A + B) = 2 \sin A \sin B, \quad (4.4.20)$$

$$\cos(A - B) + \cos(A + B) = 2 \cos A \cos B, \quad (4.4.21)$$
we have (using \(A = \theta, B = \theta + \alpha\)) from (4.4.19)

\[
\frac{1}{2} \cos \alpha = \frac{1}{2} \cos (2\theta + \alpha) = \cos \alpha, \quad \text{so} \quad (4.4.22)
\]

\[
\cos \alpha + \cos (2\theta + \alpha) = 0, \quad (4.4.23)
\]

and now, using \(A = \alpha + \theta, B = \theta,\)

\[
2 \cos(\alpha + \theta) \cos \theta = 0. \quad (4.4.24)
\]

So \(c_p = c_e\) if (and only if) \(\theta + \alpha = \pm \pi/2\) or \(\theta = \pm \pi/2\). Recalling the constraint (4.4.12), the condition for \(c_p = c_e\) which applies is

\[
\theta = \pi/2 - \alpha. \quad (4.4.25)
\]

If the same argument is applied to the inequality

\[
\sin \theta \sin (\theta + \alpha) < \cos \alpha \quad (4.4.26)
\]

instead of (4.4.19), we reach the conclusion that \(c_p < c_e\) when \(\theta < \pi/2 - \alpha\), and the reverse inequalities are proven the same way.

\[\square\]

4.4.3 Discretized Shear Band

The spatially discretized system (2.1.17)–(2.1.18) is a system of ordinary differential equations, and as such, it is unclear how to describe traveling wave solutions, let alone determine the speed of such a wave.

However, given parameters \((E, \alpha, \phi, N)\), we can use the results of Section 4.2 to determine the stress \(\tau_{\text{nsb}}\) outside a shear band; we can then use this stress to determine the plastic wavespeed of the continuous problem, if it exists, and compare it to the elastic wavespeed. The periodic solution of the spatially discretized elastoplastic antiplane shear problem has clear fronts which propagate across the domain, and comparison of the speeds of these fronts with the plastic and elastic wavespeeds of the continuous problem may give insight into their nature.

Away from the shear band, the shear band solution has a stress of \(\tau = \begin{pmatrix} \cos \theta \\ -\sin \theta \end{pmatrix}\), where \(\theta\) is given to first order in the mesh size \(\Delta x = 1/N\) by (2.1.27), which we restate here for convenience:

\[
\theta = \alpha + \frac{1}{\cot \phi + \cot(2\alpha)} \Delta x. \quad (4.4.27)
\]
From this, and the previous subsection, we can find an equation for the value of $\alpha_{\text{crit}}$ (to first order in $\delta$) that generates a shear band with equal plastic and elastic wave speeds:

**Lemma 2.** Given $\phi$ and $N$, the value of $\alpha$ which results in equal plastic and elastic wavespeeds $c_p = c_e$ is, to first order in $\Delta x$,

$$\alpha_{\text{crit}} = \frac{\pi}{4} - \frac{\Delta x}{2} \tan \phi. \quad (4.4.28)$$

**Proof.** From (4.4.14), we have

$$\theta + \alpha_{\text{crit}} = \pi/2. \quad (4.4.29)$$

Using (4.4.27), this becomes

$$2\alpha_{\text{crit}} + \frac{1}{\cot \phi + \cot(2\alpha_{\text{crit}})} \Delta x = \pi/2, \text{ or} \quad (4.4.30)$$

$$\alpha_{\text{crit}} = \frac{\pi}{4} - \frac{\Delta x}{2} \frac{1}{\cot \phi + \cot(2\alpha_{\text{crit}})}. \quad (4.4.31)$$

This defines $\alpha_{\text{crit}}$ implicitly. But we can do better: by replacing $\alpha_{\text{crit}}$ in the right hand side with the right hand side itself, we get

$$\alpha_{\text{crit}} = \frac{\pi}{4} - \frac{\Delta x}{2} \frac{1}{\cot \phi + \cot(\pi/2 - \Delta x \left( \frac{1}{\cot \phi + \cot(2\alpha_{\text{crit}})} \right))}. \quad (4.4.32)$$

Expanding the Taylor series of cotangent,

$$\alpha_{\text{crit}} = \frac{\pi}{4} - \frac{\Delta x}{2} \frac{1}{\cot \phi + 0 - O(\Delta x)}. \quad (4.4.33)$$

Since $\frac{1}{x + O(\Delta x)} = \frac{1}{x} + O(\Delta x)$, to first order in $\Delta x$,

$$\alpha_{\text{crit}} = \frac{\pi}{4} - \frac{\Delta x}{2} \tan \phi. \quad (4.4.34)$$

From (4.4.14), the corresponding $\theta$, to first order, is

$$\theta = \frac{\pi}{4} + \frac{\Delta x}{2} \tan \phi. \quad (4.4.35)$$
Significance of Critical $\alpha$

Plastic wave speed exceeding the elastic wave speed is sometimes associated with the loss of uniqueness of solutions, or with nonexistence of a continuous solution for some initial conditions [21]. The space discretization was adopted in order to create a well-posed system from the ill-posed continuous one, so avoiding the problematic case $c_p > c_e$ is appropriate. Lemma 2 shows that this can be avoided in the context of shear bands by constraining $\alpha$ so that $0 \leq \alpha < \frac{\pi}{4} - \frac{\Delta \varphi}{2} \tan \phi$. (For $\alpha = 0$, (4.4.17) gives $c_p = |\sin \theta| c_e < c_e$, so $c_p < c_e$ for $\alpha < \alpha_{\text{crit}}$.)
Chapter 5

Elastoplastic Transitions

5.1 Introduction

The equations of motion for elastoplastic antiplane shear can be written

\[
\begin{pmatrix}
  v_t \\
p_t \\
q_t
\end{pmatrix}
= \begin{pmatrix}
p_x \\
E(v_x - \chi(p^2 + q^2)(pv_x + qb)_+(p + q \tan \alpha)) \\
E(b - \chi(p^2 + q^2)(pv_x + qb)_+(-p \tan \alpha + q))
\end{pmatrix},
\]

(5.1.1)

In these equations, \( \chi(1) = 1, \chi(z) = 0 \) for \( z < 1 \). When \( p^2 + q^2 = 1 \) and \( pv_x + qb > 0 \), the deformation is plastic, otherwise it is elastic. In the plastic case, \( q = \sqrt{1 - p^2} \) and the third equation is redundant.

We seek solutions to these equations which are continuous, but may have discontinuous derivatives. We desire solutions with propagating fronts, which may be elastic or plastic waves, but could also be elastoplastic boundaries representing a transition between states. In this chapter, we shall see that following the approach used in [21] for the elastoplastic system described in Section 2.3 will yield meaningful results in this case also. For \( v, p, \) and \( q \) to be continuous across an elastoplastic interface, their tangential derivatives have to be equal on each side of the interface. This condition, together with (5.1.1) and the yield condition, leads to interface conditions constraining the space of allowable elastoplastic transitions. Together with knowledge of the elastic and plastic wave speeds, these conditions permit us to determine the local behavior of a solution to (5.1.1) with piecewise linear initial data. The solution consists of piecewise linear wedges separated by fronts moving with constant speed, suggestive of our numerical results for the periodic solution.
5.1.1 Interface Conditions at Elastoplastic Boundaries

Consider an interface between elastic and plastic states, across which the variables \( v, p, \) and \( q \) are continuous. We denote the interface curve by \( x = s(t) \), the variables on the elastic side as \( v^E, p^E, \) and \( q^E \), and the plastic variables as \( v^P, p^P, \) and \( q^P \) (although \( q^P \) is redundant, since \( (q^P)^2 + (p^P)^2 = 1 \)). Along the interface, \( v^E = v^P, \) etc., so where these quantities are undifferentiated, we label them simply \( v, p, q \). Continuity of tangential derivatives along the interface is expressed by

\[
\frac{d}{dt} \begin{pmatrix} v^E(s(t), t) \\ p^E(s(t), t) \\ q^E(s(t), t) \end{pmatrix} = \frac{d}{dt} \begin{pmatrix} v^P(s(t), t) \\ p^P(s(t), t) \\ q^P(s(t), t) \end{pmatrix},
\]

i.e.,

\[
\dot{s} \begin{pmatrix} v_x^E \\ p_x^E \\ q_x^E \end{pmatrix} + \begin{pmatrix} v_x^E \\ p_x^E \\ q_x^E \end{pmatrix} = \dot{s} \begin{pmatrix} v_x^P \\ p_x^P \\ q_x^P \end{pmatrix} + \begin{pmatrix} p_x^P \\ p_x^P \\ p_x^P \end{pmatrix} + \begin{pmatrix} p_x^P \\ p_x^P \\ p_x^P \end{pmatrix}.
\]

Applying (5.1.1), this becomes

\[
\dot{s} \begin{pmatrix} v_x^E \\ p_x^E \\ q_x^E \end{pmatrix} + \begin{pmatrix} v_x^E \\ p_x^E \\ q_x^E \end{pmatrix} = \dot{s} \begin{pmatrix} v_x^P \\ p_x^P \\ q_x^P \end{pmatrix} + \begin{pmatrix} p_x^P \\ p_x^P \\ p_x^P \end{pmatrix} + \begin{pmatrix} E(v_x^P - (p_x^P + qb)(p + q \tan \alpha)) \\ E(b - (p_x^P + qb)(q - p \tan \alpha)) \end{pmatrix}.
\]

If we regard the values of \( p \) and \( q \) on the interface as known, this system would appear to have 7 unknown functions: \( v_x, p_x, \) and \( q_x \) on each side of the interface, and \( \dot{s} \). The yield condition eliminates one of these: since \( p^2 + q^2 \) is constant on the plastic side, \( q_x^P \) can be written in terms of \( p_x^P \):

\[
pp_x^P + qq_x^P = 0, \quad p_x^P = \frac{q}{p} q_x^P.
\]

Therefore, we eliminate \( q_x^P \) (and \( Eb \)) from the last condition of (5.1.4), resulting in

\[
\dot{s}qq_x^E = -5pp_x^P - Eq(pv_x^P + qb)(q - p \tan \alpha).
\]

Thus the interface conditions for elastic-plastic transitions are

\[
\dot{s} \begin{pmatrix} v_x^E \\ p_x^E \\ q_x^E \end{pmatrix} + \begin{pmatrix} v_x^E \\ p_x^E \\ q_x^E \end{pmatrix} = \dot{s} \begin{pmatrix} v_x^P \\ p_x^P \\ q_x^P \end{pmatrix} + \begin{pmatrix} p_x^P \\ p_x^P \\ p_x^P \end{pmatrix} + \begin{pmatrix} p_x^P \\ p_x^P \\ p_x^P \end{pmatrix} + \begin{pmatrix} E(v_x^P - (p_x^P + qb)(p + q \tan \alpha)) \\ E(b - (p_x^P + qb)(q - p \tan \alpha)) \end{pmatrix}.
\]
5.2 Scheme for description of elastoplastic transitions in elasto-plastic antiplane shear

Key to the development in 2.3 are the observations of a scale invariance in the evolution equations, and that transitions between elastic and plastic states of scale-invariant solutions would occur along radial lines. Both of these assertions are not generally true for elastoplastic antiplane shear. We will consider the implications of these properties not holding, but first we point out areas of the elastoplastic antiplane shear model that parallel the elastoplastic rod model precisely: the elastic state and the $(\tau \cdot \nabla v)$ part of the yield condition.

As in the case of the elastoplastic rod model, the elastic state admits a linear solution. The elastic evolution equations for $v$ and $p$ ($v^E_t = p^E_x$, $p^E_t = E v^E_x$) are obviously compatible with a linear solution, while with linear initial condition $q^E(x, 0) = q_0(x)$, the elastic $q$ is explicitly linear: $q^E_t = E b$, so $q^E(x, t) = q_0(x) + E b t$.

The yield condition $p^2 + q^2 = 1$ will not fall along a radial line for a piecewise linear solution, but the loading condition $\tau \cdot \nabla v = 0$ will. Consider a scale invariant solution $(v, p, q)^T = t(f_1(x/t), f_2(x/t), f_3(x/t))^T$. Then the expression $\tau \cdot \nabla v \geq 0$, i.e., $pv_x + qb \geq 0$, becomes

$$t(f_2(x/t) f'_1(x/t) + f_3(x/t) b) \geq 0.$$  (5.2.1)

This expression has the form $tg(x/t) \geq 0$, where $g(\xi) = f_2(\xi) f'_1(\xi) + f_3(\xi) b$. Thus, $\tau \cdot \nabla v \geq 0$ is unchanged by the scaling transformation $x \rightarrow \eta x$, $t \rightarrow \eta t$, $\eta > 0$. Consequently, along each radial line this condition will either hold at each point, or fail at each point.

5.2.1 Solving the equations (5.1.7)

Given the linearity of elastic solutions, let us suppose the spatial derivatives $(v^E_x, p^E_x, q^E_x)^T$ on the elastic side of an interface are constant and known. Then we have three nonlinear equations (5.1.7) in the three unknowns $v^P_x, p^P_x$, and $s$. This system can be rewritten

$$\dot{s} v^P_x + p^P_x = \dot{s} v^E_x + p^E_x \quad (5.2.2)$$
$$\dot{s} p^P_x + v^P_x E (1 - p^2 - pq \tan \alpha) = \dot{s} p^E_x + E (v^E_x + b (pq + q^2 \tan \alpha)) \quad (5.2.3)$$
$$\dot{s} p^P_x + v^P_x E (q^2 - pq \tan \alpha) = -s q^E_x + E b (q^2 \tan \alpha - \frac{q^3}{p}). \quad (5.2.4)$$
We can eliminate \( p_x^P \) and \( v_x^P \) to solve for \( \dot{s} \). Since along the interface \( p^2 + q^2 = 1 \), subtracting (5.2.3) from (5.2.4) leads to
\[
\dot{s} \left( \frac{q}{p} p_x^E + p_x^E \right) = -Eb\left( \frac{q^3}{p} + pq \right) - Ev_x^E, \tag{5.2.5}
\]
from which we obtain
\[
\dot{s} = -\frac{E(p v_x^E + qb)}{p p_x^E + q q_x^E}. \tag{5.2.6}
\]
We note that in hindsight, this value of \( \dot{s} \) is intuitive: we can derive (5.2.6) by consideration of the elastic form of (5.1.1), plus the condition \( p^2 + q^2 = 1 \) along the curve \( s(t) \). The condition \( p^2 + q^2 = 1 \) implies
\[
\frac{d}{dt} (p(s(t), t)^2 + q(s(t), t)^2) = 0, \tag{5.2.7}
\]
therefore
\[
2(p(s(t), t)(p_x^E \dot{s} + p_t^E) + q(s(t), t)(q_x^E \dot{s} + q_t^E)) = 0. \tag{5.2.8}
\]
Thus,
\[
\dot{s}(p p_x^E + q q_x^E) + (p p_t^E + q q_t^E) = 0, \tag{5.2.9}
\]
so we conclude
\[
\dot{s} = -\frac{p p_t^E + q q_t^E}{p p_x^E + q q_x^E}. \tag{5.2.10}
\]
Using the evolution equation (5.1.1) for \( p \) and \( q \) in the elastic state, we see that (5.2.10) is the same as (5.2.6).

With \( \dot{s} \) in hand, the nonlinearities are removed, and the remaining plastic unknowns \( v_x^P \) and \( p_x^P \) are easily found: eliminating \( v_x^P \) from (5.2.2), (5.2.3) and solving for \( p_x^P \), we find
\[
p_x^P = p_x^E - \frac{E \dot{s}}{E(q^2 - pq \tan \alpha)} - \frac{s^2}{s^2} (p v_x^E + qb)(p + q \tan \alpha), \tag{5.2.11}
\]
and then we can conclude from (5.2.2)
\[
v_x^P = v_x^E + \frac{p_x^E - p_x^P}{\dot{s}} = v_x^E + \frac{E}{E(q^2 - pq \tan \alpha)} - \frac{s^2}{s^2} (p v_x^E + qb)(p + q \tan \alpha). \tag{5.2.12}
\]
Thus, if the solution on the elastic side of a transition is specified, the speed of the interface is determined by (5.2.6), \( p_x^P \) is given by (5.2.11), and \( v_x^P \) is given by (5.2.12).
5.2.2 Locus of elastic states

On the other hand, if we specify the plastic state, and look for a transition to an elastic state, there is a 1-d locus of admissible transitions, analogous to a shock curve. To determine this locus, we fix \( v_x^P \) and \( p_x^P \), and seek equations for \( v_x^E \), \( p_x^E \), and \( q_x^E \), which for convenience we rewrite as \( X \), \( Y \), and \( Z \), respectively. The expression for \( \dot{s} \) from (5.2.6) is still valid, so we rewrite it as

\[
\dot{s} = -\frac{E(pX + qb)}{pY + qZ}. \tag{5.2.13}
\]

With the new variables, (5.2.3) becomes

\[
\dot{p}_x^P + E(v_x^P(q^2 - pq \tan \alpha) - b(pq + q^2 \tan \alpha)) - \dot{s}Y = EX. \tag{5.2.14}
\]

Substituting expression (5.2.13), and denoting

\[
h = v_x^P(q^2 - pq \tan \alpha) - b(pq + q^2 \tan \alpha) \tag{5.2.15}
\]

results in

\[-E(pX + qb)p_x^P + Eh(pY + qZ) + E(pX + qb)Y = EX(pY + qZ) \]

i.e., \( E(pX + qb)(Y - p_x^P) = E(X - h)(pY + qZ) \). \tag{5.2.17}

Now, (5.2.2) is written

\[
\dot{s}v_x^P + p_x^P - \dot{s}X = Y. \tag{5.2.18}
\]

Similarly, substituting (5.2.13) into (5.2.18) produces an expression for \( pY + qZ \):

\[-E(pX + qb)v_x^P + p_x^P(pY + qZ) + E(pX + qb)X = Y(pY + qZ), \tag{5.2.19}
\]

and so

\[
pY + qZ = \frac{E(pX + qb)(X - v_x^P)}{Y - p_x^P}. \tag{5.2.20}
\]

Using this expression, we can eliminate \( Z \) from (5.2.17). Substituting (5.2.20) into (5.2.17), we eliminate \( pY + qZ \), resulting in an equation relating \( X \) and \( Y \) only.

\[
E(pX + qb)(Y - p_x^P)^2 = E^2(pX + qb)(X - h)(X - v_x^P), \tag{5.2.21}
\]

so either \( pX + qb = 0 \) (in other words, \( \tau \cdot \nabla v = 0 \)), or

\[
(Y - p_x^P)^2 = E(X - h)(X - v_x^P). \tag{5.2.22}
\]
At this point, it is worthwhile to note that by expanding \( h \) from (5.2.15) we can write this elastic locus as

\[
(Y - p_x^P)^2 = E(X - v_x^P)^2 + E(pv_x^P + qb)(p + q \tan \alpha)(X - v_x^P).
\]

(5.2.23)

If we denote

\[
m = (pv_x^P + qb)(p + q \tan \alpha),
\]

(5.2.24)

(5.2.23) becomes

\[
(Y - p_x^P)^2 = E(X - v_x^P)^2 + Em(X - v_x^P),
\]

(5.2.25)

or

\[
E(X - v_x^P + m/2)^2 - (Y - p_x^P)^2 = Em^2/4.
\]

(5.2.26)

Thus, the locus is a hyperbola for \( X \) and \( Y \), with \((v_x^P, p_x^P)\) as one of the vertices, as shown in Figure 5.1. In addition, the slopes of the asymptotes of this hyperbola are \( \pm \sqrt{E} \), the elastic waves speeds. The interface speed \( \dot{s} \) varies along the hyperbola: solving (5.2.2) for \( \dot{s} \) gives

\[
\dot{s} = -\frac{p_x^P - p_x^E}{v_x^P - v_x^E},
\]

(5.2.27)

so the interface speed is the negative of the slope of the secant line connecting the vertex \((v_x^P, p_x^P)\) to the point on the hyperbola \((v_x^E, p_x^E)\).

Thus, for a fixed \( p, q, v_x^P, p_x^P \), (5.2.26) describes the hyperbola of admissible values for \( X = v_x^E, Y = p_x^E \). Then \( Z = q_x^E \) is given by (5.2.20), and the interface speed \( \dot{s} \) is given by (5.2.27).

**5.2.3 Constraints on Elastic States Near an Elastic-Plastic Interface**

Our goal is to construct a solution to the equations of elastoplastic antiplane shear with piecewise linear initial data, by selecting an appropriate combination of elastic and plastic waves together with elastoplastic transitions. We must determine which initial conditions are amenable to this approach. In addition, we must determine general criteria for admissibility of elastic and plastic states which may arise after \( t = 0 \).

The equations (5.1.7) were generated by considering tangential derivatives of \( v, p \) and \( q \) along the interface; similarly, (5.2.10) came from the tangential derivative of \( p^2 + q^2 \). The derivative normal to the interface provides information as well: since \( p^2 + q^2 \leq 1 \), the normal derivative of \( p^2 + q^2 \) into the interface from an elastic region must be nondecreasing.
Figure 5.1: Hyperbola of elastoplastic transitions from a given plastic state \((v^p_x, p^p_x)\). The bold part of the hyperbola represents left-propogating transitions \((\dot{s} < 0)\).

(The normal and tangential derivatives of \(p^2 + q^2\) in the plastic region are both zero, of course; the plastic form of the evolution equations are constructed to preserve \(p^2 + q^2 = 1\), so these derivatives give us no information about the interface.)

For an elastic state on the right of an elastic-plastic interface \(x = s(t)\), the inward normal derivative of \(p^2 + q^2\) is

\[
2(p(-p^E_x + \dot{s}p^E_t) + q(-q^E_x + \dot{s}q^E_t)) \geq 0,
\]

which, after substituting the value of \(\dot{s}\) from (5.2.6), simplifies to

\[
pp^E_x + qq^E_x = \tau \cdot \tau^E_x \leq 0. \tag{5.2.29}
\]

For an elastic state on the left of an interface, the inward normal has opposite sign, so the inequality is reversed:

\[
pp^E_x + qq^E_x = \tau \cdot \tau^E_x \geq 0. \tag{5.2.30}
\]

If the elastic-plastic interface is a straight line through the origin, the two cases can be described by a single inequality. A region in the \((x, t)\) plane deforming elastically for \(t > 0\) to the right of an elastic-plastic interface \(x = \dot{s}t\) satisfies the inequality (using (5.2.6))

\[
(x/t) \geq \dot{s} = -\frac{E(\tau \cdot \nabla v^E)}{\tau \cdot \tau^E_x}. \tag{5.2.31}
\]
Multiplying both sides by $\tau \cdot \tau_x^E$, using (5.2.29), results in

$$(x/t)(\tau \cdot \tau_x^E) \leq -E(\tau \cdot \nabla v). \tag{5.2.32}$$

If the elastic region is on the left of the interface, it satisfies the reverse inequality of (5.2.31), but since the sign of $\tau \cdot \tau_x^E$ is also switched (from (5.2.30)), the inequality (5.2.32) holds in this case also.

**Appropriate Initial Conditions**

We consider a test problem with continuous $v, p, q$ and a jump in their derivatives, perhaps like

$$v(x, 0) = \begin{cases} v_x^L x, & \text{for } x \leq 0, \\ v_x^R x, & \text{for } x > 0 \end{cases}$$

$$p(x, 0) = \begin{cases} p_x^L x, & \text{for } x \leq 0, \\ p_x^R x, & \text{for } x > 0 \end{cases} \tag{5.2.33}$$

$$q(x, 0) = \begin{cases} q_x^L x, & \text{for } x \leq 0, \\ q_x^R x, & \text{for } x > 0 \end{cases}$$

The coefficients $v_x^L, v_x^R, p_x^L, p_x^R, q_x^L, q_x^R$ are understood to be constant. We plan to proceed with reconciling the jump in the derivatives with elastic-plastic transitions, together with elastic and plastic waves.

Immediately there are issues we must address. First, the conditions (5.2.33) violate the constraint $p^2 + q^2 \leq 1$ for some finite $x$, but this difficulty is easily dispensed with, since we are concerned with $x$ near 0. For example, we can modify the initial conditions so that they are constant outside some neighborhood of $x = 0$.

In addition, the yield locus $p^2 + q^2 = 1$ must be achieved for any elastic-plastic transition; the initial conditions (5.2.33) do not achieve the yield locus in a neighborhood of $x = 0$. To resolve this, and to allow for elastic-plastic interfaces, we raise the values of $p$...
and $q$ at $x = 0$ to the yield locus:

\[
\begin{align*}
v(x, 0) &= \begin{cases} v_x^L x, & \text{for } x \leq 0, \\ v_x^R x, & \text{for } x > 0 \end{cases} \\
p(x, 0) &= \begin{cases} p_0 + p_x^L x, & \text{for } x \leq 0, \\ p_0 + p_x^R x, & \text{for } x > 0 \end{cases} \\
q(x, 0) &= \begin{cases} q_0 + q_x^L x, & \text{for } x \leq 0, \\ q_0 + q_x^R x, & \text{for } x > 0, \end{cases}
\end{align*}
\] (5.2.34)

where $p_0^2 + q_0^2 = 1$. To insure that these initial conditions do not violate the yield condition for all $x \neq 0$, we constrain $p^2(x, 0) + q^2(x, 0)$:

\[
(p_0 + xp_x)^2 + (q_0 + xq_x)^2 = 1 + 2x(p_0p_x + q_0q_x) + x^2(p_x^2 + q_x^2) \leq 1, \quad (5.2.35)
\]

which implies

\[
x(p_0p_x + q_0q_x) = x(\tau_0 \cdot \tau_x) \leq 0. \quad (5.2.36)
\]

Conversely, if (5.2.36) holds strictly, then for $|x| < |\tau_0 \cdot \tau_x|/|\tau_x \cdot \tau_x|$, 

\[
2x(\tau_0 \cdot \tau_x) + x^2(\tau_x \cdot \tau_x) < 2x(\tau_0 \cdot \tau_x) + |x||\tau_0 \cdot \tau_x| = x(\tau_0 \cdot \tau_x) < 0, \quad (5.2.37)
\]

so (5.2.35) holds in a neighborhood of $x = 0$. Note that (5.2.36) is another expression of the constraints on elastic states (5.2.29) and (5.2.30).

From the initial conditions (5.2.34), subject to the constraint (5.2.36) on both left and right, we will construct a local solution to the antiplane shear equations 5.1.1 by an appropriate combination of waves and elastic-plastic transitions. Our construction will only describe local behavior; not only because of the problem for $x$ far from zero mentioned earlier, but also because we will assume elastic-plastic transitions are approximately linear, that is,

\[
\dot{s} = -\frac{E(pv_x^E + qb)}{pp_x^E + qq_x^E} \approx -\frac{E(p_0v_x^E + q_0b)}{p_0p_x^E + q_0q_x^E}. \quad (5.2.38)
\]

Such an assumption is only acceptable for small values of $t$. 
5.3 Wave Curves

We restrict ourselves to considering only transitions propagating to the left; the right-hand case is symmetric. Our goal is to determine a locus of possible derivative values \((v_x, p_x, q_x)\) that could hold along the vertical axis \(x = 0\) as a result of these left-propagating transitions.

5.3.1 Characterizing Elastoplastic Transitions

First, we wish to distinguish the sections of the hyperbola (5.2.26) that indicate a transition in increasing time from elastic to plastic (for fronts moving to the left, this means the elastic state is on the left of the front, and the plastic on the right), and from plastic to elastic. The two types will be denoted \(E-P\) and \(P-E\), respectively. For this purpose, it is convenient to refer to Figure 5.1. As seen in Subsection 5.2.2, (5.2.27) gives the speed \(\dot{s}\) of an elastic-plastic interface as the negative of the slope of the line connecting the vertex \((v^E_x, p^E_x)\) to the point \((v^E_x, p^E_x)\) on the hyperbola. Therefore, only the bold parts of the hyperbola in Figure 5.1 represent left-propagating transitions \((\dot{s} < 0)\).

Equation (5.2.27) gives the speed of the interface, but does not take into account whether the states are loading or unloading. To distinguish elastic to plastic \((E-P)\) from plastic to elastic \((P-E)\) transitions, we have the following principle:

**Lemma 3.** In an \(E-P\) transition, \(pv^E_x + qb \geq 0\), whereas \(pv^E_x + qb \leq 0\) in a \(P-E\) transition.

**Proof.** The time evolution of the norm of the stress is given by \(\partial_t(\|\tau\|^2) = 2pp_t + 2qq_t = 2E(pv_x + qb)\). If, on the elastic side of an \(E-P\) front, near the front \(pv^E_x + qb < 0\), then \(p^2 + q^2\) is decreasing, so \(p^2 + q^2\) is not 1 at the front, which is a contradiction.

On the other hand, for a \(P-E\) front, the elastic side must satisfy \(pv^E_x + qb < 0\) along the front: at the front \(p^2 + q^2 = 1\), so an elastic state evolving from the front must have \(pv^E_x + qb \leq 0\). (If both \(p^2 + q^2 = 1\) and \(pv_x + qb > 0\), then the state is plastic.)

Now we interpret Lemma 3 in terms of the hyperbola of Figure 5.1. Given \(p\) (here assumed to be positive) and \(q\), we can split the hyperbola into \(E-P\) fronts, for which \(v^E_x > -\frac{q}{p}b\), to the right of the vertical line in the figure, and \(P-E\) fronts, for which the complementary inequality \(v^E_x < -\frac{q}{p}b\) holds, on the left of the line. In a plastic region, the solution must satisfy the condition for continued loading \(pv^E_x + qb > 0\), so the vertex \((v^P_x, p^P_x)\) is always on the side of the \(E-P\) fronts, i.e. to the right of the vertical line when \(p > 0\).
Figure 5.1 shows the vertical line \( v_x = -\frac{q}{p}b \) between loading and unloading states intersecting the hyperbola. There is no guarantee \( \textit{a priori} \) that this line intersects the hyperbola, or, if it does, which branch it intersects. Also, the vertices of the hyperbola (5.2.26) are \((v_x^p, p_x^p)\) and \((v_x^P - m, p_x^P)\), where \( m \) is given by (5.2.24); if \( m > 0 \), the vertices are as shown in Figure 5.1, but if \( m < 0 \), the plastic state \((v_x^P, p_x^P)\) is the left vertex, opposite of the orientation in the figure. Since \( m = (pv_x^P + qb)(p + q \tan \alpha) \), the sign of \( m \) is the same as the sign of \( p + q \tan \alpha \). In subsection 5.3.2, we will consider all these cases. We note, however, from our numerical results that the case \( m > 0 \) is most relevant for the periodic solution data away from the shock, and the line separating loading and unloading intersects the left branch of the hyperbola, as shown in Figure 5.1. Therefore, this is the important case for comparing the periodic results with this analysis.

If the line does intersect the hyperbola, we can find the point of intersection, and therefore (via (5.2.27)) the front speed associated with the change in type, from E–P to P–E.

**Lemma 4.** The line \( pX + qb = 0 \) intersects the hyperbola (5.2.26) if and only if \( q^2 - pq \tan \alpha > 0 \). If they do intersect, the slope of the secant line between the vertex \((v_x^P, p_x^P)\) and either point of intersection is the plastic wavespeed.

**Proof.** Substituting \( X = -\frac{q}{p}b \) into (5.2.26), we find

\[
E(-\frac{q}{p}b - v_x^P + m/2)^2 - (Y - p_x^P)^2 = Em^2/4. \tag{5.3.1}
\]

Thus,

\[
(Y - p_x^P)^2 = E(-(-\frac{q}{p}b + v_x^P) + m/2)^2 - Em^2/4 \tag{5.3.2}
\]

\[
= E[(\frac{q}{p}b + v_x^P)^2 - m(\frac{q}{p}b + v_x^P)] \tag{5.3.3}
\]

\[
= E(pv_x^P + qb)(\frac{q}{p}b + v_x^P - m). \tag{5.3.4}
\]

Expanding the expression for \( m \) from (5.2.24) results in

\[
(Y - p_x^P)^2 = \frac{E(pv_x^P + qb)}{p}((\frac{pv_x^P + qb}{p}) - (pv_x^P + qb)(p + q \tan \alpha)) \tag{5.3.5}
\]

\[
= \frac{E(pv_x^P + qb)^2}{p^2}(1 - p^2 - pq \tan \alpha) \tag{5.3.6}
\]

\[
= \frac{E(pv_x^P + qb)^2}{p^2}(q^2 - pq \tan \alpha). \tag{5.3.7}
\]
Thus, if $q^2 - pq \tan \alpha < 0$, there is no real value of $Y$ corresponding to $X = -\frac{q}{p} b$ on the hyperbola: the line and hyperbola do not intersect. On the other hand, if $q^2 - pq \tan \alpha > 0$, we can solve for $Y$, finding the points of intersection:

$$Y - p^P_x = \pm \frac{\sqrt{E(pv^P_x + qb)}}{p} \sqrt{q^2 - pq \tan \alpha}. \quad (5.3.8)$$

The slope of the secant line is now straightforward:

$$\frac{Y - p^P_x}{X - v^P_x} = \frac{\pm \sqrt{E(pv^P_x + qb)\sqrt{q^2 - pq \tan \alpha}}}{-\frac{q}{p} b - v^P_x} \quad (5.3.9)$$

$$= -\frac{\pm \sqrt{E} \sqrt{q^2 - pq \tan \alpha}}{pv^P_x + qb} \quad (5.3.10)$$

$$= \mp \sqrt{E} \sqrt{q^2 - pq \tan \alpha}, \quad (5.3.11)$$

which are the plastic wave speeds (see Subsection 4.4).

Thus, for appropriate values of $p$ and $q$, the types of front are divided into fast E–P, P–E, and slow E–P types: the fast E–P fronts are faster than elastic waves, the slow E–P fronts are slower than plastic waves, and the P–E fronts have speeds in between the two wave speeds. For example, for left-moving transitions, in the Figure 5.1 the fast E–P transitions lie above the vertex $(v^P_x, p^P_x)$ on the right branch of the hyperbola, the P–E transitions are the lower part of the left branch to the left of the line $v_x = -q b / p$, and the slow E–P transitions are the remainder of the lower half of the left branch. The lower half of the right branch and upper half of the left branch correspond to right-moving transitions.

### 5.3.2 Characterizing the Hyperbola for Various $(p, q)$

The various cases for the hyperbola of admissible elastic states are determined by the values of $p$ and $q$. Since, at the interface, $p^2 + q^2 = 1$, we define the angle $\theta$ by the formulae

$$p = \cos \theta, \quad q = -\sin \theta. \quad (5.3.12)$$

As mentioned above, the sign of $m$ is the sign of $p + q \tan \alpha$; with $\theta$ defined as above, $m = 0$ when $\cos \theta - \sin \theta \tan \alpha = 0$, or

$$\cot \theta = \tan \alpha. \quad (5.3.13)$$
Figure 5.2: Disposition of the hyperbola of elastic states connectable to a given plastic state. The marked point is the plastic state, \((v^P_x, p^P_x)\), and the vertical line is \(v_x = -qb/p\). Elastic to plastic (E-P) transitions are on the same side of the vertical line as the plastic state; plastic to elastic (P-E) transitions are on the other side of the line.

This equation’s solutions are \(\theta = \pi/2 - \alpha\) and \(\theta = 3\pi/2 - \alpha\), and these are the values of \(\theta\) at which the vertex \((v^P_x, p^P_x)\), labeled by a dot in Figure 5.2, switches from the right arm of the hyperbola to the left.

Next, we consider where the line \(pv_x + qb = 0\) falls relative to the hyperbola. If \(1 < p^2 + pq \tan \alpha\), then \(v^P_x - m < -qb/p\), so the vertex \((v^P_x - m, p^P_x)\) is to the left of the line. The vertex switches orientation relative to the line \(pv_x + qb = 0\) when \(1 = p^2 + pq \tan \alpha\), which occurs if \(p = 1\) \((\theta = \pm \pi/2)\) or \(q - p \tan \alpha = 0\), in which case \(\theta = \pi - \alpha\) or \(\theta = 2\pi - \alpha\).

Lastly, it should be noted that since \(pv^P_x + qb > 0\), for \(p > 0\), \(v^P_x > -qb/p\), while for \(p < 0\), \(v^P_x < -qb/p\). Therefore, the vertex \((v^P_x, p^P_x)\) is to the right of the line when \(p > 0\), and to the left of the line when \(p < 0\).

We can now determine the form of the hyperbola for all values of \(\theta\): Figure 5.2 displays all the cases. Notably, if \(p + q \tan \alpha = 0\), the hyperbola collapses to a degenerate case, as it does for \(pv^P_x + qb = 0\). The latter case signifies that the “plastic” state is in fact a borderline case between elastic and plastic state, so it is not surprising that the only further transitions are elastic waves. Also, there are certainly values of \((p, q)\) so that the line falls between the two branches of the hyperbola; in this case, there will be no “slow” E-P transitions, only the faster E-P transitions and slower P-E transitions.
5.3.3 Constructing the Solution

Here we describe how Figure 5.1 is used to construct solutions of the initial value problem (5.1.1), (5.2.34). We begin by assuming that the initial condition (5.2.34) is elastic on the left ($x < 0$). (As stated in [21], solutions for plastic initial conditions can be found as the limit of elastic conditions.) As mentioned in Subsection 5.2.3, the initial condition must satisfy (5.2.36), $x(\tau_0 \cdot \tau_x) \leq 0$. Therefore, for $x < 0$, the initial condition must satisfy $\tau_0 \cdot \tau_x^L \geq 0$. In addition, we assume that $0 < \theta < \pi/2 - \alpha$, so that, as shown in figure 5.2, the elastic wavespeed is faster than the plastic wavespeed, and the latter is well defined.

Our goal is to determine the locus of $(v_x, p_x)$ values that could result along the line $x = 0$ from the initial conditions on the ray $x \leq 0$, taking into account any elastoplastic transitions, and elastic or plastic waves. Since the initial state is elastic, we first compare the speed of elastic characteristics with the speed associated with the radial line on which the linearized yield condition

$$(x/t)(\tau_0 \cdot \tau_x^E) = -E(\tau_0 \cdot \nabla v). \quad (5.3.14)$$

is first satisfied. If an elastic wave is faster, the material may undergo a jump in the derivatives along that characteristic to another elastic state: we will denote such solutions as “EE”. The designation “EP” will denote the cases for which yield is reached before an elastic wave can occur. The second elastic state of an “EE” case may reach yield, in which case we describe the solution as “EEP”, or it may remain elastic for arbitrarily small values of $x/t$ (and remain denoted “EE”). Similarly, the plastic intermediate state of the “EP” class can either change state back to an elastic state (“EPE”) or undergo a jump in derivatives along a plastic characteristic to another plastic state (“EPP”). For any initial condition, a wave curve can be constructed from all possible values of $(v_x, \sigma_x)$ along the vertical axis $x = 0$.

To determine if an initial condition is of the “EE” or “EP” type, we compare the speed of the transition from elastic to plastic with the elastic wave speed. The speed of the onset of yield is, from (5.2.6),

$$\dot{s} = -(E(\tau_0 \cdot \nabla v))/(\tau_0 \cdot \tau_x), \quad (5.3.15)$$

while the (negative) elastic speed is $-c_E = -\sqrt{E}$. If

$$\sqrt{E}(\tau_0 \cdot \nabla v^L) > \tau_0 \cdot \tau_x^L, \quad (5.3.16)$$
then the onset of yield will occur before any elastic wave \(|\dot{s}| > c_E\), and an elastoplastic (EP) transition is required. We denote the linear approximation of the plastic state after the transition by the spatial derivatives \(U_I = (v_L^I, p_L^I, q_L^I)\), the \(I\) standing for “intermediate.”

This E–P transition is determined by the values of \(v_L^I, p_L^I,\) and \(q_L^I\), as shown in Subsection 5.2.1. The intermediate plastic state \(U_I\) can either jump to another plastic state (EPP) or back to an elastic state (EPE, through a P–E transition); Figure 5.3 shows the two transition types. The plastic states lie along the plastic wave curve: \([p_x] = c_p[v_x]\). However, since all plastic states have to satisfy \(p v_x + qb \geq 0\), the line defined by the plastic wave curve terminates at \(v_x = -qb/p\). Conveniently, the P–E transitions are precisely the part of the hyperbola on the other side of this line, and at \(v_x = -qb/p\), the hyperbola and plastic wave curve meet. The union of the plastic wave curve and the P–E part of the hyperbola are all the values of \((p_x, v_x)\) along the line \(x = 0\) that the initial condition (5.2.34) (for \(x < 0\)) can be connected to through elastoplastic transitions and plastic waves; this wave curve is the bold curve in Figure 5.3. Note that the initial state \((v_L^L, p_L^L)\) is on the hyperbola (as any elastic state with a transition to or from \((v_L^I, p_L^I)\) must), but \((v_L^L, p_L^L)\) is not on the final wave curve. The wave curve is the possible pairs \((v_x, p_x)\) that can occur along the line \(x = 0\) as a result of waves and elastoplastic transitions from the initial conditions characterized by \((v_L^L, p_L^L)\); since this initial left-hand state reaches yield in finite time, it will not be the state along \(x = 0\).

One might ask what the corresponding values of \(q_x\) along \(x = 0\) might be. Within a plastic region, the value of \(q_x\) is determined by the value of \(p_x\) (see Section 5.1); within an elastic region, the characteristics of \(q\) are vertical lines. Therefore, an (EE) solution has \(q_x = q_L^L\) throughout the left quadrant, but \(q_x\) may jump along the line \(x = 0\); for an (EEP) or (EPP) solution, the value of \(q_x\) is determined by the value of \(p_x\) along \(x = 0\), and for an (EPE) solution, the value of \(q_x\) is determined by the value of \(p_x\) within the plastic wedge, but, as in the (EE) case, \(q_x\) may jump along the line \(x = 0\). Therefore, we can concern ourselves only with matching left and right values of \((v_x, p_x)\) along \(x = 0\), and \(q_x\) will either jump appropriately (if the state is elastic along \(x = 0\)), or will be forced to be continuous across \(x = 0\) because \(p_x\) is (if the state is plastic.)

If (5.3.16) does not hold, then an elastic wave will occur before any elastoplastic transition:

\[
|\dot{s}| = E \frac{\tau_0 \cdot \nabla v^L}{\tau_0 \cdot \tau^L_x} < \sqrt{E} = c_e,
\]
Figure 5.3: Projection of the wave curve of states that can be connected to a left state \((v^L_x, p^L_x, q^L_x)\) which satisfies (5.3.16).

so the intermediate state \(U_I\) will be an elastic state (EE), along the elastic wave curve of the initial state, as shown in Figure 5.4. The intermediate elastic state either undergoes an elastoplastic transition to a plastic state in the wedge \(-\sqrt{E} \leq x/t \leq 0\) (EEP), or has no further transitions (since the elastic wave has already passed.) The speed of a transition from \(U_I\) to a plastic state is, from (5.3.15),

\[
\dot{s} = -E \frac{\tau_0 \cdot \nabla v^I}{\tau_0 \cdot \tau^I_x}. \tag{5.3.18}
\]

If \(\dot{s} < 0\), then the intermediate state will reach yield. This condition occurs if the intermediate state is loading:

**Lemma 5.** An intermediate elastic state \((v^I_x, p^I_x)\) will be followed by a transition to a plastic state if and only if \(\tau_0 \cdot \nabla v^I > 0\).

**Proof.** The equation (5.3.16) does not hold, or else the initial state \((v^L_x, P^L_x)\) would have reached yield before an elastic wave. Therefore

\[
\sqrt{E}(\tau_0 \cdot \nabla v^L) < \tau_0 \cdot \tau^L_x. \tag{5.3.19}
\]
In addition, \((v_x^L, p_x^L)\) is on the elastic wave curve of \((v_x^I, p_x^I)\), so
\[
p_x^I - p_x^L = \sqrt{E}(v_x^I - v_x^L). \tag{5.3.20}
\]
Therefore,
\[
\sqrt{E}(\tau_0 \cdot \nabla v^I) = \sqrt{E}(\tau_0 \cdot \nabla v^L) + \sqrt{E}p_0(v_x^I - v_x^L)
= \sqrt{E}(\tau_0 \cdot \nabla v^L) + p_0(p_x^I - p_x^L) \quad \text{(by (5.3.20))} \tag{5.3.21}
\]
\[
< \tau_0 \cdot \tau_x^L + p_0(p_x^I - p_x^L) = \tau_0 \cdot \tau_x^I. \quad \text{(by (5.3.19).)}
\]
Therefore, \(\tau_0 \cdot \tau_x^I > 0\) if \(\tau_0 \cdot \nabla v^I > 0\), in which case \(\dot{s} < 0\) by (5.3.18), so a plastic transition will follow.

Conversely, if \(\tau_0 \cdot \nabla v^I < 0\), then the material is unloading, so an E–P transition will not occur, by Lemma 3.

If the solution is of the (EEP) type, the final plastic state is determined by the intermediate elastic state by equations (5.2.11)–(5.2.12), just as an intermediate plastic state was determined by the original elastic state in the (EPP) and (EPE) cases. The transition from the intermediate elastic state to the final plastic state occurs through a slow E–P transition, since the transition occurs after the elastic wave, as (5.3.17) indicates. Slow E–P transitions are not just slower than \(c_e\), but are in fact slower than the plastic wave speed \(c_p\) as well, which restricts the domain of admissible intermediate elastic states:

**Lemma 6.** The elastic states \((v_x^E, p_x^E, q_x^L)\) for which \(\dot{s} = -c_p\) fall on a straight line. For a given initial condition \((v_x^L, p_x^L, q_x^L)\) for which (5.3.17) holds, the allowable intermediate states for an (EEP) solution lie on the segment of the elastic wave curve (5.3.20) bounded by the line \(\tau_0 \cdot \nabla v = 0\) on the left and the line \(\dot{s} = -c_p\) on the right.

**Proof.** The speed of an elastoplastic transition from \((v_x^E, p_x^E)\) is given by (5.3.18):
\[
\dot{s} = -E \frac{\tau_0 \cdot \nabla v^E}{\tau_0 \cdot \tau_x^E} = -c_p, \tag{5.3.22}
\]
so
\[
\tau_0 \cdot \tau_x^E = E \frac{\tau_0 \cdot \nabla v^E}{c_p}, \tag{5.3.23}
\]
which can be expanded to
\[
p_x^E = \frac{E}{c_p} v_x^E + \frac{E q_0}{c_p p_0} b - \frac{q_0}{p_0} q_x^L. \tag{5.3.24}
\]
Figure 5.4: Projection of the wave curve of states that can be connected to a left state \((v_x^L, p_x^L, q_x^L)\) which does not satisfy (5.3.16).

Lemma 5 gives that intermediate elastic states leading to (EEP) solutions are loading \((\tau_0 \cdot \nabla v > 0)\); after observing that \(s = 0\) for states along the line \(\tau_0 \cdot \nabla v = 0\), it is clear that only the states \((v_x^E, p_x^E)\) between the lines \(v_x^E = -\frac{q_0}{p_0} b\) and (5.3.24) are followed by slow E–P transitions, which concludes the proof.

5.4 Comparison of Analytic and Numerical Results

The construction of piecewise linear approximate solutions to (5.1.1) with initial conditions (5.2.34) was achieved by freezing the values of \(p = p_0\) and \(q = q_0\) and linearizing the yield condition. The results are approximations to the actual solution, valid for small \(|x|\) and \(t\). To evaluate the meaningfulness of these approximate solutions, we use the spatial discretization (2.1.17)–(2.1.18) to generate numerical solutions of (5.1.1) subject to initial conditions (5.2.34). These results are displayed as contour plots in Figures 5.5–5.6. The parameters used for the figures are \(\alpha = \pi/4\), \(E = .21\), \(b = -\sin(\pi/8)\), \(p_0 = \cos(\pi/6)\), and \(q_0 = -\sin(\pi/6)\). The initial conditions are \(v_x^L = 3, p_x^L = 1, q_x^L = 0\) on the left, and \(v_x^R = 1.3, p_x^R = -1, q_x^R = 1\) on the right. The left-hand side initial condition satisfies (5.3.16), so we
expect a fast E–P transition, and a (EPP) or (EPE) type solution on the left; the right-hand condition satisfies (5.3.17), so we expect an (EE) or (EEP) type solution on the right. The figures show that the numerical solution approximates an (EPP) solution on the left, and an (EEP) solution on the right; the plastic state along $x = 0$ is the end state in common to both sides. The transitions between states are clearly visible in the figures, although there are small oscillations in the final plastic state and intermediate elastic state on the left. There is no change in $q_x$ across the elastic wave on the right, because the elastic form of the equation for $q_x$ (5.1.1) decouples from $v_x$ and $p_x$.

The Figures 5.7–5.8 show the values of $\|\tau\|$ and the derivatives $(v_x, p_x, q_x)$ for the same numerical solution at representative times $t = 0, t = 0.2, t = 0.4, t = 0.6$. In the approximate solution, these derivatives are piecewise constant; the numerical results are approximately linear in the intermediate plastic state on the left, and $q_x$ in particular is approximately constant. The intermediate elastic state on the right and the final plastic state are noisy and not clearly distinguishable from each other, particularly in the graph of $p_x$. Nevertheless, the oscillations are small in amplitude, and the sharp jumps in the variables at transitions indicates that the propagation of discontinuities in the derivatives $(v_x, p_x, q_x)$ along straight lines in $(x, t)$ space described by the approximate solution is reflected in numerical simulations of the nonlinear equations (5.1.1).
Figure 5.6: Contour plots of $x$-derivatives of $p$ and $q$.

Figure 5.7: Snapshots of $\|\tau\|$ and $v_x$ at $t=0, 0.2, 0.4, 0.6$. 
Figure 5.8: Snapshots of $p_x$ and $q_x$ at $t=0, 0.2, 0.4, 0.6$. 
Chapter 6

Periodic Solution

6.1 Introduction

For values of the shear modulus above a critical value, $E_{crit}$, the shear band solution is stable. For smaller $E$, the shear band is unstable, and the system converges over time to a periodic solution with complex behavior. The jump from the steady state solution to a periodic solution suggests a Hopf bifurcation.

To visualize this periodic solution, the system of ordinary differential equations (2.1.17)–(2.1.18) is solved with periodic boundary conditions. The periodic solution that develops has a shear band which grows and shrinks over the period; the value of $x$ at which this shear band appears depends on the initial condition. The initial condition used for our numerical experiments is the shear band solution, with a shear band at $x = 0.5$; because of this initial condition, the shear band that appears in the periodic solution that develops is also at $x = 0.5$.

Contour plots of the periodic solution reveal a structure of elastic waves and elastoplastic transitions, not unlike the analysis of the previous section. Plots of the solution in phase space clarify transitions between elastic and plastic behavior, and the interplay between the variables $p$, $q$ and $w$. In addition, long-term phase plots, over many periods, justify the supposition that these solutions really are periodic. Phase plots also demonstrate the effect of mesh refinement on the structure of the periodic solution. In later sections, we will discuss the effect of parameters and mesh size on the periodic solution; until that time, all figures are generated using a common set of parameters: $N = 40$, $E = 0.1$, $\alpha = \pi/4$. 
6.2 General Description

Figure 6.1 shows a contour plot of the velocity perturbation $w(x,t)$; Figure 6.2, showing the graph of $w(x,t)$ as a function of $x$ for various times $t$, corresponds to vertical cross-sections of Figure 6.1. Recall that the velocity in the shear band solution is skew-symmetric about the shear band, and note that in the periodic solution $w$ is likewise skew-symmetric. The periodic solution has an intermittent shear band that appears and disappears, as illustrated in Figures 6.1 and 6.2. The diamond pattern of traveling steep changes in $w$ is repeated in the contour plot of $p$, seen in Figure 6.3. The value of $p$ is symmetric about the shear band, as is $q$, shown in Figure 6.4. Whether the shear band jump in $w$ is present or not, $q$ exhibits a sharp spike at the value of $x$ corresponding to the interior of the shear band.

Inspection of Figure 6.4 reveals that $q$ decays linearly with time over most of the domain, corresponding to the elastic evolution equation $q_t = Eb$. This motivates the labeling of the distinct regions shown in Figures 6.1 and 6.3 by $P_1$, $E_1$, $E_2$, and $E_3$, as seen in Figure 6.3.

However, the domains of elastic and plastic deformation are more complicated than the diamond pattern in Figures 6.1 and 6.3 suggest, as is shown by Figure 6.5, which is a contour plot of $p^2 + q^2$ over one period of the solution. The region labeled $E_3$ is
Figure 6.2: Snapshots of $w$ as a function of $x$ at fixed values of $t$

Figure 6.3: Contour Plot of $p$, $N=40$, $E=0.1$, $\alpha = \pi/4$. 
Figure 6.4: Contour Plot of $q$, $N=40$, $E=0.1$, $\alpha = \pi/4$

Figure 6.5: Contour plot of $\|\tau\|^2$, with regions labeled, $N=40$, $E=0.1$, $\alpha = \pi/4$. 
predominantly elastic, but near the shock, the transition to plasticity occurs before the sharp transition in $p$ and $w$. Also, although the material is generally plastic in the region $P_1$, there are “islands” of elasticity in $P_1$. The largest of these appears to be a plastic to elastic transition, propagating towards the shock with a speed slower than the previous transition from elastic to plastic (which, away from the shock, marks the transition from $E_3$ to $P_1$.) We will investigate these features further in the following section. To resolve the ambiguity introduced by considering Figure 6.5, we will continue to demarcate the four regions $P_1$, $E_1$, $E_2$, and $E_3$ by the sharp transitions in $p$ and $w$, keeping in mind that $P_1$ and $E_3$ contain both plastic and elastic domains.

The regions $P_1$, $E_1$, $E_2$, and $E_3$ (and their mirror images) cover the entire domain except the location of the shear band itself. Along $x = 0.5$, where the shear band appears and disappears, comparison of Figure 6.5 with Figure 6.6 reveals that the shear band is plastic, but when there is no jump in $w$, the material is elastic at $x = 0.5$.

6.3 Phase plots

6.3.1 Plotting $p$ vs. $q$

Choosing a particular value of $x$, we can construct a graph of $p$ vs. $q$ at that $x$ as time progresses. Since the yield locus is $p^2 + q^2 = 1$, the material is plastic when $p$ and $q$ lie
Figure 6.7: $p$ versus $q$, $x = 0.25$, $N = 40$, $E = 0.1$, $\alpha = \pi/4$
onumber

on the unit circle. An example of such a phase plot is Figure 6.7. The straighter arc section is a portion of the yield locus, and the small loop from the yield locus is the brief return to elasticity within region $P_1$ observed in Figure 6.5. The trajectory of $p$ vs. $q$ is generally traced counterclockwise as time increases, and parts of the trajectory corresponding to the regions of Figure 6.3 are also labeled here.

Of course, choosing a different value for $x$ results in a different phase plot: Figure 6.8 shows the phase plot of $p$ vs. $q$ for a value of $x$ adjacent to the shear band, $x = .475$. Note that $E_1$ is much longer, and $E_2$ is much narrower, corresponding to the transition times in Figure 6.6. In all elastic states, $q_t = Eb$ is constant, so the height of an elastic state in Figure 6.8 corresponds to the time spent in that state as shown in Figure 6.6. We observe that the time spent in the region $E_3$ for a given $x$ is approximately the same as the time spent in region $E_1$, and the value of $p$ is approximately equal in those regions as well. For $x = .475$, $E_3$ cannot be entirely elastic; Figure 6.8 shows that if $E_3$ took up the same height as $E_1$, it would cross the yield locus. Instead, the region $E_3$ is predominantly plastic for $x = .475$, but the value of $p$ is approximately the same as the value of $p$ in region $E_1$, ...
for a comparable period of time. The jump in $p$ labeled $P_1$ is brief at $x = .475$, just as the jump down is at $E_2$.

In what follows, we explore effects of varying $\alpha$, $N$, and $E$. To illustrate the results clearly, we show plots of the variables at a fixed value of $x = .25$. Note that at $x = .25$, the material spends roughly equal time in each of the four regions.

### 6.3.2 Phase Plots with $w$

The four regions are most clearly distinguished in the phase plots of $p$ vs. $w$, as seen in Figure 6.9. This plot is traced out clockwise as time increases. The transition from $E_3$ to $P_1$ is plastic; the other sides of the diamond shape are elastic. Figure 6.10 shows how the plastic state $P_1$ and the elastic state $E_2$ are only brief spikes near the shear band.

### 6.3.3 Long-time Phase Plots

Although the contour plots of section 6.2 appear to show a periodic solution, graphs in phase space over a long time frame are more compelling evidence that the solution is
Figure 6.9: $p$ versus $w$, $x = .25$, $N = 40$, $E = 0.1$, $\alpha = \pi/4$

Figure 6.10: $p$ versus $w$, $x = .475$, $N = 40$, $E = 0.1$, $\alpha = \pi/4$
Figure 6.11: $p$ vs. $q$, $x = .25$, $N = 40$, $E = 0.1$, $\alpha = \pi/4$, over many periods

Figure 6.12: Magnification of Figure 6.11
in fact periodic, and not slowly varying over time or otherwise nonrepeating. Figure 6.11 shows $p$ vs. $q$ over 19,800 seconds. Over that time, the graph is the same as Figure 6.7. The period for the solution is approximately 7 seconds, as can be seen in Figure 6.1. Thus, the solution remains cyclic for more than two thousand periods. Figure 6.11 is in fact composed of many individual data points, although it appears to be a connected graph. In order to resolve the individual points, data points over only 100 periods were used in Figure 6.12.

### 6.4 Effect of Parameter Values on the Periodic Solution

#### 6.4.1 Effect of Shear Modulus $E$

For values of $E$ above a critical value $E_{\text{crit}}$, the steady state shear band is stable. This steady state is marked by the red cross in Figures 6.13 and 6.14. For $E$ below $E_{\text{crit}}$, the periodic solution appears. Further reducing $E$ reduces the amplitude of the oscillation of $p$ and $q$, as is also shown in Figure 6.13. The amplitude of $w$, however, remains fairly constant as $E$ is reduced, as is shown in Figure 6.14. The steady state shear band is approximately
Figure 6.14: Phase plot of $p$ vs. $w$, $N = 40$, $\alpha = \pi/4$.

Figure 6.15: Dependence of period on $E$, $N = 40$, $\alpha = \pi/4$. 
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Figure 6.16: Contour of $\|\tau\|^2$, $N = 40$, $E = .01$, $\alpha = \pi/4$.

at the center of the diamond pattern of the $p$ vs. $w$ phase plot, regardless of $E$.

Reducing $E$ rapidly increases the period of the solution; the dependence is non-linear (Figure 6.15). In a wide range of values for $E$, the solution still exhibits distinct wedge-shaped regions of approximately constant $w$ and $p$, and the transitions between elastic and plastic states shown in Figure 6.16 is very similar to Figure 6.5, even though the value of $E$ is an order of magnitude smaller.

6.4.2 Effect of Parameter $\alpha$

Changing the parameter $\alpha$ can result in the dissolving of the distinct regions $E_1$–$E_3$ and $P_1$, as the Figure 6.17 shows. For $\alpha = \pi/4 - 0.1 \approx 0.685$, the jump in $w$ across the shear band never completely disappears, although it grows and shrinks in strength. Most of the domain is plastic (Figure 6.18), contrasting with the case $\alpha = \pi/4$, for which most of the domain was elastic.

Less drastic changes in $\alpha$ demonstrate how the breakdown of the regions $E_1$–$E_3$ and $P_1$ takes place. Figure 6.19 shows that as $\alpha$ decreases, the changes in values of $p$
Figure 6.17: Contour of $w$, $N = 40$, $E = .1$, $\alpha = \pi/4 - .1$.

Figure 6.18: Contour of $||\tau||^2$, $N = 40$, $E = .1$, $\alpha = \pi/4 - .1$. The broad white space is at yield.
Figure 6.19: Phase plot of $p$ vs. $q$, $N = 40$, $E = 0.1$.

Figure 6.20: Change in the shape of the solution between $\alpha = \pi/4 - .087$ and $\alpha = \pi/4 - .088$, $N = 40$, $E = 0.1$. 
become much less sharp. There is a sudden change in the character of the solution between
\( \alpha = \pi/4 - 0.05 \) and \( \alpha = \pi/4 - 0.1 \); Figure 6.20 shows that this transition is very sharp,
occurring between \( \alpha = \pi/4 - 0.087 \) and \( \alpha = \pi/4 - 0.088 \). Even before the sharp transition,
however, the behavior of the solution is complicated; for example, although in Figure 6.19 it appears that the maximum values of \( p \) and \( q \) increase with decreasing \( \alpha \) before the sharp change, it is clear from comparison with Figure 6.20, which has the same scale, that the maximum \( (p,q) \) of the solution for \( \alpha = \pi/4 - 0.05 \) is larger than that of the solution for \( \alpha = \pi/4 - 0.088 \). As \( \alpha \) decreases, the small elastic oscillations in \( p \) shrink, and the height of the curve diminishes, indicating that less time is spent in the elastic state. (Recall that, in the elastic state, \( q_t = Eb \), so the change in \( q \) over the course of the elastic state is directly proportional to the time spent in that state.)

Figure 6.21 shows the effect of small variations of \( \alpha \) on the \( p \) vs. \( w \) phase plot. The steady state shear band that is stable for \( E > E_{crit} \) (and \( \alpha = \pi/4 \)) is indicated by the green cross. The most noticeable effect of changing \( \alpha \) is the shift in the center of the diamond shape. This shift is the result of a shift in the steady state solution. As \( \alpha \) decreases,
Figure 6.22: Phase plot of $p$ vs. $w$ for lower values of $\alpha$, $N = 40$, $E = 0.1$.

Figure 6.23: Steady state shear band solution as a function of $\alpha$, $N = 40$, $E = 0.1$. 
the steady state solution moves along the yield locus in the direction of increasing \( p \); over the range of \( \alpha \) considered, the steady state value of \( p \) is linear in \( \alpha \) (Figure 6.23). The diamond shaped trajectories in Figure 6.21 are each roughly centered at the corresponding steady state solution. This trend continues in Figure 6.22, but in addition we see that the small loops separate from the corners and eventually vanish. The smooth path of the case \( \alpha = \pi/4 - 0.1 \) suggests a sinusoidal oscillation, which is reflected in the graph of \( w \) over time in Figure 6.24. That figure also shows that changing \( \alpha \) has only a small effect on the period of the solution.

Attempts to analyze the periodic solution have been complicated by its elaborate structure; the possibility that for some parameter values the solution may be linearly approximated is alluring. In light of this, the case \( \alpha = \pi/4 - 0.2 \) is particularly interesting, because the solution with this value of \( \alpha \) appears to be piecewise linear in Figure 6.24. There are small oscillations in \( w \), as can be seen in Figure 6.25; far away from the shock, at \( x = 1 \), the magnitude of \( w \) is so small that the solution is on the same order as these oscillations. Curiously, for \( x = 1 \), the components of stress are very well behaved; they are
Figure 6.25: Velocity $w$ as a function of time for various values of $x$, $\alpha = \pi/4 - .2$, $N = 40$, $E = 0.1$.

Figure 6.26: Stress component $p$ as a function of time for various values of $x$, $\alpha = \pi/4 - .2$, $N = 40$, $E = 0.1$. 
most nonlinear near the shock, as can be seen in Figure 6.26. Nevertheless, for this value of \( \alpha \), the variables appear to be well approximated by piecewise linear functions over most of the domain, which suggests trying to analyze the behavior of fronts between linear solutions might reveal the structure of the periodic solution, for this or a nearby value of \( \alpha \).

### 6.4.3 Effect of Mesh Size

The parameters \( E \) and \( \alpha \), as seen in the previous subsections, can be changed independently of one another. The number of points in the space discretization, \( N \), is not independent of \( E \), since \( E_{\text{crit}} \) decreases as \( N \) increases. This dependence is roughly linear; to first order in \( \Delta x = 1/N \), the case \( \alpha = \pi/4 \) gave us equation (2.1.31): \( E_{\text{crit}} = 4/(a^2N) \). Therefore, in our investigation of the effect of changing \( N \), we use a value of \( E = 0.033 \); this satisfies \( E < E_{\text{crit}} \) for all values of \( N \) we consider.

The effect of changing the mesh size \( N \) on \((w, p, q)\) in phase space is shown in Figures 6.27 and 6.28. The difference in solutions as the mesh is refined is not as dramatic as the changes that resulted from changing the parameters \( E \) and \( \alpha \). The elastic oscillation...
Figure 6.28: Phase plots of $p$ vs. $w$, $\alpha = \pi/4$, $E = 0.033$, $N = 20, 40, 80, 120$.

In the $p$ vs. $q$ graph shown in Figure 6.27 increase in frequency as $N$ increases, but their amplitude decreases. There is an illusion that further mesh refinement might lead to convergence to a nearly piecewise constant solution. However, if the mesh is refined further while $E$ is held constant, eventually $E$ will be greater than $E_{\text{crit}}$, and the steady state shear band will be the stable solution.

### 6.5 Bifurcation diagram

The transition from the steady state shear band solution to the periodic solution is not a smooth one, as Figure 6.29 shows. As the shear modulus drops below $E_{\text{crit}}$, the amplitude of the solution instantaneously jumps to a finite value. After $E_{\text{crit}}$, the dependence of the amplitude on $E$ is generally smooth, although some corners are visible, for example near $E = 0.15$ on the curve for $\alpha = \pi/4 - .16$.

If the periodic solution is used as the initial condition for increasing values of $E$, then we find that the periodic solution continues to be stable until a second critical value of the shear modulus for $E_{\text{return}} > E_{\text{crit}}$. This hysteretic behavior suggests a Hopf bifurcation, and an unstable solution connecting the zero-amplitude steady shear band solution at $E_{\text{crit}}$
Figure 6.29: Peak-to-peak amplitude of the periodic solution at $x = 0.5$ as a function of shear modulus $E$. The size of the mesh is $N = 20$.

and the periodic solution at $E_{\text{return}}$. 

Peak-to-peak amplitude at $x = 0.2$
Chapter 7

Conclusion

The antiplane shear problem is intended to illuminate the key features of continuum models of granular flow. While antiplane shear is a more approachable problem than a general model of granular flow, it still presents complex phenomena, and a challenge to both numerical simulation and analysis.

7.1 Closing Remarks

7.1.1 Hopper Flow

Although the continuum equations of granular flow as they are typically formulated are dynamically ill-posed, engineers have based their designs on simulations of the corresponding steady state models regardless of this mathematical concern. However, attempts to model changes in Rankine states, for example in a cone terminating in a standpipe, are of practical importance. The Hugoniot locus includes shocks from one Rankine state to another, but such shocks do not satisfy the Lax entropy condition. Another condition for admissibility must be found if shocks are to provide the basis for Rankine state-switching solutions.

7.1.2 Elastoplastic Transitions and Periodic Solution

The construction of piecewise linear solutions to elastoplastic equations in [21] can be used to generate approximate solutions to antiplane shear with piecewise linear initial data. Comparison with numerical solution of the antiplane shear equations shows
that the elastoplastic fronts, elastic waves, and plastic waves that form the approximate solution are also the predominant features of the numerical solution. Traveling fronts are also prominent in the periodic solution of antiplane shear which arises for low values of the shear modulus $E$. At a fixed point in space, this periodic solution oscillates through four states of approximately constant $p$ and $w$, three elastic and one plastic. In the conclusion of [19], Schaeffer conjectures that the elastoplastic form of general granular flow models have time-periodic solutions, and that these solutions reflect the experimental fact that flow in silos is often pulsating. The periodic solution in antiplane shear is evidence for the conjecture that more general elastoplastic models may also have periodic solutions.

### 7.2 Future Work

The construction of solutions formed from waves and elastoplastic transitions in Chapter 5 is based on piecewise linear initial data. The periodic solution contains a shear band for part of its period, which approaches a discontinuity as $N \to \infty$. If the construction of piecewise linear solutions can be combined with understanding of how the shear band affects traveling waves, the periodic solution may be completely describable in terms of traveling fronts, using the approach of Chapter 5.

The periodic solution found in antiplane shear suggests that more general elastoplastic models of granular flow may also have periodic solutions; finding such solutions would confirm the conjecture of [19]. An obstacle to such a search is the choice of suitable boundary conditions.

In the long term, the goal of investigating the antiplane shear model is to gain insight into the general continuum equations of granular flow; once the antiplane shear model’s behavior is well understood, we must apply that knowledge as a steppingstone toward better analysis of real-world problems in granular materials.
Bibliography


